ALMOST-EVERYWHERE CONVERGENCE OF BOCHNER–RIESZ MEANS ON HEISENBERG-TYPE GROUPS

by

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Abstract

In this thesis, we prove a result regarding almost-everywhere convergence of Bochner–Riesz means on Heisenberg-type (H-type) groups, a class of 2-step nilpotent Lie groups that includes the Heisenberg groups H_m. We broadly follow the method developed by Gorges and Müller [24] for the case of Heisenberg groups, which in turn extends techniques used by Carbery, Rubio de Francia and Vega [8] to prove a result regarding Bochner–Riesz means on Euclidean spaces. The implicit results in both papers, which reduce estimates for the maximal Bochner–Riesz operator from L^p to weighted L^2 spaces and from the maximal operator to the non-maximal operator, have been stated as stand-alone results, as well as simplified and extended to all stratified Lie groups. We also develop formulae for integral operators for fractional integration on the dual of H-type groups corresponding to pure first and second layer weights on the group, which are used to develop 'trace lemma' type inequalities for H-type groups. Estimates for Jacobi polynomials with one parameter fixed, which are relevant to the application of the second layer fractional integration formula, are also given.

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CHAPTER 1

INTRODUCTION

The study of Bochner–Riesz means is a classical topic in harmonic analysis. Recall that the Bochner–Riesz means of any function $f \in \mathscr{S}(\mathbb{R}^n)$, where $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz class on $\mathbb{R}^n$, are defined, for $r, \lambda \in (0, \infty)$, by

$$T_r^\lambda f := (1 - r\Delta)_+^\lambda f = [(1 - 4\pi r|\cdot|^2)_+^\lambda \widehat{f}]^\vee,$$

where $\Delta := -\sum_{j=1}^n \partial_j^2$ is the Euclidean Laplacian defined on $\mathscr{S}(\mathbb{R}^n)$ and where g_+ denotes the positive part of the function g (that is, $g_+(x) := \max\{g(x), 0\}$). Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we may then extend T_r^λ to an operator defined for $f \in L^p(\mathbb{R}^n)$. The associated maximal Bochner–Riesz operator is then given by

$$T_*^\lambda f := \sup_{r>0} |(1 - r\Delta)_+^\lambda f|.$$

A question of interest is the range of λ for which T_r^λ and T_*^λ are bounded on $L^p(\mathbb{R}^n)$; the Bochner–Riesz conjecture (respectively, maximal Bochner–Riesz conjecture) are conjectures on what the best possible range of λ is such that these operators are bounded. It is conjectured that, for $0 < \lambda \leqslant \frac{1}{2}(n-1)$, the operator T_r^λ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\frac{n-1}{n}\left(\frac{1}{2} - \frac{\lambda}{n-1}\right) < \frac{1}{p} < \frac{n+1}{n}\left(\frac{1}{2} + \frac{\lambda}{n-1}\right). \tag{1.0.1}$$

We refer the reader to Chapter IX of [55], [16] and [57] for more information and background on this problem. The case $n = 2$ of the Bochner–Riesz conjecture has been proved by Carleson and Sjölin [9], while for $n \geqslant 3$ the problem remains open, albeit with some progress made. The best result known to this author is by Bourgain and Guth [4], where the conjecture is proved for $\max\{p, p'\} \geqslant \tilde{p}$, where

$$\tilde{p} = 2 + \frac{12}{4d - 3 - k}, \quad \text{where } d \equiv k (\text{mod } 3), k \in \{-1, 0, 1\}.$$

For $p \geqslant 2$ it is expected that T_*^λ is bounded in the same range as T_r^λ and this has been shown by Carbery to be true for $n = 2$ [7]. Boundedness of T_*^λ for $p \geqslant 2$ and $n \geqslant 3$ has only been shown in reduced ranges. Christ proved boundedness with the additional assumption $\lambda \geqslant (n-1)/2(n+1)$ in [14] with improvements made by Lee ([36], [37]). In particular, in [37], boundedness of T_*^λ on $L^p(\mathbb{R}^n)$ is shown for

$$\frac{n - 2\lambda}{2n} < \frac{1}{p} \leqslant \max\left\{\frac{1}{\tilde{p}}, \frac{n}{2(n+2)}\right\}$$

where now

$$\tilde{p} = 2 + \frac{12}{4d - 6 - k}, \quad \text{where } d \equiv k (\text{mod } 3), k \in \{0, 1, 2\}.$$

A weaker result than L^p boundedness of T_*^λ is that of almost-everywhere convergence of $T_r^\lambda f(x)$ to $f(x)$ as $r \to 0$ for all $f \in L^p$. While the maximal Bochner–Riesz conjecture remains open, almost-everywhere convergence has been demonstrated in the range (1.0.1) for $p \geqslant 2$. We state this result by Carbery, Rubio de Francia and Vega [8] here.

Theorem (Carbery, Rubio de Francia and Vega)**.** *Let $\lambda > 0$ and $2 \leqslant p \leqslant \infty$. Suppose that*

$$\frac{n-1}{n}\left(\frac{1}{2} - \frac{\lambda}{n-1}\right) < \frac{1}{p} \leqslant \frac{1}{2}.$$

Then for all $f \in L^p(\mathbb{R}^n)$ we have that $T_r^\lambda f(x)$ converges to $f(x)$ almost-everywhere as $r \to 0$.

As the Laplacian on $\mathbb{R}^n$ is a positive self-adjoint operator, it has a spectral resolution which may be used to define Bochner–Riesz operators. As such, we may extend the notion of Bochner–Riesz operators to other positive self-adjoint operators on $L^2(X)$ for some measure space X. In particular, we will be concerned with (homogeneous left-invariant) sub-Laplacians on stratified Lie groups. Similar almost-everywhere convergence results can then be shown for these new operators. For instance, Gorges and Müller [24] extend the result of Carbery, Rubio de Francia and Vega [8] to the setting of Heisenberg groups H_m (which may be identified with $\mathbb{C}^m \times \mathbb{R}$). Similarly to the Euclidean case, we define the Bochner–Riesz means on $f \in L^p(H_m)$ as $T_r^\lambda f := (1-rL)_+^\lambda f$, where L is the sub-Laplacian on H_m. Gorges and Müller then show the following.

Theorem (Gorges and Müller)**.** *Consider a Heisenberg group H_m. Set $Q = 2m+2$ and $D = 2m+1$. Let $\lambda > 0$ and $2 \leqslant p \leqslant \infty$. Suppose that*

$$\frac{Q-1}{Q}\left(\frac{1}{2} - \frac{\lambda}{D-1}\right) < \frac{1}{p} \leqslant \frac{1}{2}. \tag{1.0.2}$$

Then for all $f \in L^p(H_m)$, we have that $T_r^\lambda f(z,u)$ converges almost-everywhere to $f(z,u)$ as $r \to 0$.

We remark that the quantities represented by Q and D, namely the homogeneous and topological dimension of H_m respectively, make sense for any stratified Lie group and are both equal to n for $\mathbb{R}^n$. Setting $Q = D = n$ in (1.0.2) recovers the condition of the Carbery, Rubio de Francia and Vega result. We also have the following result by Mauceri and Meda [45], which is valid for any stratified group and concerns boundedness of the maximal Bochner–Riesz operator on such groups.

Theorem (Mauceri and Meda)**.** *Let G be a stratified group of homogeneous dimension Q and L a sub-Laplacian on G. Let $\lambda > 0$ and $2 \leqslant p \leqslant \infty$. If*

$$\frac{1}{2} - \frac{\lambda}{Q-1} < \frac{1}{p} \leqslant \frac{1}{2}. \tag{1.0.3}$$

then the operator T_*^λ defined by

$$T_*^\lambda f := \sup_{r>0} |T_r^\lambda f|$$

extends to a bounded operator on L^p. In particular, for all $f \in L^p(G)$, we have that $T_r^\lambda f(x)$ converges almost-everywhere to $f(x)$ as $r \to 0$.

Our intention is to extend the work of Gorges and Müller to apply to a more general class of groups called Heisenberg-type (henceforth H-type) groups. This is a class of Lie groups that includes H_m and may be identified with $\mathbb{C}^m \times \mathbb{R}^n$. We will further extend some of the techniques used by Gorges and Müller to prove their result, simplifying their proofs and generalising them to any stratified group. These generalisations are stated as standalone results that may be of separate interest.

The main result is a theorem similar to that of Gorges and Müller, which gives almost-everywhere convergence in a slightly reduced range of p but is valid for all H-type groups (where, using exponential coordinates, we may identify an H-type group G with $\mathbb{C}^m \times \mathbb{R}^n$, where the factors $\mathbb{C}^m$ and $\mathbb{R}^n$ correspond respectively to the 'first layer' and 'second layer' of the Lie algebra $\mathfrak{g}$).

Theorem 1.1. *Let G be an H-type group. Let $\lambda > 0$ and $2 \leqslant p \leqslant \infty$. Let $Q = 2m + 2n$ be the homogeneous dimension of G and $D = 2m + n$ be the Euclidean dimension of G. If*

$$\frac{Q - \frac{2}{3}}{Q}\left(\frac{1}{2} - \frac{\lambda}{D-1}\right) < \frac{1}{p} \leqslant \frac{1}{2}$$

then for all $f \in L^p(G)$

$$T_r^\lambda f(z,u) \to f(z,u) \text{ almost-everywhere as } r \to 0.$$

The proof is given in Chapter 8. Our techniques also yield a 'mixed L^p' result. Given that we may identify G with $\mathbb{C}^m \times \mathbb{R}^n$, we then define $L^{(p,q)}(G) := L^p(\mathbb{C}^m, L^q(\mathbb{R}^n))$. By considering integral operators corresponding to multiplication on the group side by a pure first layer weight, we arrive at the following result.

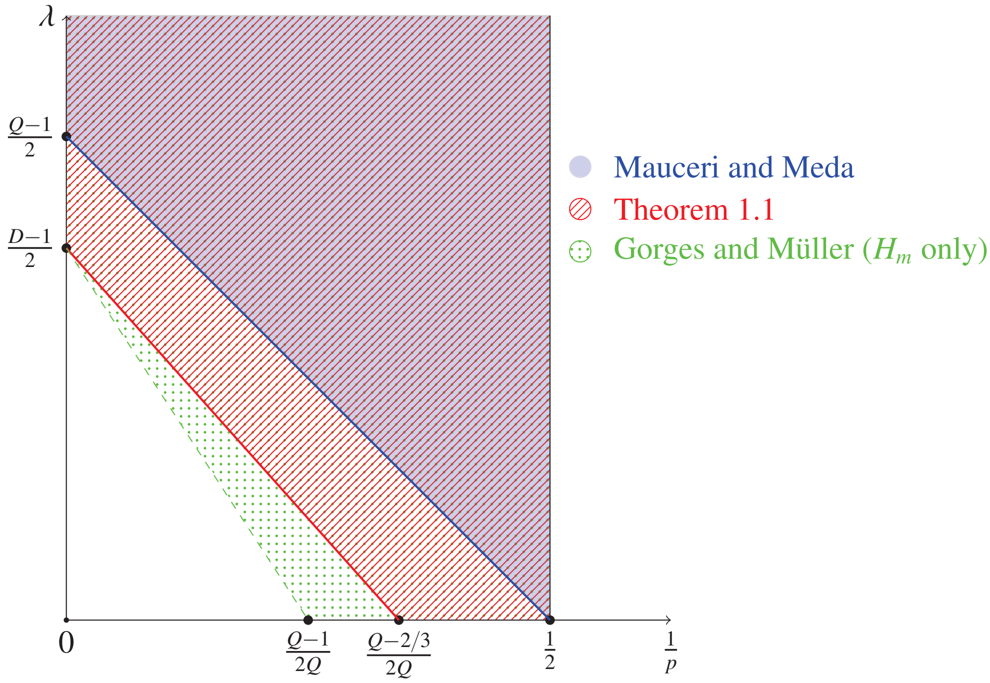


Figure 1.1: Almost-everywhere convergence on H-type groups occurs in the shaded region (which extends upwards infinitely in the λ direction). The diagram also depicts the results of Gorges and Müller (valid for H_m only) and Mauceri and Meda.

Theorem 1.2. *Let G be an H-type group, let $\lambda > 0$ and $2 \leqslant p \leqslant \infty$. If*

$$\frac{1}{2} - \frac{\lambda}{D-1} < \frac{1}{q} \leqslant \frac{1}{2}$$

and

$$\left(\frac{2m-1}{2m}\right)\frac{1}{q} < \frac{1}{p} \leqslant \frac{1}{q}$$

then for all $f \in L^{(p,q)}(G)$ we have

$$T_r^\lambda f(z,u) \to f(z,u) \text{ almost-everywhere as } r \to 0.$$

Observe that, while these results improve upon the almost-everywhere convergence result of Mauceri and Meda by allowing us to replace the homogeneous dimension Q in (1.0.3) with the topological dimension D (by setting $p = q$ in Theorem 1.2), Mauceri and Meda's result proves something stronger, namely L^p boundedness of the maximal Bochner–Riesz operator T_*^λ. This improvement in the range of p (replacing Q by D) can also be obtained for L^p boundedness of T_*^λ and in more general groups than just H-type. In general, it is known that an improvement can be found for all 2-step Lie groups. We refer to Lemma 2.7 for more details of the constant $\eta(L)$ described in this result, but note here that for H-type groups it is known that $\eta(L) = D$, while for general 2-step Lie groups Martini and Müller [41] have proven that $D \leqslant \eta(L) < Q$. The constant $\eta(L)$ arises from considerations of the boundedness of operators $m(L)$ on $L^p(G)$ for $1 < p < \infty$; for a stratified Lie group G, a sub-Laplacian L thereon and a Borel function m, the operator $m(L)$ will be bounded on $L^p(G)$ for $1 < p < \infty$ provided the multiplier m satisfies a Mikhlin–Hörmander type condition for some Sobolev exponent $s > \frac{\eta(L)}{2}$ (compare [41]).

Theorem 1.3. *Let G be a stratified group, L be a sub-Laplacian and let $\eta(L)$ be the constant described in Lemma 2.7. Let $\lambda > 0$ and $p \geqslant 2$. If*

$$\frac{1}{2} - \frac{\lambda}{\eta(L) - 1} < \frac{1}{p} \leqslant \frac{1}{2}$$

then

$$\|T_*^\lambda\|_{L^p \to L^p} < \infty$$

and so for all $f \in L^p(G)$,

$$T_r^\lambda f(x) \to f(x) \text{ almost everywhere as } r \to 0.$$

We refer to Chapter 3 for the proof.

Methods other than the techniques we will use have also yielded results of this nature. For instance, Theorem A of a paper by P. Chen, S. Lee, A. Sikora and L. Yan [12] shows that the maximal Bochner–Riesz operator T_r^* is bounded on L^p for

$$2 \leqslant p < q'$$

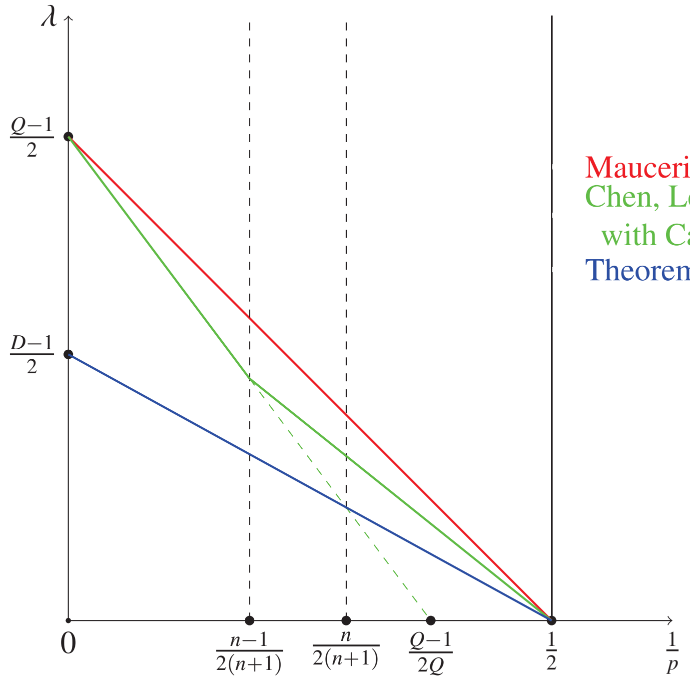


Figure 1.2: A diagram showing various L^p boundedness regions for T_*^λ for an H-type group. The operator T_*^λ is bounded in the region between the λ-axis, the black line $\frac{1}{p} = \frac{1}{2}$ and above the solid coloured line depicting the estimate used.

provided that

$$\frac{Q-1}{Q}\left(\frac{1}{2} - \frac{\lambda}{Q-1}\right) < \frac{1}{q'} \leqslant \frac{1}{2}$$

(cf. the result of Carbery, Rubio de Francia and Vega and note that this is also an improvement on the result of Mauceri and Meda) and provided that a suitable $L^q \to L^{q'}$ restriction estimate can be proved. Such restriction estimates were proved by V. Casarino and P. Ciatti [10] with the additional constraint that

$$\frac{1}{q'} \leqslant \frac{n-1}{2(n+1)}. \tag{1.0.4}$$

Firstly, note that if $n = 1$ (meaning that G is a Heisenberg group) then this condition is never verified except when $\frac{1}{q'} = 0$, that is at $q = 1, q' = \infty$. This matches the result of Müller, [47], which states that the only restriction estimate of this type available is the trivial $L^1 \to L^\infty$ one.

Secondly, we note from [10] that the constraint (1.0.4) is sharp. Unfortunately, in this region of validity of the restriction estimate, the result given by Chen, Lee, Sikora and Yan is not as good as Theorem 1.3. By comparing the results, we can show that the result of Chen, Lee, Sikora and Yan is never better than Theorem 1.3 at or before the restriction constraint (1.0.4). One can in fact show that the Chen, Lee, Sikora and Yan estimate becomes superior only for $\frac{1}{p} > \frac{n}{2(n+1)}$. Since (1.0.4) is sharp, then it would seem that methods other than restriction are more suitable here.

In both the paper by Carbery et al. and the paper by Gorges and Müller, the key result is obtained by considering instead L^p to L^2_{loc} boundedness of the maximal Bochner–Riesz operator. Furthermore, we need only prove such a boundedness result for the 'local' maximal Bochner–Riesz operator which we define as

$$T^\lambda_\bullet f := \sup_{0<r<1} |T^\lambda_r f|. \tag{1.0.5}$$

In particular, for a stratified group G, if we can show that $\|\chi_K T^\lambda_\bullet\|_{L^p\to L^2} < \infty$ for all compact sets $K \subseteq G$, then this is sufficient to prove almost-everywhere convergence of $T^\lambda_r f(x)$ to $f(x)$ as $r \to 0$. The proof is a standard 3ϵ argument which we reproduce in Chapter 4 as Lemma 4.1 for the reader's convenience.

Rather than considering the whole maximal operator $T^\lambda_\bullet$, we consider a classical decomposition of the multiplier. In particular, as found in, for example, [8], for $\zeta > 0$ and $\mathbb{D} = \{2^{-k} : k \in \mathbb{N}_0\}$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), we may write

$$(1-\zeta)^\lambda_+ = \sum_{\delta\in\mathbb{D}} \delta^\lambda m_\delta(\zeta), \tag{1.0.6}$$

where for every $j \in \mathbb{N}_0, \delta \in \mathbb{D}$ the functions m_δ satisfy

$$\operatorname{supp}(m_\delta) \subseteq [1-\delta, 1] \text{ and } \|m^{(j)}_\delta\|_\infty \lesssim \delta^{-j}. \tag{1.0.7}$$

We remark that there is an abuse of notation here: the functions m_δ in (1.0.6) depend on λ, but satisfy (1.0.7) with implicit constants independent of λ. For this reason, other authors suppress the dependence on λ of the functions m_δ in their notation, and we follow this convention.

We also define the (global) maximal operator

$$M_\delta^* f := \sup_{r>0} |m_\delta(rL) f| \tag{1.0.8}$$

and the local maximal operator

$$M_\delta^\bullet f := \sup_{0<r<1} |m_\delta(rL) f| \tag{1.0.9}$$

corresponding to the above dyadic decomposition. By the triangle inequality, if we can prove that the operators $M_\delta^\bullet$ (respectively M_δ^*) are bounded such that the operator norm doesn't grow too rapidly, then we can prove boundedness of the operator $T_\bullet^\lambda$ (respectively T_*^λ).

Thus, we will be concerned with proving an estimate of the form

$$\|\chi_K M_\delta^\bullet f\|_{L^p \to L^2} \lesssim \delta^A, \tag{1.0.10}$$

for some $A \in \mathbb{R}$ which may depend on p, on the operators $M_\delta^\bullet$, where again, the implicit constants depend only on those in (1.0.7) and on the choice of group G and sub-Laplacian L. Such an estimate will imply boundedness of $T_\bullet^\lambda$ provided A is not so negative that the exponent λ of δ in (1.0.6) cannot compensate for it (i.e., so long as $\lambda + A > 0$). Furthermore, in order to obtain Theorem 1.1 by interpolation, it suffices to consider just the 'vertices' of the trapezoid depicted in Figure 1.1. Among these, the vertex on the vertical axis, which corresponds to the estimate

$$\|\chi_K M_\delta^\bullet f\|_{L^\infty \to L^2} \lessapprox \delta^{-(D-1)/2}$$

(where notation $\lessapprox$ denotes $\lesssim C(\epsilon)\delta^{-\epsilon}$ for some non-negative function C of ϵ and arbitrarily small $\epsilon > 0$) can be dealt with in a relatively standard way using available estimates for functions of a sub-Laplacian (these estimates in fact yield an $L^\infty \to L^\infty$ estimate for the 'global' maximal operator M_δ^*, from which follow both this estimate on

$M_\delta^\bullet$ and the stronger L^p boundedness result of Theorem 1.3). The estimate corresponding to the vertex on the horizontal axis,

$$\|\chi_K M_\delta^\bullet f\|_{L^{2Q/(Q-2/3)}\to L^2} \lessapprox 1,$$

follows from weighted L^2 estimates for $M_\delta^\bullet$ and requires instead a much more delicate analysis. Through this thesis, we will reduce the problem of proving such an estimate down to the problem of proving a better 'Sobolev trace' inequality given by Theorem 7.1.

To explain the idea, in addition to the sub-Laplacian L, we also fix an orthonormal basis $U_1, \ldots, U_n$ of the second layer $\mathfrak{g}_2$ of the Lie algebra of the H-type group G, which has coordinates $(z, u) \in \mathbb{C}^m \times \mathbb{R}^n$. As discussed in [48], the operators L, and U_j/i all commute, so they admit a joint spectral resolution which allows us to make sense of expressions such as $m(L, U_1/i, \ldots, U_n/i)$. We define the pseudo-differential operator

$$\Lambda := (-(U_1^2 + \ldots + U_n^2))^{1/2} \tag{1.0.11}$$

and the spectral cut-off operator $M_{\delta,j}$ by

$$M_{\delta,j} := \chi_{[1-\delta,1]}(L)\chi_{[2^j,2^{j+1})}(2\pi L/\Lambda).$$

We wish to prove, for $\delta \leqslant 1/4$ and integers $1 \leqslant j \leqslant J$, such that $2^{J-1} \leqslant 10\delta^{-1} \leqslant 2^J$, the estimate

$$\|M_{\delta,j}f\|_2^2 \lessapprox (2^{-j}\delta)^{1/3}\|f\|^2_{L^2(1+|\cdot|_K^{2/3})}, \tag{1.0.12}$$

where $|(z, u)|_K = (|z|^4 + 16|u|^2)^{1/4}$ ($|z|$ denotes the usual norm on Euclidean/complex spaces).

Theorem 7.1 is a minor modification of this inequality, made for technical reasons. A version of this statement (with different exponents on the constant and weight) also appears in Gorges and Müller's paper as Lemma 7, arising as a replacement on Heisenberg groups for the Euclidean estimate

$$\|\chi_{[1-\delta,1]}(\Delta)f\|_2^2 \lessapprox \delta\|f\|_{L^2(|\cdot|)}^2. \tag{1.0.13}$$

The method of Carbery, Rubio de Francia and Vega reduces the proof of the Euclidean case to an estimate such as this. It is this method that was adapted by Gorges and Müller for use on Heisenberg groups (see Figure 1.3 for a graphical comparison between applying this method to $\mathbb{R}^n$ and an H-type group G).

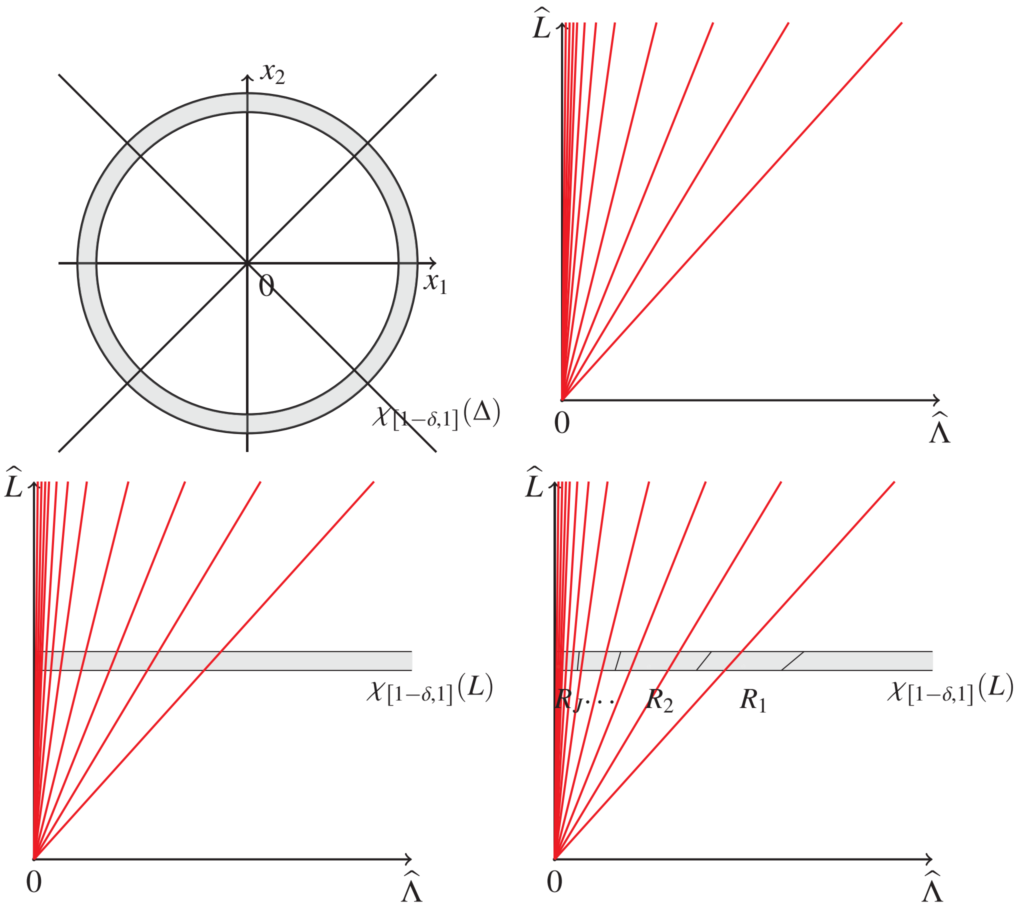
Let us briefly recall how one may prove (1.0.13). If we take the Fourier transform of the contents of the norms on both sides, then by Plancherel's Theorem the left-hand side of (1.0.13) becomes the square of the L^2 norm of the restriction of $\widehat{f}$ to the unit sphere, thickened inwards by a width $\simeq \delta$, which leads to the δ constant appearing. In particular, the integral over this annulus can be bounded above by a Sobolev norm of order $\frac{1}{2}$ by Lemma 3 of [8] with constant $\simeq \delta|\ln(\delta)|$, which gives (1.0.13).

The method used by Gorges and Müller to prove their Lemma 7 involved considering negative and fractional powers of a difference-differential operator defined on the Fourier-dual space to the Heisenberg group which corresponds on the group side to the multiplication operator $f(z,u) \mapsto (|z|^2 - 4iu)f(z,u)$, and in Chapter 7 we attempt to follow a similar idea in order to prove Theorem 7.1 (with $(|z|^2 - 4iu)$ replaced by $|(z,u)|_K^4 = |z|^4 + 16|u|^2$, since the former no longer makes sense for $n > 1$). In the case of a Heisenberg group, the method employed by Gorges and Müller to define these fractional powers is to solve a first-order ordinary differential equation to obtain a Green's function for the difference-differential operator in question. The obtained Green's function is then modified to give an integral kernel for arbitrary powers of the difference-differential operator. The integral operator that arises from this has a polynomial kernel which can be manipulated to obtain the desired estimates.

The 'Green's Function' method appears to break down when attempting it on a more general H-type group. Instead, we calculate integral kernels for such operators directly from their definition in terms of weights and the group Fourier transform by exploiting,

Figure 1.3

Diagram demonstrating the similarity of the idea of using trace lemmas. On $\mathbb{R}^n$, the function $\widehat{\chi_{[1-\delta,1]}(\Delta)f} = \chi_{[1-\delta,1]}(|\xi|^2)\hat{f}(\xi)$ is a function of $\mathbb{R}^n$ supported on a sphere of thickness δ. On G, $\widehat{M_{\delta,j}f}$ (where $M_{\delta,j} = R_j\chi_{[1-\delta,1]}(L)$ with $R_j = \chi_{[2^j,2^{j+1})}(2\pi L/\Lambda)$) instead is a function of $(\mu,k) \in \mathbb{R}^n \times \mathbb{N}_0$ supported on the 'Heisenberg fan' (depicted by the red lines) where $\widehat{M_{\delta,j}}$ becomes a cutoff in k and $|\mu|$.
We also observe that, in the Euclidean case, we could use only trace lemmas that use 'partial weights'; e.g., on $\mathbb{R}^2$, by using only the weight $|x_1|$ in the east and west quadrants and $|x_2|$ in the north and south quadrants, the trace lemma arising from using the full weight $|x|$ on the whole annulus may be obtained. Similarly, on G, we use a trace lemma which uses only a first layer weight in the region defined by the cutoff $M_{\delta,J}$ and a trace lemma which uses a second layer weight in the regions defined by $M_{\delta,j}$ for $j < J$.



Decompositions of Fourier transforms of a function amongst other things, identities for special functions (in particular, Laguerre and Jacobi polynomials). Once explicit formulae for these integral kernels are known, then, if we assume that the function on the group f is radial (that is, depends only on $|z|, u$), then the proof of the 'trace lemma' (1.0.12) essentially reduces to an application of Schur's test. For general functions f, a more sophisticated technique is required. This technique is based on complex interpolation and is analogous to the method used by Gorges and Müller.

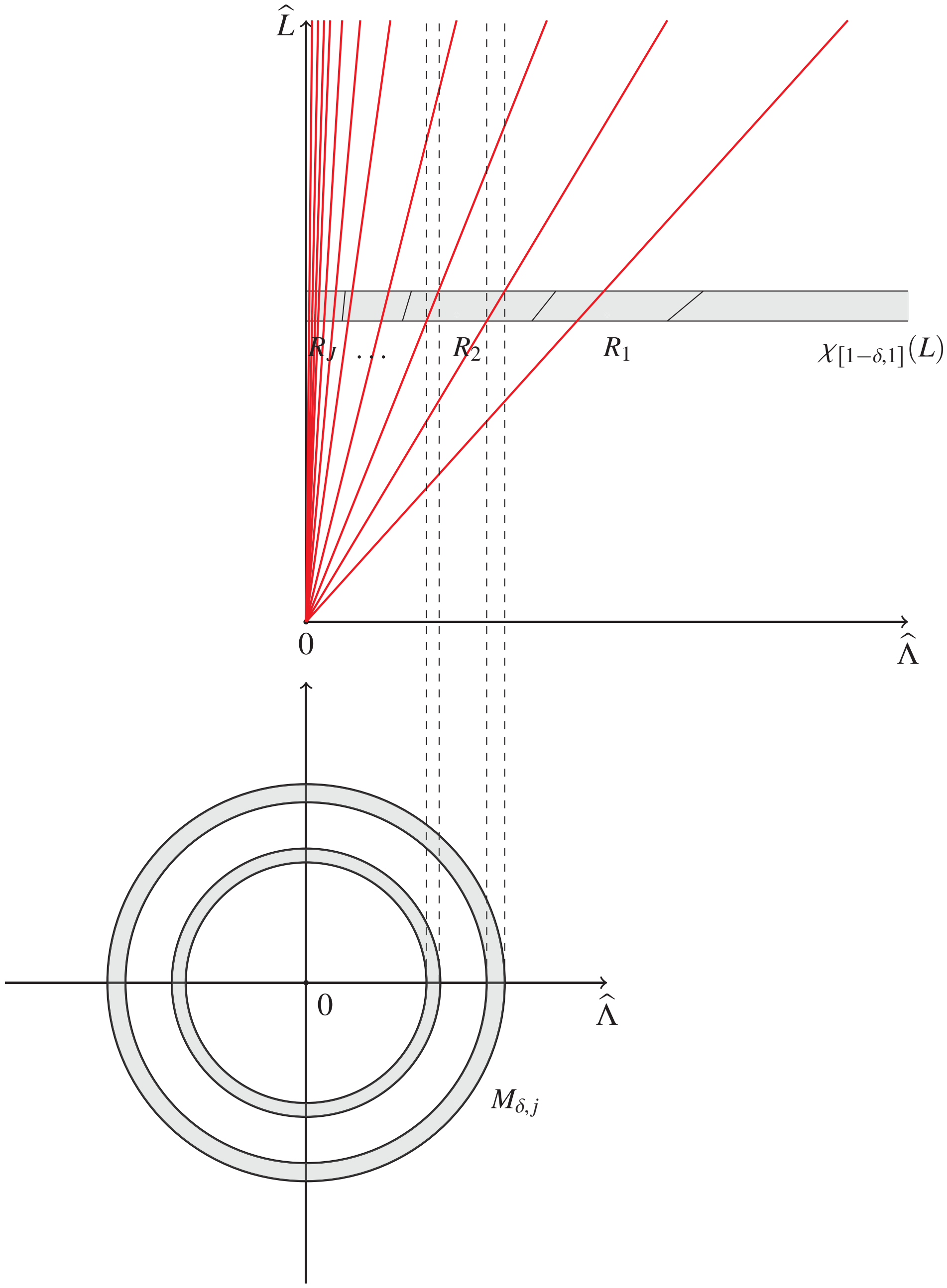
While our method can be used to easily recover the formula obtained by Gorges and Müller in the case of Heisenberg groups, when the second layer has dimension $n > 1$, the formulae for negative fractional powers of the full weight $|\cdot|_K$ become significantly more complicated and, in particular, are harder to estimate. For this reason, noting that $|(z, u)|_K \simeq |z| + |u|^{1/2}$, we instead consider the 'fractional integration operators' corresponding to multiplication on the group side by negative powers of $|z|$ and $|u|$, i.e., pure first and second layer weights. While the resulting formulae remain substantially more complicated than those used by Gorges and Müller in the H_m case, we nevertheless manage to estimate them and use them to deduce (1.0.12).

Discussed in Section 7.2 is work on deriving and obtaining estimates from such kernels related to using a pure first layer weight. These kernels are significantly simpler to calculate and allow us to obtain our mixed L^p space result, but the resulting pure L^p estimate, found by setting $p = q$ in Theorem 4.5, does not exceed the 'Mauceri and Meda type' result given by Theorem 1.3.

As can be seen from Figure 1.4, the cutoff $M_{\delta,j}$ can be thought of as producing a number of disjoint annuli on the 'Euclidean dual' of the second layer. As such, we attempted to use purely Euclidean methods (specifically, a refinement of Lemma 3 of Carbery, Rubio de Francia and Vega [8] where we consider multiple annuli at once instead of a single annulus) to solve the problem. However, it turns out that these methods are not sufficient to obtain an estimate of the type (1.0.12) for the full range $1 \leqslant j \leqslant J$. Instead, integral formulae are found that allow us to calculate explicitly the integral kernel for fractional integration involving second layer weights in terms of Jacobi poly-

Figure 1.4

While $M_{\delta,j}$ consists of cutoffs on the Fourier transform side in k and $(2k+m)|\mu|$, combining both together and disregarding k produces a cutoff in μ which is supported in annuli (in the μ variable only, corresponding to a Euclidean Laplacian on the second layer) which, for $j < J$, are disjoint.



Demonstration of how annuli emerge from Fourier cutoffs nomials. While these provide a full result, it should be noted that these formulae are trickier to estimate, in particular requiring a number of estimates on Jacobi polynomials which had to be collected from the literature. Furthermore, some of these estimates were not initially in the right form for us to use. The developments of purely Euclidean methods are included as they may be of independent interest, and they demonstrate clearly the idea of using Schur's test. Comparing these results also shows that using the full 'joint spectral cut-off' $M_{\delta,j}$ is more efficient than 'neglecting' a part of this to reduce to a Euclidean problem.

In Chapter 2, we recall the definitions of stratified (and in particular H-type) groups and discuss important features of analysis on them such as the standard measure and distances, the definition of sub-Laplacians and the functional calculus of them and some A_2 theory. We then define the group Fourier transform in the particular case of H-type groups, demonstrate a number of key features of how it interacts with convolution and simplify it in the case of radial functions. Finally, we introduce a number of weights we will be working with and see how they interact with convolution (so-called Leibniz rules) and the Fourier transform.

In Chapter 3, we prove Theorem 1.3, which uses some more recent developments on 2-step stratified groups to improve upon the result of Mauceri and Meda in this case.

In Chapter 4 we motivate the study of weighted L^2 estimates of maximal operators $M_\delta^\bullet := \sup\limits_{0<s<1} |m_\delta(sL)|$ coming from the decomposition of the Bochner–Riesz multiplier into pieces $m_\delta(L)$ by demonstrating that, if we can find particular weighted L^2 estimates, then these yield the almost-everywhere convergence results. The analogous results for mixed L^p spaces are included here, and both of these results that reduce the problem to weighted L^2 estimates are given in terms of an arbitrary stratified group and sublinear operator.

In Chapter 5 we reduce this problem further, by showing that, for a certain class of weights w which will include the ones we will use, the weighted L^2 estimates of the maximal operators $M_\delta^\bullet$ can be obtained from estimating the weighted norm of just

$m_\delta(L)$, rather than the maximal version. Specifically, we show

$$\|M_\delta^\bullet\|^2_{L^2(w)\to L^2(w)} \lessapprox (1 + \|m_\delta(L)\|_{L^2(w)\to L^2(w)})(1 + \|\tilde{m}_\delta(L)\|_{L^2(w)\to L^2(w)}),$$

where $\tilde{m}_\delta$ is defined using m_δ and also satisfies (1.0.7). As in the previous chapter, the results in this chapter use only general theory of stratified groups, rather than structure specific to H-type groups, and so this reduction is valid for analysing the Bochner–Riesz multiplier defined on any stratified group. In particular, the work in this chapter simplifies a similar discussion in the work of Gorges and Müller [24]. While the idea of reducing from estimates on the maximal operator to those for the nonmaximal operators is already present in both the works of Carbery, Rubio de Francia and Vega [8] and Gorges and Müller [24], an explicit estimate as above does not seem to appear in either work. This estimate, which may be of independent interest, allows us to greatly streamline the ”reduction-to-trace-lemma” argument as presented in Gorges and Müller's work.

In Chapter 6, we show how the weighted estimates of the operator norm of $m_\delta(L)$ follows from a 'trace lemma' as discussed above. These trace lemmas are finally discussed in Chapter 7. As noted previously, while the main result of this chapter is to prove Theorem 7.1, this chapter includes certain 'inferior' results, as they may be of independent interest (work using purely Euclidean methods) or they demonstrate the main ideas used to prove Theorem 7.1 in a simpler context (work using pure first layer weights and work assuming that functions are radial).

Chapter 8 then explains how the key results proven throughout link together to prove Theorems 1.1 and 1.2.

We conclude with Chapter 9, which contains the results regarding estimates of Jacobi polynomials that are used in the work on second layer weights in Chapter 7. The first result of this chapter is a lemma showing how such polynomials arise from an integral involving two Laguerre polynomials against an exponential and power weight. A number of useful, uniform results found in existing literature precede a more delicate pointwise result, derived from an asymptotic expansion of Jacobi polynomials in terms of Bessel functions.

Appendix A includes a number of results generally found elsewhere in the literature which do not apply specifically to stratified groups or the functional calculus of sub-Laplacians thereon, which are reproduced for the reader's convenience.

1.1 Notation and Conventions

We briefly summarise some notation and conventions that will be of standard use in this thesis. The letter 'G' will be reserved for the group under consideration at the time of its use. The symbol 'L' will be reserved for sub-Laplacians, while 'D' and 'Q' will respectively be reserved for denoting the topological/Euclidean and homogeneous dimensions of the group G under consideration, respectively (these will be defined explicitly in the following chapter, cf. (2.1.4), (2.1.2) and (2.1.5)).

We adopt the convention that $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of strictly positive integers and that $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ is the set of non-negative integers. We also use the convention that $\mathbb{R}^+ = (0, \infty)$ is the set of strictly positive real numbers and that $\mathbb{R}_0^+ = [0, \infty)$ is the set of non-negative real numbers. We will denote the complex conjugate of a complex number z by $\bar{z}$.

The symbol 'δ' will be reserved for an element of the set $\mathbb{D} = \{2^{-k} : k \in \mathbb{N}_0\}$, while '$m_\delta$' will denote one of the functions defined by (1.0.6) and (1.0.7).

For positive quantities A, B we will write '$A \lesssim B$' and say that 'A is majorised by B' or 'B majorises A' to mean that there exists a non-negative constant C such that $A \leqslant CB$ (analogously $A \gtrsim B$ to mean $A \geqslant CB$). We shall write '$A \simeq B$' to mean that both $A \lesssim B$ and $A \gtrsim B$. We may write $A \lesssim_\beta B$ to mean $A \leqslant C(\beta)B$ where $C > 0$ has dependence on β (and similarly for '$\simeq$'). By convention, such constants C will not depend on δ with any such dependence being written explicitly instead. The exception to this is if we can prove an estimate of the form $A \lesssim_\epsilon \delta^{-\epsilon} B$ for arbitrarily small $\epsilon > 0$, in which case we shall instead write $A \lessapprox B$. If there is additional dependence on δ in the bound which is independent of ϵ, it will be written explicitly, as before. Since the function $\ln(x)$ grows slower than any positive power of x, then $A \lesssim |\ln(\delta)|$ implies that $A \lessapprox B$.

Given two vector spaces X, Y and an operator $T : X \to Y$, we shall denote by '$T^\dagger$' the adjoint operator $T^\dagger : Y \to X$. Similarly, if we have a dual pair $(X, Y, \langle \cdot, \cdot \rangle)$ and an operator $T : X \to X$, we shall denote by '$T^\dagger$' the adjoint operator $T^\dagger : Y \to Y$ such that $\langle Tx, y \rangle = \langle x, T^\dagger y \rangle$ for all $x, \in X, y \in Y$.

CHAPTER 2

ANALYSIS ON H-TYPE GROUPS

2.1 Stratified Lie Groups

We briefly recall a number of standard definitions and results. For details, we refer the reader to [22]. We first recall the definition of a stratified Lie group, and do so by starting with the Lie algebra. Given a Lie algebra $\mathfrak{g}$ with Lie bracket $[\cdot,\cdot]$, we say that $\mathfrak{g}$ is graded with step $k \in \mathbb{N}$ if there exist subspaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ (called layers) such that

$$\mathfrak{g} = \bigoplus_{j=1}^{k} \mathfrak{g}_j$$

and where $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$ and $\mathfrak{g}_a = \{0\}$ for all $a > k$. Moreover, we say that $\mathfrak{g}$ is stratified if $[\mathfrak{g}_a, \mathfrak{g}_1] = \mathfrak{g}_{a+1}$, so that $\mathfrak{g}_1$ generates the Lie algebra.

We then endow $\mathfrak{g}$ with a group structure. Recall, for a Lie group G with Lie algebra $\mathfrak{g}$, given the exponential map $\exp : \mathfrak{g} \to G$, the Baker–Campbell–Hausdorff formula is given by

$$\exp(x)\exp(y) = \exp(x + y + \tfrac{1}{2}[x, y] + \ldots).$$

We then define the group multiplication law on $\mathfrak{g}$ as

$$xy = x + y + \tfrac{1}{2}[x, y] + \ldots, \text{ for all } x, y \in \mathfrak{g},$$

where the right-hand side is given by the Baker–Campbell–Hausdorff formula. Note that since the Lie algebra $\mathfrak{g}$ is stratified with step k, then all terms of at least k nested Lie brackets will be zero, so this group law is a polynomial. We refer to $\mathfrak{g}$ with this group law as the Lie group G. It can be checked that the Lie algebra of G is in fact $\mathfrak{g}$. For $x \in G$ we may write

$$x = (x_1, \dots, x_k) \tag{2.1.1}$$

where $x_j \in \mathfrak{g}_j$ $(1 \leqslant j \leqslant k)$. It is easily seen from the group multiplication law that we have the group inverse law (written using the notation of (2.1.1))

$$x^{-1} = (-x_1, \dots, -x_k).$$

We will equip G with its Haar measure, which is given by Lebesgue measure on the Lie algebra $\mathfrak{g} \simeq \mathbb{R}^D$, where D is the topological dimension of G given by

$$D := \sum_{j=1}^{k} \dim(\mathfrak{g}_j). \tag{2.1.2}$$

From now on, we exclusively use D to refer to the topological dimension of a stratified Lie group G. Since the Lebesgue measure is left- and right-invariant, then stratified Lie groups are unimodular, so we may consider L^p spaces. Recall that, for a measure space (S, Σ, μ), and $1 \leqslant p < \infty$, we define $L^p(S, \mu)$ as the space of functions $f : S \to \mathbb{C}$ for which $\int_S |f|^p d\mu < \infty$, with the usual extension to $p = \infty$. When the measure on the space is a weighted Lebesgue measure (which will be true for e.g. G or $\mathbb{R}^d$) with weight w then we will write $L^p(S, w)$ instead (or omit the w if there is no weight). Furthermore, we will omit writing the space when it is the group G (i.e. we will abbreviate $L^p(G, w(x)dx)$ to $L^p(w)$).

We define the convolution of functions $f, g \in L^1(G)$ as

$$f * g(x) := \int_G f(xy^{-1})g(y)dy.$$

We also define the involution of a function $f \in L^1(G)$ by

$$f^*(x) := \overline{f(x^{-1})}.$$

The following relation regarding convolutions is an immediate consequence of the fact that G is unimodular.

Lemma 2.1. *Let G be a stratified Lie group and let $f, g, h \in L^1(G)$. Then,*

$$\int_G f(x)\overline{g * h(x)}dx = \int_G f * h^*(x)\overline{g(x)}dx. \tag{2.1.3}$$

Proof. By the substitution $w = xy$ and the fact that G is a unimodular group, we have

$$\begin{aligned}\int_G f(x)\overline{g * h(x)}dx &= \int_G\int_G f(x)\overline{g(xy^{-1})h(y)}dydx \\ &= \int_G\int_G f(wy)h^*(y^{-1})\overline{g(w)}dydw = \int_G f * h^*(x)\overline{g(x)}dx.\end{aligned}$$

□

Recall that the Lie algebra $\mathfrak{g}$ of a group G may also be thought of as the space of left-invariant vector fields on G. If we fix an inner product $\langle \cdot, \cdot \rangle$ on the first layer $\mathfrak{g}_1$ and take an orthonormal basis $X_1, \ldots, X_d$ of $\mathfrak{g}_1$, then we define a sub-Laplacian on G (compare e.g. [22], [59]) as

$$L := -\sum_{j=1}^{d} X_j^2. \tag{2.1.4}$$

Note that there is not a unique sub-Laplacian on a stratified Lie group; this definition depends on the choice of inner product.

We may also consider the sub-Laplacian L via its spectral decomposition. Note that L is positive and essentially self-adjoint on $\mathscr{S}(G)$ (compare [22], page 56). Its closure, which is again denoted by L, is self-adjoint on $L^2(G)$. Hence, L has a spectral decomposition

$$L = \int_0^\infty \lambda dE(\lambda).$$

For $m \in C_c(\mathbb{R}_0^+)$ we can then define a functional calculus for L by defining operators

$$m(L) := \int_0^\infty m(\lambda) dE(\lambda).$$

Since L is left-invariant, then so is the operator $m(L)$. Thus, by the Schwartz Kernel Theorem, $m(L)$ is a convolution operator (compare [22], page 208). That is, there exists $K \in \mathscr{S}'(G)$ such that $m(L)f = f * K$.

On any stratified Lie group G we have a family of dilations δ_r defined for $x \in G$ and $r > 0$ by

$$\delta_r x = \delta_r(x_1, \ldots, x_k) = (rx_1, r^2 x_2, \ldots, r^k x_k).$$

This means that G is a homogeneous group of homogeneous dimension

$$Q := \sum_{j=1}^{k} j \dim(\mathfrak{g}_j). \tag{2.1.5}$$

From now on, we reserve Q to be used for the homogeneous dimension of the group G in question at the time.

A metric that occurs naturally on Lie groups is the Carnot–Carathéodory distance, which we will denote by $d_{CC}(x, y)$. The construction of this distance is described in Section III.4 of [59]. In particular, note that it is left-invariant and induces a homogeneous norm $|y^{-1}x|_{CC} = d_{CC}(x, y)$ (that is, for every $r > 0$ and $x \in G$ we have that $|\delta_r x|_{CC} = r|x|_{CC}$). In particular, $|x|_{CC} = d_{CC}(x, 0)$. It should be noted that the construction of d_{CC} depends on the choice of inner product on the first layer of the lie algebra $\mathfrak{g}_1$ of the group G, so when speaking of d_{CC} we will mean 'the Carnot–Carathéodory distance corresponding to the inner product on $\mathfrak{g}_1$', but we shall in general suppress this dependence in our notation. When speaking of balls in a stratified group, we will use the notation $B(x, r)$ to refer to open balls with respect to the Carnot–Carathéodory distance.

Recall that $\overline{B}_d(x,r)$ denotes the closed ball $\{y \in G : d(x,y) \leqslant r\}$. We omit the subscript d when we are using the Carnot–Carathéodory distance.

A second homogeneous norm that may be defined on a stratified group is given by

$$|x|_S := \sum_{j=1}^{k} |x_j|^{1/j},$$

where $|x_j|$ is the Euclidean norm of $x_j \in \mathfrak{g}_j \simeq \mathbb{R}^{\dim(\mathfrak{g}_j)}$. This norm is equivalent to the Carnot–Carathéodory norm, in the sense that there exist $A, B > 0$ such that

$$A|x|_S \leqslant |x|_{CC} \leqslant B|x|_S.$$

We now recall a number of geometric properties of spaces. A metric space (X,d) is geometrically doubling if there exists a constant $M > 0$ such that for all $x \in X$ and for all $r > 0$ the open ball

$$B_d(x,r) := \{y \in X : d(x,y) < r\}$$

may be covered by at most M disjoint balls of radius $\frac{r}{2}$. A measure λ on a metric space (X,d) is doubling if there exists $C > 0$ such that for all $x \in X$ and for all $r > 0$ we have

$$\lambda(B_d(x,2r)) \leqslant C\lambda(B_d(x,r)).$$

Recall that a metric measure space (X,d,λ) with a doubling measure is automatically a geometrically doubling metric space (compare e.g. Section 2 of [31], [15]). We prove that the Lebesgue measure on stratified Lie groups is doubling, and so stratified Lie groups are also geometrically doubling.

Lemma 2.2. *The Lebesgue measure on a stratified Lie group G is doubling.*

Proof. Let $\lambda(S)$ denote the Lebesgue measure of a set $S \subseteq G$. By left-invariance and homogeneity of the Carnot–Carathéodory norm,

$$\lambda(B(x,r)) = \lambda(B(0,r)) = \int\limits_{B(0,r)} dy = r^Q \int\limits_{B(0,1)} dy \simeq r^Q.$$

Hence, the Lebesgue measure is a doubling measure. □

Furthermore, recall that a weight on G is a non-negative locally integrable function $w : G \to \mathbb{R}_0^+$. The set of weights $A_2(G)$ is the set of weights for which the Hardy–Littlewood maximal function on G is bounded on $L^2(w)$. An equivalent characterisation is that $w \in A_2(G)$ if and only if

$$\sup_{\substack{x\in G\\ r>0}} r^{-2Q} \int\limits_{B(x,r)} w(y)dy \int\limits_{B(x,r)} w(y)^{-1}dy < \infty. \tag{2.1.6}$$

Then we have the following results (comparable to Euclidean results found in Chapter V of [55]).

Lemma 2.3. *Let G be a stratified Lie group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous norm on G. Then the weights $|\cdot|^a$ and $(1+|\cdot|)^a$ are A_2 weights for $|a| < Q$.*

Proof. Clearly the constant function $f(x) \equiv 1 \in A_2$ and A_2 is closed under addition and taking the reciprocal. Since $(1+|\cdot|)^a \simeq 1+|\cdot|^a$, for $a \geqslant 0$, then it suffices to prove that $|\cdot|^a \in A_2$ for $0 < a < Q$.

Let

$$I(x,r) := r^{-2Q} \int\limits_{B(x,r)} |y|^a dy \int\limits_{B(x,r)} |y|^{-a} dy.$$

We must argue in two cases. First, suppose that $r \leqslant \frac{1}{2}|x|$. Note that this implies that $0 \notin B(x,r)$. Therefore,

$$\sup_{y\in B(x,r)} |y|^a = (|x|+r)^a, \qquad \inf_{y\in B(x,r)} |y|^a = (|x|-r)^a.$$

Thus

$$I(x,r) \lesssim r^{-2Q}(|x|+r)^a r^Q (|x|-r)^{-a} r^Q = \left(\frac{|x|+r}{|x|-r}\right)^a \leqslant \left(\frac{\frac{3}{2}|x|}{\frac{1}{2}|x|}\right)^a = 3^a.$$

Now suppose that $r > \frac{1}{2}|x|$. In this case, observe that $B(0,5r) \supseteq B(x,r)$. Therefore, since $r^{-2Q} \lesssim (5r)^{-2Q}$, where the constant is independent of r, and since both integrands are non-negative, it suffices to consider the case $x = 0$. Indeed,

$$\begin{aligned} I(x,r) &= r^{-2Q} \int\limits_{B(x,r)} |y|^a dy \int\limits_{B(x,r)} |y|^{-a} dy \\ &\lesssim (5r)^{-2Q} \int\limits_{B(0,5r)} |y|^a dy \int\limits_{B(0,5r)} |y|^{-a} dy = I(0,5r). \end{aligned}$$

Hence, we need only show that $I(0,r)$ is bounded uniformly in r. We now use Proposition 1.15 in [22]. In particular, let $S = \{x \in G : |x| = 1\}$. Then there exists a unique Radon measure σ on S such that for all $f \in L^1(G)$,

$$\int\limits_G f(x)dx = \int\limits_0^\infty \int\limits_S f(\delta_R y) R^{Q-1} d\sigma(y) dR.$$

Taking $f = \chi_{B(0,r)} |\cdot|^a$ gives

$$\int\limits_{B(0,r)} |y|^a dy = \int\limits_0^r \int\limits_S |\delta_R y|^a R^{Q-1} d\sigma(y) dR = \int\limits_0^r \int\limits_S R^{Q-1+a} d\sigma(y) dR \simeq \int\limits_0^r R^{Q-1+a} dR \simeq r^{Q+a}$$

Note that the condition $a > -Q$ is required for finiteness of the last integral. Hence, for $|a| < Q$ we have

$$I(0,r) \simeq r^{-2Q} r^{Q+a} r^{Q-a} = 1$$

as required. □

Lemma 2.4. *Let G be a stratified Lie group. The 'first layer' weights* $w(x) = |x_1|^a$ *and* $\tilde{w}(x) = (1+|x_1|)^a$ *are in* A_2 *for* $|a| < \dim(\mathfrak{g}_1)$.

Proof. As before, recall that $f(x) \equiv 1 \in A_2$ and A_2 is closed under addition and taking reciprocals and that $(1 + |x_1|)^a \simeq 1 + |x_1|^a$ for $a \geqslant 0$, so we need only prove that $|x_1|^a \in A_2$ for $0 < a < \dim(\mathfrak{g}_1)$.

Let

$$I(x, r) := r^{-2Q} \int\limits_{B(x,r)} |y_1|^a dy \int\limits_{B(x,r)} |y_1|^{-a} dy.$$

We must argue in two cases. First, suppose that $r \leqslant \min\{\frac{A}{2}, \frac{1}{2}\}|x_1|$, where $A > 0$ is a constant such that for all $x \in G$ we have $A \sum |x_i|^{1/i} \leqslant |x|_{CC}$. Note that this implies that $\{y \in G : |y_1| = 0\} \cap B(x, r) = \emptyset$. Indeed, observe

$$d_{CC}(x, (0, y_2, \ldots, y_k)) = |(0, y_2, \ldots, y_k)^{-1} x|_{CC} = |(x_1, z_2, \ldots, z_k)|_{CC}$$

where z_i are the expected terms from the group multiplication law (for instance, $z_2 = -y_2 + x_2 + \frac{1}{2}[0, x_2] = x_2 - y_2$). Then,

$$|(x_1, z_2, \ldots, z_k)|_{CC} \geqslant |(x_1, 0, \ldots, 0)|_{CC} \geqslant A|x_1| > r.$$

Therefore,

$$\sup_{y \in B(x,r)} |y_1| \leqslant |x_1| + r, \qquad \inf_{y \in B(x,r)} |y_1| \geqslant |x_1| - r.$$

Thus

$$\begin{aligned} I(x, r) &\lesssim r^{-2Q}(|x_1| + r)^a r^Q (|x_1| - r)^{-a} r^Q \\ &= \left(\frac{|x_1| + r}{|x_1| - r}\right)^a \leqslant \left(\frac{(1 + \min\{\frac{A}{2}, \frac{1}{2}\})|x_1|}{(1 - \min\{\frac{A}{2}, \frac{1}{2}\})|x_1|}\right)^a \lesssim 1. \end{aligned}$$

Now suppose that $r > \min\{\frac{A}{2}, \frac{1}{2}\}|x_1|$. We first consider a change of coordinates given by the left-translation $y \to (xx')y$, where $x' := (-x_1, 0, \ldots, 0)$. Note that $(xx')_1 = x_1 - x_1 = 0$.. Then, since $(xx'y)_1 = y_1$ and since dy is invariant under translations, we have

$$I(x,r) = r^{-2Q} \int\limits_{(xx')^{-1}B(x,r)} |y|^a dy \int\limits_{(xx')^{-1}B(x,r)} |y|^{-a} dy.$$

Now, let $z \in B(x,r)$, so that $(xx')^{-1}z \in (xx')^{-1}B(x,r)$. Then

$$d_{CC}((xx')^{-1}z, (x')^{-1}) = |z^{-1}xx'(x')^{-1}|_{CC} = |z^{-1}x|_{CC} = d_{CC}(z,x) < r$$

since $z \in B(x,r)$. This shows that $(xx')^{-1}B(x,r) \subseteq B((x')^{-1}, r)$. Since both integrands are non-negative, then this implies that $I(x,r) \leqslant I((x')^{-1}, r)$.

Now, since there exists $B > 0$ such that $|x|_{CC} \leqslant B \sum |x_i|^{1/i}$, then $|x'|_{CC} \leqslant B|x'| = B|x_1| < B\max\{\frac{2}{A}, 2\}r$, by assumption, so then there exists $E > 0$ depending only on A, B such that $B(0, Er) \supseteq B((x')^{-1}, r)$. Therefore, since $r^{-2Q} \lesssim (Er)^{-2Q}$, where the constant is independent of r, and since both integrands are non-negative, we have

$$\begin{aligned} I((x')^{-1}, r) &= r^{-2Q} \int\limits_{B((x')^{-1},r)} |y_1|^a dy \int\limits_{B((x')^{-1},r)} |y_1|^{-a} dy \\ &\lesssim (Er)^{-2Q} \int\limits_{B(0,Er)} |y_1|^a dy \int\limits_{B(0,Er)} |y_1|^{-a} dy = I(0, Er). \end{aligned}$$

Hence, we now need only show that $I(0,r)$ is bounded uniformly in r. We see that

$$\int\limits_{B(0,r)} |y_1|^a dy \leqslant \int\limits_{|y_i|<(r/A)^i} |y_1|^a dy_1 \ldots dy_k \simeq r^{Q+a},$$

where the condition $a > -\dim(\mathfrak{g}_1)$ is required for finiteness of the last integral. Hence, for $|a| < \dim(\mathfrak{g}_1)$ we have

$$I(0,r) \simeq r^{-2Q} r^{Q+a} r^{Q-a} = 1$$

as required. □

2.1.1 H-Type Groups

We now recall the definition of H-type groups. We start with a 2-step graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. We assume further that we have an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal. For each $\mu \in \mathfrak{g}_2^* \simeq \mathbb{R}^{\dim(\mathfrak{g}_2)}$ (the dual space of $\mathfrak{g}_2$) we define the skew-symmetric endomorphism J_μ of $\mathfrak{g}_1$ by

$$\langle J_\mu(z), z' \rangle = \mu([z, z']) \ \forall z, z' \in \mathfrak{g}_1.$$

We then say that $\mathfrak{g}$ is an H-type Lie algebra if for each $\mu \in \mathfrak{g}_2^*$ we have

$$J_\mu^2 = -|\mu|^2 \mathrm{Id}.$$

As above, we endow $\mathfrak{g}$ with the structure of a Lie group by defining the group law

$$(z, u)(y, v) = (z + y, u + v + \frac{1}{2}[z, y]),$$

which implies the inverse law

$$(z, u)^{-1} = (-z, -u).$$

Note that the dimension of $\mathfrak{g}_1$ is always even in an H-type Lie algebra. If we set $2m = \dim(\mathfrak{g}_1)$ and $n = \dim(\mathfrak{g}_2)$ then G may be identified with $\mathbb{C}^m \times \mathbb{R}^n$ in such a way that the inner product on G is identified with the standard inner product on $\mathbb{C}^m \times \mathbb{R}^n$. As the Lie bracket is antisymmetric and non-trivial, then in particular this cannot be an abelian group.

Next, we let $X_1, \dots, X_{2m}$ denote the left-invariant vector fields generated by the unit vectors $e_1, \dots, e_{2m} \in \mathbb{R}^{2m+n} \simeq T_0 G$, the tangent space of G at the group identity, where e_i denotes the vector with standard coordinates 1 in the i^{th} place and all others 0. The sub-Laplacian on G is then defined as

$$L = -\sum_{j=1}^{2m} X_j^2.$$

In addition to the Carnot–Carathéodory distance and norm, we will equip an H-type group G with the 'Koranyi norm' given by

$$|(z,u)|_K := \left(|z|^4 + 16|u|^2\right)^{1/4}, \quad (z,u) \in G.$$

As the Koranyi norm is sub-multiplicative, it induces a left-invariant metric on G given by

$$d_K(x,y) = |y^{-1}x|_K. \tag{2.1.7}$$

We will use the notation $B_K(x,r)$ to refer to balls with respect to this metric. Under the group dilations δ_r, the Koranyi norm is a homogeneous norm.

Note that the two metrics d_K and d_{CC} are equivalent. That is, there exists some constant $A > 0$ depending only on G such that, for all $x, y \in G$,

$$A^{-1} d_K(x,y) \leqslant d_{CC}(x,y) \leqslant A d_K(x,y). \tag{2.1.8}$$

2.1.2 The Functional Calculus of Sub-Laplacians

Here we briefly state a number of results concerning the functional calculus of sub-Laplacians L on stratified groups. The majority of proofs will be omitted, with references given to where they may be found.

Lemma 2.5. *Let G be a stratified Lie group and L be a sub-Laplacian. Suppose a function $m : \mathbb{R}_0^+ \to \mathbb{R}$ satisfies*

$$\|m\|_{k,k'}^* := \sup_{\substack{\lambda \in \mathbb{R}_0^+ \\ j=0,\ldots,k}} (1+\lambda)^k |m^{(j)}(\lambda)| < \infty$$

for k, k' sufficiently large. Then the convolution kernel K of the operator $m(L)$ satisfies the estimate

$$|K(x)| \lesssim \frac{\|m\|^*_{k,k'}}{(1 + |x|_{CC})^{Q+1}}. \tag{2.1.9}$$

The constant in '$\lesssim$' does not depend on m. In particular, this holds for $m \in \mathscr{S}(\mathbb{R}_0^+)$, the space of Schwartz functions on $\mathbb{R}_0^+$.

Proof. The proof is by combining Lemmas 1.2 and 2.4 of [30]. For a multi-index $I = (i_1, \ldots, i_{2m})$ define $X^I := X_1^{i_1} \ldots X_{2m}^{i_{2m}}$. Define

$$\|K\|_{a,b,1} := \sum_{|I| \leqslant b} \int_G |X^I K(x)|(1 + |x|_{CC})^a dx,$$

$$\|K\|_{a',0,\infty} := \sup_{x \in G} |K(x)|(1 + |x|_{CC})^{a'}.$$

By Lemma 1.2 of [30], there exist a, b and $c > 0$ that depends only on a' such that $\|K\|_{a',0,\infty} \leqslant c\|K\|_{a,b,1}$. By Lemma 2.4 of [30], given a, b, there exist k, k' sufficiently large such that if

$$\|m\|^*_{k,k'} < \infty$$

then

$$\|K\|_{a,b,1} \leqslant C\|m\|^*_{k,k'},$$

where C does not depend on m. Combining these facts, by choosing $a' = Q + 1$, we have

$$\sup_{x \in G} |K(x)|(1 + |x|_{CC})^{Q+1} \leqslant cC\|m\|^*_{k,k'}$$

and so for all $x \in G$ we have

$$|K(x)| \leqslant \frac{cC\|m\|^*_{k,k'}}{(1 + |x|_{CC})^{Q+1}}$$

as required. Note that if $m \in \mathscr{S}$ then $\|m\|^*_{k,k'}$ is bounded for arbitrarily large k, k'. □

A property of the sub-Laplacian L which we will use is that it has the 'finite propagation speed' property for the solution of the wave equation. This property is stated in a form that will be useful to us in Lemma 2.6.

Lemma 2.6. *Let G be a stratified group and L be a sub-Laplacian. Let $t \in \mathbb{R}$, let K denote the convolution kernel of the operator* $\cos(t\sqrt{L})$. *Then*

$$supp(K) \subseteq \overline{B}(0, |t|),$$

where we recall that $\overline{B}(x, r)$ is the closed ball centred at x of radius r with respect to the Carnot–Carathéodory distance.

Proof. The original notion was proved in [46]. For the case of Lie groups, the result is also shown in Section 8.2 of [54]. □

The following result is a collection of results of a well-studied problem concerning bounding the kernel of an operator of the form $m(L)$.

Lemma 2.7. *Let G be a stratified group and L be a sub-Laplacian. There exists a finite constant N such that for every $b > \frac{N}{2}$, for every compact set $U \subseteq \mathbb{R}$, for all functions $m \in C_c^\infty(\mathbb{R})$ with $supp(m) \subseteq U$, we have*

$$\int_G |K(x)|dx \lesssim \|m\|_{L_b^2(\mathbb{R})},$$

where K is the convolution kernel of $m(L)$, where the implicit constant in '$\lesssim$' may depend on b and U and where $\|\cdot\|_{L_b^2(\mathbb{R})}$ denotes the norm

$$\|m\|_{L_b^2(\mathbb{R})}^2 := \int_{\mathbb{R}} \left|(1 + |x|)^b \widehat{m}(x)\right|^2 dx$$

on the L^2 Sobolev space of order b. We denote by $\eta(L)$ the minimum of all such constants N. Then $\eta(L)$ is known to equal D for H-type groups, is in the range $D \leqslant \eta(L) < Q$ for a general 2-step stratified group, while for a general stratified group it is known that $D \leqslant \eta(L) \leqslant Q$.

The case of H-type groups may be inferred from [28] (see also [49] for Heisenberg groups) and may be found explicitly as Proposition 3 of [39]. The result that $\eta(L) = D$ has been proven for a number of 2-step stratified Lie groups such as those with $D \leqslant 7$ or $\dim \mathfrak{g}_2 \leqslant 2$ ([29], [38], [39], [40], [42]). The general 2-step result comes from [41]. The upper bound for an arbitrary stratified group can be found in [13], [45] while the lower bound is found in [43].

Another result regarding the convolution kernel K in the above is the following.

Lemma 2.8. *Let G be a stratified Lie group and L be a sub-Laplacian thereon. Suppose that $\varphi \in C^\infty(\mathbb{R}^+)$ and*

$$\sup_{\lambda>0} |\lambda^j \varphi^{(j)}(\lambda)| < C < \infty$$

for $0 \leqslant j \leqslant 3 + \frac{3Q}{2}$. Let K denote the convolution kernel of $\varphi(L)$. Then K and $X_k K$ are continuous on $G \backslash \{0\}$ for $1 \leqslant k \leqslant 2m$ and K satisfies the estimates

$$|K(x)| \lesssim \frac{1}{|x|_{CC}^Q}, \quad |X_k K(x)| \lesssim \frac{1}{|x|_{CC}^{Q+1}}.$$

Furthermore, the constants in $\lesssim$ depend only on C.

Proof. See the proof of Theorem 6.25 of [22]. □

Remark 2.9. In view of the results of Lemma 2.7, it is likely that the required number of derivatives $3 + \frac{3Q}{2}$ in Lemma 2.8 is not optimal. In our case, we do not require a sharper result.

Note that if we define, for any measurable function $f : G \to \mathbb{C}$,

$$f_r(x) = r^{-Q/2} f(\delta_{r^{-1/2}}(x)), \quad r > 0 \tag{2.1.10}$$

then we have the following results.

Lemma 2.10. *Let G be a stratified Lie group and L be a sub-Laplacian. Let $m \in C_c(\mathbb{R}^+)$*

and let K denote the convolution kernel of $m(L)$. Then, for $r > 0$ we have

$$m(rL)f = f * K_r = (m(L)f_{r^{-1}})_r. \tag{2.1.11}$$

Proof. This is Lemma 6.29 in [22] and its proof. □

The next lemma shows that, for operators defined on a stratified group G as in (2.1.10) satisfying a certain estimate (which the previous results show, in particular, is satisfied by operators $m(L)$ for suitably well-behaved functions m and a sub-Laplacian L on G), then the corresponding maximal operator is bounded by an analogue of the Hardy–Littlewood maximal operator defined on stratified groups. This version is analogous to the Euclidean Hardy–Littlewood maximal operator and satisfies the same L^p estimates for $p > 1$.

Lemma 2.11. *Let G be a stratified Lie group. Let T^*f be an operator defined on $L^p(G)$ by*

$$T^*f := \sup_{r>0} |f * K_r|$$

where the convolution kernel K satisfies the estimate

$$|K(x)| \leqslant \frac{C}{(1 + |x|_{CC})^{Q+\epsilon}}$$

for some $\epsilon > 0$. Then

$$T^*f(x) \lesssim CMf(x)$$

where the constant in $\lesssim$ depends only on G and ϵ and where Mf denotes the Hardy–Littlewood maximal operator on G given by

$$Mf(x) := \sup_{r>0} r^{-Q} \int\limits_{|x|_{CC} \leqslant r} |f(xy^{-1})| dy.$$

For $p > 1$ Mf satisfies the same L^p estimates as in the Euclidean case, so we have

$$\|T^* f(x)\|_p \lesssim C\|f\|_p$$

where the constant in $\lesssim$ depends only on p and G.

Proof. This is Corollary 2.5 of [22]. □

The next lemma is a result regarding a square function associated to a Littlewood-Paley decomposition for a sub-Laplacian. Here, we prove boundedness on weighted L^2 spaces with respect to A_2 weights. The result is analogous to Euclidean results found in, for example, [55]; the proof is included for the reader's convenience.

Lemma 2.12. *Let G be a stratified Lie group and L be a sub-Laplacian thereon. Let $\varphi \in C_c^\infty(\mathbb{R}^+)$ such that*

$$\sum_{l\in\mathbb{Z}} \varphi(2^{-l}\lambda) = 1, \quad \text{for } \lambda > 0$$

and let $\omega \in A_2$. Then

$$\sum_{l\in\mathbb{Z}} \|\varphi(2^{-l}L)f\|^2_{L^2(\omega)} \simeq \|f\|^2_{L^2(\omega)}. \tag{2.1.12}$$

Proof. Let $\epsilon := (\epsilon_l)_{l\in\mathbb{Z}}$ be a sequence with $\epsilon_l \in \{-1, 1\}$. Let K_ϵ be the convolution kernel of the operator

$$T_\epsilon f(x) := \sum_{l\in\mathbb{N}} \epsilon_l \varphi(2^{-l}L)f(x).$$

We will prove that, for all $x, y \in G$ we have

$$|K_\epsilon(x)| \lesssim \frac{1}{|x|^Q_{CC}}, \tag{2.1.13}$$

$$|K_\epsilon(x) - K_\epsilon(xy)| \lesssim \frac{|y|}{|x|^{Q+1}_{CC}}, \quad \text{if } |x|_{CC} \gtrsim |y|_{CC}, \tag{2.1.14}$$

$$|K_\epsilon(x) - K_\epsilon(yx)| \lesssim \frac{|y|}{|x|^{Q+1}_{CC}}, \quad \text{if } |x|_{CC} \gtrsim |y|_{CC}, \tag{2.1.15}$$

and furthermore that the implicit constants involved do not depend on ϵ.

Observe that it suffices to show only (2.1.13) and (2.1.14). Indeed, suppose we had shown that (2.1.13) and (2.1.14) hold for the kernel K_ϵ of the operator T_ϵ. Since φ is real-valued then T_ϵ is self-adjoint and $K_\epsilon = K_\epsilon^*$. But since $K_\epsilon^*(x) = \overline{K_\epsilon(x^{-1})}$ then we must have $K_\epsilon(x) = \overline{K_\epsilon(x^{-1})}$ and so from (2.1.14) we have

$$|K_\epsilon(x) - K_\epsilon(yx)| = |K_\epsilon(x^{-1}) - K_\epsilon(x^{-1}y^{-1})| \lesssim \frac{|y^{-1}|_{CC}}{|x^{-1}|_{CC}^{Q+1}} = \frac{|y|_{CC}}{|x|_{CC}^{Q+1}}$$

as required.

We see that (2.1.13) and (2.1.14) are a consequence of Lemma 2.8. From the definition of $t_\epsilon = \sum \epsilon_l \varphi(2^{-l}\cdot)$ we have that $t_\epsilon \in C^\infty(\mathbb{R}^+)$ and

$$\sup_{\lambda > 0} |t_\epsilon(\lambda)| = 1.$$

Now, note that

$$\lambda t_\epsilon'(\lambda) = \sum_{l \in \mathbb{Z}} 2^{-l} \epsilon_l \lambda \varphi'(2^{-l}\lambda).$$

Let

$$b := \sup_{\varphi(x) \neq 0} x.$$

Since φ is compactly supported we have that $2^{-l}\lambda \leqslant b$ and so $\lambda t_\epsilon'(\lambda)$ is bounded uniformly in λ. By repeating this argument, we can show that t_ϵ satisfies the hypotheses of Lemma 2.8 uniformly in ϵ and so (2.1.13) follows.

Then by the Stratified Mean Value Theorem (Theorem 1.41 of [22]) we have

$$|K_\epsilon(x) - K_\epsilon(xy)| \lesssim |y|_{CC} \sup_{\substack{|z|_{CC} \lesssim |y|_{CC} \\ 1 \leqslant j \leqslant 2m}} |X_j K_\epsilon(xz)|. \tag{2.1.16}$$

From Lemma 2.8 we then have $\sup_{1 \leqslant j \leqslant 2m} |X_j K_\epsilon(xz)| \lesssim |xz|_{CC}^{-Q-1}$. From the reverse triangle inequality we then have

$$\sup_{|z|_{CC} \lesssim |y|_{CC}} \frac{|y|_{CC}}{|xz|_{CC}^{Q+1}} \leqslant \sup_{|z|_{CC} \lesssim |y|_{CC}} \frac{|y|_{CC}}{||x|_{CC} - |z|_{CC}|^{Q+1}}.$$

Now, let C be a constant such that $|z|_{CC} \leqslant C|y|_{CC}$. We will explicitly assume that

$|y|_{CC} \leqslant \frac{1}{2C}|x|_{CC}$. In this case, for $|z|_{CC} \lesssim |y|_{CC}$ we have $|z|_{CC} \leqslant \frac{1}{2}|x|_{CC}$. Notice that

$$\sup_{|z|_{CC} \lesssim |y|_{CC}} \frac{1}{|x|_{CC} - |z|_{CC}} = \frac{1}{\inf_{|z|_{CC} \lesssim |y|_{CC}} |x|_{CC} - |z|_{CC}}.$$

This bound on $|z|_{CC}$ implies that the infimum is attained when $|z|_{CC}$ is maximised. With our restrictions, this occurs when $|z|_{CC} = \frac{1}{2}|x|_{CC}$. Thus,

$$\sup_{|z|_{CC} \lesssim |y|_{CC}} \frac{|y|_{CC}}{||x|_{CC} - |z|_{CC}|^{Q+1}} = \frac{2|y|_{CC}}{|x|_{CC}^{Q+1}} \simeq \frac{|y|_{CC}}{|x|_{CC}^{Q+1}}$$

as required.

By Lemma 2.2, G satisfies the hypotheses of Theorem 6.1 of [50], which implies that the operator T_ϵ is bounded on $L^2(\omega)$. Furthermore, as conditions (2.1.13), (2.1.14) and (2.1.15) are satisfied uniformly in $(\epsilon_l)_{l\in\mathbb{Z}}$, then the operators T_ϵ are bounded uniformly in (ϵ_l). Using Rademacher functions (see 5.2 of Section IV of [56]) with the boundedness of T_ϵ we can conclude therefore that

$$\sum_{l\in\mathbb{Z}} \|\varphi(2^{-l}L)f\|^2_{L^2(\omega)} \lesssim \sup_{\epsilon} \|T_\epsilon f\|^2_{L^2(\omega)} \lesssim \|f\|^2_{L^2(\omega)}. \tag{2.1.17}$$

To show the opposite inequality, define $T_l := \psi(2^{-l}L)$, where $\psi \in C_c^\infty(\mathbb{R})$ is such that $\psi(x) = 1$ for $x \in \operatorname{supp}(\varphi)$ and $\operatorname{supp}(\psi) \subseteq (\frac{1}{2}, \frac{9}{2})$. Using Lemma 5.5 of [38] we know that if there exists $A > 0$ such that for any choice of $(\epsilon_l)_{l\in\mathbb{Z}} \subseteq \{-1, 1\}$ we have

$$\left\| \sum_{l\in\mathbb{Z}} \epsilon_l \psi(2^{-l}L) \right\|_{L^2(\omega)\to L^2(\omega)} \leqslant A \tag{2.1.18}$$

then

$$\left\| \sum_{l\in\mathbb{Z}} \psi(2^{-l}L)\varphi(2^{-l}L)f \right\|^2_{L^2(\omega)} \lesssim \left\| \left(\sum_{l\in\mathbb{Z}} |\psi(2^{-l}L)\varphi(2^{-l}L)f|^2 \right)^{1/2} \right\|^2_{L^2(\omega)}.$$

Note that we may repeat the argument used earlier to prove the boundedness of the operators T_ϵ with φ replaced by ψ to prove (2.1.18).

From the functional calculus of L we have that

$$\sum_{l\in\mathbb{Z}} \psi(2^{-l}L)\varphi(2^{-l}L) = \int_0^\infty \sum_{l\in\mathbb{Z}} \psi(2^{-l}\lambda)\varphi(2^{-l}\lambda) dE(\lambda) = \int_0^\infty 1 dE(\lambda) = Id.$$

Hence,

$$\|f\|^2_{L^2(\omega)} \lesssim \left\| \left(\sum_{l\in\mathbb{Z}} |\varphi(2^{-l}L)f|^2 \right)^{1/2} \right\|^2_{L^2(\omega)} = \sum_{l\in\mathbb{Z}} \|\varphi(2^{-l}L)f\|^2_{L^2(\omega)} \tag{2.1.19}$$

as required. □

2.2 Representation Theory and the Fourier Transform

In this section, we will recall some facts regarding analysis on H-type groups. In particular, we will define a number of coordinate systems on such groups, recall some properties of such groups as topological spaces and develop the representation theory and Fourier transform on these groups.

We let G be an arbitrary H-type group. We fix a basis of the first layer as in, for example, [1]. For each $\mu \in \mathfrak{g}_2^* \backslash \{0\} \simeq \mathbb{R}^n \backslash \{0\}$ there exists an orthonormal basis $E_1(\mu), \ldots, E_m(\mu), \tilde{E}_1(\mu), \ldots, \tilde{E}_m(\mu) \in \mathfrak{g}_1$ such that

$$J_\mu E_i(\mu) = |\mu| \tilde{E}_i(\mu) \text{ and } J_\mu \tilde{E}_i(\mu) = -|\mu| E_i(\mu).$$

For brevity, from now on we shall write

$$E_{k+m}(\mu) := \tilde{E}_k(\mu) \text{ for } k = 1, \ldots, m. \tag{2.2.1}$$

For convenience, for $x, y \in \mathbb{R}^d$, for any d, we write $xy = x \cdot y = \sum_j x_j y_j$. We can now write an element $z \in \mathfrak{g}_1$ as

$$z = \sum_{k=1}^{2m} z_k(\mu) E_k(\mu).$$

This decomposition naturally applies to G and gives us global coordinates for G for each $\mu \in \mathfrak{g}_2^*\backslash\{0\}$. We define

$$z^{(re)}(\mu) = (z_1^{(re)}(\mu), \ldots, z_m^{(re)}(\mu)) := (z_1(\mu), \ldots, z_m(\mu)), \tag{2.2.2}$$

$$z^{(im)}(\mu) = (z_1^{(im)}(\mu), \ldots, z_m^{(im)}(\mu)) := (z_{m+1}(\mu), \ldots, z_{2m}(\mu)), \tag{2.2.3}$$

$$z^{(\mathbb{R})}(\mu) := (z_1(\mu), \ldots, z_{2m}(\mu)) \tag{2.2.4}$$

$$z(\mu) = (z_1^{(\mathbb{C})}(\mu), \ldots, z_m^{(\mathbb{C})}(\mu)) := (z_1^{(re)}(\mu) + iz_1^{(im)}(\mu), \ldots, z_m^{(re)}(\mu) + iz_m^{(im)}(\mu)). \tag{2.2.5}$$

Sometimes we will wish to write this basis or the coordinates it defines in terms of a different μ. Let $M(\mu, \mu_1)$ be the change of coordinates matrix with entries

$$(M(\mu, \mu_1))_{j,k} := m_{j,k}(\mu, \mu_1) \tag{2.2.6}$$

such that

$$z^{(\mathbb{R})}(\mu_1)^T = M(\mu, \mu_1) z^{(\mathbb{R})}(\mu)^T. \tag{2.2.7}$$

Lemma 2.13. *Let $N(\mu, \mu_1)$ be the change-of-basis matrix such that*

$$E_j(\mu_1) = \sum_{k=1}^{2m} n_{j,k}(\mu, \mu_1) E_k(\mu). \tag{2.2.8}$$

Then $N(\mu, \mu_1) = M(\mu, \mu_1)$.

Proof. Let $z \in \mathbb{C}^n$. Then

$$z = \sum_{k=1}^{2m} z_k(\mu_1) E_k(\mu_1) = \sum_{k=1}^{2m} \sum_{j=1}^{2m} z_k(\mu_1) n_{j,k}(\mu, \mu_1) E_k(\mu). \tag{2.2.9}$$

Observe from (2.2.9) that

$$N(\mu, \mu_1)^T = M(\mu_1, \mu) = M(\mu, \mu_1)^{-1}.$$

Hence, since $M(\mu, \mu_1)$ is an orthogonal matrix, we have

$$N(\mu, \mu_1) = (M(\mu, \mu_1)^{-1})^T = M(\mu, \mu_1).$$

□

It will often be convenient to consider functions that depend only on $|z|$ and u. We shall call such functions radial. That is, $f(z, u)$ is radial if there exists a function F such that $f(z, u) = F(|z|, u)$.

The group Fourier transform of $f \in L^1(G)$ is the operator-valued function given by

$$[\widehat{f}(\mu)]\varphi(x) = \int_G f(g)\pi_\mu(g)\varphi(x)dg$$

where π_μ is the irreducible unitary representation $G \to \mathscr{L}(L^2(\mathbb{R}^m))$, given by the bounded linear operators on $L^2(\mathbb{R}^m)$ defined as

$$[\pi_\mu(z, u)\varphi](x) = e^{2\pi i(\mu u + |\mu|(z^{(im)}(\mu)x + \frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}\varphi(z^{(re)}(\mu) + x). \tag{2.2.10}$$

The following basic properties of the Fourier transform hold (see, for instance, [20]). Note that $f^*(x) := \overline{f(x^{-1})}$ and $T^\dagger$ denotes the adjoint operator.

Lemma 2.14. *We have the identities*

$$\widehat{f * g}(\mu) = \widehat{f}(\mu)\widehat{g}(\mu) \tag{2.2.11}$$

$$\widehat{f^*}(\mu) = [\widehat{f}(\mu)]^\dagger \tag{2.2.12}$$

$$\langle f, g\rangle_G := \int_G f(z, u)\overline{g(z, u)}dzdu = \int_{\mathbb{R}^n} tr(\widehat{f}(\mu)[\widehat{g}(\mu)]^\dagger)|\mu|^m d\mu =: \langle \widehat{f}, \widehat{g}\rangle \tag{2.2.13}$$

$$\|f\|_2^2 = \int_{\mathbb{R}^n} \|\widehat{f}(\mu)\|_{HS}^2 |\mu|^m d\mu. \tag{2.2.14}$$

From this, we can obtain an estimate on the L^2 norm of convolution of functions.

Lemma 2.15. *We have*

$$\|f * g\|_2^2 \leqslant \|f\|_2^2 \sup_{\mu \in \mathbb{R}^n} \|\widehat{g}(\mu)\|_{L^2 \to L^2}^2. \tag{2.2.15}$$

Proof. From (2.2.14) we have

$$\begin{aligned} \|f * g\|_2^2 = \int_{\mathbb{R}^n} \|\widehat{f * g}(\mu)\|_{HS}^2 |\mu|^m d\mu &\leqslant \int_{\mathbb{R}^n} \|\widehat{f}(\mu)\|_{HS}^2 (\sup_{\mu \in \mathbb{R}^n} \|\widehat{g}(\mu)\|_{L^2 \to L^2}^2) |\mu|^m d\mu \\ &= \|f\|_2^2 \sup_{\mu \in \mathbb{R}^n} \|\widehat{g}(\mu)\|_{L^2 \to L^2}^2. \end{aligned}$$

□

As in [24], we consider re-normalised Hermite functions. We start by defining Hermite functions on the real line by

$$H_k(x) := (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}, \quad x \in \mathbb{R}, k \in \mathbb{N}_0.$$

These are then normalised by

$$h_k(x) := (2^k k! \sqrt{\pi})^{-1/2} H_k(x), \quad x \in \mathbb{R}.$$

We can then define normalised Hermite functions on $\mathbb{R}^d$ by

$$h_\alpha(x) := \prod_{j=1}^{d} h_{\alpha_j}(x_j), \quad x \in \mathbb{R}^d, k \in \mathbb{N}_0^d.$$

These normalised Hermite functions form an orthonormal basis of $L^2(\mathbb{R}^d)$. We then renormalise these Hermite functions by defining

$$h_\alpha^\mu(x) := (2\pi|\mu|)^{d/4} h_\alpha((2\pi|\mu|)^{1/2} x), \quad x \in \mathbb{R}^d. \tag{2.2.16}$$

For each $\mu \in \mathbb{R}^d$ the family $(h_\alpha^\mu)_{\alpha \in \mathbb{N}_0^d}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$.

For a weight ω we define the operator ∂_ω by

$$\partial_\omega \widehat{f}(\mu) := \widehat{\omega f}(\mu). \tag{2.2.17}$$

We will specifically be interested in these operators for the following weights. We define

$$\begin{aligned}
\zeta_{\mu,j}(z,u) &= z_j^{(\mathbb{C})}(\mu) \\
\overline{\zeta_{\mu,j}}(z,u) &= \overline{z_j^{(\mathbb{C})}(\mu)} \\
\rho(z,u) &= |z| \\
\psi_l(z,u) &= u_l \\
\psi(z,u) &= |u|.
\end{aligned} \tag{2.2.18}$$

In order to calculate operators, such as $\partial_{\zeta_{\mu,j}}$ and $\partial_{\overline{\zeta_{\mu,j}}}$, we must first understand how these Hermite functions interact with differentiation with respect to and multiplication by components of their inputs. This is realised via the identities

$$2(2\pi|\mu|)^{1/2} x_j h_\alpha^\mu(x) = 2\alpha_j{}^{1/2} h_{\alpha-e_j}^\mu(x) + (2\alpha_j + 2)^{1/2} h_{\alpha+e_j}^\mu(x) \tag{2.2.19}$$

and

$$2(2\pi|\mu|)^{-1/2} \partial_{x_j} h_\alpha^\mu(x) = 2\alpha_j{}^{1/2} h_{\alpha-e_j}^\mu(x) - (2\alpha_j + 2)^{1/2} h_{\alpha+e_j}^\mu(x). \tag{2.2.20}$$

Having such a basis of $L^2(\mathbb{R}^m)$ will allow us to consider the 'matrix components' of the group-Fourier transform of a function. For $f \in L^1(G)$, $\mu \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$ these are defined by

$$\widehat{f}(\mu,\alpha,\beta) := \langle \widehat{f}(\mu) h_\alpha^\mu, h_\beta^\mu \rangle_{\mathbb{R}^m} := \int_G f(g) \langle \pi_\mu(g) h_\alpha^\mu, h_\beta^\mu \rangle_{\mathbb{R}^m} dg. \tag{2.2.21}$$

With these matrix components, similarly to (2.2.12), there is a relation between the matrix components of a function and its involution.

Lemma 2.16. *For $f \in L^1(G)$ we have the relation*

$$\widehat{f^*}(\mu,\alpha,\beta) = \overline{\widehat{f}(\mu,\beta,\alpha)}. \tag{2.2.22}$$

Proof. By definition (2.2.21) and (2.2.12) we have

$$\widehat{f^*}(\mu,\alpha,\beta) = \langle \widehat{f^*}(\mu)h^{\mu}_{\alpha}, h^{\mu}_{\beta}\rangle_{\mathbb{R}^m} = \overline{\langle \widehat{f}(\mu)h^{\mu}_{\beta}, h^{\mu}_{\alpha}\rangle_{\mathbb{R}^m}} = \overline{\widehat{f}(\mu,\beta,\alpha)}.$$

□

We also have an identity on the components of a convolution.

Lemma 2.17. *For $f, g \in L^1(G)$ we have*

$$\widehat{f * g}(\mu,\alpha,\beta) = \sum_{\gamma \in \mathbb{N}_0^m} \widehat{g}(\mu,\alpha,\gamma)\widehat{f}(\mu,\gamma,\beta).$$

Proof. Since the Hermite functions h^{μ}_{α} form an orthonormal basis of $\mathbb{R}^m$, we use (2.2.11) to see that

$$\begin{aligned}
\widehat{f * g}(\mu,\alpha,\beta) &= \langle \widehat{f}(\mu)\widehat{g}(\mu)h^{\mu}_{\alpha}, h^{\mu}_{\beta}\rangle_{\mathbb{R}^m} \\
&= \langle \widehat{f}(\mu) \sum_{\gamma \in \mathbb{N}_0^m} \langle \widehat{g}(\mu)h^{\mu}_{\alpha}, h^{\mu}_{\gamma}\rangle_{\mathbb{R}^m} h^{\mu}_{\gamma}, h^{\mu}_{\beta}\rangle_{\mathbb{R}^m} \\
&= \sum_{\gamma \in \mathbb{N}_0^m} \langle \widehat{g}(\mu)h^{\mu}_{\alpha}, h^{\mu}_{\gamma}\rangle_{\mathbb{R}^m} \langle \widehat{f}(\mu)h^{\mu}_{\gamma}, h^{\mu}_{\beta}\rangle_{\mathbb{R}^m} \\
&= \sum_{\gamma \in \mathbb{N}_0^m} \widehat{g}(\mu,\alpha,\gamma)\widehat{f}(\mu,\gamma,\beta).
\end{aligned}$$

□

It may sometimes be necessary to consider the above expression subject to operators that may map α or β to multi-indices contained in $\mathbb{Z}^m$ rather than $\mathbb{N}_0^m$. We therefore extend the function $\widehat{f}(\mu,\alpha,\beta)$ from $\mathbb{R}^n \times \mathbb{N}_0^m \times \mathbb{N}_0^m$ to $\mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{Z}^m$ by setting

$$\widehat{f}(\mu,\alpha,\beta) := 0 \quad \text{for all } (\alpha,\beta) \notin \mathbb{N}_0^m \times \mathbb{N}_0^m.$$

One can show that these Hermite functions are eigenfunctions of $\widehat{L}(\mu)$, the group-Fourier transform of the sub-Laplacian, with eigenvalue

$$c(|\alpha|)|\mu| := 2\pi(2|\alpha| + m)|\mu|. \tag{2.2.23}$$

That is,

$$\widehat{L}(\mu)h^{\mu}_{\alpha} = c(|\alpha|)|\mu|h^{\mu}_{\alpha}.$$

The Fourier transform is also compatible with the spectral decomposition and functional calculus of L, and so we have

$$\widehat{m(L)f}(\mu,\alpha,\beta) = m(c(|\alpha|)|\mu|)\widehat{f}(\mu,\alpha,\beta). \tag{2.2.24}$$

As noted in, for example, [52], the matrix components with respect to the Hermite basis have a connection to special Hermite functions. Although written in the case of Heisenberg groups, Chapter IV of [52] may be extended to H-type groups, which we do here. For $a, b \in \mathbb{N}_0$ and $p, q \in \mathbb{R}$ the 1-dimensional special Hermite functions are defined by

$$\Phi^{\mu}_{a,b}(p,q) := \int\limits_{\mathbb{R}} e^{2\pi i xq|\mu|}h^{\mu}_{a}(x + \tfrac{1}{2}p)h^{\mu}_{b}(x - \tfrac{1}{2}p)dx.$$

That is, they are the Fourier-Wigner transform of Hermite functions. We then expand this definition to higher dimensions. For $\alpha, \beta \in \mathbb{N}^d_0$ and $x, y \in \mathbb{R}^d$ we define

$$\Phi^{\mu}_{\alpha,\beta}(x,y) := \prod_{j=1}^{d} \Phi^{\mu}_{\alpha_j,\beta_j}(x_j, y_j).$$

Now, we can write our matrix components in terms of special Hermite functions. Indeed, we have the following lemma.

Lemma 2.18. *For $f \in L^1(G)$ we have*

$$\widehat{f}(\mu,\alpha,\beta) = \int\limits_{G} e^{2\pi i\mu u} f(z,u)\Phi^{\mu}_{\alpha,\beta}(z^{(re)}(\mu), z^{(im)}(\mu))dzdu.$$

Proof. We have

$$\begin{aligned}
\widehat{f}(\mu,\alpha,\beta) &= \int_G f(z,u)\langle \pi_\mu(z,u)h^\mu_\alpha, h^\mu_\beta\rangle_{\mathbb{R}^m} dzdu \\
&= \int_G \int_{\mathbb{R}^m} f(z,u) e^{2\pi i(\mu u + |\mu|(z^{(im)}(\mu)x + \frac{1}{2}z^{(im)}(\mu)z^{(re)}(\mu)))} h^\mu_\alpha(x + z^{(re)}(\mu)) h^\mu_\beta(x) dxdzdu \\
&= \int_G e^{2\pi i\mu u} f(z,u) \int_{\mathbb{R}^m} e^{2\pi i|\mu|z^{(im)}(\mu)x} h^\mu_\alpha(x + \tfrac{1}{2}z^{(re)}(\mu)) h^\mu_\beta(x - \tfrac{1}{2}z^{(re)}(\mu)) dxdzdu \\
&= \int_G e^{2\pi i\mu u} f(z,u) \Phi^\mu_{\alpha,\beta}(z^{(re)}(\mu), z^{(im)}(\mu)) dzdu.
\end{aligned}$$

□

In the particular case that f is radial, certain simplifications occur. We find that in such a case, off-diagonal matrix components are zero, and furthermore that they are dependent only on the magnitude of α. In order words, we have

$$\widehat{f}(\mu,\alpha,\beta) = \delta_{\alpha,\beta}\widehat{f}(\mu, |\alpha|e_1, |\alpha|e_1). \tag{2.2.25}$$

In this case, we adopt the notation

$$\widehat{f}(\mu,k) = \widehat{f}(\mu, ke_1, ke_1)$$

and (2.2.23) becomes

$$|\mu|c(k) = |\mu|2\pi(2k+m).$$

That is, $\widehat{f}(\mu,\alpha,\beta)$ reduces to a function defined on $\mathbb{R}^n \times \mathbb{N}_0$. We recall that Laguerre polynomials of type $a > -1$ and degree k are defined by

$$L^a_k(x) := \frac{1}{k!} e^x x^{-a} \frac{d^k}{dx^k}(e^{-x}x^{k+a}), \quad x \in \mathbb{R}.$$

Similarly to Hermite functions, Laguerre polynomials satisfy a number of recurrence relations, linking polynomials of different types and degrees, as well as their derivatives and polynomials multiplied by their inputs. Given the number of identities available, for clarity of reading these will be stated where they are first needed.

Now, special Hermite functions can be expressed in terms of Laguerre functions. In the case that f is radial, we can then rewrite the matrix components as the inner product of the Euclidean inverse Fourier transform of the function $f(z,u)$ taken in the u variable only with a Laguerre function. The remainder of this section will be devoted to proving this fact.

To begin with, we will demonstrate some vector fields that may be used to shift the indices of special Hermite functions.

Lemma 2.19. *For $w = p + iq \in \mathbb{C}^d$ let $w_j = p_j + iq_j$. Define the following vector fields on $\mathbb{C}^d$:*

$$\begin{aligned}
Z_j &= \left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial w_j} + \left(\frac{\pi|\mu|}{2}\right)^{1/2} \overline{w_j} \\
\tilde{Z}_j &= \left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial w_j} - \left(\frac{\pi|\mu|}{2}\right)^{1/2} \overline{w_j} \\
\overline{Z_j} &= \left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial \overline{w_j}} + \left(\frac{\pi|\mu|}{2}\right)^{1/2} w_j \\
\overline{\tilde{Z}_j} &= \left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial \overline{w_j}} - \left(\frac{\pi|\mu|}{2}\right)^{1/2} w_j
\end{aligned}$$

Then we have

$$Z_j \Phi^\mu_{\alpha,\beta}(p,q) = \sqrt{2\alpha_j}\Phi^\mu_{\alpha-e_j,\beta}(p,q) \tag{2.2.26}$$

$$\tilde{Z}_j \Phi^\mu_{\alpha,\beta}(p,q) = \sqrt{2\beta_j+2}\Phi^\mu_{\alpha,\beta+e_j}(p,q) \tag{2.2.27}$$

$$\overline{Z_j} \Phi^\mu_{\alpha,\beta}(p,q) = -\sqrt{2\beta_j}\Phi^\mu_{\alpha,\beta-e_j}(p,q) \tag{2.2.28}$$

$$\overline{\tilde{Z}_j} \Phi^\mu_{\alpha,\beta}(p,q) = -\sqrt{2\alpha_j+2}\Phi^\mu_{\alpha+e_j,\beta}(p,q) \tag{2.2.29}$$

Proof. Since special Hermite functions in dimensions $d > 1$ are defined as products of 1-dimensional special Hermite functions, it suffices to consider the case $d = 1$. Noting that $2x = (x + \frac{1}{2}p) + (x - \frac{1}{2}p)$ we have that

$$\left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial p}\Phi^{\mu}_{a,b}(p,q) = (2\pi|\mu|)^{-1/2} \int_{\mathbb{R}} e^{2\pi i x q|\mu|}(h^{\mu}_a)'(x + \tfrac{1}{2}p)h^{\mu}_b(x - \tfrac{1}{2}p) - e^{2\pi i x q|\mu|}h^{\mu}_a(x + \tfrac{1}{2}p)(h^{\mu}_b)'(x - \tfrac{1}{2}p)dx$$

and

$$i\left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial q}\Phi^{\mu}_{a,b}(p,q) = -(2\pi|\mu|)^{1/2} \int_{\mathbb{R}} e^{2\pi i x q|\mu|}(x + \tfrac{1}{2}p)h^{\mu}_a(x + \tfrac{1}{2}p)h^{\mu}_b(x - \tfrac{1}{2}p) + e^{2\pi i x q|\mu|}h^{\mu}_a(x + \tfrac{1}{2}p)(x - \tfrac{1}{2}p)h^{\mu}_b(x - \tfrac{1}{2}p)dx$$

Note that from (2.2.19) and (2.2.20) we have

$$(2\pi|\mu|)^{1/2} x h^{\mu}_a(x) + (2\pi|\mu|)^{-1/2}\frac{d}{dx}h^{\mu}_a(x) = \sqrt{2a}h^{\mu}_{a-1}(x) \tag{2.2.30}$$

and

$$(2\pi|\mu|)^{1/2} x h^{\mu}_a(x) - (2\pi|\mu|)^{-1/2}\frac{d}{dx}h^{\mu}_a(x) = \sqrt{2a+2}h^{\mu}_{a+1}(x). \tag{2.2.31}$$

Thus, we deduce that

$$\left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial w}\Phi^{\mu}_{a,b}(p,q) = \frac{\sqrt{2a}}{2}\Phi^{\mu}_{a-1,b}(p,q) + \frac{\sqrt{2b+2}}{2}\Phi^{\mu}_{a,b+1}(p,q) \tag{2.2.32}$$

and

$$\left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial \overline{w}}\Phi^{\mu}_{a,b}(p,q) = -\frac{\sqrt{2a+2}}{2}\Phi^{\mu}_{a+1,b}(p,q) - \frac{\sqrt{2b}}{2}\Phi^{\mu}_{a,b-1}(p,q). \tag{2.2.33}$$

Furthermore, noting that $p = (x + \frac{1}{2}p) - (x - \frac{1}{2}p)$, we have

$$(2\pi|\mu|)^{1/2} p\Phi^{\mu}_{a,b}(p,q) = (2\pi|\mu|)^{1/2} \int_{\mathbb{R}} e^{2\pi i x q|\mu|}(x + \tfrac{1}{2}p)h^{\mu}_a(x + \tfrac{1}{2}p)h^{\mu}_b(x - \tfrac{1}{2}p) - e^{2\pi i x q|\mu|}h^{\mu}_a(x + \tfrac{1}{2}p)(x - \tfrac{1}{2}p)h^{\mu}_b(x - \tfrac{1}{2}p)dx.$$

Using integration by parts we have that

$$(2\pi|\mu|)^{1/2} iq\Phi^{\mu}_{a,b}(p,q) = -(2\pi|\mu|)^{-1/2} \int_{\mathbb{R}} e^{2\pi i x q|\mu|}(h^{\mu}_a)'(x + \tfrac{1}{2}p)h^{\mu}_b(x - \tfrac{1}{2}p)$$
$$+ e^{2\pi i x q|\mu|}h^{\mu}_a(x + \tfrac{1}{2}p)(h^{\mu}_b)'(x - \tfrac{1}{2}p)dx.$$

We can combine these as before to obtain

$$(2\pi|\mu|)^{1/2} w\Phi^{\mu}_{a,b}(p,q) = \sqrt{2a+2}\Phi^{\mu}_{a+1,b}(p,q) - \sqrt{2b}\Phi^{\mu}_{a,b-1}(p,q) \tag{2.2.34}$$

and

$$(2\pi|\mu|)^{1/2} \overline{w}\Phi^{\mu}_{a,b}(p,q) = \sqrt{2a}\Phi^{\mu}_{a-1,b}(p,q) - \sqrt{2b+2}\Phi^{\mu}_{a,b+1}(p,q). \tag{2.2.35}$$

The desired formulae follow from combining these expressions so that the left-hand side becomes one of the operators in question. □

The next tool we will develop will be the connection between special Hermite functions and Laguerre functions. First, we show that special Hermite functions with matching indices are Laguerre functions. The calculations may be found in [58], however, we are required to renormalise them for use with our Hermite functions. It suffices to consider only the 1-dimensional case.

Lemma 2.20. *Let $k \in \mathbb{N}_0$ and $p, q \in \mathbb{R}$. Then*

$$\Phi^{\mu}_{k,k}(p,q) = L_k(\pi|\mu|(p^2+q^2))e^{-\frac{\pi|\mu|(p^2+q^2)}{2}}. \tag{2.2.36}$$

Proof. We start with Mehler's formula [58] given, for $|w| < 1$, by

$$\sum_{k\in\mathbb{N}_0} h^{\mu}_k(x)h^{\mu}_k(y)w^k = (2\pi|\mu|)^{1/2}\pi^{-1/2}(1-w^2)^{-1/2}e^{-2\pi|\mu|\frac{1}{2}\frac{1+w^2}{1-w^2}(x^2+y^2)+2\pi|\mu|\frac{2w}{1-w^2}xy}. \tag{2.2.37}$$

If we set $x = z + \frac{1}{2}p$ and $y = z - \frac{1}{2}p$ this becomes

$$\sum_{k\in\mathbb{N}_0} h_k^\mu(z+\tfrac{1}{2}p)h_k^\mu(z-\tfrac{1}{2}p)w^k = (2\pi|\mu|)^{1/2}\pi^{-1/2}(1-w^2)^{-1/2}e^{-2\pi|\mu|\frac{1+w}{1-w}\frac{p^2}{4}+2\pi|\mu|\frac{1-w}{1+w}z^2}.$$

We then take the inverse Fourier transform of this equation. The left-hand side becomes

$$\sum_{k\in\mathbb{N}_0}\int_{\mathbb{R}} e^{2\pi izq|\mu|}h_k^\mu(z+\tfrac{1}{2}p)h_k^\mu(z-\tfrac{1}{2}p)w^k dz = \sum_{k\in\mathbb{N}_0}\Phi_{k,k}^\mu(p,q)w^k.$$

The right-hand side becomes

$$\begin{aligned}&\int_{\mathbb{R}} e^{2\pi izq|\mu|}(2\pi|\mu|)^{1/2}\pi^{-1/2}(1-w^2)^{-1/2}e^{-2\pi|\mu|\frac{1+w}{1-w}\frac{p^2}{4}+2\pi|\mu|\frac{1-w}{1+w}z^2}\\ &\qquad = (2\pi|\mu|)^{1/2}\pi^{-1/2}(1-w^2)^{-1/2}e^{-2\pi|\mu|\frac{1+w}{1-w}\frac{p^2}{4}}\int_{\mathbb{R}} e^{2\pi izq|\mu|}e^{2\pi|\mu|\frac{1-w}{1+w}z^2}dz. \qquad (2.2.38)\end{aligned}$$

We recall that

$$\int_{\mathbb{R}} e^{2\pi ixy}e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}e^{\pi^2y^2/a}$$

and so (2.2.38) becomes

$$\begin{aligned}&(2\pi|\mu|)^{1/2}\pi^{-1/2}(1-w^2)^{-1/2}\sqrt{\frac{\pi(1+w)}{2\pi|\mu|(1-w)}}e^{-2\pi|\mu|\frac{1+w}{1-w}\frac{p^2}{4}-\pi^2q^2|\mu|^2\frac{1+w}{1-w}\frac{1}{2\pi|\mu|}}\\ &\qquad = (1-w)^{-1}e^{-2\pi|\mu|\frac{1+w}{1-w}\frac{p^2+q^2}{4}} = (1-w)^{-1}e^{\frac{-w}{1-w}\pi|\mu|(p^2+q^2)}e^{\frac{\pi|\mu|(p^2+q^2)}{2}}.\end{aligned}$$

From equation (1.1.45) in [58] This may be expressed in terms of Laguerre functions as

$$\sum_{k\in\mathbb{N}_0} L_k(\pi|\mu|(p^2+q^2))w^k e^{-\frac{\pi|\mu|(p^2+q^2)}{2}}.$$

Comparing coefficients, we see that

$$\Phi_{k,k}^\mu(p,q) = L_k(\pi|\mu|(p^2+q^2))e^{-\frac{\pi|\mu|(p^2+q^2)}{2}}.$$

□

We can now combine these two Lemmas to obtain a general formula expressing special Hermite functions as a product of Laguerre functions. We will explicitly deal with the 1-dimensional case. As usual, we can take the product of these to obtain expressions for higher dimensional cases.

Lemma 2.21. *Let $z = x + iy \in \mathbb{C}$ and let $a, h \in \mathbb{N}_0$. The following formulae hold:*

$$\Phi^{\mu}_{a+h,a}(x, y) = e^{-\frac{\pi|\mu||z|^2}{2}} \left(\frac{a!}{(a+h)!}\right)^{1/2} \left((\pi|\mu|)^{1/2} z\right)^h L^h_a(\pi|\mu||z|^2) \tag{2.2.39}$$

and

$$\Phi^{\mu}_{a,a+h}(x, y) = e^{-\frac{\pi|\mu||z|^2}{2}} \left(\frac{a!}{(a+h)!}\right)^{1/2} \left(-(\pi|\mu|)^{1/2} \bar{z}\right)^h L^h_a(\pi|\mu||z|^2). \tag{2.2.40}$$

Proof. Note that it suffices to prove only (2.2.39) as

$$\Phi^{\mu}_{a,a+h}(x, y) = \overline{\Phi^{\mu}_{a+h,a}(-x, -y)}.$$

We proceed by induction on h. The case $h = 0$ is proven in Lemma 2.20. Assume (2.2.39) holds for arbitrary a and some $h - 1$ and define

$$\overline{Z} = \left(\frac{2}{\pi|\mu|}\right)^{1/2} \frac{\partial}{\partial \bar{z}} + \left(\frac{\pi|\mu|}{2}\right)^{1/2} z.$$

In this case, by (2.2.28) we have

$$\Phi^{\mu}_{a+h,a}(x, y) = \frac{-1}{\sqrt{2a+2}} \overline{Z} \Phi^{\mu}_{a+h,a+1}(x, y).$$

Then, using (2.2.39) with indices $a, h - 1$, $\frac{d}{dx} L^a_k(x) = -L^{a+1}_{k-1}(x)$, $\partial_{\bar{z}}|z|^2 = z$ and noting that $\partial_{\bar{z}} z = 0$, we calculate

$$\begin{aligned}
\overline{Z}\Phi^{\mu}_{a+h,a+1}(x,y) &= \left(\left(\frac{2}{\pi|\mu|}\right)^{1/2}\frac{\partial}{\partial \bar{z}} + \left(\frac{\pi|\mu|}{2}\right)^{1/2} z\right)\left(\frac{(a+1)!}{(a+h)!}\right)^{1/2} \\
&\qquad \left((\pi|\mu|)^{1/2} z\right)^{h-1} L^{h-1}_{a+1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} \\
&= -\left(\frac{(a+1)!}{(a+h)!}\right)^{1/2}\left((\pi|\mu|)^{1/2} z\right)^{h-1}\left(\frac{2}{\pi|\mu|}\right)^{1/2} \pi|\mu| z L^{h}_{a}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} \\
&\quad -\left(\frac{(a+1)!}{(a+h)!}\right)^{1/2}\left((\pi|\mu|)^{1/2} z\right)^{h-1}\left(\frac{2}{\pi|\mu|}\right)^{1/2} L^{h-1}_{a+1}(\pi|\mu||z|^2)\frac{\pi|\mu| z}{2} e^{-\frac{\pi|\mu||z|^2}{2}} \\
&\quad +\left(\frac{(a+1)!}{(a+h)!}\right)^{1/2}\left((\pi|\mu|)^{1/2} z\right)^{h-1}\left(\frac{\pi|\mu|}{2}\right)^{1/2} z L^{h-1}_{a+1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} \\
&= -\left(\frac{(a+1)!}{(a+h)!}\right)^{1/2}\left((\pi|\mu|)^{1/2} z\right)^{h} \sqrt{2} L^{h}_{a}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}}.
\end{aligned}$$

Combining this and (2.2.28) we have

$$\begin{aligned}
\Phi^{\mu}_{a+h,a}(x,y) &= \frac{-1}{\sqrt{2a+2}}\overline{Z}\Phi^{\mu}_{a+h,a+1}(x,y) \\
&= e^{-\frac{\pi|\mu||z|^2}{2}}\left(\frac{a!}{(a+h)!}\right)^{1/2}\left((\pi|\mu|)^{1/2} z\right)^{h} L^{h}_{a}(\pi|\mu||z|^2)
\end{aligned}$$

as required. □

Before we prove the next theorem, we will make an observation regarding the choice of coordinates in Lemma 2.18. While it is necessary to use our μ-dependent coordinate system in general, if f is a radial function (so there exists a function F such that $f(z,u) = F(|z|,u)$ then we may in fact use standard coordinates. Define $M(\mu)$ as the $\mathbb{R}$-linear operator on $\mathbb{C}^m$ such that

$$M(\mu)z = z(\mu), \qquad \text{for } z \in \mathbb{C}^m.$$

Recall that $M(\mu)$ is multiplication of the canonical coordinates z by an orthogonal matrix. In particular, this matrix (which we again denote by $M(\mu)$) is invertible and has determinant either 1 or -1. Furthermore, the Jacobian matrix of $M(\mu)$ is the matrix $M(\mu)$. Also, for standard coordinates $z = x + iy$ set $\tilde{\Phi}^{\mu}_{\alpha,\beta}(z) = \Phi^{\mu}_{\alpha,\beta}(x,y)$. Then, by the standard change of coordinates formula we have

$$\begin{aligned}
&\int_G e^{2\pi i\mu u} f(z,u)\Phi^{\mu}_{\alpha,\beta}(z^{(re)}(\mu), z^{(im)}(\mu))dz^{(re)}(\mu)dz^{(im)}(\mu)du \\
&\qquad = \int_G e^{2\pi i\mu u} F(|z|,u)\tilde{\Phi}^{\mu}_{\alpha,\beta}(M(\mu)z)dz^{(re)}(\mu)dz^{(im)}(\mu)du \\
&\qquad = \int_G e^{2\pi i\mu u} F(|M(\mu)^{-1}z|,u)\tilde{\Phi}^{\mu}_{\alpha,\beta}(z)|\det(M(\mu))|dzdu \\
&\qquad = \int_G e^{2\pi i\mu u} f(z,u)\tilde{\Phi}^{\mu}_{\alpha,\beta}(z)dzdu.
\end{aligned}$$

Hence, for any calculations with the assumption that f is a radial function, we can assume we are working in the standard coordinates of z, which from now on we will always denote by $z = x + iy$. This gives us the following result.

Lemma 2.22. *For radial functions f we have*

$$\begin{aligned}
&\int_G e^{2\pi i\mu u} f(z,u)\Phi^{\mu}_{\alpha,\beta}(z^{(re)}(\mu), z^{(im)}(\mu))dzdu \\
&\qquad = \int_G e^{2\pi i\mu u} f(z,u)\Phi^{\mu}_{\alpha,\beta}(x,y)dzdu = \int_G e^{2\pi i\mu u} f(z,u)\tilde{\Phi}^{\mu}_{\alpha,\beta}(z)dzdu,
\end{aligned}$$

where $z = x + iy$ are the standard coordinates of z.

With this, we can now show that radial functions have diagonal matrix components.

Theorem 2.23. *Let f be radial. Then*

$$\widehat{f}(\mu,\alpha,\beta) = \delta_{\alpha,\beta}\widehat{f}(\mu,\alpha,\alpha).$$

Proof. For $z \in \mathbb{C}^m$ let $z_{j,\theta} = (z_1, \ldots, e^{i\theta}z_j, \ldots, z_m)$. Since f is radial, then $f(z,u) =$

$f(z_{j,\theta}, u)$ for any $\theta \in \mathbb{R}$. Then from Lemma 2.18 and Lemma 2.22 we have

$$\begin{aligned}\widehat{f}(\mu,\alpha,\beta) &= \int_G e^{2\pi i\mu u} f(z,u)\tilde{\Phi}^{\mu}_{\alpha,\beta}(z)dzdu \\ &= \int_G e^{2\pi i\mu u} f(z_{j,\theta},u)\tilde{\Phi}^{\mu}_{\alpha,\beta}(z)dzdu \\ &= \int_G e^{2\pi i\mu u} f(z,u)\tilde{\Phi}^{\mu}_{\alpha,\beta}(z_{j,-\theta})dzdu\end{aligned}$$

By (2.2.39) and (2.2.40) we have that

$$\tilde{\Phi}^{\mu}_{\alpha,\beta}(z_{j,-\theta}) = e^{-i\theta(\alpha_j-\beta_j)}\tilde{\Phi}^{\mu}_{\alpha,\beta}(z)$$

and hence

$$\widehat{f}(\mu,\alpha,\beta) = e^{-i\theta(\alpha_j-\beta_j)}\widehat{f}(\mu,\alpha,\beta).$$

Clearly if $\alpha_j \neq \beta_j$ it follows that $\widehat{f}(\mu,\alpha,\beta) = 0$. As j was chosen arbitrarily, it follows that $\widehat{f}(\mu,\alpha,\beta)$ is zero for $\alpha \neq \beta$. □

The next step is to prove that the matrix components of radial functions depend only on the magnitude of the index.

Lemma 2.24. *Recall the vector fields $\overline{Z_j}, \overline{\tilde{Z}_j}$ defined in Lemma 2.19 by*

$$\overline{Z_j} = \left(\frac{2}{\pi|\mu|}\right)^{1/2}\frac{\partial}{\partial\overline{z_j}} + \left(\frac{\pi|\mu|}{2}\right)^{1/2} z_j, \quad \overline{\tilde{Z}_j} = \left(\frac{2}{\pi|\mu|}\right)^{1/2}\frac{\partial}{\partial\overline{z_j}} - \left(\frac{\pi|\mu|}{2}\right)^{1/2} z_j.$$

Given indices j and k, then, if f is a radial function, we find that

$$\overline{Z_j}\overline{\tilde{Z}_k}f = \overline{Z_k}\overline{\tilde{Z}_j}f.$$

Proof. First, write $f(z,u) = F(|z|,u)$. Then, for any j we have that

$$\frac{\partial}{\partial\overline{z_j}}F(|z|,u) = \frac{z_j}{2|z|}F'(|z|,u). \tag{2.2.41}$$

Clearly the purely-differential operators commute with each other, as do the pure multiplier operators. Furthermore, multipliers and differential operators commute if they have different indices. Since there is nothing to prove otherwise, let $j \neq k$. Then,

$$\begin{aligned}
\overline{Z_j}\overline{\tilde{Z}_k}f &= \left(\left(\frac{2}{\pi|\mu|}\right)^{1/2}\frac{\partial}{\partial z_j} + \left(\frac{\pi|\mu|}{2}\right)^{1/2} z_j\right)\left(\left(\frac{2}{\pi|\mu|}\right)^{1/2}\frac{\partial}{\partial z_k} - \left(\frac{\pi|\mu|}{2}\right)^{1/2} z_k\right) f \\
&= \left(\frac{2}{\pi|\mu|}\right)\frac{\partial}{\partial \overline{z_j}}\frac{\partial}{\partial \overline{z_k}}f - \frac{\pi|\mu|}{2}z_j z_k f + z_j\frac{\partial}{\partial \overline{z_k}}f - z_k\frac{\partial}{\partial \overline{z_j}}f.
\end{aligned}$$

Using (2.2.41) we note that

$$z_j\frac{\partial}{\partial \overline{z_k}}f = \frac{z_j z_k}{2|z|}f' = z_k\frac{\partial}{\partial \overline{z_j}}f.$$

The same result applies with j and k interchanged. Thus,

$$\begin{aligned}
\overline{Z_j}\overline{\tilde{Z}_k}f &= \left(\frac{2}{\pi|\mu|}\right)\frac{\partial}{\partial \overline{z_j}}\frac{\partial}{\partial \overline{z_k}}f - \frac{\pi|\mu|}{2}z_j z_k f + z_j\frac{\partial}{\partial \overline{z_k}}f - z_k\frac{\partial}{\partial \overline{z_j}}f \\
&= \left(\frac{2}{\pi|\mu|}\right)\frac{\partial}{\partial \overline{z_k}}\frac{\partial}{\partial \overline{z_j}}f - \frac{\pi|\mu|}{2}z_k z_j f + z_k\frac{\partial}{\partial \overline{z_j}}f - z_j\frac{\partial}{\partial \overline{z_k}}f \\
&= \overline{Z_k}\overline{\tilde{Z}_j}f.
\end{aligned}$$

□

We now have the tools we need to show the first of the main results of this section.

Theorem 2.25. *If f is radial, then $\widehat{f}(\mu,\alpha,\beta) = \delta_{\alpha,\beta}\widehat{f}(\mu,|\alpha|e_1,|\alpha|e_1)$.*

Proof. Since f is radial, its matrix components are diagonal by Theorem 2.23. Using integration by parts twice and Lemma 2.19 we have that

$$\begin{aligned}
[\overline{Z_k}\overline{\tilde{Z}_j}f]^\frown(\mu,\alpha-e_j,\alpha+e_k) &= \int_G e^{2\pi i\mu u} Z_k\tilde{Z}_j f(z,u)\tilde{\Phi}^{\mu}_{\alpha-e_j,\alpha+e_k}(z)dzdu \\
&= \int_G e^{2\pi i\mu u} f(z,u) Z_k\tilde{Z}_j\tilde{\Phi}^{\mu}_{\alpha-e_j,\alpha+e_k}(z)dzdu \\
&= 2\sqrt{\alpha_j(\alpha_k+1)}\widehat{f}(\mu,\alpha,\alpha)
\end{aligned}$$

and similarly

$$[\overline{Z_j}\overline{\tilde{Z}_k}f]^\wedge(\mu,\beta-e_k,\beta+e_j) = 2\sqrt{\beta_k(\beta_j+1)}\widehat{f}(\mu,\beta,\beta).$$

Now, fix $j \neq k$ and take α,β such that $\alpha + e_k = \beta + e_j$. By Lemma 2.24 and the choice of α,β the left-hand sides are equal. By our choice of α,β the coefficients of the right-hand sides are equal. Hence, we must have that $\widehat{f}(\mu,\alpha,\alpha) = \widehat{f}(\mu,\beta,\beta)$. Since j,k were arbitrary, we can use this to pass between any two multi-indices with the same size and, upon doing so, the matrix components do not change value. □

Now, our use of the notation $\widehat{f}(\mu,k)$ makes sense for radial functions. Our final goal for this section will be to show that, in such a case we may write these matrix components in terms of Laguerre functions.

Theorem 2.26. *For $f \in \mathscr{S}(G)$ radial we have*

$$\widehat{f}(\mu,k) = \binom{k+m-1}{k}^{-1} \int_G e^{2\pi i\mu u} f(z,u) L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dz du. \tag{2.2.42}$$

Proof. Fix $k \in \mathbb{N}_0$ and let $\alpha = (k,0,\ldots,0) \in \mathbb{N}_0^m$. We start with Lemma 2.18,

$$\widehat{f}(\mu,k) = \int_G e^{2\pi i\mu u} f(z,u)\tilde{\Phi}^\mu_{\alpha,\alpha}(z) dz du$$

Now, from page 30 of [58] we have

$$\sum_{|\alpha|=k} \tilde{\Phi}^\mu_{\alpha,\alpha}(z) = L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}}. \tag{2.2.43}$$

This may be proved by induction on m using Lemma 2.20 and the well-known formula

$$L_k^{a+b+1}(x+y) = \sum_{j=0}^k L_j^a(x) L_{k-j}^b(y).$$

Since f is radial, then any matrix components $\widehat{f}(\mu,\beta,\beta)$ with $|\beta| = k$ are equal to

$\widehat{f}(\mu, \alpha, \alpha)$ by Theorem 2.25. Thus we obtain

$$\begin{aligned}\widehat{f}(\mu, k) &= \binom{k+m-1}{k}^{-1} \sum_{|\beta|=k} \widehat{f}(\mu, \beta, \beta) \\ &= \binom{k+m-1}{k}^{-1} \int_G e^{2\pi i \mu u} f(|z|, u) \sum_{|\beta|=k} \tilde{\Phi}^{\mu}_{\beta,\beta}(z) dz du\end{aligned}$$

which, using (2.2.43), becomes

$$\binom{k+m-1}{k}^{-1} \int_G e^{2\pi i \mu u} f(|z|, u) L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dz du.$$

□

This now gives us an alternative Plancherel theorem and inversion formula for radial functions, which may also be found in [48]. Specifically, if $f \in L^2(G)$ is radial, then

$$\|f\|_2^2 = C(m, n) \int_{\mathbb{R}^n \setminus \{0\}} \sum_{k \in \mathbb{N}_0} \binom{k+m-1}{k} |\widehat{f}(\mu, k)|^2 |\mu|^m d\mu \tag{2.2.44}$$

and if $f \in \mathscr{S}(G)$ is radial then

$$f(z, u) = C(m, n) \int_{\mathbb{R}^n \setminus \{0\}} \sum_{k \in \mathbb{N}_0} \widehat{f}(\mu, k) e^{-2\pi i \mu u} L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} |\mu|^m d\mu \tag{2.2.45}$$

2.3 Leibniz Rules and Difference-Differential Operators

In this section, we will develop some of the analysis that occurs on the (Fourier-) dual spaces of H-type groups. We will calculate formulae for the difference-differential operators that arise on the dual spaces that correspond to multiplying by some of the weights in (2.2.18) on the group side. Similarly to the Leibniz rule that arises from ordinary differentiation on a product of functions on $\mathbb{R}$, these difference-differential operators will have formulae explaining how they act upon a product of functions on the dual space, which corresponds to a convolution of functions on the group G multiplied by a weight.

It is on the group side that these 'Leibniz' rules will be understood.

The work in the section will extend Leibniz rule formulae given in [24] from Heisenberg groups H_m to H-type groups (compare also [17], [23], [44], [49]). The notion of such formulae on stratified groups has been widely studied, for instance in Proposition 5.2.10 of [19], although we need more specific formulae for our application to H-type groups than are found in this much more general treatment of Leibniz rules.

We begin this section by calculating the effects of the operators $\partial_{\zeta_{\mu,j}}$ and $\partial_{\overline{\zeta_{\mu,j}}}$ (recall (2.2.17) and (2.2.18)) upon matrix components of the Fourier transform of a function.

Theorem 2.27. *For $f \in \mathscr{S}(G)$ we have*

$$(2\pi|\mu|)^{1/2}\partial_{\zeta_{\mu,j}}\widehat{f}(\mu,\alpha,\beta) = (2\alpha_j + 2)^{1/2}\widehat{f}(\mu,\alpha + e_j,\beta) - 2\beta_j{}^{1/2}\widehat{f}(\mu,\alpha,\beta - e_j) \quad (2.3.1)$$

and

$$(2\pi|\mu|)^{1/2}\partial_{\overline{\zeta_{\mu,j}}}\widehat{f}(\mu,\alpha,\beta) = 2\alpha_j{}^{1/2}\widehat{f}(\mu,\alpha - e_j,\beta) - (2\beta_j + 2)^{1/2}\widehat{f}(\mu,\alpha,\beta + e_j). \quad (2.3.2)$$

Proof. Fix $\mu \in \mathbb{R}^n \backslash \{0\}$ and $f \in \mathscr{S}(G)$. Then by adding and subtracting x_j we have

$$\begin{aligned}
&(2\pi|\mu|)^{1/2}(z_j(\mu) + i\tilde{z}_j(\mu))e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)\\
&\quad = (2\pi|\mu|)^{1/2}e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}(z_j(\mu) + x_j)h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)\\
&\qquad - (2\pi|\mu|)^{1/2}e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}x_jh^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)\\
&\qquad + (2\pi|\mu|)^{1/2}i\tilde{z}_j(\mu)e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x).
\end{aligned}$$

Now, by the chain rule,

$$\begin{aligned}
&(2\pi|\mu|)^{1/2}i\tilde{z}_j(\mu)e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)\\
&\quad = (2\pi|\mu|)^{-1/2}\partial_j\left(e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)\right)\\
&\qquad - (2\pi|\mu|)^{-1/2}e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}\partial_jh^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)\\
&\qquad - (2\pi|\mu|)^{-1/2}e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)\partial_{x_j}h^{\mu}_{\beta}(x)
\end{aligned}$$

Hence, we have

$$(2\pi|\mu|)^{1/2}(z_j(\mu) + i\tilde{z}_j(\mu))e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x) \quad (2.3.3)$$
$$= e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))} \cdot$$
$$\left((2\pi|\mu|)^{1/2}(z_j(\mu) + x_j) - (2\pi|\mu|)^{-1/2}\partial_{x_j}\right) h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)$$
$$- e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))} \cdot$$
$$h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)\left((2\pi|\mu|)^{1/2}z_j(\mu) + (2\pi|\mu|)^{-1/2}\partial_{x_j}\right) h^{\mu}_{\beta}(x)$$
$$+ (2\pi|\mu|)^{-1/2}\partial_{x_j}\left(e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)\right).$$

Now, we note that

$$\int_{\mathbb{R}^m} \partial_{x_j}\left(e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x)\right) dx$$
$$= e^{2\pi i(\mu u+\frac{1}{2}|\mu|z^{(re)}(\mu)z^{(im)}(\mu))} \int_{\mathbb{R}} \frac{d}{dx_j} e^{2\pi i|\mu|\tilde{z}_j(\mu)x_j} h^{\mu}_{\alpha_j}(z_j(\mu) + x_j)h^{\mu}_{\beta_j}(x_j)dx_j$$
$$\cdot \prod_{k\neq j}\int_{\mathbb{R}} e^{2\pi i|\mu|\tilde{z}_k(\mu)x_k}h^{\mu}_{\alpha_k}(z_k(\mu) + x_k)h^{\mu}_{\beta_k}(x_k)dx_k = 0.$$

Hence, taking the integral of (2.3.3) and using (2.2.30) and (2.2.31), we can conclude that

$$(2\pi|\mu|)^{1/2}\partial_{\zeta_{\mu,j}}\widehat{f}(\mu,\alpha,\beta) = (2\alpha_j + 2)^{1/2}\widehat{f}(\mu,\alpha + e_j,\beta) - 2\beta_j{}^{1/2}\widehat{f}(\mu,\alpha,\beta - e_j).$$

The proof for (2.3.2) follows analogously by considering

$$(2\pi|\mu|)^{1/2}(z_j(\mu) - i\tilde{z}_j(\mu))e^{2\pi i(\mu u+|\mu|(z^{(im)}(\mu)x+\frac{1}{2}z^{(re)}(\mu)z^{(im)}(\mu)))}h^{\mu}_{\alpha}(z^{(re)}(\mu) + x)h^{\mu}_{\beta}(x).$$

□

By combining these operators and summing over j we thus obtain a formula for the difference-differential operator ∂_{ρ^2} (recall (2.2.18)) for radial functions, specifically

$$\partial_{\rho^2}\widehat{f}(\mu,k) = \frac{1}{\pi|\mu|}[(2k+m)\widehat{f}(\mu,k) - k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)]. \tag{2.3.4}$$

We can also combine the $\partial_{\zeta_{\mu,j}}$ operators so that the μ in $\partial_{\zeta_{\mu,j}}$ does not have to match the μ in $\widehat{f}(\mu,\alpha,\beta)$.

Lemma 2.28. *Let $\mu_1,\mu \in \mathbb{R}^n\backslash\{0\}$. Then there exist constants $C_{i,j,k}(\mu,\mu_1)$ $(i=1,2)$ such that*

$$\partial_{\zeta_{\mu_1,j}}\widehat{f}(\mu,\alpha,\beta) = \sum_{k=1}^{m} C_{1,j,k}(\mu,\mu_1)\partial_{\zeta_{\mu,k}}\widehat{f}(\mu,\alpha,\beta) + C_{2,j,k}\partial_{\overline{\zeta_{\mu,k}}}\widehat{f}(\mu,\alpha,\beta) \tag{2.3.5}$$

where $|C_{i,j,k}(\mu,\mu_1)|$ is bounded uniformly in i,j,k,μ,μ_1. An analogous formula holds for $\partial_{\overline{\zeta_{\mu_1,j}}}\widehat{f}(\mu,\alpha,\beta)$.

Proof. Recall the definition of $m_{j,k}(\mu,\mu_1)$ given by (2.2.6). By linearity of the Fourier transform we have

$$\begin{aligned}\partial_{\zeta_{\mu_1,j}}\widehat{f}(\mu,\alpha,\beta) = &\sum_{k=1}^{m} m_{j,k}(\mu,\mu_1)\frac{\partial_{\zeta_{\mu,k}}\widehat{f}(\mu,\alpha,\beta)+\partial_{\overline{\zeta_{\mu,k}}}\widehat{f}(\mu,\alpha,\beta)}{2}\\ &+\sum_{k=1}^{m} m_{j,k+m}(\mu,\mu_1)\frac{\partial_{\zeta_{\mu,k}}\widehat{f}(\mu,\alpha,\beta)-\partial_{\overline{\zeta_{\mu,k}}}\widehat{f}(\mu,\alpha,\beta)}{2i}\\ &+i\sum_{k=1}^{m} m_{j+m,k}(\mu,\mu_1)\frac{\partial_{\zeta_{\mu,k}}\widehat{f}(\mu,\alpha,\beta)+\partial_{\overline{\zeta_{\mu,k}}}\widehat{f}(\mu,\alpha,\beta)}{2}\\ &+i\sum_{k=1}^{m} m_{j+m,k+m}(\mu,\mu_1)\frac{\partial_{\zeta_{\mu,k}}\widehat{f}(\mu,\alpha,\beta)-\partial_{\overline{\zeta_{\mu,k}}}\widehat{f}(\mu,\alpha,\beta)}{2i}.\end{aligned}$$

Now, recall from Lemma 2.13 that $M(\mu,\mu_1)$ is an orthogonal matrix. In particular, since $M(\mu,\mu_1)M(\mu,\mu_1)^T = Id$ then each entry $m_{j,k}(\mu,\mu_1)$ must have absolute value at most 1. □

We can also derive a formula for the operator ∂_{ψ_j} (recall (2.2.18)) acting on radial functions.

Theorem 2.29. *For $f \in \mathscr{S}(G)$ radial we have*

$$4\pi i \partial_{\psi_j} \widehat{f}(\mu,k) = 2\frac{\partial}{\partial \mu_j}\widehat{f}(\mu,k) + \frac{\mu_j}{|\mu|^2}[m\widehat{f}(\mu,k) + k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)]. \quad (2.3.6)$$

Proof. We differentiate (2.2.42) with respect to μ_j. Now, from the product rule and the identity $\frac{d}{dx}L_k^a(x) = -L_{k-1}^{a+1}(x)$ we find that

$$\begin{aligned}
&\partial_{\mu_j}\binom{k+m-1}{k}^{-1}\int_G e^{2\pi i\mu u} f(z,u) L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dzdu \\
&\quad = 2\pi i \binom{k+m-1}{k}^{-1}\int_G u_j e^{2\pi i\mu u} f(z,u) L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dzdu \\
&\quad\quad - \pi\frac{\mu_j}{|\mu|}\binom{k+m-1}{k}^{-1}\int_G |z|^2 e^{2\pi i\mu u} f(z,u) L_{k-1}^{m}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dzdu \\
&\quad\quad\quad - \frac{\pi}{2}\frac{\mu_j}{|\mu|}\binom{k+m-1}{k}^{-1}\int_G |z|^2 e^{2\pi i\mu u} f(z,u) L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dzdu.
\end{aligned}$$

Note we may freely pass the derivative through the integral as neither z nor u depends on μ. We multiply the middle line by $|\mu|/|\mu|$ to obtain

$$\begin{aligned}
&\partial_{\mu_j}\binom{k+m-1}{k}^{-1}\int_G e^{2\pi i\mu u} f(z,u) L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dzdu \\
&\quad = 2\pi i \binom{k+m-1}{k}^{-1}\int_G u_j e^{2\pi i\mu u} f(z,u) L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dzdu \\
&\quad - \frac{\mu_j}{|\mu|^2}\binom{k+m-1}{k}^{-1}\int_G e^{2\pi i\mu u} f(z,u) (\pi|\mu||z|^2) L_{k-1}^{m}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dzdu \\
&\quad - \frac{\pi}{2}\frac{\mu_j}{|\mu|}\binom{k+m-1}{k}^{-1}\int_G |z|^2 e^{2\pi i\mu u} f(z,u) L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dzdu.
\end{aligned}$$

Now, for Laguerre functions we have the identity

$$xL_{k-1}^{m}(x) = (k+m-1)L_{k-1}^{m-1}(x) - kL_k^{m-1}(x).$$

Using matrix component notation and noting that

$$(k+m-1)\binom{k+m-1}{k}^{-1} = k\binom{k+m-2}{k-1}^{-1},$$

we have that

$$\partial_{\mu_j}\widehat{f}(\mu,k) = 2\pi i\partial_{\psi_j}\widehat{f}(\mu,k) + k\frac{\mu_j}{|\mu|^2}\widehat{f}(\mu,k) - k\frac{\mu_j}{|\mu|^2}\widehat{f}(\mu,k-1) - \frac{\pi}{2}\frac{\mu_j}{|\mu|}\partial_{\rho^2}\widehat{f}(\mu,k).$$

Using (2.3.4) this expands to

$$\begin{aligned}2\pi i\partial_{\psi_j}\widehat{f}(\mu,k)+&k\frac{\mu_j}{|\mu|^2}\widehat{f}(\mu,k) - k\frac{\mu_j}{|\mu|^2}\widehat{f}(\mu,k-1)- \\ &\frac{\pi}{2}\frac{\mu_j}{|\mu|}\frac{1}{\pi|\mu|}[(2k+m)\widehat{f}(\mu,k) - k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)] \\ &= 2\pi i\partial_{\psi_j}\widehat{f}(\mu,k) + k\frac{\mu_j}{|\mu|^2}\widehat{f}(\mu,k) - k\frac{\mu_j}{|\mu|^2}\widehat{f}(\mu,k-1)- \\ &\frac{\mu_j}{2|\mu|^2}[(2k+m)\widehat{f}(\mu,k) - k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)]\end{aligned}$$

After multiplying by 2, this simplifies to

$$\begin{aligned}&4\pi i\partial_{\psi_j}\widehat{f}(\mu,k) + 2k\frac{\mu_j}{|\mu|^2}\widehat{f}(\mu,k) - 2k\frac{\mu_j}{|\mu|^2}\widehat{f}(\mu,k-1)- \\ &\frac{\mu_j}{|\mu|^2}[(2k+m)\widehat{f}(\mu,k) - k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)] \\ &= 4\pi i\partial_{\psi_j}\widehat{f}(\mu,k) - \frac{\mu_j}{|\mu|^2}[m\widehat{f}(\mu,k) + k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)]\end{aligned}$$

Hence,

$$4\pi i\partial_{\psi_j}\widehat{f}(\mu,k) = 2\partial_{\mu_j}\widehat{f}(\mu,k) + \frac{\mu_j}{|\mu|^2}[m\widehat{f}(\mu,k) + k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)]$$

as required. □

We remark that we could have also calculated (2.3.4) using this method of using (2.2.42). Indeed, doing so leads to the same formula.

We can also compound these formulae. Indeed, calculating the squares of these operators, ∂_{ρ^4} and $\partial_{\psi_j^2}$ will be useful.

Lemma 2.30. *For radial functions $f \in \mathscr{S}(G)$ we have*

$$\begin{aligned}\partial_{\rho^4}\widehat{f}(\mu,k) = \frac{1}{(\pi|\mu|)^2}\Big[&((2k+m)^2+(k+m)(2k+1)-k)\widehat{f}(\mu,k) \\ &- 2k(2k+m-1)\widehat{f}(\mu,k-1) - 2(k+m)(2k+m+1)\widehat{f}(\mu,k+1) \\ &+ k(k-1)\widehat{f}(\mu,k-2) + (k+m)(k+1+m)\widehat{f}(\mu,k+2)\Big].\end{aligned} \tag{2.3.7}$$

and

$$\begin{aligned}-16\pi^2\partial_{\psi_j^2} = {}& 4\partial_{\mu_j}^2\widehat{f}(\mu,k) + 4\frac{\mu_j}{|\mu|^2}[m\partial_{\mu_j}\widehat{f}(\mu,k) + k\partial_{\mu_j}\widehat{f}(\mu,k-1) \\ &- (k+m)\partial_{\mu_j}\widehat{f}(\mu,k+1)] \\ &+ 2\left(\frac{1}{|\mu|^2} - \frac{2\mu_j^2}{|\mu|^4}\right)[m\widehat{f}(\mu,k) + k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)] \\ &+ \frac{\mu_j^2}{|\mu|^4}\Big[(m^2-(k+m)(2k+1)+k)\widehat{f}(\mu,k) + 2km\widehat{f}(\mu,k-1) \\ &- 2m(k+m)\widehat{f}(\mu,k+1) + k(k-1)\widehat{f}(\mu,k-2) \\ &+ (k+m)(k+m+1)\widehat{f}(\mu,k+2)\Big].\end{aligned} \tag{2.3.8}$$

which combine to give

$$\begin{aligned}&\partial_{|\cdot|_K^4}\widehat{f}(\mu,k) \\ &= \frac{2}{(\pi|\mu|)^2}\Big[(4k^2+m(4k-n+3))\widehat{f}(\mu,k) \\ &\quad - k(2k+2m+n-3)\widehat{f}(\mu,k-1) - (k+m)(2k-n+3)\widehat{f}(\mu,k+1)\Big] \\ &\quad - \frac{4}{\pi^2}\Delta_\mu\widehat{f}(\mu,k) - \frac{4}{(\pi|\mu|)^2}(\mu\cdot\nabla_\mu)[m\widehat{f}(\mu,k) + k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)].\end{aligned} \tag{2.3.9}$$

Proof. Since $|(z,u)|_K^4 = |z|^4 + 16\sum|u_j|^2$, then by linearity, we may calculate the operators ∂_{ρ^4} and $\partial_{\psi_j^2}$ and sum them to prove (2.3.9). Note also that these are simply the operators $\partial^2_{\rho^2}$ and $\partial^2_{\psi_j}$. So, from (2.3.4), by applying ∂_{ρ^2} to $\partial_{\rho^2}\widehat{f}(\mu,k)$ we have

$$\partial_{\rho^4}\widehat{f}(\mu,k) = \frac{1}{\pi|\mu|}[(2k+m)\partial_{\rho^2}\widehat{f}(\mu,k) - k\partial_{\rho^2}\widehat{f}(\mu,k-1) - (k+m)\partial_{\rho^2}\widehat{f}(\mu,k+1)].$$

From (2.3.4) this expands to

$$\begin{aligned}\frac{1}{(\pi|\mu|)^2}\Big[&(2k+m)[(2k+m)\widehat{f}(\mu,k) - k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)] \\ &- k[(2k+m-2)\widehat{f}(\mu,k-1) - (k-1)\widehat{f}(\mu,k-2) - (k+m-1)\widehat{f}(\mu,k)] \\ &- (k+m)[(2k+m+2)\widehat{f}(\mu,k+1) - (k+1)\widehat{f}(\mu,k) \\ &\qquad\qquad - (k+m+1)\widehat{f}(\mu,k+2)]\Big].\end{aligned}$$

Collecting terms together gives (2.3.7).

Similarly, for $16\partial_{\psi_j^2}$, by (2.3.6) we have

$$\begin{aligned}-16\pi^2\partial_{\psi_j^2}\widehat{f}(\mu,k) = 2\partial_{\mu_j}(4\pi i\partial_{\psi_j}\widehat{f}(\mu,k)) + \frac{\mu_j}{|\mu|^2}[&m4\pi i\partial_{\psi_j}\widehat{f}(\mu,k) \\ &+ k4\pi i\partial_{\psi_j}\widehat{f}(\mu,k-1) - (k+m)4\pi i\partial_{\psi_j}\widehat{f}(\mu,k+1)].\end{aligned}$$

We now calculate each of these terms separately. By (2.3.6) the first term expands to

$$\begin{aligned}&2\big[2\partial^2_{\mu_j}\widehat{f}(\mu,k) + \frac{\mu_j}{|\mu|^2}[m\partial_{\mu_j}\widehat{f}(\mu,k) + k\partial_{\mu_j}\widehat{f}(\mu,k-1) - (k+m)\partial_{\mu_j}\widehat{f}(\mu,k+1)] \\ &+ \left(\frac{1}{|\mu|^2} - \frac{2\mu_j^2}{|\mu|^4}\right)[m\widehat{f}(\mu,k) + k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)]\big].\end{aligned}$$

The second term expands to

$$2m\frac{\mu_j}{|\mu|^2}\partial_{\mu_j}\widehat{f}(\mu,k) + m\frac{\mu_j^2}{|\mu|^4}[m\widehat{f}(\mu,k) + k\widehat{f}(\mu,k-1) - (k+m)\widehat{f}(\mu,k+1)]\big].$$

The third term expands to

$$2k\frac{\mu_j}{|\mu|^2}\partial_{\mu_j}\widehat{f}(\mu,k-1)$$
$$+ k\frac{\mu_j^2}{|\mu|^4}[m\widehat{f}(\mu,k-1)+(k-1)\widehat{f}(\mu,k-2)-(k-1+m)\widehat{f}(\mu,k)].$$

Finally, the last term expands to

$$-2(k+m)\frac{\mu_j}{|\mu|^2}\partial_{\mu_j}\widehat{f}(\mu,k+1)$$
$$-(k+m)\frac{\mu_j^2}{|\mu|^4}[m\widehat{f}(\mu,k+1)+(k+1)\widehat{f}(\mu,k)-(k+1+m)\widehat{f}(\mu,k+2)].$$

Summing these together yields (2.3.8).

Summing (2.3.8) over j (noting that we must divide all the terms corresponding to ∂_{ψ_j} by π^2 and that $\sum \mu_j^2 = |\mu|^2$) and subtracting this from (2.3.7) gives (2.3.9). □

We now proceed to calculate Leibniz rules for our weights. A simple calculation demonstrates the following result.

Lemma 2.31. *For functions $f, g \in L^1(G)$ we have*

$$\rho^2(f*g) = (\rho^2 f)*g + f*(\rho^2 g) + \sum_{j=1}^{m}(\zeta_{\mu,j}f)*(\overline{\zeta_{\mu,j}}g) + \sum_{j=1}^{m}(\overline{\zeta_{\mu,j}}f)*(\zeta_{\mu,j}g). \quad (2.3.10)$$

The rule for ψ_l requires more work and we shall spend the remainder of this section developing it. We use the notation $(X)_l$ to denote the l^{th} component of the vector X.

Lemma 2.32. *Let $f, g \in L^1(G)$ and define $c_{\mu,k,j}^{(l)} := ([E_k(\mu), E_j(\mu)])_l/4i$, where $\{E_k(\mu)\}$*

is the basis described in (2.2.1). *Then*

$$\psi_l(f*g) = (\psi_l f)*g + f*(\psi_l g) \tag{2.3.11}$$
$$+\sum_{k=1}^{m}\sum_{j=1}^{m}(c^{(l)}_{\mu,k,j+m} + c^{(l)}_{\mu,k+m,j} + ic^{(l)}_{\mu,k,j} + ic^{(l)}_{\mu,k+m,j+m})(\zeta_{\mu,k}f)*(\zeta_{\mu,j}g)$$
$$+(-c^{(l)}_{\mu,k,j+m} + c^{(l)}_{\mu,k+m,j} + ic^{(l)}_{\mu,k,j} - ic^{(l)}_{\mu,k+m,j+m})(\zeta_{\mu,k}f)*(\overline{\zeta_{\mu,j}}g)$$
$$+(c^{(l)}_{\mu,k,j+m} - c^{(l)}_{\mu,k+m,j} + ic^{(l)}_{\mu,k,j} - ic^{(l)}_{\mu,k+m,j+m})(\overline{\zeta_{\mu,k}}f)*(\zeta_{\mu,j}g)$$
$$+(-c^{(l)}_{\mu,k,j+m} - c^{(l)}_{\mu,k+m,j} + ic^{(l)}_{\mu,k,j} + ic^{(l)}_{\mu,k+m,j+m})(\overline{\zeta_{\mu,k}}f)*(\overline{\zeta_{\mu,j}}g).$$

Proof. Recall that

$$(z,u)(y,v) = (z+y, u+v+\frac{1}{2}[z,y]).$$

and hence

$$\psi_l(z,u) = \psi_l((z,u)(y,v)^{-1}) + \psi_l(y,v) + \frac{1}{2}([z,y])_l, \tag{2.3.12}$$

The first two terms on the right hand side are in a suitable form that we can apply this to a convolution of functions, however, the term coming from the Lie bracket associated to the group G is not. By bilinearity of the Lie bracket we may expand this term as (recall (2.2.2) and (2.2.3))

$$[z,y] = \sum_{k=1}^{m}\sum_{j=1}^{m}\frac{1}{4}z_k^{(re)}(\mu)y_j^{(re)}(\mu)[E_k(\mu),E_j(\mu)] + \frac{1}{4i}z_k^{(re)}(\mu)y_j^{(im)}(\mu)[E_k(\mu),E_{j+m}(\mu)]$$
$$+\frac{1}{4i}z_k^{(im)}(\mu)y_j^{(re)}(\mu)[E_{k+m}(\mu),E_j(\mu)] - \frac{1}{4}z_k^{(im)}(\mu)y_j^{(im)}(\mu)[E_{k+m}(\mu),E_{j+m}(\mu)].$$

Since G is an H-type group, each of these Lie brackets will be equal to either a multiple of some basis vector U_j of the second layer or to zero if the E_js commute, but a priori we have no way of knowing exactly what each Lie bracket evaluates to. Hence, our terms will contain the 'structure constants' $c^{(l)}_{\mu,k,j}$. This is also still not in a suitable form to apply to convolutions. Indeed, we must rewrite this in terms of products of $\rho^2((z,u)(y,v)^{-1})$, $\zeta_{\mu,j}((z,u)(y,v)^{-1})$ and $\zeta_{\mu,j}(y,v)$, that is, $|z(\mu)-y(\mu)|^2$, $z_j^{(\mathbb{C})}(\mu)-y_j^{(\mathbb{C})}(\mu)$

and $y_j^{(\mathbb{C})}(\mu)$ and their conjugates. The first term may be expanded out as

$$\begin{aligned}
z_k^{(re)}(\mu)y_j^{(re)}(\mu) &= (z_k^{(\mathbb{C})}(\mu) + \overline{z_k^{(\mathbb{C})}(\mu)})(y_j^{(\mathbb{C})}(\mu) + \overline{y_j^{(\mathbb{C})}(\mu)}) \\
&= (z_k^{(\mathbb{C})}(\mu) - y_k^{(\mathbb{C})}(\mu))y_j^{(\mathbb{C})}(\mu) + (z_k^{(\mathbb{C})}(\mu) - y_k^{(\mathbb{C})}(\mu))\overline{y_j^{(\mathbb{C})}(\mu)} \\
&\quad + (\overline{z_k^{(\mathbb{C})}(\mu) - y_k^{(\mathbb{C})}(\mu)})y_j^{(\mathbb{C})}(\mu) + (\overline{z_k^{(\mathbb{C})}(\mu)} - \overline{y_k^{(\mathbb{C})}(\mu)})\overline{y_j^{(\mathbb{C})}(\mu)} \\
&\quad + y_k^{(\mathbb{C})}(\mu)y_j^{(\mathbb{C})}(\mu) + y_k^{(\mathbb{C})}(\mu)\overline{y_j^{(\mathbb{C})}(\mu)} + \overline{y_k^{(\mathbb{C})}(\mu)}y_j^{(\mathbb{C})}(\mu) + \overline{y_k^{(\mathbb{C})}(\mu)y_j^{(\mathbb{C})}(\mu)}.
\end{aligned}$$

The other terms may be expanded similarly. Combining these terms and rearranging, we obtain the following rule for ψ_l:

$$\begin{aligned}
\psi_l(f * g) = (\psi_l f)*g + f * (\psi_l g) \qquad (2.3.13) \\
+ \sum_{k=1}^{m}\sum_{j=1}^{m}(c_{\mu,k,j+m}^{(l)} + c_{\mu,k+m,j}^{(l)} + ic_{\mu,k,j}^{(l)} + ic_{\mu,k+m,j+m}^{(l)})(\zeta_{\mu,k}f) * (\zeta_{\mu,j}g) \\
+ (-c_{\mu,k,j+m}^{(l)} + c_{\mu,k+m,j}^{(l)} + ic_{\mu,k,j}^{(l)} - ic_{\mu,k+m,j+m}^{(l)})(\zeta_{\mu,k}f) * (\overline{\zeta_{\mu,j}}g) \\
+ (c_{\mu,k,j+m}^{(l)} - c_{\mu,k+m,j}^{(l)} + ic_{\mu,k,j}^{(l)} - ic_{\mu,k+m,j+m}^{(l)})(\overline{\zeta_{\mu,k}}f) * (\zeta_{\mu,j}g) \\
+ (-c_{\mu,k,j+m}^{(l)} - c_{\mu,k+m,j}^{(l)} + ic_{\mu,k,j}^{(l)} + ic_{\mu,k+m,j+m}^{(l)})(\overline{\zeta_{\mu,k}}f) * (\overline{\zeta_{\mu,j}}g) \\
+ (c_{\mu,k,j+m}^{(l)} + c_{\mu,k+m,j}^{(l)} + ic_{\mu,k,j}^{(l)} + ic_{\mu,k+m,j+m}^{(l)})f * (\zeta_{\mu,k}\zeta_{\mu,j}g) \\
+ (-c_{\mu,k,j+m}^{(l)} + c_{\mu,k+m,j}^{(l)} + ic_{\mu,k,j}^{(l)} - ic_{\mu,k+m,j+m}^{(l)})f * (\zeta_{\mu,k}\overline{\zeta_{\mu,j}}g) \\
+ (c_{\mu,k,j+m}^{(l)} - c_{\mu,k+m,j}^{(l)} + ic_{\mu,k,j}^{(l)} - ic_{\mu,k+m,j+m}^{(l)})f * (\overline{\zeta_{\mu,k}}\zeta_{\mu,j}g) \\
+ (-c_{\mu,k,j+m}^{(l)} - c_{\mu,k+m,j}^{(l)} + ic_{\mu,k,j}^{(l)} + ic_{\mu,k+m,j+m}^{(l)})f * (\overline{\zeta_{\mu,k}\zeta_{\mu,j}}g).
\end{aligned}$$

We remark that there are some symmetries in the last four lines. By using the antisymmetry of the structure constants in j and k and the commutativity of $\zeta_{\mu,k}$ and $\zeta_{\mu,j}$ when they are both on the same side of the convolution operator (that is, $f * (\zeta_{\mu,k}\zeta_{\mu,j}g) = f * (\zeta_{\mu,j}\zeta_{\mu,k}g)$), we may eliminate these lines as follows.

First, suppose that $k \neq j$. Then

$$\begin{aligned}
&(c^{(l)}_{\mu,k,j+m} + c^{(l)}_{\mu,k+m,j} + ic^{(l)}_{\mu,k,j} + ic^{(l)}_{\mu,k+m,j+m}) f * (\zeta_{\mu,k}\zeta_{\mu,j} g) \\
&\qquad + (c^{(l)}_{j,k+m} + c^{(l)}_{j+m,k} + ic^{(l)}_{j,k} + ic^{(l)}_{j+m,k+m}) f * (\zeta_{\mu,k}\zeta_{\mu,j} g) \\
&= (c^{(l)}_{\mu,k,j+m} + c^{(l)}_{j+m,k} + c^{(l)}_{\mu,k+m,j} + c^{(l)}_{j,k+m} + ic^{(l)}_{\mu,k,j} + \\
&\qquad\qquad ic^{(l)}_{j,k} + ic^{(l)}_{\mu,k+m,j+m} + ic^{(l)}_{j+m,k+m}) f * (\zeta_{\mu,k}\zeta_{\mu,j} g) = 0.
\end{aligned}$$

On the other hand, if $k = j$, then we have

$$(c^{(l)}_{\mu,k,k+m} + c^{(l)}_{\mu,k+m,k} + ic^{(l)}_{\mu,k,k} + ic^{(l)}_{\mu,k+m,k+m}) f * (\zeta_{\mu,k}\zeta_{\mu,k} g) = 0.$$

Hence, the sum over k and j of the fifth line in (2.3.13) is zero. A similar analysis shows that the last line in (2.3.13) also sums to zero. We can also similarly eliminate lines six and seven, however we must consider them together. In particular, first let $k \neq j$. Then

$$\begin{aligned}
&(-c^{(l)}_{\mu,k,j+m} + c^{(l)}_{\mu,k+m,j} + ic^{(l)}_{\mu,k,j} - ic^{(l)}_{\mu,k+m,j+m}) f * (\zeta_{\mu,k}\overline{\zeta_{\mu,j}} g) \\
&\qquad + (c^{(l)}_{j,k+m} - c^{(l)}_{j+m,k} + ic^{(l)}_{j,k} - ic^{(l)}_{j+m,k+m}) f * (\overline{\zeta_{\mu,j}}\zeta_{\mu,k} g) \\
&= (-c^{(l)}_{\mu,k,j+m} - c^{(l)}_{j+m,k} + c^{(l)}_{\mu,k+m,j} + c^{(l)}_{j,k+m} + \\
&\qquad\qquad ic^{(l)}_{\mu,k,j} + ic^{(l)}_{j,k} - ic^{(l)}_{\mu,k+m,j+m} - ic^{(l)}_{j+m,k+m}) f * (\zeta_{\mu,k}\overline{\zeta_{\mu,j}} g) = 0.
\end{aligned}$$

Furthermore, if $k = j$ then we have

$$(-c^{(l)}_{\mu,k,k+m} + c^{(l)}_{\mu,k+m,k}) f * (\zeta_{\mu,j}\overline{\zeta_{\mu,j}} g) + (c^{(l)}_{\mu,k,k+m} - c^{(l)}_{\mu,k+m,k}) f * (\overline{\zeta_{\mu,k}}\zeta_{\mu,k} g) = 0.$$

These cancellations give the desired formula. □

We should remark that the left-hand sides of (2.3.10) and (2.3.11) are independent of μ. It therefore follows that these formulae hold regardless of the choice of μ.

In general, no further cancellation in (2.3.11) (beyond the $k = j$ terms again being zero) is possible, as the non-commutative nature of convolution on H-type groups means that terms corresponding to swapping j and k around do not cancel out. We can, however, essentially ignore the structure constants, provided we are dealing with positive things. For brevity, we employ some new notation. Define

$$\zeta_{\mu,p,0} := \zeta_{\mu,p} \quad \text{and} \quad \zeta_{\mu,p,1} := \overline{\zeta_{\mu,p}}. \tag{2.3.14}$$

Corollary 2.33. *Let $f, g \in L^1(G)$. Then*

$$|\psi_l(f * g) - (\psi_l f) * g - f * (\psi_l g)| \lesssim \sum_{p=1}^{m} \sum_{q=1}^{m} \sum_{\alpha,\beta \in \{0,1\}} |(\zeta_{\mu,p,\alpha} f) * (\zeta_{\mu,q,\beta} g)| \tag{2.3.15}$$

where the implicit constant here does not depend on μ.

Proof. First, recall the change-of-coordinates matrix $M(\mu_1, \mu)$ defined by (2.2.6) for any $\mu, \mu_1 \in \mathbb{R}^n \backslash \{0\}$. Then, by Lemma 2.13, the matrix representing the change of basis from $X_j(\mu_1)$ to $X_j(\mu)$ is given again by $M(\mu_1, \mu)$. So, for each $j = 1, \ldots, 2m$ we have

$$E_j(\mu) = \sum_{k=1}^{2m} m_{k,j}(\mu_1, \mu) E_k(\mu_1). \tag{2.3.16}$$

Then, by bilinearity of the Lie bracket we have that

$$\begin{aligned}
[E_j(\mu), E_k(\mu)] &= \left[\sum_{p=1}^{2m} m_{p,j}(\mu_1, \mu) E_p(\mu_1), \sum_{q=1}^{2m} m_{q,k}(\mu_1, \mu) E_q(\mu_1) \right] \\
&= \sum_{p=1}^{2m} \sum_{q=1}^{2m} m_{p,j}(\mu_1, \mu) m_{q,k}(\mu_1, \mu) [E_p(\mu_1), E_q(\mu_1)].
\end{aligned}$$

Recall further that the matrix entries $m_{a,b}(\mu_1, \mu)$ are bounded in absolute value by 1, uniformly in a, b, μ_1, μ. We now choose (arbitrarily) to set $\mu_1 = (1, \ldots, 1)$. Then, for any $\mu \in \mathbb{R}^n \backslash \{0\}$ and for any $1 \leqslant j, k \leqslant 2m$ we have

$$|c^{(l)}_{\mu,j,k}| \lesssim \sum_{p=1}^{2m} \sum_{q=1}^{2m} |c^{(l)}_{\mu_1,p,q}|. \tag{2.3.17}$$

Observe that this bound is uniform in μ, j, k, indeed depending only on μ_1 which we have fixed to depend only on G. Consequently, by the triangle inequality, we obtain the 'Leibniz inequality' (2.3.13) as required. □

Corollary 2.34. *Let* $f, g \in L^1(G)$. *Then*

$$
\begin{aligned}
|\psi_l^2(f * g)| \lesssim & |(\psi_l^2 f) * g| + |(\psi_l f) * (\psi_l g)| + |f * (\psi_l^2 g)| \\
+ & \sum_{s=1}^{m}\sum_{t=1}^{m}\sum_{\alpha,\beta\in\{0,1\}} \Big(|(\zeta_{\mu,s,\alpha}\psi_l f) * (\zeta_{\mu,t,\beta} g)| + |(\zeta_{\mu,s,\alpha} f) * (\zeta_{\mu,t,\beta}\psi_l g)| \\
& + \sum_{p=1}^{m}\sum_{q=1}^{m}\sum_{\gamma,\epsilon\in\{0,1\}} |(\zeta_{\mu,p,\gamma}\zeta_{\mu,s,\alpha} f) * (\zeta_{\mu,q,\epsilon}\zeta_{\mu,t,\beta} g)| \Big).
\end{aligned}
\tag{2.3.18}
$$

Proof. The result follows by applying (2.3.11) twice, recalling that multiplication operators here commute, and using (2.3.17) to uniformly bound the constants. □

CHAPTER 3

PROOF OF THEOREM 1.3

In this chapter, we prove Theorem 1.3. This theorem is valid in a smaller range of p than Theorem 1.1, but where it is valid, it proves something stronger than almost-everywhere convergence, instead proving full L^p boundedness of the maximal Bochner–Riesz operator.

Recall (1.0.6) and (1.0.8). Let K_δ be the convolution kernel of $m_\delta(L)$. Note that, by Lemma 2.10,

$$m_\delta(rL)f = f * (K_\delta)_r. \tag{3.0.1}$$

Note that for each $\delta \in \mathbb{D}$, the operator M_δ^* is bounded on L^p by Lemma 2.5 and Lemma 2.11, so it suffices to consider only the terms with $\delta \leqslant \frac{1}{4}$.

Lemma 3.1. *Let G be a stratified Lie group with sub-Laplacian L and let $\eta(L)$ be the constant described in Lemma 2.7. Let K_δ be the convolution kernel of $m_\delta(L)$. For every $b > \frac{\eta(L)-1}{2}$, $\delta \leqslant \frac{1}{2}$ and $r > 0$ we have*

$$\int_G |(K_\delta(x))_r| dx \lesssim \|m_\delta\|_{L^2_{b+1/2}(\mathbb{R})} \lesssim \delta^{-b} \tag{3.0.2}$$

and

$$\|m_\delta(rL)f\|_\infty \lesssim \delta^{-b}\|f\|_\infty. \tag{3.0.3}$$

where the implicit constant in '$\lesssim$' may depend on b but does not depend on r or δ.

Proof. It suffices to assume $r = 1$. Indeed,

$$\begin{aligned}\int_G |(K_\delta(x))_r| dx &= r^{-Q/2} \int_G |K_\delta((\delta_{r^{-1/2}}(x))| dx \\ &= r^{-Q/2} r^{Q/2} \int_G |K_\delta(x)| dx \\ &= \int_G |K_\delta(x)| dx.\end{aligned}$$

The first inequality in (3.0.2) then follows from Lemma 2.7. The second inequality comes from estimating this Sobolev norm. In particular, let $j := b + \frac{1}{2} \in \mathbb{N}_0$. Then from the Euclidean Fourier transform we have

$$\|m_\delta\|^2_{L^2_{b+1/2}(\mathbb{R})} := \int_{\mathbb{R}} \left|(1+|x|)^j \widehat{m_\delta}(x)\right|^2 dx \lesssim \int_{\mathbb{R}} \sum_{k=0}^{j} \left|x^k \widehat{m_\delta}(x)\right|^2 dx \simeq \int_{\mathbb{R}} \sum_{k=0}^{j} \left|m_\delta{}^{(k)}(x)\right|^2 dx.$$

From the support and boundedness conditions on derivatives of m_δ we have

$$\int_{\mathbb{R}} \left|\sum_{k=0}^{j} m_\delta{}^{(k)}(x)\right|^2 dx \lesssim \int_{1-\delta}^{1} \left(\sum_{k=0}^{j} \delta^{-k}\right)^2 dx = \delta \left(\frac{\delta^{-j}-1}{\delta^{-1}-1}\right)^2 \leqslant \frac{\delta^{-2b}}{(\delta^{-1}-1)^2}.$$

Since $\delta \leqslant \frac{1}{2}$ then $\delta^{-1} - 1 \geqslant 1$ and so the above estimate is bounded by δ^{-2b}. Taking square roots gives the required estimate for $b + \frac{1}{2} \in \mathbb{N}_0$ and interpolation gives the estimate for any $b > \frac{\eta(L)-1}{2}$. That is, for non-integer $b > \frac{\eta(L)-1}{2}$ write $b + \frac{1}{2} = a + \frac{1}{2} + \theta$ for $a + \frac{1}{2} \in \mathbb{N}, 0 < \theta < 1$. Then

$$\|m_\delta\|_{L^2_{b+1/2}(\mathbb{R})} \leqslant \|m_\delta\|^{1-\theta}_{L^2_{a+1/2}(\mathbb{R})} \|m_\delta\|^{\theta}_{L^2_{a+3/2}(\mathbb{R})} \lesssim \delta^{-a(1-\theta)} \delta^{-(a+1)\theta} = \delta^{-(a+\theta)} = \delta^{-b}.$$

We then obtain (3.0.3) from Young's inequality for convolutions and (3.0.2). □

An immediate consequence of Lemma 3.1 is the result

$$\|M^*_\delta\|_{L^\infty \to L^\infty} \lesssim \delta^{-b} \tag{3.0.4}$$

for $b > \frac{\eta(L)-1}{2}$.

We now proceed to consider boundedness on L^2. First, we will need the following pointwise almost-everywhere equality for the global and local maximal operators of a function, $M^*_\delta f(x)$ and $M^\bullet_\delta f(x)$, resulting from the Fundamental Theorem of Calculus.

Lemma 3.2. *We have, almost-everywhere in G,*

$$|M^*_\delta f(x)|^2 \leqslant 2\delta^{-1} \int_0^\infty |m_\delta(tL)f(x)||\tilde{m}_\delta(tL)f(x)|\frac{dt}{t} \tag{3.0.5}$$

and

$$|M^\bullet_\delta f(x)|^2 \leqslant 2\delta^{-1} \int_0^1 |m_\delta(tL)f(x)||\tilde{m}_\delta(tL)f(x)|\frac{dt}{t}. \tag{3.0.6}$$

where

$$\tilde{m}_\delta(\zeta) := \delta\zeta m'_\delta(\zeta), \quad \zeta > 0. \tag{3.0.7}$$

Proof. Consider $(m_\delta(rL)f(x))^2$ as a function of $r > 0$. Then, from the functional calculus of L, we have

$$\begin{aligned}
\frac{d}{dr}(m_\delta(rL)f(x))^2 &= \frac{d}{dr}\left(\int_0^\infty m_\delta(r\lambda)f(x)dE(\lambda)\right)^2 \\
&= 2\int_0^\infty m_\delta(r\lambda)f(x)dE(\lambda)\frac{d}{dr}\left(\int_0^\infty m_\delta(r\lambda)f(x)dE(\lambda)\right) \\
&= 2\int_0^\infty m_\delta(r\lambda)f(x)dE(\lambda)\int_0^\infty \lambda m'_\delta(r\lambda)f(x)dE(\lambda) \\
&= 2\delta^{-1}m_\delta(rL)f(x)\tilde{m}_\delta(rL)f(x).
\end{aligned}$$

Hence, by the Fundamental Theorem of Calculus we have, almost-everywhere in G and for $r > 0$,

$$\begin{aligned} m_\delta(rL)f(x)^2 &= m_\delta(rL)f(x)^2 - m_\delta(0L)f(x)^2 \\ &= 2\delta^{-1}\int_0^r m_\delta(tL)f(x)\tilde{m}_\delta(tL)f(x)\frac{dt}{t}. \end{aligned}$$

Note that $m_\delta(0L)f = 0$ from the functional calculus of L, since $m_\delta(0) = 0$. Hence, taking absolute values, taking these inside the integral and taking the supremum, we obtain the required results. □

We now use the preceding lemma to bound the $L^2 \to L^2$ operator norms of the local and global maximal operators $M_\delta^\bullet$ and M_δ^*.

Lemma 3.3. *We have*

$$\|M_\delta^\bullet\|_{L^2\to L^2} \lesssim \|M_\delta^*\|_{L^2\to L^2} \lesssim 1.$$

Proof. The first inequality is obvious. Let $X, \widehat{L}$ be, respectively, the space and operator in Proposition A.2. Then from Proposition A.2, we have

$$\begin{aligned} \left\| \int_0^\infty |m_\delta(tL)f|^2 \frac{dt}{t} \right\|_{L^1} &= \int_0^\infty \int_G |m_\delta(tL)f(x)|^2 dx \frac{dt}{t} \\ &= \int_0^\infty \int_X |m_\delta(t\widehat{L}(\zeta))|^2 |\widehat{f}(\zeta)|^2 d\mu(\zeta)\frac{dt}{t} \\ &\lesssim \delta \|f\|_2^2, \end{aligned} \tag{3.0.8}$$

where the last inequality follows from the conditions on m_δ. In particular, from the support condition, recalling that $\widehat{L}$ is a multiplier operator and then the L^∞ bound on m_δ, we have

$$\int_0^\infty |m_\delta(t\widehat{L}(\zeta))|^2 \frac{dt}{t} \leqslant \int_{(1-\delta)/\widehat{L}(\zeta)}^{1/\widehat{L}(\zeta)} |m_\delta(t\widehat{L}(\zeta))|^2 \frac{dt}{t} = \int_{1-\delta}^1 |m_\delta(s)|^2 \frac{ds}{s} \lesssim \int_{1-\delta}^1 \frac{ds}{s}.$$

Since $\delta \leqslant \frac{1}{2}$ then in the above integral $\frac{1}{s} \leqslant \frac{1}{1-\delta} \leqslant 2$, so

$$\int_{1-\delta}^{1} \frac{ds}{s} \lesssim \delta.$$

Note that this also holds with m_δ replaced by $\tilde{m}_\delta$.

Now, integrating (3.0.5) from Lemma 3.2 over G yields

$$\|M_\delta^* f\|_2^2 \lesssim \delta^{-1} \int_G \int_0^\infty |m_\delta(tL)f(x)||\tilde{m}_\delta(tL)f(x)| \frac{dt}{t} dx.$$

Now, by applying the Cauchy-Schwarz inequality to the double integral followed by (3.0.8), we see that

$$\delta^{-1} \int_G \int_0^\infty |m_\delta(tL)f(x)||\tilde{m}_\delta(tL)f(x)| \frac{dt}{t} dx$$
$$\leqslant \delta^{-1} \left\| \int_0^\infty |m_\delta(tL)f|^2 \frac{dt}{t} \right\|_{L^1}^{1/2} \left\| \int_0^\infty |\tilde{m}_\delta(tL)f|^2 \frac{dt}{t} \right\|_{L^1}^{1/2} \lesssim \|f\|_2^2.$$

Hence,

$$\|M_\delta^* f\|_2^2 \lesssim \|f\|_2^2.$$

□

Finally, we present the proof of Theorem 1.3.

Proof of Theorem 1.3. By complex interpolation of the results (3.0.4) and Lemma 3.3, we get

$$\|M_\delta^*\|_{L^p \to L^p} \lesssim \delta^{(2/p-1)b}$$

for p, b satisfying

$$b > \frac{\eta(L) - 1}{2} \text{ and } p \geqslant 2.$$

In particular, note from [6] that the Riesz–Thorin interpolation theorem extends to sub-linear operators such as M_δ^*. Set $\theta = \frac{2}{p}$. Then $0 \leqslant \theta \leqslant 1$ and for

$$\frac{1}{p_\theta} := \frac{\theta}{2} + \frac{1-\theta}{\infty}$$

we get

$$\|M_\delta^*\|_{L^{p_\theta} \to L^{p_\theta}} \lesssim \delta^{-b(1-\theta)}.$$

But from the definition of θ we have

$$\frac{1}{p_\theta} = \frac{\frac{2}{p}}{2} = \frac{1}{p}$$

so that $p_\theta = p$. Hence,

$$\|M_\delta^*\|_{L^p \to L^p} \lesssim \delta^{-b(1-\theta)} = \delta^{-b(1-2/p)} = \delta^{(2/p-1)b}.$$

Now, for each $\delta \in \mathbb{D}$ the operator M_δ^* is bounded on L^p by Lemmas 2.5 and 2.11, so in the following sum it suffices to consider only the terms with $\delta \leqslant \frac{1}{4}$, as there are only finitely many terms with $\delta > \frac{1}{4}$.

By the triangle inequality, we have

$$\begin{aligned}
\|T_*^\lambda f\|_p &= \left\| \sup_{r>0} \left| \sum_{\delta \in \mathbb{D}} \delta^\lambda m_\delta(rL) f \right| \right\|_p \\
&\leqslant \sum_{\delta \in \mathbb{D}} \delta^\lambda \|M_\delta^* f\|_p \\
&\lesssim \sum_{\delta \in \mathbb{D}} \delta^\lambda \delta^{(2/p-1)b} \|f\|_p.
\end{aligned}$$

Thus, we see that T_*^λ is bounded on L^p provided

$$\lambda + \left(\frac{2}{p} - 1\right) b > 0.$$

Since $b > (\eta(L)-1)/2$ then

$$\lambda + \left(\frac{2}{p} - 1\right) b > \lambda - \left(\frac{1}{2} - \frac{1}{p}\right)(\eta(L)-1) > 0$$

where the second inequality is satisfied by assumption, concluding the proof. $\square$

There are other ways in which this result may be proved. For instance, in [45], by replacing Lemma 1.2 with Lemma 3.1 and suitably modifying Lemma 2.2, Theorem 2.6(ii) and Corollary 2.8(ii) one may prove L^p boundedness of the maximal Bochner–Riesz operator in the same range we have done.

We should note that in the range $\lambda > (\eta(L)-1)(\frac{1}{2} - \frac{1}{p})$, we have proved L^p boundedness of the maximal Bochner–Riesz operator T_*^λ. In the larger range in which we will prove almost-everywhere convergence, we will only be able to prove a weaker, but still sufficient, statement, namely boundedness from L^{p_0} to L^2_{loc} of the local maximal Bochner–Riesz operator $T_\bullet^\lambda$.

CHAPTER 4

ALMOST-EVERYWHERE CONVERGENCE VIA WEIGHTED ESTIMATES

In this chapter, we motivate the study of weighted L^2 estimates of the maximal operators $M_\delta^\bullet$. We begin with the proof of the 'standard' 3-epsilon argument used to reduce the problem of almost-everywhere convergence to proving finiteness of $\|\chi_K T_\bullet^\lambda\|_{L^p\to L^2}$.

Lemma 4.1. *Let G be a stratified group, let $\lambda > 0$ and let $p \geqslant 1$. If*

$$\|\chi_K T_\bullet^\lambda\|_{L^p\to L^2} < \infty$$

for all compact sets $K \subseteq G$, then for all $f \in L^p$ we have

$$T_r^\lambda f(x) \to f(x) \textit{ almost-everywhere as } r \to 0.$$

Proof. Let $f \in L^p(G)$ be arbitrary. We may find a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(G)$ such that $\|f_n - f\|_p \to 0$. Also, by Proposition A.2, we have

$$\|T_r^\lambda f_n - f_n\|_2^2 = \int |(1 - r\widehat{L}(\zeta))_+^\lambda \widehat{f}_n(\zeta) - \widehat{f}_n(\zeta)|^2 d\mu(\zeta).$$

Since $(1 - r\widehat{L}(\zeta))_+^\lambda \widehat{f}_n(\zeta) - \widehat{f}_n(\zeta) \to 0$ as $r \to 0$, then by Dominated convergence, for each fixed n we have,

$$\|T_r^\lambda f_n - f_n\|_2 \to 0 \quad \text{as } r \to 0.$$

Now, we have that

$$|T_r^\lambda f - f| \leqslant |T_r^\lambda f - T_r^\lambda f_n| + |T_r^\lambda f_n - f_n| + |f_n - f|.$$

The last term is trivial. By assumption, $f_n \to f$ in L^p, so there is a subsequence of (f_n) for which we have convergence point-wise almost-everywhere. We now assume we are working on this subsequence.

To deal with the first term, first we fix $K \subseteq G$ compact. We note that, by linearity of T_r^λ, the definition of the maximal operator and assumption, we have

$$\|\chi_K(T_r^\lambda f - T_r^\lambda f_n)\|_2 = \|\chi_K T_r^\lambda (f - f_n)\|_2 \leqslant \|\chi_K T_\bullet^\lambda (f - f_n)\|_2 \lesssim \|f - f_n\|_p.$$

Hence, by again extracting a subsequence we have that $\chi_K |T_\bullet^\lambda (f - f_n)| \to 0$ pointwise almost-everywhere as $n \to \infty$ inside the compact set K. Define S_K as the subset of K of full measure in which we have pointwise convergence.

Now, fix $n \in \mathbb{N}$ and let $k > \frac{Q}{2}$. Note that the operators T_r^λ and $(1 + L)^k$ commute and that if $f_n \in C_c^\infty$ then $(1 + L)^k f_n \in C_c^\infty$. By Theorem 5.15 in [21] we have that

$$\|T_r^\lambda f_n - f_n\|_\infty \lesssim \|(1 + L)^k (T_r^\lambda f_n - f_n)\|_2 = \|T_r^\lambda ((1 + L)^k f_n) - (1 + L)^k f_n\|_2.$$

Since $(1 + L)^k f_n \in C_c^\infty$ then we know from above that the L^2 norm tends to 0 as $r \to 0$, so for each fixed n we have $T_r^\lambda f_n \to f_n$ uniformly as $r \to 0$.

Now, fix $x \in S_K$ and $\epsilon > 0$. We may choose n sufficiently large so that for all $r > 0$

$$\chi_K |T_r^\lambda f(x) - T_r^\lambda f_n(x)|, |f_n(x) - f(x)| < \epsilon$$

and hence

$$\chi_K |T_r^\lambda f(x) - f(x)| \leqslant |T_r^\lambda f_n(x) - f_n(x)| + 2\epsilon.$$

Now, for this n we may choose R such that, for $r < R$ we have

$$|T_r^\lambda f_n(x) - f_n(x)| < \epsilon$$

by uniform convergence. Hence for $r < R$ we have

$$\chi_K |T_r^\lambda f(x) - f(x)| \leqslant 3\epsilon.$$

As $x \in S_K$ and ϵ were arbitrary, we have pointwise almost-everywhere convergence as $r \to 0$ inside the compact set K. Now, we note that G is σ-compact. If we consider the expression of G as a countable union of compact sets, then we have $T_r^\lambda f \to f$ pointwise almost-everywhere as $r \to 0$ inside each of these compact sets. The union of the null-subset of each compact set in which convergence fails is still a null-set, so we have convergence pointwise almost-everywhere in G. □

The next result demonstrates how we may use Hölder's inequality to reduce the problem from locally bounding an operator from some L^{p_0} to L^2 to instead proving a weighted and unweighted L^2 bound.

Lemma 4.2. *Let G be a stratified group. Suppose T is a sublinear operator that is bounded on $L^2((1 + |\cdot|)^{-A})$ for $A = 0, a$ for some $a > 0$. Then we can show that*

$$\|\chi_K T\|_{L^{p_0} \to L^2} \lesssim \max_{A \in \{0,a\}} \|T\|_{L^2((1+|\cdot|)^{-A}) \to L^2((1+|\cdot|)^{-A})}$$

where K is an arbitrary compact set and p_0 satisfies

$$\frac{1}{2} \geqslant \frac{1}{p_0} > \frac{Q - a}{2Q}.$$

The implicit constant may depend on K but does not depend on T.

Proof. Let

$$B := \{x \in G : |x| \leqslant 1\}.$$

Using Hölder's inequality we see that for $\frac{1}{2} \geqslant \frac{1}{p_0} > \frac{Q-a}{2Q}$ we have

$$\|\chi_{G\setminus B} f\|_{L^2(|\cdot|^{-a})} + \|\chi_B f\|_2 \lesssim \|f\|_{p_0}.$$

Also, for $\alpha \in \mathbb{R}$ and any compact set K, we have

$$\|\chi_K T f\|_2 \simeq \|\chi_K T f\|_{L^2((1+|\cdot|)^{-\alpha})}.$$

Writing

$$f(x) = \chi_B(x) f(x) + \chi_{G\setminus B}(x) f(x)$$

and using sublinearity of T we have

$$\|\chi_K T f\|_2^2 \lesssim \|\chi_K T \chi_B f\|_2^2 + \|\chi_K T \chi_{G\setminus B} f\|_2^2$$

Combining these facts and our initial estimate, we can deduce

$$\begin{aligned}
\|\chi_K T f\|_2^2 &\lesssim \|\chi_K T \chi_B f\|_2^2 + \|\chi_K T \chi_{G\setminus B} f\|_2^2 \\
&\simeq \|\chi_K T \chi_B f\|_2^2 + \|\chi_K T \chi_{G\setminus B} f\|_{L^2((1+|\cdot|)^{-a})}^2 \\
&\leqslant \|T\|_{L^2 \to L^2} \|\chi_B f\|_2^2 + \|T\|_{L^2((1+|\cdot|)^{-a}) \to L^2((1+|\cdot|)^{-a})} \|\chi_{G\setminus B} f\|_{L^2((1+|\cdot|)^{-a})}^2 \\
&\leqslant \max_{A \in \{0,a\}} \|T\|_{L^2((1+|\cdot|)^{-A}) \to L^2((1+|\cdot|)^{-A})} \|f\|_{p_0}^2.
\end{aligned}$$

□

This leads to the following convergence result.

Theorem 4.3. *Let G be a stratified group and L be a sub-Laplacian on G. Suppose that*

$$\|M_\delta^\bullet\|_{L^2((1+|\cdot|)^{-A}) \to L^2((1+|\cdot|)^{-A})} \lessapprox 1$$

for $A = 0, a$. *Let* $\lambda > 0$, *let* $\eta(L)$ *be as in Lemma 2.7. If*

$$\frac{Q-a}{Q}\left(\frac{1}{2} - \frac{\lambda}{\eta(L)-1}\right) < \frac{1}{p} \leqslant \frac{1}{2}$$

then for all $f \in L^p(G)$ *we have*

$$T_r^\lambda f(x) \to f(x) \text{ almost-everywhere as } r \to 0.$$

Proof. In view of Lemma 4.1, it suffices to prove that

$$\|\chi_K T_\bullet^\lambda f\|_2 \lesssim \|f\|_p,$$

for p, λ as stated and K an arbitrary compact set.

Recall (1.0.6), (1.0.7) and (3.0.1). Recall from (3.0.4) that $\|M_\delta^\bullet\|_{L^\infty \to L^\infty} \lesssim \delta^{-b}$ for $b > \frac{\eta(L)-1}{2}$. Since $\chi_K M_\delta^\bullet f$ has compact support, then we can further deduce that

$$\|\chi_K M_\delta^\bullet f\|_2 \lesssim \|M_\delta^\bullet f\|_\infty \lesssim \delta^{-b}\|f\|_\infty.$$

From Lemma 4.2 we know that $\|\chi_K M_\delta^\bullet\|_{L^{p_0} \to L^2} \lesssim_\epsilon \delta^{-\epsilon}$ for arbitrary $\epsilon > 0$ and for $\frac{1}{2} \geqslant \frac{1}{p_0} > \frac{Q-a}{2Q}$. Then by complex interpolation, we get

$$\|\chi_K M_\delta^\bullet\|_{L^p \to L^2} \lessapprox \delta^{(p_0/p-1)b - \epsilon p_0/p}$$

for p_0, p, b satisfying

$$b > \frac{\eta(L)-1}{2}, \quad p \geqslant p_0 \quad \text{and} \quad \frac{1}{2} \geqslant \frac{1}{p_0} > \frac{Q-a}{2Q}.$$

Specifically, set $\theta = \frac{p_0}{p}$. Then $0 \leqslant \theta \leqslant 1$ and for

$$\frac{1}{p_\theta} := \frac{\theta}{p_0} + \frac{1-\theta}{\infty}$$

we get

$$\|\chi_K M_\delta^\bullet\|_{L^{p_\theta}\to L^2} \lessapprox \delta^{-b(1-\theta)}\delta^{-\epsilon\theta}.$$

But from the definition of θ we have

$$\frac{1}{p_\theta} = \frac{\frac{p_0}{p}}{p_0} = \frac{1}{p}$$

so that $p_\theta = p$. The corresponding operator bound is then

$$\|\chi_K M_\delta^\bullet\|_{L^{p_\theta}\to L^2} \lessapprox \delta^{-b(1-\theta)}\delta^{-\epsilon\theta} = \delta^{-b(1-p_0/p)}\delta^{-\epsilon p_0/p} = \delta^{(p_0/p-1)b-\epsilon p_0/p}.$$

Now, for each $\delta \in \mathbb{D}$ the operator M_δ^* is bounded on L^p by Lemmas 2.5 and 2.11, so it suffices to consider only the terms with $\delta \leqslant \frac{1}{4}$.

By the triangle inequality, we have

$$\begin{aligned}\|\chi_K T_\bullet^\lambda f\|_2 &= \left\|\chi_K \sup_{0<r<1}\left|\sum_{\delta\in\mathbb{D}} \delta^\lambda m_\delta(rL) f\right|\right\|_2 \\ &\leqslant \sum_{\delta\in\mathbb{D}} \delta^\lambda \|M_\delta^\bullet f\|_2 \\ &\lesssim \sum_{\delta\in\mathbb{D}} \delta^\lambda \delta^{(p_0/p-1)b-\epsilon p_0/p}\|f\|_p.\end{aligned}$$

Thus, we have the desired estimate provided

$$\lambda + \left(\frac{p_0}{p} - 1\right) b - \epsilon\frac{p_0}{p} > 0.$$

This fact follows from the bounds on p, p_0 as follows. First, if

$$\frac{Q-a}{2Q} < \frac{1}{p} \leqslant \frac{1}{2}$$

then we may take p_0, ϵ such that $p_0 = p$ and $\lambda > \epsilon > 0$. Then

$$\lambda + \left(\frac{p_0}{p} - 1\right) b - \epsilon\frac{p_0}{p} = \lambda - \epsilon > 0$$

as required.

Otherwise, by rearranging the lower bound on $\frac{1}{p}$ we have

$$\frac{Q-a}{Q}\left(\frac{1}{2}-\frac{\lambda}{\eta(L)-1}\right)<\frac{1}{p} \iff \lambda\frac{Q-a}{2Q}\frac{2}{\eta(L)-1}+\frac{1}{p}-\frac{Q-a}{2Q}>0$$
$$\iff \lambda+\frac{\eta(L)-1}{2}\left(\frac{2Q}{Q-a}\frac{1}{p}-1\right)>0.$$

Note that the set of points $(\alpha,\beta,\epsilon)\in\mathbb{R}^3$ that satisfies

$$\lambda+\beta\left(\frac{\alpha}{p}-1\right)+\epsilon\frac{\alpha}{p}>0$$

is open and contains the point $\left(\frac{2Q}{Q-a},\frac{\eta(L)-1}{2},0\right)$. Hence we may choose $b>\frac{\eta(L)-1}{2}$, $p_0<\frac{2Q}{Q-1}$ and $\epsilon>0$ such that

$$\lambda+\left(\frac{p_0}{p}-1\right)b-\epsilon\frac{p_0}{p}>0.$$

as required. In particular, such a choice of p_0 is valid since $p\geqslant\frac{2Q}{Q-a}$ implies that $p>p_0$. □

In order to apply this result, we require the estimate

$$\|M_\delta^\bullet\|_{L^2((1+|\cdot|)^{-A})\to L^2((1+|\cdot|)^{-A})}\lessapprox 1$$

for $A=0$ and for some $a>0$. While $A=0$ is relatively trivial, the problem of obtaining this for $A=a$ is more involved.

Rather than considering $|\cdot|$, we can obtain a result of a similar flavour by replacing $|\cdot|$ by a pure first-layer weight $|x_1|$ (which, as with H-type groups, we will denote by ρ. In order to state the relevant analogues of Lemma 4.2 and Theorem 4.3, we must introduce some new terminology.

For a stratified Lie group G, we define

$$\|f\|_{(p,q)} := \left(\int_{\mathfrak{g}_1} \left(\int_{\mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_k} |f(x_1, \ldots, x_k)|^q dx_2 \ldots dx_k \right)^{p/q} \ldots dx_1 \right)^{1/p} \tag{4.0.1}$$

for $1 \leqslant p, q < \infty$ and the usual extension to p or q being ∞ (a 'mixed Lebesgue norm' where we distinguish between the first layer and the others). The space of functions for which $\|f\|_{(p,q)} < \infty$ is denoted by $L^{(p,q)}(G)$. In the particular case that G is an H-type group, we will write $L^{(p,q)}(G) := L^p(\mathbb{C}^m, L^q(\mathbb{R}^n))$. Clearly $L^{(p,p)}(G) = L^p(G)$. Where the group in question is unambiguous, we will omit the '(G)' from the notation.

Lemma 4.4. *Let G be a stratified group with* $\dim(\mathfrak{g}_1) = d$ *and let* $\rho(x) = |x_1|$ *for* $x \in G$*. Suppose T is a sublinear operator that is bounded on* $L^2((1+\rho)^{-A})$ *for* $A = 0, a$ *for some* $a > 0$*. Then we can show that*

$$\|\chi_K T\|_{L^{(p_0,2)} \to L^2} \lesssim \max_{A \in \{0,a\}} \|T\|_{L^2((1+\rho)^{-A}) \to L^2((1+\rho)^{-A})}$$

where K is an arbitrary compact set and p_0 satisfies

$$\frac{1}{2} \geqslant \frac{1}{p_0} > \frac{d-a}{2d}.$$

The implicit constant may depend on K but does not depend on T.

Proof. Let

$$B := \{x \in G : \rho \leqslant 1\}.$$

Using Hölder's inequality we see that

$$\int_{\mathfrak{g}_1 \setminus B} |f(x)|^2 |x_1|^{-a} dx_1 \leqslant \left(\int_{\mathfrak{g}_1} |f(x)|^{p_0} dx_1 \right)^{2/p_0} \left(\int_{\mathfrak{g}_1 \setminus B} |x_1|^{-qa} dx_1 \right)^{1/q} \tag{4.0.2}$$

where $\frac{2}{p_0} + \frac{1}{q} = 1$ and we must necessarily have $qa > d$ for finiteness of the right-hand side. Hence,

$$\begin{aligned}\|\chi_{G\setminus B}f\|_{L^2(\rho^{-a})} &= \left(\int\limits_{\mathfrak{g}_2\oplus\cdots\oplus\mathfrak{g}_k}\int\limits_{\mathfrak{g}_1\setminus B}|f(x)|^2|x_1|^{-a}dx_1dx_2\dots dx_k\right)^{1/2}\\ &\leqslant \left(\int\limits_{\mathfrak{g}_1\setminus B}|x_1|^{-qa}dx_1\right)^{1/2q}\left(\int\limits_{\mathfrak{g}_2\oplus\cdots\oplus\mathfrak{g}_k}\left(\int\limits_{\mathfrak{g}_1}|f(x)|^{p_0}dx_1\right)^{2/p_0}dx_2\dots dx_k\right)^{1/2}\\ &\simeq \|f\|_{(p_0,2)}.\end{aligned}$$

where $\frac{1}{p_0} > \frac{d-a}{2d}$. Thus,

$$\|\chi_{G\setminus B}f\|_{L^2(\rho^{-a})} + \|\chi_B f\|_2 \lesssim \|f\|_{(p_0,2)}.$$

Furthermore, for $a \in \mathbb{R}$ and any compact set K, we have

$$\|\chi_K Tf\|_2 \simeq \|\chi_K Tf\|_{L^2((1+\rho)^{-a})}.$$

Writing

$$f(x) = \chi_B(x)f(x) + \chi_{G\setminus B}(x)f(x)$$

and using sublinearity of T we have

$$\|\chi_K Tf\|_2^2 \lesssim \|\chi_K T\chi_B f\|_2^2 + \|\chi_K T\chi_{G\setminus B}f\|_2^2$$

Combining these facts and our initial estimate, we can deduce

$$\begin{aligned}\|\chi_K Tf\|_2^2 &\lesssim \|\chi_K T\chi_B f\|_2^2 + \|\chi_K T\chi_{G\setminus B}f\|_2^2\\ &\simeq \|\chi_K T\chi_B f\|_2^2 + \|\chi_K T\chi_{G\setminus B}f\|_{L^2((1+\rho)^{-a})}^2\\ &\leqslant \|T\|_{L^2\to L^2}\|\chi_B f\|_2^2 + \|T\|_{L^2((1+\rho)^{-a})\to L^2((1+\rho)^{-a})}\|\chi_{G\setminus B}f\|_{L^2(\rho^{-a})}^2\\ &\leqslant \max_{A\in\{0,a\}}\|T\|_{L^2((1+\rho)^{-A})\to L^2((1+\rho)^{-A})}\|f\|_{(p_0,2)}^2.\end{aligned}$$

□

Theorem 4.5. *Let G be a stratified group with sub-Laplacian L. Suppose*

$$\|M_\delta^\bullet\|_{L^2((1+\rho)^{-A})\to L^2((1+\rho)^{-A})} \lessapprox 1$$

for $A = 0, a$. *Let* $\lambda > 0$ *and recall the definition of* $\eta(L)$ *from Lemma 2.7. If*

$$\frac{1}{2} - \frac{\lambda}{\eta(L)-1} < \frac{1}{r} \leqslant \frac{1}{2}$$

and

$$\left(\frac{d-a}{d}\right)\frac{1}{q} < \frac{1}{p} \leqslant \frac{1}{q}$$

then for all $f \in L^{(p,q)}(G)$ *we have*

$$T_r^\lambda f(x) \to f(x) \textit{ almost-everywhere as } r \to 0.$$

Proof. In view of Lemma 4.1 (with the L^p condition replaced by $L^{(p,q)}$), it suffices to prove that

$$\|\chi_K T_\bullet^\lambda f\|_2 \lesssim \|f\|_{(p,q)},$$

for p, q, λ as stated and K an arbitrary compact set.

Recall (1.0.6), (1.0.7) and (3.0.1). From Young's inequality for convolutions and Lemma 3.1, we have, for $b > \frac{\eta(L)-1}{2}$, that

$$\|m_\delta(rL)f\|_\infty \lesssim \|(K_\delta)_r\|_1\|f\|_\infty \lesssim \delta^{-b}\|f\|_{(\infty,\infty)}.$$

Since $\chi_K M_\delta^\bullet f$ has compact support, then we can further deduce that

$$\|\chi_K M_\delta^\bullet f\|_2 \lesssim \|M_\delta^\bullet f\|_{(\infty,\infty)} \lesssim \delta^{-b}\|f\|_{(\infty,\infty)}.$$

From Lemma 4.4, since $M_\delta^\bullet$ is sublinear, we know that $\|\chi_K M_\delta^\bullet\|_{L^{(p_0,2)}\to L^2} \lesssim_\epsilon \delta^{-\epsilon}$ for arbitrary $\epsilon > 0$ and for $\frac{1}{2} \geqslant \frac{1}{p_0} := \frac{q}{2p} > \frac{d-a}{2d}$. Since $L^\infty = L^{(\infty,\infty)}$, then by complex interpolation, we get

$$\|\chi_K M_\delta^\bullet\|_{L^{(p,2p/p_0)} \to L^2} \lessapprox \delta^{(p_0/p-1)b - \epsilon\frac{p_0}{p}}$$

for p_0, p, b satisfying

$$b > \frac{\eta(L)-1}{2}, \quad p \geqslant p_0 \quad \text{and} \quad \frac{1}{2} \geqslant \frac{1}{p_0} > \frac{d-a}{2d}.$$

Specifically, set $\theta = \frac{p_0}{p}$. Then $0 \leqslant \theta \leqslant 1$ and for

$$\frac{1}{p_\theta(1)} := \frac{\theta}{p_0} + \frac{1-\theta}{\infty} \quad \text{and} \quad \frac{1}{p_\theta(2)} := \frac{\theta}{2} + \frac{1-\theta}{\infty}$$

we get

$$\|\chi_K M_\delta^\bullet\|_{L^{(p_\theta(1),p_\theta(2))} \to L^2} \lessapprox \delta^{-b(1-\theta)}\delta^{-\epsilon\theta}.$$

But from the definition of θ we have

$$\frac{1}{p_\theta(1)} = \frac{\frac{p_0}{p}}{p_0} = \frac{1}{p} \quad \text{and} \quad \frac{1}{p_\theta(2)} = \frac{\frac{p_0}{p}}{2} = \frac{1}{r},$$

so that $p_\theta(1) = p$ and $p_\theta(2) = r$. The corresponding operator bound is then

$$\|\chi_K M_\delta^\bullet\|_{L^{(p_\theta(1),p_\theta(2))} \to L^2} \lessapprox \delta^{-b(1-\theta)}\delta^{-\epsilon\theta} = \delta^{-b(1-p_0/p)}\delta^{-\epsilon p_0/p} = \delta^{(p_0/p-1)b - \epsilon p_0/p}.$$

Now, observe that the Hardy–Littlewood Maximal function is bounded from $L^{(p,q)}$ to L^2_{loc} by Lemmas 2.4 and 4.4 and then following the same interpolation argument we have just used. Hence, 2.11 is easily modified to give the required boundedness (the hypotheses of Lemma 2.11 are satisfied due to Lemma 2.5).

Hence, for each $\delta \in \mathbb{D}$ the operator $M_\delta^\bullet$ is bounded on $L^{(p,q)}$, so it suffices to consider only the terms with $\delta \leqslant \frac{1}{4}$. By the triangle inequality, we have

$$\begin{aligned}
\|\chi_K T^\lambda_\bullet f\|_2 &= \left\| \chi_K \sup_{0<r<1} \left| \sum_{\delta \in \mathbb{D}} \delta^\lambda m_\delta(rL) f \right| \right\|_2 \\
&\leqslant \sum_{\delta \in \mathbb{D}} \delta^\lambda \|M^\bullet_\delta f\|_2 \\
&\lesssim \sum_{\delta \in \mathbb{D}} \delta^\lambda \delta^{(p_0/p-1)b-\epsilon p_0/p} \|f\|_{(p,q)}.
\end{aligned}$$

Hence, we have the desired estimate provided

$$\lambda + \left(\frac{p_0}{p} - 1\right) b - \epsilon \frac{p_0}{p} > 0,$$

that is,

$$\lambda + \left(\frac{2}{r} - 1\right) b - \epsilon \frac{2}{q} > 0,$$

First, clearly if $q = 2$, we can choose $\lambda > \epsilon > 0$. So suppose now that $q > 2$. We rearrange the lower bound on $\frac{1}{q}$ to see that

$$\begin{aligned}
\frac{1}{q} > \frac{1}{2} - \frac{\lambda}{\eta(L) - 1} &\iff \lambda > \left(\frac{1}{2} - \frac{1}{q}\right)(\eta(L) - 1) \\
&\iff \lambda + \left(\frac{2}{q} - 1\right) \frac{\eta(L) - 1}{2} > 0.
\end{aligned}$$

Now, the set of points $(b, \epsilon) \in \mathbb{R}^2$ such that

$$\lambda + (\tfrac{2}{q} - 1)b - \epsilon \tfrac{2}{q} > 0$$

is open and contains the point $(\frac{\eta(L)-1}{2}, 0)$. Hence, we can find $b > \frac{\eta(L)-1}{2}$ and $\epsilon > 0$ such that

$$\lambda + (\tfrac{2}{q} - 1)b - \epsilon \tfrac{2}{q} > 0$$

as required. □

CHAPTER 5

THE SQUARE FUNCTION ARGUMENT

In this chapter, we consider any stratified Lie group G with sub-Laplacian L. We will show that, in some cases, in order to obtain weighted L^2 estimate for the local maximal operator $M_\delta^\bullet$, it suffices to prove an analogous estimate on just $m_\delta(L)$. The idea is implicit in [8] and [24]. Here, we explicitly prove this as a stand-alone result in this general setting.

Definition 5.1. *The set of admissible weights $Adm(G)$ is the set of non-negative locally integrable functions $w \in A_2(G)$ such that there exists some non-negative function w_0 and some $a \geqslant 0$ such that $w \simeq (1 + w_0)^{-a}$, w_0 is 1-homogeneous with respect to group dilations and w_0 is Hölder continuous with respect to the distance d_{CC}.*

These properties will be necessary in order to prove the following result, which is the main result of this chapter (cf. Lemma 3.2 for the definition of $\tilde{m}_\delta$).

Theorem 5.2. *For $w \in Adm(G)$ we have*

$$\|M_\delta^\bullet\|^2_{L^2(w)\to L^2(w)} \lessapprox (1 + \|m_\delta(L)\|_{L^2(w)\to L^2(w)})(1 + \|\tilde{m}_\delta(L)\|_{L^2(w)\to L^2(w)}). \tag{5.0.1}$$

This chapter will be devoted to the proof of results leading up to the proof of Theorem 5.2. The homogeneity property of weights $w \in Adm(G)$ will be used in Lemma 5.4, the fact that $w \in A_2(G)$ will be required for Lemma 5.7, while Hölder continuity is used in Lemma 5.8.

If we define the operator $\mathscr{T}_\delta : f \mapsto (m_\delta(tL)f)_{t\in(0,1)}$ (i.e. $\mathscr{T}_\delta$ maps a function f to a family of functions $(m_\delta(tL)f)_{t\in(0,1)}$), then, in order to prove Theorem 5.9 (from which the proof of Theorem 5.2 follows by a scaling argument) it suffices to prove

$$\|\mathscr{T}_\delta\|_{L^2(w)\to L^2((0,1),dt/t)L^2(w)} \lessapprox \sqrt{\delta}(1 + \sup_{t\in(0,1)} \|m_\delta(tL)\|_{L^2(w)\to L^2(w)}). \tag{5.0.2}$$

Observe that the adjoint operator $\mathscr{T}_\delta^\dagger$ is defined, for families of functions $(\varphi_s)_{s\in(0,1)}$ by

$$\mathscr{T}_\delta^\dagger(\varphi_s)_s = \int_0^1 m_\delta(sL)\varphi_s \frac{ds}{s}. \tag{5.0.3}$$

Hence, (5.0.2) is equivalent to

$$\|\mathscr{T}_\delta^\dagger\|_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \lessapprox \sqrt{\delta}(1 + \sup_{t\in(0,1)} \|m_\delta(sL)\|_{L^2(1/w)\to L^2(1/w)}). \tag{5.0.4}$$

We start by proving (5.0.2) in the unweighted case. This is a simple consequence of the spectral theorem, as the next statement shows.

Theorem 5.3. *Let L be a self-adjoint operator on a Hilbert space H and $(\varphi_s)_{s\in I}$ be a family of elements of H defined on $I \subset \mathbb{R}^+$. Let $m \in C_c(\mathbb{R}^+)$ be a function with support of $\frac{ds}{s}$-measure $\simeq \delta$. If we define*

$$\mathscr{T}_m(\varphi_s)_s := \int_I m(sL)\varphi_s \frac{ds}{s},$$

then

$$\begin{aligned}\|\mathscr{T}_m(\varphi_s)_s\|_H^2 &\lesssim \delta \int_I \|m(sL)\varphi_s\|_H^2 \frac{ds}{s}\\ &\leqslant \delta \sup_{s\in I} \|m(sL)\|_{H\to H}^2 \|(\varphi_s)_s\|_{L^2(I,ds/s);H}^2\end{aligned}$$

Proof. By Proposition A.2,

$$\|\mathscr{T}_m(\varphi_s)_s\|_H^2 = \left\| \int_I m(sL)\varphi_s \frac{ds}{s} \right\|_H^2 = \int \left| \int_I m(s\widehat{L}(\zeta))\widehat{\varphi}_s(\zeta)\frac{ds}{s}\right|^2 d\mu(\zeta). \tag{5.0.5}$$

Let $A := \operatorname{supp}(m)$. Then by the Cauchy-Schwarz inequality and Fubini's Theorem, (5.0.5) is bounded above by

$$\begin{aligned}
&\int \int_I |m(s\widehat{L}(\zeta))\widehat{\varphi}_s(\zeta)|^2 \frac{ds}{s} \int_I \chi_A(s\widehat{L}(\zeta))\frac{ds}{s} d\mu(\zeta) \\
&\quad \simeq \delta \int_I \int |m(s\widehat{L}(\zeta))\widehat{\varphi}_s(\zeta)|^2 d\mu(\zeta)\frac{ds}{s} \\
&\quad = \delta \int_I \|m(sL)\varphi_s\|_H^2 \frac{ds}{s},
\end{aligned}$$

yielding the first required inequality. The second follows trivially. □

In particular, taking L to be the sub-Laplacian on G, $I = (0,1)$ and m to be m_δ, Theorem 5.3 gives

$$\|\mathscr{T}_\delta^\dagger(\varphi_s)_s\|_2^2 \lesssim \delta \int_0^1 \|m_\delta(sL)\varphi_s\|_2^2 \frac{ds}{s} \lesssim \delta \sup_{s\in(0,1)} \|m_\delta(sL)\|_{L^2(G)\to L^2(G)}^2 \|(\varphi_s)_s\|_{L^2((0,1),ds/s)L^2(G)}^2, \tag{5.0.6}$$

which implies (5.0.2) for $a = 0$.

Recall that $\mathscr{T}_\delta^\dagger$ is defined in terms of an integral over the interval $[0,1]$. We now use a Littlewood–Paley decomposition and a dilation argument to bound the operator norm of $\mathscr{T}_\delta^\dagger$ by operator norms of analogously defined operators involving an integral over the interval $[\frac{1}{8}, 1]$.

Lemma 5.4. *For $w \in Adm(G)$ we have*

$$\begin{aligned}
\|\mathscr{T}_\delta^\dagger\|_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)}^2 \lesssim{}& \|\mathscr{T}_{\delta,0}^\dagger\|_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)}^2 \\
&+ \|\mathscr{T}_{\delta,0}^\dagger\|_{L^2((0,1),ds/s)L^2(G)\to L^2(G)}^2
\end{aligned}$$

where

$$\mathscr{T}^{\dagger}_{\delta,k}(\varphi_s)_s = \int_{2^{-k-3}}^{2^{-k}} m_\delta(sL)\varphi_s \frac{ds}{s}. \tag{5.0.7}$$

Proof. We first choose $\psi \in C^\infty(\mathbb{R})$ with $\mathrm{supp}(\psi) \subseteq (1,4)$ and

$$1 = \sum_{k\in\mathbb{Z}} \psi(2^{-k}s), \quad s > 0.$$

Now, by considering support conditions we observe that

$$m_\delta(tL)\psi(2^{-k}L) = 0 \text{ for } t \notin [2^{-3-k}, 2^{-k}]. \tag{5.0.8}$$

Specifically, $\psi(2^{-k}\zeta) = 0$ if $\zeta \notin (2^k, 2^{k+2})$. As we are assuming that $\delta \leqslant \frac{1}{2}$ then $m_\delta(t\zeta) = 0$ for $t\zeta \notin [2^{-1}, 2^0]$. Combining these gives (5.0.8).

Using Lemma 2.12, the fact that $w \in A_2$ and (5.0.8) (note that for $k \leqslant -4$ the intervals $[2^{-k-k}, 2^{-k}]$ and $[0,1]$ are disjoint) we deduce

$$\begin{aligned}
\|\mathscr{T}^{\dagger}_{\delta}(\varphi_s)_s\|^2_{L^2(1/w)} &= \left\| \int_0^1 m_\delta(sL)\varphi_s \frac{ds}{s} \right\|^2_{L^2(1/w)} \\
&\simeq \sum_{k\in\mathbb{Z}} \left\| \int_0^1 \psi(2^{-k}L) m_\delta(sL)\varphi_s \frac{ds}{s} \right\|^2_{L^2(1/w)} \\
&= \sum_{k=-3}^{\infty} \left\| \int_{2^{-3-k}}^{2^{-k}} \psi(2^{-k}L) m_\delta(sL)\varphi_s \frac{ds}{s} \right\|^2_{L^2(1/w)} \\
&= \sum_{k=-3}^{\infty} \|\mathscr{T}^{\dagger}_{\delta,k}\psi(2^{-k}L)\varphi_s\|^2_{L^2(1/w)}.
\end{aligned}$$

We now use a dilation argument. Define F_r as in (2.1.10). Then using Lemma 2.10 we see that

$$m_\delta(2^{-k}sL)f = (f_{2^k} * K_s)_{2^{-k}} = (m_\delta(sL)f_{2^k})_{2^{-k}}.$$

Then, by making various substitutions we deduce that

$$
\begin{aligned}
\|\mathscr{T}^{\dagger}_{\delta,k}(g_s)_s\|^2_{L^2(1/w)} &= \int_G \left| \int_{1/8}^{1} (m_\delta(sL)(g_s)_{2^k})_{2^{-k}}(x)\frac{ds}{s}\right|^2 \frac{dx}{w(x)} \\
&= 2^{Qk} \int_G \left| \int_{1/8}^{1} m_\delta(sL)(g_s)_{2^k}(\delta_{2^{k/2}}(x))\frac{ds}{s}\right|^2 \frac{dx}{w(x)} \\
&\simeq 2^{Qk} \int_G \left| \int_{1/8}^{1} m_\delta(sL)(g_s)_{2^k}(x)\frac{ds}{s}\right|^2 (1+2^{-k/2}w_0(x))^a 2^{-Qk/2}dx \\
&\simeq 2^{Qk/2}2^{-ka/2} \left\| \int_{1/8}^{1} m_\delta(sL)(g_s)_{2^k}\frac{ds}{s}\right\|^2_{L^2(w_0^a)} \\
&\qquad + 2^{Qk/2} \left\| \int_{1/8}^{1} m_\delta(sL)(g_s)_{2^k}\frac{ds}{s}\right\|^2_{L^2(G)} \\
&\lesssim 2^{Qk/2}2^{-ka/2}\|\mathscr{T}^{\dagger}_{\delta,0}((g_s)_{2^k})_s\|^2_{L^2(1/w)} + 2^{Qk/2}\|\mathscr{T}^{\dagger}_{\delta,0}((g_s)_{2^k})_s\|^2_{L^2(G)}.
\end{aligned}
$$

Now, observe that, for $k \geqslant 0$ and $a \geqslant 0$,

$$
\begin{aligned}
\|(g_s)_{2^k}\|^2_{L^2(1/w)} &\simeq \int_G |2^{-Qk/2}g_s(\delta_{2^{-k/2}}(x))|^2(1+w_0(x))^a dx \\
&= 2^{-Qk}\int_G |g_s(x)|^2(1+2^{k/2}w_0(x))^a 2^{Qk/2}dx \\
&\lesssim 2^{-Qk/2}2^{ka/2}\|g_s\|^2_{L^2(1/w)}.
\end{aligned}
\tag{5.0.9}
$$

Then we have

$$
\begin{aligned}
&2^{Qk/2}2^{-ka/2}\|\mathscr{T}^{\dagger}_{\delta,0}((g_s)_{2^k})_s\|^2_{L^2(1/w)} \\
&\qquad \leqslant 2^{Qk/2}2^{-ka/2}\|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)}\|((g_s)_{2^k})_s\|^2_{L^2((0,1),ds/s)L^2(1/w)} \\
&\qquad \leqslant \|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)}\|(g_s)_s\|^2_{L^2((0,1),ds/s)L^2(1/w)}.
\end{aligned}
$$

For $k \in \{-1,-2,-3\}$ we simply have

$$\begin{aligned}
\|(g_s)_{2^k}\|^2_{L^2(1/w)} \simeq \int_G |2^{-Qk/2} g_s(\delta_{2^{-k}/2}(x))|^2 (1+w_0(x))^a dx \\
= 2^{-Qk} \int_G |g_s(x)|^2 (1 + 2^{\frac{k}{2}} w_0(x)^a) 2^{Qk/2} dx \\
\lesssim 2^{-Qk/2} \|g_s\|^2_{L^2(1/w)}
\end{aligned}$$

and

$$\begin{aligned}
&2^{Qk/2} 2^{-ka/2} \|\mathscr{T}^{\dagger}_{\delta,0}((g_s)_{2^k})_s\|^2_{L^2(1/w)} \\
&\quad \lesssim 2^{Qk/2} \|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \|((g_s)_{2^k})_s\|^2_{L^2((0,1),ds/s)L^2(1/w)} \\
&\quad \leqslant \|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \|(g_s)_s\|^2_{L^2((0,1),ds/s)L^2(1/w)},
\end{aligned}$$

since $2^{-ka/2} \leqslant \sqrt{8}^a$. Hence,

$$\begin{aligned}
&2^{Qk/2} 2^{-ka/2} \|\mathscr{T}^{\dagger}_{\delta,0}(g_s)_{2^k}\|^2_{L^2(1/w)} + 2^{Qk/2} \|\mathscr{T}^{\dagger}_{\delta,0}(g_s)_{2^k}\|^2_{L^2(G)} \\
&\quad \lesssim \|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \|(g_s)_s\|^2_{L^2((0,1),ds/s)L^2(1/w)} \\
&\qquad + \|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(G)\to L^2(G)} \|(g_s)_s\|^2_{L^2((0,1),ds/s)L^2(G)} \\
&\quad \leqslant \big(\|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \\
&\qquad + \|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(G)\to L^2(G)}\big) \|(g_s)_s\|^2_{L^2((0,1),ds/s)L^2(1/w)},
\end{aligned}$$

where we use that $\frac{1}{w} \geqslant 1$.

Finally, by applying Lemma 2.12 again and choosing $(g_s)_s = \psi(2^{-k}L)(\varphi_s)_s$, we have

$$\sum_{k=-3}^{\infty} \|\mathscr{T}_{\delta,k}^{\dagger}\psi(2^{-k}L)(\varphi_s)_s\|_{L^2(1/w)}^2$$
$$\lesssim \left(\|\mathscr{T}_{\delta,0}^{\dagger}\|_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)}^2 + \|\mathscr{T}_{\delta,0}^{\dagger}\|_{L^2((0,1),ds/s)L^2(G)\to L^2(G)}^2\right)$$
$$\cdot \sum_{k=-3}^{\infty} \|\psi(2^{-k}L)(\varphi_s)_s\|_{L^2((0,1),ds/s)L^2(1/w)}^2$$
$$\lesssim \left(\|\mathscr{T}_{\delta,0}^{\dagger}\|_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)}^2\right.$$
$$\left. + \|\mathscr{T}_{\delta,0}^{\dagger}\|_{L^2((0,1),ds/s)L^2(G)\to L^2(G)}^2\right)\|(\varphi_s)_s\|_{L^2((0,1),ds/s)L^2(1/w)}^2.$$

This completes the proof. □

In what follows, we will wish to bound the operator norm of an operator $m(sL)$ on the unweighted space $L^2(G)$ by its norm as an operator on a weighted space $L^2(w)$. For weights w where w and $\frac{1}{w}$ are both weights (such as $A_2(G)$ weights, and hence weights $w \in Adm(G)$), this is an easy consequence of the fact that operators of the form $m(sL)$ are self-adjoint on $L^2(G)$.

Lemma 5.5. *For $0 < s \leqslant 1$, any $m \in C_c(\mathbb{R}^+)$ and any weight w such that $\frac{1}{w}$ is also a weight, we have*

$$\|m(sL)\|_{L^2(w)\to L^2(w)} = \|m(sL)\|_{L^2(1/w)\to L^2(1/w)} \tag{5.0.10}$$

and so

$$\|m(sL)\|_{L^2(G)\to L^2(G)} \leqslant \|m(sL)\|_{L^2(w)\to L^2(w)}. \tag{5.0.11}$$

Proof. Let $X_1 = L^2(w)$ and $X_2 = L^2(1/w)$. Since $m(sL)$ is self-adjoint on $L^2(G)$, then

$$\langle m(sL)f, g\rangle_{X_1,X_2} = \langle m(sL)f, g\rangle = \langle f, m(sL)g\rangle = \langle f, m(sL)g\rangle_{X_1,X_2}.$$

Thus, if we consider the operator $m(sL) : X_1 \to X_1$, then the adjoint operator $[m(sL)]^{\dagger} : X_2 \to X_2$ is again $m(sL)$, and so

$$\|m(sL)\|_{L^2(w)\to L^2(w)} = \|m(sL)\|_{L^2(1/w)\to L^2(1/w)},$$

which is (5.0.10).

The second result follows from the first result by interpolation between the spaces X_1 and X_2 at $\theta = \frac{1}{2}$. Specifically, $(L^2(w), L^2(1/w))_{1/2} = L^2(G)$, which yields

$$\|m(sL)\|_{L^2(G)\to L^2(G)} \leqslant \|m(sL)\|^{1/2}_{L^2(w)\to L^2(w)} \|m_\delta(sL)\|^{1/2}_{L^2(1/w)\to L^2(1/w)}.$$

Applying (5.0.10) to this gives (5.0.11). □

The following is an immediate application of Lemma 5.5.

Corollary 5.6. *For any weight w such that $\frac{1}{w}$ is also a weight, we have*

$$\|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(G)\to L^2(G)} \lesssim \delta \sup_{s\in(0,1)} \|m_\delta(sL)\|^2_{L^2(1/w)\to L^2(1/w)}.$$

Proof. From (5.0.6), we have

$$\|\mathscr{T}^{\dagger}_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(G)\to L^2(G)} \lesssim \delta \sup_{s\in(0,1)} \|m_\delta(sL)\|^2_{L^2(G)\to L^2(G)}.$$

Applying (5.0.11) with $m = m_\delta$ completes the proof. □

Let $\chi \in C^\infty(\mathbb{R})$ satisfy

$$\operatorname{supp}(\chi) \subseteq (-2,2), \quad \chi(\lambda) = \chi(-\lambda) \text{ for } \lambda \in \mathbb{R}, \quad \chi(\lambda) = 1 \text{ for } \lambda \in (-1,1).$$

Define, for $\lambda \in \mathbb{R}$, $n_\delta(\lambda) = m_\delta(\lambda^2)$ and

$$\widehat{n^I_\delta}(\lambda) := \chi(\delta^2\lambda)\widehat{n_\delta}(\lambda), \quad \widehat{n^{II}_\delta}(\lambda) := (1-\chi(\delta^2\lambda))\widehat{n_\delta}(\lambda).$$

Then, via the spectral decomposition of $m_\delta(tL)$ we have

$$m_\delta(tL) = n^I_\delta(\sqrt{tL}) + n^{II}_\delta(\sqrt{tL}).$$

The following lemma then tells us that the 'contribution' to $m_\delta(tL)$ of $n^{II}_\delta(\sqrt{tL})$ is small.

Lemma 5.7. *Define*

$$\Psi_\delta^{II}(\varphi_s)_s := \int_{1/8}^{1} n_\delta^{II}(\sqrt{sL})\varphi_s \frac{ds}{s}.$$

Then we have the estimates

$$\|\Psi_\delta^{II}(\varphi_s)_s\|_{L^2(1/w)}^2 \lesssim_k \delta^k \|(\varphi_s)_s\|_{L^2((0,1),ds/s)L^2(1/w)}^2$$

and

$$\sup_{s\in(0,1)} \|n_\delta^{II}(\sqrt{sL})f\|_{L^2(1/w)}^2 \lesssim_k \delta^k \|f\|_{L^2(1/w)}^2$$

for all $k \in \mathbb{N}$ *and for all* $w \in A_2$.

Proof. Let $n_\delta = n_\delta^+ + n_\delta^-$ where $\mathrm{supp}(n_\delta^+) \subseteq (0,\infty)$ and $\mathrm{supp}(n_\delta^-) \subseteq (-\infty, 0)$. Then each Schwartz semi-norm of $N_\delta(\lambda) := n_\delta^+(\delta\lambda + 1)$ is bounded independently of δ. Indeed, first note that $\mathrm{supp}(n_\delta^+) \subseteq [\sqrt{1-\delta}, 1]$ so

$$\mathrm{supp}(N_\delta) \in [-\delta^{-1}(1 - \sqrt{1-\delta}), 0]. \tag{5.0.12}$$

Now,

$$\frac{d^b}{d\lambda^b} N_\delta(\lambda) = \frac{d^b}{d\lambda^b} n_\delta^+(\delta\lambda + 1) = \delta^b (n_\delta^+)^{(b)}(\delta\lambda + 1). \tag{5.0.13}$$

Then

$$\|(n_\delta^+)^{(b)}(\lambda)\|_\infty = \left\| \frac{d^b}{d\lambda^b} m_\delta(\lambda^2) \right\|_\infty \lesssim \sum_{0\leqslant a,c\leqslant b} \left\| \lambda^a m_\delta^{(c)}(\lambda^2) \right\|_\infty \lesssim \|m_\delta^{(b)}(\lambda^2)\|_\infty, \tag{5.0.14}$$

by (1.0.7). Thus, by combining (5.0.13), (5.0.14) and (1.0.7) we have

$$\|N_\delta^{(b)}\|_\infty \lesssim \delta^b \delta^{-b} = 1.$$

So, by using (5.0.12) to majorise $|\lambda|$ by 1, we have

$$\begin{aligned}\|N_\delta\|_{a,b} &= \sup_{\lambda\in[-\delta^{-1}(1-\sqrt{1-\delta}),0]} \left|\lambda^a \frac{d^b}{d\lambda^b} N_\delta(\lambda)\right| \\ &\lesssim \|N_\delta^{(b)}\|_\infty \lesssim 1.\end{aligned}$$

As the Fourier transform maps Schwartz functions to Schwartz functions, the same estimate applies to $\widehat{N_\delta}$. Now, each Schwartz semi-norm of the function

$$\begin{aligned}(1-\chi(\delta\lambda))\widehat{n_\delta^+}(\delta^{-1}\lambda) &= (1-\chi(\delta\lambda))\int_{\mathbb{R}} e^{-2\pi i\delta^{-1}\lambda\zeta} n_\delta^+(\zeta)d\zeta \\ &= (1-\chi(\delta\lambda))\int_{\mathbb{R}} e^{-2\pi i\lambda(\eta+\delta^{-1})} n_\delta^+(\delta\eta+1)\delta d\eta \\ &= (1-\chi(\delta\lambda))\delta e^{-2\pi i\lambda\delta^{-1}}\widehat{N_\delta}(\lambda), \quad \lambda\in\mathbb{R},\end{aligned}$$

is majorised by δ^k for every $k\in\mathbb{N}$ with a constant depending on the corresponding Schwartz semi-norm and N. Since differentiation does not increase supports and since $1-\chi(\delta\cdot)$ is supported outside the ball $B_{\mathbb{R}}(0,\delta^{-1})$, then in what follows we have $\delta^{-1}\leqslant|\lambda|$. We have that

$$\begin{aligned}\|(1-\chi(\delta\cdot))\delta e^{-2\pi i(\cdot)\delta^{-1}}\widehat{N_\delta}(\cdot)\|_{a,b} &= \delta^k\|(1-\chi(\delta\cdot))\delta e^{-2\pi i(\cdot)\delta^{-1}}\delta^{-k}\widehat{N_\delta}(\cdot)\|_{a,b} \\ &\lesssim \delta^k \sum_{\substack{m,n,l\geqslant 0\\ m+n+l=b}} \|(1-\chi(\delta\cdot))^{(m)}\delta\left(e^{-2\pi i(\cdot)\delta^{-1}}\right)^{(n)}\widehat{N_\delta}^{(l)}(\cdot)\|_{a,0} \\ &\leqslant \delta^k \sum_{\substack{m,n,l\geqslant 0\\ m+n+l=b}} \delta^m\|(1-\chi)^{(m)}(\delta\cdot)\,\delta^{1-n-k}\widehat{N_\delta}^{(l)}(\cdot)\|_{a,0} \\ &\lesssim \delta^k \sum_{\substack{m,n,l\geqslant 0\\ m+n+l=b}} \delta^m\|(1-\chi)^{(m)}(\delta\cdot)\widehat{N_\delta}^{(l)}(\cdot)\|_{a+k+n-1,l} \lesssim \delta^k.\end{aligned}$$

Note that the δ factors are arranged so as to keep the power of δ inside the semi-norm negative, so that we may estimate them by multiplication by the function argument. The above arguments also apply to n_δ^-. By writing

$$\begin{aligned}
\|\widehat{\delta n_\delta^{II}(\delta\cdot)}\|_{a,b} &= \|\widehat{n_\delta^{II}}(\delta^{-1}\cdot)\|_{a,b} \\
&= \|(1-\chi(\delta))[\widehat{n_\delta^+}(\delta^{-1}\cdot) + \widehat{n_\delta^-}(\delta^{-1}\cdot)]\|_{a,b} \\
&\leqslant \|(1-\chi(\delta))\widehat{n_\delta^-}(\delta^{-1}\cdot)\|_{a,b} + \|(1-\chi(\delta))\widehat{n_\delta^+}(\delta^{-1}\cdot)\|_{a,b}
\end{aligned}$$

we see that every Schwartz semi-norm of $\delta n_\delta^{II}(\delta\cdot)$ is bounded by any given power of δ. Note that n_δ^{II} is even. By Lemma A.1 we see that $n_\delta^{II}(\sqrt{\cdot})$ is Schwartz, and furthermore that each of its Schwartz semi-norms can be bounded by any given power of δ. Hence $n_\delta^{II}(\sqrt{\cdot})$ satisfies the hypotheses of Lemma 2.5, where the quantity $\|m\|_{k,k'}^*$ may majorised by any positive power of δ. Hence we may apply Lemma 2.11. Hence we have, for all $t > 0$ and for all $N \in \mathbb{N}$, the estimate

$$|n_\delta^{II}(\sqrt{sL})f(x)| \lesssim_k \delta^k Mf(x) \quad \text{a.e.}$$

where M denotes the Littlewood-Hardy maximal operator on G and the implicit constant in $\lesssim$ does not depend on t or δ but may depend on N. Since $\frac{1}{w} \in A_2$ and M is bounded on $L^2(1/w)$, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\|\Psi_\delta^{II}(\varphi_s)_s\|_{L^2(1/w)}^2 &= \int_G \left| \int_{1/8}^1 n_\delta^{II}(\sqrt{sL})\varphi_s(x)\frac{ds}{s}\right|^2 \frac{dx}{w(x)} \\
&\lesssim \int_G \int_{1/8}^1 |n_\delta^{II}(\sqrt{sL})\varphi_s(x)|^2 \frac{ds}{s}\frac{dx}{w(x)} \\
&\lesssim_k \delta^k \int_G \int_{1/8}^1 |M\varphi_s(x)|^2 \frac{ds}{s}\frac{dx}{w(x)} \\
&\lesssim \delta^k \|(\varphi_s)_s\|_{L^2((0,1),ds/s)L^2(1/w)}^2,
\end{aligned} \tag{5.0.15}$$

yielding the first desired result.

Similarly,

$$
\begin{aligned}
\|n_\delta^{II}(\sqrt{sL})f\|^2_{L^2(1/w)} &= \int_G |n_\delta^{II}(\sqrt{sL})f(x)|^2 \frac{dx}{w(x)} \\
&\lesssim_k \delta^k \int_G |Mf(x)|^2 \frac{dx}{w(x)} \\
&\lesssim \delta^k \|f\|^2_{L^2(1/w)}.
\end{aligned}
$$

Hence,

$$\|n_\delta^{II}(\sqrt{sL})\|^2_{L^2(1/w)\to L^2(1/w)} \lesssim_k \delta^k$$

and the second result follows by taking this supremum over $s \in (0,1)$. □

We shall now combine the results established in this chapter to prove (5.0.4), which we recall is equivalent to (5.0.2).

Lemma 5.8. *We have* (5.0.4) *for $w \in Adm(G)$, that is,*

$$\|\mathscr{T}_\delta^\dagger\|_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \lessapprox \sqrt{\delta}(1 + \sup_{s\in(0,1)} \|m_\delta(sL)\|_{L^2(1/w)\to L^2(1/w)}).$$

Proof. By Fourier inversion, the convolution kernel $K_{\delta,t}$ of $n_\delta^I(\sqrt{tL})$ satisfies

$$K_{\delta,t}(x) = \int_0^\infty \chi(\delta^2 s)\widehat{n_\delta^I}(s)\cos(s\sqrt{tL})\delta_0(x)ds, \tag{5.0.16}$$

where $\cos(s\sqrt{tL})\delta_0(x)$ is the convolution kernel of the operator $\cos(s\sqrt{tL})$. By finite propagation speed (Lemma 2.6) for $t \leqslant 1$ the support of $\cos(s\sqrt{tL})\delta_0(x)$ is contained within the ball $\overline{B}(0,s)$ and hence, due to the bump function χ, we have

$$\mathrm{supp}(K_{\delta,t}) \subseteq \overline{B}(0, 2\delta^{-2}). \tag{5.0.17}$$

Define

$$\varphi_{s,l} := \varphi_s \chi_{[2^{l-1},2^l)}(1 + w_0(\cdot)), \text{ for } l \in \mathbb{N}.$$

Note that

$$\varphi_s = \sum_{l \in \mathbb{N}} \varphi_{s,l}$$

and

$$A_l := \operatorname{supp}(\varphi_{s,l}) = \{x \in G : 2^{l-1} \leqslant 1 + w_0(x) \leqslant 2^l\} \tag{5.0.18}$$

for $l \geqslant 1$.

We wish to prove that, for k, j sufficiently large and far apart, we have

$$\operatorname{supp}\left(\int_{1/8}^{1} n_\delta^I(\sqrt{sL})\varphi_{s,k}\frac{ds}{s}\right) \cap \operatorname{supp}\left(\int_{1/8}^{1} n_\delta^I(\sqrt{sL})\varphi_{s,j}\frac{ds}{s}\right) = \varnothing. \tag{5.0.19}$$

Fix $1 < k < j$. By definition, we have that

$$I_k(x) := \int_{1/8}^{1} n_\delta^I(\sqrt{sL})\varphi_{s,k}(x)\frac{ds}{s} = \int_{1/8}^{1}\int_G \varphi_{s,k}(xy^{-1})K_{\delta,s}(y)dy\frac{ds}{s},$$

and analogously for j. Since $xy^{-1} \in A_k$ and since $y \in \overline{B}(0, 2\delta^{-2})$ by (5.0.17), then we have that I_k is supported in a $2\delta^{-2}$-neighbourhood of the set A_k. Hence, for any $z \in \operatorname{supp}(I_k)$, there exists $x \in A_k$ such that $d(x,z) \leqslant 2\delta^{-2}$.

Recall that w_0 is Hölder continuous, so there exist $C, \alpha > 0$ such that $|w_0(x) - w_0(z)| \leqslant C d_{CC}(x,z)^\alpha$. Hence,

$$\frac{|1 + w_0(x)|}{|1 + w_0(z)|} \leqslant \frac{|1 + w_0(z)| + |w_0(x) - w_0(z)|}{|1 + w_0(z)|} \leqslant 1 + C d_{CC}(x,z)^\alpha. \tag{5.0.20}$$

In particular, this shows that if $d_{CC}(x,z) \leqslant R$, then

$$1 + w_0(x) \lesssim R^\alpha(1 + w_0(z)).$$

Choose $R = 2\delta^{-2}$. Then, for every $z \in \operatorname{supp}(I_k)$, there exists $x \in A_K$ such that

$$(2\delta^{-2})^{-\alpha}(1 + w_0(x)) \leqslant 1 + w_0(z) \leqslant (2\delta^{-2})^{\alpha}(1 + w_0(x)).$$

Thus,

$$\operatorname{supp}(I_k) \subseteq \{x \in G : (2\delta^{-2})^{-\alpha} 2^{k-1} \leqslant 1 + w_0(x) \leqslant (2\delta^{-2})^{\alpha} 2^k\}.$$

Clearly, if we choose k, j so that $2^k, 2^j \geqslant (2\delta^{-2})^{\alpha}$, then $\operatorname{supp}(I_k)$ will intersect at most finitely many $\operatorname{supp}(I_j)$. We have $k \simeq \alpha \ln(2\delta^{-2})$ many summands below this. Note that since $\ln(\delta^{-1})$ grows slower than $\delta^{-\epsilon}$ then in particular, $A \lesssim \ln(\delta^{-1})B$ implies that $A \lessapprox B$. Hence, the following sum is locally a sum of finitely many objects and so

$$\begin{aligned} \|\Psi_\delta^I(\varphi_s)_s\|_{L^2(1/w)}^2 &= \left\| \sum_{l \in \mathbb{N}_0} \int_{1/8}^{1} n_\delta^I(\sqrt{sL})\varphi_{s,l} \frac{ds}{s} \right\|_{L^2(1/w)}^2 \\ &\lessapprox \sum_{l \in \mathbb{N}_0} \left\| \int_{1/8}^{1} n_\delta^I(\sqrt{sL})\varphi_{s,l} \frac{ds}{s} \right\|_{L^2(1/w)}^2 \\ &\simeq \sum_{l \in \mathbb{N}_0} 2^{la} \|\Psi_\delta^I(\varphi_{s,l})_s\|_{L^2(G)}^2. \end{aligned}$$

Now, taking L to be the sub-Laplacian on G, $I = (\frac{1}{8}, 1)$ and m to be $n_\delta^I(\sqrt{\cdot})$ in Theorem 5.3 gives

$$\|\Psi_\delta^I(\varphi_{s,l})_s\|_{L^2(G)}^2 \lesssim \delta \int_{1/8}^{1} \|n_\delta^I(\sqrt{sL})\varphi_{s,l}\|_{L^2(G)}^2 \frac{ds}{s}. \tag{5.0.21}$$

Thus,

$$\begin{aligned} \|\Psi_\delta^I(\varphi_s)_s\|_{L^2(1/w)}^2 &\lessapprox \sum_{l \in \mathbb{N}_0} 2^{la} \|\Psi_\delta^I(\varphi_{s,l})_s\|_{L^2(G)}^2 \\ &\lesssim \sum_{l \in \mathbb{N}_0} 2^{la} \delta \int_{1/8}^{1} \|n_\delta^I(\sqrt{sL})\varphi_{s,l}\|_{L^2(G)}^2 \frac{ds}{s} \\ &\lesssim \delta \sum_{l \in \mathbb{N}_0} \int_{1/8}^{1} \|n_\delta^I(\sqrt{sL})\varphi_{s,l}\|_{L^2(1/w)}^2 \frac{ds}{s}. \end{aligned}$$

Now by Lemma 5.4 and Corollary 5.6, we have

$$\begin{aligned}\|\mathscr{T}_\delta^\dagger\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} &\lesssim \|\mathscr{T}_{\delta,0}^\dagger\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \\ &\quad + \|\mathscr{T}_{\delta,0}^\dagger\|^2_{L^2((0,1),ds/s)L^2(G)\to L^2(G)} \\ &\lesssim \|\mathscr{T}_{\delta,0}^\dagger\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \\ &\quad + \delta \sup_{s\in(0,1)} \|m_\delta(sL)\|^2_{L^2(1/w)\to L^2(1/w)}.\end{aligned}$$

Furthermore, by (5.0.21) and Lemma 5.7,

$$\begin{aligned}\|\mathscr{T}_{\delta,0}^\dagger(\varphi_s)_s\|^2_{L^2(1/w)} &\lesssim \|\Psi_\delta^I(\varphi_s)_s\|^2_{L^2(1/w)} + \|\Psi_\delta^{II}(\varphi_s)_s\|^2_{L^2(1/w)} \\ &\lesssim_N \delta \sum_{l\in\mathbb{N}_0} \int_{1/8}^{1} \|n_\delta^I(\sqrt{sL})\varphi_{s,l}\|^2_{L^2(1/w)} \frac{ds}{s} + \delta^N \|(\varphi_s)_s\|^2_{L^2((0,1),ds/s)L^2(1/w)}\end{aligned}$$

Since $n_\delta^I(\sqrt{sL})$ is bounded, then

$$\begin{aligned}&\delta \sum_{l\in\mathbb{N}_0} \int_{1/8}^{1} \|n_\delta^I(\sqrt{sL})\varphi_{s,l}\|^2_{L^2(1/w)} \frac{ds}{s} \\ &\qquad \leqslant \delta \sum_{l\in\mathbb{N}_0} \int_{1/8}^{1} \|n_\delta^I(\sqrt{sL})\|^2_{L^2(1/w)\to L^2(1/w)} \|(\varphi_{s,l})_s\|^2_{L^2(1/w)} \frac{ds}{s} \\ &\qquad \leqslant \delta \sup_{s\in(0,1)} \left(\|n_\delta^I(\sqrt{sL})\|^2_{L^2(1/w)\to L^2((1+|\cdot|)^a)}\right) \|(\varphi_s)_s\|^2_{L^2((1/8,1),ds/s)L^2(1/w)}.\end{aligned}$$

Observe that

$$\|(\varphi_s)_s\|^2_{L^2((1/8,1),ds/s)L^2(1/w)} \leqslant \|(\varphi_s)_s\|^2_{L^2((0,1),ds/s)L^2(1/w)}.$$

By Lemma 5.7 we have

$$\begin{aligned}\sup_{s\in(0,1)} \|n_\delta^I(\sqrt{sL})\|^2_{L^2(1/w)\to L^2(1/w)} &= \sup_{s\in(0,1)} \|m_\delta(sL) - n_\delta^{II}(\sqrt{sL})\|^2_{L^2(1/w)\to L^2(1/w)} \\ &\leqslant \sup_{s\in(0,1)} \|m_\delta(sL)\|^2_{L^2(1/w)\to L^2(1/w)} \\ &\qquad + \sup_{s\in(0,1)} \|n_\delta^{II}(\sqrt{sL})\|^2_{L^2(1/w)\to L^2(1/w)} \\ &\lesssim_N \sup_{s\in(0,1)} \|m_\delta(sL)\|^2_{L^2(1/w)\to L^2(1/w)} + \delta^N\end{aligned}$$

Combining, we therefore have, for all $N \in \mathbb{N}$,

$$\|\mathscr{T}^\dagger_{\delta,0}\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \lessapprox_N \delta(\sup_{s\in(0,1)} \|m_\delta(sL)\|^2_{L^2(1/w)\to L^2(1/w)} + \delta^N + \delta^{N-1})$$

and so

$$\|\mathscr{T}^\dagger_{\delta}\|^2_{L^2((0,1),ds/s)L^2(1/w)\to L^2(1/w)} \lessapprox_N \delta(\sup_{s\in(0,1)} \|m_\delta(sL)\|^2_{L^2(1/w)\to L^2(1/w)} + 1),$$

which is equivalent to (5.0.4) as required. □

Now that (5.0.2) is proved, we may combine it with Lemma 3.2 to obtain the penultimate result of this section, which will allow us to bound $\|M_\delta^\bullet\|^2_{L^2(w)\to L^2(w)}$ using $\sup_{0<r<1} \|m_\delta(rL)\|_{L^2(w)\to L^2(w)}$ and the corresponding operator norm for $\tilde{m}_\delta(rL)$.

Theorem 5.9. *For $w \in Adm(G)$ we have*

$$\|M_\delta^\bullet\|^2_{L^2(w)\to L^2(w)} \lessapprox (1 + \sup_{0<r<1} \|m_\delta(rL)\|_{L^2(w)\to L^2(w)})(1 + \sup_{0<r<1} \|\tilde{m}_\delta(rL)\|_{L^2(w)\to L^2(w)}). \tag{5.0.22}$$

Proof. Recall from Lemma 3.2, we have

$$|M_\delta^\bullet f(x)|^2 \leqslant 2\delta^{-1} \int_0^1 |m_\delta(tL)f(x)||\tilde{m}_\delta(tL)f(x)|\frac{dt}{t}.$$

Taking the weighted L^2 norm and applying the Cauchy-Schwarz inequality gives

$$\|M_\delta^\bullet f(x)\|_{L^2(w)}^2 \leqslant 2\delta^{-1}\|m_\delta(tL)f\|_{L^2((0,1),dt/t)L^2(w)}\|\tilde{m}_\delta(tL)f\|_{L^2((0,1),dt/t)L^2(w)}.$$

Applying (5.0.2) completes the proof (note that since $\tilde{m}_\delta$ satisfies the same support conditions and norm estimates as m_δ, then (5.0.2) holds when m_δ is replaced by $\tilde{m}_\delta$). □

Finally, we use a dilation argument to deduce Theorem 5.2 from Theorem 5.9.

Proof of Theorem 5.2. If we can show that

$$\sup_{0<s<1} \|m_\delta(sL)\|_{L^2(w)\to L^2(w)} \lesssim \|m_\delta(L)\|_{L^2(w)\to L^2(w)},$$

and similarly for $\tilde{m}_\delta$, then applying this to Theorem 5.9 completes the proof. Recall from (5.0.10) that

$$\|m_\delta(sL)\|_{L^2(w)\to L^2(w)} = \|m_\delta(sL)\|_{L^2(1/w)\to L^2(1/w)},$$

so it suffices to prove that $\|m_\delta(sL)\|_{L^2(1/w)\to L^2(1/w)} \lesssim \|m_\delta(L)\|_{L^2(1/w)\to L^2(1/w)}$ with the constant in $\lesssim$ independent of s.

By Lemma 2.10, we have

$$m_\delta(sL)f = (m_\delta(L)(f)_{s^{-1}})_s.$$

Now,

$$
\begin{aligned}
\|m_\delta(sL)f\|^2_{L^2(1/w)} &\simeq \int_G |(m_\delta(L)f_{s^{-1}})_s(x)|^2(1+w_0(x))^a dx \\
&= s^{-Q}\int_G |m_\delta(L)f_{s^{-1}}(\delta_{s^{-1/2}}(x))|^2(1+w_0(x))^a dx \\
&= s^{-Q}\int_G |m_\delta(L)f_{s^{-1}}(x)|^2(1+s^{1/2}w_0(x))^a s^{Q/2} dx \\
&\simeq s^{-Q/2}s^{a/2}\|m_\delta(L)f_{s^{-1}}\|^2_{L^2(w_0^a)} + s^{-Q/2}\|m_\delta(L)f_{s^{-1}}\|^2_{L^2(G)} \\
&\lesssim s^{-Q/2}s^{a/2}\|m_\delta(L)f_{s^{-1}}\|^2_{L^2(1/w)} + s^{-Q/2}\|m_\delta(L)f_{s^{-1}}\|^2_{L^2(G)} \\
&\leqslant s^{-Q/2}s^{a/2}\|m_\delta(L)\|^2_{L^2(1/w)\to L^2(1/w)}\|f_{s^{-1}}\|^2_{L^2(1/w)} \\
&\qquad + s^{-Q/2}\|m_\delta(L)\|^2_{L^2(G)\to L^2(G)}\|f_{s^{-1}}\|^2_{L^2(G)}.
\end{aligned}
$$

Now, since $a \geqslant 0$,

$$
\begin{aligned}
\|f_{s^{-1}}\|^2_{L^2(1/w)} &\simeq \int_G |s^{Q/2}f(\delta_{s^{1/2}}(x))|^2(1+w_0(x))^a dx \\
&= s^Q\int_G |f(x)|^2(1+s^{-1/2}w_0(x))^a s^{-Q/2} dx \lesssim s^{Q/2}s^{-a/2}\|f\|^2_{L^2(1/w)},
\end{aligned}
$$

where, in the penultimate step, we have used that $0 < s < 1$. Hence, since $\frac{1}{w} \geqslant 1$,

$$
\begin{aligned}
&s^{-Q/2}s^{a/2}\|m_\delta(L)\|^2_{L^2(1/w)\to L^2(1/w)}\|f_{s^{-1}}\|^2_{L^2(1/w)} + s^{-Q/2}\|m_\delta(L)\|^2_{L^2(G)\to L^2(G)}\|f_{s^{-1}}\|^2_{L^2(G)} \\
&\qquad \leqslant \|m_\delta(L)\|^2_{L^2(1/w)\to L^2(1/w)}\|f\|^2_{L^2(1/w)} + \|m_\delta(L)\|^2_{L^2(G)\to L^2(G)}\|f\|^2_{L^2(G)} \\
&\qquad \leqslant \left(\|m_\delta(L)\|^2_{L^2((1/w)\to L^2(1/w} + \|m_\delta(L)\|^2_{L^2(G)\to L^2(G)}\right)\|f\|^2_{L^2(1/w)}
\end{aligned}
$$

By (5.0.11), this is bounded by

$$
2\|m_\delta(L)\|^2_{L^2((1/w)\to L^2(1/w)}\|f\|^2_{L^2(1/w)},
$$

as required. The calculation for $\tilde{m}_\delta(sL)$ is essentially identical. □

CHAPTER 6

REDUCTION TO TRACE LEMMAS

From now on, we will be working with an H-type group G and sub-Laplacian L thereon. In this chapter, we will show how proving suitable 'trace lemma' results allows us to deduce boundedness of the operators $m_\delta(L)$ on weighted L^2 spaces, which, as shown in the previous chapter, is sufficient to deduce boundedness of the corresponding maximal operators $M_\delta^\bullet$.

We begin by further decomposing our multiplier. We choose $J \in \mathbb{N}$ such that

$$2^{J-1} \leqslant 10\delta^{-1} \leqslant 2^J \tag{6.0.1}$$

and define operators R_j on $L^2(G)$ by

$$\widehat{R_j f}(\mu,\alpha,\beta) := \chi_{[2^j,2^{j+1})}(c(|\alpha|))\hat{f}(\mu,\alpha,\beta), \qquad \text{for } f \in L^2(G),\, j = 1,\dots,J-1, \tag{6.0.2}$$

and

$$\widehat{R_J f}(\mu,\alpha,\beta) := \chi_{[2^J,\infty)}(c(|\alpha|))\hat{f}(\mu,\alpha,\beta), \qquad \text{for } f \in L^2(G). \tag{6.0.3}$$

The following result motivates the subsequent developments.

Proposition 6.1. *Suppose that for a weight $w(z,u)$ and $N > 1$ we have, for all $1 \leqslant j \leqslant J$ and for all $0 < \delta \leqslant \frac{1}{2}$, the estimates*

$$\|R_j m_\delta(L) f\|_2^2 \lessapprox C\|f\|_{L^2(w)}^2 \tag{6.0.4}$$

and

$$\|R_j m_\delta(L) f\|^2_{L^2(w^N)} \lessapprox C^{1-N}\|f\|^2_{L^2(w)} + \|f\|^2_{L^2(w^N)} \tag{6.0.5}$$

where $C = C(j, \delta) > 0$. *Then this implies that*

$$\|m_\delta(L)\|^2_{L^2(w)\to L^2(w)} \lessapprox 1. \tag{6.0.6}$$

Proof. We let $S := \{(z,u) \in G : Cw(z,u) < 1\}$. Now, note that, for $(z,u) \in S$ we have $w(z,u) < C^{-1}$ and so $w(z,u)^{N-1} < C^{1-N}$. Hence,

$$\|\chi_S f\|^2_{L^2(w^N)} \lesssim C^{1-N}\|\chi_S f\|^2_{L^2(w)}$$

which turns (6.0.5) into

$$\|R_j m_\delta(L)\chi_S f\|^2_{L^2(w^N)} \lessapprox C^{1-N}\|\chi_S f\|^2_{L^2(w)}. \tag{6.0.7}$$

We can now interpolate this with (6.0.4) at $\theta = \frac{1}{N}$ to give

$$\|R_j m_\delta(L)\chi_S f\|^2_{L^2(w)} \lessapprox \|\chi_S f\|^2_{L^2(w)}. \tag{6.0.8}$$

For $(z,u) \notin S$ we have $C^{-1} \leqslant w(z,u)$ and so $C^{1-N}w(z,u) \leqslant w(z,u)^N$. Hence,

$$C^{1-N}\|\chi_{G\setminus S} f\|^2_{L^2(w)} \lesssim \|\chi_{G\setminus S} f\|^2_{L^2(w^N)}$$

which turns (6.0.5) into

$$\|R_j m_\delta(L)\chi_{G\setminus S} f\|^2_{L^2(w^N)} \lessapprox \|\chi_{G\setminus S} f\|^2_{L^2(w^N)}. \tag{6.0.9}$$

We can now interpolate this with the trivial L^2 estimate at $\theta = \frac{1}{N}$ to give

$$\|R_j m_\delta(L)\chi_{G\setminus S} f\|^2_{L^2(w)} \lessapprox \|\chi_{G\setminus S} f\|^2_{L^2(w)}. \tag{6.0.10}$$

We can sum (6.0.8) and (6.0.10) to obtain

$$\|R_j m_\delta(L)f\|^2_{L^2(w)} \lessapprox \|f\|^2_{L^2(w)}. \tag{6.0.11}$$

Finally, since the number of R_j is $J \simeq |\ln(\delta)|$, then this estimate holds if $R_j m_\delta$ is replaced by just m_δ, giving the required result. □

The aim of this chapter will then be to reduce proving the estimates we need, that is (6.0.4) and (6.0.5), to proving the following two 'trace lemmas'. We set

$$b(k) := c(k+\gamma) \text{ for } \gamma \in \{-1,0,1\},$$

and define operators $M^\gamma_{\delta,j}$, for $\gamma = 0, 1, -1$, $j = 1, \ldots, J-1$ and $f \in \mathscr{S}(G)$, by

$$\widehat{M^\gamma_{\delta,j} f}(\mu,\alpha,\beta) := \chi_{[1-\delta,1]}(b(|\alpha|)|\mu|)\chi_{[2^j,2^{j+1})}(b(|\alpha|))\widehat{f}(\mu,\alpha,\beta)$$

and for $j = J$, with γ, f as before, by

$$\widehat{M^\gamma_{\delta,J} f}(\mu,\alpha,\beta) := \chi_{[1-\delta,1]}(b(|\alpha|)|\mu|)\chi_{[2^J,\infty)}(b(|\alpha|))\widehat{f}(\mu,\alpha,\beta).$$

Then the 'trace lemmas' we need will be the estimates

$$\|M^\gamma_{\delta,j} f\|^2_2 \lessapprox (2^{-j})^a \|f\|^2_{L^2(1+\rho^a)}, \tag{6.0.12}$$

and

$$\|M^\gamma_{\delta,j} f\|^2_2 \lessapprox (2^{-j}\delta)^{a/2} \|f\|^2_{L^2(1+|\cdot|^a)}, \tag{6.0.13}$$

for $1 \leqslant j \leqslant J$ and $\gamma = 0, 1, -1$.

Lemma 6.2. *Suppose* (6.0.12) *for* $1 \leqslant j \leqslant J$, $\gamma \in \{-1,0,1\}$ *and for some* $a \in (0,2]$. *Then we can deduce the estimates* (6.0.4) *and* (6.0.5) *with* $w = 1 + \rho^a$ *and* $C = (2^{-j})^a$ *for* $N = \frac{4}{a}$.

Lemma 6.3. *Suppose* (6.0.13) *for* $1 \leqslant j \leqslant J$, $\gamma \in \{-1,0,1\}$ *and for some* $a \in (0,2]$. *Then we can deduce the estimates* (6.0.4) *and* (6.0.5) *with* $w = 1+|\cdot|^a$ *and* $C = (2^{-j}\delta)^{a/2}$ *for* $N = \frac{4}{a}$.

The proofs of these lemmas are long and technical, so we pause to explain the idea of the proof before going into details.

First, for Lemma 6.2, it is obvious that the estimate (6.0.12) with $\gamma = 0$ yields (6.0.4). Meanwhile, the left-hand side of (6.0.5) will be bounded by the unweighted L^2 norm, and the $L^2(\rho^4)$ norm (observe that $\rho^{aN} = \rho^4$). This is the same as the unweighted L^2 norm of $\rho^2 R_j m_\delta(L)f$, and since $R_j m_\delta(L)$ is a convolution operator, then we can use the previously-developed Leibniz rules to expand this out and estimate the pieces so-obtained separately.

The proof of Lemma 6.3 is essentially analogous to Lemma 6.2. In particular, note that

$$\|R_j m_\delta(L)f\|^2_{L^2(|\cdot|^4)} = \|R_j m_\delta(L)f\|^2_{L^2(\rho^4)} + 16\sum_{l=1}^{n} \|R_j m_\delta(L)f\|^2_{L^2(\psi_l^2)},$$

where N was chosen such that $aN = 4$. We then see that the first term is bounded by the same working done for Lemma 6.2. The other term will then follow from analogous working, modified as necessary by replacing the Leibniz rule for ρ^2 and formula for ∂_{ρ^2} developed in Section 2.3 with the analogous rule for ψ_l and formula for ∂_{ψ_l}.

As we have noted, $R_j m_\delta(L)$ is a convolution operator and we will be using the previously developed Leibniz rules. We therefore set $K_{\delta,j}$ to be the convolution kernel of $R_j m_\delta(L)$ and furthermore recall that by (2.2.24) we have

$$\widehat{K_{\delta,j}}(\mu,k) = \begin{cases} \chi_{[2^j,2^{j+1})}(c(k))m_\delta(|\mu|c(k)) \text{ for } j = 1,\ldots,J-1, \\ \chi_{[2^J,\infty)}(c(k))m_\delta(|\mu|c(k)) \text{ for } j = J. \end{cases} \tag{6.0.14}$$

First, some technical lemmas that will be required. We will need a lemma regarding matrix operators. This is a specific case of Schur's Test for matrices, which may be found in [26]. Note that if we let # denote the counting measure, then for an index set I

we write $l^p(I) := L^p(I,\#)$. We also call a matrix M 'finitely off-diagonal' if all entries are 0 except for those on a finite number of diagonals.

Lemma 6.4. *Denote by $\mathscr{L}(H)$ the space of bounded linear operators on a Hilbert space H. Let $T \in \mathscr{L}(H)$ and let $M := (m_{\alpha,\beta})_{\alpha,\beta}$ be the matrix obtained from isometrically isomorphically identifying $\mathscr{L}(H)$ with $\mathscr{L}(l^2(I))$ by fixing a choice of basis $\{e_\alpha\}_{\alpha\in I}$ and identifying T with $(m_{\alpha,\beta})_{\alpha,\beta} := (\langle Te_\alpha, e_\beta\rangle_H)_{\alpha,\beta}$. If M is finitely off-diagonal, then*

$$\|T\|^2_{H\to H} \lesssim \sum_\beta \|(m_{\alpha,\beta})_\alpha\|^2_{l^\infty} < \infty$$

where the constant in $\lesssim$ depends on the number of non-zero diagonals in M.

Proof. Note that

$$\|T\|_{H\to H} = \|M\|_{l^2(I)\to l^2(I)}.$$

Fix a sequence $x = (x_\alpha)_{\alpha\in I}$. Then, since all sums over β are finite,

$$\|Mx\|^2_{l^2(I)} = \sum_\alpha \left|\sum_\beta m_{\alpha,\beta}x_\beta\right|^2 \lesssim \sum_\alpha\sum_\beta |m_{\alpha,\beta}x_\beta|^2 \leqslant \sum_\beta \|(m_{\alpha,\beta})_\alpha\|_{l^\infty(I)}\|x_\alpha\|^2_{l^2(I)}.$$

□

This next lemma demonstrates how the operators ∂_{ρ^2} and ∂_{ψ_l} interact with the Fourier transform of $K_{\delta,j}$, in particular demonstrating how the support shifts and L^∞ norm of $\widehat{K_{\delta,j}}$ increases. The different constants appearing after application of the two different operators ∂_{ρ^2} and ∂_{ψ_l} highlights their differing behaviours.

Lemma 6.5. *Let*

$$H_{\delta,j}(\mu,k) := \chi_{[2^j,2^j+1)}(c(k))\chi_{[1-\delta,1]}(c(k)|\mu|)$$

for $j < J$ and

$$H_{\delta,J}(\mu,k) := \chi_{[2^J,\infty)}(c(k))\chi_{[1-\delta,1]}(c(k)|\mu|).$$

For $1 \leqslant j \leqslant J$*, we have*

$$|\partial_{\rho^2}\widehat{K_{\delta,j}}(\mu,k)| \lesssim 2^{2j}(H_{\delta,j}(\mu,k-1)+H_{\delta,j}(\mu,k)+H_{\delta,j}(\mu,k+1)) \tag{6.0.15}$$

and

$$|\partial_{\psi_l}\widehat{K_{\delta,j}}(\mu,k)| \lesssim \delta^{-1}2^{j}(H_{\delta,j}(\mu,k-1)+H_{\delta,j}(\mu,k)+H_{\delta,j}(\mu,k+1)). \tag{6.0.16}$$

For $j = J$ *we have that*

$$(H_{\delta,J}(\mu,k-1)+H_{\delta,J}(\mu,k)+H_{\delta,J}(\mu,k+1)) \lesssim \chi_{[2^{J-1},\infty)}(c(k))\chi_{[1-A\delta,1+A\delta]}(c(k)|\mu|)$$

for $A = 1+\frac{4\pi}{10}$.

Proof. We will first demonstrate the calculations for the $|\partial_{\rho^2}\widehat{K_{\delta,j}}(\mu,k)|$ case. We must argue the cases $1 \leqslant j \leqslant J-1$ and $j = J$ separately.

Let $j < J$. By applying (2.3.4) and estimating each summand individually, noting from support and boundedness conditions of $\chi_{[2^j,2^{j+1})}$ and m_δ (see (1.0.7)) and the definition of J (see (6.0.1)) that $k \simeq c(k) \simeq |\mu|^{-1} \simeq 2^j$ we see that

$$\begin{aligned}
|\partial_{\rho^2}\widehat{K_{\delta,j}}(\mu,k)| \leqslant &\frac{1}{\pi|\mu|}[|(2k+m)\chi_{[2^j,2^{j+1})}(c(k))m_\delta(|\mu|c(k))| \\
&+|k\chi_{[2^j,2^{j+1})}(c(k-1))m_\delta(|\mu|c(k-1))| \\
&+|(k+m)\chi_{[2^j,2^{j+1})}(c(k+1))m_\delta(|\mu|c(k+1))|] \\
&\lesssim 2^j[2^jH_{\delta,j}(\mu,k)+2^jH_{\delta,j}(\mu,k-1)+2^jH_{\delta,j}(\mu,k+1)] \\
&\lesssim 2^{2j}(H_{\delta,j}(\mu,k-1)+H_{\delta,j}(\mu,k)+H_{\delta,j}(\mu,k+1)).
\end{aligned} \tag{6.0.17}$$

For $j = J$, recall that by (2.3.4) and by (6.0.14) we have

$$\partial_{\rho^2}\widehat{K_{\delta,J}}(\mu,k) = \frac{1}{\pi|\mu|}[(2k+m)\chi_{[2^J,\infty)}(c(k))m_\delta(|\mu|c(k)) \\ - k\chi_{[2^J,\infty)}(c(k-1))m_\delta(|\mu|c(k-1)) \\ - (k+m)\chi_{[2^J,\infty)}(c(k+1))m_\delta(|\mu|c(k+1))]. \tag{6.0.18}$$

We argue three further sub-cases. First, suppose $c(k+1) \in [2^J,\infty)$ but $c(k) \notin [2^J,\infty)$. Then $k \simeq c(k) \simeq |\mu|^{-1} \simeq 2^{J-1}$ and (1.0.7) implies that

$$|\partial_{\rho^2}\widehat{K_{\delta,J}}(\mu,k)| = \frac{1}{\pi|\mu|}(k+m)\chi_{[2^J,\infty)}(c(k+1))|m_\delta(|\mu|c(k+1))| \lesssim 2^{2J}H_{\delta,j}(\mu,k+1).$$

Similarly, suppose $c(k) \in [2^J,\infty)$ but $c(k-1) \notin [2^J,\infty)$. Then $k \simeq c(k) \simeq |\mu|^{-1} \simeq 2^J$ and so from (1.0.7) we have

$$\begin{aligned}|\partial_{\rho^2}\widehat{K_{\delta,J}}(\mu,k)| \leqslant & \frac{1}{\pi|\mu|}[(2k+m)\chi_{[2^J,\infty)}(c(k))|m_\delta(|\mu|c(k))| \\ & + (k+m)\chi_{[2^J,\infty)}(c(k+1))|m_\delta(|\mu|c(k+1))|] \\ \lesssim & \, 2^J[2^J H_{\delta,j}(\mu,k) + 2^J H_{\delta,j}(\mu,k+1)] \\ = & \, 2^{2J}(H_{\delta,j}(\mu,k) + H_{\delta,j}(\mu,k+1)).\end{aligned}$$

For the final case, where $c(k-1) \in [2^J,\infty)$ we expand $\partial_{\rho^2}\widehat{K_{\delta,j}}(\mu,k)$ as a Taylor series. For some $0 \leqslant \theta_1, \theta_2 \leqslant 4\pi$ we have, by Taylor's Theorem,

$$m_\delta(|\mu|c(k-1)) = m_\delta(|\mu|c(k)) + |\mu|m_\delta'(|\mu|c(k))(-4\pi) + |\mu|^2 m_\delta''(|\mu|(c(k)+\theta_1))(-4\pi)^2$$

and

$$m_\delta(|\mu|c(k+1)) = m_\delta(|\mu|c(k)) + |\mu|m_\delta'(|\mu|c(k))(4\pi) + |\mu|^2 m_\delta''(|\mu|(c(k)+\theta_2))(4\pi)^2.$$

Substituting these into (6.0.18), and noting that under our assumptions all the characteristic functions evaluate to 1, gives

$$\begin{aligned}\partial_{\rho^2}\widehat{K_{\delta,J}}(\mu,k) &= \frac{1}{\pi|\mu|}[(2k+m)m_\delta(|\mu|c(k)) - k(m_\delta(|\mu|c(k)) \\ &\quad + |\mu|m_\delta'(|\mu|c(k))(-4\pi) + |\mu|^2 m_\delta''(|\mu|(c(k)+\theta_1))(-4\pi)^2) \\ &\quad - (k+m)(m_\delta(|\mu|c(k)) + |\mu|m_\delta'(|\mu|c(k))(4\pi) \\ &\quad + |\mu|^2 m_\delta''(|\mu|(c(k)+\theta_2))(4\pi)^2)] \\ &= -(4m)m_\delta'(|\mu|c(k)) \\ &\quad - 16k\pi|\mu|m_\delta''(|\mu|(c(k)+\theta_1)) - 16\pi(k+m)|\mu|m_\delta''(|\mu|(c(k)+\theta_2)).\end{aligned} \tag{6.0.19}$$

Under our assumptions and with the support condition of m_δ given in (1.0.7), we have $k \simeq c(k) \simeq |\mu|^{-1}$ and $2^J \simeq \delta^{-1}$. Using also the boundedness conditions in (1.0.7) and the facts that $c(k) \leqslant c(k) + \theta_i \leqslant c(k+1)$ for $i = 1, 2$ and that for k this large, $H_{\delta,j}(\mu,k) + H_{\delta,j}(\mu,k+1) \gtrsim \widehat{K_{\delta,J}}(\mu,k+\theta_i)$, this implies that

$$\begin{aligned}|\partial_{\rho^2}\widehat{K_{\delta,J}}(\mu,k)| &\lesssim \delta^{-1} + 1\delta^{-2} + 1\delta^{-2} \\ &\simeq 2^{2J}(H_{\delta,j}(\mu,k) + H_{\delta,j}(\mu,k+1))\end{aligned}$$

as required.

Now, for the ∂_{ψ_l}, for $j < J$ case, note that $\frac{\mu_l}{|\mu|^2} \lesssim \frac{1}{|\mu|}$. Considering (2.3.6), it therefore suffices to show that $|\partial_{\mu_l}\widehat{K_{\delta,j}}(\mu,k)| \lesssim 2^j\delta^{-1}H_{\delta,j}(\mu,k)$. The bound on the remaining terms then follows as in the ∂_{ρ^2} case.

By (6.0.14) and (1.0.7) we have,

$$\begin{aligned}|\partial_{\mu_l}\widehat{K_{\delta,j}}(\mu,k)| &= |\chi_{[2^j,2^{j+1})}(c(k))c(k)\tfrac{\mu_l}{|\mu|}m_\delta'(|\mu|c(k))| \\ &\lesssim |H_{\delta,j}(\mu,k)c(k)\delta^{-1}| \lesssim 2^j\delta^{-1}H_{\delta,j}(\mu,k).\end{aligned}$$

For $j = J$, if $c(k)$ is close to 2^J then similar working to the ∂_{ρ^2} case gives us the desired bound. We assume now that $c(k-1) \in [2^J, \infty)$. In this case, by (2.3.6) and (6.0.14), noting that we have assumed all the characteristic functions are identically 1, we have

$$\partial_{\psi_l}\widehat{K_{\delta,j}}(\mu,k) = 2c(k)\frac{\mu_l}{|\mu|}m_\delta'(|\mu|c(k)) + \frac{\mu_l}{|\mu|^2}[(m)m_\delta(|\mu|c(k)) \\ + km_\delta(|\mu|c(k-1)) - (k+m)m_\delta(|\mu|c(k+1))].$$

We now substitute the Taylor series expansions to obtain

$$\frac{\mu_l}{|\mu|}c(k)m_\delta'(|\mu|c(k)) - \frac{\mu_l}{|\mu|^2}[(2k+m)|\mu|m_\delta'(|\mu|c(k))(4\pi)] \\ + \mu_l[km_\delta''(|\mu|(c(k)+\theta_1))(4\pi)^2 - (k+m)m_\delta''(|\mu|(c(k)+\theta_2))(4\pi)^2)]$$

Now, note that $|\mu_l|k \lesssim |\mu|c(k) \simeq 1$, so by (1.0.7) the second-order terms are majorised in absolute value by $\delta^{-2} \simeq 2^J\delta^{-1}$ as required. For the first-order terms we have

$$\frac{\mu_l}{|\mu|}2c(k)m_\delta'(|\mu|c(k)) - \frac{\mu_l}{|\mu|^2}[(2k+m)|\mu|m_\delta'(|\mu|c(k))(4\pi)] \\ = \frac{\mu_l}{|\mu|}[2c(k)m_\delta'(|\mu|c(k)) - 2c(k)m_\delta'(|\mu|c(k))] = 0.$$

As before, note that everything remains supported on $H_{\delta,J}(\mu,k-1) + H_{\delta,J}(\mu,k) + H_{\delta,J}(\mu,k+1)$.

Finally, note that in the case $j = J$, for $\gamma \in \{-1,0,1\}$, we have $c(k+\gamma) \geqslant 2^J \geqslant 10\delta^{-1}$. Since $c(k+\gamma)|\mu| \leqslant 1$ then we have $|\mu| < \frac{\delta}{10}$. Hence, if $1-\delta \leqslant c(k+\gamma)|\mu| = c(k)|\mu| + 4\pi\gamma|\mu| \leqslant 1$, then $1-\delta-4\pi\gamma|\mu| \leqslant c(k)|\mu| \leqslant 1-4\pi\gamma|\mu|$. Clearly we can therefore choose $A = 1+\frac{4\pi}{10}$ such that $1-A\delta \leqslant c(k)|\mu| \leqslant 1+A\delta$. This completes the proof. □

We now state the proofs of Lemmas 6.2 and 6.3.

Proof of Lemma 6.2. Clearly, (6.0.12) with $\gamma = 0$ implies (6.0.4).

For (6.0.5), we have, recalling that $N = \frac{4}{a}$,

$$\|R_j m_\delta(L)f\|^2_{L^2((1+\rho^a)^N)} \simeq \|R_j m_\delta(L)f\|_2^2 + \|R_j m_\delta(L)f\|^2_{L^2(\rho^4)}.$$

Since $1 \leqslant 1 + |z|^4$ then clearly it suffices to prove the appropriate bound for the second term. Now,

$$\|R_j m_\delta(L) f\|_{L^2(\rho^4)}^2 = \|\rho^2 (f * K_{\delta,j})\|_2^2.$$

If we apply (2.3.10) to $\rho^2(f * K_{\delta,j})$, then using the triangle inequality, we will get

$$\|R_j m_\delta(L) f\|_{L^2(\rho^4)}^2 \lesssim \|f * (\rho^2 K_{\delta,k})\|_2^2 + \|(\rho^2 f) * K_{\delta,j}\|_2^2 \\ + \|\rho^2 (f * K_{\delta,j}) - (\rho^2 f) * K_{\delta,j} - f * (\rho^2 K_{\delta,j})\|_2^2.$$

Note that we trivially have

$$\|(\rho^2 f) * K_{\delta,j}\|_2^2 \lesssim \|f\|_{L^2(\rho^4)}^2$$

since $R_j m_\delta(L)$ is bounded on L^2.

Next, recall from (2.3.10) that, for some fixed $\mu_1 \in \mathbb{R}^n$,

$$\begin{aligned} &\|\rho^2 (f * K_{\delta,j}) - (\rho^2 f) * K_{\delta,j} - f * (\rho^2 K_{\delta,j})\|_2^2 \\ &\qquad = \left\| \sum_{k=1}^{m} (\zeta_{\mu_1,k} f) * (\overline{\zeta_{\mu_1,j}} K_{\delta,j}) + (\overline{\zeta_{\mu_1,k}} f) * (\zeta_{\mu_1,j} K_{\delta,j}) \right\|_2^2 \\ &\qquad \lesssim \sum_{k=1}^{m} \|(\zeta_{\mu_1,k} f) * (\overline{\zeta_{\mu_1,j}} K_{\delta,j})\|_2^2 + \|(\overline{\zeta_{\mu_1,k}} f) * (\zeta_{\mu_1,j} K_{\delta,j})\|_2^2. \end{aligned}$$

For brevity, we will only display calculations involving $\partial_{\zeta_{\mu,k}}$ as the analogous results for $\partial_{\overline{\zeta_{\mu,k}}}$ are essentially identical. Now, by (2.3.1) we have

$$\partial_{\zeta_{\mu,p}} \widehat{K_{\delta,j}}(\mu,\alpha,\beta) = \sqrt{\frac{2\alpha_p + 2}{2\pi|\mu|}} \widehat{K_{\delta,j}}(\mu,\alpha + e_p,\beta) - \sqrt{\frac{2\beta_p}{2\pi|\mu|}} \widehat{K_{\delta,j}}(\mu,\alpha,\beta - e_p). \quad (6.0.20)$$

Since $K_{\delta,j}$ is radial, then by Theorem 2.25, $\widehat{\zeta_{\mu,p} K_{\delta,j}}(\mu,\alpha,\beta) = 0$ if $\beta \neq \alpha + e_p$. Furthermore, for $1 \leqslant j < J$, by (6.0.14), on the support of $\partial_{\zeta_{\mu,p}} \widehat{K_{\delta,j}}(\mu,\alpha,\beta)$ we have that

$|\alpha| \simeq |\beta| \simeq |\mu|^{-1} \simeq 2^j$. If instead $j = J$ then we have

$$\partial_{\zeta_{\mu,p}} \widehat{K_{\delta,j}}(\mu, \alpha, \alpha + e_p) = 2\sqrt{\frac{2\alpha_p + 2}{2\pi|\mu|}}(\widehat{K_{\delta,j}}(\mu, \alpha + e_p, \alpha + e_p) - \widehat{K_{\delta,j}}(\mu, \alpha, \alpha))$$

Expanding $\widehat{K_{\delta,j}}(\mu, \alpha + e_p, \alpha + e_p) = \chi_{[2^J,\infty)}(c(|\alpha| + 1))m_\delta(|\mu|c(|\alpha| + 1))$ as a Taylor series, we get, for some $0 \leqslant \theta \leqslant 4\pi$,

$$\begin{aligned}
&\chi_{[2^J,\infty)}(c(|\alpha| + 1))m_\delta(|\mu|c(|\alpha| + 1)) - \chi_{[2^J,\infty)}(c(|\alpha|))m_\delta(|\mu|c(|\alpha|)) \\
&= \chi_{[2^J,\infty)}(c(|\alpha| + 1))m_\delta(|\mu|c(|\alpha|) + |\mu|m'_\delta(|\mu|c(|\alpha|))(4\pi) \\
&\quad + |\mu|^2 m''_\delta(|\mu|(c(|\alpha|) + \theta))(4\pi)^2) - \chi_{[2^J,\infty)}(c(|\alpha|))m_\delta(|\mu|c(|\alpha|)).
\end{aligned}$$

This gives us

$$\left|\widehat{\zeta_{\mu,p}K_{\delta,j}}(\mu, \alpha, \beta)\right| \lesssim 2^j \text{ for } 1 \leqslant j \leqslant J.$$

By (2.3.5) we have

$$\left|\widehat{\zeta_{\mu_1,p}K_{\delta,j}}(\mu, \alpha, \beta)\right| \lesssim 2^j \text{ for } 1 \leqslant j \leqslant J$$

for any $\mu, \mu_1 \in \mathbb{R}^n$. Furthermore, if $|\alpha - \beta| \geqslant 2$ then $\widehat{\zeta_{\mu_1,p}K_{\delta,j}}(\mu, \alpha, \beta) = 0$ (because $\widehat{\zeta_{\mu_1,p}K_{\delta,j}}(\mu, \alpha, \beta)$ is a sum of terms which will all be zero if α and β differ in more than one digit or by more than 1 in one digit). Lemma 6.4 then gives us that

$$\|\widehat{\zeta_{\mu_1,p}K_{\delta,j}}(\mu)\|_{L^2(\mathbb{R}^m)\to L^2(\mathbb{R}^m)} \lesssim 2^j \text{ for } 1 \leqslant j \leqslant J \tag{6.0.21}$$

for any $\mu, \mu_1 \in \mathbb{R}^n$.

Fix some μ_1. Then by Plancherel's theorem we have

$$\begin{aligned}
\|(\zeta_{\mu_1,k}f) * (\zeta_{\mu_1,p}K_{\delta,j})\|_2^2 &= \int_{\mathbb{R}^n} \|\widehat{\zeta_{\mu_1,p}K_{\delta,j}}(\mu)\widehat{\zeta_{\mu_1,k}f}(\mu)\|_{HS}^2 |\mu|^m d\mu \\
&\leqslant \int_{\mathbb{R}^n} \|\widehat{\zeta_{\mu_1,p}K_{\delta,j}}(\mu)\|_{L^2(\mathbb{R}^m)\to L^2(\mathbb{R}^m)}^2 \|\widehat{\zeta_{\mu_1,k}f}(\mu)\|_{HS}^2 |\mu|^m d\mu \\
&\lesssim 2^{2j}\|f\|_{L^2(\rho^2)}^2.
\end{aligned}$$

This gives us the estimate

$$\|\rho^2(f * K_{\delta,j}) - (\rho^2 f) * K_{\delta,j} - f * (\rho^2 K_{\delta,j})\|_2^2 \lesssim 2^{2j}\|f\|_{L^2(\rho^2)}^2. \tag{6.0.22}$$

Next, for $j < J$, by applying Plancherel's theorem, using (6.0.15) and then Plancherel's theorem again, we get

$$\begin{aligned}\|f * (\rho^2 K_{\delta,j})\|_2^2 &= \int_{\mathbb{R}^n} \sum_{\alpha,\beta} |\widehat{|z|^2 K_{\delta,j}}(\mu, |\alpha|)\widehat{f}(\mu,\alpha,\beta)|^2 |\mu|^m d\mu \\ &\lesssim \int_{\mathbb{R}^n} \sum_{\alpha,\beta} \sum_{\gamma\in\{-1,0,1\}} |2^{2j}\widehat{M_{\delta,j}^{\gamma}f}(\mu,\alpha,\beta)|^2|\mu|^m d\mu \\ &\lesssim 2^{4j}(\|M_{\delta,j}^0 f\|_2^2 + \|M_{\delta,j}^1 f\|_2^2 + \|M_{\delta,j}^{-1} f\|_2^2)\end{aligned}$$

Applying (6.0.12) gives us

$$\|f * (\rho^2 K_{\delta,j})\|_2^2 \lessapprox 2^{(4-a)j}\|f\|_{L^2(1+\rho^a)}^2 \tag{6.0.23}$$

as required. For $j = J$, using Plancherel's Theorem and considering support conditions, (6.0.15) gives us, for some $A > 0$

$$\begin{aligned}\|f * (\rho^2 K_{\delta,J})\|_2^2 &= \int_{\mathbb{R}^n} \sum_{\alpha,\beta} |\widehat{|z|^2 K_{\delta,J}}(\mu,|\alpha|)\widehat{f}(\mu,\alpha,\beta)|^2|\mu|^m d\mu \\ &\lesssim \int_{\mathbb{R}^n} \sum_{\alpha,\beta} 2^{4J}\chi_{[1-A\delta,1]}(|\mu|c(|\alpha|))\chi_{[2^J,\infty)}(c(|\alpha|))|\widehat{f}(\mu,\alpha,\beta)|^2|\mu|^m d\mu \\ &\lesssim 2^{4J}\|\chi_{[1-A\delta,1]}(L)R_J f\|_2^2\end{aligned} \tag{6.0.24}$$

Again, applying (6.0.12) to (6.0.24) gives

$$\|f * (\rho^2 K_{\delta,J})\|_2^2 \lessapprox 2^{(4-a)J}\|f(z,u)\|_{L^2(1+\rho^a)}^2.$$

Combining everything gives

$$\|R_j m_\delta(L) f\|^2_{L^2(\rho^4)} \lessapprox 2^{(4-a)j}\|f\|^2_{L^2(1+\rho^a)} + \|f\|^2_{L^2(\rho^4)} + 2^{2j}\|f\|^2_{L^2(\rho^2)}.$$

Now, if $a = 2$ then the proof is complete. Otherwise, let

$$S := \{(z,u) \in G : \rho^2(z,u) \leqslant 2^{2j}\}$$

and write $f = \chi_S f + (1-\chi_S)f$. Then

$$2^{2j}\|f\|^2_{L^2(\rho^2)} = 2^{2j}\|\chi_S f\|^2_{L^2(\rho^2)} + 2^{2j}\|(1-\chi_S)f\|^2_{L^2(\rho^2)}$$

where (since for $a \in (0,2)$, $\rho^2(z,u) \leqslant 2^{2j}$ implies that $\rho^2(z,u)^{1-a/2} \leqslant 2^{(2-a)j}$)

$$2^{2j}\|\chi_S f\|^2_{L^2(\rho^2)} \leqslant 2^{(4-a)j}\|f\|^2_{L^2(1+\rho^a)}$$

and

$$2^{2j}\|(1-\chi_S)f\|^2_{L^2(\rho^2)} \leqslant \|f\|^2_{L^2(\rho^4)}.$$

This concludes the proof. □

Proof of Lemma 6.3. The proof is analogous to Lemma 6.2. In particular, note that

$$\|R_j m_\delta(L) f\|^2_{L^2(|\cdot|^4)} = \|R_j m_\delta(L) f\|^2_{L^2(\rho^4)} + 16\sum_{l=1}^{n}\|R_j m_\delta(L) f\|^2_{L^2(\psi_l^2)}.$$

Then the proof of Lemma 6.2 bounds the first term (note that $2^j \leqslant \sqrt{2^j\delta^{-1}}$, and the proof may be modified by replacing various uses of the Leibniz rule (2.3.10) for ρ^2 with the analogous rule (2.3.15) for ψ_l, uses of (6.0.15) with (6.0.16) and uses of (6.0.12) with (6.0.13). Note that these will produce instances of constants $\sqrt{2^j\delta^{-1}}$ instead of 2^j at various stages of the modified proof, but these will combine in the same way as the constants 2^j do in the proof of Lemma 6.2. □

CHAPTER 7

THE TRACE LEMMAS

First, recall that $c(k)$ is defined in (2.2.23). Recall that we define

$$b(k) := c(k+\gamma) \text{ for } \gamma \in \{-1, 0, 1\}, \tag{7.0.1}$$

and $M^{\gamma}_{\delta,j}$, for $\gamma = 0, 1, -1$, $j = 1, \ldots, J-1$ and $f \in \mathscr{S}(G)$, by

$$\widehat{M^{\gamma}_{\delta,j} f}(\mu,\alpha,\beta) := \chi_{[1-\delta,1]}(b(|\alpha|)|\mu|)\chi_{[2^j,2^{j+1})}(b(|\alpha|))\widehat{f}(\mu,\alpha,\beta) \tag{7.0.2}$$

and for $j = J$, with γ, f as before, by

$$\widehat{M^{\gamma}_{\delta,J} f}(\mu,\alpha,\beta) := \chi_{[1-\delta,1]}(b(|\alpha|)|\mu|)\chi_{[2^J,\infty)}(b(|\alpha|))\widehat{f}(\mu,\alpha,\beta). \tag{7.0.3}$$

The main result of this section is the following:

Theorem 7.1. *We have, for all* $1 \leqslant j \leqslant J$*, for all* $\gamma \in \{-1, 0, 1\}$ *and for* $a \in [0, \frac{2}{3}]$,

$$\|M^{\gamma}_{\delta,j} f\|_2^2 \lessapprox (2^{-j}\delta)^{a/2} \|f\|^2_{L^2(1+|\cdot|^a_K)}. \tag{7.0.4}$$

The proof of this ‘trace lemma’ is technical, and so in this chapter we present a number of other results whose development lead up to this, before concluding with the proof as Section 7.3.3.

The first section of this chapter, Section 7.1, deals with some of the general theory that is used; in particular, we obtain formulae for the integral kernels that correspond to fractional integration on the dual of H-type groups corresponding to fractional powers of pure first or second layer weights on the group side.

The $j = J$ case of Theorem 7.1 requires only pure first layer weights and does not invoke any fractional integration formulae, and is a consequence of Theorem 7.2 below. The ideas used to prove the $1 \leqslant j < J$ cases of Theorem 7.1 were first developed in the context of pure first layer weights, due to the relative simplicity of the formula for the integral kernel of fractional integration corresponding to such weights. These first layer results are presented as they demonstrate the ideas behind the proof of the $1 \leqslant j < J$ cases of Theorem 7.1 in a simpler context. First, the radial case is shown, as this leads to many further simplifications, before the method that allows us to remove this assumption is demonstrated. This culminates in the following result, which may be used to prove an almost-everywhere convergence result for functions in mixed L^p spaces.

Theorem 7.2. *We have, for all $1 \leqslant j \leqslant J$, for all $\gamma \in \{0, 1, -1\}$ and for $a \in [0, 1]$,*

$$\|M^{\gamma}_{\delta,j} f\|_2^2 \lessapprox (2^{-j})^a \|f\|^2_{L^2(1+\rho^a)}.$$

Finally, the work to prove Theorem 7.1 for $j < J$ is given in Section 7.3. Although the ideas behind the proof of Theorem 7.1 are the same as in the pure first-layer case, the integral kernels are very different and are more complicated to study. When using a pure first layer weight, the quantity we ultimately are required to estimate is a sum of ratios of gamma functions. When moving to second layer weights, we are instead required to estimate Jacobi polynomials. As in the first layer case, we first prove the result for radial functions only. Unlike in the first layer case, the radial case of Theorem 7.1, for $j < J$ only, is slightly stronger than the full result, in that the final trace lemma contains only a pure second layer weight ψ rather than the full weight $|\cdot|_K$.

Proposition 7.3. *Let f be a radial function. We have, for all $1 \leqslant j < J$, for all*

$\gamma \in \{-1, 0, 1\}$ *and* $a \in [0, \frac{2}{3}]$,

$$\|M^{\gamma}_{\delta,j} f\|_2^2 \lessapprox (2^{-j}\delta)^{a/2} \|f\|^2_{L^2(1+\psi^a)}. \tag{7.0.5}$$

The reason for this is the need to apply a Leibniz rule for the second layer weight ψ in the general case that does not occur in the radial case. Recall that applying a second layer Leibniz rule ((2.3.11) or (2.3.18)) generates both first and second layer terms, and bounding both of these may only be done by using the full weight.

We remarked earlier that the $j = J$ case of Theorem 7.1 is a consequence of Theorem 7.2. Recall that we defined J such that $2^{-J} \simeq \delta$, so in particular $(2^{-J}\delta)^{a/2} \simeq (2^{-J})^a$. That is, for $j = J$ the constants appearing in Theorems 7.1 and 7.2 are essentially the same. Away from $j = J$, the two constants will instead differ non-trivially, reflecting the differing nature of using first and second layer weights. Recall (2.3.4) and (2.3.6). Observe that (2.3.4) acts only on the discrete 'Fourier' variable k while (2.3.6) acts on both 'Fourier' variables μ and k. Recall also that the cutoff $M^{\gamma}_{\delta,j}$ is the product of one cutoff of length 2^j in the discrete variable only and a second cutoff of length δ on $b(k)|\mu|$. It should perhaps therefore not be surprising to see the type of constants that we do see in Theorems 7.1 and 7.2.

There are some indications that a stronger trace lemma than Theorem 7.1 holds. For $1 \leqslant j \leqslant J$ and $\gamma = 0, 1, -1$, the estimate

$$\|M^{\gamma}_{\delta,j} f\|_2^2 \lessapprox \sqrt{2^{-j}\delta} \|f\|^2_{L^2(1+|\cdot|_K)}. \tag{7.0.6}$$

(which, if proven, would give Theorem 7.1 for $a \in [0, 1]$ and a consequent improvement in Theorem 1.1) is fully proven in the specific case of Heisenberg groups in [24]. Thus, given (7.0.6), one could use our previous reduction results to obtain a simplification of the proof of almost-everywhere convergence result in [24]. On general H-type groups, with the methods we employ, we could only manage to verify (7.0.6) for the restricted range of j given by $1 \leqslant j \leqslant \frac{3J}{4}$ and for $j = J$ (see Remark 7.31). Given the estimates available for Jacobi polynomials, it seems likely that this result is impossible to prove using pure second layer weights (see Remark 7.25) and would instead require analysis of integral kernels arising from a weight involving both first and second layer components. This would be different to the Euclidean case, where we recall from Figure 1.3 that partial weights are sufficient to recover the full result.

Before discussing the proofs of the 'trace lemmas', Theorems 7.1 and 7.2, we shall prove a small lemma that will be of use in both the first and second layer cases.

Lemma 7.4. *Let $k \in \mathbb{Z}$ and $x \in \mathbb{N}_0$. If $c(k) > 0$, then $\frac{c(k+x)}{c(k)} \in [1, 1+2x]$. If additionally $c(k-x) > 0$, then $\frac{c(k-x)}{c(k)} \in [\frac{1}{1+2x}, 1]$.*

Proof. For the first inclusion, since $c(k) > 0$ then $c(k) \geqslant 2\pi$, so we have

$$1 \leqslant \frac{c(k+x)}{c(k)} = 1 + \frac{4\pi x}{c(k)} = 1 + \frac{4\pi x}{2\pi} \leqslant 1 + 2x. \tag{7.0.7}$$

If $c(k-x) > 0$, then let $l := k - x$. Then $c(l) > 0$, so applying the first result of this Lemma gives

$$\frac{c(k)}{c(k-x)} = \frac{c(l+x)}{c(l)} \in [1, 1+2x], \tag{7.0.8}$$

which gives the second result. □

7.1 Fractional Integration of Radial Functions on the Dual

In this section, we will define fractional powers of difference-differential operators on the dual space arising from multiplying by weights on the group side via their heat kernels.

Let $f(|z|, u)$ by a radial function on G and let $\omega(|z|, u)$ be a radial weight. Then by definition we have (cf. (2.2.42))

$$\partial_\omega \widehat{f}(\mu,k) = \binom{k+m-1}{k}^{-1} \int\limits_G \omega(|z|,u) f(|z|,u) e^{2\pi i \mu u} L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dz du. \tag{7.1.1}$$

which, by (2.2.45) and our identification of G as $\mathbb{C}^m \times \mathbb{R}^n$ is then equal to

$$\binom{k+m-1}{k}^{-1} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{C}^m} \omega(|z|,u) C(m,n) \int\limits_{\mathbb{R}^n \setminus \{0\}} \sum_{l\in\mathbb{N}_0} \widehat{f}(\nu,l) e^{-2\pi i \nu u} L_l^{m-1}(\pi|\nu||z|^2) e^{-\frac{\pi|\nu||z|^2}{2}} |\nu|^m d\nu$$

$$\cdot\, e^{2\pi i \mu u} L_k^{m-1}(\pi|\mu||z|^2) e^{-\frac{\pi|\mu||z|^2}{2}} dz du.$$

If we define

$$\omega(|z|,\widehat{\nu}) := \int\limits_{\mathbb{R}^n} \omega(|z|,u) e^{-2\pi i u \nu} du, \tag{7.1.2}$$

(note that this is the partial Euclidean Fourier transform of $\omega(|z|,u)$ in the u variable only) and set $t = |z|^2$, so that $dz = t^{m-1} dt$, this becomes

$$\frac{C(m,n)}{\binom{k+m-1}{k}} \int\limits_{\mathbb{R}^n} \sum_{l\in\mathbb{N}_0} \int\limits_0^\infty \omega(\sqrt{t}, \widehat{\nu-\mu}) \widehat{f}(\nu,l) L_l^{m-1}(\pi|\nu|t) L_k^{m-1}(\pi|\mu|t) e^{-\frac{\pi(|\nu|+|\mu|)t}{2}} t^{m-1} dt |\nu|^m d\nu.$$

Hence, we have that $\partial_\omega \widehat{f}(\mu,k)$ is an integral operator

$$\partial_\omega \widehat{f}(\mu,k) = \int\limits_{\mathbb{R}^n} \sum_{l\in\mathbb{N}_0} \widehat{f}(\nu,l) K_\omega(\nu,l;\mu,k) \binom{l+m-1}{l} |\nu|^m d\nu. \tag{7.1.3}$$

with integral kernel

$$K_\omega(\nu,l;\mu,k) := \frac{C(m,n)}{\binom{k+m-1}{k}\binom{l+m-1}{l}} \int\limits_0^\infty \omega(\sqrt{t}, \widehat{\nu-\mu})$$

$$L_l^{m-1}(\pi|\nu|t) L_k^{m-1}(\pi|\mu|t) e^{-\frac{\pi(|\nu|+|\mu|)t}{2}} t^{m-1} dt. \tag{7.1.4}$$

Now, if we wish to prove an estimate of the form

$$\|\omega T f\|_2^2 \lesssim C_\omega(\delta)\|f\|_2^2$$

for radial functions $f \in L^2(G)$, some radial weight $\omega(|z|, u)$ and operator T where $\widehat{Tf}(\mu, k) = t(\mu, k)\widehat{f}(\mu, k)$ and t is real valued, then by taking the Fourier transform, this is equivalent to

$$\|\partial_\omega(tg)\|_2^2 \lesssim C_\omega(\delta)\|g\|_2^2$$

for functions $g(\mu, k) \in L^2(\mathbb{R}^n\backslash\{0\} \times \mathbb{N}_0, \binom{k+m-1}{k}\#(k)|\mu|^m d\mu)$. Now, by the Cauchy-Schwarz inequality and then Lemma A.3, we have

$$\begin{aligned}
\|\partial_\omega(tg)\|_2^2 &= \langle \partial_\omega(tg), \partial_\omega(tg)\rangle \\
&= \langle t\partial_{\omega^2}(tg), g\rangle \\
&\leqslant \|t\partial_{\omega^2}(tg)\|_2 \|g\|_2 \\
&\leqslant \|g\|_2^2 \left(\sup_{k,\mu} \int_{\mathbb{R}^n} \sum_{l\in\mathbb{N}_0} t(\mu, k)|K_{\omega^2}(\nu, l; \mu, k)|t(\nu, l)\binom{l+m-1}{l}|\nu|^m d\nu\right)^{1/2} \\
&\quad \cdot \left(\sup_{l,\nu} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{N}_0} t(\mu, k)|K_{\omega^2}(\nu, l; \mu, k)|t(\nu, l)\binom{k+m-1}{k}|\mu|^m d\mu\right)^{1/2}. \quad (7.1.5)
\end{aligned}$$

Note that since $\omega^2 > 0$, then the operator $f \mapsto \omega^2 f$ is self-adjoint on $L^2(G)$. Consequently, the operator ∂_{ω^2} is self-adjoint on $L^2(\mathbb{R}^n\backslash\{0\} \times \mathbb{N}_0, \binom{k+m-1}{k}\#(k)|\mu|^m d\mu)$, whence

$$K_{\omega^2}(\nu, l; \mu, k) = \overline{K_{\omega^2}(\mu, k; \nu, l)}.$$

It follows that both integrals in (7.1.5) are equal, and so this reduces to

$$\|\partial_\omega tg\|_2^2 \leqslant \|g\|_2^2 \sup_{k,\mu} \int_{\mathbb{R}^n} \sum_{l\in\mathbb{N}_0} t(\mu, k)|K_{\omega^2}(\nu, l; \mu, k)|t(\nu, l)\binom{l+m-1}{l}|\nu|^m d\nu \qquad (7.1.6)$$

If we assume that $\omega(|z|, u)$ is a function of only $|z|$ or u then simplifications occur in the formula for $K_\omega(\nu, l; \mu, k)$.

Lemma 7.5. *If $\omega(|z|, u) = w(|z|)$, then*

$$K_\omega(\nu, l; \mu, k) = \frac{C(m,n)\delta_\mu(\nu)}{\binom{k+m-1}{k}\binom{l+m-1}{l}} \int_0^\infty w(\sqrt{t}) L_l^{m-1}(\pi|\nu|t) L_k^{m-1}(\pi|\nu|t) e^{-\pi|\nu|t} t^{m-1} dt. \quad (7.1.7)$$

Proof. Observe that in this case,

$$\omega(|z|, \widehat{\nu}) = \int_{\mathbb{R}^n} w(|z|) e^{-2\pi i u\nu} du = w(|z|)\delta_0(\nu).$$

The result is then immediate from (7.1.4). □

Lemma 7.6. *If $\omega(|z|, u) = w(u)$, then, for $k \geqslant l$ we have*

$$K_\omega(\nu, l; \mu, k) \simeq \frac{\widehat{w}(\nu - \mu)}{\binom{l+m-1}{l}(|\nu| + |\mu|)^m} \left(2\tfrac{|\nu|}{|\mu|+|\nu|} - 1\right)^{k-l} P_l^{(k-l,m-1)}\left(1 - 2\left(1 - 2\tfrac{|\nu|}{|\mu|+|\nu|}\right)^2\right) \quad (7.1.8)$$

and for $k < l$ we have

$$K_\omega(\nu, l; \mu, k) \simeq \frac{\widehat{w}(\nu - \mu)}{\binom{k+m-1}{k}(|\nu| + |\mu|)^m} \left(1 - 2\tfrac{|\nu|}{|\mu|+|\nu|}\right)^{l-k} P_k^{(l-k,m-1)}\left(1 - 2\left(1 - 2\tfrac{|\nu|}{|\mu|+|\nu|}\right)^2\right), \quad (7.1.9)$$

where $P_n^{(a,b)}$ are Jacobi polynomials.

Proof. Since $\omega(|z|, \widehat{\nu - \mu}) = \widehat{w}(\nu - \mu)$, then (7.1.4) becomes

$$K_\omega(\nu, l; \mu, k) = \frac{C(m,n)\widehat{w}(\nu - \mu)}{\binom{k+m-1}{k}\binom{l+m-1}{l}} \int_0^\infty L_l^{m-1}(\pi|\nu|t) L_k^{m-1}(\pi|\mu|t) e^{-\frac{\pi(|\nu|+|\mu|)t}{2}} t^{m-1} dt.$$

If we set $u = \pi t(|\nu| + |\mu|)$, then

$$K_\omega(\nu, l; \mu, k) \simeq \frac{C(m,n)\widehat{w}(\nu - \mu)}{\binom{k+m-1}{k}\binom{l+m-1}{l}(|\nu| + |\mu|)^m} \int_0^\infty L_l^{m-1}\left(\tfrac{|\nu|}{|\nu|+|\mu|}u\right) L_k^{m-1}\left(\tfrac{|\mu|}{|\nu|+|\mu|}u\right) e^{-u/2} u^{m-1} du.$$

The result is then immediate using Lemma 9.1, specifically (9.0.3). □

7.2 First Layer Trace Lemmas

7.2.1 Proof of Theorem 7.2 for $j = J$

Lemma 7.7. *For $A = 1 + \frac{4\pi}{10}$ let*

$$\Xi(t) := \int_t^\infty \int_s^\infty \chi_{[1-A\delta,1+A\delta]}(r)drds. \tag{7.2.1}$$

Furthermore let

$$\Omega(\mu,k) := \Xi(c(k)|\mu|). \tag{7.2.2}$$

Then there exists $A' > 0$ such that

$$\partial_{\rho^2}\Omega(\mu,k) + A'\delta \geqslant 0$$

and if $2^j \leqslant c(k)$, then

$$\chi_{[1-A\delta,1+A\delta]}(c(k)|\mu|) \lesssim 2^{-j}\delta^{-1}\partial_{\rho^2}\Omega(\mu,k) + A'2^{-j}. \tag{7.2.3}$$

Proof. If $k \geqslant 1$ then, recalling that $c(k \pm 1) = c(k) \pm 4\pi$, by (2.3.4) we have

$$\begin{aligned}\partial_{\rho^2}\Omega(\mu,k) = \frac{k}{\pi|\mu|}\left[2\Xi(c(k)|\mu|) - \Xi(c(k)|\mu| - 4\pi|\mu|) - \Xi(c(k)|\mu| + 4\pi|\mu|)\right] \\ + \frac{m}{\pi|\mu|}\left[\Xi(c(k)|\mu|) - \Xi(c(k)|\mu| + 4\pi|\mu|)\right].\end{aligned}$$

Now,

$$\begin{aligned}&\frac{k}{\pi|\mu|}\left[2\Xi(c(k)|\mu|) - \Xi(c(k)|\mu| - 4\pi|\mu|) - \Xi(c(k)|\mu| + 4\pi|\mu|)\right] \\ &\quad = 16\pi|\mu|k\int_0^1 s\int_{-1}^1 \chi_{[1-A\delta,1+A\delta]}(c(k)|\mu| - 4\pi sr|\mu|)drds \geqslant 0,\end{aligned}$$

and

$$c(k)|\mu| + 4\pi|\mu| \geqslant c(k)|\mu| \geqslant c(k) - 4\pi|\mu| \geqslant 0.$$

The $k = 0$ estimate is trivial. Furthermore, by the Fundamental Theorem of Calculus,

$$\left|\frac{m}{\pi|\mu|}\left[\Xi(c(k)|\mu|) - \Xi(c(k)|\mu| + 4\pi|\mu|)\right]\right| \lesssim \|\Xi'\|_\infty \leqslant \sup_s \left|\int_s^\infty \chi_{[1-A\delta,1+A\delta]}(r)dr\right| \simeq \delta$$

for all $k \in \mathbb{N}_0$ (recall that $c \simeq 1$). This implies that $\partial_{\rho^2}\Omega(\mu, k) + A'\delta \geqslant 0$.

Now, suppose $1 - \frac{A}{2}\delta \leqslant c(k)|\mu| \leqslant 1 + A\delta$ and $2^j \leqslant c(k)$. Then

$$\begin{aligned}
16\pi|\mu|k \int_0^1 s \int_{-1}^1 &\chi_{[1-A\delta,1+A\delta]}(c(k)|\mu| - 4\pi sr|\mu|)drds \\
&\gtrsim \int_{1/2}^1 \int_{-1}^1 \chi_{[1-A\delta,1+A\delta]}(c(k)|\mu| - 4\pi sr|\mu|)drds \\
&\geqslant \int_{1/2}^1 \int_0^{2^j\delta/8\pi} drds = \frac{2^j\delta}{16\pi}.
\end{aligned}$$

Similarly, if $1 - A\delta \leqslant c(k)|\mu| \leqslant 1 - \frac{A}{2}\delta$, then

$$\begin{aligned}
16\pi|\mu|k \int_0^1 s \int_{-1}^1 &\chi_{[1-A\delta,1+A\delta]}(c(k)|\mu| - 4\pi sr|\mu|)drds \\
&\gtrsim \int_{1/2}^1 \int_{-1}^1 \chi_{[1-A\delta,1+A\delta]}(c(k)|\mu| - 4\pi sr|\mu|)drds \\
&\geqslant \int_{1/2}^1 \int_{-2^j\delta/8\pi}^0 drds = \frac{2^j\delta}{16\pi}.
\end{aligned}$$

This implies that $\partial_{\rho^2}\Omega(\mu, k) \gtrsim 2^j\delta$ for $1 - A\delta \leqslant c(k)|\mu| \leqslant 1 + A\delta$ and so proves (7.2.3). □

In what follows, we consider $\varphi, \varphi_0 \in C_c^\infty(\mathbb{R})$ be such that $\operatorname{supp}(\varphi) \subseteq (1,3)$ and

$$1 = \sum_{k\in\mathbb{N}_0} \varphi_k^2(t) \text{ for } t > 0, \text{ where } \varphi_k(t) := \varphi(2^{-k}t) \text{ for } k \geqslant 1. \tag{7.2.4}$$

Furthermore, define $\Lambda_k f(z,u) := \varphi_k(|z|)f(z,u)$.

Theorem 7.8. *Let K denote the kernel of $\Xi(L)$ (where Ξ is defined as in (7.2.1)). Then, for all $f \in \mathscr{S}(G)$ we have*

$$\left| e^{a^2} \int_G |z|^{1-\epsilon} K(z,u)|z|^a \overline{[(\Lambda_k f)^* * (\Lambda_k f)](z,u)} dz du \right| \lessapprox A\delta \|2^{ak/2} f\|_2^2$$

for all $a \in \mathbb{C}$ such that $0 \leqslant Re(a) \leqslant 2$.

Proof. By interpolation, it suffices to prove this for $a = i\theta, 2 + i\theta$ for all $\theta \in \mathbb{R}$. Set $\Theta(t) = \Xi(t^2)$. Then we have, by Euclidean Fourier Inversion the formula

$$K = \int_{\mathbb{R}} \widehat{\Theta}(\xi) e^{2\pi i \xi \sqrt{L}} \delta_0 d\xi = 2 \int_0^\infty \widehat{\Theta}(\xi) P_\xi d\xi. \tag{7.2.5}$$

where $P_\xi := \cos(2\pi\xi\sqrt{L})\delta_0$ is the convolution kernel of $\cos(2\pi\xi\sqrt{L})$. Recall that we have the finite propagation speed property (Lemma 2.6), so that

$$\operatorname{supp}(P_\xi) \subseteq \overline{B}(0, 2\pi\xi). \tag{7.2.6}$$

Choose $\varphi \in C_c^\infty(\mathbb{R})$ supported in $(2\pi, 4\pi)$ such that

$$\int_{2\pi}^{4\pi} \varphi(t) t^{\epsilon-2} dt = 1$$

and define $\varphi_\theta := t^{i\theta}\varphi(t)$ for $t \in \mathbb{R}$. Then

$$\int_0^\infty \varphi_\theta\left(\frac{1}{t}\right) t^{i\theta-\epsilon} dt = \int_0^\infty \varphi\left(\frac{1}{t}\right) t^{-\epsilon} dt = \int_{2\pi}^{4\pi} \varphi(u) u^{\epsilon-2} du = 1. \tag{7.2.7}$$

Now, define

$$w_\xi^\theta(z) := \int_0^\xi \varphi_\theta\left(\frac{|z|}{s}\right) s^{i\theta-\epsilon} ds, \text{ for } z \in \mathbb{C}^m. \tag{7.2.8}$$

Then by setting $t = \frac{s}{|z|}$ we have from (7.2.7) the formula

$$w_\xi^\theta(z) = \int_0^{\xi/|z|} \varphi_\theta\left(\frac{1}{t}\right) t^{i\theta-\epsilon}|z|^{i\theta-\epsilon}|z| dt = |z|^{1-\epsilon+i\theta} \tag{7.2.9}$$

provided $|z| \leqslant 2\pi\xi$ (as $\varphi_\theta(\frac{1}{t})$ is supported in $(\frac{1}{4\pi}, \frac{1}{2\pi})$).

Recall that a Herz-Schur multiplier is a function K such that for all $F \in \mathscr{S}'(G)$ the operator T_F on L^2 defined by $T_F f := f * F$ satisfies

$$\|T_F\|_{L^2\to L^2} = \|T_{KF}\|_{L^2\to L^2}. \tag{7.2.10}$$

Note that the characters $\chi_\eta : (z, u) \mapsto e^{2\pi i\eta z}$ are Herz-Schur multipliers for all $\eta \in \mathbb{R}^{2m}$, since

$$f * (\chi_\eta F) = \chi_\eta \left[(\chi_{-\eta} f) * F\right]$$

and since $|\chi_\eta| = 1$.

Now, note that by Hölder's inequality we have

$$\|\widehat{\varphi_\theta(|\cdot|)}\|_{L^1(\mathbb{R}^{2m})} \lesssim \|(1 + |\cdot|^2)^{m+1}\widehat{\varphi_\theta(|\cdot|)}\|_{L^2(\mathbb{R}^{2m})} = \|(1-\Delta)^{m+1}\varphi_\theta(|\cdot|)\|_{L^2(\mathbb{R}^{2m})},$$

where Δ is the usual Euclidean Laplacian and $|\cdot|$ here is the usual Euclidean distance on $\mathbb{R}^{2m}$. Since φ is supported in $(2\pi, 4\pi)$ then

$$\begin{aligned} |\partial_k^{2m+2}\varphi_\theta(|x|)| = |\partial_k^{2m+2}(|x|^{i\theta}\varphi(|x|))| &\lesssim \sum_{\substack{a+b=2m+2\\ a,b\in\mathbb{N}_0}} |\partial_k^a(|x|^{i\theta})\partial_k^b(\varphi(|x|))| \\ &\lesssim \sum_{b=1}^{2m+2} (1+|\theta|)^{2m+2}|x|^{i\theta}|\partial_k^b(\varphi(|x|))| \\ &\lesssim (1+|\theta|)^{2m+2}\chi_{(2\pi,4\pi)}(|x|) \end{aligned}$$

and thus

$$\|(1-\Delta)^{m+1}\varphi_\theta(|\cdot|)\|_{L^2(\mathbb{R}^{2m})} \lesssim (1+|\theta|)^{2m+2}.$$

Hence, we conclude that

$$\int_{\mathbb{R}^{2m}} |\widehat{\varphi_\theta(|\cdot|)}(\eta)| d\eta \lesssim (1+|\theta|)^{2m+2}. \tag{7.2.11}$$

By Euclidean Fourier inversion, we have

$$\varphi_\theta\left(\frac{|z|}{s}\right) = \int_{\mathbb{R}^{2m}} \widehat{\varphi_\theta(|\cdot|)}(\eta)\chi_{\eta/s}(z,u)d\eta, \text{ for } (z,u) \in G, s \in \mathbb{R}^+$$

and so, using (7.2.11) with Fubini's theorem we have

$$\begin{aligned} T_{\varphi_\theta\left(\frac{|z|}{s}\right)F} f(z,u) &= \int_{\mathbb{R}^{2m}} \widehat{\varphi_\theta(|\cdot|)}(\eta) \int_G f((z,u)(y,v)^{-1})\chi_{\eta/s}(y,v)F(y,v)dydvd\eta \\ &\lesssim (1+|\theta|)^{2m+2} T_{\chi_{\eta/s}F} f(z,u) \end{aligned}$$

which implies that

$$\|T_{\varphi_\theta(|z|/s)F}\|_{L^2\to L^2} \lesssim (1+|\theta|)^{2m+2}\|T_F\|_{L^2\to L^2}.$$

We also know by a similar argument using (7.2.8) that

$$\|T_{w_\xi^\theta F}\|_{L^2\to L^2} \lesssim \int_0^\xi s^{-\epsilon}\|T_{\varphi_\theta(|z|/s)F}\|_{L^2\to L^2} ds \lesssim |\xi|^{1-\epsilon}(1+|\theta|)^{2m+2}\|T_F\|_{L^2\to L^2}. \tag{7.2.12}$$

Set $K_\theta(z,u) := |z|^{1-\epsilon+i\theta}K(z,u)$. Then using (7.2.5), (7.2.6) and (7.2.9), we obtain

$$K_\theta = 2\int_0^\infty \widehat{\Theta}(\xi)|z|^{1-\epsilon+i\theta}P_\xi d\xi = 2\int_0^\infty \widehat{\Theta}(\xi)w_\xi^\theta P_\xi d\xi$$

Hence, using (7.2.12) we see that

$$\|T_{K_\theta}\|_{L^2\to L^2} \leqslant 2\int_0^\infty |\widehat{\Theta}(\xi)|\|T_{w_\xi^\theta P_\xi}\|_{L^2\to L^2} d\xi \tag{7.2.13}$$
$$\lesssim (1+|\theta|)^{2m+2}\int_0^\infty |\widehat{\Theta}(\xi)||\xi|^{1-\epsilon} d\xi$$

Now, by considering support conditions, we have that

$$|\Theta(t)| := \left|\int_{t^2}^\infty\int_s^\infty \chi_{[1-A\delta,1+A\delta]}(r)drds\right| \lesssim A\delta\int_{t^2}^\infty \chi_{(-\infty,1]}(s)ds \leqslant A\delta\chi_{[-1,1]}(t)$$

and

$$|\Theta''(t)| \lesssim |\Xi'(t^2)| + |t^2\Xi''(t^2)| := \left|\int_{t^2}^\infty \chi_{[1-A\delta,1+A\delta]}(r)dr\right| + |t^2\chi_{[1-A\delta,1+A\delta]}(t^2)|$$
$$\lesssim A\delta\chi_{[-1,1]}(t) + \chi_{[1-A\delta,1+A\delta]}(t^2).$$

This implies that

$$|\widehat{\Theta}(\xi)| \leqslant \int_{\mathbb{R}} |\Theta(t)|dt \lesssim \delta$$

and

$$|\widehat{\Theta''}(\xi)| = \left|\int_{\mathbb{R}} \Theta''(t)e^{-2\pi it\xi}dt\right| \lesssim A\delta\int_{\mathbb{R}} \chi_{[-1,1]}(t)dt + \int_{\mathbb{R}} \chi_{[1-A\delta,1+A\delta]}(t^2)dt \lesssim \delta$$

and so

$$(1+|\xi|^2)|\widehat{\Theta}(\xi)| \lesssim \delta + |\widehat{\Theta''}(\xi)| \lesssim \delta \tag{7.2.14}$$

After dividing through by $(1+|\xi|^2)$, by substitution into (7.2.13) this yields

$$||T_{K_\theta}\|_{L^2\to L^2} \lesssim \delta(1+|\theta|)^{2m+2}\int_0^\infty \frac{\xi^{1-\epsilon}}{(1+\xi)^2}d\xi \lesssim \delta(1+|\theta|)^{2m+2}. \tag{7.2.15}$$

Now, let $a = i\theta$. Then, using Lemma 2.1, the Cauchy-Schwarz inequality, the fact that the involution $f \mapsto f^*$ is an isometry on L^2 and (7.2.15) we have

$$\begin{aligned}
&\left| e^{a^2} \int_G |z|^{1-\epsilon} K(z,u)|z|^a \overline{[(\Lambda_k f)^* * (\Lambda_k f)](z,u)} dzdu \right| \\
&\lesssim e^{-\theta^2} \int_G |[K_\theta * (\Lambda_k f)^*](z,u)||(\Lambda_k f)^*(z,u)| dzdu \\
&\lesssim e^{-\theta^2} \|K_\theta * (\Lambda_k f)^*\|_2 \|(\Lambda_k f)\|_2 \\
&\lesssim \delta(1+|\theta|)^{2m+2} e^{-\theta^2} \|\Lambda_k f\|_2^2 \lesssim \delta \|f\|_2^2.
\end{aligned}$$

If $a = 2 + i\theta$ then using a similar argument with (2.3.10) we have

$$\begin{aligned}
&\left| e^{a^2} \int_G |z|^{1-\epsilon} K(z,u)|z|^a \overline{[(\Lambda_k f)^* * (\Lambda_k f)](z,u)} dzdu \right| \\
&\simeq e^{-\theta^2} \left| \int_G K_\theta(z,u)|z|^2 \overline{[(\Lambda_k f)^* * (\Lambda_k f)](z,u)} dzdu \right| \\
&\lesssim \delta(1+|\theta|)^{2m+2} e^{-\theta^2} (\|\nu \Lambda_k f\|_2 \|\Lambda_k f\|_2 + \|\nu^{1/2} \Lambda_k f\|_2^2) \lesssim \delta \|2^k f\|_2^2.
\end{aligned}$$

Since $|2^{i\theta k/2}| = 1$ and $|2^{(2+i\theta)k/2}| = 2^k$ then interpolation concludes the proof. □

Although we will only require the $j = J$ case of the following lemma, is simple to extend the proof to cover $j < J$ and so we do so here.

Lemma 7.9. *For* $1 \leqslant j \leqslant J$ *we have*

$$\|\chi_{[1-A\delta,1+A\delta]}(L) R_j f\|_2^2 \lessapprox 2^{-j} \|f\|_{L^2(1+\rho)}^2.$$

Proof. By the Plancherel theorem, for $1 \leqslant j < J$ (recalling the definition of R_j in

(6.0.3) and the estimate (6.0.1)) and then Lemma 7.7 we see that

$$\|\chi_{[1-A\delta,1+A\delta]}(L)R_j\Lambda_k f\|_2^2 = \sum_{\substack{2^j\leqslant c(|\alpha|)<2^{j+1}\\ \alpha,\beta\in\mathbb{N}_0^m}} \int_{\mathbb{R}^n} \chi_{[1-A\delta,1+A\delta]}(c(|\alpha|)|\mu|)|\widehat{\Lambda_k f}(\mu,\alpha,\beta)|^2|\mu|^m d\mu$$

$$\lesssim 2^{-j}\delta^{-1}\left|\sum_{\alpha,\beta}\int_{\mathbb{R}^n}\partial_{\rho^2}\Omega(\mu,|\alpha|)|\widehat{\Lambda_k f}(\mu,\alpha,\beta)|^2|\mu|^m d\mu\right| + 2^{-j}\|f\|_2^2. \tag{7.2.16}$$

This also applies to the case $j = J$, but with the summation now over $2^J \leqslant c(|\alpha|)$. Now, observe that, by (2.2.13), Lemma 2.17 and Lemma 2.16 (since $\widehat{K}(\mu,k) = \Omega(\mu,k)$ by (7.2.2)), we have

$$\int_G |z|^2K(z,u)\overline{[(\Lambda_k f)^* * (\Lambda_k f)](z,u)}dzdu$$

$$= \int_{\mathbb{R}^n}\operatorname{tr}(\partial_{\rho^2}\widehat{K}(\mu)\big[\,[(\Lambda_k f)^* * (\Lambda_k f)]\,\widehat{}\,(\mu)\big]^\dagger)|\mu|^m d\mu$$

$$= \int_{\mathbb{R}^n}\sum_{\alpha}\partial_{\rho^2}\widehat{K}(\mu,|\alpha|)\,[(\Lambda_k f)^* * (\Lambda_k f)]\,\widehat{}\,(\mu,\alpha,\alpha)|\mu|^m d\mu$$

$$= \int_{\mathbb{R}^n}\sum_{\alpha,\beta}\partial_{\rho^2}\widehat{K}(\mu,|\alpha|)|\widehat{\Lambda_k f}(\mu,\alpha,\beta)|^2|\mu|^m d\mu.$$

Hence, (7.2.16) is majorised by

$$2^{-j}\|f\|_2^2 + 2^{-j}\delta^{-1}\left|\int_G |z|^2K(z,u)\overline{[(\Lambda_k f)^* * (\Lambda_k f)](z,u)}dzdu\right|.$$

Combining this with Theorem 7.8 (where we set $a = 1+\epsilon$) gives

$$\|\chi_{[1-A\delta,1+A\delta]}(L)R_J\Lambda_k f\|_2^2 \lesssim 2^{-j}\|2^{k(1+\epsilon)/2}f\|_2^2. \tag{7.2.17}$$

Then, we have

$$\|\chi_{[1-A\delta,1+A\delta]}(L)R_J f\|_2^2 \lesssim \left[\sum_{k\in\mathbb{N}_0} \|\chi_{[1-A\delta,1+A\delta]}(L)R_J\Lambda_k^2 f\|_2\right]^2$$
$$\lesssim 2^{-j}\left[\sum_{k\in\mathbb{N}_0} \|2^{k(1+\epsilon)/2}\Lambda_k f\|_2\right]^2.$$

Due to the support condition of φ_0 we have the estimate

$$2^{k(1+2\epsilon)}|\Lambda_k f|^2 \lesssim (1+|z|)^{1+2\epsilon}|\Lambda_k f|^2 \tag{7.2.18}$$

and so by the Cauchy-Schwarz inequality we have

$$2^{-j}\left[\sum_{k\in\mathbb{N}_0} \|2^{k(1+\epsilon)/2}\Lambda_k f\|_2\right]^2 \leqslant 2^{-j}\left(\sum_{k\in\mathbb{N}_0} \|2^{k(1+2\epsilon)/2}\Lambda_k f\|_2^2\right)\left(\sum_{k\in\mathbb{N}_0} 2^{-2k\epsilon}\right) \tag{7.2.19}$$
$$\lessapprox 2^{-j}\int_G |f(z,u)|^2(1+|z|)^{1+2\epsilon}dzdu.$$

Interpolation with the trivial L^2 estimate then completes the proof. □

Proof of Theorem 7.2, $j = J$. By interpolation, it suffices to prove Theorem 7.2 for $a = 1$. Recall from Lemma 6.5 that

$$M_{\delta,J}^{-1} + M_{\delta,J}^{0} + M_{\delta,J}^{1} \lesssim R_J\chi_{[1-A\delta,1+A\delta]}(L)$$

for $A = 1 + \frac{4\pi}{10}$. Then applying Lemma 7.9 for $j = J$ completes the proof. □

Remark 7.10. Observe that Lemma 7.9 is essentially a proof of Theorem 7.2 for $j < J$ but only in the case $\gamma = 0$ (since $\|M_{\delta,j}^0 f\|_2 \leqslant \|\chi_{[1-A\delta,1+A\delta]}(L)R_j f\|_2$). For $j < J$, the annuli defined by $M_{\delta,j}^{\gamma}$ are disjoint for fixed j and different choices of γ and far enough apart that they cannot be covered by a thickened annulus $\chi_{[1-A\delta,1+A\delta]}(L)R_j$ for c independent of δ.

7.2.2 Proof of Theorem 7.2 for $j < J$ and Radial Functions

We first present a proof of Theorem 7.2 valid specifically in the case that f is a radial function. The methods employed for this case follow from an approach using Schur's Lemma, similar to the method employed in [8]. The general case is presented in the section after this.

First, the following estimate will be useful.

Lemma 7.11. *Let $x \in \mathbb{N}_0$, let $m \in \mathbb{N}$ and let $a \in (1, 2m)$. Then*

$$\sum_{p=0}^{x} (x-p+1)^{a-2}(p+1)^{m-a/2-1} \lesssim (1+x)^{m+a/2-2}. \tag{7.2.20}$$

If instead $a = 1$ then we have

$$\sum_{p=0}^{x} (x-p+1)^{a-2}(p+1)^{m-a/2-1} \lesssim \ln(2+x)(1+x)^{m+a/2-2}. \tag{7.2.21}$$

The implicit constants here may depend on m, a but do not depend on x.

Proof. The case of $x = 0$ is trivial, so in what follows we assume that $x > 0$. Since this and our assumption means that $x \in \mathbb{N}$, then in what follows, $x + 1 \simeq x$. We must consider several cases. First, we assume that $m \geqslant 2$.

Assume further that $1 < a < 2$. Then $a - 2 < 0$ and $m - a/2 - 1 > 0$. That is, the summands in (7.2.20) are monotone increasing, so that

$$\begin{aligned} &\sum_{p=0}^{x} (x-p+1)^{a-2}(p+1)^{m-a/2-1} \\ &\lesssim (x+1)^{a-2} + \int_0^x (x-p)^{a-2} p^{m-a/2-1} dp + x^{m-a/2-1} \\ &\lesssim x^{a-2} + x^{a/2-2+m} \int_0^1 (1-s)^{a-2} s^{m-a/2-1} ds + x^{m-a/2-1} \\ &\simeq x^{a-2} + x^{a/2-2+m} + x^{m-a/2-1}. \end{aligned} \tag{7.2.22}$$

Note that the integral is finite since $m - a/2 - 1 > 0$ and $a < 2$ implies that $a - 2 > -1$. Observe that

$$a/2 - 2 + m > m - a/2 - 1 \iff a > 1$$

and

$$a/2 - 2 + m > a - 2 \iff a < 2m.$$

Thus, in this case we have $x^{a-2} + x^{a/2-2+m}x^{m-a/2-1} \simeq x^{m+a/2-2}$.

If $2m - 2 < a < 2m$ then both terms in the summands are now monotone decreasing. The calculation is then identical to the previous case.

Now suppose $2 \leqslant a \leqslant 2m - 2$. Then $a - 2 > 0$ and $m - a/2 - 1 > 0$, so that

$$\sum_{p=0}^{x}(x - p + 1)^{a-2}(p + 1)^{m-a/2-1} \leqslant x(x + 1)^{a-2}(x + 1)^{m-a/2-1} \simeq x^{m+a/2-2}.$$

If $a = 1$ we have instead

$$\begin{aligned}
&\sum_{p=0}^{x}(x - p + 1)^{-1}(p + 1)^{m-3/2} \\
&\quad\lesssim (x + 1)^{-1} + \int_{0}^{x}(x - p + 1)^{-1}p^{m-3/2}dp + x^{m-3/2} \\
&\quad\leqslant x^{m-3/2}\int_{0}^{1}(1 - s + \tfrac{1}{x})^{-1}ds + x^{m-3/2} \\
&\quad\simeq \ln(2 + x)x^{m-3/2} + x^{m-3/2} \lesssim \ln(2 + x)x^{m-3/2}.
\end{aligned}$$

We now assume $m = 1$, which necessarily means $1 \leqslant a < 2$. Then $a - 2 < 0$ and $m - a/2 - 1 = -a < 0$. Define

$$f(p) := (x - p + 1)^{a-2}(p + 1)^{m-a/2-1}.$$

Then

$$
\begin{aligned}
f'(p) :&= (2-a)(x-p+1)^{a-3}(p+1)^{-a/2} \\
&\qquad - \tfrac{a}{2}(x-p+1)^{a-2}(p+1)^{-a/2-1} \\
&= (x-p+1)^{a-3}(p+1)^{-a/2-1} \\
&\qquad \cdot \left[(2-a)(p+1) - \tfrac{a}{2}(x-p+1)\right] \\
&= (x-p+1)^{a-3}(p+1)^{-a/2-1} \\
&\qquad \cdot \left[p(2-a) + 2 - 3a/2 - ax/2\right].
\end{aligned}
$$

Now, $\mathrm{sgn}(f'(p)) = \mathrm{sgn}(p(2-a/2) + 2 - 3a/2 - ax/2)$. In particular,

$$
f'(p) = 0 \iff p = \frac{(x+3)a/2 - 2}{2 - a/2}.
$$

Furthermore, clearly $f'(p)$ is negative for values of p below this and positive for values above this. Hence, either f is monotone in the range $p \in [0, x]$ or else it has a minimum. Hence,

$$
\begin{aligned}
&\sum_{p=0}^{x} (x-p+1)^{a-2}(p+1)^{m-a/2-1} \\
&\qquad \lesssim (x+1)^{a-2} + \int_0^x (x-p)^{a-2} p^{m-a/2-1} dp + x^{m-a/2-1},
\end{aligned}
$$

and so the calculations in (7.2.22) apply here and give the desired result.

For $a = 1$ we instead have

$$\begin{aligned}
&\sum_{p=0}^{x}(x-p+1)^{-1}(p+1)^{-1/2}\\
&\quad\lesssim \int_0^x (x-p+1)^{-1}p^{-1/2}dp + x^{-1/2}\\
&\quad\leqslant x^{-1/2}\int_0^1 (1-s+\tfrac{1}{x})^{-1}s^{-1/2}ds + x^{-1/2}\\
&\quad= \frac{2\coth^{-1}(\sqrt{1+x^{-1}})}{\sqrt{1+x^{-1}}}x^{-1/2} + x^{-1/2}.
\end{aligned}$$

Since $(1+x^{-1})^{-1/2} \leqslant 1$, since

$$2\coth^{-1}(\sqrt{1+x^{-1}}) = \ln\left(\frac{\sqrt{1+x^{-1}}+1}{\sqrt{1+x^{-1}}-1}\right) \lesssim \ln(2+x)$$

and since $-\frac{1}{2} = 1 - \frac{1}{2} - 1 = m + \frac{a}{2} - 2$, then we are done. □

In order to prove Theorem 7.2, it suffices to prove the case $a = 1$. We will not be able to prove this case directly, but in the following lemma we show that we have the result for $a > 1$ but arbitrarily close to 1. Interpolation may then be used to obtain the $a = 1$ case of Theorem 7.2.

Lemma 7.12. *Recall* (7.0.1) *and* (7.0.2). *For radial functions* $f(z,u) \in L^2(G)$, $1 \leqslant j < J$, $\gamma \in \{-1,0,1\}$ *and* $1 < a < 2m$ *we have*

$$\|M^{\gamma}_{\delta,j}f\|_2^2 \lesssim 2^{-j}\|f\|^2_{L^2(\rho^a)}. \tag{7.2.23}$$

Proof. First, by duality and then taking the Fourier transform, (7.2.23) is equivalent to

$$\|\partial_{\rho^{-a/2}}\widehat{M^{\gamma}_{\delta,j}}g\|_2^2 \lesssim 2^{-j}\|g\|_2^2 \tag{7.2.24}$$

for $g(\mu,k) \in L^2(\mathbb{R}^n\backslash\{0\} \times \mathbb{N}_0, \binom{k+m-1}{k}\#(k)|\mu|^m d\mu)$. Thus, by (7.1.6), we have

$$\|\partial_{\rho^{-a/2}}\widehat{M^{\gamma}_{\delta,j}}g\|_2^2 \leqslant \|g\|_2^2 \sup_{k,\mu} \int_{\mathbb{R}^n} \sum_{l\in\mathbb{N}_0} \widehat{M^{\gamma}_{\delta,j}}(\mu,k) K_{\rho^{-a}}(\nu,l;\mu,k)\widehat{M^{\gamma}_{\delta,j}}(\nu,l)\binom{l+m-1}{l}|\nu|^m d\nu$$

Now, let $\omega(\sqrt{t},u) = w(\sqrt{t}) = t^{-a/2}$. Then, from Lemma 7.1.7, we have

$$K_{\rho^{-a}}(\nu,l;\mu,k) = \frac{C(m,n)\delta_\mu(\nu)}{\binom{k+m-1}{k}\binom{l+m-1}{l}} \int_0^\infty L_l^{m-1}(\pi|\nu|t)L_k^{m-1}(\pi|\nu|t)e^{-\pi|\nu|t}t^{m-1-a/2}dt,$$

which rescales to

$$K_{\rho^{-a}}(\nu,l;\mu,k) \simeq \frac{\delta_\mu(\nu)|\nu|^{a-m}}{\binom{k+m-1}{k}\binom{l+m-1}{l}} \int_0^\infty L_l^{m-1}(t)L_k^{m-1}(t)e^{-t}t^{m-1-a/2}dt. \tag{7.2.25}$$

We may then use the identity

$$L_n^{\beta}(x) = \sum_{j=0}^{n}\binom{a+n-j-1}{n-j}L_j^{\beta-a}(x) \tag{7.2.26}$$

on both of the Laguerre polynomials, followed by the orthogonality property of Laguerre polynomials, to obtain

$$\begin{aligned}
&\int_0^\infty L_k^{m-1}(t)L_l^{m-1}(t)e^{-t}t^{m-1-a/2}dt \\
&\qquad = \sum_{p=0}^{k}\sum_{q=0}^{l}\binom{a/2+k-p-1}{k-p}\binom{a/2+l-q-1}{l-q}\frac{\Gamma(p+m-a/2)}{p!}\delta_{p,q} \\
&\qquad = \sum_{p=0}^{\min\{k,l\}}\frac{\Gamma(a/2+k-p)\Gamma(a/2+l-p)\Gamma(p+m-a/2)}{\Gamma(k-p+1)\Gamma(l-p+1)(\Gamma(a/2))^2 p!}.
\end{aligned}$$

Hence, we have

$$K_{\rho^{-a}}(\nu,l;\mu,k) \simeq \frac{\delta_\mu(\nu)|\nu|^{a/2-m}}{\binom{k+m-1}{k}\binom{l+m-1}{l}} \sum_{p=0}^{\min\{k,l\}} \frac{\Gamma(a/2+k-p)\Gamma(a/2+l-p)\Gamma(p+m-a/2)}{\Gamma(k-p+1)\Gamma(l-p+1)(\Gamma(a/2))^2 p!}$$

In particular, we are required to estimate

$$\sup_{k,\mu}\int_{\mathbb{R}^n}\sum_{l\in\mathbb{N}_0}\chi_{[1-\delta,1]}(b(k)|\mu|)\chi_{[2^j,2^{j+1})}(b(k))\frac{\delta_\mu(\nu)|\nu|^{a/2-m}}{\binom{k+m-1}{k}\binom{l+m-1}{l}}$$
$$\sum_{p=0}^{\min\{k,l\}}\frac{\Gamma(a/2+k-p)\Gamma(a/2+l-p)\Gamma(p+m-a/2)}{\Gamma(k-p+1)\Gamma(l-p+1)(\Gamma(a/2))^2p!}$$
$$\chi_{[1-\delta,1]}(b(l)|\nu|)\chi_{[2^j,2^{j+1})}(b(l))\binom{l+m-1}{l}|\nu|^m d\nu$$

Notice that, since $k,l\in[2^j,2^{j+1})$ and $j<J$, then $\chi_{[1-\delta,1]}(b(k)|\mu|)\chi_{[1-\delta,1]}(b(l)|\mu|)=0$ for $k\neq l$ (since $b(k+1)|\mu|$ is more than $\delta-$far away from $b(k)|\mu|$). Hence, this expression is equivalent to

$$\sup_{\substack{b(k)|\mu|\in[1-\delta,1],\\ b(k)\in[2^j,2^{j+1})}}\frac{|\mu|^{a/2}}{\binom{k+m-1}{k}}\sum_{p=0}^{k}\left[\left(\frac{\Gamma(a/2+k-p)}{\Gamma(k-p+1)}\right)^2\frac{\Gamma(p+m-a/2)}{\Gamma(p+1)}\right]. \tag{7.2.27}$$

By Stirling's formula, we have

$$\sum_{p=0}^{k}\left[\left(\frac{\Gamma(k-p+a/2)}{\Gamma(k-p+1)}\right)^2\frac{\Gamma(p+m-a/2)}{\Gamma(p+1)}\right]$$
$$\simeq\sum_{p=0}^{k}\left[\left(\frac{(k-p+a/2)^{k-p+a/2-1/2}e^{k-p+1}}{(k-p+1)^{k-p+1-1/2}e^{k-p+a/2}}\right)^2\frac{(p+m-a/2)^{p+m-a/2-1/2}e^{p+1}}{(p+1)^{p+1-1/2}e^{p+m-a/2}}\right]. \tag{7.2.28}$$

Notice that any instances of e^p and e^k cancel out, so we may take out the remaining powers of e as an implicit constant. Furthermore,

$$\frac{(p+m-a/2)^{p+m-a/2-1/2}}{(p+1)^{p+1-1/2}} \tag{7.2.29}$$
$$=(p+1)^{m-a/2-1}\left(\frac{p+m-a/2}{p+1}\right)^{p+m-a/2-1/2}$$
$$=(p+1)^{m-a/2-1}\left(1+\frac{m-a/2-1}{p+1}\right)^{p+1}\left(1+\frac{m-a/2-1}{p+1}\right)^{m-a/2-3/2}$$
$$\leqslant(p+1)^{m-a/2-1}e^{m-a/2-1}(m-a/2)^{m-a/2-3/2}\simeq(p+1)^{m-a/2-1}.$$

Hence, (7.2.28) is majorised by

$$\sum_{p=0}^{k}(k-p+1)^{2a/2-2}(p+1)^{m-a/2-1}. \tag{7.2.30}$$

By Lemma 7.11, for all $1 < a < 2m$, we have that this is majorised by $k^{m+a/2-2} \simeq (2^j)^{m+a/2-2}$, so (7.2.27) is majorised by

$$(2^{-j})^{a/2+m-1}(2^{-j})^{2-a/2-m} = 2^{-j} \tag{7.2.31}$$

as required. □

Corollary 7.13. *For radial functions $f \in L^2(G)$ and $1 \leqslant j < J$, we have*

$$\|M_{\delta,j}^{\gamma} f\|_2^2 \lessapprox 2^{-j}\|f\|_{L^2(\rho)}^2. \tag{7.2.32}$$

Proof. By Lemma 7.12, with $a = 1+\epsilon$ we have

$$\|M_{\delta,j}^{\gamma} f\|_2^2 \lesssim (2^{-j})\|f\|_{L^2(\rho^{1+\epsilon})}^2. \tag{7.2.33}$$

We interpolate this with the trivial estimate at $\theta = \frac{1}{1+\epsilon}$ and use $\frac{1}{1+\epsilon} = 1 - \frac{\epsilon}{1+\epsilon}$ to get

$$\|M_{\delta,j}^{\gamma} f\|_2^2 \lesssim_\epsilon (2^{-j})^{1-\frac{\epsilon}{1+\epsilon}}\|f\|_{L^2(\rho)}^2. \tag{7.2.34}$$

Since $\epsilon' := \frac{\epsilon}{1+\epsilon}$ may be made arbitrarily small, this completes the proof. □

7.2.3 Proof of Theorem 7.2 for $j < J$

Let $K_{\delta,j}^{\gamma}$ denote the convolution kernel of $M_{\delta,j}^{\gamma}$. Let φ_r, where $r \in \mathbb{N}_0$, be the functions defined in (7.2.4) and let $\Lambda_r f(z,u) := \varphi_r(|z|)f(z,u)$. We also fix some $\Psi \in C_c^\infty(\mathbb{R}^+)$ with $\Psi(x) = 1$ for $x \in [\frac{1}{6}, 3]$. We recall from (2.2.13), Lemma 2.17 and Lemma 2.16

that

$$\langle f,g\rangle_G = \langle \widehat{f},\widehat{g}\rangle = \int\limits_{\mathbb{R}^n} tr(\widehat{f}(\mu)[\widehat{g}(\mu)]^\dagger)|\mu|^m d\mu \tag{7.2.35}$$
$$= \int\limits_{\mathbb{R}^n} \sum_{\alpha\in\mathbb{N}_0^m} \langle \widehat{f}(\mu)[\widehat{g}(\mu)]^\dagger h_\alpha^\mu, h_\alpha^\mu\rangle_{\mathbb{R}^m} |\mu|^m d\mu$$
$$= \int\limits_{\mathbb{R}^n} \sum_{\alpha,\beta\in\mathbb{N}_0^m} \widehat{f}(\mu,\beta,\alpha)\overline{\widehat{g}(\mu,\beta,\alpha)}|\mu|^m d\mu.$$

Observe that, if one of f, g is a radial function, then the only non-zero terms would be the diagonal ones, where $\alpha = \beta$. Thus, if one of f, g is radial, then

$$\langle \widehat{f},\widehat{g}\rangle = \int\limits_{\mathbb{R}^n} \sum_{\alpha\in\mathbb{N}_0^m} \widehat{f}(\mu,\alpha,\alpha)\overline{\widehat{g}(\mu,\alpha,\alpha)}|\mu|^m d\mu. \tag{7.2.36}$$

Furthermore, from Lemma 2.17 and the Cauchy-Schwarz inequality, we have

$$\int\limits_{\mathbb{R}^n} \sum_{\alpha\in\mathbb{N}_0^m} |\widehat{f*g}(\mu,\alpha,\alpha)||\mu|^m d\mu \leqslant \int\limits_{\mathbb{R}^n} \sum_{\alpha,\beta\in\mathbb{N}_0^m} |\widehat{g}(\mu,\alpha,\beta)||\widehat{f}(\mu,\beta,\alpha)||\mu|^m d\mu \leqslant \|f\|_2\|g\|_2. \tag{7.2.37}$$

Lemma 7.14. *Recall* (7.0.1) *and* (7.0.2)*. Suppose we have*

$$|e^{a^2}\langle \partial_{\rho^a}\widehat{K_{\delta,j}^\gamma}, [\Psi(L)[(\Lambda_r f)*(\Lambda_r f)^*]]\widehat{\ }\rangle| \lessapprox 2^{-j}\|2^{(a+1)r/2}f\|_2^2 \tag{7.2.38}$$

for $-1 \leqslant \mathrm{Re}(a) \leqslant 1$ *and for all* $r \in \mathbb{N}_0$*, where the implicit constant does not depend on* r *or* $\theta := \mathrm{Im}(a)$*. Then for* $f(z,u) \in L^2(G)$*,* $1 \leqslant j < J$ *and* $\gamma \in \{-1,0,1\}$ *we have*

$$\|M_{\delta,j}^\gamma f\|_2^2 \lessapprox 2^{-j}\|f\|^2_{L^2(1+\rho)}. \tag{7.2.39}$$

That is, proving (7.2.38) *would prove Theorem 7.2 for* $j < J$*,* $a = 1$*.*

Proof. Choose $a = 0$ in (7.2.38). Then, by assumptions and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\|M^{\gamma}_{\delta,j}f\|_2 &\leqslant \sum_{r=0}^{\infty}\|M^{\gamma}_{\delta,j}\Lambda_r^2 f\|_2\\
&= \sum_{r=0}^{\infty}|\langle \partial_{\rho^0}\widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r^2 f)^* * (\Lambda_r^2 f)]]\widehat{\ }\rangle|^{1/2}\\
&\lessapprox 2^{-j/2}\sum_{r=0}^{\infty}\|2^{r/2}\Lambda_r f\|_2\\
&\leqslant 2^{-j/2}\left(\sum_{r=0}^{\infty}2^{r(1+\epsilon)}\|\Lambda_r f\|_2^2\right)^{1/2}\left(\sum_{r=0}^{\infty}2^{-r\epsilon}\right)^{1/2}\\
&\lesssim_\epsilon 2^{-j/2}\|f\|_{L^2((1+\rho)^{1+\epsilon})}.
\end{aligned}
$$

Interpolation with the trivial L^2 estimate completes the proof. Note that we may introduce $\Psi(L)$ in this way, since $\widehat{K^{\gamma}_{\delta,j}}(\mu,k) = \chi_{[1-\delta,1]}(b(k)|\mu|)\chi_{[2^j,2^{j+1})}(b(k))$, $\widehat{\Psi}(\mu,k) = \Psi(c(k)|\mu|)$, we have that $[1-\delta,1] \subseteq [\frac{1}{2},1]$ and

$$
\frac{1}{3} \leqslant \frac{b(k)}{c(k)} \leqslant 3,
$$

by Lemma 7.4. □

Hence, by interpolation, it remains only to prove (7.2.38) for $a = -1 + i\theta$ and $a = 1 + i\theta$, with $\theta \in \mathbb{R}$. We begin by uniformly bounding $e^{-\theta^2}\partial_{\rho^{-a}}\widehat{K^{\gamma}_{\delta,j}}$.

Lemma 7.15. *Let $j < J$ and $I \subseteq (0,\infty)$ be a compact interval. For $a \in \mathbb{C}$ with $1 < \mathrm{Re}(a) < 2m$ and $\theta := \mathrm{Im}(a)$ we have*

$$
\sup_{\substack{k,\mu\\ c(k)|\mu|\in I}} |e^{-\theta^2}\partial_{\rho^{-a}}\widehat{K^{\gamma}_{\delta,j}}(\mu,k)| \lesssim 2^{-j}. \tag{7.2.40}
$$

For $a \in \mathbb{C}$ with $a = 1 + i\theta$ we have

$$
\sup_{\substack{k,\mu\\ c(k)|\mu|\in I}} |e^{-\theta^2}\partial_{\rho^{-a}}\widehat{K^{\gamma}_{\delta,j}}(\mu,k)| \lessapprox 2^{-j}. \tag{7.2.41}
$$

Furthermore, the implicit constants are independent of θ but will depend on the choice *of interval I.*

Proof. Recall that the integral kernel $K_{\rho^{-a}}$ of $\partial_{\rho^{-a}}$ is given by (7.2.25) and note that the identity (7.2.26) holds for $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > -1$ by analytic continuation. Hence, we have

$$K_{\rho^{-a}}(\nu, l; \mu, k) \simeq \frac{\delta_\mu(\nu)|\nu|^{a/2-m}}{\binom{k+m-1}{k}\binom{l+m-1}{l}} \sum_{p=0}^{\min\{k,l\}} \frac{\Gamma(a/2+k-p)\Gamma(a/2+l-p)\Gamma(p+m-a/2)}{\Gamma(k-p+1)\Gamma(l-p+1)(\Gamma(a/2))^2 p!}$$

In particular, we are required to estimate

$$\begin{aligned} A := \sup_{\substack{k,\mu \\ c(k)|\mu|\in I}} \Bigg| e^{-\theta^2} \int_{\mathbb{R}^n} \sum_{l\in\mathbb{N}_0} \frac{\delta_\mu(\nu)|\nu|^{a/2-m}}{\binom{k+m-1}{k}\binom{l+m-1}{l}} \chi_{[1-\delta,1]}(b(l)|\nu|)\chi_{[2^j,2^{j+1})}(b(l)) \qquad (7.2.42) \\ \sum_{p=0}^{\min\{k,l\}} \frac{\Gamma(a/2+k-p)\Gamma(a/2+l-p)\Gamma(p+m-a/2)}{\Gamma(k-p+1)\Gamma(l-p+1)(\Gamma(a/2))^2 p!} \binom{l+m-1}{l} |\nu|^m d\nu \Bigg| \\ = \sup_{\substack{k,\mu \\ c(k)|\mu|\in I}} \Bigg| e^{-\theta^2} \sum_{b(l)\in[2^j,2^{j+1})} \frac{\chi_{[1-\delta,1]}(b(l)|\mu|)|\mu|^{a/2}}{\binom{k+m-1}{k}} \\ \cdot \sum_{p=0}^{\min\{k,l\}} \frac{\Gamma(a/2+k-p)\Gamma(a/2+l-p)\Gamma(p+m-a/2)}{\Gamma(k-p+1)\Gamma(l-p+1)(\Gamma(a/2))^2 p!} \Bigg| \end{aligned}$$

Recall that, for $b(l) \in [2^j, 2^{j+1})$ with $j < J$, the annuli

$$\{\mu \in \mathbb{R}^n : \tfrac{1-\delta}{b(l)} \leqslant |\mu| \leqslant \tfrac{1}{b(l)}\}$$

are disjoint. Therefore, for fixed $|\mu|$, there exists at most one l_μ with $b(l_\mu) \in [2^j, 2^{j+1})$ such that $\chi_{[1-\delta,1]}(b(l_\mu)|\mu|) \neq 0$. Furthermore, since $c(k)|\mu| \in I$, then $1 + k \simeq |\mu|^{-1} \simeq 1 + l \simeq 2^j$, allowing us to estimate

$$|\mu|^{\mathrm{Re}(a/2)} \binom{k+m-1}{k}^{-1} \simeq |\mu|^{\mathrm{Re}(a/2)}(1+k)^{1-m} \simeq (2^{-j})^{\mathrm{Re}(a/2)+m-1}.$$

Hence,

$$A \simeq \sup_{k,\mu} \left| e^{-\theta^2} (2^{-j})^{\mathrm{Re}(a/2)+m-1} \sum_{p=0}^{\min\{k,l_\mu\}} \frac{\Gamma(a/2+k-p)\Gamma(a/2+l_\mu-p)\Gamma(p+m-a/2)}{\Gamma(k-p+1)\Gamma(l_\mu-p+1)(\Gamma(a/2))^2 p!} \right|. \tag{7.2.43}$$

From (1.4.16) in [35] we have the estimate

$$|\Gamma(z)|^{-1} \lesssim e^{\pi|\mathrm{Im}(z)|/2}, \text{ for } \mathrm{Re}(z) \geqslant 0,$$

so that

$$\left| \frac{e^{-\theta^2}}{(\Gamma(a/2))^2} \right| \lesssim e^{-\theta^2} e^{\pi|\theta|} \lesssim 1. \tag{7.2.44}$$

Furthermore, for $\mathrm{Re}(z) > 0$, we have

$$|\Gamma(z)| \leqslant \Gamma(\mathrm{Re}(z)),$$

which is easily seen from the definition of the Gamma function. Hence,

$$\left| \frac{\Gamma(k-p+a/2)}{\Gamma(k-p+1)} \right| \leqslant \left| \frac{\Gamma(k-p+\mathrm{Re}(a/2))}{\Gamma(k-p+1)} \right| \lesssim (k-p+1)^{\mathrm{Re}(a/2)-1}. \tag{7.2.45}$$

The calculation with the term where k is replaced by l_μ is identical. Similarly, (cf (7.2.29))

$$\left| \frac{\Gamma(p+m-1)}{\Gamma(p+1)} \right| \leqslant \left| \frac{\Gamma(p+m-\mathrm{Re}(a/2))}{\Gamma(p+1)} \right| \lesssim (p+1)^{m-\mathrm{Re}(a/2)-1}. \tag{7.2.46}$$

Hence, combining (7.2.44), (7.2.45) and (7.2.46) we have

$$\begin{aligned} e^{-\theta^2} & \left| \sum_{p=0}^{\min\{k,l_\mu\}} \frac{\Gamma(a/2+k-p)\Gamma(a/2+l_\mu-p)\Gamma(p+m-a/2)}{\Gamma(k-p+1)\Gamma(l_\mu-p+1)(\Gamma(a/2))^2 p!} \right| \\ & \lesssim \sum_{p=0}^{\min\{k,l_\mu\}} (k-p+1)^{\mathrm{Re}(a/2)-1} (l_\mu-p+1)^{\mathrm{Re}(a/2)-1} (p+1)^{m-\mathrm{Re}(a/2)-1}, \end{aligned} \tag{7.2.47}$$

Now, first suppose that $1 \leqslant \mathrm{Re}(a) < 2$. Define $x = x(k, l_\mu) := \min\{k, l_\mu\}$ and observe that $1 + x \simeq 2^j$, since both $1 + k$ and $1 + l_\mu$ are of this order. Then (7.2.47) is bounded above by

$$\sum_{p=0}^{x} (x - p + 1)^{2\,\mathrm{Re}(a/2)-2}(p+1)^{m-\mathrm{Re}(a/2)-1} \tag{7.2.48}$$

since $1 \leqslant \mathrm{Re}(a) < 2$ implies that $\mathrm{Re}(a) - 2 < 0$.

If $2 \leqslant \mathrm{Re}(a) < 2m$ then we define $y = y(k, l_\mu) := \max\{k, l_\mu\}$. Since $\mathrm{Re}(a) - 2 > 0$ then

$$\sum_{p=0}^{x} (k - p + 1)^{\mathrm{Re}(a)/2-1}(l_\mu - p + 1)^{\mathrm{Re}(a)/2-1}(p+1)^{m-\mathrm{Re}(a)/2-1}$$
$$\leqslant \sum_{p=0}^{x} (y - p + 1)^{\mathrm{Re}(a)-2}(p+1)^{m-a/2-1}.$$

By Lemma 7.11, we therefore have that (7.2.47) is majorised by

$$\begin{aligned} j(1+x)^{m+\mathrm{Re}(a)/2-2} &\text{ for } \mathrm{Re}(a) = 1 \\ (1+x)^{m+\mathrm{Re}(a)/2-2} &\text{ for } \mathrm{Re}(a) \in (1,2) \\ (1+y)^{m+\mathrm{Re}(a)/2-2} &\text{ for } \mathrm{Re}(a) \in [2,2m). \end{aligned}$$

Since $1 + x \simeq 1 + y \simeq 2^j$, then for all $1 < \mathrm{Re}(a) < 2m$, we have that (7.2.43) is majorised by

$$(2^{-j})^{\mathrm{Re}(a)/2+m-1}(2^{-j})^{2-\mathrm{Re}(a)/2-m} = 2^{-j} \tag{7.2.49}$$

and for $\mathrm{Re}(a) = 1$ we have that (7.2.43) is majorised by

$$j(2^{-j})^{\mathrm{Re}(a)/2+m-1}(2^{-j})^{2-\mathrm{Re}(a)/2-m} \lessapprox 2^{-j} \tag{7.2.50}$$

as required. □

The $\mathrm{Re}(a) = -1$ case of (7.2.38) is then an easy corollary.

Corollary 7.16. *We have*

$$|e^{a^2}\langle \partial_{\rho^a}\widehat{K^\gamma_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \lessapprox 2^{-j}\|2^{(a+1)r/2} f\|_2^2 \tag{7.2.51}$$

for $a = -1 + i\theta$, where the implicit constant does not depend on θ.

Proof. First, note that $|e^{a^2}| \simeq |e^{-\theta^2}|$ and that $|2^{(a+1)r/2}| = 1$. Thus,

$$
\begin{aligned}
\mathscr{I} :=& |e^{a^2}\langle \partial_{\rho^a}\widehat{K^\gamma_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \\
\simeq & \left| e^{-\theta^2} \int_{\mathbb{R}^n} \sum_{\alpha\in\mathbb{N}_0^m} \partial_{\rho^{-1/2+i\theta/2}}\widehat{K^\gamma_{\delta,j}}(\mu,|\alpha|)\overline{\Psi(|\mu|c(|\alpha|))[(\Lambda_r f) * (\Lambda_r f)^*]\widehat{\ }(\mu,\alpha,\alpha)}|\mu|^m d\mu \right|
\end{aligned}
$$

Since Ψ is compactly supported, then $c(|\alpha|)|\mu|$ is in some compact interval, so applying Lemma 7.15 with $a = 1 - i\theta$ and (7.2.37) gives the estimate

$$
\begin{aligned}
\mathscr{I} &\lessapprox 2^{-j} \int_{\mathbb{R}^n} \sum_{\alpha\in\mathbb{N}_0^m} |\Psi(|\mu|c(|\alpha|))[(\Lambda_r f) * (\Lambda_r f)^*]\widehat{\ }(\mu,\alpha,\alpha)||\mu|^m d\mu \\
&\lesssim 2^{-j} \int_{\mathbb{R}^n} \sum_{\alpha\in\mathbb{N}_0^m} |[(\Lambda_r f) * (\Lambda_r f)^*]\widehat{\ }(\mu,\alpha,\alpha)||\mu|^m d\mu \\
&\lesssim \|\Lambda_r f\|_2^2 \lesssim \|f\|_2^2
\end{aligned}
$$

Since we used Lemma 7.15 then the implicit constants do not depend on θ; they will depend on Ψ, however, we fixed Ψ at the start of this section, so we are done. □

The $\mathrm{Re}(a) = 1$ case of (7.2.38) now follows by applying the Leibniz rule (2.3.10) and estimating the terms it produces.

Lemma 7.17. *We have*

$$
|e^{a^2}\langle \partial_{\rho^a}\widehat{K^\gamma_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \lessapprox 2^{-j}\|2^{(a+1)r/2} f\|_2^2 \tag{7.2.52}
$$

for $a = 1 + i\theta$, where the implicit constant does not depend on θ.

Proof. Note that $|e^{a^2}| = |e^{1-\theta^2+i2\theta}| \simeq e^{-\theta^2}$ and $|2^{(a+1)r/2}| = 2^r$. Let K denote the convolution kernel of $\Psi(L)$. Now, (recall (2.3.10))

$$\begin{aligned}
&|e^{(1+i\theta)^2}\langle \partial_{\rho^{1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[f * f^*]]\widehat{\ }\rangle| \\
&\qquad \simeq |e^{-\theta^2}\langle \partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}, [\rho^2([f * f^*] * K)]\widehat{\ }\rangle| \\
&\qquad \leqslant |e^{-\theta^2}\langle \partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}, [(\rho^2[f * f^*]) * K]\widehat{\ }\rangle| \\
&\qquad\quad + |e^{-\theta^2}\langle \partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}, [[f * f^*] * (\rho^2 K)]\widehat{\ }\rangle| \\
&\qquad\quad + \sum_{p=1}^{m} |e^{-\theta^2}\langle \partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}, [(\zeta_{\mu_1,p}[f * f^*]) * (\overline{\zeta_{\mu_1,p}}K)]\widehat{\ }\rangle| \\
&\qquad\quad + \sum_{p=1}^{m} |e^{-\theta^2}\langle \partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}, [(\overline{\zeta_{\mu_1,p}}[f * f^*]) * (\zeta_{\mu_1,p}K)]\widehat{\ }\rangle|,
\end{aligned}$$

for some fixed $\mu_1 \in \mathbb{R}^n$. We first estimate the terms involving $(\zeta_{\mu,p}[f * f^*]) * (\overline{\zeta_{\mu,p}}K)$. Note that the case with $(\overline{\zeta_{\mu,p}}[f * f^*]) * (\zeta_{\mu,p}K)$ instead will be essentially identical, with uses of (2.3.2) replaced by (2.3.1).

Using (2.3.2) and the fact that $\widehat{K}$ is radial, we see that $\widehat{\overline{\zeta_{\mu,p}}K}(\mu,\alpha,\beta) = 0$ except for when $\beta = \alpha - e_p$. For $\alpha \neq 0$ we have

$$\begin{aligned}
|\widehat{\overline{\zeta_{\mu,p}}K}(\mu,\alpha,\alpha - e_p)| &= \left|\sqrt{\frac{\alpha_p}{\pi|\mu|}}\widehat{K}(\mu,\alpha - e_p,\alpha - e_p) - \sqrt{\frac{\alpha_p}{\pi|\mu|}}\widehat{K}(\mu,\alpha,\alpha)\right| \\
&= \left|\sqrt{\frac{\alpha_p}{\pi|\mu|}}\widehat{K}(\mu,|\alpha| - 1) - \sqrt{\frac{\alpha_p}{\pi|\mu|}}\widehat{K}(\mu,|\alpha|)\right|. \qquad (7.2.53)
\end{aligned}$$

For $\alpha = 0$ we have

$$|\widehat{\overline{\zeta_{\mu,p}}K}(\mu,\alpha,\alpha - e_p)| = \sqrt{\frac{\alpha_p}{\pi|\mu|}}|\widehat{K}(\mu,|\alpha|)|$$

which is clearly 0 for $\alpha = 0$. This shows that the support of $\widehat{\overline{\zeta_{\mu,p}}K}(\mu,\alpha,\beta)$ does not change in any significant way from $\widehat{K}(\mu\,\alpha,\beta)$ (in particular, if this is non-zero, then $c(|\alpha|)|\mu| \simeq 1$). By (2.3.5) we have that this is also true of $\widehat{\overline{\zeta_{\mu_1,p}}K}(\mu,\alpha,\beta)$ for any $\mu,\mu_1 \in \mathbb{R}^n$. Furthermore $\widehat{\overline{\zeta_{\mu_1,p}}K}(\mu,\alpha,\beta) = 0$ if $|\alpha - \beta| \geqslant 2$ (since $\widehat{\overline{\zeta_{\mu_1,p}}K}(\mu,\alpha,\beta)$ is a sum of terms which will all be zero if α and β differ in more than one component or by more than 1 in one component). A simple calculation shows that

$$\zeta_{\mu,j}(f * g) = (\zeta_{\mu,j}f) * g + f * (\zeta_{\mu,j}g) \tag{7.2.54}$$

with an analogous formula for $\overline{\zeta_{\mu,j}}$. Now, using the above support conditions of $\widehat{\overline{\zeta_{\mu_1,p}}K}$ with Lemma 7.15 and Young's inequality,

$$\begin{aligned}
&|e^{-\theta^2}\langle \partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}, [(\zeta_{\mu_1,p}[f * f^*]) * (\overline{\zeta_{\mu_1,p}}K)]\widehat{\ }\rangle| \\
&\quad = \left| e^{-\theta^2}\int_{\mathbb{R}^n}\sum_{\alpha\in\mathbb{N}_0^m}\partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}(\mu,|\alpha|)\sum_{\beta\in\mathbb{N}_0^m}\overline{\widehat{\zeta_{\mu_1,p}[f * f^*]}(\mu,\beta,\alpha)\widehat{\overline{\zeta_{\mu_1,p}}K}(\mu,\alpha,\beta)}|\mu|^m d\mu\right| \\
&\quad \lessapprox 2^{-j}\int_{\mathbb{R}^n}\sum_{\alpha\in\mathbb{N}_0^m}\left|[(\zeta_{\mu_1,p}f) * (f^* * (\overline{\zeta_{\mu_1,p}}K))]\widehat{\ }(\mu,\alpha,\alpha)\right| |\mu|^m d\mu \\
&\qquad + 2^{-j}\int_{\mathbb{R}^n}\sum_{\alpha\in\mathbb{N}_0^m}\left|[f * ((\zeta_{\mu_1,p}f^*) * (\overline{\zeta_{\mu_1,p}}K))]\widehat{\ }(\mu,\alpha,\alpha)\right| |\mu|^m d\mu \\
&\quad \leqslant 2^{-j}\left(\|\zeta_{\mu_1,p}f\|_2\|f^* * (\overline{\zeta_{\mu_1,p}}K)\|_2 + \|f\|_2\|(\zeta_{\mu_1,p}f^*) * (\overline{\zeta_{\mu,p}}K)\|_2\right) \\
&\quad \leqslant 2^{-j}\left(\|\zeta_{\mu_1,p}f\|_2\|f^*\|_2\|\overline{\zeta_{\mu_1,p}}K\|_1 + \|f\|_2\|\zeta_{\mu_1,p}f^*\|_2\|\overline{\zeta_{\mu,p}}K\|_1\right).
\end{aligned}$$

Since we chose Ψ to be smooth and compactly supported, then K is Schwartz, and so the weighted L^1 norms are uniformly bounded. (c.f. the proof of Lemma 2.5, where the weighted L^1 norm of K is bounded by $\|K\|_{1,0,1}$ which in turn is bounded by the supremum of its Schwartz semi-norms). If we replace f by $\Lambda_r f$ in these calculations, then we have that $\zeta_{\mu_1,p} \lesssim 2^r$, so this is majorised by the right-hand side of (7.2.52) as required.

Next, we have

$$\begin{aligned}
\mathscr{I} &:= |e^{-\theta^2}\langle \partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}, [(\rho^2[f * f^*]) * K]\widehat{\ }\rangle| \\
&\leqslant \int_{\mathbb{R}^n}\sum_{\alpha\in\mathbb{N}_0^m}|e^{-\theta^2}\partial_{\rho^{-1+i\theta}}\widehat{K^{\gamma}_{\delta,j}}(\mu,|\alpha|)||\Psi(c(|\alpha|)|\mu|)||(\rho^2[f * f^*])\widehat{\ }(\mu,\alpha,\alpha)||\mu|^m d\mu
\end{aligned}$$

Since $c(|\alpha|)|\mu| \in \operatorname{supp}(\Psi)$, which is a compact interval, then we can apply Lemma 7.15

to majorise this by

$$2^{-j} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |(\rho^2 [f * f^*])\hat{}(\mu, \alpha, \alpha)| |\mu|^m d\mu.$$

By (2.3.10) and (7.2.37), we have

$$\begin{aligned}
\mathscr{I} \lessapprox{}& 2^{-j} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[(\rho^2 f) * f^*]\hat{}(\mu, \alpha, \alpha)| |\mu|^m d\mu \\
&+ 2^{-j} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[f * (\rho^2 f^*)]\hat{}(\mu, \alpha, \alpha)| |\mu|^m d\mu \\
&+ 2^{-j} \sum_{j=1}^{m} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[(\zeta_{\mu_1, j} f) * (\overline{\zeta_{\mu_1, j}} f^*)]\hat{}(\mu, \alpha, \alpha)| |\mu|^m d\mu \\
&+ 2^{-j} \sum_{j=1}^{m} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[(\overline{\zeta_{\mu_1, j}} f) * (\zeta_{\mu_1, j} f^*)]\hat{}(\mu, \alpha, \alpha)| |\mu|^m d\mu \\
\lessapprox{}& 2^{-j} \left(\|\rho^2 f\|_2 \|f\|_2 + \|\rho f\|_2^2 \right).
\end{aligned}$$

As in the previous case, replacing f with $\Lambda_r f$ yields the desired result.

For the final case, using (2.3.4) we first see that

$$|\partial_{\rho^2} \widehat{K}(\mu, k)| = \frac{1}{\pi |\mu|} |(2k + m)\Psi(c(k)|\mu|) - k\Psi(c(k-1)|\mu|) - (k+m)\Psi(c(k+1)|\mu|)|$$

From this, we see that the support of $\partial_{\rho^2} \widehat{K}(\mu, k)$ is essentially the same as that of $\widehat{K}(\mu, k)$. In particular, we have that $c(k)|\mu|$ is contained in some compact interval depending only on Ψ. Hence, as in previous cases,

$$\begin{aligned}
\mathscr{K} :={}& |e^{-\theta^2} \langle \partial_{\rho^{-1+i\theta}} \widehat{K_{\delta,j}^{\gamma}}, [[f * f^*] * (\rho^2 K)]\hat{}\rangle| \\
\leqslant{}& \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |e^{-\theta^2} \partial_{\rho^{-1+i\theta}} \widehat{K_{\delta,j}^{\gamma}}(\mu, |\alpha|)| |[[f * f^*] * (\rho^2 K)]\hat{}(\mu, \alpha, \alpha)| |\mu|^m d\mu \\
={}& \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |e^{-\theta^2} \partial_{\rho^{-1+i\theta}} \widehat{K_{\delta,j}^{\gamma}}(\mu, |\alpha|)| |\partial_{\rho^2} \widehat{K}(\mu, |\alpha|)| |[f * f^*]\hat{}(\mu, \alpha, \alpha)| |\mu|^m d\mu.
\end{aligned}$$

Since we have shown that $|\partial_{\rho^2} \widehat{K}(\mu, |\alpha|)|$ is supported in a compact interval and uniformly bounded in μ and $|\alpha|$, then applying Lemma 7.15 shows that

$$\mathscr{K} \lessapprox 2^{-j} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[f * [f^* * (\rho^2 K)]]\widehat{\ }(\mu, \alpha, \alpha)||\mu|^m d\mu.$$

Applying (7.2.37) bounds the integral by $\|f\|_2 \|f^* * (\rho^2 K)\|_2$. By Young's inequality, this is bounded by $\|f\|_2^2 \|\rho^2 K\|_1$, and as in previous cases, since K is Schwartz, this completes the proof. □

Corollary 7.18. *We have, for all* $1 \leqslant j < J$ *and for all* $\gamma \in \{-1, 0, 1\}$,

$$\|M_{\delta,j}^{\gamma} f\|_2^2 \lessapprox 2^{-j} \|f\|_{L^2(1+\rho)}^2. \tag{7.2.55}$$

Proof. Interpolation of Lemmas 7.16 and 7.17 yields the assumptions of Lemma 7.26, the application of which completes the proof. □

Proof of Theorem 7.2. By interpolation, it suffices to prove Theorem 7.2 for $a = 1$. The $1 \leqslant j < J$ case follows from Corollary 7.18, while the $j = J$ case was proved in Section 7.2.1. □

7.3 The Second Layer Trace Lemma

7.3.1 Estimates from Euclidean Methods

In this section, we will refine Lemma 3 in [8] in order to obtain better estimates in our situation, where multiple annuli are considered at once, rather than just one as in [8]. First, we develop some technical estimates.

Lemma 7.19. *Let* $x, y \in \mathbb{R}^n$. *Then* $|x - y|$ *is equivalent to the distance*

$$d_N(x, y) := ||y| - |x|| + \min\{|y|, |x|\}\theta,$$

where $\theta \in [0, \pi)$ *is the angle between the points* x *and* y.

Proof. First, write $x = r\rho$ and $y = t\tau$, where $r, t > 0$ and $\rho, \tau \in S^{n-1}$. Then

$$|x - y| = |r\rho - t\tau| = t\left|\frac{r}{t}\rho - \tau\right|.$$

The value of this quantity is invariant under rotations around the origin, so we rotate so that $\rho \mapsto (1, 0, \ldots, 0)$. This sends τ to some σ where the angle between $(1, 0, \ldots, 0)$ and σ is again θ. If we set $s := r/t$ and $P_1 := (s, 0, \ldots, 0)$ then we have shown that

$$|x - y| = t|P_1 - \sigma|.$$

Hence, if we can prove that $|P_1 - \sigma|$ is equivalent to $d_N(P_1, \sigma)$, then this would imply that

$$|x - y| \simeq t(|1 - \tfrac{r}{t}| + \min\{1, \tfrac{r}{t}\}\theta) = d_n(x, y).$$

Clearly $d(s, \sigma) \leqslant d_N(s, \sigma)$, so it suffices to find some $C > 0$ independent of s, σ such that $d_N(s, \sigma) \leqslant Cd(s, \sigma)$.

First, assume that $s \geqslant 1$. Additionally assume that the angle between σ and P_1 is at most $\frac{\pi}{2}$. If we project the point σ onto the line OP_1 (where O is the origin) to the point P_2, then $P_1P_2\sigma$ forms a right-angled triangle whose hypotenuse is $d(s, \sigma)$. Furthermore, the side P_1P_2 clearly has length greater than $|1-s|$, and in turn the hypotenuse is longer than this side length, so that $|1 - s| \leqslant d(s, \sigma)$. Also, the side $P_2\sigma$ has length $\sin\theta$ while the distance on the surface of S^{n-1} between the points σ and $(1, 0 \ldots, 0)$, which is the point where the side P_1P_2 intersects S^{n-1}, has length θ, using basic geometry. Since we assumed that $\theta \leqslant \frac{\pi}{2}$ then $\theta \simeq \sin\theta < d(s, \sigma)$. Hence, in this case, we have $d_N(s, \sigma) \lesssim d(s, \sigma)$

If instead the angle between σ and P_1 is larger than $\frac{\pi}{2}$, we consider the triangle $P_1O\sigma$. Since the side $P_1\sigma$ is opposite an angle larger than $\frac{\pi}{2}$, then this is the largest side, and by definition it has length $d(s, \sigma)$. Similar to before, the side OP_1 has length $s > |1 - s|$, while the side $O\sigma$ has length 1. Since the arc on the surface of S^{n-1} from $(1, 0, \ldots, 0)$ to σ has length $\theta \leqslant \pi$. Since we have individually shown that $|1 - s| \leqslant$

$d(s,\sigma)$ and $\theta \leqslant \pi d(s,\sigma)$, then $d_N(s,\sigma) \leqslant (\pi+1)d(s,\sigma)$.

For the case $0 < s < 1$ the process is essentially the same as above, except that we instead project the point σ down onto the sphere of radius s. Together, these results show that $d_N(s,\sigma) \simeq d(s,\sigma)$. $\square$

Lemma 7.20. *We define*

$$F(s) := \int\limits_{S^{n-1}} \frac{1}{|(s,0,\ldots,0)-\sigma|^{n-2\beta}} d\sigma. \tag{7.3.1}$$

Then for all $s > 0$, we have

$$F(s) \lesssim \begin{cases} (1+s)^{1-n}|1-s|^{2\beta-1} & \text{for } \beta < \frac{1}{2} \\ (1+\ln_+ \frac{1}{|1-s|})(1+s)^{2\beta-n} & \text{for } \beta = \frac{1}{2} \\ (1+s)^{2\beta-n} & \text{for } \beta > \frac{1}{2}. \end{cases} \tag{7.3.2}$$

Proof. Using Lemma 7.19 we have

$$F(s) := \int\limits_{S^{n-1}} \frac{1}{|(s,0,\ldots,0)-\sigma|^{n-2\beta}} d\sigma \simeq \int\limits_0^\pi \frac{\sin^{n-2}\theta}{\big||1-s| + \min\{1,s\}\theta\big|^{n-2\beta}} d\theta,$$

where we have used that the surface area of S^{n-2} is $c_n \sin^{n-2}\theta$ (this may be seen by splitting the sphere S^{n-1} into cylinders of small lengths $d\theta$; each cylinder will have a radius $\sin\theta$ by basic geometry, so will have surface area $c_n \sin^{n-2}\theta d\theta$).

Now, notice that the numerator is symmetric on $(0,\pi)$, while the denominator is larger on $(\frac{\pi}{2},\pi)$ than on $(0,\frac{\pi}{2})$. Since $\theta \simeq \sin\theta$ on $(0,\frac{\pi}{2})$, then $F(s)$ is majorised by

$$\int\limits_0^{\pi/2} \frac{\theta^{n-2}}{\big||1-s| + \min\{1,s\}\theta\big|^{n-2\beta}} d\theta.$$

We again consider a number of different cases in order to estimate this. First, suppose

$0 < s \leqslant \frac{1}{2}$ or $s \geqslant \frac{3}{2}$. In this case, since θ is bounded, then

$$|1-s| \geqslant \frac{1}{2} \gtrsim \min\{1,s\}\theta,$$

giving

$$F(s) \lesssim \int_0^{\pi/2} \theta^{n-2} d\theta |1-s|^{2\beta-n} \lesssim |1-s|^{2\beta-n}. \tag{7.3.3}$$

Now, we assume $\frac{1}{2} < s < \frac{3}{2}$. In this case, $|1-s| \leqslant \frac{1}{2}$ and $\min\{1,s\} \simeq 1$. We have

$$\begin{aligned} F(s) &\simeq \int_0^{\pi/2} \frac{\theta^{n-2}}{\big||1-s|+\theta\big|^{n-2\beta}} d\theta \qquad (7.3.4) \\ &\leqslant \int_0^{|1-s|} \frac{\theta^{n-2}}{\big|2|1-s|\big|^{n-2\beta}} d\theta + \int_{|1-s|}^{\pi/2} \frac{\theta^{n-2}}{|2\theta|^{n-2\beta}} d\theta \\ &\simeq \left(|1-s|^{2\beta-1} + K_\beta\right) \end{aligned}$$

where

$$K_\beta = \begin{cases} |1-s|^{2\beta-1} & \text{for } \beta < \frac{1}{2} \\ \big|\ln|1-s|\big| & \text{for } \beta = \frac{1}{2} \\ 1 & \text{for } \beta > \frac{1}{2}. \end{cases} \tag{7.3.5}$$

Note now that in all three cases, $K_\beta \geqslant |1-s|^{2\beta-1}$, so $F(s) \lesssim K_\beta$.

We now prove our claim. If $\beta < \frac{1}{2}$, then, for $s \leqslant \frac{1}{2}$ we have $(1+s) \simeq 1$ and so this is approximately $\simeq |1-s|^{2\beta-1} \simeq |1-s|^{2\beta-n}$, which we know majorises $F(s)$ from (7.3.3). If $\frac{1}{2} \leqslant s \leqslant \frac{3}{2}$ then $(1+s) \simeq 1$, so this is approximately $|1-s|^{2\beta-1}$, which we know majorises $F(s)$ by (7.3.4). Finally, if $s > \frac{1}{2}$ then $(1+s) \simeq |1-s|$, so this is approximately $|1-s|^{1-n+2\beta-1} = |1-s|^{2\beta-n}$, which majorises $F(s)$ by (7.3.3). Similar workings prove the other cases. □

Lemma 7.21. *Define* $A(r,\delta) = \left\{x \in \mathbb{R}^n : \left|\frac{|x|}{r} - 1\right| < \delta\right\}$. *For* $0 < \beta < \frac{n}{2}$, $0 < \delta < \frac{1}{2}$,

$p(k) = \aleph k + \beth$ *for* $k \in \mathbb{N}$ *and* $\aleph, \beth \in \mathbb{R}$ *where* $\aleph > \max\{0, -\beth\}$ *and* $2^j \lesssim \delta^{-1}$ *we have*

$$\int_{\bigcup_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)} |f(x)|^2 dx \lesssim C_\beta(\delta) \|f\|^2_{\dot{L}^2_\beta}, \tag{7.3.6}$$

where

$$C_\beta(\delta) = \begin{cases} \max\{2^{(1-2\beta)j}\delta, (2^{-j}\delta)^{2\beta}\} & \text{for } \beta < \frac{1}{2} \\ \delta|\ln\delta| & \text{for } \beta = \frac{1}{2} \\ 2^{(1-2\beta)j}\delta & \text{for } \frac{1}{2} < \beta < \frac{n}{2}. \end{cases}$$

The constant in $\lesssim$ *may depend on* $\aleph$ *but not on* $\beth$.

Proof. By taking Euclidean Fourier transforms, (7.3.6) is equivalent to

$$\int_{\bigcup_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)} |\widehat{f}(x)|^2 dx \lesssim C_\beta(\delta) \int_{\mathbb{R}^n} |f(x)|x|^{2\beta} dx, \tag{7.3.7}$$

which, by duality, is equivalent to

$$\int_{\mathbb{R}^n} |\widehat{g}(x)|^2 \frac{dx}{|x|^{2\beta}} \lesssim C_\beta(\delta) \int_{\bigcup_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)} |g(x)|^2 dx \tag{7.3.8}$$

for g supported in $\bigcup_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)$. Now, by Lemma A.3,

$$\begin{aligned} \int_{\mathbb{R}^n} |\widehat{g}(x)|^2 \frac{dx}{|x|^{2\beta}} &= \int_{\mathbb{R}^n} (\widehat{g * g^*})(x) \frac{dx}{|x|^{2\beta}} \simeq \int_{\mathbb{R}^n} g * g^*(x) \frac{1}{|x|^{n-2\beta}} dx \\ &= \int_{\bigcup_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)} \int_{\bigcup_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)} g(x)\overline{g(y)} \frac{1}{|x-y|^{n-2\beta}} dx dy \\ &\leqslant \|g\|_2^2 \sup_x \int_{\bigcup_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)} \frac{1}{|x-y|^{n-2\beta}} dy. \end{aligned}$$

We rotate so that $x = (r, 0, \ldots, 0)$. Then

$$\sup_x \int\limits_{\bigcup_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)} \frac{1}{|x-y|^{n-2\beta}} dy \leqslant \sup_{r>0} \sum_{p(k)\in[2^j,2^{j+1})} \int\limits_{A(1/p(k),\delta)} \frac{1}{|(r,0,\ldots,0)-y|^{n-2\beta}} dy$$

$$= \sup_{r>0} \sum_{p(k)\in[2^j,2^{j+1})} \int\limits_{A(1,\delta)} \frac{p(k)^{-2\beta}}{|(p(k)r,0,\ldots,0)-y|^{n-2\beta}} dy$$

We now seek to estimate, for $t > 0$, the integral

$$\int\limits_{A(1,\delta)} \frac{1}{|(t,0,\ldots,0)-y|^{n-2\beta}} dy.$$

By switching to spherical coordinates, this integral is equal to

$$\int_{1-\delta}^{1+\delta} \int_{S^{n-1}} \frac{1}{|(t,0,\ldots,0)-r\sigma|^{n-2\beta}} d\sigma r^{n-1} dr \tag{7.3.9}$$

$$= \int_{1-\delta}^{1+\delta} \int_{S^{n-1}} \frac{1}{r^{n-2\beta}|(t/r,0,\ldots,0)-\sigma|^{n-2\beta}} d\sigma r^{n-1} dr$$

$$= \int_{1-\delta}^{1+\delta} \int_{S^{n-1}} \frac{1}{|(t/r,0,\ldots,0)-\sigma|^{n-2\beta}} d\sigma r^{2\beta-1} dr.$$

By definition, this is (cf. Lemma 7.20)

$$\int_{1-\delta}^{1+\delta} F(\tfrac{t}{r}) r^{2\beta-1} dr \tag{7.3.10}$$

Using Lemma 7.20, for $\beta < \frac{1}{2}$, (7.3.10) is majorised by

$$\int_{1-\delta}^{1+\delta} (1+\tfrac{t}{r})^{1-n} |1-\tfrac{t}{r}|^{2\beta-1} r^{2\beta-1} dr = \int_{1-\delta}^{1+\delta} |r-t|^{2\beta-1} \frac{r^{n-1}}{(r+t)^{n-1}} dr.$$

First, suppose $t \geqslant 2$. Then t is much larger than r, so we can approximate this integral by $\delta t^{2\beta-n}$.

Now, suppose $t < 2$ and $|1-t| > 2\delta$. Then either $t \in (0, 1-2\delta)$ or $t \in (1+2\delta, 2)$. In the first case, we have (recall that $2\beta - 1 < 0$)

$$|r-t|^{2\beta-1} = (r-t)^{2\beta-1} \leqslant (1-\delta-t)^{2\beta-1}.$$

Since $t < 1-2\delta$ then $1-\delta-t \simeq 1-t$. The second case is analogous. Hence, we may majorise the integral by $\delta|1-t|^{2\beta-1}$.

Conversely, if $|1-t| \leqslant 2\delta$, then

$$\int_{1-\delta}^{1+\delta} |r-t|^{2\beta-1} \frac{r^{n-1}}{(r+t)^{n-1}} dr \lesssim \int_{1-t-\delta}^{1-t+\delta} |r|^{2\beta-1} dr \leqslant \int_{-3\delta}^{3\delta} |r|^{2\beta-1} dr \lesssim \delta^{2\beta}.$$

Hence, for $\beta < \frac{1}{2}$, and $t < 2$, (7.3.10) is majorised by $\delta(\delta + |1-t|)^{2\beta-1}$.

For $t \geqslant 2$ note that $t \simeq 1+t \simeq \delta + |1-t|$ and so $\delta t^{2\beta-n} \simeq \delta(1+t)^{1-n}(\delta+|1-t|)^{2\beta-1}$, while for $0 < t < 2$, we have $(1+t) \simeq 1$. Hence, for $\beta < \frac{1}{2}$, we have

$$\int_{1-\delta}^{1+\delta} F(\tfrac{t}{r}) r^{2\beta-1} dr \lesssim \delta(1+t)^{1-n}(\delta + |1-t|)^{2\beta-1}. \tag{7.3.11}$$

Hence, we need to estimate

$$\begin{aligned} &\sup_{r>0} \sum_{p(k)\in[2^j,2^{j+1})} p(k)^{-2\beta} \delta(\delta + |1-p(k)r|)^{2\beta-1}(1+p(k)r)^{1-n} \\ &\lesssim \sup_{r>0} \delta 2^{-2\beta j} \sum_{p(k)\in[2^j,2^{j+1})} (\delta + |1-p(k)r|)^{2\beta-1} \end{aligned}$$

Notice that the summand is increasing on $p(k) \leqslant \frac{1}{r}$ and decreasing on $p(k) \geqslant \frac{1}{r}$. We will wish to estimate this sum by an integral; in order to do this, we must estimate separately the terms $\frac{1}{r} - 1 \leqslant p(k) \leqslant \frac{1}{r} + 1$ and all other terms.

First, note that

$$\tfrac{1}{r} - 1 \leqslant p(k) \leqslant \tfrac{1}{r} + 1 \iff 1-r \leqslant p(k)r \leqslant 1+r \iff |p(k)r - 1| \leqslant r.$$

Since $2\beta - 1 < 0$, then in this range,

$$(\delta + |1 - p(k)r|)^{2\beta-1} \lesssim \delta^{2\beta-1}.$$

Note that since the length of the interval $[\frac{1}{r} - 1, \frac{1}{r} + 1]$ is 2 and $p(k \pm 1) = b(k) \pm \aleph$, then there are at most $1 + 2\lfloor 1/\aleph \rfloor$ summands in this interval (in particular, if $\aleph > 1$ then there is only one summand in this interval).

Outside of this range, we have that $(\delta + |1 - p(k)r|)^{2\beta-1}$ is monotone increasing for $p(k) < \frac{1}{r} - 1$ and monotone decreasing for $p(k) > \frac{1}{r} + 1$. Hence

$$\begin{aligned}
&\sum_{\substack{p(k)\in[2^j,2^{j+1}) \\ |p(k)r-1|>r}} (\delta + |1 - p(k)r|)^{2\beta-1} \\
&\quad = \sum_{\substack{p(k)\in[2^j,2^{j+1}) \\ p(k)<\frac{1}{r}-1}} (\delta + |1 - p(k)r|)^{2\beta-1} + \sum_{\substack{p(k)\in[2^j,2^{j+1}) \\ p(k)>\frac{1}{r}+1}} (\delta + |1 - p(k)r|)^{2\beta-1} \\
&\quad \leqslant \int\limits_{p(k)\in[2^j,\frac{1}{r}]} (\delta + |1 - p(k)r|)^{2\beta-1} dk + \int\limits_{p(k)\in[\frac{1}{r},2^{j+1})} (\delta + |1 - p(k)r|)^{2\beta-1} dk \\
&\quad = \int\limits_{p(k)\in[2^j,2^{j+1})} (\delta + |1 - p(k)r|)^{2\beta-1} dk
\end{aligned}$$

Since $|1 - p(k)r| > r \geqslant 2^{-j} > \delta$ here, we have

$$\int\limits_{p(k)\in[2^j,2^{j+1})} (\delta + |1 - p(k)r|)^{2\beta-1} dk \lesssim \int\limits_{p(k)\in[2^j,2^{j+1})} |1 - p(k)r|^{2\beta-1} dk$$

If $r \geqslant 2^{-j}$ then $p(k)r \geqslant 1$, and we have

$$\begin{aligned}
\int\limits_{p(k)\in[2^j,2^{j+1})} |1 - p(k)r|^{2\beta-1} dk &= \int\limits_{p(k)\in[2^j,2^{j+1})} (p(k)r - 1)^{2\beta-1} dk \\
&= \frac{(p(k)r - 1)^{2\beta}}{\aleph r} \Big|_{p(k)=2^j}^{p(k)=2^{j+1}} \\
&\simeq r^{-1}[(2^{j+1}r - 1)^{2\beta} - (2^j r - 1)^{2\beta}]
\end{aligned}$$

since $p(k)$ is linear in k. We have analogous workings in the cases $r \leqslant 2^{-j-1}$ and $2^{-j-1} < r < 2^{-j}$. All of the obtained quantities can be majorised uniformly by 2^j, hence,

$$\sup_{r>0} \sum_{p(k)\in[2^j,2^{j+1})} (\delta + |1 - p(k)r|)^{2\beta-1} \leqslant 2^j + \delta^{2\beta-1}. \tag{7.3.12}$$

Thus,

$$\sup_x \int\limits_{\bigcup\limits_{p(k)\in[2^j,2^{j+1})} A(1/p(k),\delta)} \frac{1}{|x-y|^{n-2\beta}} dy \lesssim 2^{-2\beta j}\delta(2^j + \delta^{2\beta-1}) \leqslant \max\{2^{(1-2\beta)j}\delta, (2^{-j}\delta)^{2\beta}\}.$$

For $\beta = \frac{1}{2}$, (7.3.10) is majorised by

$$\int_{1-\delta}^{1+\delta} (1 + \ln_+ \frac{1}{|1-\frac{t}{r}|})(1+\tfrac{t}{r})^{1-n} dr.$$

Notice that $\ln_+ \frac{1}{|1-t/r|} = 0$ for $t \geqslant 2 + 2\delta$, since in this case we have

$$\frac{1}{|1-\frac{t}{r}|} \leqslant \frac{1}{\frac{t}{1+\delta} - 1}$$

which is less-than or equal to 1 if and only if $t \geqslant 2 + 2\delta$. In this case, our integral is majorised by $\delta(1+t)^{1-n}$. For $0 < t \leqslant 2$ we can take $(1+t)^{1-n} \simeq 1$, so we need only deal now with the logarithmic term.

First, we estimate when $\ln_+ \frac{1}{|1-\frac{t}{r}|} \leqslant 1$. This is the case provided

$$\tfrac{1}{|1-\frac{t}{r}|} \leqslant e.$$

First, assume $1 + \delta < t \leqslant 2 + 2\delta$. Then

$$\frac{1}{\left|1-\frac{t}{r}\right|} = \frac{1}{\frac{t}{r} - 1} \leqslant e \iff e^{-1} \leqslant \frac{t}{r} - 1 \iff r(e^{-1} + 1) \leqslant t.$$

Hence, $\ln_+ \frac{1}{|1-\frac{t}{r}|} \leqslant 1$ for $(1+\delta)(e^{-1}+1) \leqslant t \leqslant 2 + 2\delta$. Analogously, we have this for $0 < t \leqslant (1-\delta)(1-e^{-1})$. In both of these ranges, we have $t \lesssim 1$ so that our integral is majorised by δ.

Finally, suppose $(1-\delta)(1-e^{-1}) < t < (1+\delta)(1+e^{-1})$. Then $(1+\frac{t}{r}) \simeq 1$ and $1+\ln_+ \frac{1}{|1-\frac{t}{r}|} \simeq \ln \frac{1}{|1-\frac{t}{r}|}$. Note that

$$\int_a^b \ln \frac{1}{\frac{t}{r}-1} dr = r \ln r + (t-r)\ln(t-r)|_a^b \text{ for } t \geqslant b$$

and

$$\int_a^b \ln \frac{1}{1-\frac{t}{r}} dr = r \ln r + (t-r)\ln(r-t)|_a^b \text{ for } t \leqslant a.$$

Hence, for $(1-\delta)(1-e^{-1}) \leqslant t \leqslant 1-\delta$ we have

$$\begin{aligned}\int_{1-\delta}^{1+\delta} \ln \frac{1}{1-\frac{t}{r}} dr &= r \ln r + (t-r)\ln(r-t)|_{1-\delta}^{1+\delta} \\ &= \ln \frac{1+\delta}{1-\delta} + \delta \ln(1-\delta^2) + (t-1)\ln \frac{1-t+\delta}{1-t-\delta} - \delta \ln(\delta^2-(1-t)^2) \\ &\simeq \delta + (t-1)\ln \frac{1-t+\delta}{1-t-\delta} - \delta \ln((1-t)^2-\delta^2) \\ &\lesssim \delta - \delta \ln((1-t)^2-\delta^2)\end{aligned}$$

as the remaining term is negative. Analogously, for $1+\delta \leqslant t \leqslant (1+\delta)(1+e^{-1})$ we have

$$\begin{aligned}\int_{1-\delta}^{1+\delta} \ln \frac{1}{\frac{t}{r}-1} dr &= r \ln r + (t-r)\ln(t-r)|_{1-\delta}^{1+\delta} \\ &\simeq \delta + (t-1)\ln \frac{t-1-\delta}{t-1+\delta} - \delta \ln((t-1)^2-\delta^2) \\ &\lesssim \delta - \delta \ln((t-1)^2-\delta^2).\end{aligned}$$

Finally, for $1-\delta \leqslant t \leqslant 1+\delta$ we have

$$\begin{aligned}\int_{1-\delta}^{1+\delta} \ln \frac{1}{|\frac{t}{r}-1|} dr &= \int_{1-\delta}^{t} \ln \frac{1}{1-\frac{t}{r}} dr + \int_{t}^{1+\delta} \ln \frac{1}{\frac{t}{r}-1} dr \\ &= (r\ln r + (t-r)\ln(t-r))|_{1-\delta}^{t} + (r\ln r + (t-r)\ln(r-t))|_{t}^{1+\delta} \\ &\simeq \delta + (t-1)\ln\frac{\delta+1-t}{\delta-(1-t)} - \delta\ln(\delta^2-(1-t)^2) \\ &\lesssim \delta - \delta\ln(\delta^2-(1-t)^2).\end{aligned}$$

We note that $-\delta \ln|(1-t)^2-\delta^2|$ is maximised at $t=1$, with value $\simeq \delta|\ln\delta|$, decaying to δ as t approaches $(1-\delta)(1-e^{-1})$ and $(1+\delta)(1+e^{-1})$. Therefore, we can majorise (7.3.10) for $\beta = \frac{1}{2}$ and for all $t>0$ by $\delta(1+t)^{1-n} + \chi_{[(1-\delta)(1-e^{-1}),(1+\delta)(1+e^{-1})]}(t)\delta|\ln\delta|$.

Hence, we need to estimate

$$\begin{aligned}&\sup_{r>0} \sum_{p(k)\in[2^j,2^{j+1})} p(k)^{-1}(\delta(1+t)^{1-n} + \chi_{[(1-\delta)(1-e^{-1}),(1+\delta)(1+e^{-1})]}(p(k)r)\delta|\ln\delta|) \\ &\lesssim \sup_{r>0} \delta 2^{-j} \sum_{p(k)\in[2^j,2^{j+1})} (1+p(k)r)^{1-n} + \chi_{[(1-\delta)(1-e^{-1}),(1+\delta)(1+e^{-1})]}(p(k)r)|\ln\delta|\end{aligned}$$

Since $1-n \leqslant 0$ this is clearly maximised when the largest number of $p(k)r$ are contained in $[(1-\delta)(1-e^{-1}),(1+\delta)(1+e^{-1})]$. If we choose $r = 2^{-j}(1-e^{-1})$ then this is true for all $2^j \leqslant p(k) \leqslant 2^{j+1}$, so this sum is majorised by $\delta|\ln\delta|$.

For $\beta > \frac{1}{2}$, (7.3.10) is majorised by

$$\int_{1-\delta}^{1+\delta} (1+\tfrac{t}{r})^{2\beta-n} r^{2\beta-1} dr \simeq \int_{1-\delta}^{1+\delta} (1+t)^{2\beta-n} dr \simeq \delta(1+t)^{2\beta-n}$$

Hence, we need to estimate

$$\sup_{r>0} \sum_{p(k)\in[2^j,2^{j+1})} p(k)^{-2\beta}\delta(1+p(k)r)^{2\beta-n} \lesssim \sup_{r>0} \delta 2^{-2\beta j} \sum_{p(k)\in[2^j,2^{j+1})} (1+p(k)r)^{2\beta-n}$$

Since $2\beta - n < 1$, then the summands would be maximised by taking $r=0$. As there are $\simeq 2^j$ summands, this yields

$$\delta 2^{(1-2\beta)j}.$$

$\square$

Lemma 7.22. *Recall the definition of $M^{\gamma}_{\delta,j}f$ (cf.* (7.0.1) *and* (7.0.2)*) and let* $0 < a < 2$. *For $\gamma \in \{-1, 0, 1\}$ and $1 \leqslant j < J(1-\frac{a}{2}) + C$ we have*

$$\|M^{\gamma}_{\delta,j}f\|_2^2 \lesssim (2^{-j}\delta)^{a/2}\|f\|^2_{L^2(\psi^{a/2})}.$$

For $J(1-\frac{a}{2}) + C \leqslant j \leqslant J-1$ we have

$$\|M^{\gamma}_{\delta,j}f\|_2^2 \lesssim 2^{(1-a/2)j}\delta\|f\|^2_{L^2(\psi^{a/2})}.$$

Note that the implicit constant depends on C.

Proof. We wish to apply Lemma 7.21 to $M^{\gamma}_{\delta,j}f$ with $\beta = \frac{a}{4}$ and $p(k) = b(k)$. Define $H_\gamma = 2\pi L/\Lambda + 4\pi\gamma Id$, where Id is the identity operator. Then

$$\begin{aligned}
\|M^{\gamma}_{\delta,j}f\|_2^2 &= \|\chi_{[1-\delta,1]}(H_\gamma\Lambda/2\pi)\chi_{[2^j,2^{j+1})}(H_\gamma)f\|_2^2 \\
&= \sum_{b(k)\in[2^j,2^{j+1})} \|\chi_{[1-\delta,1]}(b(k)\Lambda/2\pi)\chi_{\{b(k)\}}(H_\gamma)f\|_2^2 \\
&\leqslant \sum_{b(k)\in[2^j,2^{j+1})} \int_G \|\chi_{[1-\delta,1]}(b(k)|u|)\widehat{f_z}\|^2_{L^2(\mathbb{R}^n)}dz \\
&= \int_{\mathbb{C}^m} \sum_{b(k)\in[2^j,2^{j+1})} \int_{A(b(k)^{-1},\delta)} |\widehat{f_z}(z,u)|^2 du dz \\
&= \int_{\mathbb{C}^m} \int_{\bigcup_{b(k)\in[2^j,2^{j+1})} A(b(k)^{-1},\delta)} |\widehat{f_z}(z,u)|^2 du dz
\end{aligned}$$

where $\widehat{f_z}$ denotes the Euclidean Fourier transform of the function $f(z,u)$ in the u-variable only. The last line here makes sense since the annuli $A(b(k)^{-1},\delta)$ are disjoint. Indeed, consider annuli $A(b(k)^{-1},\delta)$ and $A(b(k+1)^{-1},\delta)$. If the upper boundary of the smaller annulus is strictly smaller than the lower bound of the larger annulus, then they are disjoint. This condition is given by $\frac{1+\delta}{b(k+1)} < \frac{1-\delta}{b(k)}$, or equivalently

$$(1+\delta)b(k) < (1-\delta)b(k+1).$$

Since $b(k+1) = b(k) + 4\pi$, then rearranging gives the condition

$$\delta(4\pi + 2b(k)) < 4\pi,$$

which is true since $b(k) < 2^J$.

Hence, Lemma 7.21 with $\beta = \frac{a}{4}$ shows that this is majorised by

$$\max\{2^{(1-a/2)j}\delta, (2^{-j}\delta)^{a/2}\} \int_{\mathbb{C}^m} \|\widehat{f}_z\|^2_{\dot{L}^2_{a/4}}\, dz$$
$$= \max\{2^{(1-a/2)j}\delta, (2^{-j}\delta)^{a/2}\} \int_G |f(z,u)|^2 |u|^{a/2} dz du.$$

That is,

$$\|M^{\gamma}_{\delta,j} f\|_2^2 \lesssim \max\{2^{(1-a/2)j}\delta, (2^{-j}\delta)^{a/2}\} \|f\|^2_{L^2(\psi^{a/2})}.$$

Now,

$$(2^{-j}\delta)^{a/2} \gtrsim 2^{(1-a/2)j}\delta \iff 2^j \lesssim \delta^{a/2-1}.$$

Recall that $2^{-J} \simeq \delta$, so the result follows. □

Remark 7.23. While this result does prove Theorem 7.1 (in fact something slightly stronger, as the result proved would only use a pure second layer weight) for a partial range of j ($1 \leqslant j \leqslant \frac{2J}{3}$), and can be used to prove the stronger result (7.0.6) in the range $1 \leqslant j \leqslant \frac{J}{2}$, observe that it cannot be used to prove a trace lemma

$$\|M^{\gamma}_{\delta,j} f\|_2^2 \lesssim (2^{-j}\delta)^{a/2} \|f\|^2_{L^2(\psi^{a/2})}$$

for the full range $1 \leqslant j \leqslant J$ for any exponent $a > 0$. This necessitates the following developments that make use of the H-type group structure.

7.3.2 An Improved Estimate for Radial Functions

In the previous section, we essentially ignored the H-type group structure, which gave simpler formulae to deal with, in order to obtain the result. Here, we instead use the more 'involved' formulae developed in Lemma 7.6, with some of the tools developed in the previous section and estimates for Jacobi polynomials developed in Chapter 9 to obtain an improvement of Lemma 7.22, but only for radial functions.

Lemma 7.24. *For radial functions f and $a \in (0, \min\{2, n\})\backslash\{\frac{2}{3}\}$, if $j \leqslant \frac{3J(2-a)}{4}$ and $j < J$ then*

$$\|M^{\gamma}_{\delta,j}f\|_2^2 \lesssim (2^{-j}\delta)^{a/2}\|f\|^2_{L^2(\psi^{a/2})}, \tag{7.3.13}$$

while for $\frac{3J(2-a)}{4} \leqslant j < J$ then

$$\|M^{\gamma}_{\delta,j}f\|_2^2 \lesssim (2^{-j})^{a/2-2/3}\delta\|f\|^2_{L^2(\psi^{a/2})}. \tag{7.3.14}$$

This result holds for $a = \frac{2}{3}$ provided '$\lesssim$' is replaced by '$\lessapprox$'.

Proof. First, by duality and then taking the Fourier transform, (7.3.13) and (7.3.14) are equivalent to

$$\|\partial_{\psi^{-a/4}}\widehat{M^{\gamma}_{\delta,j}}g\|_2^2 \lesssim C_{\delta,j}(a)\|g\|_2^2 \tag{7.3.15}$$

for $g(\mu,k) \in L^2(\mathbb{R}^n\backslash\{0\} \times \mathbb{N}_0, \binom{k+m-1}{k}\#(k)|\mu|^m d\mu)$ and where $C_{\delta,j}(a)$ denotes the constants stated above. By (7.1.6) we need to estimate

$$I := \sup_{k,\mu}\int_{\mathbb{R}^n}\sum_{l\in\mathbb{N}_0}\chi_{[1-\delta,1]}(b(k)|\mu|)\chi_{[2^j,2^{j+1})}(b(k))|K_{\psi^{-a/2}}(\nu,l;\mu,k)| \tag{7.3.16}$$

$$\chi_{[1-\delta,1]}(b(l)|\nu|)\chi_{[2^j,2^{j+1})}(b(l))\binom{l+m-1}{l}|\nu|^m d\nu,$$

Now, for $\omega(|z|,u) = w(u) = |u|^{-a/2}$, where $0 < a < n$, recall that $\widehat{w}(\nu-\mu) \simeq |\nu-\mu|^{a/2-n}$.

Hence we have (c.f. (7.1.8) and (7.1.9)), for $k \geqslant l$, the formula

$$K_{\psi^{-a/2}}(\nu, l; \mu, k) \simeq \frac{|\nu-\mu|^{a/2-n}}{\binom{l+m-1}{l}(|\nu|+|\mu|)^m}\left(2\tfrac{|\nu|}{|\mu|+|\nu|}-1\right)^{k-l} P_l^{(k-l,m-1)}\left(1-2\left(1-2\tfrac{|\nu|}{|\mu|+|\nu|}\right)^2\right), \tag{7.3.17}$$

and for $l > k$,

$$K_{\psi^{-a/2}}(\nu, l; \mu, k) \simeq \frac{|\nu-\mu|^{a/2-n}}{\binom{k+m-1}{k}(|\nu|+|\mu|)^m}\left(1-2\tfrac{|\nu|}{|\mu|+|\nu|}\right)^{l-k} P_k^{(l-k,m-1)}\left(1-2\left(1-2\tfrac{|\nu|}{|\mu|+|\nu|}\right)^2\right). \tag{7.3.18}$$

We substitute (7.3.17) and (7.3.18) into (7.3.16) and change to spherical coordinates, setting $\nu = r\rho$, where $r \in (0, \infty)$ and $\rho \in S^{n-1}$, and similarly let $\mu = s\sigma$. Then (7.3.16) becomes

$$I \simeq \sup_{\substack{b(k)s\in[1-\delta,1]\\ b(k)\in[2^j,2^{j+1})}} \sum_{b(l)\in[2^j,2^{j+1})} \int_{(1-\delta)/b(l)}^{1/b(l)} \int_{S^{n-1}} \frac{1}{|r\rho - s\sigma|^{n-a/2}} d\rho \tag{7.3.19}$$
$$\left|\frac{r-s}{r+s}\right|^{|k-l|} \left|P_{\min\{k,l\}}^{(|k-l|,m-1)}\left(1-2\left(\tfrac{r-s}{r+s}\right)^2\right)\right| \frac{r^{m+n-1}}{(r+s)^m} \frac{\binom{l+m-1}{l}}{\binom{\max\{k,l\}+m-1}{\max\{k,l\}}} dr.$$

We rotate so that $\sigma \mapsto (1, 0, \ldots, 0)$ and rescale to get

$$I \simeq \sup_{\substack{b(k)s\in[1-\delta,1]\\ b(k)\in[2^j,2^{j+1})}} \sum_{b(l)\in[2^j,2^{j+1})} \int_{(1-\delta)/b(l)}^{1/b(l)} \int_{S^{n-1}} \frac{1}{|\rho - (\frac{s}{r}, 0, \ldots, 0)|^{n-a/2}} d\rho$$
$$\left|\frac{r-s}{r+s}\right|^{|k-l|} \left|P_{\min\{k,l\}}^{(|k-l|,m-1)}\left(1-2\left(\tfrac{r-s}{r+s}\right)^2\right)\right| \frac{r^{m+a/2-1}}{(r+s)^m} \frac{\binom{l+m-1}{l}}{\binom{\max\{k,l\}+m-1}{\max\{k,l\}}} dr. \tag{7.3.20}$$

By Lemma 7.20, the integral over S^{n-1} is majorised by

$$\begin{cases} (1+\frac{s}{r})^{1-n}|1-\frac{s}{r}|^{a/2-1} & \text{for } a < 2 \\ (1+\ln_+\frac{1}{|1-s/r|})(1+\frac{s}{r})^{a/2-n} & \text{for } a = 2 \\ (1+\frac{s}{r})^{a/2-n} & \text{for } a > 2. \end{cases} \tag{7.3.21}$$

Notice that s, r are both of a similar size. Specifically, we have $s \in [\frac{1-\delta}{b(k)}, \frac{1}{b(k)}]$ and $r \in [\frac{1-\delta}{b(l)}, \frac{1}{b(l)}]$ where $b(k), b(l) \in [2^j, 2^{j+1})$. Hence (recall we assume $\delta \leqslant \frac{1}{4}$),

$$\frac{3}{8} \leqslant \frac{(1-\delta)2^j}{2^{j+1}} \leqslant \frac{(1-\delta)b(k)}{b(l)} \leqslant \frac{r}{s} \leqslant \frac{b(k)}{(1-\delta)b(l)} \leqslant \frac{2^{j+1}}{(1-\delta)2^j} \leqslant \frac{8}{3}. \tag{7.3.22}$$

Rearranging this shows that $\frac{s}{r}$ has the same bounds, so we may take $\frac{r}{s} \simeq \frac{s}{r} \simeq 1$, and so $1 + \frac{s}{r} \simeq 1$. Hence, (7.3.21) is equivalent to

$$\begin{cases} |1 - \frac{s}{r}|^{a/2-1} & \text{for } a < 2 \\ 1 + \ln_+ \frac{1}{|1-s/r|} & \text{for } a = 2 \\ 1 & \text{for } a > 2. \end{cases}$$

Recall that $r \in [\frac{1-\delta}{b(l)}, \frac{1}{b(l)}]$ and $s \in [\frac{1-\delta}{b(k)}, \frac{1}{b(k)}]$. Since $\frac{3}{8} \leqslant \frac{s}{r} \leqslant \frac{8}{3}$, then

$$\frac{3}{11} \leqslant \frac{r}{r+s} = \frac{1}{1+\frac{s}{r}} \leqslant \frac{8}{11}. \tag{7.3.23}$$

In addition to this, note that $k \simeq l$, so that

$$\frac{\binom{l+m-1}{l}}{\binom{k+m-1}{k}} \simeq \frac{l^{m-1}}{k^{m-1}} \simeq 1.$$

Hence, since $a < 2$, we have

$$\begin{aligned} I \lesssim \sup_{\substack{b(k)s \in [1-\delta,1] \\ b(k) \in [2^j, 2^{j+1})}} \Bigg(& \sum_{\substack{l \leqslant k \\ b(l) \in [2^j, 2^{j+1})}} \int_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1} |\tfrac{r-s}{r+s}|^{k-l} |P_l^{(k-l,m-1)}(1 - 2(\tfrac{s-r}{r+s})^2)| dr \\ & + \sum_{\substack{l > k \\ b(l) \in [2^j, 2^{j+1})}} \int_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1} |\tfrac{s-r}{r+s}|^{l-k} |P_k^{(l-k,m-1)}(1 - 2(\tfrac{s-r}{r+s})^2)| dr \Bigg) \end{aligned}$$

Fix $k \in \mathbb{N}_0$. First, note that if $l = k$ then by Theorem 9.2,

$$\int_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1} |\tfrac{r-s}{r+s}|^{k-l} |P_l^{(k-l,m-1)}(1-2(\tfrac{s-r}{r+s})^2)| dr \lesssim \int_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1} dr.$$

where the factor $(\frac{1+x}{2})^{\beta/2}$ in Theorem 9.2 is $(1-(\frac{r-s}{r+s})^2)^{(m-1)/2} \simeq 1$, since

$$0 \leqslant \left|\frac{r-s}{r+s}\right| \leqslant C < 1,$$

where C is a constant depending on the choice of interval I. Note that C must be strictly less than 1 because both $r, s > 0$ and are in compact intervals away from 0.

Since $a > 0$ then

$$\begin{aligned} \int_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1} dr &= \int_{(1-\delta)/b(l)}^{s} (s-r)^{a/2-1} dr + \int_{s}^{1/b(l)} (r-s)^{a/2-1} dr \\ &= -\frac{2}{a}(s-r)^{a/2}\Big|_{(1-\delta)/b(l)}^{s} + \frac{2}{a}(r-s)^{a/2}\Big|_{s}^{1/b(l)} \\ &= \frac{2}{a}\left(s - \frac{1-\delta}{b(l)}\right)^{a/2} + \frac{2}{a}\left(\frac{1}{b(l)} - s\right)^{a/2} \\ &\simeq \frac{2}{a}\left(\frac{1}{b(l)} - \frac{1-\delta}{b(l)}\right)^{a/2} \simeq (2^{-j}\delta)^{a/2}. \end{aligned} \tag{7.3.24}$$

Now, assume $l \neq k$. Note that,

$$|r-s| \simeq \left|\frac{b(k)-b(l)}{b(k)b(l)}\right| \simeq 2^{-2j}|k-l|.$$

By Theorem 9.7, for r, s in the given ranges, we have

$$|\tfrac{r-s}{r+s}|^{|k-l|} P_{\min\{k,l\}}^{(|k-l|,m-1)}(1-2(\tfrac{s-r}{r+s})^2)| \lesssim |k-l|^{-1/3}. \tag{7.3.25}$$

Thus,

$$\int\limits_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1}|\tfrac{r-s}{r+s}|^{|k-l|}|P^{(|k-l|,m-1)}_{\min\{k,l\}}(1-2(\tfrac{s-r}{r+s})^2)|dr$$

$$\lesssim (2^{-j})^{2(a/2-1)} \int\limits_{(1-\delta)/b(l)}^{1/b(l)} |k-l|^{a/2-4/3}dr$$

$$\simeq (2^{-j})^{a-2}|k-l|^{a/2-4/3} \int\limits_{(1-\delta)/b(l)}^{1/b(l)} dr \simeq |k-l|^{a/2-4/3}(2^{-j})^{a-1}\delta.$$

Thus, provided $a < \frac{2}{3}$,

$$I \lesssim (2^{-j}\delta)^{a/2} + (2^{-j})^{a-1}\delta \sup_{\substack{b(k)s\in[1-\delta,1] \\ b(k)\in[2^j,2^{j+1})}} \sum_{\substack{b(l)\in[2^j,2^{j+1}) \\ k\neq l}} |k-l|^{a/2-4/3}$$

$$\lesssim (2^{-j}\delta)^{a/2} + (2^{-j})^{a-1}\delta$$

$$\simeq (2^{-j}\delta)^{a/2}.$$

For $a = \frac{2}{3}$ we instead have

$$I \lesssim (2^{-j}\delta)^{1/3} + (2^{-j})^{-1/3}\delta \sup_{\substack{b(k)s\in[1-\delta,1] \\ b(k)\in[2^j,2^{j+1})}} \sum_{\substack{b(l)\in[2^j,2^{j+1}) \\ k\neq l}} |k-l|^{-1}$$

$$\lessapprox (2^{-j}\delta)^{1/3} + 2^{j/3}\delta \simeq (2^{-j}\delta)^{1/3}.$$

Finally, if $\frac{2}{3} < a < \min\{2, n\}$ then,

$$I \lesssim (2^{-j}\delta)^{a/2} + (2^{-j})^{a-1}\delta \sup_{\substack{b(k)s\in[1-\delta,1] \\ b(k)\in[2^j,2^{j+1})}} \sum_{\substack{b(l)\in[2^j,2^{j+1}) \\ k\neq l}} |k-l|^{a/2-4/3}$$

$$\lesssim (2^{-j}\delta)^{a/2} + (2^{-j})^{a-1}\delta(2^j)^{a/2-1/3}$$

$$= (2^{-j}\delta)^{a/2} + (2^{-j})^{a/2-2/3}\delta.$$

Note that in this case,

$$
\begin{aligned}
(2^{-j})^{a/2-2/3}\delta \lesssim (2^{-j}\delta)^{a/2} &\iff 2^{2j/3}\delta \lesssim \delta^{a/2} \\
&\iff 2^{2j/3} \lesssim \delta^{a/2-1} \simeq 2^{J(1-a/2)} \\
&\iff j \leqslant \frac{3J(2-a)}{4} + C,
\end{aligned}
\tag{7.3.26}
$$

where C is some numerical constant independent of a, j, δ, which completes the proof.

□

Remark 7.25. We remark that if we had a better estimate for these Jacobi polynomials than the one given by Corollary 9.7, then this would improve this result. In fact, if we were able to use the better bound $u^{-1/2}|x - x_{tr}|^{-1/4}$ in Theorem 9.6 for points far from the 'transition point' (cf (9.0.15)), then this would be enough. We can see from (9.0.25) that for points far enough from the transition point, this estimate is enough for us to replace $\alpha^{-1/3}$ by $\alpha^{-1/2}$. Redoing the proof of the previous Theorem with this modification would in fact give (7.0.6) for radial functions. Unfortunately, in the above calculations, it turns out that we are in fact very close to the transition point. Note that the restriction of the ranges of r and s mean that $r = r(\theta) := \frac{1-\theta\delta}{b(l)}$ and $s = s(\sigma) := \frac{1-\sigma\delta}{b(k)}$ for $\theta, \sigma \in [0, 1]$. One may show that the function

$$
\left| \frac{r(\theta) - s(\sigma)}{r(\theta) + s(\sigma)} \right|
$$

is maximised at

$$
\left| \frac{r(0) - s(1)}{r(0) + s(1)} \right|
$$

and minimised at

$$
\left| \frac{r(1) - s(0)}{r(1) + s(0)} \right|.
$$

Then, recalling the definition of x_{tr} from (9.0.15),

$$
\begin{aligned}
\aleph :&= x_{tr} - \left(1 - 2\sup\left(\frac{r-s}{r+s}\right)^2\right) \\
&= 1 - \frac{2|k-l|^2}{(k+l+m)^2} - \left(1 - 2\frac{(4\pi|k-l| - 1 - \delta b(l))^2}{(4\pi(k+l+m+2\gamma) - \delta b(l))^2}\right) \\
&= -\frac{2|k-l|^2}{(k+l+m)^2} + \frac{2(4\pi|k-l| + \delta b(l))^2}{(4\pi(k+l+m+2\gamma) - \delta b(l))^2} \\
&= \frac{2(4\pi|k-l| + \delta b(l))^2(k+l+m)^2 - 2|k-l|^2(4\pi(k+l+m+2\gamma) - \delta b(l))^2}{(4\pi(k+l+m+2\gamma) - \delta b(l))^2(k+l+m)^2} \\
&= \frac{(32\pi^2|k-l|^2 + 16\pi|k-l|\delta b(l) + \delta^2 b(l)^2)(k+l+m)^2}{(4\pi(k+l+m+2\gamma) - \delta b(l))^2(k+l+m)^2} \\
&\qquad - \frac{|k-l|^2(32\pi^2(k+l+m+2\gamma)^2 - 16\pi(k+l+m+2\gamma)\delta b(l) + 2\delta^2 b(l)^2)}{(4\pi(k+l+m+2\gamma) - \delta b(l))^2(k+l+m)^2} \\
&= \frac{(16\pi|k-l|\delta b(l) + \delta^2 b(l)^2)(k+l+m)^2 + 16\pi|k-l|^2(k+l+m+2\gamma)\delta b(l)}{(4\pi(k+l+m+2\gamma) - \delta b(l))^2(k+l+m)^2} \\
&\qquad - \frac{|k-l|^2(64\pi^2(k+l+m)(2\gamma) + 32\pi^2(2\gamma)^2 + 2\delta^2 b(l)^2)}{(4\pi(k+l+m+2\gamma) - \delta b(l))^2(k+l+m)^2}
\end{aligned}
$$

Then in particular,

$$
|\aleph| \lesssim \frac{|k-l|(k+l+m)^2 + |k-l|^2(k+l+m)}{(k+l+m)^4} = \frac{|k-l|}{(k+l+m)^2} + \frac{|k-l|^2}{(k+l+m)^3}.
$$

That is, recalling the definition of u from (9.0.12) and noting that $2u = k+l+m$,

$$
u^2|\aleph| \simeq |k-l| + u^{-1}|k-l|^2 \simeq |k-l|, \tag{7.3.27}
$$

since $k, l \in \mathbb{N}_0$ and we use these estimates for $|k-l| \geqslant 1$. An analogous calculation may be done with $\sup\left(\frac{r-s}{r+s}\right)$ replaced by $\inf\left(\frac{r-s}{r+s}\right)$. Since $|k-l| \geqslant 1$, then this shows that, in this case, the bound given by Theorem 9.6 is really only the bound $|k-l|^{-1/3}$ in Corollary 9.7; we cannot get anything better out of this result.

7.3.3 Proof of Theorem 7.1

Note that, in this section, we must assume that $G \neq H_1$, the Heisenberg group with first layer dimension $2m = 2$. This is due to a technical constraint on one of the estimates for Jacobi polynomials we will use (Corollary 9.5) which requires that $m > 1$. However, as this is a Heisenberg group, this case is already covered by [24].

Let $K^{\gamma}_{\delta,j}$ denote the convolution kernel of $M^{\gamma}_{\delta,j}$. Let φ_r, where $r \in \mathbb{N}_0$, be the functions defined in (7.2.4) and let $\Lambda_r f(z,u) := \varphi_r(|(z,u)|_K) f(z,u)$ (recall that $|(z,u)|_K = (|z|^4 + 16|u|^2)^{1/4}$. We also fix some $\Psi \in C_c^\infty(\mathbb{R}^+)$ with $\Psi(x) = 1$ for $x \in [\frac{1}{6}, 3]$. Furthermore, recall (7.2.35), (7.2.36) and (7.2.37).

Lemma 7.26. *Suppose, for some $c, d \in \mathbb{R}$ and for all $r \in \mathbb{N}_0$, we have*

$$|\langle \partial_{\psi^0} \widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ } \rangle| \lessapprox (2^{-j}\delta)^d \|2^{cr/2} f\|_2^2 \tag{7.3.28}$$

where the implicit constant does not depend on r. Then for $f(z,u) \in L^2(G)$, $1 \leqslant j < J$ and $\gamma \in \{-1, 0, 1\}$ we have

$$\|M^{\gamma}_{\delta,j} f\|_2^2 \lessapprox (2^{-j}\delta)^d \|f\|^2_{L^2(1+|\cdot|_K^c)}. \tag{7.3.29}$$

Proof. Using the assumption,

$$\begin{aligned}
\|M^{\gamma}_{\delta,j} f\|_2 &\leqslant \sum_{r=0}^{\infty} \|M^{\gamma}_{\delta,j} \Lambda_r^2 f\|_2 \\
&= \sum_{r=0}^{\infty} |\langle \partial_{\psi^0} \widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r^2 f) * (\Lambda_r^2 f)^*]]\widehat{\ } \rangle|^{1/2} \\
&\lessapprox (2^{-j}\delta)^{d/2} \sum_{r=0}^{\infty} 2^{rc/2} \|\Lambda_r f\|_2 \\
&\leqslant (2^{-j}\delta)^{d/2} \left(\sum_{r=0}^{\infty} 2^{(c+\epsilon)r/2} \|\Lambda_r f\|_2^2 \right)^{1/2} \left(\sum_{r=0}^{\infty} 2^{-\epsilon r} \right)^{1/2} \\
&\lesssim_\epsilon (2^{-j}\delta)^{d/2} \|f\|_{L^2(1+|\cdot|_K^{c+\epsilon})}
\end{aligned}$$

The verification that $\Psi(L)$ can be introduced in this way follows from Lemma 7.4. Interpolation with the trivial estimate completes the proof. ◻

Thus, all that remains is to prove (7.3.28). We will not prove it directly, but rather by interpolation. Indeed suppose, for some $a \in (0, 1]$ we can prove

$$|\langle \partial_{\psi^{b/2}} \widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \lessapprox (2^{-j}\delta)^{a/2} \|2^{(b+a)r/2} f\|_2^2 \tag{7.3.30}$$

for all complex numbers b with $\mathrm{Re}(b) \in \{-a, 4-a\}$ and for all $r \in \mathbb{N}_0$, where the implicit constant does not depend on r or $\theta := \mathrm{Im}(b)$. Then, interpolation gives (7.3.28) with $d = \frac{a}{2}, c = a$. The following Lemma will be necessary to do this.

Lemma 7.27. *Assume that $G \neq H_1$. Let $j < J$ and $I = [A, B] \subseteq (0, \infty)$ be a compact interval. For $a \in \mathbb{C}$ with $\mathrm{Re}(a) \in (0, \frac{2}{3})$, $j < J$, or for $a \in \mathbb{C}$ with $\mathrm{Re}(a) \in (\frac{2}{3}, \min\{2, n\})$, $j < J$ and $j \leqslant \frac{3J(2-a)}{4} + C$, we have*

$$\sup_{\substack{\mu,k \\ c(k)|\mu| \in I}} |\partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu, k)| \lesssim (2^{-j}\delta)^{\mathrm{Re}(a)/2}. \tag{7.3.31}$$

For $a \in \mathbb{C}$ with $\mathrm{Re}(a) \in (\frac{2}{3}, \min\{2, n\})$, $\frac{3J(2-a)}{4} + C < j < J$ we have

$$\sup_{\substack{\mu,k \\ c(k)|\mu| \in I}} |\partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu, k)| \lesssim (2^{-j})^{\mathrm{Re}(a)/2 - 2/3} \delta. \tag{7.3.32}$$

The result (7.3.31) also holds for $\mathrm{Re}(a) = \frac{2}{3}$ if we replace $\lesssim$ with $\lessapprox$. The implicit constants in all cases will depend on the choice of interval I but are independent of $\theta = \mathrm{Im}(a)$.

Proof. Recall that the integral kernel $K_{\psi^{-a/2}}$ of $\partial_{\psi^{-a/2}}$ is given by (7.3.17) and (7.3.18) (note that, while these formulae are stated for $0 < a < n$, they are in fact valid for complex a with $0 < \mathrm{Re}(a) < n$). However, these formulae also show that the absolute value of $K_{\psi^{-a/2}}$ is independent of $\mathrm{Im}(a)$. Thus, we may assume that a is real in what follows (and in general, replace a with $\mathrm{Re}(a)$). Thus, we are required to estimate

$$\mathscr{I} := \sup_{\substack{\mu,k \\ c(k)|\mu|\in I}} \int_{\mathbb{R}^n} \sum_{l\in\mathbb{N}_0} \frac{|\nu-\mu|^{a/2-n}}{\binom{\min\{k,l\}+m-1}{\min\{k,l\}}(|\nu|+|\mu|)^m} |2\tfrac{|\nu|}{|\mu|+|\nu|}-1|^{|k-l|}$$
$$|P^{(|k-l|,m-1)}_{\min\{k,l\}}(1-2(1-2\tfrac{|\nu|}{|\mu|+|\nu|})^2)|\chi_{[1-\delta,1]}(b(l)|\nu|)\chi_{[2^j,2^{j+1})}(b(l))\binom{l+m-1}{l}|\nu|^m d\nu.$$

This rearranges to

$$\mathscr{I} = \sup_{\substack{\mu,k \\ c(k)|\mu|\in I}} \sum_{b(l)\in[2^j,2^{j+1})} \int_{|\nu|\in[(1-\delta)/b(l),1/b(l)]} |\nu-\mu|^{a/2-n}|2\tfrac{|\nu|}{|\mu|+|\nu|}-1|^{|k-l|}$$
$$\cdot\, |P^{(|k-l|,m-1)}_{\min\{k,l\}}(1-2(1-2\tfrac{|\nu|}{|\mu|+|\nu|})^2)| \frac{\binom{l+m-1}{l}}{\binom{\min\{k,l\}+m-1}{\min\{k,l\}}} \frac{|\nu|^m}{(|\nu|+|\mu|)^m} d\nu.$$

As in Lemma 7.24, we change to spherical coordinates, rotate and rescale (cf (7.3.19) and (7.3.20), with the same notation) and then apply Lemma 7.20 (cf (7.3.21)) to get

$$\mathscr{I} \leqslant \sup_{\substack{s,k \\ c(k)s\in I}} \sum_{b(l)\in[2^j,2^{j+1})} \int_{(1-\delta)/b(l)}^{1/b(l)} (1+\tfrac{s}{r})^{1-n}|r-s|^{a/2-1}$$
$$\left|\frac{r-s}{r+s}\right|^{|k-l|} |P^{(|k-l|,m-1)}_{\min\{k,l\}}(1-2(\tfrac{r-s}{r+s})^2)| \frac{\binom{l+m-1}{l}}{\binom{\min\{k,l\}+m-1}{\min\{k,l\}}} \frac{r^m}{(r+s)^m} dr.$$

For brevity, we define

$$\mathscr{K} := \mathscr{K}(k,l,s) = \int_{(1-\delta)/b(l)}^{1/b(l)} (1+\tfrac{s}{r})^{1-n}|r-s|^{a/2-1} \tag{7.3.33}$$
$$\left|\frac{r-s}{r+s}\right|^{|k-l|} |P^{(|k-l|,m-1)}_{\min\{k,l\}}(1-2(\tfrac{r-s}{r+s})^2)| \frac{\binom{l+m-1}{l}}{\binom{\min\{k,l\}+m-1}{\min\{k,l\}}} \frac{r^m}{(r+s)^m} dr.$$

Fix $k \in \mathbb{N}_0$. First, note that if $l = k$ then $s \simeq r$, so that

$$(1+\tfrac{s}{r})^{1-n} \simeq 1, \quad \binom{l+m-1}{l}\binom{\min\{k,l\}+m-1}{\min\{k,l\}}^{-1} = 1, \quad \frac{r^m}{(r+s)^m} \simeq 1.$$

Then by Theorem 9.2,

$$\mathscr{K} \simeq \int\limits_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1} |\tfrac{r-s}{r+s}|^0 |P_l^{(0,m-1)}(1-2(\tfrac{s-r}{r+s})^2)| dr \lesssim \int\limits_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1} dr, \tag{7.3.34}$$

where the factor $(\frac{1+x}{2})^{\beta/2}$ is $(1-(\frac{r-s}{r+s})^2)^{(m-1)/2} = (\frac{4rs}{(r+s)^2})^{(m-1)/2} \simeq 1$, since $r \simeq s$. If $s \in [\frac{1-\delta}{b(l)}, \frac{1}{b(l)}]$ then, since $a > 0$ then (7.3.34) is integrable and majorised by $(2^{-j}\delta)^{a/2}$ as in (7.3.24). For $s < \frac{1-\delta}{b(l)}$, since $a \in (0,2)$ then $a/2 - 1 < 0$ so that $|r-s|^{a/2-1}$ is an increasing function of s. Analogously, for $s > \frac{1}{b(l)}$, $|r-s|^{a/2-1}$ is a decreasing function of s. Hence, both of these cases are majorised by the $s \in [\frac{1-\delta}{b(l)}, \frac{1}{b(l)}]$ case.

Now, assume $l \neq k$ (that is, $|k-l| \geqslant 1$). We consider multiple cases. First, suppose that $|k-l| \leqslant c_1(\min\{k,l\} + \frac{m}{2})$, where $c_1 > 1$ is a constant to be specified later. Then, $1+k \simeq 1+l$ so that $r \simeq s$, $(1+\frac{s}{r})^{1-n} \simeq 1$, $\binom{l+m-1}{l}\binom{\min\{k,l\}+m-1}{\min\{k,l\}}^{-1} \simeq 1$ and $\frac{r^m}{(r+s)^m} \simeq 1$. Thus,

$$\mathscr{K} \simeq \int\limits_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1} \left|\frac{r-s}{r+s}\right|^{|k-l|} |P_{\min\{k,l\}}^{(|k-l|,m-1)}(1-2(\tfrac{r-s}{r+s})^2)| dr. \tag{7.3.35}$$

Suppose further that $|\frac{r-s}{r+s}| \geqslant 4|\frac{k-l}{k+l+m}|$, so that $|r-s| \gtrsim 2^{-2j}|k-l|$ (note that this extra condition does impose a further constraint on r, which we may disregard in the integral as the integrand is positive). Note that

$$1 - 2\left(\frac{s-r}{r+s}\right)^2 = -1 + \frac{8rs}{(r+s)^2} \geqslant -1 + \epsilon, \tag{7.3.36}$$

for some $\epsilon \in (0,2)$ which is independent of r, s since $r \simeq s$.

Then, by (7.3.35) and the first part of Theorem 9.6 (that is, (9.0.16)),

$$\mathscr{K} \lesssim \int\limits_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1}((k+l+m)^2|(\tfrac{k-l}{k+l+m})^2 - (\tfrac{r-s}{r+s})^2| + |k-l|^{4/3})^{-1/4} dr$$

Now, since $|\frac{r-s}{r+s}| \geqslant 4|\frac{k-l}{k+l+m}|$, then

$$|(\tfrac{k-l}{k+l+m})^2 - (\tfrac{r-s}{r+s})^2| \simeq (\tfrac{r-s}{r+s})^2$$

and

$$((k+l+m)^2(\tfrac{r-s}{r+s})^2 + |k-l|^{4/3})^{-1/4} \simeq ((k+l+m)^2(\tfrac{r-s}{r+s})^2)^{-1/4} \simeq 2^{-j}|r-s|^{-1/2}.$$

Hence,

$$\mathscr{K} \lesssim \int_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1}2^{-j}|r-s|^{-1/2}dr \lesssim 2^{-j(a-1)}|k-l|^{a/2-3/2}\delta.$$

Next, assume that $|\frac{r-s}{r+s}| \leqslant \frac{1}{4}|\frac{k-l}{k+l+m}|$. Note that this exactly implies the conditions of (9.0.17), and hence applying that to (7.3.35) gives

$$\mathscr{K} \lesssim 2^{-|k-l|} \int_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1}dr \lesssim 2^{-|k-l|}(2^{-j}\delta)^{a/2}.$$

Finally, assume that $\frac{1}{4}|\frac{k-l}{k+l+m}| \leqslant |\frac{r-s}{r+s}| \leqslant 4|\frac{k-l}{k+l+m}|$, so that $|r-s| \simeq 2^{-2j}|k-l|$. Thus, we again have (7.3.36), so applying Corollary 9.7 to (7.3.35) gives

$$\begin{aligned}\mathscr{K} &\lesssim \int_{(1-\delta)/b(l)}^{1/b(l)} |r-s|^{a/2-1}|k-l|^{-1/3}dr \\ &\simeq 2^{-j(a-2)}|k-l|^{a/2-4/3} \int_{(1-\delta)/b(l)}^{1/b(l)} dr \\ &\simeq 2^{-j(a-1)}|k-l|^{a/2-4/3}\delta.\end{aligned}$$

Thus, for $k \neq l$ and $|k-l| \leqslant c_1(\min\{k,l\} + \frac{m}{2})$ we have that

$$\begin{aligned}\mathscr{K} &\lesssim 2^{-j(a-1)}|k-l|^{a/2-3/2}\delta + 2^{-j(a-1)}|k-l|^{a/2-4/3}\delta + 2^{-|k-l|}(2^{-j}\delta)^{a/2} \\ &\lesssim 2^{-j(a-1)}\delta|k-l|^{a/2-4/3} + 2^{-|k-l|}(2^{-j}\delta)^{a/2}.\end{aligned}$$

If this were summed over $b(l) \in [2^j, 2^{j+1})\backslash\{b(k)\}$, then this is majorised by

$$\begin{cases} 2^{-j(a-1)}\delta + (2^{-j}\delta)^{a/2}, & a \in (0, \frac{2}{3}) \\ j2^{-j(a-1)}\delta + (2^{-j}\delta)^{a/2}, & a = \frac{2}{3} \\ 2^{-j(a-1)}\delta 2^{j(a/2-1/3)} + (2^{-j}\delta)^{a/2}, & a \in (\frac{2}{3}, \min\{2,n\}), \end{cases} \tag{7.3.37}$$

which is majorised by

$$\begin{cases} (2^{-j}\delta)^{a/2}, & a \in (0, \frac{2}{3}) \\ j(2^{-j}\delta)^{a/2}, & a = \frac{2}{3} \\ 2^{-j(a/2-2/3)}\delta + (2^{-j}\delta)^{a/2}, & a \in (\frac{2}{3}, \min\{2,n\}), \end{cases} \tag{7.3.38}$$

Now, we assume that $l \neq k$ with $|k-l| > c_1(\min\{k,l\} + \frac{m}{2})$. We consider two cases. First, let $l < k$ so that $k > c_1(l + \frac{m}{2}) + l$. Then,

$$k + \frac{m}{2} > (c_1+1)\left(l + \frac{m}{2}\right),$$

so by Lemma 7.4, since $b(l) > 0$,

$$c(k) > (c_1+1)c(l) \geqslant \frac{1}{3}(c_1+1)b(l).$$

Hence, $s \leqslant \frac{B}{c(k)} < \frac{3B}{(2c_1+1)b(l)} \leqslant \frac{3B}{(2c_1+1)(1-\delta)}r \leqslant \frac{4B}{(2c_1+1)}r$. For c_1 sufficiently large, this means that,

$$|r-s| \simeq r \simeq \frac{1}{b(l)} \simeq 2^{-j},$$

$$\frac{\binom{l+m-1}{l}}{\binom{\min\{k,l\}+m-1}{\min\{k,l\}}} = 1,$$

$$\frac{r^m}{(r+s)^m} \simeq 1$$

and

$$1 + \frac{s}{r} \simeq 1.$$

Hence, recalling (7.3.33),

$$\mathscr{K} \simeq \int_{(1-\delta)/b(l)}^{1/b(l)} r^{a/2-1} \left|\frac{r-s}{r+s}\right|^{k-l} |P_l^{(k-l,m-1)}(1-2(\tfrac{r-s}{r+s})^2)|dr.$$

We apply Corollary 9.5 and the fact that $|r-s| \simeq |r+s| \simeq r$ to get

$$\begin{aligned}\mathscr{K} &\lesssim \int_{(1-\delta)/b(l)}^{1/b(l)} r^{a/2-1}(1-(\tfrac{r-s}{r+s})^2)^{-m/2+1/4} \left|\frac{r-s}{r+s}\right|^{-1/2} \\ &\qquad (k+1)^{-1/4}(l+1)^{-1/12}\left(\frac{l+1}{k+1}\right)^{(m-1)/2} dr \\ &\lesssim (k+1)^{-1/4}\left(\frac{l+1}{k+1}\right)^{(m-1)/2} 2^{-j(a/2-1+1/12)} \int_{(1-\delta)/b(l)}^{1/b(l)} (1-(\tfrac{r-s}{r+s})^2)^{-m/2+1/4} dr.\end{aligned}$$

Now,

$$1 - \left(\frac{r-s}{r+s}\right)^2 = \frac{4rs}{(r+s)^2} \simeq \frac{s}{r}$$

so

$$\mathscr{K} \lesssim 2^{-j(a/2-1+1/12)} \int_{(1-\delta)/b(l)}^{1/b(l)} \left(\frac{l+1}{k+1}\right)^{(m-1)/2} \left(\frac{r}{s}\right)^{m/2-1/4} (k+1)^{-1/4} dr.$$

Recall that $(k+1)s \simeq (l+1)r \simeq 1$ and $(l+1) \simeq 2^j$ so that

$$\begin{aligned}
\mathscr{K} &\lesssim 2^{-j(a/2-1+1/12)} \int\limits_{(1-\delta)/b(l)}^{1/b(l)} \frac{(l+1)^{m/2-1/2} r^{m/2-1/4}}{(k+1)^{m/2-1/2+1/4} s^{m/2-1/4}} dr \\
&\simeq 2^{-j(a/2-1+1/12)} (l+1)^{-1/4} \int\limits_{(1-\delta)/b(l)}^{1/b(l)} dr \\
&\simeq 2^{-j(a/2-1+1/12+1/4+1)} \delta = 2^{-j(a/2+1/3)} \delta. \qquad (7.3.39)
\end{aligned}$$

Now, let $l > k$, so that $l > c_1(k + \frac{m}{2}) + k$ and thus

$$l + \frac{m}{2} > (c_1 + 1)(k + \frac{m}{2}).$$

Then, using Lemma 7.4,

$$3b(l) \geqslant c(l) > (c_1 + 1)c(k).$$

Now, $r \leqslant \frac{1}{b(l)} < \frac{3}{(2c_1+1)c(k)} \leqslant \frac{3}{(2c_1+1)A} s$. Thus, for c_1 sufficiently large,

$$|r - s| \simeq s \simeq |r + s| \simeq (1 + k)^{-1},$$

$$1 + \frac{s}{r} \simeq \frac{s}{r}$$

and

$$\frac{\binom{l+m-1}{l}}{\binom{\min\{k,l\}+m-1}{\min\{k,l\}}} \frac{r^m}{(r+s)^m} \simeq \frac{(l+1)^{m-1} r^m}{(k+1)^{m-1} s^m} \simeq \frac{r}{s}.$$

So,

$$\mathscr{K} \simeq \int\limits_{(1-\delta)/b(l)}^{1/b(l)} \left(\frac{r}{s}\right)^n s^{a/2-1} \left|\frac{r-s}{r+s}\right|^{l-k} |P_k^{(l-k,m-1)}(1 - 2(\tfrac{r-s}{r+s})^2)| dr.$$

As before, we apply Corollary 9.5 to get

$$\mathscr{K} \lesssim (l+1)^{-1/4}(k+1)^{-1/12}\left(\frac{k+1}{l+1}\right)^{(m-1)/2} s^{a/2-1} \int\limits_{(1-\delta)/b(l)}^{1/b(l)} \left(\frac{r}{s}\right)^{n}\left(\frac{s}{r}\right)^{m/2-1/4} dr,$$

where this time,

$$1-\left(\frac{r-s}{r+s}\right)^2 = \frac{4rs}{(r+s)^2} \simeq \frac{r}{s}$$

Thus, since $\frac{a}{2}-\frac{2}{3}-n<0$,

$$\begin{aligned}
\mathscr{K} &\lesssim s^{a/2-2/3-n} \int\limits_{(1-\delta)/b(l)}^{1/b(l)} r^n dr \\
&\simeq (1+k)^{n+2/3-a/2} 2^{-j(n+1)}\delta \\
&\lesssim 2^{-j(-n-2/3+a/2+n+1)}\delta \\
&\lesssim 2^{-j(a/2+1/3)}\delta. \qquad (7.3.40)
\end{aligned}$$

When summed over $b(l) \in [2^j, 2^{j+1})\backslash\{b(k)\}$, up to constants, these last two results (7.3.39) and (7.3.40) give $2^{-j(a/2-2/3)}\delta$; note that for $a \in (0, \min\{2, n\})$, this is always dominated by (7.3.38).

Finally, note that

$$(2^{-j})^{a/2-2/3}\delta \lesssim (2^{-j}\delta)^{a/2} \iff j \leqslant \frac{3J(2-a)}{4} + C,$$

completing the proof. □

Lemma 7.28. *Let $R \in (0,1]$. Suppose that, for all $a = -R + i\theta \in \mathbb{C}$, for all integers $1 \leqslant j < J$, for all $\gamma \in \{-1,0,1\}$ and for all compact intervals $I \subseteq (0,\infty)$ we have*

$$\sup_{\substack{\mu,k \\ c(k)|\mu| \in I}} |\partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu,k)| \lessapprox (2^{-j}\delta)^{R/2}, \qquad (7.3.41)$$

where the implicit constant does not depend on θ. Then,

$$|\langle \partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \lessapprox (2^{-j}\delta)^{R/2}\|f\|_2^2 \tag{7.3.42}$$

where the implicit constant does not depend on θ.

Proof. First,

$$\begin{aligned}
&|\langle \partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \\
&\simeq \left| \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} \partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu, |\alpha|) \overline{\Psi(|\mu| c(|\alpha|))[(\Lambda_r f) * (\Lambda_r f)^*]\widehat{\ }(\mu, \alpha, \alpha)} |\mu|^m d\mu \right| =: \mathscr{I}
\end{aligned}$$

Applying our assumption and (7.2.37) gives the estimate

$$\begin{aligned}
\mathscr{I} &\lessapprox (2^{-j}\delta)^{R/2} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |\Psi(|\mu| c(|\alpha|))[(\Lambda_r f) * (\Lambda_r f)^*]\widehat{\ }(\mu, \alpha, \alpha)||\mu|^m d\mu \\
&\lesssim (2^{-j}\delta)^{R/2} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[(\Lambda_r f) * (\Lambda_r f)^*]\widehat{\ }(\mu, \alpha, \alpha)||\mu|^m d\mu \\
&\lesssim (2^{-j}\delta)^{R/2} \|\Lambda_r f\|_2^2 \lesssim (2^{-j}\delta)^{1/3} \|f\|_2^2
\end{aligned}$$

□

Lemma 7.29. *Let $R \in (0, 1]$. Suppose that, for all $a = -R + i\theta \in \mathbb{C}$, for all integers $1 \leqslant j < J$, for all $\gamma \in \{-1, 0, 1\}$ and for all compact intervals $I \subseteq (0, \infty)$ we have*

$$\sup_{\substack{\mu,k \\ c(k)|\mu| \in I}} |\partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu, k)| \lessapprox (2^{-j}\delta)^{R/2}, \tag{7.3.43}$$

where the implicit constant does not depend on θ. Then,

$$|\langle \partial_{\psi^{(4-a)/2}} \widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \lessapprox (2^{-j}\delta)^{R/2}\|2^{2r} f\|_2^2 \tag{7.3.44}$$

where the implicit constant does not depend on θ.

Proof. Let K denote the convolution kernel of $\Psi(L)$. Now, (recall (2.3.15))

$$\begin{aligned}&|\langle \partial_{\psi^{(4-a)/2}} \widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[f * f^*]]\widehat{\ }\rangle| \\ &\qquad = |\langle \partial_{\rho^{-a}} \widehat{K^{\gamma}_{\delta,j}}, [\psi^2([f * f^*] * K)]\widehat{\ }\rangle|\end{aligned}$$

The exact expansion of this follows from (2.3.18); we estimate each of the terms separately.

First, we have

$$\begin{aligned}&|\langle \partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}, [(\psi_l^2[f * f^*]) * K]\widehat{\ }\rangle| \\ &\quad \leqslant \int\limits_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |\partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu, |\alpha|)||\Psi(c(|\alpha|)|\mu|)||(\psi_l^2[f * f^*])\widehat{\ }(\mu, \alpha, \alpha)||\mu|^m d\mu\end{aligned}$$

We then apply our assumptions to majorise this by

$$(2^{-j}\delta)^{R/2} \int\limits_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |(\psi_l^2[f * f^*])\widehat{\ }(\mu, \alpha, \alpha)||\mu|^m d\mu.$$

In order to bound this, we apply (2.3.18). Taking the inverse Fourier transform, our integral is majorised by

$$(2^{-j}\delta)^{R/2} \left(\|\psi_l^2 f\|_2 \|f\|_2 + \|\psi_l f\|_2^2 + \|\rho\psi_l f\|_2 \|\rho f\|_2 + \|\rho^2 f\|_2^2 \right).$$

As in the previous case, replacing f with $\Lambda_r f$ yields the desired result.

The next term to bound is

$$|\langle \partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}, [(\psi_l[f * f^*]) * (\psi_l K)]\widehat{\ }\rangle|$$

First, we calculate $\partial_{\psi_l} \widehat{K}(\mu, k)$. From (2.3.6) we have

$$4\pi i \partial_{\psi_l} \widehat{K}(\mu, k) = 2\partial_{\mu_l} \widehat{K}(\mu, k) + \frac{\mu_l}{|\mu|^2}[m\widehat{K}(\mu, k) + k\widehat{K}(\mu, k-1) - (k+m)\widehat{K}(\mu, k+1)]$$

Then, by (2.3.11) and Young's inequality,

$$
\begin{aligned}
&|\langle \partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}, [(\psi_l[f * f^*]) * (\psi_l K)]^{\wedge}\rangle| \\
&= \left| \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} \partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu, |\alpha|) \sum_{\beta \in \mathbb{N}_0^m} \overline{\psi_l[f * f^*](\mu,\beta,\alpha) \widehat{\psi_l K}(\mu,\alpha,\beta)} |\mu|^m d\mu \right| \\
&\lessapprox (2^{-j}\delta)^{R/2} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[(\psi_l f) * (f^* * (\psi_l K))]^{\wedge}(\mu,\alpha,\alpha)|\, |\mu|^m d\mu \\
&\quad + (2^{-j}\delta)^{R/2} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[f * ((\psi_l f^*) * (\psi_l K))]^{\wedge}(\mu,\alpha,\alpha)|\, |\mu|^m d\mu \\
&\quad + \sum_{p,q=1}^{m} \sum_{\gamma,\epsilon=0,1} (2^{-j}\delta)^{R/2} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[(\zeta_{\mu_1,p,\gamma} f) * ((\zeta_{\mu_1,q,\epsilon} f^*) * (\psi_l K))]^{\wedge}(\mu,\alpha,\alpha)|\, |\mu|^m d\mu \\
&\leqslant (2^{-j}\delta)^{R/2} \Bigg(\|\psi_l f\|_2 \|f^* * (\psi_l K)\|_2 + \|f\|_2 \|(\psi_l f^*) * (\psi_l K)\|_2 \\
&\qquad\qquad + \sum_{p,q=1}^{m} \sum_{\gamma,\epsilon=0,1} \|\zeta_{\mu_1,p,\gamma} f\|_2 \|(\zeta_{\mu_1,q,\epsilon} f^*) * (\psi_l K)\|_2 \Bigg) \\
&\leqslant (2^{-j}\delta)^{R/2} \Bigg(\|\psi_l f\|_2 \|f^*\|_2 \|\psi_l K\|_1 + \|f\|_2 \|\psi_l f^*\|_2 \|\psi_l K\|_1 \\
&\qquad\qquad + \sum_{p,q=1}^{m} \sum_{\gamma,\epsilon=0,1} \|\zeta_{\mu_1,p,\gamma} f\|_2 \|\zeta_{\mu_1,q,\epsilon} f^*\|_2 \|\psi_l K\|_1 \Bigg).
\end{aligned}
$$

Since we chose Ψ to be smooth and compactly supported, then K is Schwartz, and so the weighted L^1 norms are uniformly bounded. (c.f. the proof of Lemma 2.4, where the weighted L^1 norm of K is bounded by $\|K\|_{1,0,1}$ which in turn is bounded by the supremum of its Schwartz semi-norms).

Next,

$$
\begin{aligned}
&|\langle \partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}, [[f * f^*] * (\psi_l^2 K)]^{\wedge}\rangle| \\
&\leqslant \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |\partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu, |\alpha|)| |[[f * f^*] * (\psi_l^2 K)]^{\wedge}(\mu,\alpha,\alpha)| |\mu|^m d\mu \\
&= \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |\partial_{\psi^{-a/2}} \widehat{K^{\gamma}_{\delta,j}}(\mu, |\alpha|)| |\partial_{\psi_l^2} \widehat{K}(\mu, |\alpha|)| |[f * f^*]^{\wedge}(\mu,\alpha,\alpha)| |\mu|^m d\mu =: \mathscr{I}.
\end{aligned}
$$

From Lemma 2.30, we can see that if $\partial_{\psi_l^2} \widehat{K}(\mu, k)$ is non-zero, then k, μ must satisfy

$c(k+x)|\mu| \in \operatorname{supp}(\Psi) = [A, B]$ for some $x \in \{-2, -1, 0, 1, 2\}$, which means that $c(k)|\mu| \in [\frac{A}{5}, 5B]$, by Lemma 7.4. Applying our assumption shows that

$$\mathscr{I} \lessapprox (2^{-j}\delta)^{R/2} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[f * [f^* * (\psi_l^2 K)]]^\wedge(\mu, \alpha, \alpha)||\mu|^m d\mu.$$

Applying (7.2.37) bounds the integral by $\|f\|_2 \|f^* * (\psi_l^2 K)\|_2$. By Young's inequality, this is bounded by $\|f\|_2^2 \|\psi_l^2 K\|_1$, and as in previous cases, since K is Schwartz, If we replace f by $\Lambda_r f$ in these calculations, this is majorised by the right-hand side of (7.2.52) as required.

The remaining cases are entirely analogous. Using similar workings to those already done, with (2.3.1), (2.3.2) and (2.3.6) as necessary (and using Lemma 2.28 with the latter two formulae), shows that any combination of these that will fall on $\widehat{K}$ due to (2.3.18) as difference-partial-differential operators will shift the support no more than $\partial_{\psi_l^2}$, that is, we will always have that we may assume that $c(|\alpha|+x)|\mu| \in \operatorname{supp}(\Psi)$ for $x \in \{-2, -1, 0, 1, 2\}$ (note that while $\partial_{\zeta_{\mu_1,p,\gamma}} \widehat{K}(\mu, \alpha, \beta)$ is no longer radial, it is still supported where $|\alpha - \beta| \leqslant 1$ and is a linear combination of radial terms $\widehat{K}(\mu, k+x)$). Using calculations analogous to those above with the fact that K is Schwartz, we will always be able to show that the various remaining summands S_i occurring due to (2.3.18) satisfy

$$S_i \lessapprox (2^{-j}\delta)^{R/2} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{N}_0^m} |[X(f * f^*) * (YK)]^\wedge(\mu, \alpha, \alpha)|\, |\mu|^m d\mu$$

where X, Y are each some combination of at most either two of some $\zeta_{\mu,p,\gamma}$ or one with a ψ_l. We may use analogous calculations to those already done and existing Leibniz rules to expand out $X(f * f^*)$ and then use (7.2.37) to bound this by

$$\|Uf\|_2 \|(Vf^*) * (YK)\|_2$$

where U, V are again some combination of some $\zeta_{\mu,p,\gamma}$ and ψ_l, consisting of either one each of some $\zeta_{\mu,p,\gamma}$ and ψ_l total, or up to three of some $\zeta_{\mu,p,\gamma}$ with one of U and V

consisting of one $\zeta_{\mu,p,\gamma}$ and the other of either one or two $\zeta_{\mu,p,\gamma}$. Using Young's inequality for convolutions and the fact that K is Schwartz, we majorise by

$$\|Uf\|_2\|Vf^*\|_2$$

which, upon replacing f by $\Lambda_r f$ gives the desired result. □

These results lead to the following 'trace lemma' result.

Corollary 7.30. *We have, for all* $1 \leqslant j < J$ *and for all* $\gamma \in \{-1, 0, 1\}$,

$$\|M^{\gamma}_{\delta,j}f\|_2^2 \lessapprox (2^{-j}\delta)^{1/3}\|f\|^2_{L^2(1+|\cdot|_K^{2/3})}. \tag{7.3.45}$$

Proof. As noted earlier, we must defer to Lemma 7 of [24] if $G = H_1$. Otherwise, using Lemma 7.27 with $\mathrm{Re}(a) = \frac{2}{3}$ to satisfy the assumptions of Lemmas 7.28 and 7.29 yields

$$|\langle \partial_{\psi^{-1/3}}\widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \lessapprox (2^{-j}\delta)^{1/3}\|f\|_2^2 \tag{7.3.46}$$

and

$$|\langle \partial_{\psi^{2-1/3}}\widehat{K^{\gamma}_{\delta,j}}, [\Psi(L)[(\Lambda_r f) * (\Lambda_r f)^*]]\widehat{\ }\rangle| \lessapprox (2^{-j}\delta)^{1/3}\|2^{2r}f\|_2^2. \tag{7.3.47}$$

As remarked after the proof of Lemma 7.26, interpolation of these leads to the assumptions of Lemma 7.26, the application of which completes the proof. □

Remark 7.31. If, we instead consider the result of Lemma 7.27 with $a = 1$, then the results of this section instead combine to prove a 'stronger' estimate

$$\|M^{\gamma}_{\delta,j}f\|_2^2 \lessapprox (2^{-j}\delta)^{1/2}\|f\|^2_{L^2(1+|\cdot|_K)}$$

but on a reduced range of j, specifically $1 \leqslant j \leqslant \frac{3}{4}J$. The 'endpoint' case $j = J$ then follows from Theorem 7.2. This leaves a 'middle region' $\frac{3}{4}J < j < J$ where pure first or second layer weights do not appear to be sufficient to prove this estimate. This is notably different to the Euclidean case (see Figure 1.3) where no such region exists.

Proof of Theorem 7.1. By interpolation, it suffices to prove Theorem 7.1 for $a = \frac{2}{3}$. First, we see that the $j = J$ case of Theorem 7.1 follows by interpolating the estimate of Theorem 7.2 with the trivial L^2 estimate

$$\|M^{\gamma}_{\delta,j} f\|_2^2 \lesssim \|f\|_2^2$$

at $\theta = \frac{2}{3}$. Combining this with Corollary 7.30 completes the proof. □

CHAPTER 8

PROOF OF MAIN THEOREMS

We conclude by explaining exactly how the various results we have proven throughout link together to form the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. First, we see that Theorem 7.1 verifies the conditions of Lemma 6.3 for $a = \frac{2}{3}$. This in turn verifies the conditions of Proposition 6.1 for $C = (2^{-j}\delta)^{1/3}$, $N = 6$ and $w = 1 + |\cdot|_K^{2/3}$ and so proves that

$$\|m_\delta(L)\|^2_{L^2(1+|\cdot|_K^{2/3}) \to L^2(1+|\cdot|_K^{2/3})} \lessapprox 1.$$

Then, by Theorem 5.2, we have the same result for $M_\delta^\bullet$. That is,

$$\|M_\delta^\bullet\|^2_{L^2(1+|\cdot|_K^{2/3}) \to L^2(1+|\cdot|_K^{2/3})} \lessapprox 1.$$

We then apply this result and Lemmas 2.7 and 3.3 to Theorem 4.3, which completes the proof. □

Proof of Theorem 1.2. First, we see that Theorem 7.2 verifies the conditions of Lemma 6.2 for $a = 1$. This in turn verifies the conditions of Proposition 6.1 for $C = 2^{-j}$, $N = 4$ and $w = 1 + \rho$ and so proves that

$$\|m_\delta(L)\|^2_{L^2(1+\rho) \to L^2(1+\rho)} \lessapprox 1.$$

Then, by Theorem 5.2, we have the same result for $M_\delta^\bullet$. That is,

$$\|M_\delta^\bullet\|^2_{L^2(1+\rho)\to L^2(1+\rho)} \lessapprox 1.$$

We then apply this result and Lemmas 2.7 and 3.3 to Theorem 4.5, which completes the proof. □

CHAPTER 9

JACOBI POLYNOMIALS

As observed in Chapter 7, when calculating integral kernels for second layer or 'full' weights on H-type groups, we consistently encounter an integral over the positive half-line of a pair of Laguerre polynomials against an exponential and polynomial weight. With a pure first layer weight, a delta appeared which ensured that, after rescaling, the integrand was of the form $L_k^\gamma(u)L_l^\gamma(u)e^{-u}u^\gamma$. Laguerre polynomials with this weight are orthogonal; this integral is well-known and easy to calculate.

In the second layer case, this delta does not appear. As such, we are driven to investigate more general integrals of the form

$$\int_0^\infty L_l^\gamma(at)L_k^\gamma(bt)e^{-ct}t^\gamma dt \tag{9.0.1}$$

Lemma 9.1. *Let $a, b, c > 0$. Let $\gamma > -1$ and let $l \leqslant k \in \mathbb{N}_0$. Then, if $a + b = c$,*

$$\int_0^\infty L_l^\gamma(at)L_k^\gamma(bt)e^{-ct}t^\gamma dt = \frac{\Gamma(k+l+\gamma+1)b^l a^k}{l!k!c^{k+l+\gamma+1}} \tag{9.0.2}$$

Else,

$$\int_0^\infty L_l^\gamma(at)L_k^\gamma(bt)e^{-ct}t^\gamma dt = \tag{9.0.3}$$

$$\frac{\Gamma(k+\gamma+1)(c-b)^{k-l}(a+b-c)^l}{k!c^{k+\gamma+1}}P_l^{(k-l,\gamma)}\left(1-2\frac{(c-a)(c-b)}{c(c-a-b)}\right).$$

Proof. By entry (35) on page 175 of [2], (9.0.1) is precisely equal to

$$\frac{\Gamma(k+l+\gamma+1)(c-a)^l(c-b)^k}{l!k!c^{k+l+\gamma+1}}\ {}_2F_1\left[-l,-k;-l-k-\gamma;\frac{c(c-a-b)}{(c-a)(c-b)}\right] \quad (9.0.4)$$

where we recall that the hypergeometric functions ${}_2F_1[m,n;p;z]$ are defined by

$${}_2F_1[m,n;p;z] := \sum_{q\in\mathbb{N}_0} \frac{(m)_q(n)_q z^q}{(p)_q q!} \quad (9.0.5)$$

for $|z|<1$ and by analytic continuation elsewhere, and where $(x)_n := x(x+1)...(x+n-1)$.

First, note that if $a+b=c$ then $c-a-b=0$. Since ${}_2F_1[a,b;c;0]=1$ then (9.0.2) follows by recalling that $c-a=b$ and $c-b=a$.

Otherwise, first, we assume $l \leqslant k$. Note that while (9.0.4) is undefined if $c=a$ or $c=b$, both sides of (9.0.3) are, and in this case the result follows by a limiting argument. By identity 15.8.6 of [61] we have

$${}_2F_1\left[-l,-k;-l-k-\gamma;z\right] = \frac{(-k)_l}{(-l-k-\gamma)_l}(-z)^l\ {}_2F_1\left[-l,k+\gamma+1;1+k-l;\frac{1}{z}\right].$$

Note that

$$\frac{(-k)_l}{(-l-k-\gamma)_l} = \frac{k!\Gamma(k+\gamma+1)}{(k-l)!\Gamma(k+l+\gamma+1)}.$$

Hence, we have that (9.0.1) is equal to

$$\frac{(c-b)^{k-l}}{l!c^{k+\gamma+1}}\frac{\Gamma(k+\gamma+1)}{(k-l)!}(a+b-c)^l\ {}_2F_1\left[-l,k+\gamma+1;1+k-l;\frac{(c-a)(c-b)}{c(c-a-b)}\right]. \quad (9.0.6)$$

Now, by [25] we have that Jacobi polynomials are defined by

$$P_n^{(\alpha,\beta)}(x) := \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\ {}_2F_1\left[-n,n+\alpha+\beta+1,\alpha+1,\frac{1-x}{2}\right], \quad (9.0.7)$$

for $\alpha,\beta > -1$. Notice that if we take $n = l$, $\alpha = k - l$ and $\beta = \gamma$ then we have

$$
{}_2F_1\left[-l, k+\gamma+1; 1+k-l; \frac{(c-a)(c-b)}{c(c-a-b)}\right] = \\
\frac{\Gamma(l+1)\Gamma(k-l+1)}{\Gamma(k+1)} P_l^{(k-l,\gamma)}\left(1 - 2\frac{(c-a)(c-b)}{c(c-a-b)}\right).
$$

Hence, (9.0.1) is equal to

$$
\frac{\Gamma(k+\gamma+1)(c-b)^{k-l}(a+b-c)^l}{k!c^{k+\gamma+1}} P_l^{(k-l,\gamma)}\left(1 - 2\frac{(c-a)(c-b)}{c(c-a-b)}\right).
$$

□

It will therefore be required of us to know estimates for these polynomials. Such estimates have been studied in many contexts as, for instance, they are solutions to certain differential equations and they appear in the formula for Zonal harmonics. Here, we intend to prove some estimates for the Jacobi polynomials that appear in our formulae.

We first note some uniform, weighted bounds that are available in the literature. The first is Theorem 5.1 of [33].

Theorem 9.2. *For all $x \in [-1,1]$, for all $\beta \geqslant 0$ and $\alpha \geqslant \beta - \lfloor\beta\rfloor$ and for all $n \in \mathbb{N}_0$,*

$$
\left(\frac{1+x}{2}\right)^{\beta/2} |P_n^{(\alpha,\beta)}(x)| \leqslant \binom{n+\alpha}{n}. \tag{9.0.8}
$$

In particular, this estimate holds whenever $\alpha,\beta \in \mathbb{N}_0$.

The next may be found as equation (2) in [25]

Theorem 9.3. *For all $x \in [-1,1]$, for all $\alpha,\beta \geqslant 0$ and for all $n \in \mathbb{N}_0$,*

$$
\left(\frac{1-x}{2}\right)^{\alpha/2+1/4}\left(\frac{1+x}{2}\right)^{\beta/2+1/4} |P_n^{(\alpha,\beta)}(x)| \\
\lesssim (2n+\alpha+\beta+1)^{-1/4}\left(\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\right)^{1/2}. \tag{9.0.9}
$$

The last of our results from existing literature is Theorem 2 of [32].

Theorem 9.4. *For all $x \in [-1,1]$, for all $\alpha \geqslant \beta \geqslant \frac{1+\sqrt{2}}{4}$ and for all $n \in \mathbb{N}_0$ with $n \geqslant 6$,*

$$\begin{aligned}\left(\frac{1-x}{2}\right)^{\alpha/2+1/4}\left(\frac{1+x}{2}\right)^{\beta/2+1/4}&|P_n^{(\alpha,\beta)}(x)| \\ &< 3^{1/3}\alpha^{1/6}\left(1+\frac{\alpha}{n}\right)^{1/12}\left(\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\right)^{1/2}.\end{aligned} \tag{9.0.10}$$

Here, we intend to prove some more specialised estimates; better than the above estimates but only on a restricted range of indices α, β, n. First, a corollary to the previous two Theorems.

Corollary 9.5. *Let $\alpha, n \in \mathbb{N}_0$, $\beta \in \mathbb{N}$ and $c > 0$ with $\alpha \geqslant c(1+n)$. Then, for $x \in [-1,1]$, we have*

$$\left(\frac{1-x}{2}\right)^{\alpha/2+1/4}\left(\frac{1+x}{2}\right)^{\beta/2+1/4}|P_n^{(\alpha,\beta)}(x)| \lesssim_{\beta,c} (\alpha+1)^{-1/4}(n+1)^{-1/12}\left(\frac{n+1}{\alpha+1}\right)^{\beta/2}, \tag{9.0.11}$$

where the implicit constant depends on β, c but not on α, n.

Proof. First, if $\alpha \geqslant \beta \geqslant 1$ and $n \geqslant 6$, then this is an obvious corollary of Theorem 9.4. For $\alpha \geqslant \beta \geqslant 1$ and $0 \leqslant n \leqslant 5$, then $n+1 \simeq 1$ and (9.0.11) is a corollary of Theorem 9.3. If $\alpha < \beta$ then, for all $n \in \mathbb{N}_0$,

$$1 + n \leqslant \frac{\beta}{c} \simeq_{\beta,c} 1$$

and we again obtain our result from Theorem 9.3. □

The above estimates are essentially weighted L^∞ estimates for Jacobi polynomials in certain regions. We will also prove a more point-wise estimate that takes advantage of the nature of Jacobi polynomials; in particular, they have a ‘transition point’, away from which much better estimates may be obtained. We proceed similarly to Proposition 3.5 of [11] in order to prove such an estimate.

First, we fix some notation. In this chapter only, the letter u will be reserved for

$$u := u(\alpha, \beta, n) = n + \frac{\alpha + \beta + 1}{2}. \tag{9.0.12}$$

We also define

$$\tilde{\alpha} := \frac{\alpha}{u}, \quad \tilde{\beta} := \frac{\beta}{u}. \tag{9.0.13}$$

According to [18], the transition point for the weighted Jacobi polynomial $(x-1)^{\alpha/2}(x+1)^{\beta/2}P_n^{(\alpha,\beta)}(x)$ in the case where α is fixed and, for some small positive number ϵ we have $\epsilon \leqslant \tilde{\beta} < 2$, is given by

$$\frac{\tilde{\beta}^2}{2} - 1.$$

If instead we are interested in large α and fixed β, which we are, then using the well-known relation of Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x). \tag{9.0.14}$$

gives that the transition point would be found at

$$x_{tr} := 1 - \frac{\tilde{\alpha}^2}{2}. \tag{9.0.15}$$

We now state our more refined estimate for Jacobi polynomials.

Theorem 9.6. *Let $\alpha, \beta, n \in \mathbb{N}_0$ and $\epsilon \in (0, 2), c \geqslant 1$ with $1 \leqslant \alpha \leqslant c(1 + n)$. Then, for $x \in [-1 + \epsilon, 1]$ we have*

$$\left| \left(\frac{1-x}{2} \right)^{\alpha/2} P_n^{(\alpha,\beta)}(x) \right| \lesssim_{\beta,c,\epsilon} (u^2|x - x_{tr}| + \alpha^{4/3})^{-1/4}, \tag{9.0.16}$$

where the constant depends on β, c, ϵ but not on α, n.

For $1 - x \leqslant \frac{1-x_{tr}}{16}$ we have

$$\left| \left(\frac{1-x}{2} \right)^{\alpha/2} P_n^{(\alpha,\beta)}(x) \right| \lesssim_{\beta,c,\epsilon} 2^{-\alpha}, \tag{9.0.17}$$

where the constant depends on β, c, ϵ but not on α, n.

Proof. By (9.0.14), we have that

$$|P_n^{(\alpha,\beta)}(x)| = |P_n^{(\beta,\alpha)}(-x)|.$$

By assumption, the conditions of Section 3 of [18] are satisfied. Notice that when deriving (3.16) of [18], the tool used is Theorem 3 of [5]. This theorem is a modification of Theorem 1 of [5]; while Theorem 1 is designed for use with real variables, Theorem 3 is set up to allow for complex variables. As we are only interested in real variables, it will simplify matters to instead consider the results of [18] where Theorem 1 is applied instead of Theorem 3. It can be seen that the results are essentially identical, with only the error terms differing. Hence, the suitably modified version of equation (3.49) of [18], with $N = 0$, states that

$$|P_n^{(\beta,\alpha)}(y)| = \left| \frac{D_1^{(0)}(u,\tilde{\alpha})}{\zeta^{1/2}(1-y)^{\beta/2+1/4}(1+y)^{\alpha/2}} \left(\frac{\zeta-\tilde{\alpha}^2}{y-y_{tr}}\right)^{1/4} \left(2\zeta^{1/2}J_\alpha(u\zeta^{1/2}) + \epsilon_{1,1}(u,\tilde{\alpha},\zeta)\right)\right|$$

where we are now interested in $y = -x$, $y_{tr} = -x_{tr} = \frac{\tilde{\alpha}^2}{2} - 1$, $-1 < y \leqslant 1 - \epsilon$, where $D_1^{(0)}(u,\tilde{\alpha})$ satisfies

$$D_1^{(0)}(u,\tilde{\alpha}) = 2^{(\alpha+\beta)/2-1}\left(\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\right)^{1/2}(1+O(u^{-1})),$$

where J_α is the Bessel function of the first kind

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}, \tag{9.0.18}$$

where $\epsilon_{1,1}$ is an error term which we will bound soon and where ζ is the implicitly defined change of variables given by

$$G(\zeta) := \int_{\tilde{\alpha}^2}^{\zeta} \frac{(\tau-\tilde{\alpha}^2)^{1/2}}{2\tau} d\tau = \int_{y_{tr}}^{y} \frac{(t-y_{tr})^{1/2}}{(1-t)^{1/2}(1+t)} dt =: F(y), \quad (y_{tr} \leqslant y \leqslant 1-\epsilon), \tag{9.0.19}$$

$$G_1(\zeta) := \int_{\zeta}^{\tilde{\alpha}^2} \frac{(\tilde{\alpha}^2-\tau)^{1/2}}{2\tau} d\tau = \int_{y}^{y_{tr}} \frac{(y_{tr}-t)^{1/2}}{(1-t)^{1/2}(1+t)} dt =: F_1(y), \quad (-1 < y \leqslant y_{tr}). \tag{9.0.20}$$

From our choice of α, n we have

$$\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(n+1)} \simeq_{\beta,c} \frac{n^{\beta}}{(n+\alpha)^{\beta}} \simeq_{\beta} 1,$$

so it is clear that $D_1^{(0)}(u,\tilde{\alpha}) \simeq_{\beta,c} 2^{\alpha/2}$. Hence, since for $-1 < y \leqslant 1-\epsilon$ we have that $1-y \simeq_{\epsilon} 1$,

$$\left| \left(\frac{1+y}{2}\right)^{\alpha/2} P_n^{(\beta,\alpha)}(y) \right| \simeq_{\beta,c,\epsilon} \left| \left(\frac{\zeta-\tilde{\alpha}^2}{y-y_{tr}}\right)^{1/4} \left(J_{\alpha}(u\zeta^{1/2}) + \frac{\epsilon_{1,1}(u,\tilde{\alpha},\zeta)}{2\zeta^{1/2}} \right) \right|$$

As stated in the equation below equation (3.11) of [5], the error term satisfies

$$\epsilon_{1,1}(u,\tilde{\alpha},\zeta) = \zeta^{1/2} E_{\alpha}^{-1}(u\zeta^{1/2}) M_{\alpha}(u\zeta^{1/2}) O(u^{-1}) \tag{9.0.21}$$

where E_{α}, M_{α} are functions defined below equation (3.4) in [5] (see also [51], section 12.1.3). Furthermore, this is uniform provided ζ remains in a bounded interval. Now,

$$\begin{aligned} \int_{4}^{\max\{4,\zeta\}} \frac{(\tau-4)^{1/2}}{2\tau} d\tau &\leqslant \int_{\tilde{\alpha}^2}^{\zeta} \frac{(\tau-\tilde{\alpha}^2)^{1/2}}{2\tau} d\tau \\ &= G(\zeta) = F(y) = \int_{y_{tr}}^{y} \frac{(t-y_{tr})^{1/2}}{(1-t)^{1/2}(1+t)} dt \\ &\leqslant \int_{-1}^{1} \frac{1}{(1-t^2)^{1/2}} dt = \pi = \int_{4}^{\zeta_1} \frac{(\tau-1)^{1/2}}{2\tau} d\tau \end{aligned}$$

for some $\zeta_1 \in (4,\infty)$ independent of all parameters, so that $\zeta \in [0,\zeta_1]$. It is then clear from [51] that we have the pointwise estimate

$$|J_\alpha| \leqslant E_\alpha^{-1} M_\alpha \leqslant M_\alpha \tag{9.0.22}$$

and furthermore, by Lemma 2 in Appendix B of [5], the quantity

$$u^{1/2}|\zeta - \tilde{\alpha}^2|^{1/4} M_\alpha(u\zeta^{1/2})$$

is uniformly bounded. Thus, we have

$$\left| \left(\frac{1+y}{2}\right)^{\alpha/2} P_n^{(\beta,\alpha)}(y) \right| \lesssim_{\beta,c,\epsilon} u^{-1/2}|y - y_{tr}|^{-1/4}. \tag{9.0.23}$$

Now, define I to be the interval of the y satisfying

$$\frac{2}{3}(1 + y_{tr}) \leqslant 1 + y \leqslant \frac{3}{2}(1 + y_{tr}). \tag{9.0.24}$$

We first observe that, for $y \notin I$ we have

$$\begin{aligned} u^{-1/2}|y - y_{tr}|^{-1/4} &= u^{-1/2}|(1+y) - (1+y_{tr})|^{-1/4} \\ &= u^{-1/2}|1 + y_{tr}|^{-1/4} \left| \frac{1+y}{1+y_{tr}} - 1 \right|^{-1/4} \\ &\lesssim u^{-1/2}|1 + y_{tr}|^{-1/4} \quad \text{since we are outside of } I \\ &= \left(\frac{\alpha^2}{2}\right)^{-1/4} \\ &\simeq \alpha^{-1/2} \leqslant \alpha^{-1/3}. \end{aligned} \tag{9.0.25}$$

Hence, in this region, we are done, as the $u^2|y - y_{tr}|$ term already dominates.

Otherwise, in order to complete the proof, we first show that

$$\left(\frac{\zeta - \tilde{\alpha}^2}{y - y_{tr}}\right)^{1/4} \simeq 1 \tag{9.0.26}$$

for y close to y_{tr}. Note that the assumptions on α mean that y_{tr} is bounded away from 1. First, assume

$$(1 + y_{tr}) \leqslant 1 + y \leqslant \frac{3}{2}(1 + y_{tr})$$

so (9.0.19) is applicable. Now, recalling that $y_{tr} = \frac{\tilde{\alpha}^2}{2} - 1$ and that $\epsilon \leqslant 1 - y \leqslant 1 - t \leqslant 1 - y_{tr} \leqslant 2$,

$$F(y) \leqslant \frac{1}{1 + y_{tr}} \int_{y_{tr}}^{y} \frac{(t - y_{tr})^{1/2}}{(1 - t)^{1/2}} dt \leqslant \frac{2}{\tilde{\alpha}^2 \sqrt{\epsilon}} \int_{y_{tr}}^{y} (t - y_{tr})^{1/2} dt = \frac{4}{3\sqrt{\epsilon}} \frac{(y - y_{tr})^{3/2}}{\tilde{\alpha}^2}$$

and

$$F(y) \geqslant \frac{2}{3} \frac{1}{1 + y_{tr}} \int_{y_{tr}}^{y} \frac{(t - y_{tr})^{1/2}}{(1 - t)^{1/2}} dt \geqslant \frac{4}{3\sqrt{2}} \frac{1}{\tilde{\alpha}^2} \int_{y_{tr}}^{y} (t - y_{tr})^{1/2} dt = \frac{8}{9\sqrt{2}} \frac{(y - y_{tr})^{3/2}}{\tilde{\alpha}^2}.$$

Together, these two inequalities and the fact that under our current assumptions,

$$y - y_{tr} = (1 + y) - (1 + y_{tr}) \lesssim 1 + y_{tr} \lesssim \tilde{\alpha}^2 \tag{9.0.27}$$

prove that

$$F(y) \simeq_{\epsilon} \frac{(y - y_{tr})^{3/2}}{\tilde{\alpha}^2} \lesssim (y - y_{tr})^{1/2}. \tag{9.0.28}$$

Furthermore,

$$G(\zeta) \leqslant \frac{1}{2\tilde{\alpha}^2} \int_{\tilde{\alpha}^2}^{\zeta} (\tau - \tilde{\alpha}^2)^{1/2} d\tau = \frac{1}{3} \frac{(\zeta - \tilde{\alpha}^2)^{3/2}}{\tilde{\alpha}^2}.$$

Now, assume

$$\tilde{\alpha}^2 \leqslant \zeta \leqslant 2\tilde{\alpha}^2. \tag{9.0.29}$$

This allows us to deduce

$$G(\zeta) \geqslant \frac{1}{4\tilde{\alpha}^2} \int_{\tilde{\alpha}^2}^{\zeta} (\tau - \tilde{\alpha}^2)^{1/2} d\tau = \frac{1}{6} \frac{(\zeta - \tilde{\alpha}^2)^{3/2}}{\tilde{\alpha}^2}.$$

Together, this shows that

$$\frac{(\zeta-\tilde{\alpha}^2)^{3/2}}{\tilde{\alpha}^2} \simeq G(\zeta) = F(y) \simeq_\epsilon \frac{(\zeta-\tilde{\alpha}^2)^{3/2}}{\tilde{\alpha}^2}, \tag{9.0.30}$$

which proves (9.0.26) under this assumption on ζ. If instead

$$\zeta \geqslant 2\tilde{\alpha}^2,$$

then

$$G(\zeta) \geqslant \frac{1}{2\zeta}\int_{\tilde{\alpha}^2}^{\zeta}(\tau-\tilde{\alpha}^2)^{1/2}d\tau = \frac{1}{3}\frac{(\zeta-\tilde{\alpha}^2)^{3/2}}{\zeta} \gtrsim (\zeta-\tilde{\alpha}^2)^{1/2}.$$

Combining this with (9.0.28) proves the upper bound of (9.0.26) (that is, that the left-hand side is majorised by 1), since

$$\left(\frac{\zeta-\tilde{\alpha}^2}{y-y_{tr}}\right)^{1/4} \lesssim_\epsilon \frac{G(\zeta)}{F(y)} = 1.$$

This, combined with (9.0.27) gives that

$$\zeta-\tilde{\alpha}^2 \lesssim_\epsilon y-y_{tr} \lesssim \tilde{\alpha}^2$$

and so

$$\zeta \leqslant C\tilde{\alpha}^2 \tag{9.0.31}$$

for some $C > 1$. Repeating the above argument, with (9.0.29) replaced with (9.0.31) leads again to (9.0.26). Analogous calculations may be performed, this time with the assumption

$$\frac{2}{3}(1+y_{tr}) \leqslant 1+y \leqslant (1+y_{tr}),$$

which makes (9.0.20) applicable instead of (9.0.19).

Combined, both of these cases prove (9.0.26), so, in this region close to the transition point, we have

$$\left|\left(\frac{1+y}{2}\right)^{\alpha/2} P_n^{(\beta,\alpha)}(y)\right| \lesssim_{\beta,c,\epsilon} \left|J_\alpha(u\zeta^{1/2}) + \frac{\epsilon_{1,1}(u,\tilde{\alpha},\zeta)}{2\zeta^{1/2}}\right|.$$

Now, from (9.0.21) we have that

$$\frac{\epsilon_{1,1}(u,\tilde{\alpha},\zeta)}{2\zeta^{1/2}} \lesssim E_\alpha^{-1}(u\zeta^{1/2})M_\alpha(u\zeta^{1/2}).$$

Note that $E_\alpha(x)^{-1}M_\alpha(x) \simeq J_\alpha(x)$ for all $0 < x \leqslant X_\alpha$, where X_α is defined above (3.4) in [5] and Section 12.1.3 of [51] and, by Ex 1.1 (section 12.1.3) of the latter reference, satisfies

$$X_\alpha = \alpha + C\alpha^{1/3} + O(\alpha^{-1/3})$$

where $C > 0$ is a fixed constant which may be inferred from [51]. Thus, there exists some α_0 such that, for $\alpha \geqslant \alpha_0$ we have

$$X_\alpha = \alpha + C\alpha^{1/3} + O(\alpha^{-1/3}) \geqslant \alpha + \frac{C}{2}\alpha^{1/3}.$$

Now, if $u\zeta^{1/2} \leqslant \alpha + \frac{C}{2}\alpha^{1/3}$ then

$$\left|\left(\frac{1+y}{2}\right)^{\alpha/2} P_n^{(\beta,\alpha)}(y)\right| \lesssim_{\beta,c,\epsilon} |J_\alpha(u\zeta^{1/2})|. \tag{9.0.32}$$

For $\alpha, y > 0$ Bessel functions satisfy the uniform estimate (cf [34])

$$|J_\alpha(y)| \lesssim \alpha^{-1/3}, \tag{9.0.33}$$

so we are done. If not, then $u\zeta^{1/2} \geqslant \alpha + \frac{C}{2}\alpha^{1/3}$ and recall that we are in the region where

$$|\zeta - \tilde{\alpha}^2| \simeq |y - y_{tr}|$$

and

$$\zeta \simeq \tilde{\alpha}^2.$$

Hence,

$$|\zeta - \tilde{\alpha}^2| = |\zeta^{1/2} + \tilde{\alpha}||\zeta^{1/2} - \tilde{\alpha}| \simeq \tilde{\alpha}|\zeta^{1/2} - \tilde{\alpha}|$$

and

$$\alpha^{1/3} \lesssim |u\zeta^{1/2} - \alpha| \simeq \frac{u^2|y - y_{tr}|}{\alpha}$$

so that

$$\alpha^{-1/3} \gtrsim u^{-1/2}|y - y_{tr}|^{-1/4}.$$

Finally, for $1 \leqslant \alpha < \alpha_0$ we have that M_α is a continuous function on $(0, \infty)$ and from equations (1.23) and (1.24) on page 437 of [51] we have, for $\alpha \geqslant 1$, that $M_\alpha(x)$ is bounded as $x \to 0$ and $x \to \infty$. Hence, M_α is a bounded function. Since

$$\left| J_\alpha(u\zeta^{1/2}) + \frac{\epsilon_{1,1}(u, \tilde{\alpha}, \zeta)}{2\zeta^{1/2}} \right| \lesssim M_\alpha(u\zeta^{1/2})$$

and since we have only finitely many $\alpha < \alpha_0$, then we are done with (9.0.15).

In order to prove (9.0.17), first note that this is equivalent under our change of variables ($x \to y$) to proving

$$\left| \left(\frac{1 + y}{2} \right)^{\alpha/2} P_n^{(\beta,\alpha)}(y) \right| \lesssim_{\beta,c,\epsilon} 2^{-\alpha}, \tag{9.0.34}$$

for $1 + y \leqslant \frac{1+y_{tr}}{64}$. Now, from (9.0.20),

$$\int_{\zeta}^{\tilde{\alpha}^2} \frac{(\tilde{\alpha}^2 - \tau)^{1/2}}{2\tau} d\tau = \int_{y}^{y_{tr}} \frac{(y_{tr} - t)^{1/2}}{(1-t)^{1/2}(1+t)} dt$$
$$= \int_{1+y}^{1+y_{tr}} \frac{(1 + y_{tr} - t)^{1/2}}{(2-t)^{1/2} t} dt$$
$$= \int_{1+y}^{\tilde{\alpha}^2/2} \frac{(\tilde{\alpha}^2/2 - t)^{1/2}}{(2-t)^{1/2} t} dt$$
$$= \int_{2(1+y)}^{\tilde{\alpha}^2} \frac{(\tilde{\alpha}^2 - t)^{1/2}}{(4-t)^{1/2} t} dt \geqslant \int_{2(1+y)}^{\tilde{\alpha}^2} \frac{(\tilde{\alpha}^2 - t)^{1/2}}{2t} dt.$$

Since the integrand is non-negative, then this is only possible if

$$\zeta \leqslant 2(1+y).$$

Now, by assumption for (9.0.17),

$$\zeta \leqslant 2(1+y) \leqslant 2\frac{1+y_{tr}}{16} \leqslant \frac{\tilde{\alpha}^2}{16},$$

and hence

$$u\zeta^{1/2} \leqslant \frac{\alpha}{4}.$$

From (9.0.18) we can deduce, for all $\alpha \geqslant -\frac{1}{2}$ and for all $z \in \mathbb{R}$,

$$|J_\alpha(z)| \leqslant \frac{|z/2|^\alpha}{\Gamma(\alpha+1)} \sum_{m=0}^{\infty} \frac{(-x^2/4)^m}{m!} \leqslant \frac{|z/2|^\alpha}{\Gamma(\alpha+1)}.$$

Thus, by (9.0.32) and Stirling's formula,

$$
\begin{aligned}
\left| \left(\frac{1+y}{2} \right)^{\alpha/2} P_n^{(\beta,\alpha)}(y) \right| &\lesssim_{\beta,c,\epsilon} |J_\alpha(u\zeta^{1/2})| \\
&\leqslant \frac{|u\zeta^{1/2}/2|^\alpha}{\alpha!} \\
&\lesssim \frac{\alpha^\alpha 2^{-\alpha} 4^{-\alpha}}{\alpha^{1/2}\alpha^\alpha e^{-\alpha}} \\
&= \alpha^{-1/2} 2^{-\alpha} \left(\frac{e}{4} \right)^\alpha \lesssim 2^{-\alpha},
\end{aligned}
$$

which proves (9.0.17). □

Corollary 9.7. *Let $\alpha, \beta, n \in \mathbb{N}_0$ and $\epsilon > 0, c \geqslant 1$ with $1 \leqslant \alpha \leqslant c(1+n)$. Then, for $x \in [-1+\epsilon, 1]$, we have*

$$
\left| \left(\frac{1-x}{2} \right)^{\alpha/2} P_n^{(\alpha,\beta)}(x) \right| \lesssim_{\beta,c,\epsilon} \alpha^{-1/3}, \tag{9.0.35}
$$

where the constant depends on β, c, ϵ but not on α, n.

Proof. An obvious consequence of Theorem 9.6. □

APPENDIX A

RESULTS FROM FUNCTIONAL ANALYSIS

This section contains a number of estimates related to general Functional Analysis. These results have been proven in various literature and are stated here for the reader's convenience, and as such the proofs will generally be omitted in favour of references to the relevant sources.

Define $\mathscr{S}(\mathbb{R}^+)$ as the space of functions $m : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\|m\|^*_{k,k'} := \sup_{\lambda\in\mathbb{R}^+, j=0,\dots,k} (1+\lambda)^k |m^{(j)}(\lambda)| < \infty$$

for all $k, k' \in \mathbb{N}$. Note that this is a Fréchet space. Let $\mathscr{S}_{even}(\mathbb{R})$ denote the set of even Schwartz functions on $\mathbb{R}$ defined as a subspace of $\mathscr{S}(\mathbb{R})$ with its usual Fréchet structure.

Lemma A.1. *Define the operator* $T : \mathscr{S}_{even}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}^+)$ *by*

$$Tf(x) = f(\sqrt{x}).$$

Then T is a well-defined continuous operator.

Proof. Let $f \in \mathscr{S}_{even}(\mathbb{R})$. By [60], $Tf \in C^\infty(\mathbb{R}^+)$. Clearly if $0 \leqslant x \leqslant 1$ then for all $a, b \in \mathbb{N}_0$ we have

$$\sup_{0\leqslant x\leqslant 1} |x^a (Tf)^{(b)}(x)| \leqslant \sup_{0\leqslant x\leqslant 1} |(Tf)^{(b)}(x)| < \infty.$$

Also, note that differentiating $f(\sqrt{x})$ gives a sum of terms $x^m f^{(n)}(\sqrt{x})$ where $m < 0$.

Hence, we have

$$\sup_{x>1} |x^a (Tf)^{(b)}(x)| \lesssim \sum_{k=1}^{b} \sup_{x>1} |\sqrt{x}^{2a} f^{(k)}(\sqrt{x})| < \infty.$$

Hence T is well-defined. Continuity follows from an application of the Closed-Graph Theorem (note that all the spaces we are considering here are Fréchet spaces). In particular, observe that a sequence of even functions $(f_n)_n$ will converge to an even function and that convergence in $\mathscr{S}_{even}(\mathbb{R})$ includes uniform and thus pointwise convergence, which will give pointwise convergence of the corresponding sequence $(Tf_n)_n$, implying that the graph of T is closed. □

A common tool used in harmonic analysis is to use the Fourier transform to convert a differential operator, or an operator defined via the functional calculus of such an operator, into a multiplier. The Spectral Theorem states this idea in a much more general context as follows.

Proposition A.2. *Let H be a separable Hilbert space and L be a self-adjoint operator on H. We may find a measure space (X, μ) and a unitary isometric isomorphism $V : H \to L^2(\mu)$ that intertwines L with a multiplication operator $\widehat{L}$, i.e. a real-valued multiplier operator with $VLV^{-1}f = \widehat{L}f$ for all $f \in L^2(\mu)$ (equivalently $VLg = \widehat{L}Vg$ for $g \in L^2(H)$). Additionally, the functional calculus is preserved, so that $Vm(L)V^{-1} = m(\widehat{L})$, where $m(\widehat{L})$ is again a multiplication operator.*

Proof. The existence of V is by Theorem 1.47 of [20] and Theorem 13.30 in [53]. Preservation of the functional calculus is by Theorem 1.51 of [20]. □

Finally, we state here Schur's Test, since this is the underlying idea behind much of the work in Chapter 7.

Lemma A.3. *(Schur's Test) Let X, Y be measure spaces and let T be an integral opera-*

tor with Schwartz kernel $K(x,y)$, with $x \in X$ and $y \in Y$. Then

$$\|T\|^2_{L^2 \to L^2} \leqslant \left(\sup_{x \in X} \int_Y |K(x,y)| dy \right) \left(\sup_{y \in Y} \int_X |K(x,y)| dx \right).$$

Proof. May be found e.g. as Theorem 5.2 in [27]. □

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