SIMPLE LIE ALGEBRAS AND THEIR CLASSIFICATION

by

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Abstract

This thesis is about the classification of simple Lie algebras. We start with some basic structure theory given in Chapter 2. Thereafter in Chapters 3 to 5 we give the complete classification over a field of characteristic zero. In Chapter 6 we discuss the notion of Chevalley basis which is important for the definition of classical Lie algebras over a field of positive characteristic. We end up in Chapter 7 with providing the basic notions about the positive characteristic case as well as stating the latest results on this matter.

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Chapter 1

Introduction

Studying continuous transformation groups in the end of nineteenth century, Sophus Lie discovered new algebraic structures now known as Lie algebras. They were not only interesting on their own right but also played an important role in twentieth century mathematical physics. Furthermore, mathematicians discovered that every Lie algebra could be associated to a continuous group, or a Lie group, which in turn considerably expanded the theory. Today, more than a century after Lie's discovery, we have a vast algebraic theory studying objects like Lie algebras, Lie groups, Root systems, Weyl groups, Linear algebraic groups, etc. It is called Lie theory and the intensive current research indicates its importance in modern mathematics.

In this thesis we discuss the classification of simple Lie algebras. It depends on the characteristic of the field and the complete classification for arbitrary characteristic is yet unknown. While the characteristic zero case was completely resolved many years ago, there are still open questions about the classification in positive characteristic. More precisely, the characteristics 2 and 3 seem to be very difficult and not much is known besides some examples of simple Lie algebras. Despite the difficulties, however, the classification of simple Lie algebras over fields of characteristic strictly greater than 3 has been recently completed. The aim of this thesis is to introduce the reader to the classification of simple

Lie algebras done so far. We give in full the classification in characteristic zero and outline the basics, required to state the classification theorems for a positive characteristic known so far.

For the sake of coherence, we start from the very beginning and first briefly discuss the basic structure theory of Lie algebras. This will hopefully enable any reader with basic algebraic background to follow the text. We particularly lay stress on the root space decomposition and root systems since they are the tools needed for the classification of simple Lie algebras in characteristic zero. In addition, we describe root systems of the classical Lie algebras A_l, B_l, C_l and D_l; construct the exceptional ones and show that the corresponding Dynkin diagrams are connected in each case, i.e. they are all simple. Using this we state and fully proof Theorem 5.2.1.

In the end we discuss the Chevalley basis and the Chevalley algebras. Using them we define classical Lie algebras when $p > 0$. Furthermore, we briefly describe a new family of simple Lie algebras arising in positive characteristic. They are called Lie algebras of Cartan type. Having defined these we state the classification theorem for $p > 5$. We end up by stating one further result aiming to introduce the reader to one of the latest results in the field.

Chapter 2

Basic Structure Theory

We start with basic definitions and a brief discussion on the structure of Lie algebras. Although most of the results in this chapter hold for arbitrary characteristic we assume $\operatorname{char}\mathbb{F} = 0$.

2.1 First Definitions

An *algebra* over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ together with a bilinear map

$$\begin{aligned} A \times A &\longrightarrow A \\ (x, y) &\longmapsto xy \end{aligned}$$

usually called the product of x and y. The algebra A is said to be *associative* if $(xy)z = x(yz)$ for all $x, y, z \in A$ and *unital* if there is an element 1_A in A such that $1_A x = x = x 1_A$ for all elements in A. An example for a unital associative algebra is the vector space of all linear transformations (endomorphisms) of the vector space V. We denote this algebra by $\operatorname{End}(V)$. The product in $\operatorname{End}(V)$ is given by the composition of maps and the identity transformation is the identity element in $\operatorname{End}(V)$. Similarly, we can consider $\mathrm{M}(n, \mathbb{F})$, the set of $n \times n$ matrices over $\mathbb{F}$. Clearly, it is a unital associative algebra and obviously the multiplication in $\mathrm{M}(n, \mathbb{F})$ is the standard matrix multiplication with identity element the identity matrix.

An algebra L over a field $\mathbb{F}$ with bilinear operation denoted by $(x, y) \longmapsto [xy]$ and called the *bracket* or *commutator* of x and y, is called a *Lie algebra* over $\mathbb{F}$ if the following axioms are satisfied:

(i) $[xx] = 0$ for all $x \in L$

(ii) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for all $x, y, z \in L$

Axiom (ii) is called the *Jacobi identity*. There is an alternative axiom to our first axiom. For $x, y \in L$ by dint of first axiom we have $0 = [x+y, x+y] = [xx]+[xy]+[yx]+[yy] = [xy] + [yx]$ which implies the anticommutativity property $[xy] = -[yx]$. Conversely, let $[xy] = -[yx]$. Then $[xy]+[yx] = 0$ is equivalent to $[x, y+x]-[xx]+[x+y, x]-[xx] = 0$. Now by anticommutativity we have that $[x, y+x]+[x+y, x] = 0$ and therefore $-[xx]-[xx] = 0$. The last relation implies that $[xx] = 0$ for all $x \in L$ if and only if $\operatorname{char} \mathbb{F} \neq 2$. Thus we could use anticommutativity instead our first axiom provided the characteristic of $\mathbb{F}$ is not 2. We also naturally define a *subalgebra* of L as a subspace K of L with $[xy] \in K$ whenever $x, y \in K$.

Consider now $\mathrm{End}(V)$ which is a unital associative algebra. Define the bracket of x and y by $[xy] = xy - yx$. With this operation $\mathrm{End}(V)$ becomes a Lie algebra over $\mathbb{F}$. Indeed, $[x+y, z] = (x+y)z - z(x+y) = xz - zx + yz - zy = [xz] + [yz]$ and similarly $[x, y+z] = [xy] + [xz]$. Also, $[\alpha x, \beta y] = \alpha x \beta y - \beta y \alpha x = \alpha\beta[xy]$ and thus the bracket operation on $\mathrm{End}(V)$ is bilinear. Clearly, $[xx] = xx - xx = 0$ for all $x \in L$. Finally, we have $[x[yz]] + [y[zx]] + [z[xy]] = [x(yz - zy)] + [y(zx - xz)] + [z(xy - yx)]$. We also easily get that $[x(yz - zy)] = xyz - xzy - yzx + zyx$, $[y(zx - xz)] = yzx - yxz - zxy + xzy$ and $[z(xy - yx)] = zxy - zyx - xyz + yxz$. We now easily see that these verify that Jacobi identity holds. So $\mathrm{End}\,(V)$ is a Lie algebra. In order to distinguish the new algebra structure from the old associative one we write $\mathfrak{gl}(V)$ for $\mathrm{End}\,(V)$ viewed as a Lie algebra and call it *general linear algebra*. We also use $\mathfrak{gl}(n, \mathbb{F})$. Any subalgebra of $\mathfrak{gl}(V)$ is called a *linear Lie algebra*.

Let now L be Lie algebra. Then by a *derivation* of L we mean a linear map $\delta : L \longrightarrow L$ satisfying the product rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in L$. We immediately see that the collection $\operatorname{Der} L$ of all derivations of L is a vector subspace of $\operatorname{End} L$. Moreover, if $\delta, \delta' \in \operatorname{Der} L$ then we have the following $[\delta, \delta'](ab) = (\delta\delta' - \delta'\delta)(ab) = \delta\delta'(ab) - \delta'\delta(ab) = \delta(\delta'(a)b + a\delta'(b)) - \delta'(\delta(a)b + a\delta(b)) = \delta\delta'(a)b + a\delta\delta'(b) - \delta'\delta(a)b - a\delta'\delta(b)$. On the other hand we have that $[\delta, \delta'](a)b + a[\delta, \delta'](b) = \delta\delta'(a)b - \delta'\delta(a)b + a\delta\delta'(b) - a\delta'\delta(b)$. Therefore $[\delta, \delta'] \in \operatorname{Der} L$. So $\operatorname{Der} L$ is a subalgebra of $\mathfrak{gl}(L)$. Now if $x \in L$, then the map $y \longmapsto [xy]$ is clearly an endomorphism of L. We denote it $\operatorname{ad} x$. If we now rewrite the Jacobi identity in the following way $[x[yz]] = [[xy]z] + [y[xz]]$ we easily see that $\operatorname{ad} x \in \operatorname{Der} L$. Derivations of this form are called *inner* and all others *outer*. The map $L \longrightarrow \operatorname{Der} L$ sending x to $\operatorname{ad} x$ is called the *adjoint representation* of L and plays a vital role in Lie theory.

2.2 Classical Lie Algebras

As an important examples of linear Lie algebras we now consider so called *classical* Lie algebras. These are the four families A_l, B_l, C_l and D_l, which are described in this section. They all are clearly subalgebras of $\mathfrak{gl}(n, \mathbb{F})$ and hence are linear. In all cases below, we consider the endomorphisms of a vector space V with particular properties.

A_l: Let $\dim V = l + 1$. Then by $\mathfrak{sl}(V)$, or $\mathfrak{sl}(l + 1, \mathbb{F})$, we denote the set of all endomorphisms of V having trace zero. Recalling $Tr(x+y) = Tr(x)+Tr(y)$ and $Tr(xy) = Tr(yx)$ we immediately see that $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$. We call $\mathfrak{sl}(V)$ the *special linear algebra*. The basis for $\mathfrak{sl}(n, \mathbb{F})$ is e_{ij} $(i \neq j)$ along with $h_i = e_{ii} + e_{i+1,i+1}$. Here by e_{ij} we understand the $(l + 1) \times (l + 1)$ matrix with one at i, j position and zeros elsewhere. Clearly, the number of e_{ij}'s is $(l + 1)^2 - (l + 1)$ and the number of h_i's is l and thus the dimension of $\mathfrak{sl}(V)$ is $(l + 1)^2 - 1 = l(l + 2)$.

For example, the basis of $\mathfrak{sl}(2,\mathbb{F})$ is given by the matrices:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

C_l: Let $\dim V = 2l$ with basis $v_1, .., v_{2l}$. Define a nondegenerate skew-symmetric form f on V by the matrix

$$s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$$

We define the *symplectic Lie algebra* to be the set of all endomorphisms x of V satisfying $f(xv, w) = -f(v, xw)$. We denote it by $\mathfrak{sp}(V)$, or $\mathfrak{sp}(2l, \mathbb{F})$. For $x, y \in L$ we have

$$f([xy]v, w) = f(x\,y\,v - y\,x\,v, w) = f(x\,y\,v, w) - f(y\,x\,v, w) = -f(y\,v, x\,w) + \\ + f(x\,v, y\,w) = f(v, y\,x\,w) - f(v, x\,y\,w) = f(v, [yx]w) = -f(v, [xy]w).$$

Hence $\mathfrak{sp}(V)$ is closed under the bracket operation.

In matrix terms the condition for

$$x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

to be symplectic is that $sx = -x^t s$, i.e., that $n^t = n$, $p^t = p$ and $m^t = -q$. Here x^t is the transpose of x. Notice also that $m, n, p, q \in \mathfrak{gl}(l, \mathbb{F})$. The basis for $\mathfrak{sp}(V)$ is given as follows. Take the diagonal matrices $e_{ii} - e_{l+i,l+i}$ $(1 \le i \le l)$, l in total. Add to them all $e_{ij} - e_{l+j,l+i}$ $(1 \le i \ne j \le l)$, $l^2 - l$ in number. For n we use the matrices $e_{i,l+i}$ $(1 \le i \le l)$

and $e_{i,l+j} + e_{j,l+i}$ $(1 \leq i < j \leq l)$, a total of $l + 1/2l(l-1)$, and similarly for p. Adding up, we find that $\dim \mathfrak{sp}(2l, \mathbb{F}) = 2l^2 + l$.

B_l: Let $\dim V = 2l+1$ and take f to be a nondegenerate symmetric bilinear form on V whose matrix is

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$$

Then the *orthogonal algebra* $\mathfrak{o}(V)$, or $\mathfrak{o}(2l+1, \mathbb{F})$, consists of all endomorphisms of V satisfying the same condition as for C_l, i.e. $f(xv, w) = -f(v, xw)$. For a basis in this case we first take l diagonal matrices $e_{ii} - e_{l+i,l+i}$ $(2 \leq i \leq l+1)$. Add the $2l$ matrices involving only row one and column one: $e_{1,l+i+1} - e_{i+1,1}$ and $e_{1,i+1} - e_{l+i+1,1}$ $(1 \leq i \leq l)$. Corresponding to $q = -m^t$, take $e_{i+1,j+1} - e_{l+j+1,l+i+1}$ $(1 \leq i \neq j \leq l)$. For n take $e_{i+1,l+j+1} - e_{j+1,l+i+1}$ $(1 \leq i < j \leq l)$, and for p, $e_{i+l+1,j+1} - e_{j+l+1,i+1}$ $(1 \leqslant j < i \leqslant l)$. The total number of basis elements is $2l^2 + l$, i.e. the same dimension as that of C_l.

D_l: This last linear algebra is constructed similarly to B_l except that $\dim V = 2l$. It is also an orthogonal algebra and s has the form

$$s = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$$

Now $\dim \mathfrak{o}(2l, \mathbb{F}) = 2l^2 - l$.

We also should mention several important subalgebras of $\mathfrak{gl}(n, \mathbb{F})$ which are often useful. Let $\mathfrak{t}(n, \mathbb{F})$ be the set of *upper triangular matrices*, i.e $(a_{ij}) = 0$ if $i > j$. Let $\mathfrak{n}(n, \mathbb{F})$ be the *strictly upper triangular matrices* $a_{ij} = 0$ if $j \geq j$. Finally, let $\mathfrak{d}(n, \mathbb{F})$ be the set of all *diagonal matrices*. It is trivial to show that each of these is closed under the bracket and so they are subalgebras of $\mathfrak{gl}(n, \mathbb{F})$.

2.3 Ideals, Homomorphisms and Representations

One way to analyze the structure of a Lie algebra is to look at its ideals. A subspace I of a Lie algebra L is called an *ideal* of L if $[xy] \in I$ for all $x \in I$ and $y \in L$. Trivial examples for ideals are the zero subspace of L and L itself. A non-trivial and important example is the *centre* of L defined by $Z(L) = \{z \in L \mid [xz] = 0 \text{ for all } x \in L\}$. Using the Jacobi identity we easily see that $Z(L)$ is an ideal of L. Furthermore, a Lie algebra L is called *abelian* if and only if $Z(L) = L$.

Another very important example is the *derived subalgebra* of L. It consists of all linear combinations of the commutators $[xy]$ and we denote it by $[LL] = L'$. It is clearly an ideal. In fact, the derived subalgebra is analogous to the commutator subgroup of a group.

We also mention two useful facts which easily follow from the bilinearity of the bracket and the Jacobi identity. First of all, the sum of any two ideals I, J is an ideal of L which we denote $I + J = \{x + y \mid x \in I, y \in J\}$. Second of all, $[IJ] = \{\sum_i [x_i y_i] \mid x_i \in I, y_i \in J\}$ is an ideal of L and clearly the derived subalgebra is a special case of this construction.

We now define the main object of this thesis. If a Lie algebra L has no ideals except 0 and itself and is *non-abelian* we call it *simple*. Clearly, simple is equivalent to $Z(L) = 0$ and $L = [LL]$.

Example 2.3.1 An example for a simple Lie algebra is $L = \mathfrak{sl}(2, \mathbb{F})$ for $\operatorname{char} \mathbb{F} \neq 2$. To see this, take the standard basis for $\mathfrak{sl}(2, \mathbb{F})$ given in former section. A straightforward computations show that the multiplication table of L is completely determined by $[xy] = h$, $[hx] = 2x$ and $[hy] = -2y$. Moreover, these relations say that x, y, h are eigenvectors for $\operatorname{ad} h$ corresponding to the eigenvalues $2, -2, 0$ respectively. Since the characteristic is not 2 these eigenvalues are distinct. Now suppose that $I \neq 0$ is an ideal of L and let $z = ax + by + ch$ be an arbitrary nonzero element of I. Now, $\operatorname{ad} x(ax + by + ch) = b \operatorname{ad} x(y) + c \operatorname{ad} x(h) = bh - 2cx$. Further, $\operatorname{ad} x(bh + 2cx) = -2bx$. Thus applying $\operatorname{ad} x$ twice to z we have $-2bx \in I$. Similarly, applying $\operatorname{ad} y$ twice we get $-2ay \in I$. Therefore, if a or b are nonzero, I contains either y or x. So now if $x \in I$ then $[xy] \in I$ and hence h also lies in I. But $[hy]$ lies in our ideal as well, and thus $I = L$. Similarly, if $a \neq 0$ then we start with $y \in I$ and by the same argument we have $I = L$. Finally, if $a = b = 0$,then $0 \neq ch \in I$, so $h \in I$ and again by our multiplication relations it follows that $I = L$. We now conclude that L is simple.

Every linear transformation $\phi : L \longrightarrow L'$ which preserves the bracket is called a Lie algebras *homomorphism*. Preserving bracket means that $\phi([xy]) = [\phi(x)\phi(y)]$ for all $x, y \in L$. A bijective homomorphism is called an *isomorphism*. A *representation* of a Lie algebra L is a homomorphism $\phi : L \longrightarrow \mathfrak{gl}(V)$. We have already discussed the adjoint representation given by the map $\operatorname{ad} : L \longrightarrow \operatorname{Der} L \subset \mathfrak{gl}(L)$. We know it is a liner transformation and want to show that it preserves the bracket. We see this using the anticommutativity and Jacobi identity, i.e. $[\operatorname{ad} x, \operatorname{ad} y](z) = \operatorname{ad} x \operatorname{ad} y(z) - \operatorname{ad} y \operatorname{ad} x(z) = \operatorname{ad} x([yz]) - \operatorname{ad} y([xz]) = [x[yz]] - [y[xz]] = [x[yz]] + [[xz]y] = [[xy]z] = \operatorname{ad}[xy](z)$. The natural question arising here is about the kernel of ad . Clearly, $\operatorname{Ker}(\operatorname{ad}) = \{x \in L \mid \operatorname{ad} x = 0\}$ which is nothing but $[xy] = 0$ for all $y \in L$. Thus, $\operatorname{Ker}(\operatorname{ad}) = Z(L)$. So if L is simple then $Z(L) = 0$ and $\operatorname{ad} : L \longrightarrow \operatorname{Der} L \subset \mathfrak{gl}(L)$ is a monomorphism. This means that *any simple Lie algebra is isomorphic to a linear Lie algebra.* Yet more evidence for the importance of linear Lie algebras.

2.4 Solvability and Nilpotency

Both *solvability* and *nilpotency* play a decisive role in understanding structure and properties of Lie algebras. First, we define a sequence of ideals of L by $L^{(0)} = L, L^{(1)} = [LL], L^{(2)} = [L^{(1)}L^{(1)}] ... L^{(k)} = [L^{(k-1)}L^{(k-1)}]$. This sequence is called the *derived series* of L. We call L *solvable* if $L^{(n)} = 0$ for some n. For example, the abelian algebras are clearly solvable whereas simple algebras are definitely nonsolvable. Indeed, by definition the simple Lie algebra L satisfies $[LL] = L$ and thus $L^{(n)} = L$ for every positive integer n, whence L cannot be solvable.

The first crucial application of solvability leads to the definition of an important class of Lie algebras. Let L be an arbitrary Lie algebra and let S be a maximal solvable ideal denoted $\mathrm{Rad}\, L$. It is unique (for a proof see [5]) and is called the *radical* of L. If $L \neq 0$ and $\mathrm{Rad}\, L = 0$, then L is called *semisimple*. For instance, a simple Lie algebra is semisimple since L has no ideals except itself and 0 and L is nonsolvable.

We also define another sequence of ideals of L by $L^0 = L, L^1 = [LL], L^2 = [LL^1], ..., L^k = [LL^{k-1}]$ and call it the *descending central series* or *lower central series*. L is called *nilpotent* if $L^n = 0$ for some n. A trivial example for a nilpotent Lie algebra is any abelian algebra as $[LL] = 0$. We also have that $L^{(i)} \subset L^i$ for all i, so every nilpotent Lie algebra is automatically solvable.

2.5 Simple Ideals and Inner Derivations

We need to say a bit more about ideals and inner derivations. We state two important results in this section. For proofs see [5].

A Lie algebra L is said to be the *direct sum* of ideals $I_1, ..., I_t$ if $L = I_1 \oplus \cdots \oplus I_t$ is a direct sum in the sense of vector spaces. (Indeed ideals of L are also a subspaces of L). We automatically have then that $[I_i I_j] \subset I_i \cap I_j = 0$ for all $i \neq j$. We then have the following

Theorem 2.5.1 *Let L be semisimple. Then there exist ideals $L_1, ..., L_t$ of L which are simple (as Lie algebras), such that $L = L_1 \oplus \cdots \oplus L_t$. Every simple ideal of L coincides with one of the L_i.*

We stated this theorem as it says that simple Lie algebras are building blocks of semisimple Lie algebras. This means that the complete classification of simple Lie algebras will naturally provide the classification of semisimple Lie algebras.

Regarding the inner derivations, for any $x, y \in L$ and $\delta \in \mathrm{Der} L$ we have that $[\delta, \mathrm{ad}\, x](y) = \delta(\mathrm{ad}\, x)(y) - \mathrm{ad}\, x(\delta(y)) = \delta([x, y]) - [x, \delta y] = [\delta x, y] = \mathrm{ad}\,(\delta x)$, i.e. $[\delta, \mathrm{ad}\, x] = \mathrm{ad}\,(\delta x)$. This along with the fact that adL is an ideal in $\mathrm{Der} L$ for any Lie algebra L yield the following theorem.

Theorem 2.5.2 *If L is semisimple, then* $\mathrm{ad}\, L = \mathrm{Der}\, L$*.*

2.6 Killing Form and Cartan's Criteria

The *Killing form* on a Lie algebra L is the symmetric bilinear form defined by $\kappa(x, y) = tr(\mathrm{ad}\, x\, \mathrm{ad}\, y)$ for $x, y \in L$. In addition, the Killing form is also associative in the sense of $\kappa([xy], z) = \kappa(x, [yz])$. This follows straightforwardly from the fact that for any endomorphisms a, b, c we have $tr([a, b], c) = tr(a, [b, c])$ and that $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a homomorphism.

We now state two criteria which are important for the structure theory. Due to lack of space, however, we give them without proof. For their proofs see [4],[5]. First of them is the solvability criterion, or

Theorem 2.6.1 *(Cartan's First Criterion)*
L is solvable if and only if $\kappa(x, y) = 0$ for all $x \in L$ and $y \in [LL]$.

The second one is the so called *semisimplicity criterion*

Theorem 2.6.2 *(Cartan's second criterion)*
Let L be a nonzero Lie algebra. Then L is semisimple if and only if its Killing form is nondegenerate.

2.7 Jordan-Chevalley Decomposition

We now introduce one very useful tool for dealing with linear transformations. We know from Linear Algebra that the Jordan normal form for an endomorphism x is its block diagonal matrix expression, such that every diagonal block is of the form:

$$\begin{pmatrix} a & 1 & 0 & 0 & ... & 0 \\ 0 & a & 1 & 0 & ... & 0 \\ 0 & 0 & a & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & a & 1 \\ 0 & 0 & 0 & ... & 0 & a \end{pmatrix}$$

It is easy to see that the matrix above can be split into sum of the $\mathrm{diag}(a, ..., a)$ and the matrix having one's just above the diagonal and zeros elsewhere. The first thing we observe is that raising the latter to n-th power gives the zero matrix, and hence it is nilpotent. We also have that $\mathrm{diag}(a, ..., a)$ commutes with our nilpotent matrix. Thus our endomorphism x is represented as a sum of a diagonal and a nilpotent matrices which commute.

Let V be finite dimensional vector space. We then call $x \in \mathrm{End}\, V$ *semisimple* if and only if it is diagonalizable. An important fact is that two commuting semisimple endomorphisms can be simultaneously diagonalized and hence their sum (difference) is semisimple as well. We can now decompose every endomorphism x as follows $x = x_s + x_n$. Here x_s and x_n are called the *semisimple part* and the *nilpotent part* of x respectively. We call this the *Jordan-Chevalley decomposition.*

Uniqueness of Jordan-Chevalley decomposition is guaranteed by the following.

Proposition 2.7.1 *Let V be a finite dimensional vector space over F, $x \in \operatorname{End} V$. Then there exists unique $x_s, x_n \in \operatorname{End} V$ satisfying the conditions $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent, x_s and x_n commute.*

A further useful fact is the following.

Lemma 2.7.2 *let $\mathfrak{U}$ be a finite dimensional F-algebra. Then* $\operatorname{Der}\mathfrak{U}$ *contains the semisimple and the nilpotent parts (in* $\operatorname{End}\mathfrak{U}$*) of all its elements.*

The reader may refer to [5] for the proofs.

We want to generalize the notion of decomposing an endomorphism to a semisimple and a nilpotent part. In other words, we introduce the *abstract* Jordan decomposition. Consider an arbitrary semisimple Lie Algebra L. By Lemma 2.7.2 we have that $\operatorname{Der} L$ contains the semisimple and nilpotent parts in $\operatorname{End} L$ of all its elements. Clearly, the map $L \longrightarrow \operatorname{ad} L$ is injective. We also have by Theorem 2.5.2 that $\operatorname{Der} L$ coincides with $\operatorname{ad} L$ and thus each $x \in L$ determines unique elements $s, n \in L$ such that $\operatorname{ad} x = \operatorname{ad} s + \operatorname{ad} n$ is nothing but the usual Jordan decomposition of $\operatorname{ad} x$ (in $\operatorname{End} L$). This means that $x = s + n$ with $[sn] = 0$, s is *ad-semisimple* (i.e., $\operatorname{ad} s$ is semisimple) and n is *ad-nilpotent.*

2.8 Maximal Toral Subalgebras and Cartan Decomposition

As we saw in previous section, Jordan-Chevalley decomposition implies that there are subalgebras of L consisting of semisimple elements. We call such subalgebras *toral.* Also, we call the maximal toral subalgebra of L a *Cartan subalgebra* of L and abbreviate it CSA. For the sake of precision, we should mention that in literature CSA of a Lie algebra L is defined to be a nilpotent subalgebra which equals it's normalizer in L. In zero characteristic, however, CSA's of L are exactly the maximal toral subalgebras, provided L semisimple. This fact is sufficient for us, so this is why we adopt it as a definition.

For details see [5]. The very first property of toral subalgebras is given by the following lemma. For a proof see [5].

Lemma 2.8.1 *A toral subalgebra of L is abelian.*

Now fix a *maximal toral subalgebra* H of L. Since H is abelian, $\mathrm{ad}_L H$ is a commuting family of semisimple endomorphisms of L. A standard result from linear algebra implies that $\mathrm{ad}_L H$ is *simultaneously diagonalizable* and therefore L is a direct sum of the subspaces $L_\alpha = \{x \in L \mid [hx] = \alpha(h)x \; for \; all \; h \in H\}$. Here α ranges over the dual space H^* of H. The set of all nonzero $\alpha \in H^*$ for which $L_\alpha \neq 0$ is denoted by Φ. The elements of Φ play central role and are called *roots* of L relative to H. We notice that they are finite in number. When $\alpha = 0$ then L_0 is simply the centralizer of H. Clearly, H is contained in $C_L(H)$ by dint of the lemma above. Thus we can write $L = C_L(H) \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$. We call this a *root space decomposition* of L. In the literature it is also known as the *Cartan decomposition.* The first observation about the root space decomposition is the following

Proposition 2.8.2 *For all α , $\beta \in H^*$, $[L_\alpha L_\beta] \subset L_{\alpha+\beta}$. If $x \in L_\alpha$, $\alpha \neq 0$, then* $\mathrm{ad}\, x$ *is nilpotent. If α,$\beta \in H^*$ and $\alpha + \beta \neq 0$, then L_α is orthogonal to L_β, relative to the Killing form κ of L.*

Proof. Taking $x \in L_\alpha$, $y \in L_\beta$ and using Jacobi identity we get $\mathrm{ad}\, h([xy]) = [[hx]y] + [x[hy]] = \alpha(h)[xy] + \beta(h)[xy] = (\alpha+\beta)(h)[xy]$. This means that $[xy] \neq 0$ is an eigenvector for $\mathrm{ad}\, h \in H$ with eigenvalue $\alpha(h) + \beta(h)$ and therefore $[L_\alpha L_\beta] \subset L_{\alpha+\beta}$. Now the second assertion follows from this we have just proved. Finally, since $\alpha + \beta \neq 0$ we certainly have $h \in H$ such that $(\alpha + \beta)(h) \neq 0$. Now taking $x \in L_\alpha$, $y \in L_\beta$ and using the associativity of the Killing form we get $\alpha(h)\kappa(x, y) = \kappa([hx], y) = -\kappa([xh], y) = -\kappa(x, [hy]) = -\beta(h)\kappa(x, y)$ which is the same as $(\alpha + \beta)(h)\kappa(x, y) = 0$ and hence $\kappa(x, y) = 0$. □

We immediately observe that the restriction of the Killing form of L to $L_0 = C_L(H)$ is nondegenerate. Indeed, on the one hand L is semisimple and by semisimplicity criterion the Killing form on L is nondegenerate. On the other hand, take an element $z \in L_0$ which is orthogonal to L_0. Then by proposition L_0 is orthogonal to all L_α and thus $\kappa(z, L) = 0$ forces $z = 0$.

Proposition 2.8.3 *Let H be a maximal toral subalgebra of L. Then $H = C_L(H)$.*

We refer [5] for a proof. The important thing is that this proposition along with its preceding argument lead to

Corollary 2.8.4 *The restriction of κ to H is nondegenerate.*

The importance of this corollary is in the following. Write H^* for the dual space of H. To $\phi \in H^*$ corresponds a unique element $t_\phi \in H$ satisfying $\phi(h) = \kappa(t_\phi, h)$ for all $h \in H$. This means that we identify H with H^* and in particular we have that Φ corresponds to the subset $\{t_\alpha \mid \alpha \in \Phi\}$ of H. Our goal is to clarify the fact that Φ characterizes L completely. This is why we discuss this matter in detail in the next chapter.

At the end of this section we state without proof one useful fact. In the literature it is known as the *Orthogonality properties.*

Proposition 2.8.5 *(Orthogonality properties)*

(a) Φ spans H^*.

(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

(c) Let $\alpha \in \Phi$, $x \in L_\alpha$, $y \in L_{-\alpha}$. Then $[xy] = \kappa(x, y)t_\alpha$.

(d) If $\alpha \in \Phi$, then $[L_\alpha L_{-\alpha}]$ is one dimensional with basis t_α.

(e) $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$, for $\alpha \in \Phi$.

(f) If $\alpha \in \Phi$ and x_α is any non-zero element of L_α, then there exists $y_\alpha \in L_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha y_\alpha]$ form a standard basis for a subalgebra of L isomorphic to $\mathfrak{sl}(2, \mathbb{F})$.

(g) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$; $h_\alpha = -h_{-\alpha}$.

It not only shows how can we obtain information about the Cartan Decomposition via calculating the Killing form, but is also important for the construction of the Chevalley basis (see Chapter 6). For a proof see [5].

Chapter 3

Root Systems

3.1 Reflections in Euclidean Space

Before properly defining root systems we need to say a couple of words about reflections in Euclidean space. Let E be an Euclidean space. Then a *reflection* in E is an invertible linear transformation which fixes pointwise some *hyperplane* and sends all orthogonal vectors to that hyperplane to their negatives. We recall that a hyperplane is a subspace of E of codimension one. Evidently reflections preserve the inner product on E. We call such transformations *orthogonal.* Any nonzero vector α determines a reflection σ_α with a *reflecting hyperplane* $P_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$. By bilinearity of the form β it follows that every nonzero vector which is proportional to α generates the same reflection. Finally, we can use the following explicit formula for a reflection along the vector α:

$$\sigma_\alpha(\beta) = \beta - \tfrac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

A straightforward calculations show that this linear transformation sends α to $-\alpha$ and fixes pointwise the points of P_α and so σ_α is a reflection. We also usually use the abbreviation $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$. Notice that $\langle \beta, \alpha \rangle$ is linear only in the first variable.

3.2 Root Systems

A subset Φ of the Euclidean space E is called a *root system* in E if the following axioms are satisfied:

(R1) Φ *is finite, spans* E*, and does not contain* 0.

(R2) *If* $\alpha \in \Phi$*, the only multiples of* α *in* Φ *are* $\pm\alpha$.

(R3) *If* $\alpha \in \Phi$*, the reflection* σ_α *leaves* Φ *invariant.*

(R4) *If* $\alpha, \beta \in \Phi$*, then* $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Since, both axioms (R1) and (R2) imply that $\Phi = -\Phi$, sometimes in the literature (R2) is omitted and then a "root system" is referred to as a *"reduced root system"*. But here we adopt all four axioms. The dimension of E is called the *rank* of the root system Φ.

A subset Δ of Φ is called a *base* if:

(B1) Δ *is basis of* E *(as a vector space).*

(B2) *Each root* β *can be uniquely written as* $\beta = \sum\limits_{\alpha \in \Delta} k_\alpha \alpha$ *with all* k_α *nonnegative or all nonpositive.*

The roots in Δ are called *simple*. Trivially, (B1) implies that $|\Delta| = \dim E$. We also define the *height* of a root (relative to Δ) by $ht\beta = \sum_{\alpha \in \Delta} k_\alpha$. If all $k_\alpha \geqslant 0$ (respectively all $k_\alpha \leqslant 0$) we call β *positive* (respectively *negative*) and write $\beta \succ 0$ (respectively $\beta \prec 0$). Finally, we denote the collections of positive and negative roots by Φ^+ and Φ^- respectively and clearly $\Phi^- = -\Phi^+$.

Example 3.2.1 Let us give an important example which we shall use later. We work with $\mathbb{R}^{l+1}$ endowed with the Euclidean inner product. Let $\{\varepsilon_i\}$ be the standard basis of $\mathbb{R}^{l+1}$, define $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leqslant i < j \leqslant l+1\}$ and let $E = \operatorname{Span} \Phi$. We now will show that Φ is a root system for E. Clearly, Φ is finite as $\mathbb{R}^{l+1}$ is a finite dimensional vector space. By definition of Φ we also have that 0 does not lie in it and thus axiom (R1) holds. Our definition of Φ automatically satisfies (R2). Pick now two elements from

Φ, say $(\varepsilon_i - \varepsilon_j)$ and $(\varepsilon_m - \varepsilon_n)$. Observe first that all elements in Φ are of length $\sqrt{2}$. Then we have the following $\sigma_{(\varepsilon_i - \varepsilon_j)}(\varepsilon_m - \varepsilon_n) = (\varepsilon_m - \varepsilon_n) - (\varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j)(\varepsilon_i - \varepsilon_j)$ since $(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j) = 2$. It is easy to see that $(\varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j) \in \{0, \pm 1\}$, whence the reflection $\sigma_{(\varepsilon_i - \varepsilon_j)}$ permutes the elements of Φ. Furthermore, by the same argument it follows that $\langle(\varepsilon_m - \varepsilon_n, \varepsilon_i - \varepsilon_j)\rangle \in \mathbb{Z}$ and therefore Φ is a root system for E.

Now let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leqslant i \leqslant l$. Then we claim that $\Delta = \{\alpha_1, ..., \alpha_l\}$ is a base for Φ. Take $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathbb{R}^{l+1}$ and an element of Φ, say $\beta = \varepsilon_i - \varepsilon_j$. We can rewrite it in the following way $\beta = (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_j) = \cdots = \varepsilon_i - \varepsilon_{i+1} + \cdots + \varepsilon_{j-1} - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ and so Δ spans Φ. Finally, $\sum \lambda_i \alpha_i = \sum \lambda_i(\varepsilon_i - \varepsilon_{i+1}) = 0$ forces $\lambda_i = 0$,for $1 \leqslant i \leqslant l$, since $\{\varepsilon_i\}$ is the standard basis in $\mathbb{R}^{l+1}$. Therefore α_i are linearly independent and hence Δ is a basis for E. Thus (B1) holds. (B2) is automatically satisfied by the definition of Φ and hence Δ is a base for the root system Φ. Furthermore, the definition of Φ also says that the positive roots are $+(\varepsilon_i - \varepsilon_{i+1})$.

Now we prove an important lemma, sometimes called the *Finiteness Lemma*.

Lemma 3.2.2 *Let Φ be a root system for the Euclidean space E and let $\alpha, \beta \in \Phi$ be two non-proportional roots. Then $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$.*

Proof. Our axiom (R4) guarantees that the result will be an integer. We then only need to establish the bounds. Recall that for any two non-zero vectors $v, w \in E$ the angle between them θ is such that $(v, w) = (v, v)(w, w)cos\theta$. Then, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 4\frac{(\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta)} = 4cos^2\theta \leqslant 4$. Suppose now that $cos^2\theta = 1$. Then θ is an integer multiple of π and α and β must be colinear. This contradiction completes the proof. □

This lemma shows us the very few possibilities for the integers $\langle \alpha, \beta \rangle$. Take two roots α, β in a root system Φ so that they are not proportional. Choose the labeling $(\beta, \beta) \geqslant (\alpha, \alpha)$ and hence we can write the following $|\langle \beta, \alpha \rangle| = \frac{2|(\beta,\alpha)|}{(\alpha,\alpha)} \geqslant \frac{2|(\alpha,\beta)|}{(\beta,\beta)} = |\langle \alpha, \beta \rangle|$. Thus we summarize the results in the following table:

Table 3.1:

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
0	0	$\frac{\pi}{2}$	undetermined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

Lemma 3.2.3 *Let α, β be two nonproportional roots in a root system Φ. If $(\alpha, \beta) > 0$, then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root.*

Proof. We only need to prove the first assertion of the lemma as the second immediately follows from the first when applied to $-\beta$ in place of β. Clearly, both (α, β) and $\langle\alpha, \beta\rangle$ have the same sign and the table above shows that either $\langle\alpha, \beta\rangle$ or $\langle\beta, \alpha\rangle$ equals 1. If $\langle\alpha, \beta\rangle = 1$, then $\sigma_\beta(\alpha) = \alpha - \beta \in \Phi$ by axiom (R3). Similarly, if $\langle\beta, \alpha\rangle = 1$, then $\beta - \alpha \in \Phi$, whence $\sigma_{\beta-\alpha}(\beta - \alpha) = \alpha - \beta \in \Phi$. □

A vitally important result is the following theorem whose proof can be found in either [4] or [5].

Theorem 3.2.4 *Every root system has a base.*

In the end of this section we discuss the reducibility of root systems. We say that a root system Φ is *irreducible* if it cannot be expressed as a disjoint union of two non-empty sets $\Phi_1 \cup \Phi_2$ such that $(\alpha, \beta) = 0$ for $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. We now will prove a little lemma, which shows that classification of the irreducible root systems gives us the amount of information we need to classify simple Lie algebras.

Lemma 3.2.5 *Let Φ be e a root system in the real vector space E. Then we may write Φ as a disjoint union $\Phi = \Phi_1 \cup \cdots \cup \Phi_k$, where each Φ_i is an irreducible root system in the space E_i spanned by Φ_i, and $E = E_1 \oplus \cdots \oplus E_k$.*

Proof. If Φ is irreducible we have nothing to prove. If it is not, we simply need to partition Φ. For this purpose we only need to define an equivalence relation on Φ and the equivalence classes will give us the desired partition. Define $\sim$ on Φ by $\alpha \sim \beta$ if there exist $\gamma_1, \gamma_2, ..., \gamma_s \in \Phi$ with $\alpha = \gamma_1$ and $\beta = \gamma_s$ such that $(\gamma_i, \gamma_{i+1}) \neq 0$ for $1 \leqslant i \leqslant s$. It follows immediately from our construction that each Φ_i is irreducible. Now, clearly reflexivity holds. The symmetry property automatically holds because of the symmetry of the inner product. Finally, let $(\alpha, \beta) \neq 0$. But we also have that $(\alpha, \sigma_\alpha(\beta)) = (\alpha, \beta - \langle \beta, \alpha \rangle \alpha) = (\alpha, \beta) - \langle \beta, \alpha \rangle (\alpha, \alpha) = -(\alpha, \beta)$. In other words $(\alpha, \sigma_\alpha(\beta)) \neq 0$ and $\alpha \sim \sigma_\alpha(\beta)$. Since α and β are from the same equivalent class and by (R3), σ_α leaves this class invariant, we have the transitivity property.

The last assertion follows from the fact that every root must appear in some E_i and therefore the sum of E_i spans E. Suppose that $v_1 + \cdots + v_k = 0$ for $v_i \in E_i$. Taking inner product with v_j we get $(v_i, v_j) = 0 \Leftrightarrow v_j = 0$ for all j and we deduce that $E = E_1 \oplus \cdots \oplus E_k$. □

3.3 Weyl Group, Cartan Matrix and Dynkin Diagrams

For a root system Φ of E we denote by W the subgroup of $GL(E)$ generated by the reflections σ_α for $\alpha \in \Phi$. R(3) implies that W permutes the set Φ, which by $R(1)$ is finite and spans E. Therefore we identify W with a finite subgroup of $Sym(\Phi)$. W is called the *Weyl group* of Φ and is of vital importance. We start our discussion about Weyl groups with the following lemma whose proof may be found in [5].

Lemma 3.3.1 *Let Φ be a root system in E, with Weyl group W. If $\sigma \in GL(E)$ leaves Φ invariant,then $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$, and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.*

One can ask what happens when we have two distinct root systems Φ, Φ' in respective Euclidean spaces E, E'. We say that the pairs (E, Φ) and (E', Φ') are *isomorphic* if there exists a vector space isomorphism $\phi : E \longrightarrow E'$ sending Φ onto Φ' with $\langle \phi(\beta), \phi(\alpha) \rangle = \langle \beta, \alpha \rangle$ for each pair of roots $\alpha, \beta \in \Phi$. Now, we have $\sigma_{\phi(\alpha)}(\phi(\beta)) = \phi(\beta) - \langle \phi(\beta), \phi(\alpha) \rangle \phi(\alpha) = \phi(\beta) - \phi(\langle \beta, \alpha \rangle \alpha) = \phi(\beta - \langle \beta, \alpha \rangle \alpha) = \phi(\sigma_\alpha(\beta))$. We can now conclude that this isomorphism of root systems induces a natural isomorphism of Weyl groups given by $\sigma \longmapsto \phi \sigma \phi^{-1}$. By dint of lemma above this is an automorphism of E leaving Φ invariant. Thus, in particular, $W \leqslant Aut(\Phi)$.

Let Δ be a base in a root system Φ. Fix $(\alpha_1, ..., \alpha_l)$ to be an order of the elements of Δ. The *Cartan matrix* of Φ is defined to be $l \times l$ matrix with ij-th entry $\langle \alpha_i, \alpha_j \rangle$. Now by Lemma 3.3.1 for any root β we have $\langle \sigma_\beta(\alpha_i), \sigma_\beta(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle$ and by the discussion in the former paragraph we see that the Cartan matrix depends only on the ordering of the base and not on the base itself. Furthermore, by axiom (R4) we have that the entries of the Cartan matrix are integers. We call them *Cartan integers*.

Example 3.3.2 We first give the Cartan matrices for the root system of rank 2. Four root systems of rank 2 are known. They arise for $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$ and $\frac{5\pi}{6}$, where θ is the angle between the simple roots. Details may be found in [4] or [5]. Important for us, however, is the fact that these values of θ appear in Table 3.1. and hence we have the following Cartan matrices:

$$A_1 \times A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \text{ and } \quad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Example 3.3.3 Let the root system be as in Example 3.2.1. Now with respect to the ordered base $(\alpha_1, \alpha_2, ..., \alpha_l)$ we have the following $l - tuples$: $\alpha_1 = (1, -1, 0, ..., 0)$, $\alpha_2 = (0, 1, -1, 0, ..., 0) \cdots \alpha_l = (0, ..., 0, 1, -1)$. Now, clearly $(\alpha_i, \alpha_i) = 2$ for all $1 \leqslant i \leqslant l$. It is also easy to see that most of the entries $\langle \alpha_i, \alpha_j \rangle$ are zeros since every element of the base Δ has only two non-zero coordinates. We easily get that $\langle \alpha_1, \alpha_2 \rangle = -1$ as the length of any simple root α_i is $\sqrt{2}$. All the other entries in the first row are zeros. Similarly, $\langle \alpha_2, \alpha_1 \rangle = -1$ and $\langle \alpha_2, \alpha_3 \rangle = -1$ and all other zeros. Thus, the Cartan matrix is given by

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

One can also record the information given in the Cartan matrix using graphs. Define the graph $\mathit{\Gamma} = \mathit{\Gamma}(\Phi)$ associated to the root system Φ as follows. The vertices of $\mathit{\Gamma}$ are labeled by the simple roots of Δ. Between the vertices labeled by simple roots α and β, we draw $d_{\alpha\beta}$ many lines, where $d_{\alpha\beta} = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$. If $d_{\alpha\beta} > 1$ then we also draw an arrow pointing from the longer root to the shorter root. This graph is called the *Dynkin diagram* of Φ. Now, the Dynkin diagram is independent on choice of base since the Cartan matrix is. We recall that the graph with the same vertices and edges, but without arrows, is known as the *Coxeter graph* of Φ. Finally, in view of our discussion in the end of former section we have that irreducible root systems are represented by connected Dynkin diagrams. This fact plays an essential role in the classification.

Example 3.3.4 As first examples we give the Dynkin diagrams of root systems of rank 2. It is clear that only type $A_1 \times A_1$ is reducible and hence its Dynkin diagram must be disconnected. It is a straightforward task to calculate that $d_{\alpha\beta} = 1, 2, 3$ for types A_2, B_2 and G_2 respectively. Taking into account the lengths of the roots in all three cases we draw their Dynkin diagrams as follows

$A_1 \times A_1$ A_2 B_2 G_2

3.4 Isomorphism of Root Systems

In Section 3.3 we already defined the isomorphism of root systems. If $\varphi : \Phi \to \Phi'$ is the root system isomorphism then clearly the following conditions hold:

(a) $\varphi(\Phi) = \Phi'$

(b) $\langle \alpha, \beta \rangle = \langle \varphi(\alpha), \varphi(\beta) \rangle$ for any two $\alpha, \beta \in \Phi$.

Now it follows immediately that isomorphic root systems have the same Dynkin diagram. In fact, when we defined the Cartan matrix we used precisely condition (b) and we had that it depends only on the ordering adopted in the chosen base Δ. We now want to prove the converse statement.

Proposition 3.4.1 *Let Φ and Φ' be root systems in the real vector spaces E ans E' respectively. If the Dynkin diagrams of Φ and Φ' are the same, then the root systems are isomorphic.*

Proof. We first choose bases $\Delta = \{\alpha_1, ..., \alpha_l\}$ and $\Delta' = \{\alpha_1', ..., \alpha_l'\}$ in Φ and Φ' respectively. Let Δ and Δ' be such that for all i, j we have that $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i', \alpha_j' \rangle$. This automatically gives us the existence of a linear map $\varphi : E \to E'$ which maps α_i to α_i'. Moreover, it is trivial, that this map satisfies condition (b). We thus only need to prove that $\varphi(\Phi) = \Phi'$.

Pick a vector v from E and a simple root α_i. Now, expressing v as a linear combination of the simple roots and using the linearity of the inner product in its first component we get: $\langle \varphi(v), \alpha_i' \rangle = \langle \varphi(\sum v_i \alpha_i), \alpha_i' \rangle = \sum v_i \langle \varphi(\alpha_i), \alpha_i' \rangle = \sum v_i \langle \alpha_i', \alpha_i' \rangle = \sum v_i \langle \alpha_i, \alpha_i \rangle = \langle v, \alpha_i \rangle$. Next, using $\langle \varphi(v), \alpha_i' \rangle = \langle v, \alpha_i \rangle$ we obtain $\varphi(\sigma_{\alpha_i}(v)) = \varphi(v - \langle v, \alpha_i \rangle \alpha_i) = \varphi(v) -$

$\langle v, \alpha_i \rangle \varphi(\alpha_i) = \varphi(v) - \langle \varphi(v), \alpha_i' \rangle \alpha_i' = \sigma_{\alpha'}(\varphi(v))$. Since the Weyl group of Φ is generated by the simple reflections σ_{α_i}, we have that the image under φ of the orbit of $v \in E$ under the action of the Weyl group of Φ is contained in the orbit of $\varphi(v)$ under the action of the Weyl group of Φ'. This combined with the fact that the Weyl group permutes the roots yield that $\{\sigma(\alpha) : \sigma \in W, \alpha \in \Delta\}$, where W is the Weyl group of Φ. Since $\varphi(\Delta) = \Delta'$ we conclude that $\varphi(\Phi) \subseteqq \Phi'$.

Applying the same argument for the inverse of φ we have that $\varphi^{-1}(\Phi') \subseteqq \Phi$. Hence $\varphi(\Phi) = \Phi'$, as required. □

This proposition is very important for the classification as it shows that a root system is essentially determined by its Dynkin diagram.

We now go further and state a crucial isomorphism theorem. Before stating it we will need one fact about the generators of a semisimple Lie algebra. In fact, as stated in the following proposition, this is set is not only a small one but also consists of the root spaces of L.

Proposition 3.4.2 *Let L be a semisimple Lie algebra, H a maximal toral subalgebra of L, Φ the root system of L relative to H. Fix a base Δ of Φ. Then L is generated by the root spaces $L_\alpha, L_{-\alpha}$.*

We omit the proof of this proposition. The reader may find it in [5]. We mention only that this proposition is equivalent to the statement that a semisimple Lie algebra L is generated by arbitrary non-zero root vectors $x_\alpha \in L_\alpha$ and $y_\alpha \in L_{-\alpha}$. If these root vectors also obey $[x_\alpha y_\alpha] = h_\alpha$, then we call the set $\{x_\alpha, y_\alpha, h_\alpha\}$ the *standard set of generators* for L.

Now, let us consider the two pairs of a simple Lie algebra and a maximal toral subalgebra (L, H) and (L', H'). Let Φ and Φ' be the corresponding root systems. Our aim is to see if an isomorphism of the two root systems will induce a Lie algebra isomorphism $L \to L'$

sending H onto H'. If this is possible then it will be enough to classify irreducible root systems. Fortunately this is the case and we have the following theorem.

Theorem 3.4.3 *Let L, L' be simple Lie algebras over $\mathbb{F}$, with respective maximal toral subalgebras H, H' and corresponding root systems Φ, Φ'. Suppose there is an isomorphism $\Phi \to \Phi'$ defined by $\alpha \mapsto \alpha'$, inducing $\pi : H \to H'$. Fix a base $\Delta \subset \Phi$, so $\Delta' = \{\alpha' \mid \alpha \in \Delta\}$ is a base of Φ'. For each $\alpha \in \Delta, \alpha' \in \Delta'$, choose an arbitrary nonzero $x_\alpha \in L_\alpha$ and $x'_{\alpha'} \in L_{\alpha'}$. Then there exist a unique isomorphism $\pi : L \to L'$ extending $\pi : H \to H'$.*

The proof is long and it can be found in [5]. Nevertheless, we shall make two comments about this statement. First, we briefly comment on the motivation. Indeed, since Φ spans H^* and Φ' spans H'^*, we have that that the isomorphism $\Phi \to \Phi'$ extends uniquely to an isomorphism of vector spaces $\psi : H^* \to H'^*$ which in turn induces the isomorphism $\pi : H \to H'$. So thereafter, bearing in mind the Cartan decomposition, one wishes to find an extension which sends L_α onto $L'_{\alpha'}$ and this is exactly what the theorem tells us. Second, choosing an arbitrary nonzero $x_\alpha \in L_\alpha$ and $x'_{\alpha'} \in L_{\alpha'}$ is equivalent to choosing and arbitrary Lie algebra isomorphism $\pi_\alpha : L_\alpha \to L'_{\alpha'}$. Thus our unique isomorphism will extend all π_α as well.

Chapter 4

The Classical Lie Algebras and Their Root Systems

4.1 General Strategy

Without loss of generality we shall work over $\mathbb{C}$ in this entire subsection. We consider the classical Lie algebras $\mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$ for $n \geqslant 2$. We want to find their root systems and to show that their Dynkin diagrams are connected. Thus we will prove the following important result:

Theorem 4.1.1 *If L is a classical Lie algebra other than $\mathfrak{so}(2, \mathbb{C})$ and $\mathfrak{so}(4, \mathbb{C})$, then L is simple.*

This theorem actually is a great pace towards the classification theorem in characteristic zero. We will also show that the root systems will give us all possible isomorphisms between different classical Lie algebras and thus we will have a complete classification of classical Lie algebras.

Let L be a classical Lie algebra. Now it follows from definitions that in each case L has a subalgebra of diagonal matrices, say H. The maps $\operatorname{ad} h$ for $h \in H$ are diagonalisable and therefore H is toral. Consider the α-eigenspaces L_α of H for $\alpha \in \Phi$. We can write

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha. \tag{4.1.2}$$

We now state a lemma which guarantees that this is exactly the Cartan decomposition. For a proof see [4].

Lemma 4.1.3 *Let $L \subseteqq \mathfrak{gl}(n, \mathbb{C})$ and H be as above. Suppose that for all non-zero $h \in H$ there is some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$. Then H is a Cartan subalgebra of L.*

To show that the classical Lie algebras are simple we first need to show that they are semisimple. We use the following criterion:

Proposition 4.1.4 *Let L be a complex Lie algebra with Cartan subalgebra H. Let $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ be the direct sum decomposition of L into simultaneous eigenspaces for the elements of* $\operatorname{ad} H$*, where Φ is the set of non-zero $\alpha \in H^*$ such that $L_\alpha \neq 0$. Suppose that the following condition hold:*

(i) For each $0 \neq h \in H$, there is some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$.

(ii) For each $\alpha \in \Phi$, the space L_α is $1-dimensional$.

(iii) If $\alpha \in \Phi$, then $-\alpha \in \Phi$, and if L_α is spanned by x_α, then $[[x_\alpha, x_{-\alpha}], x_\alpha] \neq 0$.

Then L is semisimple.

Proof. We first mention the fact that a Lie algebra L is semisimple if and only if it has no non-zero abelian ideals (for details see [4],[5]). Thus it suffices to prove that L has no non-zero abelian ideals. Let A be an abelian ideal of L. H acts diagonalisably on A since $[H, A] \subseteqq A$ and by hypothesis H acts diagonalisably on L. We can then decompose our ideal as $A = (A \cap H) \oplus \bigoplus_{\alpha \in \Phi}(A \cap L_\alpha)$.

If we can proof that $A \subseteqq H$ the job will be done because if A contains some non-zero element h, then condition (i) implies $[h, x_\alpha] = \alpha(h) x_\alpha \in L_\alpha$ for some $\alpha \in \Phi$. Also we have that $[h, x_\alpha] \in A$ and thus $[h, x_\alpha] \in L_\alpha \cap A$. This contradicts the root space decomposition of L and hence $A \subsetneqq H$ is equivalent to $A = 0$.

We can easily see that $A \subsetneqq H$ means that $A = A \cap H$ and now we only need to prove that $L_\alpha \cap A = 0$. Suppose that for some root α we have $L_\alpha \cap A \neq 0$. Then condition (ii) yields that $L_\alpha \subsetneqq A$. This together with the fact that A is ideal imply that $[L_\alpha, L_{-\alpha}] \subsetneqq A$ and thus A contains the element $h = [x_\alpha, x_{-\alpha}]$ where x_α spans L_α and $x_{-\alpha}$ spans $L_{-\alpha}$. Finally, the commutativity of A gives us $[h, x_\alpha] = 0$ which contradicts condition (iii). □

We note here that as $[L_\alpha, L_{-\alpha}] \subseteq L_0$, condition (iii) holds if and only if $\alpha([L_\alpha, L_{-\alpha}]) \neq 0$. Therefore to show that this condition holds it suffices to verify that $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for the pair of roots $\pm\alpha$. This will significantly reduce the amount of calculations.

Now let L be a semisimple Lie algebra with Cartan subalgebra H. First, we identify the root system. Second, having found a base for Φ, we determine the Cartan matrix for β, γ in the base . In order to find the Cartan integers (the Cartan matrix entries) we will use the identity $\langle \beta, \gamma \rangle = \beta(h_\gamma)$, where h_γ is part of the standard basis of the subalgebra $\mathfrak{sl}(\gamma)$ associated to the root γ. We will actually see that h_γ can be easily computed. Finally, using the Cartan matrix we construct the associated Dynkin diagram which is in fact our goal.

We thereafter need to show that L is simple. The following proposition tells us what is enough to be shown.

Proposition 4.1.5 *Let L be a complex semisimple Lie algebra with Cartan subalgebra H and root system Φ. If Φ is irreducible, then L is simple.*

Proof. Suppose for a contradiction that L has a proper non-zero ideal I. Using the root space decomposition we write $L = H \bigoplus_{\alpha \in \Phi} L_\alpha$. Recall that H contains only semisimple elements and hence it acts diagonalizably on I. Moreover, each root space L_α is $1-$

dimensional and thus we can decompose I as follows $I = H_1 \bigoplus_{\alpha\in\Phi_1} L_\alpha$ with $H_1 \subset H$ and $\Phi_1 \subset \Phi$. With respect to the Killing form we can define $I^\perp$ which is the perpendicular space to I. Let $y \in I^\perp$ and $z \in L$. Then for $x \in I$ with respect to the Killing form we have $\kappa(x, [z, y]) = \kappa([x, z], y) = 0$ as $[x, z] \in I$ and $y \in I^\perp$. Hence, $I^\perp$ is an ideal of L and we can also write its root space decomposition in the following way $I^\perp = H_2 \bigoplus_{\alpha\in\Phi_2} L_\alpha$. Now, as L is semisimple, by Theorem 2.6.2 the Killing form is nondegenerate. This implies that $I \cap I^\perp = \varnothing$, whence $I^\perp \oplus I = L$. Thus, we must have that $H_1 \oplus H_2 = H$, $\Phi_1 \cap \Phi_2 = \varnothing$ and $\Phi_1 \cup \Phi_2 = \Phi$.

If either Φ_1 or Φ_2 is empty, then $L_\alpha \subseteqq I$ for all $\alpha \in \Phi$ and thus L is generated by its root spaces, whence $L = I$ which contradicts the choice of I. Otherwise, for Φ_1 and Φ_2 non-empty, pick $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ and observe that $\langle \alpha, \beta \rangle = \alpha(h_\beta) = 0$ since $\alpha(h_\beta)e_\alpha = [h_\beta, e_\alpha]$ is an element of $I^\perp \cap I$ which is the zero space. Therefore, $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ and hence Φ must be reducible. □

We now summarize all the facts from the above in order to outline what we shall do in practice. In brief our programme will be the following:

(1)*Find the subalgebra H of diagonal matrices in L and determine the decomposition 4.1.2. This will show that the conditions* (*i*) *and* (*ii*) *of Proposition 4.1.4 hold.*

(2)*Check that $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for each root $\alpha \in \Phi$.*

By Proposition 4.1.4 and Lemma 4.1.3 we now have that L is semisimple and that H is a Cartan subalgebra of L.

(3) *Find a base for Φ.*

(4) *For γ, β in the base we want to find h_γ and e_β and thus we will be able to determine the Dynkin diagram of our root system, from which we can verify that Φ is irreducible and hence L is simple.*

4.2 The $\mathfrak{sl}(l+1,\mathbb{C})$

(1) The root space decomposition of $L = \mathfrak{sl}(l+1,\mathbb{C})$ is $L = H \bigoplus_{i\neq j} L_{\varepsilon_i - \varepsilon_j}$. Here $\varepsilon_i(h)$ is the $i-th$ entry of h and the root space $L_{\varepsilon_i-\varepsilon_j}$ is clearly spanned by e_{ij}. Thus $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leqslant i < j \leqslant l+1\}$.

(2) Now it suffices to do the calculations for the basis of $\mathfrak{sl}(l+1,\mathbb{C})$. For $i < j$ we have $[e_{ij}, e_{ji}] = e_{ii} - e_{jj} = h_{ij}$. Furthermore we have $[h_{ij}, e_{ij}] = 2e_{ij} \neq 0$ and thus $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for each root $\alpha \in \Phi$.

(3) In Example 3.2.1 we showed that the root system $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leqslant i < j \leqslant l+1\}$ has a base $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leqslant i \leqslant l\}$.

(4) We have already computed the Cartan matrix for this root system. We have simply the following

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } |i-j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

We shall also notice that from (2) follows that standard basis elements for the subalgebras $\mathfrak{sl}(\alpha_i)$ can be chosen as $e_{\alpha_i} = e_{i,i+1}, f_{\alpha_i} = e_{i+1,i}, h_{\alpha_i} = e_{ii} - e_{i+1,i+1}$. We say that the root system of $\mathfrak{sl}(l+1,\mathbb{C})$ has *type* A_l and the Dynkin diagram is

This diagram is connected, so L is simple.

4.3 The $\mathfrak{so}(2l+1,\mathbb{C})$

Recall first that $\mathfrak{so}(2l+1,\mathbb{C})$ is represented by the block matrices of the type

$$\begin{pmatrix} 0 & c^t & -b^t \\ b & m & p \\ -c & q & -m^t \end{pmatrix}$$

with $p=-p^t$ and $q=-q^t$. As usual, let H be the set of diagonal matrices in L. We label the matrix entries from 0 to $2l$ and thus every element $h \in H$ can be written in the form $h=\sum_{i=1}^{l} a_i(e_{ii}-e_{i+l,i+l})$, where $0, a_1, ..., a_l, -a_1, ..., -a_l$ are exactly the diagonal entries of h.

(1) We first start by finding the root spaces for H and then using them we find the root space decomposition of L. Now consider the subspace of L spanned by the matrices whose non-zero entries lie only in the positions labeled by b and c. Now using our labeling and looking at the block matrix above we easily see that this subspace has a basis $b_i = e_{i,0}-e_{0,l+i}$ and $c_i = e_{0,i}-e_{l+i,0}$ for $1 \leqslant i \leqslant l$. We do the following calculation:

$$\begin{aligned}
[h, b_i] &= [\sum_{i=1}^{l} a_i(e_{ii}-e_{i+l,i+l}), e_{i,0}-e_{0,l+i}] = \\
&= \sum_{i=1}^{l} a_i([e_{ii}, e_{i,0}] - [e_{ii}, e_{0,l+i}] - [e_{l+i,l+i}, e_{i,0}] + [e_{l+i,l+i}, e_{0,l+i}]) = \\
&= \sum_{i=1}^{l} a_i(e_{ii}-e_{i+l,i+l}) = a_i b_i,
\end{aligned}$$

where we use the following relations

$$[e_{ii}, e_{i,0}] = e_{i0},\ [e_{ii}, e_{0,l+i}] = 0,\ [e_{l+i,l+i}, e_{i,0}] = 0, \text{ and } [e_{l+i,l+i}, e_{0,l+i}] = -e_{l+i,l+i}.$$

Similarly, we get $[h, c_i] = -a_i c_i$. Further, we extend to a basis of L by the matrices:

$m_{ij} = e_{ij} - e_{l+j,l+i}$ for $1 \leqslant i \neq j \leqslant l$,

$p_{ij} = e_{i,l+j} - e_{j,l+i}$ for $1 \leqslant i < j \leqslant l$,

$q_{ji} = p_{ij}^t = e_{l+j,i} - e_{l+i,j}$ for $1 \leqslant i < j \leqslant l$.

We now calculate the following relations:

$[h, m_{ij}] = (a_i - a_j)m_{ij}$,

$[h, p_{ij}] = (a_i + a_j)p_{ij}$,

$[h, q_{ji}] = -(a_i + a_j)q_{ji}$.

We can now list the roots. For $1 \leqslant i \leqslant l$, let $\varepsilon_i \in H^*$ be the map sending h to a_i, its entry position i. Thus we can summarize all roots as follows:

root	ε_i	$-\varepsilon_i$	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$
eigenvector	b_i	c_i	$m_{ij}(i \neq j)$	$p_{ij}(i < j)$	$q_{ji}(i < j)$

(2) It suffices to show that $[h_\alpha, x_\alpha] \neq 0$, where $h_\alpha = [x_\alpha, x_{-\alpha}]$. We do this in three steps. First, for $\alpha = \varepsilon_i$, we have $h_i = [b_i, c_i] = e_{ii} - e_{l+i,l+i}$ and by (1) we have $[h_i, b_i] = b_i$. Second, for $\alpha = \varepsilon_i - \varepsilon_j$ and $i < j$, we have $h_{ij} = [m_{ij}, m_{ji}] = (e_{ii} - e_{l+i,l+i}) - (e_{jj} - e_{l+j,l+j})$ and again by (1) we obtain $[h_{ij}, m_{ij}] = 2m_{ij}$. Finally, for $\alpha = \varepsilon_i + \varepsilon_j$ and $i < j$, we have $k_{ij} = [p_{ij}, q_{ji}] = (e_{ii} - e_{l+i,l+i}) + (e_{jj} - e_{l+j,l+j})$, whence $[k_{ij}, p_{ij}] = 2p_{ij}$.

(3) The base for our root system is given by $\Delta = \{\alpha_i : 1 \leqslant i < l\} \cup \{\beta_l\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\beta_l = \varepsilon_l$. For $1 \leqslant i < l$ we see that

$\varepsilon_i = \alpha_i + \alpha_{i+1} \cdots + \alpha_{l-1} + \beta_l$

and for $1 \leqslant i < j \leqslant l$,

$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$,

$\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{l-1} + \beta_l)$.

Now using our table of roots we see that if $\gamma \in \Phi$ then either γ or $-\gamma$ appears above as a non-negative linear combination of elements of Δ. Since $\dim H$ is the same as the number of elements of Δ, precisely l ,we conclude that Δ is a base for Φ.

(4) For $i < j$ take $e_{\alpha_i} = m_{i,i+1}$ and by (2) follows $h_{\alpha_i} = h_{i,i+1}$. Taking $e_{\beta_l} = b_l$ we see that $h_\beta = 2(e_{ll} - e_{2l,2l})$. For $1 \leqslant i, j \leqslant l$, we calculate that

$$[h_{\alpha_j}, e_{\alpha_i}] = \begin{cases} 2e_{\alpha_i} & \text{if } i = j; \\ -e_{\alpha_i} & \text{if } |i-j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Hence

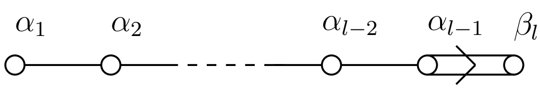
$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } |i-j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, by calculating $[h_{\beta_l}, e_{\alpha_i}]$ and $[h_{\alpha_i}, e_{\beta_l}]$, we find that

$$\langle \alpha_i, \beta_l \rangle = \begin{cases} -2 & \text{if } i = l-1; \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \beta_l, \alpha_i \rangle = \begin{cases} -1 & \text{if } i = l-1; \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the Dynkin diagram of Φ is :



and since it is connected, Φ is irreducible and so L is simple. The root system of $\mathfrak{so}(2l+1, \mathbb{C})$ is said to have type B_l.

4.4 The $\mathfrak{so}(2l, \mathbb{C})$

Recall first that we write all the elements of this classical algebra as block matrices:

$$\begin{pmatrix} m & p \\ q & -m^t \end{pmatrix}$$

where $p = -p^t$ and $q = -q^t$.

We observe that for $l = 1$ our Lie algebra is one dimensional so by definition is neither simple nor semisimple. In particular, the classical Lie algebra $\mathfrak{so}(2, \mathbb{C})$ is neither simple or semisimple (it is one dimensional and hence abelian). Again H is the set of diagonal matrices in L and we do the same labeling as in the former case. Thus we can use the calculations above by simply ignoring the row and column of matrices labeled by 0.

(1) We now simply copy the second half of the calculations for $\mathfrak{so}(2l+1, \mathbb{C})$ and we simply have the following roots:

root	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$
eigenvector	$m_{ij}(i \neq j)$	$p_{ij}(i < j)$	$q_{ji}(i < j)$

(2) The calculations done above immediately yield that $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for each root α.

(3) We now claim that the base for our root system is $\Delta = \{\alpha_i : 1 \leqslant i < l\} \cup \{\beta_l\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\beta_l = \varepsilon_{l-1} - \varepsilon_l$. For $1 \leqslant i < j \leqslant l$ we have the following:

$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$

$\varepsilon_i + \varepsilon_j = (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-2}) + (\alpha_j + \alpha_{j+1} + \cdots + \alpha_{l-1} + \beta_l)$.

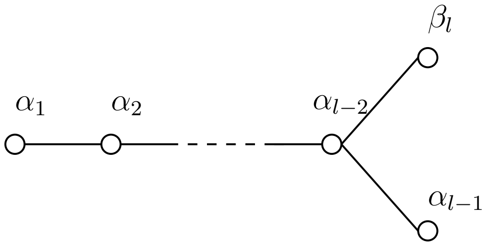
Then if $\gamma \in \Phi$ then either γ or $-\gamma$ is a non-negative $\mathbb{Z}$-linear combination of elements of Δ. Therefore, Δ is a base for our root system.

(4) Now we calculate the Cartan integers. The work already done for $\mathfrak{so}(2l+1,\mathbb{C})$ gives us the Cartan numbers $\langle\alpha_i,\alpha_j\rangle$ for $i,j<l$. To calculate the remaining ones we take $e_{\beta_l}=p_{l-1,l}$ and use (2) from $\mathfrak{so}(2l+1,\mathbb{C})$. Thus we obtain that $h_{\beta_l}=(e_{l-1,l-1}-e_{2l-1,2l-1})+(e_{l,l}-e_{2l,2l})$. Hence

$$\langle\alpha_j,\beta_l\rangle=\begin{cases}-1 & \text{if } j=l-2;\\ 0 & \text{otherwise,}\end{cases}$$

$$\langle\beta_l,\alpha_j\rangle=\begin{cases}-1 & \text{if } j=l-2;\\ 0 & \text{otherwise.}\end{cases}$$

If $l=2$, then the base has only two orthogonal roots α_1 and β_2, so in this case , Φ is reducible and hence $\mathfrak{so}(4,\mathbb{C})$ is not simple. If $l\geqslant 3$, then our calculations show that the Dynkin diagram of Φ is



As this diagram is connected, the Lie algebra is simple. When $l=3$, the Dynkin diagram is the same as A_3, the root system of $\mathfrak{sl}(4,\mathbb{C})$, so we have that $\mathfrak{so}(6,\mathbb{C})\cong\mathfrak{sl}(4,\mathbb{C})$. For $l\geqslant 4$, the root system of $\mathfrak{so}(2l,\mathbb{C})$ is said to have type D_l.

So far we have that only $\mathfrak{so}(2,\mathbb{C})$ and $\mathfrak{so}(4,\mathbb{C})$ are not simple. Therefore, it now remains to show that $\mathfrak{sp}(2l,\mathbb{C})$ is simple and thus to complete the proof of 4.1.1, we only have to prove that the symplectic algebra is simple.

4.5 The $\mathfrak{sp}(2l,\mathbb{C})$

We recall that we write the elements of this algebra as block matrices as follows:

$$\begin{pmatrix} m & p \\ q & -m^t \end{pmatrix}$$

where $p = p^t$ and $q = q^t$. The first observation to make is that for $l = 1$ we have $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$, since we have 2×2 matrices with entries numbers and not block matrices. Thus, without loss of generality we will assume that $l \geqslant 2$. As above, H is the set of diagonal matrices in L. We also use the same labeling of the matrix entries so $h = \sum_{i=1}^{l} a_i(e_{ii} - e_{i+l,i+l})$.

(1) Take the following basis for the root space of L:

$m_{ij} = e_{ij} - e_{l+j,l+j}$ for $1 \leqslant i \neq j \leqslant l$,

$p_{ij} = e_{i,l+j} + e_{j,l+j}$ for $1 \leqslant i < j \leqslant l$, $p_{ii} = e_{i,l+i}$ for $1 \leqslant i \leqslant l$,

$q_{ji} = p_{ij}^t = e_{l+j,i} + e_{l+i,j}$ for $1 \leqslant i < j \leqslant l$, $q_{ii} = e_{l+i,i}$ for $1 \leqslant i \leqslant l$.

Calculations show that:

$[h, m_{ij}] = (a_i - a_j)m_{ij}$,

$[h, p_{ij}] = (a_i + a_j)p_{ij}$,

$[h, q_{ji}] = -(a_i + a_j)q_{ji}$.

Clearly, for $i = j$ the eigenvalues for p_{ij} and q_{ji} are $2a_i$ and $-2a_i$ respectively. So we can now list the roots:

root	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$	$2\varepsilon_i$	$-2\varepsilon_i$
eigenvector	$m_{ij}(i \neq j)$	$p_{ij}(i < j)$	$q_{ji}(i < j)$	p_{ii}	q_{ii}

(2) Now we must check that $[h, x_\alpha] \neq 0$ with $h = [x_\alpha, x_{-\alpha}]$ holds for each root α. It has been done for $\alpha = \varepsilon_i - \varepsilon_j$ for $\mathfrak{so}(2l+1, \mathbb{C})$. If $\alpha = \varepsilon_i + \varepsilon_j$, then $x_\alpha = p_{ij}$ and $x_{-\alpha} = q_{ji}$ and $h = (\varepsilon_{ii} - \varepsilon_{l+i,l+i}) + (\varepsilon_{jj} - e_{l+j,l+j})$ for $i \neq j$, and $h = (\varepsilon_{ii} - \varepsilon_{l+i,l+i})$ for $i = j$. We then have $[h, x_\alpha] = 2x_\alpha$ in both cases.

(3) Choose $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leqslant i \leqslant l-1$ as before, and $\beta_l = 2\varepsilon_l$. Our claim now is that $\{\alpha_1, ..., \alpha_{l-1}, \beta_l\}$ is a base for the root system Φ of $\mathfrak{sp}(2l, \mathbb{C})$. For $1 \leqslant i < j \leqslant l$ we have:

$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1},$

$\varepsilon_i + \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \beta_l,$

$2\varepsilon_i = 2(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1}) + \beta_l.$

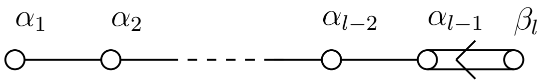
Thus using the same arguments as above we conclude that $\{\alpha_1, ..., \alpha_{l-1}, \beta_l\}$ is the base of Φ.

(4) In the end we need to calculate the Cartan integers. The numbers $\langle \alpha_i, \alpha_j \rangle$ are already known. Taking $e_{\beta_l} = p_{ll}$ we find that $h_{\beta_l} = e_{l,l} - e_{2l,2l}$ and so

$$\langle \alpha_i, \beta_l \rangle = \begin{cases} -1 & \text{if } i = l-1; \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \beta_l, \alpha_j \rangle = \begin{cases} -2 & \text{if } i = l-1; \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram of this root system is



which is connected, so L is simple. The root systems of $\mathfrak{sp}(2l, \mathbb{C})$ is said to have *type*

C_l.

4.6 More on Root Systems and Isomorphisms

We need to say something about isomorphisms of root systems since it is crucial for the classification. We actually need two vitally important facts before we move to classification. We will only state them, but for proofs the reader may refer to [4].

Theorem 4.6.1 *Let L be a complex semisimple Lie algebra. If Φ_1 and Φ_2 are the root systems associated to two Cartan subalgebras of L, then they are isomorphic.*

The importance of this fact lies in the consequence that two Lie algebras L_1 and L_2 with non-isomorphic root systems (with respect to some Cartan subalgebras) can not be isomorphic. Thus, classifying root systems is the same as classifying the corresponding Lie algebras and virtually this approach is used in the classification theorem. Furthermore, this result does the most work needed to prove the following proposition.

Proposition 4.6.2 *The only isomorphisms between classical Lie algebras are:*
(1) $\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$*; root systems of type* A_1*,*
(2) $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$*; root system of type* $A_1 \times A_1$*,*
(3) $\mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(4,\mathbb{C})$*; root systems of types* B_2 *and* C_2*,*
(4) $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$*; root systems of type* D_3 *and* A_3*.*

Chapter 5

Classification Theorem

5.1 The Ideology

Before we classify the simple Lie algebras over a field of characteristic 0 we discuss the main idea of the proof. We first observe that B_l and C_l differ in the relative numbers of short and long simple roots. We also know that the only difference between a Coxeter graph and a Dynkin diagram is in the arrows appearing in the latter. Therefore the Dynkin diagrams for B_l and C_l can be derived from same Coxeter graph. So the idea is to classify first the possible Coxeter diagrams, and then to look at the resulting Dynkin diagrams.

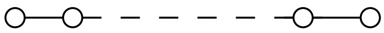
We work with unit vectors and for the sake of maximum flexibility we need the following assumptions. Let E be an Euclidean space of arbitrary dimension and $\mathfrak{U} = \{\varepsilon_1, ..., \varepsilon_n\}$ be the set of n linearly independent unit vectors which satisfy $(\varepsilon_i, \varepsilon_j) \leqslant 0$ and $4(\varepsilon_i, \varepsilon_j)^2 = 0, 1, 2$ or 3 for $i \neq j$. We call $\mathfrak{U}$ an *admissible* set. Indeed, the best example for an admissible set is any base of a root system with elements normed to 1. We attach a graph Γ to the set $\mathfrak{U}$ in the very same way as in Section 3.3. Thus our task is to determine all the connected Coxeter graphs associated with admissible sets of vectors.

5.2 The Theorem

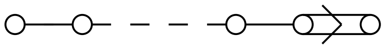
The classification of simple Lie algebras of characteristic 0 is given by the following

Theorem 5.2.1 *If Φ is an irreducible root system of rank l then its Dynkin diagram is one of the following:*

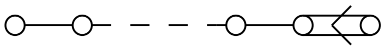
$A_l (l \geqslant 1)$:



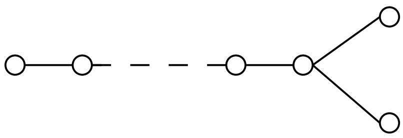
$B_l (l \geqslant 2)$:



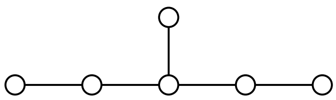
$C_l (l \geqslant 2)$:



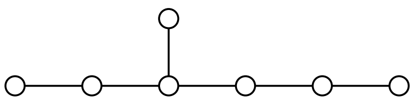
$D_l (l \geqslant 4)$:



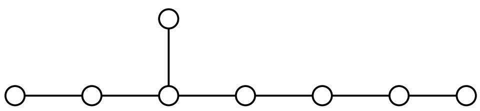
E_6:



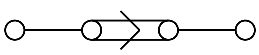
E_7:



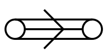
E_8:



F_4:



G_2:



Proof. The exceptional algebras are discussed in Appendix B. We prove the theorem in ten steps.

(1) *If some of the* ε_i *is discarded, the remaining ones still form an admissible set, whose graph is obtained from* Γ *by omitting the corresponding vertices and all incident edges.*

This step is trivial as all the remaining vectors are unit, linearly independent and satisfy the same properties of the inner product as in the original admissible set.

(2) *The number of pair of vertices in* Γ *connected by at least one edge is strictly less than* n.

To see this set $\varepsilon = \sum_{i=1}^{n} \varepsilon_i$. Clearly, $\varepsilon \neq 0$ and $(\varepsilon, \varepsilon) = (\sum_{i=1}^{n} \varepsilon_i, \sum_{j=1}^{n} \varepsilon_j) = n + 2\sum_{i<j}(\varepsilon_i, \varepsilon_j)$. Let the pair of distinct vertices i and j be joined, i.e. $(\varepsilon_i, \varepsilon_j) \neq 0$. Then $4(\varepsilon_i, \varepsilon_j)^2 = 1, 2$ or 3, which yields $2(\varepsilon_i, \varepsilon_j) \leqslant -1$. Now, having that $(\varepsilon, \varepsilon) > 0$ we deduce that the number of such pairs is at most $n - 1$.

(3) Γ *contains no cycles*

Suppose that there is a cycle Γ' in Γ. Then by (1) it must be the graph of an admissible subset $\mathfrak{U}'$ of $\mathfrak{U}$. Then clearly $n = \operatorname{Card} \mathfrak{U}'$ and Γ' violates (2).

(4) *No more than three edges can originate at a given vertex of* Γ

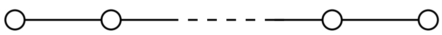
Let $\varepsilon, \eta_1, ..., \eta_k \in \mathfrak{U}$ be all distinct. Suppose that $(\varepsilon, \eta_i) < 0$, i.e. $\eta_1, ..., \eta_k$ are connected to ε by 1,2 or 3 edges. By previous step no two η's can be connected, so $(\eta_i, \eta_j) = 0$ for $i \neq j$. By definition $\mathfrak{U}$ and we can choose a unit vector $\eta_0 \in span(\varepsilon, \eta_1, ..., \eta_k)$ such that $(\eta_0, \varepsilon) \neq 0$ and $(\eta_0, \eta_i) = 0$ for all $1 \leqslant i \leqslant k$. Since $\varepsilon = \sum_{i=0}^{k}(\varepsilon, \eta_i)\eta_i$ and ε and η_i's are unit vectors we easily get that $\sum_{i=0}^{k}(\varepsilon, \eta_i)^2 = 1$, whence $\sum_{i=1}^{k} 4(\varepsilon, \eta_i)^2 < 4$. We are done since

$4(\varepsilon, \eta_i)^2$ is nothing but the number of edges joining ε and η_i's in Γ.

(5) *The only connected graph Γ of an admissible set $\mathfrak{U}$ which contains a triple edge is the Coxeter graph of G_2.*

Indeed this follows at once from (4).

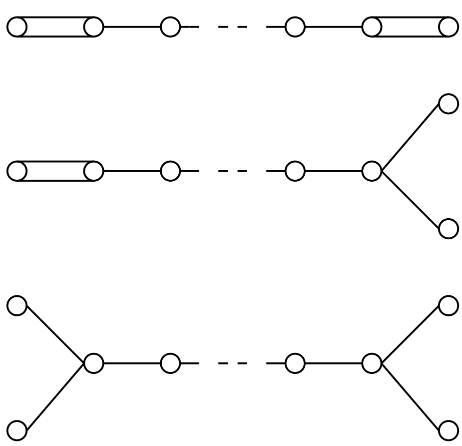
(6) *Let $\{\varepsilon_1, ..., \varepsilon_k\} \subset \mathfrak{U}$ be a simple chain in Γ, i.e. a subgraph of Γ of the form:*



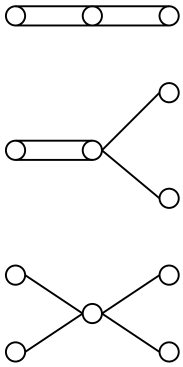
If $\mathfrak{U}' = (\mathfrak{U} - \{\varepsilon_1, ..., \varepsilon_k\}) \cup \{\varepsilon\}$, $\varepsilon = \sum_{i=1}^{k} \varepsilon_i$, then $\mathfrak{U}'$ is admissible.

Notice first that the graph of $\mathfrak{U}'$ is obtained from Γ by shrinking the simple chain to a point. Now, clearly $\mathfrak{U}'$ is linearly independent. Since $\mathfrak{U}$ is admissible, using the same arguments as in (2) we have that $2(\varepsilon_i, \varepsilon_{i+1}) \leqslant -1$ $(1 \leqslant i \leqslant k-1)$ and thus $(\varepsilon, \varepsilon) = k + 2\sum_{i<j}(\varepsilon_i, \varepsilon_j) = k - (k-1) = 1$. Therefore ε is a unit vector. Since no cycles are allowed (by (3)), for any $\eta \in \mathfrak{U} - \{\varepsilon_1, ..., \varepsilon_k\}$ we have either $(\eta, \varepsilon) = 0$ or $(\eta, \varepsilon) = (\eta, \varepsilon_i)$ for some $i \in [1, k]$. Thus $\mathfrak{U}'$ is admissible.

(7) *Γ contains no subgraph of the form:*

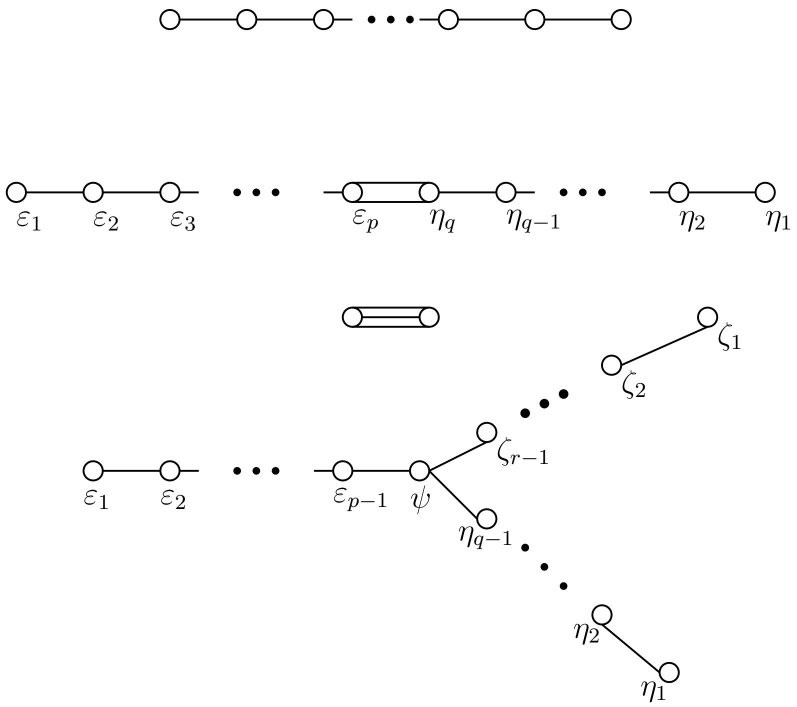


Suppose for a contradiction that one of these graphs occurred in Γ. First, by (1) it would be a graph of an admissible set. Second, by (6) we can replace the simple chain in each case by e single vertex and so we reduce the graphs to:



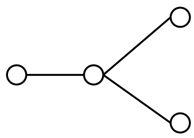
Now clearly (4) is violated.

(8) *Any connected graph Γ of an admissible set has one of the following forms:*



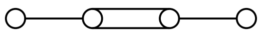
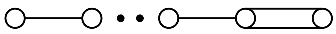
Indeed, only the graph of G_2 contains a triple edge by (5).Then any connected graph containing more than one double edge would contain a subgraph

which is forbidden by (7) and therefore at most one double edge occurs. Furthermore, if Γ has a double edge, it cannot also have a *branch point*, i.e. there is no subgraph of the form



Since (3) forbids cycles it follows that the second graph pictured is the only possibility for a graph containing double edge. Finally, suppose that Γ has only single edges. Again as no cycles are allowed, if Γ has no a branch point it must be a simple chain and this is the case of first graph picture. (7) yields that Γ can contain at most one branch point and hence the fourth graph is the remaining possibility.

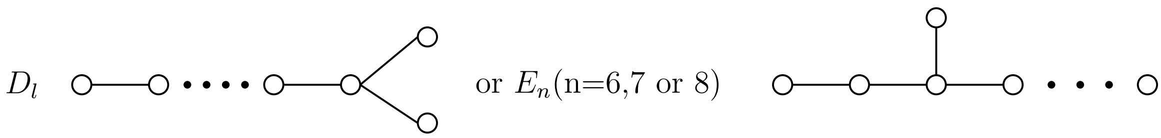
(9) *The only connected Γ of the second type in (8) is either the Coxeter graph F_4 or the Coxeter graph $B_l(=C_l)$, i.e.:*

F_4  or $B_l(=C_l)$ 

Set $\varepsilon = \sum_{i=1}^{p} i\varepsilon_i$, $\eta = \sum_{i=1}^{q} i\eta_i$. By hypothesis we have that $2(\varepsilon_i, \varepsilon_{i+1}) = -1 = 2(\eta_i, \eta_{i+1})$ and other pairs being orthogonal. We calculate that
$(\eta, \eta) = (\sum_{i=1}^{q} i\eta_i, \sum_{i=1}^{q} i\eta_i) = \sum_{i=1}^{q} i^2 - \sum_{i=1}^{q-1} i(i+1) = q^2 - \sum_{i=1}^{q-1} i = q^2 - \dfrac{(q-1)q}{2} = \dfrac{q(q+1)}{2}$.
We similarly get that $(\varepsilon, \varepsilon) = \frac{p(p+1)}{2}$. We also have that $4(\varepsilon_p, \eta_q) = 2$ and thus $(\varepsilon, \eta)^2 = (\sum_{i=1}^{p} i\varepsilon_i, \sum_{i=1}^{q} i\eta_i)^2 = p^2q^2(\varepsilon_p, \eta_q)^2 = \dfrac{1}{2}p^2q^2$. Now, by the Schwartz inequality we have $(\varepsilon, \eta)^2 < (\varepsilon, \varepsilon)(\eta, \eta)$ which is the same as $\frac{1}{2}p^2q^2 < \frac{1}{4}p(p+1)q(q+1)$. Notice that the inequality is strict since ε and η are independent. As $p, q \in \mathbb{Z}^+$ we can rewrite the last

inequality as $2pq < (p+1)(q+1)$ which is easily converted into $(p-1)(q-1) < 2$. The last inequality restricts us only to two possibilities. First, one is $p = q = 2$ which gives us the graph F_4. The second one is either $p = 1$ (q-arbitrary) or $q = 1$ (p-arbitrary) which certainly is $B_l(= C_l)$.

(10) *The only connected Γ of the fourth type in (8) is either the Coxeter graph D_l or the Coxeter graph E_l, i.e.:*



Set now $\varepsilon = \sum i\varepsilon_i$, $\eta = \sum i\eta_i$ and $\zeta = \sum i\zeta_i$. For the sake of brevity we omit the first half of the proof since it relies entirely on the same arguments and calculations as in (4) and (9). More details can be found in [5]. Assuming this, however, we could derive the inequality $(*)\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. We first observe that if p, q, or r equals 1 then we have the graph type A_l. Using appropriate labeling we now can write $\frac{1}{p} \leqslant \frac{1}{q} \leqslant \frac{1}{r} \leqslant \frac{1}{2}$. Therefore, the inequality $(*)$ implies that $\frac{3}{2} \geqslant \frac{3}{r}$, whence $r = 2$. Then $\frac{1}{p}+\frac{1}{q} > \frac{1}{2}$, $\frac{2}{q} > \frac{1}{2}$ and $2 \leqslant q < 4$. Now clearly for $q = 3$ we have that $p < 6$ and for $q = 2$, $p > 0$. Therefore the possible triples (p, q, r) are: $(p, 2, 2)$, $(3, 3, 2)$, $(4, 3, 2)$ and $(5, 3, 2)$ which clearly determine D_l, E_6, E_7 and E_8 respectively.

We have just completed the proof of the classification theorem in characteristic zero. This proof also shows that except B_l and C_l, the Coxeter graphs of an admissible set of vectors in Euclidean space uniquely determine their Dynkin diagrams.Indeed, the constructions done in Section 4.3 and Section 4.4 allow us precisely to determine the Dynking diagrams for B_l and C_l. □

Chapter 6

Chevalley Basis

In this chapter we describe in detail how to construct a basis of L which will play a crucial role in the next chapter. We assume that L is a semisimple Lie algebra over the algebraically closed field F of characteristic 0, H a Cartan subalgebra, and Φ the root system.

6.1 Pairs of Roots

We first need some facts about pairs of roots α, β for which $\alpha + \beta$ is also a root. For this reason we first define the notion of α-strings.

Consider first a pair of non-proportional roots α, β. The *α-string through β* are all the roots of the form $\beta + i\alpha$ for some $i \in \mathbb{Z}$. Let $r, q \in \mathbb{Z}^+$ be the largest integers for which $\beta - r\alpha, \beta + q\alpha \in \Phi$. Is there a possibility our $\alpha-$string to be broken? In other words, is there an integer in the interval $-r \leqslant i \leqslant q$ such that $\beta + i\alpha \notin \Phi$? Suppose this is true. Then we can find p, s in this interval, say $p < s$, such that $\beta + p\alpha$ and $\beta + s\alpha$ are in Φ, but $\beta + (p+1)\alpha$ and $\beta + (s-1)\alpha$ are not roots. Clearly, $\beta + (p+1)\alpha = (\beta + p\alpha) + \alpha$ and $\beta + p\alpha$ and α are two non-proportional roots, so by Lemma 3.2.3 $(\alpha, \beta + p\alpha) \leqslant 0$. This implies $(\beta, \alpha) < -p(\alpha, \alpha)$. By the same argument we have that $(\beta, \alpha) > -s(\alpha, \alpha)$. We chose $p < s$ and as $(\alpha, \alpha) > 0$ it is clear that this is an absurd. Hence, we conclude that our α-string is never broken. The moral of the story is that for all integers in the interval $-r \leqslant i \leqslant q$ all $\beta + i\alpha$ are roots.

We finish our discussion about α-strings with the answer of the question how reflections act on an $\alpha - string$. Take a reflection σ_α. Clearly, $\sigma_\alpha(\beta + q\alpha) = \beta - \langle \beta, \alpha \rangle \alpha - q\alpha = \beta - (\langle \beta, \alpha \rangle + q)\alpha$. By axiom (R3) and by Table 3.1 we have that $(\langle \beta, \alpha \rangle + q) \in \mathbb{Z}$, call it r, and $(\beta - r\alpha) \in \Phi$. Therefore, the $\alpha - string$ is invariant under σ_α. Geometrically, the action of σ_α simply reverses our string. Furthermore, we have the length of our string to be $r - q = \langle \beta, \alpha \rangle$ and from possible values for $\langle \beta, \alpha \rangle$ at once follows that the root strings are of length at most 4.

The next proposition depends only on the root system Φ and is the first step in our construction.

Proposition 6.1.1 *Let α, β be linearly independent roots, $\beta - r\alpha, ..., \beta, ..., \beta + q\alpha$ the $\alpha - string$ through β.Then:*
(a) $\langle \beta, \alpha \rangle = r - q$.
(b) At most two roots lengths occur in this string.
(c)If $\alpha + \beta \in \Phi$, then $r + 1 = \frac{q(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)}$.

Proof. We prove only part (c) as it is important for the proof of Chevalley Theorem. For a proof of (a) and (b) see [5]. Using the relation from (a) we have the following:

$$(r+1) - \frac{q(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)} = q + \frac{2(\beta,\alpha)}{(\alpha,\alpha)} + 1 - \frac{q(\alpha,\alpha)}{(\beta,\beta)} - \frac{2q(\alpha,\beta)}{(\beta,\beta)} - q = (\langle \beta, \alpha \rangle + 1)(1 - \frac{q(\alpha,\alpha)}{(\beta,\beta)}).$$

Now it suffices to show that the right hand side is zero. Call $A = (\langle \beta, \alpha \rangle + 1)$ and $B = (1 - \frac{q(\alpha,\alpha)}{(\beta,\beta)})$. Since our roots are non-proportional we distinguish two cases.

Case i: $(\alpha, \alpha) \geqslant (\beta, \beta)$ clearly implies that $|\langle \beta, \alpha \rangle| \leqslant |\langle \alpha, \beta \rangle|$. Recall that $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle =$

$0, 1, 2,$ or 3 (see Table 3.1). Now the inequality above forces $\langle \beta, \alpha \rangle = -1, 0,$ or 1 and clearly in the first case $A = 0$. Otherwise, $(\beta, \alpha) \geqslant 0$, so $(\beta + \alpha, \beta + \alpha)$ is strictly larger than both (α, α) and (β, β). Since $\alpha + \beta \in \Phi$, part (b) yields $(\alpha, \alpha) = (\beta, \beta)$. Similarly, $(\beta + 2\alpha, \beta + 2\alpha) > (\beta + \alpha, \beta + \alpha)$, and again by (b) we have that $\beta + 2\alpha \notin \Phi$, that is $q = 1$, forcing $B = 0$.

Case ii: Suppose now that $(\alpha, \alpha) < (\beta, \beta)$. Then by part (b) we have that $(\alpha + \beta, \alpha + \beta)$ is equal to either (α, α) or (β, β) which yields $(\alpha, \beta) < 0$ in either case. We also have $(\alpha - \beta, \alpha - \beta) > (\beta, \beta) > (\alpha, \alpha)$ and by (b) $\alpha - \beta$ is not a root. We therefore infer that $r = 0$. Similarly to the previous case we have that $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 0, 1, 2,$ or 3, but now we have $|\langle \alpha, \beta \rangle| < |\langle \beta, \alpha \rangle|$, so $\langle \alpha, \beta \rangle = -1, 0$ or 1. Clearly, $(\alpha, \beta) < 0$ implies $\langle \alpha, \beta \rangle < 0$ and thus $\langle \alpha, \beta \rangle = -1$. Finally, by (a) we have that $q = -\langle \beta, \alpha \rangle = \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \beta \rangle} = \frac{(\beta, \beta)}{\alpha, \alpha}$, whence $B = 0$. $\square$

6.2 Construction of Chevalley Basis

We start with one lemma whose proof can be found in [5]. Also, for a definition of h_α see Proposition 2.8.5.

Lemma 6.2.1 *Let α, β be independent roots. Choose $x_\alpha \in L_\alpha$, $x_{-\alpha} \in L_{-\alpha}$ for which $[x_\alpha x_{-\alpha}] = h_\alpha$, and let $x_\beta \in L_\beta$ be arbitrary. Then if $\beta - r\alpha, ..., \beta, ..., \beta + q\alpha$ the $\alpha -$string through β, we have: $[x_{-\alpha}[x_\alpha x_\beta]] = q(r + 1)x_\beta$.*

We also need the following proposition, especially its part (c). We omit the proof since it is very long. The reader may again refer [5].

Proposition 6.2.2 *It is possible to choose root vectors $x_\alpha \in L_\alpha (\alpha \in \Phi)$ satisfying:*

(a) $[x_\alpha, x_{-\alpha}] = h_\alpha$.

(b) If $\alpha, \beta, \alpha + \beta \in \Phi$, $[x_\alpha x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$, then $c_{\alpha\beta} = -c_{\alpha,-\beta}$.

(c) For any choice of root vectors as above, the scalars $c_{\alpha\beta}$ $(\alpha, \beta, \alpha + \beta \in \Phi)$ *satisfy:* $c_{\alpha\beta}^2 = q(r+1)\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)}$, *where* $\beta - r\alpha, ..., \beta, ..., \beta + q\alpha$ *the* $\alpha -$ *string through* β.

We are now ready to construct a *Chevalley basis* of L. This is by definition any basis $\{x_\alpha, \alpha \in \Phi; h_i, 1 \leqslant i \leqslant l\}$ for which x_α satisfy (a) and (b) of the preceding proposition, while $h_i = h_{\alpha_i}$ for some base $\Delta = \{\alpha_1, ..., \alpha_l\}$ of Φ.

Theorem 6.2.3 *(Chevalley) Let* $\{x_\alpha, \alpha \in \Phi; h_i, 1 \leqslant i \leqslant l\}$ *be a Chevalley basis of* L. *Then the resulting structure constants lie in* $\mathbb{Z}$. *More precisely:*

(a) $[h_i h_j] = 0, 1 \leqslant i, j \leqslant l$.

(b) $[h_i x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha, 1 \leqslant i \leqslant l, \alpha \in \Phi$.

(c) $[x_\alpha x_{-\alpha}] = h_\alpha$ *is a* $\mathbb{Z} -$ *linear combination of* $h_1, ..., h_l$.

(d) If α, β *are independent roots,* $\beta - r\alpha, ..., \beta, ..., \beta + q\alpha$ *the* $\alpha -$ *string through* β, *then* $[x_\alpha x_\beta] = 0$ *if* $q = 0$, *while* $[x_\alpha x_\beta] = \pm(r+1)x_{\alpha+\beta}$ *if* $\alpha + \beta \in \Phi$.

Proof. (a) We clearly have the following:

$$[h_i h_j] = [h_{\alpha_i} h_{\alpha_j}] = \frac{4t_{\alpha_i}t_{\alpha_j}}{\kappa(t_{\alpha_i}, t_{\alpha_i})\kappa(t_{\alpha_j}, t_{\alpha_j})} - \frac{4t_{\alpha_j}t_{\alpha_i}}{\kappa(t_{\alpha_j}, t_{\alpha_j})\kappa(t_{\alpha_i}, t_{\alpha_i})} = \frac{4[t_{\alpha_i}t_{\alpha_j}]}{\kappa(t_{\alpha_j}, t_{\alpha_j})\kappa(t_{\alpha_i}, t_{\alpha_i})}.$$

Since denominator is a product of two Killing forms it is clearly a number. Part (e) of Proposition 2.8.5 guarantees that it is nonzero. Since t_α is an element of a toral algebra (see Section 2.8) we have that $[t_{\alpha_i} t_{\alpha_j}] = 0$ and hence $[h_{\alpha_i} h_{\alpha_j}] = 0$.

(b) The proof is simply the following calculation

$$[h_i, x_\alpha] = [\frac{2t_{\alpha_i}}{\kappa(t_{\alpha_i}, t_{\alpha_i})}, x_\alpha] = \frac{2}{\kappa(t_{\alpha_i}, t_{\alpha_i})}[t_{\alpha_i}, x_\alpha] = \frac{2\kappa(t_{\alpha_i}, t_\alpha)}{\kappa(t_{\alpha_i}, t_{\alpha_i})}x_\alpha = \langle \alpha_i, \alpha \rangle x_\alpha.$$

(c) Define the *dual* roots of α by $\alpha^* := \frac{2\alpha}{(\alpha,\alpha)}$. Via easy calculations one can show that the dual roots form a root system with base $\Delta^* = \{{\alpha_1}^*, ..., {\alpha_l}^*\}$. Now, using the Killing form identification of H with H^* (see Section 2.8) and part (g) of Proposition 2.8.5 we see that

h_α corresponds to α^*. Hence the statement follows as each α^* is a $\mathbb{Z}$-linear combination of Δ^*.

(d) By Proposition 6.2.2 (b) we have that $[x_\alpha, x_\beta] = c_{\alpha\beta}x_{\alpha+\beta}$ and clearly $c_{\alpha\beta} = 0$ for $q = 0$, whence $[x_\alpha, x_\beta] = 0$. Otherwise, by Proposition 6.1.1 (c) we have that $q = \frac{(r+1)(\beta,\beta)}{(\alpha+\beta,\alpha+\beta)}$, so $c_{\alpha\beta} = \pm(r+1)$ and the statement follows. □

6.3 Uniqueness

We need a brief discussion on the uniqueness. Clearly, once Δ is fixed, the h_i are completely determined. It is possible, however, to vary the choice of the x_α.

Define the function $\eta : \Phi \to \mathbb{F}$ and assume that $\eta(\alpha)x_\alpha$ is a basis element. Then we have $[\eta(\alpha)x_\alpha, \eta(-\alpha)x_{-\alpha}] = \eta(\alpha)\eta(-\alpha)[x_\alpha, x_{-\alpha}] = \eta(\alpha)\eta(-\alpha)h_\alpha$. In order to satisfy (a) of Proposition 6.2.2 we must have $\eta(\alpha)\eta(-\alpha) = 1$ $(*)$. Also, for $\alpha, \beta, \alpha+\beta \in \Phi$ we have $[\eta(\alpha)x_\alpha, \eta(\beta)x_\beta] = \eta(\alpha)\eta(\beta)[x_\alpha, x_\beta] = c_{\alpha\beta}\eta(\alpha)\eta(\beta)x_{\alpha+\beta} = c'_{\alpha\beta}\eta(\alpha+\beta)x_{\alpha+\beta}$, where $c'_{\alpha\beta} = c_{\alpha\beta}\frac{\eta(\alpha)\eta(\beta)}{\eta(\alpha+\beta)}$. Now, to satisfy (b) of Proposition 6.2.2, we must have $c'_{\alpha\beta} = -c'_{-\alpha,-\beta}$. Thus, using $(*)$ we obtain that our function η must also satisfy $\eta(\alpha)\eta(\beta) = \pm\eta(\alpha+\beta)$ $(**)$. Therefore, any function $\eta : \Phi \to \mathbb{F}$ satisfying $(*)$ and $(**)$ can be used to modify the choice of the x_α.

6.4 Reduction Modulo a Prime

In the end of this chapter we show how to construct Lie algebras over prime fields using a given Lie algebra L and its Chevalley basis.

Let L be a Lie algebra with Chevalley basis $\{x_\alpha, h_i\}$. Consider the $\mathbb{Z}$-span of $\{x_\alpha, h_i\}$. It is a Lie algebra over $\mathbb{Z}$ under the bracket operation inherited from L. Indeed, the closure follows at once from Chevalley Theorem. Denote this new Lie algebra $L(\mathbb{Z})$. We now go further and consider the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Then the tensor product $L(\mathbb{F}_p) = L(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is a vector space over $\mathbb{F}_p$ with basis $\{x_\alpha \otimes 1, h_i \otimes 1\}$. Furthermore,

$L(\mathbb{F}_p)$ is a Lie algebra with bracket operation induced by that in $L(\mathbb{Z})$. Finally, the multiplication table is again given by Chevalley Theorem, but this time with integers modulo p.

Now, if $\mathbb{K}$ is any field extension of $\mathbb{F}_p$, we similarly define $L(\mathbb{K}) = L(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{K}$, where the right hand side is essentially the same as $L(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{K}$. Thus $L(\mathbb{K})$ inherits both basis and Lie algebra structure from $L(\mathbb{F}_p)$. We therefore associate a Lie algebra over $\mathbb{K}$ with the pair $(L, \mathbb{K})$. $L(\mathbb{K})$ is called a *Chevalley algebra* and its structure resembles that of L. Moreover, by the arguments given in the previous section and Theorem 3.4.3 we have that the Chevalley algebra $L(\mathbb{K})$ depends (up to isomorphism) only on the pair $L(\mathbb{K})$.

Chapter 7

Simple Lie Algebras over Fields of Positive Characteristics

In this final chapter we briefly discuss (without proofs) the construction of two types of simple Lie algebras over positive characteristic. From now on we consider $\mathbb{F}$ an algebraically closed field of positive characteristic.

7.1 First Examples in Prime characteristic

The theory of Lie algebras over a field $\mathbb{F}$ of positive characteristic started in late $1930's$. Jacobson[6], Witt and Zassenhaus[18] were the first to consider constructing and studying Lie algebras over a field of prime characteristic.

The first example for a Lie algebra over a field of prime characteristic which has no analogues in characteristic 0, was given by Witt. Witt himself never published his example and it first appeared along with its generalization in Zassenhaus work [18]. It is a p-dimensional Lie algebra with basis $\{e_{-1}, e_0, e_1, ..., e_{p-2}\}$ and bracket defined by the following relations:

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j} & \text{if } -1 \leqslant i+j \leqslant p-2; \\ 0 & \text{otherwise} \end{cases}$$

We want to convince ourselves that the operation above really defines a Lie algebra. Clearly, $[e_i, e_i] = 0$ as the coefficient of e_{i+i} is zero. Furthermore, the relation above gives us $[e_i, e_j] + [e_j, e_i] = 0$ for all $-1 \leqslant i \neq j \leqslant p-2$. Thus, by taking an arbitrary element $x = \sum_{i=-1}^{p-2} \lambda_i e_i$ in L and using the bilinearity of the bracket we have $[x, x] = 0$, for all $x \in L$. We now have to check the Jacobi identity. Again by bilinearity, it suffices to check it for the basis vectors. If $-1 \leqslant j+k \leqslant p-2$ and $-1 \leqslant i+j+k \leqslant p-2$, then $[e_i, [e_j, e_k]] = (j-k)[e_i, e_{j+k}] = (j-k)(i-j-k)e_{i+j+k}$. Similarly, we get $[e_j, [e_k, e_i]] = (k-i)(j-k-i)e_{j+k+i}$ and, $[e_k, [e_i, e_j]] = (i-j)(k-i-j)e_{k+i+j}$. Clearly, $e_{i+j+k} = e_{j+k+i} = e_{k+i+j}$ and therefore it now suffices to show that $(j-k)(i-j-k) + (k-i)(j-k-i) + (i-j)(k-i-j) = 0$. This is true and hence Jacobi identity holds. We denote the Witt algebra by $W(1; \underline{1})$.

We now consider an example which motivates the concept of *divided power mappings* and allows us to generalize the notion of Witt algebras. It was first considered by A.I.Kostrikin and I.R.Safarevich in [7]. More on this matter can be found in [17]. It leads, however, to the general definition of Witt algebras and this is exactly what we are interested in.

The set up is the following. We start with the polynomial ring $\mathbb{C}[X_1, ..., X_m]$ and let $X_i^{(a_i)} := \frac{1}{a_i!} X_i^{a_i}$. For $a, b \in \mathbb{N}^m$ we use the following notations: $X^{(a)} := \prod_{i=1}^{m} X_i^{(a_i)}$, $\binom{a}{b} := \prod_{i=1}^{m} \binom{a_i}{b_i}$, $|a| := \sum_{i=1}^{m} a_i$ and $a \geqslant b$ if and only if $a_i \geqslant b_i$ for all i.

The first thing to observe is the following calculation

$$\begin{aligned} X^{(a)}X^{(b)} &= \prod_{i=1}^{m} X_i^{(a_i)} \prod_{j=1}^{m} X_j^{(b_j)} = \prod_{k=1}^{m} X_k^{(a_k)} X_k^{(b_k)} = \prod_{k=1}^{m} \frac{1}{a_k!}\frac{1}{b_k!} X_k^{a_k} X_k^{b_k} = \\ &= \prod_{k=1}^{m} \frac{1}{a_k! b_k!} X_k^{(a_k+b_k)} = \prod_{k=1}^{m} \frac{(a_k+b_k)!}{a_k! b_k!} \frac{X_k^{a_k+b_k}}{(a_k+b_k)!} = \\ &= \prod_{i=1}^{m} \frac{(a_i+b_i)!}{a_i! b_i!} \prod_{j=1}^{m} \frac{X_j^{a_j+b_j}}{(a_j+b_j)!} = \prod_{i=1}^{m} \binom{a_i+b_i}{a_i} X^{(a+b)} = \binom{a+b}{a} X^{(a+b)} \end{aligned}$$

, i.e. $X^{(a)}X^{(b)} = \binom{a+b}{a} X^{(a+b)}$. Also if $\varepsilon_i = (0, ..., 1, ..., 0)$, with 1 at i-th position, we have that $\partial_i X_i^{(a_i)} = \frac{1}{a_i!}\partial_i X_i^{a_i} = \frac{1}{a_i!} X_i^{a_i - \varepsilon_i}$ for all i and $a_i \neq 0$. Hence the partial derivatives satisfy the following relation $\partial_i X^{(a)} = X^{(a-\varepsilon_i)}$ for all $a_i \neq 0$.

Let now $1 \leqslant i \leqslant m$, $r, s \geqslant 0$ and $\mathbb{F}$ is of characteristic p. Define the following relations $x_i^{(0)} = 1$ and $x_i^{(r)} x_i^{(s)} = \binom{r+s}{r} x_i^{(r+s)}$. They generate a commutative associative algebra denoted by $\mathcal{O}(m)$. Put $x_i := x_i^{(1)}$ and $x^{(a)} := x_{(1)}^{(a_1)} ... x_m^{(a_m)}$ for $a \in \mathbb{N}^m$. Then $\{x^{(a)} \mid 0 \leqslant a, a \in \mathbb{N}^m\}$ is a basis for $\mathcal{O}(m)$. Next define $\mathcal{O}(m)_{(j)} := span\{x^{(a)} \mid |a| \geqslant j\}$. Thus $(\mathcal{O}(m))_{(j)}$ is a descending chain of ideals. Let $\mathcal{O}((m))$ denote the completion of $\mathcal{O}(m)$. Then for any m-tuple $\underline{n} = (n_1, ..., n_m) \in \mathbb{N}^m \cup \{\infty\}$ we set $\mathcal{O}(m; \underline{n}) := span\{x^{(a)} \mid 0 \leqslant a_i < p^{n_i}\}$ where p is the characteristic of the field and $p^{\infty} = \infty$. Now we similarly denote by $\mathcal{O}((m; \underline{n}))$ the completion of $\mathcal{O}(m; \underline{n})$.

Due to the defining relations $\mathcal{O}(m; \underline{n})$ and $\mathcal{O}((m; \underline{n}))$ are subalgebras of $\mathcal{O}(m)$ and $\mathcal{O}((m))$ respectively. If $\underline{n} = (\infty, ..., \infty)$, then $\mathcal{O}(m; \underline{n}) = \mathcal{O}(m)$ as well as $\mathcal{O}((m; \underline{n})) = \mathcal{O}((m))$. Observe that $\dim \mathcal{O}(m; \underline{n}) = p^{|n|}$ if $\underline{n} \in \mathbb{N}^m$.

For each i let ∂_i be the derivation of $\mathcal{O}(m)$ defined by $\partial_i(x_j^{(r)}) = \delta_{ij} x_j^{(r-1)}$. This condition clearly implies that $\partial_i(\mathcal{O}(m)_{(j)}) \subset \mathcal{O}(m)_{(j-1)}$. We notice also that ∂_i is a continuous derivation of $\mathcal{O}((m))$. This means that $\partial_i(\sum \alpha_a x^{(a)}) = \sum \alpha_a x^{a-\varepsilon_i}$ is true for infinite sums. We finally set:

$$W(m) := \sum_{i=1}^{m} \mathcal{O}(m)\partial_i \quad W((m)) := \sum_{i=1}^{m} \mathcal{O}((m))\partial_i$$

and

$$W(m;\underline{n}) := \sum_{i=1}^{m} \mathcal{O}(m;\underline{n})\partial_i \quad W((m;\underline{n})) := \sum_{i=1}^{m} \mathcal{O}((m;\underline{n}))\partial_i$$

These are called *Witt* algebras. Note that $\dim W(m;\underline{n}) = mp^{|n|}$ if $\underline{n} \in \mathbb{N}^m$.

7.2 Classical Lie Algebras

We already dealt with classical Lie algebras in characteristic 0. Now the idea is to construct new Lie algebras from the classical and exceptional Lie algebras over $\mathbb{C}$ by use of Chevalley basis and reduction modulo p. We again call those *classical* Lie algebras, and throughout this entire section we use the term classical in this sense.

Let L be a finite dimensional simple Lie algebra over $\mathbb{C}$ with Killing form κ and a base of simple roots $\Delta = \{\alpha_1, ..., \alpha_l\}$. Fix a Chevalley basis $\{x_\alpha, h_i\}$ of L. Then we construct the associated Chevalley algebra $L(\mathbb{F})$ as in Section 6.4. We recall that this construction is independent on the choice of the Cartan subalgebra and the base Δ. Different choices simply give rise to isomorphic Lie algebras.

According to Table 3.1. for $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$ we have the following possible values

$\langle \alpha, \beta \rangle$	0	1	-1
$\langle \beta, \alpha \rangle$	0	1,2,3	$-1, -2, -3$

Recall also that $0 \leqslant r + q \leqslant 3$ (see Section 6.1). The only problem now is that $L(\mathbb{F})$ may fail to be simple. For $p > 3$, $L(\mathbb{F})$ can only have central ideals. More importantly, it happens that $L(\mathbb{F})$ is simple except when $L(\mathbb{F}) \cong A_l$, $l \equiv -1(\mod p)$. In this case $L(\mathbb{F})$ has one dimensional central ideal $C = \mathbb{F}(h_1 + 2h_2 + \cdots + lh_l)$, and clearly $L(\mathbb{F})/C$

($\cong \mathfrak{psl}(l+1)$)is simple. Details may be found in [15]. Thus the *simple classical* Lie algebras are: $A_l\,(p \nmid l+1)$, $\mathfrak{psl}(l+1)\,(p \mid l+1)$, B_l, C_l, D_l, G_2, F_4, E_6, E_7, E_8.
We notice, that by abuse of the characteristic 0 notation, exceptional types are also included in characteristic p. Moreover, this construction is valid for any $p > 3$.

There is also an axiomatic approach to the classical Lie algebras due to G.B.Seligman and W.H.Mills which is in the following

Theorem 7.2.1 *Let $\mathbb{F}_p$ be a field of characteristic $p > 3$. A Lie algebra L over $\mathbb{F}_p$ is a direct sum of simple classical Lie algebras if and only if:*
(1) the center of L is 0,
(2) $L^{(1)} = L$,
(3) L has an abelian CSA H, relative to which
(a) $L = \sum L_\alpha$, where $[h, x] = \alpha(h)x$ for all $x \in L_\alpha, h \in H$,
(b) if $\alpha \neq 0$ is a root, $[L_\alpha, L_{-\alpha}]$ is $1-$ dimensional,
(c) if α and β are roots, and if $\beta \neq 0$, then not all $\alpha + k\beta (k \in \mathbb{F}_p)$ are roots.

The proof can be found in [15]. We have already seen part (3) for characteristic 0 (See Proposition 4.1.3).

7.3 Lie Algebras of Cartan Type

Differential forms are essential to define Lie algebras of Cartan type. This is why we give a brief discussion on them in Appendix C.

In [16] Skryabin proved that $W((m))$ is a free $\mathcal{O}((m))$-module with basis $\partial_1, ..., \partial_m$. This allows us to define differential forms. Set $\Omega^0((m)) = \mathcal{O}((m))$ and $\Omega^1((m)) = \mathrm{Hom}_{\mathcal{O}((m))}(W((m)), \mathcal{O}((m)))$. Thereafter, the aim is to give $\Omega^1((m))$ an $\mathcal{O}((m))$-module structure. We can do this naturally via the following equation:

$$(f\omega)(D) = f\omega(D) \quad \text{for all} \quad f \in \mathcal{O}((m)), \omega \in \Omega^1((m)) \,\text{and}\, D \in W((m)).$$

We also give the canonical $W((m))$-module structure via:

$$(D\omega)(D') = D(\omega(D')) - \omega([D, D']) \quad \text{for all} \quad D, D' \in W((m)) \,\text{and}\, \omega \in \Omega^1((m)).$$

We also have that:

$$D(f\omega) = (Df)\omega + f(D\omega) \quad \text{for all} \quad D \in W((m)), f \in \mathcal{O}((m)), \omega \in \Omega^1((m)).$$

The last equation helps us to view every element D of $W((m))$ as a derivation. For this purpose we need simply to set the r-fold exterior power over $\mathcal{O}((m))$, i.e. $\Omega^r((m)) = \bigwedge^r \Omega^1((m))$. We also use the following notation $\Omega((m)) = \bigoplus \Omega^r((m))$. In particular, for $r = 2$, we can define the derivation via:

$$D(\omega_1 \wedge \omega_2) = D(\omega_1) \wedge \omega_2 + \omega_1 \wedge D(\omega_2) \quad \text{for all} \quad \omega_1, \omega_2 \in \Omega((m)).$$

We also define the linear mapping $d : \Omega^0((m)) \to \Omega^1((m))$ via the equation $(df)(D) = D(f)$ for all $f \in \mathcal{O}((m)), D \in W((m))$. Since we endowed $\Omega^1((m))$ with an $\mathcal{O}((m)) -$ *module* structure, every $\lambda \in \Omega^1((m))$ is determined by its action on $\partial_1, ..., \partial_m$. Using that $\lambda(\partial_i) \in \mathcal{O}((m))$ and $\partial_i \in W((m))$ we have $(\sum_j \lambda(\partial_j) dx_j)(\partial_i) = \sum_j \lambda(\partial_j)((dx_j)(\partial_i)) = \lambda(\partial_i)$ and therefore we infer that $\lambda = \sum_j \lambda(\partial_j) dx_j$. We have thus shown that $\Omega^1((m))$ is a free module with basis $\{dx_1, ..., dx_m\}$. Further, we use the canonical $W((m))$-module structure and the definition of the linear mapping d to obtain the property $D(df) = dD(f)$.

It directly follows from the following calculation, for all $D, E \in W((m))$:

$$(D(df))(E) = D((df)(E)) - (df)([D,E]) = D(E(f)) - [D,E](f) = E(D(f)) = d(D(f))(E)$$

We next claim that $df^{(k)} = f^{(k-1)}df$ holds for all $f \in \mathcal{O}((m))_{(1)}$, $k \geqslant 1$ and $D \in W((m))$. Indeed, as $W((m))$ is an $\mathcal{O}((m))$-module, we proof the claim via the calculation

$$(df^{(k)} - f^{(k-1)}df)(D) = (df^{(k)})(D) - f^{(k-1)}df(D) = D(f^{(k)}) - f^{(k-1)}D(f) = 0.$$

Set $d(fdg) = df \wedge dg$ and extend it inductively via $d(\omega_1 \wedge \omega_2) = d(\omega_1) \wedge \omega_2 + (-1)^{deg(\omega_1)}\omega_1 \wedge d(\omega_2)$ for all $\omega_1, \omega_2 \in \Omega((m))$. Then $d : \Omega((m)) \to \Omega((m))$ is a linear operator of degree 1 which,for all $f \in \mathcal{O}((m)), D \in W((m)), \omega \in \Omega((m))$, satisfy the following:
(1) $d^2\omega = 0$, (2) $(df)D = D(f)$, (3) $D(d\omega) = d(D\omega)$, (4) $d(f\omega) = (df) \wedge \omega + fd\omega$.

Differential forms of particular interest in prime characteristic are the following:
$\omega_S = dx_1 \wedge \cdots \wedge dx_m$, $m \geqslant 3$
$\omega_H = \sum_{i=1}^{r} dx_i \wedge dx_{i+r}$, $m = 2r$
$\omega_K = dx_m + \sum_{i=1}^{2r} \sigma(i)x_i dx_{i'}$, $m = 2r+1$,
where in the last equation the following notations are referred:

$$i' = \begin{cases} i+r & \text{if } 1 \leqslant i \leqslant r; \\ i-r & \text{if } r+1 \leqslant i \leqslant 2r. \end{cases} \quad \text{and} \quad \sigma(i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant r; \\ -1 & \text{if } r+1 \leqslant i \leqslant 2r. \end{cases}$$

Notice that for $m = 2, r = 1$ we have $\omega_S = \omega_H$. Now, this differential forms give rise to the following Lie subalgebras of $W((m))$:

$$S((m)) = \{D \in W((m)) \mid D(\omega_S) = 0\},$$

$$CS((m)) = \{D \in W((m)) \mid D(\omega_S) \in \mathbb{F}\omega_S\},$$
$$H((2r)) = \{D \in W((2r)) \mid D(\omega_H) = 0\},$$
$$CH((2r)) = \{D \in W((2r)) \mid D(\omega_H) \in \mathbb{F}\omega_{H)}\},$$
$$K((2r+1)) = \{D \in W((2r+1)) \mid D(\omega_K) \in \mathcal{O}((2r+1))\omega_K\}$$

These algebras are not always simple, but direct computations show that their suitable commutator subalgebras are simple (See [17]). Now, for $X \in \{S, CS, H, CH, K\}$ and all $m \in \mathbb{N}$, $\underline{n} \in (\mathbb{N} \cup \{\infty\})^\infty$ we define $X((m; \underline{n})) := X((m)) \cap W((m; \underline{n}))$ and $X(m; \underline{n}) := X((m)) \cap W(m; \underline{n})$ Notice that $X(m; \underline{n}) = X(m)$ and $X((m; \underline{n})) = X((m))$ for $\underline{n} = (\infty, ..., \infty)$ and $X((m; \underline{n})) = X(m; \underline{n})$ for any $\underline{n} \in \mathbb{N}^m$. Then we have that $X((m; \underline{n}))$ is finite dimensional if and only if $\underline{n} \in \mathbb{N}^m$.

We need one more definition before we define Lie algebras of Cartan type. A *grading* of a Lie algebra L is a decomposition $L = \bigoplus_{j \in \mathbb{Z}} L_j$, such that $[L_i, L_j] \subset L_{i+j}$ for all $i, j \in \mathbb{Z}$. We recall that $[L_i, L_J] = \langle\, [x, y] \mid x \in L_i, y \in L_j \rangle$. Lie algebras admitting such decomposition are called *graded*.

Example 7.3.1 For example let us consider $L = \mathfrak{n}(4, \mathbb{F})$, i.e. the Lie algebra of 4×4 strictly upper-triangular matrices. Then the grading of L is given by $L = \bigoplus_{j=0}^{3} L_j$ where L_0 is the zero algebra and L_1, L_2 and L_3 are generated by

$$L_1 = \left\langle \begin{pmatrix} 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \lambda, \mu, \nu \in \mathbb{F} \right\rangle, L_2 = \left\langle \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \lambda, \mu \in \mathbb{F} \right\rangle$$

and

$$L_3 = \left\langle \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{F} \right\rangle$$

A graded Lie algebra L is called a *graded Lie algebra of Cartan type* X, if there are $X \in \{W, S, CS, H, CH, K\}$, $m \in \mathbb{N}$, $\underline{n} \in \mathbb{N}^m$ such that $X(m;\underline{n})^{(\infty)} \subset L \subset X(m;\underline{n})$ as X-graded subalgebras. The X-gradation is the corresponding gradation of W. It happens that all algebras of Cartan type are obtained from W, S, CS, H, CS, K and due to lack of time we stop here. For further generalizations see [17].

7.4 A Word on Classification

The classical Lie algebras along with Lie algebras of Cartan type exhaust all the possibilities for finite simple Lie algebras when $p > 5$. This has been recently proven in [13] by A. Premet and H. Strade. Their original statement was

Theorem 7.4.1 *Any finite dimensional simple Lie algebra over an algebraically closed field of characteristic* $p > 5$ *is either classical or of Cartan type.*

In characteristic 5 a further family of simple Lie algebras is known. Discovered by G.Melikian [8], they are now known as *Melikian algebras* and appear only in this characteristic. More importantly, no zero characteristic analogues are yet known, what makes them very important for the ultimate classification in positive characteristic.

In 2008 A.Premet and H.Strade, concluded their series of papers([9],[10],[11],[12],[13] and [14]) and succeeded to prove in [14] the following classification theorem:

Theorem 7.4.2 *Every simple finite dimensional Lie Algebra over an algebraically closed field of characteristic* $p > 3$ *is of classical, Cartan, or Melikian type.*

The proof is very long and technical, but indisputably this theorem is an enormous progress. Characteristics 2 and 3 remain a huge open problem which is claimed to be very subtle and delicate.

Appendix A

Bilinear Forms

We used the notion of bilinear form several times in the main text. For this reason we shortly recapitulate (without proofs) all definitions and facts relevant to this thesis.

Let V be a finite dimensional vector space over a field $\mathbb{F}$. Then a *bilinear form* on V is a map:

$$\beta : V \times V \rightarrow \mathbb{F}$$

such that

$$(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 (v_1, w) + \alpha_2 (v_2, w),$$

$$(v, \beta_1 w_1 + \beta_2 w_2) = \beta_1 (v, w_1) + \beta_2 (v, w_2),$$

for all $v, w, v_i, w_i \in V$ and $\alpha_i, \beta_i \in \mathbb{F}$.

The most trivial example for bilinear form is the dot product in any Euclidean space. Another example is quadratic forms, but they are not of interest in this thesis.

It is convenient to represent bilinear forms by matrices. For this purpose we only need to fix a basis on V, say $\{v_1, ..., v_n\}$. Then the matrix of β is $A_\beta = (a_{ij})$, with entries $a_{ij} := (v_i, v_j)$. A_β is usually called the *Gram matrix* for β. If we choose another basis, say $\{w_1, ..., w_n\}$, the matrix A'_β is nothing but $P^t A P$, where $P = (p_{ij})$ and is defined by

$w_j = \sum_{i=1}^{n} p_{ij} v_i$. The converse is also true, so given an $n \times n$ matrix we can define a bilinear form on V.

Define $S = \{x \in V \mid \beta(x, y) = 0 \text{ for all } y \in V\}$ and call it the *radical* of β. If $S = 0$, then β is called *nondegenerate*. By bilinearity we have that the radical is a subspace of V. In fact, S is $V^{\perp}$.

Here we recall the fact that even dimensionality is a necessary condition for existence a nondegenerate bilinear form satisfying $f(v, w) = -f(w, v)$. This follows from the following well-known facts:

(i) a bilinear form is nondegenerate if and only if its Gram matrix has non-zero determinant.

(ii) the determinant of a skew-symmetric $(2l + 1) \times (2l + 1)$ *matrix is always zero.*

Appendix B

Exceptional Lie Algebras

In this chapter we describe some new types of root systems which are associated with so called *exceptional Lie algebras*. In each case we use the following set up. Let E be a subspace of $\mathbb{R}^m$ and ε_i be the vector with 1 in i-th position and 0 elsewhere. Similarly to chapter 4 we take as many simple roots as possible from the set $\{\alpha_1, ..., \alpha_{m-1}\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. For these elements we have

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } |i - j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

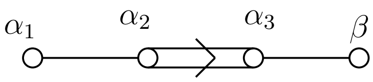
B.1 Type G_2

Let $E = \{v = \sum_{i=1}^{3} c_i \varepsilon_i \in \mathbb{R}^3 : \sum c_i = 0\}$, let $I = \{m_1\varepsilon_1 + m_2\varepsilon_2 + m_3\varepsilon_3 \in \mathbb{R}^3 : m_1, m_2, m_3 \in \mathbb{Z}\}$, and let $R = \{\alpha \in I \cap E : (\alpha, \alpha) = 2 \text{ or } (\alpha, \alpha) = 6\}$. This choice is motivated by the fact that the ratio of the length of a long root to the length of short root in this case is

$\sqrt{3}$. By direct calculation we have that this root system is given by the set: $R = \{\pm(\varepsilon_i - \varepsilon_j) \mid i \neq j\} \cup \{\pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \mid \{i,j,k\} = \{1,2,3\}\}$. This gives 12 roots in total as expected. To find a base, we need to find two roots in R of different lengths making an angle of $\frac{5\pi}{6}$. One such choice is $\alpha = \varepsilon_1 - \varepsilon_2$ and $\beta = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1$.

B.2 Type F_4

Fortunately, this type of root systems can be constructed by simply extending the root system of B_3. We start with given roots $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$ and ε_3, and look for a root $\beta \in \mathbb{R}^4$ so that $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3, \beta\}$ is a base for F_4. A straightforward computation shows that the length of the root α_2 is $\sqrt{2}$, while the length of α_3 is 1. Furthermore, another simple calculation gives us $\langle \alpha_2, \alpha_3 \rangle = -2$ and $\langle \alpha_3, \alpha_2 \rangle = -1$ which implies that there must be two edges between α_2 and α_3. Thus the Dynkin diagram is the following:



Δ must span $\mathbb{R}^4$ since we want it to be a base of our root system. From linear algebra clearly follows that the first three roots in Δ are linearly independent and we now only need an appropriate β. Easily we observe that the only possibilities are $\beta = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \pm \varepsilon_4$. We now set $R = \{\pm\varepsilon_i : 1 \leqslant i \leqslant 4\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i \neq j \leqslant 4\} \cup \{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$. First thing to be checked are certainly the axioms (R1) up to (R4). Since this is routine we omit this kind of calculations. What remains to check is that:

$\beta_1 = \varepsilon_1 - \varepsilon_2$,

$\beta_2 = \varepsilon_2 - \varepsilon_3$,

$\beta_3 = \varepsilon_3$,

$\beta_4 = \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)$,

really defines a base for R. Let $\sum_{i=1}^{4} c_i\beta_i = 0$. Then from $\sum_{i=1}^{4} c_i\beta_i = (c_1 - \frac{1}{2}c_4)\varepsilon_1 + (-c_1 -$

$\frac{1}{2}c_4 + c_2)\varepsilon_2 + (-c_2 + c_3 - \frac{1}{2}c_4)\varepsilon_3 + \frac{1}{2}c_4\varepsilon_4$, and the fact that $\varepsilon_i's$ are the standard basis in $\mathbb{R}^4$. Thus $\beta_i's$ clearly form another basis of $\mathbb{R}^4$ and hence B(1) holds. Using similar calculations we verify the axiom B(2) and we are done.

We also easily count that R consists of 48 elements, so we need to find 24 positive roots. Indeed, each ε_i is a positive root and they are 4 in total. If $1 \leqslant i < j \leqslant 3$ then both $\varepsilon_i - \varepsilon_j$ and $\varepsilon_i + \varepsilon_j$ are positive as well. Their number is 6. Furthermore, $\varepsilon_4 \pm \varepsilon_i$ for $1 \leqslant i \leqslant 3$ are another 6 positive roots. The rest positive roots are 3 of the form $\beta_4 + \varepsilon_j$, 3 of the form $\beta_4 + \varepsilon_j + \varepsilon_k$, β_4 itself and $\beta_4 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$.

B.3 Type E

Root systems of type E are the root systems of types E_6, E_7 and E_8 respectively. It is very convenient to describe first the root systems of type E_8 and then to find the root systems of type E_6 and E_7 inside it. We now rather give only an outline how to construct these since many technicalities and long calculations appear. For details one could refer [2].

Let $E = \mathbb{R}^8$ and let $R = \{\pm\varepsilon_i \pm \varepsilon_j : i < j\} \cup \{\frac{1}{2}\sum_{i=1}^{8} \pm\varepsilon_i\}$, where in the second set an even number of + signs are chosen.

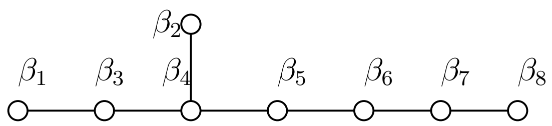
Assuming that R is a root system, we claim that $\{\beta_1, \beta_2, ..., \beta_8\}$ is its base, where

$\beta_1 = \frac{1}{2}(-\varepsilon_1 - \varepsilon_8 + \sum_{i=2}^{7} \varepsilon_i)$,

$\beta_2 = -\varepsilon_1 - \varepsilon_2$,

$\beta_i = \varepsilon_{i-2} - \varepsilon_{i-1}$ for $3 \leqslant i \leqslant 8$.

Verifying B(1) is yet straightforward although long calculations. To check B(2) we need first to verify that the roots $\pm\varepsilon_i - \varepsilon_j$ for $i < j$ can be written as linear combination of the simple roots with positive coefficients. It turns out that the remaining positive roots are those of the form $\frac{1}{2}(-\varepsilon_8 + \sum_{i=1}^{7} \pm\varepsilon_i)$. The labeled Dynking diagram for E_8 is:



Finally, the base for a root system of type E_7 is obtained by omitting the root β_8. Similarly, $\{\beta_1, ..., \beta_6\}$ is the base for E_6.

Appendix C

Differential Forms

We give here the notion of a *differential form*. We discuss only the definitions and properties required for the definition of Cartan type algebras. For details on this matter the reader can refer to [1], [3].

Taking a vector $(x_1, ..., x_n) \in \mathbb{R}^n$ we define Ω^* to be the algebra over $\mathbb{R}$ generated by $dx_1, ..., dx_n$ with relations:

(i) $(dx_i)^2 = 0$

(ii) $dx_i dx_j = -dx_j dx_i$ for $i \neq j$.

Clearly, the basis of Ω^* is 1, dx_i, $dx_i dx_j$, $dx_i dx_j dx_k$, ..., $dx_1 dx_2 ....dx_n$. Notice, that in order to avoid repetitions we require $i < j$ for $dx_i dx_j$ as well as $i < j < k$ for $dx_i dx_j dx_k$ and so on. If $n = 3$, then $1, dx_1, dx_2, dx_3, dx_1dx_2, dx_1dx_3, dx_2dx_3, dx_1dx_2dx_3$ is the basis and clearly the dimension of Ω^* as a vector space is 2^3. Easy combinatorics shows that in general $\dim \Omega^* = 2^n$. Let C^∞ be the set of all smooth functions. Then we define C^∞ *differential forms* on $\mathbb{R}^n$ as the elements of $\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*$. In other words, any differential form $\omega \in \Omega^*(\mathbb{R}^n)$ is uniquely represented by $\omega = \sum f_{i_1...i_q} dx_{i_1} ...dx_{i_q}$ with coefficients $f_{i_1...i_q}$ being C^∞ function. We call ω a *q-form*. Also for brevity one usually writes $\omega = \sum f_I dx_I$. Looking back at the generators we easily see that if one dx_i is discarded the remaining generators will generate an algebra of smaller dimension with the same properties as in Ω^*. If we keep discarding generator after generator we shall obtain smaller and smaller subalgebras of Ω^*. Thus $\Omega^*(\mathbb{R}^n)$ is naturally graded, i.e.

$$\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^{n} \Omega^q(\mathbb{R}^n),$$

where $\Omega^q(\mathbb{R}^n)$ consists of C^∞ q-forms on $\mathbb{R}^n$.

We also define a differential operator :

$$d : \Omega^q(\mathbb{R}^n) \longrightarrow \Omega^{q+1}(\mathbb{R}^n)$$

satisfying the following

$$(a) \quad \text{if } f \in \Omega^0(\mathbb{R}^n), \text{ then } df = \sum \frac{\partial f}{\partial x_i} dx_i$$
$$(b) \quad \text{if } \omega = \sum f_I dx_I, \text{ then } d\omega = \sum df_I dx_I.$$

Let $\tau = \sum f_I d\, x_I$ and $\omega = \sum g_J d\, x_J$. Then we define the *wedge product* of τ and ω as $\tau \wedge \omega = \sum f_I g_J d\, x_I\, d\, x_J$. It is also known as an *exterior multiplication.* Clearly, for any scalars $a, b \in \mathbb{R}$ we have that $(a\,\tau)(b\,\omega) = \sum a\, b f_I g_J d\, x_I\, d\, x_J = a\, b \sum f_I g_J d\, x_I\, d\, x_J = a\, b \tau \wedge \omega$. Also, for $\omega' = \sum g_K d\, x_K$ we have that $\tau \wedge \omega + \tau \wedge \omega' = \sum f_I g_J d\, x_I\, d\, x_J + \sum f_I g_K d\, x_I\, d\, x_K = \sum f_I d\, x_I (g_J d\, x_J + g_K d\, x_K) = \tau \wedge (\omega + \omega')$ and so the wedge product is bilinear. It is also anticommutative with $\tau \wedge \omega = (-1)^{(deg\tau)(deg\omega)} \omega \wedge \tau$. To see this we only need to apply relation (i) $(deg\tau)(deg\omega)$ times.

The first property of the operator d is the following

Proposition C.0.1 *d is an anti-derivation, i.e. for any two differential forms τ and ω we have $d(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg \tau} \tau \wedge (d\omega)$.*

Proof. Let $\tau = \sum f_I d\, x_I$ and $\omega = \sum g_J d\, x_J$. For 0-forms, or functions, we know that $d\,(fg) = (d\, f)g + f(d\, g)$. Then the anticommutativity of the wedge product yields

$$d\,(\tau \wedge \omega) = \sum d\,(f_I g_J) d\, x_I d\, x_J = \sum (d\, f_I) g_J d\, x_I d\, x_J + \sum f_I (d\, g_J) d\, x_I d\, x_J =$$
$$= \sum g_J (d\, f_I d\, x_I) d\, x_J + \sum f_I (-1)^{deg\tau} (d\, g_J d\, x_J) d\, x_I = (d\,\tau) \wedge \omega + (-1)^{deg\tau} \tau \wedge (d\,\omega). \qquad \square$$

Proposition C.0.2 $d^2 = 0$

Proof. We first prove this for functions. For any function f we have that $d^2(f) = d(df) = d(\sum_i \frac{\partial f}{\partial x_i} d\,x_i) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} d\,x_j d\,x_i$. Thus $d^2 f = 0$, since $d\,x_i$ and $d\,x_j$ are skew-symmetric, and the mixed partial derivatives are equal. Further, $d^2\omega = d^2(\sum f_I d\,x_I) = d(\sum d\,f_I d\,x_I) = \sum d\,(d\,f_I d\,x_I)$. Now the right hand side equals zero by former calculation and the antiderivation property of d. □

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