AXIAL ALGEBRAS

by

Felix Gabriel Rehren

A thesis submitted to
the University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY
IN PURE MATHEMATICS

School of Mathematics
28th Febuary 2015

Abstract

Axial algebras are nonassociative algebras controlled by fusion rules for idempotents. We have three main results. Firstly, we give a classification of axial algebras with fusion rules of Jordan type, with parameter α, in terms of 3-transposition groups. When alpha is $\frac{1}{2}$, we also classify the related Jordan algebras. Secondly, we develop a structure theory for Matsuo algebras, especially using large associative subalgebras, and apply it to the special case of the Dynkin diagram of type A_n, which has relations to vertex operator algebras. Thirdly, we generalize dihedral axial algebras of Ising type, with parameters α, β, coming from the Monster sporadic simple group. This also helps determine the role that the parameters play in the larger theory, where indeed the Griess algebra turns out to be a special point.

Contents

Tables and figures

Chapter 1. Introduction

This thesis is about a novel theory of nonassociative algebras.

Algebras combine the notions of linearity and multiplication. Their linearity is controlled by the underlying ring or field; their multiplication is controlled by relations. Cornerstone relations are commutativity ($ab = ba$) and associativity ($(ab)c = a(bc)$), but for example in Lie algebras associativity is replaced by the Jacobi identity, a more complicated relation on words. Algebras whose associativity is replaced by a word relation were a central topic in mathematics in the 20th Century, including the classical theory of Jordan algebras.

We replace the associativity by a fundamentally different control:

fusion rules describing the multiplication of eigenvectors of idempotents.

If structure constants describe the multiplication of elements, fusion rules describe the multiplication of submodules.

Fusion rules are an instance of decomposing an algebra with respect to the representation theory of a subalgebra. A familiar example comes from Lie theory: the modules of $\mathfrak{sl}_2$ have highest weights in $\mathbb{Z}$, so any $\mathfrak{sl}_2$-subalgebra induces a $\mathbb{Z}$-grading on the entire algebra. We use idempotents, elements e satisfying $ee = e$ and generating 1-dimensional subalgebras, to similarly gain structural information—especially when the fusion rules induce automorphisms of the algebra.

We make three main investigations.

In Chapter 2, we describe algebras generated by idempotents satisfying the first nontrivial $\mathbb{Z}/2$-graded example, $\Phi(\alpha)$, of fusion rules. This gives an algebraic interpretation to 3-transposition groups and their incidence geometries. We also classify Jordan algebras related to 3-transposition groups. This chapter is partly based on work undertaken jointly with S. Shpectorov and J. Hall, and with T. de Medts.

In Chapter 3, we show that special classes of idempotents, in $\Phi(\alpha)$-axial algebras coming from simply-laced root systems, have certain good properties analogous to those in associative algebras. We find a connection between algebraic combinatorics and vertex algebras and analyse two special cases in detail.

Chapter 4 is concerned with a generalisation $\Phi(\alpha, \beta)$ of $\Phi(\alpha)$, also generalising the fusion rules coming from the Monster sporadic group. We investigate how the choice of constants α, β affects the theory, especially the celebrated Sakuma's theorem, by generalising the so-called Norton-Sakuma algebras involving $\Phi({}^1\!/\!{}_4, {}^1\!/\!{}_{32})$.

A more detailed background, summary of this thesis and discussion follow.

1.1 Thanks

I would like to thank the School of Mathematics, its faculty and staff especially including Janette, for its many years of generous support.

I very much enjoyed the supervision of Sergey Shpectorov, towards whom I feel deep gratitude. Working with him, Tom de Medts and Jon Hall, Alonso Castillo-Ramirez and Oli Gray were the most enjoyable times of my project.

Among my friends I would particularly like to thank Alessandro, Alonso, and Glen; Jamie, Laurence, Lorna, Lucy and Oli.

Letztens bedanke ich mich bei meinen Verwandten und meinem Vater.

It's the thought that counts, and it's the Count that thoughts—ah ..

Sesame Street.

1.2 Background

In the late 1960s, B. Fischer developed the theory [F71] of 3-transposition groups[1] that greatly influenced finite group theory for the rest of the century. One of its great successes was the conjectural existence of the Monster group, the largest of the sporadic simple finite groups, which is a 6-transposition group.

The first realisation of the Monster was constructed by R. Griess in early 1981 [G82] as the automorphism group of a 196884-dimensional algebra. This *Griess algebra* is commutative and nonassociative, but has a bilinear associating form and has been described as 'more than 99% associative' [C85]. Coincidences among character degrees of the Monster and certain coefficients arising in modular forms[2] led to a conjectured connection to number theory, christened *moonshine* by J. H. Conway[3]. I. Frenkel, J. Lepowsky and A. Meurman provided a partial explanation [FLM98] by constructing an infinite-dimensional module $V^\natural$ for the Monster with some number-theoretic properties as an affinization of the Griess algebra. R. Borcherds' subsequent introduction of *vertex (operator) algebras* [B86] realized $V^\natural$ as part of a family of algebraic structures that includes representations of affine Kac-Moody Lie algebras and gave a foundation to moonshine. The same structures were also of interest to physicists under the name of *WZW models*. Investigations into the connection between finite groups, number theory and string theory continue to be a hot topic today;[4] see for example [DGO14] for a recent survey.

Vertex algebras are intimately related to the Virasoro algebra, whose remarkable

[1]Apparently he claimed that the only groups generated by involutions whose pairwise products have order at most 3 are the symmetric groups, was corrected by R. Carter and in response classified such groups.

[2]a key observation, due to J. McKay, is that $196884 = 196883 + 1$

[3]relating to the bottle of Jack Daniels' offered by A. Ogg, and the lunacy of the connection

[4]The most recent discovery is *Mathieu moonshine*, relating $K3$ surfaces to five sporadic groups also contained in the Monster.

representation theory [W93] is parametrised by the *central charge* $c \in \mathbb{R}$. The rôle of identity in a vertex algebra is played by a Virasoro subalgebra, which (like $\mathfrak{sl}_2$ in a Lie algebra) induces a $\mathbb{Z}$-grading, called *weight*, on the vertex algebra. A particular class of vertex algebras[5], including $V^\natural$, whose weight-2 subspaces are closed under multiplication and possess an associating form $(,)$, attracted group theorists. We restrict our discussion to these vertex algebras. M. Miyamoto made the profound insight that *every* idempotent, not just the identity, in a weight-2 subspace affords the action of a Virasoro algebra:

1.2.1 Theorem (Miyamoto's Theorem, [M96] Lemma 5.1, using [W93]). *If e is an idempotent in the weight-2 subalgebra of a vertex algebra, it is Seress*[6]*. If the central charge $\mathrm{cc}(e) = \frac{1}{2}(e, e)$ is less than 1, then $\mathrm{cc}(e) = c_p = 1 - \frac{6}{(p+2)(p+3)}$ for some integer $p \geq 1$, and moreover e has possible eigenvalues 1 and $h^p_{r,s}$,*[7] *for*

$$h^p_{r,s} = \frac{((p+2)r - (p+3)s)^2 - 1}{8(p+2)(p+3)}, \qquad 1 \leq r \leq s \leq p+1. \tag{1.1}$$

In turn, the Virasoro algebra's representation theory places significant restrictions on the modules (with respect to the adjoint action) of the idempotent in the vertex algebra, in particular inducing an automorphism of order 2, see [W93] and [M96] Theorem 4.2. Thus the local property of being an idempotent yields global information. In fact the automorphisms induced by idempotents of central charge $\frac{1}{2}$ in the Griess algebra, which is the weight-2 subalgebra of $V^\natural$, form the $(2A)$-conjugacy class of involutions in the Monster [C85].

Miyamoto's student S. Sakuma deepened the investigation [M03] into vertex

[5] *OZ-type* vertex algebras: those having only nonnegative weight, and weight-0,1 subspaces which are 1,0-dimensional respectively. Hence the acronym for One, Zero.

[6] that is, $0 \otimes M \cong M$ for 0 the trivial module, M any module; or $0 \star \phi = \{\phi\}$ for any ϕ; *c.f.* Section 2.1

[7] We have slightly modified the definition usual in the literature. Normally $e \in A$ is taken to satisfy $ee = 2e$, and our $h^p_{r,s}$ are half of the highest weights usually denoted $h^p_{s,r}$ of L_0 where the L_i are the Fourier coefficients or modes of e and generate the Virasoro algebra. For example, the highest weights for L_0 of central charge $\frac{1}{2}$ are $0, \frac{1}{2}, \frac{1}{16}$, but the possible eigenvalues of e with $ee = e$ and $\mathrm{cc}(e) = \frac{1}{2}$ are $1, 0, \frac{1}{4}, \frac{1}{32}$. The fusion rules for 1 are the same as those for 0.

algebras generated by idempotents to prove a spectacular result [S07]: any two central charge $\frac{1}{2}$ idempotents must generate a vertex algebra of one of only 9 isomorphism types, all realised in $V^\natural$. In the Griess algebra, subalgebras generated by two such elements are controlled by the conjugacy class of the product of their two involutions in the Monster—and these are exactly the conjugacy classes implicated in J. McKay's observation labelling the affine E_8-diagram in Figure 1.1. These dihedral subalgebras had already gained S. Norton's attention [C85] in the Griess algebra; Sakuma showed they were a general feature of vertex algebras.

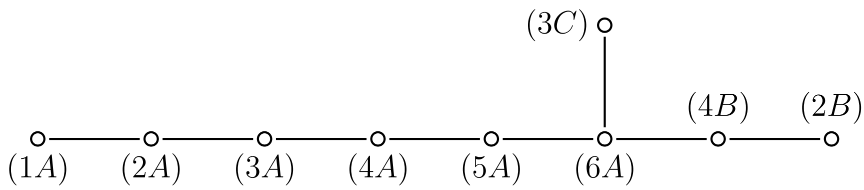

Figure 1.1: McKay's affine E_8 observation[8]

A further, radical, step was taken by A. A. Ivanov in axiomatising the properties of weight-2 subalgebras of vertex algebras generated by central charge $\frac{1}{2}$ idempotents, which includes the Griess algebra, under the name *Majorana* [I09]. The significance of this was immediately clear after Sakuma's theorem was reproven in the new context [IPSS10], without the use of vertex algebra machinery, but again leading to the same list of the 9 isomorphism types of the so-called Norton-Sakuma algebras. Ivanov et al pursued a programme of finding further Majorana algebras as representations of subgroups of the Monster, with results for many finite simple groups, currently up to M_{11} [S12]. These suddenly put the Griess algebra into a tractable family of nonassociative algebras.

[8]If $r_{(\mathrm{n}X)}$ are 9 roots generating affine E_8, with the pairwise relationships indicated in the figure, then $\sum_{(\mathrm{n}X)} nr_{(\mathrm{n}X)}$ is in the radical of the Killing form.

The algorithms that were developed to compute such algebras, for example [Ax13], eventually led[9] us to a new point of view: *axial algebras*, that is, algebras generated by idempotents satisfying prescribed fusion rules. The first goal is to understand the algebras with *Ising fusion rules* in Table 1.2. The specialisation $\Phi(1/4, 1/32)$ describes the multiplication of representations of the central charge $\frac{1}{2}$ Virasoro algebra, which arises in the celebrated exact solution of the 2-dimensional Ising lattice model of magnetic interaction in statistical mechanics [M92], and also plays a crucial rôle in the Griess algebra.

$\star$	1	0	α	β
1	$\{1\}$	$\emptyset$	$\{\alpha\}$	$\{\beta\}$
0		$\{0\}$	$\{\alpha\}$	$\{\beta\}$
α			$\{1, 0\}$	$\{\beta\}$
β				$\{1, 0, \alpha\}$

Table 1.2: The Ising fusion rules $\Phi(\alpha, \beta)$

The theory of vertex algebras suggests many things to consider from the axial point of view, and has been especially advanced by the work of C. Dong, Miyamoto and their collaborators. We briefly mention some results which touch our work. The paper [DLMN96] gives a remarkably simple description of certain lattice vertex algebras[10] in terms of the kind of nonassociative algebra we study in detail in

[9] by R. Parker's maxim, "if a computer does something interesting with it, it must be interesting", reversed: "if the computer doesn't use it, it's superfluous"

[10] a lattice vertex algebra V_L is defined for any even integral lattice L. The Kummer involution of L that acts by sending $x \mapsto -x$ for $x \in L$ induces to an automorphism θ of V_L; if the shortest elements of L have length 4, then the θ-fixed-point subspace V_L^θ is an OZ-type vertex algebra; this construction goes back to [FLM98]. The most important examples of L are therefore $\sqrt{2}\mathrm{X}_r$, where X_r is the root lattice of a simply-laced root system, and Λ the Leech lattice

Chapters 2, 3, namely

1.2.2 Theorem ([DLMN96] Theorem 3.1)**.** *The Matsuo algebra* $M^{1/2}_{1/4}(\mathcal{G}^{\pm})$ *on* $\mathcal{G}^{\pm}$ *with central charge* $\frac{1}{2}$ *and eigenvalue* $\frac{1}{4}$*, modulo its radical, is realised as the weight-*2 *subalgebra of a vertex algebra when* $\mathcal{G}$ *is the Fischer space of a simply-laced root system, that is,* $\mathrm{A}_n, \mathrm{D}_n$*,* $n \in \mathbb{N}$*, or* $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$.

The structure of the lattice vertex algebra of $\sqrt{2}\mathrm{A}_n$ is dissected in [LY04a]. Framed vertex algebras, especially related to the Griess algebra, are treated in [DGH98]. Analogues of the Norton-Sakuma vertex algebras, for $\mathrm{E}_6, \mathrm{E}_7$ replacing E_8, are introduced in [HLY12]. There are many more significant results in this area, part of a vast literature often between mathematics and physics. The texts [G03], [DFMS97] (Chapters B.7, 8, 10 and C.13-18) and [FBZ01] may together form the basis of an introduction.

The final piece of the picture is given by Jordan algebras. First proposed by P. Jordan as a framework to capture the algebras of observables in quantum mechanics, the American school under N. Jacobson gave a thoroughly satisfactory account of finite-dimensional Jordan algebras in the second half of the 20th Century; the infinite case was famously dealt with by E. Zelmanov [McC04]. A Jordan algebra is a commutative algebra which is not necessarily associative, but instead satisfies $a(a(ba)) = (aa)(ba)$. We are grateful to A. Elduque for pointing us to the realisation that the fusion rules of any idempotent in a Jordan algebra are $\Phi(1/2)$, where $\Phi(\alpha) \subseteq \Phi(\alpha, \beta)$ are the *Jordan* fusion rules of Table 2.2, contained in the Ising fusion rules. Via *capacity* and the *Peirce decomposition*, idempotents lead directly to a satisfactory and rich structure theory for Jordan algebras. The axial algebras with fusion rules $\Phi(1/2)$ are in general wilder, but for example Griess's algebra for the Monster contains a 300-dimensional Jordan subalgebra [G03].

1.3 In this text

Chapter 2 is foundational for the other two chapters, which are independent of each other. It begins with an exposition, Section 2.1, of the well-known graphs and algebras which will play a key rôle in the sequel. In particular, this includes *linear* 3*-graphs* and the subclass of *Fischer spaces*. *Matsuo algebras* are algebras on linear 3-graphs. *Fusion rules* are a bookkeeping device controlling multiplication of eigenvectors of idempotent elements called *axes*. An *axial algebra* is an algebra generated by axes. We also discuss the particularly straightforward case of axes in associative algebras, which are generalised by the *Seress* property of fusion rules.

Section 2.2 is taken up entirely with the proof that a linear 3-graph is a Fischer space if and only if its points are $\Phi(\alpha)$-axes, where $\Phi(\alpha) \subseteq \Phi(\alpha, \beta)$ are the *Jordan fusion rules*. This problem, giving an algebraic characterisation of a geometric condition, was suggested by C. Parker; an alternate proof is in [HRS14].

In Section 2.3, we classify algebras generated by two primitive $\Phi(\alpha)$-axes, called $\Phi(\alpha)$-dihedral algebras for short, in

Theorem (2.3.1)**.** *A* $\Phi(\alpha)$*-dihedral algebra, over an everywhere faithful ring containing* $\frac{1}{2}, \alpha, \alpha^{-1}$ *and* $(\alpha-1)^{-1}$*, is* $(2B)$*, a quotient of* $(3C_\alpha)$*, or, when* $\alpha = \frac{1}{2}$*, a quotient of* $(3J_\kappa)$ *for some* κ*.*

This follows the work, joint with Hall and Shpectorov, published in [HRS14]. Furthermore we find all the idempotents in these $\Phi(\alpha)$-dihedral algebras, which also generalises some results from [CR13].

Section 2.4 begins the discussion of automorphisms for $\mathbb{Z}/2$-graded fusion rules, and $\Phi(\alpha)$ in particular. We also introduce the subclass of *triangulating* cases, a condition which is always satisfied when $\alpha \neq \frac{1}{2}$, and prove

Theorem (2.4.7). *If A is generated by primitive triangulating $\Phi(\alpha)$-axes, then A is a quotient of a Matsuo algebra.*

Our treatment is a variation of that given in [HRS14].

The final Section 2.5 considers the case $\Phi(1/2)$, which is related to Jordan algebras. It is based on work undertaken with Tom de Medts. In particular, we prove

Theorem (2.5.5, 2.5.8, 2.5.9). *The Matsuo algebra with eigenvalue $\alpha = \frac{1}{2}$ of an irreducible 3-transposition group (G, D), over a field $\mathbb{F}$ containing $\frac{1}{2}$, is a Jordan algebra if and only if $G = \mathrm{Sym}(n)$ or, when also $\mathbb{F} \ni \frac{1}{3}$, $G = 3^2 : 2$.*

In Chapter 3, the invariants and structural results we develop for Matsuo algebras come from algebraic-combinatorial data but have particular application to vertex algebras. Section 3.1 introduces *doubles* and *boundaries* of graphs, *very regular* embeddings and recalls standard topics including the Perron-Frobenius theorem. Here Theorem 3.1.5 is the basis for a conjecture that suitable maximal embeddings of 3-transposition groups have transitive action on their 'boundary transpositions', by giving proofs in many cases.

In Section 3.2, we determine the restrictions on the eigenvalue α for the existence, positivity of eigenvalues and Seress property of identity elements for Matsuo algebras. Section 3.3 introduces *coset axes*, which are idempotents coming from inclusions of unital subalgebras. A key property is *coincidence-freeness*: to avoid, in a suitable sense, coincidences between α and eigenvalues of the adjacency matrices of the underlying graphs.

Theorem (3.3.6, 3.3.7). *The coset axis of a very regular $\mathcal{K} \subseteq \mathcal{H}$ in $M_\alpha(\mathcal{G})$ is Seress when $\mathcal{K} \subseteq \mathcal{H}$ is coincidence-free for α. In particular, this is always the case when α is transcendental over a field of characteristic 0.*

Given a graph $\mathcal{G}$, there are only finitely many eigenvalues to avoid, so we are 'almost always' in the coincidence-free case.

Based on the simultaneous decomposition of eigenspaces, Section 3.3 also determines the relationships between decompositions of the identity (*e.g.,* coming from coset axes of parabolic chains of 3-transposition groups) and maximal, globally-associative subalgebras, here called *tori*. This relates to classical work on the Griess algebra [MN93], Peirce decompositions [P81] and framed vertex algebras [DGH98]. Using the results of the later Section 3.5, this allows us to establish the existence of tori in many classes of examples including lattice vertex algebras.

Section 3.4 recalls the bilinear Frobenius form. As well as proving general results on Frobenius forms, including the radical, we give some specific formulae and identities. Altogether with Section 3.2 this proves

Theorem (3.2.5, 3.4.8, 3.4.9). *If $\mathcal{G}$ is a regular linear 3-graph, then $M_\alpha(\mathcal{G})$ is semisimple and $\mathrm{id}_\mathcal{G}$ is Seress in any larger algebra if $\alpha \neq -\frac{2}{\lambda}$ for any $\lambda \in \mathrm{Spec}(\mathcal{G})$.*

Finally, in Section 3.5 all of our previous results are applied to describe a torus in $M^c_\alpha(\mathcal{A}^\pm_n)$. We first give several results on graphs and adjacency matrices, especially for the graphs of $\mathcal{A}_n$ and $\mathcal{D}_n$. In Propositions 3.5.6, 3.5.12 we give the eigenvalues of coset axes in the Matsuo algebras $M_\alpha(\mathcal{R}^\pm)$ for $\mathrm{R} = \mathrm{A}_n, \mathrm{D}_n$ when α is coincidence-free. (The specialisation of $\alpha = \frac{1}{4}$, in Proposition 3.5.7, classifies the highest weights of conformal vectors from commutants occurring in the weight-2 subspace of the lattice vertex algebras $V_{\sqrt{2}\mathrm{A}_n}$, recovering the results of [Y01].) Using Theorems 1.2.1, 1.2.2, we find the fusion rules of coset axes in $M_\alpha(\mathcal{A}^\pm_n)$ in Lemma 3.5.11, by first passing to a larger axial algebra; this method is further developed in the next chapter. Lastly,

Theorem (3.5.13, 3.5.14). *The fusion rules of $\mathrm{id}_{\mathcal{D}_m}$ in $M_\alpha(\mathcal{D}_n)$ for $3 \leq m < n$, over $\mathbb{F}(\alpha)$ with α transcendental, induce Miyamoto involutions which are not transpositions in the automorphism group.*

Chapter 4 principally investigates $\Phi(\alpha, \beta)$-dihedral algebras. The only previously known examples of these are the Norton-Sakuma algebras, presented in Section 4.1. We also generalise Theorem 2.3.1 to show that an algebra generated by a $\Phi(\alpha)$-axis and a $\Phi(\beta)$-axis must be one of the possibilities in the same theorem. Lastly, we prove Proposition 4.1.3 showing that the order of products of Miyamoto automorphisms is determined in the respective dihedral subalgebra: this is key to proving 6-transposition statements later.

The following Section 4.2 generalises an argument from [HRS15], showing that

Theorem (4.2.4)**.** *There exists a ring R and an m-generated Φ-axial algebra A, such that, for any m-generated Φ-axial algebra B over S a ring, S is an associative algebra over R and B is a quotient of a ring extension of A.*

In other words, for any given fusion rules Φ, we can look for a universal object that classifies the m-generated Φ-axial algebras (with or without form).

Under certain additional conditions on the ring, in Section 4.3 we deduce that any $\Phi(\alpha, \beta)$-dihedral algebra is spanned by 8 elements, and we completely determine the structure constants for this multiplication. The computational proof essentially goes back to the method in [S07]. This is a major step in finding a classifying theorem for $\Phi(\alpha, \beta)$-dihedral algebras, but we do not fully determine the ring, instead giving a finite set of generators.

In Section 4.4 we introduce *(weak) covers*, which are larger generalisations of axial algebras via smaller ideals in the universal object previously constructed. Also recall that the covers of $(1A), (2B), (2A)$ and $(3C)$ are effectively known by Section 2.3. Using the fusion rules and the result of Section 4.3, we deduce

Theorem (4.4.2)**.** *The weak covers of the Norton-Sakuma algebras $(3A)$, $(4A)$, $(4B)$, $(5A)$ and $(6A)$, are given by Table 4.2. The algebras are Frobenius and satisfy a global 6-transposition property.*

We conjecture that these weak covers are actually the largest possible covers.

The theorem uses calculations from Section 4.5. The information gathered there is summarised in Table 4.2 and, in less detail, in Figure 1.3: the key result is that $\Phi(\alpha,\beta)$-analogues of the Norton-Sakuma algebras do not in general exist for all pairs α,β, but do exist along certain curves in α,β. In particular, the point $(1/4, 1/32)$ is the only common point of intersection, and these are precisely the values for which the Griess algebra is defined.

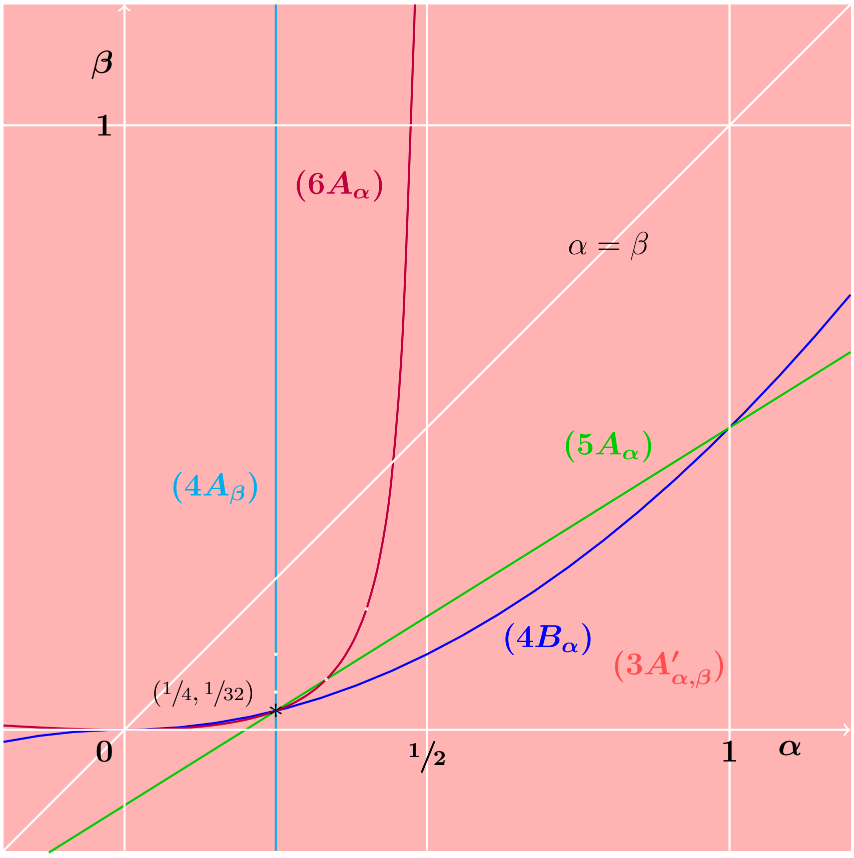

Figure 1.3: The Norton-Sakuma-like algebras for $(\alpha,\beta) \in \mathbb{R}^2$

1.4 Discussion

Axial algebras are at the confluence of nonassociative algebras, vertex algebras and transposition groups. Here we discuss an open question in each direction.

As a class of nonassociative algebras, to what extent does the theory of Φ-axial algebras depend on the specific features of the fusion rules Φ? In particular, the theory is very well-behaved when it coincides with vertex algebras or Jordan algebras. Consider group algebras for groups as analogous to axial algebras for transposition groups. Their theory of idempotents (especially those coming from group characters) is also well-developed; *c.f.* [P79]. How wild is our general case?

Throughout Chapter 3, especially in Sections 3.2 and 3.3, we see the importance of the Seress property, which forms the basis of our analogy to associative algebras. Miyamoto's Theorem 1.2.1 states that all idempotents in the weight-2 subalgebra of a vertex algebra, called c-conformal vectors for c their central charge, are Seress. Theorem 2.5.4 shows that all idempotents in Jordan algebras are Seress.

Proposition 3.3.10 uses the Seress property to establish in particular cases that the commutant $C_A(B)$ of a subalgebra B of A is again a subalgebra. This is well-known if A, B are vertex algebras. For A, B Jordan algebras it holds when B is separable and finite-dimensional by [J68], VIII.3 Theorem 8. In the Griess algebra this was proven by Norton [C85].

Basic extension questions have also not yet been settled. To summarise:

1.1 Question. *Suppose that A, B are algebras generated by sets $\mathcal{A}, \mathcal{B}$ of Seress axes.*

i. Are all idempotents in A Seress?

ii. When $B \subseteq A$, is the commutant $\mathrm{C}_A(B)$ a subalgebra?

iii. If $ef = 0$ for any $e \in \mathcal{A}, f \in \mathcal{B}$, is $AB = \langle \mathcal{A} \cup \mathcal{B} \rangle = A \oplus B$?

The inclusion of axial algebras in vertex algebras is an open problem:

1.2 Question. *Is there a simple criterion of an axial algebra A which guarantees the existence of a vertex algebra V whose weight-2 subalgebra is A?*

The most interesting known example at this time comes from the group $3^2 : 2$, whose Fischer space is $\mathcal{P}_3$. Matsuo showed, by finding a contradiction to Miyamoto's Theorem 1.2.1, that the Matsuo algebra $M^{1/2}_{1/4}(\mathcal{P}_3)$ cannot be in the weight-2 subspace of a vertex algebra [M03]. On the other hand, [CL14] furnishes an explicit construction of a vertex algebra V whose weight-2 subalgebra is $M^{1/2}_{1/32}(\mathcal{P}_3)$ over $\mathbb{R}$. We observe that both of these algebras are quotients of the same algebra $M^{1/2}_{\alpha}(\mathcal{P}_3)$ over $\mathbb{R}[\alpha]$, by the ring specialisations $\alpha \mapsto \frac{1}{4}$ and $\alpha \mapsto \frac{1}{32}$.

The best-understood vertex algebras are those built from $\frac{1}{2}$-conformal vectors, corresponding to $\Phi({}^1\!/\!{}_4, {}^1\!/\!{}_{32})$-axes. (The two examples for $3^2 : 2$ concern realisations with eigenvalues $\frac{1}{4}$ and $\frac{1}{32}$ respectively.) This includes lattice vertex algebras. Extending Theorem 1.2.2, and that $\frac{1}{2}$-conformal vectors are generalised by the c_p-conformal vectors for $c_p = 1 - \frac{6}{(p+2)(p+3)}$, in combination with our work on tori in Chapter 3 (such as Proposition 3.5.8), together with O. Gray we conjecture

For X_r a root system, d the dimension of its Lie algebra, $\hat{\mathrm{X}}r_\ell$ its affine untwisted Kac-Moody Lie algebra at level ℓ and $h^\vee$ its dual Coxeter number, $N^{c_{\mathrm{X}_r}}_{h_{\mathrm{X}_r}}(\mathcal{A}_n)$ can be realised as the weight-2 subalgebra of a vertex algebra for any n and

$$c_{\mathrm{X}_r} = \mathrm{cc}(\mathrm{C}_{\hat{\mathrm{X}}r_1 \oplus \hat{\mathrm{X}}r_1}(\hat{\mathrm{X}}r_2)) = \frac{2d}{(h^\vee+1)(h^\vee+2)}, \qquad h_{\mathrm{X}_r} = \frac{1}{h^\vee+2}. \tag{1.2}$$

In the case $\mathrm{X}_r = \mathrm{A}_1$, we have $(c_{\mathrm{A}_1}, h_{\mathrm{A}_1}) = ({}^1\!/\!{}_2, {}^1\!/\!{}_4)$, and $N^{1/2}_{1/4}(\mathcal{A}_n)$ is inside the ordinary lattice vertex algebra of $\sqrt{2}\mathrm{A}_n$; together with I. Frenkel's rank-level duality and the commutant construction mentioned earlier, this forms the basis of the celebrated Goddard-Kent-Olive construction [GKO85]. For $\mathrm{X}_r = \mathrm{E}_8$, the pair $(c_{\mathrm{E}_8}, h_{\mathrm{E}_8})$ is $({}^1\!/\!{}_2, {}^1\!/\!{}_{32})$. For all root systems other than $\mathrm{A}_1, \mathrm{E}_8$, this would be a new realisation.

The final question is also perhaps the deepest. We recall three facts:

i. The McKay observation: the nine conjugacy classes $(1A), \ldots, (6A)$ in the Monster involved in the Norton-Sakuma algebras label affine E_8;
ii. (Sakuma's Theorem) The Norton-Sakuma algebras are precisely those vertex algebras generated by any two $\frac{1}{2}$-conformal vectors;
iii. The construction (1.2) for E_8 leads to $c_{\mathrm{E}_8} = \frac{1}{2}$-conformal vectors.

So far, this seems to be a story about vertex algebras, but we saw in [IPSS10] and [HRS15] that Sakuma's theorem has an exactly analogous statement, Theorem 4.1.4, in terms of axial algebras. Furthermore, the results of Chapter 4 and in particular Theorem 4.4.2 show that the point $(\alpha, \beta) = (^1\!/\!_4, ^1\!/\!_{32})$ is distinguished among $\Phi(\alpha, \beta)$-dihedral algebras and crucially admits the Monster's 6-transposition property, and the axial analysis may be easier than that in vertex algebras.

The evidence collected in the inspiring paper [HLY12] by G. Höhn, C. H. Lam and H. Yamauchi suggests a further link and, in fact, a wide-ranging conjecture extending McKay's observation. They identify the vertex algebras generated by two c_{X_r}-conformal vectors in the natural vertex algebra representations of the Baby Monster (a 4-transposition group) and the Fischer group Fi_{24} (a 3-transposition group) when $\mathrm{X}_r = \mathrm{E}_7$ and E_6 respectively; these vertex algebras label the nodes of the affine Dynkin diagrams $\hat{\mathrm{E}}_7, \hat{\mathrm{E}}_6$ up to diagram automorphism. This is a remarkable analogy of the moonshine triple of the Norton-Sakuma algebras, the Monster, and E_8. However a classification (like Sakuma's theorem) of algebras generated by two c_{X_r}-conformal vectors is not yet known for $\mathrm{X}_r \neq \mathrm{A}_1, \mathrm{E}_8$.

1.3 Question. *What are the analogues of Sakuma's theorem for root systems* X_r *other than* E_8*? Do the dihedral algebras label the nodes of* $\hat{\mathrm{X}}_r$*, and is there a finite transposition group whose conjugacy classes control the dihedral algebras in its natural vertex algebra representation?*

1.5 Notation

In general, uppercase Latin letters are algebraic structures, and lowercase Latin letters their elements; lowercase Greek letters are scalars, and uppercase Greeks their collections; script letters are graphs.

The following is a glossary of commonly-used symbols.

$\mathbb{F}$	a field
R	a ring; always associative, commutative, unital
A	an algebra, where multiplication is juxtaposition
a, b, e, f	idempotents or Φ-*axes*, see Section 2.1
$\mathrm{ad}(x)$	the adjoint map $A \to A, y \mapsto xy$ for $x \in A$
$\mathrm{Spec}(x)$	the set of eigenvalues of x, or of $\mathrm{ad}(x)$
Φ	*fusion rules*: subset of $\mathbb{F}$ or R together with $\star\colon \Phi \times \Phi \to 2^{\Phi}$
$\alpha, \beta, \phi, \psi$	the *eigenvalues* lying in Φ
κ, λ	constants controlling algebra structure
$\tau(a)$	*Miyamoto involution* of a, see Section 2.4
A^e_ϕ, x^e_ϕ	ϕ-eigenspace of e in A, and projection of x onto A^e_ϕ
$a \circ b$	symmetric element, see Sections 2.3, 4.3
$\lambda^a(x)$	defined by $x^a_1 = \lambda^a(x)a$, *w.r.t.* projection of x onto $A^a_1 = \langle a \rangle$; see (4.8)
$M_\alpha(\mathcal{G})$	the *Matsuo algebra* on $\mathcal{G}$, see Section 2.1
$(\mathrm{n}X), (\mathrm{n}X_\alpha)$	$\Phi(\alpha)$- or $\Phi(\alpha, \beta)$-dihedral algebras; Section 2.3, Theorems 4.1.4, 4.4.2
$\mathcal{G}, \mathcal{H}, \mathcal{R}$	a graph or Fischer space, esp. if G a 3-tr.-group or R a root system
$\mathcal{G}^{\pm}$	double of $\mathcal{G}$, see Section 3.1
$\mathcal{G}/\mathcal{H}$	boundary graph, see Section 3.1
$\mathrm{ad}(\mathcal{G})$	adjacency matrix of the graph $\mathcal{G}$
$k_{\mathcal{G}}, k^{\mathcal{H}}_{\mathcal{G}}$	valency of $\mathcal{G}$, boundary graph $\mathcal{G}/\mathcal{H}$, if regular

Table 1.4: Notation

Chapter 2. Matsuo algebras

A Matsuo algebra is an algebraic structure which captures the combinatorial information of 3-transposition groups. We will show that Matsuo algebras are the generic case of $\Phi(\alpha)$-axial algebras; we also discuss the pathological case $\alpha = \frac{1}{2}$, which allows Jordan algebras. In this chapter, we introduce and classify $\Phi(\alpha)$-axial algebras and Jordan algebras subject to containing 3-transposition groups.

2.1 Preliminaries

In this section, we recall or introduce many of the concepts important for the rest of this thesis. In particular, these are graphs $\mathcal{G}$ including Fischer spaces; the Matsuo algebra on $\mathcal{G}$; general fusion rules Φ and Φ-axial algebras; the Jordan fusion rules $\Phi(\alpha)$; and basic properties of idempotents, including their behaviour in associative algebras, which is the starting point of our generalisation in the rest of this work.

A *graph* is a pair $(\mathcal{G}, \mathcal{L})$, where $\mathcal{G}$ is a set of *points* and $\mathcal{L} \subseteq 2^{\mathcal{G}}$ a set of *lines*. Usually $\mathcal{G}$ alone refers to $(\mathcal{G}, \mathcal{L})$. An n-*graph* is a graph $\mathcal{G}$ for which any line $\ell \in \mathcal{L}$ has size n. We consider 2-graphs and 3-graphs in this text.

A graph $\mathcal{G}$ is *linear* if two distinct lines of $\mathcal{G}$ intersect in at most one point.

In a graph $\mathcal{G}$, for two distinct points $x, y \in \mathcal{G}$ we write that $x \sim y$ if x and y are collinear, that is, if there exists a line containing x and y, and $x \not\sim y$ otherwise. The graph $\mathcal{G}$ partitions with respect to x as $\{x\} \cup x^{\sim} \cup x^{\not\sim}$, where

$$x^{\sim} = \{y \in \mathcal{G} \mid x \sim y\} \text{ and } x^{\not\sim} = \{y \in \mathcal{G} \mid x \not\sim y\}. \tag{2.1}$$

Note that, in a linear 3-graph $\mathcal{G}$, for any two collinear points $x, y \in \mathcal{G}$ there exists a unique line ℓ connecting x and y, and a unique element denoted $x \wedge y \in \mathcal{G}$ such that $\ell = \{x, y, x \wedge y\}$. Linear 3-graphs are also known as partial (Steiner) triple systems.

A *subspace* of $\mathcal{G}$ is a subset $\mathcal{H} \subseteq \mathcal{G}$ such that any line containing two points of $\mathcal{H}$ lies entirely inside $\mathcal{H}$. The subspace $\langle P \rangle$ of $\mathcal{G}$ generated by a set of points $P \subseteq \mathcal{G}$ is the intersection of all subspaces of $\mathcal{G}$ containing P.

The *dual* $\mathcal{G}^{\vee}$ of a graph $(\mathcal{G}, \mathcal{L})$ is the graph with point set $\mathcal{L}$ and line set $\{\{\ell \in \mathcal{L} \mid x \in \ell\} \mid x \in \mathcal{G}\}$. The *affine plane* $\mathcal{P}_n$ *of order* n is the graph with point set $\mathbb{F}_n^2$ and lines $\{U + v \mid U \leq \mathbb{F}_n^2, \dim U = 1, v \in \mathbb{F}_n^2\}$. Two examples are given in Figure 2.1.

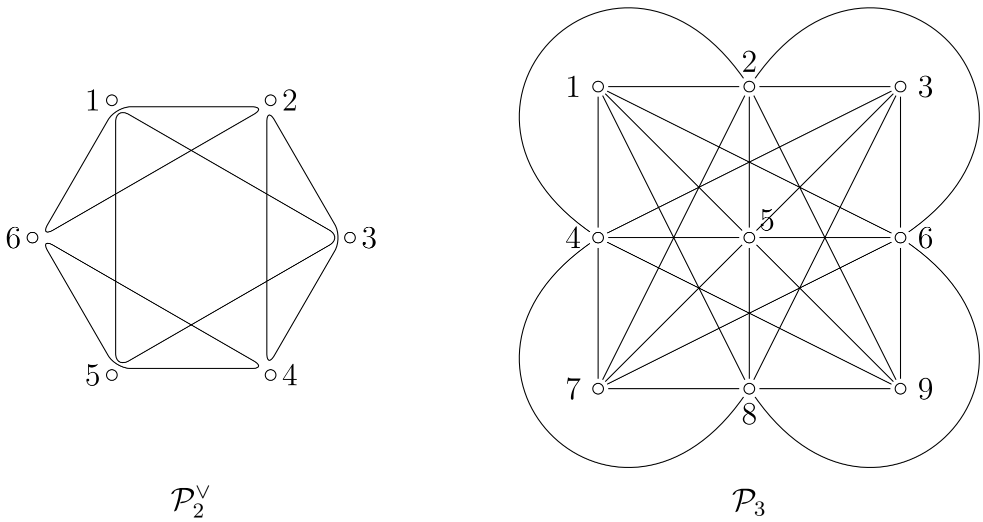

Figure 2.1: The dual affine plane $\mathcal{P}_2^{\vee}$ of order 2 and the affine plane $\mathcal{P}_3$ of order 3

We rephrase the definition of a Fischer space from [A97] as

Definition. A *Fischer space* is a linear 3-graph for which, if ℓ_1, ℓ_2 are any two distinct intersecting lines, the subspace $\langle \ell_1 \cup \ell_2 \rangle$ is isomorphic to the dual affine plane $\mathcal{P}_2^{\vee}$ of order 2 or to the affine plane $\mathcal{P}_3$ of order 3 from Figure 2.1.

Some important Fischer spaces come from root systems. Suppose that V is a vector space isomorphic to $\mathbb{R}^n$ with $(,)$ the Euclidean bilinear form. For $v \in V$, set $v^\perp = \{w \in V \mid (v, w) = 0\}$.

Definition. A *(crystallographic) root system* in V is a spanning set R satisfying

i. for $r \in \mathrm{R}$, $\kappa \in \mathbb{R}$, $\kappa r \in \mathrm{R}$ if and only if $\kappa = \pm 1$;

ii. for $r \in \mathrm{R}$, R is closed under reflection in $r^\perp$: for all $s \in \mathrm{R}$,

$$\sigma_r(s) = s - 2\frac{(r, s)}{(r, r)} r \in \mathrm{R}; \tag{2.2}$$

iii. for $r, s \in \mathrm{R}$, $2\frac{(r,s)}{(r,r)} \in \mathbb{Z}$.

The *Weyl group* $W(\mathrm{R})$ of R is the group generated by $\sigma_r, r \in \mathrm{R}$.

If all $r \in \mathrm{R}$ have the same length, that is, (r, r) is the same for all r, then R is *simply-laced*. A root system R is indecomposable if there is no proper partition $\mathrm{R} = \mathrm{R}' \cup \mathrm{R}''$ into root systems such that $(\mathrm{R}', \mathrm{R}'') = 0$. The simply-laced indecomposable root systems are the root systems A_n for $n \geq 1$, D_n for $n \geq 4$, $\mathrm{E}_6, \mathrm{E}_7$ and E_8, where X_d spans $\mathbb{R}^d$. By convention, A_0 is the empty set in $\mathbb{R}^0$.

For R a root system, write $\mathrm{R} = \mathrm{R}_+ \cup \mathrm{R}_-$ for some partition of roots R such that $\mathrm{R}_- = -\mathrm{R}_+$.

A Fischer space $\mathcal{G}$ is said to have *symplectic type* if its connected subspaces generated by three points can always be embedded in $\mathcal{P}_2^\vee$. Hence $\mathcal{P}_3$ cannot be embedded in a Fischer space of symplectic type, and all Fischer spaces coming from root systems are of symplectic type.

2.1.1 Lemma. *Suppose* R *is a simply-laced root system and* $\mathcal{R}$ *is the graph with point set* R_+ *and lines* $\{r, s, t\} \subseteq \mathrm{R}_+$ *spanning a root system of type* A_2*, for distinct roots* r, s *and* $t = r - s$ *or* $t = s - r$*. Then* $\mathcal{R}$ *is a Fischer space of symplectic type.*

Proof. Suppose that R is a simply-laced root system spanning V. As defined, $\mathcal{R}$ is a 3-graph. To show that $\mathcal{R}$ is linear, suppose ℓ_1, ℓ_2 are lines intersecting in two linearly independent points r, s. Then r, s span a root system A_2 in a subspace $U \subseteq V$ of dimension 2. Then $U \cap \mathrm{R}_+$ has size 3, so the third point in both ℓ_1 and ℓ_2 is uniquely determined, so that $\ell_1 = \ell_2$. Thus $\mathcal{R}$ is linear.

Now we show that any subspace spanned by two lines is contained in $\mathcal{P}_2^\vee$. Suppose that ℓ_1, ℓ_2 are two distinct intersecting lines, say $\ell_1 = \{r, s, t\}$ and $\ell_2 = \{r, u, v\}$, so that ℓ_1, ℓ_2 each span a copy of A_2. Therefore $U = \langle \ell_1 \cup \ell_2 \rangle$ in V is 3-dimensional and indecomposable, and as A_3 is the only simply-laced root system spanning a 3-dimensional space, $\ell_1 \cup \ell_2$ must span a copy of A_3. Observe that A_3 contains 6 positive roots and has 4 subspaces A_2, so that it is straightforward to see that the subspace $\langle \ell_1 \cup \ell_2 \rangle$ in $\mathcal{R}$ is isomorphic to $\mathcal{P}_2^\vee$ with 6 points and 4 lines. □

Whenever R is a simply-laced root system, we will denote its Fischer space by $\mathcal{R}$.

An *algebra* is a module over a ring R with a linear distributive multiplication. We will exclusively consider commutative algebras.

Rings in this thesis are always commutative and associative with 1. For the most part we will take R to be a field $\mathbb{F}$, a polynomial ring over $\mathbb{F}$, or a quotient of a polynomial ring of $\mathbb{Z}$. For this reason, for example working over a field or working in a free module, we almost always work in the following special situation. (We are not aware of similar definitions in the literature.)

Definition. A ring R is *everywhere faithful* for its module M if, for any nonzero $m \in M$, the annihilator ideal $\{r \in R \mid rm = 0\}$ of its action is always trivial.

We make two useful observations about such a ring:

2.1.2 Lemma. *If R is everywhere faithful on a nonzero module, then R is a domain.*

Proof. Suppose that $a, b \in R$ satisfy $ab = 0$. Take an arbitrary nonzero $m \in M$, M a nonzero module. Then $a(bm) = (ab)m$ by associativity of R, hence $a(bm) = 0$. As m is nonzero, its R-annihilator ideal is trivial, so $b = 0$ or $bm \neq 0$. In the latter case, a lies in the trivial R-annihilator of bm. In conclusion, either $a = 0$ or $b = 0$, so R contains no nontrivial zero divisors. This means that R is a domain. □

2.1.3 Lemma. *A (unital) domain R contains exactly two idempotents: 0 and 1.*

Proof. Let $e \in R$ be an arbitrary idempotent, so that $ee = e$. Then $e(1-e) = e1 - ee = e - e = 0$. As R contains no zero divisors, either $e = 0$ or $1 - e = 0$. □

We study the following algebra in detail in Chapters 2 and 3. Recall the notation that collinear x, y span a line $\{x, y, x \wedge y\}$.

Definition ([M03]). Let $R \ni \frac{1}{2}$ be a ring, $\alpha \in R$ and $\mathcal{G}$ a linear 3-graph. The *Matsuo algebra* $M_\alpha(\mathcal{G})$ is the free R-module with basis $\mathcal{G}$ with multiplication defined by,

$$\text{for } x, y \in \mathcal{G}, \quad xy = \begin{cases} x & \text{if } x = y \\ 0 & \text{if } x \not\sim y \\ \frac{\alpha}{2}(x + y - x \wedge y) & \text{if } x \sim y. \end{cases} \tag{2.3}$$

We will view $\mathcal{G}$ as embedded in $M_\alpha(\mathcal{G})$. Hence any $x \in \mathcal{G}$ is an idempotent, that is, $xx = x$. What can be said about the eigenvectors of x?

To avoid degeneracy, from now on *for the rest of the text* we assume $\alpha \neq 1, 0$.

2.1.4 Lemma. *The eigenspaces of $x \in \mathcal{G}$ in $M_\alpha(\mathcal{G})$ over $R = \mathbb{F}$ a field are*

$$\langle x \rangle, \text{ its } 1\text{-eigenspace,} \tag{2.4}$$

$$\langle y + x \wedge y - \alpha x \mid y \sim x \rangle \oplus \langle y \mid y \not\sim x \rangle, \text{ its } 0\text{-eigenspace, and} \tag{2.5}$$

$$\langle y - x \wedge y \mid y \sim x \rangle, \text{ its } \alpha\text{-eigenspace.} \tag{2.6}$$

The algebra $M_\alpha(\mathcal{G})$ decomposes as a direct sum of these eigenspaces for any $x \in \mathcal{G}$.

Proof. We show that $\mathrm{ad}(x)$ acts diagonalisably on $A = M_\alpha(\mathcal{G})$ by decomposing any vector into a sum of eigenvectors. Let $y \in A$ be arbitrary; the points of $\mathcal{G}$ form a basis for A, so by linearity we may assume $y \in \mathcal{G}$. If $x = y$ then y is a 1-eigenvector. If $x \not\sim y$ then $xy = 0$, so y is a 0-eigenvector. Otherwise $xy = \frac{\alpha}{2}(x + y - x \wedge y)$, and

$$y = y_1 + y_0 + y_\alpha, \quad y_1 = \frac{\alpha}{2}x, \quad y_0 = \frac{1}{2}(y + x \wedge y - \alpha x), \quad y_\alpha = \frac{1}{2}(y - x \wedge y), \tag{2.7}$$

where $xy_\phi = \phi y$ for $\phi \in \{1, 0, \alpha\}$: the latter two cases are

$$x(y - x \wedge y) = \frac{\alpha}{2}(x + y - x \wedge y - x - x \wedge y + y) = \alpha(y - x \wedge y), \tag{2.8}$$

$$x(y + x \wedge y - \alpha x) = \frac{\alpha}{2}(x + y - x \wedge y + x + x \wedge y - y) - \alpha x = \alpha x - \alpha x = 0. \tag{2.9}$$

Thus A has a basis of eigenvectors for $\mathrm{ad}(x)$, for any $x \in \mathcal{G}$, and the eigenvectors are those given. □

The key question that will occupy us is how eigenvectors multiply.

Definition. *Fusion rules* are a pair $(\Phi, \star)$, consisting of a set $\Phi \subseteq R$, called *eigenvalues*, in a ring R and a mapping $\star \colon \Phi \times \Phi \to 2^\Phi$. We also use Φ to refer to $(\Phi, \star)$.

For example, $\Phi(\alpha)$ are the *Jordan fusion rules* with eigenvalues $\{1, 0, \alpha\} \subseteq R$ for $\alpha \neq 1, 0$ and $\star$ symmetric as given by Table 2.2.

$\star$	1	0	α
1	$\{1\}$	$\emptyset$	$\{\alpha\}$
0		$\{0\}$	$\{\alpha\}$
α			$\{1, 0\}$

Table 2.2: The Jordan fusion rules $\Phi(\alpha)$

An element x in an algebra A induces an endomorphism $\mathrm{ad}(x) \in \mathrm{End}(A)$ given by left-multiplication $a \mapsto xa$. The α-eigenspace of $\mathrm{ad}(x)$ in A is denoted $A^x_\alpha = \{a \in A \mid xa = \alpha a\}$. By extension, if $\Psi \subseteq R$ is a set, we write $A^x_\Psi = \bigoplus_{\alpha\in\Psi} A^x_\alpha$ and $A^x_\emptyset = 0$.

If A is a vector space over a field $\mathbb{F}$, the eigenvalues of any endomorphism are always uniquely defined, but this is not the case in general if A is an R-module. In general, the endomorphism $\mathrm{ad}(x) \in \mathrm{End}(A)$ is said to be Φ-*diagonalisable* if A is the sum of ϕ-eigenspaces A^x_ϕ of $\mathrm{ad}(x)$ and these have pairwise trivial intersection, so that $A^x_\phi \cap A^x_\psi = 0$ if $\phi \neq \psi$. When A is a vector space, this is equivalent to the statement that the matrix representation of $\mathrm{ad}(x)$ with respect to some basis be a diagonalisable matrix. We say that x is diagonalisable, or Φ-diagonalisable, if $\mathrm{ad}(x)$ is diagonalisable, or Φ-diagonalisable. We always call the eigenvalues, -vectors, -spaces of $\mathrm{ad}(x)$ the eigenvalues, -vectors, -spaces of x.

If $x \in A$ is Φ-diagonalisable and $a \in A$, we write a^x_ϕ for the projection of a to the ϕ-eigenspace of x.

Definition. A Φ-diagonalisable idempotent e in an algebra A is a Φ-*axis*, or *axis*, if the multiplication of eigenvectors satisfies the fusion rules Φ:

$$\text{for all } x \in A^e_\phi, y \in A^e_\psi, \quad xy \in A^e_{\phi\star\psi} = \bigoplus_{\chi\in\phi\star\psi} A^e_\chi. \tag{2.10}$$

The last equation, if e is a Φ-axis, can thus be rewritten as $A^e_\phi A^e_\psi \subseteq A^e_{\phi\star\psi}$.

Definition. An algebra A is a Φ-*axial algebra* if it is generated by Φ-axes.

An idempotent $e \in A$ lies in its own 1-eigenspace A^e_1; e is *primitive* if e spans A^e_1.

The direct sum $A \oplus B$ of two algebras A, B over a ring R is the direct sum $\{(a,b) \mid a \in A, b \in B\}$ of their modules together with their pointwise products $(a_1,b_1)(a_2,b_2) = (a_1a_2, b_1b_2)$ for $a_1, a_2 \in A, b_1, b_2 \in B$.

2.1.5 Lemma. *Suppose that A, B are everywhere-faithful R-algebras. If $f \in A \oplus B$ is a primitive idempotent then $f \in A$ or $f \in B$.*

Proof. By assumption, $f = x + y$ for $x \in A$, $y \in B$ and $xy = 0$. From $f = ff$ we deduce $x+y = xx+0+yy$, so that $x = xx$ and $y = yy$. Now $xf = x(x+y) = xx+0 = x$, so that $x \in (A \oplus B)^f_1$. Primitivity of f implies that $(A \oplus B)^f_1 = \langle f \rangle \cong R$. This R is a domain by Lemma 2.1.2, and its only idempotents are 0 and 1 by Lemma 2.1.3, so the only idempotents in $\langle f \rangle$ are 0 and f. Therefore either $x = 0$ and $f = y$ or $x = f$ and $y = 0$. □

The following property is a generalisation of associativity, as we shall see, and is important throughout the sequel.

Definition. A fusion rule Φ is *Seress* if for all $\phi \in \Phi$ we have that $1 \star \phi, 0 \star \phi \subseteq \{\phi\}$, and in particular $1 \star 1 = \{1\}, 0 \star 0 = \{0\}, 1 \star 0 = 0 \star 1 = \emptyset$.

2.1.6 Lemma. *A Φ-diagonalisable idempotent $e \in A$ associates with its $1,0$-eigenspace $A^e_{1,0}$ in A,* i.e.,

$$e(xz) = (ex)z \textit{ for all } x \in A, z \in A^e_{1,0}, \tag{2.11}$$

if and only if e satisfies Seress fusion rules on Φ.

Proof. Suppose that $e \in A$ is a Φ-axis. Let $x, z \in A$ be arbitrary. By linearity, we may take $x \in A^e_\phi$ for some $\phi \in \Phi$. Then $ex = \phi x$ and in particular $(ex)z = \phi xz$.

Suppose that $\star$ are the smallest fusion rules on Φ satisfied by e, that is, for any $\phi, \psi, \chi \in \Phi$, $\chi \in \phi \star \psi$ if and only if $A^e_\chi \cap (A^e_\phi A^e_\psi) = (A^e_\phi A^e_\psi)^e_\chi \neq 0$. Observe that $xz \in A^e_\phi$ for any $x \in A^e_\phi, z \in A^e_{1,0}$ and $\phi \in \Phi$, if and only if $(\Phi, \star)$ is Seress. Furthermore $xz \in A^e_\phi$ if and only if $e(xz) = \phi xz$, that is, $e(xz) = (ex)z$. □

A further example of fusion rules is given by $\Phi_{\text{ass}} = \{1, 0\}$ with

$$1 \star 1 = \{1\}, \quad 0 \star 0 = \{0\}, \quad 1 \star 0 = 0 \star 1 = \emptyset. \tag{2.12}$$

2.1.7 Lemma. *A idempotent e in an associative algebra A over a field is a Φ_{ass}-axis.*

Proof. Let $x \in A$ be arbitrary; then $x = ex + (x - ex)$, and $e^n x = e(e(\cdots e(ex)\cdots))$ can also be written as $(e \cdots e)x = ex$, so that ex is a 1-eigenvector of e and $x - ex$ is a 0-eigenvector. Therefore A is spanned by $1, 0$-eigenvectors.

Suppose that $x, x' \in A_1^e$ and $z, z' \in A_0^e$. We see that $e(xx') = (ex)x' = xx'$ so $A_1^e A_1^e \subseteq A_1^e$, that is, $1 \star 1 \subseteq \{1\}$, and $e(zz') = (ez)z' = 0$ so $0 \star 0 \subseteq \{0\}$. For the mixed case, observe that $e(xz) = (ex)z = xz$ and $e(xz) = (ez)x = 0$, so $0 \star 1 = 1 \star 0 \subseteq \{1\} \cap \{0\} = \emptyset$. Therefore e is a Φ_{ass}-axis. □

2.1.8 Lemma. *An algebra A generated by Φ_{ass}-axes $\mathcal{A}$ is associative.*

Proof. Since Φ_{ass} is Seress, it follows that any $a \in \mathcal{A}$ associates with $A_{1,0}^a = A$ by Lemma 2.1.6. Therefore the associator

$$\text{Ass}(A) = \{x \in A \mid (xy)z = x(yz) \text{ for all } y, z \in A\} \tag{2.13}$$

of A contains $\mathcal{A}$. Observe that the associator is a subalgebra: for $\lambda \in R$ the underlying ring, $x \in \text{Ass}(A)$ implies $\lambda x \in \text{Ass}(A)$, and, for $w, x \in \text{Ass}(A)$ and $y, z \in A$,

$$((wx)y)z = (w(xy))z = w((xy)z) = w(x(yz)) = (wx)(yz), \tag{2.14}$$

so that $wx \in \text{Ass}(A)$. Since $\text{Ass}(A)$ is a subalgebra of A containing the generators of A, we have $A = \text{Ass}(A)$ and so A is associative. □

2.2 Characterising Fischer spaces

We prove that the eigenspaces of Lemma 2.1.4 of a Matsuo algebra $M_\alpha(\mathcal{G})$ on a graph $\mathcal{G}$ multiply according to the $\mathbb{Z}/2$-graded Jordan fusion rules $\Phi(\alpha)$ if and only if $\mathcal{G}$ is a Fischer space. Recall that $\alpha \in \mathbb{F} \setminus \{1, 0\}$ and $\frac{1}{2} \in \mathbb{F}$ throughout.

2.2.1 Theorem. *Suppose that $\mathcal{G}$ is a Fischer space. Then every $x \in \mathcal{G}$ is a $\Phi(\alpha)$-axis in the Matsuo algebra $M_\alpha(\mathcal{G})$ over $\mathbb{F}$.*

Proof. We have to prove that the eigenvectors of $x \in \mathcal{G}$ satisfy the fusion rules $\Phi(\alpha)$. That x is diagonalisable with eigenvalues $\{1, 0, \alpha\}$ follows by Lemma 2.1.4.

Since it also follows by Lemma 2.1.4 that the 1-eigenspace of x is spanned by x, the fusion rules $1 \star 1 = \{1\}, 1 \star 0 = \emptyset$ and $1 \star \alpha = \{\alpha\}$ are immediately seen to be satisfied. It remains to prove that $0 \star 0 = \{0\}, 0 \star \alpha = \{\alpha\}$ and $\alpha \star \alpha = \{1, 0\}$.

We first show that $0 \star 0 = \{0\}$; it breaks up into three cases, since there are two kinds of 0-eigenvectors for x. The first case is $y, z \in x^{\not\sim}$. Then if $y = z$, $yz = y \in x^{\not\sim}$; if $y \not\sim z$, $yz = 0$; and if $y \sim z$, then $yz = \frac{\alpha}{2}(y + z - y \wedge z)$. If the last possibility is realised and $x \not\sim y \wedge z$ then $x(yz) = 0$ and we are done. We can also rule out $x \sim y \wedge z$ by a general observation: a point x cannot be collinear to only one point in a line ℓ not containing x, for $\{x\} \cup \ell$ generate $\mathcal{P}_2^\vee$ or $\mathcal{P}_3$, and x is collinear to either two or three points in ℓ in these cases.

Suppose the second case: $y \in x^\sim$ and $z \in x^{\not\sim}$. If $z \not\sim y, x \wedge y$ then $z(y + x \wedge y - \alpha x) = 0$. As before, it is not possible that $z \sim y$ and $z \not\sim x \wedge y$, or vice-versa. If $z \sim y, x \wedge y$, then as all points are collinear in $\mathcal{P}_3$, x, y, z must lie in a subspace $\mathcal{P}_2^\vee$. Thus

$$z(y + x \wedge y - \alpha x) = \alpha z + \frac{\alpha}{2}((y + x \wedge y) - (z \wedge y + z \wedge (x \wedge y))) \tag{2.15}$$

and this, when multiplied by x, yields $0 + \frac{\alpha}{2}(\alpha x - \alpha x) = 0$. Hence $x((y + y \wedge x - \alpha x)z) = 0$.

Suppose finally that $y, z \in x^{\sim}$. Observe

$$(y + x \wedge y - \alpha x)(z + x \wedge z - \alpha x) = yz + y(x \wedge z) + z(x \wedge y) + (x \wedge z)(x \wedge y) - \alpha^2 x; \tag{2.16}$$

if $y = z$ or $y = x \wedge z$, this reduces to the obvious 0-eigenvector

$$= (1 + \alpha)(y + x \wedge y - \alpha x). \tag{2.17}$$

Otherwise, x, y, z lie in $\mathcal{P}_2^{\vee}$ or $\mathcal{P}_3$. Equation (2.16) can be grouped as

$$= yz + (x \wedge y)(x \wedge z) + \big[(x \wedge y)z + (x \wedge z)y\big]_{\mathcal{P}_3} - \alpha^2 x \tag{2.18}$$

where the square-bracketed terms are zero in $\mathcal{P}_2^{\vee}$;

$$\begin{aligned} &= \frac{\alpha}{2}\big(y + x \wedge y + z + x \wedge z\big) - \alpha^2 x \\ &\quad - \frac{\alpha}{2}\big(y \wedge z + (x \wedge y) \wedge (x \wedge z)\big) \\ &\quad + \frac{\alpha}{2}\big[y + x \wedge y + z + x \wedge z - ((x \wedge y) \wedge zx(x \wedge z) \wedge y)\big]_{\mathcal{P}_3}. \end{aligned} \tag{2.19}$$

The first line always is a 0-eigenvector; the second line is also a 0-eigenvector in $\mathcal{P}_2^{\vee}$, and the second plus third lines are 0-eigenvectors in $\mathcal{P}_3$. Therefore $x((y + x \wedge y - \alpha x)(z + x \wedge z - \alpha x)) = 0$ and we have that $0 \star 0 = \{0\}$.

We next show that $0 \star \alpha = \{\alpha\}$. There are two cases. Suppose firstly that $y \in x^{\sim}, z \in x^{\not\sim}$. Then either $z \not\sim y, x \wedge y$, so $(y - x \wedge y)z = 0$, or we are in $\mathcal{P}_2^{\vee}$. Then

$$(y - x \wedge y)z = \frac{\alpha}{2}\big(z + y - y \wedge z - z - x \wedge y + (x \wedge y) \wedge z\big) = \frac{\alpha}{2}(y - x \wedge y) + \frac{\alpha}{2}(y \wedge z - x \wedge (y \wedge z)) \tag{2.20}$$

is clearly an α-eigenvector, where we used that, in $\mathcal{P}_2^{\vee}$, $z \wedge (x \wedge y) = x \wedge (y \wedge z)$. Suppose secondly that $y, z \in x^{\sim}$. We again bracket off terms only occurring in $\mathcal{P}_3$. We see $(y - x \wedge y)(z + x \wedge z - \alpha x)$

$$= yz - (x \wedge y)(x \wedge z) - \alpha^2 (y - x \wedge y) + \big[y(x \wedge z) - z(x \wedge y)\big]_{\mathcal{P}_3} \tag{2.21}$$

$$= \alpha^2(x \wedge y - y) + \frac{\alpha}{2}\big(y - x \wedge y + z - x \wedge z\big) + \big[y \wedge z - (x \wedge y) \wedge (x \wedge z) \\ + y - x \wedge y - (z - x \wedge z) + x \wedge z - (x \wedge z) \wedge y + x \wedge y - (x \wedge y) \wedge z\big]_{\mathcal{P}_3} \tag{2.22}$$

and we again recognise these to be α-eigenvectors on each line. Here, as $y \wedge z = (x \wedge y) \wedge (x \wedge z)$ in $\mathcal{P}_2^\vee$, we wrote $y \wedge z - (x \wedge y) \wedge (x \wedge z)$ into the $\mathcal{P}_3$-bracket. This shows $0 \star \alpha = \{\alpha\}$.

The final case is $\alpha \star \alpha = \{1, 0\}$. Suppose $y, z \in x^{\sim}$. Then

$$(y - x \wedge y)(y - x \wedge y) = y + x \wedge y - \alpha(y + x \wedge y) + \alpha x. \tag{2.23}$$

$$\begin{aligned}(y - x \wedge y)(z - x \wedge z) &= yz + (x \wedge y)(x \wedge z) + \big[(x \wedge y)z + (x \wedge z)y\big]_{\mathcal{P}_3} \\ &= \frac{\alpha}{2}\big(y + x \wedge y + z + x \wedge z - (y \wedge z + (x \wedge y) \wedge (x \wedge z))\big) \\ &\quad + \frac{\alpha}{2}\big[y + x \wedge y + z + x \wedge z - ((x \wedge y) \wedge z + (x \wedge z) \wedge y)\big]_{\mathcal{P}_3}.\end{aligned} \tag{2.24}$$

Observe that $y + x \wedge y$ is a sum of 1- and 0-eigenvectors for x. Also $y \wedge z = (x \wedge y) \wedge (x \wedge z)$ is a 0-eigenvector in $\mathcal{P}_2^\vee$. In $\mathcal{P}_3$, $\{x, y \wedge z, (x \wedge y) \wedge (x \wedge z)\}$ and $\{x, (x \wedge y) \wedge z, (x \wedge z) \wedge y\}$ are lines. This allows us to conclude the final case. □

2.2.2 Theorem. *Suppose that $\mathcal{G}$ is a linear 3-graph for which every $x \in \mathcal{G}$ is a $\Phi(\alpha)$-axis in $M_\alpha(\mathcal{G})$. Then $\mathcal{G}$ is a Fischer space.*

Proof. The fusion rules $1 \star \phi = \{\phi\}$ for $\phi = 1, \alpha$ and $1 \star 0 = \emptyset$ are trivially seen to be satisfied. It turns out that any of the remaining fusion rules $0 \star 0 = \{0\}, 0 \star \alpha = \{\alpha\}$ and $\alpha \star \alpha = \{1, 0\}$ are enough to imply that $\mathcal{G}$ is a Fischer space. The case with the most straightforward combinatorial argument is $0 \star \alpha = \{\alpha\}$, so this is the one we will exhibit.

Suppose that $y \sim x$ and $z \not\sim x$. If $z \not\sim y, x \wedge y$ then $z(y - x \wedge y) = 0$, trivially satisfying the fusion rule. If z is collinear with exactly one of $y, x \wedge y$, say with y but not with $x \wedge y$, then $z(y - x \wedge y) = \frac{\alpha}{2}(z + y - y \wedge z)$ is a product of 0- and α-eigenvectors.

This is not an α-eigenvector of x, as $x(z+y-y\wedge z) = \alpha(x+y-x\wedge y) - x(y\wedge z)$ has no term in z on the right unless $x\wedge(y\wedge z) = z$, which contradicts that $x \not\sim z$. Therefore we can rule this case out.

We are left with the situation where $z \sim y, x\wedge y$.

$$\begin{aligned} z(y - x\wedge y) &= \frac{\alpha}{2}(z + y - z\wedge y - z - x\wedge y + (x\wedge y)\wedge z) \\ &= \frac{\alpha}{2}(y - x\wedge y - (y\wedge z - (x\wedge y)\wedge z)) \end{aligned} \tag{2.25}$$

Evidently $y - x\wedge y$ is an α-eigenvector, so necessarily $y\wedge z - (x\wedge y)\wedge z$ is too. That is,

$$\alpha(y\wedge z - (x\wedge y)\wedge z) = x(y\wedge z) - x((x\wedge y)\wedge z) \tag{2.26}$$

and now there are several cases. If both terms on the righthand side are 0, then $y\wedge z = (x\wedge y)\wedge z$, but this is not possible: it implies that there are distinct lines $\{y, z, y\wedge z\}$ and $\{x\wedge y, z, (x\wedge y)\wedge z\}$ intersecting in more than one point, contradicting the linearity of $\mathcal{G}$. If just one term on the right is nonzero, there remains a nonzero contribution of x, whereas x has no part on the left. Therefore neither terms are 0 and x is collinear to $y\wedge z$ and $(x\wedge y)\wedge z$. So, continuing equation (2.26),

$$= \frac{\alpha}{2}(x + y\wedge z - x\wedge(y\wedge z) - x - (x\wedge y)\wedge z + x\wedge((x\wedge y)\wedge z). \tag{2.27}$$

Collecting terms and rescaling, using that α is invertible, we find

$$y\wedge z - (x\wedge y)\wedge z = x\wedge((x\wedge y)\wedge z) - x\wedge(y\wedge z). \tag{2.28}$$

It remains impossible for $y\wedge z$ to equal $(x\wedge y)\wedge z$ or $x\wedge(y\wedge z)$. Therefore $y\wedge z = x\wedge((x\wedge y)\wedge z)$ and $(x\wedge y)\wedge z = x\wedge(y\wedge z)$. These identifications completely determine the subspace generated by the lines $\{x, y, x\wedge y\}$ and $\{y, z, y\wedge z\}$ as isomorphic to $\mathcal{P}_2^\vee$.

The other possibility is for $y, z \in x^\sim$. If $z, x\wedge z \not\sim y, x\wedge y$ then $(y - x\wedge y)(z + x\wedge z - \alpha x) = \alpha^2(y - x\wedge y)$ is an α-eigenvector. If $z \sim y$ and $z \not\sim x\wedge y$, then z is in the rôle of x in the previous paragraph and therefore $\{x, y, x \wedge y\}$ and $\{x, z, x \wedge z\}$ generate $\mathcal{P}_2^\vee$. We suppose otherwise, and can therefore assume that $y, x \wedge y \sim z, x \wedge z$. So

$$\begin{aligned}(y - x \wedge y)(z + x \wedge z - \alpha x) &= (y - x \wedge y)(z + x \wedge z) - \alpha^2(y - x \wedge y)\\ &= yz + y(x \wedge z) - (x \wedge y)z - (x \wedge y)(x \wedge z) - \alpha^2(y - x \wedge y)\\ &= \frac{\alpha}{2}\big(2(y - x \wedge y) - (y \wedge z - (x \wedge y) \wedge (x \wedge z))\\ &\quad + ((x \wedge y) \wedge z - y \wedge (x \wedge z))\big) - \alpha^2(y - x \wedge y).\end{aligned} \tag{2.29}$$

Getting rid of the α-eigenvectors $y - x \wedge y$, we deduce that

$$\begin{aligned}&\alpha\left((x \wedge y) \wedge (x \wedge z) - y \wedge z + (x \wedge y) \wedge z - y \wedge (x \wedge z)\right)\\ &\quad = x\left((x \wedge y) \wedge (x \wedge z) - y \wedge z + (x \wedge y) \wedge z - y \wedge (x \wedge z)\right).\end{aligned} \tag{2.30}$$

We may assume that all the products on the righthand side are nonzero, since otherwise we return to the case $z \not\sim x, y \sim x$ already dealt with whose conclusion was $\mathcal{P}_2^\vee$. Thus $y \wedge z, (x \wedge y) \wedge z, y \wedge (x \wedge z), (x \wedge y) \wedge (x \wedge z) \in x^\sim$. Now $x \wedge ((x \wedge y) \wedge (x \wedge z))$, for example, does not appear on the right and therefore, to cancel on the left, it must be equal to one of $x, y \wedge z, x \wedge ((x \wedge y) \wedge z)$ or $y \wedge (x \wedge z)$. Of course it is not equal to x. The situation and the argument so far is symmetric under interchange of y and z, as is $x \wedge ((x \wedge y) \wedge (x \wedge z))$; $x \wedge ((x \wedge y) \wedge z)$ and $y \wedge (x \wedge z)$ are not, so this leaves $x \wedge ((x \wedge y) \wedge (x \wedge z)) = y \wedge z$ as the only possibility. Likewise $x \wedge ((x \wedge y) \wedge z) = x \wedge (y \wedge (x \wedge z))$ and we can identify the subspace generated by $\{x, y, x \wedge y\}$ and $\{x, z, x \wedge z\}$ as $\mathcal{P}_3$.

This means that for an arbitrary pair of lines intersecting in precisely one point, the subspace they generate is isomorphic either to $\mathcal{P}_2^\vee$ or $\mathcal{P}_3$. $\square$

2.3 A Sakuma theorem

We saw that, among the Matsuo algebras on linear 3-graphs, the $\Phi(\alpha)$-axial property characterises Fischer spaces. More generally, what can be said about an arbitrary algebra which has a set of $\Phi(\alpha)$-axial generators? The first answer, for two generators, is given by Theorem 2.3.1. We also classify the idempotents in all possibilities.

We use the notation $(\mathrm{n}X)$ to indicate the isomorphism type of certain algebras, and conflate the name $(\mathrm{n}X)$ with an instance of such an algebra. We introduce some isomorphism types now, namely those of Theorem 2.3.1, over a ring $R \ni \frac{1}{2}$.

The algebra $(1A)$ is the 1-dimensional algebra generated by an idempotent e. The algebra $(2B)$ is the direct sum $\langle e\rangle \oplus \langle f\rangle$ over R of two $(1A)$-algebras.

The algebra $(3C_\alpha)$, for $\alpha \neq 1, 0$, is spanned by $\{e, f, g\}$ with multiplication given by, for $\{x, y, z\} = \{e, f, g\}$, $xx = x$ and $xy = \frac{\alpha}{2}(x + y - z)$, as in Table 2.3.

$(3C_\alpha)$	e	f	g
e	e	$\frac{\alpha}{2}(e+f-g)$	$\frac{\alpha}{2}(e+g-f)$
f		f	$\frac{\alpha}{2}(f+g-e)$
g			g

Table 2.3: The algebra $(3C_\alpha)$

The algebras $(3C_{-1}^{\times})$ and $(3J_0^{\times})$ are spanned by $\{e, f\}$ with products $ee = e$, $ef = \alpha(e + f)$, $ff = f$ when $\alpha = -1$ and $\alpha = \frac{1}{2}$ respectively.

The algebra $(3J_\kappa)$, for $\alpha = \frac{1}{2}$ and $\kappa \in R$, is spanned by $\{e, f, e \circ f\}$ (*c.f.* Lemma 2.3.2 for $e \circ f$) and multiplication from Table 2.4.

An algebra generated by two primitive Φ-axes is called a Φ-*dihedral algebra*.

$(3J_\kappa)$	e	f	$e \circ f$
e	e	$\frac{1}{2}e + \frac{1}{2}f + e \circ f$	κe
f		f	κf
$e \circ f$			$\kappa(e \circ f)$

Table 2.4: The algebra $(3J_\kappa)$

2.3.1 Theorem. *Suppose that R is a ring containing $\frac{1}{2}, \alpha, \alpha^{-1}, (\alpha-1)^{-1}$ and A is a $\Phi(\alpha)$-dihedral everywhere faithful R-algebra. Then A is isomorphic to one of $(1A)$, $(2B)$, $(3C_\alpha)$, or $(3C_{-1}^{\times})$, or to $(3J_\kappa)$, $(3J_0^{\times})$ if $\alpha = \frac{1}{2}$ and $\kappa \in R$.*

We prove Theorem 2.3.1 after the general and useful result

2.3.2 Lemma. *Suppose that e, f are Φ-axes with $\alpha \in \Phi$. Set $e \circ f = ef - \alpha e - \alpha f$. Then $(e \circ f)^e_\alpha = 0 = (e \circ f)^f_\alpha$.*

Proof. Recall that $e = \sum_{\phi \in \Phi} e^f_\phi$. As, with respect to f, $ef = \sum_{\phi \in \Phi} \phi e^f_\phi$,

$$e \circ f = \sum_{\phi \in \Phi} \phi e^f_\phi - \sum_{\phi \in \Phi} \alpha e^f_\phi - \alpha f = (1-\alpha)e^f_1 - \alpha f - \alpha e^f_0 + \sum_{\phi \in \Phi \setminus \{1,0\}} (\phi - \alpha) e^f_\phi. \tag{2.31}$$

Of course $f \in A^f_1$ and $e^f_\phi \in A^f_\phi$. The coefficient of e^f_ϕ in the above expression is $\phi - \alpha$, so the coefficient of e^f_α is 0 and $(e \circ f)^f_\alpha = 0$. Likewise $(e \circ f)^e_\alpha = 0$. □

Proof of Theorem 2.3.1. If f is in the span of e then $\langle e, f \rangle = \langle e \rangle$, and, as R acts everywhere faithfully, $\langle e \rangle = (1A) \cong R$ (likewise if e is in the span of f). From now on we suppose that neither axis e, f is in the span of the other axis.

Our strategy is to use relations in the algebra, and that R acts everywhere faithfully, to derive relations in the ring. We first show a symmetry of coefficients in the decompositions of e and f with respect to each other. Namely, set λ_1, λ_2 such that $e^f_1 = \lambda_1 f$ and $f^e_1 = \lambda_2 e$. These coefficients are unique because e spans A^e_1 so

$f_1^e = re$ for some $r \in R$; if $s \in R$ satisfies $f_1^e = se$, then $(r-s)e = re - se = f_1^e - f_1^e = 0$. As e is nonzero, by assumption its R-annihilator is 0, so $r - s = 0$ and $r = s$, showing uniqueness of λ_2. The same argument applies of course for λ_1.

By Lemma 2.3.2, $e \circ f \in A^e_{\{1,0\}} \cap A^f_{\{1,0\}}$ and furthermore

$$e(e\circ f) = (e\circ f)_1^e = (1-\alpha)f_1^e - \alpha e = ((1-\alpha)\lambda_2 - \alpha)e, \quad f(e\circ f) = ((1-\alpha)\lambda_1 - \alpha)f. \quad \textbf{(2.32)}$$

Applying Lemma 2.1.6,

$$(ef)(e\circ f) = e(f(e\circ f)) = ((1-\alpha)\lambda_1 - \alpha)ef \quad \textbf{(2.33)}$$

$$= f(e(e\circ f)) = ((1-\alpha)\lambda_2 - \alpha)ef. \quad \textbf{(2.34)}$$

By rearranging and using the invertibility of $1-\alpha$ we have $(\lambda_1 - \lambda_2)ef = 0$. If $ef = 0$ then $\langle e, f\rangle = \langle e\rangle \oplus \langle f\rangle = (2B)$.

Suppose from now on that $ef \neq 0$. As the ring acts everywhere faithfully, we have $ef \neq 0$ and $\lambda_1 = \lambda_2$, and we will now denote λ_1 by λ. This also shows that $\{e, f, e\circ f\}$ is closed under multiplication and spans A.

Suppose now that ef is in the span of e and f. Then, as $ef \neq 0$, we have $A = A_1^e \oplus A_\alpha^e = A_1^f \oplus A_\alpha^f$. By Lemma 2.3.2 $e\circ f \in A^e_{1,0} \cap A^f_{1,0} = A^e_1 \cap A^f_1$. This intersection is 0, so $e \circ f = 0$ and $ef = \alpha e + \alpha f$. By our assumption that $\alpha - 1$ is invertible, $e(f + \frac{\alpha}{\alpha-1}e) = \alpha(f + \frac{\alpha}{\alpha-1}e)$ exhibits an α-eigenvector for e. The fusion rule $\alpha \star \alpha = \{1, 0\}$ reduces to $\alpha \star \alpha = \{1\}$ and, as $f_\alpha^e \neq 0$, we deduce that the coefficient of f in

$$(f + \frac{\alpha}{\alpha-1}e)(f + \frac{\alpha}{\alpha-1}e) = f + \frac{\alpha^2}{(\alpha-1)^2}e + \frac{2\alpha^2}{\alpha-1}(e+f) \quad \textbf{(2.35)}$$

is 0. Therefore α is a root of $1 + \frac{2\alpha^2}{\alpha-1}$, equivalently, of $2\alpha^2 + \alpha - 1 = (2\alpha-1)(\alpha+1)$. As $\frac{1}{2} \in R$ we know $2 \neq 0$ and so $\alpha = \frac{1}{2}$ or $\alpha = -1$. These are the cases $(3J_0^\times), (3C_{-1}^\times)$.

Finally, we consider the remaining case: that e, f, ef (and likewise $e, f, e\circ f$) are linearly independent. Observe that $x = (\alpha - \lambda)e + \alpha f + e\circ f$ is an α-eigenvector for

e, and $z = ((1-\alpha)\lambda - \alpha)e - e \circ f$ is a 0-eigenvector. As A is 3-dimensional and by assumption decomposes into a direct sum of eigenspaces, we have $A = A_1^e \oplus A_0^e \oplus A_\alpha^e$. We compute

$$\begin{aligned} xx = &\big((\alpha-\lambda)^2 + 2\alpha^2(\alpha-\lambda) + 2(\alpha-\lambda)((1-\alpha)\lambda-\alpha)\big)\, e \\ &+ \big(2\alpha^2(\alpha-\lambda) + \alpha^2 + 2\alpha((1-\alpha)\lambda-\alpha)\big)\, f \\ &+ (((1-\alpha)\lambda-\alpha) + 2\alpha(\alpha-\lambda))\, e \circ f. \end{aligned} \tag{2.36}$$

Since $e, e \circ f \in A^e_{\{1,0\}}$, we have that $xx \in A^e_{\{1,0\}}$ if and only if $(xx)^e_\alpha = 0$ if and only if the coefficient of f in the above expression is 0. The coefficient factors as $\alpha(2\alpha-1)(\alpha-2\lambda)$. As $\alpha \neq 0$ and furthermore $\alpha^{-1} \in R$ means α is not a 0-divisor, this gives two possibilities.

If $2\alpha - 1 = 0$, then $\alpha = \frac{1}{2}$. This is the case $(3J_\kappa)$, with $\kappa = (1-\alpha)\lambda - \alpha = \frac{1}{2}(\lambda - 1)$.

If $\alpha = 2\lambda$, this is the case $(3C_\alpha)$: namely, set $g = f_1^e + f_0^e - f_\alpha^e$. Then it is easy to check $ef = \frac{\alpha}{2}(e + f - g)$ and the rest of the structure constants in Table 2.3. Also, $\lambda = -\frac{\alpha}{2}(\alpha+1)$, so $x(e \circ f) = -\frac{\alpha}{2}(\alpha+1)x$ for all $x \in A$. □

Observe that both cases at the end of the proof may be satisfied simultaneously: this gives $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{4}$ and $\kappa = -\frac{3}{8}$, so that $(3C_{1/2}) \cong (3J_{-3/8})$.

We now classify the idempotents of the previously-named algebras, *over a field* $\mathbb{F}$.

In $\langle e \rangle = (1A) \cong \mathbb{F}$, there are 2 idempotents: 0 and $e = \mathrm{id}_{(1A)}$.

2.3.3 Proposition. *i. In* $(2B)$ *there are* 4 *idempotents:* $0, e, f, \mathrm{id} = e + f$.

ii. In $(3C^\times_{-1})$ *there are* 4 *idempotents:* $0, e, f, -e-f$.

iii. The idempotents in $(3J_0^\times)$ *are* 0 *and* $e_\mu = \mu e + (1-\mu)f$ *for* $\mu \in \mathbb{F}$.

Proof. Let $\alpha = -1, 0$ or $\frac{1}{2}$, and let A be spanned by e, f with multiplication $ef = \alpha(e+f)$, corresponding to the cases $(3C^\times_{-1})$, $(2B)$ and $(3J_0^\times)$. Then write $x = \mu e + \nu f$

and solve for $xx = x$:

$$\mu e + \nu f = \mu(\mu + 2\alpha\nu)e + \nu(\nu + 2\alpha\mu)f, \tag{2.37}$$

and since $\mu\nu = 0$ would leave only the possibilities $0, e, f$, we assume that $\mu\nu \neq 0$. Therefore $1 = \mu + 2\alpha\nu = \nu + 2\alpha\mu$, and by substituting we find $\nu(1-2\alpha)(1+2\alpha) = 1-2\alpha$. Therefore either $\alpha = \frac{1}{2}$ or $\nu = \frac{1}{1+2\alpha}$. In the former case, we find $\mu = 1 - \nu$, and in the latter, $\mu = \frac{1}{1+2\alpha} = \nu$. Substituting α gives the advertised possibilities. $\square$

Moreover, in $(3J_0^\times)$, $e_\mu e_\nu = e_{(\mu+\nu)/2}$ and $e_\mu - e_\nu \in \langle e - f \rangle$ for all $\mu, \nu \in R$.

2.3.4 Theorem. *i. In $(3C_\alpha)$ for $\alpha \neq -1, \frac{1}{2}$ there are 8 idempotents:*

$$0, \quad e, f, g = f_1^e + f_0^e - f_\alpha^e, \quad \mathrm{id} - e, \mathrm{id} - f, \mathrm{id} - g, \quad \mathrm{id} = \frac{1}{1+\alpha}(e + f + g). \tag{2.38}$$

ii. In $(3C_{-1})$ there are 4 idempotents: $0, e, f, g = -e - f + 2s$.

iii. The idempotents in $(3J_\kappa)$ are, for $\kappa \neq 0$, $\pi \in \mathbb{F}$ and $\mu_\kappa(\pi), \bar{\mu}_\kappa(\pi)$ solutions of $\mu^2 - \mu(1 - 2\kappa\pi) - \frac{1}{2}\pi(\kappa\pi - 1) = 0$ with respect to μ,

$$0, \quad e_\pi = \mu_\kappa(\pi)e + \bar{\mu}_\kappa(\pi)f + \pi(e \circ f), \quad \frac{1}{\kappa}e \circ f - e_\pi, \quad \frac{1}{\kappa}e \circ f = \mathrm{id}\,. \tag{2.39}$$

iv. The idempotents in $(3J_0)$ are, for $\pi \in \mathbb{F}$ and $\mu_0(\pi), \bar{\mu}_0(\pi)$ solutions of $\mu^2 - \mu + \frac{1}{2}\pi = 0$ with respect to μ, 0 and $e_\pi = \mu_0(\pi)e + \bar{\mu}_0(\pi)f + \pi(e \circ f)$.

Proof. Suppose that $A = \langle e, f \rangle$ for e, f $\Phi(\alpha)$-axes such that A is spanned by $\{e, f, e \circ f\}$. Then for $x = \mu e + \nu f + \pi e \circ f$, $xx = x$ implies that

$$Q_\mu = \mu(\mu + 2\kappa\pi + 2\alpha\nu - 1) \tag{2.40}$$

$$Q_\nu = \nu(\nu + 2\kappa\pi + 2\alpha\mu - 1) \tag{2.41}$$

$$Q_\pi = \pi(\kappa\pi - 1) + 2\mu\nu \tag{2.42}$$

all vanish.

When $\mu\nu\pi = 0$, there are no additional idempotents: if $\pi = 0$, the only solutions to $(\mu e + \nu f)^2 = \mu e + \nu f$ have $\mu\nu = 0$, as ef is linearly independent of e, f and has coefficient $2\mu\nu$, and $2 \neq 0$. If $\mu\nu = 0$, say, without loss of generality, that $\mu \neq 0$, then

$$(\mu e + \pi e \circ f)^2 = \mu(\mu + 2\kappa\pi)e + \kappa\pi^2 e \circ f = \mu e + \pi e \circ f, \tag{2.43}$$

for $\kappa = (1-\alpha)\lambda - \alpha$, so $\pi = 0$ or, if $\kappa \neq 0$, $\pi = \frac{1}{\kappa}$. If $\pi = \frac{1}{\kappa}$, then $\mu = \mu(\mu + 2)$; if $\mu \neq 0$, then $\mu = -1$. This corresponds to the idempotent $e' = \mathrm{id}_A - e$, when $\kappa \neq 0$.

Under the assumption that $\mu\nu \neq 0$,

$$Q'_\mu = \mu + 2\kappa\pi + 2\alpha\nu - 1, \quad Q'_\nu = \nu + 2\kappa\pi + 2\alpha\mu - 1 \tag{2.44}$$

also vanish. Observe $Q'_\mu = Q'_\nu$ if and only if $\alpha = \frac{1}{2}$, so we find two cases.

First assume that $\alpha = \frac{1}{2}$. Then substituting from $Q'_\mu = Q'_\nu$ into Q_π we obtain

$$\mu^2 - \mu(1 - 2\kappa\pi) - \frac{1}{2}\pi(\kappa\pi - 1) = 0 = \nu^2 - \nu(1 - 2\kappa\pi) - \frac{1}{2}\pi(\kappa\pi - 1). \tag{2.45}$$

So μ, ν are roots of the same quadratic equation. Since also $\mu\nu = -\frac{1}{2}\pi(\kappa\pi - 1)$, μ, ν are distinct roots of the equation (noting that $\pi(\kappa\pi - 1)$ and $2\kappa\pi - 1$ cannot both vanish), we write $\mu = \mu_\kappa(\pi)$ and $\nu = \bar{\mu}_\kappa(\pi)$ for the two roots. The case $\kappa = 0$ follows.

Suppose now that $\alpha \neq \frac{1}{2}$. If $\alpha = -1$, we observe that $\mu = \nu = -1, \pi = 2$ is the only possibility for $\mu\nu \neq 0$, which corresponds to g. Otherwise, observe that if α is specialised to any value other than $\frac{1}{2}$ then Q'_μ, Q'_ν, Q_π are independent irreducible polynomials by inspection, and therefore $\mathbb{F}[\mu, \nu, \pi]/(Q'_\mu, Q'_\nu, Q_\pi)$ is 0-dimensional. Also, Q'_μ, Q'_ν, Q_π are of degree 2. Then by Bezout's lemma [CLO96], Section 8.7, there are at most 2^3 solutions satisfying Q'_μ, Q'_ν, Q_π. But we have already given 8 idempotents when $\alpha \neq -1$, whence these are all solutions. □

In fact, we can say more about these idempotents. We continue to work over a field, although it is possible that the assumptions could be weakened.

2.3.5 Lemma. *The idempotents in* $(3C_\alpha)$*, for* $\alpha \neq \frac{1}{2}, -1$*, are* 0*,* id*, and the primitive* $\Phi(\alpha)$*-axes* e_1, e_2, e_3 *and* $\Phi(1-\alpha)$*-axes* $e_i' = \mathrm{id} - e_i$*.*

Proof. By Theorem 2.3.4 we have 0 and id in $A = (3C_\alpha)$ for $\alpha \neq -1, \frac{1}{2}$, and $e, f, g \in A$ are $\Phi(\alpha)$-axes. It only remains to show that $e' = \mathrm{id} - e$ is a $\Phi(1-\alpha)$-axis in A. This follows from the fact that, if $x \in A$ is a ϕ-eigenvector of e, then $e'x = (\mathrm{id} - e)x = x - \phi x = (1-\phi)x$, so x is a $1-\phi$-eigenvector of e'. Hence $\mathrm{Spec}(e') = 1 - \mathrm{Spec}(e) = \{1, 0, 1-\alpha\}$. As a consequence, if $x \in A$ decomposes into a sum of $\Psi \subseteq \Phi(\alpha)$-eigenvectors of e then it decomposes into a sum of $1-\Psi$-eigenvectors of e'. Primitivity of e' follows from the fact that $\dim_{\mathbb{F}} A_0^e = 1$.

For $\phi, \psi \in \Phi(\alpha)$, we write $(1-\phi) \star' (1-\psi) = \{1 - \nu \mid \nu \in \phi \star \psi\}$ where $\star$ gives the fusion rules of $\Phi(\alpha)$; the eigenvalues $\{1, 0, 1-\alpha\}$ together with $\star'$ give the fusion rules $\Phi(1-\alpha)$. For example, $(1-\alpha) \star' (1-\alpha) = 1 - \alpha \star \alpha = 1 - \{1, 0\} = \{1, 0\}$. $\square$

2.3.6 Lemma. *For* $\alpha \neq -1, 2$*, we have* $(3C_\alpha) = (3C_{1-\alpha})$*. Also,* $(3C_{-1}^\times) \subseteq (3C_2)$*.*

Proof. If $\alpha = \frac{1}{2}$, the statement is vacuously satisfied. It follows by Lemma 2.3.5 that if $\alpha \neq -1, \frac{1}{2}$ there are exactly three $\Phi(1-\alpha)$-axes e', f', g' in $A = (3C_\alpha)$. We calculate

$$e'f' = (\mathrm{id} - e)(\mathrm{id} - f) = \mathrm{id} - e - f - ef = \frac{1}{1+\alpha}(e+f+g) - e - f - \frac{\alpha}{2}(e+f-g) = \frac{\alpha}{2}(e'+f'-g').$$

By Lemma 2.3.5, e', f' are primitive $\Phi(1-\alpha)$-axes, and $e'f' \neq 0$. By Theorem 2.3.1, e', f' generate a subalgebra $B = (3C_{1-\alpha})$ in A, or possibly $B = (3C_{1-\alpha}^\times)$ if $1 - \alpha = -1$, that is, $\alpha = 2$. This B contains three $\Phi(1-\alpha)$-axes by Lemma 2.3.5, so $g' \in B$. If $\alpha \neq 2$ then necessarily $B = (3C_{1-\alpha})$ and e', f', g' are linearly independent, whence $B = A$. If $\alpha = 2$ then by Theorem 2.3.4 A is unital and B is not unital, so $B \neq A$. This means that B is not 3-dimensional, so $B \neq (3C_{-1})$ and instead $B = (3C_{-1}^\times)$. $\square$

2.4 Automorphisms

Here we discuss the Miyamoto automorphisms of $\Phi(\alpha)$-axes, and use them to classify $\Phi(\alpha)$-axial algebras as Matsuo algebras in the special *triangulating* case. This forms our algebraic characterisation of 3-transposition groups.

Suppose that A is generated by primitive Φ-axes $\mathcal{A}$ and let $\mathcal{A}^\circ = \{a \in \mathcal{A} \mid A = A_1^a + A_0^a\}$. Then $A = A^\circ \oplus A'$, where $A^\circ = \bigoplus_{a\in\mathcal{A}^\circ}\langle a\rangle$ is an associative algebra and A' is the subalgebra generated by $\mathcal{A}' = \mathcal{A} \smallsetminus \mathcal{A}^\circ$. (This follows by Lemma 2.1.5, showing that, for $a \in \mathcal{A}'$, $a\mathcal{A}^\circ = 0$ so the sum is direct, and Lemma 2.1.8 states that A° is associative.) As A° is understood, we now focus on the axes in $\mathcal{A}'$ with a nontrivial α-eigenspace: the so-called *nondegenerate* $\Phi(\alpha)$-axes.

Definition. Suppose that the fusion rules Φ are $\mathbb{Z}/2$-graded, so that Φ partitions as $\Phi_+ \cup \Phi_-$ and, for $\varepsilon, \varepsilon' \in \{+,-\}$, $\phi \in \Phi_\varepsilon$, $\phi' \in \Phi_{\varepsilon'}$, $\phi \star \phi' \subseteq \Phi_{\varepsilon\varepsilon'}$. Then the *Miyamoto involution* associated to a Φ-axis $a \in A$ is the linear automorphism $\tau(a) \in \mathrm{Aut}(A)$ defined by

$$x^{\tau(a)} = \begin{cases} x & \text{if } x \in A^a_{\Phi_+}, \\ -x & \text{if } x \in A^a_{\Phi_-}. \end{cases} \tag{2.46}$$

The fusion rules $\Phi(\alpha)$ are $\mathbb{Z}/2$-graded by $\{1,0\}\cup\{\alpha\}$, with $\Phi(\alpha)_- = \{\alpha\}$. Therefore, by application of Lemma 2.1.4, for $x, y \in \mathcal{G}$ considered as points of $M_\alpha(\mathcal{G})$,

$$y^{\tau(x)} = \begin{cases} x \wedge y & \text{if } x \sim y, \\ y & \text{otherwise}. \end{cases} \tag{2.47}$$

This observation generalises:

2.4.1 Lemma. *Suppose that $t \in \mathrm{Aut}(A)$, Φ are fusion rules and $a \in A$ is a Φ-axis. Then a^t is again a Φ-axis. Furthermore, if Φ is $\mathbb{Z}/2$-graded then $\tau(a)^t = \tau(a^t)$.*

Proof. Observe that $a^t = (aa)^t = a^t a^t$ is a nonzero idempotent. Viewing $t \in \mathrm{Aut}(A)$ as an endomorphism $t \in \mathrm{End}(A)$, for any $x \in A$ we have

$$\mathrm{ad}(a)^t x = t\,\mathrm{ad}(a)t^{-1}x = (\mathrm{ad}(a)x^{t^{-1}})^t = (ax^{t^{-1}})^t = a^t x \tag{2.48}$$

so $\mathrm{ad}(a)^t = \mathrm{ad}(a^t)$. Since $\mathrm{ad}(a)$ affords a diagonalisable decomposition of A and t is linear, $\mathrm{ad}(a^t) = \mathrm{ad}(a)^t$ again affords a diagonalisable decomposition. In particular, if $x \in A^a_\phi$, then $a^t x^t = (ax)^t = \phi x^t$, so $x^t \in A^{a^t}_\phi$ and therefore $(A^a_\phi)^t \subseteq A^{a^t}_\phi$. Similarly, if $x \in A^{a^t}_\phi$ then $a^t x = \phi x$ so $\phi x^{t^{-1}} = (a^t x)^{t^{-1}} = ax^{t^{-1}}$, whence $(A^{a^t}_\phi)^{t^{-1}} \subseteq A^a_\phi$, *i.e.,* $A^{a^t}_\phi \subseteq (A^a_\phi)^t$, showing that $A^{a^t}_\phi = (A^a_\phi)^t$ and furthermore the eigenvalues of a^t are precisely the eigenvalues of a and lie in Φ. Moreover, the fusion rules are also transported: suppose that $x \in A^a_\psi$ and $y \in A^a_\phi$. Then $x^t y^t = (xy)^t$ is in $(A^a_{\phi\star\psi})^t = A^{a^t}_{\phi\star\psi}$, so $A^{a^t}_\phi A^{a^t}_\psi \subseteq A^{a^t}_{\phi\star\psi}$. Therefore a^t is a Φ-axis.

Suppose that Φ is $\mathbb{Z}/2$-graded and thus a has a Miyamoto involution $\tau(a)$. If $\tau(a)$ acts as $\varepsilon = \pm 1$ on A^a_ϕ, then $\tau(a^t)$ acts by ε on $A^{a^t}_\phi = (A^a_\phi)^t$. As also $\tau(a)^t$ acts by ε on $(A^a_\phi)^t$, it follows that $\tau(a)^t = \tau(a^t)$. □

The fact recorded in Theorem 2.3.4 that $(3C_\alpha), \alpha \neq \frac{1}{2}, (3C^\times_{-1})$ contain finitely many idempotents, and $(3J_\kappa), \kappa \in \mathbb{F}, (3J^\times_0)$ do not necessarily, is a qualitative dichotomy also seen in the automorphism groups; compare the following to Lemma 2.5.3.

2.4.2 Proposition. $\mathrm{Aut}(3C_\alpha) \cong \mathrm{Aut}(3C^\times_{-1}) \cong \mathrm{Sym}(3)$ *for* $\alpha \neq \frac{1}{2}$ *over a field* $\mathbb{F}$.

Proof. Let $A \cong (3C_\alpha)$ or $(3C^\times_{-1})$, and $\mathcal{G}$ be the graph whose points are $\Phi(\alpha)$-axes in A with two vertices joined by an edge if they generate A. The axes are classified by Proposition 2.3.3, Theorem 2.3.4 and Lemma 2.3.5. When $\alpha \neq \frac{1}{2}$, this means $\mathcal{G}$ is the complete graph on three vertices. By Lemma 2.4.1, any automorphism t of A also has an action on $\mathcal{G}$. Since the $\Phi(\alpha)$-axes span A, t is determined by its action on $\mathcal{G}$. The automorphism group of $\mathcal{G}$ is $\mathrm{Sym}(3)$, so $\mathrm{Aut}(A)$ can be embedded in $\mathrm{Sym}(3)$. On the other hand, $\mathrm{Sym}(3)$ is clearly realised acting on A, so $\mathrm{Aut}(A) = \mathrm{Sym}(3)$. □

Any two distinct idempotents $e, f \in A \cong (3C_{1/2})$ generate A if $ef \neq 0$, but it will turn out that in general $|\tau(e)\tau(f)|$ is unbounded. However e, f satisfy the presentation in Table 2.3 if and only if $|\tau(e)\tau(f)| = 3$. Equivalently, there exists an idempotent $g \in A$ such that

$$f_1^e + f_0^e - f_\alpha^e = f^{\tau(e)} = g = e^{\tau(f)} = e_1^f + e_0^f - e_\alpha^f. \tag{2.49}$$

In the sequel, we will pay special attention to this case.

We will therefore say that two idempotents e, f are *triangulating* if e, f generate $(1A), (2B), (3C_{-1}^\times)$, or generate $(3C_\alpha)$ satisfying the presentation in Table 2.3. If $\alpha \neq \frac{1}{2}$, this condition is vacuously satisfied, unless the characteristic of the underlying field is 3; then $-1 = 2 = \frac{1}{2}$ so that for e, f generating $(3C_{-1}^\times)$ we also require them to satisfy the presentation of Section 2.3. We also say that a collection $\mathcal{A}$ of $\Phi(\alpha)$-axes is *triangulating* if any pair $e, f \in \mathcal{A}$ is triangulating. The points in $\mathcal{G}$ of a Matsuo algebra $M_\alpha(\mathcal{G})$ are of course triangulating.[1]

2.4.3 Lemma. *Let $\mathcal{A}$ be a generating, triangulating set of $\Phi(\alpha)$-axes for A. If $a, b \in \mathcal{A}$ are nondegenerate and $t = \tau(a) = \tau(b)$, then $a = b$.*

Proof. Since $t \neq 1$ by the assumption of nondegeneracy, and $\mathcal{A}$ is generating, there exists $c \in \mathcal{A}$ such that $c^t \neq c$. Set $B_a = \langle a, c \rangle$ and $B_b = \langle b, c \rangle$, both isomorphic to $(3C)$. By assumption, $\langle t, \tau(c) \rangle$ acts as $\mathrm{Sym}(3)$ on both B_a, B_b, giving

$$a^{\tau(c)} = c^{\tau(a)} = c^{\tau(b)} = b^{\tau(c)} \tag{2.50}$$

and therefore $a = b$. □

2.4.4 Lemma. *For a nondegenerate $\Phi(\alpha)$-axis $a \in A$, $\tau(a)$ has order 2. If a, b are*

[1] In [HRS14], an algebra is called a 3-*transposition algebra* if generated by a triangulating set of idempotents.

nondegenerate $\Phi(\alpha)$-axes, then

$$|\tau(a)\tau(b)| = \begin{cases} 1 & \textit{if } a, b \textit{ generate } (1A), \\ \leq 2 & \textit{if } a, b \textit{ generate } (2B), \\ 3 & \textit{if } a, b \textit{ generate } (3C_\alpha) \textit{ for } \alpha \neq \frac{1}{2}, (3C_{-1}^{\times}), \\ & \qquad \textit{or } (3C_{1/2}) \textit{ satisfying the presentation of Table 2.3.} \end{cases} \tag{2.51}$$

Proof. If $\langle a, b\rangle \cong (1A)$, then $a = b$, so that $\tau(a) = \tau(b)$ and $\tau(a)\tau(b) = \tau(a)^2 = 1$.

If $\langle a, b\rangle \cong (2B)$, then $b \in A_0^a$ so $b^{\tau(a)} = b$. But therefore $\tau(b)^{\tau(a)} = \tau(b)$ by Lemma 2.4.1, so $\tau(a), \tau(b)$ commute, and thus $|\tau(a)\tau(b)| \leq 2$. If a, b are part of a triangulating set of axes, then by Lemma 2.4.3 we have that $\tau(a) \neq \tau(b)$. Thus the product $\tau(a)\tau(b)$ is an involution in this case.

Suppose that we are in the final case. Then there exists $c \in \langle a, b\rangle$ such that $a^{\tau(b)} = c = b^{\tau(a)}$, whence $\tau(a)^{\tau(b)} = \tau(c) = \tau(b)^{\tau(a)}$ and therefore

$$(\tau(a)\tau(b))^3 = \tau(b)^{\tau(a)}\tau(a)^{\tau(b)} = \tau(c)\tau(c) = 1. \tag{2.52}$$

On the other hand, $a^{\tau(a)\tau(b)} = c$, so $\tau(a)\tau(b) \neq 1$ and has order 3. □

In other words, Lemma 2.4.4 states that if e and f are triangulating then $|\tau(e)\tau(f)| \leq 3$. The Miyamoto involutions generate a subgroup of the automorphism group, and their property of bounded pairwise product size has a well-known description, first introduced by Fischer:

Definition ([A97])**.** A 3-*transposition group* is a pair (G, D) where G is a group, and

i. $D \subseteq G$ is a generating set of involutions closed under conjugation, and
ii. $|cd| \leq 3$ for all $c, d \in D$.

The classification of 3-transposition groups was accomplished in special cases by Fischer [F71] and in generality by H. Cuypers and Hall [CH95]. A celebrated interpretation of 3-transposition groups in a combinatorial setting is

2.4.5 Theorem (Buekenhout's Geometric Characterisation, [B]). *Fischer spaces without totally disconnected points are in bijection with the* 3*-transposition groups up to centre.* □

Here we give a sketch of the bijection. If (G, D) is a 3-transposition group, let $\mathcal{G}$ be the graph with point set D and lines $\{c, d, e\}$ if $\langle c, d, e\rangle \cong \mathrm{Sym}(3)$. If $\mathcal{G}$ is a Fischer space, then let $\tau(x) \in \mathrm{Aut}(\mathcal{G})$ for $x \in \mathcal{G}$ be the unique automorphism fixing x and $x^{\not\sim}$, and exchanging any two elements $y, x \wedge y$ for $y \in x^{\sim}$. Then $D = \{\tau(x) \mid x \in \mathcal{G}\}, G = \langle D\rangle$ gives a 3-transposition group (G, D).

If Φ is $\mathbb{Z}/2$-graded, a set of Φ-axes $\mathcal{A}$ is *closed* if $\mathcal{A}^{\tau(a)} = \mathcal{A}$ for any $a \in \mathcal{A}$.

We can now prove results about general algebras generated by certain classes of $\Phi(\alpha)$-axes. We have an intermediary lemma before the general case.

2.4.6 Lemma. *If $\mathcal{A}$ is a closed triangulating set of primitive nondegenerate $\Phi(\alpha)$-axes then the Miyamoto involutions of $\mathcal{A}$ generate a* 3*-transposition group.*

Proof. Let $D = \{\tau(a) \mid a \in \mathcal{A}\} \subseteq \mathrm{Aut}(A)$ and set G to be the subgroup of $\mathrm{Aut}(A)$ generated by D. Then $D^G = D$ as $D^{\tau(a)} = D$ for any $a \in \mathcal{A}$. Then (G, D) is a 3-transposition group: every a^t, for $a \in \mathcal{A}$ and $t \in D$, is conjugate by the automorphism t to a nondegenerate $\Phi(\alpha)$-axis a, therefore is itself a nondegenerate $\Phi(\alpha)$-axis by Lemma 2.4.1. Thus $\tau(a^t)$ is an involution. By definition, D generates G and is closed under D-conjugation, hence under G-conjugation. Finally, Lemma 2.4.4 provides the bound $|cd| \leq 3$ for $c, d \in D$. □

2.4.7 Theorem. *An algebra A generated by a closed triangulating set $\mathcal{A}$ of primitive $\Phi(\alpha)$-axes is a quotient of a Matsuo algebra $M_\alpha(\mathcal{G})$ for $\mathcal{G}$ a Fischer space.*

Proof. Write $\mathcal{A} = \mathcal{A}^\circ \cup \mathcal{A}'$, where $\mathcal{A}^\circ = \{e \in \mathcal{A} \mid A^e_\alpha = 0\}$ are the degenerate axes in $\mathcal{A}$ and $\mathcal{A}'$ are the nondegenerate axes. We saw that $A = A^\circ \oplus A'$, for

$A^\circ = \bigoplus_{a\in\mathcal{A}^\circ}\langle a\rangle$ an associative algebra. Observe that $A^\circ = M_\alpha(\mathcal{H})$ where $\mathcal{H}$ is the totally disconnected Fischer space with $|\mathcal{A}^\circ|$ points. It only remains to show that $A' = \langle \mathcal{A}'\rangle$ is a quotient of a Matsuo algebra on a Fischer space $\mathcal{G}$, so that A is a quotient of $M_\alpha(\mathcal{H}) \oplus M_\alpha(\mathcal{G}) = M_\alpha(\mathcal{H}\cup\mathcal{G})$.

We therefore assume that all axes in $\mathcal{A}$ are nondegenerate. By Lemma 2.4.6, the Miyamoto involutions D of the axes $\mathcal{A}$ generate a 3-transposition group (G, D). By Lemma 2.4.3, the map $\tau\colon a \mapsto \tau(a)$ is injective on $\mathcal{A}$, so $|D| = |\mathcal{A}|$. Since the axes in $a \in \mathcal{A}$ are nondegenerate, for any $a \in \mathcal{A}$ there exists a $b \in \mathcal{A}$ such that $a \neq b, ab \neq 0$ and $|\tau(a)\tau(b)| = 3$. Thus $\tau(a) \notin \mathrm{Z}(G)$ and in particular $D \cap \mathrm{Z}(G) = \emptyset$. Let $\mathcal{D}$ be the Fischer space of (G, D) afforded by Theorem 2.4.5 on the points D. As any a, b are triangulating, there exists (according to the presentation of $\langle a, b\rangle$ given in Section 2.3) an axis $c \in \mathcal{A}$ such that $ab = \frac{\alpha}{2}(a + b - c)$ and $\tau(c) = \tau(a)^{\tau(b)}$. As τ is injective, this c is uniquely defined and in particular, for $\tau(a), \tau(b) \in \mathcal{D}$, $\tau(c) = \tau(a) \wedge \tau(b)$ in the Fischer space. Therefore $\mathcal{A}$ spans A and the multiplication among axes in $\mathcal{A}$ satisfies (2.3), so A is a quotient of $M_\alpha(\mathcal{D})$. $\square$

2.5 Jordan algebras

We notice in Theorem 2.5.4 that the fusion rules of $\Phi(1/2)$-axes occur also as fusion rules of idempotents in Jordan algebras. Here we will establish that the Matsuo algebra $M_\alpha(\mathcal{G})$ is a Jordan algebra if and only if $\mathcal{G}$ is $\mathcal{A}_n$ or $\mathcal{P}_3$. We assume throughout that $\alpha = \frac{1}{2} \in \mathbb{F}$, and later that $\frac{1}{3} \in \mathbb{F}$, where $\mathbb{F}$ is the field over which we work.

Definition ([A47]). A *Jordan algebra* J is a unital nonassociative commutative algebra over a field $\mathbb{F}$ in which, for all $a, b \in J$, $(ab)(aa) = a(b(aa))$.

If A is an associative $\mathbb{F}$-algebra then A^+ with the same underlying vector space and *Jordan product* $x \bullet y = \frac{1}{2}(xy + yx)$ is a Jordan algebra [A47].

2.5.1 Proposition. *The algebra* $(3J_\kappa)$ *is a Jordan algebra for all* $\kappa \in \mathbb{F}$.

Proof. We give an explicit isomorphism to a known Jordan algebra when $\kappa \neq 0$. Let V_κ be a vector space spanned by $\{v_1, v_2\}$ with symmetric bilinear form $\langle , \rangle$ defined by $\langle v_1, v_1 \rangle = \frac{1}{4} = \langle v_2, v_2 \rangle$ and $\langle v_1, v_2 \rangle = \kappa - \frac{1}{4}$. Then $J_\kappa = \mathbb{F}1 \oplus V_\kappa$ is a Jordan algebra when we extend $\langle 1, V_\kappa \rangle = 0, \langle 1, 1 \rangle = 1$ and $(\nu 1 + v)(\mu 1 + w) = (\nu\mu + \langle v, w \rangle)1 + \mu v + \nu w$; this is given as Spin Factor Example 3.3.3 in [McC04].

For $\kappa \neq 0$, the isomorphism $\phi \colon (3J_\kappa) \to J_\kappa$ is given by

$$e^\phi = \frac{1}{2}1 + v_1, \quad f^\phi = \frac{1}{2}1 + v_2, \quad (e \circ f)^\phi = \kappa 1. \tag{2.53}$$

For $A = (3J_0)$, we verify manually that $\operatorname{ad}(aa)\operatorname{ad}(a) = \operatorname{ad}(a)\operatorname{ad}(aa)$ for all $a \in A$. An arbitrary $a \in A$ can be written as $a = \lambda e + \mu f + \nu(e \circ f)$, with $\lambda, \mu, \nu \in \mathbb{F}$. As $\operatorname{ad}(e \circ f) = 0$, by linearity of ad we can assume $\nu = 0$. Then $aa = \lambda^2 e + \mu^2 f + \lambda\mu(e + f + 2e \circ f)$. Again $\operatorname{ad}(e \circ f)$ commutes with everything, so to check $\operatorname{ad}(aa)$ commutes with $\operatorname{ad}(a)$ it suffices to consider the terms e, f: for $b = \lambda(\lambda+\mu)e + \mu(\lambda+\mu)f$,

$\mathrm{ad}(aa)\,\mathrm{ad}(a) = \mathrm{ad}(a)\,\mathrm{ad}(aa)$ if and only if $\mathrm{ad}(b)\,\mathrm{ad}(a) = \mathrm{ad}(a)\,\mathrm{ad}(b)$. By the linearity of ad, and expressions for $\mathrm{ad}(e), \mathrm{ad}(f)$ which may be derived from Table 2.4, we see that this is indeed the case. Thus A satisfies $(aa)(ba) = ((aa)b)a$ for all $a, b \in A$. $\square$

We solve the isomorphism problem for $(3J_\kappa)$ over the reals $\mathbb{R}$:

2.5.2 Lemma. *There are* 4 *isomorphism types of* $(3J_\kappa)$ *over* $\mathbb{R}$, $\kappa \in \mathbb{R}$, *corresponding to* $\kappa = 0$, $0 < \kappa < \frac{1}{2}$, $\kappa = \frac{1}{2}$, *and* $\kappa \notin [0, \frac{1}{2}]$.

Proof. The classification of isomorphism types among $(3J_\kappa)$, for $\kappa \neq 0$, is equivalent to the classification of symmetric bilinear 2-dimensional forms over $\mathbb{F}$ using the isomorphism into $\mathbb{F}1 \oplus V$ of the proof of Proposition 2.5.1. The form has Gram matrix

$$\begin{pmatrix} \frac{1}{4} & \kappa - \frac{1}{4} \\ \kappa - \frac{1}{4} & \frac{1}{4} \end{pmatrix}. \tag{2.54}$$

Over the reals, that is, when $\mathbb{F} = \mathbb{R}$, there are three symmetric bilinear forms which are possible. These correspond to the different possibilities for eigenvalues of the Gram matrix: its determinant is $\mu^2 - \frac{1}{2}\mu - \kappa(\kappa - \frac{1}{2})$, with roots $\frac{1}{4} \pm (\kappa - \frac{1}{4})$, that is, κ and $\frac{1}{2} - \kappa$. At least one root is strictly positive. Write

$$R_0 = \{0\}, \quad R_1 = \left(0, \frac{1}{2}\right), \quad R_2 = \left\{\frac{1}{2}\right\}, \quad R_3 = (-\infty, 0) \cup \left(\frac{1}{2}, \infty\right). \tag{2.55}$$

Then $R_0, \dots, R_3$ is a partition of $\mathbb{R}$, and it is well-known that for any $\kappa, \kappa' \in R_i$, $i = 1, 2, 3$, the forms parametrised by κ and κ' are equivalent, hence $(3J_\kappa) \cong (3J_{\kappa'})$, and moreover if $\gamma \in R_j$, $j = 1, 2, 3, j \neq i$ then $(3J_\kappa) \ncong (3J_\gamma)$. Here $\kappa \in R_1, R_2, R_3$ corresponds to the so-called pure, zero and mixed signature form respectively.

The algebra $A = (3J_\kappa)$ for any $\kappa \in R_1 \cup R_2 \cup R_3$, that is, $\kappa \neq 0$, has zero annihilator

$$\mathrm{Ann}(A) = \{x \in A \mid xy = 0 \text{ for all } y \in A\}. \tag{2.56}$$

For suppose that $x = \lambda e + \mu f + \nu e \circ f \in \mathrm{Ann}(A)$. Then $0 = xe = \lambda e + \mu(\frac{1}{2}e + \frac{1}{2}f + e \circ f) + \nu\kappa e$

implies that $\mu = 0$ by the linear independence of e, f and $e \circ f$; likewise $0 = xf$ implies that $\lambda = 0$. Now $x = \nu s$ and therefore $x \in \mathrm{Ann}(A)$ if and only if $\nu = 0$, that is, $x = 0$, or $\kappa = 0$. On the other hand $e \circ f \in \mathrm{Ann}(3J_0) \neq 0$, so $(3J_0)$ is not isomorphic to any $(3J_\kappa)$ for $\kappa \neq 0$. □

It is also well-known that there are similarly, up to equivalence, three symmetric bilinear forms in 2 dimensions over a finite field, referred to as $+, 0$ or $-$-type.

What can we say about the automorphism groups?

2.5.3 Lemma. *For $\kappa \in \mathbb{F} \setminus \{0\}$, $\mathrm{Aut}(3J_\kappa)$ contains an algebraic group, has size at least $\mathrm{char}\,\mathbb{F}$, and is infinite in characteristic 0.*

Proof. By the proof of Proposition 2.5.1, $(3J_\kappa) \cong \mathbb{F}1 \oplus V$ when $\kappa \neq 0$, where $\dim V = 2$ and V has a bilinear form $\langle , \rangle \colon V \to \mathbb{F}$ such that $\langle v, v \rangle = \frac{1}{4} = \langle w, w \rangle$ with $\langle v, w \rangle = \kappa + \frac{1}{4}$. Any equivalence of $\langle , \rangle$ induces an automorphism of A, so $\mathrm{Aut}(V, \langle , \rangle)$ contains an (orthogonal) algebraic group.

Let $A = (3J_0^\times)$ generated by $\Phi(1/2)$-axes e, f, and set $e_\nu = \nu e + (1-\nu) f$. Then

$$e_\nu e_\nu = \nu^2 e + (1-\nu)^2 f + 2\frac{1}{2}\nu(1-\nu)(e+f) = \nu e + (1-\nu) f = e_\nu, \tag{2.57}$$

and in fact (by direct verification, or by Proposition 2.5.1 and Theorem 2.5.4 i.) e_ν is a $\Phi(1/2)$-axis for all ν. Furthermore $e_\nu e_\mu = e_{\frac{1}{2}(\nu+\mu)}$ and the order of $\tau(e_\nu)\tau(e_\mu)$ can be arbitrarily large if $\nu \neq \mu$; in particular, if $\rho = \tau(e)\tau(f)$, then $f^\rho = 2e - f$ and $f^{\rho^n} = 2ne + (1-2n) f = e_{2n}$, and $e_{2n} \neq e_0 = f$ for any n less than $\mathrm{char}\,\mathbb{F}$. Therefore $\mathrm{Aut}(3J_0^\times)$ has size at least $\mathrm{char}\,\mathbb{F}$, or, if $\mathrm{char}\,\mathbb{F} = 0$, is infinite.

Write $\hat{A} = (3J_0)$ and let $\hat{e}, \hat{f}$ be two generating $\Phi(1/2)$-axes in $\hat{A}$ such that under the quotient $\hat{A} \to A$ by the ideal $I = \langle \hat{e} \circ \hat{f} \rangle \subseteq \hat{A}$ the image of $\hat{e}, \hat{f}$ is e, f respectively. Then $\tau(\hat{e}), \tau(\hat{f}) \in \mathrm{Aut}(\hat{A})$ fix I and therefore have an action on $\hat{A}/I \cong A$ matching the action of $\tau(e), \tau(f) \in \mathrm{Aut}(A)$ respectively. Thus $\tau(\hat{e})\tau(\hat{f})$ has strictly larger order than $\tau(e)\tau(f)$, and so $\mathrm{Aut}(3J_0)$ has size at least that of $\mathrm{Aut}(3J_0^\times)$. □

Therefore some of our examples of $\Phi(1/2)$-axial algebras are Jordan algebras. The following results give a largely satisfactory answer to the question of which Jordan algebras are axial, that is, generated by $\Phi(1/2)$-axes. Recall that $x \in A$ is nilpotent if there exists some integer n such that $x^n = 0$.

2.5.4 Theorem. *Suppose that J is a Jordan algebra over a field $\mathbb{F}$.*

i. ([A47] Theorem 6) *Idempotents in J are $\Phi(1/2)$-axes.*

ii. ([A47] Lemma 4) *If $a \in J$ is not nilpotent, then the subalgebra $\mathbb{F}[a]$, and hence J, contains a nonzero idempotent.*

iii. ([A47] Theorems 4, 5) *The subset $\operatorname{Rad}(J)$ of all nilpotent elements of J is an ideal, and $J/\operatorname{Rad}(J)$ is semisimple.*

iv. ([J68] Chap. VIII, Sect. 3, Lemma 2) *If J is finite-dimensional, semisimple and $\mathbb{F}$ is algebraically closed, then J is spanned by idempotents.* □

On the other hand, in light of Section 2.4, we ask: which Matsuo algebras are Jordan? By Lemmas 2.4.3, 2.4.4, this implies that the automorphism group of the Jordan algebra J contains a 3-transposition subgroup.

Recall that matrix transposition is the map which takes a matrix $M = (x_{ij})_{1\leq i,j\leq n}$ to $M^t = (x_{ji})_{1\leq i,j\leq n}$. A matrix is symmetric if it is fixed by transposition.

2.5.5 Theorem. *The Matsuo algebra $M_{1/2}(\mathcal{A}_n)$ over $\mathbb{F} \ni \frac{1}{2}$ is isomorphic to the Jordan algebra over $\mathbb{F}$ of symmetric $(n+1)\times(n+1)$ matrices whose rows have sum 0.*

For the proof of this theorem, we first present a construction of Jordan algebras. Suppose that R is a root system; recall that this means R spans $V = \mathbb{R}^n$ with Euclidean form $(,)$. We will give alternative constructions of the simply-laced root system over $\mathbb{F}^{n+1}$ for any $\mathbb{F}$, based on the integral lattices of the root system, following [C05]. By assumption $\frac{1}{2} \in \mathbb{F}$, the field $\mathbb{F}$ is not of characteristic 2 and therefore $-1 \neq 1$.

For a nonsingular vector $v \in V$, write $m_v = \frac{1}{vv^t} v^t v$ for the projection matrix of the 1-dimensional subspace $\langle v \rangle \subseteq V$. The collection $\{m_a \mid a \in \mathrm{R}\}$ generates, with the Jordan product $m_a \bullet m_b = \frac{1}{2}(m_a m_b + m_b m_a)$, a Jordan algebra $A^+(\mathrm{R})$. As $m_a = m_{-a}$, it suffices to take the projection matrices for a set R_+ of positive roots in R.

Let $V = \mathbb{F}^{n+1}$ with standard ordered basis $v_0, \ldots, v_n$. Then

$$(\mathrm{A}_n)_+ = \{a_{ij} = v_i - v_j \mid 0 \leq j < i \leq n\}, \tag{2.58}$$

and the projection matrices are, for e_{ij} the $(n+1) \times (n+1)$ matrix with 0 everywhere except a 1 in position i, j,

$$m_{a_{ij}} = \frac{1}{2}(e_{ii} - e_{ij} - e_{ji} + e_{jj}). \tag{2.59}$$

2.5.6 Lemma. *Suppose that $a, b \in \mathrm{R}$ are two roots of equal length and m_a, m_b are the associated projection matrices. Then*

$$m_a \bullet m_b = \begin{cases} m_a & \textit{if } a = \pm b, \text{ i.e., } \langle a, b \rangle \cong \mathrm{A}_1 \\ 0 & \textit{if } (a,b) = 0, \text{ i.e., } \langle a, b \rangle \cong \mathrm{A}_1 \times \mathrm{A}_1 \\ \frac{1}{4}(m_a + m_b - m_c) & \textit{otherwise: } \langle a, b \rangle \cong \mathrm{A}_2, c = \pm(a-b). \end{cases} \tag{2.60}$$

Proof. Projections are idempotents, so that $m_a \bullet m_a = m_a^2 = m_a$ for all $a \in \mathrm{R}$.

Suppose that $(a, b) = 0$. Then there exists a basis $\{v_1, \ldots, v_n\}$ of V extending $\{a, b\}$ with $v_1 = a, v_2 = b$. If $v \in V$, $v = \sum_{1 \leq i \leq n} \lambda_i v_i$, then $m_a v = \lambda_1 v_1, m_b v = \lambda_2 v_2$ and therefore $(m_a m_b) v = 0$ for all $v \in V$. Thus $m_a m_b = 0$ and $m_a \bullet m_b = 0$.

Suppose now that $a, b, c \in \mathrm{A}_2 \subseteq \mathrm{R}$ with m_a, m_b, m_c distinct. Without loss of generality we may assume that $R = \mathrm{A}_2$ in $\mathbb{F}^3$ as in the above construction, and

$$\mathrm{R}_+ = \{(1,-1,0), (1,0,-1), (0,1,-1)\}, \tag{2.61}$$

$$\{m_a, m_b, m_c\} = \left\{ \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right\}. \tag{2.62}$$

Indeed $m_a \bullet m_b = \frac{1}{4}(m_a + m_b - m_c)$ in this representation, and hence in general. $\square$

All roots in R have equal length if and only if R is simply-laced. Therefore the multiplication rule in A^+ is the multiplication rule of a Matsuo algebra when the underlying root system R is simply-laced. In particular, A^+ is then spanned by $\mathcal{A} = \{m_a \mid a \in \mathrm{R}_+\}$. It remains to determine whether $\mathcal{A}$ is linearly independent.

2.5.7 Lemma. *The algebra* $A^+(\mathrm{A}_n)$ *has dimension* $\frac{1}{2}n(n+1)$.

Proof. There are $\frac{1}{2}n(n+1)$ positive roots in A_n. Clearly the projection matrices of the positive roots are all linearly independent, since a projection m has $m_{i,j} \neq 0$ if and only if it corresponds to the positive root $v_i - v_j$, so distinct projections have nonzero entries in distinct positions. $\square$

Proof of Theorem 2.5.5. Let A_n be embedded in $V \cong \mathbb{F}^{n+1}$ as described in (2.58). By the previous Lemmas 2.5.6, 2.5.7, $A^+(\mathrm{A}_n)$ is a Jordan algebra of dimension $\frac{1}{2}n(n+1)$ which, since it satisfies the same multiplication, is a quotient of the Matsuo algebra $M_{1/2}(\mathcal{A}_n)$. On the other hand, $M_{1/2}(\mathcal{A}_n)$ has dimension $|(\mathrm{A}_n)_+| = \frac{1}{2}n(n+1) = d$, and therefore $A^+(\mathrm{A}_n) \cong M_{1/2}(\mathcal{A}_n)$. From the description in (2.59), the matrices are symmetric, and their column (equivalently, row) sums are 0. We see that any symmetric matrix with 0 row sum is in the span of these projection matrices. $\square$

For the case of the Matsuo algebra of $\mathcal{P}_3$ from Figure 2.1, we will need the following definition. Let E be the quadratic étale extension $E = \mathbb{F}[x]/(x^2+3)$ of $\mathbb{F}$ a field of characteristic not 2 or 3, and let $\sigma \in \mathrm{Gal}(E/\mathbb{F})$ be its non-trivial Galois automorphism. Notice that E might or might not be a field, depending on whether -3 is a square in $\mathbb{F}$. Write $E = \mathbb{F}[\zeta]$ with $\zeta^2 = -3$, so in particular $\zeta^\sigma = -\zeta$.

The Jordan algebra $H_3(E, *)$ consists of 3×3 matrices over E fixed by $*$, where $*$ is the involution on $\mathrm{Mat}_3(E)$ given by conjugate transposition, *i.e.,* $(x_{ij})^* = (x_{ji}^\sigma)$.

2.5.8 Theorem. *The Matsuo algebra $M_{1/2}(\mathcal{P}_3)$ over $\mathbb{F} \ni \frac{1}{6}$ is isomorphic to the Jordan algebra $H_3(E, *)$.*

Proof. Let $A = M_{1/2}(\mathcal{P}_3)$. For each $i \in \{1, \ldots, 9\}$, we let p_i be the generator of the Matsuo algebra corresponding to the point i in Figure 2.1. Our proof has five steps: we establish an identity element in A; find idempotents for lines in $\mathcal{P}_3$; calculate their eigenspaces and intersections; recall the multiplication of $H(E, *)$; and verify an explicit isomorphism of the two Peirce decompositions.

The algebra A is unital, with $\mathrm{id} = \frac{1}{3}\sum_{i=1}^{9} p_i$: Let $z = \sum_{i=1}^{9} p_i$. By symmetry and linearity, it suffices to verify that $zp_1 = 3p_1$. Namely,

$$zp_1 = p_1 + \tfrac{1}{4}\sum_{i=2}^{9}(p_1 + p_j - p_1 \wedge p_j) = 3p_1 + \tfrac{1}{4}\sum_{i=2}^{9}(p_j - p_1 \wedge p_j) = 3p_1$$

since each of the 8 elements $p_2, \ldots, p_9$ occurs once with each sign in the sum.

We need the idempotents associated to lines of $\mathcal{P}_3$. Let L be any of the 12 lines of $\mathcal{P}_3$. Then define

$$e_L = -\tfrac{1}{3}\sum_{i \in L} p_i + \tfrac{1}{3}\sum_{i \notin L} p_i \quad \text{and} \quad f_L = \mathrm{id} - e_L = \tfrac{2}{3}\sum_{i \in L} p_i. \tag{2.63}$$

For each line L of $\mathcal{P}_3$, the element e_L is an idempotent of A. We can equivalently show that f_L is idempotent. Without loss of generality, $L = \{1, 2, 3\}$. Then

$$\begin{aligned} f_L^2 &= \tfrac{4}{9}(p_1 + p_2 + p_3)^2 = \tfrac{4}{9}(p_1 + p_2 + p_3) + 2 \cdot \tfrac{4}{9} \cdot \tfrac{1}{4} \sum_{1 \le i < j \le 3} (p_i + p_j - p_i \wedge p_j) \\ &= \tfrac{4}{9}(p_1 + p_2 + p_3) + \tfrac{2}{9}(p_1 + p_2 + p_3) = \tfrac{2}{3}(p_1 + p_2 + p_3) = f_L. \end{aligned}$$

Observe that in fact f_L is the identity of the subalgebra spanned by points in L.

If L and M are two parallel lines in $\mathcal{P}_3$, then e_L and e_M are orthogonal idempotents, *i.e.*, $e_L e_M = 0$. Notice that $e_L e_M = 0$ if and only if $f_L f_M = f_L + f_M - \mathrm{id}$. Without loss of generality, we may assume that $L = \{1,2,3\}$ and $M = \{4,5,6\}$. Then indeed

$$\begin{aligned} f_L f_M &= \tfrac{4}{9}\sum_{i=1}^{3}\sum_{j=4}^{6} p_i p_j = \tfrac{1}{9}\sum_{i=1}^{3}\sum_{j=4}^{6}(p_i + p_j - p_i \wedge p_j) \\ &= \tfrac{1}{9}\Big(3\sum_{i=1}^{6} p_i - 3\sum_{i=7}^{9} p_i\Big) = \tfrac{1}{3}\sum_{i=1}^{6} p_i - \tfrac{1}{3}\sum_{i=7}^{9} p_i = f_L + f_M - \mathrm{id}\,. \end{aligned}$$

We will now describe the eigenspaces corresponding to each e_L. Let L be a line of $\mathcal{P}_3$. The element e_L has is primitive, with eigenvalues $1, 0$ and $\frac{1}{2}$:

$$\begin{aligned} A_1^{e_L} &= \{\lambda e_L \mid \lambda \in \mathbb{F}\}, \\ A_0^{e_L} &= \Big\{\sum_{i\in L}\lambda_i p_i + \lambda\Big(\sum_{i\in M} p_i - \sum_{i\in N} p_i\Big) \mid \lambda_i, \lambda \in \mathbb{F}\Big\}, \\ A_{1/2}^{e_L} &= \Big\{\Big(\sum_{i\in M}\lambda_i p_i + \sum_{j\in N}\mu_j p_j\Big) \mid \textstyle\sum_i \lambda_i = 0, \sum_j \mu_j = 0\Big\}. \end{aligned}$$

To prove this, we may assume that $L = \{1,2,3\}$, $M = \{4,5,6\}$ and $N = \{7,8,9\}$. Let $x = \sum_{i=1}^{9}\lambda_i p_i$ be an arbitrary element of A. Then

$$\begin{aligned} xe_L = -\tfrac{1}{6}(\lambda_4 + \cdots + \lambda_9)(p_1 + p_2 + p_3) + \tfrac{1}{2}(\lambda_4 p_4 + \cdots + \lambda_9 p_9) \\ + \tfrac{1}{6}(\lambda_4 + \lambda_5 + \lambda_6)(p_7 + p_8 + p_9) + \tfrac{1}{6}(\lambda_7 + \lambda_8 + \lambda_9)(p_4 + p_5 + p_6). \end{aligned}$$

It is straightforward to verify these eigenvectors for e_L; since the dimensions of these three subspaces are 4, 4 and 1 respectively, they together span all of A, and hence we have found all eigenvectors. Since $\dim A_1^{e_L} = 1$, the idempotent e_L is primitive.

As a consequence, we get a decomposition for A, which will *ex post facto* be a Peirce decomposition. Namely, let $\{L_1, L_2, L_3\}$ be a parallel set of lines in $\mathcal{P}_3$, and denote the idempotent e_{L_i} by e_i respectively. Let $A_{ii} = A_1^{e_i} = \langle e_i\rangle$ for each i, and let $A_{ij} = A_{1/2}^{e_i} \cap A_{1/2}^{L_j}$ for each $i \neq j$. Then for any $\{i,j,k\} = \{1,2,3\}$, we have

$$A_{ij} = \Big\{\sum_{\ell\in L_k}\lambda_\ell p_\ell \mid \textstyle\sum_\ell \lambda_\ell = 0\Big\} \tag{2.64}$$

so $\dim A_{ij} = 2$, and hence

$$A = A_{11} \oplus A_{22} \oplus A_{33} \oplus A_{12} \oplus A_{13} \oplus A_{23}. \tag{2.65}$$

We now recall some facts for $H_3(E, *)$ from [J68], p. 125–126. Let e_{ij} be the usual matrix units in $\mathrm{Mat}_3(E)$. Define, for $x \in E$,

$$x[ij] = xe_{ij} + x^\sigma e_{ji} \in \mathrm{Mat}_3(E) \tag{2.66}$$

for all i, j; in particular, $x[ii] = (x + x^\sigma)e_{ii}$ for all i, and $x[ji] = x^\sigma[ij]$ for all i, j. Recall that the multiplication in J is given by

$$2x[ij] \cdot y[jk] = xy[ik] \qquad \text{for all } i, j, k \text{ distinct,} \tag{2.67}$$

$$2x[ii] \cdot y[ij] = (x + x^\sigma)y[ij] \qquad \text{for all } i \neq j, \tag{2.68}$$

$$2x[ij] \cdot y[ij] = xy^\sigma[ii] + xy^\sigma[jj] \qquad \text{for all } i \neq j, \tag{2.69}$$

$$2x[ii] \cdot y[ii] = (x + x^\sigma)(y + y^\sigma)[ii] \qquad \text{for all } i, \tag{2.70}$$

$$x[ij] \cdot y[k\ell] = 0 \qquad \text{if } \{i, j\} \cap \{k, \ell\} = \emptyset. \tag{2.71}$$

Finally, let $J_{ij} = \{x[ij] \mid x \in E\}$ for all i, j, so in particular

$$J = J_{11} \oplus J_{22} \oplus J_{33} \oplus J_{12} \oplus J_{13} \oplus J_{23}. \tag{2.72}$$

The final step in our proof is to establish directly the isomorphism.

Consider the decomposition of A in (2.65). Let η be the $\mathbb{F}$-vector space isomorphism from A to J given on each of the six subspaces by

$$e_i \mapsto e_{ii} = \tfrac{1}{2}[ii] \quad \text{for all } i,$$

$$\lambda p_1 + \mu p_2 - (\lambda + \mu)p_3 \mapsto \big(\tfrac{3}{4}(\lambda + \mu) + \tfrac{1}{4}(\lambda - \mu)\zeta\big)[23],$$

$$\lambda p_4 + \mu p_5 - (\lambda + \mu)p_6 \mapsto \big(\tfrac{3}{4}(\lambda + \mu) + \tfrac{1}{4}(\mu - \lambda)\zeta\big)[13],$$

$$\lambda p_7 + \mu p_8 - (\lambda+\mu)p_9 \mapsto \big(\tfrac{3}{4}(\lambda+\mu) + \tfrac{1}{4}(\lambda-\mu)\zeta\big)[12],$$

for all $\lambda, \mu \in \mathbb{F}$. We verify that η is an isomorphism of algebras by examining the cases (2.67) through (2.71) one by one.

For case (2.67), assume that $i = 1$, $j = 2$ and $k = 3$; the other possibilities for i, j, k are completely similar. So let $x_{12} = \lambda p_7 + \mu p_8 - (\lambda + \mu)p_9 \in A_{12}$ and $y_{23} = \lambda' p_1 + \mu' p_2 - (\lambda' + \mu')p_3 \in A_{23}$ be arbitrary. Then

$$\begin{aligned}
2x_{12}y_{23} &= \tfrac{1}{2}\big(-\lambda\lambda' p_4 - \lambda\mu' p_6 + \lambda(\lambda'+\mu')p_5 - \mu\lambda' p_6 - \mu\mu' p_5 + \mu(\lambda'+\mu')p_4 \\
&\qquad + (\lambda+\mu)\lambda' p_5 + (\lambda+\mu)\mu' p_4 - (\lambda+\mu)(\lambda'+\mu')p_6\big) \\
&= \tfrac{1}{2}\big(-\lambda\lambda' + 2\mu\mu' + \lambda\mu' + \mu\lambda'\big)p_4 + \tfrac{1}{2}\big(2\lambda\lambda' - \mu\mu' + \lambda\mu' + \mu\lambda'\big)p_5 \\
&\qquad + \tfrac{1}{2}\big(-\lambda\lambda' - \mu\mu' - 2\lambda\mu' - 2\mu\lambda'\big)p_6,
\end{aligned}$$

so

$$\eta(2x_{12}y_{23}) = \big(\tfrac{3}{8}(\lambda\lambda' + \mu\mu' + 2\lambda\mu' + 2\mu\lambda') + \tfrac{3}{8}(\lambda\lambda' - \mu\mu')\zeta\big)[13]. \qquad \textbf{(2.73)}$$

On the other hand,

$$\begin{aligned}
&\big(\tfrac{3}{4}(\lambda+\mu) + \tfrac{1}{4}(\lambda-\mu)\zeta\big)\cdot\big(\tfrac{3}{4}(\lambda'+\mu') + \tfrac{1}{4}(\lambda'-\mu')\zeta\big) \\
&\quad = \big(\tfrac{9}{16}(\lambda+\mu)(\lambda'+\mu') - \tfrac{3}{16}(\lambda-\mu)(\lambda'-\mu')\big) + \tfrac{3}{16}\big((\lambda+\mu)(\lambda'-\mu') + (\lambda-\mu)(\lambda'+\mu')\big)\zeta \\
&\quad = \big(\tfrac{3}{8}(\lambda\lambda' + \mu\mu' + 2\lambda\mu' + 2\mu\lambda') + \tfrac{3}{8}(\lambda\lambda' - \mu\mu')\zeta\big);
\end{aligned}$$

we conclude that $\eta(2x_{12}y_{23}) = 2\eta(x_{12})\eta(y_{23})$.

The multiplication rule (2.68) is that, for $y \in E$, $y[ij]$ is a $\frac{1}{2}$-eigenvector for e_{ii}. Since A_{ij} is contained in the $\frac{1}{2}$-eigenspace of e_i, it follows that $\eta(e_i y_{ij}) = \eta(e_i)\eta(y_{ij})$ for all $i \neq j$ and all $y_{ij} \in A_{ij}$.

We now check (2.69), and again we assume that $i = 1$ and $j = 2$ since the other cases are completely similar. So let $x_{12} = \lambda p_7 + \mu p_8 - (\lambda + \mu)p_9 \in A_{12}$ and

$y_{12} = \lambda' p_7 + \mu' p_8 - (\lambda' + \mu') p_9 \in A_{23}$ be arbitrary. Then $2x_{12}y_{12} =$

$$\big((\lambda\lambda' + \mu\mu') + \tfrac{1}{2}(\lambda\mu' + \mu\lambda')\big)(p_7 + p_8 + p_9) = \big(\tfrac{3}{2}(\lambda\lambda' + \mu\mu') + \tfrac{3}{4}(\lambda\mu' + \mu\lambda')\big)(e_1 + e_2).$$

On the other hand,

$$\begin{aligned} &\big(\tfrac{3}{4}(\lambda + \mu) + \tfrac{1}{4}(\lambda - \mu)\zeta\big) \cdot \big(\tfrac{3}{4}(\lambda' + \mu') + \tfrac{1}{4}(\lambda' - \mu')\zeta\big)^{\sigma} \\ &\qquad = \big(\tfrac{3}{4}(\lambda + \mu) + \tfrac{1}{4}(\lambda - \mu)\zeta\big) \cdot \big(\tfrac{3}{4}(\lambda' + \mu') - \tfrac{1}{4}(\lambda' - \mu')\zeta\big) \\ &\qquad = \big(\tfrac{3}{4}(\lambda\lambda' + \mu\mu') + \tfrac{3}{8}(\lambda\mu' + \mu\lambda')\big) + \tfrac{3}{8}(\lambda\mu' - \mu\lambda')\zeta, \end{aligned}$$

and hence, allowing us to conclude that $\eta(2x_{12}y_{12}) = 2\eta(x_{12})\eta(y_{12})$,

$$\big(\tfrac{3}{4}(\lambda+\mu) + \tfrac{1}{4}(\lambda-\mu)\zeta\big)\big(\tfrac{3}{4}(\lambda'+\mu') + \tfrac{1}{4}(\lambda'-\mu')\zeta\big)^{\sigma}[ii] = \big(\tfrac{3}{2}(\lambda\lambda'+\mu\mu') + \tfrac{3}{4}(\lambda\mu'+\mu\lambda')\big)e_{ii}.$$

Case (2.70) is an immediate consequence of the definition of $x[ii] = (x + x^{\sigma})e_{ii}$ combined with the fact that e_i and e_{ii} are idempotents.

Finally, to deal with case (2.71), we have to verify that $A_{ij}A_{k\ell} = 0$ as soon as $\{i, j\} \cap \{k, \ell\} = \emptyset$. If $i = j$ and $k = \ell$, then this is an immediate consequence of the fact that e_i and e_j are orthogonal idempotents. If $i = j$ and $k \neq \ell$, then $A_{k\ell}$ is contained in the $\frac{1}{2}$-eigenspace of both e_k and e_ℓ, and hence in the 0-eigenspace of $1 - e_k - e_\ell = e_i$; it follows that $A_{ii}A_{k\ell} = 0$. □

If we knew in advance that $M_{1/2}(\mathcal{P}_3)$ was a Jordan algebra, the calculations in the proof of Theorem 2.5.8 could be replaced by an application of Jacobson's Strong Coordinatization Theorem, [J68], Theorem 5, p. 133. The idempotents e_1, e_2 and e_3 are *strongly connected*, and the *coordinatizing algebra*, an algebra structure on A_{ij} for $i \neq j$, is isomorphic to the $\mathbb{F}$-algebra E; *c.f.* [J68], Lemma 3, p. 135.

In the next proof, we will require two well-known definitions and facts. The *non-commuting graph* on a subset $D \subseteq G$ is the graph $\mathcal{D}$ with points D and lines $\{c, d\}$

for $c, d \in D$ with $[c, d] \neq 1$. If D is a generating set of involutions closed under conjugation, then $\mathcal{D}$ has the same point set as the Fischer space of (G, D), and two points are connected in the noncommuting graph if and only if connected in the Fischer space, but lines are sets of size 2 in $\mathcal{D}$.

The *Coxeter group* $\operatorname{Cox}(\mathcal{G})$ of a simply-laced graph $\mathcal{G}$ is the transposition group (G, D), where D' is a set of generators in bijection with the points of $\mathcal{G}$, G is the group generated by D' modulo the relations $d^2 = 1$ for $d \in D'$, $|cd| = 2$ for $c, d \in D'$ not connected in $\mathcal{G}$ and $|cd| = 3$ for $c, d \in D'$ connected in $\mathcal{G}$, and we set $D = D'^G$.

We will also need some 3-transposition groups; as reference, we use [H93].

The group $W_k(\hat{\mathrm{A}}_n)$, for $k = 2, 3$, $n \in \mathbb{N}$, and $\hat{\mathrm{A}}_n$ the affine extension of the root system A_{n-1}, is defined as follows. Let G be the $\mathbb{F}_k$-linear permutation representation of $\operatorname{Sym}(n+1)$, that is, the semidirect product of $\operatorname{Sym}(n+1)$ with the module $\mathbb{F}_k^{n+1}$, where the action is permutation of the standard ordered basis $\{v_1, \dots, v_{n+1}\}$ of $\mathbb{F}_k^{n+1}$. Let D be the image of the conjugacy class $(1, 2)^{\operatorname{Sym}(n+1)}$ of $\operatorname{Sym}(n+1)$ embedded in G. Then $W_k(\hat{\mathrm{A}}_n)$ is the quotient $(\bar{G}, \bar{D})$ of (G, D) by the diagonal $\langle v_1 + \cdots + v_{n+1} \rangle$.

Let $\mathcal{C}$ be the complete graph on $\{a, b, c, d\}$, and $\mathcal{C}'$ the graph obtained from $\mathcal{C}$ by deleting the edge $\{b, c\}$. Then set

$$G_4 = \operatorname{Cox}(\mathcal{C}')/\big((a^b d)^3 = (a^c d)^3 = (a^{bc} d)^3 = 1\big), \tag{2.74}$$

$$G_5 = \operatorname{Cox}(\mathcal{C})/\big((b^c d)^3 = (a^b c)^3 = (a^b d)^3 = (a^c d)^3 = (a^{bd} c)^3 = (a^{cd} b)^3 = (a^{dc} b)^3 = 1\big). \tag{2.75}$$

Let D_i be the image of the Coxeter involutions closed under conjugation in the above quotient for $i = 4, 5$. Then D_i generates G_i and (G_i, D_i) is a 3-transposition group. We note that $G_4 \cong 2^{1+6} : SU_3(2)'$ and G_5 is M. Hall's $3^{10} : 2$ [H93].

2.5.9 Theorem. *Let J be a finite-dimensional Jordan algebra over $\mathbb{F}$ which is also a Matsuo algebra $M_{1/2}(\mathcal{G})$ for $\mathcal{G}$ connected. Then $\mathcal{G} = \mathcal{P}_3$ or $\mathcal{G} = \mathcal{A}_n$.*

Proof. A connected Fischer space of rank 1 is a single point, and its Matsuo algebra $(1A)$ is Jordan. A connected Fischer space of rank 2 is a line, with Matsuo algebra $M_{1/2}(\mathrm{A}_2)$. In rank 3, by definition the only connected Fischer spaces are $\mathcal{P}_2^\vee$ and $\mathcal{P}_3$. As $\mathcal{P}_2^\vee \cong \mathrm{A}_3$, $M_{1/2}(\mathcal{P}_2^\vee)$ is a Jordan algebra by Theorem 2.5.5. By Theorem 2.5.8, $A = M_{1/2}(\mathcal{P}_3)$ is the Jordan algebra of 3×3 matrices with Jordan product.

The rank 4 Fischer spaces are classified by [H93], Proposition 2.9. They are the Fischer space A_4 and the quotients of the Fischer spaces of the 3-transposition groups $W_k(\hat{\mathrm{A}}_3), k = 2, 3$ and $(G_i, D_i), i = 4, 5$, defined above. It follows by Theorem 2.5.5 that the Matsuo algebra of A_4 is Jordan. This is the only one out of the five groups which gives a Jordan algebra. For the others, if a, b, c, d are any 4 generating transpositions of G, G a quotient of $W_2(\hat{\mathrm{A}}_3), W_3(\hat{\mathrm{A}}_3), G_4$ or G_5 and $D = a^G \cup b^G \cup c^G \cup d^G = a^G$, then for $x = a + b + c$ in the algebra $M_{1/2}(\mathcal{G})$, $(xx)(dx) \neq ((xx)d)x$, whence A is not Jordan. We show that A is not Jordan for $G = W_k(\hat{\mathrm{A}}_3)$ and $k = 2, 3$ by the explicit example: set $a', b', c', d' =$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \end{pmatrix}, \tag{2.76}$$

which are generators of $G' = k^4 : \mathrm{Sym}(4)$, and if $n = \left(\begin{array}{c|c} I_4 & 0 \\ \hline 1 & 1 \end{array}\right)$, then $G = G'/\langle n \rangle$ and $a = a'/\langle n \rangle$ and likewise define b, c, d. For $(G, D) = W_2(\hat{\mathrm{A}}_3)$, the coefficient of a in $((xx)d)x$ is $\frac{3}{8}$ and the coefficient of a in $(xx)(dx)$ is $\frac{7}{16}$. For $(G, D) = W_3(\hat{\mathrm{A}}_3)$, the respective coefficients are $\frac{13}{32}$ and $\frac{7}{16}$. We see that $\frac{3}{8}, \frac{13}{32} \neq \frac{7}{16}$ in any characteristic (not 2 by assumption), and similar inequalities hold in the quotient cases. In each case this shows that the Jordan identity does not hold.

Abusively, let now a, b, c, d stand for the images of a, b, c, d under the quotient

$\mathrm{Cox}(\mathcal{C}') \to G_4$ or $\mathrm{Cox}(\mathcal{C}) \to G_5$. Then $x = a + b + c$ in the algebra again gives $((xx)d)x \neq (xx)(dx)$. In both cases, the idempotent corresponding to a^{cdb} has a nonzero contribution, namely with coefficient $-\frac{1}{32}$, on only the lefthand side. Therefore the Matsuo algebras for (G_4, D_4) and (G_5, D_5) are not Jordan.

Suppose that (G, D) is a transposition group whose Fischer space $\mathcal{G}$ has rank r at least 5, such that the Matsuo algebra $A = M_{1/2}(G, D)$ is Jordan. If $T \subseteq D$ is a generating set for G and $\mathcal{T}$ is the noncommuting graph on T, then G is a quotient of the Coxeter group on $\mathcal{T}$. Suppose that the subspace spanned by $T' = \{d_1, \ldots, d_4\}$ has rank 4 in $\mathcal{G}$. By the above, $\langle T' \rangle \cong \mathrm{Sym}(5)$ and $\mathcal{T}'$ is a line with 4 nodes, since the subalgebra of A generated by $d_1, \ldots, d_4$ must itself be Jordan. Therefore if $T = \{d_1, \ldots, d_r\} \subseteq D$ are a (connected, since the noncommuting graph on D is connected) set of generators for G, then no vertex has valency 3 in $\mathcal{T}$. Therefore $\mathcal{T}$ is either a line or a loop, corresponding to A_r or $\hat{\mathrm{A}}_{r-1}$. By Theorem 2.5.5, $M_{1/2}(\mathrm{A}_r)$ is Jordan. Suppose $\mathcal{T}$ is $\hat{\mathrm{A}}_{r-1}$. Then G is a quotient of $W_k(\hat{\mathrm{A}}_{r-1})$ [H93]. But $W_k(\hat{\mathrm{A}}_{r-1})$ admits an embedding of $W_k(\hat{\mathrm{A}}_3)$ for all $r \geq 5$: for

$$a' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \oplus I_{r-4}, \quad b' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \oplus I_{r-4}, \quad c' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \oplus I_{r-4},$$

$$d' = \left(\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{r-3} & 0 \\ \hline 1 & -1 & 0 & 1 \end{array}\right), \quad n = \left(\begin{array}{c|c} I_{r-1} & 0 \\ \hline 1 & 1 \end{array}\right),$$

we have that $W_k(\hat{\mathrm{A}}_{r-1})$ is the quotient of $k^r : \mathrm{Sym}(r)$ by $\langle n' \rangle$, and a, b, c, d the images of a', b', c', d' in $W_k(\hat{\mathrm{A}}_{r-1})$ generate a subgroup $W_k(\hat{\mathrm{A}}_3)$. Therefore the Matsuo algebra of $W_k(\hat{\mathrm{A}}_3)$ is a subalgebra of A, but it is not Jordan, so A is not Jordan. Hence the only possibility is that $\mathcal{T}$ is A_r. $\square$

Chapter 3. Lattice algebras

In this section we try to recover good behaviour, and good subalgebras, for Matsuo algebras by using properties of their underlying graphs and their field of definition. We prove a variety of general statements and eventually specialise to the cases of A_n and D_n, with a view towards vertex algebras and towards automorphisms.

3.1 Preliminaries

This section introduces the some graph-theoretic constructions and also relates them to the 3-transposition groups which are their source. We present a conjecture on the action of maximal parabolic subgroups on transpositions and prove it in many cases, which we will need in the sequel for our results on Matsuo algebras.

We now introduce a particular combinatorial property of embeddings of Fischer spaces which will be important to us later.

We say that a graph $\mathcal{G}$ is *k-regular* for $k \in \mathbb{N}$ if

$$\text{for all } x \in \mathcal{G}, \quad |x^{\sim}| = |\{y \in \mathcal{G} \mid x \sim y\}| = k. \tag{3.1}$$

If $\mathcal{G}$ is k-regular for some $k \in \mathbb{N}$, we say that $\mathcal{G}$ is *regular*, and set $k_{\mathcal{G}} \in \mathbb{N}$ such that $\mathcal{G}$ is $k_{\mathcal{G}}$-regular.

3.1.1 Proposition. *Suppose that $\mathcal{G}$ is a connected Fischer space. Then $\mathcal{G}$ is regular.*

Proof. Let $\tau(\mathcal{G}) = \{\tau(x) \mid x \in \mathcal{G}\} \subseteq \mathrm{Aut}(\mathcal{G})$, and $G = \langle \tau(\mathcal{G}) \rangle$. If $\mathcal{G}$ is a single point, then $\mathcal{G}$ is 0-regular. Otherwise, since $\mathcal{G}$ is connected, every point lies on at least one line, so that $\tau(x)$ is nontrivial for all $x \in \mathcal{G}$, and therefore $|\tau(x)| = 2$. Now G acts transitively on $\mathcal{G}$, because $\mathcal{G}$ connected means that for any $x, x' \in \mathcal{G}$ there exist $x_0 = x, x_1, \ldots, x_{n-1}, x_n = x'$ such that $x_i \sim x_{i+1}$ for $0 \leq i < n$ and thus $x_i \wedge (x_i \wedge x_{i+1}) = x_{i+1}$, so $x_i^{\tau(x_i \wedge x_{i+1})} = x_{i+1}$ and for $t = \tau(x_0 \wedge x_1)\tau(x_1 \wedge x_2) \cdots \tau(x_{n-1} \wedge x_n)$ we have $x^t = x'$. This action moves edges to edges, and so in particular $(x^\sim)^t = (x^t)^\sim = x'^\sim$. Therefore $|x'^\sim| = |x^\sim|$ for all $x, x' \in \mathcal{G}$, and $\mathcal{G}$ is $|x^\sim|$-regular for any $x \in \mathcal{G}$. $\square$

In the cases where we have an embedding of linear 3-graphs $\mathcal{H} \subseteq \mathcal{G}$ we also have a notion of a *boundary graph* $\mathcal{G}/\mathcal{H}$: the graph with point set

$$\mathcal{H}^\sim = \{x \in \mathcal{G} \setminus \mathcal{H} \mid x \sim y \text{ for some } y \in \mathcal{H}\} \tag{3.2}$$

and lines $\{x, x'\}$ if $x \wedge x' \in \mathcal{H}$.

We will be interested in cases where

Definition. If $\mathcal{G}, \mathcal{H}$ are connected regular 3-graphs, $\mathcal{H}$ is maximal in $\mathcal{G}$ and $\mathcal{G}/\mathcal{H}$ is also a connected regular graph, then the embedding $\mathcal{H} \subseteq \mathcal{G}$ is *very regular*.

Define $k_\mathcal{H}^\mathcal{G}$ such that $\mathcal{G}/\mathcal{H}$ is $k_\mathcal{H}^\mathcal{G}$-regular. Observe that $k_\mathcal{H}^\mathcal{G} = |x^\sim \cap \mathcal{H}|$ for $x \in \mathcal{H}^\sim$ in this case.

We will look for examples of very regular embeddings coming from 3-transposition groups. A subgroup H of a 3-transposition group (G, D) is *parabolic* if H is generated by $H \cap D$. This H is *maximal parabolic* if it is a proper subgroup maximal among parabolic subgroups of (G, D).

We also say that a maximal parabolic subgroup H is *very regular* in a 3-transposition group (G, D) if the induced embedding of graphs $\mathcal{H} \subseteq \mathcal{G}$ is very regular.

By extension from (3.2),

$$\mathcal{H}^{\not\sim} = \{x \in \mathcal{G} \smallsetminus \mathcal{H} \mid x \not\sim y \text{ for any } y \in \mathcal{H}\}. \tag{3.3}$$

If $\mathcal{H} \subseteq \mathcal{G}$ then $\mathcal{H}^{\not\sim} = \emptyset$, that is, $\mathcal{G} = \mathcal{H} \cup \mathcal{H}^{\sim}$. For if $x \in \mathcal{H}^{\not\sim}$, then $\mathcal{H} \subset \langle x, \mathcal{H} \rangle \subseteq \mathcal{G}$ either is a strict inclusion of subspaces, contradicting that $\mathcal{H}$ is maximal, or $\mathcal{G} = \langle x, \mathcal{H} \rangle = \langle x \rangle \oplus \mathcal{H}$, contradicting that $\mathcal{G}$ is connected.

3.1.2 Lemma. *Any maximal parabolic subgroup H of (G, D) is the subgroup generated by $M \cap D$ for M a maximal subgroup of G containing H.*

Proof. Suppose that H is generated by $H \cap D$, and let M be a maximal subgroup of G containing H. Then $H \subseteq M$ implies $H \cap D \subseteq M \cap D$, and $H = \langle H \cap D \rangle \subseteq \langle M \cap D \rangle \subseteq M \neq G$. If H is maximal among parabolic subgroups, then necessarily $H = \langle M \cap D \rangle$. □

3.1.3 Lemma. *Maximal parabolic subgroups H of (G, D) are in bijection with maximal subspaces $\mathcal{H}$ of the Fischer space $\mathcal{G}$ of (G, D).*

Proof. Suppose that H is a parabolic subgroup and let $\mathcal{H}$ be the Fischer space of $(H, H \cap D)$, viewed as a subspace of $\mathcal{G}$. There exists a point $x \in \mathcal{G} \smallsetminus \mathcal{H}$ such that $\langle x, \mathcal{H} \rangle \neq \mathcal{G}$ (strictly) contains $\mathcal{H}$ if and only if there exists an transposition $d \in D$ such that $\langle d, H \rangle \neq G$ (strictly) contains H. □

For brevity, when (G, D) is a 3-transposition group and D is a single conjugacy class in G, $H \subseteq G$ is parabolic and $H \cap D$ is a single conjugacy class in H, we say that H is a *connected* subgroup of (G, D).

3.1.4 Conjecture. *Whenever $\mathcal{G}$ is the Fischer space of a 3-transposition group (G, D) and $\mathcal{H}$ is the Fischer space of a connected maximal parabolic subgroup H of (G, D), we conjecture that $\mathcal{H} \subseteq \mathcal{G}$ is very regular.*

Our evidence is collected in Theorem 3.1.5.

In the cases where G is a Weyl group, we form the associated 3-transposition group (G, D) by taking D to be the conjugacy class of reflections of roots. When $G = W_k(\hat{\mathrm{A}}_n)$, G is a quotient of a Weyl group (as defined in Section 2.5) and D is the image of the conjugacy class of reflections in the quotient. By $3^n : 2$ we understand the elementary abelian group 3^n extended by an inverting involution, unless otherwise indicated. In all groups of shape $3^m : 2$, the transpositions are the unique class of involutions.

3.1.5 Theorem. *The connected maximal parabolic subgroups H of (G, D) induce very regular Fischer spaces $\mathcal{H} \subseteq \mathcal{G}$ when (G, D) is, for any $n \in \mathbb{N}$, the Weyl group of* A_n*,* D_n*,* $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$*, or* $W_k(\hat{\mathrm{A}}_n)$ *for* $k = 2, 3$*, or* $3^n : 2$*, or M. Hall's* $3^{10} : 2$.

Proof. Recall that (G, D) for A_n is $G = \mathrm{Sym}(n+1), D = (1,2)^G$. Let $E \subseteq D$ and $S \subseteq \{1, \ldots, n+1\}$ be the support of E, that is, the smallest subset S of $\{1, \ldots, n+1\}$ such that any transposition $e \in E$ is of the form (s_1, s_2) for some $s_1, s_2 \in S$. Then partition S into orbits $S_1, \ldots, S_n$ of $\langle E \rangle$. Observe that $\langle E \rangle \cong \mathrm{Sym}(|S_1|) \times \cdots \times \mathrm{Sym}(|S_n|)$ and therefore E does not satisfy the hypothesis of connectedness unless $S = S_1$ is a single orbit. Furthermore if $|S|$ is less than n then H is not maximal. Therefore a connected maximal parabolic subgroup H of G has support $\{1, \ldots, j-1, j+1, \ldots, n+1\}$ for some j and $H \cong \mathrm{Sym}(n)$. In these cases let $d = (1, j)$, or $d = (1, 2)$ if $j = 1$, so that $d \in D \setminus (D \cap H)$. We see that $D = (H \cap D) \cup d^H$.

As $W(\mathrm{D}_n) \cong W_2(\hat{\mathrm{A}}_{n-1})$ by [H93], we cover it below as part of $W_k(\hat{\mathrm{A}}_{n-1})$.

The cases for $W(\mathrm{E}_n), n = 6, 7, 8$, were checked in [MAGMA] with the computational assistance of Raul Moragues Moncho.

Suppose that (G, D) comes from $W_k(\hat{\mathrm{A}}_n)$ when $k = 2, 3$ and $n \geq 3$. There are two possibilities for a parabolic subgroup H such that $H \cap D$ is a single conjugacy class: either H is isomorphic to $\mathrm{Sym}(n)$ or to $W_k(\hat{\mathrm{A}}_{n-1})$. Along the same lines as in the proof of Theorem 2.5.9, we use a representation of G as a matrix group. Let

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-1}, \quad g_2 = \begin{pmatrix} 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-2}, \quad \ldots, \quad g_{n-1} = I_{n-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \end{pmatrix},$$

$$g_{n+1} = \left(\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{r-3} & 0 \\ \hline 1 & -1 & 0 & 1 \end{array}\right), \quad h = \left(\begin{array}{c|c} I_{n+1} & 0 \\ \hline 1 & 1 \end{array}\right) \text{ over } \mathbb{F}_k. \tag{3.4}$$

Then $G \cong \langle g_1, \ldots, g_{n+1} \rangle / \langle h \rangle$ and D is the set of conjugates of $\{g_i \langle n \rangle\}_{1 \leq i \leq n+1}$. We also set $\hat{G} = \langle g_1, \ldots, g_{n+1} \rangle$ and $\hat{D}$ the set of conjugates of $\{g_i\}_{1 \leq i \leq n+1}$. Now $H \cong W_k(\hat{\mathrm{A}}_{n-1})$ if and only if, up to conjugation, $H = \hat{H} / \langle h \rangle$ for $\hat{H} = \langle g_1, \ldots, g_{n-1}, g_{n+1} \rangle$. Then it is clear that in $\hat{G}$, $\hat{D} = (\hat{H} \cap \hat{D}) \cup g_n^{\hat{H}}$. The same property descends to the quotient, so that $D = (H \cap D) \cup (g_n \langle n \rangle)^H$. This shows that $\mathcal{G}/\mathcal{H}$ is connected, so $\mathcal{H} \subseteq \mathcal{G}$ is very regular. The other possibility is that $H = \langle g_1, \ldots, g_n \rangle / \langle n \rangle$. In this case, when $k = 2$ we see that $W_2(\hat{\mathrm{A}}_{n-1}) \cong W(\mathrm{D}_n)$, and as we will see, $\mathcal{D}_n = \mathcal{A}_n^{\pm}$ by Lemma 3.5.2 and $\mathcal{G}$ is very regular in $\mathcal{G}^{\pm}$ by Lemma 3.1.6, so that this possibility is covered. However we can observe that in general in $\hat{G}$, the orbit of g_{n+1} under the action of $\hat{H} = \langle g_1, \ldots, g_n \rangle$ has size $\frac{1}{2} n(n+1)$ if $k = 2$ and $n(n+1)$ if $k = 3$, so that $\hat{H}$ is transitive on the transpositions in $\hat{G}$ outside $\hat{H}$. This again holds in the quotient H.

When $G = 3^n : 2$, there is only one conjugacy class D of involutions. Observe that any subset of involutions of G generates a subgroup $H \cong 3^m : 2$ for some m. Then H is maximal if $m = n - 1$. In this case, if t, s are two transpositions in $D \smallsetminus H$, then $\langle t, s \rangle \cap H = \{t^s\}$ as $t^s \notin H$ would contradict maximality, so $\mathcal{G}/\mathcal{H}$ is connected. This shows that it is also regular by transitivity.

That the statement holds for M. Hall's $3^{10} : 2$ was checked in [GAP] using the presentation in (2.75). $\square$

An important family of very regular embeddings can be constructed as follows. If $\mathcal{G}$ is a 2-graph, its *double*, denoted $\mathcal{G}^{\pm}$, is the 2-graph with point set $\{x^+, x^- \mid x \in \mathcal{G}\}$ and lines $\{x^\varepsilon, y^\eta\}$ for $x \sim y$ in $\mathcal{G}$ and $\varepsilon, \eta \in \{+, -\}$. If $\mathcal{G}$ is a linear 3-graph, its *double* $\mathcal{G}^{\pm}$ is the 3-graph with point set $\{x^+, x^- \mid x \in \mathcal{G}\}$ and lines $\{x^\varepsilon, y^\eta, (x \wedge y)^{\varepsilon\eta}\}$ for $x \sim y$ in $\mathcal{G}$ and $\varepsilon, \eta \in \{+, -\}$. We always identify $\mathcal{G}$ with $\mathcal{G}^+ = \{x^+ \mid x \in \mathcal{G}\}$ embedded in $\mathcal{G}^{\pm}$. Note that $\mathcal{G}^{\pm}$ is not necessarily a linear 3-graph, as it is possible in general that two lines intersect in exactly two points.

3.1.6 Lemma. *If $\mathcal{H}$ is a connected regular 3-graph, then $\mathcal{H}^+ \subseteq \mathcal{H}^{\pm}$ is very regular, and $k_{\mathcal{H}}^{\mathcal{H}^{\pm}} = k_{\mathcal{H}}$.*

Proof. Observe that $\mathcal{H} = \mathcal{H}^+ \subseteq \mathcal{H}^{\pm}$ is maximal. The point set underlying $\mathcal{H}^{\pm}/\mathcal{H}$ is $\mathcal{H}^- \subseteq \mathcal{H}^{\pm}$, and for any two $x^-, y^- \in \mathcal{H}^-$, if $x \sim y$ then we have $x^- \wedge y^- = x^+ \wedge y^+ \in \mathcal{H}^+$. As $\mathcal{H}$ is connected, if $x \not\sim y$ there exists a path $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$, and $(x_i^- \wedge x_{i+1}^-) \in \mathcal{H}^+$, so that therefore $x_0^- \sim x_1^- \sim \cdots \sim x_n^-$ in $\mathcal{H}^-$, and $\mathcal{H}^{\pm}/\mathcal{H}^+$ is connected. It follows from the definition that $\mathcal{H}^+ \cap (x^-)^\sim = \mathcal{H}^+ \cap x^\sim$, so $\mathcal{H}^{\pm}/\mathcal{H}^+$ is $|\mathcal{H}^+ \cap x^\sim| = k_{\mathcal{H}}$-regular. □

Finally, we introduce some useful concepts from algebraic combinatorics. Suppose that $\mathcal{G}$ is a (finite) graph and order its vertices as $x_1, \dots, x_n$. The *adjacency matrix* of $\mathcal{G}$ is the $n \times n$-matrix $\mathrm{ad}(\mathcal{G}) = (m_{ij})_{1 \le i,j \le n}$ whose entries are

$$m_{ij} = \begin{cases} 1 & \text{if } x_i \sim x_j \\ 0 & \text{otherwise} . \end{cases} \tag{3.5}$$

The adjacency matrix allows us to derive eigenvalues from a graph. In particular, we write $\mathrm{Spec}(\mathcal{G})$ for the multiset of eigenvalues of $\mathrm{ad}(\mathcal{G})$.

3.1.7 Lemma. *The eigenvalues of $\mathcal{G}^{\pm}$ for $\mathcal{G}$ a graph are $2\,\mathrm{Spec}(\mathcal{G}) \cup \{0^{|\mathcal{G}|}\}$, and the multiplicities of 2ϕ, $\phi \in \mathrm{Spec}(\mathcal{G})$, are preserved.*

Proof. Fix an ordering $\{x_1, \dots, x_n\}$ of $\mathcal{G}$ and $\{x_1^+, \dots, x_n^+, x_1^-, \dots, x_n^-\}$ of $\mathcal{G}^\pm$. Let $x = \sum_{1 \le i \le n} \phi_i x_i$ be an α-eigenvector of $\mathrm{ad}(\mathcal{G})$. Then $\hat{x} = \sum_{1 \le i \le n} \phi_i (x_i^+ + x_i^-)$ and $\check{x} = \sum_{1 \le i \le n} \phi_i (x_i^+ - x_i^-)$. Take $\hat{x}$ is a 2α-eigenvector, and $\check{x}$ is a 0-eigenvector, of $\mathrm{ad}(\mathcal{G}^\pm)$. This affords $2n$ linearly independent eigenvectors for $\mathrm{ad}(\mathcal{G}^\pm)$, which has size $2n \times 2n$, so these must be all eigenvectors. □

We write I_n for the $n \times n$ identity matrix, and 0_n for the $n \times n$ all-zero matrix. Recall the direct sum of matrices is $A \oplus B = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$.

3.1.8 Theorem (Perron-Frobenius, [GR01] Theorem 8.8.1)**.** *For the adjacency matrix A of a connected graph, there exists a real positive eigenvalue ρ of A such that $|\phi| \le |\rho|$ for all eigenvalues ϕ of A, and the ρ-eigenspace of A is 1-dimensional. If furthermore the graph is k-regular, then $\rho = k$.* □

3.2 Identity elements

This section is concerned with establishing that parabolic subalgebras are almost always unital, and establishing the eigenvalues and the Seress property of their identity elements. The exceptions derive from coincidences between the parameter α and the spectrum of the underlying graphs.

Throughout this chapter, $\mathcal{H} \subseteq \mathcal{G}$ are Fischer spaces, and our algebras are over a field $\mathbb{F}$, both unless otherwise stated. A particularly useful case, for which some of our later assumptions on α hold automatically, is when $\mathbb{F} = \mathbb{F}'(\alpha)$, where $\mathbb{F}'$ is a field over which α is transcendental.

3.2.1 Proposition. *Suppose that $\mathcal{G}$ is a connected Fischer space. Then $M_\alpha(\mathcal{G})$ (over $\mathbb{F}$) is unital if $\alpha \neq -\frac{2}{k_\mathcal{G}}$, with identity*

$$\mathrm{id}_\mathcal{G} = \frac{1}{1 + \frac{1}{2}\alpha k_\mathcal{G}} \sum_{x \in \mathcal{G}} x. \tag{3.6}$$

Proof. We show that, for $x \in \mathcal{G}$,

$$x \sum_{y \in \mathcal{G}} y = (1 + \frac{1}{2}\alpha k_\mathcal{G})x. \tag{3.7}$$

Recall the notation $x^\sim = \{y \in \mathcal{G} \mid x \sim y\}$ and $x^\nsim = \{y \in \mathcal{G} \mid x \nsim y\}$. Note that $\mathcal{G} = \{x\} \cup x^\sim \cup x^\nsim$, and $|x^\sim| = k_\mathcal{G}$. Then

$$x \sum_{y \in \mathcal{G}} y = xx + x \sum_{y \in x^\sim} y + x \sum_{y \in x^\nsim} y = x + \frac{\alpha}{2} \sum_{y \in x^\sim} (x + y - x \wedge y) + 0 = (1 + \frac{1}{2}\alpha k_\mathcal{G})x, \tag{3.8}$$

where the last equality follows since, as y ranges over $x^\sim$, so does $x \wedge y$: that is, $\{x \wedge y \mid y \in x^\sim\} = x^\sim$ and $\sum_{y \in x^\sim}(y - x \wedge y) = \sum_{y \in x^\sim} y - \sum_{y \in x^\sim} x \wedge y = 0$. $\square$

This result generalises to a nonconnected Fischer space $\mathcal{G}$. If $\mathcal{G} = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_n$ is a partition into pairwise disconnected Fischer spaces, then $\mathrm{id}_\mathcal{G} = \sum_i \mathrm{id}_{\mathcal{G}_i}$, provided each $M_\alpha(\mathcal{G}_i)$ is unital.

3.2.2 Lemma. *Suppose that $\mathcal{H} \subseteq \mathcal{G}$ is very regular and $\alpha \neq -\frac{2}{k_{\mathcal{H}}}$. Then $\mathrm{id}_{\mathcal{H}}$ in $A = M_\alpha(\mathcal{G})$ acts diagonalisably on the subspaces spanned by*

$\mathcal{H}$ with eigenvalue 1, and

$\mathcal{H} \cup \mathcal{H}^{\sim}$ with further eigenvalues $\dfrac{\alpha}{2+\alpha k_{\mathcal{H}}}(k_{\mathcal{H}}^{\mathcal{G}} - \mathrm{Spec}(\mathcal{G}/\mathcal{H}))$.

Proof. By Proposition 3.2.1, $\langle \mathcal{H} \rangle_{\mathbb{F}}$ is a subspace of the 1-eigenspace of $\mathrm{id}_{\mathcal{H}}$.

Take $y \in \mathcal{H}^{\sim}$, where $\mathcal{H}^{\sim}$ is defined in (3.2). Then $y \notin \mathcal{H}$ and

$$\mathrm{id}_{\mathcal{H}}\, y = \frac{1}{1+\frac{1}{2}\alpha k_{\mathcal{H}}} \sum_{x \in y^{\sim} \cap \mathcal{H}} \frac{\alpha}{2}(x + y - x \wedge y). \tag{3.9}$$

If $x \in \mathcal{H}$ and $y \in x^{\sim} \setminus \mathcal{H}$ then $y, x \wedge y \in \mathcal{H}^{\sim}$, so that $\mathrm{id}_{\mathcal{H}}$ fixes the subspace spanned by $\mathcal{H} \cup \mathcal{H}^{\sim}$. Furthermore, as $k_{\mathcal{G}}^{\mathcal{H}} = |y^{\sim} \cap \mathcal{H}|$,

$$\mathrm{id}_{\mathcal{H}}\, y = \frac{\alpha k_{\mathcal{G}}^{\mathcal{H}}}{2+\alpha k_{\mathcal{H}}} y + \frac{\alpha}{2+\alpha k_{\mathcal{H}}} \sum_{x \in y^{\sim} \cap \mathcal{H}} (x - x \wedge y). \tag{3.10}$$

Observe that $x \in \mathcal{H}$ and $x \wedge y \in \mathcal{H}^{\sim}$ (for, if $x \wedge y \in \mathcal{H}$, then as $\mathcal{H}$ is a subspace we would have $x \wedge (x \wedge y) = y \in \mathcal{H}$). Now suppose that $e \in \langle \mathcal{H} \cup \mathcal{H}^{\sim} \rangle_{\mathbb{F}}$ is an eigenvector for $\mathrm{id}_{\mathcal{H}}$. Write $e_{\sim}$ for the projection of e to $\langle \mathcal{H}^{\sim} \rangle_{\mathbb{F}}$ and $e_0 = e - e_{\sim}$. Then

$$\mathrm{id}_{\mathcal{H}}\, e = e_0 + \mathrm{id}_{\mathcal{H}}\, e_{\sim} = \phi e \tag{3.11}$$

for some ϕ, and, using (3.10), the projection of $\mathrm{id}_{\mathcal{H}}\, e_{\sim}$ to $\langle \mathcal{H}^{\sim} \rangle_{\mathbb{F}}$ is

$$\frac{\alpha}{2+\alpha k_{\mathcal{H}}}(k_{\mathcal{G}}^{\mathcal{H}} I_{|\mathcal{H}|} + \mathrm{ad}(\mathcal{G}/\mathcal{H}))e_{\sim} = \phi e_{\sim}, \tag{3.12}$$

where $\mathrm{ad}(\mathcal{G}/\mathcal{H})y = \sum_{x \in y^{\sim} \cap \mathcal{H}} x \wedge y$ is extended $\mathbb{F}$-linearly to $\langle \mathcal{H}^{\sim} \rangle_{\mathbb{F}}$. Therefore if $e_{\sim} \neq 0$, then ϕ is an eigenvalue of

$$\frac{\alpha}{2+\alpha k_{\mathcal{H}}}(k_{\mathcal{G}}^{\mathcal{H}} I_{|\mathcal{H}|} - \mathrm{ad}(\mathcal{G}/\mathcal{H})). \tag{3.13}$$

Therefore ϕ is in $\frac{\alpha}{2+\alpha k_{\mathcal{H}}}(k_{\mathcal{G}}^{\mathcal{H}} - \mathrm{Spec}(\mathcal{G}/\mathcal{H}))$. By comparing dimensions, the eigenspaces of $\mathrm{id}_{\mathcal{H}}$ span $\langle \mathcal{H} \cup \mathcal{H}^{\sim} \rangle_{\mathbb{F}}$, so $\mathrm{id}_{\mathcal{H}}$ is diagonalisable. □

3.2.3 Lemma. *Suppose that $\mathcal{H} \subseteq \mathcal{G}$ and $\mathcal{H}$ is very regular in any subspace $\mathcal{G}' \subseteq \mathcal{G}$ in which $\mathcal{H}$ is maximal. If $\alpha \neq -\frac{2}{k_{\mathcal{H}}}$, then $\mathrm{id}_{\mathcal{H}}$ is diagonalisable in $M_\alpha(\mathcal{G})$.*

Proof. By Proposition 3.2.1, the subalgebra of $M_\alpha(\mathcal{G})$ spanned by $\mathcal{H}$ has an identity $\mathrm{id}_{\mathcal{H}}$.

Let $x \in \mathcal{G} \setminus \mathcal{H}$ be arbitrary and set $\mathcal{G}' = \langle x, \mathcal{H} \rangle$. If $x \notin \mathcal{H}^{\sim}$, then $\mathrm{id}_{\mathcal{H}} x = 0$; otherwise, $\mathrm{id}_{\mathcal{H}}$ acts on $\mathcal{G}'$ diagonalisably by Lemma 3.2.2. Now $\mathcal{G} \setminus \mathcal{H}$ can be partitioned in $\mathcal{G}'_1, \mathcal{G}'_2, \ldots, \mathcal{G}'_r$ and $\mathcal{H}^{\not\sim}$ where each $\mathcal{G}'_i$ is a subgraph of $\mathcal{G}$ in which $\mathcal{H}$ is maximal. That $\mathcal{G}'_i \cap \mathcal{G}'_j = \mathcal{H}$ if $i \neq j$ follows from the fact that, if $y \in (\mathcal{G}'_i \cap \mathcal{G}'_j) \setminus \mathcal{H}$ then $\mathcal{G}'_i = \langle \mathcal{H}, y \rangle = \mathcal{G}'_j$ by maximality, so the $\mathcal{G}'_i$ have pairwise trivial intersection. Thus $\mathrm{id}_{\mathcal{H}}$ acts diagonalisably on a basis of $M_\alpha(\mathcal{G})$. □

3.2.4 Lemma. *If $\mathcal{H} \subseteq \mathcal{G}$ is very regular and $\alpha \neq -\frac{2}{\phi}$ for $\phi \in \mathrm{Spec}(\mathrm{ad}(\mathcal{H}))$, then the 1-eigenspace of $\mathrm{id}_{\mathcal{H}}$ in $M_\alpha(\mathcal{G})$ is $\mathcal{H}$ and its 0-eigenspace is 1-dimensional.*

Proof. The eigenvalues of $\mathrm{id}_{\mathcal{H}}$ on $\mathcal{G}^{\pm}$ are classified by Lemma 3.2.2. Evidently $\mathcal{H} \subseteq A_1^{\mathrm{id}_{\mathcal{H}}}$. The Perron-Frobenius eigenspace of $\mathrm{ad}(\mathcal{G}/\mathcal{H})$ is 1-dimensional with eigenvalue $k_{\mathcal{H}}^{\mathcal{G}}$, so it gives a 1-dimensional eigenspace $\langle z \rangle$ of eigenvalue 0 for $\mathrm{id}_{\mathcal{H}}$. By Theorem 3.1.8, this is the only $k_{\mathcal{H}}^{\mathcal{G}}$-eigenvector of $\mathrm{ad}(\mathcal{G}/\mathcal{H})$, and therefore the only 0-eigenvector of $\mathrm{id}_{\mathcal{H}}$. It only remains to consider other 1-eigenvectors. The only solution, when $\phi \neq k_{\mathcal{H}}$, to

$$\frac{\alpha}{2 + \alpha k_{\mathcal{H}}}(k_{\mathcal{H}} - \phi) = 1 \tag{3.14}$$

is $\alpha = -\frac{2}{\phi}$. □

We say that an element $x \in A$ is *Seress* if it acts diagonalisably and the (smallest) fusion rules satisfied by its eigenspaces are Seress.

Recall from Section 3.1 the definition of a very regular embedding of Fischer spaces.

3.2.5 Theorem. *If $\mathcal{H} \subseteq \mathcal{G}$ are very regular Fischer spaces and $\alpha \neq -\frac{2}{\phi}$ for any $\phi \in \mathrm{Spec}(\mathrm{ad}(\mathcal{H}))$, then $\mathrm{id}_{\mathcal{H}}$ exists and is Seress in $M_\alpha(\mathcal{G})$. Furthermore $\mathrm{id}_{\mathcal{H}}$ is Seress in $M_\alpha(\mathcal{G})$ if $\mathcal{H} \subseteq \mathcal{G}'$ is very regular for every $\mathcal{G}'$ such that $\mathcal{H} \subseteq \mathcal{G}'$ is maximal and $\mathcal{G}' \subseteq \mathcal{G}$.*

Proof. By Proposition 3.2.1, $\mathrm{id}_{\mathcal{H}}$ exists. Lemma 3.2.2 showed that $\mathrm{id}_{\mathcal{H}}$ acts diagonalisably. We use the classification of 1- and 0-eigenvectors of Lemma 3.2.4 to prove that $1 \star \phi \subseteq \{\phi\} \supseteq 0 \star \phi$ and $1 \star 0 = \emptyset$ for eigenvectors of $\mathrm{id}_{\mathcal{H}}$ in $A = M_\alpha(\mathcal{G})$, first for the case when $\mathcal{H} \subseteq \mathcal{G}$ is very regular.

Suppose that $\mathcal{H} \subseteq \mathcal{G}$ is very regular. Since under our hypotheses the 1-eigenspace of $\mathrm{id}_{\mathcal{H}}$ is $\mathcal{H}$, which is closed under multiplication, it is obvious that $1 \star 1 = \{1\}$.

We will use three facts. Firstly, for any $h \in \mathcal{H}, x \in A$, by application of (2.47),

$$hx = \frac{\alpha}{2}(\kappa_h h + x - x^{\tau(h)}) \text{ for some } \kappa_h \in \mathbb{F}. \tag{3.15}$$

Secondly, if $t \in \mathrm{Aut}(A) \subseteq \mathrm{End}(A)$ fixes $x \in A$, then t centralises $\mathrm{ad}(x) \in \mathrm{End}(A)$ and the eigenspaces A^x_ϕ of x. Thirdly, if $h \in \mathcal{H}$ then, as $\mathcal{H}$ is closed under $\wedge$, $\tau(h)$ permutes the points of $\mathcal{H}$ and therefore fixes $\mathrm{id}_{\mathcal{H}}$.

To show that $1 \star \phi = \{\phi\}$ for $\phi \neq 1$, suppose that $h \in \mathcal{H}$ and y is a ϕ-eigenvector of $\mathrm{id}_{\mathcal{H}}$ in $\mathcal{G}$. Set $y = y_{\mathcal{H}} + y'$, for $y_{\mathcal{H}} \in \langle \mathcal{H} \rangle$ the projection of y onto $\mathcal{H}$ and y' in the span of $\mathcal{H}^\sim$. Now as y is a ϕ-eigenvector for $\mathrm{id}_{\mathcal{H}}$, $\mathrm{id}_{\mathcal{H}}\, y = \phi y$ is again a ϕ-eigenvector. On the other hand, using Proposition 3.2.1 and (3.15),

$$\mathrm{id}_{\mathcal{H}}\, y = \frac{1}{1 + \frac{1}{2}k_{\mathcal{H}}\alpha} \sum_{h \in \mathcal{H}} hy = \frac{\alpha}{2 + \alpha k_{\mathcal{H}}} \sum_{h \in \mathcal{H}} (\kappa_h h + y - y^{\tau(h)}). \tag{3.16}$$

Noting that $y^{\tau(h)} \in A^{\mathrm{id}_{\mathcal{H}}}_\phi$, we have that $\sum_{h \in \mathcal{H}} \kappa_h h$ may be expressed as a sum of ϕ-eigenvectors. On the other hand, any $h \in \mathcal{H}$ is a 1-eigenvector and $A^{\mathrm{id}_{\mathcal{H}}}_1 \cap A^{\mathrm{id}_{\mathcal{H}}}_\phi = 0$, so that $\sum_{h \in \mathcal{H}} \kappa_h h = 0$. As the points $h \in \mathcal{H}$ are linearly independent, this means $\kappa_h = 0$ for all $h \in \mathcal{H}$. Therefore $hy = \frac{\alpha}{2}(y - y^{\tau(h)}) \in A^{\mathrm{id}_{\mathcal{H}}}_\phi$.

To show that $1 \star 0 = \emptyset$, observe that the 0-eigenspace of $\mathrm{id}_{\mathcal{H}}$ is 1-dimensional by

Lemma 3.2.4, and therefore fixed by any automorphism t fixing $\mathrm{id}_{\mathcal{H}}$. In particular, $\tau(h)$ fixes $y \in A_0^{\mathrm{id}_{\mathcal{H}}}$, so by the previous paragraph, $hy = \frac{\alpha}{2}(y - y) = 0$.

Therefore a 0-eigenvector z of $\mathrm{id}_{\mathcal{H}}$ in $M_\alpha(\mathcal{G})$ is also a 0-eigenvector of any $h \in \mathcal{H}$. By Lemma 2.1.6, for any $x \in A$ we have $h(xz) = (hx)z$. As id_H is a linear combination of $h \in \mathcal{H}$, we conclude $\mathrm{id}_{\mathcal{H}}(xz) = (\mathrm{id}_{\mathcal{H}}\, x)z$. Thus $\mathrm{id}_{\mathcal{H}}$ and z associate, and using the other direction of Lemma 2.1.6 this implies that $0 \star \phi = \{\phi\}$ for all $\phi \neq 1$.

We now tackle the general case of connected $\mathcal{H}$ in some $\mathcal{G}$ such that $\mathcal{H} \subseteq \mathcal{G}'$ is very regular in every $\mathcal{G}' \subseteq \mathcal{G}$ for which $\mathcal{H} \subseteq \mathcal{G}'$ is maximal. The 1-eigenspace of $\mathrm{id}_{\mathcal{H}}$ in $M_\alpha(\mathcal{G})$ is still spanned by $\mathrm{id}_{\mathcal{H}}$ and, by the same argument as that in the proof of Lemma 3.2.3, a ϕ-eigenvector can be decomposed into a sum of ϕ-eigenvectors lying in $M_\alpha(\mathcal{G}')$ for $\mathcal{H} \subseteq \mathcal{G}'$ very regular—unless $\phi = 0$, in which case the 0-eigenspace also includes $\mathcal{H}^{\not\sim}$. Therefore the fusion rules $1 \star \phi = \{\phi\}$, at least for $\phi \neq 0$, are satisfied.

Suppose that $z \in \mathcal{H}^{\not\sim}$. Then for all $x \in \mathcal{H}$, $x \not\sim z$ so $xz = 0$. Lemma 3.2.4 states that, for $\alpha \neq -\frac{2}{\phi}$, $A_1^{\mathrm{id}_{\mathcal{H}}} = \langle \mathcal{H} \rangle$, so this completes the proof that also $1 \star 0 = \emptyset$ in $M_\alpha(\mathcal{G})$.

To show that $0 \star \phi = \{\phi\}$ in $M_\alpha(\mathcal{G})$, we repeat our observation that the 0-eigenvectors of $\mathrm{id}_{\mathcal{H}}$ are 0-eigenvectors of $h \in \mathcal{H}$, which are Seress, so that by linearity $\mathrm{id}_{\mathcal{H}}$ associates with its 0-eigenspace and, using Lemma 2.1.6, therefore $0 \star \phi = \{\phi\}$ for all $\phi \neq 1$. $\square$

Finally, as a simple consequence we will need in a later section, we prove a well-kown fact which also has (simpler) geometric and group-theoretic proofs.

3.2.6 Lemma. *Suppose that $\mathcal{H} \subseteq \mathcal{G}$ satisfies the hypotheses of Theorem 3.2.5, and that $x, y \in \mathcal{G}$ are collinear. If $x, y \in \mathcal{H}^{\not\sim}$, then $x \wedge y \in \mathcal{H}^{\not\sim}$.*

Proof. Suppose that $x, y \in \mathcal{H}^{\not\sim}$. Then x, y are 0-eigenvectors for $\mathrm{id}_{\mathcal{H}}$. Since $\mathrm{id}_{\mathcal{H}}$ is Seress, xy is again a 0-eigenvector of $\mathrm{id}_{\mathcal{H}}$. As $xy = \frac{\alpha}{2}(x + y - x \wedge y)$, $x \wedge y$ must also be a 0-eigenvector. The 0-eigenvectors of $\mathrm{id}_{\mathcal{H}}$ are classified in $\mathcal{G}'$ for any $\mathcal{G}' \subseteq \mathcal{G}$ such that $\mathcal{H} \subseteq \mathcal{G}'$ is very regular, by Lemma 3.2.4, so that either $x \wedge y \in \mathcal{H}^{\not\sim}$ or $x \wedge y \in \mathcal{H}^{\sim}$ and there exists $\mathcal{H} \subseteq \mathcal{G}' \ni x \wedge y$. In this latter case, the only 0-eigenvector of $\mathrm{id}_{\mathcal{H}}$ in the span of $\mathcal{G}'$ has full support in $\mathcal{G}'$ by the Perron-Frobenius Theorem 3.1.8, so that $\mathcal{G}' = \mathcal{H} \cup \{x \wedge y\}$, contradicting that $\mathcal{G}'$ is connected. Therefore $x \wedge y \in \mathcal{H}^{\not\sim}$. $\square$

3.3 Coset axes and tori

We discuss the special idempotents which are the difference of two identity elements, coming from an embedding of subgroups. We aim to establish facts (such as the Seress property) about large associative subalgebras, called tori, of which a tractable class comes from chains of subgroups. Tori generalise Peirce decompositions of Jordan algebras. On the way, we also introduce commutants and decompositions of the identity. Again, coincidences in the field need to be avoided.

A subalgebra B of the Matsuo algebra $A = M_\alpha(\mathcal{G})$ is *parabolic* if B is spanned by some subset $\mathcal{H} \subseteq \mathcal{G}$. Throughout, $\mathcal{G}$ is a Fischer space.

Definition. Suppose that $C \subseteq B \subseteq A$ are unital parabolic subalgebras. The *coset axis* $e_{B/C}$ is $\mathrm{id}_B - \mathrm{id}_C$.

In the next few lemmas, we study when $e_{B/C}$ is primitive or Seress in A.

3.3.1 Lemma. *The coset axis $e_{B/C}$ is an idempotent.*

Proof. Since $C \subseteq B$, $\mathrm{id}_C \in A_1^{\mathrm{id}_B}$ so $\mathrm{id}_B \mathrm{id}_C = \mathrm{id}_C$ and therefore

$$(\mathrm{id}_B - \mathrm{id}_C)(\mathrm{id}_B - \mathrm{id}_C) = \mathrm{id}_B - 2\,\mathrm{id}_B \mathrm{id}_C + \mathrm{id}_C = \mathrm{id}_B - \mathrm{id}_C\,. \quad \square \tag{3.17}$$

3.3.2 Lemma. *Suppose that $C \subseteq B$ are unital subalgebras of A, and that id_B is Seress in A. Then $e_{B/C}$ is diagonalisable.*

Proof. We can rewrite the property of having a simultaneous decomposition with respect to the operators $\mathrm{ad}(\mathrm{id}_B), \mathrm{ad}(\mathrm{id}_C)$ in several ways:

$$\begin{gathered}[\mathrm{ad}(\mathrm{id}_C), \mathrm{ad}(\mathrm{id}_B)] = 0,\\ \mathrm{ad}(\mathrm{id}_C)\,\mathrm{ad}(\mathrm{id}_B) = \mathrm{ad}(\mathrm{id}_B)\,\mathrm{ad}(\mathrm{id}_C),\\ \mathrm{id}_C(\mathrm{id}_B\, x) = \mathrm{id}_B(\mathrm{id}_C\, x) \text{ for all } x \in A.\end{gathered} \tag{3.18}$$

But this follows if $C \subset B \subseteq A$ are parabolic subalgebras, since $\mathrm{id}_C \in A_1^{\mathrm{id}_B}$ and id_B is Seress. In this case, $\mathrm{ad}(e_{B/C})$ is the sum of two commuting diagonalisable operators, namely $\mathrm{ad}(\mathrm{id}_B)$ and $-\mathrm{ad}(\mathrm{id}_C)$, so $\mathrm{ad}(e_{B/C})$ is also diagonalisable. □

Let $\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{G}$ be Fischer spaces and set $e_{\mathcal{H}/\mathcal{K}} = e_{\langle\mathcal{H}\rangle/\langle\mathcal{K}\rangle}$ in $M_\alpha(\mathcal{G})$. When is $e_{\mathcal{H}/\mathcal{K}}$ primitive?

3.3.3 Lemma. *The coset axis $e_{\mathcal{G}/\mathcal{H}}$ is primitive in $M_\alpha(\mathcal{G})$ only if $\mathcal{H}$ is maximal in $\mathcal{G}$.*

Proof. We show that the 0-eigenspace of id_C inside B has dimension greater than 1 if $C \subseteq B$ is not a maximal parabolic subalgebra. This implies that the 1-eigenspace of $e_{B/C}$ has dimension greater than 1.

Suppose that $\mathcal{H} \subseteq \mathcal{G}$. If $\mathcal{H}$ is not maximal inside $\mathcal{G}$, we have some $x, y \in \mathcal{G}$ such that $\mathcal{H} \subset \langle\mathcal{H}, x\rangle \subset \langle\mathcal{H}, x, y\rangle$ and $\mathcal{H} \subset \langle\mathcal{H}, y\rangle \subset \langle\mathcal{H}, x, y\rangle$. As $\langle\mathcal{H}, x\rangle \neq \langle\mathcal{H}, y\rangle$, the 0-eigenvector z_1 of $\mathrm{id}_{\mathcal{H}}$ in $\langle\mathcal{H}, x\rangle$ is linearly independent of the 0-eigenvector z_2 of $\mathrm{id}_{\mathcal{H}}$ in $\langle\mathcal{H}, y\rangle$, since z_1 and z_2, coming from Perron-Frobenius, have full support. Now $z_1, z_2 \in \mathcal{G}$ show that the 0-eigenspace of $\mathrm{id}_{\mathcal{H}}$ is at least 2-dimensional in $\mathcal{G}^+$. □

3.3.4 Proposition. *The coset axis $e_{\mathcal{G}/\mathcal{H}}$ is primitive in $M_\alpha(\mathcal{G})$ if $\mathcal{H} \subseteq \mathcal{G}$ is very regular and, for any eigenvalues ϕ, ψ of $\mathrm{id}_{\mathcal{G}}, \mathrm{id}_{\mathcal{H}}$, $\phi - \psi = 1$ implies that $\phi = 1, \psi = 0$.*

Proof. By Lemma 3.3.2, eigenvectors of $e_{B/C}$ are simultaneous eigenvectors of id_B and id_C, and thus the eigenvalues of $e_{B/C}$ are a subset of $\mathrm{Spec}(\mathrm{id}_B) - \mathrm{Spec}(\mathrm{id}_C)$. That $e_{B/C}$ is primitive means its 1-eigenspace is 1-dimensional. Now if x is a 1-eigenvector for $e_{B/C}$, it is a ϕ, ψ-eigenvector for $\mathrm{id}_B, \mathrm{id}_C$ with $\phi - \psi = 1$. Our assumption is that this implies $\phi = 1, \psi = 0$. Therefore, if the 0-eigenspace of id_C is n-dimensional in B, then the 1-eigenspace of $e_{B/C}$ is also n-dimensional. Hence $e_{B/C}$ is primitive if and only if $n = 1$. When $C = M_\alpha(\mathcal{H}) \subseteq M_\alpha(\mathcal{G}) = B$ and $\mathcal{H} \subseteq \mathcal{G}$ is very regular, Lemma 3.2.4 shows that this is the case. □

The next, very general, lemma will help us determine when $e_{B/C}$ is Seress.

3.3.5 Lemma. *Suppose that* $e, f \in A$ *are Seress idempotents. If* $A_{1,0}^{e-f} \subseteq A_{1,0}^{e} \cap A_{1,0}^{f}$, *then* $e - f$ *is Seress.*

Proof. Suppose that $z \in A_{1,0}^{e-f}$ and $x \in A$. Then $(e-f)(zx) = e(zx) - f(zx)$ and, using that $z \in A_{1,0}^{e} \cap A_{1,0}^{f}$ and Lemma 2.1.6, $= (ex)z - (fx)z = ((e-f)x)z$, so that $e - f$ and z associate for all $z \in A_{1,0}^{e-f}$. Using the other direction of Lemma 2.1.6, this implies that $e - f$ is Seress. □

3.3.6 Lemma. *The coset axis* $e_{B/C}$ *is Seress in* A *when, for any simultaneous* ϕ, ψ*-eigenvector of* $\mathrm{id}_B, \mathrm{id}_C$ *in* A, $\phi - \psi \in \{1, 0\}$ *implies that* $\psi = 0$.

Proof. By Lemma 3.3.2, it suffices to consider simultaneous eigenvectors. By our assumptions on the eigenvalues of $\mathrm{id}_B, \mathrm{id}_C$, the conditions of Lemma 3.3.5 are satisfied, since x is a 1-eigenvector for $e_{B/C}$ if and only if $\mathrm{id}_B\, x = 1, \mathrm{id}_C\, x = 0$, and x is a 0-eigenvector for $e_{B/C}$ if and only if $\mathrm{id}_B\, x = 0 = \mathrm{id}_C\, x$. □

Because of the useful statements of Proposition 3.3.4 and Lemma 3.3.6, we say that α, or $M_\alpha(\mathcal{G})$, is *(eigenvalue) coincidence-free* for an embedding of subspaces $\mathcal{K} \subseteq \mathcal{H}$ in $\mathcal{G}$, whenever the condition holds that

$$\text{if } \phi \in \mathrm{Spec}(\mathrm{id}_{\mathcal{H}}), \psi \in \mathrm{Spec}(\mathrm{id}_{\mathcal{K}}) \text{ such that } \phi - \psi \in \{1, 0\} \text{ then } \psi = 0. \tag{3.19}$$

This somewhat technical condition on α is often realised; indeed, an 'ideal' case is

3.3.7 Lemma. *Suppose that* $\mathbb{F}$ *has characteristic* 0 *and* α *is transcendental over* $\mathbb{F}$. *Then any very regular embedding* $\mathcal{K} \subset \mathcal{H}$ *in* $\mathcal{G}$ *is coincidence-free in* $M_\alpha(\mathcal{G})$ *over* $\mathbb{F}(\alpha)$.

Proof. Suppose that $\phi \in \mathrm{Spec}(\mathrm{id}_{\mathcal{H}}), \psi \in \mathrm{Spec}(\mathrm{id}_{\mathcal{K}})$ and $\phi - \psi \in \{1, 0\}$. By Lemma 3.2.2, the eigenvalues of $\mathrm{id}_{\mathcal{H}}, \mathrm{id}_{\mathcal{K}}$ are known. The only eigenvalues in $\mathbb{F}$ are $1, 0$, so a 'coincidence of eigenvalues' occurs only when ϕ, ψ are both not equal to 1 or 0.

Therefore $\phi = \frac{\alpha\phi'}{2+\alpha k_{\mathcal{H}}}$ and $\psi = \frac{\alpha\psi'}{2+\alpha k_{\mathcal{K}}}$ for some nonzero $\phi', \psi' \in \mathbb{Q}$. Then $\phi - \psi \neq 1$ as its denominator has a constant term, whereas its numerator has degree 1 in α, so the two cannot cancel. As $\mathcal{K}$ is a strict subgraph of $\mathcal{H}$ and both are regular, $k_{\mathcal{K}}$ is strictly smaller than $k_{\mathcal{H}}$, and in particular $k_{\mathcal{K}} \neq k_{\mathcal{H}}$ since the characteristic is 0. Therefore $\frac{1}{2+\alpha k_{\mathcal{K}}}$ and $\frac{1}{2+\alpha k_{\mathcal{H}}}$ are linearly independent over $\mathbb{F}$, and so $\phi - \psi \neq 0$. □

The *commutant*[1] $\mathrm{C}_A(B)$ of a subset B of A is the subspace of all elements $x \in A$ such that $xB = \{xb \mid b \in B\} = \{0\}$.

3.3.8 Lemma. *If $C \subseteq B$ are unital subalgebras of A, then $e_{B/C} \in \mathrm{C}_B(C) \subseteq A_1^{e_{B/C}}$, and if $\mathrm{C}_B(C)$ is a subalgebra then $\mathrm{id}_{\mathrm{C}_B(C)} = e_{B/C}$.*

Proof. For $x \in C \subseteq B$, $e_{B/C}x = (\mathrm{id}_B - \mathrm{id}_C)x = x - x = 0$, so $e_{B/C} \in \mathrm{C}_B(C)$.

Suppose that $x \in \mathrm{C}_A(B)$. Then $x \in A$ so $\mathrm{id}_A\, x = x$ and, as $\mathrm{id}_B \in B$, $x\,\mathrm{id}_B = 0$. Therefore $(\mathrm{id}_A - \mathrm{id}_B)x = x - 0 = x$, so that $\mathrm{C}_A(B) \subseteq A_1^{e_{B/C}}$. As $\mathrm{id}_A - \mathrm{id}_B \in \mathrm{C}_A(B)$ and the identity is unique if $\mathrm{C}_B(C)$ is an algebra, it is equal to $\mathrm{id}_{\mathrm{C}_A(B)}$. □

When $\mathcal{H} \subseteq \mathcal{K}$ is very regular, Proposition 3.3.4 states that $e_{\mathcal{H}/\mathcal{K}}$ is primitive, so that the commutant $\mathrm{C}_{M_\alpha(\mathcal{H})}(\langle\mathcal{K}\rangle) = \langle e_{\mathcal{H}/\mathcal{K}}\rangle$ is 1-dimensional.

We digress to consider when the commutant is closed under multiplication.

3.3.9 Proposition. *If $e \in A$ is Seress, then the commutant $\mathrm{C}_A(\{e\}) = A_0^e$ is a subalgebra of A. The commutant $\mathrm{C}_A(B)$ of a subalgebra $B \subseteq A$ is a subalgebra of A if B is spanned by Seress elements.*

Proof. That $\mathrm{C}_A(\{e\}) = A_0^e$ is immediate by definition. If e is Seress, then the fusion rules in A of its eigenvectors satisfy $0 \star 0 = \{0\}$, so $A_0^e A_0^e \subseteq A_0^e$ is a subalgebra of A.

[1] or *annihilator*; the terminology *commutant* comes from the vertex algebra literature

Let $\{e_1, \ldots, e_n\}$ be a spanning set of Seress elements for B. Then it is clear that

$$\mathrm{C}_A(B) \subseteq \bigcap_i \mathrm{C}_A(\{e_i\}), \tag{3.20}$$

and moreover since the e_i span B, by linearity of the condition $xB = 0$, the above is an equality. Therefore $\mathrm{C}_A(B)$ is an intersection of subalgebras of A, so $\mathrm{C}_A(B)$ is itself a subalgebra. □

3.3.10 Proposition. *Suppose that $\mathcal{H} \subseteq \mathcal{G}$ and that the induced parabolic subalgebra $B = \langle \mathcal{H} \rangle \subseteq A = M_\alpha(\mathcal{G})$ is unital. Then $\mathrm{C}_A(B) = A_0^{\mathrm{id}_B}$, and in particular it is closed under multiplication if id_B is Seress.*

Proof. Suppose that $x \in \mathrm{C}_A(B)$; by definition, as $\mathrm{id}_B \in B$, we have $x\,\mathrm{id}_B = 0$, so $x \in A_0^{\mathrm{id}_B}$ and hence $\mathrm{C}_A(B) \subseteq A_0^{\mathrm{id}_B}$. Conversely, in the proof of Theorem 3.2.5, we showed that $x\,\mathrm{id}_B = 0$ only if $xh = 0$ for all points $h \in \mathcal{H} \subseteq B$. Therefore $A_0^{\mathrm{id}_B} \subseteq \mathrm{C}_A(B)$. □

A *decomposition of the identity* id_A in a unital algebra A is a collection $\{e_1, \ldots, e_n\}$ of idempotents in A such that $e_i e_j = \delta_{ij} e_i$ and $\mathrm{id}_A = e_1 + \cdots + e_n$. It is *maximal* if none of the e_j can be replaced by two idempotents e_{j1}, e_{j2} such that $e_{j1} + e_{j2} = e_j$ and $\{e_1, \ldots, e_{j-1}, e_{j1}, e_{j2}, e_{j+1}, \ldots, e_n\}$ is again a decomposition of the identity.

For convenience, we generalise the terminology we recently introduced. If $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ is a chain of unital subalgebras, and for any eigenvalues ϕ_i of id_{A_i} in A we have

$$\phi_{i+1} - \phi_i = \begin{cases} 1 & \text{if and only if } \phi_{i+1} = 1, \phi_i = 0, \\ 0 & \text{if and only if } \phi_{i+1} = 0 = \phi_i, \end{cases} \tag{3.21}$$

for any $0 \leq i \leq n$, then we say that the chain is *(eigenvalue) coincidence-free* in A. Lemma 3.3.7 asserts that this is the case in $M_\alpha(\mathcal{G})$, if any graphs $\mathcal{K} \subseteq \mathcal{H}$ lying in $\mathcal{G}$

are very regular whenever they are maximal, whenever α is transcendental over a subfield of the field of definition. Notice also that, in any $M_\alpha(\mathcal{G})$, there can be only finitely many values of α for which some embedding $\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{G}$ has coincidences.

When all the idempotents e_i in a decomposition $\{e_i\}_{1\le i\le n}$ of the identity are primitive, or Seress, we simply refer to the decomposition itself as primitive or Seress respectively. Observe that if $e = e_1 + e_2$ and $e_ie_j = \delta_{ij}$ for $i, j = 1, 2$, then $e_1, e_2 \in A_1^e$. Conversely, if A_1^e is 1-dimensional, then e cannot admit a decomposition into idempotents as e is the only (nonzero) idempotent in $A_1^e = \langle e\rangle$ over the field $\mathbb{F}$. Therefore if a decomposition of the identity is primitive, it must also be maximal.

3.3.11 Lemma. *If* $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ *is a coincidence-free chain of unital subalgebras then* $\{e_i = e_{A_i/A_{i-1}}\}_{1\le i\le n}$ *is a Seress decomposition of* id_A.

Proof. The e_i are idempotents by Lemma 3.3.1, diagonalisable as id_{A_i} is Seress by the coincidence-free assumption together with Lemma 3.3.2, and, by Lemma 3.3.8, $A_{i-1} \subseteq A_0^{e_i}$ so $e_ie_j = 0$ for $i \neq j$. Lemma 3.3.6 shows that all the e_i are Seress. □

To extend the previous lemma to Fischer spaces, we say that $0 = \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{G}_n = \mathcal{G}$ is a *maximal parabolic chain* of Fischer spaces if each $\mathcal{G}_i$ is maximal inside $\mathcal{G}_{i+1}$. We say that the chain is *coincidence-free* for α if the chain of subalgebras $A_0 \subseteq \cdots \subseteq A_n$, with A_i spanned by $\mathcal{G}_i$ in $M_\alpha(\mathcal{G})$, is coincidence-free. We see from Proposition 3.3.4 that in this case, for $e_i = \mathrm{id}_{\mathcal{G}_i} - \mathrm{id}_{\mathcal{G}_{i-1}}$, the e_i are primitive and Seress.

Previously we have mostly considered decompositions of A with respect to a single axis e. Observe that $\langle e\rangle$ is an associative subalgebra—in fact, a copy of the underlying field. The rôle of $\langle e\rangle$ is generalised by $\langle e_1, \ldots, e_n\rangle$ for pairwise annihilating Seress idempotents $e_1, \ldots, e_n$.

An *associative subalgebra* of A is a subalgebra $B \subseteq A$ which is an associative algebra, that is, for all $a, b, c \in B$, we have $(ab)c = a(bc)$. A *globally associative subalgebra* of A is a subalgebra $B \subseteq A$ which is associative with all of A, that is, for all $a \in A$, $b, b' \in B$, we have $b(ab') = (ba)b'$.

3.3.12 Lemma. *An associative subalgebra B of A spanned by Seress idempotents is globally associative.*

Proof. Let $e, f \in B$ be Seress idempotents. Since B is associative, $f \in A^e_{1,0}$, and in particular, since e is Seress, $e(af) = (ea)f$ for all $a \in A$. As B is spanned by such elements and associativity is linear, B itself is globally associative. □

Definition. A *torus* T in A is a maximal globally associative subalgebra of A.[2]

3.3.13 Proposition. *If $\{e_i\}_{1\leq i\leq n}$ is a decomposition of the identity in A, then $T = \sum_{i=1}^{n}\langle e_i\rangle$ is an associative subalgebra of A. If $\{e_i\}_{1\leq i\leq n}$ is primitive, then T is a maximal associative subalgebra. If $\{e_i\}_{1\leq i\leq n}$ is furthermore Seress, its span T is a torus.*

Proof. If $\{e_i\}_{1\leq i\leq n}$ is a decomposition of the identity, it is clear that $T = \langle e_i\rangle_{1\leq i\leq n}$ is an associative subalgebra of A. Suppose that $x \in A$ is associative with T, that is, $e_i(xx) = (e_ix)x$ and $e_i(e_jx) = (e_ie_j)x$ for all $1 \leq i, j \leq n$. Since the only eigenvalues of idempotents in an associative algebra over a field are 1 and 0 (Lemma 2.1.7), x decomposes in the associative subalgebra $\langle e_i, x\rangle$ as $x^i_1 + x^i_0$ for each e_i such that $e_ix^i_1 = x^i_1$ and $e_ix^i_0 = 0$, whence also $e_ix = x^i_1$. As $\mathrm{id}_A = \sum_i e_i \in T$, we have

$$x = \mathrm{id}_A\, x = \sum_i e_ix = \sum_i x^i_1. \tag{3.22}$$

[2] A torus is also called a *frame* in the context of vertex algebras, especially when spanned by conformal vectors of central charge $\frac{1}{2}$. Our terminology comes from the analogy with Cartan subalgebras in Lie algebras, which has a formal realisation for Jordan algebras via algebraic groups by [S97], for example in $M_{1/2}(\mathcal{A}_n)$, *c.f.* Theorem 2.5.5.

When the e_i are primitive, $x_1^i \in \langle e_i \rangle$ for all i and therefore $x \in \sum_i \langle e_i \rangle = T$. Thus T is a maximal associative subalgebra. That T is globally associative when $\{e_i\}_{1 \leq i \leq n}$ is Seress is Lemma 3.3.12. □

3.3.14 Lemma. *If T is a maximal associative subalgebra in a unital $\mathbb{R}$-algebra A and A is positive-definite with respect to an associating form $(,)$, then T contains a decomposition of the identity, primitive in T.*

Proof. Semisimplicity follows from the fact that T contains no nilpotent elements: if $x \in T$ is nilpotent with $x^{k+1} = 0$, then

$$(x^k, x^k) = (x^k x^k, \mathrm{id}_A) = (x^{2k}, \mathrm{id}_A) = (0, \mathrm{id}_A) = 0. \tag{3.23}$$

By positive-definiteness, $x^k = 0$, so by induction $x = 0$. Thus the Jacobson radical is 0 and T is semisimple. By Wedderburn's theorem, as T is semisimple, commutative and associative, T is isomorphic to a direct sum of finite field extensions T_j of $\mathbb{R}$.

Since every element $t \in T$ is diagonalisable over $\mathbb{R}$ but has an action on T_j, T_j must be 1-dimensional, so $T_j \cong \mathbb{R}$ and $T \cong \mathbb{R}^r$. Each of the summands T_j of T has an identity f_j, which is an idempotent, and $f_j f_k = \delta_{j,k} f_j$. Observe that $\mathrm{id}_T = \sum_{1 \leq j \leq r} f_j$.

As id_A is associative with any subalgebra of A, id_A is contained in any maximal associative subalgebra of A. Therefore $\mathrm{id}_A = \mathrm{id}_T = \sum_{1 \leq j \leq r} f_j$, f_j as before. The f_j are primitive in T since their 1-eigenspace is precisely $T_j \cong \mathbb{R}$. □

Suppose that $\{t_i\}_{1 \leq i \leq n}$ is a basis for a torus $T \subseteq A$ such that each $\mathrm{ad}(t_i)$ is diagonalisable on A. Since T is globally associative, the $\mathrm{ad}(t_i)$ pairwise commute, so that they can be simultaneously diagonalised. With respect to a basis $\{a_j\}_{1 \leq j \leq m}$ of simultaneous T-eigenvectors of A, we write

$$\psi_i = (\psi_{i1}, \psi_{i2}, \ldots, \psi_{in}) \text{ such that } t_j a_i = \psi_{ij} a_i. \tag{3.24}$$

Let $\hat{\Psi}$ be the multiset $\{\psi_i\}_{1 \leq i \leq r}$ of simultaneous eigenvalue tuples ψ_i; note that it is possible for $\psi_i = \psi_{i'}$ for some i, i'. We write Ψ for the set underlying $\hat{\Psi}$. Also write $A^T_{\psi_i}$ for the subspace of A spanned by $a_{i'}$ with $\psi_{i'} = \psi_i$.

If the t_j are idempotents with fusion rules Φ_j, then $\psi_{ij} \in \Phi_j$ for all i, and the multiplication of the basis $\{a_i\}_{1\le i\le m}$ is partially controlled by Φ_j. Namely,

$$a_i a_j \in \sum_{\psi_{\ell j} \in \psi_{ij} \star \psi_{kj}} A^T_{\psi_\ell}. \tag{3.25}$$

Therefore Ψ is a subset of $\Phi = \Phi_1 \times \Phi_2 \times \cdots \times \Phi_n$. This Φ has a product over $\mathbb{Z}$ given by $\star_1 \times \cdots \times \star_n$, that is, the pointwise or direct product of the fusion rules. It is possible that Ψ is not closed under this product. However Ψ is by construction closed under a (sub)product $\star$, defined by

$$\psi_i \star \psi_j = \{\psi_k \mid a_k \in A^T_{\psi_i} A^T_{\psi_j}\}. \tag{3.26}$$

The fusion rules of a torus are analogous to, and indeed generalise, the Peirce decomposition of a Jordan algebra; *c.f.* [McC04], Section 6.1 and Chapter 8. To give an idea of the structure of these fusion rules we give the following two results.

3.3.15 Lemma. *If $\{e_i\}_{1\le i\le n}$ is a decomposition of identity in A and $x \in A$ a simultaneous eigenvector with eigenvalues $\phi = (\phi_i)_{1\le i\le n}$ then $\sum_{i=1}^n \phi_i = 1$.*

Proof. Since $\mathrm{id}_A = \sum_{i=1}^n e_i$ it follows that

$$x = \mathrm{id}_A\, x = \sum_{i=1}^n e_i x = \sum_{i=1} \phi_i x = \left(\sum_{i=1}^n \phi_i\right) x \tag{3.27}$$

and therefore the sum of eigenvalues is 1. $\square$

3.3.16 Lemma. *Suppose that T is a torus spanned by $t_1, \ldots, t_n$, Seress idempotents in A. For any $u, v \in A$ which are simultaneous $\psi = (\psi_1, \ldots, \psi_n), \nu = (\nu_1, \ldots, \nu_n)$-eigenvectors for $t_1, \ldots, t_n$ respectively, if $\phi_i\psi_i = 0$ for all $1 \le i \le n$ then $uv = 0$.*

Proof. Suppose that u, v and ψ, ν satisfy the hypotheses. Then for any t_i we have $uv \in A^{t_i}_{\phi_i} A^{t_i}_{\psi_i}$; as $\phi_i\psi_i = 0$, one of ϕ_i or ψ_i is 0. By the Seress property of the fusion rules of t_i, $\phi_i \star \psi_i = \{\phi_i\}$ or $\{\psi_i\}$, depending on whether $\psi_i = 0$ or $\phi_i = 0$ respectively. This means $\phi_i \star \psi_i = \{\phi_i + \psi_i\}$ in all cases. Therefore $uv \in A^{t_i}_{\phi_i+\psi_i}$, and

$$uv = \mathrm{id}_A(uv) = \sum_i t_i(uv) = \sum_i (\phi_i + \psi_i)uv. \tag{3.28}$$

As $\sum_i \phi_i = 1 = \sum_i \psi_i$ by Lemma 3.3.15, we have $\sum_i (\phi_i + \psi_i) = 2$. Therefore $uv = 2uv$ so $uv = 0$ (since our underlying field is assumed to have characteristic not 2). $\square$

3.4 Central charge

The existence of a Frobenius form, which is a common feature of all well-known examples of axial algebras, allows us to prove some general statements about the algebra, related to the trace form, a Casimir element, and the radical. The form plays an important rôle in vertex algebras, so we also establish some formulae for later use.

The Matsuo algebra $A = M_\alpha(\mathcal{G})$ admits a bilinear form $\langle , \rangle_c$ with parameter $c \in \mathbb{F}$, whose definition is also due to [M03]: for $x, y \in \mathcal{G}$,

$$\langle x, y\rangle_c = \begin{cases} 2c & \text{if } x = y \\ c\alpha & \text{if } x \sim y \\ 0 & \text{otherwise .} \end{cases} \tag{3.29}$$

3.4.1 Lemma. *The bilinear form $\langle , \rangle_c$ is symmetric and associating, that is, it satisfies $\langle x, yz\rangle_c = \langle xy, z\rangle_c$ for all $x, y, z \in A$, on $A = M_\alpha(\mathcal{G})$ for $\mathcal{G}$ a Fischer space.*

Proof. By linearity, it suffices to consider points in the spanning set $\mathcal{G}$. From the definition it is clear that $\langle x, y\rangle_c = \langle y, x\rangle_c$ for all $x, y \in \mathcal{G}$. Let $x, y, z \in \mathcal{G}$ be arbitrary. Then $\langle x, y, z\rangle$ generates a subspace $\mathcal{H}$ of rank at most 3, so $\mathcal{H}$ lies inside $\mathcal{P}_2^\vee$ or $\mathcal{P}_3$ from Figure 2.1. Verifying that $\langle x, yz\rangle_c = \langle xy, z\rangle_c$ for any $x, y, z \in \mathcal{P}_2^\vee$ or $x, y, z \in \mathcal{P}_3$ is a straightforward case of calculating. We exhibit two of the base cases. If $x \wedge y = z$,

$$\langle x, yz\rangle_c = \frac{\alpha}{2}\langle x, y + z - x\rangle_c = \alpha(\alpha - 1)c = \frac{\alpha}{2}\langle x + y - z, z\rangle_c = \langle xy, z\rangle_c \tag{3.30}$$

as required. If $x \not\sim y, z$ then $x \not\sim y \wedge z$ and

$$\langle x, yz\rangle_c = \frac{\alpha}{2}\langle x, y + z - y \wedge z\rangle_c = 0 = \langle 0, z\rangle_c = \langle xy, z\rangle_c. \quad \square \tag{3.31}$$

The bilinear form $\langle , \rangle_c$ on $M_\alpha(\mathcal{G})$ exhibits an important general property:

Definition. An algebra A over a ring R is ***Frobenius*** if there exists a nonzero symmetric bilinear form $(,)$ on A which is *associating*:

$$\text{for all } x, y, z \in A, \quad (xy, z) = (x, yz). \tag{3.32}$$

We write $M^c_\alpha(\mathcal{G})$ for the algebra $M_\alpha(\mathcal{G})$ together with the bilinear form $\langle,\rangle_c$.

For x an idempotent, the *central charge* $\mathrm{cc}(x)$ of e is $\frac{1}{2}\langle x, x\rangle$. Therefore for $x \in \mathcal{G} \subset M^c_\alpha(\mathcal{G})$, $\mathrm{cc}(x) = c$. We also set, if A is a unital algebra with identity id, $\mathrm{cc}(A) = \mathrm{cc}(\mathrm{id})$.

3.4.2 Lemma. *Let R be a ring. Suppose that A is a Frobenius R-algebra with form $(,)$. Then the eigenspaces of an element $a \in A$ are $(,)$-perpendicular if the pairwise difference of their eigenvalues is invertible in R.*

Proof. Suppose that $a \in A$ has ϕ, ψ-eigenvectors x, y respectively. Then $ax = \phi x, ay = \psi y$ and $\phi(x, y) = (ax, y) = (x, ay) = \psi(x, y)$, so $(\phi - \psi)(x, y) = 0$. If $\phi - \psi$ is invertible, then $(x, y) = 0$. □

3.4.3 Lemma. *A Frobenius form $(,)$ on A is symmetric if A is generated by idempotents.*

Proof. Any $a \in A$ can be written as a linear combination of products in the idempotent generators. Let $a, b \in A$ be arbitrary, $e \in A$ an idempotent and $a' \in A$ such that $a = a'e$. Using the Frobenius property, $(a, b) = (a'e, b) = (ea', b) = (e, a'b)$. Similarly $(b, a) = (b, a'e) = (ba', e) = (a'b, e)$. Then

$$(e, a'b) = (ee, a'b) = (e, e(a'b)) = (e, (a'b)e) = (e(a'b), e) = (a'b, ee) = (a'b, e), \tag{3.33}$$

so that $(a, b) = (b, a)$ for all $a, b \in A$. □

In the literature, all known examples of axial algebras A over a field $\mathbb{F}$ generated by a set $\mathcal{A}$ of Φ-idempotents admit for any $c \in \mathbb{F}$ exactly one Frobenius form $(,)$ such that $(a, a) = 2c$ for all $a \in \mathcal{A}$. In the Matsuo case, we have

3.4.4 Proposition. *The form in* (3.29) *is the unique Frobenius form* $(,)$ *on* $M_\alpha(\mathcal{G})$ *satisfying* $(x,x) = 2c$ *for all* $x \in \mathcal{G}$*,* $\mathcal{G}$ *a Fischer space.*

Proof. To check (3.29), let $a, b \in \mathcal{G}$ be arbitrary. There are three cases: $a = b$, $a \not\sim b$ and $a \sim b$. When $a = b$ we know that $(a,a) = 2c$ as required.

Observe that if $ax = 0$ for $a \in A$ an idempotent, $x \in A$, and $(,)$ is Frobenius on A, then x is a 0-eigenvector and therefore $(a,x) = 0$ by Lemma 3.4.2.

Finally suppose $a \sim b$. Then $(a,b) = (aa,b) = (a,ab) = \frac{\alpha}{2}((a,a) + (a, b - a \wedge b))$. Then $b - a \wedge b$ is an α-eigenvector for a, so by Lemma 3.4.2, $(a, a - a \wedge b) = 0$ and $(a,b) = \frac{\alpha}{2}2c = \alpha c$ as required.

Together with Lemma 3.4.1, this shows that $(,) = \langle , \rangle_c$ is the unique Frobenius form for c. □

Note that the form of (3.29) is not in general the same as the trace form

$$\langle a, b \rangle_{\mathrm{tr}} = \mathrm{tr}(\mathrm{ad}(a)\,\mathrm{ad}(b)). \tag{3.34}$$

For example, in $M^c_\alpha(\mathcal{A}_2)$, the Gram matrix of the form of (3.29) and the trace form are

$$c \begin{pmatrix} 2 & \alpha & \alpha \\ \alpha & 2 & \alpha \\ \alpha & \alpha & 2 \end{pmatrix}, \quad \begin{pmatrix} \alpha^2+1 & \alpha & \alpha \\ \alpha & \alpha^2+1 & \alpha \\ \alpha & \alpha & \alpha^2+1 \end{pmatrix}. \tag{3.35}$$

The respective eigenvalues are $(2-\alpha)c, 2(\alpha+1)c$ and $\alpha^2 - \alpha + 1, (\alpha+1)^2$, so these matrices are not equivalent in general.

The relation between an arbitrary Frobenius form $(,)$ on an algebra A, especially when A is the Griess algebra or occurs as the weight-2 subspace of a vertex algebra, and the trace form on A has been well-studied [M01]. In particular, *Norton's trace* *formula* is said to hold in a unital algebra A if there exist constants κ_1, κ_2 such that

$$\langle a, b\rangle_{\mathrm{tr}} = k_1(a, \mathrm{id}_A)(b, \mathrm{id}_A) + k_2(a, b) \quad \text{for all } a, b \in A. \tag{3.36}$$

We show a specialisation. Following [M01], let $\{x_1, \dots, x_n\}$ be a basis of A and $\{x_1^\vee, \dots, x_n^\vee\} \subseteq A$ chosen such that $(x_i, x_j^\vee) = \delta_{ij}$, so that the $x_i^\vee$ form a *dual basis* with respect to $(,)$. The following element is an analogue of a Casimir operator:

$$K_2 = \sum_{i=1}^{n} x_i x_i^\vee. \tag{3.37}$$

The algebra A is said to be *of class* $\mathcal{S}^2$ if K_2 is a multiple of id_A.

3.4.5 Proposition. *If A is a unital algebra of dimension d and class $\mathcal{S}^2$, with Frobenius form $(,)$ and central charge c, then $K_2 = \frac{d}{2c}\,\mathrm{id}_A$ and $\mathrm{tr}\,\mathrm{ad}(a) = \frac{d}{2c}(a, \mathrm{id}_A)$ for any $a \in A$. If $\mathcal{G}$ is a connected Fischer space and $\alpha \neq -\frac{2}{k_\mathcal{G}}$, then $M_\alpha(\mathcal{G})$ is of class $\mathcal{S}^2$.*

Proof. Recall that for any dual basis, $\mathrm{tr}\,\mathrm{ad}(a) = \sum_i (a x_i^\vee, x_i) = \sum_i (a, x_i^\vee x_i) = (a, K_2)$. Observe that $\mathrm{tr}\,\mathrm{ad}(\mathrm{id}_A) = d$ and $(\mathrm{id}_A, \mathrm{id}_A) = 2c$, so that $(\mathrm{id}_A, K_2) = d$ and $K_2 \in \langle \mathrm{id}_A \rangle$ implies that $K_2 = \frac{d}{2c}\,\mathrm{id}_A$, and the first statement follows.

For the second statement, let $\{x_1, \dots, x_n\}$ be the points of $\mathcal{G}$, which form a basis of $A = M_\alpha(\mathcal{G})$. Therefore $G = \mathrm{Aut}(\mathcal{G})$ has an embedding in $\mathrm{Aut}(A)$. Since $\mathcal{G}$ is connected, G acts transitively on the points of $\mathcal{G}$. Thus fixed-point subspace F of G acting on A is 1-dimensional, namely it is spanned by $\sum_{x \in \mathcal{G}} x$. If $\alpha \neq -\frac{2}{k_\mathcal{G}}$ then Proposition 3.2.1 asserts that id_A exists and is a multiple of $\sum_{x \in \mathcal{G}} x$, so $F = \langle \mathrm{id}_A \rangle$. Finally, we observe that K_2, defined using the basis $\{x_1, \dots, x_n\}$ of points in $\mathcal{G}$, is fixed by G, as for any $g \in G$ we have

$$K_2^g = \sum_{x_i \in \mathcal{G}} (x_i x_i^\vee)^g = \sum_{x_i \in \mathcal{G}} x_i^g (x_i^\vee)^g = \sum_{x_i \in \mathcal{G}} x_i^g (x_i^g)^\vee = \sum_{x_i^{g^{-1}} \in \mathcal{G}} x_i x_i^\vee = \sum_{x_i \in \mathcal{G}} x_i x_i^\vee = K_2. \tag{3.38}$$

Thus $K_2 \in F$, so it is a multiple of id_A. $\square$

The *radical* of a bilinear form $(,)$ on A is

$$\mathrm{Rad}(A) = \{x \in A \mid (x, y) = 0 \text{ for all } y \in A\}. \tag{3.39}$$

3.4.6 Lemma. *The radical* $\mathrm{Rad}(A)$ *of a Frobenius algebra is an ideal.*

Proof. Evidently $\mathrm{Rad}(A)$ is closed under linear combinations. Suppose that $x \in \mathrm{Rad}(A)$ and $y \in A$. Then, for all $z \in A$, $(xy, z) = (x, yz) = 0$, so that $xy \in \mathrm{Rad}(A)$. □

3.4.7 Lemma. *Let A be a Frobenius $\mathbb{F}$-algebra generated by a set $\mathcal{A}$ of primitive idempotents such that any $a \in \mathcal{A}$ has finitely many eigenvalues on A and $(a, a) \neq 0$. If $I \subseteq A$ is an ideal not containing any $a \in \mathcal{A}$, then I lies in the radical of A.*

Proof. Take x in an ideal $I \subseteq A$ and $a \in A$ a primitive idempotent with n distinct eigenvalues $\Phi \subseteq A$. Write $\lambda^a(x) = \frac{(a,x)}{(a,a)}$. For all $i \geq 1$, $a^i x = a(a(\cdots(ax)\cdots))$ lies in I. We can find an expression for $x_1^a = \lambda^a(x)a \in I$ by solving the n linear equations given by $a^i x = \lambda^a(x)a + \sum_{\phi \in \Phi \setminus \{1\}} \phi^i x_\phi^a$, $1 \leq i \leq n$. Therefore if $\lambda^a(x) \neq 0$ for some $x \in I$, then $a \in I$. Thus $a \notin I$ implies $(a, x) = 0$ for all $x \in I$, so $(a, I) = 0$.

Suppose that A is generated by primitive idempotents $\mathcal{A}$ and take $y \in A, x \in I$ arbitrary; without loss of generality, y is a monomial in $\mathcal{A}$, so either $y = a \in \mathcal{A}$ or $y = y_1y_2$ for y_1, y_2 monomials in $\mathcal{A}$. If $y = y_1y_2$, then $(y, x) = (y_1y_2, x) = (y_1, y_2x)$, and $y_2x \in I$. By induction on the length of the monomial y, there exists $a \in \mathcal{A}$ such that $(y, x) = (a, x')$ for some $x' \in I$. By assumption $a \notin I$ and the previous paragraph, $(y, x) = 0$. Therefore $(A, I) = 0$. □

Recall that an algebra A is *simple* if it has no proper nontrivial ideals. The algebra A is *semisimple* if A is a direct sum of simple algebras.

3.4.8 Proposition. *If $\mathcal{G}$ is a connected linear 3-graph and $A = M_\alpha^c(\mathcal{G})_\mathbb{F}$ over a field $\mathbb{F}$ has zero radical, then A is simple.*

Proof. Suppose that I is an ideal in A. We show that if I contains a $\Phi(\alpha)$-axis a from a generating set $\mathcal{A} \subseteq \mathcal{G}$, then I contains the connected component of a in $\mathcal{G}$: for suppose that $b \in \mathcal{G}$ with $ab \neq 0$. Then $ab, a(ab) \in I$ and $\{a, ab, a(ab)\}$ span the subalgebra $\langle a, b\rangle$, which contains a third idempotent $c \in \mathcal{G}$ such that $\{a, b, c\}$ is a line in $\mathcal{G}$. Therefore, if I contains any point of $\mathcal{G}$ it contains all the lines incident to that point, and in particular I contains connected components of $\mathcal{G}$ with which it intersects. If $\mathcal{G}$ is connected, then the existence of any $a \in \mathcal{G} \cap I$ implies that $\mathcal{G} \subseteq I$ and as A is the span of $\mathcal{G}$ this means $I = A$, so no proper ideal of A can contain any point from $\mathcal{G}$. Together with Lemma 3.4.7 this means that every proper ideal of A lies in the radical of A, which is zero by assumption, so that A has no proper nontrivial ideals, hence is simple. □

The key property we used is that the subalgebra $\langle a, b\rangle$ is spanned by $a, ab, a(ab)$ when a, b are $\Phi(\alpha)$-axes. The above result holds similarly whenever $\mathcal{A}$ is any generating set of Φ-axes such that $\mathcal{A}$ cannot be partitioned into $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$ with $ab = 0$ for all $a \in \mathcal{A}'$ and $b \in \mathcal{A}''$ and, for any $a, b \in \mathcal{A}$ with $ab \neq 0$ we have that $b \in \{a, ab, a(ab), a^3b, \ldots, a^nb\}$ for $n = |\Phi|$.

3.4.9 Lemma. *The radical of $M_\alpha^c(\mathcal{G})$, for $c \neq 0$, is the $-\frac{2}{\alpha}$-eigenspace of $\mathrm{ad}(\mathcal{G})$. If $c = 0$ then $\mathrm{Rad}(M_\alpha^c(\mathcal{G})) = A$.*

Proof. The Gram matrix of $\langle , \rangle_c$, for a given ordering $\{x_1, \ldots, x_n\}$ of $\mathcal{G}$, is $(g_{ij})_{1\leq i,j\leq n}$ with $g_{ii} = 2c$ and $g_{ij} = 0$ if $x_i \not\sim x_j$ and $= \alpha c$ if $x_i \sim x_j$. Its determinant is $c^n p(\alpha)$, for $p(\alpha)$ a polynomial in $\mathbb{Z}[\alpha]$. If $c = 0$, then $\langle , \rangle_0$ is the 0 form. Otherwise, the zeroes of $\langle , \rangle_c$ are the same as those of $\langle , \rangle_1$, whose Gram matrix is $2I_n + \alpha\,\mathrm{ad}(\mathcal{G})$. Now x is in the radical if and only if $(2I_n + \alpha\,\mathrm{ad}(\mathcal{G}))x = 0$ if and only if x is an $\mathrm{ad}(\mathcal{G})$-eigenvector, with eigenvalue ϕ say, and $2 + \alpha\phi = 0$, that is, $\phi = -\frac{\alpha}{2}$. □

Finally we give some useful formulae.

3.4.10 Lemma. *If* $B \subseteq A$ *is a subalgebra, then* $\mathrm{cc}(\mathrm{C}_A(B)) = \mathrm{cc}(A) - \mathrm{cc}(B)$.

Proof. By Lemma 3.3.8, $\mathrm{id}_{\mathrm{C}_A(B)} = \mathrm{id}_A - \mathrm{id}_B$, and so we compute

$$\langle \mathrm{id}_A - \mathrm{id}_B, \mathrm{id}_A - \mathrm{id}_B \rangle = \langle \mathrm{id}_A, \mathrm{id}_A \rangle + \langle \mathrm{id}_B, \mathrm{id}_B \rangle - 2\langle \mathrm{id}_A, \mathrm{id}_B \rangle. \tag{3.40}$$

By the Frobenius property,

$$\langle \mathrm{id}_A, \mathrm{id}_B \rangle = \langle \mathrm{id}_A, \mathrm{id}_B\, \mathrm{id}_B \rangle = \langle \mathrm{id}_A\, \mathrm{id}_B, \mathrm{id}_B \rangle = \langle \mathrm{id}_B, \mathrm{id}_B \rangle \tag{3.41}$$

and therefore

$$\langle \mathrm{id}_A - \mathrm{id}_B, \mathrm{id}_A - \mathrm{id}_B \rangle = \langle \mathrm{id}_A, \mathrm{id}_A \rangle - \langle \mathrm{id}_B, \mathrm{id}_B \rangle, \tag{3.42}$$

that is, $\mathrm{cc}(\mathrm{C}_A(B)) = \mathrm{cc}(A) - \mathrm{cc}(B)$. □

3.4.11 Corollary (Proposition 3.2.1)**.** *We have*

$$\mathrm{cc}(\mathrm{id}_{\mathcal{H}^\pm}) = \frac{2c|\mathcal{H}|}{1 + \alpha k_{\mathcal{H}}}, \quad \mathrm{cc}(\mathrm{id}_{\mathcal{H}}) = \frac{2c|\mathcal{H}|}{2 + \alpha k_{\mathcal{H}}}. \quad \square \tag{3.43}$$

3.4.12 Corollary (Lemmas 3.4.10, Corollary 3.4.11)**.** *We have*

$$\mathrm{cc}(e_{\mathcal{G}/\mathcal{H}}) = \frac{2c(|\mathcal{G}| - |\mathcal{H}| + \alpha(k_{\mathcal{H}}|\mathcal{G}| - k_{\mathcal{G}}|\mathcal{H}|))}{(1 + \alpha k_{\mathcal{H}})(1 + \alpha k_{\mathcal{G}})}, \quad \mathrm{cc}(e_{\mathcal{G}^\pm/\mathcal{G}}) = \frac{2c|\mathcal{G}|}{(1 + \alpha k_{\mathcal{G}})(2 + \alpha k_{\mathcal{G}})}. \quad \square \tag{3.44}$$

3.5 The cases $\mathrm{A}_n, \mathrm{D}_n$

In this section, after some combinatorial results, we capitalise on all our previous work. In particular, we use the previous formulae and for example the Seress property to describe coset axes, and therefore a torus, in the Matsuo algebras for A_n and D_n. For A_n, this leads to an observation in vertex algebras; for D_n, we also find some new automorphisms.

The four following results are combinatorial preliminaries.

3.5.1 Lemma. *The boundary graph $\mathcal{A}_n/\mathcal{A}_{n-1}$ is K_n, the complete graph on n points.*

Proof. Recall that the Miyamoto involutions of points $x \in \mathcal{A}_n$ generate the symmetric group $\mathrm{Sym}(n+1)$ on $n+1$ letters. Taking the embedding $H = \mathrm{Sym}(n) \subseteq \mathrm{Sym}(n+1) = G$ that corresponds to $\mathcal{A}_{n-1} \subseteq \mathcal{A}_n$ gives that H has support $\{1, \dots, n\}$ and G has support $\{1, \dots, n+1\}$ in the standard permutation realisation of G. Then if $s, t \in G \setminus H$ are transpositions, they each move two letters in $\{1, \dots, n+1\}$. If s moves two letters in $\{1, \dots, n\}$ then $s \in H$, so s moves $n+1$; the same goes for t. We can therefore write $s = (i, n+1)$ and $t = (j, n+1)$ for $1 \leq i, j \leq n$. Then $s^t = (i, j)$ lies in H. This shows that the points $x, y \in \mathcal{A}_n$ corresponding to s, t satisfy $x \wedge y \in \mathcal{A}_{n-1}$. As s, t were arbitrary, any two points in $\mathcal{A}_n/\mathcal{A}_{n-1}$ are connected. □

3.5.2 Lemma. *The double graph of $\mathcal{A}_n$ is $\mathcal{D}_{n+1}$.*

Proof. Suppose that $\{x_1, \dots, x_{\frac{1}{2}n(n+1)}\}$ are the points in $\mathcal{A}_n$, inducing transpositions $\{t_1, \dots, t_{\frac{1}{2}n(n+1)}\}$ in $G(\mathcal{A}_n)$. Then there are transpositions $s_1, \dots, s_n$ among them satisfying the Coxeter presentation for $G(\mathcal{A}_n)$ in Figure 3.1.

Let $x_1^+, \dots, x_m^+, x_1^-, \dots, x_m^-$ be the points of $\mathcal{A}_{n+1}^{\pm}$ and t_i^ε the transposition $\tau(x_i^\varepsilon)$ of x_i^ε in the permutation representation. Then it follows that $S = \{t_1^-, t_1^+, t_2^+, t_3^+, \dots, t_r^+\}$,

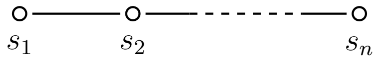

Figure 3.1: Coxeter presentation for $\mathcal{A}_n$

transpositions induced from the points of $\mathcal{A}_n^{\pm}$, satisfies the Coxeter presentation for $G(\mathcal{D}_{n+1})$ in Figure 3.2.

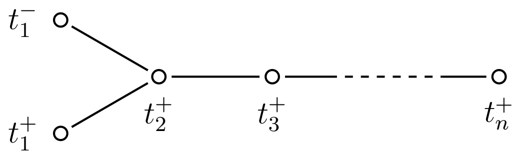

Figure 3.2: Coxeter presentation for $\mathcal{D}_{n+1}$

Moreover, S generates $G = G(\mathcal{A}_n^{\pm})$, so G is a quotient of $G(\mathcal{D}_{n+1})$. In fact a counting argument shows that $G = G(\mathcal{D}_{n+1})$, since $G(\mathcal{D}_{n+1})$ has $n(n+1)$ transpositions and $G(\mathcal{A}_n)$ has the same number, namely $2 \cdot \frac{1}{2}n(n+1)$. The corresponding points $x_1^-, x_1^+, x_2^+, \ldots, x_r^+$ generate $\mathcal{A}_n^{\pm}$, therefore $\mathcal{D}_{n+1} \cong \mathcal{A}_n^{\pm}$. $\square$

3.5.3 Lemma. *The double graph $(\mathcal{G}/\mathcal{H})^{\pm}$ of $\mathcal{G}/\mathcal{H}$, for $\mathcal{G}, \mathcal{H}$ linear 3-graphs, is $\mathcal{G}^{\pm}/\mathcal{H}^{\pm}$.*

Proof. The naive bijection works out: take $x^{\varepsilon} \in (\mathcal{G}/\mathcal{H})^{\pm}$. Then $x \in \mathcal{G}/\mathcal{H}$ and is uniquely identified with a point x' in $\mathcal{G} \setminus \mathcal{H}$, for which there exists $y' \in \mathcal{G} \setminus \mathcal{H}$ with $x' \wedge y' \in \mathcal{H}$. Now $x'^{\varepsilon}, y'^{\varepsilon} \in \mathcal{G}^{\pm} \setminus \mathcal{H}^{\pm}$ and $x'^{\varepsilon} \wedge y'^{\varepsilon} \in \mathcal{H}^{\varepsilon\varepsilon} \subseteq \mathcal{H}^{\pm}$, so $x'^{\varepsilon} \in \mathcal{G}^{\pm}/\mathcal{H}^{\pm}$. Therefore $(\mathcal{G}/\mathcal{H})^{\pm}$ has the same cardinality as $\mathcal{G}^{\pm}/\mathcal{H}^{\pm}$. Indeed identifying $y' \in \mathcal{G} \setminus \mathcal{H}$ in the above argument with $y \in \mathcal{G}/\mathcal{H}$ shows that this bijection also preserves lines $x \sim y$, so that we have an isomorphism of graphs. $\square$

3.5.4 Lemma. *If $\mathcal{G}$ is a nontrivial linear 3-graph containing no isolated points, then $\mathcal{G}^{\pm}/\mathcal{G}^{+}$ is the same as the 2-graph underlying $\mathcal{G}$.*

Proof. Let $x^-, y^- \in \mathcal{G}^{\pm} \setminus \mathcal{G}^{+}$ be arbitrary. Then $x^- \sim y^-$ if and only if $x \sim y$ by definition, and if so, then $x^- \wedge y^- = (x \wedge y)^{--} = (x \wedge y)^{+} \in \mathcal{G}^{+}$. Furthermore since $\mathcal{G}$

contains no isolated points, every $x^- \in \mathcal{G}^-$ is connected to at least one other point $y^- \in \mathcal{G}^-$. Therefore the point set of $\mathcal{X} = \mathcal{G}^\pm/\mathcal{G}^+$ is $\mathcal{G}^-$, and $\mathcal{X}$ has lines $\{x^-, y^-\}$ exactly when $\{x, y, x \wedge y\}$ is a line in $\mathcal{G}$. $\square$

We now specialise to specific graphs. Recall the notation $k_\mathcal{G}, k_\mathcal{G}^\mathcal{H}$ from Section 3.1.

3.5.5 Lemma. *We record*

$$\begin{gathered}\mathrm{Spec}(\mathrm{ad}(K_n)) = \{(n-1)^1, -1^{n-1}\}, \quad k_{\mathcal{A}_{n+1}}^{\mathcal{A}_{n\geq 1}} = n-1,\\ \mathrm{Spec}(\mathrm{ad}(\mathcal{A}_1)) = \{0^1\}, \quad \mathrm{Spec}(\mathrm{ad}(\mathcal{A}_2)) = \{2^1, -1^2\},\\ \mathrm{Spec}(\mathrm{ad}(\mathcal{A}_{n\geq 3})) = \{(2n-2)^1, (n-3)^n, -2^{(n+1)(n-2)/2}\},\\ \mathrm{Spec}(\mathrm{ad}(\mathcal{D}_{n\geq 4})) = \{(4n-8)^1, (2n-8)^{n-1}, -4^{n(n-3)/2}, 0^{(n-1)n/2}\}.\end{gathered} \tag{3.45}$$

Proof. These facts are folklore; we refer to [HS15] for details. For $\mathcal{D}_n$, we can also deduce the values using Lemma 3.5.2 from those of $\mathcal{A}_n$. $\square$

Let $\mathcal{G}_n$ be a family of Fischer spaces. Set $k_n = k_{\mathcal{G}_n}$ and $k_n^m = k_{\mathcal{G}_n}^{\mathcal{G}_m}$. Also write $\mathrm{id}_n = \mathrm{id}_{\mathcal{G}_n}$, $\hat{\mathrm{id}}_n = \mathrm{id}_{\mathcal{G}_n^\pm}$, $e_i = \mathrm{id}_i - \mathrm{id}_{i-1}$ and $\hat{e}_i = \hat{\mathrm{id}}_i - \mathrm{id}_i$.

3.5.6 Proposition. *In $A = M_\alpha^c(\mathcal{A}_n^\pm)$ over $\mathbb{F}(\alpha)$, α transcendental, for $4 \leq i < n$,*

$$\begin{aligned}\mathrm{Spec}(e_i) = \{1, 0, \eta_\alpha(i), &1 - \eta_\alpha(i-1), \eta_\alpha(i) - \eta_\alpha(i-1),\\ &\hat{\eta}_\alpha(i) - \eta_\alpha(i-1), \hat{\eta}_\alpha(i) - \hat{\eta}_\alpha(i-1)\},\end{aligned} \tag{3.46}$$

$$\mathrm{Spec}(\hat{e}_i) = \{1, 0, 1 - \eta_\alpha(i-1), 1 - \hat{\eta}_\alpha(i-1)\}. \tag{3.47}$$

$$\textit{for } \eta_\alpha(i) = \frac{\alpha(i+1)}{2 + 2\alpha(i-1)}, \quad \hat{\eta}_\alpha(i) = \frac{\alpha i}{1 + \alpha(i-1)}. \tag{3.48}$$

$$\mathrm{cc}_\alpha^c(e_i) = \frac{c}{2}\frac{i(2 + \alpha(i-3))}{(1 + \alpha(i-1))(1 + \alpha(i-2))}, \tag{3.49}$$

$$\mathrm{cc}_\alpha^c(\hat{e}_i) = \frac{c}{2}\frac{i(i+1)}{(1 + 2\alpha(i+1))(1 + \alpha(i+1))}. \tag{3.50}$$

Proof. It follows from Lemma 3.2.2, and substitutions from Lemma 3.5.5, that the eigenvalues of $\mathrm{id}_{\mathcal{A}_i}$ in A are

$$\begin{gathered}\mathrm{Spec}(\mathrm{id}_{\mathcal{A}_0}) = \{0\}, \quad \mathrm{Spec}(\mathrm{id}_{\mathcal{A}_{i=1,2}}) = \{1, 0, \eta_\alpha(i)\}, \\ \mathrm{Spec}(\mathrm{id}_{\mathcal{A}_{i\geq 3}}) = \{1, 0, \eta_\alpha(i), \hat{\eta}_\alpha(i)\}.\end{gathered} \tag{3.51}$$

By observations on the inclusions of eigenspaces and the fact that, for commuting matrices x, y, $\mathrm{Spec}(x - y) = \mathrm{Spec}(x) - \mathrm{Spec}(y)$, we deduce the spectrum of e_i and $\hat{e}_i$. Namely, denote $A^{\mathrm{id}_{\mathcal{A}_i}}_{\phi_\alpha(i)}$ by A^i_ϕ; then $A^{i-1}_1 \subseteq A^i_1$ is clear, $A^i_0 \subseteq A^{i-1}_0$ implies that an eigenvalue $0 - \phi$ is only realised for $\phi = 0$, and $A^i_{\hat{\eta}} \subseteq A^{i-1}_{\eta,\hat{\eta}}$. We deduce the central charges from Corollary 3.4.12 with specialisations from Lemma 3.5.5. □

In view of Theorem 1.2.1, where we define $c_i, h^i_{r,s}$, and Theorem 1.2.2, we can now determine the highest weights of the Virasoro algebra at central charge c_i inside the weight-2 subalgebra of the vertex algebra $V^\theta_{\sqrt{2}\mathrm{A}_n}$, and therefore, by Corollary 3.3 of [Y01], inside $V_{\sqrt{2}\mathrm{A}_n}$:

3.5.7 Proposition. *The specialisation for $\alpha = \frac{1}{4}$, $c = \frac{1}{2}$ of Proposition 3.5.6 is*

$$\mathrm{cc}^{1/2}_{1/4}(e_i) = 1 - \frac{6}{(i+2)(i+3)} = c_i, \quad \mathrm{cc}^{1/2}_{1/4}(\hat{e}_i) = \frac{2i}{i+3}. \tag{3.52}$$

$$0 = h^i_{1,1}, \tag{3.53}$$

$$\eta_{1/4}(i) = \frac{1}{2}\frac{i+1}{i+3} = h^i_{3,1}, \tag{3.54}$$

$$1 - \eta_{1/4}(i-1) = \frac{1}{2}\frac{i+4}{i+2} = h^i_{1,3}, \tag{3.55}$$

$$\eta_{1/4}(i) - \eta_{1/4}(i-1) = \frac{1}{(i+2)(i+3)} = h^i_{3,3}, \tag{3.56}$$

$$\hat{\eta}_{1/4}(i) - \eta_{1/4}(i-1) = \frac{1}{2}\frac{i(i-1)}{(i+2)(i+3)} = h^i_{5,3}, \tag{3.57}$$

$$\hat{\eta}_{1/4}(i) - \hat{\eta}_{1/4}(i-1) = \frac{3}{(i+2)(i+3)} = h^i_{5,5}. \tag{3.58}$$

Proof. We calculate these directly; in particular,

$$\eta_{1/4}(i) = \frac{1}{2}\frac{i+1}{i+3}, \quad \hat{\eta}_{1/4}(i) = \frac{i}{i+3}. \quad \square \tag{3.59}$$

By allowing values of α other than $\frac{1}{4}$, we generalise the results of [Y01]. In particular, in light of Theorem 1.2.2 and the fact that the highest weights of $\mathrm{Vir}(^1/_2)$ are $\frac{1}{4}$, and $\frac{1}{32}$, the following proposition is analogous to Proposition 3.5.7:

3.5.8 Proposition. *e_2 in $M_{1/32}(\mathcal{A}_n^{\pm})$ has central charge $\frac{21}{22} = c_9$ and eigenvalues*

$$\eta_{1/32}(2) = \frac{1}{22} = h^9_{5,1}, \quad 1 - \eta_{1/32}(1) = \frac{31}{32} = h^9_{4,1}, \quad \eta_{1/32}(2) - \eta_{1/32}(1) = \frac{5}{352} = h^9_{4,4}. \tag{3.60}$$

Proof. These are specialisations of

$$\mathrm{cc}^{1/2}_{1/32}(e_i) = \frac{2i(61+i)}{(31+i)(30+i)}, \quad \eta_{1/32}(i) = \frac{1}{2}\frac{i+1}{i+31}, \quad \hat{\eta}_{1/32}(i) = \frac{i}{i+31}. \quad \square \tag{3.61}$$

To compute the fusion rules of the idempotents e_i, we need new tools:

3.5.9 Lemma. *Suppose that A is an algebra over a ring R and $f\colon R \to S$ is a ring surjection with kernel K inducing a surjection of algebras $f\colon A \to A/(KA)$. If $e \in A$ is a Φ-axis then $f(e) \in \operatorname{im} f$ has fusion rules $f(\Phi)$.*

Proof. If $e \in A$ is a Φ-axis then A is spanned (over R) by e-eigenvectors $x_1, \dots, x_n$ with eigenvalues $\alpha_1, \dots, \alpha_n$. Also $B = \operatorname{im} f$ is spanned over S by $f(x_1), \dots, f(x_n)$. Now $f(e)f(x_i) = f(ex_i) = f(\alpha_i x_i) = f(\alpha_i)f(x_i)$ for $\alpha_i \in \Phi$, so $f(x_i)$ is a $f(\alpha_i)$-eigenvector of $f(e)$. Thus $f(e)$ only has eigenvalues in $f(\Phi)$.

Likewise, for any $1 \le i, j \le n$ there exist ϕ_k with $1 \le k \le n$ such that $x_i x_j = \sum_k \phi_k x_k$. As e is a Φ-axis, ϕ_k is nonzero only if $\alpha_k \in \alpha_i \star \alpha_j$. Therefore $(f(x_i)f(x_j)) = \sum_k f(\phi_k)f(x_k)$ and, as f is a ring homomorphism, $f(\phi_k)$ is nonzero only if ϕ_k is nonzero, so in particular $f(\phi_k)$ is nonzero only if $\alpha_k \in \alpha_i \star \alpha_j$. The fusion rules on

$f(\Phi)$ are just pointwise evaluations of the fusion rules on Φ, and $f(\alpha_k)$ is nonzero only if $f(\alpha_k) \in f(\alpha_i) \star f(\alpha_j)$, so eigenvectors of $f(e)$ satisfy the fusion rules $f(\Phi)$. □

3.5.10 Corollary. *In the situation of Lemma 3.5.9, if f is injective when restricted to Φ, then the fusion rules on $f(\Phi)$ determine the fusion rules on Φ.* □

Because of Proposition 3.5.7, Corollary 3.5.10 and the statement of the next lemma, we name the functions for the eigenvalues of Proposition 3.5.6 as

$$\hbar^i_{1,1}(\alpha) = 0, \quad \hbar^i_{3,1}(\alpha) = \eta_\alpha(i), \quad \hbar^i_{1,3}(\alpha) = 1 - \eta_\alpha(i-1), \tag{3.62}$$

$$\hbar^i_{3,3}(\alpha) = \eta_\alpha(i) - \eta_\alpha(i-1), \quad \hbar^i_{5,3}(\alpha) = \hat{\eta}_\alpha(i) - \eta_\alpha(i-1), \quad \hbar^i_{5,5}(\alpha) = \hat{\eta}_\alpha(i) - \hat{\eta}_\alpha(i-1).$$

3.5.11 Lemma. *The fusion rules on $\Phi_i = \{1, 0, \hbar^i_{3,1}, \hbar^i_{3,3}, \hbar^i_{5,3}, \hbar^i_{5,5}\}$ for e_i in Proposition 3.5.6, in $M_\alpha(\mathcal{A}^\pm_n)$ over $\mathbb{R}(\alpha)$ with α transcendental, are in Table 3.3.*

$\star$	1	0	$\hbar^i_{3,1}$	$\hbar^i_{1,3}$	$\hbar^i_{3,3}$	$\hbar^i_{5,3}$	$\hbar^i_{5,5}$
1	$\{1\}$	$\emptyset$	$\{\hbar^i_{3,1}\}$	$\{\hbar^i_{1,3}\}$	$\{\hbar^i_{3,3}\}$	$\{\hbar^i_{5,3}\}$	$\{\hbar^i_{5,5}\}$
0		$\{0\}$	$\{\hbar^i_{3,1}\}$	$\{\hbar^i_{1,3}\}$	$\{\hbar^i_{3,3}\}$	$\{\hbar^i_{5,3}\}$	$\{\hbar^i_{5,5}\}$
$\hbar^i_{3,1}$			$\{1, 0, \hbar^i_{3,1}\}$	$\{\hbar^i_{3,3}\}$	$\{\hbar^i_{1,3}, \hbar^i_{3,3}, \hbar^i_{5,3}\}$	$\{\hbar^i_{3,3}, \hbar^i_{5,3}\}$	$\{\hbar^i_{5,5}\}$
$\hbar^i_{1,3}$				$\{1, 0, \hbar^i_{1,3}\}$	$\{\hbar^i_{3,1}, \hbar^i_{3,3}\}$	$\{\hbar^i_{5,3}, \hbar^i_{5,5}\}$	$\{\hbar^i_{5,3}, \hbar^i_{5,5}\}$
$\hbar^i_{3,3}$					Φ_i	$\{\hbar^i_{3,1}, \hbar^i_{3,3}, \hbar^i_{5,3}, \hbar^i_{5,5}\}$	$\{\hbar^i_{3,3}, \hbar^i_{5,3}, \hbar^i_{5,5}\}$
$\hbar^i_{5,3}$						Φ_i	$\{\hbar^i_{1,3}, \hbar^i_{3,3}, \hbar^i_{5,3}, \hbar^i_{5,5}\}$
$\hbar^i_{5,5}$							Φ_i

Table 3.3: Fusion rules for e_i in $M_\alpha(\mathcal{A}^\pm_n)$

Proof. By Theorem 1.2.2 we know that $M_{1/4}(\mathcal{A}^\pm_n)_\mathbb{R}$ is realised as the weight-2 subalgebra of a vertex algebra, so by Theorem 1.2.1 any idempotent e is a conformal vector, the fusion rules Φ' of which are recorded in [W93], Theorem 4.3, when $\mathrm{cc}(e) < 1$. By Proposition 3.5.7, $\mathrm{cc}(e_i) = \mathrm{cc}^{1/2}_{1/4}(e_i) = c_i$, which is indeed less than 1 for all i. Let

$R = \mathbb{R}[\alpha, (i\alpha+1)^{-1} \mid 1 \le i < n] \subseteq \mathbb{R}(\alpha)$ and $M_\alpha(\mathcal{A}_n^\pm)_R$ be the Matsuo algebra $M_\alpha(\mathcal{A}_n^\pm)$ over R. Then $M_{1/4}(\mathcal{A}_n^\pm)_\mathbb{R}$ is the image of $M_\alpha(\mathcal{A}_n^\pm)_R$ under the map $f_{1/4}\colon R \to \mathbb{R}, \alpha \mapsto \frac{1}{4}$ which is a injection on $\mathrm{Spec}(e_i)$ for each i; this follows by comparing the polynomials in Proposition 3.5.7. By Proposition 3.2.1, as $k_{\mathcal{A}_n} = 2n-2$ and $(\alpha k_{\mathcal{A}_n} - 2)$ is invertible in R, $M_\alpha(\mathcal{A}_n^\pm)_R$ is unital. Thus $e_i \in M_\alpha(\mathcal{A}_n^\pm)_R$ are well-defined. Using Corollary 3.5.10, we can deduce the fusion rules on $\mathrm{Spec}(e_i)$ over R from the fusion rules given by [W93], Theorem 4.3, for e_i using the inverse map of $f_{1/4}$ restricted to $f_{1/4}(\mathrm{Spec}(e_i))$. □

We now present similar results about eigenvalues for a parabolic chain of subalgebras of $A = M_\alpha(\mathcal{D}_n^\pm)$ coming from $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \cdots \subseteq \mathcal{D}_n \subseteq \mathcal{D}_n^\pm$.

3.5.12 Proposition. *In $M_\alpha^c(\mathcal{D}_n^\pm)$ over $\mathbb{F}(\alpha)$ with α transcendental, for $i \ge 3$,*

$$\begin{aligned}\mathrm{Spec}(e_i) = \{1, 0, &\theta_\alpha(i), 1-\theta_\alpha(i-1), \theta_\alpha(i)-\theta_\alpha(i-1),\\ &\theta'_\alpha(i), 1-\theta'_\alpha(i-1), \theta'_\alpha(i)-\theta'_\alpha(i-1),\\ &\hat\theta_\alpha(i)-\hat\theta_\alpha(i-1), \hat\theta_\alpha(i)-\theta_\alpha(i-1),\\ &\hat\theta'_\alpha(i)-\hat\theta'_\alpha(i-1), \hat\theta'_\alpha(i)-\theta'_\alpha(i-1), \},\end{aligned} \tag{3.63}$$

$$\mathrm{Spec}(\hat e_i) = \{1, 0, 1-\theta_\alpha(i), 1-\hat\theta_\alpha(i), 1-\theta'_\alpha(i), 1-\hat\theta'_\alpha(i)\} \tag{3.64}$$

$$\begin{aligned}\theta_\alpha(i) &= \frac{\alpha i}{1+2\alpha(i-2)}, \quad \theta'_\alpha(i) = \frac{\alpha(i-1)}{1+2\alpha(i-2)},\\ \hat\theta_\alpha(i) &= \frac{2\alpha(i-1)}{1+2\alpha(i-2)}, \quad \hat\theta'_\alpha(i) = \frac{2\alpha(i-2)}{1+2\alpha(i-2)}.\end{aligned} \tag{3.65}$$

$$\mathrm{cc}_\alpha(e_i) = 4c\frac{(i-1)(2\alpha i - 8\alpha + 1)}{(4\alpha i - 12\alpha + 1)(4\alpha i - 8\alpha + 1)}, \tag{3.66}$$

$$\mathrm{cc}_\alpha(\hat e_i) = c\frac{i(i-1)}{(4\alpha i - 8 + 1)(2\alpha i - 4\alpha + 1)}. \tag{3.67}$$

Proof. It follows from from Lemma 3.2.2, and substitutions from Lemma 3.5.5, that the eigenvalues of $\mathrm{id}_{\mathcal{D}_i}$ in A are

$$\mathrm{Spec}(\mathrm{id}_{\mathcal{D}_i}) = \{1, 0, \theta_\alpha(i), \theta'_\alpha(i), \hat\theta_\alpha(i), \hat\theta'_\alpha(i)\} \tag{3.68}$$

By inclusions of eigenspaces as in the proof of Proposition 3.5.6 together with Lemma 3.5.2, and the fact that, for commuting matrices x, y, $\mathrm{Spec}(x-y) = \mathrm{Spec}(x) - \mathrm{Spec}(y)$, we deduce the spectrum of e_i and $\hat{e}_i$. We deduce the central charges from Corollary 3.4.12 with substitutions from Lemma 3.5.5. $\square$

In the final results of this section, we consider idempotents and involutions of $M_\alpha(\mathcal{D}_n)$. Recall that $\mathcal{D}_{n+1} = \mathcal{A}_n^\pm$ by Lemma 3.5.2, so this continues the study of $M_\alpha(\mathcal{A}_n^\pm)$. In fact, we previously looked at coset axes of the chain

$$\mathrm{A}_0^+ \subseteq \mathrm{A}_1^+ \subseteq \cdots \subseteq \mathrm{A}_n^+ \text{ in } A_n^\pm; \tag{3.69}$$

now we focus on $\mathrm{id}_{\mathrm{A}_i^\pm}$ in the same algebra $M_\alpha(\mathcal{A}_n^\pm)$.

3.5.13 Proposition. *The fusion rules of* $\mathrm{id}_{\mathcal{D}_i}$ *in* $M_\alpha(\mathcal{D}_m)$*,* $3 \leq i < m$*, are* $\mathbb{Z}/2$*-graded.*

Proof. The eigenvalues of $x = \mathrm{id}_{\mathcal{D}_n}$ are $\{1, 0, \theta_\alpha(i), \theta'_\alpha(i)\}$. We will show that $\Phi_+ \cup \Phi_0 - = \{1, 0, \theta_\alpha(i)\} \cup \{\theta'_\alpha(i)\}$ is a $\mathbb{Z}/2$-graded partition of the fusion rules. We first observe that the $\theta'_\alpha(i)$-eigenvectors are of the form $x^+ - x^-$ for $x \in \mathcal{A}_i^\sim \subseteq \mathcal{A}_m$ using the identification $\mathcal{D}_m = \mathcal{A}_m^\pm$ from Lemma 3.5.2. We can verify by direct computation that $\mathrm{id}_{\mathcal{D}_i}(x^+ - x^-) = \theta'_\alpha(i)(x^+ - x^-)$. Furthermore note that the quotient graph of $\mathcal{D}_m$ by $\{x^+ - x^- \mid x \in \mathcal{A}_{m-1}\}$ is exactly $\mathcal{A}_{m-1}^\pm / \mathcal{A}_{m-1} \cong \mathcal{A}_{m-1}$ (see Lemma 3.5.4), and the image of $\mathrm{id}_{\mathcal{D}_i}$ under this map is a scalar multiple of $\mathrm{id}_{\mathcal{A}_{i-1}}$. Every vector which is annihilated in the quotient is a $\theta'_\alpha(i)$-eigenvector, so in particular no $\theta_\alpha(i)$-eigenvector is mapped to 0. As $\mathrm{id}_{\mathcal{A}_{i-1}}$ has only 3 distinct eigenvalues in $M_\alpha(\mathcal{A}_{m-1})$ by Lemma 3.2.2, and the image of $1, 0$-eigenvectors are again $1, 0$-eigenvectors, it follows that the $\theta_\alpha(i)$-eigenspace of $\mathrm{id}_{\mathcal{D}_i}$ is mapped to the $\eta_\alpha(i-1)$-eigenspace of $\mathrm{id}_{\mathcal{A}_{i-1}}$ and the $\theta'_\alpha(i)$-eigenspace is completely annihilated, so that all $\theta'_\alpha(i)$-eigenvectors lie in the span of $\{x^+ - x^- \mid x \in \mathcal{A}_i^\sim\}$.

Let $t = \tau(\mathrm{id}_{\mathcal{D}_i})$ be the map

$$x \mapsto \begin{cases} x^{\varepsilon} & \text{if } x \in \mathcal{A}_i \cup \mathcal{A}_i^{\not\sim}, \\ x^{-\varepsilon} & \text{if } x \in \mathcal{A}_i^{\sim}. \end{cases} \tag{3.70}$$

Observe that t inverts the $\theta'_\alpha(i)$-eigenspace of $\mathrm{id}_{\mathcal{D}_i}$ and fixes the other eigenspaces. By showing that t is an automorphism of $A = M_\alpha(\mathcal{G})$, together with Theorem 3.2.5 which states that the fusion rules are Seress, we show that the fusion rules of $\mathrm{id}_{\mathcal{D}_i}$ are a subset of Table 3.4, which is $\mathbb{Z}/2$-graded.

$\star$	1	0	$\theta_\alpha(i)$	$\theta'_\alpha(i)$
1	$\{1\}$	$\emptyset$	$\{\theta_\alpha(i)\}$	$\{\theta'_\alpha(i)\}$
0		$\{0\}$	$\{\theta_\alpha(i)\}$	$\{\theta'_\alpha(i)\}$
$\theta_\alpha(i)$			$\{1, 0, \theta_\alpha(i)\}$	$\{\theta'_\alpha(i)\}$
$\theta'_\alpha(i)$				$\{1, 0, \theta_\alpha(i)\}$

Table 3.4: Fusion rules of $\mathrm{id}_{\mathcal{D}_i}$

Again identify $\mathcal{D}_m$ as $\mathcal{A}_{m-1}^{\pm}$. Let $\varepsilon, \eta \in \{+, -\}$ and $x, y \in \mathcal{A}_{m-1} \subseteq \mathcal{A}_{m-1}^{\pm}$. We will consider the product $\wedge$ on collinear points x^ε, y^η from the subspaces $\mathcal{D}_i, \mathcal{D}_i^{\sim}$ and $\mathcal{D}_i^{\not\sim}$.

If $x^\varepsilon, y^\eta \in \mathcal{D}_i$ then $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i$, since $\mathcal{D}_i$ is a closed subspace. If $x^\varepsilon, y^\eta \in \mathcal{D}_i^{\not\sim}$ then $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i^{\not\sim}$ by Lemma 3.2.6. If $x^\varepsilon \in \mathcal{D}_i^{\sim}, y^\eta \in \mathcal{D}_i$ then $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i^{\sim}$, as $y \sim (x \wedge y)$ rules out $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i^{\not\sim}$ and $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i$ would force $x^\varepsilon \in \mathcal{D}_i$, a contradiction. If $x^\varepsilon \in \mathcal{D}_i^{\sim}, y^\eta \in \mathcal{D}_i^{\not\sim}$ then $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i^{\sim}$, as $y \sim (x \wedge y)$ rules out $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i$ and $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i$ would force $x^\varepsilon \in \mathcal{D}_i^{\not\sim}$, a contradiction.

Finally, suppose that $x^\varepsilon, y^\eta \in \mathcal{D}_i^{\sim}$. We show that $x^\varepsilon \wedge y^\eta \in \mathcal{D}_i \cup \mathcal{D}_i^{\not\sim}$. It is sufficient to show that for $x, y \in \mathcal{A}_{i-1}^{\sim}$ in $\mathcal{A}_m$ we have $x \wedge y \in \mathcal{A}_{i-1} \cup \mathcal{A}_{i-1}^{\not\sim}$. Suppose that the points of $\mathcal{A}_{i-1}$ are labelled by transpositions in $\mathrm{Sym}(i)$ with support $\{1, \dots, i\}$

inside $\mathrm{Sym}(m+1)$ with support $\{1,\ldots,m+1\}$. Then x, y are labelled $(i_x, j_x), (i_y, j_y)$ respectively with $i_x, i_y \in \{1,\ldots,i\}$ and $j_x, j_y \in \{i+1,\ldots,m+1\}$. That $x \sim y$ implies that either $i_x = i_y$ or $j_x = j_y$. Thus $x \wedge y$ is labelled (j_x, j_y) or (i_x, i_y) respectively, and hence $x \wedge y \in \mathcal{A}_{i-1} \cup \mathcal{A}_{i-1}^{\not\sim}$.

To show that t is an automorphism of $M_\alpha(\mathcal{G})$, by linearity it suffices to show that for any $x^\varepsilon, y^\eta \in \mathcal{G}$ we have

$$(x^\varepsilon)^t (y^\eta)^t = (x^\varepsilon y^\eta)^t. \tag{3.71}$$

When $x \not\sim y$, both sides are seen to be 0. By a case-by-case analysis for x^ε, y^η coming from the subspaces $\mathcal{D}_i, \mathcal{D}_i^\sim$ and $\mathcal{D}_i^{\not\sim}$, using our information on $\wedge$ calculated previously, we see that (3.71) is satisfied in all cases, for example, when $x^\varepsilon, y^\eta \in \mathcal{D}_i^\sim$,

$$\begin{aligned}(x^\varepsilon)^t (y^\eta)^t = x^{-\varepsilon} y^{-\eta} = \frac{\alpha}{2}(x^{-\varepsilon} + y^{-\eta} - x^{-\varepsilon} \wedge y^{-\eta}), \\ (x^\varepsilon y^\eta)^t = \frac{\alpha}{2}(x^\varepsilon + y^\eta - x^\varepsilon \wedge y^\eta)^t = \frac{\alpha}{2}(x^{-\varepsilon} + y^{-\eta} - x^\varepsilon \wedge y^\eta),\end{aligned} \tag{3.72}$$

and as $x^{-\varepsilon} \wedge y^{-\eta} = x^\varepsilon \wedge y^\eta$, we have the desired equality.

Therefore t is an automorphism, and is the Miyamoto involution of $\mathrm{id}_{\mathcal{D}_i}$. □

3.5.14 Lemma. *The Miyamoto involution of* $\mathrm{id}_{\mathcal{D}_i} \in M_\alpha(\mathcal{D}_m)$*, for* $3 \le i < m$*, inverts the* $\theta'_\alpha(i)$*-eigenspace, has an action on* $\mathcal{D}_m$*, and is not a transposition.*

Proof. It follows from the proof of Proposition 3.5.13 that $\tau(\mathrm{id}_{\mathcal{D}_i})$ acts by swapping points in $\mathcal{D}_m$ which are not collinear. On the other hand, for any $x \in \mathcal{D}_m$ we know by (2.47) that $\tau(x)$ acts on $\mathcal{D}_m$ by permuting collinear points. Therefore $\tau(\mathrm{id}_{\mathcal{D}_i})$ is not in the conjugacy class of any $\tau(x)$ for $x \in \mathcal{D}_m$.

However the action of $\tau(\mathrm{id}_{\mathcal{D}_i})$ on $\mathcal{D}_m$ induces an action on $W(\mathcal{D}_m)$ since by Lemma 2.4.1 we have $\tau(x^{\tau(\mathrm{id}_{\mathcal{D}_i})}) = \tau(x)^{\tau(\mathrm{id}_{\mathcal{D}_i})}$. Therefore $\tau(\mathrm{id}_{\mathcal{D}_i})$ is an automorphism of $W(\mathcal{D}_m)$. That it is involutory follows by the observation that its permutations are composed of disjoint 2-cycles in points of $\mathcal{D}_i$. □

Chapter 4. Ising algebras

Recall the Ising fusion rules $\Phi(\alpha, \beta)$ from Table 1.2. Analogously to Section 2.3, we are interested in $\Phi(\alpha, \beta)$-dihedral algebras, that is, those generated by two primitive $\Phi(\alpha, \beta)$-axes. The situation turns out to be much more complicated; short of a full classification, we find the generalisations of the Norton-Sakuma algebras.

4.1 Preliminaries

In this section, we revisit some base cases of the classification. Theorem 4.1.1 deals with a degenerate $\Phi(\alpha), \Phi(\beta)$ case; Theorem 4.1.4 and Table 4.1 recall the Norton-Sakuma algebras. We also prove the Proposition 4.1.3, useful for statements about the involutions of our idempotents.

We assume throughout this chapter that $\alpha, \beta \neq 1, 0$. The hypothesis of Theorem 2.3.1 is that a, b are $\Phi(\alpha)$-axes. One possibility when a, b are $\Phi(\alpha, \beta)$-axes is that a is a $\Phi(\alpha)$-axis and b is a $\Phi(\beta)$-axis, which we consider separately to the general case:

4.1.1 Theorem. *Let A be a everywhere faithful R-algebra, over a ring R containing $\frac{1}{2}, \alpha, \beta, \alpha^{-1}, (1-\alpha)^{-1}, \beta^{-1}, (1-\beta)^{-1}$, containing $e \in A$ a primitive $\Phi(\alpha)$-axis and $f \in A$ a primitive $\Phi(\beta)$-axis, with $\alpha \neq \beta$ and $ef \neq 0$. Then $\langle e, f\rangle \cong (3C_\alpha)$ and $\beta = 1 - \alpha$.*

Proof. If e, f are primitive, $ef \neq 0$, and $e \neq f$ then the respective α, β-eigenspaces

A^e_α, A^f_β of e, f are nonzero. Set $s = ef - \beta e - \alpha f$. As in Lemma 2.3.2 with s taking the rôle of $e \circ f$, $s \in A^e_{\{1,0\}} \cap A^f_{\{1,0\}}$, since

$$s = ((1-\beta)\lambda^f(e) - \alpha)f - \beta e^f_0 = ((1-\alpha)\lambda^e(f) - \beta)e - \alpha f^e_0 \tag{4.1}$$

for $e^f_1 = \lambda^f(e)f$ and $f^e_1 = \lambda^e(f)e$.

If ef is in the span of $\{e, f\}$, then the subalgebra $B = \langle e, f\rangle$ decomposes as $B^e_1 \oplus B^e_\alpha = B = B^f_1 \oplus B^f_\beta$. Hence $s \in A^e_1 \cap A^f_1 = 0$, so $s = 0$. Therefore $ef = \beta e + \alpha f$. In this case, $x = \beta e + (\alpha - 1)f$ is an α-eigenvector for e, that is, $x \in A^e_\alpha$, and $y = (\beta - 1)e + \alpha f \in A^f_\beta$. From $\alpha \star \alpha = \{1\}$ we deduce that the coefficient of f in xx is 0, and from

$$xx = \alpha\beta^2 e + (\alpha - 1)(2\alpha\beta + \alpha - 1)f \tag{4.2}$$

we deduce that $(\alpha-1)(2\alpha\beta+\alpha-1) = 0$. Since $\alpha - 1$ is invertible, $2\alpha\beta+\alpha-1 = 0$. On the other hand, we deduce that $2\alpha\beta + \beta - 1 = 0$ from $\beta \star \beta = \{1\}$ and the coefficient of e in yy. Put together, $\alpha = \beta$, which goes against our assumption. Therefore e, f, ef are linearly independent, and $s \neq 0$.

Now observe $es = ((1-\alpha)\lambda^e(f) - \beta)e$ and $fs = ((1-\beta)\lambda^f(e) - \alpha)f$. Since $s \in A^e_{\{1,0\}} \cap A^f_{\{1,0\}}$, by Lemma 2.1.6, $(ef)s = e(fs) = f(es)$, so that $((1-\alpha)\lambda^e(f)-\beta)ef = ((1-\beta)\lambda^f(e) - \alpha)ef$, and therefore we give the name κ to

$$\kappa = (1-\alpha)\lambda^e(f) - \beta = (1-\beta)\lambda^f(e) - \alpha. \tag{4.3}$$

We also deduce that $\{e, f, s\}$ is closed under multiplication.

Observe that $x = (\beta - \lambda^e(f))e + \alpha f + s \in A^e_\alpha$, that is, x is an α-eigenvector for e, and $y = (\alpha - \lambda^f(e))f + \beta e + s \in A^f_\beta$. Since $e, s \in A^e_{\{1,0\}}$, $f \notin A^e_{\{1,0\}}$ and $\alpha \star \alpha = \{1, 0\}$ implies that $xx \in A^e_{\{1,0\}}$, when xx is expressed as a sum of e, f, s the coefficient of f

must be 0. As

$$xx = (\beta-\lambda^e(f))(\beta-\lambda^e(f)+2\kappa+2\alpha\beta)e+\alpha(\alpha+2\kappa+2\alpha(\beta-\lambda^e(f)))f+(\kappa+2\alpha(\beta-\lambda^e(f)))s, \tag{4.4}$$

we deduce $\alpha(\alpha + 2\kappa + 2\alpha(\beta - \lambda^e(f))) = 0$. Similarly the coefficient $\beta(\beta + 2\kappa + 2\beta(\alpha - \lambda^f(e)))$ of e in yy must be 0. As α, β are invertible, we obtain the two formulae

$$\alpha + 2\kappa + 2\alpha(\beta - \lambda^e(f)) = 0 = \beta + 2\kappa + 2\beta(\alpha - \lambda^f(e)). \tag{4.5}$$

Taking differences, $\alpha(1 - 2\lambda^e(f)) = \beta(1 - 2\lambda^f(e))$ and in particular

$$\lambda^e(f) = \frac{\alpha - \beta + 2\beta\lambda^f(e)}{2\alpha}, \quad \lambda^f(e) = \frac{\beta - \alpha + 2\alpha\lambda^e(f)}{2\beta}. \tag{4.6}$$

From substituting each into (4.3), we respectively obtain $\lambda^f(e) = \frac{1}{2}(\alpha + 1)$ and $\lambda^e(f) = \frac{1}{2}(\beta + 1)$. Therefore $\kappa = \frac{1}{2}(1 - \alpha - \beta - \alpha\beta)$. Now substituting this into either side of (4.5), we obtain $0 = 1 - \alpha - \beta$.

Therefore $\beta = 1-\alpha$. Making this substitution into (4.6) and then substituting the expression for $\lambda^f(e)$ into (4.3), we deduce that $\lambda^e(f) = 1 - \frac{\alpha}{2}$ after some cancellations. Therefore $\kappa = -\frac{\alpha}{2}(1 - \alpha)$ and $\lambda^f(e) = \frac{1}{2}(1 + \alpha)$.

Assume without loss of generality, by swapping α and β if necessary, that $\alpha \neq -1$ and set $g = e^{\tau(f)}$. Then $g = e^f_1 + e^f_0 - e^f_{1-\alpha}$ where

$$e^f_1 = \frac{1}{2}(1 + \alpha)f, \quad e^f_{1-\alpha} = e - \frac{1}{2}f + \frac{1}{1-\alpha}s, \quad e^f_0 = -\frac{\alpha}{2}f - \frac{1}{1-\alpha}s. \tag{4.7}$$

Set $f' = e + g - \frac{2}{\alpha}eg$. We observe that f', g are again idempotents and $\Phi(\alpha)$-axes. Therefore the algebra generated by e, f is the algebra generated by e, f' is $(3C_\alpha)$, as $\alpha \neq -1, \frac{1}{2}$, using Theorem 2.3.1. □

The above result shows that two axes with Jordan fusion rules always occur as the axes in $(3C_\alpha)$ classified in Theorem 2.3.4 and Lemma 2.3.5.

The choice of notation $\lambda^e(f)$ in the preceding proof is deliberate (and *c.f.* λ^a in the proof of Lemma 3.4.7 as well as λ in the proof of Theorem 2.3.1): for any primitive idempotent $e \in A$ in an algebra A over an everywhere faithful ring R, we define a function

$$\lambda^e \colon A \to R \quad \text{satisfying, for all } x \in A, \quad x_1^e = \lambda^e(x)e. \tag{4.8}$$

This is well-defined since e spans its 1-eigenspace $A_1^e \ni x_1^e$ and the R-annihilator of e is 0, so the coefficient $r \in R$ satisfying $x_1^e = re$ is unique: $r = \lambda^e(x)$.

4.1.2 Lemma. *Suppose that $t \in \mathrm{Aut}(A)$ and that $a \in A$ is a Φ-axis. If $x \in A$ then $\lambda^a(x) = \lambda^{a^t}(x^t)$.*

Proof. We see that $(A_1^a)^t = A_1^{a^t}$ by Lemma 2.4.1. Therefore $\lambda^{a^t}(x^t)a^t = (x^t)_1^{a^t} = (x_1^a)^t = (\lambda^a(x)a)^t = \lambda^a(x)a^t$. $\square$

Observe that the fusion rules $\Phi(\alpha, \beta)$ admit a $\mathbb{Z}/2$-grading into $\{1, 0, \alpha\} \cup \{\beta\}$. In particular, $\Phi(\alpha, \beta)_+$ is exactly the Jordan fusion rules $\Phi(\alpha)$ of Chapter 2. We will need this latter observation, together with Theorem 2.3.1, later on.

For $\mathbb{Z}/2$-graded fusion rules Φ and two Φ-axes a_0, a_1, we write $T = \langle \tau(a_0), \tau(a_1) \rangle$ and $\rho = \tau(a_0)\tau(a_1)$. Then T is a dihedral group and we set

$$a_{2i} = {a_0}^{\rho^i}, \quad a_{2i+1} = {a_1}^{\rho^i}. \tag{4.9}$$

The following result majorly generalises Lemma 4.1 in [S07], which was restricted to $\Phi(1/4, 1/32)$-axes in vertex algebras:

4.1.3 Proposition. *Suppose that Φ are $\mathbb{Z}/2$-graded fusion rules, and a_0, a_1 are Φ-axes in A with $a_i^{\tau(a_j)} \neq a_i$ for $\{i, j\} = \{0, 1\}$. Then $|a_0^T| = |a_1^T|$ and $\rho^{|a_0^T \cup a_1^T|} = 1$ as an automorphism of A.*

Proof. The conjugacy classes of two generating involutions in a dihedral group have equal size, so $|\tau(a_0)^T| = |\tau(a_1)^T|$. Furthermore $|\rho| = |\tau(a_0)^T \cup \tau(a_1)^T|$. As

$\tau(a_i)^T = \{\tau(a) \mid a \in a_i^T\}$ we also have $|\tau(a_i)^T| \leq |a_i^T|$. Let $\mathcal{A} = \{a_i \mid i \in \mathbb{Z}\} = a_0^T \cup a_1^T$ and n the smallest positive integer such that $a_0 = a_n$. If no such n exists then $\mathcal{A}$ is infinite and both a_0^T and a_1^T are infinite (as ρ has infinite order, one of $\tau(a_0)^T$ and $\tau(a_1)^T$ must be infinite, and the sizes of these orbits coincides, so both are infinite). Suppose instead that there is such a finite n.

If n is odd then $n = 2m+1$ and $a_0 = a_{2m+1} = a_1^{\rho^m}$. Therefore $a_1 \in a_0^T$, so $a_0^T = a_1^T$ has size n and

$$\tau(a_1) = \tau(a_0)^{\rho^{-m}} = \rho^m \tau(a_0) \rho^{-m} = \tau(a_0)\rho^{-2m}, \tag{4.10}$$

where the last step used that $\rho^{\tau(a_i)} = \rho^{-1}$. Multiplying by $\tau(a_0)$ on the left, we have $\rho = \tau(a_0)\tau(a_1) = \rho^{-2m}$, that is, $\rho^{1+2m} = \rho^n = 1$ as required.

If n is even, we consider the cases for both a_0 and a_1. So, say $a_0 = a_{n_0}, a_1 = a_{n_1}$; by the above paragraph, both n_0 and n_1 must be even, so $n_i = 2m_i$ for $m_{i=0,1}$ an integer. That is: $a_0 = a_0^{\rho^{m_0}}, a_1 = a_1^{\rho^{m_1}}$. Therefore

$$\tau(a_0) = \rho^{-m_0}\tau(a_0)\rho^{m_0} = \tau(a_0)\rho^{2m_0} \text{ and } \tau(a_1) = \rho^{-m_1}\tau(a_1)\rho^{m_1} = \tau(a_1)\rho^{2m_1}, \tag{4.11}$$

which means that ρ has order dividing $2m_0$ and $2m_1$. As ρ also has orbits of size m_0, m_1, this means that ρ has order lying in $\{m_0, 2m_0\} \cap \{m_1, 2m_1\}$. If $|\rho| = m_0 = m_1$ or $|\rho| = 2m_0 = 2m_1$, then $m_0 = m_1$ and the statement of the proposition is satisfied.

Out of the two remaining cases, consider without loss of generality the case $2m_0 = m_1$. By assumption, $m_0 \geq 2$ as $a_2 = a_0^{\tau(a_1)} \neq a_0$. That is, there are twice as many distinct axes a_i for i odd as there are distinct axes a_i for i even. However, the action of $\{\tau(a_{2i})\}_{i\in\mathbb{Z}}$ on $\{a_{2j+1}\}_{j\in\mathbb{Z}}$ is transitive, so that $\rho_2 = \tau(a_0)\tau(a_2)$ if $m_0 \geq 2$ has an orbit of size $2m_0$, which is impossible as ρ_2 has order at most $m_0 = |\tau(a_0)^{\langle\rho_2\rangle} \cup \tau(a_2)^{\langle\rho_2\rangle}|$. □

Since $a_0^T, a_1^T \subseteq B = \langle a_0, a_1\rangle$, to bound the order of $\rho \in \mathrm{Aut}(A)$ it therefore suffices to determine the order of the action of ρ on the subalgebra B of A.

The following was proven in [HRS15], following the groundbreaking work in [S07, IPSS10].

4.1.4 Theorem (Sakuma's theorem, Theorems 5.10, 8.7 [HRS15])**.** *Any* $\Phi(1/4, 1/32)$-*dihedral Frobenius* $\mathbb{Q}$-*algebra is a quotient of the direct sum of the Norton-Sakuma algebras over* $\mathbb{Q}$. □

The nontrivial Norton-Sakuma algebras $(\mathrm{n}X)$ over $\mathbb{Z}[1/2, 1/3]$ are given by Table 4.1, together with the formulas

$$a_i a_i = a_i, \quad a_i = a_{i \bmod n}, \quad (a_i, a_i) = 1 \tag{4.12}$$

$$a_i a_{i+1} = \frac{1}{64}(a_i + a_{i+1}) + s, \quad a_i a_{i+2} = \frac{1}{64}(a_i + a_{i+2}) + \begin{cases} s_2 & \text{if } i \text{ is even,} \\ s_2^f & \text{if } i \text{ is odd.} \end{cases} \tag{4.13}$$

Under the isomorphism type $(\mathrm{n}X)$, we give a spanning set and notes; the other column contains all the products necessary to calculate in the algebra.

The trivial Norton-Sakuma algebra generated by two axes a_0, a_1 is $(1A)$, the 1-dimensional algebra in which $a_0 = a_1$.

Description	Products
$(2B)$	$a_0a_1 = 0$
a_0, a_1	$(a_0, a_1) = 0$
$(2A)$	$a_ia_j = \frac{1}{8}(a_i + a_j - a_k)$
a_0, a_1, a_2	$(a_i, a_j) = \frac{1}{8}$
$(3C)$	$a_ia_j = \frac{1}{64}(a_i + a_j - a_k)$
a_0, a_1, a_2	$(a_i, a_j) = \frac{1}{64}$
$(3A)$	$a_is = \frac{7}{2^{11}}(4a_i + a_{i+1} + a_{i+2}) + \frac{7}{32}s$
a_0, a_1, a_2, s	$ss = \frac{147}{2^{16}}(a_0 + a_1 + a_2) - \frac{63}{2^{11}}s$
	$(a_i, a_j) = \frac{13}{256}$
$(4A)$	$a_is = \frac{1}{2^{11}}(7(a_{i+1} + a_{i-1}) - 2a_i) + \frac{7}{32}s$
a_{-1}, a_0, a_1, a_2, s	$ss = \frac{21}{2^{15}}(a_{-1} + a_0 + a_1 + a_2) - \frac{11}{2^9}s$
$\langle a_i, a_{i+2}\rangle \cong (2B)$	$(a_i, a_{i+1}) = \frac{1}{32}$
$(4B)$	$a_ia_{i+2} = -\frac{1}{8}(a_{i+1} + a_{i-1}) - 8s$
a_{-1}, a_0, a_1, a_2, s	$a_is = \frac{1}{2^{11}}(7(a_{i+1} + a_{i-1}) - 26a_i) + \frac{7}{32}s$
$\langle a_i, a_{i+2}\rangle \cong (2A)$	$ss = \frac{7}{2^{15}}(a_{-1} + a_0 + a_1 + a_2) - \frac{3}{2^9}s$
	$(a_i, a_{i+1}) = \frac{1}{64}$
$(5A)$	$a_is = \frac{7}{2^{11}}(a_{i+1} + a_{i-1} - 2a_i) + \frac{7}{32}s$
$a_{-2}, a_{-1}, a_0, a_1, a_2, s$	$ss = \frac{35}{2^{17}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$
$s_2 = s$	$(a_i, a_j) = \frac{3}{128}$
	$a_ia_{i+2} = \frac{1}{64}(3(a_i + a_{i+2}) + a_{i+4} - a_{i+1} - a_{i+3} - a_{i+5}) + \bar{s}_2$
$(6A)$	$a_ia_{i+3} = \frac{1}{12}(2(a_i + a_{i+3}) - a_{i+1} - a_{i+2} - a_{i+4} - a_{i+5}) - 8s - \frac{8}{3}\bar{s}_2$
$a_{-2}, a_{-1}, a_0, a_1, a_2,$	$a_is = \frac{1}{2^{11}}(7(a_{i+1} + a_{i-1}) - 20a_i) + \frac{7}{32}s$
$a_3, s, \bar{s}_2$	$a_i\bar{s}_2 = \frac{5}{2^{10}3}a_i - \frac{23}{2^{11}3}(a_{i+1} + a_{i-1}) + \frac{7}{2^93}(a_{i+2} + a_{i+4} - \frac{1}{2}a_{i+3})$
$\bar{s}_2 = \frac{1}{2}(s_2 + s_2^f)$	$\quad -\frac{3}{32}s + \frac{7}{48}\bar{s}_2$
$\langle a_i, a_{i+3}\rangle \cong (2A)$	$ss = \frac{49}{2^{17}3}(a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3) - \frac{17}{2^{10}}s - \frac{7}{2^{11}3}\bar{s}_2$
$\langle a_i, a_{i+2}\rangle \cong (3A)$	$s\bar{s}_2 = -\frac{21}{2^{17}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3) + \frac{21}{2^{10}}s + \frac{15}{2^{11}}\bar{s}_2$
	$\bar{s}_2\bar{s}_2 = \frac{107}{2^{17}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3) - \frac{9}{2^{10}}s - \frac{77}{2^{11}}\bar{s}_2$
	$(a_i, a_{i+1}) = \frac{5}{256}$

Table 4.1: The nontrivial Norton-Sakuma algebras

4.2 The universal algebra

In this section, we make the formal construction of a certain universal algebra: an algebra of which all Φ-axial algebras, or in our case of interest all $\Phi(\alpha, \beta)$-dihedral algebras, are quotients. We proceed by constructing a chain of increasingly specialised universal objects, starting from a free magma.

Recall that all our algebras are commutative and not necessarily associative.

Let $\{\alpha_3, \ldots, \alpha_n\}$ be a collection of symbols, $\alpha_1 = 1, \alpha_2 = 0$, and set $\Phi = \{1, 0\} \cup \{\alpha_3, \ldots, \alpha_n\}$. For R a ring, recall that $R[x, x^{-1}]$ means $R[x, y]/(xy - 1)$. Let

$$R'' = R''(\Phi) = \mathbb{Z}[\alpha_i, (\alpha_i - \alpha_j)^{-1} \mid \alpha_i, \alpha_j \in \Phi, i \neq j]. \quad \textbf{(4.14)}$$

Suppose that R is a ring and S is an associative R-algebra. For an R-algebra A, the *scalar extension by* S of A is the S-algebra $A \otimes_R S$ with product $(x \otimes s)(y \otimes s') = (xy) \otimes (ss')$.

Let $\mathcal{A} = \{a_1, \ldots, a_m\}$ be an ordered collection and $\mathcal{M}'$ the nonassociative magma on $\mathcal{A}$, that is, the collection of all bracketings of nonempty words on $\mathcal{A}$ together with a multiplication given by juxtaposition. In the category of R-algebras with m marked generators, where morphisms are algebra homomorphisms mapping the marked generators to marked generators and preserving the ordering, $R\mathcal{M}'$ is an initial object: there exists exactly one morphism from $R\mathcal{M}'$ to any other object A in the category. This mapping $R\mathcal{M}' \to A$ is given by evaluating the word $w \in \mathcal{M}'$ as a word in A.

Furthermore set $\mathcal{M}$ to be the commutative nonassociative magma on idempotents $\mathcal{A}$, that is, $\mathcal{M}$ is $\mathcal{M}'$ modulo the relations $a_i a_i = a_i$ for all i and $uv = vu$ for all words u, v. Then $R\mathcal{M}$ is an initial object in the category of commutative (nonassociative)

R-algebras generated by m marked idempotents.

Let $\lambda^a(w)$ be a symbol for all $a \in \mathcal{A}$ and $w \in \mathcal{M}$, and set

$$R' = R''[\lambda^a(w) \mid a \in \mathcal{A}, w \in \mathcal{M}] = R''[\lambda^{\mathcal{A}}(\mathcal{M})]. \tag{4.15}$$

This formal function $\lambda^a \colon \mathcal{M} \to R', w \mapsto \lambda^a(w)$ takes the rôle of λ^e in (4.8).

4.2.1 Lemma. *Suppose that A is an everywhere faithful R-algebra, generated by a set $\mathcal{A}$ of primitive idempotents with eigenvalues Φ. If R is an associative $R'' = R''(\Phi)$-algebra, then R is an associative R'-algebra in a unique way such that $w_1^a = \lambda^a(w)a$ for all $a \in \mathcal{A}, w \in \mathcal{M}$.*

Proof. We use the multiplication in A to identify the mapping of $\lambda^a(w) \in R'$ into R. Let $\hat{R}$ be the polynomial ring $R[\lambda^{\mathcal{A}}(\mathcal{M})]$ of R extended by indeterminates $\lambda^a(w)$ for all $a \in \mathcal{A}, w \in \mathcal{M}$ (*c.f.* (4.15)). Then there exists a unique mapping of R' into $\hat{R}$: as R is an associative R''-algebra, there exists $\tilde{\psi} \colon R'' \to R$; this extends canonically to $\hat{\psi} \colon R''[\lambda^{\mathcal{A}}(\mathcal{M})] \to R[\lambda^{\mathcal{A}}(\mathcal{M})]$, that is, $\hat{\psi} \colon R' \to \hat{R}$, by setting $\tilde{\psi}(\lambda^a(w)) = \lambda^a(w)$. Let J be the ideal of $\hat{R}$ generated by $\lambda^a(w) - \lambda^a(\tilde{w})$ for all $a \in \mathcal{A}$ and $w \in \mathcal{M}$, where $\tilde{w} \in A$ is the evaluation of the word $w \in \mathcal{M}$ and the second λ^a is the mapping $A \to R \subseteq \hat{R}$ of (4.8). Then set $\psi \colon R' \to R$ to be the map $r \mapsto \hat{\psi}(r)/J$. As $\tilde{\psi}, \hat{\psi}$ are ring homomorphisms, so is ψ, and ψ makes R an associative R'-algebra. $\square$

Let A' be the quotient of $R'\mathcal{M}$ modulo the ideal generated by

$$\Big(\prod_{\alpha_i \in \Phi \smallsetminus \{1\}} (\mathrm{ad}(a) - \alpha_i \,\mathrm{id})\Big)(w - \lambda^a(w)a) \quad \text{for all } a \in \mathcal{A}, w \in A''. \tag{4.16}$$

Then A' is generated by primitive *diagonalisable* idempotents.

4.2.2 Lemma. *There exists a unique morphism from A' to any R'-algebra A generated by m primitive diagonalisable idempotents with eigenvalues Φ.*

Proof. Since A is commutative and m-generated by idempotents, there is a unique mapping from A'' to A, given by the I_A be the evaluation ideal in A'' induced by A: as in the proof of Lemma 4.2.1, for $w \in A''$, write $\tilde{w}$ for the evaluation of the word w in A, and $\bar{w}$ for the evaluation of the word $\tilde{w}$ in A''. Then I_A is generated by $w - \bar{w}$ for $w \in A''$. That the m generators in A are primitive diagonalisable idempotents with eigenvalues Φ implies that the ideal in (4.16) is contained in I_A and therefore the unique mapping from A'' to A factors through a morphism from A' to A. $\square$

Let $\mathcal{C}' = \mathcal{C}'_{R''}$ be the category whose objects are pairs (R, A) such that R is a ring and an associative R''-algebra, $\mathcal{A}$ in A are primitive diagonalisable idempotents with (ordered) eigenvalues $\bar{\Phi} = \{1, 0, \bar{\alpha}_2, \ldots, \bar{\alpha}_n\}$, and A is an everywhere faithful R-algebra generated by $\mathcal{A}$. (For convenience, we always write $\mathcal{A}$ for the generating set.) Morphisms in the category from (R_1, A_1) to (R_2, A_2) are pairs (ϕ, ψ) such that $\phi\colon R_1 \to R_2$ is a R''-algebra homomorphism and $\psi\colon A_1 \to A_2$ is an algebra homomorphism mapping generators to generators and compatible with ϕ, in the sense that $(rx)^\psi = r^\phi x^\psi$ for all $r \in R_1, x \in A_1$.

4.2.3 Lemma. *(R', A') is an initial object in $\mathcal{C}'$.*

Proof. That is, for any (R, A) in $\mathcal{C}'$ there exists a single morphism $(\phi, \psi)\colon (R', A') \to (R, A)$. This follows since R is an R'-algebra, by Lemma 4.2.1, by a unique nontrivial map $\phi\colon R' \to R$ which maps $R' \mapsto (R' \otimes_{\mathbb{Z}} R)/I_R$, where I_R is the ideal $(\alpha_i - \bar{\alpha}_i \mid 3 \leq i \leq n)$ encoding the identification of $\bar{\alpha}_i$ with α_i. Furthermore A' is initial among R'-algebras generated by m marked primitive idempotents acting diagonalisably with eigenvalues Φ by Lemma 4.2.2. $\square$

Suppose that (J, I) is a pair with J an ideal of R and I an ideal of A. The pair (J, I) *match* if $JA \subseteq I$ and $\lambda^a(x) \in J$ for all $x \in I, a \in \mathcal{A}$. If (J, I) match and I contains none of the generators in $\mathcal{A}$, then $(R/J, A/I)$ is again an element of $\mathcal{C}'$, since

$\mathcal{A}$ is preserved, R/J is again an associative algebra over R'' (and hence R') acting everywhere faithfully on A/I, and A/I is generated by m primitive diagonalisable idempotents with eigenvalues Φ.

Take an arbitrary object (R, A) and let (ϕ, ψ) be the morphism from (R', A') to (R, A). Then ψ is a surjection in the sense that $RA'^{\psi} = A$. We can write ϕ as the composition $\phi_s \circ \phi_i$ of an injection $R' \mapsto R' \otimes_{\mathbb{Z}} R$ with a surjection $(R' \otimes_{\mathbb{Z}} R)/J_A$, for $J_A = \ker \phi_s$ an ideal. Let $I_A = \ker \psi$. The ideals (J_R, I_A) match by the condition that ψ, ϕ_s have to be compatible. Write $\mathcal{C}'(J, I)$ for the subcategory of $\mathcal{C}'$ whose objects are algebras (R, A) such that $J_R \subseteq J, I_A \subseteq I$. In $\mathcal{C}'(J, I)$, the object $(R'/J, A'/I)$ is again initial.

So far, Φ has only been a collection of eigenvalues. Now suppose that Φ comes with fusion rules $\star$. We write down two special ideals J', I': let I' be generated by

$$\prod_{\alpha \in \alpha_j \star \alpha_k} (\mathrm{ad}(a_i) - \alpha \, \mathrm{id}) \Big(\prod_{\alpha \in \Phi \smallsetminus \{\alpha_j\}} (\mathrm{ad}(a_i) - \alpha \, \mathrm{id}) x \cdot \prod_{\alpha \in \Phi \smallsetminus \{\alpha_k\}} (\mathrm{ad}(a_i) - \alpha \, \mathrm{id}) y \Big) \tag{4.17}$$

for all $x, y \in A$ and $\alpha_j, \alpha_k \in \Phi$ and $a_i \in \mathcal{A}$, and set $J' = (\lambda^a(x) \mid a \in \mathcal{A}, x \in I')$. Then (J', I') match. The ideals J', I' are minimal with respect to the axioms of Φ-axial algebras, that is, if (R, A) are Φ-axial R'-algebras then they are in C' and the unique morphism $(\phi, \psi)\colon (R', A') \to (R, A)$ necessarily has $J' \subseteq \ker \phi$ and $I' \subseteq \ker \psi$. Therefore $\mathcal{C} = \mathcal{C}'(J', I')$ is the category of m-generated Φ-axial algebras and has initial object $(R_U, U) = (R'/J', A'/I')$. We have

4.2.4 Theorem. *For fusion rules $\Phi = \{1, 0, \alpha_3, \ldots, \alpha_n\}$, $m \in \mathbb{N}$, and R'' from* (4.14)*, there exists an algebra U over a ring R_U such that if S is an associative R''-algebra and B is an m-generated Φ-axial S-algebra, there exist matching ideals $J_B \subseteq R_U \otimes_{\mathbb{Z}} S, I_B \subseteq U \otimes_{\mathbb{Z}} S$ such that $U \otimes_{\mathbb{Z}} S/I_B \cong B$ as an algebra over $R_U \otimes_{\mathbb{Z}} S/J_B \cong S$.* □

We say that the R_U-algebra U is the *universal m-generated Φ-axial algebra*.

Note that, if we replaced the underlying ring R'' in (4.14) with a larger ring $\hat{R}''$, Theorem 4.2.4 continues to hold with minor modification. In particular, set

$$R_0 = \mathbb{Z}[1/2, \alpha, \beta, \alpha^{-1}, \beta^{-1}, (\alpha-\beta)^{-1}, (\alpha-2\beta)^{-1}, (\alpha-4\beta)^{-1}, \lambda_1, \lambda_1^f, \lambda_2, \lambda_2^f]. \qquad (4.18)$$

The upcoming Theorem 4.3.7 gives the multiplication table for an algebra A_{R_0} over R_0. The universal $\Phi(\alpha, \beta)$-dihedral algebra U that exists by Theorem 4.2.4 is a quotient of A_{R_0}, although we will not completely determine U in this text.

4.3 The multiplication table

Here we determine the structure constants of the universal $\Phi(\alpha, \beta)$-dihedral algebra, by repeatedly exploiting the fusion rules $\Phi(\alpha, \beta)$. In particular, we show that, under mild assumptions on the ring, a certain set of 8 elements spans the algebra.

For fusion rules Φ, let R' be the polynomial ring

$$R' = R'(\Phi) = \mathbb{Z}[\alpha_i, (\alpha_i - \alpha_j)^{-1} \mid \alpha_i, \alpha_j \in \Phi, i \neq j], \tag{4.19}$$

From now on, we only consider the Ising fusion rules $\Phi = \Phi(\alpha, \beta)$. Suppose for the rest of the section that R is an associative R'-algebra, for so that $\alpha, \beta \in R$ and $\alpha, \beta, \alpha - 1, \beta - 1, \alpha - \beta$ are invertible in R, and suppose that furthermore $\frac{1}{2} \in R$.

Suppose that A is a everywhere faithful R-algebra generated by Φ-axes a_0, a_1. In this section, culminating in Theorem 4.3.7, we give a spanning set and multiplication table for the algebra A, under some further conditions on R.

A second product on A, $\circ \colon A \times A \to A$,

$$x \circ y = xy - \beta(x + y) \text{ for } x, y \in A, \tag{4.20}$$

will play a critical rôle for us because of the $\mathbb{Z}/2$-grading on Φ. Recall this already played a rôle in Lemma 2.3.2.

Recall that, for a Φ-axis $a \in A$, $\tau(a) \in \mathrm{Aut}(A)$ is the automorphism acting as -1 on the β-eigenspace of a and fixing all other eigenspaces.

Let B be the subset of A with elements

$$B = \{a_{-2}, a_{-1}, a_0, a_1, a_2, s, s_2, s_2^f\}, \tag{4.21}$$

$$\text{for } s = a_0 \circ a_1, \quad s_2 = a_0 \circ a_2, \quad s_2^f = a_{-1} \circ a_1. \tag{4.22}$$

In this section we compute the products between elements of B, recorded in lemmas; other computations will be part of the rolling text.

Four elements of the ring will also have a special rôle to play. Namely, we write

$$\lambda_1 = \lambda^{a_0}(a_1), \quad \lambda_1^f = \lambda^{a_1}(a_0), \quad \lambda_2 = \lambda^{a_0}(a_2), \quad \lambda_2^f = \lambda^{a_1}(a_{-1}). \tag{4.23}$$

Note that, since $\alpha, \beta, \lambda_1, \lambda_1^f, \lambda_2, \lambda_2^f$ lie in the ring R, the automorphisms of A and in particular $\tau(a_0), \tau(a_1)$ fix them.

The superscript f notation refers to the map f, called the *flip*, interchanging a_0, a_1. In general f is not known to be an automorphism, but it turns out to be in some special cases. Observe that it has an action on the ring and on the algebra.

Recall that, by the definition of a Φ-axis a, any element $x \in A$ may be written as

$$x = x_1^a + x_0^a + x_\alpha^a + x_\beta^a = x_1 + x_0 + x_\alpha + x_\beta, \tag{4.24}$$

where the second time we have omitted the superscript a because this is clear from context. In this section, all eigenvector decompositions will be with respect to a or a_0 unless indicated otherwise. We deduce that

$$x_\alpha = \frac{1}{\alpha}(ax - \lambda^a(x)a - \beta x_\beta), \tag{4.25}$$

$$x_\beta = \frac{1}{2}(x - x^{\tau(a)}). \tag{4.26}$$

The latter follows since $\tau(a)$ inverts the β-eigenspace, so we have that $x_\beta = \frac{1}{2}(x - x^{\tau(a)})$. The former is found by using the previous equations in $ax = \lambda^a(x)a + \alpha x_\alpha + \beta x_\beta$ and rearranging. Furthermore

$$a^2x = a(ax) = \lambda^a(x)(1-\alpha)a + \alpha ax + \beta(\beta - \alpha)x_\beta, \tag{4.27}$$

using the substitution (4.25) in the expression $a^2x = \lambda^a(x)a + \alpha^2 x_\alpha + \beta^2 x_\beta$.

Recall (from Lemma 2.3.2) that $a \circ x$ lies in $A^a_{\{1,0,\alpha\}}$, and thus is fixed by $\tau(a)$, for any $x \in A$:

$$a \circ x = \lambda^a(x)a + \alpha x_\alpha + \beta x_\beta - \beta(x+a), \tag{4.28}$$

$$= (\lambda^a(x) - \beta)a - \beta x_0 + (\alpha - \beta)x_\alpha. \tag{4.29}$$

4.3.1 Lemma. *We have*

$$\begin{aligned} a_0 s &= (\alpha-\beta)s + (\lambda_1(1-\alpha) + \beta(\alpha-\beta-1))a_0 + \frac{1}{2}(\alpha-\beta)\beta\big(a_1 + a_1^{\tau(a_0)}\big), \\ a_1 s &= (\alpha-\beta)s + (\lambda_1^f(1-\alpha) + \beta(\alpha-\beta-1))a_1 + \frac{1}{2}(\alpha-\beta)\beta\big(a_0 + a_0^{\tau(a_1)}\big), \\ a_0 s_2 &= (\alpha-\beta)s_2 + (\lambda_2(1-\alpha) + \beta(\alpha-\beta-1))a_0 + \frac{1}{2}(\alpha-\beta)\beta\big(a_2 + a_2^{\tau(a_0)}\big) \end{aligned} \tag{4.30}$$

from

$$a(a \circ x) = (\alpha-\beta)a \circ x + (\lambda^a(x)(1-\alpha) + \beta(\alpha-\beta-1))a + (\alpha-\beta)\beta x_+. \tag{4.31}$$

Proof. The equation (4.31) can be deduced starting from the definition of $\circ$:

$$\begin{aligned} a(a \circ x) &= a(ax) - \beta(aa + ax) \\ &= (\alpha-\beta)ax + (\lambda^a(x)(1-\alpha) - \beta)a + \beta(\beta-\alpha)x_\beta \end{aligned}$$

by (4.27), and then rewriting ax using $\circ$ as

$$= (\alpha-\beta)a \circ x + (\alpha-\beta)\beta(a+x) + (\lambda^a(x)(1-\alpha) - \beta)a - (\alpha-\beta)\beta x_\beta.$$

This gives the result after collecting terms and writing $x_+ = x - x_\beta$. From (4.31) we deduce $a_0 s, a_0 s_2$ by substituting a_1, a_2 for x respectively. Then $a_1 s$ follows by swapping a_0, a_1. □

We define a further commutative product

$$*_a \colon A \times A \to A, \quad x *_a y = (x \circ a)y + (y \circ a)x \tag{4.32}$$

for our computations. With respect to the Φ-axis a, for $x, y \in A$,

$$(x *_a y)_{1,0} = -2\beta(x_+y_+)_{1,0} + 2\alpha x_\alpha y_\alpha + ((\lambda^a(x) - \beta)\lambda^a(y) + (\lambda^a(y) - \beta)\lambda^a(x))a, \quad (4.33)$$

$$(x *_a y)_\alpha = (\alpha - 2\beta)(x_+y_+)_\alpha + \alpha(\lambda^a(x) - \beta)y_\alpha + \alpha(\lambda^a(y) - \beta)x_\alpha. \quad (4.34)$$

To deduce this, we use the previous calculations for $(x \circ a)_{1,0}$ and $(x \circ a)_\alpha$. From the fusion rules, since $(x \circ a)_\beta = 0$, we have that

$$\begin{aligned}(x *_a y)_{1,0} &= ((x \circ a)y)_{1,0} + ((y \circ a)x)_{1,0} \\ &= (x \circ a)_{1,0}y_{1,0} + (y \circ a)_{1,0}x_{1,0} + (x \circ a)_\alpha y_\alpha + (y \circ a)_\alpha x_\alpha \\ &= -2\beta x_{1,0}y_{1,0} + 2(\alpha - \beta)x_\alpha y_\alpha + ((\lambda^a(x) - \beta)\lambda^a(y) + (\lambda^a(y) - \beta)\lambda^a(x))a,\end{aligned}$$

which gives (4.33) when we use that $(x_+y_+)_{1,0} = x_{1,0}y_{1,0} + x_\alpha y_\alpha$. For (4.34),

$$\begin{aligned}(x *_a y)_\alpha &= ((x \circ a)y)_\alpha + ((y \circ a)x)_\alpha \\ &= (x \circ a)_{1,0}y_\alpha + (x \circ a)_\alpha y_{1,0} + (y \circ a)_{1,0}x_\alpha + (y \circ a)_\alpha x_{1,0} \\ &= (\alpha - 2\beta)(x_{1,0}y_\alpha + x_\alpha y_{1,0}) + \alpha(\lambda^a(x) - \beta)y_\alpha + \alpha(\lambda^a(y) - \beta)x_\alpha,\end{aligned}$$

and after using $x_{1,0}y_\alpha + x_\alpha y_{1,0} = (x_+y_+)_\alpha$ we have the answer.

We also find that, for $x, y \in A$, when $\alpha - 2\beta$ is invertible,

$$(x_+y_+)_\alpha = \frac{1}{\alpha - 2\beta}\big((x *_a y)_\alpha + \alpha\beta(x_\alpha + y_\alpha) - \alpha(\lambda^a(x)y_\alpha + \lambda^a(y)x_\alpha)\big), \quad (4.35)$$

$$x_\alpha y_\alpha = \frac{1}{2\alpha}\big((x *_a y)_{1,0} + 2\beta(x_+y_+)_{1,0} + (\beta\lambda^a(x) + \beta\lambda^a(y) - 2\lambda^a(x)\lambda^a(y))a\big). \quad (4.36)$$

The expression (4.35) is just a rearrangement of (4.34). For (4.36), in turn rearranging from (4.33),

$$\begin{aligned}2\alpha x_\alpha y_\alpha + (2\lambda^a(x)\lambda^a(y) - \beta(\lambda^a(x) + \lambda^a(y)))a &= (x *_a y)_{1,0} + 2\beta(x_+y_+)_{1,0} \\ &= (x *_a y)_+ - (x *_a y)_\alpha + 2\beta((x_+y_+)_+ - (x_+y_+)_\alpha).\end{aligned}$$

As $(x_+y_+)_+ = x_+y_+$, rearranging gives the final claim.

4.3.2 Lemma. *We have that, if $\alpha - 2\beta$ is invertible, then $ss =$*

$$
\frac{1}{2}\frac{\alpha-\beta}{\alpha-2\beta}\Big[\big(4(1-2\alpha)\lambda_1^2+2(\alpha^2+\alpha\beta-4\beta)\lambda_1+\alpha\beta\lambda_2-\beta(\alpha^2+9\alpha\beta-4\beta^2-4\beta)\big)\,a_0
$$
$$
+\big(2(-\alpha^2+6\alpha\beta+\alpha-4\beta)\lambda_1-\beta(10\alpha\beta-4\beta^2+\alpha-6\beta)\big)\,(a_1)_+ +\beta(\alpha-4\beta)(\alpha-\beta)(a_2)_+
$$
$$
+\frac{2}{\alpha-\beta}\left(\alpha(3\alpha-2\beta-1)\lambda_1-\beta(6\alpha^2-10\alpha\beta+4\beta^2-\alpha)\right)s-\alpha\beta s_2+\beta(\alpha-2\beta)s_2^f\Big], \tag{4.37}
$$

derived from the equation, for any $x, y \in A$,

$$
\begin{aligned}
(a\circ x)(a\circ y) = {} & \left(\frac{\alpha}{2}-\beta\right)(x *_a y)_+ - \frac{\alpha^2}{2(\alpha-2\beta)}(x *_a y)_\alpha + \beta(\alpha-\beta)x_+y_+ \\
& + \alpha^2\left(1+\frac{\beta}{\alpha-2\beta}\right)((\lambda^a(y)-\beta)x_\alpha + (\lambda^a(x)-\beta)y_\alpha) \\
& + (\lambda^a(x)\lambda^a(y)(1-\alpha) + (\lambda^a(x)+\lambda^a(y))\left(\frac{\alpha}{2}-1\right)\beta + \beta^2)a.
\end{aligned} \tag{4.38}
$$

Proof. Note that

$$
a\circ x+\beta x_+ = ax-\beta a-\beta x+\beta x_+ = ax-\beta x_\beta-\beta a = ax_+-\beta a = (\lambda^a(x)-\beta)a+\alpha x_\alpha. \tag{4.39}
$$

Now we can compute in two ways:

$$
\begin{aligned}
(a\circ x+\beta x_+)(a\circ y+\beta y_+) &= (a\circ x)(a\circ y)+\beta((a\circ x)y_+ + (a\circ y)x_+)+\beta^2 x_+y_+ \\
&= (a\circ x)(a\circ y)+\beta(x *_a y)_+ + \beta^2 x_+y_+
\end{aligned}
$$

since $a\circ x = (a\circ x)_+$, so $(a\circ x)y_+ = ((a\circ x)y)_+$, on the one hand; on the other,

$$
\begin{aligned}
&= ((\lambda^a(x)-\beta)a+\alpha x_\alpha)((\lambda^a(y)-\beta)a+\alpha y_\alpha) \\
&= \alpha^2 x_\alpha y_\alpha + \alpha^2((\lambda^a(x)-\beta)y_\alpha + (\lambda^a(y)-\beta)x_\alpha) + (\lambda^a(x)-\beta)(\lambda^a(y)-\beta)a
\end{aligned}
$$

and using (4.36),

$$
\begin{aligned}
&= \alpha\left(\frac{1}{2}(x *_a y)_{1,0}+\beta(x_+y_+)_{1,0}\right)+\alpha^2((\lambda^a(y)-\beta)x_\alpha+(\lambda^a(x)-\beta)y_\alpha) \\
&\quad + \left(\lambda^a(x)\lambda^a(y)(1-\alpha)+\beta\left(\frac{\alpha}{2}-1\right)(\lambda^a(x)+\lambda^a(y))+\beta^2\right)a.
\end{aligned}
$$

So we may rearrange for our desired term:

$$\begin{aligned}(a\circ x)(a\circ y) = &\left(\frac{\alpha}{2}-\beta\right)(x *_a y)_+ - \frac{\alpha}{2}(x *_a y)_\alpha + \beta(\alpha-\beta)x_+y_+ - \alpha\beta(x_+y_+)_\alpha \\ &+ \alpha^2((\lambda^a(y)-\beta)x_\alpha + (\lambda^a(x)-\beta)y_\alpha) \\ &+ (\lambda^a(x)\lambda^a(y)(1-\alpha) + (\lambda^a(x)+\lambda^a(y))\left(\frac{\alpha}{2}-1\right)\beta + \beta^2)a\end{aligned}$$

and finally, using (4.35),

$$\begin{aligned}= &\left(\frac{\alpha}{2}-\beta\right)(x *_a y)_+ - \alpha\left(\frac{1}{2}+\frac{\beta}{\alpha-2\beta}\right)(x *_a y)_\alpha + \beta(\alpha-\beta)x_+y_+ \\ &+ \alpha^2\left(1+\frac{\beta}{\alpha-2\beta}\right)(\lambda^a(y)x_\alpha + \lambda^a(x)y_\alpha) - \left(\frac{\alpha^2\beta^2}{\alpha-2\beta}+\alpha^2\beta\right)(x_\alpha+y_\alpha) \\ &+ (\lambda^a(x)\lambda^a(y)(1-\alpha) + (\lambda^a(x)+\lambda^a(y))\left(\frac{\alpha}{2}-1\right)\beta + \beta^2)a,\end{aligned}$$

so we arrive at our conclusion (4.38) after collecting terms.

Now also note that, with respect to the Φ-axis a, for any idempotent $e \in A$,

$$e_+e_+ = \frac{1}{2}e\circ e^{\tau(a)} + \left(\frac{1}{2}+\beta\right)e_+. \tag{4.40}$$

By definition, $e_+ = \frac{1}{2}(e+e^{\tau(a)})$. Hence, multiplying out, $e_+e_+ = \frac{1}{4}(e+e^{\tau(a)}) + \frac{1}{2}ee^{\tau(a)}$, and we rewrite the expression using the definition of $\circ$. For the product ss, we specialise: for any $x \in A$,

$$\begin{aligned}(a\circ x)(a\circ x) = &\left(\frac{\alpha}{2}-\beta\right)(x *_a x)_+ + \beta(\alpha-\beta)x_+x_+ \\ &+ (\lambda^a(x)^2(1-\alpha) + \lambda^a(x)(\alpha-2)\beta + \beta^2)a \\ &+ 2\alpha^2\left(1+\frac{\beta}{\alpha-2\beta}\right)(\lambda^a(x)-\beta)x_\alpha - \frac{\alpha^2}{2(\alpha-2\beta)}(x *_a x)_\alpha.\end{aligned} \tag{4.41}$$

We need a number of auxiliary expressions. Observe that (4.25) can be rewritten as,

$$x_\alpha = \frac{1}{\alpha}(a\circ x + \beta(a+x) - \lambda^a(x)a - \beta x_\beta) = \frac{1}{\alpha}(a\circ x + (\beta-\lambda^a(x))a + \beta x_+). \tag{4.42}$$

By application of (4.31) with a_1 in place of a, substituting $\frac{1}{2}(a_0 + a_2)$ for $a_{0+}^{a_1}$,

$$a_1 *_{a_0} a_1 = 2sa_1 = 2(\alpha - \beta)s + 2(\lambda_1^f(1-\alpha) + \beta(\alpha - \beta - 1))a_1 + (\alpha - \beta)\beta(a_0 + a_2). \tag{4.43}$$

We immediately deduce, using $s_+ = s$ and $a_+ = a$, then using (4.28) and (4.42),

$$(a_1 *_{a_0} a_1)_+ = 2(\alpha - \beta)s + 2(\lambda_1^f(1-\alpha) + \beta(\alpha - \beta - 1))(a_1)_+ + (\alpha - \beta)\beta(a_0 + (a_2)_+), \tag{4.44}$$

$$\begin{aligned}(a_1 *_{a_0} a_1)_\alpha &= 2(\alpha - \beta)s_\alpha + 2(\lambda_1^f(1-\alpha) + \beta(\alpha - \beta - 1))(a_1)_\alpha + (\alpha - \beta)\beta((a_0)_\alpha + (a_2)_\alpha) \\ &= \frac{2}{\alpha}((\alpha-\beta)^2 + \lambda_1^f(1-\alpha) + \beta(\alpha - \beta - 1))(s + (\beta - \lambda_1)a_0 + \beta(a_1)_+) \\ &\quad + (\alpha - \beta)\frac{\beta}{\alpha}(s_2 + (\beta - \lambda_2)a_0 + \beta(a_2)_+).\end{aligned} \tag{4.45}$$

Finally, we substitute a_1 in place of x in (4.41), and substitute the expressions we collected for $(a_1 *_{a_0} a_1)_+$ in (4.44), for $(a_1)_+(a_1)_+$ in (4.40), for $(a_1)_\alpha$ in (4.42), and for $(a_1 *_{a_0} a_1)_\alpha$ in (4.45), to find $ss =$

$$\begin{aligned}&\left(\frac{\alpha}{2} - \beta\right)\left[2(\alpha-\beta)s + 2(\lambda_1^f(1-\alpha) + \beta(\alpha-\beta-1))(a_1)_+ + (\alpha-\beta)\beta(a + a^{\tau(b)}{}_+)\right] \\ &+ \beta(\alpha - \beta)\left(\frac{1}{2}s_2^f + \left(\frac{1}{2} + \beta\right)(a_1)_+\right) + (\lambda_1^2(1-\alpha) + \lambda_1(\alpha - 2)\beta + \beta^2)a_0 \\ &+ 2\alpha^2\left(1 + \frac{\beta}{\alpha - 2\beta}\right)\frac{\lambda_1 - \beta}{\alpha}(s + (\beta - \lambda_1)a_0 + \beta(a_1)_+) - \frac{\alpha^2}{2(\alpha - 2\beta)}\left[\frac{2}{\alpha}((\alpha-\beta)^2 + \lambda_1^f(1-\alpha)\right. \\ &\left. + \beta(\alpha - \beta - 1))(s + (\beta - \lambda_1)a_0 + \beta(a_1)_+) + (\alpha - \beta)\frac{\beta}{\alpha}(s_2 + (\beta - \lambda_2)a_0 + \beta(a_2)_+)\right]\end{aligned} \tag{4.46}$$

$$
\begin{aligned}
= &\Bigg[\left(\frac{\alpha}{2}-\beta\right)(\alpha-\beta)\beta+\lambda_1^2(1-\alpha)+\lambda_1(\alpha-2)\beta+\beta^2+2\alpha\left(1+\frac{\beta}{\alpha-2\beta}\right)(\lambda_1-\beta)(\beta-\lambda_1) \\
&-\frac{\alpha}{2(\alpha-2\beta)}\left(\beta(\alpha-\beta)(\beta-\lambda^a(a^{\tau(b)}))+2((\alpha-\beta)^2+\lambda_1^f(1-\alpha)+\beta(\alpha-\beta-1))(\beta-\lambda_1)\right)\Bigg]a_0 \\
&+\Bigg[(\alpha-2\beta)(\lambda_1^f(1-\alpha)+\beta(\alpha-\beta-1))+\beta(\alpha-\beta)\left(\frac{1}{2}+\beta\right)+2\alpha\left(1+\frac{\beta}{\alpha-2\beta}\right)(\lambda_1-\beta)\beta \\
&-\frac{\alpha}{\alpha-2\beta}((\alpha-\beta)^2+\lambda_1^f(1-\alpha)+\beta(\alpha-\beta-1))\Bigg](a_1)_+ \qquad \textbf{(4.47)} \\
&+\left[\beta(\alpha-\beta)\left(\frac{\alpha}{2}-\beta\right)-\frac{\alpha^2}{2(\alpha-2\beta)}(\alpha-\beta)\frac{\beta^2}{\alpha}\right](a_2)_+-\frac{\alpha\beta}{2(\alpha-2\beta)}(\alpha-\beta)s_2+\frac{\beta(\alpha-\beta)}{2}s_2^f \\
&+\left[(\alpha-\beta)(\alpha-2\beta)+2\alpha\left(1+\frac{\beta}{\alpha-2\beta}\right)(\lambda_1-\beta)-\frac{\alpha((\alpha-\beta)^2+\lambda_1^f(1-\alpha)+\beta(\alpha-\beta-1))}{\alpha-2\beta}\right]s
\end{aligned}
$$

which is the equation given in the statement, once its terms have been collected. □

From now on, for the rest of this section, we will assume that $\alpha-2\beta$ is invertible.

4.3.3 Lemma. *If $\alpha-4\beta$ is invertible, then $a_3=$*

$$
\begin{aligned}
&a_{-2}+\frac{1}{\beta(\alpha-\beta)(\alpha-4\beta)}\Bigg(\left(4\alpha\beta\lambda_1-2(\alpha^2-(4\beta+1)\alpha+4\beta)\lambda_1^f-\alpha^2\beta-\alpha\beta(5\beta-1)+6\beta^2\right)a_{-1} \\
&+\frac{1}{\alpha-\beta}\Big(4(3\alpha^2-(4\beta-1)\alpha+2\beta)\lambda_1^2+(4\alpha(\alpha-1)\lambda_1^f-6\alpha^3-2(3\beta-1)\alpha^2+2\alpha\beta(8\beta+1)-8\beta^2)\lambda_1 \\
&-4\alpha\beta(\beta-1)\lambda_1^f-2\alpha\beta(\alpha-\beta)\lambda_2+2\beta\alpha^3+\beta(6\beta-1)\alpha^2-\alpha\beta^2(12\beta-1)+2\beta^3(2\beta-1)\Big)a_0 \\
&+\frac{1}{\alpha-\beta}\Big(4\alpha(\alpha-1)\lambda_1\lambda_1^f-4\alpha\beta(\beta-1)\lambda_1+4(3\alpha^2-4\alpha(4\beta-1)+2\beta){\lambda_1^f}^2-2(3\alpha^3+\alpha^2(3\beta-1) \\
&-\alpha\beta(8\beta+1)+4\beta^2)\lambda_1^f+2\alpha\beta(\alpha-\beta)\lambda_2+2\alpha^3\beta+\alpha^2\beta(6\beta-1)-\alpha\beta^2(12\beta+1)+2\beta^3(2\beta+1)\Big)a_1 \\
&+\Big(2(\alpha^2-\alpha(4\beta+1)+4\beta)\lambda_1-4\alpha\beta\lambda_1^f+\alpha^2\beta+\alpha\beta(5\beta+1)-6\beta^2\Big)a_2 \\
&+\frac{1}{\alpha-\beta}\left(4\alpha(\alpha-2\beta+1)\lambda_1+4\alpha(-\alpha+2\beta-1)\lambda_1^f\right)s\Bigg)+\frac{4}{\alpha-4\beta}\left(s_2^f-s_2\right), \qquad \textbf{(4.48)}
\end{aligned}
$$

and therefore the R-span of B is stable under T. Thus we deduce expressions for

$$a_2s,\quad a_{-1}s,\quad a_{-2}s,\quad a_1s_2^f,\quad a_{-1}s_2^f,\quad a_2s_2,\quad a_{-2}s_2.$$

Proof. Lemma 4.3.2 calculates ss using eigenspace decompositions and fusion rules with respect to a_0, and by repeating the computation with the rôles of a_0 and a_1 swapped, we obtain an expression for $(ss)^f$ from (4.37). On the other hand, s is symmetric in a_0 and a_1 so that ss is invariant under interchange of a_0 and a_1. The equation (4.48) follows from the equality $(ss)^f = ss$. Since $a_3 = (a_{-2})^f$, we see the desired term in the expression for $(ss)^f$:

$$\begin{aligned}
&\frac{1}{2}\frac{\alpha-\beta}{\alpha-2\beta}\Big[\Big(4(1-2\alpha){\lambda_1^f}^2 + 2(\alpha^2+\alpha\beta-4\beta)\lambda_1^f + \alpha\beta\lambda_2^f/we - \beta(\alpha^2+9\alpha\beta-4\beta^2-4\beta)\Big)a_1 \\
&+\left((-\alpha^2+6\alpha\beta+\alpha-4\beta)\lambda_1^f - \frac{1}{2}\beta(10\alpha\beta-4\beta^2+\alpha-6\beta)\right)(a_0+a_2) \qquad (4.49)\\
&+\frac{1}{2}\beta(\alpha-4\beta)(\alpha-\beta)(a_{-1}+a_3) \\
&+\frac{2}{\alpha-\beta}\Big(\alpha(3\alpha-2\beta-1)\lambda_1^f - \beta(6\alpha^2-10\alpha\beta+4\beta^2-\alpha)\Big)s - \alpha\beta s_2^f + \beta(\alpha-2\beta)s_2\Big].
\end{aligned}$$

Rearranging yields the claim for a_3.

Now observe that

$$B^f = \{a_3, a_2, a_1, a_0, a_{-1}, s, s_2^f, s_2\}. \qquad (4.50)$$

Out of these, the only term not already in B is a_3. Our above expression for this term shows that RB and RB^f coincide. As $a_4 = a_{-3}^f$, that is, $a_3^{\tau(a_0)}$ with the rôles of a_0, a_1 reversed, we have that $B^{\tau(a_1)} \subseteq RB^f = RB$ also, and since $B^{\tau(a_0)} = B$ this proves $RB^T = RB$.

Now all of the terms in (4.3.3) are in the T, f-orbit of $a_0 s$ or $a_0 s_2$:

$$\begin{aligned}
a_2 s &= (a_0 s)^{\tau(a_1)}, & a_1 s_2^f &= (a_0 s_2)^f,\\
a_{-1} s &= (a_0 s)^{\tau(a_1) f}, & a_{-1} s_2^f &= (a_0 s_2)^{f\tau(a_0)},\\
a_{-2} s &= (a_0 s)^{\tau(a_1)\tau(a_0)}, & a_2 s_2 &= (a_0 s_2)^{\tau(a_1)},\\
& & a_{-2} s_2 &= (a_0 s_2)^{\tau(a_1)\tau(a_0)}.
\end{aligned}$$

□

From now on we also assume that $\alpha - 4\beta$ is invertible.

4.3.4 Lemma. *We have that* $a_0 s_2^f =$

$$\frac{1}{\alpha-2\beta}\bigg(\frac{1}{\alpha-\beta}\Big(2(3\alpha-(4\beta-1)\alpha+2\beta)\lambda_1^2+2\alpha(\alpha-1)\lambda_1\lambda_1^f+2(-2\alpha^3+\alpha^2+\alpha\beta(2\beta-1))\lambda_1$$
$$+4\beta(-\alpha^2+\alpha(\beta+1)-\beta)\lambda_1^f-\alpha\beta(\alpha-\beta)\lambda_2+4\alpha^3\beta-2\beta(2\beta+1)\alpha^2+2\alpha\beta^2(\beta+1)-2\beta^4\Big)a_0$$
$$-(2\alpha\lambda_1+2(\alpha-1)\lambda_1^f-2\alpha^2+(\beta+\frac{1}{2})\alpha-2\beta^2+\beta)\big(\beta(a_1+a_{-1})+s\big)+(\alpha-\beta)\beta^2\big(a_2+a_{-2}\big)$$
$$+2\beta(\alpha-\beta)s_2\bigg), \tag{4.51}$$

and from this follow expressions, using the T-invariance in Lemma 4.3.3, for

$$a_2 s_2^f,\quad a_{-2}s_2^f,\quad a_1 s_2,\quad a_{-1}s_2 \quad \textit{and} \quad a_{-2}a_1,\quad a_{-2}a_2,\quad a_{-1}a_2.$$

Proof. Recall from (4.42) and (4.26) that

$$(a_1)_0 = \frac{1}{\alpha}\big((1-\alpha)\lambda_1-\beta)a_0+(\alpha-\beta)\frac{1}{2}(a_1+a_{-1})-s\big). \tag{4.52}$$

First we write down, using Lemma 4.3.3,

$$(a_1)_0(a_1)_0 = \frac{1}{2\alpha(\alpha-2\beta)}\bigg(-\big(2\alpha(\alpha-2\beta+1)\lambda_1^2+2(\alpha-1)\lambda_1\lambda_1^f+2(-\alpha^2-2\alpha\beta+2\beta^2+\beta)\lambda_1$$
$$-2\beta(\alpha-1)\lambda_1^f-\beta(\alpha-\beta)\lambda_2+3\beta\alpha^2-3\beta^3-2\beta^2\big)a_0 \tag{4.53}$$
$$+(\alpha-\beta)\big(4\beta\lambda_1+2(\alpha-1)\lambda_1^f-(2\beta-1)\alpha-4\beta^2\big)(a_1)_+ +\beta(\alpha-\beta)(a_2)_+$$
$$(2(\alpha-\beta)\lambda_1+(\alpha-1)\lambda_1^f-2\alpha^2+2\alpha\beta+\beta)s-\beta(\alpha-\beta)s_2+(\alpha-\beta)(\alpha-2\beta)s_2^f\bigg).$$

From the fusion rule $0\star 0=0$ we deduce that $a_0((a_1)_0(a_1)_0)=0$. The only product we do not already know is $a_0 s_2^f$, so substituting and rearranging gives the result.

The second set of equations follows from the fact that T is transitive on $a_{-2}, a_{-1}, a_0, a_1, a_2$, and we have expressions for $a_0 x$ for all $x \in B$, so we can take any product $a_i x$ and find a representative $a_0 x'$ in the T-orbit of $a_i x$ with an expression for this product in RB. As RB is T-closed, this allows us to calculate any $a_i x$. □

4.3.5 Lemma. *We can find an expression for ss_2 in RB, and hence for ss_2^f too. We also have expressions for $s_2 s_2$ and $s_2^f s_2^f$ in RB.*

Proof. We will derive the first equality starting from the equation

$$a_0((a_1)_0(a_2)_\alpha + (a_1)_0(a_2)_0) = \alpha(a_1)_0(a_2)_\alpha, \tag{4.54}$$

which follows from the fusion rules. The key is that the contributions of ss_2 from each of the terms $(a_1)_0(a_2)_\alpha$ and $(a_1)_0(a_2)_0$ cancel on the lefthand side, but not on the righthand side.

We know $(a_1)_0$ from (4.52). In the same way we calculate

$$(a_2)_0 = \frac{1}{\alpha}\big((1-\alpha)\lambda_2 - \beta)a_0 + (\alpha-\beta)\frac{1}{2}(a_2 + a_{-2}) - s_2\big), \tag{4.55}$$

and also get, from (4.42),

$$(a_2)_\alpha = \frac{1}{\alpha}\big((\beta - \lambda_2)a_0 + \frac{1}{2}\beta(a_2 + a_{-2}) + s_2\big). \tag{4.56}$$

The previous lemmas are enough to calculate the products, so arriving at the answer is a matter of rearranging the copious terms.

Then $ss_2^f = (ss_2)^f$. The third and forth products promised follow from

$$a_0((a_2)_0(a_2)_\alpha + (a_2)_0(a_2)_0) = \alpha(a_2)_0(a_2)_\alpha \tag{4.57}$$

using the same method. □

4.3.6 Lemma. *We can express $s_2s_2^f$ as a sum of terms in RB.*

Proof. Recall that idempotents are preserved by automorphisms, so that $a_3a_3 = a_3$. The expression afforded in Lemma 4.3.3, and knowing all other products, allows us to express $s_2s_2^f$. □

Altogether the previous sequence of lemmas proves

4.3.7 Theorem. *Suppose that R is an associative $R'[(\alpha-2\beta)^{-1},(\alpha-4\beta)^{-1}]$-algebra, for R' from (4.19). If A is an everywhere faithful $\Phi(\alpha,\beta)$-dihedral R-algebra then A*

is spanned by

$$B = \{a_{-2}, a_{-1}, a_0, a_1, a_2, s, s_2, s_2^f\}.$$

with the multiplication table described by the previous Lemmas 4.3.2 to 4.3.6. □

In particular, when U is the universal Φ-dihedral algebra over ring R_U from Theorem 4.2.4 with the extra condition that $\alpha - 2\beta, \alpha - 4\beta$ be invertible, then R_U is a quotient of the polynomial ring

$$\mathbb{Z}[{}^1\!/\!_2, \alpha, \beta, \alpha^{-1}, \beta^{-1}, (\alpha-\beta)^{-1}, (\alpha-2\beta)^{-1}, (\alpha-4\beta)^{-1}] \tag{4.58}$$

and U satisfies the multiplication of Theorem 4.3.7 over R_U.

We define a form $(,)$ on the algebra of Theorem 4.3.7 by setting $(a_i, a_i) = 1$ and extending the definition using relations that would be satisfied if $(,)$ is Frobenius. Firstly, $\lambda^{a_i}(a_j) = \frac{(a_i,a_j)}{(a_i,a_i)} = (a_i, a_j)$. Secondly,

$$(a_i, s) = (a_i, a_i a_{i+1} - \beta(a_i + a_{i+1})) = (a_i, a_{i+1}) - \beta(a_i, a_i) - \beta(a_i, a_{i+1}) = \lambda_1(1-\beta) - \beta. \tag{4.59}$$

We likewise compute the remaining values of the form up to action by T and f.

$$(a_i, s_2) = \lambda_2(1-\beta) - \beta, \text{ for } i = -2, 0, 2 \tag{4.60}$$

$$(a_i, s_2) = (\beta\lambda_2 - (2\beta - 1)\lambda_1 - \beta), \text{ for } i = -1, 1 \tag{4.61}$$

$$(s, s) = \frac{1}{2}(\beta(\alpha-\beta)\lambda_2 - 2(\alpha-1)\lambda_1^2 + 2(\alpha + 2\beta^2 - 4\beta)\lambda_1 - \beta\alpha + 5\beta^2), \tag{4.62}$$

$$\begin{aligned}(s, s_2) = \frac{1}{2}\frac{(\alpha-2\beta)(\alpha-\beta)}{(\alpha-4\beta)(\alpha-2\beta)(\alpha-\beta)}\big(&\beta(-6\alpha\lambda_1 + 4\alpha^2 - (2\beta+3)\alpha + 4\beta^2 + 6\beta)\lambda_2 \\ &+ 8(2\alpha-1)\lambda_1^3 - 4(\alpha+2)(2\alpha-1)\lambda_1^2 + 2(4\alpha^2 + (2\beta^2-1)\alpha - 8\beta^3 + 12\beta^2 - 4\beta)\lambda_1 \\ &- 4\beta\alpha^2 + \beta(6\beta+1)\alpha - 20\beta^3 + 2\beta^2\big),\end{aligned} \tag{4.63}$$

$$
\begin{aligned}
(s_2, s_2) = \frac{1}{2}\frac{1}{\beta(\alpha-4\beta)^2(\alpha-\beta)}\big(&-2\beta(\alpha-4\beta)(\alpha-\beta)(\alpha^2-(3\beta+1)\alpha+4\beta)\lambda_2^2 \\
&+(4(\alpha^4-(9\beta+2)\alpha^3+(22\beta^2+19\beta+1)\alpha^2+2\beta(8\beta^2-21\beta-4)\alpha-8\beta^3+16\beta^2)\lambda_1^2 \\
&+2\beta(2(2\beta-1)\alpha^3-\beta(74\beta+11)\alpha^2+2\beta(2\beta^2+47\beta+4)\alpha-32\beta^2)\lambda_1+2\beta(2\beta+1)\alpha^4 \\
&+2\beta^2(6\beta-11)\alpha^3+\beta^2(12\beta^2+88\beta-1)\alpha^2+4\beta^3(14\beta^2-61\beta+1)\alpha-48\beta^6+2^7\beta^5 \\
&+12\beta^4)\lambda_2+16(2\alpha-1)(\alpha^2-(6\beta+1)\alpha+4\beta)\lambda_1^4-8(2\alpha-1)(\alpha^3-(6\beta-1)\alpha^2 \\
&-(2\beta+1)(5\beta+2)\alpha+4\beta^3+6\beta^2+8\beta)\lambda_1^3-4(-3\alpha^4+(20\beta^2+17\beta+4)\alpha^3 \\
&-(8\beta^3-40\beta^2+6\beta+1)\alpha^2+2\beta(2\beta^2-40\beta-1)\alpha+28\beta^2)\lambda_1^2+2\beta(6(6\beta+1)\alpha^3 \\
&+(58\beta^2-33\beta-2)\alpha^2-2\beta(2\beta^2+43\beta-2)\alpha+32\beta^2)\lambda_1-4\beta^2\alpha^4-2\beta^2(2\beta+1)\alpha^3 \\
&-\beta^2(84\beta^2-22\beta-1)\alpha^2+4\beta^3(34\beta^2+7\beta-1)\alpha-80\beta^6-12\beta^4\big), \qquad \textbf{(4.64)}
\end{aligned}
$$

$$
\begin{aligned}
(s_2, s_2^f) = \frac{1}{2}\frac{1}{(\alpha-4\beta)^2(\alpha-\beta)}\big(&-2\beta\alpha(\alpha-4\beta)(\alpha-\beta)\lambda_2^2+(8(\alpha^3-3\beta\alpha^2+\beta(8\beta+1)\alpha-4\beta^2)\lambda_1^2 \\
&-2((26\beta+3)\alpha^3-(70\beta^2+27\beta+1)\alpha^2+2\beta(46\beta^2+27\beta+3)\alpha-48\beta^3-8\beta^2)\lambda_1 \\
&+16\beta\alpha^4-2\beta(22\beta+3)\alpha^3+\beta(72\beta^2-2\beta+1)\alpha^2-4\beta^3(22\beta-7)\alpha+80\beta^5-32\beta^4 \\
&-4\beta^3)\lambda_2 32\alpha(2\alpha-1)\lambda_1^4+8(2\alpha-1)(6\alpha^2-(10\beta-3)\alpha+12\beta^2+2\beta)\lambda_1^3 \\
&-4(8\alpha^4-20(\beta-1)\alpha^3+(24\beta^2-32\beta-15)\alpha^2+2(26\beta^2+14\beta+1)\alpha-32\beta^2-4\beta)\lambda_1^2 \\
&+2(16\alpha^4+2(2\beta^2-9\beta-4)\alpha^3-(36\beta^3-14\beta^2+8\beta-1)\alpha^2+2\beta(48\beta^3-2\beta^2 \\
&+27\beta+1)\alpha-64\beta^5+64\beta^4-64\beta^3-8\beta^2)\lambda_1-16\beta\alpha^4+4\beta(11\beta+2)\alpha^3 \\
&-\beta(72\beta^2+8\beta+1)\alpha^2+4\beta^3(22\beta-5)\alpha-80\beta^5+32\beta^4+4\beta^3\big). \qquad \textbf{(4.65)}
\end{aligned}
$$

By this definition, $(,)$ does not make the algebra Frobenius; for example, $(a_{-1}, a_2) \neq (a_{-1}, a_{-1}a_2)$. However, some quotients of $(,)$ will turn out to be Frobenius in the sequel.

4.4 A further fusion rule

Not all fusion rules have been enforced yet. We describe, given a $\Phi(\alpha,\beta)$-dihedral algebra, how to find its generalisation, called an axial cover, by finding smaller ideals, coming from the fusion rules, in the universal algebra previously described. We also introduce the extra assumption that the coefficient functions λ^e, λ^f are symmetric.

Suppose that R is a ring satisfying the assumption in Theorem 4.3.7, so that R is an associative algebra over

$$R_0 = \mathbb{Z}[\alpha, \beta, \alpha^{-1}, \beta^{-1}, (\alpha-\beta)^{-1}, (\alpha-2\beta)^{-1}, (\alpha-4\beta)^{-1}]. \tag{4.18}$$

Let A_R be the free R-module on $B = \{a_{-2}, a_{-1}, a_0, a_1, a_2, s, s_2, s_2^f\}$ together with the multiplication from Theorem 4.3.7. Then $a_0, a_1 \in A_R$ are not necessarily $\Phi(\alpha,\beta)$-axes, since their eigenvectors do not satisfy the fusion rules in general, as we will see in Lemma 4.4.4. Therefore A_R is not necessarily a $\Phi(\alpha,\beta)$-dihedral algebra.

However, Theorem 4.3.7 shows that any $\Phi(\alpha,\beta)$-dihedral algebra over a ring R satisfies the multiplication rules given in Section 4.3, and therefore is a quotient of A_R. In particular, Theorem 4.2.4 asserts that there exists a ring R_U, which is a quotient of R_0 by some ideal $J^{R_0}_{\Phi(\alpha,\beta)}$, and an algebra U over R_U which is the universal $\Phi(\alpha,\beta)$-dihedral algebra. Hence U is a quotient of A_{R_U} by some ideal $I^{R_0}_{\Phi(\alpha,\beta)}$. (We use the subscript $\Phi(\alpha,\beta)$ in our notation for these ideals to indicate that they come solely from the fusion rules.) While actually finding U over R_U is beyond our reach, we work with A_{R_0} over R_0 as an approximation to the universal object.

Short of classifying all the $\Phi(\alpha,\beta)$-dihedral algebras, we can use our results to significantly generalise the known $\Phi(\alpha,\beta)$-dihedral algebras by pulling back certain ideals, as we will now explain.

Namely, suppose that $(\mathrm{n}X)_R$ is a $\Phi(\alpha,\beta)$-dihedral everywhere-faithful R-algebra, where R is an associative algebra over R_0. Then, by Theorem 4.2.4, there exist matching ideals

$$J^R_{(\mathrm{n}X)} \subseteq R_U \otimes_{\mathbb{Z}} R, \quad I^R_{(\mathrm{n}X)} \subseteq U_R = (R_U \otimes_{\mathbb{Z}} R)U \tag{4.66}$$

such that

$$R \cong (R_U \otimes_{\mathbb{Z}} R)/J^R_{(\mathrm{n}X)}, \quad (\mathrm{n}X) \cong U_R/I^R_{(\mathrm{n}X)}. \tag{4.67}$$

Recall that, for the two ideals $J^R_{(\mathrm{n}X)}, I^R_{(\mathrm{n}X)}$ to match, we must have $J^R_{(\mathrm{n}X)}U_R \subseteq I^R_{(\mathrm{n}X)}$, and $\lambda^{a_i}(x) \in J^R_{(\mathrm{n}X)}$ for $i = 0, 1$ and any $x \in I^R_{(\mathrm{n}X)}$. Note that, if R is a domain, then $J^R_{(\mathrm{n}X)}$ is a prime ideal.

Suppose we have matching ideals

$$J'^R_{(\mathrm{n}X)} \subseteq J^R_{(\mathrm{n}X)} \subseteq R_U \otimes_{\mathbb{Z}} R, \quad I'^R_{(\mathrm{n}X)} \subseteq I^R_{(\mathrm{n}X)} \subseteq U_R = (R_U \otimes_{\mathbb{Z}} R)U \tag{4.68}$$

such that $(\mathrm{n}X') = U_{(R_U \otimes_{\mathbb{Z}} R)/J'^R_{(\mathrm{n}X)}}/I'^R_{(\mathrm{n}X)}$ is a $\Phi(\alpha,\beta)$-dihedral everywhere faithful $(R_U \otimes_{\mathbb{Z}} R)/J'^R_{(\mathrm{n}X)}$-algebra. We say that $(\mathrm{n}X')$ is a *weak (axial) cover* of $(\mathrm{n}X)$, as $(\mathrm{n}X)$ is a quotient of $(\mathrm{n}X')$.

Definition. Let $\hat{J}^R_{(\mathrm{n}X)} = \bigcap J'^R_{(\mathrm{n}X)}$, $\hat{I}^R_{(\mathrm{n}X)} = \bigcap I'^R_{(\mathrm{n}X)}$ over all matching ideals $J'^R_{(\mathrm{n}X)}, I'^R_{(\mathrm{n}X)}$ as above. The $(R_U \otimes_{\mathbb{Z}} R)/\hat{J}^R_{(\mathrm{n}X)}$-algebra $U_R/\hat{I}^R_{(\mathrm{n}X)}$ is the *(axial) cover* of $(\mathrm{n}X)$.

Note that it is possible that there are infinite descending chains of such ideals $J'^R_{(\mathrm{n}X)}, I'^R_{(\mathrm{n}X)}$, so it is not *a priori* clear that their intersection, or the axial cover, is well-defined.

If the ideals $\hat{J}^R_{(\mathrm{n}X)}, \hat{I}^R_{(\mathrm{n}X)}$ are strictly smaller than $J^R_{(\mathrm{n}X)}, I^R_{(\mathrm{n}X)}$, this means that $(\mathrm{n}X)$ is subject to additional constraints other than those coming from the fusion rules. Then $(\mathrm{n}X)$ is a proper quotient of its axial cover, which is its largest generalisation as an axial algebra. Our specific application will be to the Norton-Sakuma algebras $(\mathrm{n}X)$, listed in Table 4.1.

Since R_U and therefore U_R are not available to work with, we will use our approximation A_{R_0} as follows. We still consider a fixed $\Phi(\alpha,\beta)$-dihedral algebra $(\mathrm{n}X)$ over an everywhere faithful ring R which is an associative R_0-algebra. Then

$$A_{R_0\otimes_{\mathbb{Z}}R} \xrightarrow{I^R_{\Phi(\alpha,\beta)}} U_R \xrightarrow{I^R_{(\mathrm{n}X)}} (\mathrm{n}X), \tag{4.69}$$

$$R_0 \otimes_{\mathbb{Z}} R \xrightarrow{J^R_{\Phi(\alpha,\beta)}} (R_U \otimes_{\mathbb{Z}} R)/J^R_{\Phi(\alpha,\beta)} \xrightarrow{J^R_{(\mathrm{n}X)}} R \tag{4.70}$$

shows, in the top line, the relation among the algebras, and in the bottom line the relation among their rings. Instead of finding

$$\hat{I}^R_{(\mathrm{n}X)} \subseteq U_R \text{ and } \hat{J}^R_{(\mathrm{n}X)} \subseteq (R_U \otimes_{\mathbb{Z}} R), \tag{4.71}$$

we will try to find ideals

$$\bar{I}^R_{(\mathrm{n}X)} \subseteq A_R \text{ and } \bar{J}^R_{(\mathrm{n}X)} \subseteq (R_0 \otimes_{\mathbb{Z}} R) \tag{4.72}$$

such that

$$\bar{I}^R_{(\mathrm{n}X)}/I^R_{\Phi(\alpha,\beta)} = \hat{I}^R_{(\mathrm{n}X)} \text{ and } \bar{J}^R_{(\mathrm{n}X)}/J^R_{\Phi(\alpha,\beta)} = \hat{J}^R_{(\mathrm{n}X)}. \tag{4.73}$$

Notice that $A_R/\bar{I}^R_{(\mathrm{n}X)}$ as an $(R_0 \otimes_{\mathbb{Z}} R)/\bar{J}^R_{(\mathrm{n}X)}$-algebra is exactly the axial cover of $(\mathrm{n}X)$.

As it turns out, a subset of the fusion rules will be sufficient to generate $\bar{I}^R_{(\mathrm{n}X)}$ and $\bar{J}^R_{(\mathrm{n}X)}$, which will considerably shorten our work. In practice, we can calculate $\bar{J}^R_{(\mathrm{n}X)}$ as follows. As the cover is everywhere faithful over its ring $(R_0 \otimes_{\mathbb{Z}} R)/\bar{J}^R_{(\mathrm{n}X)}$, the ring is a domain and $\bar{J}^R_{(\mathrm{n}X)}$ must be a prime ideal. Thus we have

4.4.1 Lemma. *For any $p \in J^R_{\Phi(\alpha,\beta)}$, let $p_{(\mathrm{n}X)}$ be the smallest factor of p contained in $\bar{J}^R_{(\mathrm{n}X)}$. Then $\bar{J}^R_{(\mathrm{n}X)} = (p_{(\mathrm{n}X)} \mid p \in J^R_{\Phi(\alpha,\beta)})$.* □

Note that if $(\mathrm{n}X)$ is a $\Phi(\bar{\alpha},\bar{\beta})$-axial algebra then $\alpha-\bar{\alpha}, \beta-\bar{\beta} \in \hat{J}^R_{(\mathrm{n}X)}$. Therefore if $p(\alpha)$ is an irreducible polynomial in $\bar{J}^R_{(\mathrm{n}X)} \subseteq \hat{J}^R_{(\mathrm{n}X)}$ and coprime to $\alpha-\bar{\alpha}$ then the ideal $(p(\alpha)) + (\alpha - \bar{\alpha})$ is equal to (1), the entire ring, which implies $\hat{J}^R_{(\mathrm{n}X)} = (1)$ and $(\mathrm{n}X)$ is the trivial algebra. The same argument applies with respect to β. This is a useful restriction on relations inside $\hat{J}^R_{(\mathrm{n}X)}$.

In the following Section 4.5, we find weak covers of the Norton-Sakuma algebras $(\mathrm{n}X)$, recorded in Table 4.1, for $(\mathrm{n}X)$ one of $(3A), (4A), (4B), (5A), (6A)$. The relation, arising from the fusion rules, in the coming Lemma 4.4.4 is conjectured to be 0; this would be clear if the relevant terms are linear independent *in the cover*. If an argument to this effect can be supplied, as we expect, it would show that the weak covers we find are in fact 'full' covers, except for $(3A)$ which requires an extra assumption. In summary, we prove

4.4.2 Theorem. *Suppose that the underlying everywhere faithful ring is an associative algebra over R_0 from* (4.18)*. The weak covers of the Norton-Sakuma algebras $(\mathrm{n}X)$, for $n \geq 4$, and $(3A)$ are given by Table 4.2. The algebras are Frobenius and satisfy a global 6-transposition property.* □

Alg.	$\lvert\rho\rvert$	dim.	Parameters	Quotients	page
$(3A'_{\alpha,\beta})$	3	4	$\alpha \neq \frac{1}{2}$, $\lambda_1 = \lambda_2 = \frac{3\alpha^2+\alpha(3\beta-1)-2\beta}{4(2\alpha-1)}$	Lemma 4.5.18	149
$(4A_\beta)$	4	5	$\alpha = \frac{1}{4}$, $\beta \neq \frac{1}{8}, \frac{1}{16}$, $\lambda_1 = \beta$, $\lambda_2 = 0$	$\beta = \frac{1}{2}$	147
$(4B_\alpha)$	4	5	$\alpha \notin$ (4.102), $\beta = \frac{\alpha^2}{2}$, $\lambda_1 = \frac{\alpha^2}{4}$, $\lambda_2 = \frac{\alpha}{2}$	$\alpha = -1$	146
$(5A_\alpha)$	5	6	$\beta = \frac{1}{8}(5\alpha - 1)$, $\lambda_1 = \lambda_2 = \frac{3(5\alpha-1)}{32}$	$\alpha = \frac{7}{3}$	136
$(6A_\alpha)$	6	8	$\alpha \neq -4 \pm 2\sqrt{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}$, $\beta = \frac{\alpha^2}{4(1-2\alpha)}$ $\lambda_1 = \frac{-\alpha^2(3\alpha-2)}{16(2\alpha-1)^2}$, $\lambda_2 = \frac{\alpha(21\alpha^2-18\alpha+4)}{16(2\alpha-1)^2}$	$\alpha = \frac{3}{2}, \frac{4}{7}, \frac{1\pm\sqrt{97}}{24}$	139

Table 4.2: The covers of the Norton-Sakuma algebras

We now provide the eigenvectors and fusion rules used to find the ideals $\bar{I}^R_{(\mathrm{n}X)}, \bar{J}^R_{(\mathrm{n}X)}$.

4.4.3 Lemma. *In the algebra A_R (coming from Theorem 4.3.7),*

$$
\begin{aligned}
&A_1^{a_0} = \langle a_0 \rangle. \\
&A_0^{a_0} \textit{ contains } z = ((1-\alpha)\lambda_1 - \beta)a_0 + \frac{1}{2}(\alpha-\beta)(a_1 + a_{-1}) - s, \\
&\qquad zz = -\frac{1}{4}\frac{1}{\alpha - 2\beta}\Big(\beta\alpha(\alpha-\beta)^2(a_{-2}+a_2) \\
&\qquad\qquad + \alpha(\alpha-\beta)(2\lambda_1^f - \alpha - 4\beta\lambda_1 + 4\beta^2 - 2\alpha\lambda_1^f + 2\alpha\beta)(a_{-1}+a_1) \\
&\qquad\qquad + 2\alpha(-2\lambda_1\lambda_1^f + 2\beta\lambda_1^f + 2\beta\lambda_1 - 2\beta^2 + \beta^2\lambda_2 + 4\beta^2\lambda_1 - 3\beta^3 + 2\alpha\lambda_1\lambda_1^f + 2\alpha\lambda_1^2 \\
&\qquad\qquad\qquad - \alpha\beta\lambda_2 - 2\alpha\beta\lambda_1^f - 4\alpha\beta\lambda_1 - 2\alpha^2\lambda_1 + 3\alpha^2\beta - 4\alpha\beta\lambda_1^2 + 2\alpha^2\lambda_1^2)a_0 \\
&\qquad\qquad + 4\alpha(\lambda_1^f - \beta + 2\beta\lambda_1 - \alpha\lambda_1^f - 2\alpha\lambda_1 - 2\alpha\beta + 2\alpha^2)s \\
&\qquad\qquad + 2\beta\alpha(\alpha-\beta)s_2 - 2\alpha(\alpha-\beta)(\alpha-2\beta)s_2^f\Big), \textit{ and} \\
&\qquad z_2 = ((1-\alpha)\lambda_2 - \beta)a + \frac{1}{2}(\alpha-\beta)(a_2+a_{-2}) - s_2. \\
&A_\alpha^a \textit{ contains } x = (\beta - \lambda_1)a_0 + \frac{1}{2}\beta(a_1+a_{-1}) + s, \quad x_2 = (\beta-\lambda_2)a_0 + \frac{1}{2}\beta(a_2+a_{-2}) + s_2. \\
&A_\beta^a \textit{ contains } y = a_1 - a_{-1}, \quad y_2 = a_2 - a_{-2}.
\end{aligned} \tag{4.74}
$$

Proof. Since we have the multiplication table, it is a routine calculation to check that the vectors listed are eigenvectors of the appropriate eigenvalues. That a_0 spans $A_1^{a_0}$ comes from the assumption of primitivity. □

Similarly, eigenvectors of a_1 can be easily calculated by interchanging the rôles of a_0 and a_1 in the above.

4.4.4 Lemma. *The coefficient of a_{-1} in $a_0(z_2z_2)$ is*

$$
\begin{aligned}
&\frac{-2}{\beta(\alpha-4\beta)^3} \cdot ((\alpha-1)\lambda_1 + \beta(2\beta - 2\alpha + 1)) \\
&\qquad \cdot (2(\alpha(\alpha-1) - 6\alpha\beta + 4\beta)\lambda_1 + (10\beta+1)\alpha\beta - 2\beta^2(2+3\beta)) \\
&\qquad \cdot (4(2\alpha-1)\lambda_1^2 - 2(\alpha^2 + 6\beta\alpha - 4\beta)\lambda_1 - \alpha\beta\lambda_2 + 2\alpha^2\beta + 4(\alpha-1)\beta^2).
\end{aligned} \tag{4.75}
$$

The other nonzero coefficients are those of a_{-2}, a_0, a_1, a_2.

Proof. We establish the formula by direct computation using the results of Section 4.3. That the coefficient is unique follows by our assumption that we are working over an everywhere faithful ring. □

Finally, to simplify our calculations, we introduce an assumption on the coefficients.

We will from now on, for simplicity of calculation, assume that $\lambda_1 = \lambda_1^f$ and $\lambda_2 = \lambda_2^f$. This assumption is realised in at least three different situations:

4.4.5 Lemma. *Suppose that* $a_0 = a_{2n+1}$ *for some* n*. Then* $\lambda^{a_i}(a_j) = \lambda^{a_j}(a_i)$ *for all* i, j*, so that* $\lambda_i = \lambda_i^f$*, and* $s_2 = s_2^f$*.*

Proof. The group $T = \langle \tau(a_0), \tau(a_1) \rangle$ acts transitively on pairs $\{a_i, a_j\}$ with $i - j \equiv_2 0$ and on pairs $\{a_i, a_j\}$ with $i - j \equiv_2 1$. If $a_0 = a_{2n+1}$, then $\tau(a_{n+1}) \in T$ swaps $a_0 = a$ and $a_1 = b$, and $\{a_0, a_2\}$ is swapped with $\{a_1, a_{-1}\}$. Therefore using Lemma 4.1.2, $\lambda^{a_i}(a_j) = \lambda^{a_j}(a_i)$. Also $\tau(a_{n+1})$ interchanges s_2, s_2^f, but T is generated by $\tau(a_0), \tau(a_1)$ which both fix s_2, so $\tau(a_{n+1})$ must fix s_2 and therefore $s_2 = s_2^f$. □

Secondly, if the algebra is Frobenius, then $\lambda^a(b) = \frac{(a,b)}{(a,a)}a$ for any primitive nonsingular a; in particular, if $(a_0, a_0) = (a_1, a_1)$ then $\lambda_1 = \lambda_1^f$ and furthermore, by the same argument as Lemma 7.4 in [HRS15], $\lambda_2 = \lambda_2^f$. (In fact, it can be shown that $\lambda^{a_i}(a_j)$ depends only on $|i - j|$.)

Finally, Theorem 4.1.1 which classifies the $\Phi(\alpha, \beta)$-dihedral algebras in a degenerate case, shows that if $\lambda^a(b) \neq \lambda^b(a)$ then $b' = \mathrm{id} - b$ is a $\Phi(\alpha)$-axis, a and b' generate the algebra and $\lambda^a(b') = \lambda^{b'}(a)$. In other words, the property $\lambda^a(b) = \lambda^b(a)$ follows from the axioms in $\Phi(\alpha)$-dihedral algebras if one allows the swap b' for b.

We are not sure if similarly $\lambda^a(b) = \lambda^b(a)$ is a consequence of the axioms for Ising-axial algebras, but no counterexamples are known.

4.5 The covers of $(\mathrm{n}X)$

This final section provides the computations to determine by via ideals the axial covers, that is, in our sense the largest generalisations, of the Norton-Sakuma algebras, which, for $(\alpha, \beta) = (1/4, 1/32)$, occur in the Griess algebra.

Recall that we assume that each of the Norton-Sakuma algebras $(\mathrm{n}X)$ from Table 4.1 is defined over a ring R, which is an associative R_0-algebra for

$$R_0 = \mathbb{Z}[1/2, \alpha, \beta, \alpha^{-1}, \beta^{-1}, (\alpha-\beta)^{-1}, (\alpha-2\beta)^{-1}, (\alpha-4\beta)^{-1}, \lambda_1, \lambda_2]. \tag{4.76}$$

We first consider $(5A)$. Only the first factor in (4.75) is 0 in $(5A)$ and therefore this factor is in $J_{(5A)}$. Thus in $(R_0 \otimes_{\mathbb{Z}} R)/J^R_{(5A)}$, as $\alpha - 1$ is invertible,

$$\lambda_1 = \frac{(2\alpha - 2\beta - 1)\beta}{\alpha - 1}. \tag{4.77}$$

After substituting (4.77), the only terms with nonzero coefficients in $a_0(z_2z_2)$ are a_{-2}, a_0, a_2, and the coefficient of a_0 in $a_0(z_2z_2)$ is

$$\begin{aligned}&\frac{(\alpha-2\beta)^2}{(\alpha-1)^3(\alpha-4\beta)^2(\alpha-\beta)}\\ &\quad\cdot(2\alpha-1)(4\alpha^2-5(2\beta+1)\alpha+6\beta+1)((\alpha-1)\lambda_2-2\alpha\beta+\beta(2\beta+1))\\ &\quad\cdot(1+\lambda_2+4\beta-4\alpha-8\beta^2-2\alpha\lambda_2+3\alpha^2-16\beta^3+24\alpha\beta^2+\alpha^2\lambda_2-8\alpha^2\beta).\end{aligned} \tag{4.78}$$

We deduce several possibilities. We cannot have a nontrivial relation among the elements a_{-2}, a_0, a_2 in the cover of $(5A)$, because they are linearly independent in $(5A)$, unless the coefficients in the relation all lie in the ideal. Among the coefficients, the possibilities are as follows. Firstly, $\beta = \frac{4\alpha^2-5\alpha+1}{2(5\alpha-3)}$. Secondly, $\lambda_2 = \frac{(2\alpha-2\beta-1)\beta}{\alpha-1} = \lambda_1$. Thirdly, $\lambda_2 = \frac{8\alpha^2\beta-24\alpha\beta^2+16\beta^3-3\alpha^2+8\beta^2+4(\alpha-\beta)-1}{(\alpha-1)^2}$. In $(5A)$, only the second possibility is satisfied, so in the cover of $(5A)$, we have $\lambda_2 = \frac{(2\alpha-2\beta-1)\beta}{\alpha-1} = \lambda_1$. Then the coefficient of a_2 in $a_0(z_2 z_2)$ becomes

$$\begin{aligned}&\frac{1}{4}\frac{-\beta(2\beta-1)(3\alpha-4\beta-1)(-1-8\beta+5\alpha)(2\beta-\alpha-3\alpha\beta+\alpha^2)}{(\alpha-1)^3(\alpha-4\beta)^2}\\ &\quad\cdot(-2\beta+\alpha-8\beta^2+11\alpha\beta-4\alpha^2+16\alpha\beta^2-13\alpha^2\beta+3\alpha^3).\end{aligned} \tag{4.79}$$

We again have new possibilities.

If $\beta = \frac{1}{4}(3\alpha-1)$, then $\alpha \neq 1, 0, \beta, 2\beta, 4\beta$ and $\beta \neq 0, 1$ means $\alpha \neq -1, -\frac{1}{3}, 0, \frac{1}{2}, 1, \frac{5}{3}$. This leads to $(3C_\beta)$, as it turns out: after substitution, $\lambda_1 = \frac{1}{8}(3\alpha-1) = \frac{1}{2}\beta$, and by the fusion rules $A_\alpha^{a_0}$ is killed. But we do not pursue this, since it is not satisfied by $(\alpha, \beta) = (1/4, 1/32)$.

The possibilities $\beta = \frac{\alpha(1-\alpha)}{2-3\alpha}$ and $(16\alpha-8)\beta^2+(-13\alpha^2+11\alpha-2)\beta+3\alpha^3-4\alpha^2+\alpha = 0$ are not realised in $(5A)$.

If $\beta = \frac{1}{8}(5\alpha-1)$, then $\alpha \neq 1, 0, \beta, 2\beta, 4\beta$ and $\beta \neq 0, 1$ means $\alpha \neq -\frac{1}{3}, 0, \frac{1}{5}, \frac{1}{3}, 1, \frac{9}{5}$. This is satisfied by $(\alpha, \beta) = (1/4, 1/32)$. We now compute the action of a_2 on the subspace Q spanned by

$$g = s_2 - s_2^f, \quad h = \beta(a_{-2}+a_{-1}+a_0+a_1+a_2) + \frac{2}{\beta}s + \frac{1}{\beta}(s_2+s_2^f). \tag{4.80}$$

This Q is fixed by $\mathrm{ad}(a_2)$, $a_2 \notin Q$, and

$$N = \mathrm{ad}(a_2)|_Q = \frac{1}{8(\alpha-1)}\begin{pmatrix} 9\alpha^2-1 & 9\alpha^2-1 \\ \frac{3-27\alpha+57\alpha^2-17\alpha^3}{1-3\alpha} & -\alpha^2-8\alpha+1 \end{pmatrix}. \tag{4.81}$$

Therefore Q must decompose into a direct sum of $0, \alpha$ or β-eigenvectors for a_2. Now suppose that μ is an eigenvalue of N, so that $\det(N-\mu I_2) = 0$. That is,

$$\frac{1}{32}\frac{1}{(1-\alpha)}(1-4\alpha+32\mu^2-32\alpha\mu-11\alpha^2-32\alpha\mu^2+32\alpha^2\mu+30\alpha^3) = 0. \tag{4.82}$$

We can substitute $\mu = 0, \alpha, \beta = \frac{1}{8}(5\alpha-1)$ respectively and solve the resulting equation. If $\mu = 0$, then the numerator factors as $(\alpha-1)(2\alpha-1)(3\alpha+1)(5\alpha-1)$; the only legitimate possibility is $\alpha = \frac{1}{2}$ with $\beta = \frac{3}{16}$, which will be a special case for us.

If $\mu = \alpha$, the numerator becomes $-60\alpha^4+74\alpha^3-5\alpha^2-10\alpha+2$, which is irreducible.

If $\mu = \frac{1}{8}(5\alpha-1) = \beta$, the numerator becomes $-\frac{1}{64}(5\alpha-1)(768\alpha^3-832\alpha^2-69\alpha+129)$.

Note that $\alpha = \frac{1}{5}$ was already ruled out. So we are left with the additional assumptions that

$$(-60\alpha^4 + 74\alpha^3 - 5\alpha^2 - 10\alpha + 2)(768\alpha^3 - 832\alpha^2 - 69\alpha + 129) \neq 0. \tag{4.83}$$

When additionally $(\alpha, \beta) \neq (1/2, 3/16)$, we have to quotient by Q, which gives the multiplication in Table 4.3:

Description	Products
$a_{-2}, a_{-1}, a_0, a_1, a_2, s$	$a_i a_{i+2} = \frac{1}{16}(5\alpha - 1)(a_i + a_{i+2} - a_{i+1} - a_{i+3} - a_{i+4}) - s$
$a_i = a_{i \bmod 5}$	$a_i s = \frac{1}{64}(1 - 5\alpha)(1 + 3\alpha)a_i - \frac{1}{2^7}(1 + 3\alpha)(1 - 5\alpha)(a_{i-1} + a_{i+1})$ $+\frac{1}{8}(1 + 3\alpha)s$
	$ss = \frac{1}{2^{11}}(7\alpha - 3)(1 + 3\alpha)(1 - 5\alpha)(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$ $+\frac{5}{2^7}(1 - 5\alpha)(1 + 3\alpha)s$
	$(a_i, a_j) = \frac{3}{32}(5\alpha - 1)$

Table 4.3: The algebra $(5A_\alpha)$

4.5.1 Theorem. *The algebra* $(5A_\alpha)$ *over* $R = \mathbb{Z}[1/2, \alpha]$ *with basis* $a_{-2}, a_{-1}, a_0, a_1, a_2, s$ *and multiplication from Table 4.3 is* $\Phi(\alpha, \frac{1}{8}(5\alpha - 1))$*-dihedral, and is the cover of* $(5A)$ *when additionally* (4.83) *is invertible.*

Proof. A is generated by a_0, a_1, so we only need to check the fusion rules for a_0, a_1. We do this in [GAP], using Lemma 4.4.3 and the obvious symmetry between the a_i.

We check manually, that is, in [GAP], that in the case $(\alpha,\beta) = (1/2, 3/16)$, the vector space Q is also killed and the resulting algebra has the same presentation. □

4.5.2 Lemma. *The algebra* $(5A_\alpha)$ *is Frobenius; the Frobenius form is degenerate for* $\alpha = \frac{7}{3}$*, and positive-definite if realised over* $R = \mathbb{R}$ *for* $\frac{1}{5} < \alpha < \frac{7}{3}$*.*

Proof. We check by direct computation in [GAP] that the form on $(5A_\alpha)$ induced as a quotient of the form in Section 4.3 is Frobenius. We also use [GAP] to calculate the Gram matrix, the determinant of which is

$$\frac{-5^6}{2^{36}}(3\alpha-7)^5(3\alpha+1)^2(5\alpha-1). \tag{4.84}$$

This is positive in $\mathbb{R}$ exactly when $(3\alpha-7)(5\alpha-1) < 0$; the root $(5\alpha-1)$ is not admissible since $\beta = 0$ when $\alpha = \frac{1}{5}$, and $\beta = \alpha$ for $\alpha = -\frac{1}{3}$ ruling out $3\alpha+1 = 0$. The remaining root is $(\alpha,\beta) = (7/3, 4/3)$. □

4.5.3 Lemma. *We have* $|\tau(a_0)\tau(a_1)| = 5$*.*

Proof. $\tau(a_0)$ and $\tau(a_1)$ act as the permutation matrices corresponding to $(1,5)(2,4)(3)(6)$ and $(1,2)(3,5)(4)(6)$ on $(5A_\alpha)$, respectively. Since a_0 and a_1 are in the same orbit under T, we have that $|\tau(a_0)\tau(a_1)| = 5$ everywhere by Proposition 4.1.3. □

We now consider $(6A)$. Only the second factor in (4.75) is 0 in $(6A)$:

$$2(\alpha(\alpha-1) - 6\alpha\beta + 4\beta)\lambda_1 + (10\beta+1)\alpha\beta - 2\beta^2(2+3\beta) = 0. \tag{4.85}$$

Since $\dim_{\mathbb{Q}}(6A)_{\mathbb{Q}} = 8$, the key relations for the cover come from the ring, that is, $\bar{I}^R_{(6A)} = \bar{J}^R_{(6A)}A_R$. Suppose that the coefficient of λ_1 in (4.85) is 0, that is, $\alpha^2 - 6\alpha\beta - \alpha + 4\beta = 2(2-3\alpha)\beta + \alpha(1-\alpha) = 0$. As $\alpha = \frac{2}{3}$ is not a solution to this equation, we can rearrange to find

$$\beta = \frac{\alpha(\alpha-1)}{2(2-3\alpha)} \tag{4.86}$$

(which is not in $\bar{J}^R_{(6A)}$) and substituting this into (4.85),

$$7\alpha^2 - 12\alpha + 4 = 0. \tag{4.87}$$

This irreducible polynomial is coprime to $\alpha - \frac{1}{4}$, so that $(\alpha - {}^1\!/\!{}_4)(7\alpha^2 - 12\alpha + 4) = (1)$ and therefore if $7\alpha^2 - 12\alpha + 4 \in \bar{J}^R_{(6A)}$ then $J^R_{(6A)} = \bar{J}^R_{(6A)} + (\alpha - {}^1\!/\!{}_4) + (\beta - {}^1\!/\!{}_{32}) = (1)$ is not maximal. So α is not a root of $7\alpha^2 - 12\alpha + 4$, that is, $\alpha \neq \frac{1}{7}(4 \pm \sqrt{2})$.

Therefore $\alpha^2 - 6\alpha\beta - \alpha + 4\beta \neq 0$ and

$$\lambda_1 = \frac{(10\beta^2 + \beta)\alpha - (4\beta + 6)\beta^2}{2(\alpha^2 - 6\alpha\beta - \alpha + 4\beta)}. \tag{4.88}$$

After making the substitution, we again calculate $a_0(z_2z_2)$, and find that the coefficient of a_2 is

$$\frac{\beta(\alpha-\beta)(1-2\beta)}{4(\alpha-4\beta)(\alpha^2-(6\beta+1)\alpha+4\beta)^2}\cdot \Big(-2\alpha(\alpha^2-(6\beta+1)\alpha+4\beta)^2\lambda_2 + 3\alpha^6 - (3\beta+4)\alpha^5 - (60\beta^2+6\beta-1)\alpha^4 + \beta(2\beta+1)(98\beta+9)\alpha^3 - 2\beta(88\beta^3 + 2^2 3\cdot 11\beta^2 + 30\beta + 1)\alpha^2 + 8\beta^2(8\beta^3 + 20\beta^2 + 12\beta + 1)\alpha - 32\beta^5 - 32\beta^4 - 8\beta^3\Big). \tag{4.89}$$

We deduce an expression for λ_2 (its coefficient being not 0 when $\alpha \neq \frac{1}{7}(4 \pm \sqrt{2})$). Substituting, we find that the coefficient of a_2 in $a_0(xx_2)$ is

$$\frac{-\beta^2(2\beta-1)\alpha(\alpha-2\beta)(\alpha^2+8\beta\alpha-4\beta)}{4(\alpha-4\beta)(\alpha^2-(6\beta+1)\alpha+4\beta)}. \tag{4.90}$$

Here $\beta = \frac{\alpha^2}{4(1-2\alpha)}$ is the only zero factor in $J^R_{(6A)}$. After specialising, we find no more relations. We get

4.5.4 Theorem. *The algebra* $(6A_\alpha)$ *over* $R = \mathbb{Z}[{}^1\!/\!{}_2, \alpha, \alpha^{-1}, (1-2\alpha)^{-1}, (2-5\alpha)^{-1}]$*, with basis* $a_{-2}, a_{-1}, a_0, a_1, a_2, s, s_2, s_2^f$ *and multiplication from Table 4.4 is a* $\Phi(\alpha, \frac{\alpha^2}{4(1-2\alpha)})$*-dihedral algebra and the cover of* $(6A)$ *when additionally* $7\alpha^2 - 12\alpha + 4$ *is invertible.*

Proof. We only need to check the fusion rules for a_0, a_1. We do this in [GAP], using Lemma 4.4.3. □

Description	Products
$a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, s, \bar{s}_2$	$a_i a_{i+2} = \frac{1}{8}\frac{\alpha(3\alpha-1)}{(2\alpha-1)} a_{i+4} - \frac{1}{8}\frac{\alpha(3\alpha-1)}{(2\alpha-1)}(a_{i+1} + a_{i+3} + a_{i+5})$
$\bar{s}_2 = \frac{1}{2}(s_2 + s_2^f)$	$+\frac{1}{8}\frac{(\alpha-1)\alpha}{(2\alpha-1)}(a_i + a_{i+2}) + \bar{s}_2$
$a_i = a_{i \bmod 6}$	$a_i a_{i+3} = \frac{1}{2}\frac{\alpha(3\alpha-1)}{(5\alpha-2)}(a_i + a_{i+3}) - \frac{1}{2}\frac{\alpha(2\alpha-1)}{(5\alpha-2)}(a_{i+1} + a_{i+2} + a_{i+4} + a_{i+5})$
$\langle a_i, a_{i+3}\rangle \cong (3C_\alpha)$	$+4\frac{(2\alpha-1)}{\alpha} s - 4\frac{2\alpha-1}{5\alpha-2}\bar{s}_2$
$\langle a_i, a_{i+2}\rangle \cong (3A_{\alpha,\beta})$	$a_i s = -\frac{1}{32}\frac{\alpha^3(9\alpha-4)}{(2\alpha-1)^2}(a_{i-1} + a_{i+1}) - \frac{1}{16}\frac{\alpha^2(3\alpha-2)}{(2\alpha-1)} a_i + \frac{1}{4}\frac{\alpha(9\alpha-4)}{(2\alpha-1)} s$
	$a_i \bar{s}_2 = \frac{1}{32}\frac{\alpha^2(9\alpha-4)}{(5\alpha-2)}(a_{i+4} + a_{i+2}) - \frac{1}{32}\frac{\alpha^2(3\alpha-1)(43\alpha^2-37\alpha+8)}{(2\alpha-1)^2(5\alpha-2)}(a_{i-1} + a_{i+1})$
	$-\frac{1}{32}\frac{\alpha(87\alpha^3-3^3 5\alpha^2+60\alpha-8)}{(2\alpha-1)(5\alpha-2)} a_i - \frac{1}{32}\frac{\alpha^2(3\alpha-1)(9\alpha-4)}{(2\alpha-1)(5\alpha-2)} a_{i+3}$
	$+\frac{1}{4}\frac{(3\alpha-1)(5\alpha-2)}{(2\alpha-1)} s + \frac{1}{4}\frac{\alpha(9\alpha-4)}{(5\alpha-2)}\bar{s}_2$
	$ss = -\frac{1}{2^8}\frac{\alpha^4(3\alpha-1)(9\alpha-4)^2}{(2\alpha-1)^3(5\alpha-2)}(a_i + a_{i+1} + a_{i+2} + a_{i+3} + a_{i+4} + a_{i+5})$
	$+\frac{1}{16}\frac{\alpha^2(39\alpha^2-22\alpha+2)}{(2\alpha-1)^2} s - \frac{1}{32}\frac{\alpha^4(9\alpha-4)}{(2\alpha-1)^2(5\alpha-2)}\bar{s}_2$
	$s\bar{s}_2 = -\frac{1}{2^8}\frac{\alpha^3(3\alpha-1)(5\alpha-2)(9\alpha-4)}{(2\alpha-1)^3}(a_i + a_{i+1} + a_{i+2} + a_{i+3} + a_{i+4} + a_{i+5})$
	$+\frac{3}{16}\frac{\alpha^2(3\alpha-1)(9\alpha-4)}{(2\alpha-1)^2} s + \frac{1}{32}\frac{\alpha^2(3\alpha-2)(5\alpha-2)}{(2\alpha-1)^2}\bar{s}_2$
	$\bar{s}_2\bar{s}_2 = \frac{1}{32}\frac{\alpha(3\cdot 173\alpha^4-2^6 13\alpha^3+2\cdot 3\cdot 5\cdot 17\alpha^2-2^4 3^2\alpha+16)}{(2\alpha-1)^2(5\alpha-2)}\bar{s}_2$
	$-\frac{1}{2^8}\frac{\alpha^2(3\alpha-1)}{(2\alpha-1)^3(5\alpha-2)}(3\cdot 7\cdot 29\alpha^4 - 2^3 131\alpha^3 + 2^2 173\alpha^2 - 2^4 13\alpha + 24)\cdot$
	$(a_i + a_{i+1} + a_{i+2} + a_{i+3} + a_{i+4} + a_{i+5}) + \frac{3}{16}\frac{\alpha(3\alpha-1)^2(5\alpha-2)}{(2\alpha-1)^2} s$
	$(a_i, a_{i+1}) = \frac{1}{16}\frac{\alpha^2}{(2\alpha-1)^2}(2 - 3\alpha)$

Table 4.4: The algebra $(6A_\alpha)$

4.5.5 Lemma. *The algebra* $(6A_\alpha)$ *is Frobenius; the form is degenerate for* $\alpha = \frac{3}{2}, \frac{4}{7}, \frac{1}{24}(1 \pm \sqrt{97})$*, and positive-definite over* $\mathbb{R}$ *for* $\alpha \in (\frac{1}{2}, \frac{4}{7}) \cup (\frac{2}{3}, 1) \cup (\frac{1}{24}(1-\sqrt{97}), \frac{1}{24}(1+\sqrt{97}))$.

Proof. We check by direct computation in [GAP] that the form on $(5A_\alpha)$ induced as a quotient of the form in Section 4.3 is Frobenius. We also use [GAP] to calculate the Gram matrix, the determinant of which is

$$\frac{-\alpha^8}{2^{31}(2\alpha-1)^{17}}(\alpha-1)^3(3\alpha-2)(3\alpha-1)^2(5\alpha-2)^2(7\alpha-4)^5(\alpha^2+4\alpha-2)^4(12\alpha^2-\alpha-2). \quad \textbf{(4.91)}$$

This is positive exactly when $(2\alpha-1)(\alpha-1)(3\alpha-2)(7\alpha-4)(12\alpha^2-\alpha-2) < 0$. The roots of the equation which are not already ruled out in Theorem 4.5.4 correspond to (α, β) being $(3/2, -9/32)$, $(4/7, -4/7)$ and $\alpha = \frac{1}{24}(1 \pm \sqrt{97})$. □

4.5.6 Lemma. *On* $(6A_\alpha)$, $|\tau(a_0)\tau(a_1)| = 3$*, the flip is an automorphism and* $|\tau(a_0)\tau(a_1)| \leq 6$ *in any larger algebra.*

Proof. On $(6A_\alpha)$, $\tau(a_0)$ is the permutation matrix of $(1,5)(2,4)(3)(6)(7)(8)$ using our normal basis, and $\tau(a_1)$ fixes a_1 and a_{-2}, s, s_2, s_2^f, but a_{-1} is mapped to

$$a_3 = a_{-2} + a_0 + a_2 - a_{-1} - a_1 + 4\frac{(1-2\alpha)}{\alpha(3\alpha-1)}(s_2 - s_2^f), \quad \textbf{(4.92)}$$

and a_0 is swapped with a_2. Therefore, for $\kappa = 4\frac{1-2\alpha}{\alpha(3\alpha-1)}$,

$$\tau(a_0)\tau(a_1) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 & \kappa & -\kappa \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \textbf{(4.93)}$$

and $(\tau(a_0)\tau(a_1))^3$ is the identity matrix. By Proposition 4.1.3, $\tau(a_0)\tau(a_1)$ has order at most $|a_0^T \cup a_1^T| = 6$. □

We now consider the two cases $(4A), (4B)$. Only the third factor in (4.75) is 0 in $(4A)$ and in $(4B)$ and therefore, in these cases, as $\alpha - 1$ is invertible,

$$\lambda_2 = \frac{1}{\alpha\beta}(4(2\alpha-1)\lambda_1^2 - 2(\alpha^2+6\beta\alpha-4\beta)\lambda_1 + 2\alpha^2\beta + 4(\alpha-1)\beta^2). \tag{4.94}$$

4.5.7 Lemma. *If* $(\alpha, \beta, \lambda_1)$ *is not a root of* (4.97)*, then* $a_0 = a_4$.

Proof. Our aim is first to show that $a_2 = a_{-2}$, by showing that

$$k = a_2 - a_{-2} \in A_{1,0,\alpha}^{a_0} \oplus \langle kk \rangle, \tag{4.95}$$

that is, we find constants $\kappa, \kappa_1, \kappa_z, \kappa_{zz}, \kappa_{z_2}, \kappa_x, \kappa_{x_2}$ such that

$$\kappa k = \kappa_1 a_0 + \kappa_z z + \kappa_{zz} zz + \kappa_{z_2} z_2 + \kappa_x x + \kappa_{x_2} x_2 + kk, \tag{4.96}$$

which implies that $\kappa k = 0$, because $k \in A_\beta^{a_0}$ so $kk \in A_{1,0,\alpha}^{a_0}$ along with the other summands, and the diagonalisability of a_0 implies that $A_\beta^{a_0} \cap A_{1,0,\alpha}^{a_0} = 0$.

We multiply kk using the multiplication table. That suitable $\kappa, \kappa_1, \kappa_z, \kappa_{zz}, \kappa_{z_2}, \kappa_x, \kappa_{x_2}$ exist follows from the fact that the coefficients of a_{-1} and a_1 in $a_2 a_{-2}$ are equal, so that there are only 7 parameters; now the chosen set is linearly independent in those seven variables. Even though the actual equations are lengthy, the working is straightforward:

$$\kappa_{zz} = -\frac{kk|_{s_2^f}}{zz|_{s_2^f}},$$

$$(kk)' = kk + \kappa_{zz} zz,$$

$$\kappa_{z_2} - \kappa_{x_2} = (kk)'|_{s_2},$$

$$\frac{1}{2}(\alpha-\beta)\kappa_{z_2} + \frac{1}{2}\beta\kappa_{x_2} = -\frac{1}{2}((kk)'|_{a_{-2}} + (kk)'|_{a_2}),$$

$$\kappa_{x_2} = \frac{2}{\alpha}(-\frac{1}{2}((kk)'|_{a_{-2}} + (kk)'|_{a_2}) - \frac{1}{2(\alpha-\beta)}(kk)'|_{s_2}),$$

$$\kappa_{z_2} = \kappa_{x_2} + (kk)'|_{s_2},$$

$$(kk)'' = kk + \kappa_{z_2} z_2 + \kappa_{x_2} x_2,$$

$$\kappa_z - \kappa_x = (kk)''|_s,$$

$$\frac{1}{2}(\alpha-\beta)\kappa_z + \frac{1}{2}\beta\kappa_x = -\frac{1}{2}((kk)''|_{a_{-1}} + (kk)''|_{a_1}),$$

$$\kappa_x = \frac{2}{\alpha}(-\frac{1}{2}((kk)''|_{a_{-1}} + (kk)''|_{a_1}) - \frac{1}{2(\alpha-\beta)}(kk)''|_s),$$

$$\kappa_z = \kappa_x + (kk)''|_s,$$

$$(kk)''' = (kk)'' + \kappa_z z + \kappa_x x,$$

$$\kappa_1 = (kk)'''|_{a_0}.$$

Note that the only division is by $zz|_{s_2^f} = -2\alpha(\alpha-\beta)(\alpha-2\beta)$, which is invertible by assumption. Then $\kappa =$

$$\frac{-2(-2(\alpha^2-6\alpha\beta-\alpha+4\beta)\lambda_1+10\alpha\beta^2-4\beta^3+\alpha\beta-6\beta^2)}{\alpha\beta^2(\alpha-\beta)^2(\alpha-2\beta)(\alpha-4\beta)^3}\cdot$$

$$\Big(-(8(\alpha-1)(\alpha-2\beta)^2(3\alpha^2-8\alpha\beta-\alpha+4\beta)\lambda_1^2-2(4\alpha^6-\alpha^5\beta-96\alpha^4\beta^2+2^2 73\alpha^3\beta^3$$
$$-2^8\alpha^2\beta^4-6\alpha^5+7\alpha^4\beta+2\cdot3\cdot19\alpha^3\beta^2-2^7 3\alpha^2\beta^3+2^4 23\alpha\beta^4+2\alpha^4-2\alpha^3\beta-40\alpha^2\beta^2$$
$$+2^7\alpha\beta^3-2^7\beta^4)\lambda_1+8\alpha^6\beta-22\alpha^5\beta^2-20\alpha^4\beta^3+2^2 3\cdot11\alpha^3\beta^4-2^3 3\cdot5\alpha^2\beta^5-32\alpha\beta^6$$
$$-8\alpha^5\beta+5\alpha^4\beta^2+2^2 31\alpha^3\beta^3-2^4 23\alpha^2\beta^4+2^5 11\alpha\beta^5+2\alpha^4\beta+4\alpha^3\beta^2-56\alpha^2\beta^3+2^7\alpha\beta^4$$
$$-2^7\beta^5)\lambda_2+16(\alpha-1)(\alpha^4-8\alpha^3\beta+12\alpha^2\beta^2-\alpha^3+14\alpha^2\beta-16\alpha\beta^2-4\alpha\beta+4\beta^2)\lambda_1^3$$
$$-4(3\alpha^5\beta-60\alpha^4\beta^2+2^2 3\cdot11\alpha^3\beta^3-48\alpha^2\beta^4+3\alpha^5-17\alpha^4\beta+2\cdot103\alpha^3\beta^2-2^3 3\cdot13\alpha^2\beta^3$$
$$+64\alpha\beta^4-5\alpha^4+28\alpha^3\beta-2^2 53\alpha^2\beta^2+2^2 5\cdot11\alpha\beta^3-16\beta^4+2\alpha^3-12\alpha^2\beta+64\alpha\beta^2-48\beta^3)\lambda_1^2$$
$$-2\beta(14\alpha^5\beta-8\alpha^4\beta^2-56\alpha^3\beta^3+32\alpha^2\beta^4-11\alpha^5+33\alpha^4\beta-2\cdot3^2 17\alpha^3\beta^2+2^3 59\alpha^2\beta^3-$$
$$2^7\alpha\beta^4+14\alpha^4-80\alpha^3\beta+2^6 7\alpha^2\beta^2-2^6 7\alpha\beta^3+64\beta^4-6\alpha^3+40\alpha^2\beta-2^3 19\alpha\beta^2+96\beta^3)\lambda_1$$
$$+2\alpha^5\beta^2-28\alpha^4\beta^3-60\alpha^3\beta^4+2^3 17\alpha^2\beta^5-32\alpha\beta^6+5\alpha^4\beta^2-48\alpha^3\beta^3+2^2 7\cdot11\alpha^2\beta^4$$
$$-2^6 5\alpha\beta^5+64\beta^6-4\alpha^3\beta^2+32\alpha^2\beta^3-2^4 7\alpha\beta^4+64\beta^5\Big) \quad \textbf{(4.97)}$$

is an irreducible polynomial. So assuming that (α,β,λ_1) is not a root of (4.97), we have that $a_2 = a_{-2}$.

By applying $\tau(a_1)$ to both sides, it follows that $a_2^{\tau(a_1)} = a_{-2}^{\tau(a_1)}$, that is, $a_0 = a_4$. □

Now $a_2^{\tau(a_0)} = a_{-2} = a_2$, so that $a_2 \in A_{1,0,\alpha}^{a_0}$, and likewise $a_0 \in A_{1,0,\alpha}^{a_2}$ and $a_{-1} \in A_{1,0,\alpha}^{a_1}, a_1 \in A_{1,0,\alpha}^{a_{-1}}$. Therefore we may apply Theorem 2.3.1 to the subalgebras $\langle a_0, a_2\rangle$ and $\langle a_1, a_{-1}\rangle$. In particular, either $\lambda_2 = \frac{\alpha}{2}$ or $\lambda_2 = 0$, corresponding to $\lambda_2 - \frac{\alpha}{2} \in J_{(4B)}$ and $\lambda_2 \in J_{(4A)}$ respectively. In other words, we identify a $(2B)$-subalgebra in the cover of $(4A)$ and a $(3C_\alpha)$-subalgebra in the cover of $(4B)$. We take the two cases separately.

4.5.8 Lemma. *$\beta - \frac{\alpha^2}{2}$ and $\lambda_1 - \frac{\beta}{2}$ are in $\bar{J}^R_{(4B)}$.*

Proof. After making the substitution $a_{-2} = a_2$, we calculate $0 = a_0(zz_2) =$

$$= \frac{1}{4}\frac{(\alpha-\beta)}{(\alpha-4\beta)}\Big((8(3\beta-1)(2\alpha-1)\lambda_1^2 - 2(3(4\beta-1)\alpha^2 + (2\beta-1)(10\beta-1)\alpha - 12\beta^2 + 4\beta)\lambda_1$$
$$+2\beta\alpha^3 + \beta(11\beta-3)\alpha^2 + \beta(8\beta^2 - 12\beta + 1)\alpha - 4\beta^3 + 2\beta^2)\Big)(a_{-1} - a_1), \quad (4.98)$$

and in particular there is no linear dependence between a_{-1} and a_1 in $(4B)$, so the coefficient must be 0. Therefore we have a quadratic formula for λ_1. We also have a different quadratic formula for λ_1 from combining (4.94) and $\lambda_2 = \frac{\alpha}{2}$. The resultant of the numerators of the two quadratics with respect to λ is

$$8(2\alpha-1)\beta(2\beta-1)(2\beta-\alpha^2)(2\beta+\alpha^2-1)(4\alpha^3-(18\beta+1)\alpha^2+4\beta(4\beta+1)\alpha-4\beta^2), \quad (4.99)$$

and the only factor which is 0 when $(\alpha, \beta) = (1/4, 1/32)$ is $2\beta - \alpha^2$, that is, $\beta = \frac{\alpha^2}{2}$. Now substituting $\beta = \frac{\alpha^2}{2}$ back into both of the quadratic formulae for λ_1 yields that the only common factor is $4\lambda_1 - 2\alpha^2$, which is also the only factor of either which is 0 in $(4B)$. This gives $\lambda_1 = \frac{\alpha^2}{4}$. □

4.5.9 Lemma. *If $\beta = \frac{\alpha^2}{2}$ and $\lambda_1 = \frac{\alpha^2}{4}$ then the subspace K spanned by*

$$\begin{aligned} y_2 &= a_2 - a_{-2} \in A^{a_0}_\beta \\ x &= \alpha^3(a_0 + a_2) + \alpha^2(a_{-1} + a_1) + 4s + 2\alpha s_2 \in A^{a_0}_\alpha \\ x^f &= \alpha^2(a_0 + a_2) + \alpha^3(a_{-1} + a_1) + 4s + 2\alpha s_2^f \end{aligned} \quad (4.100)$$

is killed by the fusion rules in $\bar{I}^R_{(4B)}$.

Proof. K is closed under the action of $\mathrm{ad}(a_2)$:

$$\begin{aligned}
(\alpha-1)(2\alpha-1)^2 a_2 y_2 &= \frac{1}{2}(\alpha-3)(2\alpha-1)\alpha^2 y_2 - 2x + 2(\alpha-2)(\alpha-1)x^f \\
(\alpha-1)(2\alpha-1)^2 a_2 x &= \frac{1}{4}\alpha^5(\alpha+1)(2\alpha-1)y_2 + \alpha(\alpha^4-\alpha^3+3\alpha^2-3\alpha+1)x \\
&\quad - \alpha(\alpha-1)^2(\alpha^2-\alpha+1)x^f \\
(\alpha-1)(2\alpha-1)^2 a_2 x^f &= -\frac{1}{4}\alpha^3(2\alpha-1)(\alpha^3-6\alpha^2+6\alpha-2)y_2 \\
&\quad - \frac{1}{2}\alpha(2\alpha^4-10\alpha^3+9\alpha^2-\alpha-1)x + \alpha(\alpha-1)^3(\alpha+1)x^f.
\end{aligned} \tag{4.101}$$

The determinant of $\mathrm{ad}(a_2)|_K - \mu I_3$ is must be 0 for $\mu = 0, \alpha, \frac{\alpha^2}{2}$, since a_2 is a $\Phi(\alpha, {}^{\alpha^2}\!/_2)$-axis and $a_2 \notin K$. These cases respectively correspond to

$$\begin{aligned}
\alpha(2\alpha^5 - 4\alpha^4 - 6\alpha^3 + 13\alpha^2 - 7\alpha + 1) &= 0 \\
66\alpha^6 - 124\alpha^5 + 26\alpha^4 + 71\alpha^3 - 57\alpha^2 + 17\alpha - 2 &= 0 \\
\alpha(16\alpha^7 - 24\alpha^6 - 22\alpha^5 + 78\alpha^4 - 91\alpha^3 + 57\alpha^2 - 18\alpha + 2) &= 0
\end{aligned} \tag{4.102}$$

with common additional factor $-\frac{\alpha^3}{4}(\alpha-1)^{-1}(2\alpha-1)^{-4}$. The equations have three, two and two real roots respectively. Note that $\alpha = {}^1\!/_4$ is not a solution to any of them. Therefore if α is not a solution to (4.102) then K is killed by the fusion rules. □

4.5.10 Theorem. *The algebra* $(4B_\alpha)$ *over* $R = \mathbb{Z}[{}^1\!/_2, \alpha, \alpha^{-1}]$*, with basis* a_{-1}, a_0, a_1, a_2, s *and multiplication from Table 4.5 is a* $\Phi(\alpha, {}^{\alpha^2}\!/_2)$*-dihedral algebra is the cover of* $(4B)$ *when additionally* (4.102) *is invertible.*

Proof. The algebra is generated by a_0, a_1, so we only need to check the fusion rules for a_0, a_1. We do this in [GAP]. This is easily done using Lemma 4.4.3. □

4.5.11 Lemma. *The algebra* $(4B_\alpha)$ *is Frobenius; the form is positive-definite over* $\mathbb{R}$ *for all* α*, and degenerate for* $\alpha = -1$*.*

Proof. We check by direct computation in [GAP] that the form on $(4B_\alpha)$ induced as a quotient of the form in Section 4.3 is Frobenius. We also use [GAP] to calculate

Description	Products
a_{-1}, a_0, a_1, a_2, s	$a_i s = -\frac{1}{4}\alpha^2(1-\alpha+\alpha^2)a_{-1} + \frac{1}{8}(2-\alpha)\alpha^3(a_0+a_2) + \frac{1}{2}(2-\alpha)\alpha s$
$a_i = a_{i \bmod 4}$	$ss = \frac{1}{8}(-2+\alpha)(\frac{1}{2}-1+2\alpha)\alpha^4(a_{-1}+a_0+a_1+a_2)$
$\langle a_i, a_{i+2}\rangle \cong (3C_\alpha)$	$+\frac{1}{2}\alpha^2(\frac{1}{2}1-6\alpha+2\alpha^2)s$
	$(a_i, a_j) = \frac{\alpha^2}{4}$

Table 4.5: The algebra $(4B_\alpha)$

the Gram matrix, the determinant of which is

$$\frac{\alpha^4}{2^8}(\alpha-2)^4(\alpha+1)^2. \tag{4.103}$$

This is always positive in $\mathbb{R}$. Its roots are $\alpha = -1$. □

We now return our attention to the other case: $(4A)$.

4.5.12 Lemma. *$s_2 + \beta(a_0+a_2), s_2^f + \beta(a_1+a_{-1}) \in \bar{I}^R_{(4A)}$ and $\lambda_1 - \beta \in \bar{J}^R_{(4A)}$.*

Proof. From (4.94), when $\lambda_2 = 0$ we get

$$(\lambda_1-\beta)(2(2\alpha-1)\lambda_1 - \alpha^2 - 2\alpha\beta + 2\beta) = 0. \tag{4.104}$$

But $\lambda_1 - \beta = 0$ in $(4A)$ and the other factor is not 0 in $(4A)$.

From $0 = \lambda_2 = \lambda_1^{a_0}(a_2) = \lambda_1^{a_1}(a_{-1})$ we deduce that $a_0a_2 = 0 = a_1a_{-1}$. By substitution the statement follows. □

4.5.13 Lemma. *$\alpha - \frac{1}{4} \in \bar{J}^R_{(4A)}$.*

Proof. From Lemma 4.5.12 we substitute $\lambda_1 = \beta, \lambda_2 = \frac{\alpha}{2}, \alpha = 1/4, \beta = 1/32$ into (4.97) and see that this is not a solution. Therefore $a_{-2} - a_2 \in \bar{I}^R_{(4A)}$. After substituting similarly, we have that $z = -\alpha\beta a_0 + \frac{1}{2}(\alpha - \beta)(a_{-1} + a_1) - s$ is a 0-eigenvector for a_0. Then from the fusion rule $0 \star 0 = \{0\}$,

$$a_0(zz) = -\alpha(\alpha - \frac{1}{4})(\alpha - \beta)(\beta(a_{-1} + a_1) + 2s) = 0 \tag{4.105}$$

and as there can be no further relations in $\bar{I}^R_{(4A)}$ since $(4A)$ is 5-dimensional, we must have $\alpha - \frac{1}{4} \in \bar{J}^R_{(4A)}$. □

4.5.14 Theorem. *The algebra* $(4A_\beta)$ *over* $R = \mathbb{Z}[\frac{1}{2}, \beta]$ *with basis* a_{-1}, a_0, a_1, a_2, s *and multiplication from Table 4.6 is a* $\Phi({}^1\!/\!{}_4, \beta)$*-dihedral algebra, and is the cover of* $(4A)$.

Description	Products
a_{-1}, a_0, a_1, a_2, s	$a_is = -\beta^2 a_i + \frac{1}{8}(1-4\beta)\beta(a_{i+1} + a_{i+3}) + \frac{1}{4}(1-4\beta)s$
$a_i = a_{i \bmod 4}$	$ss = \frac{1}{32}\beta(4\beta-1)(8\beta-1)(a_{-1} + a_0 + a_1 + a_2) + \frac{1}{4}\beta(3-8\beta)s$
$\langle a_i, a_{i+2}\rangle \cong (2B)$	$(a_i, a_j) = \beta$

Table 4.6: The algebra $(4A_\beta)$

Proof. A is generated by a_0, a_1, so we only need to check the fusion rules for a_0, a_1. We do this in [GAP]. This is easily done using Lemma 4.4.3. □

4.5.15 Lemma. *The algebra* $(4A_\beta)$ *is Frobenius; the form is positive-definite over* $\mathbb{R}$ *for* $0 < \beta < \frac{1}{2}$*, and degenerate for* $\beta = \frac{1}{2}$.

Proof. We check by direct computation in [GAP] that the form on $(4A_\beta)$ induced as a quotient of the form in Section 4.3 is Frobenius. We also use [GAP] to calculate the Gram matrix, the determinant of which is

$$\frac{\beta}{8}(1-2\beta)^3. \tag{4.106}$$

This is positive in $\mathbb{R}$ when $\beta(1-2\beta)$ is positive. Its only acceptable root is $\beta = \frac{1}{2}$. □

4.5.16 Lemma. *On* $(4A_\beta)$ *and* $(4B_\alpha)$, $|\tau(a_0)\tau(a_1)| = 2$, *and* $|\tau(a_0)\tau(a_1)| \leq 4$ *in any larger algebra.*

Proof. $\tau(a_0)$ and $\tau(a_1)$ are the permutation matrices corresponding to $(1,3)(2)(4)(5)$ and $(1)(2,4)(3)(5)$, respectively, on both $(4A_\beta)$ and $(4B_\alpha)$. By Proposition 4.1.3, the order of $|\tau(a_0)\tau(a_1)|$ is bounded by $2+2=4$, the size of $a_0^T \cup a_1^T$. □

We finally consider $(3A)$. We see that (4.75) is not zero in $(3A)$, so that we have a relation in the algebra. Instead, for the case of the cover of $(3A)$ only, we have to start by making an extra assumption. Namely, suppose also that $a_3 = a_0$.

Then $a_{-2} = a_1, a_{-1} = a_2$, so that $s_2 = s_1 = s_2^f$ and $\lambda_2 = \lambda_1$. By Lemma 4.4.5, also $\lambda_1^f = \lambda_1$ (so this assumption implies the earlier one), and we now write λ for λ_1.

From Theorem 4.3.7 we therefore deduce a multiplication table on $\{a_0, a_1, a_2, s\}$.

The eigenvectors of a_0 of eigenvalue not 1 are, from Lemma 4.4.3:

$$\begin{aligned} z &= ((1-\alpha)\lambda_1 - \beta)a_0 + \frac{1}{2}(\alpha-\beta)(a_1+a_2) - s, \\ x &= (\beta - \lambda_1)a_0 + \frac{1}{2}\beta(a_1+a_2) + s, \\ y &= a_1 - a_2. \end{aligned} \tag{4.107}$$

Now $0 \star 0 = \{0\}$ means $zz \in A_0^{a_0}$ and $a_0(zz) = 0$. We calculate that

$$\begin{aligned} a_0(zz) = &-\frac{(\alpha-\beta)(-4(2\alpha-1)\lambda + 3\alpha^2 + (3\beta-1)\alpha - 2\beta)}{4(\alpha-2\beta)(\alpha-4\beta)} \cdot \\ &\Big(2(-(\alpha^2 - 2(\beta+1)\alpha - 4\beta^2 + 4\beta)\lambda + \beta\alpha^2 - \beta(3\beta+1)\alpha - 2\beta^3 + 2\beta^2)a_0 \\ &\quad + \beta(-4(\alpha-2\beta)\lambda + \alpha^2 - 2\beta\alpha - 4\beta^2)a_1 + \beta\alpha(\alpha-4\beta)a_2 + 2(-2(\alpha-2\beta)\lambda + \alpha^2 - 3\beta\alpha - 2\beta^2)s\Big). \end{aligned} \tag{4.108}$$

The coefficients in the ring must be 0 since there cannot be a nontrivial relation among the spanning set, since $(3A)$ is 4-dimensional. Apart from $\alpha - \beta$, the only factor of the coefficient of a_2 in the above is $-4(2\alpha-1)\lambda + 3\alpha^2 + (3\beta-1)\alpha - 2\beta$, so this must be 0, and indeed this is a common factor for all the coefficients. We obtain a relation satisfied by $(3A)$:

$$4(2\alpha-1)\lambda = 3\alpha^2 + \alpha(3\beta-1) - 2\beta. \tag{4.109}$$

If $\alpha = 1/2$, then the lefthand side is 0 and we find $0 = 3/4+3/2\beta-1/2-2\beta = 1/4-1/2\beta$, that is, $\beta = 1/2$, which contradicts that $\alpha \neq \beta$. Therefore $\alpha \neq 1/2$ and we have an expression for λ in α, β. After substitution we find Table 4.7.

Description	Products
$(3A'_{\alpha,\beta})$	$a_i s = \frac{1}{4(1-2\alpha)}(3\alpha^3 - 5\alpha^2\beta + 8\alpha\beta^2 - 4\alpha^2 + 7\alpha\beta - 4\beta^2 + \alpha - 2\beta)a_i$
a_0, a_1, a_2, s	$+\frac{1}{2}\beta(\alpha-\beta)(a_{i+1} + a_{i+2}) + (\alpha-\beta)s$
$a_i = a_{i \bmod 3}$	$ss = \frac{(\alpha-\beta)}{8(1-2\alpha)}(3\alpha^3 - 13\alpha^2\beta + 16\alpha\beta^2 - 4\alpha^2 + 11\alpha\beta - 8\beta^2 + \alpha - 2\beta)$
	$\cdot(a_0 + a_1 + a_2)$
	$+\frac{1}{4(2\alpha-1)}(9\alpha^3 - 27\alpha^2\beta + 12\alpha\beta^2 - 6\alpha^2 + 13\alpha\beta - 6\beta^2 + \alpha)s$
	$(a_i, a_j) = \frac{1}{4}\frac{1}{(2\alpha-1)}(3\alpha^2 + (3\beta-1)\alpha - 2\beta)$

Table 4.7: The algebra $(3A'_{\alpha,\beta})$

4.5.17 Theorem. *The algebra* $(3A'_{\alpha,\beta})$ *over* $R = \mathbb{Z}[1/2, \alpha, \beta, (1-2\alpha)^{-1}]$ *with basis* a_0, a_1, a_2, s *and multiplication table 4.7 is a* $\Phi(\alpha,\beta)$*-dihedral algebra. It is a weak cover of* $(3A)$.

Proof. A is generated by a_0, a_1, so we only need to check the fusion rules for a_0, a_1. We do this in [GAP]. For a_0 this is easily done using (4.107). For a_1, this follows from the evident $\mathrm{Sym}(3)$-symmetry of the multiplication table with respect to a_0, a_1, a_2. □

4.5.18 Lemma. *The algebra* $(3A'_{\alpha,\beta})$ *is Frobenius; the form is positive-definite over* $\mathbb{R}$ *for*

$$(2\alpha-1)(3\alpha-\beta-1)(3\alpha^2+3\alpha\beta-\beta-1)(3\alpha^2+(3\beta-9)\alpha-2\beta+4)<0, \qquad (4.110)$$

and degenerate when $(3\alpha-\beta-1)(3\alpha^2+3\alpha\beta-\beta-1)(3\alpha^2+(3\beta-9)\alpha-2\beta+4)=0$.

Proof. We check by direct computation in [GAP] that the form on $(3A'_{\alpha,\beta})$ induced as a quotient of the form in Section 4.3 is Frobenius. We also use [GAP] to calculate the Gram matrix, the determinant of which is

$$\frac{-\alpha^2}{2^9(2\alpha-1)^5}(3\alpha-\beta-1)(3\alpha^2+3\alpha\beta-\beta-1)(3\alpha^2+(3\beta-9)\alpha-2\beta+4)^3. \qquad (4.111)$$

□

4.5.19 Lemma. *We have* $|\tau(a_0)\tau(a_1)|=3$.

Proof. $\tau(a_0),\tau(a_1)$ are the permutation matrices of $(1)(2,3)(4)$ and $(1,3)(2)(4)$ respectively on $(3A'_{\alpha,\beta})$. Proposition 4.1.3 implies that the order of $\tau(a_0)\tau(a_1)$ is bounded by 3 everywhere. □

Bibliography

[A47] A. A. Albert, *A Structure Theory for Jordan Algebras*, Annals Math 48: 546–567, 1947.

[A97] M. Aschbacher, 3-*Transposition Groups*, CUP 1997.

[Ax13] F. Rehren, *Axial algebras—a GAP package*, `github.com/felixrehren/axials`, 2013.

[B86] R. E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Nat. Acad. Sci. U.S.A. 83: 3068–3071, 1986.

[B] F. Buekenhout, *La géométrie des groupes de Fischer*, unpublished.

[C05] R. Carter, *Lie Algebras of Finite and Affine Type*, CUP 2005.

[CR13] A. Castillo-Ramirez, *Idempotents of the Norton-Sakuma algebras*, J. Group Th. 16: 419–444, 2013. `arXiv:1310.0285`

[CL14] H.-Y. Chen, C. H. Lam, *An explicit Majorana representation of the group* $3^2 : 2$ *of* $(3C)$*-pure type*, Pacific J. Math. 271: 148–173, 2014. `arXiv:1305:7306`

[C85] J. Conway, *A simple construction for the Fischer-Griess monster group*, Invent. Math. 79: 513–540, 1985.

[CLO96] D.A. Cox, J. Little, D. O'Shea, *Ideals, Varieties, and Algorithms*, Springer, New York, 1996.

[CH95] H. Cuypers, J. I. Hall, *The* 3*-transposition groups with trivial center*, J. Algebra 178: 149–193, 1995.

[DGO14] J. F. R. Duncan, M. J. Griffin, K. Ono, *Moonshine*, `arXiv:1411.6571`.

[DFMS97] P. Di Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics, Springer, 1997.

[DGH98] C. Dong, R. Griess, G. Höhn, *Framed vertex operator algebras, codes and the moonshine module*, Comm. Math Phys. 193: 407–448, 1998. arXiv:q-alg/9707008

[DLMN96] C. Dong, H. Li, G. Mason, S. P. Norton, *Associative subalgebras of the Griess algebra and related topics*, in *The Monster and Lie algebras* (proceedings, ed. J. Ferrar, K. Harada), Ohio State / de Gruyter, 1998. arXiv:q-alg/9607013

[F71] B. Fischer, *Finite groups generated by 3-transpositions. I*, Invent. Math. 13: 232–246, 1971.

[FBZ01] E. Frenkel, D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Mathematical Surveys and Monographs 88, AMS, 2001.

[FLM98] I. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics 134, Academic Press, 1998.

[GAP] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.7.5, 2014.

[GKO85] P. Goddard, A. Kent, D. Olive, *Virasoro algebras and coset space models*, Physics Letters 152: 88–92, 1985.

[GR01] C. Godsil, G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, Springer, 2001.

[G82] R. L. Griess, *The friendly giant*, Invent. Math. 69: 1–102, 1982.

[G03] R. L. Griess, *GNAVOA, I. Studies in groups, nonassociative algebras and vertex operator algebras*, in Vertex Operator Algebras in Mathematics and Physics (ed. S. Berman, Y. Billig, Y-Z. Huang, J. Lepowsky), Fields Institute Communications 39: 71–89, AMS, 2003.

[H93] J. I. Hall, *The general theory of 3-transposition groups*, Math. Proc. Camb. Phil. Soc. 114: 269–294, 1993.

[HRS15] J. I. Hall, F. Rehren, S. Shpectorov, *Universal Axial Algebras and a Theorem of Sakuma*, J. Algebra 421: 394–424, 2015. arXiv:1311.0217

[HRS14] J. I. Hall, F. Rehren, S. Shpectorov, *Primitive axial algebras of Jordan type*, J. Algebra, forthcoming. `arXiv:1403.1898`

[HS15] J. I. Hall, S. Shpectorov, *Eigenvalues draft*, preprint.

[HLY12] G. Höhn, C. H. Lam, H. Yamauchi, *McKay's E_7 observation on the Baby Monster*, Int. Math. Res. Not. 2012: 166–212, and *McKay's E_6 observation on the largest Fischer group*, Comm. Math. Phys. 310: 329–365, both at `arXiv:1002.1777`.

[I09] A. A. Ivanov, *The Monster Group and Majorana Involutions*, Cambridge University Press, 2009.

[IPSS10] A. A. Ivanov, D. Pasechnik, Á. Seress, S. Shpectorov, *Majorana Representations of the Symmetric Group of Degree 4*, J. Algebra 324: 2432–2463, 2010.

[J68] N. Jacobson, *Structure and representations of Jordan algebras*, American Mathematical Society Colloquium Publications XXXIX, AMS 1968.

[LY04a] C. H. Lam, H. Yamada, *Decomposition of the lattice vertex operator algebra $V_{\sqrt{2}A_l}$*, J. Algebra 272: 614–624, 2004.

[MAGMA] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24: 235–265, 1997.

[M01] A. Matsuo, *Norton's trace formulae for the Griess algebra of a vertex operator algebra with larger symmetry*, Comm. Math. Phys. 224: 565–591, 2001. `arXiv:math/0007169`

[M03] A. Matsuo, *3-Transposition Groups of Symplectic Type and Vertex Operator Algebras*, 2003 preprint*. `arXiv:math/0311400`

[McC04] K. McCrimmon, *A Taste of Jordan Algebras*, Springer 2004.

[MN93] W. Meyer, W. Neutsch, *Associative Subalgebras of the Griess Algebra*, J. Algebra 158: 1–17, 1993.

*preferred over the published version in J. Math. Soc. Japan 57: 639–649, 2005

[M96] M. Miyamoto, *Griess algebras and conformal vectors in vertex operator algebras*, J. Algebra 179: 523–548, 1996.

[M03] M. Miyamoto, *Vertex operator algebras generated by two conformal vectors whose τ-involutions generate S_3*, J. Algebra 268: 653–671, 2003. arXiv:math/0112031

[M92] G. Mussardo, *Off-critical statistical models: Factorized scattering theories and bootstrap program*, Phys. Rep. 218: 215–379, 1992.

[P79] D. Passman, *The Algebraic Structure of Group Rings*, Dover Publications, 2011.

[P81] B. Peirce, *Linear associative algebra*, Amer. J. Math. 4: 97–229, 1881.

[S07] S. Sakuma, *6-Transposition Property of τ-Involutions of Vertex Operator Algebras*, Int. Math. Res. Not., 030, 19 pages, 2007. arXiv:math/0608709

[S12] Á. Seress, *Construction of 2-closed M-representations*, Proc. Int. Sympos. Symbolic Algebraic Comput., 311–318, New York, AMS, 2012.

[S97] T. Springer, *Jordan Algebras and Algebraic Groups*, Classics in Mathematics 75, Springer, 1998.

[W93] W. Wang, *Rationality of Virasoro vertex operator algebras*, Int. Math. Res. Not. 71: 197-211, 1993.

[Y01] H. Yamada, *Highest weight vectors with small weights in the vertex operator algebra associated with a lattice of type* $\sqrt{2}\mathrm{A}_l$, Comm. Alg. 29: 1311–1324, 2001.

Index