COMMUTING VARIETIES AND NILPOTENT ORBITS

by

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Abstract

Let G be a reductive algebraic group over an algebraically closed field k of good characteristic, let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G, and let P be a parabolic subgroup of G with $\mathfrak{p} = \mathrm{Lie}(P)$.

We consider the commuting variety $\mathcal{C}(\mathfrak{p})$ of $\mathfrak{p}$ and obtain two criteria for $\mathcal{C}(\mathfrak{p})$ to be irreducible. In particular we classify all cases when the commuting variety $\mathcal{C}(\mathfrak{b})$ is irreducible, for $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$.

We then let G be a classical group and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be nilpotent orbits of G in $\mathfrak{g}$. We say that $\mathcal{O}_1$ and $\mathcal{O}_2$ commute if there exists a pair $(X, Y) \in \mathcal{O}_1 \times \mathcal{O}_2$ such that $[X, Y] = 0$. For $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$ or $\mathfrak{g} = \mathfrak{so}_n(k)$, we describe the orbits that commute with the regular orbit, and classify (with one exception) the orbits that commute with all other orbits in $\mathfrak{g}$. This extends previously-known results for $\mathfrak{g} = \mathfrak{gl}_n(k)$.

Finally let ϕ be a Springer isomorphism, that is, a G-equivariant isomorphism from the unipotent variety $\mathcal{U}$ of G to the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$. We show that *polynomial* Springer isomorphisms exist when G is of type G_2, but do not exist for types E_6 and E_7 for k of small characteristic.

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CONTENTS

CHAPTER 1

INTRODUCTION

Let k be an algebraically closed field, let G be an algebraic group over k and let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. The goal of this thesis is to investigate three distinct problems which fall within this setup, two of which share the common theme of pairs of commuting elements within $\mathfrak{g}$. Consequently we devote Chapter 2 to introducing the theory of algebraic groups and to describing the well-known theory of the nilpotent orbits of G on $\mathfrak{g}$.

To begin, we define the *commuting variety* of $\mathfrak{g}$ to be the set $\mathcal{C}(\mathfrak{g})$ of pairs $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ such that $[X, Y] = 0$. The study of this object can be traced back to Motzkin and Taussky [37], who in 1955 proved Theorem 1.1 below in the case of $\mathfrak{g} = \mathfrak{gl}_n$, and to Gerstenhaber who was among the first to consider $\mathcal{C}(\mathfrak{g})$ as a variety and in 1961 obtained several results concerning its structure [20]. Richardson [45] in 1979 then extended Motzkin and Taussky's result to general reductive $\mathfrak{g}$; specifically, he proved the following theorem:

Theorem 1.1. *Let* $\mathrm{char} k = 0$*, let* $\mathfrak{t}$ *be a maximal toral subalgebra of* $\mathfrak{g}$*, and let* G *be a reductive algebraic group such that* $\mathfrak{g} = \mathrm{Lie}(G)$*. Then* $\mathcal{C}(\mathfrak{g}) = \overline{G \cdot (\mathfrak{t} \times \mathfrak{t})}$*, and in particular* $\mathcal{C}(\mathfrak{g})$ *is irreducible.*

More recently in 2002 Levy [36] showed that, subject to some restrictions on the structure of G, Theorem 1.1 still holds in characteristic $p > 0$.

A refinement of the idea of the commuting variety is the *nilpotent commuting variety* defined as $\mathcal{C}^{\mathrm{nil}}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g}) \cap (\mathcal{N} \times \mathcal{N})$ where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$, and in recent years more attention has been focused on studying the irreducibility (or otherwise) of $\mathcal{C}^{\mathrm{nil}}(\mathfrak{g})$. Baranovsky in 2001 [3] proved that $\mathcal{C}^{\mathrm{nil}}(\mathfrak{gl}(V))$ is irreducible over a field of characteristic 0; furthermore, he conjectured that the irreducible components of $\mathcal{C}^{\mathrm{nil}}(\mathfrak{g})$ are of equal dimension for $\mathfrak{g}$ the Lie algebra of a general connected reductive algebraic group. This was proved, and both results extended to characteristic $p > 0$ subject to some minor restrictions, by Premet [44] in 2003.

The structure of $\mathcal{C}(\mathfrak{g})$ and $\mathcal{C}^{\mathrm{nil}}(\mathfrak{g})$ are thus both reasonably well understood. A natural next question is thus to investigate the commuting varieties of various subalgebras of $\mathfrak{g}$. Goodwin and Röhrle [26] have considered the case of $\mathcal{C}(\mathfrak{u})$ where $\mathfrak{u}$ is the nilradical of the Lie algebra of a Borel subgroup B of G and have shown that $\mathcal{C}(\mathfrak{u})$ is equidimensional if B acts on $\mathfrak{u}$ with a finite number of orbits. Bulois and Evain [14] meanwhile have studied $\mathcal{C}^{\mathrm{nil}}(\mathfrak{p})$, for $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$, via the theory of Hilbert schemes, and have proved whether or not $\mathcal{C}^{\mathrm{nil}}(\mathfrak{p})$ is irreducible for certain parabolic subalgebras $\mathfrak{p}$.

It would appear, however, that the structure of $\mathcal{C}(\mathfrak{p})$ without the nilpotency condition is much less well understood. In his thesis of 1995 Keeton [34] investigated the structure of $\mathcal{C}(\mathfrak{b})$ for a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and obtained a partial classification for when this is irreducible or reducible. In Section 3.6 we are able to use more recent results of [25], [46] and [42] to complete this classification as follows.

Theorem 1.2. *Let G be a connected reductive algebraic group over an algebraically closed field k with* $\operatorname{char} k = 0$*. Then $\mathcal{C}(\mathfrak{b})$ is irreducible if and only if the type of each simple component of G is one of the following.*

(i.) A_l *for* $l \leqslant 15$*;*

(ii.) B_l *for* $l \leqslant 6$*;*

(iii.) C_l *for* $l \leqslant 6$*;*

(iv.) D_l *for* $l \leqslant 7$*;*

(v.) G_2 *or* E_6*.*

We observe in particular that $\mathcal{C}(\mathfrak{b})$ is in most cases not irreducible.

In Chapter 3 we therefore investigate the structure of $\mathcal{C}(\mathfrak{p})$ in more detail, for an arbitrary parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. We are able to obtain criteria for $\mathcal{C}(\mathfrak{p})$ to be irreducible using two different methods, the first a generalisation of Richardson's approach, and the second based on Keeton's work. We note that this second criterion is both necessary and sufficient. We are then able to use these criteria to determine certain cases in which $\mathcal{C}(\mathfrak{p})$ is irreducible, in conjunction with work of Murray [39] and of Boos and Bulois [8].

In Chapter 4 we consider the orbits of the adjoint action of G on $\mathfrak{g}$. It is well-known that in the case that G is a classical group, these orbits are parametrised by certain partitions of the dimension of the underlying vector space which correspond to the Jordan normal forms of elements of the orbit. We describe this parametrisation in more detail in Chapter 2, and direct the reader to [32] for a full treatment.

If $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ are two orbits of this action with associated partitions λ and μ respectively, we say that $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_\mu$ if there exists a pair (X, Y) with $X \in \mathcal{O}_\lambda$ and $Y \in \mathcal{O}_\mu$ such that $[X, Y] = 0$. In general it is a difficult question to say whether two given orbits commute; however there has been recent interest in certain special cases, primarily in the case of $\mathfrak{g} = \mathfrak{gl}_n(k)$.

In 2008 Panysuhev [43] described the map $\mathcal{D}$ which associates to each orbit $\mathcal{O}$ the maximal orbit $\mathcal{D}(\mathcal{O})$ which commutes with it, and the study of this map for $\mathfrak{gl}_n(k)$ has motivated much recent work in this area. Košir and Oblak [35] in 2009 proved that $\mathcal{D}$ is idempotent, and described partitions λ such that the partition associated to $\mathcal{D}(\mathcal{O}_\lambda)$ has two parts. Oblak additionally conjectured an algorithm to determine $\mathcal{D}(\mathcal{O}_\lambda)$ for a given λ, the validity of which was shown by Basili [6]. More recent work of Iarrobino, Khatami, Van Steirtgehem and Zhao [31] has shed light on the set $\mathcal{D}^{-1}(\mathcal{O}_\lambda)$.

Further cases where two given orbits commute have been considered by Oblak [41] for $\mathfrak{gl}_n(k)$ over an algebraically closed field of characteristic 0, and by Britnell and Wildon [13] in the equivalent group-theoretic setting over finite fields. In particular it is known that the orbits $\mathcal{O}_\lambda$ and $\mathcal{O}_{(n)}$ commute if and only if λ is an almost rectangular partition; that is, any two of its parts differ by at most 1. Furthermore Oblak and Britnell and Wildon independently prove that an orbit $\mathcal{O}_\lambda$ commutes with all other orbits if and only if the largest part of λ is at most 2. We call such an orbit $\mathcal{O}_\lambda$ *universally commuting.*

Our main aim in Chapter 4 is to extend these results to the case of nilpotent orbits in $\mathfrak{g}$, where $\mathfrak{g}$ is either $\mathfrak{sp}_{2m}$ and $\mathfrak{so}_n$. In $\mathfrak{sp}_{2m}$ it is well–known that the nilpotent orbits of the adjoint action are labelled by partitions of $2m$ such that all odd parts occur with even multiplicity; similarly, nilpotent orbits in $\mathfrak{so}_n(k)$ are labelled by partitions of n where all even parts occur with even multiplicity. In Section 4.3 we show that the orbits which commute with the maximal orbit in $\mathfrak{g}$ are precisely $\mathcal{O}_\lambda$ such that λ labels an orbit in $\mathfrak{g}$ and $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(n)}$ in $\mathfrak{gl}_n(k)$.

The remainder of Chapter 4 is concerned with identifying the universally commuting orbits in $\mathfrak{sp}_{2m}(k)$ and $\mathfrak{so}_n(k)$. We begin by narrowing down the potentially universally commuting orbits to those which commute with both the maximal orbit in $\mathfrak{g}$ and another large orbit; then we examine the centralisers of orbits labelled by partitions with between two and four parts and use these as the building blocks of an inductive argument to obtain the following description of universally commuting orbits in $\mathfrak{sp}_{2m}(k)$ and $\mathfrak{so}_n(k)$.

Theorem 1.3. *(i.) The orbit $\mathcal{O}_\lambda$ is universally commuting in $\mathfrak{sp}_{2m}$ if and only if $\lambda = (2^a, 1^{2m-2a})$ with a odd and $0 \leqslant a < m$.*

(ii.) The orbit $\mathcal{O}_\lambda$ is universally commuting in $\mathfrak{so}_{2m+1}$ if and only if $\lambda = (2^a, 1^{2m-2a+1})$ with a even and $0 \leqslant a \leqslant m$.

(iii.) The orbit $\mathcal{O}_\lambda$ is universally commuting in $\mathfrak{so}_{2m}$ if $\lambda = (2^a, 1^{2m-2a})$ with a even and $0 \leqslant a < m$; or if $\lambda = (3, 2^a, 1^{2m-2a-3})$ with a even and $a < m-2$. If m is even and $\lambda = (3, 2^{m-2}, 1)$ then $\mathcal{O}_\lambda$ is universally commuting for $m \leq 6$; it is unknown whether it is universally commuting for $m \geqslant 8$. Otherwise $\mathcal{O}_\lambda$ is not universally commuting.

Finally in Chapter 5 we investigate another aspect to the study of nilpotent elements of the classical and exceptional Lie algebras, namely Springer isomorphisms. A *Springer isomorphism* is defined as an isomorphism ϕ from the unipotent variety $\mathcal{U}$ of G to the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$. These isomorphisms are named for T. A. Springer, who first proved their existence when G is a simply connected group in 1969 [49]. The result has since been strengthened to its present form; see for example [4, Cor. 9.3.4].

A natural question in this area concerns the existence of *polynomial* Springer isomorphisms. Such isomorphisms are more convenient to work with directly; so it is of interest to know whether they always exist, for example in the study of the coadjoint orbits of the unipotent radical U of a Borel subgroup of G. Consequently the main aim of Chapter 5 is to investigate when these polynomial Springer isomorphisms exist. It is well-known that we can find polynomial Springer isomorphisms for the classical groups [24, 2.1.7–8] so we focus our attentions on the exceptional cases. By using the computer algebra system GAP [19], we obtain the following results:

Theorem 1.4. *(i.) If G is of type G_2 and* $\operatorname{char} k \neq 2, 3$ *(that is,* $\operatorname{char} k$ *is good for G) then we can always find a polynomial Springer isomorphism.*

(ii.) If G is of type E_6 then a polynomial Springer isomorphism exists for the minimal representation if and only if $\operatorname{char} k > 13$.

(iii.) If G is of type E_7 then a polynomial Springer isomorphism exists for the minimal representation if and only if $\operatorname{char} k > 23$.

The question remains open for the E_8 and F_4 cases, though the above results for E_6 and E_7 suggest that it fails for E_8 at least. In any case, these results prevent the development of a general theory for fields of positive characteristic and thus render the resolution of these final cases of lesser interest.

CHAPTER 2

PRELIMINARIES

In this chapter we recall many of the basic definitions and results that we make use of throughout the rest of the thesis. For the most part we omit the proofs, noting that they can be found in the references provided. Throughout this thesis we let k be an algebraically closed field.

2.1 Classical groups and Lie algebras

We begin by defining the so-called classical groups and Lie algebras, in order to provide some examples to keep in mind when we introduce the general theory of algebraic groups.

A *Lie algebra* is a vector space $\mathfrak{g}$ over k with a bilinear operation $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ such that for all $X, Y, Z \in \mathfrak{g}$ we have $[X, X] = 0$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (the Jacobi identity). A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if it is closed under this operation.

Let V be a finite-dimensional vector space over k with $\dim V = n$. The general linear group $\mathrm{GL}(V)$ is the group comprised of all automorphisms of V. Similarly the general linear Lie algebra $\mathfrak{gl}(V)$ is comprised of all endomorphisms of V. We may represent $\mathrm{GL}(V)$ by the group $\mathrm{GL}_n(k)$ of all invertible $n \times n$ matrices with entries in k; the Lie algebra $\mathfrak{gl}(V)$ is then the set $\mathfrak{gl}_n(k)$ of all $n \times n$ matrices with entries in k, with the bracket operation $[X, Y] = XY - YX$ for all $X, Y \in \mathfrak{gl}_n(k)$.

The group $\mathrm{SL}_n(k)$ is comprised of all $n \times n$ matrices with determinant 1; similarly the Lie algebra $\mathfrak{sl}_n(k)$ is comprised of all $n \times n$ matrices with trace 0.

Suppose now that $\operatorname{char} k \neq 2$. Let $\phi : V \times V \to k$ be a bilinear form such that $\phi(v, w) = \varepsilon\phi(w, v)$ for all $v, w \in V$, and let $\varepsilon = \pm 1$. Additionally we assume that ϕ is non-degenerate, that is, for all $v \in V$ there exists $w \in V$ such that $\phi(v, w) \neq 0$. If $\varepsilon = 1$, then ϕ is called a *symmetric form*; if $\varepsilon = -1$, then ϕ is an *alternating* form.

Now let $G = \{g \in \mathrm{GL}(V) : \phi(g(v), g(w)) = \phi(v, w) \text{ for all } v, w \in V\}$, and $\mathfrak{g} = \{X \in \mathfrak{gl}(V) : \phi(X(v), w) = -\phi(v, X(w)) \text{ for all } v, w \in V\}$.

If $\varepsilon = 1$, then $G = \mathrm{O}(V, \phi)$ and $\mathfrak{g} = \mathfrak{so}(V, \phi)$. If $\varepsilon = -1$, then $G = \mathrm{Sp}(V, \phi)$ and $\mathfrak{g} = \mathfrak{sp}(V, \phi)$.

As stated this definition depends heavily on ϕ; however we may remove this by means of the following proposition. We omit the proof here, instead directing the reader to Jantzen [32, 1.3]:

Proposition 2.1.1. *Let ϕ and ϕ' be two non-degenerate bilinear forms, and let* $\dim(V) = n$. *Then we have the following:*

(i.) *If ϕ and ϕ' are symmetric, then there exists a $h \in \mathrm{GL}(V)$ such that* $\mathrm{O}(V, \phi') = h\,\mathrm{O}(V, \phi)h^{-1}$ *and* $\mathfrak{so}(V, \phi') = h\mathfrak{so}(V, \phi)h^{-1}$.

(ii.) *If ϕ and ϕ' are alternating, then there exists a $h \in \mathrm{GL}(V)$ such that* $\mathrm{Sp}(V, \phi') = h\,\mathrm{Sp}(V, \phi)h^{-1}$ *and* $\mathfrak{sp}(V, \phi') = h\mathfrak{sp}(V, \phi)h^{-1}$.

As a consequence of this result we may safely write $\mathrm{O}(V)$ for $\mathrm{O}(V, \phi)$ (and similarly for the symplectic case) without ambiguity. In particular we may fix a preferred bilinear form and maintain this throughout the remainder of the thesis.

Before proceeding further we require some additional notation. Let A be an $n \times n$ matrix; then we denote the transpose of A by A^{t}. We define the *skew-transpose* of A, denoted A^{st}, to be the reflection of A in its reverse diagonal. In other words, if $a^{\mathrm{st}}_{i,j}$ is the (i,j)th entry of the matrix A^{st}, then $a^{\mathrm{st}}_{i,j} = a_{n-j+1,n-i+1}$.

We consider first the orthogonal case. Let ϕ be the bilinear form defined by the $n \times n$ matrix $S = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$; that is, all entries on the reverse diagonal of S are equal to 1 and all remaining entries are equal to 0. Then the elements of the Lie algebra $\mathfrak{so}_n(k)$ are the $n \times n$ matrices X such that $SX = -X^{\mathrm{t}}S$. A matrix X satisfies this condition if all entries of the reverse diagonal are zero, and $X = -X^{\mathrm{st}}$. So, for example, a general element of $\mathfrak{so}_6(k)$ is of the following form:

$$X = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ b_1 & b_2 & b_3 & b_4 & 0 & -a_5 \\ c_1 & c_2 & c_3 & 0 & -b_4 & -a_4 \\ d_1 & d_2 & 0 & -c_3 & -b_3 & -a_3 \\ e_1 & 0 & -d_2 & -c_2 & -b_2 & -a_2 \\ 0 & -e_1 & -d_1 & -c_1 & -b_1 & -a_1 \end{pmatrix}$$

The symplectic Lie algebra $\mathfrak{sp}_{2m}(k)$ is defined similarly. In this case let ϕ be the bilinear form defined by the matrix

$$S = \begin{pmatrix} 0 & & \cdots & & 0 & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & & & 1 & & \\ & & -1 & & & \vdots \\ 0 & \cdot^{\cdot^{\cdot}} & & & & \\ -1 & 0 & & \cdots & & 0 \end{pmatrix}.$$

Then the elements of $\mathfrak{sp}_{2m}(k)$ are the matrices X such that $SX = -X^{\mathrm{t}}S$; these are of the form $X = \left(\begin{smallmatrix} A & B \\ C & -A^{\mathrm{st}} \end{smallmatrix}\right)$, such that A, B and C are $m \times m$ matrices, $B = B^{\mathrm{st}}$, and $C = C^{\mathrm{st}}$. So, for example, a general element X of $\mathfrak{sp}_6(k)$ is of the following form:

$$X = \left(\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & a_5 \\ c_1 & c_2 & c_3 & c_4 & b_4 & a_4 \\ \hline d_1 & d_2 & d_3 & -c_3 & -b_3 & -a_3 \\ e_1 & e_2 & d_2 & -c_2 & -b_2 & -a_2 \\ f_1 & e_1 & d_1 & -c_1 & -b_1 & -a_1 \end{array}\right).$$

2.2 Algebraic geometry

With these examples in mind we now begin to describe the general theory of algebraic groups. We first require a number of concepts from algebraic geometry; see for example the works of Hartshorne [27, Ch. 1] or of Humphreys [30, Ch. 1].

Let $k[\mathbf{T}] = k[T_1, ..., T_n]$ be the ring of polynomials in n variables with coefficients in k, and let F be a finite subset of $k[\mathbf{T}]$. An *affine algebraic variety* X is the set $X = \{x \in k^n \mid f(x) = 0 \; \forall \, f \in F\}$ of common zeroes in k^n of the polynomials in F. We denote by $I(X)$ the ideal of all polynomials which vanish on X. For an ideal I of $k[\mathbf{T}]$, the *radical* of I is the set $\sqrt{I} = \{f(T) \in k[\mathbf{T}] \mid f(\mathbf{T})^r \in I \text{ for some } r \leqslant 0\}$. We note that if $I = I(X)$, we have $I = \sqrt{I}$ [30, 1.1]; an ideal with this property is known as a *radical ideal*.

The co-ordinate ring of X, denoted $k[X]$, is defined as $k[X] = k[\mathbf{T}]/I$; the elements of $k[X]$ can be viewed as functions on X. If $X \subset k^n$ and $Y \subset k^m$ are affine algebraic varieties, a map $\phi : X \to Y$ is called a *morphism* if it is of the form $\phi(x_1, ..., x_n) = (\psi_1(x), ..., \psi_m(x))$ where the $\psi_i \in k[X]$ for all $1 \leqslant i \leqslant m$.

We define the *Zariski topology* on an affine algebraic variety X by defining a subset Z of X to be *closed* if and only if Z is the set of common zeroes of a set of polynomials in n variables over k. This does indeed satisfy the axioms of a topological space; see for example [30, 1.2].

We say that Z is *irreducible* if Z cannot be written as a union of two or more proper non-empty closed subsets of Z. There is then a unique way of writing $X = X_1 \sqcup ... \sqcup X_r$ as the disjoint union of closed irreducible subsets; these subsets X_i are said to be the *irreducible components* of X.

2.3 Algebraic groups and Lie algebras

We are now able to define algebraic groups and describe their connection to Lie algebras. The usual standard reference for Lie algebras is the book by Humphreys [29]; an overview of the theory of algebraic groups can be found in the works by Humphreys [30], Borel [9] and Springer [50].

An *linear algebraic group* G is an affine algebraic variety which has the structure of a group, such that multiplication and inversion are morphisms of varieties. There is a unique irreducible component containing the identity of G, denoted by G°. We say that G is *connected* if $G = G^\circ$; this is equivalent to being connected as a topological space. Otherwise the connected components of G are precisely the irreducible components, and are equal to the cosets of G° in G.

Finally G is said to be *simple* as an algebraic group if it contains no non-trivial closed connected normal subgroups; we note that this condition is not necessarily equivalent to being simple as an abstract group [30, 27.5].

We note that each of the groups $\mathrm{GL}(V)$, $\mathrm{Sp}(V)$ and $\mathrm{O}(V)$ defined in Section 2.1 are indeed linear algebraic groups.

Let G, G' be algebraic groups over k. A *homomorphism* of algebraic groups is a map $\phi : G \to G'$ such that ϕ is both a group homomorphism (in the usual sense) and also a morphism of varieties [30, 7.4].

A *representation* of G over a vector space V is a homomorphism $\rho : G \to \mathrm{GL}(V)$. We say that ρ is *faithful* if it is injective.

The connection between algebraic groups and Lie algebras is given in the following way. Let G be an algebraic group, and consider the co-ordinate ring $k[G]$ as defined above. A *derivation* of $k[G]$ is a map $\delta : k[G] \to k[G]$ such that for $x, y \in k[G]$ we have $\delta(x + y) = \delta(x) + \delta(y)$ and $\delta(xy) = x\delta(y) + \delta(x)y$. The set $\mathrm{Der}\, k[G]$ of all such derivations forms a vector space over k. Denote by λ_x the action of G on $k[G]$ by left translation defined by $(\lambda_x f)(y) = f(x^{-1}y)$, for $f \in k[G]$. A derivation $\delta \in \mathrm{Der}\, k[G]$ is said to be *left invariant* if it commutes with λ_x for all $x \in G$. Then the set of all left invariant derivations forms a Lie algebra under the bracket operation $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$; we denote this Lie algebra by $\mathrm{Lie}(G)$.

Alternatively, let e be the identity element of G. We define a *point derivation* of $k[G]$ at e to be a map $d : k[G] \to k$ such that for all $f, g \in k[G]$ we have $d(fg) = d(f) \cdot g(e) + f(e) \cdot d(g)$; that is, d acts similarly to a derivation of $k[G]$ followed by evaluation at e. Then the *tangent space* of G at e, denoted $\mathcal{T}(G)_e = \mathfrak{g}$, is the k-vector space of all point-derivations of $k[G]$ at e [30, §5.1].

In fact, $\mathrm{Lie}(G)$ and $\mathfrak{g}$ defined in this way are isomorphic [30, Thm. 9.1]; so we may use the notation interchangeably from now on. If $G = \mathrm{GL}_n(k)$, then $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{gl}_n(k)$ [30, §9.3]; similarly $\mathrm{Lie}(\mathrm{Sp}_n(k) = \mathfrak{sp}_n(k)$ and $\mathrm{Lie}(\mathrm{O}_n(k) = \mathfrak{so}_n(k)$.

Now let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}' = \mathrm{Lie}(G')$ be Lie algebras. A map $\psi : \mathfrak{g} \to \mathfrak{g}'$ is a *homomorphism* of Lie algebras if $\psi([X, Y]) = [\psi(X), \psi(Y)]$ for all $X, Y \in \mathfrak{g}$. A *representation* of $\mathfrak{g}$ is a homomorphism $\sigma : \mathfrak{g} \to \mathfrak{gl}(V)$ [29, 2.2].

We note that G acts on itself by conjugation; this induces an action of G

on $k[G]$ and in turn an action of G on $\mathrm{Lie}(G) \subseteq \mathrm{Der}(k[G])$. This latter action is known as the *adjoint action* [30, 10.2–3]. In the classical cases described in Section 2.1, this action takes the form of the usual matrix conjugation.

We maintain the convention of using upper case Roman letters for groups and lower case Gothic letters for their corresponding Lie algebras throughout this thesis.

2.4 Unipotency and nilpotency

Let $x \in \mathrm{GL}_n(k)$, and let $X \in \mathfrak{gl}_n(k)$. We say that x (respectively, X) is *semisimple* if it is diagonalisable over the field k. An element $X \in \mathfrak{gl}_n(k)$ is said to be *nilpotent* if there exists an $n \in \mathbb{N}$ such that $X^n = 0$. Similarly an element $x \in \mathrm{GL}_n(V)$ is *unipotent* if $(x-1)$ is nilpotent.

We now have the following important result [30, 15.1]:

Proposition 2.4.1 (Jordan-Chevalley decomposition)**.** *(i.) Let $X \in \mathfrak{gl}(V)$. Then X may be written uniquely as $X = X_s + X_n$ such that X_s is semisimple, X_n is nilpotent, and X_s commutes with X_n.*

(ii.) Let $x \in \mathrm{GL}(V)$. Then x may be written uniquely as $x = x_s x_u$ such that x_s is semisimple, x_u is unipotent, and $x_s x_u = x_u x_s$.

We now pass to the case of a general linear algebraic group, and note the following [30, 15.4]:

Proposition 2.4.2. *Let G be any linear algebraic group with $\mathfrak{g} = \mathrm{Lie}(G)$.*

(i.) Let $x \in G$. Then we may write $x = x_s x_u \in G$ such that for any representation $\rho : G \to \mathrm{GL}(V)$ there is a unique way of writing $\rho(x) = \rho(x)_s \rho(x)_u$ such that $\rho(x)_s = \rho(x_s)$ is semisimple, $\rho(x)_u = \rho(x_u)$ is unipotent, and x_s commutes with x_u.

(ii.) Let $X \in \mathfrak{g} \subseteq \mathrm{Der}(k[G]) \subseteq \mathfrak{gl}(k[G])$. Then $X = X_s + X_n$ is a Jordan decomposition of X in $\mathfrak{gl}(k[G])$ as described in Proposition 2.4.1(i.).

So we may now define $x \in G$ to be *semisimple* if $x = x_s$, and *unipotent* if $x = x_u$. Similarly we define $X \in \mathfrak{g}$ to be *semisimple* if $X = X_s$ and *nilpotent* if $X = X_n$.

We record the following important property of the Jordan-Chevalley decomposition [29, Prop. 4.2]:

Proposition 2.4.3. *Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of an algebraic group G, and let $X, Y \in \mathfrak{g}$ with Jordan-Chevalley decompositions $X = X_s + X_n$, $Y = Y_s + Y_n$. If $[X, Y] = 0$, then $[X_s, Y_s] = [X_s, Y_n] = [X_n, Y_s] = [X_n, Y_n] = 0$.*

2.5 Subgroups of algebraic groups

Throughout this thesis we will work with several specific types of subgroups of an algebraic group G. A *Borel subgroup* B is defined to be a maximal closed, connected and solvable subgroup of G. All Borel subgroups of G are conjugate [30, 21.3]. A *parabolic subgroup* P is a subgroup of G which contains a Borel subgroup. A *torus* T is an algebraic group isomorphic to the direct product of several copies of the multiplicative group k^* of k. A torus T is said to be *maximal* if it is not contained in any other torus. We note that maximal tori exist, and are conjugate [50, 6.5.5]. When G is one of our usual examples $\mathrm{GL}_n(k)$, $\mathrm{Sp}_{2n}(k)$ or $\mathrm{O}_n(k)$ as defined in Section 2.1, we have chosen a matrix representation of G so that P, B and T form subgroups of G comprised of upper triangular matrices, upper block triangular matrices (for fixed block dimensions) and diagonal matrices respectively.

We define the *radical* $R(G)$ of G to be the largest connected normal solvable subgroup of G. The subgroup $R_u(G)$ of $R(G)$ comprised of all unipotent elements of $R(G)$ is called the *unipotent radical*. An algebraic group G is called *semisimple* if it is connected and $R(G)$ is trivial; the group G is said to be *reductive* if $R_u(G)$ is trivial.

We note that a semisimple group G is the central product of a number of simple components; a reductive group G is the product of its centre $Z(G)$ and its derived subgroup (G, G), which is semisimple [30, 27.5].

We now recall that the theory of the Levi decomposition states that if P is a parabolic subgroup of an algebraic group then we can write $P = R_u(P)L$, where $R_u(P)$ is the unipotent radical of P and L is the so-called *Levi factor*. We say that a subgroup L of G is a *Levi subgroup* if it is the Levi factor of some parabolic subgroup P of G. Furthermore, any Levi subgroup is equal to the centraliser $C_G(S) = \{x \in G : sx = xs \text{ for all } s \in S\}$ of some torus S in G. In the case that P is comprised of block upper triangular matrices as described above, a Levi factor L of P is comprised of block diagonal matrices with the same block dimensions. For further details see [17, 1.15–1.22].

Similarly if $\mathfrak{g} = \mathrm{Lie}(G)$ we define a *Borel subalgebra* of $\mathfrak{g}$ to be a maximal solvable Lie subalgebra of $\mathfrak{g}$ and a *parabolic subalgebra* of $\mathfrak{g}$ to be a Lie subalgebra of $\mathfrak{g}$ that contains a Borel subalgebra of $\mathfrak{g}$. We note that a subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is a Borel subalgebra if and only if $\mathfrak{b} = \mathrm{Lie}(B)$ where B is a Borel subgroup of G; and similarly a subalgebra $\mathfrak{p}$ is a parabolic subalgebra if and only if $\mathfrak{p} = \mathrm{Lie}(P)$ for P a parabolic subgroup of G [51, Prop. 29.4.3].

2.6 Root systems

In this section we introduce the notion of the *root system* of a reductive algebraic group, which provides us with a useful tool to help classify simple algebraic groups and further investigate their structure. See for example [9, 8.17].

Let G be a reductive algebraic group, and let T be a maximal torus of G. A *character* of T is a homomorphism of algebraic groups $\chi : T \to k^*$; we denote the set of all characters of T by $X(T)$. If χ and ψ are characters of T then we define their product to be $\chi\psi(x) = \chi(x)\psi(x)$, for $x \in T$. It is then easily seen that $X(T)$ forms an abelian group under this operation.

Now let T be a maximal torus of G, and consider the adjoint action of T on $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\alpha \in X(T)$, and define $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid t \cdot X = \alpha(t)X \text{ for all } t \in T\}$. If $\mathfrak{g}_\alpha \neq 0$ then α is said to be a *weight* of G relative to T; a non-zero weight is called a *root*. We define Φ to be the set of roots of G relative to T; then Φ has the structure of an abstract root system as defined in [30, 27.1]. In particular Φ spans a vector space $E = \mathbb{R} \otimes X(T)$ [30, 27.1].

A *base* of Φ is a subset $\Pi = \{\alpha_i\} \subseteq \Phi$ such that Π forms a basis of E and each $\alpha \in \Phi$ can be written as $\sum_i c_i\alpha_i$, where the c_i are either all non-negative, or all non-positive. The elements of Π are called *simple roots*. We note that bases exist in all root systems Φ. A root α is called a *positive root* if it can be written as $\sum_i c_i\alpha_i$ where $\alpha_i \in \Pi$ for all i, and all c_i are non-negative. We denote the set of all positive roots of Φ by Φ^+, and note that each choice of Φ^+ corresponds to a Borel subgroup of G [30, 27.3].

We may represent root systems pictorially by their *Dynkin diagrams*, as explained in [29, Ch. 11]. If $\mathrm{char}\, k = 0$ then these Dynkin diagrams classify the simple Lie algebras, up to isomorphism. In this classification we obtain four families of so-called classical Lie algebras denoted by A_n, B_n, C_n and D_n and five exceptional cases E_6, E_7, E_8, F_4 and G_2.

We have a similar classification for simple algebraic groups, though in this case we require the extra information contained in the *root datum* to obtain the complete result; see [50, 7.4].

Examples of simple groups of types A_n, B_n, C_n and D_n are given by $SL_{n+1}(k)$, $SO_{2n+1}(k)$, $Sp_{2n}(k)$ and $SO_{2n}(k)$ respectively.

Now let G be a reductive algebraic group; then we may decompose G as the direct product of its centre $Z(G)$ and a number of simple groups [30, 27.5]. Consequently the Dynkin diagram of Φ is comprised of a number of connected components (in the graph–theoretic sense), each of which is of one of the classical or exceptional types described above.

Let $p = \operatorname{char} k$. We say that p is *bad* for G if $p = 2$ and Φ has a component that is not of type A, or if $p = 3$ and Φ has a component of type E_6, E_7, E_8, F_4 or G_2, or if $p = 5$ and the root system of G has a component of type E_8. We then say that p is *good* for G if it is not bad for G. For the rest of this thesis we assume that the characteristic of our field k is either zero or a good prime for whichever group G we happen to be working with, unless otherwise stated.

2.7 Partitions

The theory of partitions of a natural number plays an important role throughout this thesis. In this section we collect a number of definitions and results which will be of use later on.

Definition 2.7.1. A partition λ of a natural number n is defined to be a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ such that $\lambda_i \in \mathbb{N}$ and $\sum_{i=1}^{r} \lambda_i = n$. It will often be convenient to define $\lambda_i = 0$ for all $i > r$; that is, we may consider λ to be an infinite sequence such that only finitely many entries are non-zero. The numbers λ_i are known as the *parts* of λ. Alternatively we may write $\lambda = (1^{r_1}, 2^{r_2}, ..., n^{r_n})$ where r_i is the number of times i occurs in the partition.

We define a partial ordering (the dominance order) on a set of partitions by saying $\lambda = (\lambda_1, ..., \lambda_r) \geqslant \mu = (\mu_1, ..., \mu_s)$ if $\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i$ for all

$k \leqslant \max(r, s)$.

In what follows, when we refer to a partition λ as being maximal (respectively, minimal) in a set of partitions P we mean with respect to this ordering. We call λ *sub-maximal* (respectively, sub-minimal) if there exists a unique $\mu \in P$ with $\mu \neq \lambda$ such that $\mu \geqslant \lambda$ (respectively, $\mu \leqslant \lambda$). For example, if P is the set of all partitions of n, then the partitions (n) and $(n-1, 1)$ are maximal and sub-maximal respectively. Similarly the partitions (1^n) and $(2, 1^{n-2})$ are minimal and sub-minimal in P respectively.

Definition 2.7.2. Let $\lambda = (\lambda_1, ..., \lambda_r)$ be a partition of $n \in \mathbb{N}$.

(i.) The *size* of λ, denoted $|\lambda|$, is equal to the sum of the parts of λ; that is, $|\lambda| = \sum_{i=1}^{r} \lambda_i = n$.

(ii.) The *length* of λ, denoted $l(\lambda)$, is equal to the number of parts of λ; that is, $l(\lambda) = r$.

Definition 2.7.3. Let $\lambda = (\lambda_1, ..., \lambda_r)$ be a partition of $n \in \mathbb{N}$ and let $\mu = (\mu_1, ..., \mu_s)$ be a partition of $m \in \mathbb{N}$.

(i.) A *subpartition* of λ is a partition λ' of $n' \leqslant n$ whose parts form a subset of the set of parts of λ; that is, $\lambda' = (\lambda'_1, \lambda'_2 ..., \lambda'_s)$ with $s \leqslant r$ such that $\lambda'_1 \geqslant \lambda'_2 \geqslant ... \geqslant \lambda'_s$, and there is $j_1 < j_2 < ... < j_s$ such that $\lambda'_i = \lambda_{j_i}$ for $i \leqslant 1 \leqslant s$.

(ii.) The *sum* of two partitions λ and μ, denoted $\lambda \oplus \mu$, is the partition $\lambda \oplus \mu = (\lambda_1, ..., \lambda_r, \mu_1, ..., \mu_s)$, reordered so that the parts are non-increasing.

(iii.) The *dual partition* of λ, denoted λ^*, is the partition $\lambda^* = (\lambda_1^*, ..., \lambda_{\lambda_1}^*)$ such that λ_i^* is equal to the number of parts λ_j of λ such that $\lambda_j \geqslant i$.

We make the convention that λ_i denotes a part of the partition λ, while λ^j denotes a subpartition of λ.

The following specific types of partitions occur frequently in what follows:

Definition 2.7.4. We say that a partition $\lambda = (\lambda_1, ..., \lambda_r)$ is *rectangular* if $\lambda_1 = \lambda_i$ for all $1 \leqslant i \leqslant r$. A partition λ is said to be *almost rectangular* if $\lambda_1 - \lambda_r \leqslant 1$.

We record the following lemma for future use:

Lemma 2.7.5. *For $1 \leqslant p \leqslant n$, there is precisely one almost rectangular partition of n with p parts.*

Proof. Let $n = pq + r$, where $0 \leqslant r < p$. Then to construct an almost rectangular partition of n with p parts it is clear that each part must be either q or $q+1$, and there are r parts equal to $q+1$. Thus $((q+1)^r, q^{p-r})$ is the only possible such partition. □

Definition 2.7.6. We define $\rho_i(n)$ to be the unique almost rectangular partition of n with i parts.

For example, $\rho_3(8) = (3, 3, 2)$. We note that for any $n \in \mathbb{N}$ we have $\rho_1(n) = (n)$ and $\rho_n(n) = (1^n)$.

2.8 Nilpotent orbits and Jordan blocks

In this section we are interested in examining the properties of nilpotent elements of the Lie algebras of classical groups. In particular we consider the situation where we have a reductive algebraic group G acting on its Lie algebra $\mathfrak{g} = \text{Lie}(G)$ by the adjoint action, which in this case is given by matrix conjugation. If G is a classical group then an element $X \in \mathfrak{g}$ is nilpotent if it is nilpotent as a matrix;

that is, we have $X^n = 0$ for some $n \in \mathbb{N}$. For further details of the material in this section the reader is directed to [32].

We assume throughout the following that k is algebraically closed and of good characteristic for G, and that V is a finite dimensional vector space over k with $\dim(V) = n$. We write

$$J_r = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

for the $r \times r$ Jordan matrix with zeroes along the main diagonal.

2.8.1 Jordan blocks in type A

In this section we consider the case of $G = \mathrm{GL}(V)$ and $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{gl}(V)$. For this case we can appeal directly to the theory of the Jordan normal forms and obtain the following result.

Theorem 2.8.1. *(i.) Let $X \in \mathfrak{g}$ be nilpotent. Then there exists a basis for V such that the matrix of X with respect to this basis is* $\begin{pmatrix} J_{d_1} & & & \\ & J_{d_2} & & \\ & & \ddots & \\ & & & J_{d_r} \end{pmatrix}$. *Without loss of generality we can permute the Jordan blocks so that $d_1 \geq d_2 \geq \cdots \geq d_r > 0$. Clearly $\sum_i d_i = n = \dim(V)$, so $(d_1, ..., d_r)$ is a partition of n. We call this the Jordan type of X; the d_i in the Jordan type of X are uniquely determined by X.*

(ii.) Two nilpotent elements of $\mathfrak{g}$ are in the same G-orbit if and only if their Jordan types are the same.

As a refinement of this case, suppose now that $G = \mathrm{GL}(V)$ and $G' = \mathrm{SL}(V)$, and that $\mathfrak{g}' = \mathfrak{sl}(V)$. Then we have the following result:

Theorem 2.8.2. *Let $X, X' \in \mathfrak{g}'$. Then X and X' are in the same G'-orbit if and only if they are in the same G-orbit.*

Proof. The forward direction is trivial, since G' is a subgroup of G. For the converse, suppose that X and X' are in the same G-orbit, so that there exists $g \in G$ such that $gXg^{-1} = X'$. Since we have taken k to be algebraically closed, there exists an $a \in k$ such that $a^n = \det(g)$. Then let $h = a^{-1}g$; therefore $X' = gXg^{-1} = hXh^{-1}$ and $\det(h) = 1$. Hence X' is in the same G'-orbit as X. □

2.8.2 Jordan blocks in types B, C and D

Let G be either $\mathrm{O}(V)$ or $\mathrm{Sp}(V)$, and $\mathfrak{g}$ be either $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ respectively. We now consider the G-orbits in $\mathfrak{g}$, using results described by Jantzen [32, §1.5 & 1.6].

First we record the following lemma for later use:

Lemma 2.8.3. *Let $g \in \mathrm{GL}(V)$. Then there exists a polynomial $p(t) \in k[t]$ such that $p(g)^2 = g$.*

We may now prove the following theorem:

Theorem 2.8.4. *Two elements in $\mathfrak{g}$ belong to the same G-orbit if and only if they belong to the same $\mathrm{GL}(V)$-orbit.*

Proof. Since the bilinear form ϕ is non-degenerate, for each $f \in \mathrm{End}(V)$ there exists a unique $f^* \in \mathrm{End}(V)$ such that $\phi(f(v), w) = \phi(v, f^*(w))$.

Then it is easily seen that f^* has the following properties:

(i.) The map $* : f \mapsto f^*$ is linear;

(ii.) $(f^*)^* = f$;

(iii.) $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$;

(iv.) $(\mathrm{id}_V)^* = \mathrm{id}_V$;

(v.) If $g \in \mathrm{GL}(V)$, then $g^* \in \mathrm{GL}(V)$ and $(g^*)^{-1} = (g^{-1})^*$;

(vi.) We can write $G = \{g \in \mathrm{GL}(V) : g^* = g^{-1}\}$ and $\mathfrak{g} = \{X \in \mathfrak{gl}(V) : X^* = -X\}$.

Let $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}$ belong to the same $\mathrm{GL}(V)$-orbit, and let $g \in \mathrm{GL}(V)$ be such that $Y = gXg^{-1}$. Then we have

$$gXg^{-1} = Y = -Y^* = -(g^*)^{-1}X^*g^* = (g^*)^{-1}Xg^*.$$

It follows that $g^*gX = Xg^*g$. Let $g_1 = g^*g$, so that we have $g_1 \in \mathrm{GL}(V)$; then $g_1X = Xg_1$, and $(g_1)^* = g_1$.

Now we use Lemma 2.8.3 above to find a $h \in k[g_1] \subseteq \mathrm{End}(V)$ such that $h^2 = g_1$. Since $g_1 = h^2 \in \mathrm{GL}(V)$, we note $h \in \mathrm{GL}(V)$ also. Then we have $hX = Xh$ and $h = h^*$ since $h \in k[g_1]$. Let $h_1 = gh^{-1}$; then

$$h_1^* = (gh^{-1})^* = (h^*)^{-1}g^* = h^{-1}(g^*g)g^{-1} = h^{-1}g_1g^{-1} = hg^{-1} = (h_1)^{-1}.$$

It follows immediately from property (vi) above that $h_1 \in G$, and $h_1Xh_1^{-1} = Y$. The converse is trivial, and so the result follows. □

This theorem is enough to show that in this situation, two nilpotent elements of $\mathfrak{g}$ are in the same G-orbit if and only if they have the same partition. However it gives us no information about which possible partitions can occur. To complete the classification of the nilpotent orbits in this case we require the following theorem:

Theorem 2.8.5. *Let λ be a partition of $\dim(V)$. Then:*

(i.) If $G = \mathrm{O}(V)$, then there exists a nilpotent element in $\mathfrak{g}$ with Jordan type λ if and only if all even parts of λ occur with even multiplicity.

(ii.) If $G = \mathrm{Sp}(V)$, then there exists a nilpotent element in $\mathfrak{g}$ with Jordan type λ if and only if all odd parts of λ occur with even multiplicity.

By a slight abuse of definition we refer to partitions of the form described in part (i.) of this theorem as *orthogonal* partitions, and partitions of the form described in part (ii.) as *symplectic* partitions.

We conclude this section by remarking that the $\mathrm{O}(V)$ and $\mathrm{SO}(V)$ orbits of X in $\mathfrak{so}(V)$ will coincide in general unless the partition of X has only even parts; if this is the case then the $\mathrm{O}(V)$-orbit of X will decompose into two distinct $\mathrm{SO}(V)$ orbits. See Jantzen [32, 1.12]. So in the remainder of the thesis, if $\mathfrak{g} = \mathfrak{so}_{2m}(k)$ then we assume that $G = \mathrm{O}(V)$ in order to avoid this.

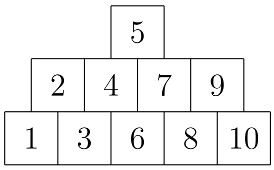
2.9 Dynkin Pyramids

In this section we describe the concept of a *Dynkin pyramid*, as first introduced by Elashvili and Kac [18]. For the formal definitions we refer the reader to [18] or [11, §6–8], and instead describe below how we construct these pyramids in practice.

2.9.1 Linear Dynkin Pyramids

First let $\mathfrak{g} = \mathfrak{gl}_n(k)$ and let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a partition of n. Then the Dynkin pyramid of λ is a diagram with λ_1 boxes in the lowest row, λ_2 boxes in the second row from the bottom, and so on until we have λ_r boxes in the top row. We align the rows so that the diagram is symmetric about a central vertical axis. We then label the boxes from 1 to n starting with the top box in the leftmost column; we work down each column from top to bottom, and label the columns from left to right. Since we are working in the context of $\mathfrak{gl}_n(k)$, we refer to this diagram as the *linear Dynkin pyramid* of λ.

This is perhaps more easily illustrated with an example; so suppose that $n = 10$ and $\lambda = (5, 4, 1)$. Then the linear Dynkin pyramid of λ is as follows:



We may use this diagram to write down a matrix $D(\lambda)$ with Jordan type λ in the following way. Let $E_{i,j}$ be the $n \times n$ matrix where the entry in the position (i, j) is equal to 1 and all other entries are equal to zero. Then $D(\lambda)$ is the sum of all $E_{i,j}$ such that the box labelled i is immediately to the left of the box labelled j in the Dynkin pyramid. So in our example, $D(5, 4, 1) = E_{1,3} + E_{2,4} + E_{3,6} + E_{4,7} + E_{6,8} + E_{7,9} + E_{8,10}$. This gives the following matrix:

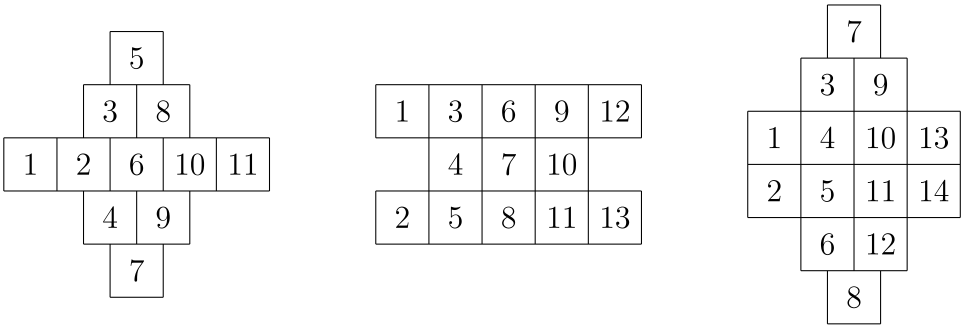
$$D(5,4,1) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2.9.2 Orthogonal Dynkin pyramids

Suppose now that $\mathfrak{g} = \mathfrak{so}_n(k)$, and let $\lambda = (\lambda_1, ..., \lambda_r)$ be an orthogonal partition of n with $\lambda_1 \geqslant ... \geqslant \lambda_r$. We recall from Section 2.1 that elements of $\mathfrak{so}_n(k)$ are antisymmetric about the reverse diagonal, and that the reverse diagonal is zero. So we need to introduce certain subtleties into the construction of Dynkin pyramids for $\mathfrak{so}_n(k)$ in order to ensure that the matrices defined by these pyramids are indeed elements of $\mathfrak{so}_n(k)$. Most notably we require the pyramid to have rotational symmetry, in order to ensure that the matrix defined from this pyramid possesses the required antisymmetry.

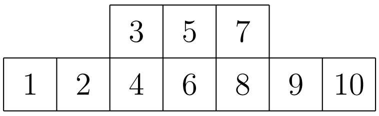
Suppose first that λ is an orthogonal partition of n with at most one part of odd multiplicity. If there exists exactly one part λ_i of odd multiplicity then we draw λ_i boxes in the central row of the Dynkin pyramid. Now let λ_j be the largest part of λ which occurs with multiplicity $2m$; then we draw m rows with λ_j boxes above the central row, and m rows with λ_j boxes beneath it. We then repeat this process with all remaining parts of λ, and number the resulting diagram in the same way as above. We refer to this as the *orthogonal Dynkin pyramid*.

We illustrate this with the orthogonal Dynkin pyramids of $n = 11$ and $\lambda = (5, 2^2, 1^2)$, with $n = 13$ and $\lambda = (5^2, 3)$; and with $n = 14$ and $\lambda = (4^2, 2^2, 1^2)$. The second example demonstrates how the central row of the pyramid need not be the longest; the third example illustrates the orthogonal Dynkin pyramid of a partition with no parts of odd multiplicity.

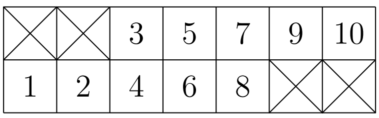


We then define the matrix $D^+(\lambda)$ to be the sum of $\epsilon E_{i,j}$ where i and j label adjacent boxes on the same row; $\epsilon = 1$ if $i + j \leqslant n$ and $\epsilon = -1$ if $i + j > n$. So $D^+(5, 2^2, 1^2) = E_{1,2} + E_{2,6} + E_{3,8} - E_{4,9} - E_{6,10} - E_{10,11}$.

If λ contains more than one part of odd multiplicity the situation is slightly more complex. Consider the example of $\lambda = (7, 3)$ in $\mathfrak{so}_{10}(k)$. Then the linear Dynkin pyramid of λ is the following:



Now to convert this into an orthogonal Dynkin pyramid we effectively 'push' the boxes labelled 9 and 10 up from the first row into the second; however we still consider boxes 8 and 9 to be adjacent, and we gain a new adjacency between boxes 7 and 9. Similarly, to ensure the rotational symmetry of the diagram, we have adjacencies between box 2 and both 3 and 4. We refer to these two rows as *skew rows* and draw crosses immediately above boxes 1 and 2, and below boxes 9 and 10, to make this clear. So the orthogonal Dynkin pyramid of $\lambda = (7, 3)$ is as follows:

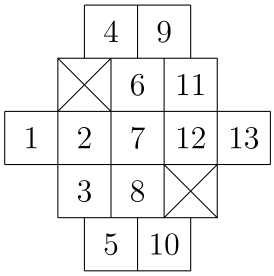


We can now write down the matrix $D^+(7,3)$ as follows, recalling that we have the extra adjacencies between 2 and 3, and between 8 and 9:

$$D^+(7,3) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It is of course possible to obtain skew rows in the orthogonal Dynkin pyramids of partitions with more than two parts. In general, if the odd parts of odd multiplicity are $\lambda'_1, ..., \lambda'_s$ in non-increasing order, we pair off parts which are adjacent in this list starting with $(\lambda'_{s-1}, \lambda'_s)$ and moving leftwards along the list. Then we draw the Dynkin pyramid of λ following the process described above, treating a skew row arising from $(\lambda'_i, \lambda'_{i+1})$ as if it were a pair (λ'_i, λ'_i) and drawing it in accordingly.

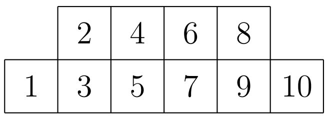
To see this more clearly, consider the example of $n = 13$ and $\lambda = (5, 3, 2^2, 1)$. We have three odd parts of odd multiplicity, so we pair off the smallest two as $(3, 1)$. We begin by drawing the central row of the pyramid with five boxes; we then add the $(3, 1)$ skew row above and below this central row, since 3 is the largest remaining part. Finally we add rows with 2 boxes in the top and bottom rows of the pyramid, and obtain the following:



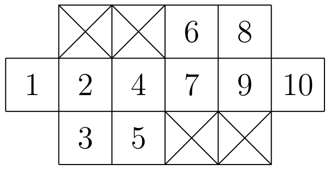
2.9.3 Symplectic Dynkin pyramids

Finally let $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$ and let $\lambda = (\lambda_1, ..., \lambda_r)$ be a symplectic partition of $2m$. As in the orthogonal case, the Dynkin pyramid must possess rotational symmetry, to ensure that the matrix defined from the pyramid has the structure of an element of $\mathfrak{sp}_{2m}(k)$ as defined in Section 2.1. So the situation where we have at least two even parts occurring with odd multiplicity needs examining more closely.

Consider the example of $m = 5$ and $\lambda = (6, 4)$. Then the linear Dynkin pyramid of λ is as follows:



Now in order to obtain the *symplectic Dynkin pyramid* of $(6, 4)$ we take the first two boxes of the short row and move them directly down to beneath the long row, without moving them horizontally. Despite this move we still consider the second and third boxes of the short row to be adjacent. As in the orthogonal case we refer to the row with 4 boxes as a *skew row*. So the symplectic Dynkin pyramid of $(6, 4)$ is the following:



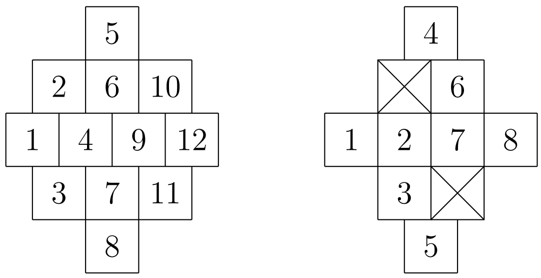
In particular we note that boxes 5 and 6 are considered to be adjacent in this pyramid.

We may now write down a matrix $D^{-}(6,4)$ with Jordan type $(6,4)$ in a similar way to the previous cases. That is, $D^{-}(6,4)$ is the sum of $\epsilon E_{i,j}$ where the box labelled i is adjacent to the box labelled j, and $\epsilon = 1$ if $i \leqslant m$ or $\epsilon = -1$ if $i \geqslant m+1$. Thus in this example we have

$$D^{-}(6,4) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As in the orthogonal case we can construct more complex pyramids in the same way. If λ has at least one part of odd multiplicity we draw the largest such part as the central row of the pyramid. We then consider the size of the largest remaining part. If this part has multiplicity at least two we draw rows of this length on the top and bottom of the pyramid; otherwise we draw a skew row of this length. Once a row is drawn we remove the corresponding part from consideration, and move on to the next largest part, repeating this process as often as necessary.

For example, below we show the symplectic Dynkin pyramids of $(4, 3^2, 1^2)$ and $(4, 2, 1^2)$ respectively:



2.9.4 Dynkin pyramids and nilpotent matrices

Now the importance of the use of pyramids is as follows. Let $E \in \mathfrak{g}$ be nilpotent. Then by the theory of nilpotent orbits as described in [32, 3.3], there is a grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j)$ of $\mathfrak{g}$ such that $E \in \mathfrak{g}(2)$ and $\mathfrak{c}_{\mathfrak{g}}(E) \subseteq \bigoplus_{j \geqslant 0} \mathfrak{g}(j) = \mathfrak{p}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. This grading is known as the *Dynkin grading.* Then if P is the parabolic subgroup of G such that $\operatorname{Lie} P = \mathfrak{p}$, we have $C_G(E) \subseteq P$.

Now let $H = G(0)$ be the Levi factor of P with $\operatorname{Lie} H = \mathfrak{g}(0)$, and let $R = R_u(P)$. Then we can write $G_E = C_G(E)$; we note that $G_E = H_E R_E$, where $H_E = C_H(E)$ and $R_E = C_R(E)$, by [32, 3.12]. Additionally we note that $R_E = R_u(C_G(E))$; consequently H_E is reductive. Let S be a maximal torus of H_E and let $T \supseteq S$ be a maximal torus of H. Now we can let $B \subseteq P$ to be a Borel subgroup of G such that $T \subseteq B$. Then $H \cap B$ is a Borel subgroup of H and $H_E \cap B$ is a Borel subgroup of H_E. Consequently we see that $G_E \cap B$ is a Borel subgroup of G_E.

So from this we see that every nilpotent element of $\mathfrak{g}_E$ is conjugate to an element of $\mathfrak{g}_E \cap B$. Additionally we have defined the pyramids in such a way that we can take T to be comprised of the diagonal matrices in G, and B to be comprised of upper triangular matrices. So when we look for possible Jordan types of nilpotent elements of $\mathfrak{g}_E$ we need only consider strictly upper triangular matrices in this centraliser.

CHAPTER 3

COMMUTING VARIETIES OF PARABOLIC SUBALGEBRAS

Let G be a connected reductive algebraic group over k, and let $\mathfrak{g} = \mathrm{Lie}(G)$. Then the *commuting variety* $\mathcal{C}(\mathfrak{g})$ of $\mathfrak{g}$ is the algebraic variety comprised of all pairs of elements $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ such that $[X, Y] = XY - YX = 0$.

The study of this object appears to originate with Motzkin and Taussky [37], who in 1955 proved Theorem 3.0.1 below in the case of $\mathfrak{g} = \mathfrak{gl}_n$, and Gerstenhaber who was among the first to consider $\mathcal{C}(\mathfrak{g})$ as a variety and in 1961 obtained several results concerning its structure; see [20]. Richardson [45] in 1979 then extended Motzkin and Taussky's result to general reductive $\mathfrak{g}$; specifically, he proved the following theorem:

Theorem 3.0.1. *Suppose that* $\mathrm{char}\, k = 0$*, and let* $\mathfrak{t}$ *be a maximal toral subalgebra of* $\mathfrak{g}$*. Then* $\mathcal{C}(\mathfrak{g}) = \overline{G \cdot (\mathfrak{t} \times \mathfrak{t})}$*, where* G *acts on* $\mathfrak{t}$ *via the adjoint action and* $G \cdot (\mathfrak{t} \times \mathfrak{t}) = (G \cdot \mathfrak{t}) \times (G \cdot \mathfrak{t})$*. In particular* $\mathcal{C}(\mathfrak{g})$ *is irreducible.*

More recently in 2002 Levy [36] showed that, subject to some restrictions on the structure of G, Theorem 3.0.1 still holds in characteristic $p > 0$.

The goal of this chapter is to investigate when $\mathcal{C}(\mathfrak{p})$ is irreducible, for a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. In the case when $\mathfrak{p} = \mathfrak{b}$ is a Borel subalgebra of

$\mathfrak{g}$ a partial classification of when this occurs is given in Keeton's thesis of 1995 [34]; but for general parabolic subalgebras $\mathfrak{p}$ this appears to be an open question. In general $\mathcal{C}(\mathfrak{p})$ is not irreducible, but we present two criteria for when it is. Firstly we attempt to generalise Richardson's approach in [45], and obtain a criterion for irreducibility that is simple to state but relies on being able to classify elements which are in $\mathfrak{p}$, a problem wihich appears little studied. We then consider an approach based on the modality of the action of a parabolic subgroup on its Lie algebra, following the ideas of Keeton [34]. Using this method we obtain a second criterion for the irreducibility of $\mathcal{C}(\mathfrak{p})$, and are able to use this to classify all cases when $\mathcal{C}(\mathfrak{b})$ is irreducible. Recent work of Boos and Bulois [8] then gives a number of cases in which $\mathcal{C}(\mathfrak{p})$ is seen to be irreducible or reducible using this criterion.

3.1 Regular semisimple elements and irreducibility

We say that an element of $\mathfrak{g}$ is *regular semisimple* if its centraliser in $\mathfrak{g}$ is a maximal toral subalgebra; we note that the regular semisimple elements form a dense open subset of $\mathfrak{g}$ [28, Thm. 2.5]. We define $\mathcal{C}_s^r(\mathfrak{p})$ to be the subvariety of $\mathcal{C}(\mathfrak{p})$ consisting of pairs of commuting semisimple elements, each of which is regular in $\mathfrak{p}$. Then $\overline{\mathcal{C}_s^r(\mathfrak{g})} = \overline{G \cdot (\mathfrak{t} \times \mathfrak{t})}$, since the subset $\mathfrak{t}^r$ consisting of all regular semisimple elements of $\mathfrak{t}$, is dense in $\mathfrak{t}$ [28, Cor. 2.3].

Now let P be a parabolic subgroup of G with corresponding parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. The goal of the following section is to seek to generalise Richardson's methods to examine the structure of $\mathcal{C}(\mathfrak{p})$.

In the case that $\mathfrak{g} = \mathfrak{gl}_n(k)$ we may take $\mathfrak{t}$ to be the subalgebra consisting of diagonal matrices in $\mathfrak{g}$. In this case, the regular semisimple elements in $\mathfrak{t}$ are precisely the diagonal matrices whose eigenvalues are all distinct. We recall that a parabolic subalgebra $\mathfrak{p}$ in this case may be easily visualised as the subalgebra comprised of block upper triangular matrices of fixed block dimensions.

We now wish to consider in what situations the following generalisation of Theorem 3.0.1 holds:

Question 3.1.1. *Is it true that $\mathcal{C}(\mathfrak{p}) = \overline{\mathcal{C}_s^r(\mathfrak{p})}$?*

We begin by establishing the following important properties of $\mathcal{C}_s^r(\mathfrak{p})$.

Lemma 3.1.2. *(i.) $\mathcal{C}_s^r(\mathfrak{p})$ is open and irreducible in $\mathcal{C}(\mathfrak{p})$;*

(ii.) $\overline{\mathcal{C}_s^r(\mathfrak{p})}$ is an irreducible component of $\mathcal{C}(\mathfrak{p})$;

(iii.) $\dim \mathcal{C}_s^r(\mathfrak{p}) = \dim \mathfrak{p} + \operatorname{rank} G$.

Proof. (i.) We first note that the variety of regular semisimple elements $\mathfrak{p}_s^r$ of $\mathfrak{p}$ is open in $\mathfrak{p}$; thus $\mathfrak{p}_s^r \times \mathfrak{p}_s^r$ is open in $\mathfrak{p} \times \mathfrak{p}$. It then follows that $\mathcal{C}_s^r(\mathfrak{p}) = \mathcal{C}(\mathfrak{p}) \cap (\mathfrak{p}_s^r \times \mathfrak{p}_s^r)$ is open in $\mathcal{C}(\mathfrak{p})$.

Now consider the morphism $\mu : P \times (\mathfrak{t}^r \times \mathfrak{t}^r) \to \mathcal{C}_s^r(\mathfrak{p})$ defined by $\mu(g, X, Y) = (g \cdot X, g \cdot Y)$. Let $(X, Y) \in \mathcal{C}_s^r(\mathfrak{p})$; then there exists a maximal torus T_1 of P such that $X \in \mathfrak{t}_1$. Since X is regular semisimple, we see that $\mathfrak{c}_\mathfrak{p}(X) = \mathfrak{t}_1$; so $Y \in \mathfrak{t}_1$ also. Now maximal tori of P are conjugate [30, Cor. 21.3A], so μ is surjective. It follows that $\mathcal{C}_s^r(\mathfrak{p})$ is irreducible.

(ii.) Since $\mathcal{C}_s^r(\mathfrak{p})$ is open and irreducible, it follows immediately that $\overline{\mathcal{C}_s^r(\mathfrak{p})}$ is an irreducible component of $\mathcal{C}(\mathfrak{p})$.

(iii.) Now suppose that $\mu(g, X, Y) = \mu(1, X', Y')$ for some $g \in P$ and $X, Y, X', Y' \in \mathfrak{t}^r$. Then $g \cdot X = X'$, so $g \cdot \mathfrak{t} = g \cdot \mathfrak{c}_\mathfrak{g}(X) = \mathfrak{c}_\mathfrak{g}(X') = \mathfrak{t}$. Hence we have $g \in N_P(\mathfrak{t}) = \{g \in P \mid g \cdot \mathfrak{t} = \mathfrak{t}\}$. It follows that the dimension of each fibre of μ is $\dim N_P(\mathfrak{t}) = \dim \mathfrak{t} = \operatorname{rank} G$. Therefore $\dim \mathcal{C}_s^r(\mathfrak{p}) = \dim P + \dim \mathfrak{t}^r + \dim \mathfrak{t}^r - \dim \mathfrak{t} = \dim \mathfrak{p} + \operatorname{rank} G$.

□

As a consequence of part (ii.) we see that $\mathcal{C}(\mathfrak{p})$ is irreducible if and only if Question 3.1.1 holds for $\mathfrak{p}$.

3.2 Distinguished and Richardson elements

We now introduce the concept of a *distinguished* nilpotent element and a *Richardson* element of a Lie algebra $\mathfrak{g}$. These definitions play an important role in the Bala–Carter classification of nilpotent orbits in the Lie algebra $\mathfrak{g}$ of an arbitrary connected reductive algebraic group, as described in [1] and [2]. We do not describe the full details of this here, and instead give only the definitions which we require in the following section. For further details we refer the reader to [32, Ch. 4].

Definition 3.2.1. Let G be an algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$. A nilpotent element $X \in \mathfrak{g}$ is said to be *distinguished in* $\mathfrak{g}$ if every torus contained in $C_G(X)$ is contained also in $Z(G)$.

Equivalently, X is distinguished in $\mathfrak{g}$ if the centraliser $C_G(X)$ contains no non-zero semisimple elements.

A nilpotent orbit in $\mathfrak{g}$ is said to be *distinguished* in $\mathfrak{g}$ if all of its elements are distinguished in $\mathfrak{g}$.

In the classical cases, the distinguished nilpotent orbits are as follows [32, 4.1–2]:

(i.) A nilpotent element $X \in \mathfrak{gl}_n(k)$ is distinguished if and only if it has Jordan type (n).

(ii.) A nilpotent element $X \in \mathfrak{so}_n(k)$ is distinguished if and only if its Jordan type is comprised of distinct odd parts.

(iii.) A nilpotent element $X \in \mathfrak{sp}_{2m}(k)$ is distinguished if and only if its Jordan type is comprised of distinct even parts.

Proposition 3.2.2. *Let P be a parabolic subgroup of G with unipotent radical U_P, and let $\mathfrak{u}_P = \mathrm{Lie}(U_P)$. Then there exists a unique nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ such that $\mathcal{O} \cap \mathfrak{u}_P$ is a single P-orbit which is open and dense in $\mathfrak{u}_P$.*

The orbit $\mathcal{O}$ is called the *Richardson orbit* corresponding to P; the elements of $\mathcal{O}$ are *Richardson elements* for P. In certain older papers, most notably for our purposes that of Richardson himself [45], the Richardson elements are referred to as *elements of parabolic type.*

In the situation of $\mathfrak{g} = \mathfrak{gl}_n(k)$, this works as follows. We recall from Section 2.5 that a parabolic subgroup P of $\mathrm{GL}_n(k)$ may be represented as the set of block upper triangular matrices with fixed block sizes; the dimensions of these blocks form a partition ν of n. From [16, Thm. 7.2.3], we then observe that the Richardson orbit corresponding to P is the nilpotent orbit denoted by the dual partition ν^* of ν. In particular we observe that every nilpotent orbit $\mathcal{O}$ of $\mathrm{GL}_n(k)$ in $\mathfrak{gl}_n(k)$ is a Richardson orbit for some parabolic subgroup P of $\mathrm{GL}_n(k)$.

3.3 An inductive approach

In this section we use an inductive method to investigate the structure of $\overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$ in more detail. We make the following inductive hypothesis:

Inductive Hypothesis 3.3.1. *Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Then $\mathcal{C}(\mathfrak{p} \cap \mathfrak{l}) = \overline{(P \cap L) \cdot (\mathfrak{t} \times \mathfrak{t})}$.*

For the remainder of this section we will work under this hypothesis.

With this established, we now turn to look at $\mathcal{C}(\mathfrak{p})$ more directly. Let $(X, Y) \in \mathcal{C}(\mathfrak{p})$, and let $X = X_s + X_n$ be the Jordan decomposition of X. By the conjugacy of maximal toral subalgebras in $\mathfrak{p}$, we may assume that $X_s \in \mathfrak{t}$. Let $\mathfrak{z}(\mathfrak{p})$ be the centre of $\mathfrak{p}$, and note that if $X_s \in \mathfrak{z}(\mathfrak{p})$ then we have that $(X_n, Y) \in \mathcal{C}(\mathfrak{p})$. So assume for the moment that this is not the case, so that we have $\mathfrak{c}_{\mathfrak{p}}(X_s) \neq \mathfrak{p}$ and $X, Y \in \mathfrak{c}_{\mathfrak{p}}(X_s)$ by Proposition 2.4.1 and 2.4.3. We then have the following lemma:

Lemma 3.3.2. *Let $(X, Y) \in \mathcal{C}(\mathfrak{p})$, let $X = X_s + X_n$ be the Jordan decomposition of X, and let $X_s \notin \mathfrak{z}(\mathfrak{p})$. Then $(X, Y) \in \overline{P \cdot \mathfrak{t} \times \mathfrak{t}}$.*

Proof. Recall that $\mathfrak{c}_{\mathfrak{g}}(X_s) = \mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$, and $\mathfrak{c}_{\mathfrak{p}}(X_s) = \mathfrak{l} \cap \mathfrak{p}$. So $(X, Y) \in \mathcal{C}(\mathfrak{p} \cap \mathfrak{l})$. By our inductive hypothesis 3.3.1, we see that $\mathcal{C}(\mathfrak{c}_{\mathfrak{p}}(X_s)) = \overline{C_P(X_s) \cdot (\mathfrak{t} \times \mathfrak{t})}$ since we note that $\mathfrak{t} \subseteq \mathfrak{c}_{\mathfrak{p}}(X_s)$. It follows that $(X, Y) \in \overline{C_P(X_s) \cdot (\mathfrak{t} \times \mathfrak{t})} \subseteq \overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$. □

As a consequence of this lemma we may now assume that $X_s \in \mathfrak{z}(\mathfrak{p})$. The following lemma allows us to simplify our calculations.

Lemma 3.3.3. *Let $S \in \mathfrak{z}(\mathfrak{p}) \subseteq \mathfrak{t}$ and $X, Y \in \mathfrak{p}$. Then $(X, Y) \in \overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$ if and only if $(X + S, Y) \in \overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$.*

Proof. We define a map $\phi_S : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p} \times \mathfrak{p}$ by $\phi_S(U, V) = (U + S, V)$. Clearly ϕ_S is an isomorphism of Lie algebras. Additionally for $p \in P$ and $T_1, T_2 \in \mathfrak{t}$ we have

$$\phi_S(p \cdot T_1, p \cdot T_2) = ((p \cdot T_1) + S, p \cdot T_2) = (p(\cdot T_1 + S), p \cdot T_2) = p \cdot \phi_S(T_1, T_2) \in P \cdot (\mathfrak{t} \times \mathfrak{t}),$$

so $P \cdot (\mathfrak{t} \times \mathfrak{t})$ is stable under ϕ_S.

Therefore $\overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$ is also ϕ_S-stable, thus proving the lemma. □

Thus we may assume that $X_s = 0$, and consider only nilpotent X. By switching the roles of X and Y in this argument we may take Y to be nilpotent also.

We now have the following lemma, an analogue of [45, Lem. 2.4]:

Lemma 3.3.4. *Suppose that we are in the situation of the inductive hypothesis 3.3.1 above. Let $(X, Y) \in \mathcal{C}(\mathfrak{p})$ with X, Y both nilpotent, and assume that there exists a non-zero semisimple element $S \in \mathfrak{c}_{\mathfrak{p}}(X)$. Then $(X, Y) \in \overline{P \cdot (\mathfrak{t}^r \times \mathfrak{t}^r)}$.*

Proof. Let $t \in k$ and set $A_t = tY + (1-t)S$; then for all $t \in k$ we have $(X, A_t) \in \mathcal{C}(\mathfrak{p})$. Denote by $\mathcal{D}$ the set of $t \in k$ such that A_t is not nilpotent, and let $\mathcal{E}$ be the set of pairs (X, A_t) such that $t \in \mathcal{D}$. Then $\mathcal{D}$ is an open subset of k under the Zariski topology, so $\mathcal{D}$ is dense in k; consequently $\overline{\mathcal{E}} = \{(X, A_t) : t \in k\}$.

Now by Lemma 3.3.3 and the inductive hypothesis, if $t \in \mathcal{D}$ then $(X, A_t) \in \overline{P \cdot (\mathfrak{t}^r \times \mathfrak{t}^r)}$. Consequently $\mathcal{E} \subseteq \overline{P \cdot (\mathfrak{t}^r \times \mathfrak{t}^r)}$. Since $\overline{P \cdot (\mathfrak{t}^r \times \mathfrak{t}^r)}$ is closed in $\mathcal{C}(\mathfrak{p})$, it follows that $\overline{\mathcal{E}} \subseteq \overline{P \cdot (\mathfrak{t}^r \times \mathfrak{t}^r)}$. So we find that $(X, A_t) \in \overline{P \cdot (\mathfrak{t}^r \times \mathfrak{t}^r)}$ for all $t \in k$; and in particular, $(X, Y) = (X, A_1) \in \overline{P \cdot (\mathfrak{t}^r \times \mathfrak{t}^r)}$. □

Thus we need only to consider the case of nilpotent elements in $\mathfrak{p}$ whose centraliser in $\mathfrak{p}$ contains no non-zero semisimple elements; we recall from Definition 3.2.1 that these are precisely the distinguished nilpotent elements in $\mathfrak{p}$.

We now move on to consider commuting pairs $(X, Y) \in \mathcal{C}(\mathfrak{p})$ of distinguished nilpotent elements, following the approach of Richardson [45, §4].

We recall from Proposition 3.2.2 that a Richardson element for a parabolic subgroup Q is an element of the unique orbit which intersects $\mathfrak{u}_Q$ in an open and dense set.

Take a parabolic subgroup Q of G with $Q \subseteq P$ and $\mathrm{Lie}(Q) = \mathfrak{q}$, and let M be a Levi subgroup of Q. Let $A = Z(M)^\circ$; then $\mathfrak{r} = \mathfrak{a} + \mathfrak{u}$ is the radical of $\mathfrak{q}$. Set $\mathfrak{a}' = \{X \in \mathfrak{a} : \mathfrak{c}_{\mathfrak{g}}(X) = \mathfrak{m}\}$ to be the set of elements of $\mathfrak{a}$ whose centralisers in $\mathfrak{g}$ are as small as possible, and let $\mathfrak{r}' = \{X \in \mathfrak{r} : \dim \mathfrak{c}_{\mathfrak{q}}(X) = \dim \mathfrak{m}\}$ be the subset of $\mathfrak{r}$ comprised of elements whose centralisers in $\mathfrak{q}$ have minimal dimension. Finally let $\mathcal{R} = \{(X, Y) \in \mathfrak{r}' \times \mathfrak{q} : Y \in C_{\mathfrak{q}}(X)\}$ be the set of commuting pairs in $\mathfrak{q} \times \mathfrak{q}$ whose first element is in $\mathfrak{r}'$.

We also require the following lemma:

Lemma 3.3.5. *Denote by* $\pi : \mathcal{R} \to \mathfrak{r}'$ *the restriction to* $\mathcal{R}$ *of the projection* $\mathfrak{r}' \times \mathfrak{p} \to \mathfrak{r}'$*. Then* π *is an open mapping.*

We omit the proof here; it can be found in Richardson [45, 4.4]. Although Richardson considers the whole of a general semisimple Lie algebra $\mathfrak{g}$, the proof also works for our present situation without modification. We are now ready to prove the main theorem of this section.

Theorem 3.3.6. *Assume that the inductive hypothesis 3.3.1 holds. Let* $\mathfrak{p}$ *be a parabolic subalgebra of* $\mathfrak{gl}_n$ *and let* $(X, Y) \in \mathcal{C}(\mathfrak{p})$ *with* X *a Richardson element for some parabolic* $\mathfrak{q} \subseteq \mathfrak{p}$*. Then* $(X, Y) \in \overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$.

Proof. Let $Q \subseteq P$ be a parabolic subgroup with unipotent radical U_Q, let $X \in \mathfrak{u}_Q = \mathrm{Lie}(U_Q)$, and let X be Richardson for Q, that is, $\overline{Q \cdot X} = \mathfrak{u}_Q$. Let $Y \in \mathfrak{p}$ be such that $(X, Y) \in \mathcal{C}(\mathfrak{p})$, let N be an open neighbourhood of (X, Y) in $\mathcal{C}(\mathfrak{p})$, and let $N' = N \cap \mathcal{R}$. Then $\pi(N')$ is an open subset of $\mathfrak{r}'$ by Lemma 3.3.5. In particular $\pi(N') \cap (\mathfrak{a}' \oplus \mathfrak{u})$ is non-empty since $\mathfrak{a}' \oplus \mathfrak{u}$ is a dense open subset of $\mathfrak{r}'$. Since the semisimple elements in $\mathfrak{a}' \oplus \mathfrak{u}$ are dense in $\mathfrak{a}' \oplus \mathfrak{u}$, we see that $\pi(N')$ contains a semisimple element.

Thus we can find a commuting pair $(S, V) \in N'$ such that S is semisimple. By the inductive argument used above, we have $(S, V) \in \overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$, hence $N \cap \overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$ is non-empty. It follows that $(X, Y) \in \overline{P \cdot (\mathfrak{t} \times \mathfrak{t})}$. □

We then obtain the following corollary.

Corollary 3.3.7. *Assume that the inductive hypothesis 3.3.1 holds. Suppose that* $\mathfrak{p}$ *is a parabolic subalgebra of* $\mathfrak{g}$ *and all distinguished nilpotent elements in* $\mathfrak{p}$ *are Richardson for some parabolic subalgebra* $\mathfrak{q} \subseteq \mathfrak{p}$*. Then* $\mathcal{C}(\mathfrak{p})$ *is irreducible.*

Proof. This follows from Theorem 3.3.6 above and our inductive hypothesis 3.3.1. □

Now Theorem 3.3.6 and Corollary 3.3.7 give us a method of determining whether Question 3.1.1 is true. In particular we see that whether $\mathcal{C}(\mathfrak{p})$ is irreducible or not is closely related to the distinguished nilpotent elements of $\mathfrak{p}$. In particular the author has shown in [21, 3.3] using this method that $\mathcal{C}(\mathfrak{p})$ is irreducible when $\mathfrak{p}$ is the parabolic subalgebra of $\mathfrak{gl}_n(k)$ defined by $(n-1,1)$ or $(1,n-2,1)$. In these cases the regular nilpotent orbit is the only orbit which is distinguished in $\mathfrak{p}$, and it is well known that this is a Richardson orbit. We omit the details here, since these cases are also covered in the more general results of Murray [39] and of Boos and Bulois [8] respectively.

In more general cases the Richardson elements for parabolic subgroups have been studied in detail (see for example [12] for $\mathfrak{gl}_n(k)$ and [7] for $\mathfrak{sp}_n(k)$ and $\mathfrak{so}_n(k)$), but less appears to be known about the distinguished nilpotent elements in a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Consequently we must first determine these in order to make use of Corollary 3.3.7; this is achievable in specific cases but no general result is presently known.

3.4 Algebraic group actions and modality

We now move on to consider an approach based on that of Keeton [34]. In order to do this we first require a number of definitions. Let G be a linear algebraic group over an algebraically closed field k with Lie algebra $\mathfrak{g}$, and let V be a variety on which G acts morphically. Let $g \in G$, $v \in V$ and let U be a subvariety of V. Then we write $g \cdot v$ for the image of v under g, and $G \cdot v = \{g \cdot v \mid g \in G\}$ for the G-orbit of v. The stabilizer of v in G is denoted $C_G(v) = \{g \in G \mid g \cdot v = v\}$. We write $g \cdot U = \{g \cdot u \mid u \in U\}$ for the image of

U under g, and $G \cdot U = \{g \cdot u \mid g \in G, u \in U\}$ for the G-saturation of U. The normalizer of U in G is denoted $N_G(U) = \{g \in G \mid g \cdot u \in U \text{ for all } u \in U\}$. Finally we set $V_j = \{v \in V \mid \dim G \cdot v = j\}$ for $j \in \mathbb{Z}_{\geqslant 0}$.

Now the *modality of G on V* is defined to be

$$\operatorname{mod}(G : V) = \max_{j \in \mathbb{Z}_{\geqslant 0}} (\dim V_j - j).$$

Informally, $\operatorname{mod}(G : V)$ is the maximum number of parameters on which a family of G-orbits depends. We note that if G acts on V with only finitely many orbits, then $\operatorname{mod}(G : V) = 0$.

Now let $\mathcal{S}(G, V)_j$ be the set of irreducible components of V_j, and let

$$\mathcal{S}(G, V) = \bigsqcup_{j \in \mathbb{Z}_{\geqslant 0}} \mathcal{S}(G, V)_j.$$

The elements of $\mathcal{S}(G, V)$ are known as the *sheets of G on V*. If G acts on V with only finitely many orbits, then the sheets of G on V coincide with the orbits of G on V.

The connection between modalities and commuting varieties is described in the following lemma [26, Lem. 2.1]:

Lemma 3.4.1. *Let G be a linear algebraic group, let $\mathfrak{g} = \operatorname{Lie} G$, and let $\operatorname{char} k = 0$. Then $\dim \mathcal{C}(\mathfrak{g}) = \dim \mathfrak{g} + \operatorname{mod}(G : \mathfrak{g})$.*

We remark that since we have assumed $\operatorname{char} k = 0$, the orbit map $G \to G \cdot X$ is separable and therefore that $\operatorname{Lie}(C_G(X)) = \mathfrak{c}_{\mathfrak{g}}(X)$ [9, Prop. 6.7]. This condition is essential for Lemma 3.4.1 to hold, and we maintain this for the remainder of this chapter. For further details in the case that $\operatorname{char} k > 0$, the reader is directed to [22].

3.5 A criterion for the irreducibility of $\mathcal{C}(\mathfrak{p})$

Armed with the definitions from the previous section, and the results of Lemma 3.1.2, we may now begin to examine when $\mathcal{C}(\mathfrak{p})$ is irreducible for a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. The work in the remainder of this chapter is joint with Simon Goodwin, and appears in [22].

We first require one further lemma.

Lemma 3.5.1. *All irreducible components of* $\mathcal{C}(\mathfrak{p})$ *have dimension at least* $\dim \mathfrak{p} + \operatorname{rank} G$.

Proof. Consider the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$. Let $X, Y \in \mathfrak{p}$, and write $X = X_1 + X_2$, $Y = Y_1 + Y_2$, where $X_1, Y_1 \in \mathfrak{l}$ and $X_2, Y_2 \in \mathfrak{u}_P$. Then $[X,Y] = [X_1, Y_1] + [X_1, Y_2] + [X_2, Y_1] + [X_2, Y_2]$; we note that $[X_1, Y_1] \in \mathfrak{l}$ and the remaining terms are in $\mathfrak{u}_P$.

Let $\mathcal{L}$ be the subvariety of $\mathfrak{p} \times \mathfrak{p}$ consisting of pairs (X, Y) such that $(X_1, Y_1) \in \mathcal{C}(\mathfrak{l})$ and $X_2, Y_2 \in \mathfrak{u}_P$. Then we have that $\mathcal{C}(\mathfrak{p}) \subseteq \mathcal{L}$, and $\mathcal{L} \cong \mathcal{C}(\mathfrak{l}) \times (\mathfrak{u}_P \times \mathfrak{u}_P)$. Since $\mathcal{C}(\mathfrak{l})$ is irreducible of dimension $\dim \mathfrak{l} + \operatorname{rank} L$, we have that $\mathcal{L}$ is irreducible of dimension $\dim \mathfrak{l} + \operatorname{rank} G + 2 \dim \mathfrak{u}_P = \dim \mathfrak{p} + \operatorname{rank} G + \dim \mathfrak{u}_P$.

Now consider the commutator map $F : \mathcal{L} \to \mathfrak{u}_P$ given by $F(X,Y) = [X,Y]$. Then $\mathcal{C}(\mathfrak{p})$ is the zero fibre of F, so the codimension in $\mathcal{L}$ of each irreducible component of $\mathcal{C}(\mathfrak{p})$ is at most $\dim \mathfrak{u}_P$, by [30, Thm. 4.1]. Therefore each irreducible component of $\mathcal{C}(\mathfrak{p})$ has dimension at least $(\dim \mathfrak{p} + \operatorname{rank} G + \dim \mathfrak{u}_P) - \dim \mathfrak{u}_P = \dim \mathfrak{p} + \operatorname{rank} G$. □

We may now state and prove the main theorem of this section.

Theorem 3.5.2. *Let* G *be a connected reductive algebraic group over the field* k*, let* P *be a parabolic subgroup of* G *and let* T *be a maximal torus of* G *contained in* P*. Denote by* $\mathcal{N}(\mathfrak{p})$ *the variety of nilpotent elements in* $\mathfrak{p}$*, and let* $\operatorname{ssrank} G$ *be the* *semisimple rank of* G*. Then the commuting variety* $\mathcal{C}(\mathfrak{p})$ *is irreducible if and only if* $\operatorname{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) < \operatorname{ssrank} H$ *for all Levi subgroups* $H \neq T$ *of* G *which contain* T*. Furthermore, if* $\mathcal{C}(\mathfrak{p})$ *is irreducible then* $\dim \mathcal{C}(\mathfrak{p}) = \dim \mathfrak{p} + \operatorname{rank} G$*.*

Proof. In order to prove this theorem we decompose $\mathcal{C}(\mathfrak{p})$ as a disjoint union of irreducible subvarieties such that the closure of some of these subvarieties are the irreducible components of $\mathcal{C}(\mathfrak{p})$. This allows us to determine when $\mathcal{C}(\mathfrak{p})$ is irreducible and is achieved by partitioning the P-orbits in $\mathfrak{p}$ in a way that generalizes the partition of G orbits in $\mathfrak{g}$ into decomposition classes as given by Borho and Kraft in [10].

Let $\hat{\mathcal{H}}$ denote the set of all Levi subgroups H of G containing T. We note that $H \in \hat{\mathcal{H}}$ is determined by the set of $\alpha \in \Phi$, where Φ is the root system of G with respect to T, such that $\mathfrak{g}_\alpha \subseteq \mathfrak{h}$. Thus we see that $\hat{\mathcal{H}}$ is a finite set. We note that different Levi subgroups in $\hat{\mathcal{H}}$ may be conjugate under P, and we choose $\mathcal{H}$ to be a subset of $\hat{\mathcal{H}}$ containing one representative from each P-conjugacy class.

Let $H \in \mathcal{H}$. We define $\mathfrak{z}(\mathfrak{h})^{\mathrm{reg}} = \{X \in \mathfrak{z}(\mathfrak{h}) \mid \mathfrak{c}_{\mathfrak{g}}(X) = \mathfrak{h}\}$. We have that $\mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$ is open in $\mathfrak{z}(\mathfrak{h})$, so $\mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$ is irreducible and $\dim \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}} = \dim \mathfrak{z}(\mathfrak{h}) = \operatorname{rank} G - \operatorname{ssrank} H$.

We have that $P \cap H$ is a parabolic subgroup of H, and we also consider $N_P(P \cap H)$. We note that $P \cap H$ has finite index in $N_P(P \cap H)$, because any coset in $N_P(P \cap H)/(P \cap H)$ can be chosen to have a representative that normalizes T, and $W = N_G(T)/T$ is finite. Thus $\dim N_P(P \cap H) = \dim(P \cap H)$.

We write $\mathcal{S}_H = \mathcal{S}(P \cap H, \mathfrak{p} \cap \mathfrak{h})$ for the set of sheets of $P \cap H$ on $\mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$ and $\mathcal{S}_{H,j} = \mathcal{S}(P \cap H, \mathcal{N}(\mathfrak{p} \cap \mathfrak{h}))_j$. So we have $\mathcal{S}_H = \bigsqcup_{j \in \mathbb{Z}_{\geqslant 0}} \mathcal{S}_{H,j}$. We note that $N_P(P \cap H)$ acts on $\mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$, and this gives rise to an action of $N_P(P \cap H)/(P \cap H)$ on $\mathcal{S}_H$.

Let $X \in \mathfrak{p}$ with Jordan decomposition $X = X_s + X_n$. Up to the adjoint action of P we may assume that $X_s \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$ for some $H \in \mathcal{H}$. Then we have that $X_n \in \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$, so $X_n \in S$ for some $S \in \mathcal{S}_H$.

Let $S \in \mathcal{S}_{H,j}$, where $j \in \mathbb{Z}_{\geqslant 0}$. We define $\mathcal{M}_{H,S} \subseteq \mathfrak{p}$ to be the variety of all $X = X_s + X_n \in \mathfrak{p}$ such that $X_s \in \mathfrak{z}(\mathfrak{h})^{\text{reg}}$ and $X_n \in S$. We have $\mathcal{M}_{H,S} \cong \mathfrak{z}(\mathfrak{h})^{\text{reg}} \times S$, so $\mathcal{M}_{H,S}$ is irreducible and $\dim \mathcal{M}_{H,S} = \operatorname{rank} G - \operatorname{ssrank} H + \dim S$.

Let $\mathcal{C}'_{H,S} = \{(X, Y) \mid X \in \mathcal{M}_{H,S}, Y \in \mathfrak{c}_{\mathfrak{p}}(X)\}$. For $X \in \mathcal{M}_{H,S}$ with Jordan decomposition $X = X_s + X_n$, we have $\mathfrak{c}_{\mathfrak{p}}(X) = \mathfrak{c}_{\mathfrak{p}}(X_s) \cap \mathfrak{c}_{\mathfrak{p}}(X_n) = \mathfrak{c}_{\mathfrak{p}\cap\mathfrak{h}}(X_n)$. Thus $\dim \mathfrak{c}_{\mathfrak{p}}(X) = \dim \mathfrak{c}_{\mathfrak{p}\cap\mathfrak{h}}(X_n) = \dim(\mathfrak{p} \cap \mathfrak{h}) - j$, where the last equality holds since we are working in the case of $\operatorname{char} k = 0$ so the orbit maps are separable. Therefore, the dimension of $\mathfrak{c}_{\mathfrak{p}}(X)$ does not depend on the choice of $X \in \mathcal{M}_{H,S}$.

Let $\mathcal{X}$ be an irreducible component of $\mathcal{C}'_{H,S}$ and consider the morphism $\pi : \mathcal{X} \to \mathcal{M}_{H,S}$ given by projecting on to the first component. The function taking $(X, Y) \in \mathcal{X}$ to the maximal dimension of an irreducible component of $\pi^{-1}(\pi((X, Y)) = \pi^{-1}(X)$ containing (X, Y) is upper semi-continuous, see for example [38, §8 Cor. 3]. Thus the set of $X \in \mathcal{M}_{H,S}$ such that $\{X\} \times \mathfrak{c}_{\mathfrak{p}}(X) \subseteq \mathcal{X}$ is closed in $\mathcal{M}_{H,S}$; here we require that $\dim \mathfrak{c}_{\mathfrak{p}}(X)$ does not depend on $X \in \mathcal{M}_{H,S}$ as proved above. Combining this with the irreducibility of $\mathcal{M}_{H,S}$ allows us to deduce that $\mathcal{C}'_{H,S}$ is irreducible.

Also we note that, for any $X \in \mathcal{M}_{H,S}$, we have

$$\begin{aligned}\dim \mathcal{C}'_{H,S} &= \dim \mathcal{M}_{H,S} + \dim \mathfrak{c}_{\mathfrak{p}}(X) \\ &= \operatorname{rank} G - \operatorname{ssrank} H + \dim S + \dim(\mathfrak{p} \cap \mathfrak{h}) - j \\ &= (\operatorname{rank} G - \operatorname{ssrank} H) + \dim(\mathfrak{p} \cap \mathfrak{h}) + (\dim S - j). \qquad (3.5.1)\end{aligned}$$

We define $\mathcal{C}_{H,S} = P \cdot \mathcal{C}'_{H,S} = \{(g \cdot X, g \cdot Y) \mid g \in P, (X, Y) \in \mathcal{C}'_{H,S}\}$ to be the P-saturation of $\mathcal{C}'_{H,S}$. Then we have that $\mathcal{C}_{H,S}$ is irreducible, as it is the image of the morphism $\phi : P \times \mathcal{C}'_{H,S} \to \mathcal{C}(\mathfrak{p})$ given by $\phi(g, (X, Y)) = (g \cdot X, g \cdot Y)$. For $S' \in \mathcal{S}_H$, we see that $\mathcal{C}_{H,S} = \mathcal{C}_{H,S'}$ if and only if S and S' lie in the same

$N_P(P \cap H)/(P \cap H)$-orbit.

We claim that

$$\dim \mathcal{C}_{H,S} = (\operatorname{rank} G - \operatorname{ssrank} H) + \dim \mathfrak{p} + (\dim S - j). \tag{3.5.2}$$

To prove this we consider the dimension of the fibres of the morphism $\phi : P \times \mathcal{C}'_{H,S} \to \mathcal{C}_{H,S}$ defined above. We note that the dimension of these fibres is constant on P-orbits, so it suffices to determine $\dim \phi^{-1}(X, Y)$ for $(X, Y) \in \mathcal{C}'_{H,S}$.

Let $(X, Y) \in \mathcal{C}'_{H,S}$. We note that if $g \in P \cap H$, then $(g, (g^{-1} \cdot X, g^{-1} \cdot Y)) \in \phi^{-1}(X, Y)$. Now suppose that $(g, X', Y')) \in \phi^{-1}(X, Y)$, so we have $X = g \cdot X'$ and $Y = g \cdot Y'$. We have Jordan decompositions $X = X_s + X_n$ and $X' = X'_s + X'_n$, where $X_s, X'_s \in \mathfrak{z}(\mathfrak{h})^{\mathrm{reg}}$, because $X, X' \in \mathcal{M}_{H,S}$. Also we have $X_s = g \cdot X'_s$, and thus $\mathfrak{c}_{\mathfrak{p}}(X_s) = g \cdot \mathfrak{c}_{\mathfrak{p}}(X'_s)$. Since $\mathfrak{c}_{\mathfrak{p}}(X_s) = \mathfrak{p} \cap \mathfrak{h} = \mathfrak{c}_{\mathfrak{p}}(X'_s)$, we have $g \in N_P(\mathfrak{p} \cap \mathfrak{h}) = N_P(P \cap H)$. Hence, we have shown that

$$P \cap H \subseteq \{g \in P \mid (g, (g^{-1} \cdot X, g^{-1} \cdot Y)) \in \phi^{-1}(X, Y)\} \subseteq N_P(P \cap H),$$

which implies that $\dim \phi^{-1}(X, Y) = \dim(P \cap H)$.

We have seen that the dimension of each fibre of ϕ is equal to $\dim(P \cap H)$. Hence, we have $\dim \mathcal{C}_{H,S} = \dim P + \dim \mathcal{C}'_{H,S} - \dim(P \cap H)$ and substituting from (3.5.1) gives (3.5.2).

Since S is a sheet for the action of $P \cap H$ on $\mathcal{N}(\mathfrak{p} \cap \mathfrak{h})$, we deduce that

$$\dim \mathcal{C}_{H,S} \leqslant (\operatorname{rank} G - \operatorname{ssrank} H) + \dim \mathfrak{p} + \operatorname{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})). \tag{3.5.3}$$

By construction we have the disjoint union

$$\mathcal{C}(\mathfrak{p}) = \bigsqcup_{\substack{H \in \mathcal{H} \\ S = \dot{\mathcal{S}}_H}} \mathcal{C}_{H,S},$$

where $\dot{\mathcal{S}}_H$ denotes a set of representatives for the $N_P(P \cap H)/(P \cap H)$-orbits in $\mathcal{S}_H$. Moreover, the closure $\overline{\mathcal{C}_{H,S}}$ of each $\mathcal{C}_{H,S}$ is closed and irreducible. Thus the irreducible components of $\mathcal{C}(\mathfrak{p})$ are given by some of the $\overline{\mathcal{C}_{H,S}}$.

We have $\mathcal{C}_{T,\{0\}}$ contains $\mathcal{C}^r_s(\mathfrak{p})$ as an open subset, so that $\overline{\mathcal{C}_{T,\{0\}}} = \overline{\mathcal{C}^r_s(\mathfrak{p})}$ is an irreducible component by Lemma 3.1.2.

Now suppose that $\operatorname{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) < \operatorname{ssrank} H$ for all $H \in \mathcal{H} \setminus \{T\}$. Then we have $\dim \overline{\mathcal{C}_{H,S}} = \dim \mathcal{C}_{H,S} < \dim \mathfrak{p} + \operatorname{rank} G$ for all $S \in \mathcal{S}_H$ and therefore $\overline{\mathcal{C}_{H,S}}$ is not an irreducible component of $\mathcal{C}(\mathfrak{p})$ by Lemma 3.5.1. Therefore, $\overline{\mathcal{C}^r_s(\mathfrak{p})}$ is the only irreducible component of $\mathcal{C}(\mathfrak{p})$ and hence $\mathcal{C}(\mathfrak{p})$ is irreducible.

Conversely, suppose that there exists a $H \in \mathcal{H} \setminus \{T\}$ such that $\operatorname{mod}(P \cap H : \mathcal{N}(\mathfrak{p} \cap \mathfrak{h})) \geqslant \operatorname{ssrank} H$. Then there exists $S \in \mathcal{S}_{H,j}$ for some $j \in \mathbb{Z}_{\geqslant 0}$ such that $\dim S - j \geqslant \operatorname{ssrank} H$. Then we have $\dim \overline{\mathcal{C}_{H,S}} = \dim \mathcal{C}_{H,S} \geqslant \operatorname{rank} G + \dim \mathfrak{p}$. Since $\mathcal{C}_{H,S} \cap \mathcal{C}^r_s(\mathfrak{p}) = \varnothing$, and $\dim \mathcal{C}_{H,S} \geqslant \dim \mathcal{C}^r_s(\mathfrak{p})$, we have $\overline{\mathcal{C}_{H,S}} \nsubseteq \overline{\mathcal{C}^r_s(\mathfrak{p})}$. Hence, $\mathcal{C}(\mathfrak{p})$ is not irreducible by Lemma 3.1.2.

Finally, it is immediate from Lemma 3.1.2 that $\dim \mathcal{C}(\mathfrak{p}) = \dim \mathfrak{p} + \operatorname{rank} G$ when $\mathcal{C}(\mathfrak{p})$ is irreducible. □

We then obtain the following immediate corollary:

Corollary 3.5.3. *Suppose that $\mathcal{C}(\mathfrak{p})$ is irreducible and let H be a Levi subgroup of G which contains T. Then $\mathcal{C}(\mathfrak{p} \cap \mathfrak{h})$ is irreducible.*

We end this section by recording a useful inductive result regarding irreducibility of $\mathcal{C}(\mathfrak{p})$.

Proposition 3.5.4. *Let P and Q be parabolic subgroups of G. Suppose that $P \leqslant Q$ and $\mathcal{C}(\mathfrak{p})$ is irreducible. Then $\mathcal{C}(\mathfrak{q})$ is irreducible.*

Proof. Let $(X, Y) \in \mathcal{C}(\mathfrak{q})$ with Jordan decompositions $X = X_s + X_n$ and $Y = Y_s + Y_n$. Then $X_n, Y_n \in \mathfrak{c}_{\mathfrak{g}}(X_s) \cap \mathfrak{c}_{\mathfrak{g}}(Y_s)$, which is the Lie algebra of the Levi subgroup $H = C_G(X_s) \cap C_G(Y_s)$ of G. By [49], there is a Springer isomorphism $\phi : \mathcal{N}(\mathfrak{h}) \to \mathcal{U}(H)$, where $\mathcal{U}(H)$ denotes the unipotent variety of H. Since ϕ is H-equivariant we deduce that $\phi(X_n)$ and $\phi(Y_n)$ commute. Therefore, $\phi(X_n)$ and $\phi(Y_n)$ lie in a Borel subgroup B_H of $Q \cap H$. Since Borel subgroups of $Q \cap H$ are conjugate, there exists $g \in Q \cap H$ such that $gB_Hg^{-1} \subseteq P \cap H$. By H-equivariance, we have that ϕ sends $\mathcal{N}(\mathfrak{h}) \cap \mathfrak{p}$ to $\mathcal{U}(H) \cap P$ and thus that $(g \cdot X, g \cdot Y) \in \mathcal{C}(\mathfrak{p})$. Since $\mathcal{C}(\mathfrak{p})$ is irreducible, we have $(g \cdot X, g \cdot Y) \in \overline{\mathcal{C}_s^r(\mathfrak{p})}$ by Lemma 3.1.2. Clearly, $\overline{\mathcal{C}_s^r(\mathfrak{p})} \subseteq \overline{\mathcal{C}_s^r(\mathfrak{q})}$, and we have that $\overline{\mathcal{C}_s^r(\mathfrak{q})}$ is stable under the adjoint action of Q. Therefore, $(X, Y) \in \overline{\mathcal{C}_s^r(\mathfrak{q})}$. Hence, $\mathcal{C}(\mathfrak{q})$ is irreducible. □

3.6 Classification of irreducibility of $\mathcal{C}(\mathfrak{b})$

Let B be a Borel subgroup of G with $\mathfrak{b} = \mathrm{Lie}(B)$, and let $\mathfrak{u} = \mathcal{N}(\mathfrak{b})$ be the nilradical of $\mathfrak{b}$. In his thesis Keeton [34, 6.3–4] provides a partial classification of the irreducibility of $\mathcal{C}(\mathfrak{b})$, using a result [34, Thm. 6.1] equivalent to Theorem 3.5.2 above for the case of $P = B$. In this section we use recent results of [25] and [42] concerning the values of $\mathrm{mod}(B : \mathfrak{u})$, along with lower bounds for $\mathrm{mod}(B : \mathfrak{u})$ established in [46, Thm. 3.1], to complete this classification.

In [25] a parametrization of the coadjoint orbits of U in $\mathfrak{u}^*$ is given for G of rank at most 8 apart from G of type E_8. It is known that $\mathrm{mod}(U : \mathfrak{u}) = \mathrm{mod}(U : \mathfrak{u}^*)$; see [47, Thm. 1.4]. Also in [25, Thm. 5.1] it is proved that $\mathrm{mod}(U : \mathfrak{u}) = \mathrm{mod}(B : \mathfrak{u}) + \mathrm{ssrank}\, G$. In this way values of $\mathrm{mod}(B : \mathfrak{u})$ for G up to rank 8 apart from G of type E_8 were determined. This extended previously known values of $\mathrm{mod}(B : \mathfrak{u})$

given in [33, Tables II & III].

The results in [42] can be used to determine $\mathrm{mod}(B : \mathfrak{u})$ for $G = \mathrm{GL}_n(k)$ and $n \leqslant 16$ in the following way. Let q be a prime power, let $U(q)$ be the subgroup of upper unitriangular matrices in $\mathrm{GL}_n(q)$ and let $\mathfrak{u}^*(q)$ be the dual space of the space $\mathfrak{u}(q)$ of strictly upper triangular matrices in $\mathfrak{gl}_n(q)$. Then $U(q)$ acts on $\mathfrak{u}^*(q)$ via the coadjoint action. The number $k(U(q), \mathfrak{u}^*(q))$ of coadjoint orbits of $U(q)$ in $\mathfrak{u}^*(q)$ is determined for $n \leqslant 16$ in [42] and this number is shown to be a polynomial in q, see [42, Thm. 1.2]. Although [42] only deals with finite fields, the methods used can be adapted to apply for other fields. In particular, this means that the calculations carried out as part of [42] can be used to determine $\mathrm{mod}(U : \mathfrak{u}^*)$ for $G = \mathrm{GL}_n(k)$ and $n \leqslant 16$; moreover, we see that $\mathrm{mod}(U : \mathfrak{u}^*)$ is equal to the degree of the polynomial in q giving $k(U(q), \mathfrak{u}^*(q))$. Combining this with the fact that $\mathrm{mod}(U : \mathfrak{u}^*) = \mathrm{mod}(U : \mathfrak{u}) = \mathrm{mod}(B : \mathfrak{u}) + \mathrm{ssrank}\, G$, we deduce the values of $\mathrm{mod}(B : \mathfrak{u})$.

Combining the results in [25], [42] and [46], gives Tables 1–5, containing the exact value or a lower bound for $\mathrm{mod}(B : \mathfrak{u})$ for G of low rank.

Type of G	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
$\mathrm{mod}(B : \mathfrak{u})$	0	0	0	0	1	1	2	3	4
Type of G	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}	
$\mathrm{mod}(B : \mathfrak{u})$	5	7	8	10	12	14	$\geqslant 16$	$\geqslant 19$	

Table 3.1: Modality of the action of B on $\mathfrak{u}$ for G of type A

Type of G	B_2	B_3	B_4	B_5	B_6	B_7	B_8
$\mathrm{mod}(B : \mathfrak{u})$	0	1	2	3	5	7	9

Table 3.2: Modality of the action of B on $\mathfrak{u}$ for G of type B

Type of G	C_3	C_4	C_5	C_6	C_7	C_8
$\mathrm{mod}(B : \mathfrak{u})$	1	2	3	5	7	9

Table 3.3: Modality of the action of B on $\mathfrak{u}$ for G of type C

Type of G	D_4	D_5	D_6	D_7	D_8
$\mathrm{mod}(B : \mathfrak{u})$	1	2	4	5	8

Table 3.4: Modality of the action of B on $\mathfrak{u}$ for G of type D

Type of G	G_2	F_4	E_6	E_7	E_8
$\mathrm{mod}(B : \mathfrak{u})$	1	4	5	10	$\geqslant 20$

Table 3.5: Modality of the action of B on $\mathfrak{u}$ for G of exceptional type

Now from Lemma 3.4.1 and Theorem 3.5.2 we see that $\mathcal{C}(\mathfrak{b})$ is irreducible if and only if $\mathrm{mod}(B : \mathfrak{u}) < \mathrm{rank}\, G$. So from the tables we immediately deduce the following theorem:

Theorem 3.6.1. *Let G be a connected reductive algebraic group over an algebraically closed field k with $\mathrm{char}\, k = 0$. Then $\mathcal{C}(\mathfrak{b})$ is irreducible if and only if the type of each simple component of the root system of G is one of the following.*

(i.) A_l *for* $l \leqslant 15$*;*

(ii.) B_l *for* $l \leqslant 6$*;*

(iii.) C_l *for* $l \leqslant 6$*;*

(iv.) D_l *for* $l \leqslant 7$*;*

(v.) G_2 *or* E_6.

3.7 Remarks on the irreducibility of $\mathcal{C}(\mathfrak{p})$

To conclude this chapter we briefly discuss some cases where we can determine whether $\mathcal{C}(\mathfrak{p})$ is irreducible (or reducible) for $P \neq B$. In general little is known about the values of $\operatorname{mod}(P : \mathcal{N}(\mathfrak{p}))$, so we can only present limited results in this direction.

First we note that by Proposition 3.5.4 and Theorem 3.6.1, we have $\mathcal{C}(\mathfrak{p})$ is irreducible whenever G satisfies the conditions in Theorem 3.6.1.

In [46] lower bounds for $\operatorname{mod}(P : \mathfrak{u}_P)$ are given, where $\mathfrak{u}_P$ is the Lie algebra of the unipotent radical of P. Of course, we have $\operatorname{mod}(P : \mathfrak{u}_P) \leqslant \operatorname{mod}(P : \mathcal{N}(\mathfrak{p}))$. From these lower bounds, we can determine many instances where $\mathcal{C}(\mathfrak{p})$ is reducible. In particular, these lower bounds are quadratic in $\operatorname{ssrank} G - \operatorname{ssrank} L$, so for a fixed value of $\operatorname{ssrank} L$, we have that $\mathcal{C}(\mathfrak{p})$ is reducible if $\operatorname{rank} G$ is sufficiently large.

If the number of P-orbits in $\mathcal{N}(\mathfrak{p})$ is finite, so that $\operatorname{mod}(P : \mathcal{N}(\mathfrak{p})) = 0$, then certainly $\mathcal{C}(\mathfrak{p})$ is irreducible. It follows from results of Murray in [39] that $\operatorname{mod}(P : \mathcal{N}(\mathfrak{p})) = 0$ for P a maximal parabolic subgroup of $\mathrm{GL}_n(k)$ such that one block has size less than 6. Recent work of Bulois and Boos in [8] gives a classification of cases where $\operatorname{mod}(P : \mathcal{N}(\mathfrak{p})) = 0$ for $G = \mathrm{GL}_n(k)$. Further, in [8, §6], there is a discussion of cases of higher modality. In particular, it is shown that for the maximal parabolic subgroup P of $\mathrm{GL}_n(k)$ with block sizes $(200, 400)$, we have that $\mathcal{C}(\mathfrak{p})$ is reducible.

CHAPTER 4

COMMUTING NILPOTENT ORBITS

We recall from Section 2.8.2, and from [32], that nilpotent orbits in $\mathfrak{gl}_n(k)$, $\mathfrak{sp}_n(k)$ and $\mathfrak{so}_n(k)$ are parametrised by partitions of n with certain properties. In this chapter we consider the following question: given two orbits $\mathcal{O}_\lambda$ with associated partition λ, and $\mathcal{O}_\mu$ with associated partition μ, does there exist $X \in \mathcal{O}_\lambda$ and $Y \in \mathcal{O}_\mu$ such that $[X, Y] = 0$?

If such a pair $(X, Y) \in \mathcal{O}_\lambda \times \mathcal{O}_\mu$ exists, we say that $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_\mu$.

In the case of $\mathfrak{gl}_n(k)$ this question has been studied by Oblak [41], and by Britnell and Wildon [13] in the group setting, among others. As remarked in [13, §4], it appears to be a difficult problem to determine whether or not an arbitrary pair of orbits commute; and even harder to show that a pair of orbits do not commute. Consequently interest has focused on orbits labelled by specific types of partitions, as described in the following discussion.

The aim of this chapter is to extend certain results of [41] and [13] to the cases of $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$ and $\mathfrak{g} = \mathfrak{so}_n(k)$. We begin by considering which orbits commute with the regular orbits in each case; these are natural targets to study since the centralisers of elements of the regular orbits possess a comparatively simple structure.

We then explore which orbits in $\mathfrak{g}$ commute with all other orbits; these orbits will be referred to as being *universally commuting*, or UC for short. In the absence of a simple way to determine whether two given orbits commute, classifying the universally commuting orbits provides a step in this direction by identifying many pairs of orbits which do indeed commute. In the case of $\mathfrak{gl}_n(k)$ there is a simple classification of UC orbits described in [13, Thm. 4.6] and [41, Thm. 2.4], which we discuss in more detail in Section 4.4; consequently it is of interest to explore how this result generalises to the symplectic and orthogonal cases. Our main results provide a classification (with a single exception) of universally commuting orbits in $\mathfrak{sp}_{2m}(k)$ and $\mathfrak{so}_n(k)$; we note that this is significantly more complex than the same problem for $\mathfrak{gl}_n(k)$.

In this chapter the only restrictions on the field k that we require are that if we are working with $\mathfrak{sp}_{2m}(k)$ or $\mathfrak{so}_n(k)$, we must have $\operatorname{char} k \neq 2$, and k must contain the elements $\sqrt{2}$ and $i = \sqrt{-1}$.

Finally, we note that throughout this chapter we spend a considerable amount of time working with centralisers of matrices arising from Dynkin pyramids as described in Section 2.9. In the case of $\mathfrak{g} = \mathfrak{gl}_n(k)$, the structure of the centraliser of a matrix in Jordan form is well-known and described in [5, Lem. 3.2]; though to use this, we must first apply the change of basis that takes a matrix in Jordan form to one described by a Dynkin pyramid. In the symplectic and orthogonal cases these centralisers are somewhat more complex, and are described in [52, Lem. 1.5.2 & 1.5.9]; we use these results implicitly throughout this chapter.

4.1 Centralisers and Dynkin pyramids

Recall that in Section 2.9 we introduced the notion of the linear Dynkin pyramid of a partition λ, and described how we may define a matrix $D(\lambda)$ from this pyramid. Similarly, if λ is a symplectic or orthogonal partition we defined the symplectic and orthogonal Dynkin pyramids of λ and defined the matrices $D^{-}(\lambda)$ and $D^{+}(\lambda)$. In this section we describe how we may use Dynkin pyramids to visualise an element of the centraliser of these matrices.

We first introduce some notation. Let $\{v_1, ..., v_n\}$ be the basis of the underlying vector space such that v_j is the n-dimensional column vector whose entries are 1 in the jth position, and 0 elsewhere. We maintain this notation for the remainder of the chapter. In particular we note that if A is an $n \times n$ matrix and $1 \leqslant j \leqslant n$, we have $Av_j = \sum_{1 \leqslant i \leqslant n} a_{ij} v_i$, where a_{ij} is the entry of A in the ith row and the jth column. By an abuse of terminology we say that A maps v_j to v_i if $a_{ij} \neq 0$, regardless of the values of the other entries of A; and that A maps v_j to zero if the jth column of A contains no non-zero entries.

Consider now the example of $\lambda = (4, 2)$ in $\mathfrak{gl}_6(k)$. We recall that the linear Dynkin pyramid of λ is as follows:

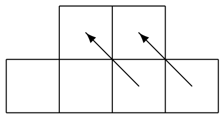
	2	4	
1	3	5	6

Now the matrix defined from this pyramid is $D(4, 2) = E_{1,3} + E_{2,4} + E_{3,5} + E_{5,6}$. By a direct calculation, or reference to [5, Lem. 3.2], we observe that a general element of the centraliser of $D(4, 2)$ in $\mathfrak{gl}_6(k)$ is of the following form:

$$A = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ & 0 & 0 & b_2 & b_3 & b_4 \\ & & 0 & a_1 & a_2 & a_4 \\ & & & 0 & 0 & b_3 \\ & & & & 0 & a_2 \\ & & & & & 0 \end{pmatrix}.$$

Consider the matrix $\tilde{A} = E_{25} + E_{46}$ obtained from A by setting $b_3 = 1$ and

$a_i = b_j = 0$ otherwise. Clearly $\tilde{A}$ maps v_6 to v_4 and v_5 to v_2; but v_4, v_3, v_2 and v_1 are all mapped to zero. We may represent this matrix on the Dynkin pyramid by drawing lines from the box labelled s to the box labelled r if and only if the entry $\tilde{a}_{rs}$ is non-zero; we say that $\tilde{a}_{rs}$ is the *coefficient* of the line. In the following, we may usually chose these coefficients to be equal to ± 1. So in the case of $\tilde{A}$ we obtain the following:



We note in particular that these two lines are parallel to each other, and of the same length. In fact this is a general property of elements of these centralisers. If in a matrix A we have $a_{ij} \neq 0$ for some (i, j), we must also have $a_{rs} \neq 0$ whenever the line on the Dynkin pyramid from s to r is parallel to and of the same length as the line from j to i, for s in the same row as j and r in the same row as i.

However we note that we are not forced to draw parallel lines between different rows of the Dynkin pyramid. For example, if we set $a_1 = 1$ in the matrix A above, this corresponds to drawing lines from box 6 to box 5, from box 5 to box 3 and from box 3 to box 1; however we do not need to draw the line from box 4 to box 2, as the start and endpoints of this line are in a different row of the pyramid to boxes 6 and 5.

The key motivation for representing matrices in this way is that it makes the Jordan type of the matrix easy to determine. If all paths along the lines on the pyramid (moving right to left, or vertically) are disjoint, then each path of length l corresponds to a part of size $(l + 1)$ in the Jordan type of the matrix. If a box of the pyramid is not the start or endpoint of a line, then we consider this box to lie on a path of length zero, thus giving rise to a part of size 1 in the Jordan type of the matrix.

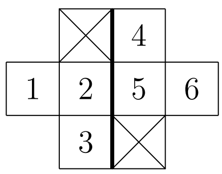
If there are two such non-disjoint paths of length l and m such that $l \geqslant m$, then we obtain a part in the Jordan type of size $l+1$ and a part of size at most m, unless $l = m$ and the paths share both a start and an endpoint. In this latter case, depending on the coefficients of the lines, we may obtain two parts of size l instead. We refer the reader to Example 4.2.4 for an example of how these latter cases work in practice.

Since the lines on the pyramid of $(4, 2)$ corresponding to $\tilde{A}$ form two disjoint paths of length 1, we conclude that $\tilde{A}$ has Jordan type $(2^2, 1^2)$. Indeed, considering the action of $\tilde{A}$ on the basis vectors $v_1, ..., v_6$ we obtain the following:

$$\begin{gathered} v_6 \mapsto \tilde{A}v_6 = v_4 \mapsto 0; \\ v_5 \mapsto \tilde{A}v_5 = v_3 \mapsto 0; \\ v_2 \mapsto \tilde{A}v_2 = 0; \\ v_1 \mapsto \tilde{A}v_1 = 0. \end{gathered}$$

Since the entries of each of the above strings are all linearly independent, each string corresponds to a block in the Jordan form of $\tilde{A}$, whose size is equal to the number of non-zero entries in the string. Further, each non-zero entry of these strings is an element of the basis with respect to which $\tilde{A}$ is in Jordan form. So we do indeed have the required Jordan type $(2^2, 1^2)$.

If we are working in $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$ or $\mathfrak{so}_n(k)$, there are some additional subtleties to be taken into account. We illustrate these with the example of $\lambda = (4, 2)$ in $\mathfrak{sp}_6(k)$. The symplectic Dynkin pyramid of λ is then the following:

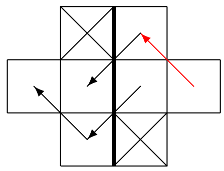


Since we recall that the boxes labelled 3 and 4 on the skew row are considered to be adjacent, the matrix $D^-(4,2)$ defined from this pyramid is $D^-(4,2) = E_{1,2} + E_{2,5} + E_{3,4} - E_{5,6}$. Then a general element of the centraliser in $\mathfrak{sp}_6(k)$ of $D^-(4,2)$ is of the following form:

$$B = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & 0 & a_5 \\ & 0 & 0 & a_2 & a_1 & 0 \\ & & 0 & c_1 & a_2 & a_3 \\ & & & 0 & 0 & -a_2 \\ & & & & 0 & -a_1 \\ & & & & & 0 \end{pmatrix}.$$

We may now represent matrices of this form by drawing lines on the Dynkin pyramid in the same way as in the linear case, with a couple of minor differences. Firstly we note that if either $a_1 \neq 0$ or $a_2 \neq 0$ then this matrix will also contain entries equal to the additive inverse of a_1 or a_2. To take this into account, suppose that there is a non-zero entry of B equal to a_i; then we colour each line that we draw on the pyramid with coefficient $\pm a_i$ either black if its coefficient is equal to a_i, or red if its coefficient is equal to $-a_i$. We describe the rules for the colour of each line later on, in conjunction with examples for which they are required.

Secondly, when the symplectic or orthogonal Dynkin pyramid contains one or more skew rows, certain lines on this pyramid may be parallel that are not parallel on the corresponding linear pyramid. Consider for example the matrix $\tilde{B} = E_{1,3} + E_{2,4} + E_{3,5} - E_{4,6}$ obtained from B by setting $a_2 = 1$ and $a_i = c_i = 0$ otherwise. Then $\tilde{B}$ may be represented on the symplectic Dynkin pyramid of $(4,2)$ by the following lines:



We note that the boxes labelled 3 and 4 on the skew row are considered to be adjacent; so the line between boxes 5 and 3, the second-to-rightmost box of each row, is considered to be parallel to the line between the rightmost boxes of each row labelled 6 and 4. The lines from 4 to 2 and from 3 to 1 are then forced by the fact that $\tilde{B}$ is symmetric (up to sign) about the reverse diagonal; this property corresponds to the lines drawn on the Dynkin pyramid possessing rotational symmetry (up to sign).

We can then read off the Jordan type of $\tilde{B}$ in exactly the same way as in the linear case; since there are two disjoint paths of length 2 and every box of the pyramid is the start or endpoint of a line, we see that $\tilde{B}$ has Jordan type (3^2).

4.2 Regular and subregular orbits in $\mathfrak{gl}_n(k)$

We begin this section by describing which orbits in $\mathfrak{gl}_n(k)$ commute with the regular orbit $\mathcal{O}_{(n)}$ and the sub-regular orbit $\mathcal{O}_{(n-1,1)}$. Propositions 4.2.2 and 4.2.3 below are due to Britnell and Wildon [13, Prop. 4.1 & 4.4] and to Oblak [40, Prop. 1]. We provide alternative proofs of these results in order to demonstrate the techniques that we will use throughout this chapter.

We recall that a partition is *almost rectangular* if its parts differ by at most 1. From Lemma 2.7.5 we observe that for any pair of natural numbers (n, i) with $i \leqslant n$, there is precisely one almost rectangular partition of n with i parts. We denote this partition by $\rho_i(n)$, and the jth part of this partition by $\rho_i(n)_j$.

Lemma 4.2.1. *Let $E = D(n)$ be the matrix defined by the linear Dynkin pyramid of the partition (n) as described in Section 2.9. Let $A = \sum_{j=1}^{n-1} a_j E^j \in \mathfrak{gl}_n(k)$, and let i be minimal such that $a_i \neq 0$. Then the Jordan type of A is $\rho_i(n)$.*

Proof. We recall that the linear Dynkin pyramid of (n) consists of a single row of boxes; the matrix $E = D(n)$ has entries equal to 1 on the first superdiagonal and zeroes elsewhere. Further, we note that the matrix E^i has entries equal to 1 on the ith superdiagonal, and zeroes elsewhere.

We now consider the action of A on the basis vectors $\{v_1, ..., v_n\}$; we find that $A^h v_n = a_i^h v_{n-hi} + \sum_{j<n-hi} b_j v_j$, for $h \in \mathbb{N}$ and $b_j \in k$. So when we repeatedly apply A to the last i basis vectors, we obtain the following:

$$
\begin{gathered}
v_n \mapsto Av_n \mapsto ... \mapsto A^{\rho_i(n)_1} v_n \mapsto 0 \\
v_{n-1} \mapsto Av_{n-1} \mapsto ... \mapsto A^{\rho_i(n)_2} v_{n-1} \mapsto 0 \\
\vdots \\
v_{n-i+1} \mapsto Av_{n-1+1} \mapsto ... \mapsto A^{\rho_i(n)_i} v_{n-i+1} \mapsto 0
\end{gathered}
$$

We note that each line above gives a block in the Jordan type of A; so the Jordan type of A has i parts each of which differ by at most 1. It follows that the Jordan type of A is $\rho_i(n)$. A basis relative to which A is in Jordan normal form is given by $\{A^j v_{n-k} : j = 0, ..., \rho_i(n)_i; k := 0, ..., i-1\}$, since the matrix that maps $\{v_n, ..., v_1\}$ to this set is upper triangular with non-zero entries on the diagonal. In particular we note that the Jordan type of A is the same as that of E^i. □

Proposition 4.2.2. *A nilpotent orbit with partition λ in $\mathfrak{gl}_n(k)$ commutes with the regular orbit $\mathcal{O}_{(n)}$ if and only if λ is almost rectangular.*

Proof. Let $E = D(n)$ be as defined in Lemma 4.2.1 above. Then the set $\{E^i : 1 \leqslant i \leqslant i-1\}$ forms a basis for $\mathfrak{c}_{\mathfrak{gl}_n(k)}(E) \cap \mathfrak{n}$ [5, Lem. 3.2]. Consequently we masy write a general element of this centraliser as $A = \sum_{j=1}^{n-1} a_j E^j$. The possible

Jordan types of A are then given by Lemma 4.2.1 above. $\square$

Proposition 4.2.3. *A nilpotent orbit with partition λ in $\mathfrak{gl}_n(k)$ commutes with the sub-regular orbit $\mathcal{O}_{(n-1,1)}$ if and only if one of the following holds:*

(i.) *λ is of the form $(\lambda', 1)$, where λ' is almost rectangular;*

(ii.) *λ is of the form $(3, 2^{r_2}, 1^{r_1})$ where $r_1 > 0$ and $2r_2 + r_1 = n - 3$;*

(iii.) *$n = 2m$ for $m \in \mathbb{N}$, and $\lambda = (2^m)$;*

(iv.) *$n = 3$ and $\lambda = (3)$.*

Before proving this formally, we describe how this works for the example of the orbit $\mathcal{O}_{(9,1)}$ in $\mathfrak{gl}_{10}(k)$, in order to demonstrate a number of techniques that will be used throughout the remainder of this chapter.

Example 4.2.4. The linear Dynkin pyramid of $(9, 1)$ is as follows:

				5				
1	2	3	4	6	7	8	9	10

We now define the matrix $D(9, 1)$ as described in Section 2.9; that is, $D(9, 1) = E_{1,2} + E_{2,3} + E_{3,4} + E_{4,6} + E_{6,7} + E_{7,8} + E_{8,9} + E_{9,10}$. Using the results of [52, Lem. 1.5.2] or [5, Lem. 3.2] we find that a general element of the centraliser of $D(9, 1)$ in $\mathfrak{gl}_{10}(k)$ is of the following form:

$$A = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & b & a_4 & a_5 & a_6 & a_7 & a_8 \\ & 0 & a_1 & a_2 & 0 & a_3 & a_4 & a_5 & a_6 & a_7 \\ & & 0 & a_1 & 0 & a_2 & a_3 & a_4 & a_5 & a_6 \\ & & & 0 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ & & & & 0 & 0 & 0 & 0 & 0 & c \\ & & & & & 0 & a_1 & a_2 & a_3 & a_4 \\ & & & & & & 0 & a_1 & a_2 & a_3 \\ & & & & & & & 0 & a_1 & a_2 \\ & & & & & & & & 0 & a_1 \\ & & & & & & & & & 0 \end{pmatrix}$$

Let E be the matrix defined from the above A by setting $a_1 = 1$ and $a_i = b = c = 0$ otherwise; that is, $E = D(9, 1)$. Additionally let X be the matrix such that $b = 1$ and $a_i = c = 0$ otherwise, and let Y be the matrix such that $c = 1$ and $a_i = b = 0$ otherwise. Then it is clear that the set $\{E^i, X, Y : 1 \leqslant i \leqslant 8\}$ forms a basis of the strictly upper triangular part of the centraliser. So write $A = \sum_{i=1}^{8} a_i E^i + bX + cY$.

Denote by A' the 9×9 matrix comprised of all unhighlighted entries of A. Then by Proposition 4.2.2, the Jordan type of A' is $\rho_i(9)$, where i is minimal such that $a_i \neq 0$. We observe from Lemma 4.2.1 that the a_j with $j > i$ have no effect on the Jordan type of A'; consequently we may assume in all that follows that we have only one non-zero a_i.

We now have four cases to consider. Suppose first that $b = c = 0$; then by Proposition 4.2.2, we see that A' has Jordan type $\rho_i(9)$, where $i \in \mathbb{N}$ is minimal such that $a_i \neq 0$. Furthermore, we note that Av_l is a linear combination of the basis elements v_j; but the coefficient of v_5 is always zero. So it remains only to consider the action of A on v_5, since there is no l such that A; but $Av_5 = 0$, so this gives rise to a part of size 1 in the Jordan type of A. Hence the Jordan type of A is $(\rho_j(9), 1)$.

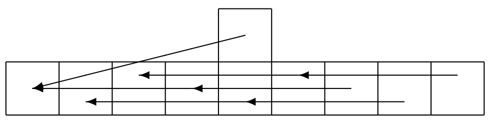
Consider for example the specific example that $a_2 = 1$ and $a_i = 0$ otherwise. By repeatedly applying A to the basis vectors $v_{10}, ..., v_1$ we obtain the following:

$$v_{10} \mapsto Av_{10} = v_8 \mapsto A^2v_{10} = v_6 \mapsto A^3v_{10} = v_3 \mapsto A^4v_{10} = v_1 \mapsto A^5v_{10} = 0;$$
$$v_9 \mapsto Av_9 = v_7 \mapsto A^2v_9 = v_4 \mapsto A^3v_9 = v_2 \mapsto A^4v_9 = 0;$$
$$v_5 \mapsto Av_5 = 0.$$

We thus observe that the Jordan type of A in this specific example is $(5, 4, 1) = (\rho_2(9), 1)$. By Lemma 4.2.1, we will obtain the same Jordan type if we were to set $a_i \neq 0$ for some $i > 2$; so we may without loss of generality consider only one non-zero a_i at a time throughout this example.

For the second case, suppose that $b \neq 0$ and $c = 0$. Clearly if all $a_i = 0$ then the only non-zero entry of the matrix A is the entry denoted by b; in this case it is clear that the Jordan type of A is $(2, 1^8)$.

To illustrate what happens when A' contains non-zero entries in this case, consider the example that $a_3 \neq 0 \neq b$ and $a_i = c = 0$ for $i \neq 3$. We illustrate the effect of the matrix A on the basis elements v_i by drawing lines on the Dynkin pyramid in the same way as described in Section 4.1. In this example, we therefore draw the following lines:

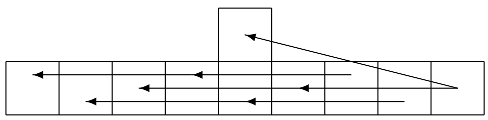


This is then equivalent to the following:

$$v_{10} \mapsto a_3v_7 \mapsto a_3^2v_3 \mapsto 0;$$
$$v_9 \mapsto a_3v_6 \mapsto a_3^2v_2 \mapsto 0;$$
$$v_8 \mapsto a_3v_4 \mapsto a_3^2v_1 \mapsto 0;$$
$$v_5 - \tfrac{b}{a_3}v_4 \mapsto 0.$$

We therefore see that the Jordan type of A is $(3^3, 1) = (\rho_3(9), 1)$, noting that $\rho_3(9)$ is the Jordan type of A'. The basis with respect to which A has Jordan normal form is $\{v_{10}, Av_{10}, A^2v_{10}, v_9, Av_9, A^2v_9, v_8, Av_8, A^2v_8, v_5 - \frac{b}{a_3}v_4\}$. In general, if i is minimal such that $a_i \neq 0$, we can use this method to show that the Jordan type of A is $(\rho_i(9), 1)$.

The case that $b = 0$ and $c \neq 0$ is for the most part similar. We illustrate this with the example of $a_3 \neq 0 \neq c$ and $a_i = b = 0$ for $i \neq 3$. By drawing lines on the Dynkin pyramid in the same way as the $b \neq 0$ example above, we obtain:



In particular, we have $v_{10} \mapsto a_3v_7 + cv_5 \mapsto a_3^2v_3 \mapsto 0$; that is to say, A maps v_{10} to 0 in two steps via v_5, but in three steps via the long block of the Dynkin pyramid. We say that the string $v_{10} \mapsto a_3v_7 \mapsto a_3^2v_3 \mapsto 0$ *subsumes* the string $v_{10} \mapsto cv_5 \mapsto 0$.

This then gives a Jordan type of $(3^3, 1) = (\rho_3(9), 1)$ with respect to the basis $\{v_{10}, Av_{10}, A^2v_{10}, v_9, Av_9, A^2v_9, v_8, Av_8, A^2v_8, v_5\}$.

As in the previous cases, we can alter the non-zero a_i to find matrices which have all Jordan types of the form $(\rho_j(9), 1)$.

Finally we consider the case that $b \neq 0 \neq c$. If $a_i = 0$ for all i, then we have $v_{10} \mapsto cv_5 \mapsto bcv_1 \mapsto 0$, and $v_j \mapsto 0$ for all $j \neq 10, 5, 1$. So here we find that A has Jordan type $(3, 1^7)$.

It remains to consider which Jordan types occur in this case for the possible choices of non-zero a_i.

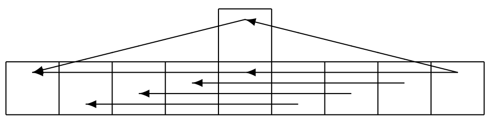
First, suppose that $a_3 \neq 0$ and $a_i = 0$ otherwise. Here the Jordan type of A' is $(3^3) = \rho_3(9)$, and we obtain

$$v_{10} \mapsto a_3 v_7 + c v_5 \mapsto a_3^2 v_3 + bc v_1 \mapsto 0;$$
$$v_9 \mapsto a_3 v_6 \mapsto a_3^2 v_2 \mapsto 0;$$
$$v_8 \mapsto a_3 v_4 \mapsto a_3^2 v_1 \mapsto 0;$$
$$v_5 - \tfrac{b}{a_3} v_4 \mapsto 0.$$

Consequently we find that the Jordan type of A is $(3^3, 1) = (\rho_3(9), 1)$.

The cases of $a_2 \neq 0$ and $a_1 \neq 0$ are treated in the same way, and in each case we obtain a Jordan type of $(\rho_i(9), 1)$. In effect, in these cases the string $v_{10} \mapsto cv_5 \mapsto bcv_1 \mapsto 0$ is subsumed into the string arising from the largest Jordan block of A'.

Next, the case of $a_4 \neq 0$ and $a_i = 0$ otherwise requires a closer look. In this case the Jordan type of A' is $\rho_4(9) = (3, 2^3)$, and we may represent this by drawing the following lines on the Dynkin pyramid:



Now we have $v_{10} \mapsto a_4 v_6 + c v_5 \mapsto (a_4^2 + bc) v_1$; so the Jordan type of A now depends on whether or not we have $a_4^2 + bc = 0$.

If it is the case that $a_4^2 + bc = 0$, then we have the following:

$$v_{10} \mapsto a_4 v_6 + c v_5 \mapsto (a_4^2 + bc) v_1 = 0;$$
$$v_9 \mapsto a_4 v_4 \mapsto 0;$$
$$v_8 \mapsto a_4 v_3 \mapsto 0;$$
$$v_7 \mapsto a_4 v_2 \mapsto 0;$$
$$v_5 \mapsto b v_1 \mapsto 0.$$

We thus see that A has Jordan type (2^5) in this case, with respect to the basis $\{v_{10}, Av_{10}, v_9, Av_9, v_8 Av_8, v_7, Av_7, v_5, Av_5\}$.

Conversely suppose that $a_4^2 + bc \neq 0$. Then we have:

$$v_{10} \mapsto a_4 v_6 + c v_5 \mapsto (a_4^2 + bc) v_1 \mapsto 0;$$
$$v_9 \mapsto a_4 v_4 \mapsto 0;$$
$$v_8 \mapsto a_4 v_3 \mapsto 0;$$
$$v_7 \mapsto a_4 v_2 \mapsto 0;$$
$$v_5 - \frac{b}{a_4} v_6 \mapsto 0.$$

Thus we find that A has Jordan type $(3, 2^3, 1)$.

Now suppose that $a_5 \neq 0$, and $a_i = 0$ otherwise. In this case the Jordan type of A' is $\rho_5(9) = (2^4, 1)$. We then find the following:

$$v_{10} \mapsto c v_5 + a_5 v_4 \mapsto bc v_1 \mapsto 0;$$
$$v_9 \mapsto a_5 v_3 \mapsto 0;$$
$$v_8 \mapsto a_5 v_2 \mapsto 0;$$
$$v_5 - \frac{b}{a_5} v_7 \mapsto 0;$$
$$v_6 \mapsto 0;$$
$$v_4 \mapsto 0.$$

In this case also, we see that A has Jordan type $(3, 2^2, 1^3)$. Finally, we can use the exact same argument to show that if $a_6 \neq 0$ (respectively, $a_7 \neq 0$ and $a_8 \neq 0$) and $a_i = 0$ otherwise, then in this case A has Jordan type $(3, 2, 1^5)$ (respectively, $(3, 1^7)$ and $(3, 1^7)$). We conclude by observing that we have obtained matrices in the centraliser of $D(9, 1)$ of each of the Jordan types described in Proposition 4.2.3.

Having shown that Proposition 4.2.3 holds in $\mathfrak{gl}_{10}(k)$, we now proceed to prove it for all cases, keeping the methods of the above example in mind.

Proof of Proposition 4.2.3. Firstly we note that in the case of $\mathfrak{gl}_3(k)$, the partition $(2, 1)$ is almost rectangular so $\mathcal{O}_{(3)}$ commutes with $\mathcal{O}_{(2,1)}$. So assume now that $n \geqslant 4$.

We begin by noting that the linear Dynkin pyramid of $(n-1, 1)$ is comprised of an upper row containing a single box, and a lower row containing $(n-1)$ boxes; the single box on the upper row is labelled $\lceil \frac{n}{2} \rceil$. Using the results of [52, Lem. 1.5.2] or [5, Lem. 3.2], we see that a general element of the centraliser of $D(n-1, 1)$ is of the form

$$A = \begin{pmatrix} A'_1 & b & A'_2 \\ & & c \\ & & A'_3 \end{pmatrix}$$

where A'_1 is a $\lfloor \frac{n-1}{2} \rfloor \times \lfloor \frac{n-1}{2} \rfloor$ block, A'_2 is a $\lfloor \frac{n-1}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$ block, and A'_3 is a $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$ block such that the matrix $A' = \begin{pmatrix} A'_1 & A'_2 \\ & A'_3 \end{pmatrix}$ is a strictly upper triangular Toeplitz matrix with entries in k; additionally b (respectively, c) is an arbitrarily chosen element of k located in the top entry of the $\lceil \frac{n}{2} \rceil$th column (respectively, the rightmost entry of the $\lceil \frac{n}{2} \rceil$th row), and all remaining entries of A are equal to zero.

We first note that since the submatrix A' is strictly upper triangular Toeplitz, it is in the centraliser of $D(n-1)$ and has Jordan type $\rho_i(n-1)$, where i is minimal such that the ith superdiagonal of A' is non-zero, by Proposition 4.2.2. Define $v'_j = v_j$ for $1 \leqslant j < \lceil \frac{n}{2} \rceil$, and $v'_j = v_{j+1}$ if $\lceil \frac{n}{2} \rceil < j \leqslant n-1$; that is, $\{v'_1, ..., v'_{n-1}\} = \{v_1, ..., v_n\} \setminus \{v_{\lceil \frac{n}{2} \rceil}\}$. Let $\mathcal{B} = \{A^j v'_{n-k} : j = 0, ..., \rho_i(n-1)_{k+1}; k := 0, ..., i-1\}$; then the set of $(n-1) \times (n-1)$ matrices $\mathcal{B}' = \{B' : B \in \mathcal{B}'\}$ forms a basis of $\mathfrak{gl}_{n-1}(k)$ with respect to which A' is in Jordan form.

As in Example 4.2.4 above, we have four possible cases for the values of the entries b and c in the matrix A above. In each case we consider the effects of these entries on the action of A on the basis elements v_n and $v_{\lceil \frac{n}{2} \rceil}$; they do not affect the other basis elements. To simplify our notation, we write $v_m = v_{\lceil \frac{n}{2} \rceil}$.

Suppose first that $b = c = 0$. Then there is no change to the action of A on v_n, and we have $v_m \mapsto 0$, which gives rise to an extra part of size 1 in the Jordan type of A. So the Jordan type of A is thus $(\rho_i(n-1), 1)$ with respect to the basis $\mathcal{B} \cup \{v_m\}$.

Next consider the case of $b \neq 0$ and $c = 0$. If $A' = 0$, then b is the only non-zero entry of A and thus the Jordan type of A is $(2, 1^{n-2})$. So assume now that this is not the case. Here we have $v_m \mapsto bv_1 \mapsto 0$; but there exists an $l \neq m$ such that $v_l \mapsto a_i v_1 \mapsto 0$. So we see that $v_m - \frac{b}{a_i} v_l \mapsto 0$, and we note that $v_m - \frac{b}{a_i} v_l$ is linearly independent of all the elements of $\mathcal{B}$. Consequently a basis with respect to which A has Jordan form is $\mathcal{B} \cup \{v_m - \frac{b}{a_i} v_l\}$, and the Jordan type of A is thus $(\rho_i(n-1), 1)$.

Now suppose that $b = 0$ and $c \neq 0$. As in the previous case, if $A' = 0$ then A has only a single non-zero entry and thus has Jordan type $(2, 1^{n-2})$. If this is not the case, then we have $v_n \mapsto Av_n = cv_m + a_i v'_{n-i} \mapsto ... \mapsto 0$. In particular we note that v_m is linearly independent from all elements of $\mathcal{B}$; thus, as in the previous case, the Jordan type of A is $(\rho_i(n-1), 1)$ with basis $\mathcal{B} \cup \{v_m\}$.

Finally, let $b \neq 0 \neq c$. If $A' = 0$, then we have $v_n \mapsto cv_m \mapsto bcv_1 \mapsto 0$ and $v_j \mapsto 0$ otherwise, so A has Jordan type $(3, 1^{n-3})$. Consider first the specific case that n is even and that $\rho_i(n-1) = (3, 2^{\frac{n-4}{2}})$. Then we have $v_n \mapsto a_i v_{m+1} + cv_m \mapsto (a_i^2 + bc)v_1$. If $a_i^2 + bc = 0$, then we note that $\mathcal{B}$ has only $n-2$ elements, and in particular none of these elements contain a term in v_1. However we have $v_m \mapsto bv_1 \mapsto 0$; then v_m and bv_1 are linearly independent from both each other and all elements of $\mathcal{B}$. It follows that $\mathcal{B} \cup \{v_m, v_1\}$ is a basis with respect to which A had Jordan normal form; and thus A has Jordan type $(2^{\frac{n}{2}})$.

Suppose now that $b \neq 0 \neq c$, that $\rho_i(n-1)_1 \geqslant 3$, and that if $\rho_i(n-1)_2 = 2$ then we have $a_i^2 + bc \neq 0$. If this is the case then no cancellations can take place as in the previous case: either the string $v_n \mapsto cv_m \mapsto bcv_1 \mapsto 0$ is subsumed, or $A^j v_n$ never contains a multiple of v_1, or if $\rho_i(n-1) = (3, 2^{\frac{n-4}{2}})$ then the condition $a_i^2 + bc \neq 0$ prevents any cancellation. So $\mathcal{B}$ has $n-1$ elements, and there exists an $l \neq m$ such that $v_l \mapsto a_i v_1$. We then observe that $v_m - \frac{b}{a_i} v_l \mapsto 0$, and that $v_m - \frac{b}{a_i} v_l$ is linearly independent from the elements of $\mathcal{B}$. It follows that in this case, the Jordan type of A is $(\rho_i(n-1), 1)$ and that A has Jordan form with respect to the basis $\mathcal{B} \cup \{v_m - \frac{b}{a_i} v_l\}$.

It remains only to consider the case of $b \neq 0 \neq c$ and $\rho_i(n-1)_1 \leqslant 2$. First we note that we have $v_n \mapsto cv_m + a_i v'_{n-i-1} \mapsto bcv_1 \mapsto 0$, as $v'_{n-i-1} \mapsto 0$. Similarly there exists an s such that $v_s \mapsto a_i v_1$; and since the largest part of the Jordan type of A' is 2, then no j exists such that $v_j \mapsto a_i v_s$. We then observe that $v_m - \frac{b}{a_i} v_s \mapsto 0$. So we have the following:

$$v_n \mapsto cv_m + a_i v'_{n-i-1} \mapsto bcv_1 \mapsto 0;$$
$$v'_{n-i-1} \mapsto 0;$$
$$v_m - \tfrac{b}{a_i} v_s \mapsto 0.$$

All the terms in these strings are linearly independent of each other, and thus this gives rise to a subpartition $(3, 1^2)$ of the Jordan type of A. If $\rho_i(n-1)$ contains more than two parts of size 2, then we will obtain $v_{n-1} \mapsto a_i v'_{n-i-2} \mapsto 0$, and a similar string for each additional part of size 2; so each of these strings gives rise to a part of size 2 in the Jordan type of A. This then accounts for all the remaining non-zero entries in the matrix A; any v_j not contained in one of the strings described above will be such that $Av_j = 0$, so gives rise to a part of size 1 in the Jordan type of A. Therefore, if $\rho_i(n-1) = (2^r, 1^{n-2r-1})$, the Jordan type of A is $(3, 2^{r-2}, 1^{n-2r+1})$.

This covers all possible cases; and we note that all the matrices we have obtained in the centraliser of $D(n-1, 1)$ are of one of the forms described in Proposition 4.2.3. □

4.3 Regular orbits in $\mathfrak{sp}_{2m}(k)$ and $\mathfrak{so}_n(k)$

We now consider which orbits commute with the regular orbit in the other classical cases. If $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$, then the regular orbit is $\mathcal{O}_{(2m)}$; similarly if $\mathfrak{g} = \mathfrak{so}_{2m+1}(k)$ then the regular orbit is $\mathcal{O}_{(2m+1)}$. However if $\mathfrak{g} = \mathfrak{so}_{2m}$, then $(2m)$ is not an orthogonal partition. So the regular orbit in this case is $\mathcal{O}_{(2m-1,1)}$.

To begin this section, we note a lemma that allows us to simplify our calculations by using results from type A in the other classical types.

Lemma 4.3.1. *Let $\mathfrak{g}$ be either $\mathfrak{sp}_n(k)$ or $\mathfrak{so}_n(k)$, and let $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ be nilpotent orbits in $\mathfrak{g}$. If $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ commute in $\mathfrak{g}$, then the orbits labelled by λ and μ*

commute in $\mathfrak{gl}_n(k)$.

Proof. Since $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ commute there exists a pair $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ such that X has Jordan type λ, Y has Jordan type μ, and $[X, Y] = 0$. But we may equally consider X and Y as elements of $\mathfrak{gl}_n(k)$; thus the orbits in $\mathfrak{gl}_n(k)$ parametrised by λ and μ commute. □

In particular, we see that any orbit which commutes with $\mathcal{O}_{(n)}$ (respectively, $\mathcal{O}_{(n-1,1)}$) in $\mathfrak{so}_n(k)$ or $\mathfrak{sp}_n(k)$ must also commute with $\mathcal{O}_{(n)}$ (respectively, $\mathcal{O}_{(n-1,1)}$) in $\mathfrak{gl}_n(k)$. So we need only consider orbits described in Proposition 4.2.2 when looking for orbits which commute with the regular orbit in $\mathfrak{sp}_{2m}(k)$ or $\mathfrak{so}_{2m+1}(k)$, and orbits described in Proposition 4.2.3 when looking for orbits which commute with the regular orbit in $\mathfrak{so}_{2m+1}(k)$.

It is sometimes useful to consider the contrapositive of Lemma 4.3.1: if the orbits in $\mathfrak{gl}_n(k)$ labelled by λ and μ do not commute, then the orbits in $\mathfrak{sp}_{2m}(k)$ or $\mathfrak{so}_n(k)$ labelled by λ and μ do not commute either.

Proposition 4.3.2. *Let* $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$. *Then a nilpotent orbit* $\mathcal{O}_\lambda$ *commutes with the regular orbit* $\mathcal{O}_{(2m)}$ *if and only if* λ *is an almost rectangular partition with an odd number of parts, or the trivial partition* (1^{2m}).

Proof. We note that the Dynkin pyramid of this partition consists of a single block, as follows:

$$\boxed{1}\boxed{2}\boxed{3}\cdots\boxed{2m}$$

The matrix $E = D^-(2m)$ corresponding to this pyramid has all entries zero apart from the first superdiagonal, where the entries are equal to 1 in the upper half of the matrix and -1 in the lower half.

Now, by [52, Lem. 1.5.8], we note that $\mathfrak{c}_\mathfrak{g}(E)$ has basis $\{E^j : j \text{ odd}\}$. So for any $A \in \mathfrak{c}_\mathfrak{g}(E)$ we may write $A = \sum_{j=1}^{2m-1} a_j E^j$, where $a_j = 0$ for all even j. Let i

be minimal such that $a_i \neq 0$. Then we recall from Lemma 4.2.1 that the Jordan type of A is equal to the Jordan type of E^i; and furthermore that the Jordan type of E^i is $\rho_i(2m)$.

Finally if $a_i = 0$ for all i, then we obtain the zero matrix with Jordan type (1^{2m}). □

We note that all almost rectangular partitions of $2m$ are symplectic, since an almost rectangular partition cannot have two distinct odd parts; thus all odd parts must occur with even multiplicity.

If $\mathfrak{g} = \mathfrak{so}_{2m+1}(k)$ the result is very similar. The proof is almost identical to that of Proposition 4.3.2, so is omitted.

Proposition 4.3.3. *Let $\mathfrak{g} = \mathfrak{so}_{2m+1}(k)$. Then a nilpotent orbit $\mathcal{O}_\lambda$ commutes with the regular orbit $\mathcal{O}_{(2m+1)}$ if and only if λ is an almost rectangular partition with an odd number of parts.*

We note that all almost rectangular partitions $\rho_i(2m+1)$ of $2m+1$ with an odd number i of parts are orthogonal; since the parts of $\rho_i(2m+1)$ sum to an odd number there must be an odd number of odd parts. The remaining parts are then all even and equal, and there are an even number of these.

In the type D case the situation is more complex. In this case the regular orbit is labelled by the partition $(2m-1, 1)$, and we obtain the following result:

Proposition 4.3.4. *Let $\mathfrak{g} = \mathfrak{so}_{2m}(k)$, and let λ be an orthogonal partition such that $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2m-1,1)}$ in $\mathfrak{gl}_{2m}(k)$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2m-1,1)}$ in $\mathfrak{g}$.*

We will prove Proposition 4.3.4 by applying a change of basis to show the connection between this situation and the case of $\mathcal{O}_{(n-1,1)}$ in $\mathfrak{gl}_n(k)$. In order to motivate this approach, we first describe two examples in which m is odd and even respectively.

Example 4.3.5. Consider first the orbit $\mathcal{O}_{(9,1)}$ in $\mathfrak{so}_{10}(k)$. The orthogonal Dynkin pyramid of $(9,1)$ is as follows:

☒	☒	☒	☒	5	7	8	9	10
1	2	3	4	6	☒	☒	☒	☒

Instead of using the standard matrix $D^+(9,1)$ we define a modified matrix $F_{(9,1)}$ from this pyramid as shown below:

$$F_{(9,1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ & & & & & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ & & & & & & 0 & -1 & 0 & 0 \\ & & & & & & & 0 & -1 & 0 \\ & & & & & & & & 0 & -1 \\ & & & & & & & & & 0 \end{pmatrix}$$

We see that the locations of non-zero entries in $F_{(9,1)}$ are the same as in $D^+(9,1)$, but these entries are equal to $\pm\frac{1}{\sqrt{2}}$ rather than ± 1 whenever there are two non-zero entries in the same row or column.

Now a general strictly upper triangular element of $\mathfrak{c}_{\mathfrak{g}}(F_{(9,1)})$ is of the following form:

$$
\tilde{A} = \begin{pmatrix}
0 & a_1 & 0 & a_3 & d' & -d' & a_5 & 0 & a_7 & 0 \\
 & 0 & a_1 & 0 & \frac{a_3}{\sqrt{2}} & \frac{a_3}{\sqrt{2}} & 0 & -a_5 & 0 & -a_7 \\
 & & 0 & a_1 & 0 & 0 & -a_3 & 0 & a_5 & 0 \\
 & & & 0 & \frac{a_1}{\sqrt{2}} & \frac{a_1}{\sqrt{2}} & 0 & a_3 & 0 & -a_6 \\
 & & & & 0 & 0 & \frac{-a_1}{\sqrt{2}} & 0 & \frac{-a_3}{\sqrt{2}} & d' \\
 & & & & & 0 & \frac{-a_1}{\sqrt{2}} & 0 & \frac{-a_3}{\sqrt{2}} & -d' \\
 & & & & & & 0 & -a_1 & 0 & -a_3 \\
 & & & & & & & 0 & -a_1 & 0 \\
 & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & 0
\end{pmatrix}
$$

We may now apply a change of basis, changing v_5 to $\frac{1}{\sqrt{2}}(v_6 - v_5)$ and v_6 to $\frac{1}{\sqrt{2}}(v_6 + v_5)$; all other v_i remain unchanged.

This then gives us the following matrix:

$$
A = \begin{pmatrix}
0 & a_1 & 0 & a_3 & -\sqrt{2}d' & 0 & a_5 & 0 & a_7 & 0 \\
 & 0 & a_1 & 0 & 0 & a_3 & 0 & -a_5 & 0 & -a_7 \\
 & & 0 & a_1 & 0 & 0 & -a_3 & 0 & a_5 & 0 \\
 & & & 0 & 0 & a_1 & 0 & a_3 & 0 & -a_5 \\
 & & & & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}d' \\
 & & & & & 0 & -a_1 & 0 & -a_3 & 0 \\
 & & & & & & 0 & -a_1 & 0 & -a_3 \\
 & & & & & & & 0 & -a_1 & 0 \\
 & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & 0
\end{pmatrix}
$$

We now note that A_1' is now, up to the signs of certain entries, of the form described in the proof of Proposition 4.2.3. If we define E, X, Y in the same way as in the proof of Proposition 4.2.3, we see that a basis of $\mathfrak{c}_{\mathfrak{so}_{10}(k)}(F)$ is $\{E, E^3, E^5, E^7, X + Y\}$. We may then use the same methods as in Proposition 4.2.3 to determine the possible Jordan types of A. In particular we find that the Jordan type of A is $(\rho_i(9), 1)$ with i odd, if $d = 0$; the Jordan type of A is $(3, 2^2, 1^3)$ if $d \neq 0 \neq a_5$, and $a_i = 0$ otherwise; and the Jordan type of A is $(3, 1^7)$ if $d \neq 0$ and $a_i = 0$ for all i.. These are the only orthogonal partitions of 10 described in Proposition 4.2.3, and thus are the only possible Jordan types of A by Lemma 4.3.1.

Example 4.3.6. The case of m even is for the most part very similar to Example 4.3.5 above, though there are some differences worth highlighting. We illustrate this with the example of $m = 6$. Let $F_{(11,1)}$ be the matrix obtained by modifying $D^+(11, 1)$ in the same way as in the $m = 5$ case above. Then a general upper triangular element of $\mathfrak{c}_{\mathfrak{so}_{12}(k)}(F)$ is of the following form:

$$\tilde{A} = \left(\begin{array}{cccccc|cccccc}
0 & a_1 & 0 & a_3 & 0 & d & a_5 & 0 & a_7 & 0 & a_9 & 0 \\
 & 0 & a_1 & 0 & a_3 & 0 & 0 & \frac{-(d+a_5)}{\sqrt{2}} & 0 & -a_7 & 0 & -a_9 \\
 & & 0 & a_1 & 0 & \frac{a_3}{\sqrt{2}} & \frac{a_3}{\sqrt{2}} & 0 & \frac{d+a_5}{\sqrt{2}} & 0 & a_7 & 0 \\
 & & & 0 & a_1 & 0 & 0 & -a_3 & 0 & \frac{-(d+a_5)}{\sqrt{2}} & 0 & -a_7 \\
 & & & & 0 & \frac{a_1}{\sqrt{2}} & \frac{a_1}{\sqrt{2}} & 0 & a_3 & 0 & \frac{d+a_5}{\sqrt{2}} & 0 \\
 & & & & & 0 & 0 & \frac{-a_1}{\sqrt{2}} & 0 & \frac{-a_3}{\sqrt{2}} & 0 & -a_5 \\
\hline
 & & & & & & 0 & \frac{-a_1}{\sqrt{2}} & 0 & \frac{-a_3}{\sqrt{2}} & 0 & -d \\
 & & & & & & & 0 & -a_1 & 0 & -a_3 & 0 \\
 & & & & & & & & 0 & -a_1 & 0 & -a_3 \\
 & & & & & & & & & 0 & -a_1 & 0 \\
 & & & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & & & 0
\end{array}\right)$$

Now, similarly to the m odd case, we apply a change of basis to take v_6 to $\frac{1}{\sqrt{2}}(v_7 - v_6)$ and v_7 to $\frac{1}{\sqrt{2}}(v_7 + v_6)$, while leaving all other v_j unchanged. This gives us the following matrix:

$$A := \begin{pmatrix}
0 & a_1 & 0 & a_3 & 0 & \frac{a_5-d}{\sqrt{2}} & \frac{a_5+d}{\sqrt{2}} & 0 & a_7 & 0 & a_9 & 0 \\
 & 0 & a_1 & 0 & a_3 & 0 & 0 & \frac{-a_5-d}{\sqrt{2}} & 0 & -a_7 & 0 & -a_9 \\
 & & 0 & a_1 & 0 & 0 & a_3 & 0 & \frac{a_5+d}{\sqrt{2}} & 0 & a_7 & 0 \\
 & & & 0 & a_1 & 0 & 0 & -a_3 & 0 & \frac{-a_5-d}{\sqrt{2}} & 0 & -a_7 \\
 & & & & 0 & 0 & a_1 & 0 & a_3 & 0 & \frac{a_5+d}{\sqrt{2}} & 0 \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_5-d}{\sqrt{2}} \\
 & & & & & & 0 & -a_1 & 0 & -a_3 & 0 & \frac{-a_5-d}{\sqrt{2}} \\
 & & & & & & & 0 & -a_1 & 0 & -a_3 & 0 \\
 & & & & & & & & 0 & -a_1 & 0 & -a_3 \\
 & & & & & & & & & 0 & -a_1 & 0 \\
 & & & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & & & 0
\end{pmatrix}$$

Let E be the matrix obtained from A above by setting $a_1 = 1$ and $a_j = b = 0$ otherwise; let X be the matrix obtained similarly by setting $d = 1$ and $a_j = 0$ for all j. Then $\{E, E^3, E^5, E^7, E^9, X\}$ forms a basis of $\mathfrak{c}_{\mathfrak{g}}(E)$ and we may write $A_1 = a_1 E + a_3 E^3 + \frac{a_5+d}{\sqrt{2}} E^5 + a_7 E^7 + a_9 E^9 + \frac{a_5-d}{\sqrt{2}} X$. Similarly to Example 4.3.5 we find that A has Jordan type $(\rho_i(11), 1)$ with i odd and $i \neq 5$, if $a_5 = d = 0$; or A has Jordan type $(3, 2^4, 1) = (\rho_5(11), 1)$ if $a_5 = 0$ and $a_i = d = 0$ otherwise. Further, we see that A has Jordan type (2^6) if $a_5 = d$ are non-zero, and $a_i = 0$ otherwise; A has Jordan type $(3, 2^2, 1^5)$ if $a_5 = -d$ are non-zero and $a_7 = 0$; and A has Jordan type $(3, 1^9)$ if $a_5 = -d$ are non-zero and $a_i = 0$ otherwise. Once again we note that these are all the orthogonal partitions of 12 described in Proposition 4.2.3.

Proof of Proposition 4.3.4. First, let m be odd. We define the matrix $F_{(2m-1,1)}$ from $D^+(2m-1,1)$ by replacing the non-zero entries of $D(2m-1,1)$ with $\pm\frac{1}{\sqrt{2}}$ in all rows and columns that contain two non-zero entries, as in Example 4.3.5. By calculating the form of a general element $\tilde{A}$ of the centraliser of $F_{(9,1)}$ and applying the change of basis given by taking v_m to $\frac{1}{\sqrt{2}}(v_{m+1}-v_m)$ and v_{m+1} to $\frac{1}{\sqrt{2}}(v_{m+1}+v_m)$, we obtain a matrix A of the following form:

$$A = \begin{pmatrix} A'_1 & -\sqrt{2}d & A'_2 \\ & & -\sqrt{2}d \\ & & A'_3 \end{pmatrix}.$$

Here, the matrix $A' = \begin{pmatrix} A'_1 & A'_2 \\ 0 & A'_3 \end{pmatrix}$ is a $(2m-1)\times(2m-1)$ matrix which centralises $(D^+(2m-1)$ in $\mathfrak{so}_{2m-1}(k)$. We may now observe that a basis for the centraliser of $F_{(2m-1,1)}$ in $\mathfrak{so}_{2m}(k)$ is given by $\{E^j, X+Y : j \text{ odd}, 1 \leqslant j \leqslant 2m-3\}$, with E, X, Y defined as in the proof of Proposition 4.2.3. In particular the matrix A is, up to the signs of certain entries, of the form described in the proof of Proposition 4.2.3.

Now we can apply Proposition 4.2.3 to find the possible Jordan types of A. To simplify the argument, assume that there exists a single $1 \leqslant i \leqslant 2m-1$ such that $a_i \neq 0$. If $d = 0$ then we use the $b = c = 0$ case in Proposition 4.2.3 and find that the Jordan type of A_1 is $(\rho_i(n-1,1)$. Conversely if $d \neq 0$ we consider the $b \neq 0 \neq c$ case. Since m is odd, we see that the partition $(3, 2^{m-2})$ of $2m-1$ has an even number of parts. So we are never in the case that gives rise to a Jordan type of (2^m); this is to be expected, since (2^m) is not an orthogonal partition in this case. Otherwise, we find that the Jordan type of A_1 is $(\rho_i(2m-1), 1)$ with

i odd, if $\rho_i(2m-1)_1 \geqslant 3$; or $(3, 2^a, 1^{2m-2a-3})$ with a even, if $\rho_i(2m-1)_1 \leqslant 2$. In particular, these are the only orthogonal partitions of $2m$ described in Proposition 4.2.3; thus the desired result follows by Lemma 4.3.1.

We may repeat this process for general even m, defining the matrix $F_{(2m-1,1)}$ in the same way as above and once again changing the basis by sending v_m to $\frac{1}{\sqrt{2}}(v_{m+1} - v_m)$ and v_{m+1} to $\frac{1}{\sqrt{2}}(v_{m+1} + v_m)$. In doing so we find that a general element of the centraliser of $F_{(2m-1,1)}$ can be written as

$$A = \begin{pmatrix} A'_1 & \frac{a_{m-1}-d}{\sqrt{2}} & A'_2 \\ & & \frac{a_{m-1}-d}{\sqrt{2}} \\ & & A'_3 \end{pmatrix}.$$

As previously $A' = \begin{pmatrix} A'_1 & A'_2 \\ 0 & A'_3 \end{pmatrix}$ is a strictly upper triangular $(2m-1) \times (2m-1)$ matrix which commutes with $D^+(2m-1)$ in $\mathfrak{so}_{2m-1}(k)$.

Now if $a_{m-1} - d = 0$ then the (highlighted) mth row and mth column of A are zero. So this gives a part of size 1 in the Jordan type of A via $v_m \mapsto 0$, and the submatrix A' formed of the unhighlighted rows and columns has Jordan type $\rho_i(2m-1)$, as previously described. So A has Jordan type $(\rho_i(2m-1), 1)$ in this case, where i is minimal such that $a_1 \neq 0$.

Conversely suppose that $a_{m-1} - d \neq 0$. We can now use the case of $b \neq 0 \neq c$ described in the proof of Proposition 4.2.3 to identify the possible Jordan types of A. Let i is minimal such that $a_i \neq 0$; then the Jordan type of A is $(\rho_i(2m-1), 1)$ if $i < m-1$, and $(3, 2^{2m-i-3}, 1^{3+2i-2m})$ if $i > m-1$ and $d = 0$.

In the case that $i = m-1$, or that $i > m-1$ and $d \neq 0$, we note that $\rho_i(2m-1) = (3, 2^{m-2})$ and observe that we have $v_{2m} \mapsto \frac{-a_{m-1}-d}{\sqrt{2}} v_{m+1} + \frac{a_{m-1}-d}{\sqrt{2}} \mapsto$

$-2a_{m-1}dv_1 \mapsto 0$. If both a_{m-1} and d are non-zero, then we obtain a part of size 3 in the Jordan type of A; furthermore, we note that the mth and $m+1$th columns of A are linear multiples of each other, giving a part of size 1 in the Jordan type of A. Finally each of the $(m-2)$ remaining non-zero entries of the superdiagonal whose entries depend on a_{m-1} and d will give a part of size 2 in the Jordan type; consequently we determine that A has Jordan type $(3, 2^{m-2}, 1)$.

Conversely, suppose that $a_{m-1} = 0$ and $d \neq 0$. Then we have $v_{2m} \mapsto -dv_{m+1} - dv_m \mapsto 0$, and $v_m \mapsto -dv_1 \mapsto 0$. Since the entries of these strings are all linearly independent, we see that we now have two parts of size 2 in the Jordan type of A, in place of the parts of size 3 and 1 in the case immediately above. We thus conclude that in this case, A has Jordan type (2^m). If instead $d = 0$ and $a_{m-1} \neq 0$, then by an almost identical argument we again obtain a Jordan type of (2^m). This completes the analysis, and we have found elements of the centraliser of $F_{(2m-1,1)}$ of all the Jordan types described in the statement of the proposition. □

Remark 4.3.7. Suppose that $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$ with $m \geqslant 3$. Then the partition $\lambda = (2^2, 1^{2(m-2)})$ is almost rectangular, so $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2m)}$ in $\mathfrak{gl}_{2m}(k)$. However it has an even number of parts so it does not commute with $\mathcal{O}_{(2m)}$ in $\mathfrak{g}$, by Lemma 4.3.2. So the converse of Lemma 4.3.1 does not hold if $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$; even if $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ commute in $\mathfrak{gl}_{2m}(k)$, they do not necessarily commute in $\mathfrak{sp}_{2m}(k)$.

Later on we will show in Proposition 4.6.2 that although the orbits $\mathcal{O}_{(2^m)}$ and $\mathcal{O}_{(2m-3,1^3)}$ commute in $\mathfrak{gl}_{2m}(k)$, they do not commute in $\mathfrak{so}_{2m}(k)$ even in the case that m is even (and thus (2^m) is an orthogonal partition).

4.4 Universally commuting orbits in $\mathfrak{gl}_n(k)$

We recall that an orbit $\mathcal{O}_\mu$ is said to be *universally commuting* (or UC) for $\mathfrak{g}$ if it commutes with all other orbits of G in $\mathfrak{g}$.

In the case of $\mathfrak{g} = \mathfrak{gl}_n(k)$ the universally commuting orbits have been classified independently by Britnell and Wildon [13, Thm. 4.6] and by Oblak [41, Thm. 2.4] as follows:

Theorem 4.4.1. *An orbit $\mathcal{O}_\lambda$ is universally commuting in $\mathfrak{g} = \mathfrak{gl}_n(k)$ if and only if one of the following holds:*

(i.) $n \leqslant 3$ and λ is any partition of n;

(ii.) $n \geqslant 4$ and λ is of the form $(2^a, 1^{n-2a})$, for $0 \leqslant a \leqslant \lfloor \frac{n}{2} \rfloor$.

We first give a proof of this theorem using methods distinct from [13] and [41], then in the remainder of this chapter use these methods to describe a generalisation of this result to the cases of $\mathfrak{g} = \mathfrak{sp}_n(k)$ and $\mathfrak{g} = \mathfrak{so}_n(k)$. Before we begin we record the following lemma, which allows us to use an inductive approach to the problem.

Lemma 4.4.2. *Let $\mathfrak{g}_n$ be either $\mathfrak{gl}_n(k)$, $\mathfrak{sp}_{2m}(k)$, or $\mathfrak{so}_n(k)$. Let λ and μ be partitions which label orbits in $\mathfrak{g}_n$, and suppose that we can write them as $\lambda = (\lambda^1, \lambda^2, ..., \lambda^t)$ and $\mu = (\mu^1, \mu^2, ..., \mu^t)$ such that, for $i = 1, ..., t$, $|\lambda^i| = |\mu^i| = n_i$ and λ^i and μ^i label orbits $\mathcal{O}_{\lambda^i}$ and $\mathcal{O}_{\mu^i}$ in $\mathfrak{g}_{n_i}$. Then if $\mathcal{O}_{\lambda^i}$ commutes with $\mathcal{O}_{\mu^i}$ in $\mathfrak{g}_{n_i}$ for all $i = 1, ..., t$, then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$ in $\mathfrak{g}_n$.*

Proof. By our definition of λ and n_i, we see that $\mathfrak{g}_n$ contains a subalgebra isomorphic to $\mathfrak{g}_{n_1} \oplus ... \oplus \mathfrak{g}_{n_t}$. By our assumption, for all $i = 1, ..., t$ there exists a pair $(E_i, F_i) \in \mathfrak{g}_{n_i} \times \mathfrak{g}_{n_i}$ with $E_i \in \mathcal{O}_{\lambda^i} \subseteq \mathfrak{g}_{n_i}$ and $F_i \in \mathcal{O}_{\mu^i} \subseteq \mathfrak{g}_{n_i}$ such that $[E_i, F_i] = 0$. So we may take $E = \sum_{i=1}^t E_i$ and $F = \sum_{i=1}^t F_i$; then $E \in \mathcal{O}_\lambda$, $F \in \mathcal{O}_\mu$, and $[E, F] = 0$ as required. □

We now use Lemma 4.4.2 and a series of further lemmas to allow us to prove Theorem 4.4.1 in the light of Propositions 4.2.2 and 4.2.3.

Lemma 4.4.3. *If λ is a partition of n that is not described in Theorem 4.4.1, then the orbit $\mathcal{O}_\lambda$ is not universally commuting in $\mathfrak{gl}_n(k)$.*

Proof. This can be seen immediately by comparing the results of Propositions 4.2.2 and 4.2.3; the only partitions λ of n such that $\mathcal{O}_\lambda$ commutes with both $\mathcal{O}_{(n)}$ and $\mathcal{O}_{(n-1)}$ are *square-zero* partitions of the form $(2^a, 1^{n-a})$ for $0 \leqslant a \leqslant \lfloor \frac{n}{2} \rfloor$, or (n) for $n = 2, 3$. $\square$

In particular, we can see from Propositions 4.2.2 and 4.2.3 that all orbits in $\mathfrak{gl}_2(k)$ and $\mathfrak{gl}_3(k)$ are universally commuting. So from now on let $n \geqslant 4$; and consider which orbits labelled by partitions of the form $(2^a, 1^{n-2a})$ commute with a given $\mathcal{O}_\lambda$.

First, let $\lambda = (\lambda_1, \lambda_2)$ be a two-part partition; and apply Lemma 4.4.2 to write $\lambda = (\lambda_1) \oplus (\lambda_2)$ as the sum of two single-part partitions. If λ_i is even, then we observe that $\mathcal{O}_{(\lambda_i)}$ commutes with $\mathcal{O}_{(2^a, 1^{\lambda_i - 2a})}$ for all $0 \leqslant a \leqslant \frac{\lambda_i}{2}$. Conversely if λ_i is odd, then $\mathcal{O}_{(\lambda_i)}$ commutes with $\mathcal{O}_{(2^a, 1^{\lambda_i - 2a})}$ for all $0 \leqslant a \leqslant \frac{\lambda_i - 1}{2}$. In particular, any square-zero partition which commutes with $\mathcal{O}_{(\lambda_i)}$ for λ_i even (respectively, odd) must contain an even (respectively, odd) number of parts of size 1.

So if at least one of λ_1 and λ_2 is even, then we can write all partitions of the form $\mu = (2^a, 1^{n - \frac{a}{2}})$ as a sum of partitions of the form $\mu^1 \oplus \mu^2 = (2^{a_1}, 1^{\lambda_1 - \frac{a_1}{2}}) \oplus (2^{a_2}, 1^{\lambda_2 - \frac{a_2}{2}})$, where $\mathcal{O}_{\mu^i}$ commutes with $\mathcal{O}_{(\lambda_i)}$ for $i = 1, 2$. It follows that $\mathcal{O}_\mu$ commutes with $\mathcal{O}_\lambda$ in this case.

However if both λ_1 and λ_2 are odd, then any square-zero partition μ^i such that $\mathcal{O}_{\mu^i}$ commutes with $\mathcal{O}_{(\lambda_i)}$ must contain at least one part of size 1. Consequently we cannot write $(2^{n/2})$ as a direct sum of two sub-partitions $\mu^1 \oplus \mu^2$ with $\mathcal{O}_{\mu^i}$ commuting with $\mathcal{O}_{(\lambda_i)}$ in the same way as above. Consequently we must introduce two further lemmas to examine this case more closely.

Lemma 4.4.4. *Let $n = 2m$; then $\mathcal{O}_{(m,m)}$ commutes with $\mathcal{O}_{(2^a,1^{2m-2a})}$ for all $0 \leqslant a \leqslant m$.*

Proof. Let $E = D(2m)$ be the matrix arising from the linear Dynkin pyramid of the partition $(2m)$. We recall from Lemma 4.2.1 that the Jordan type of E^i is $\rho_i(2m)$. In particular, E^2 has Jordan type (m^2). Let $\mu = (2^a, 1^{2m-2a})$ for $0 \leqslant a \leqslant m$; then $\mu = \rho_{2m-a}(2m)$. Hence the matrix E^{n-a} has Jordan type $(2^a, 1^{2m-2a})$, and the matrices E^2 and E^{2m-2a} clearly commute. It follows that $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$, as required. □

Lemma 4.4.5. *Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of $n = 2m$ with λ_1 and λ_2 both odd, and $\lambda_1 > \lambda_2$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^m)}$.*

Proof. We begin by recalling that the Dynkin pyramid of λ consists of a lower row consisting of λ_1 boxes, and a shorter upper row consisting of λ_2 boxes placed centrally so that the pyramid is symmetrical about a vertical axis. Since both λ_1 and λ_2 are odd, each box on the upper row is directly above a single box on the lower row. This may be seen in the example of $\lambda = (7, 3)$ in $\mathfrak{gl}_{10}(k)$:

		3	5	7		
1	2	4	6	8	9	10

We now relabel the boxes of the Dynkin pyramid as follows. We label the box in the upper row of the ith column of the pyramid by u_i; and similarly label the box in the jth column of the lower row by v_j. Since we have assumed that $\lambda_1 > \lambda_2$, some columns of the pyramid contain only a box on the lower row. If this is the case for the ith column, we set $u_i = 0$. Similarly we set $u_j = v_j = 0$ for $j \leqslant 0$. In the $(7, 3)$ example this gives the following diagram:

		u_3	u_4	u_5		
v_1	v_2	v_3	v_4	v_5	v_6	v_7

Let $d = \frac{\lambda_1 - \lambda_2}{2}$; this represents the number of columns containing only a single box at each end of the pyramid. Now let X be the map that takes $v_i \mapsto u_{i-d}$ where this is defined, and moves all other basis elements to zero. Similarly we define $Y : u_i \mapsto \ v_{i-d}$, $Z : v_i \mapsto v_{i-d}$ and $W : u_i \mapsto u_{i-d}$. Then $M = X + Z - Y - W$ commutes with $D(\lambda_1, \lambda_2)$, by [5, Lem. 3.2]. To determine the Jordan type, we consider the images of the m basis elements $v_{d+1}, ..., v_{\lambda_1}$. Let $i \in \{d+1, ..., \lambda_1\}$. If $i \leqslant 2d$, then $i - d \leqslant d$ so $u_{i-d} = 0$ and $v_{i-2d} = 0$. Thus the action of M on v_i gives $v_i \mapsto v_{i-2} \mapsto 0$. Conversely if $i > 2d$ then $u_{i-d} \neq 0$; so we have

$$v_i \mapsto v_{i-d} + u_{i-d} \mapsto v_{i-2d} + u_{i-2d} - v_{i-2d} - u_{i-2d} = 0.$$

We have therefore obtained m basis elements whose images under M are linearly independent, and which are mapped to zero by M in precisely two steps. It follows that the Jordan type of M is (2^m), with respect to the basis $\{v_{\lambda_1}, Mv_{\lambda_1}, v_{\lambda_1 - 1}, Mv_{\lambda_1 - 1}, ..., v_{d+1}, Mv_{d+1}\}$. Since M and $D(\lambda)$ commute, it follows that the orbits $\mathcal{O}_{(2^m)}$ and $\mathcal{O}_\lambda$ commute as required. □

We are now in a position to use these lemmas to complete the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. Let λ be an arbitrary partition of n. We may write λ as $(\lambda^1, ..., \lambda^t)$ where $\lambda^1, ..., \lambda^{t-1}$ are each comprised of either a single even part or two odd parts, and λ^t is of this form if n is even or comprised of a single odd part if n is odd.

Let $1 \leqslant i \leqslant t$. Then $\mathcal{O}_{\lambda^i}$ commutes with $\mathcal{O}_{(2^a, 1^{|\lambda^i| - 2a})}$ for all $0 \leqslant a \leqslant \frac{|\lambda^i|}{2}$

by Proposition 4.2.2 if λ^i is comprised of a single part, by Lemma 4.4.4 if λ^i is comprised of two equal odd parts, or by Lemma 4.4.5 if λ^i is comprised of two distinct odd parts. We remark that we have chosen the λ^i such that $|\lambda^i|$ can only be odd if $i = t$; consequently $\mathcal{O}_{\lambda^i}$ commutes with $\mathcal{O}_{(2^{|\lambda^i|/2})}$ for $i \neq t$. It now follows from Lemma 4.4.2 that $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^a,1^{n-2a})}$ for all $0 \leqslant a \leqslant \lfloor \frac{n}{2} \rfloor$. Since λ was chosen arbitrarily, the orbits $\mathcal{O}_{(2^a,1^{n-2a})}$ are thus universally commuting. Since by Lemma 4.4.3 no other orbits may possibly be universally commuting, the desired result follows. □

4.5 Universally commuting orbits in $\mathfrak{sp}_{2m}(k)$

In this section, let $n = 2m$ and let $\mathfrak{g} = \mathfrak{sp}_{2m}(k)$. Since any UC orbit $\mathcal{O}_\mu$ must commute with $\mathcal{O}_{(2m)}$, we see from Proposition 4.3.2 that μ must be almost rectangular with an odd number of parts, or be equal to (1^{2m}).

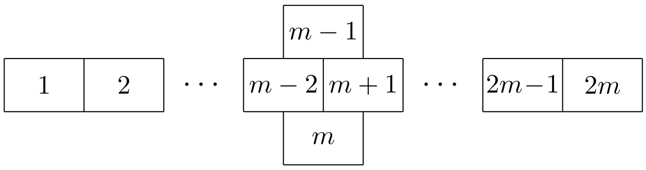
To further narrow down the potential UC orbits, we examine which orbits commute with a second symplectic orbit, namely $\mathcal{O}_{(2m-2,1^2)}$. We then move on to examine the centraliser of $\mathcal{O}_\lambda$ where λ is a two-part symplectic partition. The results thus obtained are then sufficient for us to classify all symplectic universally commuting orbits using similar methods to the proof of Theorem 4.4.1.

4.5.1 Centraliser of $(2m-2, 1^2)$ in $\mathfrak{sp}_{2m}(k)$

We begin by examining which orbits commute with $\mathcal{O}_{(2m-2,1^2)}$; we consider this orbit as the structure of its centraliser is slightly simpler than that of the subregular orbit $\mathcal{O}_{(2m-2,2)}$. We obtain the following result:

Lemma 4.5.1. *Let λ be a symplectic partition of $2m$ such that $\mathcal{O}_\lambda$ commutes with both $\mathcal{O}_{(2m)}$ and $\mathcal{O}_{(2m-2,1^2)}$. Then $\lambda = (2^a, 1^{2m-2a})$ where $a < m$ is either an* *odd natural number, or zero.*

Proof. We begin by noting that the symplectic Dynkin pyramid of $(2m-2, 1^2)$ is of the following form:



By reference to [52, Lem. 1.5.8], we note that a general element of the centraliser in $\mathfrak{sp}_{2m}(k)$ of $D^-(2m-2, 1^2)$ is of the following form:

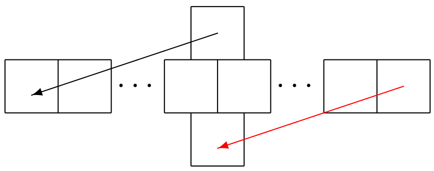
$$A = \begin{pmatrix} A'_1 & b & c & A'_2 \\ & 0 & d & c \\ & & 0 & -b \\ & & & A'_3 \end{pmatrix}.$$

Now the $(2m-2) \times (2m-2)$ submatrix $A' = \left(\begin{smallmatrix} A'_1 & A'_2 \\ & A'_3 \end{smallmatrix}\right)$ comprised of all unhighlighted entries of A commutes with $D^-(2m-2)$ in $\mathfrak{sp}_{2m-2}(k)$; that is, A' is a quasi-Toeplitz matrix whose even-numbered superdiagonals are zero. So by Lemma 4.2.1 the Jordan type of A' is $\rho_i(2m-2)$ with i odd, or $i = 2m-2$. Similarly the 2×2 submatrix $A'' = \left(\begin{smallmatrix} 0 & d \\ & 0 \end{smallmatrix}\right)$ comprised of entries which are in both a highlighted row and a highlighted column has Jordan type (2) if $d \neq 0$ and (1^2) if $d = 0$. These submatrices correspond to considering only the long row on the Dynkin pyramid, or the two short rows, respectively.

Now if $b = c = 0$ then there is no crossover between the long row and the two short rows of the pyramid. So the Jordan type λ of A is equal to the sum of the Jordan types of A' and A'', that is, $\rho_i(2m-2) \oplus \rho_j(2)$ where i is odd or

$i = 2m-2$, and $j \in \{1,2\}$. If $i = 2m-2$ then λ is either $(2, 1^{2m-2})$ if $j = 1$, or (1^{2m}) if $j = 2$. Otherwise $\rho_i(2m-2)$ has an odd number of parts, so λ will only have an odd number of parts, and thus $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2m)}$, if $j = 2$. So the Jordan type of λ is of the form $(\rho_i(2m-2), 1^2)$. But this will only be almost rectangular if $\rho_i(2m-2)_1 \leqslant 2$. So if λ is to be UC, it must be of the form $(2^a, 1^{2m-2a})$ with $a < m$.

Suppose now that $d = 0 = c$ and $b \neq 0$. Then if $a_i = 0$ for all i, we have $v_{2m} \mapsto -bv_{m+1} \mapsto 0$ and $v_m \mapsto bv_1 \mapsto 0$. This gives a Jordan type of $(2^2, 1^{2m-4})$; it can be represented by the following lines on the Dynkin pyramid, where a black line indicates a positive coefficient and a red line a negative one:



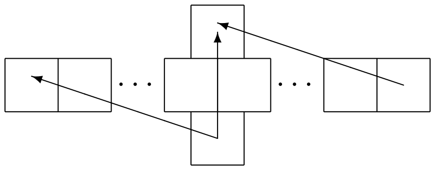
We now use similar methods to those of Proposition 4.2.3. Define $v'_j = v_j$ for $1 \leqslant j \leqslant m-2$, and $v'_j = v_{j+2}$ for $m-1 \leqslant j \leqslant 2m-2$; that is, the set $\{v'_1, ..., v'_{2m-2}\} = \{v_1, ..., v_{m-2}, v_{m+1}, ..., v_{2m}\}$ is the set of all basis elements of the underlying vector space which correspond to a box on the long row of the Dynkin pyramid.

Let i be minimal such that $a_i \neq 0$; then the set $\mathcal{B}_{A'} = \{v'_{2m-2}, Av'_{2m-2}, ..., A^{\rho_i(2m-2)_1}v'_{2m-2}, v'_{2m-3}, Av'_{2m-3}, ..., A^{\rho_i(2m-2)_2}v'_{2m-3}, ..., v'_{2m-i-1}, Av'_{2m-i-1}, ..., A^{\rho_i(2m-2)_i}v'_{2m-i-1}\}$ forms a basis for the Jordan form of A'. Following the $\mathfrak{gl}_n$ case, adding v_{m+1} and $a_iv_m - bv'_{i+1}$ gives a basis for the Jordan form of A, with type $(\rho_i(2m-2), 1^2)$. Once again this will be almost rectangular only if $\rho_i(2m-2)_1 = 2$.

The case of $d = b = 0$ and $c \neq 0$ is identical, with the roles of v_m and v_{m+1} switched. In the case that b and c are non-zero, then neglecting the action of A'

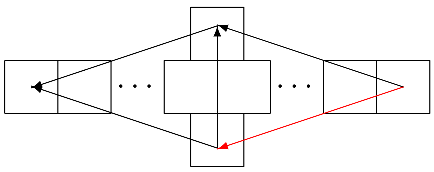
we have $v_{2m} \mapsto -bv_{m+1} + cv_m \mapsto -bcv_1 + bcv_1 = 0$. Now A' has the same Jordan basis as above, and if i is minimal with $a_i \neq 0$ we may add $a_i v_m - bv'_{i+1}$ and $a_i v_{m+1} - cv'_{i+1}$ to the set $\mathcal{B}_{A'}$ to obtain a basis for the Jordan form of A, with type $(\rho_i(2m-2), 1^2)$ as above.

Suppose now that $d \neq 0$ and $b = 0$. Then if $a_j = 0$ for all j we have $v_{2m} \mapsto cv_m \mapsto 0$ and $v_{m+1} \mapsto dv_m + cv_1 \mapsto 0$; this gives a Jordan type of $(2^2, 1^{2m-4})$ and we can represent this on the Dynkin pyramid as follows:



Now when $A' \neq 0$ then once again the Jordan form of A' has basis $\mathcal{B}_{A'}$; and by adding v_{m+1} and $Av_{m+1} = dv_m + cv_1$ we obtain a basis of the Jordan normal form of A with Jordan type $(\rho_i(2m-2), 2)$, where i is odd. But then this partition has an even number of parts; so $\mathcal{O}_{(\rho_i(2m-2),2)}$ does not commute with $\mathcal{O}_{(2m)}$. We note that this works in the same way regardless of whether or not $c \neq 0$.

Finally suppose that $d \neq 0 \neq b$, and let c be arbitrary. If we neglect the action of A' then we have $v_{2m} \mapsto -bv_{m+1} + cv_m \mapsto -bcv_1 - bdv_m + bcv_1 = -bdv_m \mapsto -b^2dv_1 \mapsto 0$, and $v_i \mapsto 0$ for all other i in the middle row. So this gives a Jordan type of $(4, 1^{2m-4})$; we denote this string by $(*)$, and can represent this on the Dynkin pyramid as follows:



Again let i be minimal such that $a_i \neq 0$. If $\rho_i(2m-2)_1 \geqslant 5$, then $(*)$ is subsumed into the Jordan block arising from this largest part. So the Jordan type of A is $(\rho_i(2m-2), 2)$ with basis $\mathcal{B}_{A'} \cup \{v_{m+1}, Av_{m+1}\}$; this clearly is not almost rectangular so does not commute with $\mathcal{O}_{(2m)}$.

If $\rho_i(2m-2)_1 = 4$, then we have two cases. In the case that $\rho_i(2m-2)_2 = 4$ also, then our argument proceeds in the same way as the case of $\rho_i(2m-2)_1 = 5$, since $A^4 v_{2m}$ is not a multiple of v_1 so no cancellation occurs. If $\rho_i(2m-2)_2 = 3$, however, the situation is slightly more complex. In this case $\rho_i(2m-2) = (4, 3^{\frac{2m-6}{3}})$; then we have $v_{2m} = v'_{2m-2} \mapsto -a_i v'_{2m-i-2} - b_{m+1} + cv_m \mapsto a_i^2 v'_{2m-2i-2} - bdv_m + (bc - bc)v_1 \mapsto (a_i^3 - b^2 d)v_1 \mapsto 0$. Additionally for all $j \neq 2m$, we have $A^3 v_j = 0$; so the largest part of the Jordan type is equal to 3. Finally we note that $v_{m+1} \mapsto dv_m + cv_1 \mapsto bdv_1 \mapsto 0$; each entry of this string is linearly independent from $A^l v'_j$ for $l \leq 2$ and $2 \leqslant j \leqslant 2m-2$. From this we deduce that the Jordan type of A is $(3^{\frac{2m}{3}})$ with respect to the basis

$$\{v_{m+1}, Av_{m+1}, A^2 v_{m+1}, v'_j, Av'_j, A^2 v'_j \mid 2m-i-1 \leqslant j \leqslant 2m-2\}.$$

However if $\frac{2m}{3}$ is an integer then it must be even; thus $\mathcal{O}_{(3^{2m/3})}$ does not commute with $\mathcal{O}_{(2m)}$.

Conversely suppose that $a_i^3 - b^2 d \neq 0$. Then we observe that $A^3 v_{2m} \neq 0$ but $A^4 v_m = 0$; so the largest part of the Jordan type of A is equal to 4, and as previously the Jordan form of A' has basis $\mathcal{B}_{A'}$. Additionally we have $v_{m+1} - \frac{bd}{a_i^2} v'_{2i+1} \mapsto dv_6 + cv_1 - \frac{bd}{a_i} v'_{i+1} \mapsto bdv_1 - bdv_1 = 0$. From this we deduce that the Jordan type of A is $(4, 3^{\frac{2m-6}{3}}, 2)$ with basis $\mathcal{B}_{A'} \cup \{v_{m+1} - \frac{bd}{a_i^2} v'_{2i+1}, A \cdot (v_{m+1} - \frac{bd}{a_i^2} v'_{2i+1})\}$. Clearly this is not almost rectangular, so does not give the Jordan type of a UC orbit.

Finally suppose $\rho_i(2m-2)_1 \leqslant 3$. Now the string denoted by $(*)$ above gives rise to a part of size 4 in the Jordan type of A. If $m = 2$ then the Jordan type of A is thus (4). Conversely if $m \geqslant 3$ then $i \geqslant 3$, so the Jordan normal form of A' has at least three blocks. Now v_{2m} and v_1 are each contained in a single block of A'; these are the only blocks which can cancel with $(*)$. So there is at least one block in the Jordan normal form of A' of size at most 2 which is preserved in the Jordan normal form of A, and thus gives rise to a part of size 2 or 1 in the Jordan type of A. Thus the Jordan type of A contains a subpartition of the form $(4, 2)$ or $(4, 1)$ and is thus not almost rectangular.

Finally we note that at no point in this process do we obtain the partition (2^m). If m is even then $\mathcal{O}_{(2^m)}$ does not commute with $\mathcal{O}_{(2m)}$, while if m is odd and the maximum part of the partition μ' of A' is 2 then the multiplicity of 2 in μ' is at most $m - 2$ since A' must commute with $(2m - 2)$ in $\mathfrak{sp}_{2m-2}(k)$.

It follows that the only partitions of $2m$ which label orbits that commute with both $\mathcal{O}_{(2m-2,1^2)}$ and $\mathcal{O}_{(2m)}$ are those of the form $(2^a, 1^{2m-a})$, with $a < m$ is either zero or odd. □

We refer to partitions of the form $(2^a, 1^{2m-2a})$, with a odd, as *odd square-zero partitions*. This is a generalisation of the terminology used by Oblak in [41], and derives from the fact that $(D(2^a, 1^{2m-2a}))^2 = 0$ for all $0 \leqslant a \leqslant m$.

4.5.2 Centralisers of two-part partitions

Having narrowed down the possible UC orbits, we now consider which orbits commute with $\mathcal{O}_\lambda$ for a two-part partition $\lambda = (\lambda_1, \lambda_2)$. We begin with the specific example of $\lambda = (m^2)$.

Proposition 4.5.2. *Let $\mu = (2^a, 1^{2m-2a})$ for $0 \leqslant a \leqslant m$. Then $\mathcal{O}_\mu$ commutes with $\mathcal{O}_{(m^2)}$.*

Proof. We begin by considering the example of $m = 5$, and then explain how this generalises to other values of m.

The Dynkin pyramid of the partition $\lambda = (5, 5)$ is as follows:

u_1	u_2	u_3	u_4	u_5
v_1	v_2	v_3	v_4	v_5

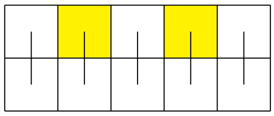
We note that as in the proof of Theorem 4.4.1 we use a slightly different method of labelling the boxes of the pyramid: we label the box of the upper row in the ith column of the pyramid as u_i, and similarly the box on the lower row in the ith column is labelled v_i. In this labelling we view $\{u_i, v_i : 1 \leqslant i \leqslant 5\}$ as a set of basis elements.

Now a general element of $\mathfrak{c}_{\mathfrak{g}}(D^{-}(5^2))$ is of the following form:

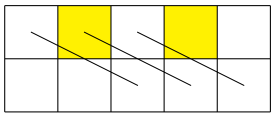
$$A = \left(\begin{array}{ccccc|ccccc}
0 & a_1 & a_2 & 0 & a_4 & a_5 & a_6 & 0 & a_8 & a_9 \\
 & 0 & 0 & a_2 & b_3 & a_4 & 0 & -a_6 & b_7 & a_8 \\
 & & 0 & a_1 & a_2 & 0 & a_4 & -a_5 & -a_6 & 0 \\
 & & & 0 & 0 & a_2 & b_3 & a_4 & 0 & a_6 \\
 & & & & 0 & a_1 & a_2 & 0 & a_4 & a_5 \\
\hline
 & & & & & 0 & 0 & -a_2 & -b_3 & -a_4 \\
 & & & & & & 0 & -a_1 & -a_2 & 0 \\
 & & & & & & & 0 & 0 & -a_2 \\
 & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & 0
\end{array}\right)$$

We may represent this on the Dynkin pyramid in the following way. We highlight the boxes labelled by u_i for i even; these denote the basis elements which A cannot map v_5 to. Then we may draw a line from v_5 to any unhighlighted box on the Dynkin pyramid, and add in all possible parallel lines; this corresponds to setting one a_i or b_i in the matrix A to be non-zero. If each box is the endpoint of at most one line, then the largest part of the Jordan type of the corresponding matrix is at most 2, and the number of lines is equal to the number of parts of size 2.

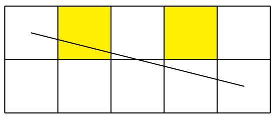
In our (5^2) example this works as follows. We obtain a matrix of Jordan type (2^5) by sending v_i to u_i; this corresponds to setting $a_1 \neq 0$ and $a_j = b_l = 0$ otherwise in the matrix A above.



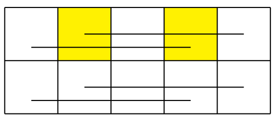
To obtain a matrix with Jordan type $(2^3, 1^4)$ we send v_i to u_{i-2}; this corresponds to setting $a_5 \neq 0$ in the matrix A.



To obtain a matrix with Jordan type $(2, 1^8)$ we send v_i to u_{i-4}; this corresponds to setting $a_9 \neq 0$ in the matrix A.

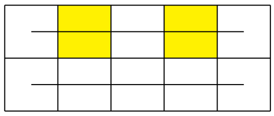


In order to obtain a matrix whose Jordan type has an even number of even parts, we move the basis elements to any other basis element within the same block; so to get a matrix of Jordan type $(2^4, 1^2)$ we take v_i to v_{i-3} and u_i to u_{i-3}. This corresponds to a_6 in the matrix above.



Finally to obtain a matrix with Jordan type $(2^2, 1^6)$ we take v_i to v_{i-4} and u_i

to u_{i-4}. This corresponds to $a_8 \neq 0$.



It is easy to see how this works for other m. If a is odd we can obtain a matrix with Jordan type $(2^a, 1^{2m-2a})$ by taking v_m to u_a. Conversely if a is even then we obtain a matrix with Jordan type $(2^a, 1^{2m-2a})$ by taking v_m to $v_{\frac{a}{2}}$ and u_m to $u_{\frac{a}{2}}$. We thus obtain elements with Jordan type $(2^a, 1^{2m-2a})$ for all $0 \leqslant a \leqslant m$ in this way. $\square$

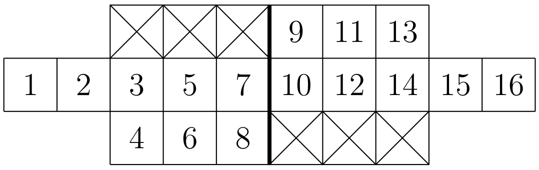
Now let $\lambda = (\lambda_1, \lambda_2)$ with λ_i even and $m < \lambda_1 < 2m$. We consider which orbits $\mathcal{O}_\mu$ commute with $\mathcal{O}_\lambda$, where $\mu = (2^a, 1^{2m-2a})$ is a symplectic square-zero partition (not necessarily odd). Using the inductive method of Lemma 4.4.2, we immediately note the following result.

Lemma 4.5.3. *Let $\lambda = (\lambda_1, \lambda_2)$ as above and $\mu = (2^a, 1^{2m-2a})$. Suppose that a is even and $a \neq m$ if $\lambda_i \equiv 0 \bmod 4$ for $i = 1, 2$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$.*

Proof. Since a is even we may write $a = a_1 + a_2$, such that a_1 and a_2 are both odd, and $0 \leqslant a_i \leqslant \frac{\lambda_i}{2}$ for $i = 1, 2$. Then $\mu^i = (2^{a_i}, 1^{\lambda_i - 2a_i})$ is an odd square-zero partition of λ_i, so $\mathcal{O}_{\mu^i}$ commutes with $\mathcal{O}_{(\lambda_i)}$ in $\mathfrak{sp}_{\lambda_i}(k)$. So we may write $\mu = \mu^1 \oplus \mu^2$, and by Lemma 4.4.2 we see that $\mathcal{O}_\mu$ commutes with $\mathcal{O}_\lambda$ in $\mathfrak{sp}_{2m}(k)$. $\square$

In the light of Lemma 4.5.3 we reduce to considering partitions of the form $\mu = (2^a, 1^{2m-2a})$ for a odd or $a = m$. In order to do this we now describe a combinatorial approach to writing down a general element of the centraliser of $D^-(\lambda_1, \lambda_2)$, where the λ_i are even and distinct. We illustrate this with the example of $(10, 6)$ in $\mathfrak{sp}_{16}(k)$.

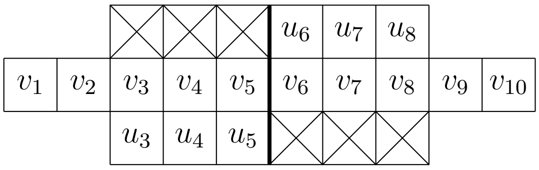
Now the Dynkin pyramid of $(10, 6)$ is as follows:



Note that we additionally draw a thick central line such that the box labelled j is to the left of this line if $j \leqslant m$ and to the right if $j \geqslant m+1$. The significance of this will become clear shortly.

We may now relabel the boxes of this pyramid in the same way as in the proof of Proposition 4.4.2. We label the box of the skew row in the ith column as u_i; if no such box exists we set $u_i = 0$. Similarly we label the box of the central row in the ith column as v_i. Let $d = \frac{\lambda_1 - \lambda_2}{2}$; then $u_i \neq 0$ for $d+1 \leqslant i \leqslant \lambda_1 - d$.

This then gives the following:



When we draw the matrix $D^-(10, 6)$ we label the first row (respectively, column) as v_1, the second as v_2, the third as v_3, the fourth as u_3, and so on.

We may now write down a general element of the centraliser of $D^-(\lambda_1, \lambda_2)$ in the following way. We begin by writing down an empty $2m \times 2m$ matrix A. We then highlight the rows and columns labelled by u_i. So in the example of $(8, 6)$ we highlight the fourth, sixth, eighth, ninth, eleventh and thirteenth rows and columns of the 16×16 matrix A.

We now note that the unhighlighted entries of A are those with co-ordinates (v_i, v_j), and that these entries form a $\lambda_1 \times \lambda_1$ submatrix of A; we denote this submatrix by A'. Similarly the entries of A with co-ordinates (u_i, u_j) lie in both a highlighted row and a highlighted column, and form a $\lambda_2 \times \lambda_2$ submatrix which we denote A''. Now A' commutes with $D^{-}(\lambda_1)$ in $\mathfrak{sp}_{\lambda_1}(k)$, and similarly A'' commutes with $D^{-}(\lambda_2)$ in $\mathfrak{sp}_{\lambda_2}(k)$. So the forms of these matrices are given in the proof of Proposition 4.3.2. These submatrices correspond to moving basis elements within the same block of the Dynkin pyramid.

In the example of $(10, 6)$, this gives us

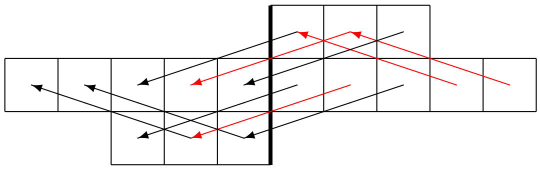
$$A' = \left(\begin{array}{ccccc|ccccc}
0 & a_1 & 0 & a_3 & 0 & a_5 & 0 & a_7 & 0 & a_9 \\
 & 0 & a_1 & 0 & a_3 & 0 & -a_5 & 0 & -a_7 & 0 \\
 & & 0 & a_1 & 0 & a_3 & 0 & a_5 & 0 & a_7 \\
 & & & 0 & a_1 & 0 & -a_3 & 0 & -a_5 & 0 \\
 & & & & 0 & a_1 & 0 & a_3 & 0 & a_5 \\
\hline
 & & & & & 0 & -a_1 & 0 & -a_3 & 0 \\
 & & & & & & 0 & -a_1 & 0 & -a_3 \\
 & & & & & & & 0 & -a_1 & 0 \\
 & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & 0
\end{array}\right)$$

and

$$A'' = \left(\begin{array}{ccc|ccc}
0 & b_1 & 0 & b_3 & 0 & b_5 \\
 & 0 & b_1 & 0 & -b_3 & 0 \\
 & & 0 & b_1 & 0 & b_3 \\
\hline
 & & & 0 & -b_1 & 0 \\
 & & & & 0 & -b_1 \\
 & & & & & 0
\end{array}\right).$$

We fill in the remaining entries of A, corresponding to moves between the blocks of the Dynkin pyramid, in the following way. On the Dynkin pyramid we draw a line between box v_{λ_1} and $u_{\lambda_1 - i}$, with coefficient $-c_i$ if the line does not cross the central line, or c_i if the line does cross the central line. We then add lines from v_j to u_{j-i} from right to left, changing the sign of c_i whenever we draw a line that crosses over the central line of the pyramid. We then draw in the line from u_{i+1} to v_1 with coefficient c_i, and fill in the remaining lines from u_j to v_{j-i} working from left to right; this time, whenever we draw a line that crosses the central line of the pyramid we change the sign of the *next* line.

For example, in the $(10, 6)$ case, we draw the following lines on the Dynkin pyramid for $i = 3$, where black lines correspond to entries of A equal to c_3 and red lines indicate entries equal to $-c_3$:



Applying this method to the $(10, 6)$ example now gives a matrix of the following form, which we denote by A:

$$
\left(\begin{array}{cccccccc|cccccccc}
0 & a_1 & & c_2 & a_3 & c_3 & & c_4 & c_5 & a_5 & c_6 & & c_7 & a_7 & & a_9 \\
 & 0 & a_1 & & & c_2 & a_3 & c_3 & c_4 & & -c_5 & -a_5 & -c_6 & & -a_7 & \\
 & & 0 & & a_1 & & & c_2 & c_3 & a_3 & -c_4 & & c_5 & a_5 & & a_7 \\
 & & & 0 & & b_1 & -c_2 & & b_3 & c_3 & & c_4 & b_5 & c_5 & -c_6 & c_7 \\
 & & & & 0 & & a_1 & & c_2 & & -c_3 & -a_3 & c_4 & & -a_5 & \\
 & & & & & 0 & & b_2 & & -c_2 & -b_3 & -c_3 & & -c_4 & -c_5 & c_6 \\
 & & & & & & 0 & & & a_1 & -c_2 & & c_3 & a_3 & & a_5 \\
 & & & & & & & 0 & b_1 & & & c_2 & b_3 & c_3 & c_4 & c_5 \\
\hline
 & & & & & & & & 0 & & -b_1 & & & -c_2 & -c_3 & -c_4 \\
 & & & & & & & & & 0 & & -a_1 & c_2 & & -a_3 & \\
 & & & & & & & & & & 0 & & -b_1 & & -c_2 & -c_3 \\
 & & & & & & & & & & & 0 & & -a_1 & & -a_3 \\
 & & & & & & & & & & & & 0 & & & -c_2 \\
 & & & & & & & & & & & & & 0 & -a_1 & \\
 & & & & & & & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & & & & & & & 0
\end{array}\right)
$$

An important point to note about the structure of this matrix is that it has a form of 'chessboard' property. We attach signs to boxes of the Dynkin pyramid by setting the central two columns to be positive, and each other column of the pyramid is of opposite sign to its neighbours. So in the $(8,6)$ example the second, fourth, fifth and seventh columns of the Dynkin pyramid are all positive, and the first, third, sixth and eighth columns are negative. We then attach the sign of each box of the pyramid to the row and column of A labelled by that box. Thus the columns (respectively, rows) denoted by u_i and v_i are adjacent to each other and of the same sign.

Consequently if we shade in every entry of the matrix whose row and column has the same sign, we obtain a pattern of blocks of size at most 2×2 which meet only at their corners; within each of these shaded blocks each entry has the same sign, and in the upper right block each of these smaller blocks has a different sign to the blocks which meet it at its corners. In our example this works as follows:

$$\left(\begin{array}{cccccccc|cccccccc}
0 & a_1 & & c_2 & a_3 & c_3 & & c_4 & c_5 & a_5 & c_6 & & c_7 & a_7 & & a_9 \\
& 0 & a_1 & & & c_2 & a_3 & c_3 & c_4 & & -c_5 & -a_5 & -c_6 & & -a_7 & \\
& & 0 & & a_1 & & & c_2 & c_3 & a_3 & -c_4 & & c_5 & a_5 & & a_7 \\
& & & 0 & & b_1 & -c_2 & & b_3 & c_3 & & c_4 & b_5 & c_5 & -c_6 & c_7 \\
& & & & 0 & & a_1 & & c_2 & & -c_3 & -a_3 & c_4 & & -a_5 & \\
& & & & & 0 & & b_1 & & -c_2 & -b_3 & -c_3 & & -c_4 & -c_5 & c_6 \\
& & & & & & 0 & & & a_1 & -c_2 & & c_3 & a_3 & & a_5 \\
& & & & & & & 0 & b_1 & & & c_2 & b_3 & c_3 & c_4 & c_5 \\
\hline
& & & & & & & & 0 & & -b_1 & & & -c_2 & -c_3 & -c_4 \\
& & & & & & & & & 0 & & -a_1 & c_2 & & -a_3 & \\
& & & & & & & & & & 0 & & -b_1 & & -c_2 & -c_3 \\
& & & & & & & & & & & 0 & & -a_1 & & -a_3 \\
& & & & & & & & & & & & 0 & & & -c_2 \\
& & & & & & & & & & & & & 0 & -a_1 & \\
& & & & & & & & & & & & & & 0 & -a_1 \\
& & & & & & & & & & & & & & & 0
\end{array}\right)$$

We can now use this to prove the following result:

Lemma 4.5.4. *Let $\lambda = (\lambda_1, \lambda_2)$ with λ_1, λ_2 even, and let $\mu = (2^a, 1^{2m-2a})$ be partitions of $2m$, with a odd or $a = m$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$.*

Proof. Let A be the general element of the centraliser of $D^-(\lambda)$ as obtained by the preceding discussion; let A_j be a matrix obtained from A by setting $a_l = b_l = c_l = 0$ for all $l \neq j$. Suppose that j is odd. Then for $1 \leqslant l \leqslant \lambda_1$, we see that $A_j v_l = \pm(a_j v_{l-j} + c_j u_{l-j})$ and $A_j u_l = \pm(b_j u_{l-j} + c_j v_{l-j})$; that is, A_j takes boxes in the lth column of the Dynkin pyramid to all boxes of the $(l-j)$th column. We recall that if $l - j \leqslant d$, then we set $u_{l-j} = 0$; similarly if $l - j \leqslant 0$ then $v_{l-j} = 0$. If there are r boxes in the lth column of the pyramid and s boxes in the $l - j$th column, then this gives rise to a shaded block of size $s \times r$ in the matrix A_j. In our $(10, 6)$ example described previously, there is a single box in the tenth column of the Dynkin pyramid and two boxes in the eighth; so this gives the 2×1 block $\left(\begin{smallmatrix} -c_3 \\ -a_3 \end{smallmatrix}\right)$ in positions $(11, 16)$ and $(12, 16)$ in A_1.

Now let $d = \frac{\lambda_1 - \lambda_2}{2}$, and suppose that $j \geqslant d$. This means that we always map v_{2m} to a column of the Dynkin pyramid which contains two boxes. We note that the matrix A_j contains $\lambda_1 - j$ shaded blocks with non-zero entries, since these blocks correspond to the leftmost $\lambda_1 - j$ columns of the Dynkin pyramid. If d is odd then m is odd also and j is minimal when $j = d$; then A_j contains $\lambda_1 - d = \frac{1}{2}(\lambda_1 + \lambda_2) = m$ blocks. Otherwise if d is even then m is also even and j is minimal when $j = d + 1$, so A_j has $\lambda_1 - (d + 1) = m - 1$ blocks.

We now set $a_j = 1$, $b_j = -1$ and $c_j = i$ where $i = \sqrt{-1}$, and calculate the Jordan type of A_j. We first see that $A_j^2 = 0$; hence the largest part of the Jordan type of A_j is at most 2. Now whenever we have a block with two columns, these columns correspond to $A_j u_r$ and $A_j v_r$ for some r, and we have chosen A_j to be such that $A_j u_r = i A_j v_r$. So $u_r - i v_r$ is in the kernel of A_j. It follows that the dimension of the image of A_j is equal to the number of blocks; and this is thus the number of parts of size 2 in the Jordan type of A_j. So A_j has Jordan type $(2^{\lambda_1 - j}, 1^{2m - 2(\lambda_1 - j)})$; and $\lambda_1 - j$ is odd such that $m \geqslant \lambda_1 - j$. Hence for any odd $0 \leqslant a \leqslant m$, there exists a j for which A_j has Jordan type $(2^a, 1^{2m-2a})$.

It remains only to construct a matrix in the centraliser of $D(\lambda_1, \lambda_2)$ with

Jordan type (2^m), if m is even. If $\lambda_i \equiv 2 \bmod 4$, then this follows from Lemmas 4.4.2 and 4.5.3. So we assume that $\lambda_i \equiv 0 \bmod 4$ for $i = 1, 2$. We now take the matrix $A = A_{d+1}$ as defined above, which has Jordan type $(2^{m-1}, 1^2)$. If $\lambda_2 = 4$, then A_{d+1} contains only a single shaded block of size 2×2, of the form $\pm \left(\begin{smallmatrix} c_3 & a_3 \\ b_3 & c_3 \end{smallmatrix}\right)$. If we set $a_3 = 1$, $c_3 = i$ and $b_3 = 0$ then the two columns of this block are no longer linearly dependent. Consequently the dimension of the kernel of A is reduced by 1; so we obtain one more part of size at least 2 in the Jordan type of A. Since A^2 is still equal to 0, the Jordan type of A is thus (2^m).

Now if $\lambda_2 > 4$, then the entry in the $(u_{d+1}, u_{\lambda_1 - d})$th position of the matrix A_{d+1} is equal to zero. On the Dynkin pyramid this corresponds to taking u_{λ_1-d} to u_{d+1}; since $u_j = 0$ for $j < d+1$ or $j > \lambda_1 - d$, no other lines parallel to this may be drawn on the pyramid. Consequently we may now set this entry of the matrix to be non-zero without changing any other entries. Then as in the previous case, the columns of the matrix labelled u_{λ_1-d} and v_{λ_1-d} are no longer linearly dependent; so we obtain one additional part of size 2 in the Jordan type of this matrix. Since the Jordan type of the original matrix A_{d+1} is $(2^{m-1}, 1^2)$, the desired result follows. $\square$

Now Lemmas 4.5.3 and 4.5.4 combine to give the following immediate corollary:

Proposition 4.5.5. *Let $\lambda = (\lambda_1, \lambda_2)$ with λ_1 and λ_2 both even, and $\mu = (2^a, 1^{2m-2a})$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$ for all $0 \leqslant a \leqslant m$.*

We now have enough information to complete the classification of universally commuting orbits in the symplectic case.

Theorem 4.5.6. *Let μ be a symplectic partition of $2m$. Then the orbit $\mathcal{O}_\mu$ is universally commuting in $\mathfrak{sp}_{2m}$ if and only if $\mu = (2^a, 1^{2m-2a})$ such that a is odd and $a \neq m$, or $a = 0$.*

Proof. We recall from Lemma 4.5.1 that if μ is not of the form $(2^a, 1^{2m-2a})$ for a odd and $a \neq m$, or $a = 0$, then $\mathcal{O}_\mu$ is not universally commuting. So it remains to show that if μ is of this form then $\mathcal{O}_\mu$ is universally commuting.

Let λ be an arbitrary symplectic partition of $2m$. Then we can decompose λ as $\lambda^1 \oplus \lambda^2 \oplus ... \oplus \lambda^t$, where $\lambda^j = (\lambda_1^j, \lambda_2^j)$ is a two-part symplectic partition of $2m_j$ for $1 \leqslant j \leqslant t-1$, and λ^t is a symplectic partition of $2m_t$ with either one or two parts. Now we write μ as $\mu^1 \oplus ... \oplus \mu^t$ where μ^j is a subpartition of μ of size $2m_j$, and $\mu^t = (2^{a_t}, 1^{2m_t-2a_t})$ with a_t odd or $a_t = 0$. Then by Propositions 4.3.2 and 4.5.5, $\mathcal{O}_{\lambda^j}$ commutes with $\mathcal{O}_{\mu^j}$ in $\mathfrak{sp}_{2m_j}$, for all $1 \leqslant j \leqslant t$. It follows that by Proposition 4.4.2, $\mathcal{O}_\mu$ commutes with $\mathcal{O}_\lambda$ in $\mathfrak{sp}_{2m}(k)$. Since we chose λ arbitrarily, it follows that $\mathcal{O}_\mu$ is universally commuting in $\mathfrak{sp}_{2m}(k)$, as required. □

4.6 Orthogonal universally commuting partitions

We now move on to consider which orthogonal partitions are universally commuting. In Propositions 4.3.3 and 4.3.4 we classified partitions which label orbits that commute with $\mathcal{O}_{(2m+1)}$ in $\mathfrak{so}_{2m+1}(k)$ and $\mathcal{O}_{(2m-1,1)}$ in $\mathfrak{so}_{2m}(k)$ respectively. So we now compare these results with those obtained below for a second orbit. In the case of $\mathfrak{so}_{2m+1}(k)$ we consider the sub-regular orbit labelled by the partition $(2m-1, 1^2)$. In the case of $\mathfrak{so}_{2m}(k)$, however, we consider the orbit labelled by $(2m-3, 1^3)$ as elements of the centraliser of $D^+(2m-3, 1^3)$ have a slightly simpler form than elements of the centraliser of $D^+(2m-3, 3)$.

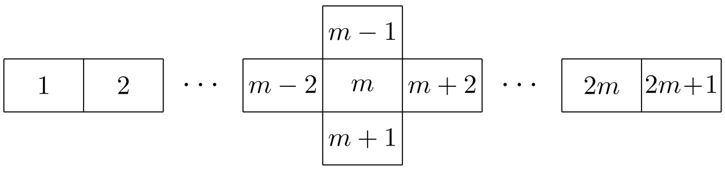
Throughout this section we consider only the case of $G = \mathrm{O}_n(k)$. We recall from Chapter 2 (see also [32, 1.12 & 3.13]) that if the Jordan type of some $X \in \mathfrak{so}_n(k)$ contains only even parts, then the $\mathrm{O}_n(k)$-orbit of X splits into two distinct $\mathrm{SO}_n(k)$-orbits. Since elements of $\mathfrak{so}_{2m}$ with Jordan type (2^m) play an important role in the following discussions, allowing $G = \mathrm{SO}_n(k)$ would introduce significant complications.

4.6.1 Centraliser of $(2m-1, 1^2)$ in $\mathfrak{so}_{2m+1}(k)$

We first consider which orbits commute with $(2m-1, 1^2)$ in $\mathfrak{so}_{2m+1}(k)$. We have the following result:

Proposition 4.6.1. *Suppose that λ is an orthogonal partition of $2m+1$ such that $\mathcal{O}_\lambda$ commutes with both $\mathcal{O}_{(2m+1)}$ and $\mathcal{O}_{(2m-1,1^2)}$. Then λ is of the form $(2^a, 1^{2m-2a+1})$, with a even.*

Proof. We begin by noting that the Dynkin pyramid of $(2m-1, 1^2)$ is of the following form:



Via reference to [52, Lem. 1.5.8] or a direct calculation, we then find that a general strictly upper triangular element of the centraliser of the matrix $D^+(2m-1, 1^2)$ is of the following form:

$$A = \begin{pmatrix} A_1' & b & \mathbf{a}_2' & c & A_3' \\ & 0 & 0 & 0 & c \\ & & 0 & 0 & \mathbf{a}_4' \\ & & & 0 & -b \\ & & & & A_5' \end{pmatrix}.$$

We note that the matrix $A' = \begin{pmatrix} A'_1 & \mathbf{a}'_2 & A'_3 \\ & 0 & \mathbf{a}'_4 \\ & & A'_5 \end{pmatrix}$ comprised of all unhighlighted entries of the above matrix form a $(2m-1) \times (2m-1)$ matrix which commutes with $(D^+(2m-1))$ in $\mathfrak{so}_{2m-1}(k)$. In particular we see from Proposition 4.3.3 that the Jordan type of A' is $\rho_i(2m-1)$, for i odd.

Now if $b = c = 0$ then it is clear that the Jordan type of A is $(\rho_i(2m-1), 1^2)$ since the only non-zero entries of A are in A'. We note that this will be almost rectangular (and thus $\mathcal{O}_{(\rho_i(2m-1),1^2)}$ commutes with $\mathcal{O}_{(2m+1)}$) if and only if $\rho_i(2m-1)_1 \leqslant 2$. If this is the case, $\rho_i(2m-1)$ must have an odd number of parts equal to 1; so the remaining parts of $\rho_i(2m-1)$ are all equal to 2 and there is an even number of these.

If $b \neq 0$ and $c = 0$, or $b = 0$ and $c \neq 0$, then we may proceed in an identical manner to the corresponding cases in Proposition 4.5.1. Once again we find that A has Jordan type $(\rho_i(2m-1), 1^2)$.

Finally suppose that $b \neq 0 \neq c$; we may approach this similarly to the equivalent case in the proof of Proposition 4.2.3. If $a_i = 0$ for all i, we obtain $v_{2m+1} \mapsto -cv_{m+2} - bv_m \mapsto -2bcv_1 \mapsto 0$; we denote this string by $(*)$.

Conversely suppose now that the a_i may be non-zero, and let $\rho_i(2m-1)$ be the Jordan type of the submatrix A' as defined above. Then if $\rho_i(2m-1)_1 = 4$, or $\rho_i(2m-1)_1 = \rho_i(2m-1)_2 = 3$, we note that A takes v_{2m+1} to zero in at least as many steps as $(*)$, with no cancellation occurring. So v_m and v_{m+2} each give rise to a part of size 1 in the Jordan type of A, which is thus $(\rho_i(2m-1), 1^2)$.

If m is even, then the partition $\rho_{m-1}(2m-1) = (3, 2^{m-2})$ is orthogonal, and A' has this as its Jordan type when $a_{m-1} \neq 0$ and $a_j = 0$ for $j < m-1$. In this situation, we see that (neglecting any lower terms that may appear) $v_{2m} \mapsto cv_{m+2} - a_{m-1}v_{m+1} - bv_m \mapsto -(2bc + a_{m-1}^2)v_1$. If $2bc + a_{m-1}^2 \neq 0$ then we obtain a part equal to 3 in the Jordan type of A, and furthermore we have $a_{m-1}v_m - bv_{m+1} \mapsto 0$ and $cv_{m+1} - a_{m-1}v_{m+2} \mapsto 0$. So a basis for the Jordan normal form of A is

$\{v_{2m+1}, Av_{2m+1}, A^2v_{2m+1}, a_{m-1}v_m - bv_{m+1}, cv_{m+1} - a_{m-1}v_{m+2}, v_{2m}, ..., v_{m+3}, Av_{2m}, ..., Av_{m+3}\}$ and its Jordan type is thus $(3, 2^{m-2}, 1^2)$.

Suppose now that $2bc + a_{m-1}^2 = 0$. Then $A^2v_m = 0$, so the maximal part of the Jordan type of A is equal to 2. Additionally we have $v_m \mapsto Av_m = bv_1 \mapsto 0$, and $cv_{m+1} - a_{m-1}v_{m+2} \mapsto 0$. So a basis with respect to which A has Jordan normal form is now $\{v_{2m+1}, Av_{2m+1}, v_m, Av_m, cv_{m+1} - a_{m-1}v_{m+2}, v_{2m}, ..., v_{m+3}, Av_{2m}, ..., Av_{m+3}\}$ and the Jordan type of A is $(2^m, 1)$.

Finally suppose that $\rho_i(2m-1)_1 = 2$. In this case we note that $v_{2m+1} \mapsto -bv_{m+2} - cv_m - a_iv_r \mapsto -2bcv_1 \mapsto 0$ for some $r < m$; so we have a part of size 3 in the Jordan type of A. Furthermore, we also have $a_iv_m - bv_r \mapsto 0$ and $cv_r - a_iv_{m+2} \mapsto 0$; so this gives rise to two parts equal to 1 in the Jordan type λ of A. Since λ is thus not almost rectangular, it follows that $\mathcal{O}_\lambda$ does not commute with $\mathcal{O}_{(2m+1)}$ in $\mathfrak{so}_{2m+1}(k)$, so is not universally commuting.

In particular, we note that the Jordan type λ of A always contains at least one part equal to 1; so we see that if $\mathcal{O}_\lambda$ commutes with both $\mathcal{O}_{(2m+1)}$ and $\mathcal{O}_{(2m-1,1^2)}$ then λ must be of the form $(2^a, 1^{2m-2a+1})$ with a even. Furthermore we have obtained all partitions of this form in the above discussion, so the desired result follows. □

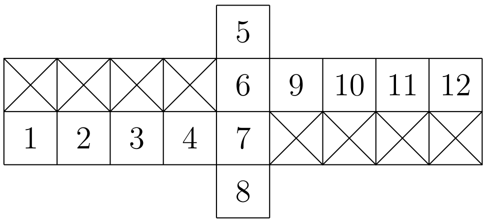
4.6.2 Centraliser of $(2m-3, 1^3)$ in $\mathfrak{so}_{2m}(k)$

It remains to narrow down the potentially universally commuting orbits in $\mathfrak{so}_{2m}(k)$. Having already considered in Proposition 4.3.4 which orbits commute with the regular orbit $\mathcal{O}_{(2m-1,1)}$, we now examine the same question for the orbit $\mathcal{O}_{(2m-3,1^3)}$. As in the previous cases we have studied, we consider the structure of the centraliser of $D^+(2m-3, 1^3)$ in $\mathfrak{so}_{2m}(k)$ to be slightly simpler than that of

$D^+(2m-3,3)$.

Proposition 4.6.2. *Suppose that λ is an orthogonal partition of $2m$ such that $\mathcal{O}_\lambda$ commutes with both $\mathcal{O}_{(2m-1,1)}$ and $\mathcal{O}_{(2m-3,1^3)}$. Then $\lambda = (2^a, 1^{2m-2a})$ with $a < m$, or $\lambda = (3, 2^a, 1^{2m-2a-3})$.*

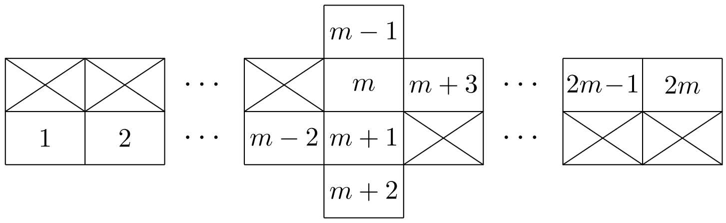
Proof. In addition to describing the general case we use the example of $(9,1^3)$ in $\mathfrak{so}_{12}(k)$ to illustrate certain details of the proof. First we note that the Dynkin pyramid of $(9,1^3)$ is as follows:



Now a general element of the centraliser of $D^+(9,1^3)$ is of the following form:

$$A = \begin{pmatrix}
0 & a_1 & 0 & a_3 & b & a_4 & -a_4 & c & -a_5 & 0 & -a_7 & 0 \\
 & 0 & a_1 & 0 & 0 & a_3 & a_3 & 0 & 0 & a_5 & 0 & a_7 \\
 & & 0 & a_1 & 0 & 0 & 0 & 0 & -2a_3 & 0 & -a_5 & 0 \\
 & & & 0 & 0 & a_1 & a_1 & 0 & 0 & 2a_3 & 0 & a_5 \\
 & & & & 0 & d & -d & 0 & 0 & 0 & 0 & -c \\
 & & & & & 0 & 0 & d & -a_1 & 0 & -a_3 & a_4 \\
 & & & & & & 0 & -d & -a_1 & 0 & -a_3 & -a_4 \\
 & & & & & & & 0 & 0 & 0 & 0 & -b \\
 & & & & & & & & 0 & -a_1 & 0 & -a_3 \\
 & & & & & & & & & 0 & -a_1 & 0 \\
 & & & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & & & 0
\end{pmatrix}.$$

For general m the Dynkin pyramid of $(2m-3,1^3)$ will be the following:



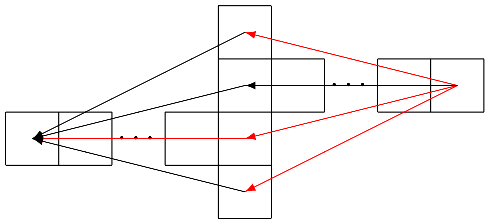
We note that, similarly to the proof of Proposition 4.6.1, the unhighlighted entries of this matrix form a $(2m-2)\times(2m-2)$ submatrix A' which has the same form as a general element of the centraliser of $D^+(2m-3,1)$ in $\mathfrak{so}_{2m-2}(k)$. Consequently much of the proof proceeds in a very similar way.

Suppose that λ is an orthogonal partition of $2m$ such that $\mathcal{O}_\lambda$ commutes with both $\mathcal{O}_{(2m-1,1)}$ and $\mathcal{O}_{(2m-3,1^3)}$ in $\mathfrak{so}_{2m}(k)$. In order to find the possible such Jordan types λ, we consider the various possibilities for the values of the entries denoted b,c,d in the matrix above. The entries equal to $\pm b$ correspond to taking $v_{2m}\mapsto bv_{m+2}$ and $v_{m-1}\mapsto bv_1$, the entries equal to $\pm c$ correspond to taking $v_{2m}\mapsto cv_{m-1}$ and $v_{m+2}\mapsto cv_1$, and those equal to $\pm d$ correspond to taking $v_{m+2}\mapsto d(v_m-v_{m+1})\mapsto 2d^2v_{m-1}$.

If $b=c=d=0$, then there is no crossover between the two central rows of the Dynkin pyramid and the top and bottom rows consisting of a single box. So the Jordan type of A is $(\lambda',1^2)$ where λ' is the Jordan type of the submatrix A'. Then comparing this with Proposition 4.3.4, we see that the only partitions λ of this form such that $\mathcal{O}_\lambda$ commutes with both $\mathcal{O}_{(2m-1,1)}$ and $\mathcal{O}_{(2m-3,1^3)}$ in $\mathfrak{so}_{2m}(k)$ are of the form $(3,2^a,1^{2m-2a-3})$ with $2m-2a-3\geqslant 3$, or $(2^a,1^{2m-2a})$ with $a<m$.

If $b\neq 0$ and $c=d=0$, or $c\neq 0$ and $b=d=0$, then once more we repeat the argument of the corresponding case in Proposition 4.5.1 to find that the Jordan type of A is again $(\lambda',1^2)$.

Suppose now that $b\neq 0\neq c$ and $d=0$. If $\lambda'_1=4$ or $\lambda'_1=\lambda'_2=3$ then by the same argument as Proposition 4.3.4 we find that the Jordan type of A is $(\lambda', 1^2)$. If $\lambda' = (3, 2^r, 1^{2m-2r-5})$ then A maps v_{2m} to v_1 in two steps both via the top and bottom rows of the Dynkin pyramid and via A', as in the following diagram:



Let i be minimal such that $a_i \neq 0$ in this case; then we have $v_{2m} \mapsto -bv_{m+2} - a_i v_{m+1} + av_m - cv_{m-1} \mapsto 2(a_i^2 - bc)v_1 \mapsto 0$. If $a_i^2 - bc \neq 0$ then we obtain a part of size 3 in the Jordan type of A; and additionally we obtain two parts of size 1 via $v_{m+1} - v_m \mapsto 0$ and $bv_{m+2} - cv_{m-1} \mapsto 0$. All remaining parts in the Jordan type of A' are preserved, thus the Jordan type of A is $(3, 2^r, 1^{2m-2r-3})$. Conversely if $a_i^2 - bc = 0$ then this gives a part of size 2 in the Jordan type of A; then we find another part of size 2 via $v_m \mapsto a_i v_1 \mapsto 0$, and a part of size 1 via $bv_{m+2} - cv_{m-1} \mapsto 0$. Since all remaining parts of the Jordan type of A are preserved, including all the parts of size 2, we see that the Jordan type of A is $(2^{r+2}, 1^{2m-2r-4})$.

Now let $d \neq 0$ and $b = c = 0$. Then if $a_i = 0$ for all i, we have $v_{m+2} \mapsto d(v_m - v_{m+1}) \mapsto 2d^2 v_{m-1} \mapsto 0$; we denote this string by $(*)$. We thus obtain a part of size 3 in the Jordan type of A, and all remaining basis elements are mapped by A to zero. Thus the Jordan type of A is $(3, 1^{2m-3})$.

Conversely suppose that the a_i may be non-zero. Suppose first that the Jordan type of A' is of the form $(\rho_i(2m-3), 1)$ or (2^{m-1}), and that A sends v_j to $\pm a_s v_m$ for some $j > m+2$. Then we have $v_j \mapsto \pm a_s(v_m + v_{m+1})$. Consequently the entries of the string $(*)$ are all linearly independent from the elements of the set $\{A^l v_j \mid j \neq m-1, m+2\}$. It follows that A has Jordan type $(\rho_i(2m-3), 3)$, with i odd, in this case. In particular we note that all partitions of the form $(3, 2^a, 1^{2m-2a-3})$ with a even can be written in this way, when $\rho_i(2m-3)_1 \leqslant 2$.

Suppose now that the Jordan type of A' is A' is $(3, 2^r, 1^{2m-2r-5})$. Then if m is even we have $v_{2m} \mapsto a_j v_m - a_j v_{m+a} \mapsto 2a_j^2 v_1 \mapsto 0$; or if m is odd we have $v_{2m} \mapsto -a_s v_m - a_t v_{m+1} \mapsto -2a_s a_t v_1 \mapsto 0$. Furthermore we have $v_{m+2} \mapsto dv_m - dv_{m+1} \mapsto 2d^2 v_1 \mapsto 0$, and no v_j maps to v_{m+2} for $j > m+2$. If m is odd and $a_t = a_s$ then we proceed as in the previous case. Otherwise the strings from v_{2m} and from v_{m+2} are both of length 3 and not all their entries are independent. Consequently these two strings do not both give rise to a part of size 3 in the Jordan type of A; we instead obtain a single part of size 3, and the remaining parts must be equal to 1 or 2. So the Jordan type of A must be of the form $(3, 2^a, 1^{2m-2a-3})$ for some even a; but we have already obtained matrices in the centraliser of $D^+(2m-3, 1^3)$ which have all possible Jordan types of this form. So no further consideration of this case is needed.

In the case that $c \neq 0 \neq d$ and $b = 0$ the situation is almost identical to the case of $d \neq 0$ and $b = c = 0$. Although we now have $v_{m+2} \mapsto v_1 \mapsto 0$ and $v_{2m} \mapsto v_{m-1} \mapsto 0$, these new substrings are subsumed into those already considered in the previous case. So as before we obtain a Jordan type of the form $(\rho_i(2m-3), 3)$.

Now let $b \neq 0 \neq d$ and $c = 0$. If $a_j = 0$ for all j then we see that $v_{2m} \mapsto -bv_{m+2} \mapsto bd(v_{m+1} - v_m) \mapsto -2bd^2 v_{m-1} \mapsto -2b^2 d^2 v_1 \mapsto 0$; denote this string by $(**)$. It follows that v_{2m} is contained in a block of size at least 5 in the Jordan form of A.

If the largest part of λ' is at least 5, then the string $v_{2m} \mapsto Av_{2m} \mapsto ... \mapsto 0$ subsumes the string $(**)$, and by cancelling v_{m+2} with some v_j such that j labels a box of the large block we obtain a block of size 3. So we have a partition of the form $(\lambda', 3)$. Comparing this to Proposition 4.3.4 we note that the orbit labelled by this partition does not commute with $\mathcal{O}_{(2m-1,1)}$.

Conversely if the largest part of λ' is at most 4, we have a single block of size 5 as described above, and the remaining basis elements form an almost rectangular partition of $(2m-5)$. Once again this clearly is not of any of the forms described in Proposition 4.3.4.

Finally, adding $c \neq 0$ gives nothing new, in the same way as it gave nothing new in the $b = 0$ case.

Now collating the cases gives the desired result. □

4.6.3 UC square-zero partitions of $2m$

As a consequence of Proposition 4.6.2, if $\mathfrak{g} = \mathfrak{so}_{2m}(k)$ we can reduce to considering partitions of the form $(2^a, 1^{2m-2a})$ or $(3, 2^a, 1^{2m-2a-3})$. The goal of this section is to prove the following result:

Theorem 4.6.3. *Let $\mu = (2^a, 1^{2m-2a})$ with $a < m$ and a even. Then $\mathcal{O}_\mu$ is universally commuting in $\mathfrak{so}_{2m}(k)$.*

In order to do this we first describe which square-zero partitions occur as Jordan types of elements in the centraliser of $D^+(\lambda)$, for orthogonal partitions λ with either two or four parts of certain forms. We then show that we can decompose an arbitrary partition ν into subpartitions so that at most one subpartition is not of one of these forms, and the result then follows by Lemma 4.4.2.

Lemma 4.6.4. *Suppose that $\lambda = (\lambda_1^{2a_1}, ..., \lambda_s^{2a_s})$ is an orthogonal partition of $2m$ such that every part occurs with even multiplicity. Let $\lambda^0 = (\lambda_1^{a_1}, ..., \lambda_s^{a_s})$ and suppose that μ^0 is an arbitrary (not necessarily orthogonal) partition of m. Let $\mu = \mu^0 \oplus \mu^0$ be an orthogonal partition of $2m$, and suppose that $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{\mu^0}$ considered as orbits of $\mathfrak{gl}_m(k)$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$ considered as orbits of $\mathfrak{so}_{2m}(k)$.*

Proof. Since all parts of λ occur with even multiplicity, the orthogonal Dynkin pyramid of λ is symmetrical about its horizontal centre line. So we may split the pyramid along this line and obtain two copies of the general linear pyramid of the partition λ^0 of m. Now we label the first copy of this subpyramid in the usual manner from 1 to m, and the second from $m+1$ to $2m$. Let E be the $2m \times 2m$ matrix obtained from the union of these two pyramids. Then $\mathfrak{c}_{\mathfrak{so}_{2m}(k)}(E)$ contains all elements of the form $F = \left(\begin{smallmatrix} A & 0 \\ 0 & -A^{\mathrm{st}} \end{smallmatrix}\right)$ where A centralises $D(\lambda^0)$ in $\mathfrak{gl}_m(k)$. So if μ^0 is a possible Jordan type of A, we see that $\mathcal{O}_{\mu^0}$ commutes with $\mathcal{O}_{\lambda^0}$ in $\mathfrak{gl}_m(k)$. It follows that the Jordan type of F is $\mu^0 \oplus \mu^0$. Consequently $\mathcal{O}_\mu$ commutes with $\mathcal{O}_\lambda$ in $\mathfrak{so}_{2m}(k)$. □

An immediate corollary of this is the following:

Lemma 4.6.5. *Let* $\lambda = (m^2)$, *and* $\mu = (2^a, 1^{2m-2a})$ *with* a *even. Then* $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_\mu$ *in* $\mathfrak{so}_{2m}(k)$.

Proof. Let $a = 2a_1$; then by Theorem 4.4.1 $\mathcal{O}_{(2^{a_1},1^{m-2a_1})}$ is universally commuting in $\mathfrak{gl}_m(k)$. In particular $\mathcal{O}_{(2^{a_1},1^{m-2a_1})}$ commutes with $\mathcal{O}_{(m)}$ in $\mathfrak{gl}_m(k)$. Consequently $\mathcal{O}_{(2^a,1^{2m-2a})}$ commutes with $\mathcal{O}_{(m^2)}$ in $\mathfrak{so}_{2m}(k)$, by Lemma 4.6.4. □

We now consider more general orthogonal two-part partitions.

Proposition 4.6.6. *Let* $\lambda = (\lambda_1, \lambda_2)$ *with* λ_1 *and* λ_2 *both odd, and let* $\mu = (2^a, 1^{2m-2a})$ *with* a *even. Then* $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_\mu$ *for all* $0 \leqslant a \leqslant m$.

Proof. We illustrate certain details of the proof with the example of $\lambda = (9,3)$ in $\mathfrak{so}_{12}(k)$. The Dynkin pyramid of $(9,3)$ is as follows:

☒	☒	☒	4	6	8	10	11	12
1	2	3	5	7	9	☒	☒	☒

As previously, we may relabel this pyramid so that the box in the top row of the ith column of the pyramid is labelled u_i, and the box in the lower row of the ith column is labelled w_i. If the ith column contains no box in the top (respectively, bottom) row of the pyramid we set $u_i = 0$ (respectively, $w_i = 0$). The Dynkin pyramid is then the following:

☒	☒	☒	u_4	u_5	u_6	u_7	u_8	u_9
w_1	w_2	w_3	w_4	w_5	w_6	☒	☒	☒

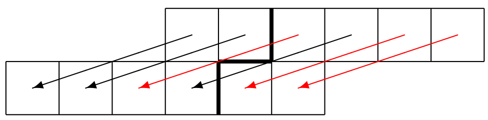
Now a general element of the centraliser of $D^+(9,3)$ is of the following form:

$$\left(\begin{array}{cccccc|cccccc}
0 & a_1 & 0 & a_3 & a_4 & a_5 & -a_5 & a_7 & a_8 & 0 & a_{10} & 0 \\
 & 0 & a_1 & 0 & 0 & a_3 & a_4 & -a_5 & a_5 & -a_7-a_8 & 0 & -a_{10} \\
 & & 0 & a_1 & a_1 & 0 & 0 & -a_3 & -a_4 & 0 & a_7+a_8 & 0 \\
 & & & 0 & 0 & d_2 & a_1-d_2 & 0 & 0 & a_4 & -a_5 & -a_8 \\
 & & & & 0 & a_1-d_2 & d_2 & 0 & 0 & a_3 & a_5 & -a_7 \\
 & & & & & 0 & 0 & -d_2 & d_2-a_1 & 0 & -a_4 & a_5 \\
\hline
 & & & & & & 0 & d_2-a_1 & -d_2 & 0 & -a_3 & -a_5 \\
 & & & & & & & 0 & 0 & -a_1 & 0 & -a_4 \\
 & & & & & & & & 0 & -a_1 & 0 & -a_3 \\
 & & & & & & & & & 0 & -a_1 & 0 \\
 & & & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & & & 0
\end{array}\right)$$

We note in particular that $v_{12} = u_9$ may be mapped by this matrix to any other basis elements apart from $v_{10} = u_7$, $v_3 = w_3$ or $v_1 = w_1$.

Now a matrix of this form maps u_{λ_1} to a linear combination of basis elements u_i and v_i. We now consider which u_i and v_i may have non-zero coefficients in this linear combination. Now using the description of the centraliser given in [52, Lem. 1.5.8] we see that if i is odd, then we may map u_{λ_1} to u_{λ_1-i} or to w_{λ_1-i}. If i is even and the (λ_1-i)th column of the Dynkin pyramid contains two boxes, then we may map u_{λ_1} to $u_{\lambda_1-i}-w_{\lambda_1-i}$. Finally if i is even and the (λ_1-i)th column of the Dynkin pyramid contains only a single box, then we may not map u_{λ_1} to this box. This pattern may be observed on the matrix above.

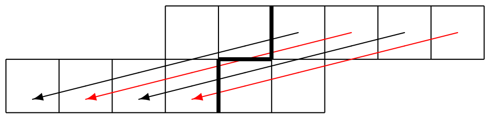
Let $d=\frac{\lambda_1-\lambda_2}{2}$, and suppose first that m is even; then $\lambda_1-\lambda_2\equiv 2 \bmod 4$ and so d is odd. Then we may map u_{λ_1} to w_{λ_1-d}, the rightmost box on the lower row of the Dynkin pyramid. We then draw in all parallel lines on the Dynkin pyramid; that is, we map u_i to w_{i-d} for all $d+1\leqslant j\leqslant\lambda_1$. In our example of $(9,3)$ this corresponds to setting $a_3\neq 0$ and $a_j=d_j=0$ otherwise, and we draw the following lines on the Dynkin pyramid:



Let F be the sum of all matrices $E_{i,j}$ such that the boxes i and j of the pyramid are joined by a line on the diagram obtained in this way. Then every box in the pyramid is now the start- or endpoint of precisely one line, so the Jordan type of the matrix F is (2^m) with respect to the basis $\{u_l, Au_l \mid d+1\leqslant l\leqslant\lambda_1\}$. The sign rules are similar to those in the symplectic case described previously; the line from u_{λ_1} has negative coefficient, and the sign changes each time we draw a line that crosses the thick central line between the boxes labelled m and $m+1$.

Furthermore, we can define a matrix with Jordan type $(2^a, 1^{2m-2a})$ for all even a by instead mapping u_i to $w_{i-(m-a)}$ and adding all parallel lines on the Dynkin pyramid.

So in our $(9,3)$ example we obtain a matrix with Jordan type $(2^4,1^4)$ by mapping u_9 to w_4, and drawing all parallel lines on the Dynkin pyramid. This corresponds to setting $a_7 \neq 0$ and $a_i = d_i = 0$ otherwise, and we obtain the following on the Dynkin pyramid:



If m is odd the partition (2^m) is not orthogonal. So instead we map u_i to w_{i-d-1} and add all parallel lines to give an element with Jordan type $(2^{m-1}, 1^2)$. Similarly we map u_i to $w_{i-(m-a)-1}$ to obtain an element of the centraliser of $D^+(\lambda_1, \lambda_2)$ with Jordan type $(2^a, 1^{m-a})$. In this way we find a matrix of Jordan type $(2^a, 1^{2m-2a})$ in the centraliser of $D^+(\lambda_1, \lambda_2)$ for all even $0 \leqslant a \leqslant m$. $\square$

We now consider partitions of the form $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with all λ_i odd, but not necessarily distinct. If there is a pair (i, j) such that $\lambda_i \not\equiv \lambda_j \bmod 4$, then $\lambda_i + \lambda_j \equiv 0 \bmod 4$. So by Proposition 4.6.6 we see that $\mathcal{O}_{(\lambda_i, \lambda_j)}$ commutes with $\mathcal{O}_{(2^a, 1^{\lambda_i+\lambda_j-2a})}$ for all even $0 \leqslant a \leqslant \frac{\lambda_i+\lambda_j}{2}$. Thus by Proposition 4.4.2, $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^a, 1^{2m-2a})}$ for all even $0 \leqslant a \leqslant m$.

So assume that $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ has four parts, all equal modulo 4. We note the following simple lemma:

Lemma 4.6.7. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ such that all the λ_i are equal modulo 4. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^a, 1^{2m-2a})}$ for all even $0 \leqslant a < m$.*

Proof. Write $\lambda = (\lambda_1, \lambda_2) \oplus (\lambda_3, \lambda_4) = \lambda^1 \oplus \lambda^2$. Then by Proposition 4.6.6 we observe that $\mathcal{O}_{\lambda^i}$ commutes with $\mathcal{O}_{(2^a, 1^{|\lambda^i|-2a})}$ for all even $0 \leqslant a \leqslant \frac{|\lambda^i|-1}{2}$. Since $|\lambda^i| \equiv 2 \bmod 4$, any partition of the form $(2^a, 1^{|\lambda^i|-2a})$ with a even must have at least two parts equal to 1. Now applying Proposition 4.4.2 gives the desired result. $\square$

Consequently we observe that whenever λ is a partition of $2m$ with four odd parts and $\mu = (2^a, 1^{2m-2a})$, then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$ in $\mathfrak{so}_{2m}(k)$, except possibly in the case when all parts of λ are equal modulo 4, and $\mu = (2^m)$. So we now focus specifically on this case.

The following two results are immediate corollaries of Lemma 4.6.4, and the result from Theorem 4.4.1 that all orbits labelled by square-zero partitions are universally commuting in $\mathfrak{gl}_n(k)$.

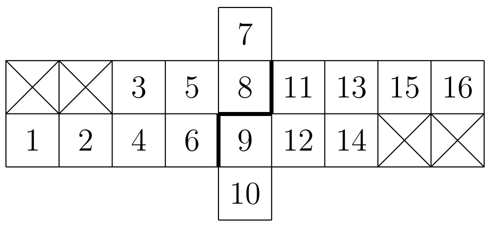
Proposition 4.6.8. *Let $\lambda = (\lambda_1^4)$ be an orthogonal partition of $2m$ with $m = 2\lambda_1$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^m)}$ in $\mathfrak{so}_{2m}(k)$.*

Proposition 4.6.9. *Let $\lambda = (\lambda_1^2, \lambda_2^2)$ be a partition of $2m = 2(\lambda_1 + \lambda_2)$, such that that $\lambda_1 \neq \lambda_2$ but $\lambda_1 \equiv \lambda_2 \bmod 4$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^m)}$ in $\mathfrak{so}_{2m}(k)$.*

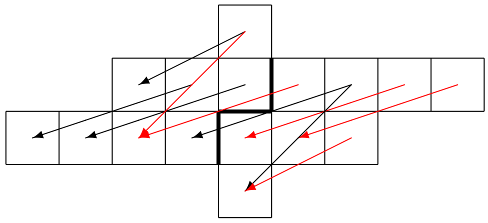
We now consider four-part partitions λ such that all parts are equal modulo 4, and at most one part occurs with multiplicity 2.

Proposition 4.6.10. *Suppose that $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_3)$ is an orthogonal partition of $2m$, with $\lambda_1 > \lambda_2 > \lambda_3$ and $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \bmod 4$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^m)}$.*

Proof. We describe the proof by means of an example, and explain how this method generalises to all other cases. Suppose that $\lambda = (1, 1, 5, 9)$; then the Dynkin pyramid of λ is as follows:



Now a direct calculation, or reference to [52, Lem. 1.5.8] shows that the matrix defined by this Dynkin pyramid commutes with the matrix $A = a(E_{1,5} + E_{2,8} - E_{4,11} + E_{6,13} - E_{9,15} - E_{12,16}) + b(E_{3,7} - E_{4,7} + E_{10,13} - E_{10,14})$. This corresponds to drawing the following lines on the Dynkin pyramid of λ, where as usual a red line indicates a negative entry in the underlying matrix and a black line indicates a positive one:



We note that on the Dynkin pyramid above, no arrows form a path of length 2, so this implies that $A^2 = 0$. Further, the image of A has dimension $m = 8$, so the matrix A has Jordan type (2^8).

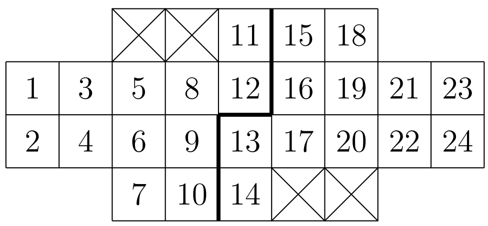
This construction generalises as follows. Select the rightmost column on the Dynkin pyramid which contains two boxes, and draw lines from both of these boxes to the rightmost box of the lowest row (in our example, from boxes 13 and 14 to box 10); then draw parallel lines until each box in the lowest row is the endpoint of a pair of lines. Then add similar lines from the top row to ensure that the diagram retains its rotational symmetry. Now draw a line from the rightmost box of the diagram to the rightmost box on the lower long row which is not already the endpoint of a line (in this case, from box 16 to box 12), and add all lines between the two long blocks that are parallel to this line. The sign rules are similar to those described in the proof of Proposition 4.6.6. For the lines between the central two rows of the pyramid (that is, the line from 16 to 12 and those parallel to it) we attach a negative sign to the rightmost line; then, working from right to left, we change the sign each time we draw a line that crosses the bold centre line of the pyramid. The lines from the rightmost column of the pyramid with two boxes to the lowest row then have opposite signs, and we use the same rules to attach signs to parallel lines to the left. Finally we attach signs to the lines originating from the top row of the pyramid in such a way as to ensure that the diagram has rotational antisymmetry.

We note that in this general construction we retain the key properties of the example described above; that is, we draw no paths of length 2 or longer on the Dynkin pyramid, and the image of the matrix defined by drawing lines on the pyramid in this way has dimension m. So we do indeed obtain an element of the centraliser of $D^+(\lambda)$ with Jordan type (2^m), as required. $\square$

Next we describe the case of a four-part partition where the largest part has multiplicity 2.

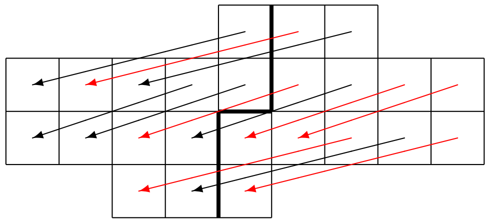
Proposition 4.6.11. *Suppose that* $\lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_3)$ *is a partition of* $n = 2m$*, with* $\lambda_1 > \lambda_2 > \lambda_3$ *and* $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \bmod 4$*. Then* $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_{(2^m)}$*.*

Proof. Here we use the example of $\lambda = (1, 5, 9, 9)$ to illustrate how we may construct a matrix in the centraliser of $D^+(\lambda)$ with the desired Jordan type. The Dynkin pyramid of λ is as follows:



By calculating the centraliser $D^+(9^2, 5, 1)$ in $\mathfrak{so}_{24}(k)$, we find that the matrix $A = a(E_{1,11} - E_{3,15} + E_{5,18} - E_{7,20} + E_{10,22} - E_{14,24}) + b(E_{2,8} + E_{4,12} - E_{6,16} + E_{9,19} - E_{21,13} - E_{23,17})$ commutes with $D^+(9^2, 5, 1)$ and has Jordan type (2^{12}).

We may represent this matrix by the following diagram:

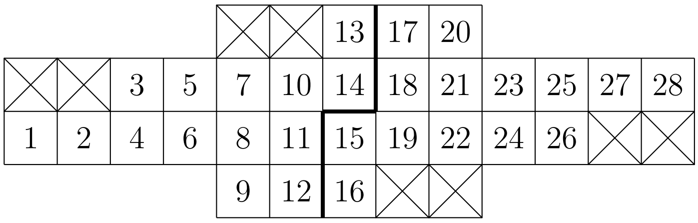


Now for a general pyramid of this form we map the rightmost box on the lower long row of the pyramid to the rightmost box on the lowest row (in our example, 24 to 14), and add parallel lines to the left until all boxes of the lowest row are endpoints of precisely one line; we then add similar lines from the top row to the leftmost boxes of the upper long block, to ensure that the diagram has the required rotational symmetry. Then we map the rightmost box on the upper long block to the rightmost block on the lower long block which is not already the endpoint of a line (in our example, 23 to 17), and add in all parallel lines to the left. The sign rules are exactly the same as described in the proof of Proposition 4.6.6; they ensure that the diagram possesses rotational antisymmetry once they are taken into account, and thus that the matrix defined in this way is indeed in $\mathfrak{so}_{2m}(k)$. Once again there are no paths of length 2 or longer on this diagram, and the image has dimension m; so the matrix corresponding to this diagram has Jordan type (2^m) and commutes with $D^+(\lambda)$. □

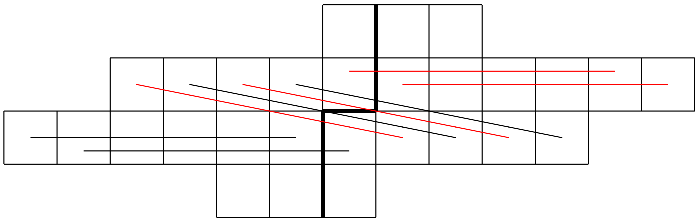
We now consider partitions λ such that the parts of λ are odd, distinct, and equal modulo 4; then we obtain the following result:

Proposition 4.6.12. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be an orthogonal partition of $2m$ such that each part is distinct and equal modulo 4. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^m)}$.*

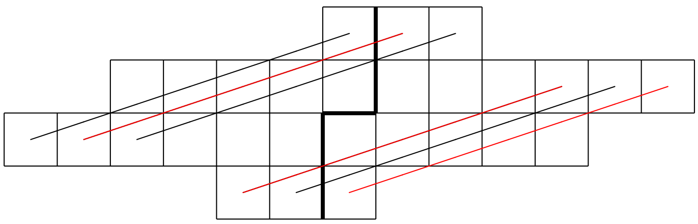
Proof. We consider the example of $\lambda = (1, 5, 9, 13)$ in $\mathfrak{so}_{28}(k)$ and use this to describe the general pattern. The Dynkin pyramid of λ is as follows:



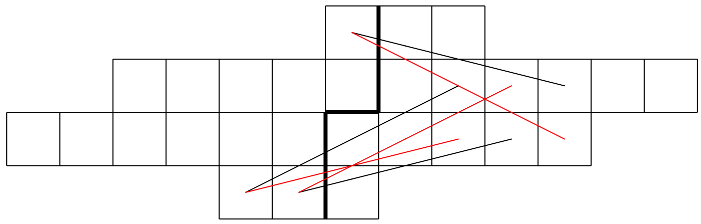
We can then obtain a matrix in the centraliser of E_λ with partition (2^m) in the following way. Firstly we draw a line between boxes 28 and 18; in general this will join box $2m$ with the leftmost box in the second row whose label is greater than m. Since the two central rows are skew rows, this also forces a line between boxes 26 and 10. We now draw in all lines parallel to these, as shown below; the black lines have coefficient 1, and the red lines have coefficient -1.



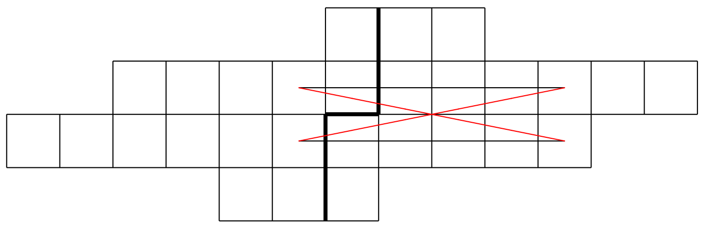
We now draw a line between boxes 28 and 16; in general this will connect box $2m$ with the last box of the lowest row. Drawing in all parallel lines on the Dynkin pyramid gives the following:



Next we draw lines from box 25 and 26 to 13 with coefficients ± 1 as illustrated below. In general, this corresponds to drawing lines from the rightmost column with two boxes to the leftmost box in the top row. Again, adding in parallel lines gives the following Dynkin pyramid. We must then add in further lines to ensure the rotational symmetry of the diagram, but these are omitted for clarity.



Finally, we link both boxes 25 and 26 with both boxes 10 and 11, with coefficient ± 1 as shown in the diagram below; we note that where this duplicates a line previously drawn, these lines have opposite sign and so cancel out. In general this corresponds to linking the rightmost column containing two boxes with the column immediately to the left of the central column. On the Dynkin pyramid this appears as follows:



This pattern is then repeated at each possible step to the left; thus it occurs four times in this example.

Now looking at the matrix whose non-zero entries are defined by the lines on these four Dynkin pyramids, we obtain the following:

$$v_{28} \mapsto -v_{18} - v_{16} \mapsto -v_4 + v_3 - v_3 + v_4 = 0;$$
$$v_{27} \mapsto -v_{14} + v_{12} \mapsto 0;$$
$$v_{26} \mapsto -v_{13} + v_{11} - v_{10} + v_{10} \mapsto -v_1 + v_1 = 0;$$
$$v_{25} \mapsto v_{13} - v_{11} + v_{10} - v_9 \mapsto v_1 - v_1 = 0;$$
$$v_{24} \mapsto v_{12} - v_8 + v_7 - v_7 \mapsto 0;$$
$$v_{23} \mapsto -v_{12} + v_8 - v_7 \mapsto 0;$$
$$v_{22} \mapsto -v_9 + v_6 - v_5 + v_5 \mapsto 0;$$
$$v_{21} \mapsto v_9 - v_6 + v_5 \mapsto 0;$$
$$v_{20} \mapsto -v_8 + v_7 + v_4 \mapsto 0;$$
$$v_{19} \mapsto v_4 - v_3 + v_3 \mapsto 0;$$
$$v_{18} \mapsto v_4 - v_3 \mapsto 0;$$
$$v_{17} \mapsto v_6 - v_5 \mapsto 0;$$
$$v_{15} \mapsto v_2 \mapsto 0;$$
$$v_{13} \mapsto v_1 \mapsto 0.$$

From this we conclude that this matrix has Jordan type (2^{14}), as required. Finally we note that this method generalises easily to other partitions of the form described in the statement of the proposition. □

We now consider one final type of partition before moving on to the general theorem.

Proposition 4.6.13. *Suppose that* $\lambda = (\lambda_1, \lambda_1, \lambda_1, \lambda_2)$ *is a partition of* $n = 2m$ *with* $\lambda_1 > \lambda_2$ *and* $\lambda_1 \equiv \lambda_2 \bmod 4$. *Then* $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_{(2^m)}$.

Proof. We consider the example of $\lambda = (5^3, 1)$ and use this to describe how our approach generalises to all partitions of this form. The Dynkin pyramid of λ is the following:

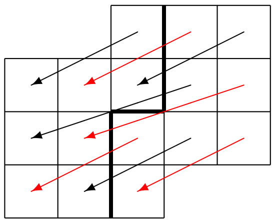
		7	11	14
1	4	8	12	15
2	5	9	13	16
3	6	10		

Now by reference to [52, Lem. 1.5.8] we see that if i labels a box on the central two rows, then an element A of the centraliser of $D^+(\lambda)$ may map v_i to any v_j with $j < i$, except if j labels a box on the other central row which is in the same column of i, or an even number of columns to the left. So in this example, v_{16} may be mapped to any v_j apart from v_1, v_8 and v_{15}. On the other hand if i labels a box on one of the skew rows, then v_i may be mapped to all v_j with $j < i$, except for $j = 2m + 1 - i$. This latter restriction is due to the fact that A is an orthogonal matrix, so all entries along its reverse diagonal are zero.

We can now obtain an element of the centraliser of $D^+(\lambda)$ with Jordan type (2^m) in the following way. Draw a line from box $2m$ to the rightmost box on the lowest row, and attach a negative sign to this line; in our example this means linking boxes 16 and 10. We then draw in all parallel lines from the lower central row to the lowest row, changing sign each time we draw a line that crosses the bold central line on the pyramid. In our example, this means linking 13 to 6 with positive sign, and 9 to 3 with negative sign. We then similarly draw lines from the top row to the upper central row, choosing signs so as to ensure that the diagram has rotational symmetry. In total we draw $\lambda_1 + \lambda_2$ lines in this way.

Next we draw a line from box $2m-1$ to the rightmost box on the lower central row which is not already the start point of a line. We note that this box will be $\frac{\lambda_1+\lambda_2}{2}$ columns to the left of the box $2m-1$; so this will always be possible, since $\lambda_1 \equiv \lambda_2$ so $\frac{\lambda_1+\lambda_2}{2}$ is odd. We then add all parallel lines between these two rows with the same sign rules as above; so we draw $\lambda_1 - \frac{\lambda_1+\lambda_2}{2}$ lines in this way. So in our example this means linking 15 to 5 and 12 to 2.

So the Dynkin pyramid now looks as follows:



Now clearly there are no paths of length 2 or longer along the arrows we have drawn, and the dimension of the image of the matrix corresponding to this diagram is equal to m. So we have constructed a matrix with Jordan type (2^m) which commutes with $D^+(\lambda)$. $\square$

Putting all this together, we are now in a position to prove Theorem 4.6.3.

Proof of Theorem 4.6.3: Let λ be an arbitrary orthogonal partition of n. We use Lemma 4.4.2 to break down λ into a sum of subpartitions $\lambda^0 \oplus \left(\bigoplus_{1\leqslant r\leqslant t} \lambda^r\right)$ as follows. Let $\Lambda = \{\lambda^r : 1 \leqslant r \leqslant t\}$ be a set containing some of these subpartitions.

To begin, let Λ be the empty set. Then for each even part λ_i, the orbit $\mathcal{O}_{(\lambda_i^2)}$ commutes with $\mathcal{O}_{(2^a,1^{2\lambda_i-2})}$ in $\mathfrak{so}_{2\lambda_i}(k)$ for even $0 \leqslant a \leqslant \lambda_i$, by Proposition 4.6.5. Additionally, for each pair of odd parts (λ_i, λ_j) with $\lambda_i \neq \lambda_j \bmod 4$, the subpartition $\mathcal{O}_{(\lambda_i,\lambda_j)}$ commutes with $\mathcal{O}_{(2^a,1^{\lambda_i+\lambda_j-2a})}$ in $\mathfrak{so}_{\lambda_i+\lambda_j}(k)$ for even $0 \leqslant a \leqslant \frac{\lambda_i+\lambda_j}{2}$, by Proposition 4.6.6. We note that in the first case $2\lambda_i \equiv 0 \bmod 4$, and similarly in the second case $\lambda_i + \lambda_j \equiv 0 \bmod 4$. Now if we can find a subpartition λ^r of either of these forms, we add it to the set Λ and remove its parts from λ. We repeat this as many times as possible.

Now we have an even number of parts of λ remaining, all equal modulo 4. Consider the subpartition λ^s consisting of the first four parts of λ. If λ^s is of the form described in one of Propositions 4.6.8, 4.6.9, 4.6.10, 4.6.11, 4.6.12 and

4.6.13, we call λ a *previously studied*, or PS, partition. When λ is PS, then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^a,1^{|\lambda^s|-2a})}$ in $\mathfrak{so}_{|\lambda^s|}(k)$ for all even $0 \leqslant a \leqslant \frac{|\lambda^s|}{2}$. So once again we add λ^s to the set Λ and remove its parts from λ.

Suppose conversely that λ^s is not PS; that is, λ^s is equal either to $(\lambda_1, \lambda_2, \lambda_2, \lambda_3)$ or $(\lambda_1, \lambda_2, \lambda_2, \lambda_2)$ with $\lambda_1 > \lambda_2 > \lambda_3$. Suppose that $\lambda \neq \lambda^s$, and let λ_5 be the first part of λ which is not in λ^s. We now consider $\lambda' = \lambda^s \oplus (\lambda_5)$.

To obtain a contradiction, assume that λ' does not contain a four-part PS subpartition. Then λ' cannot have only one or two distinct parts, since if this were the case it would contain a subpartition of the form considered in Proposition 4.6.8 or 4.6.9. Similarly if λ' had four distinct parts then it would contain a subpartition of the form described in Proposition 4.6.12. So λ' must have exactly three distinct parts, and clearly at least one part must occur with multiplicity at least 2. Now if this part were to occur with multiplicity exactly 2, then we would also find a second part in λ' with multiplicity 2, and thus a subpartition of λ' of the form described in Proposition 4.6.9. So we reduce to the case of $\lambda' = (\lambda_i, \lambda_i, \lambda_i, \lambda_j, \lambda_l)$, where we relax the requirement for the parts of λ' to be written in non-increasing order. Now if we have either $\lambda_j < \lambda_i$ or $\lambda_l < \lambda_i$ then we find a subpartition of the form considered in Proposition 4.6.13. Conversely if both $\lambda_j > \lambda_i$ and $\lambda_l > \lambda_i$ then we find a subpartition $(\lambda_j, \lambda_l, \lambda_i, \lambda_i)$ of the form considered in Proposition 4.6.10.

So we have a contradiction here, and deduce that λ' always contains a PS subpartition $\tilde{\lambda}$. So we add this subpartition $\tilde{\lambda}$ to Λ and remove its parts from λ in the same way as previously.

We repeat this process until we reach the last two (if m is odd) or four (if m is even) parts of λ; we denote the subpartition comprised of these parts by λ^0. If m is odd then we now have $\lambda^0 = (\lambda_1^0, \lambda_2^0)$ whose parts are not necessarily distinct. Then by Lemma 4.6.6 we see that $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{(2^a,1^{|\lambda|-2a})}$ for all even

$0 \leqslant a \leqslant \frac{|\lambda^0|-1}{2}$. Similarly if we have $\lambda^0 = (\lambda^0_1, \lambda^0_2, \lambda^0_3, \lambda^0_4)$ with parts not necessarily distinct, we write $\lambda^0 = (\lambda^0_1, \lambda^0_2) \oplus (\lambda^0_3, \lambda^0_4)$. By Lemmas 4.4.2 and 4.6.6 we then observe that $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{(2^a,1^{|\lambda^0|-a})}$ for all even $0 \leqslant a \leqslant \frac{|\lambda^0|-2}{2}$. It is possible that $\mathcal{O}_{\lambda^0}$ might commute with $\mathcal{O}_{(2^{|\lambda^0|/2})}$, but we do not consider this as we already know from Proposition 4.6.2 that $\mathcal{O}_{(2^m)}$ is not universally commuting.

So we may write $\lambda = \lambda^0 \oplus \left(\bigoplus_{\lambda^r \in \Lambda} \lambda^r\right)$. We now apply Lemma 4.4.2 to λ, and find that find that $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^a,1^{2m-2a})}$ for all even $0 \leqslant a < m$. Since we chose λ to be any orthogonal partition of $2m$, this means that $\mathcal{O}_{(2^a,1^{2m-2a})}$ must therefore be universally commuting for all even $0 \leqslant a < m$. □

4.6.4 UC partitions in $\mathfrak{so}_{2m+1}(k)$

We now move on to consider which orbits $\mathcal{O}_\mu$ are universally commuting in $\mathfrak{so}_{2m+1}(k)$. We recall from Proposition 4.6.1 that if $\mathcal{O}_\mu$ is UC, then μ must be of the form $(2^a, 1^{2m-2a+1})$ for a even. Now any orthogonal partition of $2m + 1$ must have an odd number of parts; so in addition to the results from the previous section we must consider which potentially UC orbits commute with $\mathcal{O}_\lambda$ in $\mathfrak{so}_{2m+1}(k)$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition of $2m + 1$ with three (not necessarily distinct) odd parts.

To begin we note the following basic results:

Lemma 4.6.14. *(i.) Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition of $2m + 1$ with three odd parts such there is a pair (λ_i, λ_j) for $i, j = 1, 2, 3$ with $\lambda_i \not\equiv \lambda_j \bmod 4$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^a,1^{2m-2a+1})}$ for all even $0 \leqslant a \leqslant m$.*

(ii.) Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a partition of $2m+1$ with $\nu_i \equiv 1 \bmod 4$ for $i = 1, 2, 3$. Then $\mathcal{O}_\nu$ commutes with $\mathcal{O}_{(2^a,1^{2m-2a+1})}$ for all even $0 \leqslant a \leqslant m$.

(iii.) Let $\pi = (\pi_1, \pi_2, \pi_3)$ be a partition of $2m+1$ with $\pi_i \equiv 3 \bmod 4$ for $i = 1, 2, 3$. Then $\mathcal{O}_\pi$ commutes with $\mathcal{O}_{(2^a,1^{2m-2a+1})}$ for all even $0 \leqslant a \leqslant m - 2$.

Proof. (i.) Let λ_i, λ_j be as in the statement of the lemma and let λ_l be the third part of λ. Then we may write $\lambda = (\lambda_i, \lambda_j) \oplus (\lambda_l)$. Then $\mathcal{O}_{(\lambda_i,\lambda_j)}$ commutes with $\mathcal{O}_{(2^a,1^{\lambda_i+\lambda_2-ja})}$ for all even $0 \leqslant a \leqslant \frac{\lambda_1+\lambda_2}{2}$, by Proposition 4.6.6; we note that since $\lambda_i + \lambda_j \equiv 0 \bmod 4$ we have $\frac{\lambda_i+\lambda_j}{2}$ even. Further, $\mathcal{O}_{(\lambda_3)}$ commutes with $\mathcal{O}_{(2^a,1^{\lambda_3-2a})}$ for all even a by Proposition 4.3.3. The desired result thus follows from Lemma 4.4.2.

(ii.) By Proposition 4.3.3 we see that $\mathcal{O}_{(\nu_i)}$ commutes with $\mathcal{O}_{(2^a,1^{\lambda_1-2a})}$ in $\mathfrak{so}_{\nu_i}(k)$ for all even $0 \leqslant a \leqslant \frac{\nu_i-1}{2}$. In particular $\mathcal{O}_{(\nu_i)}$ commutes with $\mathcal{O}_{(2^{(\nu_i-1)/2},1)}$. The result then follows by Lemma 4.4.2; since $\nu_i \equiv 1 \bmod 4$ then $2m+1 = \nu_1 + \nu_2 + \nu_3 \equiv 3 \bmod 4$, so any orthogonal square-zero partition of $2m+1$ must have at least three parts of size 1.

(iii.) Again we write $\pi = (\pi_1, \pi_2) \oplus (\pi_3)$. Then by Proposition 4.6.6 we see that $\mathcal{O}_{(\pi_1,\pi_2)}$ commutes with $\mathcal{O}_{(2^a,1^{\pi_1+\pi_2-2a})}$ in $\mathfrak{so}_{\pi_1+\pi_2}(k)$, for all even $0 \leqslant a \leqslant \frac{\pi_1+\pi_2-2}{2}$. Similarly $\mathcal{O}_{(\pi_3)}$ commutes with $\mathcal{O}_{(2^a,1^{\pi_3-2a})}$ in $\mathfrak{so}_{\pi_3}(k)$ for all even $0 \leqslant a \leqslant \frac{\pi_3-3}{2}$. The desired result then follows from Proposition 4.4.2.

□

So it now remains only to consider whether $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^m,1)}$ when $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\lambda_i \equiv 3 \bmod 4$ for $i = 1, 2, 3$. Once again we approach this via a series of lemmas for the various possible forms of λ.

Lemma 4.6.15. *Let $\lambda = (\lambda_1^3)$, with $\lambda_1 \equiv 3 \bmod 4$, be a partition of $2m+1$. Then λ commutes with $(2^m, 1)$.*

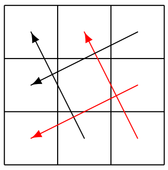
Proof. We illustrate this with the example of $\lambda = (3^3)$ in $\mathfrak{so}_9(k)$. The Dynkin pyramid of λ is as follows:

1	4	7
2	5	8
3	6	9

Then we find that a general element of the centraliser of $D^+(3^3)$ is of the following form:

$$A = \begin{pmatrix} 0 & a_1 & 0 & a_3 & a_4 & a_5 & a_6 & a_7 & 0 \\ & 0 & -a_1 & b_2 & b_3 & a_4 & b_5 & 0 & -a_7 \\ & & 0 & c_1 & b_2 & a_3 & 0 & -b_5 & -a_6 \\ & & & 0 & a_1 & 0 & -a_3 & -a_4 & -a_5 \\ & & & & 0 & -a_1 & -b_2 & -b_3 & -a_4 \\ & & & & & 0 & -c_1 & -b_2 & -a_3 \\ & & & & & & 0 & a_1 & 0 \\ & & & & & & & 0 & -a_1 \\ & & & & & & & & 0 \end{pmatrix}.$$

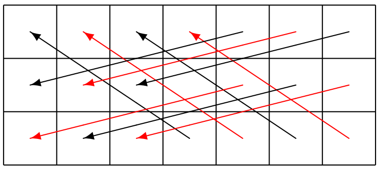
Now clearly we get a matrix with partition $(2^4, 1)$ by setting $a_5 \neq 0 \neq b_5$ and $a_i = b_i = c_i = 0$ otherwise. On the Dynkin pyramid this corresponds to drawing a line from the box labelled $2m+1$ to the box labelled by m; that is, the top box in the central column. So in our example we link boxes 9 and 4. We then draw in all parallel lines to the left of this line; in our example, joining boxes 6 and 1. Finally we link the box labelled $2m$ and the rightmost box in the lowest row which is not already the endpoint of a line- in the example boxes 8 and 3- and then draw in all parallel lines above and to the left of this line, giving a line between boxes 7 and 2 in this example. We set the lines originating in boxes $(2m+1)$ and $2m$ to have negative sign, the line originating in box $(2m-1)$ to have positive sign, and each further line has the opposite sign to the parallel line immediately to its right. Thus in the (3^3) example, the line from box 6 to box 1 has positive sign. So our Dynkin pyramid now looks like this:



It is easily observed that this generalises to larger cases, with the same sign rules. For example, in $\mathfrak{so}_{21}$ the partition (7^3) has the following Dynkin pyramid:

1	4	7	10	13	16	19
2	5	8	11	14	17	20
3	6	9	12	15	18	21

Drawing lines between boxes as described above now gives the following:



It is now easily seen that the matrix A defined in this way has partition $(2^{10}, 1)$ and commutes with $E_{(7^3)}$, as required. In general this method will allow us to draw m lines on the pyramid, none of which shares a start- or endpoint with another line. So the matrix F defined as previously by the lines on this pyramid has Jordan type $(2^m, 1)$. □

Lemma 4.6.16. *Let $\lambda = (\lambda_1, \lambda_2^2)$ be a partition of $2m+1$ with $\lambda_i \equiv 3 \bmod 4$ and $\lambda_1 > \lambda_2$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^m,1)}$.*

Proof. We illustrate this with the example of $\lambda = (7, 3^2)$. The Dynkin pyramid of λ is then as follows:

$$
\begin{array}{|c|c|c|c|c|c|c|}
\cline{3-5}
\multicolumn{2}{c|}{} & 3 & 6 & 9 & \multicolumn{2}{c}{} \\
\hline
1 & 2 & 4 & 7 & 10 & 12 & 13 \\
\hline
\multicolumn{2}{c|}{} & 5 & 8 & 11 & \multicolumn{2}{c}{} \\
\cline{3-5}
\end{array}
$$

A direct calculation then shows that a general element of the centraliser of $E_{(7,3^2)}$ is of the following form:

$$
A = \begin{pmatrix}
0 & a_1 & a_2 & 0 & a_4 & a_5 & a_6 & a_7 & a_8 & 0 & a_{10} & a_{11} & 0 \\
 & 0 & 0 & a_1 & 0 & a_2 & 0 & a_4 & -a_5 & -a_6 & -a_7 & 0 & -a_{11} \\
 & & 0 & 0 & 0 & c_3 & 0 & c_5 & c_6 & a_4 & 0 & a_7 & -a_{10} \\
 & & & 0 & 0 & 0 & a_1 & 0 & -a_2 & 0 & -a_4 & a_6 & 0 \\
 & & & & 0 & e_1 & 0 & c_3 & 0 & a_2 & -c_6 & a_5 & -a_8 \\
 & & & & & 0 & 0 & 0 & -c_3 & 0 & -c_5 & -a_4 & -a_7 \\
 & & & & & & 0 & 0 & 0 & -a_1 & 0 & 0 & -a_6 \\
 & & & & & & & 0 & -e_1 & 0 & -c_3 & -a_2 & -a_5 \\
 & & & & & & & & 0 & 0 & 0 & 0 & -a_4 \\
 & & & & & & & & & 0 & 0 & -a_1 & 0 \\
 & & & & & & & & & & 0 & 0 & -a_2 \\
 & & & & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & & & & 0
\end{pmatrix}.
$$

Now we note that by setting a_5, a_6, a_7, c_6 to be non-zero and imposing the extra condition that $a_6^2 + 2a_5a_7 = 0$ we obtain a matrix with the desired partition $(2^6, 1)$ via the following:

$$v_{13} \mapsto -a_5v_8 - a_6v_7 - a_7v_6 \mapsto -(a_6^2 + 2a_5a_7)v_1 = 0;$$

$$v_{12} \mapsto a_5v_5 + a_6v_4 + a_7v_3 \mapsto 0;$$

$$v_{11} \mapsto -c_6v_5 - a_7v_2 \mapsto 0;$$

$$v_{10} \mapsto -a_6v_2 \mapsto 0;$$

$$v_9 \mapsto c_6v_3 - a_5v_2 \mapsto 0;$$

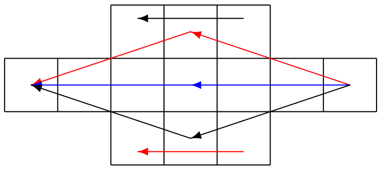
$$v_8 \mapsto a_7v_1 \mapsto 0;$$

$$a_5v_7 - a_6v_6 \mapsto 0.$$

We may for example set $(a_5, a_6, a_7, c_6) = (1, \sqrt{2}, -1, 1)$ to provide an explicit element of the centraliser of $D^+(7, 3^2)$ with this Jordan type.

For general m, we may find an element of the centraliser of $D^+(\lambda)$ with Jordan type $(2^m, 1)$ in a similar way, using the following process. On the Dynkin pyramid, begin by drawing lines from the box labelled $2m + 1$ to the boxes m, $(m + 1)$ and $(m + 2)$ of the central column of the pyramid, with coefficients $(-1, \sqrt{2}, 1)$ respectively. We then draw in all possible lines on the Dynkin pyramid that are parallel to these with the same coefficients up to their sign, which alternates between lines from horizontally adjacent boxes. Next, we draw lines from the boxes m, $(m+1)$ and $(m+2)$ to the box labelled 1 with coefficients $(-1, \sqrt{2}, 1)$; and add all possible lines parallel to these, with coefficients such that the diagram possesses the requires rotational antisymmetry. Finally we add in a line with coefficient ± 1 from the rightmost box of each short row of the Dynkin pyramid to the $(\frac{\lambda_2 - 1}{2})$th box of the same row, that is, the box of the same row which lies immediately to the left of the central column of the Dynkin pyramid. Again we then draw all parallel lines, such that the signs of the coefficients attached to these blocks alternate as we move along the row, and are chosen such that the diagram retains its rotational antisymmetry.

On the Dynkin pyramid below we draw the lines to and from the central column, as well as the lines contained solely in either of the short rows, for the example of $(7,3^2)$ in $\mathfrak{so}_{13}(k)$. We omit all other lines for clarity but note that they comprise all further possible lines that are parallel to those on the diagram.



Now on this diagram the only path along the arrows of length at least two is from v_{2m} to v_1 via the central column; but we have chosen the coefficients so that we have $v_{2m} \mapsto -v_{m+2} - \sqrt{2}v_{m+1} + v_m \mapsto 0$. We additionally have $v_{m+1} \mapsto \sqrt{2}v_1 \mapsto 0$ and $v_{m+2} + v_m \mapsto 0$. Since there are no other lines starting or ending in boxes $(2m+1)$, $(m+2)$, $(m+1)$, m or 1 we see that this gives a subpartition $(2^2,1)$ of the Jordan type of A. Furthermore, when we consider the remaining boxes on the Dynkin pyramid we observe that there are no further paths of length greater than one and half of these boxes are in the image of A. It follows that A has Jordan type $(2^m,1)$ as required. □

Remark 4.6.17. We remark that in the $(7,3^2)$ example above we can also find a matrix with partition $(2^6,1)$ by setting a_2 to be the only non-zero entry. However this only works for partitions of the form $(\lambda_1,(\lambda_1-4)^2)$, while the construction given above does indeed generalise.

Lemma 4.6.18. *Let $\lambda = (\lambda_1^2,\lambda_2)$ be a partition of $2m+1$ with $\lambda_1 > \lambda_2$ and $\lambda_i \equiv 3 \bmod 4$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(2^m,1)}$.*

Proof. Consider the example of $(7^2,3)$ in $\mathfrak{so}_{17}(k)$. The Dynkin pyramid of this partition is as follows:

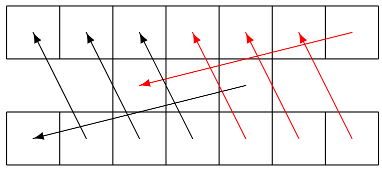
1	3	5	8	11	14	16
		6	9	12		
2	4	7	10	13	15	17

We then find that a general element of the centraliser is of the form

$$\begin{pmatrix}
0 & 0 & a_2 & a_3 & a_4 & a_5 & 0 & a_7 & a_8 & a_9 & a_{10} & a_{11} & 0 & a_{13} & a_{14} & a_{15} & 0 \\
 & 0 & b_1 & a_2 & 0 & b_4 & -a_4 & b_6 & b_7 & a_7 & 0 & b_{10} & -a_{10} & b_{12} & a_{13} & 0 & -a_{15} \\
 & & 0 & 0 & a_2 & 0 & a_3 & a_4 & a_5 & 0 & -a_7 & -a_8 & -a_9 & -a_{10} & 0 & -a_{13} & -a_{14} \\
 & & & 0 & b_1 & 0 & a_2 & 0 & b_4 & -a_4 & -b_6 & -b_7 & -a_7 & 0 & a_{10} & -b_{12} & -a_{13} \\
 & & & & 0 & 0 & 0 & a_2 & 0 & a_3 & -a_4 & -a_5 & 0 & a_7 & a_9 & a_{10} & 0 \\
 & & & & & 0 & 0 & 0 & f_3 & 0 & b_4 & 0 & a_5 & b_7 & a_8 & -b_{10} & -a_{11} \\
 & & & & & & 0 & b_1 & 0 & a_2 & 0 & -b_4 & a_4 & b_6 & a_7 & 0 & -a_{10} \\
 & & & & & & & 0 & 0 & 0 & -a_2 & 0 & -a_3 & a_4 & 0 & -a_7 & -a_9 \\
 & & & & & & & & 0 & 0 & 0 & -f_3 & 0 & -b_4 & -a_5 & -b_7 & -a_8 \\
 & & & & & & & & & 0 & -b_1 & 0 & -a_2 & 0 & -a_4 & -b_6 & -a_7 \\
 & & & & & & & & & & 0 & 0 & 0 & -a_2 & -a_3 & a_4 & 0 \\
 & & & & & & & & & & & 0 & 0 & 0 & 0 & -b_4 & -a_5 \\
 & & & & & & & & & & & & 0 & -b_1 & -a_2 & 0 & -a_4 \\
 & & & & & & & & & & & & & 0 & 0 & -a_2 & -a_3 \\
 & & & & & & & & & & & & & & 0 & -b_1 & -a_2 \\
 & & & & & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & & & & & & 0
\end{pmatrix}$$

The structure of the centraliser of a general λ of this form is similar. Now we note that by setting $a_3 \neq 0 \neq b_{10}$ and $a_i = b_i = f_i = 0$ otherwise, we get a matrix with partition $(2^8, 1)$. On the Dynkin pyramid, the entries equal to $\pm b_{10}$ correspond to drawing a line from the rightmost box on the top row to the box on the short row of the first column to the left of the central column; in this example, from 16 to 6. We then add in all possible lines parallel to this, which in our example means we join 12 to 1. We attach negative sign to the lines from the top row to the short row of the pyramid, and positive sign to the lines from the central row to the bottom row. Now we join the last box on the bottom row to the rightmost box of the top block which is not already the endpoint of a line; in this example this means joining 17 to 14. Once again we then add in all parallel lines; we attach a negative coefficient to the rightmost half of these lines, and positive sign to the remainder.

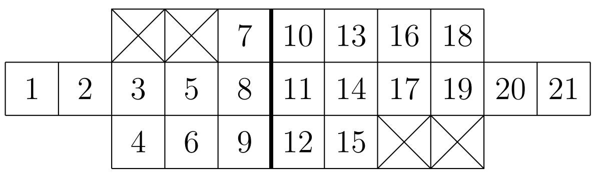
Consequently the Dynkin pyramid now looks as follows:



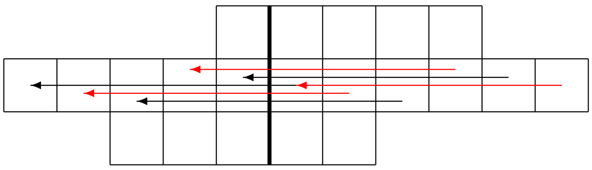
It is now clear that the matrix corresponding to this diagram has Jordan type $(2^8, 1)$ as required. Finally we note that this construction generalises easily to other cases. □

Lemma 4.6.19. *Let* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *be an orthogonal partition of* $2m + 1$ *with* $\lambda_1 > \lambda_2 > \lambda_3$ *and* $\lambda_i \equiv 3 \bmod 4$*. Then* λ *commutes with* $(2^m, 1)$*.*

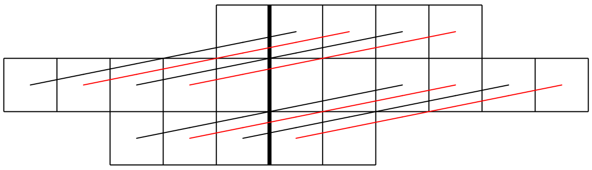
Proof. We use the example of the partition $(3, 7, 11)$ in $\mathfrak{so}_{21}(k)$ to describe the general construction of an element of the centraliser in $\mathfrak{so}_{2m+1}(k)$ of $D^+(\lambda)$. The Dynkin pyramid of $(3, 7, 11)$ is as follows:



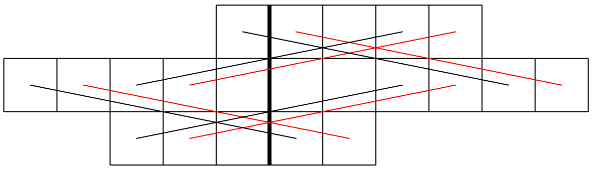
We draw a bold line immediately to the left of the central column, noting that this is not central as in many of the previous cases. We may now find an element of the centraliser of $D^+(\lambda)$ with Jordan type $(2^m, 1)$ in the following way. First we draw a line from box $2m+1$ to box $m+1$ with coefficient $-a_1$, and add all parallel lines, changing the sign of the coefficient whenever we draw a line that crosses the bold line. We maintain this sign rule for the remainder of the proof.



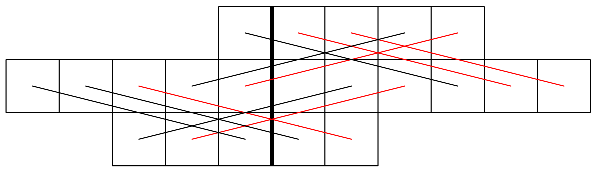
Next we draw a line from box $2m+1$ to box $m+2$ with coefficient $-a_2$, and draw all parallel lines.



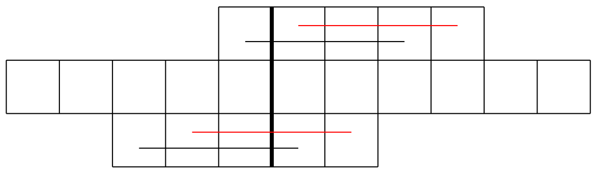
We then connect box $2m+1$ to box m with coefficient $\frac{-a_1^2}{2a_2}$. Here there is a slight subtlety when drawing the parallel lines, as illustrated in our example. Since boxes 6 and 7 are considered to be adjacent (by the properties of skew rows in the Dynkin pyramid), the lines from 20 to 7 and 19 to 6 are considered to be parallel.



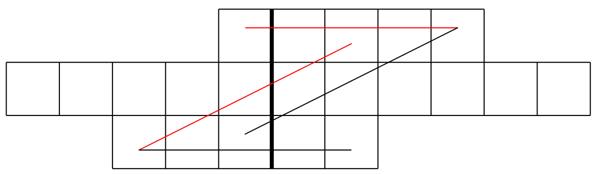
Now we draw a line from box $2m+1$ to box $m+3$ with coefficient a_3. Adding parallel lines gives the following:



We now consider lines drawn from the rightmost box of the upper row; in our example this is box 18. First we join this box to the top box in the central column of the pyramid, with coefficient b_1.



Finally we join the rightmost box of the upper row to the top box of the leftmost column which contains three boxes, with coefficient $\frac{-a_2 b_1}{a_3}$; and joining the rightmost box of the upper row with the lowest box of this column, with coefficient $\frac{a_2 b_1}{a_3}$. In our example this means joining box 18 with boxes 7 and 9, and adding further lines to ensure that the diagram possesses the required rotational antisymmetry.



Now the matrix A whose non-zero entries are defined by lines on these pyramids has Jordan type $(2^m, 1)$. Consider our example of $(11, 7, 3)$ in $\mathfrak{so}_{21}(k)$, and let $a_1 = \sqrt{-2}$, $a_2 = 2$, $a_3 = 1$ and $b_1 = 1$; then $\frac{-a_1^2}{2a_2} = 1$. Applying the matrix A to the basis elements $v_{21}, ..., v_1$ we obtain the following:

$$v_{21} \mapsto -v_{13} - v_{12} - \sqrt{-2}v_{11} - v_{10} \mapsto 0;$$
$$v_{20} \mapsto -v_{10} + v_9 + \sqrt{-2}v_8 + v_7 \mapsto 0;$$
$$v_{19} \mapsto v_7 - 2v_6 - \sqrt{-2}v_5 \mapsto 0;$$
$$v_{18} \mapsto -v_{10} + v_9 - v_8 - v_7 - 2v_5 \mapsto 0;$$
$$v_{17} \mapsto -v_6 + 2v_4 + \sqrt{-2}v_3 \mapsto 0;$$
$$v_{16} \mapsto v_7 + v_5 + 2v_3 \mapsto 0;$$
$$v_{15} \mapsto -v_6 + v_4 - v_3 - v_2 \mapsto 0;$$
$$v_{14} \mapsto v_4 - \sqrt{-2}v_2 \mapsto 0;$$
$$v_{13} \mapsto -v_4 - v_2 \mapsto 0;$$
$$v_{12} \mapsto v_4 + v_2 + v_1 \mapsto 0;$$
$$v_{11} - \sqrt{-2}v_{10} \mapsto 0.$$

We thus observe that the matrix A has Jordan type $(2^{10}, 1)$ as required. The general calculation is similar. □

We may now complete the classification of universally commuting orbits in $\mathfrak{so}_{2m+1}(k)$.

Theorem 4.6.20. *Let μ be an orthogonal partition of $2m+1$. Then $\mathcal{O}_\mu$ is universally commuting if and only if $\mu = (2^a, 1^{2m-2a+1})$ for even $0 \leqslant a \leqslant m$.*

Proof. By Proposition 4.6.1 we immediately note that $\mathcal{O}_\mu$ cannot be universally commuting if μ is not an orthogonal square-zero partition of $2m+1$.

Conversely let λ be a general orthogonal partition of $2m+1$; so λ has an odd number of parts. If λ has at least five parts, we follow the method described in the proof of Theorem 4.6.3 to progressively remove subpartitions λ^r of λ which have either two or four parts, such that $\mathcal{O}_{\lambda^r}$ commutes with $\mathcal{O}_{(2^a,1^{|\lambda^r|-2a})}$ in $\mathfrak{so}_{|\lambda^r|}(k)$, for all even $0 \leqslant a \leqslant \frac{|\lambda^r|}{2}$. As shown in the proof of Theorem 4.6.3, it is always possible to find a partition of one of these forms whenever λ has at least five parts. So we may reduce to considering subpartitions λ^0 which have either one or three parts, all equal modulo 4.

Now if $\lambda^0 = (\lambda^0_1)$ has only a single part we refer to Proposition 4.3.3 and find that $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{(2^a,1^{\lambda^0_1-2a})}$ in $\mathfrak{so}_{\lambda^0_1}(k)$ for all even $0 \leqslant a \leqslant \frac{\lambda^0_1-1}{2}$.

On the other hand, if λ^0 has three parts then it is of the form covered in one of Lemmas 4.6.15, 4.6.16, 4.6.18 or 4.6.19. In each case we find that $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{(2^a,1^{|\lambda^0|-2a})}$ in $\mathfrak{so}_{|\lambda^0|}(k)$ for all even $0 \leqslant a \leqslant \frac{|\lambda^0|-1}{2}$.

So we write $\lambda = \lambda^0 \oplus \left(\bigoplus_{\lambda^r \in \Lambda} \lambda^r\right)$, where Λ is the set of all the two-part or four-part partitions λ^r obtained above. Then applying Lemma 4.4.2 to λ we obtain the desired result. □

4.6.5 Partitions of the form $(3, 2^a, 1^b)$

In order to complete our examination of UC partitions for $\mathfrak{so}_{2m}(k)$ it now remains to consider partitions of the form $(3, 2^a, 1^{2m-3-4a})$, for a even. We begin by noting the following result of Oblak [41, Prop. 2.6]:

Lemma 4.6.21. *Suppose that $\lambda = (\lambda_1, ..., \lambda_t)$ and $\mu = (\mu_1, ..., \mu_s)$ are partitions such that $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$ in $\mathfrak{gl}_n(k)$. If $s \geqslant n - \frac{\lambda_t}{2}$, then $\mu_1 \leqslant 2$.*

We note that although Oblak considered this result only for $\mathfrak{gl}_n(k)$, Lemma 4.3.1 allows us to apply it to the case of $\mathfrak{so}_{2m}(k)$. We have the following immediate corollary:

Corollary 4.6.22 (Oblak's bound)**.** *Let $\mu = (3, 2^a, 1^{2m-4a-3})$ be an orthogonal partition of $2m$, and suppose that μ has at least $\frac{3m}{2}$ parts, that is, $2m-a-2 \geqslant \frac{3m}{2}$ (or equivalently, $a \leqslant \frac{m-4}{2}$). Then $\mathcal{O}_\mu$ is not UC in $\mathfrak{so}_{2m}(k)$.*

Proof. Consider $\lambda = (m, m)$. Then from Lemma 4.6.21 we have $n - \frac{\lambda_t}{2} = \frac{3m}{2}$; we note that this choice of λ gives us the largest possible value of λ_t. Consequently $\mathcal{O}_\lambda$ does not commute with $\mathcal{O}_\mu$, hence $\mathcal{O}_\mu$ is not UC. □

We say that an orthogonal partition of $2m$ of the form $(3, 2^a, 1^{2m-2a-3})$ satisfies Oblak's bound if $a > \frac{m-4}{2}$; thus a partition must satisfy this bound if it is to be universally commuting. A further immediate consequence of Oblak's bound is the following:

Corollary 4.6.23. *The partition $\mu = (3, 1^{2m-3})$ is not UC whenever $m > 3$.*

Proof. Since μ has no parts equal to 2, it follows immediately from Corollary 4.6.22 that μ is not UC whenever $m - 4 \geqslant 0$. □

In fact this example introduces a key difference between our analysis of the partitions $(3, 2^a, 1^{2m-3-4a})$ and $(2^b, 1^{2m-2b})$. When we considered the latter we looked for the maximal value of b such that the partition was UC. Now, however, we need to find the minimal even value of a such that $(3, 2^a, 1^{2m-3-4a})$ is UC.

To see why this is the case, consider the example of $\lambda = (7, 5)$ and $\mu = (3, 2^a, 1^{9-2a})$ in $\mathfrak{so}_{12}(k)$, for $a \in \{0, 2, 4\}$. If $a = 4$ then we can split μ into almost rectangular partitions of 7 and 5 by $(3, 2^2) \oplus (2^2, 1)$; and we have a similar result when $a = 2$. However if $a = 0$ then we note that this is not possible, and indeed a direct calculation shows that $\mathcal{O}_\lambda$ and $\mathcal{O}_\mu$ do not commute in this case. So when μ has more parts of size 2, it is closer to being (almost) rectangular and $\mathcal{O}_\mu$ is more likely to commute with a given λ.

So we now consider the partition of the form $(3, 2^a, 1^{2m-2a-3})$ such that a is even and $a > \frac{m-4}{2}$. Now if $m \equiv 0 \pmod 4$ then this is equivalent to the condition that $a \geqslant \frac{m}{2} = 2\lfloor \frac{m}{4} \rfloor$. Otherwise we see that $a \geqslant 2\lceil \frac{m-4}{4} \rceil = 2\lfloor \frac{m}{4} \rfloor$.

So we now consider which orbits $\mathcal{O}_\lambda$ commute with $\mathcal{O}_\mu$ when $\mu = (3, 2^{2\lfloor \frac{m}{4} \rfloor}, 1^{2m-3-4\lfloor \frac{m}{4} \rfloor})$. We begin with the following result:

Proposition 4.6.24. *Let $\lambda = (m, m)$ and $\mu = (3, 2^a, 1^{2m-2a-3})$, with a even. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_\mu$ if $a \geqslant 2\lfloor \frac{m}{4} \rfloor$.*

Proof. We begin by describing some examples which illustrate the general approach.

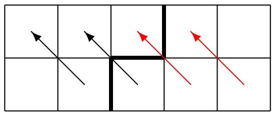
Suppose first that $m \equiv 1 \bmod 4$, and consider the example of $\lambda = (5,5)$ in $\mathfrak{so}_{10}(k)$. Then a general element of the centraliser is of the following form:

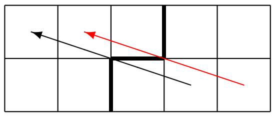
$$A = \left(\begin{array}{ccccc|ccccc}
0 & 0 & a_2 & a_3 & a_4 & & a_6 & a_7 & a_8 & 0 \\
 & 0 & b_1 & a_2 & & -a_4 & b_5 & a_6 & 0 & -a_8 \\
 & & 0 & & a_2 & a_3 & -a_4 & 0 & -a_6 & -a_7 \\
 & & & 0 & b_1 & a_2 & 0 & -a_4 & -b_5 & -a_6 \\
 & & & & 0 & 0 & -a_2 & -a_3 & a_4 & \\
\hline
 & & & & & 0 & -b_1 & -a_2 & & -a_4 \\
 & & & & & & 0 & & -a_2 & -a_3 \\
 & & & & & & & 0 & -b_1 & -a_2 \\
 & & & & & & & & 0 & \\
 & & & & & & & & & 0
\end{array}\right).$$

We see immediately from this matrix that v_{10} can be mapped to all basis elements except v_1, v_5 and v_9. Below we draw the Dynkin pyramid of $(5,5)$ and highlight the boxes corresponding to these basis elements:

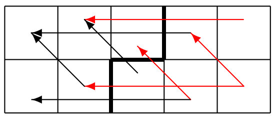
1	3	5	7	9
2	4	6	8	10

Now we can obtain an element of the centraliser with Jordan type $(2^4, 1^2)$ by mapping v_{10} to v_7, and drawing all parallel lines to the left; this corresponds to setting $a_3 \neq 0$ in the matrix above and $a_i = b_i = 0$ otherwise. Similarly we can obtain a matrix with Jordan type $(2^2, 1^6)$ by mapping v_{10} to v_3, corresponding to setting $a_7 \neq 0$ and $a_i = b_i = 0$ otherwise. We illustrate this on the Dynkin pyramids below:





Now in the first case we can obtain a matrix with Jordan type $(3, 2^2, 1^3)$ by additionally setting $a_6 \neq 0$; this corresponds to drawing horizontal lines linking the leftmost and rightmost diagonal lines. This gives the following lines on the Dynkin pyramid:

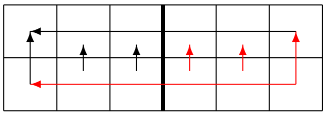


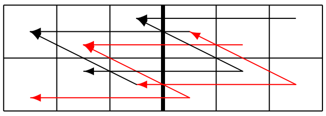
We note in particular that this maps v_{10} to v_1 in two steps via two different routes but with the same sign; so they do not cancel out and we do indeed obtain a part of size 3 in the Jordan type of the matrix. However we note that we cannot apply the same process to obtain an element of Jordan type $(3, 1^7)$ from one of $(2^2, 1^6)$, since we know from Oblak's bound that $\mathcal{O}_{(3,1^7)}$ does not commute with $\mathcal{O}_{(5^2)}$. In this case we would map v_{10} to v_8; but since this corresponds to mapping each basis element only one step left in the Dynkin pyramid, this forces v_{10} to be contained in a block of size at least 5 in the Jordan type of the corresponding matrix. It is easily seen that this approach generalises to all $m \equiv 1 \bmod 4$.

If m is even then we can use the same process. Consider the example of $m = 6$; then the Dynkin pyramid of (6^2) is as follows, where we highlight the basis elements that v_{12} cannot map to:

1	3	5	7	9	11
2	4	6	8	10	12

Now we can obtain elements in the centraliser of $D^+(6^2)$ with Jordan types (2^6), $(2^4, 1^4)$ and $(2^2, 1^8)$ by mapping v_{12} to v_{11}, v_7 and v_3 respectively. We can then use the first two of these to obtain elements with Jordan types $(3, 2^4, 1)$ and $(3, 2^2, 1^5)$ respectively, as shown in the following diagrams.





Once again it is clear how this approach generalises to all even values of m.

Finally suppose that $m \equiv 3 \bmod 4$. Then by Proposition 4.3.3 we see that $\mathcal{O}_{(m)}$ commutes in $\mathfrak{so}_m(k)$ with both $\mathcal{O}_{(3,2^{(m-3)/2})}$ and $\mathcal{O}_{(2^a,1^{m-2a})}$ for even $0 \leqslant a \leqslant \frac{m-3}{2}$. Thus by Lemma 4.4.2 we see that $\mathcal{O}_{(m^2)}$ commutes with $\mathcal{O}_{(3,2^b,1^{2m-2b-3})}$ for all even $\frac{m-3}{2} \leqslant b \leqslant m-3$.

We note that when b is minimal according to this bound we obtain the Jordan type of $(3, 2^{\frac{m-3}{2}}, 1^m)$; since $\frac{m-3}{2} \geqslant \frac{m-4}{2}$ this satisfies Oblak's bound. Furthermore if we reduce the number of parts of size 2 then we obtain a partition with at most $\frac{m-7}{2}$ parts of size 2, which thus contravenes Oblak's bound and cannot be UC. Thus we obtain all partitions that satisfy Oblak's bound by this method. $\square$

We now move on to examine more general two-part partitions. Suppose now that $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \neq \lambda_2$. We note the following result immediately:

Proposition 4.6.25. *Let* $\lambda = (\lambda_1, \lambda_2)$ *with* $\lambda_1 > \lambda_2$ *and* $\lambda_2 \equiv 3 \bmod 4$*. Then* $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_{(3,2^a,1^{2m-2a-3})}$ *whenever* $a \geqslant \frac{\lambda_2-3}{2}$*.*

Proof. We note that $\mathcal{O}_{(\lambda_2)}$ commutes with $\mathcal{O}_{(3,2^{(\lambda_2-3)/2})}$ since $(3, 2^{\frac{\lambda_2-3}{2}})$ is almost rectangular, and that $\mathcal{O}_{(\lambda_1)}$ commutes with $\mathcal{O}_{(2^b,1^{\lambda_1-2b})}$ for all $b \geqslant 0$, by Proposition 4.3.3. The result then follows by Lemma 4.4.2. □

In particular suppose that λ_2 is maximal, that is, $\lambda_2 = m - 1$. Then by the above we note that $\lambda = (m+1, m-1)$ commutes with $(3, 2^{\frac{m-4}{2}}, 1^{m+1})$ which has exactly $\frac{3m}{2}$ parts. It follows that whenever a partition λ has two parts with the smaller part equal to 3 modulo 4, then λ commutes with all partitions of the form $(3, 2^a, 1^{2m-2a-3})$ that Oblak's bound allows to be possibly UC.

So it remains for us to consider the case of $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_2 \equiv 1 \bmod 4$. We obtain the following result:

Proposition 4.6.26. *Let* $\lambda = (\lambda_1, \lambda_2)$ *with* $\lambda_1 > \lambda_2$ *and* $\lambda_2 \equiv 1 \bmod 4$*. Then* $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_{(3,2^a,1^{2m-2a-3})}$ *whenever a is even and* $a \geqslant \frac{\lambda_2-1}{2}$*.*

Proof. We prove this result by considering the structure of $\mathfrak{c}_\mathfrak{g}(D^+(\lambda))$ in more detail. To begin we look at the centraliser of $D^+(11,5)$ in $\mathfrak{so}_{16}(k)$, which exhibits many of the essential properties of this type of centraliser. In particular we note that all matrices of the following form are members of $\mathfrak{c}_\mathfrak{g}(D^+(11,5)$:

$$
\left(\begin{array}{ccc|cccccccccc|ccc}
0 & 0 & 0 & 0 & 0 & a_5 & -a_5 & a_7 & a_8 & a_9 & -a_9 & a_{11} & a_{12} & 0 & a_{14} & 0 \\
 & 0 & & & & 0 & 0 & a_5 & -a_5 & -a_7 & -a_8 & -a_9 & a_9 & -a_{11}-a_{12} & 0 & -a_{14} \\
 & & 0 & & & & & 0 & 0 & -a_5 & a_5 & a_7 & a_8 & 0 & a_{11}+a_{12} & 0 \\
\hline
 & & & 0 & & & & & & d_6 & -d_6 & a_5 & 0 & -a_8 & -a_9 & -a_{12} \\
 & & & & 0 & & & & & -d_6 & d_6 & 0 & -a_5 & -a_7 & a_9 & -a_{11} \\
 & & & & & 0 & & & & & 0 & -d_6 & d_6 & -a_5 & a_8 & a_9 \\
 & & & & & & 0 & & & 0 & & d_6 & -d_6 & a_5 & a_7 & -a_9 \\
 & & & & & & & 0 & 0 & & & & & & a_5 & -a_8 \\
 & & & & & & & & 0 & & & & & & -a_5 & -a_7 \\
 & & & & & & & & & 0 & & & & & & a_5 \\
 & & & & & & & & & & 0 & & & & & -a_5 \\
 & & & & & & & & & & & 0 & & & & 0 \\
 & & & & & & & & & & & & 0 & & & 0 \\
\hline
 & & & & & & & & & & & & & 0 & & 0 \\
 & & & & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & & & & & 0
\end{array}\right)
$$

From this example we observe that a general element of the centraliser of (λ_1, λ_2) can be written in block form as

$$
A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ & A_{2,2} & -A_{1,2}^{\mathrm{st}} \\ & & -A_{1,1}^{\mathrm{st}} \end{pmatrix}
$$

where the blocks $A_{1,1}$ and $A_{1,3}$ are $\frac{\lambda_1-\lambda_2}{2} \times \frac{\lambda_1-\lambda_2}{2}$ matrices, $A_{1,2}$ is of size $\frac{\lambda_1-\lambda_2}{2} \times 2\lambda_2$, and $A_{2,2}$ is of size $2\lambda_2 \times 2\lambda_2$. These blocks are indicated above for the example of $\lambda = (11, 5)$.

Now we can always find a part of size 3 in the Jordan type of A by setting $a_{m,2m} \neq 0 \neq a_{m+1,2m}$ to be non-zero, and $a_{i,2m} = 0$ for $i > m+1$. If $\lambda_1 \equiv 1 \bmod 4$

we have the additional condition that $a_{m,2m} = -a_{m+1,2m}$. In the $(11, 5)$ example above, these entries are denoted by a_7 and a_8. On the Dynkin pyramid this corresponds to drawing lines from box $2m$ to both boxes m and $m+1$ of the central column, and then from these boxes to box 1. Now if all other entries of the matrix A are zero we see that $v_{2m} \mapsto -a_{m+1,2m}v_{m+1} - a_{m,2m}v_m \mapsto -2a_{m,2m}a_{m+1,2m}v_1 \mapsto 0$. So we obtain a part of size at least 3 in the Jordan type of A; if $a_{i,2m} = 0$ for all $i > m + 1$ (that is, $a_{m+1,2m}$ is the lowest non-zero entry in the last column) then we obtain a part of size exactly 3. We use this throughout the rest of the proof.

In the $(11, 5)$ example above we note that $\lambda_1 - 1 = 2\lambda_2$; and neither of the blocks $A_{1,3}$ nor $A_{2,2}$ contain entries which depend on a_7 or a_8. Following this observation we now consider two separate cases.

Suppose first that $\lambda_1 - 1 \geqslant 2\lambda_2$; in this case $A_{2,2}$ will not depend on $a_{m,2m}$ and $a_{m+1,2m}$. If m is even then $A_{1,3}$ will be quasi-Toeplitz, with entries of one subdiagonal equal to $\pm(a_{m,2m} + a_{m+1,2m})$; compare what happens to a_{11} and a_{12} in the $(11, 5)$ example above. Conversely if m is odd then $A_{1,3}$ will not contain any entries depending on $a_{m,2m}$ and $a_{m+1,2m}$; see a_9 in the $(11, 5)$ example.

Now suppose that the only non-zero entries of A are those equal to $\pm a_{m,2m}$; in our $(11, 5)$ example, these are denoted by a_7 and a_8. We then find that the only non-zero entries of block $A_{1,2}$ occur as sub-blocks of size 1×2 of the form $\pm(a_{m,2m} \;\; a_{m+1,2m})$. In particular we have $\frac{\lambda_2+1}{2}$ sub-blocks of this form in $A_{1,2}$, and consequently $\frac{\lambda_2+1}{2}$ sub-blocks of size 2×1 in $-A_{1,2}^{\mathrm{st}}$. Two of these sub-blocks, those in the first row and last column, give rise to a single part of size 3 in the Jordan type of A, while each of the remainder gives a part of size 2 since the columns of the 2×1 sub-blocks are linearly dependent. So the Jordan type of A is $(3, 2^{\lambda_2-1}, 1^{2m-2\lambda_2-1})$.

This is easily seen in our $(11, 5)$ example. We have three blocks of the form $\pm(a_7 \;\; a_8)$ in $A_{1,2}$ and a further three of the form $\pm(a_7 \;\; a_8)^{\mathrm{t}}$ in $-A_{1,2}^{\mathrm{st}}$. So we have four of these blocks which do not lie in the first row or last column; hence if these are the only non-zero entries of A the Jordan type of A is $(3, 2^4, 1^5)$.

In order to obtain a matrix with more parts of size 2 in its Jordan type, we observe that we may take a 1×2 sub-block as described above in $A_{1,2}$ and add further entries to A above this sub-block so that the two columns are no longer linearly dependent. To see this more clearly, consider the matrix A obtained from our $(11, 5)$ example with $a_7 = a_8 = a_9 = 1$ and all other entries equal to zero. So instead of the 2×1 sub-blocks considered previously we now have two 2×2 sub-blocks of the form $\pm \left(\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Clearly the columns of this block are linearly independent; so this gives rise to a subpartition $(2, 2)$ of the Jordan type of A where previously we had only a single part of size 2. Consequently in our example we find that A has a Jordan type of $(3, 2^6, 1)$.

To obtain a matrix whose Jordan type has fewer parts of size 2, we consider adding non-zero entries in the block $A_{2,2}$. Again we illustrate this with our $(11, 5)$ example. Suppose first that $a_7 = -a_8 = 1$ and all other entries are zero; then add $d_6 = 1$. We now note that if we set $a_{11} = a_{12} = \frac{1}{2}$ then the 10th and 11th columns are now both linear multiples of the 14th column. Similarly if we now add $a_{14} = \frac{1}{2}$ then the 12th and 13th columns are both linear multiples of the 15th. So the only parts of size 2 in the Jordan type of A now arise from the 14th and 15th columns; thus the Jordan type of A is now $(3, 2^2, 1^9)$.

In general we may follow this process until all the columns of our matrix A are either zero, contain non-zero entries only in the first row, or are multiples of the rightmost $\frac{\lambda_2+1}{2}$ columns. Consequently the Jordan type of A can have only $\frac{\lambda_2+1}{2}$ parts of size 2 or greater; then the rightmost column gives rise to a part of size 3, and this is the only part of size at least 3 as previously discussed. So we have indeed constructed a matrix in the centraliser of $D^+(\lambda)$ with the required Jordan type $(3, 2^{\frac{\lambda_2-1}{2}}, 1^{2m-\lambda_2-2})$.

Suppose now that $\lambda_1 - 1 < 2\lambda_2$. We consider first the specific case of $m \equiv 2 \bmod 4$ and $\lambda = (m+1, m-1)$. To illustrate this case we describe the example of $\lambda = (7,5)$ in $\mathfrak{so}_{12}(k)$. A general element A of $\mathfrak{c}_{\mathfrak{g}}(D^{+}(7,5))$ is of the following form:

$$\left(\begin{array}{c|cccccccccc|c}
0 & a_1 & a_2 & a_3 & -a_3 & a_5 & a_6 & a_7 & -a_7 & a_9 & a_{10} & 0 \\
\hline
 & 0 & 0 & b_2 & a_2-b_2 & a_3 & 0 & b_6 & -a_6-b_6 & -a_7 & 0 & -a_{10} \\
 & & 0 & a_1-b_2 & b_2 & 0 & -a_3 & -a_5-b_6 & b_6 & 0 & a_7 & -a_9 \\
 & & & 0 & 0 & b_2 & a_2-b_2 & -a_3 & 0 & -b_6 & a_6+b_6 & a_7 \\
 & & & & 0 & a_1-b_2 & b_2 & 0 & a_3 & a_5+b_6 & -b_6 & -a_7 \\
 & & & & & 0 & 0 & -b_2 & -a_2+b_2 & a_3 & 0 & -a_6 \\
 & & & & & & 0 & -a_1+b_2 & -b_2 & 0 & -a_3 & -a_5 \\
 & & & & & & & 0 & 0 & -b_2 & -a_2+b_2 & a_3 \\
 & & & & & & & & 0 & -a_1+b_2 & -b_2 & -a_3 \\
 & & & & & & & & & 0 & 0 & -a_2 \\
 & & & & & & & & & & 0 & -a_1 \\
\hline
 & & & & & & & & & & & 0
\end{array}\right)$$

From this we observe that the submatrix $A_{2,2}$ is block-quasi-Toeplitz with blocks of size 2×2, and that one block superdiagonal contains entries that depend on $a_{m,2m}$ and $a_{m+1,2m}$; in this example, these correspond to a_5 and a_6.

Now in this example we note that $\frac{\lambda_1 - \lambda_2}{2} = 1$, since $(\lambda_1, \lambda_2) = (m+1, m-1)$. and as a consequence of this we find that the blocks of the first superdiagonal of $A_{2,2}$ which lies entirely in the top half of the matrix A contain entries which depend on a_5, a_6 and an additional term b_6. In particular, we observe that this superdiagonal has $\frac{m-1}{2}$ blocks.

First we note that if $a_5 = a_6 = 1$ and all other entries of A are equal to zero, then the first row and last column of A give rise to a part of size 3 in the Jordan type of A as described previously. Furthermore, each of the blocks in $A_{2,2}$ on the superdiagonal described above gives rise to a subpartition $(2,2)$ as the columns are linearly independent. So we have obtained a matrix in the centraliser of $D^+(7,5)$ with Jordan type $(3,2^4,1)$.

On the other hand, suppose now that $a_5 = a_6 = 2$ and $b_6 = -1$. Then the blocks described above are now of the form $\pm\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. Since the columns are now linearly dependent each block gives a subpartition of the Jordan type of A of the form $(2,1^2)$, and hence the Jordan type of A is now $(3,2^2,1^5)$. Finally we note that if we now add in $a_7 \neq 0$ then the eight and ninth columns, and similarly the tenth and eleventh columns, are no longer linearly dependent; so this gives an alternative matrix with Jordan type $(3,2^4,1)$. These constructions may be generalised to the case of $(m+1,m-1)$ for larger m.

Finally suppose that $\lambda = (\lambda_1,\lambda_2)$ with $\lambda_1 - 1 < 2\lambda_2$ and $\lambda_1 - \lambda_2 > 2$. We illustrate this with the example of $m = 7$ and $\lambda = (9,5)$, and note that a general element A of the centraliser of $D^+(9,5)$ in $\mathfrak{so}_{14}(k)$ is of the following form:

$$\left(\begin{array}{cc|cccccccccc|cc}
0 & a_1 & a_2 & -a_2 & a_4 & a_5 & a_6 & -a_6 & a_8 & a_9 & a_{10} & -a_{10} & a_{12} & 0 \\
 & 0 & a_1 & a_1 & a_2 & -a_2 & a_4 & a_5 & -a_6 & a_6 & -a_8 & -a_9 & 0 & -a_{12} \\
\hline
 & & 0 & & c_2 & a_1 - c_2 & a_2 & & c_6 & -a_5 - c_6 & a_6 & 0 & a_9 & a_{10} \\
 & & & 0 & a_1 - c_2 & c_2 & & -a_2 & -a_4 - c_6 & c_6 & 0 & -a_6 & a_8 & -a_{10} \\
 & & & & 0 & & c_2 & a_1 - c_2 & -a_2 & 0 & -c_6 & a_5 + c_6 & -a_6 & -a_9 \\
 & & & & & 0 & a_1 - c_2 & c_2 & 0 & a_2 & a_4 + c_6 & -c_6 & a_6 & -a_8 \\
 & & & & & & 0 & 0 & -c_2 & c_2 - a_1 & a_2 & & -a_5 & a_6 \\
 & & & & & & & 0 & c_2 - a_1 & -c_2 & & -a_2 & -a_4 & -a_6 \\
 & & & & & & & & 0 & & -c_2 & c_2 - a_1 & a_2 & -a_5 \\
 & & & & & & & & & 0 & c_2 - a_1 & -c_2 & -a_2 & -a_4 \\
 & & & & & & & & & & 0 & & -a_1 & a_2 \\
 & & & & & & & & & & & 0 & -a_1 & -a_2 \\
\hline
 & & & & & & & & & & & & 0 & -a_1 \\
 & & & & & & & & & & & & & 0
\end{array}\right)$$

We now look more closely at the block superdiagonals of the submatrix $A_{2,2}$. The longest block superdiagonal which lies entirely in the upper half of the matrix A is formed of the entries which depend on c_6 in this example, and contains $\frac{\lambda_2-1}{2} = 2$ blocks; we denote this block superdiagonal by S_1. We further note that entries equal to $a_{m,2m}$ and $a_{m+1,2m}$, in this example denoted by a_6 and $-a_6$ respectively, occur in a shorter block superdiagonal which we denote S_2. If $\lambda_1 \equiv 1 \bmod 4$, as in this example, the entries depending on $a_{m,2m}$ and $a_{m+1,2m}$ are found on the main diagonal of the blocks of S_2; if $\lambda_1 \equiv 3 \bmod 4$ then these entries occur on the reverse diagonal of each block of the S_2.

Now in the $(9,5)$ example we may construct a matrix in $\mathfrak{c}_{\mathfrak{g}}(D^+(9,5))$ with Jordan type $(3,2^2,1)$ in the following way. We begin by setting $a_6 \neq 0$ and $c_6 = -a_6$; then we add $a_8 = -a_6$ and set all remaining entries of A to be equal to zero. We now note that all non-zero columns of A can be written as a linear combination of the eighth, tenth, twelfth and fourteenth columns. The eighth and fourteenth columns are those that give rise to the part of size 3 in the Jordan type of A; the tenth and twelfth are the right–hand columns of the blocks of the superdiagonal S_1, and these thus both give rise to parts of size 2 in the Jordan type of A. Consequently the Jordan type of this matrix is $(3,2^2,1^7)$.

If we now add $a_9 = a_6$ to the matrix with Jordan type $(3,2^2,1^7)$ obtained above, however, we find that the ninth and tenth columns are now linearly independent; as are the twelfth and thirteenth columns. So this gives a matrix with Jordan type $(3,2^4,1^3)$.

Finally we observe that we may apply the same process in larger cases to obtain a matrix with Jordan type $(3,2^{\frac{\lambda_2-1}{2}},1^{2m-\lambda_2-2})$; and similarly, matrices with more parts of size 2 in their Jordan type may be obtained by adding more non-zero entries above the superdiagonal S_2. □

Now combining the results of Propositions 4.6.24, 4.6.25 and 4.6.26 immediately gives the following result:

Corollary 4.6.27. *Let* $\lambda = (\lambda_1, \lambda_2)$ *be any orthogonal two-part partition of* $2m$ *and let* $\mu = (3, 2^a, 1^{2m-2a-3})$ *with* a *even. Then* λ *commutes with* μ *whenever* $a \geqslant 2\lfloor \frac{\lambda_2}{4} \rfloor$.

We now move on to consider when $\mathcal{O}_{(3,2^a,1^{2m-2a-3})}$ commutes with $\mathcal{O}_\lambda$ where λ is a general orthogonal partition. Immediately we have the following:

Proposition 4.6.28. *Let* $\lambda = (\lambda_1, ..., \lambda_t)$ *be a general orthogonal partition of* $2m$ *with* $\lambda_1 \geqslant ... \geqslant \lambda_t$*, and let* $\mu = (3, 2^a, 1^{2m-2a-3})$ *with* a *even. Then* $\mathcal{O}_\lambda$ *commutes with* $\mathcal{O}_\mu$ *if* $2\lfloor \frac{m}{4} \rfloor \leqslant a \leqslant m - 3$.

Proof. The lower bound on a here is simply Oblak's bound. To obtain the upper bound we write $\lambda = \lambda^0 \oplus \lambda'$, where we choose λ^0 in the following way:

(i.) If λ contains a pair of even parts λ_i, let $\lambda^0 = (\lambda_i, \lambda_i)$. Then $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{(3,2^b,1^{2\lambda_i-2b-3})}$ in $\mathfrak{so}_{2\lambda_i}(k)$ for all even b such that $2\lfloor \frac{\lambda_i}{2} \rfloor \leqslant b \leqslant \lambda_i - 2$ by Proposition 4.6.24.

(ii.) If λ contains a pair of odd parts (λ_i, λ_j) such that $\lambda_i \not\equiv \lambda_j \bmod 4$, then we take $\lambda^0 = (\lambda_i, \lambda_j)$. Then $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{(3,2^b,1^{|\lambda^0|-2b-3})}$ in $\mathfrak{so}_{|\lambda^0|}(k)$ for all even b such that $2\lfloor \frac{\lambda_j}{2} \rfloor \leqslant b \leqslant \frac{\lambda_i+\lambda_j-4}{2}$ by Proposition 4.6.26.

(iii.) If $\lambda_i \equiv 3 \bmod 4$ for all $1 \leqslant i \leqslant t$, then let $\lambda^0 = (\lambda_t)$. In this case $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{(3,2^{(\lambda_t-3)/2})}$ by Proposition 4.3.3.

(iv.) If $\lambda_i \equiv 1 \bmod 4$ for all $1 \leqslant i \leqslant t$, then take $\lambda^0 = (\lambda_{t-1}, \lambda_t)$. Then $\mathcal{O}_{\lambda^0}$ commutes with $\mathcal{O}_{(3,2^b,1^{|\lambda^0|-2b-3})}$ in $\mathfrak{so}_{2\lambda_i}(k)$ for all even b such that $2\lfloor \frac{\lambda_t}{2} \rfloor \leqslant b \leqslant \frac{\lambda_{t-1}+\lambda_t-6}{2}$ by Proposition 4.6.26.

Now in all of these cases, we note that $\mathcal{O}_{\lambda'}$ commutes with $\mathcal{O}_{(2^c,1^{|\lambda'|-2c})}$ in $\mathfrak{so}_{|\lambda'|}(k)$ for all even c such that $0 \leqslant c < \frac{|\lambda'|}{2}$, by Theorem 4.6.3, or by Theorem 4.6.20 in case (iii.). The result then follows from Lemma 4.4.2. $\square$

We then obtain the following immediate corollary:

Corollary 4.6.29. *Let $\mu = (3, 2^a, 1^{2m-2a-3})$ with a even, $a \geqslant 2\lfloor \frac{m}{4} \rfloor$ (so that μ satisfies Oblak's bound), and $2m - 2a - 3 > 1$. Then $\mathcal{O}_\mu$ is UC in $\mathfrak{so}_{2m}(k)$.*

We note that the only orthogonal partition of $2m$ of the form $(3, 2^a, 1^{2m-2a-3})$ which is not described in Corollary 4.6.29 is the partition $(3, 2^{m-2}, 1)$ with m even. In order to explore this exception, let λ be a four-part partition of $2m$, with m even. It is clear that if λ contains a part λ_i such that $\lambda_i \equiv 3 \bmod 4$, then $\mathcal{O}_{(\lambda_i)}$ commutes with $\mathcal{O}_{(3,2^{(\lambda_i-3)/2})}$ in $\mathfrak{so}_{\lambda_i}(k)$ by Proposition 4.3.2. Thus $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(3,2^{m-2},1)}$ in $\mathfrak{so}_{2m}$ by Lemma 4.4.2. If λ contains a pair of even parts equal to λ_j, then we obtain a similar result using Propositions 4.6.6 and 4.6.24 to write $(3, 2^{m-2}, 1) = (3, 2^{\lambda_j-2}, 1) \oplus (2^{m-\lambda_j})$. So if λ is a four-part partition such that $\mathcal{O}_\lambda$ does not commute with $\mathcal{O}_{(3,2^{m-2},1)}$, then all parts of λ must be equal to 1 modulo 4.

Now the following lemmas allow us to narrow down further the partitions λ such that $\mathcal{O}_\lambda$ does not commute with $\mathcal{O}_{(3,2^{m-2},1)}$.

Lemma 4.6.30. *Let $\lambda = (\lambda_1^3, \lambda_2)$ be a four-part orthogonal partition of $2m$, not necessarily written in non-increasing order, with $\lambda_i \equiv 1 \bmod 4$ for $i = 1, 2$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(3,2^{m-2})}$ in $\mathfrak{so}_{2m}(k)$.*

Proof. Write $\lambda = (\lambda_1^3) \oplus (\lambda_2)$. Then we observe that both (λ_1^3) and $(3, 2^{\frac{3(\lambda_1-1)}{2}})$ are almost rectangular partitions of $3\lambda_1$. Let $E = D^+(3\lambda_1)$; then we observe that, similarly to Proposition 4.3.3, we can write $D^+(\lambda_1^3) = E^r$ and $D^+((3, 2^{\frac{3(\lambda_1-1)}{2}}) = E^s$ for some odd r, s. As these latter two matrices are both powers of E, clearly they commute. Furthermore, it is clear from Proposition 4.3.3 that the orbits $\mathcal{O}_{(\lambda_2)}$ and $\mathcal{O}_{(2^{(\lambda_2-1)/2},1)}$ commute in $\mathfrak{so}_{\lambda_2}(k)$. The desired result then follows from Lemma 4.4.2. □

Lemma 4.6.31. *Let $\lambda = (\lambda_1^2, \lambda_2^2)$ with $\lambda_1 \equiv \lambda_2 \equiv 1 \bmod 4$. Then $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(3,2^{m-2},1)}$ in $\mathfrak{so}_{2m}(k)$.*

Proof. We begin by illustrating the form of the Dynkin pyramid, using the example of $\lambda' = (9^2, 5^2)$ in $\mathfrak{so}_{28}(k)$:

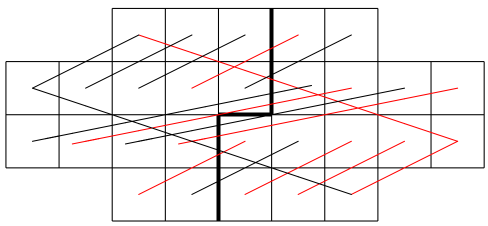
		5	9	13	17	21		
1	3	6	10	14	18	22	25	27
2	4	7	11	15	19	23	26	28
		8	12	16	20	24		

Now by considering the centraliser of the matrix $D^+(\lambda)$ as described in [52, Lem. 1.5.8], we observe that we can construct an element of the centraliser with Jordan type $(3, 2^{m-2}, 1)$ in the following way. Let $d = \frac{\lambda_1 - \lambda_2}{2}$. First we draw a line with coefficient -1 from box $2m$ to the rightmost box on the lower short row, labelled $(2m - 2d)$; we then draw in all parallel lines between these two blocks, changing sign each time we draw a line that crosses the thick central line of the pyramid. We then add in similar lines from the upper short row to the upper long row, choosing signs to ensure that the diagram has rotational antisymmetry. In this way we draw $2\lambda_2$ lines on the pyramid.

Next we draw a line from box $2m - 1$ to the rightmost box on the lower long row which is not already the start- or endpoint of a line, and add in all parallel lines using the same sign rules. In this way we draw $(\lambda_1 - \lambda_2)$ lines on the Dynkin pyramid.

Finally we draw a line with coefficient -1 from box $2m$ to the leftmost box of the upper short row, labelled $(2d + 1)$; and a line with coefficient 1 from box $(2m - 2d)$- the rightmost box of the lower short row- to 1. It is not possible to draw any further lines parallel to these, and we obtain $v_{2m} \mapsto -v_{2m-2d} - v_{2d+1} \mapsto -2v_1 \mapsto 0$, and $v_{2m-2d} - v_{2d+1} \mapsto 0$. There are now $(m-2)$ remaining lines between the remaining $(2m - 4)$ boxes of the pyramid, none of which share a start- or endpoint with another line. So each of these lines gives rise to a part of size 2 in the Jordan type of the matrix defined by this diagram. Consequently we have constructed a matrix in the centraliser of $D^+(\lambda)$ with Jordan type $(3, 2^{m-2}, 1)$.

In the example of $\lambda' = (9^2, 5^2)$ this gives the following diagram:



It is easily verified that the matrix defined from this diagram does indeed possess Jordan type $(3, 2^{12}, 1)$, and furthermore that this approach can be followed for all partitions of the form $(\lambda_1^2, \lambda_2^2)$. □

From Lemmas 4.6.30 and 4.6.31 we observe that if λ is a four-part partition such that $\mathcal{O}_\lambda$ does not commute with $\mathcal{O}_{(3,2^{m-2},1)}$, then λ must have at least three distinct parts. We obtain the following immediate corollary:

Corollary 4.6.32. *The orbit $\mathcal{O}_{(3,2^{m-2},1)}$ is universally commuting in $\mathfrak{so}_{2m}(k)$ whenever m is even and $m \leqslant 6$.*

Proof. If $m \leq 6$ then the only four-part partitions λ of $2m$ with all parts equal to 1 modulo 4 are the following:

m	2	4	6
λ	(1^4)	$(5, 1^3)$	$(9, 1^3)$ or $(5^2, 1^2)$

Clearly none of these partitions have three distinct parts; so $\mathcal{O}_\lambda$ commutes with $\mathcal{O}_{(3,2^{m-2},1)}$ in each case. □

However if $m = 8$ then we obtain the partition $(9,5,1^2)$ which cannot be dealt with using the methods described above. If, following the method of Proposition 4.6.28, we write $(9,5,1^2)$ as the direct sum of a pair of two-part subpartitions, we observe that the size of each of these subpartitions must be equivalent to 2 modulo 4. But if $2m' \equiv 2 \mod 4$, then any partition of $2m'$ of the form $(3,2^a,1^{2m'-2a-3})$ must contain at least three parts equal to 1; and any partition of $2m'$ of the form $(2^b,1^{2m'-2b})$ must contain at least two parts equal to 1. So we do not learn anything this way about whether $\mathcal{O}_{(9,5,1^2)}$ and $\mathcal{O}_{(3,2^6,1)}$ commute in $\mathfrak{so}_{16}(k)$.

On the other hand we could write $(9,5,1^2)$ as $(9,5,1) \oplus (1)$, and consider whether $\mathcal{O}_{(9,5,1)}$ commutes with $\mathcal{O}_{(3,2^6)}$ in $\mathfrak{so}_{15}(k)$. Let μ be a partition with three distinct parts all equal to 1 modulo 4; then if we could show that $\mathcal{O}_\mu$ commutes with $\mathcal{O}_{(3,2^{(|\mu|-3)/2})}$ then this would be enough to conclude that $\mathcal{O}_{(3,2^{m-2},1)}$ is universally commuting for all even m. Unfortunately it appears that these orbits do not commute in general.

Indeed, it appears that $\mathcal{O}_{(9,5,1^2)}$ does not commute with $\mathcal{O}_{(3,2^6,1)}$ in $\mathfrak{so}_{16}(k)$. However we remark that it is a very difficult question to determine when two orbits do not commute, and thus providing a definitive answer to this is beyond the methods developed in this thesis. Consequently we conclude this chapter with the following conjecture:

Conjecture 4.6.33. *Let $m \geqslant 8$ be even; then the orbit $\mathcal{O}_{(3,2^{m-2},1)}$ is not universally commuting in $\mathfrak{so}_{2m}(k)$.*

CHAPTER 5

POLYNOMIAL SPRINGER ISOMORPHISMS

5.1 Springer isomorphisms

In this final chapter we introduce the idea of a Springer isomorphism. Suppose now that G is a simple algebraic group over an algebraically closed field k of good characteristic for G, with corresponding Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. We write $\mathcal{U}$ for the unipotent variety of G, and $\mathcal{N}$ for the nilpotent variety of the Lie algebra $\mathfrak{g}$. We fix a Borel subgroup B of G, then denote the unipotent radical of B by U and write $\mathfrak{u} = \mathrm{Lie}(U)$. Finally let Φ be the root system of G and let Π be the corresponding set of simple roots. We maintain this notation throughout this chapter.

The following well-known theorem is the starting point of our work in this section

Theorem 5.1.1. *If G is of type A, assume that the covering map $SL_n(k) \to G$ is separable. Then there exists a G-equivariant isomorphism $\phi : \mathcal{U} \to \mathcal{N}$.*

Such an isomorphism ϕ is known as a Springer isomorphism after T. A. Springer, who proved this result for simply connected groups in 1968 [49]; the result has since been strengthened to its current form [4, Cor. 9.3.4].

We now note the following corollary, which is of particular use when we wish to calculate Springer isomorphisms explicitly [23, Cor. 1.2].

Corollary 5.1.2. *There exists a B-equivariant isomorphism $\phi : U \to \mathfrak{u}$, and any such ϕ is the restriction of a Springer isomorphism $\phi : \mathcal{U} \to \mathcal{N}$.*

We now consider a refinement of this approach. For G as above, we fix a faithful representation $\rho : G \to \mathrm{GL}_n(k)$ with corresponding Lie algebra representation $d\rho : \mathfrak{g} \to \mathfrak{gl}_n(k)$. We may thus identify G and $\mathfrak{g}$ with their respective images under this representation. Finally, let X be a regular nilpotent element of $\mathfrak{g}$.

We would like to find some examples of computationally pleasant Springer isomorphisms that are easy to work with directly. In particular we look for a matrix polynomial $q(X) = 1 + a_1 X + a_2 X^2 + ... + a_n X^n$ with $a_i \in k$ and $a_1 \neq 0$, such that the map $q : \mathcal{N} \to \mathcal{U}$ is a Springer isomorphism.

For the remainder of this chapter, let n be the smallest natural number such that $X^n = 0$, for all regular nilpotent elements X in $\mathfrak{g}$. We then have the following general result [48, Prop. 4.2].

Theorem 5.1.3. *Let $\mathrm{char}(k) \geq n$. Then the exponential polynomial $p : \mathcal{N} \to \mathcal{U}$ in one variable Y defined by $q(Y) = 1 + Y + \frac{1}{2}Y^2 + ... + \frac{1}{(n-1)!}Y^{n-1}$ is a Springer isomorphism.*

Useful though this theorem is, it does not hold in fields of characteristic $k < n$ where the coefficient $\frac{1}{(n-1)!}$ is not defined. Consequently more work is required to provide the whole picture that we seek. As such we now move on to examine each type of simple algebraic group individually. In Section 5.2 we consider the classical groups, then move on to look at the exceptional groups in Section 5.3.

5.2 The classical groups

For the classical groups, the existence of polynomial Springer isomorphisms is well-known even in small good characteristic. We record this here for the sake of completeness, taking $\mathrm{SL}_n(k)$, $\mathrm{Sp}_{2n}(k)$ and $\mathrm{SO}_n(k)$ as our examples of classical groups. It is easily seen that the following result holds (see for example [24, 2.1.7–8]:

Theorem 5.2.1. *(i.) If G is of type* A*, then any polynomial of the form $q(Y) = 1 + a_1 Y + ... + a_{n-1}Y^{n-1}$ with $a_i \in k$ and $a_1 \neq 0$ defines a Springer isomorphism, and all Springer isomorphisms are of this form.*

(ii.) If G is of type B, C *or* D *then there exists a polynomial Springer isomorphism of the form $q(Y) = 1 + a_1 Y + ... + a_{n-1}Y^{n-1}$ where $a_1 \neq 0$, the a_i may be chosen freely for odd $i \neq 1$, and a_i is determined by a_j for $j < i$ when i is even.*

We illustrate this with the example of $G = \mathrm{Sp}_4(k)$, so $\mathfrak{g} = \mathfrak{sp}_4(k)$. We can choose a Borel subalgebra $\mathfrak{b}$ so that elements of $\mathfrak{b}$ are of the form

$$\begin{pmatrix} a & c & e & f \\ 0 & b & d & e \\ 0 & 0 & -b & -c \\ 0 & 0 & 0 & -a \end{pmatrix}$$

for $a, ..., f \in k$. Now $\mathfrak{u}$ is the subset of $\mathfrak{b}$ such that $a = b = 0$.

In this representation B and U are the upper triangular and upper unitriangular subgroups of G respectively. By calculating the form of a 4×4 upper unitriangular matrix $u \in \mathrm{Sp}_4(k)$ we find that any $u \in U$ must be of the form

$$u = \begin{pmatrix} 1 & \alpha & \gamma & \delta \\ 0 & 1 & \beta & \gamma - \alpha\beta \\ 0 & 0 & 1 & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

again with all coefficients in the field k.

We now take as our regular nilpotent element $e = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$.

Another direct matrix calculation then shows that elements $c \in C_U(e)$ have the form

$$c = \begin{pmatrix} 1 & \alpha & -\frac{\alpha^2}{2} & \delta \\ 0 & 1 & \alpha & \frac{\alpha^2}{2} \\ 0 & 0 & 1 & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So we see that $c = p(e)$, where $p(T) = 1 + \alpha T - \frac{\alpha^2}{2}T^2 - \delta T^3$.

5.3 The exceptional groups

The situation for the exceptional groups is somewhat more intricate, as each of the five different types needs to be considered individually. In particular, even their smallest matrix representations are of large dimension, ranging from 7 for G_2 to 248 for E_8, necessitating the use of a computer to perform these calculations. In the following discussion we have used the GAP computer algebra system [19].

In the following discussion we consider the existence or otherwise of polynomial Springer isomorphisms for groups of type G_2, E_6 and E_7. First we describe the procedure we use to determine whether such a Springer polynomial exists, then we describe the results that it gives for the groups given above.

Throughout this section let char $k = p$. Let G be a group of exceptional type, and fix a representation for G. Let Φ be the root system of G with positive system Φ^+ and simple roots Π, and for each $\alpha \in \Phi$ set e_α to be the matrix in $\mathfrak{g}$ corresponding to the root α.

We now follow the well-known Chevalley process as described in [15, Ch. 4] or [30, §25] to obtain a set of generators for U. Let $t \in k$, and let $\exp(X) = \sum_{i=0}^{\infty} \frac{1}{i!} X^i$ be the usual matrix exponentiation in $\mathbb{C}$. Finally if $M = (m_{ij})$ is a matrix with entries in $\mathbb{C}$, let $\overline{M} = (\overline{m}_{ij})$ be the matrix with coefficients in k such that $\overline{m}_{ij} = m_{ij} \bmod p$. Then U is generated by $x_\alpha(t) := \overline{\exp(te_\alpha)}$, for all $\alpha \in \Phi^+$. We set $X \in \mathfrak{g}$ to be the 'standard' regular nilpotent element $\sum_{\alpha \in \Pi} e_\alpha$.

We may therefore write $x = \prod_{\alpha \in \Phi^+} \exp(t_\alpha e_\alpha)$ for all $x \in U$. Let n be the smallest integer such that $X^n = 0$. Let $x \in \mathcal{U}$; and suppose that there exists a polynomial Springer isomorphism $q : \mathcal{N} \to \mathcal{U}$ such that $q(X) = x$. It follows that there exist sets of coefficients $\{a_1, ..., a_{n-1} : a_i \in k, a_1 \neq 0\}$ and $\{t_\alpha \in k : \alpha \in \Phi^+\}$ such that

$$1 + a_1 X + ... + a_{n-1} X^{n-1} = \prod_{\alpha \in \Phi^+} \exp(t_\alpha e_\alpha) \tag{5.3.1}$$

Hence the a_i are the coefficients of our polynomial Springer isomorphism q.

We note that each side of this equation is an upper unitriangular matrix; so we obtain a system of simultaneous equations each corresponding to one position in the matrix. Solving this by hand might appear to be a daunting task, as the size of the system and the number of variables increases rapidly with the dimension of $\mathfrak{g}$. For E_6, for example, it equates to solving 351 equations in 52 variables. However, in practice, many of the equations turn out to be identical or identically zero, and it is possible to find the possible solutions by progressive substitution with the help of GAP.

Below we define a GAP function `springcoeff` which calculates the matrix defined by $Z = 1+a_1X+...+a_6X^6-\prod_{\alpha\in\Phi^+}\exp(t_\alpha e_\alpha)$. The inputs of `springcoeff` are as follows:

(i.) `A` is a list of Chevalley basis elements for the Lie algebra $\mathfrak{g}$;

(ii.) `B` is a list of indeterminates $\{a_1, ..., a_{n-1}\}$ where n is the smallest natural number such that $X^n = 0$ where X is a regular nilpotent element in $\mathfrak{g}$;

(iii.) `T` is a list of indeterminates $\{t_\alpha : \alpha \in \Phi^+\}$;

(iv.) `x` is a regular nilpotent element of $\mathfrak{g}$.

The function `springcoeff` is then defined as follows:

```
springcoeff:=function(A,B,T,x)
local exptA,px,pa,Z,i,j,k;
exptA:=[];
px:=IdentityMat(Length(A[1]));
pa:=IdentityMat(Length(A[1]));
mat:=[];
for i in [1..Length(A)] do
exptA[i]:=IdentityMat(Length(A[1]))+T[i]*A[i]+(1/2)*(T[i]*A[i])^2;
od;
for j in [1..Length(T)] do
px:=px*exptA[j];
od;
for k in [1..Length(B)] do
pa:=pa+B[k]*(x^k);
od;
Z:=pa-px;
```

```
return Z;
end;
```

We may thus attempt to solve the equation for the a_i and t_α by making entries of the output matrix `Z` identically 0. The a_i are then the coefficients of a polynomial Springer isomorphism. We describe below how this turns out to work for groups of types G_2, E_6 and E_7.

5.3.1 The G_2 case

The smallest dimensional matrix representation of a simple Lie algebra of type G_2 is of dimension 7; as the Lie algebra is of dimension 14 its Cartan subalgebra is of dimension 2 and we obtain 6 positive roots. This representation is given explicitly in Humphreys [29, 19.3], but as stated is not sufficient for our needs for two reasons. Firstly, Humphreys considers G_2 as a subalgebra of the 7-dimensional orthogonal Lie algebra with respect to the unusual form $J_H = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & I_3 \\ 0 & I_3 & 0 \end{smallmatrix}\right)$. We must therefore first transform the given representation into one with respect to the form $J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$, which will be easier to work with directly. Secondly, the basis as given is not a Chevalley basis, so we must tweak it further to ensure that all the structure constants are integers.

We thus obtain the upper triangular part of a Chevalley basis of the simple Lie algebra of type G_2 as follows, where α and β are the simple roots and e_{ij} is the matrix with coefficient 1 in position (i, j) and zeroes elsewhere:

$$\begin{aligned} e_{3\alpha+2\beta} &= e_{16} - e_{27} \\ e_{3\alpha+\beta} &= e_{15} - e_{37} \\ e_{\beta} &= -e_{23} + e_{56} \\ e_{2\alpha+\beta} &= -2e_{14} - e_{25} + e_{36} + e_{47} \\ e_{\alpha+\beta} &= e_{13} - 2e_{24} + e_{46} - e_{57} \\ e_{\alpha} &= -e_{12} - 2e_{34} + e_{45} + e_{67} \end{aligned}$$

We may now use this representation to calculate a polynomial Springer isomorphism for G_2. Let X be the regular nilpotent element in $\mathfrak{g}$ defined by $X = e_\alpha + e_\beta$. Then $X^7 = 0$; so we have at most six a_i and six t_α in equation (5.3.1). We now run the function `springcoeff` as defined in Section 5.3, which calculates the matrix $Z = 1 + a_1 X + ... + a_6 X^6 - \prod_{\alpha \in \Psi^+} \exp(t_\alpha e_\alpha)$.

Each entry of Z will be an equation in the 12 variables a_i and t_α as defined above; and Z is an upper triangular matrix so has 21 non-zero entries, not necessarily all distinct. We find that in this case Z is equal to the following matrix:

$$\begin{pmatrix} 0 & -a_1 + t_\alpha & a_2 - t_{\alpha+\beta} & * & * & * & * \\ 0 & 0 & -a_1 + t_\beta & 2y & * & * & * \\ 0 & 0 & 0 & 2a_1 + 2t_\alpha & t_\alpha^2 - 2a_2 & * & * \\ 0 & 0 & 0 & 0 & a_1 - t_\alpha & a_2 - t_{\alpha+\beta} & * \\ 0 & 0 & 0 & 0 & 0 & a_1 - t_\beta & y \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 - t_\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.3.2}$$

Here $y = a_2 + t_{\alpha+\beta} - t_\alpha t_\beta$ and entries marked $*$ are non-trivial but omitted for clarity. Observing the first superdiagonal of the matrix in (5.3.2) we see that the only possible solutions are when $t_\beta = t_\alpha = a_1$, so $p(X)$ will be regular unipotent if and only if the linear coefficient a_1 is non-zero. Substituting these values back into the matrix we then find by looking at the second superdiagonal that $t_{\alpha+\beta} = a_2 = \frac{1}{2}a_1^2$. Continuing in this manner, we obtain that for any $a, b \in k$, the polynomial

$$p(Y) := 1 + aY + \frac{1}{2}a^2Y^2 + \frac{1}{6}a^3Y^3 + \frac{1}{24}a^4Y^4 + bY^5 + \left(ab - \frac{1}{144}a^6\right)Y^6$$

is a Springer isomorphism. We note that this method fails in characteristics 2 and 3, but this is no loss as these are exactly the bad primes for G_2. In particular this gives us a polynomial Springer isomorphism when $\operatorname{char} k = 5$; this is a new result.

5.3.2 The type E case

We now consider whether there exist polynomial Springer isomorphisms for groups of type E_6 and E_7. For the sake of clarity we will describe primarily the former case in this section, but we note that the situation for E_7 is almost exactly analogous. We will not consider E_8 here; as the smallest-dimensional representation of E_8 is of dimension 248 it is computationally impractical to do so.

We describe first how we obtain a suitable representation for E_6. Comparing the Dynkin diagrams of E_6 and E_7 (see for example [29, 11.4]) it is seen that a root system of type E_7 contains a copy of a root system of type E_6. By calculating the adjoint representation of this copy of E_6 in E_7 we obtain a 27-dimensional representation for E_6 as described below.

To calculate this, let $\mathfrak{g}$ be a Lie algebra of type E_6 considered as a subalgebra of $\overline{\mathfrak{g}}$ of type E_7; let Φ be the root system of $\mathfrak{g}$, let $\mathfrak{b}$ be a Borel subalgebra and let $\mathfrak{u}$ be the nilradical of $\mathfrak{b}$. Then there is a parabolic subalgebra $\overline{\mathfrak{p}} \subseteq \overline{\mathfrak{g}}$ that has Levi factor $\bar{\mathfrak{l}} \cong \mathfrak{g} \oplus k$. Let $\overline{\Phi}$ be the root system of $\overline{\mathfrak{g}}$, and let $\overline{\Phi}$ have simple roots $\overline{\Pi} = \{\overline{\alpha}_1, ..., \overline{\alpha}_7\}$. We denote the set of simple roots of Φ by Π and identify Π with the subset $\Pi = \{\overline{\alpha}_1, ..., \overline{\alpha_6}\} \subseteq \overline{\Pi}$.

Then we can write $\overline{\Phi}^+ = \{\sum_{i=1}^7 c_i\overline{\alpha}_i : c_7 = 0\} \sqcup \{\sum_{i=1}^7 c_i\overline{\alpha}_i : c_7 > 0\} = \Phi^+ \sqcup \Phi(\overline{\mathfrak{u}})$, where $\Phi(\overline{\mathfrak{u}})$ is the set of roots of $\overline{\mathfrak{u}}$.

We now consider the adjoint action of $\mathfrak{u}$ on $\overline{\mathfrak{u}}$, which have bases $\{e_\beta : \beta \in \Phi^+\}$, $\{e_\gamma : \gamma \in \Phi(\overline{\mathfrak{u}})\}$ respectively, where e_β is the root vector corresponding to the root β. Then, for $\beta \in \Phi^+$ and $\gamma \in \Phi(\overline{\mathfrak{u}})$, we can calculate $[e_\beta, e_\gamma] = \sum_{\delta\in\Phi(\overline{\mathfrak{u}})} a^\beta_{\gamma,\delta} e_\delta$. We thus obtain the desired matrix representation of $\mathfrak{u}$ by $e_\beta \mapsto \left(a^\beta_{\gamma,\delta}\right)_{\gamma,\delta\in\Phi(\overline{\mathfrak{u}})}$.

In this case, we find that $\mathfrak{u} \subseteq \mathfrak{g}$ is generated by 36 matrices of size 27×27. From now on we denote the elements of Φ by $\alpha_1, ..., \alpha_{36}$; we note that $\alpha_i = \overline{\alpha}_i$ if and only if $i = 1, ..., 6$.

We note that we can find the 57-dimensional representation of E_7 within E_8 in exactly the same way.

The following GAP function enables us to calculate a basis for the representation of E_{n-1} within E_n, taking E_n as its input:

```
ebasis:=function(g)
local A,phi,L,I,J,K,l,rs,rp,x,i,j,k,basg;

K:=ijlist(g);
I:=K[1];
J:=K[2];
A:=[];
for i in [1..Length(I)] do
A[i]:=NullMat(Length(J),Length(J));
```

```
od;

basg:=Basis(g);
for i in [1..Length(I)] do
   for j in [1..Length(J)] do
      x:=basg[I[i]]*basg[J[j]];
      for k in [1..Length(J)] do
      A[i][j][k]:=Coefficients(basg,x)[J[k]];
od;
od;
od;
return A;
end;
```

We may now look for a polynomial Springer isomorphism in the E_6 case. First we take our 'standard' regular nilpotent element to be $X = \sum_{i=1}^{6} e_{\alpha_i}$ and note that $X^{16} \neq 0$ but $X^{17} = 0$. We also simplify the notation developed in the preceding sections by writing $t_i = t_{\alpha_i}$ for all i. So we have 52 unknowns in equation (5.3.1) in this case: a_i for $i = 1, ..., 16$ and t_j for $j = 1, ..., 36$.

We may now use the same method as described in the G_2 case above. We run the function `springcoeff` to obtain a 27×27 matrix of equations similar to equation 5.3.2 above. Observing the first superdiagonal of this new matrix we see immediately that $t_i = a_1$ for $i = 1, ..., 6$. Then the second superdiagonal gives us that $t_i = a_2$ for $i = 7, ..., 11$ and from the third superdiagonal we find that $a_2 = \frac{1}{2}a_1$. We repeat this procedure, first finding certain t_j in terms of a_i and then moving up a superdiagonal to find a_i in terms of a_1. Eventually we find that we must have $a_i = \frac{a_1^i}{i!}$ for all $i = 2, ..., 16$. In other words, the only polynomial Springer isomorphism for E_6 is the exponential polynomial $p(X) = \sum_{i=0}^{16} \frac{a_i}{i!} X^i$

with $a \neq 0$, and this is not defined for fields of characteristic 13 or smaller.

We may equally apply the same techniques to the representation of E_7 within E_8. Here we obtain 63 basis matrices of size 57×57, and for our standard regular nilpotent element $X = \sum_{i=1}^{7} e_i$ we have $X^{28} = 0$. So here we need to solve (5.3.1) for $a_1, ..., a_{27}, t_1, ..., t_{63}$. Once again we find that the only possible polynomial Springer isomorphism is the exponential polynomial $p(X) = \sum_{i=0}^{27} \frac{a_i}{i!} X^i$ with $a \neq 0$. So here no polynomial Springer isomorphism exists in characteristic 23 or smaller.

The reader will note that we have made no mention of F_4 and E_8 in the preceding discussion; as remarked earlier it is harder to obtain convenient explicit representations for these cases. Whether or not polynomial Springer isomorphisms exist for representations of minimal dimension in these cases remains an open question, though in the light of the results for E_6 and E_7 it appears unlikely that we can find a polynomial Springer isomorphism in small characteristics for E_8 at least. In any case the non-existence of polynomial Springer isomorphisms for E_6 and E_7 in small characteristics prevents the development of a general theory in small positive characteristic and thus renders the resolution of these unsolved cases of lesser interest.

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