# Root systems of Levi type for Lie algebras of affine type

by

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A thesis submitted to
The University of Birmingham
for the degree of
Doctor of Philosophy

School of Mathematics
The University of Birmingham
September 2014

# Abstract

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac–Moody Lie algebra with generalized Cartan matrix $A$. Brundan, Goodwin and independently Kostant developed a theory of root system known as Levi type root system when $A$ is a Cartan matrix so that $\mathfrak{g}(A)$ is a finite dimensional semisimple Lie algebra. This theory replicates much of the structure of usual root systems. In this thesis we build up the theory of Lie algebras to review this. Then we go on to define Levi type roots for the case where $A$ is of affine type. To describe Levi type root systems we show how these roots are related to the roots of centralizers of nilpotent elements in $\mathfrak{g}$. We also determine the normalizers of parabolic subgroups of finite and affine Weyl groups of classical types which can be viewed as the Weyl groups for so called root systems.

# Acknowledgements

First and foremost, I would like to express my deepest gratitude to my supervisor Dr. Simon Goodwin for his continuous support, patience, motivation and enthusiasm which he has given me in all time of research and writing of my PhD thesis. I have been extremely lucky to have a supervisor who cared so much about my work. This thesis would not have been possible without his support.

I would also like to thank my co-supervisor Professor Chris Parker for all his guidance and help.

I am really thankful to Dr. Dmitriy Rumynin and Dr. Kay Magaard to be in my defense and reading committee and for many valuable discussions, comments and suggestions. I also thank Dr. Daniel Loghin as the chair of my defense committee.

I am thankful to all my friends and colleagues at the department of mathematics who made my time here interesting and enjoyable.

I would like to thank my parents Ebrahim and Fariba, and my sister Mahsa for their love and support.

Finally I would like to thank my brother Rasta for all his advice and wise words he has provided me over the last several years and all his support through every decision I made.

# Contents

# Introduction

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a finite type or an untwisted affine Kac–Moody Lie algebra with generalized Cartan matrix $A$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the weight space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ with the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ or the fundamental roots $\Pi = \{\alpha_0, \ldots, \alpha_l\}$ for $\mathfrak{g}$ of finite type or affine type respectively. Let $Y \subseteq \{1, \ldots, l\}$, and $\Phi_Y = \Phi \cap (\bigoplus_{i \in Y} \mathbb{Z}\alpha_i)$.

Let $\mathfrak{g}_Y = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_Y} \mathfrak{g}_\alpha$ be the Levi factor of a standard parabolic subalgebra

$$\mathfrak{p}_Y = \mathfrak{g}_Y \oplus \mathfrak{u}_Y$$

where $\mathfrak{u}_Y = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi_Y^+} \mathfrak{g}_\alpha$ and $\Phi_Y^+ = \Phi^+ \cap \Phi_Y$. Let $\mathfrak{h}^Y = \mathfrak{z}(\mathfrak{g}_Y)$. Since $\mathfrak{h}^Y \subset \mathfrak{h}$, it acts semisimply on $\mathfrak{g}$ and we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi^Y} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x \text{ for all } t \in \mathfrak{h}^Y\}$ and $\mathfrak{g}_0 = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}^Y)$.

The subset $\Phi^Y = \{\alpha \in (\mathfrak{h}^Y)^* \mid \mathfrak{g}_\alpha \neq 0\} \subseteq (\mathfrak{h}^Y)^*$ is called the Levi type root system.

The theory of Levi type root systems when $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra is independently developed in Brundan and Goodwin [3] and Kostant [18]. This theory replicates much of the structure of usual root systems.

In this thesis we generalize much of this theory to the case when $\mathfrak{g}$ is an untwisted affine Kac–Moody Lie algebra, and $Y \subseteq \{1, \ldots, l\}$.

*Remark.* If there exists a graph automorphism $\tau$ of Dynkin diagram such that $0 \notin \tau Y$ then we use the same technique for example when $\mathfrak{g}$ is of type $\tilde{A}$. If this is not the case, then we expect the theory to be similar, but a bit more complicated.

The following theorem [18, Theorem 1.9] where $R$ is the set of Levi type roots for the finite semisimple Lie algebra $\mathfrak{g}$ plays a key role in developing the theory of Levi type root systems.

**Theorem.** *Let* $\mathfrak{r} = \bigoplus_{\nu \in R} \mathfrak{g}_\nu$*. Then* $\mathfrak{g}_\nu$ *is an irreducible* $\operatorname{ad} \mathfrak{g}_Y$*-module for any* $\nu \in R$ *and any irreducible* $\mathfrak{g}_Y$*-submodule of* $\mathfrak{r}$ *is of this form. Furthermore,* $\mathfrak{r}$ *is a multiplicity-free* $\operatorname{ad} \mathfrak{g}_Y$*-module and* $\mathfrak{r} = \bigoplus_{\nu \in R} \mathfrak{g}_\nu$ *is the unique decomposition of* $\mathfrak{r}$ *as a sum of irreducible* $\operatorname{ad} \mathfrak{g}_Y$*-modules.*

The theory of Levi type root systems in many ways replicates results in the usual root theory. For example it is established that if $\mu, \nu \in R$, and $\mu + \nu \in R$, then we have the equality $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = \mathfrak{g}_{\mu+\nu}$. Also if $\mu, \nu \in R$ and if $(\mu, \nu) < 0$, then $\mu + \nu \in R$, and if $(\mu, \nu) > 0$, then $\mu - \nu \in R$, see Theorem 2.2.2.

The theory of Levi type root systems have found applications in other areas of mathematics. For example Levi type root system has been Kostant's motivation in Borel de-Siebenthal theory, see [18] and Brundan and Goodwin's motivation in good grading of Lie algebras, see [3]. Orlik and Solomon and many others have considered underlying hyperplane arrangements, see [20]. Levi type root systems also play a role in the representation theory of the finite $W$-algebras which has attracted a lot of attention in Mathematical physics. Brundan, Goodwin, and Kleshchev set up a framework to study representation theory of finite $W$-algebras via highest weight theory, see [4]. For further applications, see [1], and [10].

For the case when $\mathfrak{g}$ is an untwisted affine Kac–Moody Lie algebra, the Proposition 4.1.1 shows that Levi type root spaces are finite dimensional.

For $\alpha \in \Phi$ we define $\alpha^Y \in \Phi^Y$ which is a restriction of $\alpha$ to $\mathfrak{h}^Y$.

The following theorem is a description of these root systems.

**Theorem.** *Let $\mathfrak{g} = \mathfrak{g}(A)$ be an untwisted affine Kac–Moody Lie algebra with a generalized Catran matrix $A$, Cartan subalgebra $\mathfrak{h}$, the set of roots $\Phi$ and the set of fundamental roots $\Pi = \{\alpha_0, \dots, \alpha_l\}$. Then for any subset $Y \subseteq \{1, \dots, l\}$ the corresponding Levi type root system is*

$$\Phi^Y = \{i\delta^Y \mid i \in \mathbb{Z}_{\neq 0}\} \cup \{\alpha + i\delta^Y \mid \alpha \in (\Phi^0)^Y, i \in \mathbb{Z}\}.$$

*where $\delta \in \Phi$ is the indecomposable imaginary root, $\Phi^0$ is the root system of the finite dimensional Lie algebra $\mathfrak{g}(A^0)$, and $(\Phi^0)^Y$ is the Levi type root system for $\mathfrak{g}(A^0)$ corresponding to $Y$.*

In fact

$$\Phi^Y = (\Phi^Y)_{\mathrm{Re}} \cup (\Phi^Y)_{\mathrm{Im}}$$

where $(\Phi^Y)_{\mathrm{Re}} = \{\alpha + i\delta^Y \mid \alpha \in (\Phi^0)^Y, i \in \mathbb{Z}\}$ and $(\Phi^Y)_{\mathrm{Im}} = \{i\delta^Y \mid i \in \mathbb{Z}_{\neq 0}\}$.

Now let $e$ be a regular nilpotent element in $\mathfrak{g}_Y$. Then we have $\mathfrak{h}^Y = \mathfrak{z}(\mathfrak{g}_Y) = \mathfrak{h} \cap \mathfrak{g}^e = \mathfrak{h}^e$ where $\mathfrak{g}^e$ and $\mathfrak{h}^e$ are centralizers of $e$ in $\mathfrak{g}$ and $\mathfrak{h}$ respectively. We also have

$$\mathfrak{g}^e = (\mathfrak{g}_0)^e \oplus \bigoplus_{\alpha \in \Phi^Y} (\mathfrak{g}_\alpha \cap \mathfrak{g}^e)$$

where $(\mathfrak{g}_0)^e = \mathfrak{g}_0 \cap \mathfrak{g}^e$. Let $\Phi^e = \{\alpha \in \Phi^Y \mid \mathfrak{g}_\alpha \cap \mathfrak{g}^e \neq 0\}$. The next proposition says that we can view $\Phi^Y$ as the "root system of $\mathfrak{g}^e$".

**Proposition.** *Let $Y \subseteq \{1, \dots, l\}$ and $e = \sum_{i \in Y} e_i \in \mathfrak{g}_Y$ be a regular nilpotent element in the Levi subalgebra $\mathfrak{g}_Y$. Then $\Phi^e = \Phi^Y$.*

The normalizers of parabolic subgroups of finite and affine Weyl groups are related to the Levi type root systems. We can consider them as the Levi type Weyl groups. For the finite Weyl groups the isomorphism classes of the normalizers of parabolic subgroups are already done by Howlett, see [11]. The normalizers of parabolic subgroups for Coxeter groups in general are done by Brink and Howlett, see [2]. In this thesis we aim to present a permutation representation for classical finite Weyl groups and affine Weyl groups from [7], [8], and [9]. We will relate the root system of Levi type $\Phi^Y$ with the normalizer of the parabolic subgroup $N_W(W_L) = W^L \ltimes W_L$ where $L$ is the partition of $\mathbb{Z}$ corresponding to $Y$.

For the case when $G = \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}[t, t^{-1}])$ and $W = W(\tilde{A}_n)$ we will show the Levi type Weyl group $W^Y$ is isomorphic to $N_G(\mathfrak{g}_Y)/G_Y$. We expect this is also true in other types.

The first chapter of this thesis is devoted to give an overview of the theory of Lie algebras. We define Lie algebras and semisimple Lie algebras in particular. We list some standard definitions and basic facts about Lie algebras. The representation theory of $\mathfrak{sl}_2$ is presented to show some properties of the root space decomposition as we study further on. The Cartan decomposition of a semisimple Lie algebra into root spaces with respect to a maximal toral subalgebra is described. We also introduce reductive Lie algebras. The rest of the first chapter is devoted to classify semisimple Lie algebras and to focus on the the idea of existence and uniqueness of a semisimple Lie algebra with a given root system.

The main content of the Chapter 2 focusses on defining Levi type root systems for finite dimensional semisimple Lie algberas. Some properties of these root systems are investigated. The rest of the chapter is devoted to relate Levi type root systems to the centralizers of nilpotent elements in semisimple Lie algebras.

In Chapter 3 we first introduce the theory of the Kac–Moody algebra associated to a generalised Cartan matrix. We obtain a trichotomy of indecomposable generalised Cartan matrices into those of finite, affine and indefinite type. Then the Weyl group and root system of a Kac–Moody algebra is discussed. Then the chapter is focused more on Kac–Moody Lie algebras of affine type and their roots. The rest of the chapter is focused on showing a method to construct the affine Kac-Moody Lie algebras from the finite dimensional simple Lie algebra via loop algebras.

In the fourth chapter we study the structure of Levi type root systems for affine Kac–Moody Lie algebras. Then we consider the centralizer of a nilpotent element and show how the centralizers of regular nilpotent elements are related to Levi type root systems. Finally we investigate some properties of these root systems.

The fifth chapter starts with some preliminaries on Coxeter groups. First we briefly study two of the most important types of Coxeter groups, finite reflection groups and affine Weyl groups including the classification of associated Coxeter graphs. We continue the general study of Coxeter groups. To describe the normalizers of parabolic subgroups of affine Weyl groups up to isomorphism as the main part of this chapter we aim to present a permutation representation for classical finite Weyl groups and affine Weyl groups.

In the final chapter we relate the Weyl group of Levi type $W^Y$ with the normalizer of the parabolic subgroup $W_Y$ of finite and affine Weyl groups. We will prove $W^Y$ known as the Levi type Weyl group is isomorphic to $N_G(\mathfrak{g}_Y)/G_Y$ when $\mathfrak{g}$ is of type $\tilde{A}_n$.

# Chapter 1

# Lie algebras

In this chapter we give an overview of the theory of Lie algebras. All the material can be found in the books of Humphreys [12] and Tauvel and Yu [24]. We do not include the references for all the results we state. In this chapter $k$ is an algebraically closed field of characteristic 0.

## 1.1 Lie algebras

**Definition 1.1.1.** A Lie algebra over $k$ is a $k$-vector space $\mathfrak{g}$, with a bilinear operation $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ denoted by $(x, y) \mapsto [x, y]$ called the bracket of $x$ and $y$, satisfying the following axioms:

(i) $[x, x] = 0$ for all $x$ in $\mathfrak{g}$

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$ (called the **Jacobi identity**).

Let $A$ be an associative algebra over $k$, then $A$ with an operation $[a, b] := ab - ba$ for all $a, b \in A$ is a Lie algebra which is denoted by $[A]$. If $V$ is a finite dimensional vector space over $k$, then $\operatorname{End} V$ is an associative algebra. We write $\mathfrak{gl}(V)$ for $[\operatorname{End} V]$ and call it the **general linear Lie algebra**. The set of all $n \times n$ matrices $\mathrm{M}(n, k)$ identified with the set of endomorphism of $V$ is a Lie algebra with $[a, b] = ab - ba$ for all $a, b \in \mathrm{M}(n, k)$ denoted by $\mathfrak{gl}(n, k)$.

Next we list some standard definitions about Lie algebras.

- A Lie algebra $\mathfrak{g}$ is called **abelian** if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.
- A **subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ such that $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$.
- A subspace $\mathfrak{a}$ of $\mathfrak{g}$ is called an **ideal** if $[x, y] \in \mathfrak{a}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$.
- A non abelian Lie algebra $\mathfrak{g}$ is **simple** if it has no ideals other than itself and 0.
- The **centre** of $\mathfrak{g}$ is defined by $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$ and is an ideal of $\mathfrak{g}$.
- The **derived subalgebra** of $\mathfrak{g}$ denoted by $\mathfrak{D}(\mathfrak{g})$ consists of all linear combinations of commutators $[x, y]$, for all $x, y \in \mathfrak{g}$.
- The **normalizer** of a subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is defined by $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{t}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{t}] \subset \mathfrak{t}\}$.
- A map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a **homomorphism** of Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ if $\varphi$ is a linear map and $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$ and it is an **isomorphism** if $\varphi$ is a bijective.
- A **representation** $(V, \varphi)$ of $\mathfrak{g}$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ where $V$ is vector space over $k$. A representation is finite dimensional if $V$ is finite dimensional and is **faithful** if it is injective.
- The map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad}\, x(y) = [x, y]$ is a Lie algebra homomorphism so it is a representation of $\mathfrak{g}$ known as the **adjoint representation**.
- If $A$ is an algebra over a field $k$, a **derivation** of $A$ is a linear map $\delta : A \to A$ satisfying $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in A$. The collection $\mathrm{Der}\, A$ of all derivations of $A$ is a subspace of $\mathrm{End}(A)$ and it is a Lie algebra with $[\delta, \delta'] = \delta\delta' - \delta'\delta$ for all $\delta, \delta' \in \mathrm{Der}\, A$.

- Let $\mathfrak{g}$ be any Lie algebra over a field $k$. A **Lie module for $\mathfrak{g}$**, or alternatively a **$\mathfrak{g}$-module**, is a finite dimensional vector space $V$ together with a map $\mathfrak{g} \times V \to V$ denoted by $(x, v) \mapsto x \cdot v$ satisfying the following conditions:

  (a) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,

  (b) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$,

  (c) $[xy] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$
  for all $x, y \in \mathfrak{g}$, $v, w \in V$and $a, b \in k$.

- Lie modules and representations are equivalent, i.e if $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, then we make $V$ a $\mathfrak{g}$-module by defining $x \cdot v = \varphi(x)(v)$ for all $x \in \mathfrak{g}$, and $v \in V$. Conversely, given a $\mathfrak{g}$-module $V$, then $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ given by $\varphi(x)v = x \cdot v$ where $x \in \mathfrak{g}$ and $v \in V$.

- Let $V$ be a $\mathfrak{g}$-module. A **submodule** of $V$ is a subspace $W$ of $V$ such that for all $x \in \mathfrak{g}$ and for all $w \in W$, we have $x \cdot w \in W$.

- A **homomorphism of $\mathfrak{g}$-modules** is a linear map $\varphi : V \to W$ such that $\varphi(x \cdot v) = x \cdot \varphi(v)$ for all $x \in \mathfrak{g}$ and $v \in V$.

- Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-module $V$ is said to be **irreducible** or **simple**, if it is nonzero and has precisely two $\mathfrak{g}$-submodules 0 and $V$. It is called **completely reducible** or **semisimple** if $V$ is a direct sum of irreducible $\mathfrak{g}$-submodules, or equivalently, if each $\mathfrak{g}$-submodule $W$ of $V$ has a complement $W'$.

Here are some examples known as **classical algebras**.

**Example 1.1.2.** Let $V$ be a vector space over a field $k$ then,

$A_\ell$: Let $\dim V = \ell + 1$. The set $\mathfrak{sl}(V)$ of all endomorphisms of $V$ having trace zero is a subalgebra of $\mathfrak{gl}(V)$ called the **special linear algebra**.

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, k)$ such that the characteristic of $k$ is not 2, is simple. Take $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as the standard basis for $\mathfrak{g}$ so that $[x, y] = h$, $[h, x] = 2x$ and $[h, y] = -2y$. Let $\mathfrak{a}$ be a nonzero ideal of $\mathfrak{g}$ and $ax + by + ch$ be a nonzero element of $\mathfrak{a}$. By applying $\operatorname{ad} x$ twice, we get $-2bx \in \mathfrak{a}$, and applying $\operatorname{ad} y$ twice, we get $-2ay \in \mathfrak{a}$. If $a$ or $b$ is nonzero, $\mathfrak{a}$ contains either $y$ or $x$ and then clearly $\mathfrak{a} = \mathfrak{g}$. On the other hand, if $a = b = 0$, then $0 \neq ch \in \mathfrak{a}$, so $h \in \mathfrak{a}$, and again $\mathfrak{a} = \mathfrak{g}$. So $\mathfrak{g}$ is simple.

Let $J = J_n$ be an $n \times n$ matrix defined by $J_{ij} = 1$ if $j = n - i + 1$, and $J_{ij} = 0$ otherwise.

$C_\ell$: Let $\dim V = 2\ell$. Let $f$ be the nondegenerate skew symmetric bilinear form on $V$ whose matrix is $S = \begin{pmatrix} 0 & J_\ell \\ -J_\ell & 0 \end{pmatrix}$. We denote by $\mathfrak{sp}(V)$, or $\mathfrak{sp}(2\ell, k)$, the **symplectic algebra**, the set of all endomorphisms $X$ of $V$ satisfying $f(X(v), w) = -f(v, X(w))$. In matrix terms, the condition for $X = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ where $M, N, P, Q \in \mathfrak{gl}(\ell, k)$ to be symplectic is that $SX = -X^{\mathrm{tr}}S$, that means $N^{\mathrm{str}} = N$, $P^{\mathrm{str}} = P$, and $M^{\mathrm{str}} = -Q$ where $(A_{ij})^{\mathrm{str}} = (A_{\ell-j+1\,\ell-i+1})$.

$B_\ell$: Let $\dim V = 2\ell + 1$, and let $f$ be the nondegenerate symmetric bilinear form on $V$ whose matrix is $S = J_{2\ell+1}$. The orthogonal algebra denoted by $\mathfrak{o}(V)$ or $\mathfrak{o}(2\ell + 1)$ consists of all endomorphisms of $V$ satisfying $f(X(v), w) = -f(v, X(w))$. In matrix terms, $X \in M(2\ell + 1, k)$ is orthogonal if $SX = -X^{\mathrm{tr}}S$, i.e. $X = -X^{\mathrm{str}}$.

$D_\ell$: This is another **orthogonal algebra**. The construction is similar to that for $B_\ell$ except that $\dim V = 2\ell$ is even.

The set $\mathfrak{b}(n, k) = \{A = (A_{ij}) \in \mathfrak{gl}(n, k) \mid A_{ij} = 0 \quad i > j\}$ called the set of **upper triangular matrices** and $\mathfrak{n}(n, k) = \{A = (A_{ij}) \in \mathfrak{gl}(n, k) \mid A_{ij} = 0 \quad i \geq j\}$ known as the **strictly upper triangular matrices** and finally $\mathfrak{d}(n, k)$ the set of **diagonal matrices** are some subalgebras of $\mathfrak{gl}(n, k)$.

Now we give some more standard definitions in the theory of Lie algebras.

- The **derived series** is defined by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ for $i \in \mathbb{Z}_{\geq 1}$. A Lie algebra $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^{(m)} = 0$ for some $m \geq 1$. The Lie algebra of upper triangular matrices is solvable.

- If $\mathfrak{g}$ is finite dimensional there is a unique solvable ideal $\operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ containing every solvable ideal of $\mathfrak{g}$ called the **radical** of $\mathfrak{g}$.

  Let $\mathfrak{a}$ and $\mathfrak{b}$ be two solvable ideals of $\mathfrak{g}$. Then $\mathfrak{a} + \mathfrak{b} = \{x + y \mid x \in \mathfrak{a}, y \in \mathfrak{b}\}$ is an ideal of $\mathfrak{g}$ and solvable see [12, Proposition 3.1]. Therefore, we can define the radical of $\mathfrak{g}$ as the sum of all the solvable ideals of $\mathfrak{g}$, hence the radical of $\mathfrak{g}$ is unique and solvable.

- A Lie algebra $\mathfrak{g}$ is called **semisimple** if $\operatorname{rad}\mathfrak{g} = 0$.

- The **descending central series** is defined by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$ for $i \in \mathbb{Z}_{\geq 1}$. A Lie algebra $\mathfrak{g}$ is said to be **nilpotent** if $\mathfrak{g}^m = 0$ for some $m \geq 1$. The Lie algebra of strictly upper triangular matrices is nilpotent.

- Let $\mathfrak{g}$ be any Lie algebra and $x \in \mathfrak{g}$. We call $x$ **ad-nilpotent** if $\operatorname{ad} x$ is a nilpotent endomorphism. Therefore, if $\mathfrak{g}$ is nilpotent, then all elements of $\mathfrak{g}$ are ad-nilpotent.

Below are two basic theorems in the theory of Lie algebras.

**Theorem 1.1.3** (**Engel's Theorem**). *Let $V$ be a finite dimensional vector space and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$ such that each element of $\mathfrak{g}$ is nilpotent. Then there exists a basis of $V$ such that the matrices of $\mathfrak{g}$ are all strictly upper triangular matrix.*

The following is a consequence of Engel's theorem.

**Corollary 1.1.4.** *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if all element of $\mathfrak{g}$ are ad-nilpotent.*

**Theorem 1.1.5** (**Lie's Theorem**). *If $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$ where $V$ is finite dimensional vector space over a field $k$, then there exists a basis of $V$ in which every matrices of $\mathfrak{g}$ is upper triangular.*

If $V$ is a vector space over an algebraically closed field $k$, there is a unique expression known as **Jordan–Chevally decomposition** for $x \in \mathrm{End}(V)$ as a sum $x = x_s + x_n$ where $x_s \in \mathrm{End}(V)$ is semisimple and $x_n \in \mathrm{End}(V)$ is nilpotent, and $x_n$ and $x_s$ commute. Moreover, there exists polynomials $p(X)$, $q(X) \in k[X]$, with no constant term, such that $x_s = p(x)$, $x_n = q(x)$. The following Lemma will be used later.

**Lemma 1.1.6.** *Let $A$ be a finite dimensional $k$-algebra. Then the semisimple and nilpotent parts of Jordan decomposition of all elements of* $\mathrm{Der}\, A$ *belong to* $\mathrm{Der}\, A$.

## 1.2 Semisimple Lie algebras

Let $\beta : V \times V \longrightarrow k$ be a bilinear form on $V$ where $V$ is a vector space over $k$. Then $\beta$ is called **nondegenerate** if $\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} \mid \beta(x, y) = 0 \text{ for all } y \in \mathfrak{g}\} = 0$. If $(V, \varphi)$ is a finite dimensional representation of $\mathfrak{g}$, then we can define a symmetric bilinear form $\beta_{\varphi}$ on $\mathfrak{g}$ associated to $\varphi$ by $\beta_{\varphi}(x, y) = \mathrm{tr}(\varphi(x)\varphi(y))$ for all $x, y \in \mathfrak{g}$. For all $x, y$ and $z \in \mathfrak{g}$ we have $\beta_{\varphi}([x, y], z) = \beta_{\varphi}(x, [y, z])$; this is known as the **associativity** or **invariance** property. The **Killing form** on $\mathfrak{g}$ is the symmetric bilinear form associated to the adjoint representation of $\mathfrak{g}$ and is denoted by $\beta$.

The following well known results will be used later.

**Theorem 1.2.1.** *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.*

**Theorem 1.2.2.** *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then,* *(a)* $\mathfrak{g}$ *is a direct sum of simple Lie algebras.*

*(b)* $\operatorname{ad}\mathfrak{g} = \operatorname{Der}\mathfrak{g}$.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the map $\mathfrak{g} \to \operatorname{ad}\mathfrak{g}$ is one to one and also we have $\operatorname{Der}\mathfrak{g} = \operatorname{ad}\mathfrak{g}$. Since $\operatorname{Der}\mathfrak{g}$ contains the semisimple and nilpotent parts of all its elements by Lemma 1.1.6, we can determine for every $x \in \mathfrak{g}$ unique elements $s, n \in \mathfrak{g}$ such that $\operatorname{ad} x = \operatorname{ad} s + \operatorname{ad} n$ as the usual Jordan decomposition of $\operatorname{ad} x$. This gives $x = s + n$ known as **abstract Jordan decomposition** of $x$ where $[s, n] = 0$ and $\operatorname{ad} s$ and $\operatorname{ad} n$ are semisimple and nilpotent respectively.

Here are some more known facts in the theory of Lie algebras.

**Theorem 1.2.3 (Weyl's theorem).** *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every finite dimensional representation of $\mathfrak{g}$ is completely reducible.*

Now we give one of applications of Weyl's theorem in the study of semisimple Lie algebras.

**Theorem 1.2.4.** *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite dimension representation of $\mathfrak{g}$. Suppose $x \in \mathfrak{g}$ has abstract Jordan decomposition $x = s + n$. Then the Jordan decomposition of $\varphi(x)$ is $\varphi(x) = \varphi(s) + \varphi(n)$.*

## 1.3 Representation theory of $\mathfrak{sl}_2$

Let $\mathfrak{g} = \mathfrak{sl}(2, k)$ with the standard basis $x, y, h$ where $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$. Let $V$ be a $\mathfrak{g}$-module of finite dimension over an algebraically closed field $k$. Since $h$ acts diagonally on the $\mathfrak{g}$-module $V$, therefore $V = \bigoplus_{\lambda \in k} V_\lambda$ where $V_\lambda = \{v \in V \mid h \cdot v = \lambda v\}$. For $\lambda \in \mathbb{C}$ when $V_\lambda \neq 0$, it is called a **weight** of $h$ in $V$ and we call $V_\lambda$ a **weight space.** We can check that if $v \in V_\lambda$, then $x \cdot v \in V_{\lambda+2}$ and $y \cdot v \in V_{\lambda-2}$. Since $V$ is finite dimensional, there exists $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. For such $\lambda$, any nonzero vector in $V_\lambda$ is called a **maximal vector** of weight $\lambda$. Here are some facts about irreducible modules for $\mathfrak{sl}(2, k)$.

**Theorem 1.3.1.** *Let $V$ be an irreducible module for $\mathfrak{g} = \mathfrak{sl}(2, k)$.*

*(a) Relative to $h$, $V$ is the direct sum of weight spaces $V_\mu$, where $\mu = m, m-2, \ldots, -(m-2), -m$, and $m + 1 = \dim V$. Moreover, $\dim V_\mu = 1$ for each $\mu$.*

*(b) $V$ has (up to nonzero scalar multiples) a unique maximal vector, whose weight (called the highest weight of $V$) is $m$.*

*(c) There exists at most one irreducible $\mathfrak{g}$-module (up to isomorphism) of each possible dimension $m + 1$, $m \geq 0$.*

Now we can use this result to prove the following result.

**Corollary 1.3.2.** *If $V$ be any finite dimensional $\mathfrak{g}$-module, then the eigenvalues of $h$ on $V$ are all integers, each occurs with its negative with equal number of times. Moreover, if we decompose $V$ into direct sum of irreducible submodules, the number of summands is exactly* $\dim V_0 + \dim V_1$.

## 1.4 Root system decomposition

For any semisimple Lie algebra $\mathfrak{g}$ there exists an element $x \in \mathfrak{g}$ whose semisimple part $x_s$ in the abstract Jordan decomposition is nonezero because otherwise $\mathfrak{g}$ is nilpotent Lie algebra which is not semisimple. A subalgebra of $\mathfrak{g}$ consisting of semisimple elements is called a **toral** subalgebra. It is known that toral subalgebras are abelian and if we let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$, then $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is a family of commuting semisimple endomorphisms. Therefore, $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is simultaneously diagonalizable and we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

known as **root space decomposition** or **Cartan decomposition** of $\mathfrak{g}$ where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$, $\alpha \in \mathfrak{h}^*$ and $\mathfrak{g}_0 = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$. The set $\Phi$ of **roots** of $\mathfrak{g}$ relative to $\mathfrak{h}$ is the set of all nonzero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$. For $\alpha \in \Phi$ an element $x \in \mathfrak{g}_\alpha$ is called a **root vector**.

Let $\beta$ be the Killing form on $\mathfrak{g}$. Then following proposition gives some properties about the root space decomposition.

**Proposition 1.4.1.** *For all $\alpha, \beta \in \Phi$;*

*(a)* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

*(b) If $x \in \mathfrak{g}_\alpha$, then* $\operatorname{ad} x$ *is nilpotent for $\alpha \neq 0$.*

*(c) $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ relative to the Killing form $\beta$ of $\mathfrak{g}$ if $\alpha + \beta \neq 0$,.*

**Corollary 1.4.2.** *The restriction of the Killing form to $\mathfrak{g}_0 = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ is nondegenerate.*

**Proposition 1.4.3.** *If $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$, then $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$.*

Now we can easily deduce that

**Corollary 1.4.4.** *The restriction of $\beta$ to $\mathfrak{h}$ is nondegenerate.*

Therefore we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\beta$. Given $\phi \in \mathfrak{h}^*$, let $t_\phi \in \mathfrak{h}$ be the unique element satisfying $\phi(h) = \beta(t_\phi, h)$ for all $h \in \mathfrak{h}$.

Using the Killing form $\beta$ we can get more information about the root space decomposition.

**Proposition 1.4.5.** *(a) $\Phi$ spans $\mathfrak{h}^*$.*

*(b) If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*

*(c) Let $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$. Then $[x, y] = \beta(x, y)t_\alpha$.*

*(d) If $\alpha \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional, with basis $t_\alpha$.*

*(e) $\alpha(t_\alpha) = \beta(t_\alpha, t_\alpha) \neq 0$ , for $\alpha \in \Phi$.*

*(f)* *If* $\alpha \in \Phi$ *and* $x_\alpha$ *is any nonzero element of* $\mathfrak{g}_\alpha$*, then there exists* $y_\alpha \in \mathfrak{g}_{-\alpha}$ *such that* $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ *span a three dimensional simple subalgebra* $S_\alpha$ *of* $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}(2,k)$ *via* $x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

*(g)* $h_\alpha = 2t_\alpha/\beta(t_\alpha, t_\alpha), \quad h_\alpha = -h_{-\alpha}$.

*(h)* *For* $\alpha \in \Phi$, $\dim \mathfrak{g}_\alpha = 1$*. In particular,* $S_\alpha = \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + \mathfrak{h}_\alpha$ $(\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}])$*, and for given nonzero* $x_\alpha \in \mathfrak{g}_\alpha$*, there exists a unique* $y_\alpha \in \mathfrak{g}_{-\alpha}$ *satisfying* $[x_\alpha, y_\alpha] = h_\alpha$.

We know the restriction of the Killing form to $\mathfrak{h}$ is nondegenerate so we can transfer the form to $\mathfrak{h}^*$ such that for $\gamma, \delta \in \Phi$ we have $(\gamma, \delta) = \beta(t_\gamma, t_\delta)$. Since $\Phi$ spans $\mathfrak{h}^*$, we can choose a basis $\alpha_1, \dots, \alpha_\ell$ of $\mathfrak{h}^*$ consisting of roots. Each element $\beta \in \Phi$ can uniquely be written as $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$, where $c_i \in \mathbb{Q}$ see [12, Section 8.5]. Moreover, the form on $\mathfrak{h}^*_{\mathbb{Q}}$ is positive definite. Now if we extend the base field from $\mathbb{Q}$ to $\mathbb{R}$, i.e. $\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{h}^*_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. This form extends canonically on $\mathfrak{h}^*_{\mathbb{R}}$ which is positive definite and turns $\mathfrak{h}^*_{\mathbb{R}}$ into an Euclidean space. Theorem below gives some properties about roots.

**Theorem 1.4.6.** *Let* $\mathfrak{g}, \mathfrak{h}, \Phi$ *and* $\mathfrak{h}^*_{\mathbb{R}}$ *be as above. Then:*

*(a)* $\Phi$ *spans* $\mathfrak{h}^*_{\mathbb{R}}$*, and* $0$ *does not belong to* $\Phi$.

*(b)* *If* $\alpha \in \Phi$*, then* $-\alpha \in \Phi$*, but no other scalar multiple of* $\alpha$ *is a root.*

*(c)* *If* $\alpha, \beta \in \Phi$*, then* $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$.

*(d)* *If* $\alpha, \beta \in \Phi$*, then* $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$.

## 1.5 Root systems

In this section let $E$ be a fixed Euclidean space. A **reflection** in $E$, is an invertible linear transformation leaving pointwise fixed some hyperplane and sending any vector orthogonal to that hyperplane to its negative. For any nonzero vector $\alpha$ there is a reflection $\sigma_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$, with reflecting hyperplane $P_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$. We use $\langle \beta, \alpha \rangle$ to abbreviate $2(\beta, \alpha)/(\alpha, \alpha)$.

**Definition 1.5.1.** A **root system** in an Euclidean space $E$ is a subset $\Phi$ of $E$ such that the following conditions are satisfied:

(a) $\Phi$ is finite, spans $E$, and does not contain 0.

(b) If $\alpha \in \Phi$, then the only multiples of $\alpha$ in $\Phi$ are $\pm\alpha$.

(c) If $\alpha \in \Phi$, then the reflection $\sigma_\alpha$ leaves $\Phi$ invariant.

(d) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Let $\Phi$ be a root system in $E$. The subgroup $W$ of $\mathrm{GL}(E)$ generated by the reflections $\sigma_\alpha$ where $\alpha \in \Phi$ is called the **Weyl group** of $\Phi$. By part (c) of the definition of a root system, $W$ permutes the set $\Phi$. Then we can identify $W$ with a subgroup of the symmetric group on $\Phi$. Therefore, $W$ is finite.

The following lemma gives a useful property about root systems.

**Proposition 1.5.2.** *Let $\alpha, \beta$ be nonproportional roots. If $(\alpha, \beta) > 0$, then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root.*

Next we recall some standard definitions and properties of root systems.

- A subset $\Pi$ of $\Phi$ is called a **base** if $\Pi$ is a basis of $E$ and each root $\beta$ can be written as $\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha$ with $k_\alpha$ all nonnegative or all nonpositive integers. This expression is unique.

- All root systems have a base, see [12, Theorem 10.1].

- The roots in $\Pi$ are called **simple** and $\operatorname{Card} \Pi = \ell$ where $\ell = \dim E$ which is called the **rank** of a root system $\Phi$.

- The hyperplanes $P_\alpha$ where $\alpha \in \Phi$ partition $E$ into finitely many regions. The connected components of $E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$ are called the (open) **Weyl chambers** of $E$.
- The Weyl chambers are in 1-1 correspondence with bases.
- The **height** of a root $\beta = \sum_{\alpha\in\Pi} k_\alpha \alpha$ relative to $\Pi$ is defined by $\operatorname{ht}\beta = \sum_{\alpha\in\Pi} k_\alpha$.
- If all integral coefficients $k_\alpha \geq 0$ we call $\beta$ **positive** and we write $\beta \succ 0$, otherwise we call it **negative** and we write $\beta \prec 0$.
- The collection of positive and negative roots relative to $\Pi$ is denoted by $\Phi^+$ and $\Phi^-$.
- A root system $\Phi$ is called **irreducible** if it cannot be partitioned into a union of two proper subsets such that each root in one set is orthogonal to each root in the other.
- If $\Phi$ is an irreducible root system then we have at most two root lengths in $\Phi$, and all roots of a given length are conjugate under $W$. These roots are regarded as either **long** or **short roots**.

Let $(\alpha_1, \ldots, \alpha_\ell)$ be an ordering of the simple roots. The matrix $(\langle \alpha_i, \alpha_j\rangle)$ is called the **Cartan matrix** of $\Phi$. The Cartan matrix is independent of the choice of $\Pi$, since $W$ acts transitively on the collection of bases. The Cartan matrix is nonsingular, as $\Pi$ is a basis of $E$. If $\alpha, \beta$ are distinct positive roots, since $\langle \alpha, \beta\rangle\langle \beta, \alpha\rangle = 0, 1, 2, \text{or}, 3$. The graph having $\ell$ vertices such that the $i$th joined to the $j$th $(i \neq j)$ by $\langle \alpha_i, \alpha_j\rangle\langle \alpha_j, \alpha_i\rangle$ edges is called the **Coxeter graph** of $\Phi$. If we have more than one root length, to recognize which root is of short length and which to long length we can add an arrow pointing to the shorter of the two roots and we call it the **Dynkin diagram** of $\Phi$.

**Theorem 1.5.3.** *If $\Phi$ is an irreducible root system of rank $\ell$, then its Dynkin diagram is one of the diagrams from Figure 1.1.*

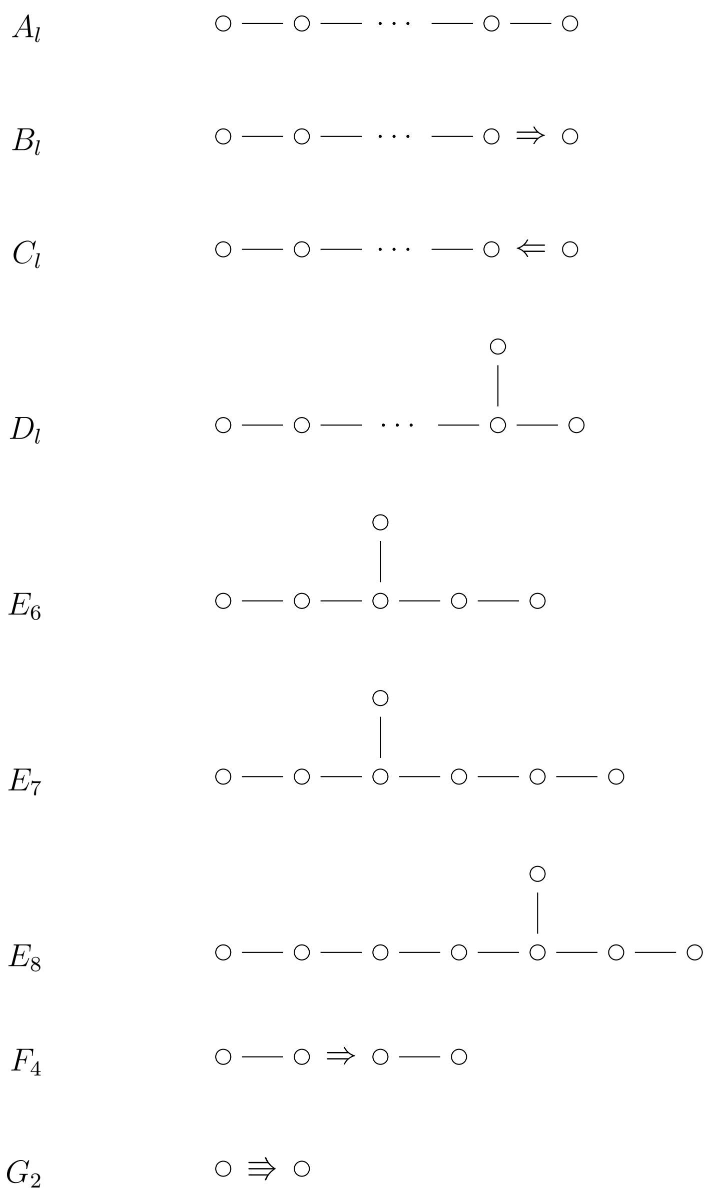


Figure 1.1: List of Dynkin diagrams

## 1.6 Reductive Lie algebras

In this section we give an overview of the structure of reductive Lie algebras. Then we introduce some subalgebras of Lie algebras such as Borel subalgebras and parabolic subalgebras.

Let $\mathfrak{g}$ be a Lie algebra. The intersection of the kernels of all finite dimensional simple representations of $\mathfrak{g}$ is an ideal called the **nilpotent radical** of $\mathfrak{g}$. The following theorem gives an equivalent alternative definition of nilpotent radicals, see [24, Theorem 19.6.2].

**Theorem 1.6.1.** *Let* $\mathfrak{n}$ *be the nilpotent radical of* $\mathfrak{g}$. *Then* $\mathfrak{n} = \operatorname{rad}\mathfrak{g} \cap \mathfrak{D}(\mathfrak{g})$. *In particular, when* $\mathfrak{g}$ *is solvable, then* $\mathfrak{n} = \mathfrak{D}(\mathfrak{g})$.

The following theorem states that any finite dimensional Lie algebra is a semidirect product of a solvable Lie algebra and a semisimple Lie algebra, see [24, Proposition 20.3.5].

**Theorem 1.6.2.** *(**Levi–Malcev Theorem**) Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{r} = \operatorname{rad}\mathfrak{g}$. *There exists a subalgebra* $\mathfrak{s}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$. *Moreover,* $\mathfrak{s}$ *is semisimple.*

**Corollary 1.6.3.** *Let* $\mathfrak{r} = \operatorname{rad}\mathfrak{g}$ *and* $\mathfrak{n}$ *the nilpotent radical of* $\mathfrak{g}$. *Then*

$$\mathfrak{n} = \mathfrak{r} \cap \mathfrak{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{r}].$$

*Proof.* Let $\mathfrak{s}$ be a complement of $\mathfrak{r}$ as in Theorem 1.6.2. Then we have $\mathfrak{D}(\mathfrak{g}) = [\mathfrak{s}, \mathfrak{s}] + [\mathfrak{s}, \mathfrak{r}] + [\mathfrak{r}, \mathfrak{r}] = \mathfrak{s} + [\mathfrak{s}, \mathfrak{r}] + [\mathfrak{r}, \mathfrak{r}] = \mathfrak{s} \oplus [\mathfrak{g}, \mathfrak{r}]$. So by Theorem 1.6.1, $\mathfrak{n} = \mathfrak{r} \cap \mathfrak{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{r}]$. □

A Lie algebra $\mathfrak{g}$ is said to be **reductive** if its adjoint representation is semisimple. The proposition below gives equivalent properties for a Lie algebra $\mathfrak{g}$ to be reductive, see [24, Proposition 20.5.4].

**Proposition 1.6.4.** *Let* $\mathfrak{r} = \operatorname{rad}\mathfrak{g}$ *and* $\mathfrak{n}$ *the nilpotent radical of* $\mathfrak{g}$. *The following conditions are equivalent:*

*(a)* $\mathfrak{g}$ *is reductive.*

*(b)* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ *where* $\mathfrak{s}$ *is semisimple and* $\mathfrak{a}$ *is abelian.*

*(c)* $\mathfrak{g}$ *has a finite-dimensional representation such that the associated bilinear form is nondegenerate.*

*(d)* $\mathfrak{g}$ *has a faithful semisimple representation of finite dimension.*

*(e)* $\mathfrak{n} = \{0\}$.

*(f)* $\mathfrak{r}$ *is the centre of* $\mathfrak{g}$.

**Corollary 1.6.5.** *A Lie algebra* $\mathfrak{g}$ *is reductive if and only if* $\mathfrak{D}(\mathfrak{g})$ *is semisimple. In particular, if* $\mathfrak{g}$ *is reductive, then* $\mathfrak{g} = \mathfrak{D}(\mathfrak{g}) \oplus \mathfrak{z}(\mathfrak{g})$.

## 1.7 Cartan subalgebras, Borel subalgebras and parabolic subalgebras

In this section we give more standard definitions in the theory of Lie algebras. Let $\mathfrak{g}$ be a Lie algebra.

- A maximal solvable subalgebra of $\mathfrak{g}$ is called a **Borel subalgebra** of $\mathfrak{g}$.

  Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a maximal toral subalgebra and $\Phi$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let $\Pi$ be a base of $\Phi$ and let $\Phi^+$ and $\Phi^-$ be the corresponding sets of positive and negative roots respectively. Now let

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha, \quad \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-.$$

  The decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is called the **triangular decomposition of** $\mathfrak{g}$ relative to $\mathfrak{h}$ and $\Phi^+$. The subalgebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent. The subalgebras $\mathfrak{b}^+$

and $\mathfrak{b}^-$ are maximal solvable subalgebras of $\mathfrak{g}$; therefore these are Borel subalgebra of $\mathfrak{g}$ relative to $\mathfrak{h}$. We call $\mathfrak{b}^+$ the **standard Borel subalgebra** relative to $\mathfrak{h}$ and $\Phi^+$.

- A subalgebra of $\mathfrak{g}$ containing a Borel subalgebra is called a **parabolic subalgebra**. A parabolic subalgebra containing $\mathfrak{b}^+$ is called a **standard parabolic subalgebra**.

- A **Cartan subalgebra** of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra which is equal to its normalizer in $\mathfrak{g}$.

  Let $\mathfrak{g}$ be semisimple, with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ relative to a maximal toral subalgebra $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian, it is nilpotent and because $[\mathfrak{h}, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ for $\alpha \in \Phi$ we have $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Therefore, in this case Cartan subalgebras exist and in fact Cartan subalgebras are the same as maximal toral subalgebras.

- Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ the nilpotent radical of $\mathfrak{p}$. A subalgebra $\mathfrak{m}$ of $\mathfrak{p}$ is called a **Levi factor** of $\mathfrak{p}$ if it is reductive in $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$.

**Theorem 1.7.1.** *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$ be the center of $\mathfrak{m}$. Then $\mathfrak{m}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$.*

## 1.8 Classification of semisimple Lie algebras

In this section we first start with describing free Lie algebras. Then we give a presentation of a semisimple Lie algebra $\mathfrak{g}$ by generators and relations which depends only on the root system $\Phi$ of $\mathfrak{g}$. Later we state the theorem about both existence and the uniqueness of a semisimple Lie algebra having $\Phi$ as a root system.

## Free Lie algebras

Let $X = \{x_i \mid i \in I\}$ be a set with an index set $I$. Let $F(X)$ be the set of all finite sums $\sum_{r\geq 0} \sum_{i_1,\dots,i_r \in I} \lambda_{i_1,\dots,i_r} x_{i_1} \dots x_{i_r}$ with $\lambda_{i_1,\dots,i_r} \in k$, $r \in \mathbb{Z}_{\geq 0}$ and all $i_1, \dots, i_r \in I$. When $r = 0$ the product $x_{i_1} \dots x_{i_r}$ is the empty product, and written as 1. The operation of addition, multiplication and scalar multiplication are in an obvious way making $F(X)$ into an associative algebra over $k$ with identity 1. Let $[F(X)]$ be the Lie algebra corresponding to the associative algebra $F(X)$ and let $\mathfrak{FL}(X)$ be the intersection of all the Lie subalgebras of $[F(X)]$ containing $X$. Then $\mathfrak{FL}(X)$ is called **the free Lie algebra** on the set $X$.

Let $X$ be a set as above. A **Lie monomial** in the elements of $X$ is a finite product of elements of $X$ bracketed by Lie brackets in any manner. A **Lie word** in the elements on $X$ is a finite linear combination of Lie monomials on $X$ with coefficients in $k$. Let $R = \{w_j \mid j \in J\}$ be a set of Lie words in the elements of $X$. We know all Lie words $w_j$ lie in $\mathfrak{FL}(X)$. Let $\langle R \rangle$ be the ideal of $\mathfrak{FL}(X)$ generated by $R$, i.e. the intersection of all ideals of $\mathfrak{FL}(X)$ containing $R$. We call the Lie algebra $\mathfrak{FL}(X)/\langle R \rangle$ **the Lie algebra generated by $X$ subject to relations $R$.**

In the following proposition we single out a set of generators for a semisimple Lie algebra $\mathfrak{g}$.

**Proposition 1.8.1.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$, and $\Phi$ the corresponding root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let $\Pi$ be a fixed base of $\Phi$. Then $\mathfrak{g}$ is generated by arbitrary nonzero root vectors $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Pi$.*

*Proof.* Let $\beta$ be an arbitrary positive root relative to $\Pi$. We know that $\beta$ can be written in the form $\beta = \alpha_1 + \ldots + \alpha_s$ where $\alpha_i \in \Pi$ and where each partial sum $\alpha_1 + \ldots + \alpha_i$ is a root. We also know that $[\mathfrak{g}_\gamma, \mathfrak{g}_\delta] = \mathfrak{g}_{\gamma+\delta}$ whenever $\gamma, \delta, \gamma + \delta \in \Phi$. By using induction on $s$, we see that $\mathfrak{g}_\beta$ lies in the subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}_\alpha$ where $\alpha \in \Pi$. Similarly if $\beta$ is negative, then $\mathfrak{g}_\beta$ lies in a subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}_{-\alpha}$ where $\alpha \in \Pi$. But

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ and $\mathfrak{h} = \sum_{\alpha\in\Phi}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, so the proposition follows. $\square$

Now let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a fixed base of $\Phi$. We know $\langle\alpha_i, \alpha_j\rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \alpha_i(h_j)$ where $h_j = h_{\alpha_j}$. Now choose a standard set of generators $x_i \in \mathfrak{g}_{\alpha_i}, y_i \in \mathfrak{g}_{-\alpha_i}$ so that $[x_i, y_i] = h_i$.

**Proposition 1.8.2.** *With the above notation,* $\mathfrak{g}$ *is generated by* $\{x_i, y_i, h_i \mid 1 \le i \le \ell\}$, *and these generators satisfy the following relations*

*(a)* $[h_i, h_j] = 0$ *for* $1 \le i, j \le \ell$.

*(b)* $[x_i, y_i] = h_i$, $[x_i, y_j] = 0$ *if* $i \ne j$.

*(c)* $[h_i, x_j] = \langle\alpha_j, \alpha_i\rangle x_j$, $[h_i, y_j] = -\langle\alpha_j, \alpha_i\rangle y_j$

*(d)* $(\operatorname{ad} x_i)^{-\langle\alpha_j,\alpha_i\rangle+1}(x_j) = 0$ *for* $i \ne j$

*(e)* $(\operatorname{ad} y_i)^{-\langle\alpha_j,\alpha_i\rangle+1}(y_j) = 0$ *for* $i \ne j$.

Now fix a root system $\Phi$ with base $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. We abbreviate the Cartan integers $\langle\alpha_i, \alpha_j\rangle$ by $c_{ij}$. Let $\mathfrak{FL}(X)$ be the free Lie algebra on the set $X = \{\tilde{x}_i, \tilde{y}_i, \tilde{h}_i \mid 1 \le i \le \ell\}$ and $\langle R\rangle$ be the ideal in $\mathfrak{FL}(X)$ generated by the set of elements $R = \{[\tilde{h}_i, \tilde{h}_j],\ [\tilde{x}_i, \tilde{y}_j] - \delta_{ij}\tilde{h}_i,\ [\tilde{h}_i, \tilde{x}_j] - c_{ji}\tilde{x}_j,\ [\tilde{h}_i, \tilde{y}_j] + c_{ji}\tilde{y}_j\}$. Let $\tilde{\mathfrak{g}} = \mathfrak{FL}(X)/\langle R\rangle$ and $x_i, y_i, h_i$ be the images of generators in $\tilde{\mathfrak{g}}$. Now we construct a module for the free Lie algebra $\mathfrak{FL}(X)$. We define a linear transformation corresponding to each of the $3\ell$ generators. Let $V$ be the tensor algebra on a vector space with basis $(v_1, \ldots, v_\ell)$. The tensors $v_{i_1} \otimes \ldots \otimes v_{i_t}$ for $t \in \mathbb{Z} \ge 1$, $i_j \in \{1, \ldots, \ell\}$ together with 1 form a basis of $V$ over $k$. Now we define automorphisms as follows

$$\tilde{h}_j \cdot 1 = 0$$

$$\tilde{h}_j \cdot v_{i_1} \ldots v_{i_t} = -(c_{i_1 j} + \ldots + c_{i_t j}) v_{i_1} \ldots v_{i_t}$$

$$\tilde{y_j} \cdot 1 = v_j$$

$$\tilde{y_j} \cdot v_{i_1} \ldots v_{i_t} = v_j v_{i_1} \ldots v_{i_t}$$

$$\tilde{x_j} \cdot 1 = 0 = \tilde{x_j} \cdot v_i$$

$$\tilde{x_j} \cdot v_{i_1} \ldots v_{i_t} = v_{i_1}(\tilde{x_j} \cdot v_{i_2} \ldots v_{i_t}) - \delta_{i_1 j}(c_{i_2 j} + \ldots + c_{i_t j}) v_{i_2} \ldots v_{i_t}$$

Then there is a unique extension to $\mathfrak{FL}(X)$ of this action by its generators which gives us a representation $\tilde{\varphi} : \mathfrak{FL}(X) \to \mathfrak{gl}(V)$. We can show $\langle R \rangle \subset \ker \tilde{\varphi}$, see [12, Proposition 18.2]. So $\tilde{\varphi}$ factors through $\tilde{\mathfrak{g}}$, making $V$ a $\tilde{\mathfrak{g}}$-module. The theorem below gives useful information about $\tilde{\mathfrak{g}}$ from the existence of this homomorphism, see [12, Theorem 18.2].

**Theorem 1.8.3.** *Let $\Phi$ be a root system with base $\{\alpha_1, \ldots, \alpha_\ell\}$. Let $\tilde{\mathfrak{g}}$ be the Lie algebra with generators $\{x_i, y_i, h_i \mid 1 \leq i \leq \ell\}$ and relations (a)–(c) of Proposition 1.8.2. Then the $h_i$ are a basis for an $\ell$-dimensional abelian subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}} \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^-$ where $\tilde{\mathfrak{n}}$, $\tilde{\mathfrak{n}}^-$ are subalgebras of $\tilde{\mathfrak{g}}$ generated by $x_i$ and $y_i$, respectively.*

The proof of the preceding theorem shows that we can describe the decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}} \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^-$ in terms of weights. For $\lambda \in \mathfrak{h}^*$ we define

$$\tilde{\mathfrak{g}}_\lambda = \{x \in \tilde{\mathfrak{g}} \mid [h, x] = \lambda(h)x \text{ for all } h \in \mathfrak{h}\}.$$

All weights $\lambda$ which occur in $\tilde{\mathfrak{g}}$ have the form $n_1\alpha_1 + \ldots + n_\ell\alpha_\ell$ with either $n_i \in \mathbb{Z}_{\geq}0$ or $n_i \in \mathbb{Z}_{\leq}0$ for all $i$.

The next theorem is the case when we impose the "finiteness" conditions (d) and (e) of Proposition 1.8.2.

**Theorem 1.8.4 (Serre's Theorem).** *Let $\Phi$ be a root system with base $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. Let $\mathfrak{g}$ be the Lie algebra generated by $3\ell$ elements $\{x_i, y_i, h_i \mid 1 \leq i \leq \ell\}$ with relations (a)–(e) listed in Proposition 1.8.2. Then $\mathfrak{g}$ is a finite dimensional semisimple algebra with maximal toral subalgebra generated by the $h_i$ and with corresponding root system $\Phi$.*

*Sketch proof.* Let $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{m}$ where $\mathfrak{m}$ is the ideal generated by elements $x_{ij} = (\operatorname{ad} x_i)^{-c_{ji}+1}(x_j)$, $y_{ij} = (\operatorname{ad} y_i)^{-c_{ji}+1}(y_j)$ with $i \neq j$. Let $\mathfrak{i}$ be the ideal of $\tilde{\mathfrak{n}}$ generated by $x_{ij}$ and $\mathfrak{j}$ be an ideal of $\tilde{\mathfrak{n}}^-$ generated by all $y_{ij}$. It can be shown that $\mathfrak{i}$ and $\mathfrak{j}$ are ideals of $\tilde{\mathfrak{g}}$ and in fact $\mathfrak{m} = \mathfrak{i} \oplus \mathfrak{j}$. Now as $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}$ we can say that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ where $\mathfrak{n}^- = \tilde{\mathfrak{n}}^-/\mathfrak{j}$ and $\mathfrak{n} = \tilde{\mathfrak{n}}/\mathfrak{i}$ and $\mathfrak{h}$ is the identified with its image under the canonical map $\tilde{\mathfrak{g}} \to \mathfrak{g}$. As $\sum Kx_i + \sum Kh_i + \sum Ky_i$ maps isomorphically into $\mathfrak{g}$ we may identify $x_i, y_i, h_i$ with elements of $\mathfrak{g}$ and we can consider decomposition of $\mathfrak{g}$ into weight spaces with respect to $\mathfrak{h}$. For each weight $\lambda \in \mathfrak{h}^*$ we define the weight space $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [h, x] = \lambda(h)x \text{ for all } h \in \mathfrak{h}\}$. The algebra $\mathfrak{g}$ is direct sum of its weight spaces and we have $\mathfrak{h} = \mathfrak{g}_0$, $\mathfrak{n} = \bigoplus_{\lambda \succ 0} \mathfrak{g}_\lambda$, and $\mathfrak{n}^- = \bigoplus_{\lambda \prec 0} \mathfrak{g}_\lambda$ moreover $\mathfrak{g}_\lambda$ is finite dimensional. The maps $\operatorname{ad} x_i$ and $\operatorname{ad} y_i$ for $1 \leq i \leq \ell$ are locally nilpotent. Hence $\tau_i = \exp(\operatorname{ad} x_i)\exp(\operatorname{ad}(-y_i))\exp(\operatorname{ad} x_i)$ is a well defined automorphism of $\mathfrak{g}$. If $\lambda, \mu \in \mathfrak{h}^*$ and $\sigma\lambda = \mu$ for $\sigma \in W$ where $W$ is the Weyl group of $\Phi$, then $\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_\mu$. To prove this we show that this is true when $\sigma = \sigma_{\alpha_i}$. This is because the automorphism $\tau_i$ of $\mathfrak{g}$ coincides on the finite dimensional space $\mathfrak{g}_\lambda + \mathfrak{g}_\mu$ with the ordinary product of exponentials, and therefore $\tau_i$ interchanges $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ and $\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_\mu$. We also for $1 \leq i \leq \ell$ have $\dim \mathfrak{g}_{\alpha_i} = 1$, while $\mathfrak{g}_{k\alpha_i} = 0$ for $k \neq 0, 1, -1$. Because each root is $W$-conjugate to a simple root so if $\alpha \in \Phi$, then $\dim \mathfrak{g}_\alpha = 1$, but $\mathfrak{g}_{k\alpha} = 0$ for $k \neq 0, 1, -1$. Furthermore $\mathfrak{g}$ is semisimple. Let $A$ be an abelian ideal of $\mathfrak{g}$. We show $A = 0$. Because $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$ and $\operatorname{ad} \mathfrak{h}$ stabilizes $A$, we have $A = (A \cap \mathfrak{h}) + \sum_{\alpha \in \Phi}(A \cap \mathfrak{g}_\alpha)$. If $\mathfrak{g}_\alpha \subset A$, then $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \subset A$, and $\mathfrak{g}_{-\alpha} \subset A$ and $A$ contains a copy of the simple algebra $\mathfrak{sl}_2$ which is absurd. So $A = A \cap \mathfrak{h} \subset \mathfrak{h}$, therefore $[\mathfrak{g}_\alpha, A] = 0$ and $A \subset \bigcap_{\alpha \in \Phi} \ker \alpha = 0$. Finally we can show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with the corresponding set of roots $\Phi$. This is because $\mathfrak{h}$ is abelian hence nilpotent and self normalizing i.e, $\mathfrak{h}$ is a Cartan subalgebra. Then obviously $\Phi$ is the corresponding set of roots. □

The following theorem states both the existence and the uniqueness of a semisimple Lie algebra having root system $\Phi$.

**Theorem 1.8.5.** *(a) Let $\Phi$ be a root system. Then there exists a semisimple Lie algebra having $\Phi$ as its root system.*

*(b) Let $\mathfrak{g}$ and $\mathfrak{g}'$ be semisimple Lie algebras, with maximal toral subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ and root systems $\Phi$ and $\Phi'$, respectively. Let $\Phi \to \Phi'$ be an isomorphism which sends a given base $\Pi$ to a base $\Pi'$. We denote by $\pi : \mathfrak{h} \to \mathfrak{h}'$ the associated isomorphism. For each $\alpha \in \Pi$ and $\alpha' \in \Pi'$ choose arbitrary nonzero $x_\alpha \in \mathfrak{g}_\alpha$ and $x'_{\alpha'} \in \mathfrak{g}'_{\alpha'}$, respectively. Then there exists a unique isomorphism $\pi : \mathfrak{g} \to \mathfrak{g}'$ extending $\pi : \mathfrak{h} \to \mathfrak{h}'$ and sending $x_\alpha$ to $x'_{\alpha'}$ for $\alpha \in \Pi$.*

*Proof.* Part (a) follows from Theorem 1.8.4. For part (b) choose $y_\alpha$, $y'_\alpha$ satisfying $[x_\alpha, y_\alpha] = h_\alpha$, $[x'_{\alpha'}, y'_{\alpha'}] = h'_{\alpha'} = \pi(h_\alpha)$ for $\alpha \in \Pi$, and $\alpha' \in \Pi'$. Since $x'_{\alpha'}, y'_{\alpha'}, h'_{\alpha'}$ satisfy the relations in Proposition 1.8.3, there is a unique homomorphism $\pi : \mathfrak{g} \to \mathfrak{g}'$ sending $x_\alpha, y_\alpha, h_\alpha$ to $x'_{\alpha'}, y'_{\alpha'}, h'_{\alpha'}$, respectively. This $\pi$ extends the given isomorphism $\mathfrak{h} \to \mathfrak{h}'$. Similarly there is a homomorphism $\pi' : \mathfrak{g}' \to \mathfrak{g}$ and the composition of these homomorphisms are the identity on the generators for $\mathfrak{g}$ or $\mathfrak{g}'$, hence $\pi$ is an isomorphism. □

# Chapter 2

# Levi type root systems and centralizers of nilpotent elements in semisimple Lie algebras

In this chapter we define root systems of Levi type corresponding to a proper parabolic subalgebra of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. We will use some theorems from the first chapter to establish properties of these root systems that replicates much of the structure of classical root systems. The materials can be found in [3] and [18]. We will also use representation theory of semisimple Lie algebras from [12, Chapter 6].

## 2.1 Root systems of Levi type for finite dimensional semisimple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ with a root system $\Phi$ relative to a maximal toral subalgebra $\mathfrak{h}$. Let $\Phi^+$ be the set of positive roots corresponding to a set of simple roots $\Pi$. For each $\alpha \in \Phi$, let $x_\alpha \in \mathfrak{g}_\alpha$ be a corresponding root vector. For any $\gamma \in \Phi$ let $\gamma = \sum_{\alpha\in\Pi} k_\alpha \alpha$ where $k_\alpha \in \mathbb{Z}$. Let $\mathfrak{u} \subseteq \mathfrak{g}$ be an ad $\mathfrak{h}$-stable subspace of

$\mathfrak{g}$. Let $\Phi(\mathfrak{u}) = \{\alpha \in \Phi \mid x_\alpha \in \mathfrak{u}\}$, $\Phi^+(\mathfrak{u}) = \Phi(\mathfrak{u}) \cap \Phi^+$ and $\Phi^-(\mathfrak{u}) = \Phi(\mathfrak{u}) \cap \Phi^-$. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$, containing $\mathfrak{h}$ such that $\Phi(\mathfrak{b}) = \Phi^+$. Assume that $\mathfrak{p}$ is some fixed standard proper parabolic subalgebra of $\mathfrak{g}$ and let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ be the unique Levi decomposition of $\mathfrak{p}$ where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$ and $\mathfrak{m}$ is a Levi factor of $\mathfrak{p}$ which contains $\mathfrak{h}$. Since $\mathfrak{m}$ is reductive, we have $\mathfrak{m} = \mathfrak{t} \oplus \mathfrak{s}$ where $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$ is the center of $\mathfrak{m}$ and $\mathfrak{s} = [\mathfrak{m}, \mathfrak{m}]$ is the unique maximal semisimple ideal in $\mathfrak{m}$, see Section 1.6.

Let $\beta$ be the Killing form on $\mathfrak{g}$ and let $\mathfrak{h}(\mathfrak{s}) = \mathfrak{h} \cap \mathfrak{s}$ which is a Cartan subalgebra of $\mathfrak{s}$.

**Proposition 2.1.1.** *The direct sum $\mathfrak{m} = \mathfrak{t} \oplus \mathfrak{s}$ is a $\beta$-orthogonal decomposition.*

*Proof.* We have $\mathfrak{t} \subseteq \mathfrak{h}$ and $\mathfrak{s} = \mathfrak{h}(\mathfrak{s}) \oplus \bigoplus_{\alpha\in\Phi(\mathfrak{s})} \mathfrak{g}_\alpha$. Also $\mathfrak{h}$ is orthogonal to $\mathfrak{g}_\alpha$ for all $\alpha \in \Phi$. We just need to check that $\mathfrak{t}$ is $\beta$-orthogonal to $\mathfrak{h}(\mathfrak{s})$. Let $x \in \mathfrak{t}$ and $y = [y_1, y_2] \in \mathfrak{h}(\mathfrak{s})$ where $y_1, y_2 \in \mathfrak{m}$. Since $x \in \mathfrak{z}(\mathfrak{m})$ and $y_1 \in \mathfrak{m}$ we have $\operatorname{ad} x \operatorname{ad} y_1 = \operatorname{ad} y_1 \operatorname{ad} x$. Then $\beta(x, y) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = \operatorname{tr}(\operatorname{ad} x(\operatorname{ad}[y_1, y_2]) = \operatorname{tr}(\operatorname{ad} x(\operatorname{ad} y_1 \operatorname{ad} y_2 - \operatorname{ad} y_2 \operatorname{ad} y_1)) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y_1 \operatorname{ad} y_2 - \operatorname{ad} x \operatorname{ad} y_2 \operatorname{ad} y_1) = \operatorname{tr}(\operatorname{ad} y_1(\operatorname{ad} x \operatorname{ad} y_2)) - \operatorname{tr}((\operatorname{ad} x \operatorname{ad} y_2) \operatorname{ad} y_1) = 0$. □

**Corollary 2.1.2.** *The direct sum $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{h}(\mathfrak{s})$ is a $\beta$-orthogonal decomposition*

Let $\overline{\mathfrak{n}}$ be the span of all $x_{-\alpha}$ for $\alpha \in \Phi(\mathfrak{n})$. Then we have a triangular decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$. Let $\mathfrak{r} = \mathfrak{n} \oplus \overline{\mathfrak{n}}$. Then we have a $\beta$-orthogonal decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{r}$. So $\beta|_{\mathfrak{m}}$ is nondegenerate. Because $\mathfrak{m} = \mathfrak{t} \oplus \mathfrak{s}$ is a $\beta$-orthogonal decomposition so $\beta|_{\mathfrak{t}}$ and $\beta|_{\mathfrak{s}}$ are nondegenerate. Moreover, we have $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{h}(\mathfrak{s})$ as a $\beta$-orthogonal decomposition and $\beta$ is nondegenerate on $\mathfrak{h}(\mathfrak{s})$.

Let $\mathfrak{h}^*$ be the dual space to $\mathfrak{h}$. We know the nondegenerate bilinear form $\beta|_{\mathfrak{h}}$ induces a nondegenerate bilinear form on $\mathfrak{h}^*$ which we also denote by $\beta|_{\mathfrak{h}^*}$. Now we can embed the dual spaces $\mathfrak{t}^*$ and $\mathfrak{h}(\mathfrak{s})^*$ in $\mathfrak{h}^*$ such that $\mathfrak{t}^*$ is the annihilator of $\mathfrak{h}(\mathfrak{s})$ and $\mathfrak{h}(\mathfrak{s})^*$ is the annihilator of $\mathfrak{t}$. Then $\beta|_{\mathfrak{h}^*}$ is nondegenerate on both $\mathfrak{t}^*$ and $\mathfrak{h}(\mathfrak{s})^*$ and $\mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{h}(\mathfrak{s})^*$ is a $\beta|_{\mathfrak{h}^*}$-orthogonal direct sum.

Let $\mathfrak{h}^*_{\mathbb{R}}$ be the real form of $\mathfrak{h}^*$ spanned over $\mathbb{R}$ by $\Phi$. Therefore $\beta|_{\mathfrak{h}^*_{\mathbb{R}}}$ is positive definite on $\mathfrak{h}^*_{\mathbb{R}}$ and we use $(\cdot,\cdot)$ to denote the $\beta|_{\mathfrak{h}^*}$. Now similarly, let $\mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$ be the real form of $\mathfrak{h}(\mathfrak{s})^*$ spanned over $\mathbb{R}$ by $\Phi(\mathfrak{s})$. Clearly $\mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$ is a real subspace of $\mathfrak{h}^*_{\mathbb{R}}$ and, if $\mathfrak{t}^*_{\mathbb{R}}$ is the $\beta|_{\mathfrak{h}^*_{\mathbb{R}}}$-orthogonal complement of $\mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$ in $\mathfrak{h}^*_{\mathbb{R}}$, then $\mathfrak{t}^*_{\mathbb{R}}$ is a real form of $\mathfrak{t}^*$ and $\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{t}^*_{\mathbb{R}} \oplus \mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$. Now for any $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$, let $\gamma = \gamma_{\mathfrak{t}} + \gamma_{\mathfrak{s}}$ where $\gamma_{\mathfrak{t}} \in \mathfrak{t}^*_{\mathbb{R}}$ and $\gamma_{\mathfrak{s}} \in \mathfrak{h}(\mathfrak{s})^*_{\mathbb{R}}$, restrictions of $\gamma$ to $\mathfrak{t}$ and $\mathfrak{s}$ respectively, are the components of $\gamma$ with respect to the decomposition so that $(\gamma_{\mathfrak{t}}, \gamma_{\mathfrak{s}}) = 0$.

Note that $\mathfrak{m}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$ by Theorem 1.7.1 and

$$\Phi(\mathfrak{r}) = \{\alpha \in \Phi \mid \alpha_{\mathfrak{t}} \neq 0\}.$$

Let $V$ be a $\mathfrak{g}$-module. Then $V$ is a $\mathfrak{t}$-module and $\mathfrak{t}$ acts semisimply on $V$. Let

$$V_\mu = \{v \in V \mid x \cdot v = \mu(x)v, \text{ for all } x \in \mathfrak{t}\}$$

where $\mu \in \mathfrak{t}^*$. The subspace $V_\mu$ is called the **$\mu$-weight space for $\mathfrak{t}$** of $V$. If $V_\mu \neq 0$, then $\mu$ is called a **$\mathfrak{t}$-weight** of $V$ and any $v \in V_\mu$ is called a **$\mu$-weight vector**. So $\mu \in \mathfrak{t}^*$ is a $\mathfrak{t}$-weight of $V$ if and only if $\mu = \gamma_{\mathfrak{t}}$ where $\gamma$ is any $\mathfrak{h}$-weight of $V$.

Now consider $\mathfrak{g}$ as a $\mathfrak{g}$-module defined by adjoint action. For $\mu \in \mathfrak{t}^*$ let

$$\mathfrak{g}_\mu = \{x \in \mathfrak{g} \mid [t,x] = \mu(t)x \text{ for all } t \in \mathfrak{t}\}$$

and

$$R' = \{\mu \in \mathfrak{t}^* \mid \mathfrak{g}_\mu \neq 0\}.$$

Let $V$ be any $\mathfrak{g}$-module, $\xi$ be a $\mathfrak{t}$-weight of $V$ and $\mu \in R'$. Then $\mathfrak{g}_\mu \cdot V_\xi \subset V_{\mu+\xi}$. Clearly

$0 \in R'$ and $\mathfrak{g}_0 = \mathfrak{m}$ so that $V_\xi$ is an $\mathfrak{m}$-module. Let $R = R' \setminus \{0\}$. So

$$R = \{\nu \in \mathfrak{t}^* \mid \nu = \varphi_{\mathfrak{t}} \text{ for some } \varphi \in \Phi(\mathfrak{r})\}$$

and moreover,

$$\mathfrak{g} = \mathfrak{m} \oplus \bigoplus_{\nu \in R} \mathfrak{g}_\nu.$$

We refer to elements in $R$ as **Levi type roots** in $\mathfrak{g}$ corresponding to a Levi factor $\mathfrak{m}$ of the parabolic subalgebra $\mathfrak{p}$. Also we call $\mathfrak{g}_\nu$ the **Levi root space** for any $\nu \in R$.

From these discussions we have disjoint unions $\Phi = \bigsqcup_{\mu \in R'} \Phi(\mathfrak{g}_\mu)$ and $\Phi(\mathfrak{r}) = \bigsqcup_{\nu \in R} \Phi(\mathfrak{g}_\nu)$. Moreover if $\nu \in R$, then $\Phi(\mathfrak{g}_\nu) = \{\alpha \in \Phi \mid \alpha_{\mathfrak{t}} = \nu\}$ and the set $\{x_\alpha \mid \alpha \in \Phi(\mathfrak{g}_\nu)\}$ is a basis of $\mathfrak{g}_\nu$. Note that $\nu \in R$ if and only if $-\nu \in R$ and $\Phi(\mathfrak{g}_{-\nu}) = -\Phi(\mathfrak{g}_\nu)$. If $\nu, \mu \in R$, then clearly $[\mathfrak{g}_\nu, \mathfrak{g}_0] \subset \mathfrak{g}_\nu$ and if $\nu + \mu \in R$ then $[\mathfrak{g}_\nu, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\nu+\mu}$. Furthermore if $\mu, \nu \in R$ and $\nu \neq -\mu$, then $(\mathfrak{g}_\mu, \mathfrak{g}_\nu) = 0$ and $\mathfrak{g}_\nu$ and $\mathfrak{g}_{-\nu}$ are nondegenerately paired by $(\cdot, \cdot)$.

Let $\tau : \mathfrak{h} \to \mathfrak{h}^*$ be the linear isomorphism defined by $\beta|_{\mathfrak{h}}$. Thus for $x \in \mathfrak{h}$ and $\mu \in \mathfrak{h}^*$,

$$\langle \mu, x \rangle = (\mu, \tau(x)) = (\tau^{-1}(\mu), x) \tag{2.1}$$

Thus if $\mathfrak{t}_{\mathbb{R}} = \tau^{-1}(\mathfrak{t}^*_{\mathbb{R}})$, then we have the following proposition.

**Proposition 2.1.3.** *(a)* $\mathfrak{t}_{\mathbb{R}}$ *is a real form of* $\mathfrak{t}$.

*(b)* $\beta$ *is real and positive definite on* $\mathfrak{t}_{\mathbb{R}}$

*(c)* $\mathfrak{t}_{\mathbb{R}} = \{x \in \mathfrak{t} \mid \nu(x) \in \mathbb{R}$ *for all* $\nu \in R\}$.

*Proof.* (a) This is clear as $\mathfrak{t}^*_{\mathbb{R}}$ is a real form of $\mathfrak{t}^*$.
(b) $\beta$ is real and positive definite on $\mathfrak{h}_{\mathbb{R}}$ then it is also on $\mathfrak{t}_{\mathbb{R}}$ as $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{h}_{\mathbb{R}}$, .
(c) Let $x \in \mathfrak{t}_{\mathbb{R}} \subset \mathfrak{t}$. We need to show $\nu(x) \in \mathbb{R}$ for all $\nu \in R \subseteq \mathfrak{t}^*$. We know $\tau(x) \in \mathfrak{t}^*_{\mathbb{R}}$ and $(\tau(x), \nu) = \nu(x)$ by definition of $\tau$. Since $\tau(x), \nu \in \mathfrak{t}^*_{\mathbb{R}}$ and $\beta$ is real on $\mathfrak{t}^*_{\mathbb{R}}$, we have

$\nu(x) \in \mathbb{R}$.

Conversely, suppose $x \in \mathfrak{t}$ and $\nu(x) \in \mathbb{R}$ for all $\nu \in R$. We need to show that $x \in \mathfrak{t}_{\mathbb{R}}$ or equivalently $\tau(x) \in \mathfrak{t}^*_{\mathbb{R}}$. By definition $(\tau(x), \nu) = \nu(x) \in \mathbb{R}$ for all $\nu \in R$. Also $\mathfrak{t}^*$ is spanned by $R$. As $\tau(x) \in \mathfrak{t}^*$ we can write $\tau(x) = \sum_{i=1}^{r} b_i\beta_i$ where $B = \{\beta_1, \dots, \beta_r\} \subseteq R$ is a basis of $\mathfrak{t}^*$ and $b_i \in \mathbb{C}$ for $1 \le i \le r$. Then we have $(\tau(x), \beta_j) = \sum_{i=1}^{r} b_i(\beta_i, \beta_j) \in \mathbb{R}$ for all $\beta_j \in B$. We know that the matrix $((\beta_i, \beta_j))_{i,j=1,\dots,r}$ is an invertible matrix over $\mathbb{R}$ as $(\cdot, \cdot)$ is nondegenerate. So $((\beta_i, \beta_j))^{-1}_{i,j=1,\dots,r}$ is a real matrix. Thus $b_i \in \mathbb{R}$ for $1 \le i \le r$. Hence we have proved $\tau(x) \in \mathfrak{t}^*_{\mathbb{R}}$ and the result follows. □

Let $\nu \in R$. Then $\ker \nu$ has codimension 1 in $\mathfrak{t}$. So $\ker \nu^{\perp} = \langle x \rangle$ is one dimensional subspace of $\mathfrak{t}$. Then $\langle \nu, x \rangle = \lambda$ for some $0 \neq \lambda \in \mathbb{R}$ by Proposition 2.1.3 part (b) and there exists a unique element $h_\nu = 2x/\lambda \in \mathfrak{t}_{\mathbb{R}}$ which is $\beta$-orthogonal to $\ker \nu$ and $\langle \nu, h_\nu \rangle = 2$ and by definition of $\tau$ we have

$$\tau(h_\nu) = 2\nu/(\nu, \nu) \tag{2.2}$$

Let $\nu \in R$ and $\mathfrak{m}(\nu) = [\mathfrak{g}_\nu, \mathfrak{g}_{-\nu}]$. Then $\mathfrak{m}(\nu)$ is an ideal of $\mathfrak{m}$ by the Jacobi identity. Let $\mathfrak{t}(\nu) = \mathfrak{m}(\nu) \cap \mathfrak{t}$ and $\mathfrak{s}(\nu) = \mathfrak{m}(\nu) \cap \mathfrak{s}$. By restriction $\mathfrak{m}(\nu)$ is an $\mathfrak{s}$-module. As $\mathfrak{s}$ is semisimple we can decompose $\mathfrak{m}(\nu)$ as a direct sum of irreducible $\mathfrak{s}$-modules. Let $\mathfrak{m}(\nu)_0$ be the direct sum of trivial $\mathfrak{s}$-submodules of $\mathfrak{m}(\nu)$. Because $\mathfrak{t}(\nu) \subseteq \mathfrak{t}$ commutes with $\mathfrak{s}$, then $\mathfrak{t}(\nu) \subseteq \mathfrak{m}(\nu)_0$. Also, since $\mathfrak{z}(\mathfrak{s}) = 0$ then $\mathfrak{s}(\nu) \cap \mathfrak{m}(\nu)_0 = \varnothing$. Thus $\mathfrak{t}(\nu)$ is equal to $\mathfrak{m}(\nu)_0$ and $\mathfrak{s}(\nu)$ is the sum of the remaining components. Consequently we have the direct sum

$$\mathfrak{m}(\nu) = \mathfrak{t}(\nu) \oplus \mathfrak{s}(\nu).$$

If $\mathfrak{s}_i$ is a simple component of $\mathfrak{s}$, then $\Phi(\mathfrak{s}_i) = -\Phi(\mathfrak{s}_i)$ and the restriction of $\beta$ to $\mathfrak{s}_i$ is nonzero and nondegenerate. If $\mathfrak{s}_i$ and $\mathfrak{s}_j$ are distinct simple components then $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$.

Moreover, $\mathfrak{s}_i$ and $\mathfrak{s}_j$ are $\beta$-orthogonal by the equality $[\mathfrak{s}_i, \mathfrak{s}_i] = \mathfrak{s}_i$ and the associativity of $\beta$.

**Proposition 2.1.4.** *Let $\nu \in R$. Then*

$$\mathfrak{t}(\nu) = \mathbb{C}h_\nu.$$

*In addition $\beta|_{\mathfrak{m}(\nu)}$ is nondegenerate and the kernel of the adjoint action of $\mathfrak{m}$ on $\mathfrak{g}_\nu$ is the orthogonal complement of $\mathfrak{m}(\nu)$ in $\mathfrak{m}$. In particular $\mathfrak{m}(\nu)$ acts faithfully on $\mathfrak{g}_\nu$.*

*Proof.* Let $\mathfrak{m}(\nu)^\perp$ be the $\beta$-orthogonal subspace to $\mathfrak{m}(\nu)$ in $\mathfrak{m}$. First we show that $\mathfrak{m}(\nu)^\perp$ is the kernel of the adjoint action of $\mathfrak{m}$ on $\mathfrak{g}_\nu$. Let $x \in \mathfrak{m}, y \in \mathfrak{g}_\nu$, and $z \in \mathfrak{g}_{-\nu}$. By associativity (invariance) of $\beta$ we have $\beta(x, [y, z]) = \beta([x, y], z)$ . Now if $x \in \mathfrak{m}(\nu)^\perp$ as $\mathfrak{g}_\nu$ and $\mathfrak{g}_{-\nu}$ are nondegenerately paired, then $[x, y] = 0$ for all $y \in \mathfrak{g}_\nu$. Therefore $\mathfrak{m}(\nu)^\perp$ is in the kernel of the adjoint action of $\mathfrak{m}$ on $\mathfrak{g}_\nu$.

Conversely, suppose $x$ is in the kernel of the adjoint action of $\mathfrak{m}$ on $\mathfrak{g}_\nu$. Then $[x, y] = 0$ for all $y \in \mathfrak{g}_\nu$. Therefore for all $z \in \mathfrak{g}_{-\nu}$ we have $= \beta(x, [y, z]) = \beta([x, y], z) = 0$. Thus $\beta(x, w) = 0$ for all $w \in \mathfrak{m}(\nu)$ and $x \in \mathfrak{m}(\nu)^\perp$. So we showed that $\mathfrak{m}(\nu)^\perp$ is the kernel of the adjoint action of $\mathfrak{m}$ on $\mathfrak{g}_\nu$. Thus we have $\ker \nu = \mathfrak{m}(\nu)^\perp \cap \mathfrak{t}$. As we mentioned before $\mathfrak{t} = \ker \nu \oplus \mathbb{C}h_\nu$. We know $\beta$ is nondegenerate on $\mathbb{C}h_\nu$ and $\mathfrak{t}$, so is on $\ker \nu$. Then it follows that $\mathfrak{t}(\nu)$ must be the one-dimensional $\beta$-orthogonal complement of $\ker \nu$ in $\mathfrak{t}$ and by definition of $h_\nu$ we have $\mathfrak{t}(\nu) = \mathbb{C}h_\nu$. Since $\mathfrak{m}(\nu) = \mathfrak{t}(\nu) \oplus \mathfrak{s}(\nu)$ where $\mathfrak{s}(\nu)$ is an ideal of $\mathfrak{s}$ and is a sum of simple components of $\mathfrak{s}$. Therefore $\beta|_{\mathfrak{m}(\nu)}$ is nondegenerate. □

In this section we will mainly be concerned with decomposing $\mathfrak{r}$ into irreducible $\mathfrak{m}$-modules. Effectively this comes down to understanding the action of $\mathfrak{s}$ on $\mathfrak{g}_\nu$ for any $\nu \in R$.

Let $\mathfrak{b}(\mathfrak{s}) = \mathfrak{h}(\mathfrak{s}) \oplus \sum_{\varphi \in \Phi^+(\mathfrak{s})} \mathbb{C}x_\varphi$ be a Borel subalgebra of $\mathfrak{s}$ where $\Phi^+(\mathfrak{s})$ is a choice of positive roots in $\Phi(\mathfrak{s})$. Let $\mathfrak{C}(\mathfrak{s}) = \{x \in \mathfrak{h}(\mathfrak{s})^*_\mathbb{R} \mid (x, \alpha) > 0 \text{ for all } \alpha \in \Phi^+(\mathfrak{s})\}$ be the fundamental Weyl chamber.

**Proposition 2.1.5.** *Let* $\xi, \eta \in \overline{\mathfrak{C}(\mathfrak{s})}$. *Then*

$$(\xi, \eta) \geq 0.$$

*Proof.* Let $I$ be an index set for the simple components $\mathfrak{s}_i$ for $i \in I$ in $\mathfrak{s}$. Then clearly $\mathfrak{C}(\mathfrak{s}) = \sum_{i \in I} \mathfrak{C}(\mathfrak{s}_i)$ where $\mathfrak{h}(\mathfrak{s}_i) = \mathfrak{h} \cap \mathfrak{s}_i$ and $\mathfrak{C}(\mathfrak{s}_i) \subset \tau(\mathfrak{h}(\mathfrak{s}_i))$ is the dominant Weyl chamber for $\mathfrak{s}_i$. If $i, j \in I$, then $\mathfrak{C}(\mathfrak{s}_i)$ and $\mathfrak{C}(\mathfrak{s}_j)$ are orthogonal with respect to $\beta|\mathfrak{h}^*$. Therefore it suffices to prove that $(\xi, \eta) \geq 0$ when $\xi, \eta$ are nonzero elements in $\mathfrak{C}(\mathfrak{s}_i)$ for $i \in I$. But we know that $\beta|_{\mathfrak{s}_i}$ is a positive multiple of the Killing form of $\mathfrak{s}_i$ and $(\xi, \eta) \geq 0$ is known for the Killing form of $\mathfrak{s}_i$ for $\xi, \eta \in \mathfrak{C}(\mathfrak{s}_i)$, see [13, Lemma 2.4]. □

**Theorem 2.1.6.** *Let* $\nu \in R$. *Then the Levi root subspace* $\mathfrak{g}_\nu$ *is an irreducible* $\mathfrak{m}$ *and irreducible* $\mathfrak{s}$*-module under the adjoint action. In fact, it is a faithful irreducible* $\mathfrak{m}(\nu)$*-module and an irreducible* $\mathfrak{s}(\nu)$*-module. In addition* $\mathfrak{r} = \bigoplus_{\nu \in R} \mathfrak{g}_\nu$ *is a multiplicity one representation of* $\mathfrak{m}$ *and the Levi root spaces are the irreducible components.*

*Proof.* Each element of $\nu \in R$ gives a different character of $\mathfrak{t}$. So corresponding $\mathfrak{m}$-modules $\mathfrak{g}_\nu$ for all $\nu \in R$ are non isomorphic. Since every $\mathfrak{s}$ submodule of $\mathfrak{g}_\nu$ is an $\mathfrak{m}$-submodule, we need to prove that $\mathfrak{g}_\nu$ is an irreducible $\mathfrak{s}$-module. The set of elements $\{x_\varphi \mid \varphi \in \Phi(\mathfrak{g}_\nu)\}$ is a basis of $\mathfrak{g}_\nu$ consisting of weight vectors for the Cartan subalgebra $\mathfrak{h}(\mathfrak{s})$. Since root spaces for $\mathfrak{h}$ have multiplicity one, the $\mathfrak{h}(\mathfrak{s})$-weights in $\mathfrak{g}_\nu$ have multiplicity one because for $\varphi, \varphi' \in \Phi(\mathfrak{g}_\nu)$, we have $\varphi_\mathfrak{t} = \varphi'_\mathfrak{t}$. Now suppose $\mathfrak{g}_\nu$ is not an irreducible $\mathfrak{s}$-module. By representation theory of semisimple Lie algebra, then there exists distinct $\varphi, \varphi' \in \Phi(\mathfrak{g}_\nu)$ such that $x_\varphi$ and $x_{\varphi'}$ are $\mathfrak{s}$-highest weight vectors. Moreover, $\varphi_\mathfrak{s}$ and $\varphi'_\mathfrak{s}$ are in $\overline{\mathfrak{C}(\mathfrak{s})}$. By Proposition 2.1.5, we have $(\varphi_\mathfrak{s}, \varphi'_\mathfrak{s}) \geq 0$. Since $\varphi = \nu + \varphi_\mathfrak{s}$ and $\varphi' = \nu + \varphi'_\mathfrak{s}$, we have $(\varphi, \varphi') > 0$. Therefore $\beta = \varphi - \varphi'$ is a root. Now since $\beta_\mathfrak{t} = 0$, we have $\beta \in \Phi(\mathfrak{m}) = \Phi(\mathfrak{s})$. Without loss of generality we may choose the ordering so that $\beta \in \Phi^+(\mathfrak{s})$. Thus $[x_\beta, x_{\varphi'}]$

is a nonzero multiple of $x_\varphi$ which contradicts the fact $x'_\varphi$ is an $\mathfrak{s}$-highest weight vector and hence the result follows. □

## 2.2 Properties of the Levi type root system $R$

We will use Theorem 2.1.6 to establish some properties of $R$.

**Lemma 2.2.1.** *Assume $\nu, \mu \in R$ and $\nu + \mu \neq 0$. Assume also that $[\mathfrak{g}_\nu, \mathfrak{g}_\mu] \neq 0$. Then $\nu + \mu \in R$ and $[\mathfrak{g}_\nu, \mathfrak{g}_\mu] = \mathfrak{g}_{\nu+\mu}$.*

*Proof.* As $[\mathfrak{g}_\nu, \mathfrak{g}_\mu]$ is a nonzero $\mathfrak{m}$-submodule of the $\mathfrak{g}_{\nu+\mu}$ and $\mathfrak{g}_{\nu+\mu}$ is irreducible $\mathfrak{m}$ module, therefore the equality follows. □

Let $p, q \in \mathbb{Z}$ where $p \leq q$. Let $[p, q]$ denote the set of integers $m$ such that $p \leq m \leq q$. A finite nonempty subset $I \subset \mathbb{Z}$ will be called an **interval** if $I = [p, q]$ for some $p, q \in \mathbb{Z}$.

**Theorem 2.2.2.** *Let $\nu \in R$ and assume $V$ is a finite dimensional $\mathfrak{g}$-module with respect to a representation $\phi$. Let $\gamma$ be a $\mathfrak{t}$-weight of $V$ and let*

$$I = \{j \in \mathbb{Z} \mid \gamma + j\nu \text{ be a } \mathfrak{t}\text{-weight of } V\},$$

*noting that $I$ is of course finite and nonempty since $0 \in I$. Then,*

*(a) There exists $p \leq 0 \leq q$ for $p, q \in \mathbb{Z}$ such that $I = I_{p,q}$. In particular, if $I$ has only one element, (i.e. $p = q = 0$) then $(\gamma, \nu) = 0$*

*(b) If $I$ has more than one element i.e. $p < q$, then $(\gamma + q\nu, \nu) > 0$ and $(\gamma + p\nu, \nu) < 0$.*

*(c) Let $m \in I_{p,q}$. If $m < q$, then*

$$\phi(\mathfrak{g}_\nu)(V_{\gamma+m\nu}) \neq 0,$$

*and if $p < m$, then*

$$\phi(\mathfrak{g}_{-\nu})(V_{\gamma+m\nu}) \neq 0$$

*Proof.* (a) Let $X = \Sigma_{j\in I} V_{\gamma+j\nu}$. Then $X$ is stable under $\phi(\mathfrak{g}_\nu)$ and $\phi(\mathfrak{g}_{-\nu})$ as well as $\phi(\mathfrak{m})$. Now if $j \in I$ and $v \in V_{\gamma+j\nu}$, then $\phi(h_\nu)|(v) = (\gamma + j\nu)(h_\nu)v = (\langle\gamma, h_\nu\rangle + 2j)v$. So $V_{\gamma+j\nu}$ is the eigenspace of $\phi(h_\nu)|$ corresponding to the eigenvalue $\langle\gamma, h_\nu\rangle + 2j$. Now since $h_\nu \in [\mathfrak{g}_\nu, \mathfrak{g}_{-\nu}]$, for any subspace $Y \subset X$ which is stable under $\phi(\mathfrak{g}_\nu)$ and $\phi(\mathfrak{g}_{-\nu})$ we have $\operatorname{tr}\phi(h_\nu)|_Y = 0$. If $I$ has one element, then $I = I_{0,0}$ and we have $V_\gamma$ as the only $\mathfrak{t}$-weight of $V$. Then $\operatorname{tr}\phi(h_\nu)|_{V_\gamma} = (\gamma, \nu) = 0$. Thus it suffices to consider the case where $I$ has more than one element. Now let $Y_1, Y_2$ be two nonzero subspaces of $X$ that are both stable under $\phi(\mathfrak{g}_\nu)$ and $\phi(\mathfrak{g}_{-\nu})$, then we have $\operatorname{tr}\phi(h_\nu)|_{Y_1} = 0$ and $\operatorname{tr}\phi(h_\nu)|_{Y_2} = 0$. So from these it follows that one cannot have that the maximal eigenvalue of $\phi(h_\nu)$ in $Y_1$ less than or equal minimal eigenvalue of $\phi(h_\nu)$ in $Y_2$. Now assume that $p, q \in I$ and $m \in \mathbb{Z}$ is such that $m \notin I$ and $p < m < q$. If we define $Y_1$ (resp. $Y_2$) to be the sum of all $V_{\gamma+j\nu}$, where $j \in I$ and $j < m$ (resp.$j > m$), the above conditions are satisfied which, as noted above, is a contradiction. Thus $I = I_{p,q}$ for some $p, q \in \mathbb{Z}$ where $q > p$ and the result follows.

(b) This follows from $\operatorname{tr}\phi(h_\nu)|_Y = 0$ where $Y = X$.

(c) Let $m \in I_{p,q}$ where $m < q$. Assume that $\phi(\mathfrak{g}_\nu)(V_{\gamma+m\nu}) = 0$. That is, $\phi(x_\alpha)(V_{\gamma+m\nu}) = 0$ for all $\alpha \in \Phi(\mathfrak{g}_\nu)$. Thus for any $v \in V_{\gamma+(m+1)\nu}$ and $\alpha \in \Phi(\mathfrak{g}_\nu)$, we have $\phi(x_\alpha)\phi(x_{-\alpha})v = 0$. Also this implies that $\phi(x_{-\alpha})v = 0$ by the representation theory of the $\mathfrak{sl}_2$. That is $\phi(\mathfrak{g}_{-\nu})(V_{\nu+(m+1)\nu}) = 0$. But then if $Y_1$ (resp. $Y_2$) is the sum of all $V_{\gamma+j\nu}$ for $j \in I_{p,q}$ where $j \leq m$ (resp. $j \geq m+1$), then $Y_1$ and $Y_2$ are satisfying the contradiction mentioned above and the result follows. □

The next theorem shows that some familiar properties of ordinary roots still hold for Levi type roots.

**Theorem 2.2.3.** *Let $\nu, \mu \in R$. If $\mu + \nu \in R$, then $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = \mathfrak{g}_{\mu+\nu}$. If $\mu - \nu \in R$, then*

$[\mathfrak{g}_\mu, \mathfrak{g}_{-\nu}] = \mathfrak{g}_{\mu-\nu}$. *Furthermore, if* $(\mu, \nu) < 0$*, then* $\mu + \nu \in R$ *and* $\mu + \nu \neq 0$*. If* $(\mu, \nu) > 0$*, then* $\mu - \nu \in R$ *and* $\mu - \nu \neq 0$.

*Proof.* If we take $V = \mathfrak{g}$ and $\phi$ the adjoint representation then by Theorem 2.2.2, and Lemma 2.2, the result follows. □

Now let $\delta_\mathfrak{n}$ be in $\mathfrak{m}^*$ such that for $x \in \mathfrak{m}$ we have $\langle \delta_\mathfrak{n}, x \rangle = \operatorname{tr} \operatorname{ad} x|_\mathfrak{n}$. Since $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are stable under $\operatorname{ad} \mathfrak{m}$, let $R = R_\mathfrak{n} \cup R|_{\bar{\mathfrak{n}}}$ such that $\mathfrak{n} = \bigoplus_{\nu \in R_\mathfrak{n}} \mathfrak{g}_\nu$ and $\bar{\mathfrak{n}} = \bigoplus_{\nu \in R_{\bar{\mathfrak{n}}}} \mathfrak{g}_\nu$. Then $R_{\bar{\mathfrak{n}}} = -R_\mathfrak{n}$.

**Lemma 2.2.4.** *We have* $\delta_\mathfrak{n} \in \mathfrak{t}^*_\mathbb{R}$*. Furthermore* $(\delta_\mathfrak{n}, \nu) > 0$*, if* $\nu \in R_\mathfrak{n}$ *and* $(\delta_\mathfrak{n}, \nu) < 0$*, if* $\nu \in R_{\bar{\mathfrak{n}}}$.

*Proof.* Since $\mathfrak{s}$ is semisimple, $\delta_\mathfrak{n}|_\mathfrak{s} = 0$ and therefore $\delta_\mathfrak{n} \in \mathfrak{t}^*$. Let $\varphi \in \Phi$. We normalize the choice of root vectors such that $(x_\varphi, x_{-\varphi}) = 1$. Let $\mathfrak{a}_\varphi$ be the $\mathfrak{sl}_2$-subalgebra corresponding to $\varphi$ and let $h_\varphi = [x_\varphi, x_{-\varphi}]$. Then $\tau(h_\varphi) = \varphi$. Let $\mathfrak{k}_\varphi = \mathfrak{a}_\varphi + \mathfrak{h}$. Since $\mathfrak{k}_\varphi$ is reductive, for some index set $P$, we have $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{u}_p$ as the decomposition of $\mathfrak{g}$ into a sum of irreducible $\operatorname{ad} \mathfrak{k}_\varphi$-submodules. We have the direct sum (with possibly 0-dimensional summands) $\mathfrak{n} = \bigoplus_{p \in P} \mathfrak{n}_p$ where $\mathfrak{n}_p = \mathfrak{n} \cap \mathfrak{u}_p$. Each $\mathfrak{u}_p$ is an irreducible $\mathfrak{a}_\varphi$-module, so for any $p \in P$ we have $\mathfrak{u}_p \cong V(m)$ for some $m \in \mathbb{Z}_{\geq 0}$. Recall that $V(m)$ is the direct sum of $V_\mu$ for $\mu = m, m-2, \ldots, -(m-2), -m$, see 1.3.1. Since each $\mathfrak{n}_p$ is stable under $\operatorname{ad} x_\varphi$ for any $p \in P$, by the representation theory of $\mathfrak{sl}_2$, it is clear that $\operatorname{tr} \operatorname{ad} h_\varphi|_{\mathfrak{n}_p} = \sum_{V_i \subseteq \mathfrak{n}_p} i \geq 0$. But there exists $p_0 \in P$ such that $\mathfrak{n}_{p_0} = \mathbb{C} x_\varphi$ so that $\operatorname{tr} \operatorname{ad} h_\varphi|_{\mathfrak{n}_{p_0}} > 0$. Thus $\delta_\mathfrak{n}(h_\varphi) > 0$ and then $(\delta_\mathfrak{n}, \varphi) > 0$. Now let $\nu \in R_\mathfrak{n}$ and let $\varphi \in \Phi(\mathfrak{g}_\nu)$ then clearly $\nu = \varphi_\mathfrak{t}$. Now since $(\delta_\mathfrak{n}, \varphi) > 0$ and the fact that $\delta_\mathfrak{n} \in \mathfrak{t}^*$ we have $(\delta_\mathfrak{n}, \nu) > 0$. The second part follows by $R_{\bar{\mathfrak{n}}} = -R_\mathfrak{n}$. Since $\mathfrak{t}^*_\mathbb{R}$ is clearly spanned by $R_\mathfrak{n}$, it follows from Proposition 2.1.3 (b) that $\delta_\mathfrak{n} \in \mathfrak{t}^*_\mathbb{R}$. □

Now we introduce the lexicographical ordering in $\mathfrak{t}^*_\mathbb{R}$ with respect to an orthogonal ordered basis of $\mathfrak{t}_{\mathbb{R}}^*$ having $\delta_{\mathfrak{n}}$ as its first element. It follows from Lemma 2.2.4, that if $R^+$ is the set of positive Levi roots with respect to this ordering, then $R^+ = R_{\mathfrak{n}}$.

*Remark* 2.2.5. Recall that $\mathfrak{t}_{\mathbb{R}}^*$ is a lexicographically ordered real Euclidean space. Now if $\xi_i \in \mathfrak{t}_{\mathbb{R}}^*$ where $i = 1, \ldots, k$ such that $(\xi_i, \xi_j) \leq 0$ for $i \neq j$, then the $\xi_i$ are linearly independent, see [12, Theorem 10.1].

Let $\ell(\mathfrak{t}) = \dim \mathfrak{t}$ and $\ell(\mathfrak{s}) = \dim \mathfrak{h}(\mathfrak{s})$ so that $\ell(\mathfrak{s})$ is the rank of $\mathfrak{s}$ and $\ell = \ell(\mathfrak{t}) + \ell(\mathfrak{s})$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple positive roots in $\Phi^+$. If $\varphi \in \Phi^+(\mathfrak{s})$ and $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1, \varphi_2 \in \Phi^+$, obviously $(\varphi_1)_{\mathfrak{t}} = -(\varphi_2)_{\mathfrak{t}}$. But then $(\varphi_1)_{\mathfrak{t}} = (\varphi_2)_{\mathfrak{t}} = 0$ by Lemma 2.2.4. Hence $\varphi_1, \varphi_2 \in \Phi^+(\mathfrak{s})$. Therefore if $\varphi \in \Phi^+(\mathfrak{s})$ is simple with respect to $\Phi^+(\mathfrak{s})$, it is simple with respect to $\Phi^+$. We may therefore order $\Pi$ so that $\alpha_{\ell(\mathfrak{t})+i} \in \Phi^+(\mathfrak{s})$, for $i = 1, \ldots, \ell(\mathfrak{s})$, and hence if $\Pi_{\mathfrak{s}} = \{\alpha_{\ell(\mathfrak{t})+1}, \ldots, \alpha_\ell\}$, then $\Pi_{\mathfrak{s}}$ is a basis of $\mathfrak{h}(\mathfrak{s})^*$. A Levi root $\nu \in R^+$ is called **simple** if $\nu$ cannot be written $\nu = \nu_1 + \nu_2$ where $\nu_1, \nu_2 \in R^+$. Let $R_{\text{simp}}$ be the set of simple Levi roots in $R^+$.

**Lemma 2.2.6.** *Assume $\xi_1, \xi_2 \in R_{\text{simp}}$ are distinct. Then $(\xi_1, \xi_2) \leq 0$ so that by Remark 2.2.5, the elements in $R_{\text{simp}}$ are linearly independent. In particular,* $\operatorname{Card} R_{\text{simp}} \leq \ell(\mathfrak{t})$.

*Proof.* Assume $(\xi_1, \xi_2) > 0$. Then by Theorem 2.2.3, $\xi_1 - \xi_2$ and $\xi_2 - \xi_1$ are in $R$. Without loss of generality let $\nu \in R^+$ where $\nu = \xi_1 - \xi_2$. Then $\xi_1 = \nu + \xi_2$ which contradicts the simplicity of $\xi_1$ and the result follows. □

As we mentioned before $\mathfrak{h}(\mathfrak{s})^*$ is the annihilator of $\mathfrak{t}$ in $\mathfrak{h}$ and $\Pi_{\mathfrak{s}} = \{\alpha_{\ell(\mathfrak{t})+1}, \ldots, \alpha_\ell\}$ is a basis of $\mathfrak{h}(\mathfrak{s})^*$. Now if $\beta_j = (\alpha_j)_{\mathfrak{t}}$ for $j = 1, \ldots, \ell(\mathfrak{t})$, then clearly $\beta_j \in R^+$ and $\beta_j$ for $j = 1, \ldots, \ell(\mathfrak{t})$, are a basis of $\mathfrak{t}_{\mathbb{R}}^*$.

For any $\nu = \sum_{j=1}^{\ell(\mathfrak{t})} k_{\alpha_j} \alpha_j$ in $R$, let $k_j = k_{\alpha_j}$ where $\varphi \in \Phi(\mathfrak{g}_\nu)$.

**Theorem 2.2.7.** *We have $R_{\text{simp}} = \{\beta_1, \ldots, \beta_{\ell(\mathfrak{t})}\}$ so that $R_{\text{simp}}$ is a basis of $\mathfrak{t}_{\mathbb{R}}^*$ and $(\beta_i, \beta_j) \leq 0$ for $i \neq j$. Moreover, for $\nu \in R^+$ we have $\nu = \sum_{j=1}^{\ell(\mathfrak{t})} k_j \beta_j$.*

*Proof.* Let $\beta_j \notin R_{\text{simp}}$ for $j \in \{1, \ldots, \ell(\mathfrak{t})\}$. Then there are $\nu_1, \nu_2 \in R^+$ such that $\beta_j = \nu_1 + \nu_2$. Then by Theorem 2.2.3, we have $[\mathfrak{g}_{\nu_1}, \mathfrak{g}_{\nu_2}] = \mathfrak{g}_{\beta_j}$. Now as $\beta_j = (\alpha_j)_{\mathfrak{t}}$ we know that $x_{\alpha_j} \in \mathfrak{g}_{\beta_j}$. Therefore there exists $\varphi_1$ and $\varphi_2 \in \Phi(\mathfrak{g}_{\nu_i})$ such that $\varphi_1 + \varphi_2 = \alpha_j$. This contradicts the simplicity of $\alpha_j$ since $\varphi_1, \varphi_2 \in \Phi^+$. As we mentioned in Lemma 2.2.6, the elements in $R_{\text{simp}}$ are linearly independent and $\operatorname{Card} R_{\text{simp}} \leq \ell(\mathfrak{t})$ hence $R_{\text{simp}} = \{\beta_1, \ldots, \beta_{\ell(\mathfrak{t})}\}$ and $R_{\text{simp}}$ is a basis of $\mathfrak{t}^*_{\mathbb{R}}$ and by Lemma 2.2.6, we have $(\beta_i, \beta_j) \leq 0$ for $i \neq j$. Now let $\nu \in R^+$ and let $\varphi \in \Phi(\mathfrak{g}_\nu)$. Then we have $\nu = \sum_{\alpha \in \Pi} n_\alpha \alpha_{\mathfrak{t}}$ and the result follows. □

## 2.3 Centralizers of nilpotent elements in semisimple Lie algebras

Let $G$ be a semisimple algebraic group over $\mathbb{C}$, let $H$ be a maximal torus and $B$ be a Borel subgroup containing $H$. We write $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{b}$ for the corresponding Lie algebras. In this section we fix a nilpotent element $e \in \mathfrak{g}$. The main reference for the material in this section is [3], [5, Chapter 5], and [15, Chapter 1-5]. We denote the centralizer of $e$ in $\mathfrak{g}$ by $\mathfrak{g}^e$. By the Jacobson-Morozov theorem, we can embed $e$ into an $\mathfrak{sl}_2$ subalgebra $\mathfrak{s} = \langle e, h, f \rangle$, so that $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Moreover, by a result of Kostant, any other such triple $(e, h', f')$ is conjugate to $(e, h, f)$ by an element of the connected centralizer $(G^e)^\circ$. Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j)$ be the $\operatorname{ad} h$-eigenspace decomposition of $\mathfrak{g}$. Let $\mathfrak{c} = \mathfrak{g}_0$ and let $C$ be the corresponding closed connected subgroup of $G$. In other words, $\mathfrak{c}$ and $C$ are the centralizers of $h$ in $\mathfrak{g}$ and $G$. Let $\mathfrak{r} = \bigoplus_{j>0} \mathfrak{g}(j)$ and let $R$ be the corresponding closed connected subgroup of $G$. It is known that $C^e$ is a maximal reductive subgroup of $G^e$, with Lie algebra $\mathfrak{c}^e$, and $R^e$ is the unipotent radical of $G^e$, with Lie algebra $\mathfrak{r}^e$. Moreover, $G^e$ is the semidirect product $C^e \ltimes R^e$ and $\mathfrak{g}^e$ is the semidirect sum $\mathfrak{c}^e \oplus \mathfrak{r}^e$. Finally, the component group $G^e/(G^e)^\circ$ is isomorphic to $C^e/(C^e)^\circ$.

Now fix a maximal torus $H$ of $G$ contained in $C$ and containing a maximal torus of

$C^e$. Let $\mathfrak{h}^e$ be the centralizer of $e$ in the Lie algebra $\mathfrak{h}$ of $H$. This is a maximal toral subalgebra of the reductive part $\mathfrak{c}^e$ of the centralizer $\mathfrak{g}^e$. Let $L$ be the centralizer of $H^e$ in $G$, and $\mathfrak{l}$ be the Lie algebra of $L$, i.e. the centralizer of $\mathfrak{h}^e$ in $\mathfrak{g}$.

By the Bala-Cater theory, see [5, Theorem 5.9.3], $\mathfrak{l}$ is a minimal Levi subalgebra of $\mathfrak{g}$ containing $e$ which is a distinguished nilpotent element of the derived subalgebra $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$.

**Lemma 2.3.1.** *The centre of* $\mathfrak{l}$ *is equal to* $\mathfrak{h}^e$.

*Proof.* We have $\mathfrak{h}^e$ as a maximal toral subalgebra of $\mathfrak{g}^e$. Also $\mathfrak{l} = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}^e)$ so $\mathfrak{h}^e \subseteq \mathfrak{z}(\mathfrak{l})$. The subalgebra $\mathfrak{z}(\mathfrak{l})$ is a toral subalgebra which centralizes $e$. So $\mathfrak{z}(\mathfrak{l}) \subseteq \mathfrak{g}^e$ and $\mathfrak{h}^e = \mathfrak{z}(\mathfrak{l})$ by maximality. □

Since $\mathfrak{h}^e$ is a subset of $\mathfrak{h}$, then it acts semisimply on $\mathfrak{g}$. Let $\mathfrak{g} = \bigoplus_{\alpha \in R \cup \{0\}} \mathfrak{g}_\alpha$ be the $\operatorname{ad} \mathfrak{h}^e$-decomposition of $\mathfrak{g}$ where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x \text{ for all } t \in \mathfrak{h}^e\}$ and $R = \{\alpha \in (\mathfrak{h}^e)^* \mid \mathfrak{g}_\alpha \neq 0\}$, the set of weights of $\mathfrak{h}^e$ on $\mathfrak{g}$. By Lemma 2.3.1 we have $\mathfrak{h}^e = \mathfrak{z}(\mathfrak{l})$ where $\mathfrak{l}$ is a levi subalgebra of $\mathfrak{g}$. Therefore $R$ is the Levi type root system corresponding to the Levi subalgebra $\mathfrak{l}$.

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Since $\mathfrak{g}^e$ is $\operatorname{ad} \mathfrak{h}^e$ stable then we have $\mathfrak{g}^e = \bigoplus_{\alpha \in \Phi^e} (\mathfrak{g}_\alpha)^e$ such that $(\mathfrak{g}_\alpha)^e = \mathfrak{g}^e \cap \mathfrak{g}_\alpha$ and $\Phi^e = \{\alpha \in (\mathfrak{h}^e)^* \mid (\mathfrak{g}_\alpha)^e \neq 0\}$ which we call "root system of $\mathfrak{g}^e$". The following theorem shows that a Levi type root system can be viewed as a root system of $\mathfrak{g}^e$.

**Lemma 2.3.2.** *The set* $\Phi^e$ *of weights of* $\mathfrak{h}^e$ *on* $\mathfrak{g}^e$ *is equal to the set* $R$ *of weights of* $\mathfrak{h}^e$ *on* $\mathfrak{g}$.

*Proof.* Let $\alpha \in R$. Then $\mathfrak{g}_\alpha$ is stable under $\operatorname{ad} \mathfrak{s}$, because $[\mathfrak{h}^e, \mathfrak{s}] = 0$. Since $\mathfrak{g}_\alpha$ is an $\mathfrak{s}$-module, by representation theory of $\mathfrak{sl}_2$ there exists a nonzero element $x \in \mathfrak{g}_\alpha$ such that $[e, x] = 0$. So $(\mathfrak{g}_\alpha)^e = \mathfrak{g}^e \cap \mathfrak{g}_\alpha \neq 0$ for all $\alpha \in R$. So $\alpha \in \Phi^e$ for all $\alpha \in R$. Therefore $\Phi^e$ the set of weights of $\mathfrak{h}^e$ on $\mathfrak{g}^e$ is equal to the set $R$ of weights of $\mathfrak{h}^e$ on $\mathfrak{g}$. □

## 2.4 Levi type Weyl group $W^Y$

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ with a root system $\Phi$ relative to a maximal toral subalgebra $\mathfrak{h}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system for $\Phi$ with the corresponding Weyl group $W$. Let $\mathfrak{h}_{\mathbb{R}}^*$ be a real form of $\mathfrak{h}^*$ spanned over $\mathbb{R}$ by $\alpha_1, \ldots, \alpha_n$. Let $\mathfrak{m}$ be a Levi subalgebra of a standard parabolic subalgebra $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ with nilpotent radical $\mathfrak{n}$. We know $\mathfrak{m}$ is reductive and $\mathfrak{m} = \mathfrak{t} \oplus \mathfrak{D}(\mathfrak{m})$ where $\mathfrak{t} = \mathfrak{z}(\mathfrak{m})$ and $\mathfrak{D}(\mathfrak{m}) = \mathfrak{s}$ is a semisimple subalgebra. Then we have $\mathfrak{s} = \mathfrak{h}(\mathfrak{s}) \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{s})} \mathfrak{g}_\alpha$. We know $\Phi(\mathfrak{s}) \subseteq \Phi$ is a closed subsystem. Let $Y = \{i \mid \alpha_i \in \Phi(\mathfrak{s})\} \subseteq \{1, \ldots, n\}$. This is a subset of index set $\{1, \ldots, n\}$ corresponding to the Levi subalgebra $\mathfrak{m}$ and we denote the subsystem $\Phi(\mathfrak{s})$ with $\Phi_Y$. Also we denote the Levi type root system $R$ corresponding to $\mathfrak{m}$ with $\Phi^Y$. Let $\mathfrak{h}(\mathfrak{s})_{\mathbb{R}}^*$ be the real form of $\mathfrak{h}(\mathfrak{s})^*$ spanned over $\mathbb{R}$ by $\alpha_i$ where $i \in Y$. Then $\mathfrak{h}_{\mathbb{R}}^* = \mathfrak{h}(\mathfrak{s})_{\mathbb{R}}^* \oplus \mathfrak{t}_{\mathbb{R}}^*$ where $\mathfrak{t}_{\mathbb{R}}^*$ is a real form of $\mathfrak{t}^*$. We introduce the **Levi type Weyl group** $W^Y$, namely, the pointwise stabilizer in $W$ of the set $\Pi_Y = \{\alpha_i \mid i \in Y\}$. This is a well known subgroup, studied particular by Howlett [11], see also [5, Section 10.4].

We have

$$W^Y = \{w \in W \mid w\alpha = \alpha \text{ for all } \alpha \in \Pi_Y\}.$$

Then $W^Y$ fixes $\mathfrak{h}(\mathfrak{s})_{\mathbb{R}}^*$ pointwise. So $W^Y$ stabilizes $\mathfrak{t}_{\mathbb{R}}^*$. Because, let $w \in W^Y$, $v \in \mathfrak{h}(\mathfrak{s})^*{}_{\mathbb{R}}$, and $u \in \mathfrak{t}_{\mathbb{R}}^*$. Then $(v, u) = 0$ and we have $(u, v) = (wv, wu) = (v, wu) = 0$. So $wu \in \mathfrak{t}^*{}_{\mathbb{R}}$. Therefore $W^Y$ acts on $\mathfrak{t}_{\mathbb{R}}^*$.

Also $W^Y$ acts on $\Phi^Y$. Because let $\alpha \in \Phi$ and let $\alpha = \alpha_Y + \alpha^Y$ where $\alpha_Y \in \Phi_Y$ and $\alpha^Y \in \Phi^Y$. Let $w \in W^Y$. Then $w\alpha = w\alpha_Y + w\alpha^Y$. We have $w\alpha \in \Phi$, and $w\alpha_Y = \alpha_Y$ which means $w\alpha^Y \in \Phi^Y$. Therefore $W^Y$ acts on $\Phi^Y$ and we can view $W^Y$ as the Weyl group of $\Phi^Y$, see also [3]. It is also known that $W^Y \cong N_W(W_Y)/W_Y$ where $W_Y$ is the group generated by reflections $\sigma_i$ for $i \in Y$, see [11, Lemma 2].

# Chapter 3

# Kac–Moody Lie algebras

In this chapter we introduce certain Lie algebras denoted by $\mathfrak{g}(A)$ associated with a generalized Cartan matrix $A$. We shall briefly explain the definitions and some of the basic properties of $\mathfrak{g}(A)$ in this chapter. All materials here can be found in [6].

## 3.1 The Lie algebra $\tilde{\mathfrak{g}}(A)$ associated with a complex matrix

First we start with some standard definitions about Kac-Moody Lie algebras.

- A **generalized Cartan matrix**, abbreviated as GCM, is an $n \times n$ matrix $A = (A_{ij})$ such that the following conditions are satisfied

  (a) $A_{ii} = 2 \quad \text{for} \quad i = 1, \dots, n$

  (b) $A_{ij} \in \mathbb{Z} \quad \text{and} \quad A_{ij} \leq 0 \quad \text{if} \quad i \neq j$

  (c) $A_{ij} = 0 \quad \text{implies} \quad A_{ji} = 0.$

- Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. A **realisation** of $A$ is a triple $(\mathfrak{h}, \Pi, \Pi^{\mathrm{v}})$ where:

  (a) $\mathfrak{h}$ is a finite dimensional vector space over $\mathbb{C}$

  (b) $\Pi^{\mathrm{v}} = \{h_1, \dots, h_n\}$ is a linearly independent subset of $\mathfrak{h}$

(c) $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent subset of $\mathfrak{h}^*$ such that $\alpha_j(h_i) = A_{ij}$ for all $i, j$.

- If $(\mathfrak{h}, \Pi, \Pi^{\vee})$ is a realisation of $A$ then $\dim \mathfrak{h} \geq 2n - \operatorname{rank} A$ see [6, Proposition 14.1].

- A **minimal realisation** of $A$ is a realisation in which $\dim \mathfrak{h} = 2n - \operatorname{rank} A$. We can show that any $n \times n$ matrix over $\mathbb{C}$ has a minimal realisation, see [6, Proposition 14.2].

- Two realisation $(\mathfrak{h}, \Pi, \Pi^{\vee})$ and $(\mathfrak{h}', \Pi', (\Pi')^{\vee})$ of $A$ are isomorphic if there is an isomorphism of vector spaces $\phi : \mathfrak{h} \to \mathfrak{h}'$ such that $\phi(h_i) = h_i'$ and $\phi^*(\alpha_i') = \alpha_i$ where $\phi^* : (\mathfrak{h}')^* \to \mathfrak{h}^*$ is the isomorphism induced by $\phi$. It is known that any two minimal realisations of an $n \times n$ matrix $A$ over $\mathbb{C}$ are isomorphic, see [6, Proposition 14.3].

Let $A$ be an $n \times n$ matrix over $\mathbb{C}$ with rank $\ell$. Let $(\mathfrak{h}, \Pi, \Pi^{\vee})$ be a minimal realisation of $A$. We define a Lie algebra $\tilde{\mathfrak{g}}(A)$ by generators and relations. Let $X = \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \cup \tilde{\mathfrak{h}}$ where $\tilde{\mathfrak{h}} = \{\tilde{x} \mid x \in \mathfrak{h}\}$ is a copy of $\mathfrak{h}$. Let $R$ be the following set of Lie words in $X$:

(a) $\tilde{x} - \lambda\tilde{y} - \mu\tilde{z}$ for all $x, y, z \in \mathfrak{h}$, $\lambda, \mu \in \mathbb{C}$ with $x = \lambda y + \mu z$

(b) $[\tilde{x}, \tilde{y}]$ for all $x, y \in \mathfrak{h}$

(c) $[e_i, f_i] - \tilde{h}_i$ for $i = 1, \ldots, n$

(d) $[e_i, f_j]$ for all $i \neq j$

(e) $[\tilde{x}, e_i] - \alpha_i(x)e_i$ for all $x \in \mathfrak{h}$ and $i = 1, \ldots, n$

(f) $[\tilde{x}, f_i] + \alpha_i(x)f_i$ for all $x \in \mathfrak{h}$ and $i = 1, \ldots, n$

We define $\tilde{\mathfrak{g}}(A) = \mathfrak{FL}(X)/\langle R \rangle$ to be the Lie algebra generated by the elements $X$ subject to relations $R$ where $\mathfrak{FL}(X)$ is the free Lie algebra on the set $X$ and $\langle R \rangle$ is the ideal of $\mathfrak{FL}(X)$ generated by the above set $R$ of Lie words.

Let $\tilde{\mathfrak{h}}$ be the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by the elements $\tilde{x}$ for all $x \in \mathfrak{h}$. Let $\tilde{\mathfrak{n}}$ be the subalgebra generated by $e_1, \dots, e_n$ and $\tilde{\mathfrak{n}}^-$ the subalgebra generated by $f_1, \dots, f_n$. In fact we have $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}$ see [6, Proposition 14.10].

Let $Q = \{\alpha = \sum_{i=1}^n k_i \alpha_i \mid k_i \in \mathbb{Z}\}$, $Q^+ = \{\alpha = \sum_{i=1}^n k_i \alpha_i \in Q \setminus \{0\} \mid k_i \in \mathbb{Z}_{\geq 0}\}$ and $Q^- = \{\alpha = \sum_{i=1}^n k_i \alpha_i \in Q \setminus \{0\} \mid k_i \in \mathbb{Z}_{\leq 0}\}$. For each $\alpha \in Q$ let $\tilde{\mathfrak{g}}_\alpha = \{y \in \tilde{\mathfrak{g}}(A) \mid [\tilde{x}, y] = \alpha(x) y \text{ for all } x \in \mathfrak{h}\}$.

**Proposition 3.1.1.** *(a)* $\tilde{\mathfrak{g}}(A) = \bigoplus_{\alpha \in Q} \tilde{\mathfrak{g}}_\alpha$.

*(b)* $\dim \tilde{\mathfrak{g}}_\alpha$ *is finite for all* $\alpha \in Q$.

*(c)* $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{h}}$.

*(d) If* $\alpha \neq 0$ *then* $\tilde{\mathfrak{g}}_\alpha = 0$ *unless* $\alpha \in Q^+$ *or* $\alpha \in Q^-$.

*(e)* $[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] \subset \tilde{\mathfrak{g}}_{\alpha+\beta}$ *for all* $\alpha, \beta \in Q$.

*Proof.* This is [6, Proposition 14.11]. □

Now $\tilde{\mathfrak{n}}$ is spanned by by Lie monomials in $e_1, \dots, e_n$ and each Lie monomial lies in some $\tilde{\mathfrak{g}}_\alpha$. Let $\alpha = k_1\alpha_1 + \ldots + k_n\alpha_n$ with $k_i \in \mathbb{Z}$ and $k_i \geq 0$. A Lie monomial lies in $\tilde{\mathfrak{g}}_\alpha$ if and only if $e_i$ appears $k_i$ times in it for each $i$. But there are only finitely many Lie monomials in which $e_i$ appears $k_i$ times for each $i$. Thus $\dim \tilde{\mathfrak{g}}_\alpha$ is finite. In particular $\dim \tilde{\mathfrak{g}}_{\alpha_i} = 1$, $\dim \tilde{\mathfrak{g}}_{-\alpha_i} = 1$, $\dim \tilde{\mathfrak{g}}_{k\alpha_i} = 0$, and $\dim \tilde{\mathfrak{g}}_{-k\alpha_i} = 0$ if $k > 1$.

**Proposition 3.1.2.** *The algebra* $\tilde{\mathfrak{g}}(A)$ *contains a unique ideal* $\mathfrak{i}$ *maximal with respect to* $\mathfrak{i} \cap \tilde{\mathfrak{h}} = 0$.

*Proof.* Let $\mathfrak{j}$ be an ideal of $\tilde{\mathfrak{g}}(A)$ with $\mathfrak{j}\cap\tilde{\mathfrak{h}} = 0$. We have $\tilde{\mathfrak{g}}(A) = \bigoplus_{\alpha\in\mathfrak{h}^*} \tilde{\mathfrak{g}}_\alpha$ and we consider $\tilde{\mathfrak{g}}(A)$ as an $\tilde{\mathfrak{h}}$-module. Then we have $\mathfrak{j} = \bigoplus_{\alpha\in\mathfrak{h}^*}(\tilde{\mathfrak{g}}_\alpha \cap \mathfrak{j})$. Each $\tilde{\mathfrak{g}}_\alpha$ with $\alpha \neq 0$ lies in $\tilde{\mathfrak{n}}$ or in $\tilde{\mathfrak{n}}^-$. Hence $\mathfrak{j} = (\tilde{\mathfrak{n}}^- \cap \mathfrak{j}) \oplus (\tilde{\mathfrak{n}} \cap \mathfrak{j})$. In particular $\mathfrak{j} \subset \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{n}}$. Now if we consider the ideal $\mathfrak{i}$ of $\tilde{\mathfrak{g}}(A)$ generated by all ideals $\mathfrak{j}$ with $\mathfrak{j}\cap\tilde{\mathfrak{h}} = 0$. All such ideals $\mathfrak{j}$ lie in $\tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{n}}$, thus $\mathfrak{i}$ lies in $\tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{n}}$. Hence $\mathfrak{i}\cap\tilde{\mathfrak{h}} = 0$. Thus $\mathfrak{i}$ is the unique maximal ideal of $\tilde{\mathfrak{g}}(A)$ with respect to $\mathfrak{i}\cap\tilde{\mathfrak{h}} = 0$. □

## 3.2 The Kac–Moody Lie algebra $\mathfrak{g}(A)$

Let $A$ be a GCM and $\tilde{\mathfrak{g}}(A)$ be the Lie algebra associated with $A$ defined as before and $\mathfrak{i}$ be the unique maximal ideal of $\tilde{\mathfrak{g}}(A)$ with $\mathfrak{i}\cap\tilde{\mathfrak{h}} = 0$. Let $\mathfrak{g}(A)$ be defined by $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{i}$. The Lie algebra $\mathfrak{g}(A)$ is called the **Kac–Moody Lie algebra** with GCM $A$. We have the natural homomorphism $\theta : \tilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$. We define $\mathfrak{n} = \theta(\tilde{\mathfrak{n}})$ and $\mathfrak{n}^- = \theta(\tilde{\mathfrak{n}}^-)$.

**Proposition 3.2.1.** $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \theta(\tilde{\mathfrak{h}}) \oplus \mathfrak{n}$. *Moreover* $\theta : \tilde{\mathfrak{h}} \to \theta(\tilde{\mathfrak{h}})$ *is an isomorphism.*

*Proof.* We know from the proof of Proposition 3.1.2 that $\mathfrak{i} = (\tilde{\mathfrak{n}}^- \cap \mathfrak{i}) \oplus (\tilde{\mathfrak{n}} \cap \mathfrak{i})$. Since $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}$ it follows that $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \theta(\tilde{\mathfrak{h}}) \oplus \mathfrak{n}$ and that $\theta : \tilde{\mathfrak{h}} \to \theta(\tilde{\mathfrak{h}})$ is an isomorphism. □

As we mentioned before there is a natural isomorphism $\mathfrak{h} \to \tilde{\mathfrak{h}}$. So combining this with $\theta$ would give us an isomorphism $\mathfrak{h} \to \theta(\tilde{\mathfrak{h}})$. Therefore we shall use this isomorphism to identify $\theta(\tilde{\mathfrak{h}})$ with $\mathfrak{h}$, and we will write $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

To show that a given Lie algebra is isomorphic to $\mathfrak{g}(A)$ the following result is useful.

**Proposition 3.2.2.** *Suppose we are given an* $n \times n$ *GCM* $A = (A_{ij})$. *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbb{C}$ *and* $\mathfrak{h}$ *be a finite dimensional abelian subalgebra of* $\mathfrak{g}$ *with* $\dim\mathfrak{h} = 2n - \operatorname{rank} A$. *Suppose* $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ *is a linearly independent subset of* $\mathfrak{h}^*$ *and* $\Pi^\text{v} = \{h_1, \ldots, h_n\}$ *a linearly independent subset if* $\mathfrak{h}$ *satisfying* $\alpha_j(h_i) = A_{ij}$. *Suppose also that* $e_1, \ldots, e_n, f_1, \ldots, f_n$ *are elements of* $\mathfrak{g}$ *satisfying*

$[e_i, f_i] = h_i$

$[e_i, f_j] = 0$ *if* $i \neq j$

$[x, e_i] = \alpha_i(x)e_i$ *for* $x \in \mathfrak{h}$

$[x, f_i] = -\alpha_i(x)f_i$ *for* $x \in \mathfrak{h}$

*Suppose that* $e_1, \ldots, e_n, f_1, \ldots, f_n$ *and* $\mathfrak{h}$ *generate* $\mathfrak{g}$ *and that* $\mathfrak{g}$ *has no nonzero ideal* $\mathfrak{j}$ *with* $\mathfrak{j} \cap \mathfrak{h} = 0$. *Then* $\mathfrak{g}$ *is isomorphic to the Kac-Moody algebra* $\mathfrak{g}(A)$.

*Proof.* The elements $e_1, \ldots, e_n, f_1, \ldots, f_n$ and $x \in \mathfrak{h}$ generate $\mathfrak{g}$ and satisfying all the defining relations of $\tilde{\mathfrak{g}}(A)$ given in the previous section. Thus there is a surjective Lie algebra homomorphism $\theta : \tilde{\mathfrak{g}}(A) \to \mathfrak{g}$ and $\mathfrak{g}$ is isomorphic to $\tilde{\mathfrak{g}}(A)/\ker\theta$. But the restriction map $\theta : \tilde{\mathfrak{h}} \to \mathfrak{h}$ is an isomorphism. Thus $\ker\theta \cap \tilde{\mathfrak{h}} = 0$. It follows that $\ker\theta \subset \mathfrak{i}$, the largest ideal of $\tilde{\mathfrak{g}}(A)$ with $\mathfrak{i} \cap \tilde{\mathfrak{h}} = 0$. In fact we have $\ker\theta = \mathfrak{i}$ since $\mathfrak{g}$ has no non-zero ideal $\mathfrak{j}$ with $\mathfrak{j} \cap \mathfrak{h} = 0$. Hence $\mathfrak{g} \cong \tilde{\mathfrak{g}}(A)/\mathfrak{i} = \mathfrak{g}(A)$. □

**Corollary 3.2.3.** *If* $A$ *is a Cartan matrix then* $\mathfrak{g}(A)$ *is the finite dimensional semisimple Lie algebra with Cartan matrix* $A$.

*Proof.* In this case we have $\operatorname{rank} A = n$, so $\dim \mathfrak{h} = n$. The finite dimensional semisimple Lie algebra satisfies all the hypothesis of the Proposition 3.2.2, so is isomorphic to the Kac-Moody algebra $\mathfrak{g}(A)$. □

This result shows that the theory of Kac-Moody algebras is an extension of the theory of finite dimensional Lie algebras. To describe more basic properties of the Kac-Moody algebra $\mathfrak{g}(A)$, we denote the image of $e_i, h_i, f_i \in \tilde{\mathfrak{g}}(A)$ under the natural homomorphism $\tilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$ by $e_i, h_i, f_i \in \mathfrak{g}(A)$.

For each $\alpha \in Q$ define $\mathfrak{g}_\alpha = \{y \in \mathfrak{g}(A) \mid [x, y] = \alpha(x)y \text{ for all } x \in \mathfrak{h}\}$. The next proposition is an analogue of Proposition 3.1.1. For the proof see [6, Proposition 14.18].

**Proposition 3.2.4.** *(a)* $\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$

*(b)* $\dim \mathfrak{g}_\alpha$ *is finite for all* $\alpha \in Q$.

*(c)* $\mathfrak{g}_0 = \mathfrak{h}$

*(d)* *If* $\alpha \neq 0$ *then* $\mathfrak{g}_\alpha = 0$ *unless* $\alpha \in Q^+$ *or* $\alpha \in Q^-$.

*(e)* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ *for all* $\alpha, \beta \in Q$.

The subgroup $\mathfrak{h}$ is called a **Cartan subalgebra** of $\mathfrak{g}(A)$. An element $\alpha \in \mathfrak{h}^*$ is called a **root** of $\mathfrak{g}(A)$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The **root system** $\Phi$ of $\mathfrak{g}(A)$ is the set of all roots of $\mathfrak{g}(A)$. Every root lies in $Q^+$ or $Q^-$. The roots in $Q^+$ are called **positive roots** and those in $Q^-$ **negative roots**. The elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the **fundamental roots** of $\mathfrak{g}$. If $\alpha$ is a root then $\mathfrak{g}_\alpha$ is called the **root space** of $\alpha$. The dimension of $\mathfrak{g}_\alpha$ is called the multiplicity of $\alpha$. When $A$ is a Cartan matrix we recall that all roots have multiplicity 1 but we will show that this not always the case when $A$ is a GCM.

**Proposition 3.2.5.** *(a)* $\dim \mathfrak{g}_{\alpha_i} = 1$ *and* $\dim \mathfrak{g}_{-\alpha_i} = 1$.

*(b)* *If* $k > 1$ *then* $\dim \mathfrak{g}_{k\alpha_i} = 0$ *and* $\dim \mathfrak{g}_{-k\alpha_i} = 0$.

*Proof.* Since $\mathfrak{g}_{\alpha_i} = \theta(\tilde{\mathfrak{g}}_{\alpha_i})$ and $\dim \tilde{\mathfrak{g}}_{\alpha_i} = 1$ we have $\dim \mathfrak{g}_{\alpha_i} \leq 1$. If $\dim \mathfrak{g}_{\alpha_i} = 0$ we have $e_i \in \mathfrak{i} = \ker \theta$. This means $[e_i, f_i] = \tilde{h}_i \in \mathfrak{i}$ which is a contradiction as $\mathfrak{i} \cap \tilde{\mathfrak{h}} = 0$. Therefore $\dim \mathfrak{g}_{\alpha_i} = 1$. Similarly we have $\dim \mathfrak{g}_{-\alpha_i} = 1$. Since $\tilde{\mathfrak{g}}_{k\alpha_i} = 0$ and $\tilde{\mathfrak{g}}_{-k\alpha_i} = 0$ for $k > 1$ this implies $\mathfrak{g}_{k\alpha_i} = 0$ and $\mathfrak{g}_{-k\alpha_i} = 0$. □

Now we give some more standard definitions.

- Two GCMs $A, A'$ are called **equivalent** if they have the same degree $n$ and there is a permutation $\sigma$ of $1, \ldots, n$ such that $A'_{ij} = A_{\sigma(i)\sigma(j)}$, for all $i, j$.

- A GCM $A$ is called **indecomposable** if it is not equivalent to a diagonal sum $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ of smaller GCMs $A_1, A_2$.

- If $A$ is a GCM so is its transpose $A^{\text{tr}}$. Moreover $A$ is indecomposable if and only if $A^{\text{tr}}$ is indecomposable.

- Let $v = (v_1, \ldots, v_n)^{\text{tr}}$ be a vector in $\mathbb{R}^n$. We write $v \geq 0$ if $v_i \geq 0$ for each $i$, and $v > 0$ if $v_i > 0$ for each $i$.

  The GCM $A$ has **finite type** if

  (a) $\det A \neq 0$

  (b) there exists $u > 0$ with $Au > 0$

  (c) $Au \geq 0$ implies $u > 0$ or $u = 0$.

  The GCM $A$ has **affine type** if

  (a) corank $A = 1$

  (b) there exists $u > 0$ such that $Au = 0$

  (c) $Au \geq 0$ implies $Au = 0$

  The GCM $A$ has **indefinite** type if

  (a) there exists $u > 0$ such that $Au < 0$

  (b) $Au \geq 0$ and $u \geq 0$ imply $u = 0$.

All the vectors in this definition are assumed to lie in $\mathbb{R}^n$, and are column vectors.

The following theorem gives a trichotomy on the set of indecomposable matrices, see [6, Theorem 15.1].

**Theorem 3.2.6.** *Let $A$ be an indecomposable GCM. Then exactly one of the three possibilities holds:*

*(a) $A$ has finite type*

*(b) $A$ has affine type*

*(c) $A$ has indefinite type.*

*Moreover, the type of $A^{\mathrm{tr}}$ is the same as the type of $A$.*

**Corollary 3.2.7.** *Let $A$ be an indecomposable GCM. Then:*

*(a) $A$ has finite type if and only if there exists $u > 0$ with $Au > 0$.*

*(b) $A$ has affine type if and only if there exists $u > 0$ with $Au = 0$.*

*(c) $A$ has indefinite type if and only if there exists $u > 0$ with $Au < 0$.*

*Proof.* (a) Let $u > 0$ and $Au > 0$. Then $A$ is not affine type as $Au \geq 0$ implies $Au = 0$. Also, $A$ is not indefinite type since $u \geq 0$ and $Au \geq 0$ implies $u = 0$. Hence $A$ has finite type.

(b) Let $u > 0$ and $Au = 0$. Then $A$ is not finite type since $\det A = 0$. Also, $A$ is not indefinite type since $u \geq 0$ and $Au \geq 0$ would imply $u = 0$. Therefore $A$ has affine type.

(c) Let $u > 0$ and $Au < 0$. Then $A(-u) > 0$ and $A$ is not finite type as this implies $-u > 0$ or $-u = 0$. Also, $A$ is not affine type since $A(-u) > 0$ implies $A(-u) = 0$. Hence $A$ has indefinite type. □

Next we give some standard definitions and facts about a special type of GCM called the symmetrisable GCM.

- A GCM $A$ is **symmetrisable** if there exists a nonsingular diagonal matrix $D$ and a symmetric matrix $B$ such that $A = DB$.

- If $A$ is a symmetrisable indecomposable GCM. Then $A$ can be expressed in the form $A = DB$ where $D = \text{diag}(d_1, \ldots, d_n)$, $B$ is symmetric, with $d_1, \ldots, d_n > 0$ in $\mathbb{Z}$ and $B_{ij} \in \mathbb{Q}$ . Also $D$ is determined by these conditions up to scalar multiple by [6, Corollary 15.16].

- It is known that if $A$ is an indecomposable GCM of finite or affine type then $A$ is symmetrisable see [6, Theorem 15.17].

- Let $A = (A_{ij})$ be a GCM with $i, j \in \{1, \ldots, n\}$ and let $J$ be a subset of $\{1, \ldots, n\}$. Let $A_J = (A_{ij})_{i,j \in J}$. Then $A_J$ is also a GCM, called a **principal minor** of $A$.

The following basic description of our trichotomy generalizes our previous description for symmetric indecomposable GCMs, see [6, Theorem 15.18].

**Theorem 3.2.8.** *Let $A$ be a symmetric indecomposable GCM. Then:*

*(a) $A$ has finite type if and only if all its principal minors have positive determinant.*

*(b) $A$ has affine type if and only if* $\det A$*=0 and all proper principal minors have positive determinants.*

*(c) $A$ has indefinite type if and only if $A$ satisfies neither of these two conditions.*

Now we see which indecomposable GCMs lie in each class of our trichotomy, see [6, Theorem 15.19].

**Proposition 3.2.9.** *Let $A$ be an indecomposable GCM. Then $A$ has finite type if and only if $A$ is a Cartan matrix.*

To determine the indecomposable GCMs of affine type first for each GCM $A$ we define an associated diagram $\Delta(A)$ called the **Dynkin diagram** of $A$. This extends the definition of the Dynkin diagram of a Cartan matrix. The vertices of $\Delta(A)$ are called

$1, \dots, n$ where $A$ is an $n \times n$ matrix. Two distinct vertices $i, j$ of $\Delta(A)$ are joined in $\Delta(A)$ depending on the pair $(A_{ij}, A_{ji})$ and the rules as follows.

(a) If $A_{ij}A_{ji} = 0$, vertices $i, j$ are not joined.

(b) If $A_{ij}A_{ji} = 1$, vertices $i, j$ are joined by a single edge.

(c) If $A_{ij}A_{ji} = 2$, $A_{ij} = -1$, $A_{ji} = -2$ vertices $i, j$ are joined by a double edge with an arrow pointing towards $j$.

(d) If $A_{ij}A_{ji} = 3$, $A_{ij} = -1$, $A_{ji} = -3$ vertices $i, j$ are joined by a triple edge with an arrow pointing towards $j$.

(e) If $A_{ij}A_{ji} = 4$, $A_{ij} = -1$, $A_{ji} = -4$ vertices $i, j$ are joined by a quadruple edge with an arrow pointing towards $j$.

(f) If $A_{ij}A_{ji} = 4$, $A_{ij} = -2$, $A_{ji} = -2$ vertices $i, j$ are joined by a double edge with two arrows pointing away from $i, j$.

(g) If $A_{ij}A_{ji} \geq 5$ vertices $i, j$ are joined by an edge with the numbers $|A_{ij}|, |A_{ji}|$ shown on it.

It is clear that the GCM $A$ is determined by its Dynkin diagram $\Delta(A)$. Moreover $A$ is indecomposable if and only if $\Delta(A)$ is connected. Now we consider a set of connected Dynkin diagrams called the affine list, which are shown in Figures 3.1 and 3.2.

$\tilde{A}_1$

$\tilde{A}_1{}'$

$\tilde{A}_l (l \geq 2)$

$\tilde{B}_l \, (l \geq 3)$

$\tilde{B}_l{}^{\mathrm{t}} (l \geq 3)$

$\tilde{C}_l (l \geq 2)$

$\tilde{C}_l^{\mathrm{t}} (l \geq 2)$

$\tilde{C}_l' (l \geq 2)$

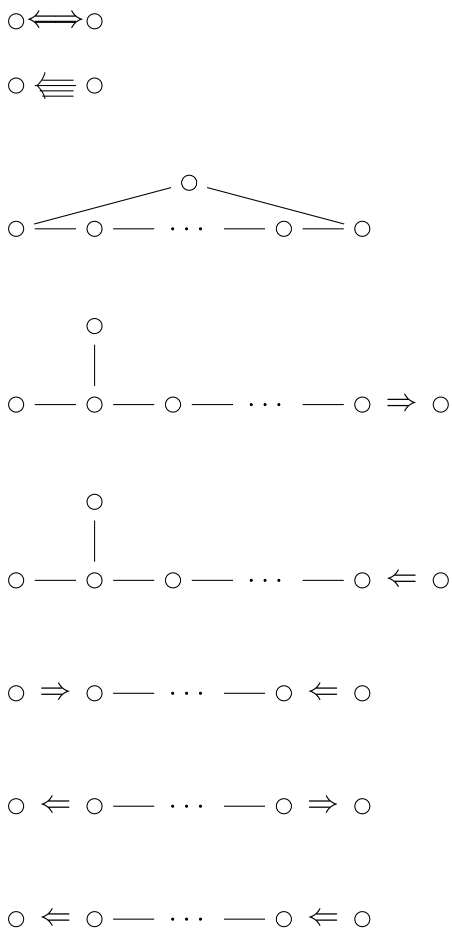

Figure 3.1: List of affine Dynkin diagrams part 1

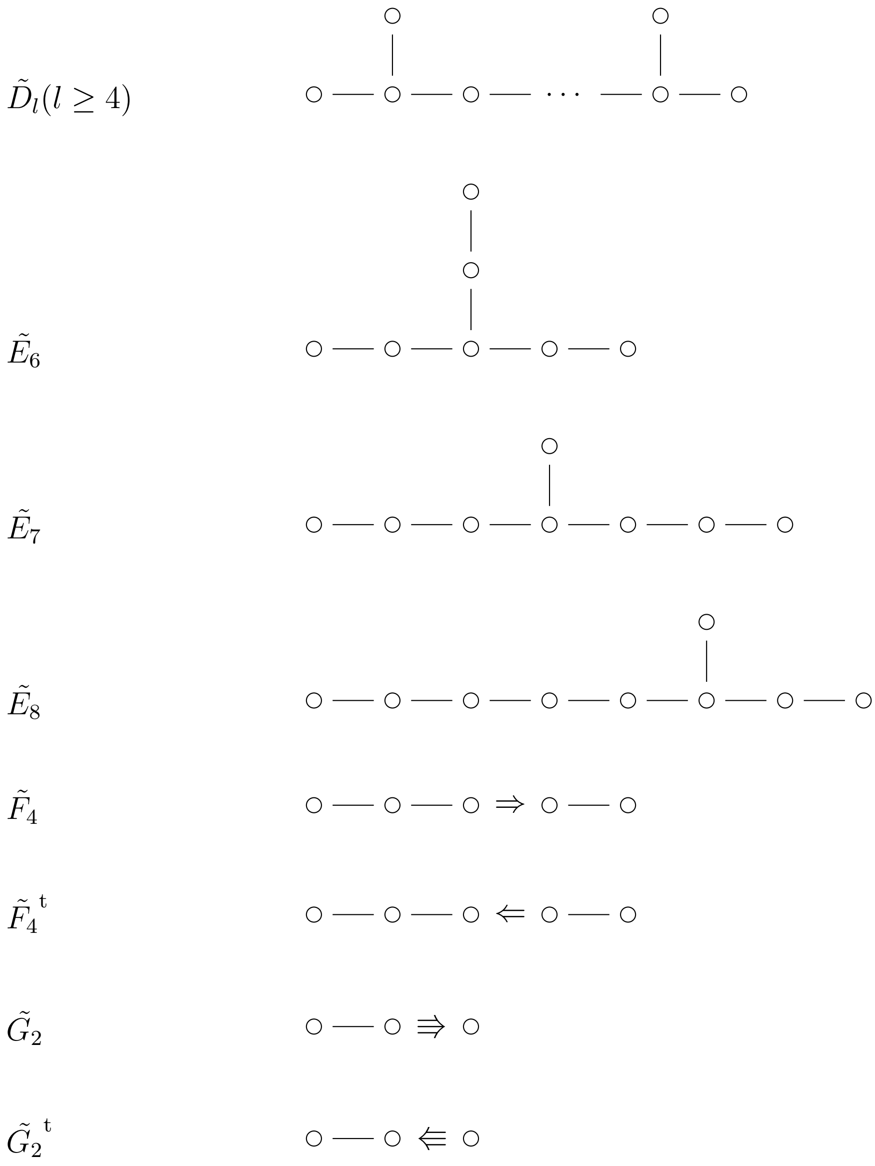


Figure 3.2: List of affine Dynkin diagrams part 2

**Proposition 3.2.10.** *Let $A$ be a GCM whose Dynkin diagram lies on the affine list. Then* $\det(A) = 0$.

*Proof.* This is [6, Proposition 15.21]. □

**Theorem 3.2.11.** *The Dynkin diagram of a GCM $A$ lies on the affine list if and only if $A$ has affine type.*

*Proof.* By Proposition 3.2.10, we have $\det A = 0$. As the Dynkin diagram of any proper principal minor has connected components on the finite list so all the proper principal minors have positive determinants. So $A$ has affine type by Theorem 3.2.8. The converse is true, see [6, Theorem 15.23]. □

**Corollary 3.2.12.** *Let $A$ be an indecomposable GCM. Then $A$ has indefinite type if and only if its Dynkin diagram $\Delta(A)$ does not appear on the finite or affine list.*

*Proof.* This is true by Theorems 3.2.6, 3.2.9, and 3.2.11. □

Now we show that there is a nondegenerate, symmetric, associative bilinear form on $\mathfrak{g}(A)$ when $A$ is symmetrisable.

Suppose $A$ is a symmetrisable GCM. Then $A = DB$ where $D$ is diagonal and $B$ is symmetric. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$. Let $(\mathfrak{h}, \Pi, \Pi^{\vee})$ be a minimal realisation of $A$, where $\Pi^{\vee} = \{h_1, \ldots, h_n\}$ is a linearly independent subset of $\mathfrak{h}$, the set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent subset of $\mathfrak{h}^*$, $\alpha_j(h_i) = A_{ij}$ and $\dim \mathfrak{h} = 2n - l$ where $l = \operatorname{rank} A$. Let $\mathfrak{h}'$ be the subspace of $\mathfrak{h}$ spanned by $h_1, \ldots, h_n$ and let $\mathfrak{h}''$ be a complementary subspace of $\mathfrak{h}'$ in $\mathfrak{h}$. Then we have $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ such that $\dim \mathfrak{h}' = n$, and $\dim \mathfrak{h}'' = n - l$. We define a bilinear form $(\cdot, \cdot) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ by the rules:

$$(h_i, h_j) = d_i d_j B_{ij} \quad i, j = 1, \ldots, n$$

$$(h_i, x) = (x, h_i) = d_i \alpha_i(x) \text{ for } x \in \mathfrak{h}''$$

$(x, y) = 0$ for $x, y \in \mathfrak{h}''$.

This is clearly a symmetric bilinear form which can be proved to be nondegenerate on $\mathfrak{h}$ as well, see [6, Proposition 16.1].

**Theorem 3.2.13.** *Suppose $A$ is a symmetrisable GCM. Then the Kac-Moody algebra $\mathfrak{g}(A)$ has a nondegenerate symmetric associative bilinear form.*

*Sketch proof.* We have $\mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$. For $\alpha = m_1\alpha_1 + \ldots m_n\alpha_n \in Q$ we define the height of $\alpha$ by $\operatorname{ht}\alpha = m_1 + \ldots + m_n$. Then $\mathfrak{g}(A) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ where $\mathfrak{g}_i$ is the direct sum of all $\mathfrak{g}_\alpha$ with $\operatorname{ht}\alpha = i$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Thus $\mathfrak{g}(A)$ can be considered as a $\mathbb{Z}$-graded Lie algebra. We define for each integer $r \geq 0$, $\mathfrak{g}(r) = \bigoplus_{-r \leq i \leq r} \mathfrak{g}_i$. Then we have $\mathfrak{h} = \mathfrak{g}(0) \subset \mathfrak{g}(1) \subset \ldots$ and $\bigcup_{r \geq 0} \mathfrak{g}(r) = \mathfrak{g}(A)$. We have already defined a symmetric bilinear form on $\mathfrak{h} = \mathfrak{g}(0)$. We shall extend this definition to give a symmetric bilinear form on $\mathfrak{g}(r)$ for $r = 1, 2, 3, \ldots$ thus eventually such a form on $\mathfrak{g}(A)$. We shall define the form on $\mathfrak{g}(r)$ by induction on $r$, assuming that is already defined on $\mathfrak{g}(r-1)$. We begin with the case $r = 1$. We have $\mathfrak{g}(1) = \left(\bigoplus_{i=1}^n \mathbb{C}f_i\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{i=1}^n \mathbb{C}e_i\right)$. We define a bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}(1)$ which is uniquely determined by the following rules:

$(\cdot, \cdot)$ agrees with the form already defined on $\mathfrak{h}$.

$(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$.

$(e_i, f_i) = (f_i, e_i) = d_i$

$(e_i, f_j) = (f_j, e_i) = 0$ for $i \neq j$.

This is clearly a symmetric bilinear form on $\mathfrak{g}(1)$. We can also show it is associative. Now suppose inductively that a symmetric bilinear form has already been defined on $\mathfrak{g}(r-1)$ and satisfies:

$(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$ for $|i|, |j| \leq r - 1$

$([x,y],z) = (x,[y,z])$ for all $x \in \mathfrak{g}_i, y \in \mathfrak{g}_j, z \in \mathfrak{g}_k$ with $|i|,|j|,k| \leq r-1$ and $i+j+k=0$.

This form can be extended to one on $\mathfrak{g}(r)$ with analogous properties. We extend the form to $\mathfrak{g}(r)$ by defining

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \text{ unless } i+j=0 \text{ for } |i|,|j| \leq r.$$

Also we need to define $(x,y) = (y,x)$ for $x \in \mathfrak{g}_r, y \in \mathfrak{g}_{-r}$. We assume $r \geq 2$ and as we know $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ with $\mathfrak{h} = \mathfrak{g}_0$, $\mathfrak{n}^- = \bigoplus_{i<0} \mathfrak{g}_i$, $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}_i$. The algebra $\mathfrak{n}^-$ is generated by $f_1, \ldots, f_n$, so each element of $\mathfrak{n}^-$ is a linear combination of Lie monomials in $f_1, \ldots, f_n$. An element of $\mathfrak{g}_{-\alpha}$ is a linear combination of Lie monomials in $f_1, \ldots, f_n$ such that the number of factors in each Lie monomial is $r$. If $r \geq 2$ each Lie monomial is the Lie product of Lie monomials of degree $s, t$ say with $s+t=r$. This means each element of $\mathfrak{g}_{-\alpha}$ can be written in the form $y = \sum_j [c_j, d_j]$ where $c_j \in \mathfrak{g}_{-u_j}$ and $d_j \in \mathfrak{g}_{-v_j}$ with $u_j > 0$, $v_j > 0$ and $u_j + v_j = r$. Note that the expression of $y$ in this form need not be unique. Let $x \in \mathfrak{g}_r$, $y \in \mathfrak{g}_{-r}$. We write $y = \sum_j [c_j, d_j]$ as above and try to define $(x,y) = \sum_j ([x, c_j], d_j)$. The right hand side is known since $[x, c_j]$ and $d_j$ lie in $\mathfrak{g}(r-1)$. However the right hand side seems to depend on the particular expression $y = \sum_j [c_j, d_j]$ for $y$ which need not be unique. We can show that the right side remains the same if a different such expression for $y$ is chosen, see [6, Theorem 16.2]. Therefore our form $(\cdot, \cdot)$ is well defined on $\mathfrak{g}(r)$, where it is bilinear, symmetric. Using the invariance of the form on $\mathfrak{g}(r-1)$ we can prove that the form is associative on $\mathfrak{g}(r)$ and finally by induction the form is associative on $\mathfrak{g}(A)$ see, [6, Theorem 16.2]. Thus we have now defined a symmetric associative bilinear form on $\mathfrak{g}(A)$. We show it is nondegenerate. Let $\mathfrak{g}^\perp = \{x \in \mathfrak{g}(A) \mid (x,y) = 0 \text{ for all } y \in \mathfrak{g}(A)\}$. Since the form is associative, $\mathfrak{g}^\perp$ is an ideal of $\mathfrak{g}(A)$. Since the form is nondegenerate on $\mathfrak{h}$ we have $\mathfrak{g}^\perp \cap \mathfrak{h} = 0$. But the Kac–Moody algebra $\mathfrak{g}(A)$ has no nonzero ideal $\mathfrak{i}$ such that $\mathfrak{i} \cap \mathfrak{h} = 0$. Therefore $\mathfrak{g}^\perp = 0$ and the form is nondegenerate on $\mathfrak{g}(A)$. □

The form constructed in Theorem 3.2.13 is called the **standard associative** form on $\mathfrak{g}(A)$. The following corollaries follow from Theorem 3.2.13.

**Corollary 3.2.14.** *For each $i \in \mathbb{Z}$ the pairing $\mathfrak{g}_i \times \mathfrak{g}_{-i} \to \mathbb{C}$ given by $x, y \to (x, y)$ is nondegenerate.*

*Proof.* Suppose $x \in \mathfrak{g}_i$ and we have $(x, y) = 0$ for all $y \in \mathfrak{g}_{-i}$. Since $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$ we have $(x, y) = 0$ for all $y \in \mathfrak{g}(A)$. Therefore, $x = 0$ because $(\cdot, \cdot)$ is nondegenerate. □

**Corollary 3.2.15.** $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *unless* $\alpha + \beta = 0$.

*Proof.* Suppose $\alpha + \beta \neq 0$ and let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then there exists $h \in \mathfrak{h}$ such that $(\alpha + \beta)(h) \neq 0$. Then by the fact that $([x, h], y) = (x, [h, y])$ this implies $-\alpha(h)(x, y) = \beta(h)(x, y)$ which means $(\alpha + \beta)(h)(x, y) = 0$. Therefore $(x, y) = 0$. □

Since the form $(\cdot, \cdot)$ is nondegenerate on $\mathfrak{h}$ it determines a bijection $\mathfrak{h}^* \to \mathfrak{h}$ given by $\alpha \mapsto h'_\alpha$ where $(h'_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$.

**Corollary 3.2.16.** *(a) Suppose $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = (x, y)h'_\alpha$.*

*(b) The pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ given by $(\cdot, \cdot)$ is nondegenerate.*

*(c) For each $x \in \mathfrak{g}_\alpha$ with $x \neq 0$ there exists $y \in \mathfrak{g}_{-\alpha}$ with $[x, y] \neq 0$.*

*Proof.* (a) Consider the element $[x, y] - (x, y)h'_\alpha \in \mathfrak{h}$. For all $h \in \mathfrak{h}$ we have $([x, y] - (x, y)h'_\alpha, h) = ([x, y], h) - (x, y)(h'_\alpha, h) = (x, [y, h]) - \alpha(h)(x, y) = 0$. Since the form is nondegenerate on $\mathfrak{h}$ we conclude that $[x, y] - (x, y)h'_\alpha = 0$.

(b) Since the form is nondegenerate on $\mathfrak{g}(A)$ and $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\beta = -\alpha$ the pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ must be nondegenerate.

(c) For each $0 \neq x \in \mathfrak{g}_\alpha$ there exists $y \in \mathfrak{g}_{-\alpha}$ with $(x, y) \neq 0$. Hence $[x, y] \neq 0$ by part (a). □

## 3.3 The Weyl group and the roots of a Kac–Moody Lie algebra

It is known that in $\mathfrak{g}(A)$ we have $(\operatorname{ad} e_i)^{1-A_{ij}} e_j = 0$ and $(\operatorname{ad} f_i)^{1-A_{ij}} f_j = 0$ for $i \neq j$, see [6, Proposition 16.10]. So the maps $\operatorname{ad} e_i$ and $\operatorname{ad} f_i$ are locally nilpotent i.e. for all $x \in \mathfrak{g}$ there exists $N \in \mathbb{N}$ such that $(\operatorname{ad} e_i)^N x = 0$. Therefore, $\exp \operatorname{ad}(e_i)$ and $\exp \operatorname{ad}(-f_i)$ can be defined as automorphisms of $\mathfrak{g}(A)$.

Let $n_i = \exp \operatorname{ad}(e_i) \circ \exp \operatorname{ad}(-f_i) \circ \exp \operatorname{ad}(e_i) \in \operatorname{Aut} \mathfrak{g}(A)$. Then we have $n_i(\mathfrak{h}) = \mathfrak{h}$ and for $x \in \mathfrak{h}$ we have $n_i(x) = x - \alpha_i(x) h_i$, see [6, Proposition 16.11].

Now let $s_i$ be the restriction of $n_i$ to $\mathfrak{h}$. Then the map $s_i : \mathfrak{h} \to \mathfrak{h}$ satisfies $s_i^2 = 1, s_i(h_i) = -h_i$, and $s_i(x) = x$ when $(h_i, x) = 0$. The maps $s_i : \mathfrak{h} \to \mathfrak{h}$ are called **fundamental reflections**. The group $W$ of nonsingular linear transformations of $\mathfrak{h}$ generated by $s_1, \ldots, s_n$ is called the **Weyl group** $W$ of $\mathfrak{g}(A)$. This group $W$ preserves the bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}$.

We may also define the action of $W$ on $\mathfrak{h}^*$ by $(w\lambda)x = \lambda(w^{-1}x)$ for $w \in W, \lambda \in \mathfrak{h}^*, x \in \mathfrak{h}$. Hence the action of $s_i$ on $\mathfrak{h}^*$ is given by $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$.

The Weyl group leaves the root system $\Phi$ of $\mathfrak{g}(A)$ invariant, i.e. if $\alpha \in \Phi$, $w \in W$ then $w(\alpha) \in \Phi$ and we have $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w(\alpha)}$.

The element $\alpha \in \Phi$ is called a **real root** if there exists $\alpha_i \in \Pi$ and $w \in W$ such that $\alpha = w(\alpha_i)$. The set of all real roots is denoted by $\Phi_{\mathrm{Re}}$. The element $\alpha \in \Phi$ is called an **imaginary root** if $\alpha$ is not real. The set of all imaginary roots is denoted by $\Phi_{\mathrm{Im}}$.

Note that if $\alpha$ is a real root, then so is $-\alpha$. For let $\alpha = w(\alpha_i)$ for some $\alpha_i \in \Pi$ and $w \in W$. Then we have $-\alpha = ws_i(\alpha_i)$. It also follows that if $\beta$ is an imaginary root so is $-\beta$. Moreover, as $\alpha = w(\alpha_i)$ and the fact that $\dim \mathfrak{g}_{w(\alpha_i)} = \dim \alpha_i = 1$, we have $\alpha$ with multiplicity 1. Also as we know if $k > 1$ then $k\alpha_i$ is not a root. So as $k\alpha = w(k\alpha_i)$, $k\alpha$ is also not a root.

**Proposition 3.3.1.** *The Weyl group $W$ preserves the set of positive imaginary roots, i.e. if $\alpha \in \Phi^+_{\mathrm{Im}}$ and $w \in W$ then $w(\alpha) \in \Phi^+_{\mathrm{Im}}$.*

*Proof.* We know that $W$ acts both on $\Phi$ and on the set $\Phi_{\mathrm{Re}}$ of real roots. Therefore $W$ acts on the set $\Phi_{\mathrm{Im}}$ of imaginary roots. We shall show that an element $w \in W$ cannot change the sign of the imaginary roots. Let $\alpha = \sum_{i=1}^{n} k_i \alpha_i$ where $k_i \geq 0$. Now at least two coefficients $k_i$ must be positive. Otherwise $\alpha$ would be a multiple of some $\alpha_i$ and hence equal to $\alpha_i$. Then $\alpha$ would be real which is a contradiction. Now $s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$ contains at least one fundamental root with positive coefficient. Hence $s_i(\alpha) \in \Phi^+_{\mathrm{Im}}$. Since $w \in W$ is a product of fundamental reflections $s_i$ we have $w(\alpha) \in \Phi^+_{\mathrm{Im}}$. □

Let $A$ be a GCM and take a real minimal realisation $(\mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^{\mathrm{v}})$. Then the **fundamental chamber** is defined as $\mathfrak{C} = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda(h_i) > 0 \text{ for } i = 1, \ldots, n\}$. Its closure is $\bar{\mathfrak{C}} = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda(h_i) \geq 0 \text{ for } i = 1, \ldots, n\}$.

**Proposition 3.3.2.** *Let $\alpha \in \Phi^+_{\mathrm{Im}}$. Then there exists $w \in W$ with $w(\alpha) \in -\bar{\mathfrak{C}}$.*

*Proof.* Consider the set of all elements $w(\alpha)$ for $w \in W$. These are all positive imaginary roots by Proposition 3.3.1. Let $\beta = \sum k_i \alpha_i$ be such a root for which $\mathrm{ht}(\beta)$ is as small as possible. Then $s_i(\beta) = \beta - \beta(h_i)\alpha_i$. Since $\mathrm{ht}\, s_i(\beta) \geq \mathrm{ht}\, \beta$ we have $\beta(h_i) \leq 0$ for all $i$. Thus $\beta \in -\bar{\mathfrak{C}}$. □

Now for $\alpha \in \Phi$ with $\alpha = \sum_{i=1}^{n} k_i \alpha_i$ we define $\mathrm{supp}\, \alpha = \{i \mid k_i \neq 0\}$. We can show that $\mathrm{supp}\, \alpha$ is connected, i.e. $\mathrm{supp}\, \alpha$ cannot be written as $\mathrm{supp}\, \alpha = J_1 \cup J_2$ with $J_1, J_2$ nonempty and $A_{ij} = 0$ for all $i \in J_1, j \in J_2$. For the proof see, [6, Proposition 16.21].

The following theorem shows that in order to understand the imaginary roots it is sufficient to understand the positive imaginary roots which lie in $-\bar{\mathfrak{C}}$, see [6, Proposition 16.23].

**Theorem 3.3.3.** *The set of positive imaginary roots of* $\mathfrak{g}(A)$ *are given by* $\Phi^+_{\mathrm{Im}} = \bigcup_{w \in W} w(K)$ *where* $K = \{\alpha \in Q^+ \mid \alpha \neq 0, \operatorname{supp}\alpha$ *is connected, and* $\alpha \in -\bar{\mathfrak{C}}\}$. *The set of all imaginary roots is* $\Phi^+_{\mathrm{Im}} \cup (-\Phi^+_{\mathrm{Im}})$.

By the above theorem and the fact that $\alpha \in K$ implies $k\alpha \in K$ for all positive integers $k$ we have,

**Corollary 3.3.4.** *Let* $\alpha \in \Phi^+_{\mathrm{Im}}$. *Then* $k\alpha \in \Phi^+_{\mathrm{Im}}$ *for all positive integers* $k$.

Now we consider the real and imaginary roots of $\mathfrak{g}(A)$ when $A$ is symmetrisable. As we know $\mathfrak{g}(A)$ has an invariant bilinear form $(\cdot,\cdot)$ which is nondegenerate on restriction to $\mathfrak{h}$, so determines an isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ under which $\lambda \mapsto h'_\lambda$, where $\lambda(x) = (h'_\lambda, x)$ for all $x \in \mathfrak{h}$. Therefore we can transfer the bilinear form to $\mathfrak{h}^*$ by defining $(\lambda, \mu) = (h'_\lambda, h'_\mu)$. In particular we can define $(\alpha, \alpha)$ for $\alpha \in \Phi$. It can be shown that if $A$ is a symmetrisable GCM, then if $\alpha$ is a real root of $\mathfrak{g}(A)$ we have $(\alpha, \alpha) > 0$ and if it is an imaginary root then $(\alpha, \alpha) \leq 0$ by [6, Proposition 16.26].

Now we mention some information about the imaginary roots in the three cases of our trichotomy, see [6, Proposition 16.27].

**Theorem 3.3.5.** *Let* $A$ *be an indecomposable GCM.*

*(a) If* $A$ *has finite type, then* $\mathfrak{g}(A)$ *has no imaginary roots.*

*(b) If* $A$ *has affine type, then there exists* $u > 0$ *with* $Au = 0$. *The vector* $u$ *is determined up to a scalar multiple. Thus there is a unique such* $u$ *whose entries are positive integers with no common factor. Let* $u = (a_1, \ldots, a_n)$. *Let* $\delta = a_1\alpha_1 + \ldots + a_n\alpha_n$. *Then the imaginary roots of* $\mathfrak{g}(A)$ *are the elements* $k\delta$ *for* $k \in \mathbb{Z}, k \neq 0$.

*(c) If* $A$ *has indefinite type, then there exists* $\alpha \in \Phi^+_{\mathrm{Im}}$ *such that* $\alpha = \sum_{i=1}^n k_i\alpha_i$ *with* $k_i > 0$ *and* $\alpha(h_i) < 0$ *for all* $i = 1, \ldots, n$.

A significant consequence of the last theorem is that if $A$ is an indecomposable GCM of affine or indefinite type then $\mathfrak{g}(A)$ has infinite dimension. Because in both cases $\mathfrak{g}(A)$ has an imaginary root $\alpha$. So $\mathfrak{g}(A)$ has infinitely many imaginary roots $k\alpha$ for $k \in \mathbb{Z}, k \neq 0$. Since $\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, therefore $\dim \mathfrak{g}(A)$ must be infinite.

## 3.4 Kac–Moody Lie algebras of affine type

Let $\mathfrak{g}(A)$ be a Kac–Moody Lie algebra where $A$ is a GCM of affine type. Let $A$ be an $n \times n$ matrix of rank $l$ then we have $n = l + 1$. We number the rows and the columns of $A$ by the integers $0, 1, \ldots, l$. We know there is a unique vector $a = (a_0, a_1, \ldots, a_l)^{\mathrm{tr}}$ whose coordinates are positive integers with no common factor such that $Aa = 0$. We choose the numbering of the vertices in such a way that node 0 is the one in black. We also show in each diagram the integer $a_i$ associated to each vertex. There is also a unique vector $(c_0, c_1, \ldots, c_l)$ whose coordinates are positive integers with no common factors such that $(c_0, c_1, \ldots, c_l)A = 0$. In fact the vector $(c_0, c_1, \ldots, c_l)$ for $A$ is the same as the vector $(a_0, a_1, \ldots, a_l)$ for the transpose $A^{\mathrm{tr}}$. So the vector $(c_0, c_1, \ldots, c_l)$ may be read off from the diagrams in the list shown in Figures 3.3 and 3.4 and in fact $c_0 = 1$ and $a_0 = 1$ unless $A$ has type $\tilde{C}'_l$ or $\tilde{A}'_1$. In these cases $a_0 = 2$.

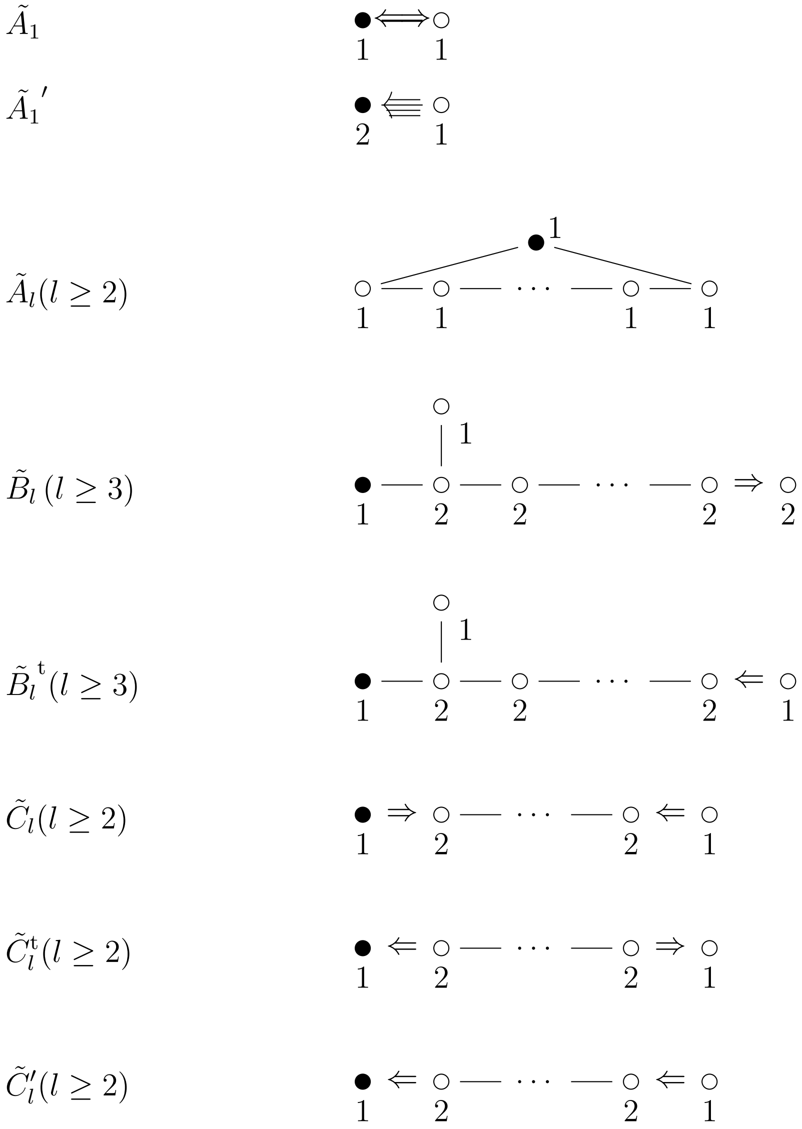

Figure 3.3: List of affine Dynkin diagrams part 1

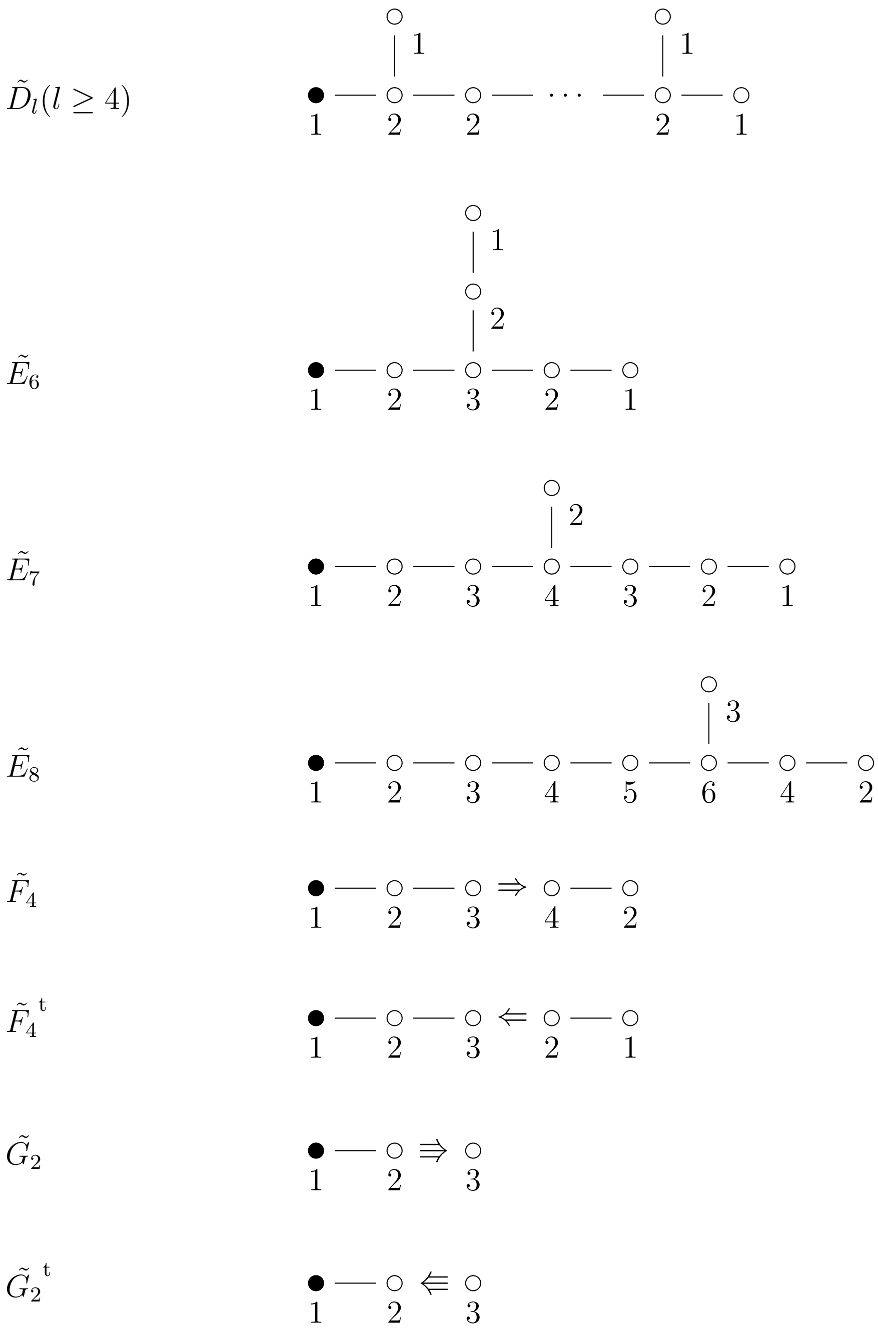


Figure 3.4: List of affine Dynkin diagrams part 2

Let $(\mathfrak{h}, \Pi, \Pi^{\text{v}})$ be a minimal realisation of $A$. Then $\dim \mathfrak{h} = 2n - l = l + 2$. $\Pi^{\text{v}} = \{h_0, h_1, \ldots, h_l\}$ is a linearly independent subset of $\mathfrak{h}$ and $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ is a linearly independent subset of $\mathfrak{h}^*$. There exists an element $d \in \mathfrak{h}$ such that $\alpha_0(d) = 1$, $\alpha_i(d) = 0$ for $i = 1, \ldots, l$. and $d$ is called a **scaling element**. It can be shown that $h_0, h_1, \ldots, h_l, d$ is a basis of $\mathfrak{h}$ see [6, Proposition 17.4].

Now we define an element $\gamma \in \mathfrak{h}^*$ determined uniquely by $\gamma(h_0) = 1$, $\gamma(h_i) = 0$, for $i = 1, \ldots, l$ and $\gamma(d) = 0$. It can be proved that $\alpha_0, \alpha_1, \ldots, \alpha_l, \gamma$ is a basis of $\mathfrak{h}^*$ see [6, Proposition 17.4].

In the following proposition we express the affine Cartan matrix $A$ in a more explicit way.

**Proposition 3.4.1.** *We have $A = DB$ where $B$ is symmetric and $D = \operatorname{diag}(d_0, d_1, \ldots, d_l)$ and $d_i = a_i/c_i$.*

*Proof.* We know that any indecomposable GCM of affine type is symmetrisable so there exists a diagonal matrix $D$ with positive diagonal entries and a symmetric matrix $B$ such that $A = DB$. Let $c = (c_0, c_1, \ldots, c_l)$ and $a^{\text{tr}} = (a_0, a_1, \ldots, a_l)$. Then $Aa = 0$ so $DBa = 0$, and therefore $Ba = 0$. Thus $a^{\text{tr}}B = 0$. Also $cA = 0$ so $(cD)B = 0$. Since $B$ has corank 1, $cD$ must be a scalar multiple of $a^{\text{tr}}$. In fact $D$ can be chosen so that $cD = a^{\text{tr}}$, that is $d_i = a_i/c_i$. □

Now the nondegenerate bilinear form which we already defined satisfies

$$(h_i, h_j) = d_i d_j B_{ij} = a_j c_j^{-1} A_{ij} \quad \text{for } i, j = 0, 1, \ldots, l$$

$$(h_0, d) = d_0 \alpha_0(d) = a_0$$

$$(h_i, d) = 0 \quad \text{for } i = 1, \ldots, l$$

$$(d, d) = 0$$

This standard invariant form on $\mathfrak{h}$ as we already mentioned defines a bijection $\mathfrak{h}^* \to \mathfrak{h}$ given by $\lambda \mapsto h'_\lambda$ where $\lambda(x) = (h'_\lambda, x)$ for all $x \in \mathfrak{h}$. Under this bijection between $\mathfrak{h}$ and $\mathfrak{h}^*$, we can show that $h_i \in \mathfrak{h}$ corresponds to $a_i c_i^{-1} \alpha_i \in \mathfrak{h}^*$ for $i = 0, 1, \ldots, l$ and $d \in \mathfrak{h}$ corresponds to $a_0 \gamma \in \mathfrak{h}^*$. So we may transfer the standard bilinear form $\mathfrak{h}$ to $\mathfrak{h}^*$ using this bijection. The form on $\mathfrak{h}^*$ is then given by

$$(\alpha_i, \alpha_j) = a_i^{-1} c_i A_{ij}$$

$$(\alpha_0, \gamma) = a_0^{-1}$$

$$(\alpha_i, \gamma) = 0 \quad i = 1, \ldots, l$$

$$(\gamma, \gamma) = 0$$

In particular, $A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Furthermore, under the given bijection between $\mathfrak{h}$ and $\mathfrak{h}^*$, we have $h_i \in \mathfrak{h}$ corresponding to $\frac{2\alpha_i}{(\alpha_i, \alpha_i)} \in \mathfrak{h}^*$.

We now define an element $c \in \mathfrak{h}$ by $c = \sum_{i=0}^{l} c_i h_i$ so that under the bijection $\mathfrak{h} \to \mathfrak{h}^*$ it corresponds to $\delta = \sum_{i=0}^{l} a_i \alpha_i$ and $h_i$ as we know corresponds to $a_i c_i^{-1} \alpha_i$. In fact it is known that $c$ lies in the centre of $\mathfrak{g}(A)$. The centre is 1-dimensional and consists of all scalar multiples of $c$, see [6, Proposition 17.8]. The element $c$ is called the **canonical central element** of $\mathfrak{g}(A)$.

## 3.5 The roots of an affine Kac–Moody Lie algebra

In this section we express the affine root systems in more explicit way.

Let $A^0$ be the matrix obtained from the affine Cartan matrix $A$ by removing the row and the column 0. Then $A^0$ is an $l \times l$ Cartan matrix of finite type. Let $\Phi^0$ be the set of roots of the finite dimensional Lie algebra $\mathfrak{g}(A^0)$ and $\Pi^0 = \{\alpha_1, \ldots, \alpha_l\}$ be a fundamental system of $\Phi^0$. Then the Weyl group $W^0$ is generated by the fundamental reflections $s_1, \ldots, s_l$. Now we consider the real roots of $\mathfrak{g}(A)$. As we know they have the form $w(\alpha_i)$

for some $w \in W$ and $i = 0, 1, \ldots, l$. We consider the squared lengths $(\alpha, \alpha)$ of the roots $\alpha \in \Phi_{\mathrm{Re}}$. Since $(w(\alpha_i), w(\alpha_i)) = (\alpha_i, \alpha_i)$ the length of any real root is equal to the length of some fundamental root. The relative length of roots can be obtained from Figures 3.1 and 3.2 using the formula $A_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ and $\frac{(\alpha_j,\alpha_j)}{(\alpha_i,\alpha_i)} = \frac{A_{ij}}{A_{ji}}$.

**Proposition 3.5.1.** *(a) If $A$ is an affine Cartan matrix of type $\tilde{A}_l, \tilde{D}_l, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ all the fundamental roots have the same length.*

*(b) If $A$ has types $\tilde{B}_l, \tilde{B}_l^t, \tilde{C}_l, \tilde{C}_l^t, \tilde{F}_4, \tilde{F}_4^t$ there are fundamental roots of two different lengths. The ratio $(\beta, \beta)/(\alpha, \alpha)$ where $\alpha$ is short and $\beta$ is long is two.*

*(c) If $A$ has type $\tilde{G}_2$ or $\tilde{G}_2^t$ there are fundamental roots of two different lengths with $(\beta, \beta)/(\alpha, \alpha) = 3$.*

*(d) If $A$ has type $\tilde{A}_1'$ there are fundamental roots of two different lengths with $(\beta, \beta)/(\alpha, \alpha) = 4$.*

*(e) If $A$ has type $\tilde{C}_l'$ there are fundamental roots of three different lengths, say $\alpha, \beta, \gamma$ with $(\beta, \beta)/(\alpha, \alpha) = 2$ and $(\gamma, \gamma)/(\beta, \beta) = 2$.*

Let $\Phi_{\mathrm{Re},s}$ be the set of short real roots, $\Phi_{\mathrm{Re,l}}$ the set of long real roots and $\Phi_{\mathrm{Re},i}$ the set of real roots of intermediate length. The following theorem characterises explicitly the set $\Phi_{\mathrm{Re}}$ of all real roots of each affine Kac–Moody algebra individually. We denote by $\Phi_s^0, \Phi_{\mathrm{l}}^0$ the set of short and long roots in $\Phi^0$. If all roots of $\Phi^0$ have the same length we write $\Phi_s^0 = \Phi^0$.

The following theorem gives more information on real roots of the affine Kac–Moody algebra $\mathfrak{g}(A)$, see [6, Theorem 17.17].

**Theorem 3.5.2.** *(a) If $A$ is one of the types $\tilde{A}_l, \tilde{B}_l, \tilde{C}_l, \tilde{D}_l, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$ then $\Phi_{\mathrm{Re}} = \{\alpha + r\delta \mid \alpha \in \Phi^0, r \in \mathbb{Z}\}$.*

(b) *If $A$ is one of the types $\tilde{B}_l^t, \tilde{C}_l^t, \tilde{F}_4^t$ then* $\Phi_{\mathrm{Re},s} = \{\alpha + r\delta \mid \alpha \in \Phi_s^0, r \in \mathbb{Z}\}$ *and* $\Phi_{\mathrm{Re,l}} = \{\alpha + 2r\delta \mid \alpha \in \Phi_{\mathrm{l}}^0, r \in \mathbb{Z}\}$.

(c) *If $A$ is of type $\tilde{G}_2^t$ then* $\Phi_{\mathrm{Re},s} = \{\alpha + r\delta \mid \alpha \in \Phi_s^0, r \in \mathbb{Z}\}$ *and* $\Phi_{\mathrm{Re,l}} = \{\alpha + 3r\delta \mid \alpha \in \Phi_{\mathrm{l}}^0, r \in \mathbb{Z}\}$.

(d) *If $A$ is of type $\tilde{C}_l'$ then* $\Phi_{\mathrm{Re},s} = \{\frac{1}{2}(\alpha + (2r-1))\delta \mid \alpha \in \Phi_l^0, r \in \mathbb{Z}\}$, $\Phi_{\mathrm{Re},i} = \{\alpha + r\delta \mid \alpha \in \Phi_s^0, r \in \mathbb{Z}\}$ *and* $\Phi_{\mathrm{Re,l}} = \{\alpha + 2r\delta \mid \alpha \in \Phi_{\mathrm{l}}^0, r \in \mathbb{Z}\}$.

(e) *If $A$ is of type $\tilde{A}_l'$ then* $\Phi_{\mathrm{Re},s} = \{\frac{1}{2}(\alpha + (2r-1)\delta \mid \alpha \in \Phi^0, r \in \mathbb{Z}\}$ *and* $\Phi_{\mathrm{Re,l}} = \{\alpha + 2r\delta \mid \alpha \in \Phi^0, r \in \mathbb{Z}\}$.

## 3.6 Realisations of affine Kac-Moody Lie algebras

In this secton we first show that every Cartan matrix of untwisted type which is defined below can be constructed from a Cartan matrix $A^0$ of finite type. Later in this section we show a method to construct the affine Kac-Moody algebra $\mathfrak{g}(A)$ from the finite dimensional simple Lie algebra $\mathfrak{g}^0 = \mathfrak{g}(A^0)$.

Let $A^0$ be an indecomposable Cartan matrix of finite type and let $\mathfrak{g}^0 = \mathfrak{g}(A^0)$ be the finite dimensional simple Lie algebra with Cartan matrix $A^0$. We can construct an $(l+1) \times (l+1)$ affine Cartan matrix $A$ from $A^0$ by adding an additional row and column, labelled by 0, as follows. Let $\theta = \sum_{i=1}^l a_i\alpha_i$ be the highest root of $\mathfrak{g}^0$ and $h_\theta = \sum_{i=1}^l c_i h_i$ be the co-root of $\theta$. Then we can define $A$ by $A_{ij} = A_{ij}^0$ if $i, j \in \{1, \ldots, l\}$, $A_{i0} = -\sum_{j=1}^l a_j A_{ij}^0$ if $i \in \{1, \ldots, l\}$, $A_{0j} = -\sum_{i=1}^l c_i A_{ij}^0$ if $j \in \{1, \ldots, l\}$, and $A_{00} = 2$. We can show that $A$ is a Cartan matrix and the type of $A$ is as follows: The type of $A$ is $\tilde{A}_l, \tilde{B}_l, \tilde{C}_l, \tilde{D}_l, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$ where $A^0$ is of type $A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$ respectively see [6, Proposition 18.1]. We call the affine Cartan matrix $A$ of type $\tilde{A}_l, \tilde{B}_l, \tilde{C}_l, \tilde{D}_l, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$ the **untwisted type**.

## Loop algebras and central extensions

Let $\mathbb{C}[t,t^{-1}]$ be the ring of Laurent polynomials $\sum_{i\in\mathbb{Z}}\zeta_i t^i$ for $\zeta_i \in \mathbb{C}$ with finitely many $\zeta_i \neq 0$. Let $\mathcal{G}(\mathfrak{g}^0) = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}^0$. Then $\mathcal{G}(\mathfrak{g}^0)$ can be made into a Lie algebra in a unique way by $[p\otimes x, q\otimes y] = pq\otimes[x,y]$ for $p,q \in \mathbb{C}[t,t^{-1}], x,y \in \mathfrak{g}^0$. This Lie algebra $\mathcal{G}(\mathfrak{g}^0)$ is called the **loop algebra** of $\mathfrak{g}^0$.

For a Lie algebra $\mathfrak{l}$ over $\mathbb{C}$, let $\tilde{\mathfrak{l}}$ be the set of elements $x+\lambda c$ with $x\in\mathfrak{l}$, $\lambda\in\mathbb{C}$ and $c$ is a symbol. Let $\kappa : \mathfrak{l}\times\mathfrak{l} \to \mathbb{C}$ be a bilinear map satisfying $\kappa(x,y) = -\kappa(y,x)$ for $x,y\in\mathfrak{l}$ and $\kappa([x,y],z)+\kappa([y,z],x)+\kappa([z,x],y) = 0$ for $x,y,z\in\mathfrak{l}$. This bilinear form $\kappa$ is called a **2-cocycle** on $\mathfrak{l}$. It is easy to show that the Lie multiplication $[x+\lambda c, y+\mu c] = [x,y]+\kappa(x,y)c$ makes $\tilde{\mathfrak{l}}$ into a Lie algebra. This Lie algebra is a 1-dimensional **central extension** of $\mathfrak{l}$, i.e. there is a surjective homomorphism $\theta : \tilde{\mathfrak{l}} \to \mathfrak{l}$ given by $\theta(x+\lambda c) = x$, such that $\dim(\ker\theta) = 1$ and $\ker\theta$ lies in the centre of $\tilde{\mathfrak{l}}$.

We can apply this idea to build a 1-dimensional central extension of $\mathcal{G}(\mathfrak{g}^0)$ by taking a 2-cocycle on $\mathcal{G}(\mathfrak{g}^0)$. Let $(\cdot,\cdot)$ be the invariant bilinear form on $\mathfrak{g}^0$ satisfying $(h_\theta,h_\theta) = 2$. This is a unique invariant bilinear form which we can obtain it by rescaling the killing bilinear form to $\mathfrak{g}^0$. As $(\theta,\theta) = 2$, hence we have $(h_\theta,h_\theta) = \left(\frac{2h_\theta{}'}{(\theta,\theta)}, \frac{2h_\theta{}'}{(\theta,\theta)}\right) = 2$. Now we define a bilinear form $(\cdot,\cdot)_t : \mathcal{G}(\mathfrak{g}^0)\times\mathcal{G}(\mathfrak{g}^0) \to \mathbb{C}[t,t^{-1}]$ by $(p\otimes x, q\otimes y)_t = pq(x,y)$. We define the residue function $\mathrm{Res} : \mathbb{C}[t,t^{-1}] \to \mathbb{C}$ by $\mathrm{Res}(\sum\zeta_i t^i) = \zeta_{-1}$. The function $\kappa : \mathcal{G}(\mathfrak{g}^0)\times\mathcal{G}(\mathfrak{g}^0) \to \mathbb{C}$ defined by $\kappa(p\otimes x, q\otimes y) = \mathrm{Res}(\frac{\mathrm{d}\,p}{\mathrm{d}\,t}\otimes x, q\otimes y)_t = \mathrm{Res}(\frac{\mathrm{d}\,p}{\mathrm{d}\,t}q(x,y))$ is a 2-cocycle on $\mathcal{G}(\mathfrak{g}^0)$, see [6, Lemma 18.3]. Therefore we can obtain the 1-dimensional central extension of $\mathcal{G}(\mathfrak{g}^0)$ given by $\tilde{\mathcal{G}}(\mathfrak{g}^0) = \mathcal{G}(\mathfrak{g}^0)\oplus\mathbb{C}c$ whose Lie multiplication is given by $[a+\lambda c, b+\mu c] = [a,b]_0 + \kappa(a,b)c$ where $a,b\in\tilde{\mathcal{G}}(\mathfrak{g}^0)$ and $[a,b]_0$ is the Lie product of $a,b\in\mathcal{G}(\mathfrak{g}^0)$. Moreover, we can adjoin to $\tilde{\mathcal{G}}(\mathfrak{g}^0)$ an element $d$ which acts on $\tilde{\mathcal{G}}(\mathfrak{g}^0)$ as a derivation which is explained below.

The map $\Delta : \mathcal{G}(\mathfrak{g}^0) \to \mathcal{G}(\mathfrak{g}^0)$ defined by $\Delta(a+\lambda c) = t\frac{\mathrm{d}\,a}{\mathrm{d}\,t}$ for $a\in\mathcal{G}(\mathfrak{g}^0)$, $\lambda\in\mathbb{C}$ is a derivation see [6, Lemma 18.4]. Now we define $\hat{\mathcal{G}}(\mathfrak{g}^0)$ by $\hat{\mathcal{G}}(\mathfrak{g}^0) = \tilde{\mathcal{G}}(\mathfrak{g}^0) \bigoplus \mathbb{C}d$ and make $\hat{\mathcal{G}}(\mathfrak{g}^0)$ into a Lie algebra by defining the Lie product as $[a + \lambda d, b + \mu d] = [a, b] + \lambda\Delta(b) - \mu\Delta(a)$. In particular we have $[(t^i \otimes x) + \lambda c + \mu d, (t^j \otimes y) + \lambda' c + \mu' d] = (t^{i+j} \otimes [x, y]) + \mu j(t^j \otimes y) - \mu' i(t^i \otimes x)) + \delta_{i,-j} i(x, y)c$ for $x, y \in \mathfrak{g}^0$, $\lambda, \mu, \lambda', \mu' \in \mathbb{C}$. The following theorem is an important result of this chapter that the Lie algebra $\hat{\mathcal{G}}(\mathfrak{g}^0)$ is isomorphic to the affine Kac-Moody algebra $\mathfrak{g}(A)$

**Theorem 3.6.1.** *Let $\mathfrak{g}^0 = \mathfrak{g}(A^0)$ be a finite dimensional simple Lie algebra. Let $A$ be the untwisted affine Cartan matrix obtained from $A^0$ as in Section 3.6. Then $\mathfrak{g}(A)$ is isomorphic to $\hat{\mathcal{G}}(\mathfrak{g}^0)$.*

*Proof.* We shall define elements $e_0, e_1, \ldots, e_l$; $f_0, f_1, \ldots, f_l$; $h_0, h_1, \ldots, h_l$ in $\hat{\mathcal{G}}(\mathfrak{g}^0)$.

We will use Proposition 3.2.2 to show that the Lie algebra $\hat{\mathcal{G}}(\mathfrak{g}^0)$ is isomorphic to $\mathfrak{g}(A)$. Let $E_1, \ldots, E_l$; $F_1, \ldots, F_l$; $H_1, \ldots, H_l$ be corresponding generators of $\mathfrak{g}^0$. We define

$$e_i = 1 \otimes E_i$$

$$f_i = 1 \otimes F_i$$

$$h_i = 1 \otimes H_i$$

for $i = 1, \ldots, l$.

Then $[e_i, f_i] = h_i$ for each $i$. We also need to define $e_0, f_0, h_0 \in \hat{\mathcal{G}}(\mathfrak{g}^0)$. To do this consider the root spaces $\mathfrak{g}^0_\theta$ and $\mathfrak{g}^0_{-\theta}$ where $\theta$ is the highest root of $\mathfrak{g}^0$. We have $\dim \mathfrak{g}^0_\theta = \dim \mathfrak{g}^0_{-\theta} = 1$ and the map $\mathfrak{g}^0_\theta \times \mathfrak{g}^0_{-\theta} \to \mathbb{C}$ given by the invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}^0$ is nondegenerate. Let $\omega^0$ be the automorphism of $\mathfrak{g}^0$ satisfying $\omega^0(E_i) = -F_i$, $\omega^0(F_i) = -E_i$, see [6, Proposition 14.17]. Then $\omega^0(\mathfrak{g}^0_\theta) = \mathfrak{g}^0_{-\theta}$. We can show that there exists elements

$F_0 \in \mathfrak{g}^0_\theta$ and $E_0 \in \mathfrak{g}^0_{-\theta}$ such that $\omega^0(F_0) = -E_0$ and $(F_0, E_0) = 1$. Now we define

$$e_0 = t \otimes E_0$$

and

$$f_0 = t^{-1} \otimes F_0.$$

Let $\mathfrak{h}^0$ be the subspace of $\mathfrak{g}^0$ generated by $h_1, \ldots, h_l$ and

$$\mathfrak{h} = (1 \otimes \mathfrak{h}^0) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

We define

$$h_0 = (1 \otimes (-H_\theta)) + c.$$

Then we have

$$[e_0, f_0] = [t \otimes E_0, t^{-1} \otimes F_0)] = (1 \otimes [E_0, F_0]) + (E_0, F_0)c.$$

and

$$[E_0, F_0] = (E_0, F_0)H'_{-\theta} = H'_{-\theta} = H_{-\theta} = -H_\theta$$

because $(\theta, \theta) = 2$ and therefore $[e_0, f_0] = h_0$.

Now we need to define $\alpha_0, \alpha_1, \ldots, \alpha_l \in \mathfrak{h}^*$. We have elements $\alpha_1, \ldots, \alpha_l \in (\mathfrak{h}^0)^*$ so we can extend these elements to $\mathfrak{h}^*$ by saying that $\alpha_i(c) = \alpha_i(d) = 0$. We also define $\theta \in \mathfrak{h}^*$ similarly by saying $\theta(c) = \theta(d) = 0$. Let $\delta$ be an element of $\mathfrak{h}^*$ defined by $\delta(x) = 0$ for all $x \in \mathfrak{h}$, $\delta(c) = 0$, and $\delta(d) = 1$. We then define $\alpha_0 = -\theta + \delta$ in $\mathfrak{h}^*$.

Next we show that $(\mathfrak{h}, \Pi, \Pi^{\text{v}})$ is a realization of $A$ where $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ and $\Pi^{\text{v}} = \{h_0, h_1, \ldots, h_l\}$.

The sets $\Pi$ and $\Pi^{\text{v}}$ are linearly independent and we can show that for all $i, j \in$

$\{0, 1, \ldots, l\}$ we have $\alpha_j(h_i) = A_{ij}$ and therefore$(\mathfrak{h}, \Pi, \Pi^{\mathrm{v}})$ is a realization of $A$. We can also check that the relations $[e_i, f_i] = h_i$, $[e_i, f_j] = 0$ for $j \neq i$, $[x, e_i] = \alpha_i(x)e_i$, and $[x, f_i] = -\alpha_i(x)f_i$ are satisfied for all $i, j \in \{0, 1, \ldots, l\}$. We now need to show that all $e_0, e_1, \ldots, e_l$; $f_0, f_1, \ldots, f_l$; $h_0, h_1, \ldots, h_l$ generate $\hat{\mathcal{G}}(\mathfrak{g}^0)$.

Let $M$ be a subalgebra of $\hat{\mathcal{G}}(\mathfrak{g}^0)$ generated by these elements. We will show that $M = \hat{\mathcal{G}}(\mathfrak{g}^0)$. Since elements $E_1, E_2, \ldots, E_l$; $F_1, F_2, \ldots, F_l$ generate $\mathfrak{g}^0$, $e_1, \ldots, e_l, f_1, \ldots, f_l$ generate $1 \otimes \mathfrak{g}^0$. Thus $1 \otimes \mathfrak{g}^0 \subset M$. Let $I^0 = \{x \in \mathfrak{g}^0 \mid t \otimes x \in M\}$. Since $e_0 = t \otimes E_0$ we have $E_0 \in I^0$ so $I^0 \neq 0$. Moreover, $I^0$ is an ideal of $\mathfrak{g}^0$ because for $x \in I^0$ and $y \in \mathfrak{g}^0$ we have $t \otimes [x, y] = [t \otimes x, 1 \otimes y]$ which is in $M$. As $\mathfrak{g}^0$ is a simple Lie algebra we have $I^0 = \mathfrak{g}^0$. Thus $t \otimes x \in M$ for all $x \in \mathfrak{g}^0$. By using the relation $[t \otimes x, t^{k-1} \otimes y] = t^k \otimes [x, y]$ and induction on $k$ we can show that $t^k \otimes x \in M$ for all $x \in \mathfrak{g}^0$ and all $k > 0$. Similarly starting from $f_0 = t^{-1} \otimes F_0$ we can show that $t^{-k} \otimes x \in M$ for all $x \in \mathfrak{g}^0$ and all $k > 0$. Now $\hat{\mathcal{G}}(\mathfrak{g}^0) = \mathfrak{h} + (1 \otimes \mathfrak{g}^0) + \sum_{k>0}(t^k \otimes \mathfrak{g}^0) + \sum_{k<0}(t^k \otimes \mathfrak{g}^0)$ hence $M = \hat{\mathcal{G}}(\mathfrak{g}^0)$. It remains to show that $\hat{\mathcal{G}}(\mathfrak{g}^0)$ has no nonzero ideal $J$ with $J \cap \mathfrak{h} = 0$.

Let $\mathcal{G} = \hat{\mathcal{G}}(\mathfrak{g}^0) = \mathfrak{h} \oplus \sum_{(i,\alpha)\neq(0,0)}(t^i \otimes (\mathfrak{g}^0)_\alpha)$ summed over $i \in \mathbb{Z}$, $\alpha \in (\mathfrak{h}^0)^*$ with $(i, \alpha) \neq (0, 0)$. We claim this is the weight space decomposition of $\mathcal{G}$ with respect to $\mathfrak{h}$. Let $h \in \mathfrak{h}$ and $x \in (\mathfrak{g}^0)_\alpha$. Then $h = h_0 + \lambda c + \mu d$ with $h_0 \in \mathfrak{h}^0$, $\lambda, \mu \in \mathbb{C}$. Thus $[h, t^i \otimes x] = [h_0 + \lambda c + \mu d, t^i \otimes x] = (t^i \otimes [h_0, x]) + \mu i(t^i \otimes x) = (\alpha(h_0) + \mu i)(t^i \otimes x) = (\alpha(h) + i\delta(h))(t^i \otimes x) = (\alpha + i\delta)(h)(t^i \otimes x)$ since $\alpha(h) = \alpha(h_0)$, and $\delta(h) = \mu$. Thus $t^i \otimes x$ is a weight vector with weight $\alpha + i\delta$. Thus we have

$$\mathcal{G} = \mathcal{G}_0 \oplus \sum_{(i,\alpha)\neq(0,0)} \mathcal{G}_{\alpha+i\delta}$$

where $\mathcal{G}_0 = \mathfrak{h}$ and $\mathcal{G}_{\alpha+i\delta} = t^i \otimes (\mathfrak{g}^0)_\alpha$.

Let $J$ be a nonzero ideal of $\mathcal{G}$ with $J \cap \mathfrak{h} = 0$. Thus we have $J = (\mathcal{G}_0 \cap J) \oplus \sum_{(i,\alpha)\neq(0,0)}(\mathcal{G}_{\alpha+i\delta} \cap J)$. Since $\mathcal{G}_0 \cap J = 0$ we have $\mathcal{G}_{\alpha+i\delta} \cap J \neq 0$ for some $(\alpha, i)$. Let

$t^i \otimes x \in J$ for some $x \in (\mathfrak{g}^0)_\alpha$ with $x \neq 0$. Then there exists $y \in (\mathfrak{g}^0)_{-\alpha}$ with $(x, y) \neq 0$. Thus $[t^i \otimes x, t^{-i} \otimes y] = [x, y] + i(x, y)c$ lies in $J \cap \mathfrak{h}$ and hence $[x, y] + i(x, y)c = 0$. Since $[x, y] \in \mathfrak{h}^0$ and $(x, y) \neq 0$ we must have $i = 0$. But this implies $[x, y] = 0$ whereas we have $[x, y] = (x, y)h'_\alpha \neq 0$. This gives us a contradiction. Thus $J \cap \mathfrak{h} = 0$ implies $J = 0$.

□

So we see from Theorem 3.6.1 that $\mathfrak{g}(A)$ can be constructed from $\mathfrak{g}^0 = \mathfrak{g}(A^0)$ by the following procedure. First we form the loop algebra $\mathcal{G}(\mathfrak{g}^0)$ of $\mathfrak{g}^0$. Then form the 1-dimensional central extension $\tilde{\mathcal{G}}(\mathfrak{g}^0)$. Finally extend this Lie algebra by a derivation to give $\hat{\mathcal{G}}(\mathfrak{g}^0)$ which is isomorphic to the affine Lie algebra $\mathfrak{g}(A)$ corresponding to the Cartan matrix $A$. Therefore, we are going to identify $\mathfrak{g} = \mathfrak{g}(A) = \hat{\mathcal{G}}(\mathfrak{g}^0)$ and we have

$$\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\Phi$ is the set of roots $\Phi = \Phi_{\text{Re}} \uplus \Phi_{\text{Im}} = \{\alpha + i\delta \mid \alpha \in \Phi^0, i \in \mathbb{Z}\} \uplus \{i\delta \mid i \in \mathbb{Z}_{\neq 0}\}$ such that $\alpha \in \Phi$ is defined by extending $\alpha \in \Phi^0$ from $\mathfrak{h}^0$ to $\mathfrak{h}$ by $\alpha(c) = \alpha(d) = 0$ and $\delta \in \mathfrak{h}^*$ is defined by $\delta(x) = 0$ for all $x \in \mathfrak{h}^0$, $\delta(c) = 0$ and $\delta(d) = 1$ and $\alpha_0 = \delta - \theta$. The nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}$ extends the form on $\mathfrak{h}^0$ such that $(\mathfrak{h}^0, \mathbb{C}c \oplus \mathbb{C}d) = 0$, $(c, c) = (d, d) = 0$ and $(c, d) = 1$. Therefore we can identify $\mathfrak{h} \cong \mathfrak{h}^*$ and $\mathfrak{h}^* = (\mathfrak{h}^0)^* \oplus (\mathbb{C}c \oplus \mathbb{C}d)^*$. We have $\alpha_1, \alpha_2, \ldots, \alpha_l \in (\mathfrak{h}^0)^*$ and $\delta \in (\mathbb{C}c \oplus \mathbb{C}d)^*$. For $\alpha \in \Phi^0$ we have $\mathfrak{g}_\alpha = (\mathfrak{g}^0)_\alpha$, and $\mathfrak{g}_{\alpha+i\delta} = t^i \otimes (\mathfrak{g}^0)_\alpha$ such that all these are 1-dimensional. Also we have $\mathfrak{g}_{i\delta} = t^i \otimes \mathfrak{h}^0$ which is $\ell$-dimensional.

The set of positive roots is

$$\Phi^+ = \left( \sum_{i=0}^{l} \mathbb{Z}_{\geq 0} \alpha_i \right) \cap \Phi = \{\alpha + i\delta \mid \alpha \in \Phi^0, i \in \mathbb{Z}_{>0}\} \cup \{\alpha \mid \alpha \in (\Phi^0)^+\}.$$

Moreover, $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$ is

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{i>0, \alpha \in \Phi^0} \mathfrak{g}_{\alpha + i\delta} \oplus \bigoplus_{\alpha \in (\Phi^0)^+} \mathfrak{g}_{\alpha}.$$

# Chapter 4

# Levi type root systems for affine Kac-Moody Lie algebras

Let $\mathfrak{g} = \mathfrak{g}(A)$ be an untwisted affine Kac–Moody Lie algebra with GCM $A$, a root system $\Phi$ and a set of fundamental roots $\Pi = \{\alpha_0, \ldots, \alpha_l\}$. In this chapter we first define the Levi type root system for $\mathfrak{g}$ corresponding to a Levi factor $\mathfrak{g}_Y$ of a parabolic subgroup $\mathfrak{p}_Y$ where $Y \subseteq \{1, \ldots, l\}$. Then we consider the centralizer of a nilpotent element and show how the centralizers of regular nilpotent elements of $\mathfrak{g}^0 = \mathfrak{g}(A^0)$ are related to Levi type root systems where $A^0$ is the finite type Cartan matrix obtained by removing the row and the column 0 from $A$. Finally we investigate some properties of these root systems.

## 4.1 Levi type root systems

Let $\mathfrak{g}$ be a Lie algebra of affine type which is identified with $\mathfrak{g} = \hat{\mathcal{G}}(\mathfrak{g}^0)$ as in Section 3.6. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the weight space decomposition with respect to $\mathfrak{h}$ and the set of fundamental roots $\Pi = \{\alpha_0, \ldots, \alpha_l\}$. For any subset $Y \subseteq \{1, \ldots, l\}$ we define $\Phi_Y = \Phi \cap (\bigoplus_{i \in Y} \mathbb{Z}\alpha_i)$, $\Phi_Y^+ = \Phi^+ \cap \Phi_Y$, $\mathfrak{g}_Y = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_Y} \mathfrak{g}_\alpha$, $\mathfrak{u}_Y = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi_Y^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}_Y = \mathfrak{g}_Y \oplus \mathfrak{u}_Y$. We call $\mathfrak{p}_Y$ the **standard parabolic subalgebra** of $\mathfrak{g}$ corresponding to the subset $Y$. The subalgebra $\mathfrak{g}_Y$ is called the **standard Levi factor** of $\mathfrak{p}_Y$ and $\mathfrak{u}_Y$ is the **nilpotent radical** of $\mathfrak{p}_Y$. Note that since $\Phi_Y \subseteq \Phi^0$, we have $\mathfrak{g}_Y = \mathfrak{g}_Y^0 \oplus \mathbb{C}c \oplus \oplus\mathbb{C}d$. Also let

$$\mathfrak{h}^Y = \mathfrak{z}(\mathfrak{g}_Y)$$

For all $\alpha \in \Phi_Y$ and all nonzero elements $x \in \mathfrak{g}_\alpha$ we have $[h, x] = \alpha(h)x \neq 0$ for some $h \in \mathfrak{h}$. Therefore, $\mathfrak{z}(\mathfrak{g}_Y) \subseteq \mathfrak{h}$ and $\mathfrak{z}(\mathfrak{g}_Y)$ consists of all those elements $h \in \mathfrak{h}$ such that for all $\alpha \in \Phi_Y$ we have $\alpha(h) = 0$. Hence,

$$\mathfrak{h}^Y = \mathfrak{z}(\mathfrak{g}_Y) = \{t \in \mathfrak{h} \mid \alpha(t) = 0 \text{ for all } \alpha \in \Phi_Y\}.$$

Since $\mathfrak{h}^Y \subset \mathfrak{h}$, it acts semisimply on $\mathfrak{g}$ and we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi^Y} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [t, x] = \alpha(t)x \text{ for all } t \in \mathfrak{h}^Y\}$ and $\mathfrak{g}_0 = \mathfrak{c}_\mathfrak{g}(\mathfrak{h}^Y)$.

The subset $\Phi^Y = \{\alpha \in (\mathfrak{h}^Y)^* \mid \mathfrak{g}_\alpha \neq 0\} \subseteq (\mathfrak{h}^Y)^*$ is called the **Levi type root system**. We also have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Phi \\ \alpha(\mathfrak{h}^Y) = 0}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^Y} \bigoplus_{\substack{\beta \in \Phi \\ \beta|_{\mathfrak{h}^Y} = \alpha}} \mathfrak{g}_\beta$$

In fact $\Phi^Y = \{\alpha|_{\mathfrak{h}^Y} \mid \alpha \in \Phi, \alpha|_{\mathfrak{h}^Y} \neq 0\}$. We also have $= \mathfrak{c}_\mathfrak{g}(\mathfrak{h}^Y) = \mathfrak{c}_\mathfrak{g}(\mathfrak{z}(\mathfrak{g}_Y)) = \mathfrak{g}_Y = \mathfrak{g}_0$.

For $\alpha \in \Phi^Y$ let $\Phi(\mathfrak{g}_\alpha) = \{\beta \in \Phi \mid \beta|_{\mathfrak{h}^Y} = \alpha\}$. Then $\Phi = \bigsqcup_{\alpha \in \Phi^Y \cup \{0\}} \Phi(\mathfrak{g}_\alpha)$.

Recall $\delta \in \mathfrak{h}^*$ in 3.6.1 defined by $\delta(x) = 0$ for all $x \in \mathfrak{h}^0$, $\delta(c) = 0$, and $\delta(d) = 1$. Let $\delta^Y = \delta|_{\mathfrak{h}^Y}$.

The following proposition shows that Levi type root spaces are finite dimensional.

**Proposition 4.1.1.** *Let $\alpha \in \Phi^Y$. Then for*

$$\mathfrak{g}_\alpha = \bigoplus_{\substack{\beta\in\Phi \\ \beta|_{\mathfrak{h}^Y}=\alpha}} \mathfrak{g}_\beta,$$

*we have $\mathfrak{g}_\alpha$ of finite dimension.*

*Proof.* For any $\beta \in \Phi$ the dimension of $\mathfrak{g}_\beta$ is finite. So equivalently we show that the set $\{\beta \in \Phi \mid \beta|_{\mathfrak{h}^Y} = \alpha \text{ for } \alpha \in \Phi^Y\}$ is a finite set. First we consider the case where $\alpha = \alpha_i|_{\mathfrak{h}^Y} \in \Phi^Y$ and $i \notin Y$. Let $\beta = \sum_{j=0}^{l} a_j\alpha_j \in \Phi$. Then $\beta|_{\mathfrak{h}^Y} = \sum_{j\notin Y} a_j\alpha_j|_{\mathfrak{h}^Y}$, because if $j \in Y$, then $\alpha_j|_{\mathfrak{h}^Y} = 0$. Note that $\beta|_{\mathfrak{h}^Y} = 0$ if and only if $\beta \in \Phi_Y$ if and only if $\beta \in \mathbb{Z}\Pi_Y$ if and only if $a_j = 0$ for $j \notin Y$. Therefore $\beta|_{\mathfrak{h}^Y} = \alpha$ if and only if $\beta = \alpha_i + \sum_{j\in Y} a_j\alpha_j$. We have $\Phi = \{\gamma + k\delta \mid \gamma \in \Phi^0, k \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}_{\neq 0}\}$ where $\delta = \sum_{j=0}^{l} d_j\alpha_j$ and $d_j \neq 0$ for all $j$. The coefficient of $\alpha_i$ in $k\delta$ is $kd_i$ and this equals to 1 for at most one value only $k = 1/d_i$. For each $\gamma \in \Phi^0$ where $\gamma = \sum_{j=1}^{l} c_j\alpha_j$ we have $\gamma + k\delta$ with coefficient $c_i + kd_i$ of $\alpha_i$. This is equal to 1 for at most one value $k = (1 - c_i)/di$. So there are finitely many elements $\beta \in \Phi$ such that $\beta|_{\mathfrak{h}^Y} = \alpha_i|_{\mathfrak{h}^Y}$.

A very similar argument will deal with arbitrary elements of $\Phi^Y$. Indeed, consider $\alpha = \sum_{j\notin Y} a_j\alpha_j|_{\mathfrak{h}^Y} \in \Phi^Y$. For $\beta \in \Phi$ we have $\beta|_{\mathfrak{h}^Y} = \alpha$ if and only if $\beta = \sum_{j\notin Y} a_j\alpha_j + \sum_{j\in Y} b_j\alpha_j$ where $b_j \in \mathbb{Z}$. First we consider the possibilities for $\beta = k\delta$ for some $k \in \mathbb{Z}$ such that $\beta|_{\mathfrak{h}^Y} = \alpha$. We pick $i \notin Y$. Then the coefficient of $\alpha_i$ in $\beta$ must be $a_i$ for $\beta|_{\mathfrak{h}^Y} = \alpha$ and is equal to $kd_i$. So we require to have $kd_i = a_i$ and therefore there is at most one possibility for $k$. Now we consider the case when $\beta = \gamma + k\delta$ for $k \in \mathbb{Z}$ and $\gamma \in \Phi^0$. The coefficient of $\alpha_i$ in $\beta$ is $c_i + kd_i$ and must be equal to $a_i$. So there exists at most one possibility for $k$. Hence there are finite number of elements $\beta \in \Phi$ such that $\beta|_{\mathfrak{h}^Y} = \alpha$ for $\alpha \in \Phi^Y$. □

The following theorem gives a description of $\Phi^Y$ for $Y \subseteq \{1, \ldots, n\}$.

**Theorem 4.1.2.** *Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra with a Cartan subalgebra $\mathfrak{h}$,* *the set of roots $\Phi$ and the set of fundamental roots $\Pi = \{\alpha_0, \ldots, \alpha_l\}$. Then for any subset $Y \subseteq \{1, \ldots, l\}$ the corresponding Levi type root system is*

$$\Phi^Y = \{i\delta^Y \mid i \in \mathbb{Z}_{\neq 0}\} \cup \{\alpha + i\delta^Y \mid \alpha \in (\Phi^0)^Y, i \in \mathbb{Z}\}.$$

*where $\Phi^0$ is the root system of the finite dimensional Lie algebra $\mathfrak{g}^0 = \mathfrak{g}(A^0)$ and $(\Phi^0)^Y$ is the Levi type root system for $\mathfrak{g}^0$ corresponding to $Y$.*

*Proof.* Let $\mathfrak{g}$ be identified with $\mathfrak{g} = \hat{\mathcal{G}}(\mathfrak{g}^0)$ where $\mathfrak{g}^0$ is a finite dimensional Lie algebra. From [3] we have $\mathfrak{g}^0 = (\mathfrak{g}^0)_0 \oplus \bigoplus_{\alpha \in (\Phi^0)^Y} (\mathfrak{g}^0)_\alpha$ and we have

$$\begin{aligned} \mathfrak{g} &= \left( (\mathfrak{g}^0)_0 \oplus \bigoplus_{\alpha \in (\Phi^0)^Y} (\mathfrak{g}^0)_\alpha \right) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \\ &= (\mathfrak{g}^0)_0 \oplus \bigoplus_{i \in \mathbb{Z}_{\neq 0}} (t^i \otimes (\mathfrak{g}^0)_0) \oplus \bigoplus_{\substack{\alpha \in {\Phi^0}^Y \\ i \in \mathbb{Z}}} (t^i \otimes (\mathfrak{g}^0)_\alpha) \oplus \mathbb{C}c \oplus \mathbb{C}d \end{aligned}$$

and $\mathfrak{g}_0 = (\mathfrak{g}^0)_0 \oplus \mathbb{C}c \oplus \mathbb{C}d$. Let $h \in \mathfrak{h}^Y = (\mathfrak{h}^0)^Y \oplus \mathbb{C}c \oplus \mathbb{C}d$ and $h = h_0 + \lambda c + \mu d$ for some $\lambda, \mu \in \mathbb{C}$. Let $x \in (\mathfrak{g}^0)_0$. Then $[h, t^i \otimes x] = [h_0 + \lambda c + \mu d, t^i \otimes x] = \mu i (t^i \otimes x) = i\delta^Y(h)(t^i \otimes x)$. Thus $(t^i \otimes (\mathfrak{g}^0)_0)$ has $\mathfrak{h}^Y$-weight $i\delta^Y$. Similarly let $x \in (\mathfrak{g}^0)_\alpha$ where $\alpha \in (\Phi^0)^Y$. Then $[h, t^i \otimes x] = [h_0 + \lambda c + \mu d, t^i \otimes x] = (\alpha(h_0) + i\delta^Y(h)) t^i \otimes x = (\alpha + i\delta^Y)(h)(t^i \otimes x)$ since $\alpha(h) = \alpha(h_0)$ and $\delta^Y(h) = \mu$. Thus $(t^i \otimes (\mathfrak{g}^0)_\alpha)$ has $\mathfrak{h}^Y$-weight $\alpha + i\delta^Y$. This gives the $\mathfrak{h}^Y$-weight decomposition and $\Phi^Y = \{i\delta^Y \mid i \in \mathbb{Z}_{\neq 0}\} \cup \{\alpha + i\delta^Y \mid \alpha \in {\Phi^0}^Y, i \in \mathbb{Z}\}$. $\square$

## 4.2 Properties of $\Phi^Y$

Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra and let $Y \subseteq \{1, \ldots, n\}$

$$\mathfrak{g} = \mathfrak{g}_Y \oplus \bigoplus_{\alpha \in \Phi^Y} \mathfrak{g}_\alpha$$

be the corresponding decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}^Y$. We know $\mathfrak{g}$ has an invariant bilinear form $(\cdot,\cdot)$ which is nondegenerate on restriction to $\mathfrak{h}$, so determines an isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ under which $\lambda \mapsto h_\lambda$, where $\lambda(x) = (h_\lambda, x)$ for all $x \in \mathfrak{h}$. Therefore we can transfer the bilinear form to $\mathfrak{h}^*$ by defining $(\lambda, \mu) = (h_\lambda, h_\mu)$. Since $\mathfrak{g}_Y$ is reductive, we have $\mathfrak{g}_Y = \mathfrak{z}(\mathfrak{g}_Y) \oplus \mathfrak{r} = \mathfrak{h}^Y \oplus \mathfrak{r}$ where $\mathfrak{r} = [\mathfrak{g}_Y, \mathfrak{g}_Y]$ is the unique semisimple ideal in $\mathfrak{g}_Y$. Let $\mathfrak{h}_Y = \mathfrak{h} \cap \mathfrak{r}$ then

$$\mathfrak{r} = \mathfrak{h}_Y \oplus \bigoplus_{\alpha \in \Phi_Y} \mathfrak{g}_\alpha$$

and $\mathfrak{h}_Y$ is a Cartan subalgebra of $\mathfrak{r}$. The bilinear form $(\cdot,\cdot)$ is nondegenerate on $\mathfrak{g}_Y$ so is on $\mathfrak{h}^Y = \mathfrak{z}(\mathfrak{g}_Y)$ and $\mathfrak{r}$. The bilinear form $(\cdot,\cdot)$ is nondegenerate on $\mathfrak{h}_Y$ and we have $\mathfrak{h} = \mathfrak{h}^Y \oplus \mathfrak{h}_Y$. Identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ through the bilinear form $(\cdot,\cdot)$, we can identify $(\mathfrak{h}^Y)^* \cong \mathrm{Ann}_{\mathfrak{h}^*}(\mathfrak{h}_Y) \subseteq \mathfrak{h}^*$ and $(\mathfrak{h}_Y)^* \cong \mathrm{Ann}_{\mathfrak{h}^*}(\mathfrak{h}^Y) \subseteq \mathfrak{h}^*$ and then $\mathfrak{h}^* = (\mathfrak{h}^Y)^* \oplus {\mathfrak{h}_Y}^*$.

Recall that in Section 3.4 we defined an element $\gamma \in \mathfrak{h}^*$ such that $\gamma(h_0) = 1$, $\gamma(h_i) = 0$ for $i = 1, \ldots, \ell$ and $\gamma(d) = 0$. Then $\alpha_0, \ldots, \alpha_\ell, \gamma$ is a basis of $\mathfrak{h}^*$ where $\ell = \operatorname{rank} A$. Now let $\mathfrak{h}^*_{\mathbb{R}}$ be the real from of $\mathfrak{h}^*$ spanned over $\mathbb{R}$ by $\alpha_0, \ldots, \alpha_\ell$ and $\gamma$. Let $(\mathfrak{h}^*_Y)_{\mathbb{R}}$ be the real from of $\mathfrak{h}^*_Y$ spanned over $\mathbb{R}$ by $\Phi_Y$. If $(\mathfrak{h}^{Y*})_{\mathbb{R}}$ is the $(\cdot,\cdot)|_{\mathfrak{h}^*_{\mathbb{R}}}$-orthogonal complement of $(\mathfrak{h}^*_Y)_{\mathbb{R}}$ in $\mathfrak{h}^*_{\mathbb{R}}$, then $(\mathfrak{h}^{Y*})_{\mathbb{R}}$ is the real form of $\mathfrak{h}^{Y*}$ and $\mathfrak{h}^*_{\mathbb{R}} = (\mathfrak{h}^{Y*})_{\mathbb{R}} \oplus ({\mathfrak{h}_Y}^*)_{\mathbb{R}}$. We can show that $(\cdot,\cdot)$ is real and nondegenerate on $(\mathfrak{h}^Y)_{\mathbb{R}}$.

Recall that for $\alpha \in \Phi^Y$ we have $\Phi(\mathfrak{g}_\alpha) = \{\beta \in \Phi \mid \beta|_{\mathfrak{h}^Y} = \alpha\}$. For $\alpha = i\delta^Y$ where $i \in \mathbb{Z}$ then $\Phi(\mathfrak{g}_{i\delta^Y}) = \{\beta \in \Phi \mid \beta|_{\mathfrak{h}^Y} = i\delta^Y\}$. Because for any $\beta \in \Phi^0(\mathfrak{r})$ where $\mathfrak{r} = [\mathfrak{g}_Y, \mathfrak{g}_Y]$ we have $\beta|_{\mathfrak{h}^Y} = 0$ therefore we have $\Phi(\mathfrak{g}_\alpha) = \{i\delta\} \cup \{\beta + i\delta \mid \beta \in \Phi^0(\mathfrak{r})\}$. Now if $\alpha = \gamma + i\delta^Y$ for some $i \in \mathbb{Z}$ and $\gamma \in (\Phi^0)^Y$, then $\Phi(\mathfrak{g}_\alpha) = \{\beta + i\delta \mid \beta \in \Phi^0(\mathfrak{g}_\gamma)\}$.

**Definition 4.2.1.** Let $\alpha \in \Phi^Y$. We call $\alpha$ a real Levi root in $\Phi^Y$ if $\Phi(\mathfrak{g}_\alpha)$ consists only of real roots in $\Phi$ and we call $\alpha$ an imaginary Levi root in $\Phi^Y$ if $\Phi(\mathfrak{g}_\alpha)$ consists of some of imaginary roots in $\Phi$.

From now on we denote the set of real roots in $\Phi^Y$ by $(\Phi^Y)_{\mathrm{Re}}$ and the set of imaginary roots in $\Phi^Y$ by $(\Phi^Y)_{\text{Im}}$ and from the argument above we can show that

$$\Phi^Y = (\Phi^Y)_{\text{Re}} \cup (\Phi^Y)_{\text{Im}}.$$

where $(\Phi^Y)_{\text{Re}} = \{\alpha + i\delta^Y \mid \alpha \in (\Phi^0)^Y, i \in \mathbb{Z}\}$ and $(\Phi^Y)_{\text{Im}} = \{i\delta^Y \mid i \in \mathbb{Z}_{\neq 0}\}$.

For $\alpha \in \Phi^Y$, is $\mathfrak{g}_\alpha$ an irreducible $\mathfrak{g}_0 = \mathfrak{g}_Y$ module? To answer this question we consider the cases where $\alpha$ is either a real or imaginary root.

First consider the case where $\alpha \in (\Phi^Y)_{\text{Re}}$ and $\alpha = \gamma + i\delta^Y$ for some $i \in \mathbb{Z}$ and $\gamma \in (\Phi^0)^Y$. Then $\Phi(\mathfrak{g}_\alpha) = \{\beta + i\delta \mid \beta \in \Phi^0(\mathfrak{g}_\gamma)\}$ and $\Phi(\mathfrak{g}_\alpha)$ consists of real roots in $\Phi$ and similar to the technique used to prove Theorem 2.1.6 we can show that $\mathfrak{g}_\alpha$ is an irreducible $\mathfrak{g}_0 = \mathfrak{g}_Y$-module.

**Proposition 4.2.2.** *For $\alpha \in (\Phi^Y)_{\text{Re}}$ we have $\mathfrak{g}_\alpha$ an irreducible $\mathfrak{g}_0 = \mathfrak{g}_Y$-module.*

Now consider the case where $\alpha \in (\Phi^Y)_{\text{Im}}$ and $\alpha = i\delta^Y$ for some $i \in \mathbb{Z}$. Then $\Phi(\mathfrak{g}_\alpha) = \{i\delta\} \cup \{\beta + i\delta \mid \beta \in \Phi^0(\mathfrak{r})\}$. If $Y = \varnothing$ then $\mathfrak{g}_\alpha$ is not an irreducible $\mathfrak{g}_0 = \mathfrak{h}$-module because $\mathfrak{g}_0 = \mathfrak{h}$ is abelian but $\dim \mathfrak{g}_\alpha = \dim \mathfrak{h}^0$ is greater than 1 unless $\dim \mathfrak{h}^0 = 1$, i.e. $\Phi$ is of type $\tilde{A}_1$.

Let $\alpha, \beta \in \Phi^Y$. If $\alpha + \beta \in \Phi^Y$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. But the equality does not happen in general. The following proposition is one of the cases where this equality occurs.

**Proposition 4.2.3.** *Assume $\alpha, \beta \in \Phi^Y$ and $\alpha + \beta \neq 0$. Assume also that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ and $\alpha + \beta \in (\Phi^Y)_{\text{Re}}$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

*Proof.* As $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is a nonzero $\mathfrak{g}_0$-submodule of the $\mathfrak{g}_{\alpha+\beta}$ and $\mathfrak{g}_{\alpha+\beta}$ is irreducible $\mathfrak{g}_0$-module, therefore the equality follows. □

## 4.3 Centralizer of a nilpotent element in an affine Lie algebra $\mathfrak{g}$

Let $\mathfrak{g}$ be an affine untwisted Kac–Moody Lie algebra and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be the weight space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\mathfrak{g}$ be identified like before as $\mathfrak{g} = \hat{\mathcal{G}}(\mathfrak{g}^0)$ where $\mathfrak{g}^0$ is the finite dimensional Lie algebra corresponding to $\mathfrak{g}$. In the next proposition we show how the centralizer of an element $e \in \mathfrak{g}^0$ is related to its centralizer in $\mathfrak{g} = \hat{\mathcal{G}}(\mathfrak{g}^0)$.

**Proposition 4.3.1.** *Let $\mathfrak{g} = \hat{\mathcal{G}}(\mathfrak{g}^0)$ be an affine Kac–Moody Lie algebra with the corresponding finite dimensional Lie algebra $\mathfrak{g}^0$. Let $e \in \mathfrak{g}^0$. Then*

$$\mathfrak{g}^e = \mathbb{C}c \oplus \mathbb{C}d \oplus ((\mathfrak{g}^0)^e \otimes \mathbb{C}[t, t^{-1}]).$$

*Proof.* Clearly the centralizer of $e$ in $\mathfrak{g} = (\mathfrak{g}^0 \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ will contain $c$ and $d$. Every element $\mathfrak{g}^0 \otimes \mathbb{C}[t, t^{-1}]$ is of a form $\sum_{i=-\infty}^{\infty}(y_i \otimes t^i)$ where $y_i \in \mathfrak{g}^0$. Then $[\sum_{i=-\infty}^{\infty}(y_i \otimes t^i), e] = [\sum_{i=-\infty}^{\infty}(y_i \otimes t^i), e \otimes 1] = \sum_{i=-\infty}^{\infty}([y_i, e] \otimes t^i) = 0$ if and only if $[y_i, e] = 0$ for all $i$. Thus $\sum y_i \otimes t^i \in (\mathfrak{g}^0)^e \otimes \mathbb{C}[t, t^{-1}]$. Hence we have $\mathfrak{g}^e = \mathbb{C}c \oplus \mathbb{C}d \oplus ((\mathfrak{g}^0)^e \otimes \mathbb{C}[t, t^{-1}])$. □

To relate the Levi type root systems to centralizer, let $e = \sum_{i\in Y} e_i \in \mathfrak{g}_Y$ be a regular nilpotent element in the Levi subalgebra $\mathfrak{g}_Y \subseteq \mathfrak{g}^0 \oplus \mathbb{C}c \oplus \mathbb{C}d$. We have $\mathfrak{h}^Y = \mathfrak{z}(\mathfrak{g}_Y) = \mathfrak{h} \cap \mathfrak{g}^e = \mathfrak{h}^e$. Since $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi^Y} \mathfrak{g}_\alpha$ and $\mathfrak{g}^e$ is $\operatorname{ad}\mathfrak{h}^e = \mathfrak{h}^Y$-stable we have $\mathfrak{g}^e = \bigoplus(\mathfrak{g}_\alpha \cap \mathfrak{g}^e)$ where $\alpha \in \Phi^Y \cup \{0\}$. Let $\Phi^e = \{\alpha \in \Phi^Y \mid \mathfrak{g}_\alpha \cap \mathfrak{g}^e \neq 0\}$. In the next proposition we prove that $\Phi^e = \Phi^Y$.

**Proposition 4.3.2.** *Let $Y \subseteq \{1, \dots, l\}$ and $e = \sum_{i\in Y} e_i \in \mathfrak{g}_Y$ be a regular nilpotent element in the Levi subalgebra $\mathfrak{g}_Y$. Then $\Phi^e = \Phi^Y$.*

*Proof.* By the Jacobson-Morozov theorem, we can embed $e$ into an $\mathfrak{sl}_2$ subalgebra $\mathfrak{s} = \langle e, h, f\rangle \subseteq \mathfrak{g}_Y$ . For each $\alpha \in \Phi^Y \cup \{0\}$, we can show that $\mathfrak{g}_\alpha$ is $\operatorname{ad}\mathfrak{s}$-stable. Indeed, let

$x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{s}$, $z \in \mathfrak{h}^Y$. Then $[z,[y,x]] = [[z,y],x] + [y,[z,x]] = 0 + \alpha(z)[y,x] = \alpha(z)[y,x]$. Hence $[y,x] \in \mathfrak{g}_\alpha$ and $\mathfrak{g}_\alpha$ is $\operatorname{ad}\mathfrak{s}$-stable. By the representation theory of $\mathfrak{sl}_2$ there exists a nonzero $x \in \mathfrak{g}_\alpha$ such that $[e,x] = 0$. Hence $(\mathfrak{g}_\alpha)^e = \mathfrak{g}^e \cap \mathfrak{g}_\alpha \neq 0$ for all $\alpha \in \Phi^Y$. Therefore, $\Phi^e = \Phi^Y$. □

This means that Levi type root system corresponding to a Levi subalgebra $\mathfrak{g}_Y$ where $Y \subseteq \{1,\dots,l\}$ is the same as the root system for decomposition of $\mathfrak{g}^e$ with respect to $\mathfrak{h}^Y = \mathfrak{h}^e = \mathfrak{z}(\mathfrak{g}_Y)$ where $e$ is the regular nilpotent element $e = \sum_{i\in Y} e_i \in \mathfrak{g}_Y^0$.

More generally we could take $e$ to be a distinguished nilpotent element and we can do the same theory.

# Chapter 5

# Normalizers of parabolic subgroups of affine Weyl groups

This chapter starts with some preliminaries on Coxeter groups. First we briefly study two of the most important types of Coxeter groups, finite (real) reflection groups and affine Weyl groups including the classification of associated Coxeter graphs. Motivated by these examples we continue the general study of Coxeter groups. To describe the normalizers of parabolic subgroups of affine Weyl groups up to isomorphism as the main part of this chapter we aim to present a permutation representation for classical finite Weyl groups and affine Weyl groups from [7]. Note that the structure of normalizers for finite Weyl groups are already known, see [11]. All the materials on Coxeter groups here are found in [13].

## 5.1 Finite reflection groups

Let $V$ be a real Euclidean space with a positive definite symmetric bilinear form $(\cdot,\cdot)$. Recall that a **reflection** is a linear transformation $s$ on $V$ sending some nonzero vector $\alpha \in V$ to its negative while fixing pointwise the hyperplane $H_\alpha$ orthogonal to $\alpha$. We write $s = s_\alpha$. There is a formula $s_\alpha \lambda = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$ for $\lambda \in V$. A finite group generated by reflections is called a **finite reflection group** which is a subgroup of $O(V)$ the group of all orthogonal transformations of $V$. Here are some basic examples:

$I_2(m)$ where $m \geq 3$: Let $V$ be the Euclidean plane and let $\mathcal{D}_m$ be the **dihedral group** of order $2m$. The group $\mathcal{D}_m$ contains $m$ rotations through multiples of $2\pi/m$ and $m$ reflections about the diagonals of the polygon. The group $\mathcal{D}_m$ is generated by reflections because a rotation through $2\pi/m$ is the same as a product of two reflections relative to a pair of adjacent diagonals which meet at an angle $\theta = \pi/m$.

$A_{n-1}$ for $n \geq 2$: Let $V = \mathbb{R}^n$ with the standard basis vectors $\epsilon_1, \ldots, \epsilon_n$. Let $\mathrm{S}_n = \mathrm{Sym}([1, n])$ be the symmetric group acting on $\mathbb{R}^n$ by permuting the subscripts. The transposition $(i, j)$ acts as a reflection which sends $\epsilon_i - \epsilon_j$ to its negative and fixes pointwise the orthogonal complement which consists of all vectors having the same $i$th and $j$th component. Since $\mathrm{S}_n$ is generated by transpositions, it is a reflection group. When $\mathrm{S}_n$ acts on $\mathbb{R}^n$ in a way we described, it fixes the line spanned by $\epsilon_1 + \ldots + \epsilon_n$ and leaves stable the orthogonal complement, the hyperplane consisting of vectors whose coordinates add up to 0. Thus $\mathrm{S}_n$ also acts on an $n-1$ dimensional Euclidean space as a group generated by reflections, fixing no point except the origin. This accounts for the subscript $n-1$ in the label $A_{n-1}$.

$B_n$ for $n \geq 2$: Again let $V = \mathbb{R}^n$ so $\mathrm{S}_n$ acts on $V$ as above. Other reflections can be defined by sending an $\epsilon_i$ to its negative and fixing all other $\epsilon_j$. These sign changes generate a group of order $2^n$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ which intersects $\mathrm{S}_n$ trivially and is normalized by $\mathrm{S}_n$. Thus the semidirect product of $\mathrm{S}_n$ and the group of sign changes gives a reflection group $W$ of order $2^n n!$

$D_n$ for $n \geq 4$: There is another reflection group acting on $\mathbb{R}^n$, a subgroup of index 2 in the group of type $B_n$. The group $\mathrm{S}_n$ normalizes the subgroup consisting of sign changes which involve an even number of signs which is generated by the reflections $\epsilon_i + \epsilon_j \mapsto -(\epsilon_i + \epsilon_j)$ where $i \neq j$. So the semidirect product is also a reflection group

Next we give some standard definitions and facts about finite reflection groups.

- A finite set $\Phi$ of vectors in $V$ is called a **root system** if for all $\alpha \in \Phi$ we have $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ and $s_\alpha \Phi = \Phi$. The elements of $\Phi$ are called **roots**.

- The group $W$ generated by all $s_\alpha$ where $\alpha \in \Phi$ is called a **reflection group associated to the root system** $\Phi$. To classify all possible reflection groups using root system we need a linearly independent subset of $\Phi$ from which $\Phi$ can be re-established.

- A subset $\Pi$ of $\Phi$ is called a **simple system** if $\Pi$ is a vector space basis for the $\mathbb{R}$-span of $\Phi$ in $V$ and if each $\alpha \in \Phi$ is a linear combination of $\Pi$ with coefficients of the same sign. The elements of $\Pi$ are called the **simple roots**.

- Simple systems exist, see [6, Theorem 1.3].

- Elements of $\Phi$ which are positive linear combination of elements of $\Pi$ denoted by $\Phi^+$ are called **positive roots**. The elements of $\Phi$ which are negative linear combination of $\Pi$ denoted by $\Phi^-$ are called **negative roots**. Clearly $\Phi^- = -\Phi^+$.

- If $\Pi$ is a simple system in $\Phi$, then $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ in $\Pi$.

- Any two simple systems in $\Phi$ are conjugate under $W$, see [6, Theorem 1.4].

- The permutation action of $W$ on simple systems is simply transitive.

- Given $\Pi$, for every $\beta \in \Phi$ there exists $w \in W$ such that $w\beta \in \Pi$.

- The group $W$ is generated by the reflections $s_\alpha$ where $\alpha \in \Pi$ see [6, Theorem 1.9].

The following theorem describes a presentation of $W$ where $m_{\alpha\beta}$ denotes the order of $s_\alpha s_\beta$ in $W$ for any roots $\alpha, \beta$.

**Theorem 5.1.1.** *Let $\Pi$ be a simple system in $\Phi$. Then $W$ is generated by the set $S = \{s_\alpha \mid \alpha \in \Pi\}$, subject only to relations $(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1$ for $\alpha, \beta \in \Pi$.*

Any group $W$ finite or infinite having a presentation

$$W = \langle s_i \mid (s_i s_j)^{m_{ij}} \text{ for } i = 1, \ldots, n \rangle$$

relative to a generating set $S = \{s_1, \ldots, s_n\}$ is called a **Coxeter group**; the pair $(W, S)$ is called a **Coxeter system**. It is known that the finite Coxeter groups are precisely the finite reflection groups. Here are Coxeter presentations of some finite Weyl groups.

(a) The Weyl group of type $A_{n-1}$ denoted by $W(A_{n-1})$:

$$W(A_{n-1}) = \langle s_0, \ldots, s_{n-1} \mid (s_i s_j)^{m_{ij}} = 1, \quad \text{for } i, j = 0, \ldots, n-1 \rangle$$

such that $m_{ii} = 1$, $m_{i\,i+1} = 3$ for $i = 0, \ldots, n-2$ and $m_{ij} = 2$ otherwise for $i, j = 0, \ldots, n-1$.

(b) The Weyl group of type $C_n$ denoted by $W(C_n)$:

$$W(C_n) = \langle s_0, \ldots, s_{n-1} \mid (s_i s_j)^{m_{ij}} = 1, \quad \text{for } i, j = 0, \ldots, n-1 \rangle$$

such that $m_{01} = 4$, $m_{ii} = 1$, $m_{i\,i+1} = 3$ for $i = 1, \ldots, n-1$ and $m_{ij} = 2$ otherwise for $i, j = 0, \ldots, n-1$.

(c) The Weyl group of type $D_n$ denoted by $W(D_n)$:

$$W(D_n) = \langle s_0, \ldots, s_{n-1} \mid (s_i s_j)^{m_{ij}} = 1, \quad \text{for } i, j = 0, \ldots, n-1 \rangle$$

such that $m_{ii} = 1$, $m_{02} = 3$, $m_{i\,i+1} = 3$ for $i = 1, \ldots, n-1$ and $m_{ij} = 2$ otherwise for $i, j = 0, \ldots, n-1$.

The notion of 'root system' we introduced in this chapter differs from that commonly used in Lie theory see Definition 1.5.1. To avoid confusion, a root system is called **crystallographic** if it satisfies the condition $\frac{2(\alpha,\beta)}{\beta,\beta} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. For crystallographic the group generated by all reflections $s_\alpha$ where $\alpha \in \Phi$ is known as the **Weyl group** of $\Phi$. Setting $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, the set $\Phi^\vee$ of all **coroots** $\alpha^\vee$ ($\alpha \in \Phi$) is also a crystallographic root system in $V$, with simple system $\Pi^\vee = \{\alpha^\vee \mid \alpha \in \Pi\}$. It is called the **inverse** or **dual** root systems. The Weyl group of $\Phi^\vee$ is $W$, with $w\alpha^\vee = w(\alpha)^\vee$. Let $\Pi$ be a fixed simple system in $\Phi$ and $S$ be the set of simple reflections $s_\alpha$ where $\alpha \in \Pi$. For any subset $I \subset S$ we define the parabolic subgroup $W_I$ to be the subgroup of $W$ generated by all $s_\alpha \in I$ and let $\Pi_I = \{\alpha \in \Pi \mid s_\alpha \in I\}$.

Now let $\Phi^+$ be the set of positive roots containing the simple system $\Pi$. For each hyperplane $H_\alpha$ we can associate the open half-spaces $A_\alpha$ and $A'_\alpha$, where $A_\alpha = \{\lambda \in V \mid (\lambda, \alpha) > 0\}$ and $A'_\alpha = -A_\alpha$. The **chamber** $\mathfrak{C}(\Pi) = \bigcap_{\alpha\in\Pi} A_\alpha$ is the intersection of open convex sets $A_\alpha$ which is itself open and convex. Let $D$ be the closure $\overline{\mathfrak{C}}$. Thus $D = \{\lambda \in V \mid (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Pi\}$. It is known that $D$ is a fundamental domain for the action of $W$ on $V$ i.e. each $\lambda \in V$ is conjugate under $W$ to one and only one point in $D$ see [6, Lemma 1.12]. Because of the simply transitive action of $W$ on simple systems, this translates into a simply transitive action of $W$ on the family $\mathfrak{C}(\Pi)$ of chambers where $\Pi$ runs over all possible simple systems. The chambers are topologically characterized as the connected components of the complement in $V$ of $\bigcup_\alpha H_\alpha$. The goal in the next section is to determine all possible finite reflection groups in terms of their Coxeter graphs. The groups satisfying crystallographic condition are especially important in Lie theory, where they arise as Weyl groups.

## 5.2 Classification of finite reflection groups

The Theorem 5.1.1 shows that $W$ is determined up to isomorphism by the set of integers $m(\alpha, \beta)$ for $\alpha, \beta \in \Pi$. To encode this information in a picture we construct a graph $\Gamma$ having elements of $\Pi$ as vertices. We join a pair of vertices corresponding to $\alpha \neq \beta$ by an edge whenever $m(\alpha, \beta) \geq 3$, and label such an edge with $m(\alpha, \beta)$. This labeled graph is called **Coxeter graph** of $W$. It determines $W$ up to isomorphism. The classification of finite reflection groups relies on the study of possible Coxeter graphs. The following proposition gives more precise criterion for reflection groups to be isomorphic in the geometric setting see [13, Proposition 2.1]:

**Proposition 5.2.1.** *For $i = 1, 2$, let $W_i$ be a finite reflection group acting on the Euclidean spaces $V_i$. Assume $W_i$ are essential meaning that the action of $W_i$ on $V_i$ has no nonzero fixed points. If $W_1$ and $W_2$ have the same Coxeter graphs, then there is an isometry of $V_1$ onto $V_2$ inducing an isomorphism of $W_1$ onto $W_2$.*

The Coxeter system $(W, S)$ is said to be **irreducible** if the Coxeter graph $\Gamma$ is connected and we also call $\Phi$ irreducible in this case. The following proposition says that the study of finite reflection groups can be reduced to the case when Coxeter graphs $\Gamma$ is connected see [13, Proposition 2.2].

**Proposition 5.2.2.** *Let $(W, S)$ have Coxeter graph $\Gamma$, with connected components $\Gamma_1, \ldots, \Gamma_r$ and let $S_1, \ldots, S_r$ be the corresponding subsets of $S$. Then $W$ is the direct product of the parabolic subgroups $W_{S_1}, \ldots, W_{S_r}$ and each $(W_{S_i}, S_i)$ for $i = 1, \ldots, r$ is irreducible.*

The next theorem limits the possibilities for crystallographic finite reflection groups, see [13, Theorem 2.7].

**Theorem 5.2.3.** *The graphs in Figure 5.1 are the only connected Coxeter graphs for crystallographic Weyl groups.*

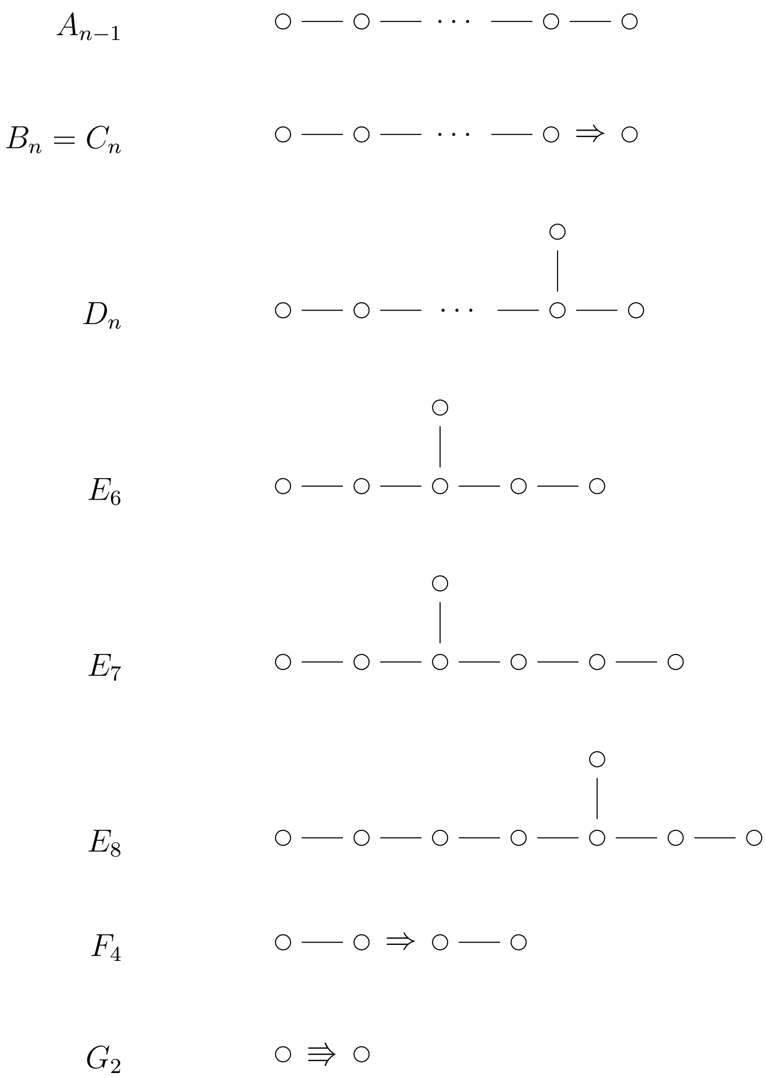


Figure 5.1: Coxeter graphs for crystallographic Weyl groups

In the next section we describe a class of infinite groups generated by affine reflections in Euclidean space which are related to Weyl groups and have a representation like that of finite reflection groups.

## 5.3 Affine reflection groups

In this section $V$ denotes a Euclidean space and $W$ is a Weyl group i.e a finite crystallographic reflection group. We want to consider not just orthogonal reflections but also **affine reflections** relative to hyperplanes that do not necessarily pass through the origin. To do this we need to introduce some standard definitions.

- The semidirect product of $\mathrm{GL}(V)$ and the group of translations by elements of $V$ is called **affine group** and it is denoted by $\mathrm{Aff}(V)$.
- For each root $\alpha \in \Phi$ and each integer $k$, we define an affine hyperplane

  $$H_{\alpha,k} = \{\lambda \in V \mid (\lambda, \alpha) = k\}.$$

  Then we define the corresponding affine reflection

  $$s_{\alpha,k}(\lambda) = \lambda - ((\lambda, \alpha) - k)\alpha^\vee$$

  where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.
- Let $\mathcal{H}$ be the collection of all hyperplanes $H_{\alpha,k}$ where $\alpha \in \Phi$ and $k \in \mathbb{Z}$ then the elements of $\mathcal{H}$ are permuted by $W$.
- The subgroup of $\mathrm{Aff}(V)$ generated by all affine reflections $s_{\alpha,k}$ where $\alpha \in \Phi$ and $k \in \mathbb{Z}$ is called the **affine Weyl group** $W_a$ of $\Phi$.

The next proposition makes the structure of $W_a$ more transparent. We define the **root lattice** $L(\Phi)$ the $\mathbb{Z}$-span of $\Phi$ and the **weight lattice** $\hat{L}(\Phi) = \{\lambda \in V \mid (\lambda, \alpha^\vee) \in$

$\mathbb{Z}$ for all $\alpha \in \Phi\}$. We also obtain lattices associated with the root system $\Phi^\vee$. Let $L = L(\Phi^\vee)$ and $\hat{L} = \hat{L}(\Phi^\vee)$, the latter is characterized by

$$\hat{L} = \{\lambda \in V \mid (\lambda, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

**Proposition 5.3.1.** *The group $W_a$ is the semidirect product of $W$ and the translation group corresponding to the coroot lattice $L = L(\Phi^\vee)$.*

To study how the group $W_a$ permutes the hyperplanes $\mathcal{H}$, we see how it permutes the collection $\mathcal{A}$ of connected components of $V^\circ = V \setminus \bigcup_{H \in \mathcal{H}} H$. Each element of $\mathcal{A}$ is called an **alcove**.

Let $\Pi$ be a set of simple roots in $\Phi$. We single out one particular alcove

$$A_\circ = \{\lambda \in V \mid 0 < (\lambda, \alpha) < 1 \text{ for all } \alpha \in \Phi^+\}.$$

In general an alcove is defined by a set of inequalities of the form $k_\alpha < (\lambda, \alpha) < k_\alpha + 1$ for $\alpha \in \Phi^+$. Since $\Phi$ is an irreducible root system, there is a unique highest root $\tilde{\alpha}$ having the property that for all positive roots $\alpha$, the element $\tilde{\alpha} - \alpha$ is a sum of simple roots. Hence we have

$$A_\circ = \{\lambda \in V \mid 0 < (\lambda, \alpha) \text{ for all } \alpha \in \Pi, \text{ and } (\lambda, \tilde{\alpha}) < 1\}.$$

The **walls** of $A_\circ$ are defined to be the hyperplanes $H_\alpha$ for $\alpha \in \Pi$ and $H_{\tilde{\alpha},1}$ and we define $S_a = \{s_\alpha \mid \alpha \in \Pi\} \cup \{s_{\tilde{\alpha},1}\}$. The next proposition shows that we can define the walls of $wA_\circ$ to be the images of these hyperplanes under $w$ for any $w \in W_a$, see [13, Proposition 4.3]. .

**Proposition 5.3.2.** *The group $W_a$ permutes the collection $\mathcal{A}$ of all alcoves transitively, and is generated by the set $S_a$ of reflections with respect to the walls of the alcove $A_\circ$*

The following theorem is one of important theorems in this chapter see [13, Theorem 4.6].

**Theorem 5.3.3.** *The pair* $(W_a, S_a)$ *is a Coxeter system.*

To construct a Coxeter graph belonging to Coxeter group $W_a$ we need to work out the order $s_\alpha s_{\tilde{\alpha},1}$ for each $\alpha \in \Pi$ in order to see what new edges and labels occur when new vertex is joined to the Coxeter graph of $W$. The resulting Coxeter graphs are those of type $\tilde{A}_{n-1}$, $\tilde{B}_n$ for $n \geq 3$, $\tilde{C}_n$ for $n \geq 2$, $\tilde{D}_n$ for $n \geq 4$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{F}_4$, and $\tilde{G}_2$ occurring in the Figure 3.1 and 3.2.

## 5.4 Permutation representation of finite and affine Weyl groups

In this section we first aim to present a permutation representation of finite Weyl groups $A_{n-1}$, $C_n$, $D_n$ and affine Weyl groups of type $\tilde{A}_{n-1}$, $\tilde{B}_n$, $\tilde{C}_n$, and $\tilde{D}_n$.

### Permutation representation of Weyl groups of type $A_{n-1}$ and $\tilde{A}_{n-1}$

A $\mathbb{Z}$-permutation is a bijective map from $\mathbb{Z}$ to itself. The Weyl group $W = W(A_{n-1})$ of type $A_{n-1}$ is the symmetric group $S_n = \mathrm{Sym}([1, n])$ and generated by transpositions $s_i = (i\, i{+}1)$ where $i = 1, ..., n-1$. This group can be seen as the group of $\mathbb{Z}$-permutations that fixes everything outside the interval $[1, n]$ and we denote it by $W = W_{A_{n-1}}$.

Let $T_n$ be a map from $\mathbb{Z}$ to itself defined by $T_n(x) = x + n$ for $x \in \mathbb{Z}$. The map $T_n$ is a translation $n$ steps to the right. A $\mathbb{Z}$-permutation $w$ commutes with $T_n$ if and only if $w(i + kn) = w(i) + kn$ for all $i, k \in \mathbb{Z}$.

Let $\widehat{W}_{\tilde{A}_{n-1}} = \mathrm{Sym}(\mathbb{Z})^{T_n}$ be the group of $\mathbb{Z}$-permutations that commute with $T_n$.

**Definition 5.4.1.** A $\mathbb{Z}$-permutation is called *locally finite* if only a finite number of values are moved from negative half-axis to the nonnegative half-axis and the same number of values are moved in the other direction.

**Lemma 5.4.2.** *A $\mathbb{Z}$-permutation $w$ commuting with $T_n$ is locally finite if and only if the following sum condition holds*

$$\sum_{i=1}^{n} w(i) = \sum_{i=1}^{n} i.$$

*Proof.* This is [7, Proposition 9].

∎

Let $W_{\tilde{A}_{n-1}}$ be the subgroup of $\widehat{W}_{\tilde{A}_{n-1}}$ of all locally finite $\mathbb{Z}$-permutations. Therefore,

$$W_{\tilde{A}_{n-1}} = \left\{ w \in \mathrm{Sym}(\mathbb{Z}) \;\middle|\; w(i+kn) = w(i) + kn \text{ for all } i,k \in \mathbb{Z} \text{ and } \sum_{i=1}^{n} w(i) = \sum_{i=1}^{n} i \right\}.$$

Since $w \in W_{\tilde{A}_{n-1}}$ is determined by $w(1), \ldots, w(n)$, it can be denoted by

$$w = \begin{pmatrix} 1 & 2 & \ldots & n \\ w(1) & w(2) & \ldots & w(n) \end{pmatrix}$$

such that $\{w(i) (\operatorname{mod} n) \mid i = 1, \ldots, n\} = [1, n]$. The group $W_{\tilde{A}_{n-1}}$ is generated by $S_0 = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 0 & 2 & \ldots & n-1 & n+1 \end{pmatrix}$ and $S_i = \begin{pmatrix} 1 & 2 & \ldots & i & i+1 & \ldots & n \\ 1 & 2 & \ldots & i+1 & i & \ldots & n \end{pmatrix}$ for $i = 1, \ldots, n-1$ and isomorphic to the Weyl group $W(\tilde{A}_{n-1})$ by sending $s_i$ to $S_i$ for $i = 1, \ldots, n$ where

$$W(\tilde{A}_{n-1}) = \langle s_0, \ldots, s_{n-1} \mid (s_i s_j)^{m_{i\,j}} = 1, \quad \text{for } i, j = 0, \ldots, n-1 \rangle.$$

such that $m_{0\,n-1} = 3$, $m_{i\,i} = 1$, $m_{i\,i+1} = 3$ for $i = 0, \ldots, n-2$ and $m_{i\,j} = 2$ otherwise for $i, j = 0, \ldots, n-1$. See [7, Theorem 14, 19 and 20].

**Proposition 5.4.3.** $W_{\tilde{A}_{n-1}} \trianglelefteq \widehat{W}_{\tilde{A}_{n-1}}$, $\widehat{W}_{\tilde{A}_{n-1}} / W_{\tilde{A}_{n-1}} \cong \mathbb{Z}$ *and* $\widehat{W}_{\tilde{A}_{n-1}} = \mathbb{Z} \ltimes W_{\tilde{A}_{n-1}}$.

*Proof.* Define $\Psi : \widehat{W}_{\tilde{A}_{n-1}} \to \mathbb{Z}$ by

$$\Psi(w) = \left( \sum_{i=1}^{n} w(i) - \sum_{i=1}^{n} i \right) / n$$

for $w \in \widehat{W}_{\tilde{A}_{n-1}}$. Since $\{w(i) \pmod n \mid i = 1, \ldots, n\} = [1, n]$ we have $\sum_{i=1}^{n} w(i) - \sum_{i=1}^{n} i = kn$ for some $k \in \mathbb{Z}$ thus $\Psi(w) \in \mathbb{Z}$. To show $\Psi$ is a group homomorphism we need to prove $\Psi(ww') = \Psi w + \Psi w'$ for all $w, w' \in \widehat{W}_{\tilde{A}_{n-1}}$ or $\sum_{i=1}^{n} ww'(i) - \sum_{i=1}^{n} i = \sum_{i=1}^{n} w(i) - \sum_{i=1}^{n} i + \sum_{i=1}^{n} w'(i) - \sum_{i=1}^{n} i$. For $w, w' \in \widehat{W}_{\tilde{A}_{n-1}}$ and $i = 1, \ldots, n$, let $w(i) = \sigma(i) + k_i n$ and $w'(i) = \tau(i) + l_i n$ where $\sigma, \tau \in \mathrm{Sym}([1, n])$ and $k_i, l_i \in \mathbb{Z}$. We have $\Psi(w) = \left(\sum_{i=1}^{n} w(i) - \sum_{i=1}^{n} i\right)/n = \left(\sum_{i=1}^{n} \sigma(i) + \sum_{i=1}^{n} k_i n - \sum_{i=1}^{n} i\right)/n = \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} k_i n - \sum_{i=1}^{n} i\right)/n = \sum_{i=1}^{n} k_i$. Similarly $\Psi(w') = \left(\sum_{i=1}^{n} w'(i) - \sum_{i=1}^{n} i\right)/n = \sum_{i=1}^{n} l_i$. We also have $ww'(i) = w(\tau(i) + l_i n) = w(\tau(i)) + l_i n = \sigma(\tau(i)) + k_{\tau(i)} n + l_i n$ so $\Psi(ww')(i) = \left(\sum_{i=1}^{n} ww'(i) - \sum_{i=1}^{n} i\right)/n = \left(\sum_{i=1}^{n} \sigma(\tau(i)) + \sum_{i=1}^{n} k_{\tau(i)} n + \sum_{i=1}^{n} l_i n - \sum_{i=1}^{n} i\right)/n$ $= \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} k_i n + \sum_{i=1}^{n} l_i n - \sum_{i=1}^{n} i\right)/n = \left(\sum_{i=1}^{n} k_i n + \sum_{i=1}^{n} l_i n\right)/n = \left(\sum_{i=1}^{n} k_i\right) + \left(\sum_{i=1}^{n} l_i\right) = \Psi(w) + \Psi(w')$. By Lemma 5.4.2 we have $W_{\tilde{A}_{n-1}} = \ker \Phi$. So $W_{\tilde{A}_{n-1}} \trianglelefteq \widehat{W}_{\tilde{A}_{n-1}}$.

If $x = \begin{pmatrix} 1 & 2 & \ldots & n \\ n+1 & 2 & \ldots & n \end{pmatrix}$, then $\Psi(x) = 1$ and $\Psi|_{\langle x \rangle}$ is surjective. We have $\langle x \rangle \cap \ker \Psi = \{1\}$ and $\langle x \rangle \cong \mathbb{Z}$. Thus $\widehat{W}_{\tilde{A}_{n-1}} = \langle x \rangle \ltimes W_{\tilde{A}_{n-1}}$ so $\widehat{W}_{\tilde{A}_{n-1}} \cong \mathbb{Z} \ltimes W_{\tilde{A}_{n-1}}$. □

## Permutation representation of Weyl groups of type $C_n$ and $\tilde{C}_n$

Let $W_{C_n}$ be the group of permutations in $\mathrm{Sym}([-n, n])$ commuting with $R_0$, the reflection with respect to 0 defined by $R_0(i) = -i$. Then

$$W_{C_n} = \{w \in \mathrm{Sym}([-n, n]) \mid w(-i) = -w(i) \text{ for all } i \in [0, n]\}.$$

The group $W_{C_n}$ is generated by $S_0 = (-1\,1)$, and $S_i = (i\,i+1)(-i\ -i-1)$ for all $i = 1, ...n-1$ which gives an isomorphism with the Weyl group $W(C_n)$ of type $C_n$

$$W(C_n) = \langle s_0, \ldots, s_{n-1} \mid (s_i s_j)^{m_{ij}} = 1, \quad \text{for } i, j = 0, \ldots, n-1 \rangle.$$

such that $m_{0\,1} = 4$, $m_{i\,i} = 1$, $m_{i\,i+1} = 3$ for $i = 1, \ldots, n-1$ and $m_{i\,j} = 2$ otherwise for $i, j = 0, \ldots, n-1$, See [7, Theorem 14, 19 and 20].

Now let $W_{\tilde{C}_n}$ be the group of permutations of $\mathbb{Z}$ commuting with all transformations in the group $\Gamma_n = \langle R_0, R_{n+1} \rangle$ where $R_0$ is the reflection with respect to 0 and $R_{n+1}$ is the reflection with respect to $n+1$ defined by $R_{n+1}(i) = 2(n+1) - i$. Then $R_0 R_{n+1}$ is translation by $k(2n+2)$. So it has infinite order. The group $\Gamma_n$ is the infinite Dihedral group with elements which are reflections in $k(n+1)$ and translations by $k(2n+2)$ for $k \in \mathbb{Z}$. Then

$$W_{\tilde{C}_n} = \{w \in \mathrm{Sym}(\mathbb{Z}) \mid w(-i) = -w(i) \text{ and } w(i) = 2n+2 - w(2n+2-i) \text{ for all } i\}.$$

There are $n$ infinite $\Gamma_n$-orbits in $\mathbb{Z}$: $\langle i \rangle = \{\pm i + k(2n+2) : k \in \mathbb{Z}\}$ for all $i \in \{1, ..., n\}$. For $w \in W_{\tilde{C}_n}$, we have $w(0) = w(-0) = -w(0) = 0$ and $w(n+1) = wR_{n+1}(n+1) = R_{n+1}w(n+1) = 2n+2-w(n+1)$ implying $w(n+1) = n+1$ and in general $w(k(n+1)) = k(n+1)$ for all $k \in \mathbb{Z}$. Also for any $j \in \mathbb{Z} \setminus \{k(n+1) \mid k \in \mathbb{Z}\}$ there exists $i \in [-n, n] \setminus \{0\}$ such that $j = \gamma(i)$ for some $\gamma \in \Gamma_n$. So for $w \in W_{\tilde{C}_n}$ we have $w(j) = w(\gamma(i)) = \gamma(w(i))$ and $w \in W_{\tilde{C}_n}$ can be represented by

$$w = \begin{pmatrix} 1 & 2 & \ldots & n \\ w(1) & w(2) & \ldots & w(n) \end{pmatrix}$$

The group $W_{\tilde{C}_n}$ is generated by $S_0 = \begin{pmatrix} 1 & 2 & \dots & n \\ -1 & 2 & \dots & n \end{pmatrix}$ and $S_n = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 1 & 2 & \dots & n-1 & n+2 \end{pmatrix}$ and $S_i = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n-1 & n \\ 1 & 2 & \dots & i+1 & i & \dots & n-1 & n \end{pmatrix}$ for $i = 1, ..., n-1$. Because all $\Gamma_n$ orbits of $\langle i \rangle$ in $\mathbb{Z}$ for $i = 1, ..., n$ are distinct, all the values of $w(i)$ for $i = 1, ..., n$ have to be in distinct $\Gamma_n$ orbits. The group $W_{\tilde{C}_n}$ is isomorphic to the Weyl group $W(\tilde{C}_n)$ of type $\tilde{C}_n$ sending $s_i$ to $S_i$ where

$$W(\tilde{C}_n) = \langle s_0, s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \quad i, j = 0, \dots, n \rangle$$

such that $m_{i\,i} = 1, \quad m_{0\,1} = 4, \quad m_{i\,i+1} = 3, \quad m_{n-1\,n} = 4, \quad m_{i\,j} = 2$ otherwise where $i, j = 1, \dots, n$. See [7, Theorem 14, 19 and 20] and Section 5.1.

## Permutation representation of Weyl groups of type $\tilde{B}_n$

Let $W_{\tilde{B}_n}$ be the subgroup of $W_{\tilde{C}_n}$ consisting of all $\mathbb{Z}$-permutations that are locally even at position 0 meaning that an even number of negative numbers are moved to positive numbers. The group $W_{\tilde{B}_n}$ is generated by $S_0 = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ -2 & -1 & \dots & n-1 & n \end{pmatrix}$, $S_n = \begin{pmatrix} 1 & \dots & n-1 & n \\ 1 & \dots & n-1 & n+2 \end{pmatrix}$ and $S_i = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n-1 & n \\ 1 & 2 & \dots & i+1 & i & \dots & n-1 & n \end{pmatrix}$ for $i = 1, ..., n-1$. This group is isomorphic to the Weyl group $W(\tilde{B}_n)$ of type $\tilde{B}_n$ where

$$W(\tilde{B}_n) = \langle s_0, s_1, \dots, s_n \mid (s_i s_j)^{m_{i\,j}} = 1 \quad i, j = 0, \dots, n \rangle$$

such that $m_{0\,1} = 2$, $m_{0\,2} = 3$, $m_{n-1\,n} = 4$, $m_{i\,i+1} = 3$ for $i = 1, \dots, n-1$ and $m_{i\,j} = 2$ otherwise for $i, j = 0, \dots, n$, see [7, Theorem 14, 19 and 20].

## Permutation representation of Weyl groups of type $D_n$ and $\tilde{D}_n$

Let $W_{D_n}$ be the subgroup of $W_{C_n}$ consisting of all permutations that are locally even at position 0. The group $W_{D_n}$ is generated by $S_0 = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ -2 & -1 & \dots & n-1 & n \end{pmatrix}$ and, $S_i = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n-1 & n \\ 1 & 2 & \dots & i+1 & i & \dots & n-1 & n \end{pmatrix}$ for $i = 1, ..., n-1$. The group $W_{D_n}$ is isomorphic to the Weyl group $W(D_n)$ of type $D_n$, where

$$W(D_n) = \langle s_0, s_1 \dots, s_{n-1} \mid (s_i s_j)^{m_{ij}} = 1 \, i, j = 1, \dots, n-1 \rangle$$

such that $m_{ii} = 1$, $m_{02} = 3$, $m_{i\,i+1} = 3$ for $i = 1, \dots, n-1$ and $m_{ij} = 2$ otherwise for $i, j = 0, \dots, n-1$ see [7, Theorem 14, 19 and 20].

Now let $W_{\tilde{D}_n}$ be a subgroup of $W_{\tilde{C}_n}$ consisting of all $\mathbb{Z}$-permutations that are locally even at both positions 0 and $n+1$. Then $W_{\tilde{D}_n}$ is generated by $S_0 = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ -2 & -1 & \dots & n-1 & n \end{pmatrix}$ and $S_n = \begin{pmatrix} 1 & 2 & \dots & n-2 & n-1 & n \\ 1 & 2 & \dots & n-2 & n+2 & n+3 \end{pmatrix}$ and $S_i = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n-1 & n \\ 1 & 2 & \dots & i+1 & i & \dots & n-1 & n \end{pmatrix}$ for $i = 1, ..., n-1$. This group is isomorphic to the Weyl group $W(\tilde{D}_n)$ of type $\tilde{D}_n$ where

$$W(\tilde{D}_n) = \langle s_0, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \quad i, j = 0, \dots, n \rangle$$

such that $m_{ii} = 1$, $m_{02} = 3$, $m_{n-2\,n} = 3$, $m_{i\,i+1} = 3$ for all $i = 1, \dots, n-1$, and $m_{ij} = 2$ otherwise for $i, j = 0, \dots, n$, see [7, Theorem 14, 19 and 20].

## 5.5 Normalizers of parabolic subgroups of affine Weyl groups

In this section we will determine the normalizers of the parabolic subgroups of finite Weyl groups (of types $A$, $B$, $C$, $D$) and affine Weyl groups (of types $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$). We implement the $\mathbb{Z}$-permutations representations of the finite and affine Weyl groups. We are interested in describing these normalizers as they can be viewed as the Weyl groups of root systems of Levi type in the next chapter.

### Normalizers of parabolic subgroups of Weyl groups of type $A_{n-1}$

Let $W = W_{A_{n-1}}$ be the Weyl group of type $A_{n-1}$ and $L = \{L_1, L_2, \ldots, L_r\}$ be a partition of $[1, n]$ such that $L_i = [a_{i-1} + 1, a_i]$ are intervals where $0 = a_0 < \ldots < a_r = n$ for $1 \leq i \leq r$. We define

$$N_{W,L} = \{w \in W \mid wL = L\}$$

$$W_L = \{w \in W \mid wL_i = L_i \text{ for all } i \in \{1, \ldots, r\}\}$$

and

$$W^L = \{w \in N_{W,L} \mid \text{ If } a, b \in L_i \text{ with } a \leq b, \text{ then } w(a) \leq w(b) \text{ for all } i \in [1, r]\}.$$

Note that $W^L$ is a subgroup and the subgroup $W_L$ is a standard parabolic subgroup of $W$ corresponding to the subset $J = \bigcup_{i=1}^{r} [a_{i-1} + 1, a_i - 1] \subseteq [1, n-1]$.

**Proposition 5.5.1.** $N_{W,L} = N_W(W_L)$.

*Proof.* First we show $N_{W,L} \subseteq N_W(W_L)$. Let $w \in N_{W,L}$ and $u \in W_L$. Then for $i = 1, \ldots, r$ we have $w^{-1}L_i \in L$ so $uw^{-1}L_i = w^{-1}L_i$. So $wuw^{-1}L_i = ww^{-1}L_i = L_i$. Therefore, $w \in N_W(W_L)$.

Now let $w \in N_W(W_L)$, we show $w \in N_{W,L}$ or $wL_i \in L$ for all $i = 1, \dots, r$. Since $w \in N_W(W_L)$, then $wW_Lw^{-1} = W_L$. Now $L = \{L_1, L_2, ..., L_r\}$ is the set of $W_L$-orbits on the set $\{1, ..., n\}$. Let $i = 1, \dots, r$ and $L_i = W_L \cdot j = \{w(j) \mid w \in W_L\}$ for some $j \in \{1, ..., n\}$. Then $wL_i = wW_L \cdot j = wW_Lw^{-1} \cdot w(j) = W_L \cdot w(j) = L_k$ for some $k \in \{1, \dots, r\}$. Hence $w \in N_{W,L}$. □

**Proposition 5.5.2.** $N_{W,L} = W^L \ltimes W_L$.

*Proof.* Clearly $W_L \trianglelefteq N_{W,L}$ by Proposition 5.5.1. We show $W^L \cap W_L = 1$. Let $w \in W^L \cap W_L$. Since $w \in W_L$, then $wL_i = L_i$ for $i = 1, \dots, r$. Let $L_i = \{j_1 < j_2 < \dots < j_t\}$ then $L_i = wL_i = \{wj_1 < wj_2 < \dots < wj_t\}$. Therefore $wj_s = j_s$ for all $j_s \in L_i$ where $s = 1, \dots, t$ and $i = 1, \dots, r$. So $w = 1$.

Now we show $N_{W,L} = W^L W_L$. Let $x \in N_{W,L}$ and $L_i = W_L \cdot j$ for some $j \in \{1, \dots, n\}$. Then $xL_i \in L$. Let $L_i = \{j_1 < j_2 < ... < j_t\}$. So $xL_i = \{xj_1, xj_2, ..., xj_t\}$. There exists $w_i \in W_L$ such that $w_ixj_1 < w_ixj_2 < ... < w_ixj_t$ for all $i = 1, \dots, r$. Hence for $w = w_1 \dots w_r \in W_L$ we have $wx \in W^L$ and $x \in W^L W_L$. □

By Propositions 5.5.1 and 5.5.2 we note that to work out the normalizers of parabolic subgroups of $W(A_{n-1})$ we need to work out the subgroup $W^L$. The following example shows how to describe these subgroups.

**Example 5.5.3.** Let $W = W(A_7)$ and $L = \{\{1\}, \{4\}, \{2, 3\}, \{5, 6\}, \{7, 8\}\}$ be a partition of $[1, 8]$. By definition $W^L = \{w \in N_{W,L} \mid$ If $a, b \in M$ with $a \leq b$, then $w(a) \leq w(b)$ for all $M \in L\}$. If $w \in W^L$ then $w$ is determined by $w(1), \dots, w(8)$. The only possibilities for $w(1)$ is 1 or 4. For $w(2)$ we can choose the least element in any of the two element sets. So $w(2)$ can be 2, 5 and 7 and $w(3)$ is determined by $w(2)$. There is just one possibility for $w(4)$. For $w(5)$ similar to $w(2)$ we can choose the least element in any two element sets apart from $\{w(2), w(3)\}$. So there are two choices for $w(5)$ and $w(6)$ is determined by $w(5)$. There is just one choice left for $w(7)$ and $w(8)$ is determined by $w(7)$. Let

$W(1)$ be subgroup of $W^L$ fixing all elements of $L$ apart from those lying in a part of $L$ of size one and let $W(2)$ be a subgroup of $W^L$ fixing all elements of $L$ apart from those lying in a part of $L$ of size two. Then $W^L = W(1) \times W(2) \cong \mathrm{Sym}([1,2]) \times \mathrm{Sym}([1,3]) = W_{A_1} \times W_{A_2}$.

Now we can argue as in the example above to prove the next theorem in general.

**Theorem 5.5.4.** *Let $W = W(A_{n-1})$. Then in general we have $W^L \cong \prod_{i=1}^{n} \mathrm{Sym}([1,n_i]) = \prod_{i=1}^{n} W(A_{n_i-1})$ with order $\prod_{i=1}^{n} n_i!$ where $n_i$ is the number of elements of partition $L$ of size $i$.*

*Proof.* Let $W(i)$ be a subgroup of $W^L$ fixing all elements of $L$ apart from those lying in a part of $L$ of size $i$ for $i = 1, \ldots, n$. Then $W(i) \cong \mathrm{Sym}([1,n_i])$ and we have $W^L \cong \prod_{i=1}^{n} W(i) \cong \prod_{i=1}^{n} \mathrm{Sym}([1,n_i]) = \prod_{i=1}^{n} W(A_{n_i-1})$. Therefore the order of $W^L$ is $\prod_{i=1}^{n} n_i!$.

□

## Normalizers of parabolic subgroups of Weyl groups of type $W_{\tilde{A}_{n-1}}$

Let $W = W_{\tilde{A}_{n-1}}$. Let $L \in \mathrm{Part}(\mathbb{Z})^{T_n}$ be a partition in $\mathbb{Z}$ invariant under $T_n$, and of the form $L = \{\ldots, [a_k+1, a_{k+1}], [a_{k+1}+1, a_{k+2}], \ldots\}$ where $\ldots a_{k-1} < a_k < a_{k+1} < \ldots$. We define $W_L$, $N_{W,L}$, and $W^L$ similar to before. The subgroup $W_L$ is a standard parabolic subgroup of $W_{\tilde{A}_{n-1}}$ corresponding to the subset $J = \bigcup([a_k+1, a_{k+1}-1] \cap [0, n-1])$. All parabolic subgroups can be obtained in this way.

The arguments proving Propositions 5.5.1 and 5.5.2 go through in this case.

In order to see what $W^L$ is isomorphic to we first work it out in the following example:

**Example 5.5.5.** Let $W = W_{\tilde{A}_4}$ and

$$L = \{\ldots, \{-3,-4\}, \{-2,-1\}, \{0\}, \{1,2\}, \{3,4\}, \{5\}, \{6,7\}, \{8,9\}, \{10\}, \ldots\}$$

be a partition of $\mathbb{Z}$ invariant under $T_5$. First we consider $\widehat{W} = \mathrm{Sym}(\mathbb{Z})^{T_5}$, the group of symmetries of $\mathbb{Z}$ invariant under $T_5$. By definition $\widehat{W}^L = \{w \in N_{\widehat{W},L} \mid$ for all $M \in$

$L$ and $a, b \in M$ if $a \leq b$ then $w(a) \leq w(b)\}$. So for $w \in \widehat{W}^L$ the only possibilities for $w(1)$ is the least element in any two element sets and $w(2)$ is determined by $w(1)$. Similarly the only possibilities for $w(3)$ is the least element in any two element sets but different from $w(1)$ modulo 5. Moreover, $w(4)$ is determined by $w(3)$ and $w(5)$ can be any element in a one element set. Let $\widehat{W}(1)$ be the subgroup of $\widehat{W}^L$ which fixes all elements of $L$ apart from those lying in parts of $L$ of size one. Let $\widehat{W}(2)$ be the subgroup of $\widehat{W}^L$ which fixes all elements of $L$ apart from those lying in parts of $L$ of size two. Clearly $\widehat{W}^L = \widehat{W}(1) \times \widehat{W}(2)$. Since every element $w \in W_{\tilde{A}_4}$ is determined by $w(1), \ldots, w(5)$ let $L^+ = \{M \in L \mid M \cap [1,5] \neq \varnothing\} = \{\{1,2\},\{3,4\},\{5\}\}$. The set $L^+$ has $\{5\}$ one element of size one. Now if we consider the group $\widehat{W}(1)$, this is a subgroup of $\mathrm{Sym}(\{\ldots,-5,0,5,\ldots\})$. Since there is a bijection between the set $\{\ldots,-5,0,5,\ldots\}$ and the set $\mathbb{Z}$, by fixing the bijection $f : \{\ldots,-5,0,5,\ldots\} \to \mathbb{Z}$ given by $f(x) = x/5$ then $\gamma_f : \mathrm{Sym}(\{\ldots,-5,0,5,\ldots\}) \to \mathrm{Sym}(\mathbb{Z})$ defined by $\gamma_f w = fwf^{-1}$ for all $w \in \mathrm{Sym}(\{\ldots,-5,0,5,\ldots\})$ is an isomorphism. For all $i \in \{\ldots,-5,0,5,\ldots\}$ we have $T_5(i) = T_1(f(i))$ and $\widehat{W}(1) \cong \mathrm{Sym}(\mathbb{Z})^{T_1} \cong \mathbb{Z}$. The set $L^+$ has $\{1,2\}$ and $\{3,4\}$ two elements of size 2. The group $\widehat{W}(2)$ is a subgroup of $\mathrm{Sym}(\Omega)$ where $\Omega = \{\ldots,\{-4,-3\},\{-2,-1\},\{1,2\},\{3,4\},\ldots\}$. There is a bijection between the set $\Omega$ and the set $\mathbb{Z}$. By fixing the bijection $g : \Omega \to \mathbb{Z}$ such that $g\{1,2\} = 1$ and $g\{a,b\} \leq g\{c,d\}$ if and only if $a \leq c$ then $\gamma_g : \mathrm{Sym}(\Omega) \to \mathbb{Z}$ defined by $\gamma_g(w) = gwg^{-1}$ for all $w \in \mathrm{Sym}(\Omega)$ is an isomorphism. Moreover, elements of $\widehat{W}(2)$ are invariant under $T_5$ and $g(T_5(\{a,b\})) = T_2(g(\{a,b\})$ for all $\{a,b\} \in \Omega$ and therefore we have $\widehat{W}(2) \cong \mathrm{Sym}(\mathbb{Z})^{T_2}$. Hence $\widehat{W}^L \cong \mathrm{Sym}(\mathbb{Z})^{T_1} \times \mathrm{Sym}(\mathbb{Z})^{T_2} \cong \mathbb{Z}^2 \ltimes (W_{\tilde{A}_1} \times W_{\tilde{A}_2})$ and $W^L \cong \mathbb{Z} \ltimes (W_{\tilde{A}_1} \times W_{\tilde{A}_2})$.

Now we can argue as in example to prove the next theorem in general.

**Theorem 5.5.6.** *Let* $W = W_{\tilde{A}_{n-1}}$. *Let* $L^+ = \{M \in L \mid \max M \in [1,n]\}$. *Let* $i_1, i_2, \ldots, i_r \in \mathbb{N}$ *be the sizes of the parts in* $L^+$ *and* $n_{i_j}$ *is the number of parts of* $L^+$ *of size* $i_j$ *for* $j \in [1,r]$. *Let* $\widehat{W} = \mathrm{Sym}(\mathbb{Z})^{T_n}$, *the group of symmetries of* $\mathbb{Z}$ *invariant under*

*$T_n$, then*

$$\widehat{W}^L \cong \prod_{j=1}^{r} \widehat{W}(i_j) \cong \prod_{j=1}^{r} \mathrm{Sym}(\mathbb{Z})^{T_{n_{i_j}}} \cong \prod_{j=1}^{r} \mathbb{Z} \ltimes W_{\tilde{A}_{n_{i_j}}} = \mathbb{Z}^r \ltimes \prod_{j=1}^{r} W_{\tilde{A}_{n_{i_j}}}$$

*where $\widehat{W}(i_j)$ is the subgroup of $\widehat{W}^L$ fixing all parts of $L$ apart from those lying in parts of size $i_j$. Therefore we have*

$$W^L = \widehat{W}^L \cap W \cong \mathbb{Z}^{r-1} \ltimes \prod_{j=1}^{r} W_{\tilde{A}_{n_{i_j}}}.$$

*Proof.* For any $i_j \in \mathbb{N}$ and $j \in [1, r]$ consider the subgroup $\widehat{W}(i_j)$ of $\widehat{W}^L$ fixing all parts of $L$ apart from those lying in parts of size $i_j$. Then the set $\Omega$ of all parts of $L$ of size $i_j$ is bijective to $\mathbb{Z}$ and $\mathrm{Sym}(\Omega)$ is isomorphic to $\mathrm{Sym}(\mathbb{Z})$ induced by this bijection. The subgroup $\widehat{W}(i_j)$ is isomorphic to a subgroup of $\mathrm{Sym}(\mathbb{Z})$. The elements of $\widehat{W}(i_j)$ are invariant under $T_n$ and in fact are invariant under $T_{n_{i_j}}$ through the isomorphism between $\mathrm{Sym}(\Omega)$ and $\mathrm{Sym}(\mathbb{Z})$. Hence $\widehat{W}^L \cong \prod_{j=1}^{r} \widehat{W}(i_j) \cong \prod_{j=1}^{r} \mathrm{Sym}(\mathbb{Z})^{T_{n_{i_j}}} \cong \prod_{j=1}^{r} \mathbb{Z} \ltimes W_{\tilde{A}_{n_{i_j}}} = \mathbb{Z}^r \ltimes \prod_{j=1}^{r} W_{\tilde{A}_{n_{i_j}}}$ by Proposition 5.4.3. Therefore we have $W^L = \widehat{W}^L \cap W \cong \mathbb{Z}^{r-1} \ltimes \prod_{j=1}^{r} W_{\tilde{A}_{n_{i_j}}}$. □

## Normalizers of parabolic subgroups of Weyl groups of type $C_n$

Let $W = W(C_n)$. Let $L = \{-L_r, \ldots, L_0, \ldots, L_r\}$ be a partition of $[-n, n]$ invariant under $R_0$ such that $L_i = [a_{i-1} + 1, a_i]$ and $-L_i = [-a_i, -a_{i-1} - 1]$ are intervals. We denote the interval containing zero by $L_0$. We define $W_L$, $N_{W,L}$, and $W^L$ similar to before. Then the subgroup $W_L$ is a parabolic subgroup of $W(C_n)$ corresponding to the subset $J = \bigcup_{i=0}^{r}([a_{i-1} + 1, a_i - 1] \cap [0, n - 1])$. We can show similarly $N_{W,L} = N_W(W_L)$ and $N_{W,L} = W^L \ltimes W_L$.

**Example 5.5.7.** Let $W = W(C_6)$ and

$$L = \{\{-6\}, \{-5\}, \{-4, -3\}, \{-2, -1\}, \{0\}, \{1, 2\}, \{3, 4\}, \{5\}, \{6\}\}$$

be a partition of $[-6, 6]$. First we work out the order of $N_{W,L}$. There are four choices for $w(6)$ which are $\pm 6, \pm 5$. So two choices are left for $w(5)$. There are eight choices for $w(4)$ which are $\pm 1, \pm 2, \pm 3, \pm 4$ and one choice for $w(3)$. There are four choices left for $w(2)$ and just one choice for $w(1)$. Hence the order of $N_{W,L}$ is 256. The order of $W^L$ is 64. As we have four choices for $w(6)$, two choices for $w(5)$, four choices for $w(4)$, and two choices for $w(2)$. There are just one choice for each $w(3)$ and $w(1)$. The group $W_L$ generated by $S_1$ and $S_3$ is the parabolic subgroup which is isomorphic to $W(A_1) \times W(A_1)$. Moreover, $W^L$ is isomorphic to $W(1) \times W(2)$. The group $W(1)$ can be viewed as the group of permutations $w$ of $\{\{\pm 5\}, \{\pm 6\}\}$ for $i = 5, 6$ such that $w(-i) = -w(i)$ so $W(1) \cong W(C_2)$. Also $W(2)$ can be viewed as permutations $w$ of $\{\pm \mathbf{1}, \pm \mathbf{3}\}$ where $\mathbf{1} = \{1, 2\}$ and $\mathbf{3} = \{3, 4\}$ such that $w(-i) = -w(i)$ for $i = \mathbf{1}, \mathbf{3}$ so $W(2) \cong W(C_2)$. Therefore $W^L$ is isomorphic to $W_{C_2} \times W_{C_2}$.

Now we can argue as in example to prove the next theorem in general.

**Theorem 5.5.8.** *Let $W = W(C_n)$. Let $L^+ = \{L_i \in L \mid L_i \subseteq [1, n]\}$ and $i_1, i_2, \ldots, i_r \in \mathbb{N}$ be the sizes of elements in $L^+$ and $n_{i_j}$ is the number of parts of $L^+$ of size $i_j$ for $j \in [1, r]$, then*

$$W^L \cong \prod_{j=1}^{r} W(i_j) \cong \prod_{j=1}^{r} W(C_{n_{i_j}})$$

*with order $\prod_{j=1}^{r} 2^{n_{i_j}} n_{i_j}!$ where $W(i_j)$ is the subgroup of $W^L$ fixing all parts of $L$ apart from those of size $i_j$.*

*Proof.* For any $i_j \in \mathbb{N}$ and $j \in [1, r]$ consider the subgroup $W(i_j)$ of $W^L$ fixing all parts of $L$ apart from those lying in parts of size $i_j$. So $W(i_j)$ is the group of permutations such that for $w \in W(i_j)$ we have $w(-L_t) = -w(L_t)$ for all $L_t$ of size $i_j$ in $L^+$. Therefore

$W(i_j) \cong W(C_{n_{i_j}})$ and $W^L \cong \prod_{j=1}^r W(i_j) \cong \prod_{j=1}^r W(C_{n_{i_j}})$. Therefore the order of $W^L$ is $\prod_{j=1}^r 2^{n_{i_j}} n_{i_j}!$. □

## Normalizers of parabolic subgroups of Weyl groups of type $\tilde{C}_n$

Let $W = W_{\tilde{C}_n}$ be the group of permutations $w$ of $\mathbb{Z}$ commuting with all transformations in $\Gamma_n = \langle R_0, R_{n+1} \rangle$. Let $L = \{\ldots [a_i + 1, a_{i+1}], [a_{i+1}, a_{i+2}], \ldots\}$ be in $\mathrm{Part}(\mathbb{Z})^{\Gamma_n}$, the set of all partitions of $\mathbb{Z}$ invariant under $\Gamma_n = \langle R_0, R_{n+1} \rangle$, where $\ldots a_{i-1} \leq a_i \leq a_{i+1} \ldots$. So if $[a_i+1, a_{i+1}] \in L$, then $[-a_{i+1}, -a_i - 1]$ and $[n+1+2(n+1-a_{i+1}), n+1+2(n+1-a_i-1)]$ are in $L$. We define $W_L$, $N_{W,L}$, and $W^L$ similar to before. Then the group $W_L$ is a parabolic subgroup of $W_{\tilde{C}_n}$ corresponding to $J = \bigcup([a_i + 1, a_{i+1} - 1] \cap [0, n])$. Similarly we can show $N_{W,L} = N_W(W_L)$ and $N_{W,L} = W^L \ltimes W_L$.

**Example 5.5.9.** Let $W = W_{\tilde{C}_8}$ and $L = \{\ldots, \{-12, -11\}, \{-10\}, \{-9\}, \{-8\}, \{-7, -6\}$ $, \{-5, -4\}, \{-3, -2\}, \{-1, 0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8\}, \{9\}, \{10\}, \{11, 12\}, \ldots\}$ be a partition of $\mathbb{Z}$ invariant under the group $\Gamma_8 = \langle R_0, R_9 \rangle$. Let $W(1)$ be a subgroup of $W^L$ fixing all parts of $L$ apart from those of size one, $W(2)$ be a subgroup of $W^L$ fixing all parts of $L$ apart from those of size two, and $W(3)$ be a subgroup of $W^L$ fixing all parts of $L$ apart from those of size three. Since the element $\{-1, 0, 1\}$ of $L$ is fixed by all element of $W^L$, then any part of $L$ of size three is also fixed by all elements of $W^L$ and $W(3) = 1$. Hence $W^L = W(1) \times W(2)$. The group $W(2)$ is isomorphic to a subgroup of $\mathrm{Sym}(\Omega)$ where $\Omega = \{\ldots, \{-12, -11\}, \{-7, -6\}, \{-5, -4\}, \{-3, -2\}, \{2, 3\}, \{4, 5\}, \{6, 7\},$
$\{11, 12\}, \ldots\}$. Let $\Theta : \Omega \to \mathbb{Z} \setminus 4\mathbb{Z}$ be a bijection such that $\Theta\{2, 3\} = 1$ and preserves the order. Then we can view $W(2)$ as a subset of $\mathrm{Sym}(\mathbb{Z})$ under the embedding $\Theta$ such that for all $w \in W(2)$ we have $w(4k) = 4k$ for all $k \in \mathbb{Z}$. The set $\Omega$ is invariant under $\Gamma_8 = \langle R_0, R_9 \rangle$ and $\Gamma_8(\{a, b\}) = \Gamma_3(\Theta(\{a, b\}))$ for all $\{a, b\} \in \Omega$. Thus $W(2) \cong \mathrm{Sym}(\mathbb{Z})^{\Gamma_3} \cong W_{\tilde{C}_3}$. Similar to the last example $W(1) \cong W_{\tilde{C}_1}$. Hence $W^L \cong W_{\tilde{C}_3} \times W_{\tilde{C}_1}$.

Now we can argue as in example to prove the next theorem in general.

**Theorem 5.5.10.** *Let $W = W_{\tilde{C}_n}$. Let $L$ be a $\mathbb{Z}$-partition invariant under the group $\Gamma_n = \langle R_0, R_{n+1}\rangle$. Let $L^+ = \{M \in L \mid M \subseteq [1,n]\}$. Let $i_1, i_2, \ldots, i_r \in \mathbb{N}$ be the sizes of elements in $L^+$ and $n_{i_j}$ is the number of parts of $L^+$ of size $i_j$ for $j \in [1,r]$, then*

$$W^L \cong \prod_{j=1}^{r} W(i_j) \cong \prod_{j=1}^{r} \mathrm{Sym}(\mathbb{Z})^{\Gamma_{n_{i_j}}} \cong \prod_{j=1}^{r} W_{\tilde{C}_{n_{i_j}}}$$

*where $W(i_j)$ is the subgroup of $W^L$ fixing all parts of $L$ apart from those of size $i_j$.*

*Proof.* For any $i_j \in \mathbb{N}$ and $j \in [1,r]$ consider the subgroup $W(i_j)$ of $W^L$ fixing all parts of $L$ apart from those lying in parts of size $i_j$. Let $\Omega$ be the set of all parts of $L$ of size $i_j$. Then there is a bijection between $\Omega$ and $\mathbb{Z}$. This induces an isomorphism from $\mathrm{Sym}(\Omega)$ to $\mathrm{Sym}(\mathbb{Z})$. Then $W(i_j)$ is isomorphic to a subgroup of $\mathrm{Sym}(\Omega)$ denoted by $\Theta$. The set $\Omega$ is invariant under $\Gamma_n = \langle R_0, R_{n+1}\rangle$ and $\Gamma_n(L_t) = \Gamma_{n_{i_j}}(\Theta(L_t))$ for all $L_t \in \Omega$. Then $W(i_j) \cong \mathrm{Sym}(\mathbb{Z})^{\Gamma_{n_{i_j}}} \cong W_{\tilde{C}_{n_{i_j}}}$. Therefore $W^L \cong \prod_{j=1}^{r} W(i_j) \cong \prod_{j=1}^{r} \mathrm{Sym}(\mathbb{Z})^{\Gamma_{n_{i_j}}} \cong \prod_{j=1}^{r} W_{\tilde{C}_{n_{i_j}}}$. □

## Normalizers of parabolic subgroups of Weyl groups of type $D_n$

The Weyl group of type $D_n$ consists of all $\mathbb{Z}$-permutations in $C_n$ that are locally even at position zero. Let $W = W_{D_n}$ and $\widehat{W} = W_{C_n}$. Let $L = \{-L_r, \ldots, L_0, \ldots, L_r\}$ be a partition of $[-n,n]$ such that $L_i = [a_i + 1, a_{i+1}]$ and $-L_i = [-a_{i+1}, -a_i - 1]$. These partitions of $[-n,n]$ will give all parabolic subgroups of $W_{D_n}$ except ones containing $S_0$ as a generator but not $S_1$ as a generator. Therefore we can get all parabolic subgroups corresponding to the subsets of $[0, n-1]$ up to graph automorphism. We define $W_L$, $N_{W,L}$, and $W^L$ similar to before. Then the group $W_L$ is a parabolic subgroup corresponding to the subset $J = \bigcup_{i=0}^{r}([a_i+1, a_{i+1}-1] \cap [0, n-1])$. Similar to the arguments in Propositions 5.5.1, and 5.5.2 we can show that $N_{W,L} = N_W(W_L)$ and $N_{W,L} = W^L \ltimes W_L$.

**Theorem 5.5.11.** *Let $W = W_{D_n}$. Let $L^+ = \{L_i \in L \mid L_i \subseteq [1,n]\}$ and $i_1, i_2, \ldots, i_r \in \mathbb{N}$*

*be the sizes of elements of* $L_+$ *and* $n_{i_j}$ *is the number of parts of* $L^+$ *of size* $i_j$ *for* $j \in [1, r]$. *Then*

$$W^L \cong (\prod_{i_j \, even} W(C_{n_{i_j}})) \times U$$

*where if* $U = (\prod_{i_j \, odd} W(C_{n_{i_j}})) \cap W \neq 1$ *then* $U$ *is a subgroup of index two in* $(\prod_{i_j \, odd} W(C_{n_{i_j}}))$.

*Proof.* To work out the isomorphism class of $W^L$, let $\widehat{W} = W_{C_n}$. We know $W^L = \widehat{W}^L \cap W$. Recall that $\widehat{W}^L \cong \prod_{j=1}^{r} \widehat{W}(i_j) \cong \prod_{j=1}^{r} W(C_{n_{i_j}})$ where $i_1, i_2, \ldots, i_r \in \mathbb{N}$ are the sizes of elements in $L^+ = \{L_i \in L \mid L_i \subseteq [1, n]\}$ and $n_{i_j}$ is the number of parts of $L_+$ of size $i_j$ for $j \in [1, r]$. Then $W^L = (\prod_{j=1}^{r} W(C_{n_{i_j}})) \cap W$. Any part of $L$ is either a positive interval or a negative interval. For the case $i_j \in \mathbb{N}$ is even the element $w \in W(C_{n_{i_j}})$ is permuting parts of $L$ of size $i_j$ so always has even number of negative numbers moved to positive numbers. Then $\widehat{W}(i_j) \subseteq W$ and $\widehat{W}(i_j) \cong W(C_{n_{i_j}})$. Hence $W^L = (\prod_{i_j \text{even}} W(C_{n_{i_j}})) \times ((\prod_{i_j \text{odd}} \widehat{W}(i_j)) \cap W) = (\prod_{i_j \text{even}} W(C_{n_{i_j}})) \times ((\prod_{i_j \text{odd}} W(C_{n_{i_j}})) \cap W)$. □

**Example 5.5.12.** Let $W = W(D_8)$ and $L = \{\{-8\}, \{-7\}, \{-6, -5\}, \{-4, -3\}, \{-2, -1, 0, 1, 2\}, \{3, 4\}, \{5, 6\}, \{7\}, \{8\}\}$ be a partition of $[-8, 8]$. Let $\widehat{W} = W(C_n)$. Then $\widehat{W}^L = \widehat{W}(1) \times \widehat{W}(2) \cong W(C_2) \times W(C_2)$ and $W^L \cong W(C_2) \times W(D_2)$.

## Normalizers of parabolic subgroups of Weyl groups of type $\tilde{B}_n$

The Weyl group of type $W_{\tilde{B}_n}$ is a subgroup of $W_{\tilde{C}_n}$ consisting of all $\mathbb{Z}$-permutations that are locally even at position zero. Let $L = \{\ldots [a_i+1, a_{i+1}], [a_{i+1}, a_{i+2}], \ldots\}$ be in $\text{Part}(\mathbb{Z})^{\Gamma_n}$, the set of all partitions of $\mathbb{Z}$ invariant under $\Gamma_n = \langle R_0, R_{n+1} \rangle$, where $a_{i-1} < a_i < a_{i+1}$ for all $i \in \mathbb{Z}$. These partitions of $\mathbb{Z}$ will give all parabolic subgroups of $W_{\tilde{B}_n}$ up to graph automorphism corresponding to the subsets of $[0, n]$. Let $W = W_{\tilde{B}_n}$. We define $W_L$, $N_{W,L}$, and $W^L$ similar to before. Then $W_L$ is a parabolic subgroup of $W_{\tilde{B}_n}$ corresponding to $J = \bigcup([a_i + 1, a_{i+1} - 1] \cap [0, n]) \subseteq [0, n]$. Similarly we can show $N_{W,L} = N_W(W_L)$ and $N_{W,L} = W^L \ltimes W_L$.

**Theorem 5.5.13.** *Let $W = W_{\tilde{B}_n}$. Let $L^+ = \{M \in L \mid M \subseteq [1,n]\}$ and $i_1, i_2, \ldots, i_r \in \mathbb{N}$ be the sizes of elements of $L^+$ and $n_{i_j}$ is the number of parts of $L_+$ of size $i_j$ for $j \in [1,r]$. Then*

$$W^L \cong (\prod_{i_j\, even} W_{\tilde{C}_{n_{i_j}}}) \times U$$

*where if $U = \prod_{i_j\, odd} W_{\tilde{C}_{n_{i_j}}} \cap W \neq 1$ then $U$ is a subgroup of index two in $\prod_{i_j\, odd} W_{\tilde{C}_{n_{i_j}}}$ such that the total number of negatives moved to positives is even.*

*Proof.* To work out what subgroup $W^L$ is isomorphic to, let $\widehat{W} = W_{\tilde{C}_n}$. Then $W^L = \widehat{W}^L \cap W$. We can describe $W^L$ in terms of $\widehat{W}^L$. Note that $\widehat{W}^L \cong \prod_{j=1}^r \widehat{W}(i_j) \cong \prod_{j=1}^r W_{\tilde{C}_{n_{i_j}}}$ where $i_1, i_2, \ldots, i_r \in \mathbb{N}$ are the sizes of elements in $L^+ = \{M \in L \mid M \subseteq [1,n]\}$ and $n_{i_j}$ is the number of parts of $L^+$ of size $i_j$ for $j \in [1,r]$. Any part of $L$ apart from the part containing zero is either a positive interval or a negative interval. For the case where $i_j \in \mathbb{N}$ is even then $\widehat{W}(i_j) \cong W_{\tilde{C}_{n_{i_j}}} \subseteq W$. Because any element $w \in \widehat{W}(i_j)$ is permuting parts of $L$ of size $i_j$ so always having even number of negative numbers moved to positive numbers. Hence $W^L$ is a subgroup of $\prod_{i_j \text{even}} W_{\tilde{C}_{n_{i_j}}} \times \prod_{i_j \text{odd}} W_{\tilde{C}_{n_{i_j}}}$ in the form of $\prod_{i_j \text{even}} W_{\tilde{C}_{n_{i_j}}} \times U$ where $U \neq 1$ is a subgroup of $\prod_{i_j \text{even}} W_{\tilde{C}_{n_{i_j}}}$ of index two such that the total number of negatives to positives is even. □

## Normalizers of parabolic subgroups of Weyl groups of type $\tilde{D}_n$

The Weyl group $W = W_{\tilde{D}_n}$ is a subgroup of $W_{\tilde{B}_n}$ consisting of all $\mathbb{Z}$-permutations that are locally even at position $n+1$. Let $L \in \text{Part}(\mathbb{Z})^{\Gamma_n}$ be a partition $L = \{\ldots [a_i + 1, a_{i+1}], [a_{i+1}, a_{i+2}], \ldots\}$ which is invariant under $\Gamma_n = \langle R_0, R_{n+1} \rangle$, where $\ldots a_{i-1} \leq a_i \leq a_{i+1} \ldots$. We define $W_L$, $N_{W,L}$, and $W^L$ similar to before. These partitions give all parabolic subgroups of $W_{\tilde{D}_n}$ up to graph automorphism. Moreover, $W_L$ is a parabolic subgroup of $W_{\tilde{D}_n}$ corresponding to a subset $J = \bigcup([a_{j-1}+1, a_j - 1] \cap [0, n+1]) \subseteq \Pi$, where $\Pi$ is the set of simple roots in $W$ of type $W(\tilde{D}_n)$.

**Theorem 5.5.14.** *Let $W = W_{\tilde{D}_n}$. Let $L^+ = \{M \in L \mid M \subseteq [1, n]\}$ and $i_1, i_2, \ldots, i_r \in \mathbb{N}$ are the sizes of elements in $L^+$. Let $n_{i_j}$ is the number of parts of $L^+$ of size $i_j$ for $j \in [1, r]$. Then*

$$W^L \cong (\prod_{i_j\, even} W_{\tilde{C}_{n_{i_j}}}) \times U$$

*where if $U \neq 1$ then it is a subgroup of $\prod_{i_j\, odd} W_{\tilde{C}_{n_{i_j}}}$ of index four such that the total number crossing $0$ and $n + 1$ is even.*

*Proof.* The argument is similar to the proof in Theorem 5.5.13. □

**Example 5.5.15.** Let $W = W_{\tilde{D}_{10}}$ and $L = \{\ldots, \{-17, -16\}, \{-15\}, \{-14\}, \{-13, -12, -11, -10, -9\}, \{-8\}, \{-7\}, \{-6, -5\}, \{-4, -3\}, \{-2, -2, 0, 1, 2\}, \{3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10, 11, 12, 13\}, \{14\}, \{15\}, \{16, 17\}, \ldots\}$ be a $\mathbb{Z}$ partition invariant under $\Gamma_{10} = \langle R_0, R_{11}\rangle$. Let $\widehat{W} = W_{\tilde{B}_{10}}$ and $\widehat{W}(1)$ be a subgroup of $\widehat{W}^L$ fixing all parts of $L$ apart from those of size one and let $\widehat{W}(2)$ be a subgroup of $\widehat{W}^L$ fixing of all parts of $L$ apart from those of size two and $\hat{W}(5)$ be a subgroup of $\hat{W}^L$ fixing all parts of $L$ apart from those of size five. Since the parts $\{-2, -2, 0, 1, 2\}$ and $\{9, 10, 11, 12, 13\}$ are fixed by all elements of $W^L$ therefore all parts of $L$ of size five are fixed. So $\widehat{W}(5) = 1$ and $\widehat{W}^L = \widehat{W}(1) \times \widehat{W}(2) \times \widehat{W}(5) = \widehat{W}(1) \times \widehat{W}(2) \cong W_{\tilde{C}_2} \times W_{\tilde{B}_2}$. Therefore $W^L = W_{\tilde{C}_2} \times W_{\tilde{D}_2}$.

# Chapter 6

# Weyl groups of Levi type

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac-Moody Lie algebra where $A$ is a GCM of affine type. As the Kac-Moody group $G(A)$, see [19] and [23] is more difficult to work with, we will restrict ourselves to $G = G(A^0)(\mathbb{C}[t, t^{-1}])$ which is a subquotient of $G(A)$. For classical type $\mathfrak{g}$ in Chapter 4 we defined the root system of Levi type denoted by $\Phi^Y$ corresponding to a Levi subalgebra $\mathfrak{g}_Y$ where $Y \subseteq \{1, \dots, n\}$. In Proposition 4.3.2 we showed that $\Phi^Y$ can be viewed as root system of $\mathfrak{g}^e$ where $e = e_Y$ is a regular nilpotent element in the Levi subalgebra $\mathfrak{g}_Y$ and we have $\mathfrak{g}^e = (\mathfrak{g}_0)^e \oplus \bigoplus_{\alpha \in \Phi^Y} (\mathfrak{g}_\alpha)^e$. In this chapter we restrict ourselves to the case when $\Phi$ is of type $\tilde{A}$. We will relate the root system of Levi type $\Phi^Y$ with the normalizer of the parabolic subgroup $N_W(W_L) = W^L \ltimes W_L$ where $L$ is the partition of $\mathbb{Z}$ corresponding to $Y$ and $N_W(W_L)$, $W^L$ and $W_L$ are defined in Section 5.5. We will prove $W^L$ known as the **Levi type Weyl group** for $\Phi^Y$ is isomorphic to $N_G(\mathfrak{g}_Y)/G_Y$.

From now on we just consider the case when $G = \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$ and we expect similar results to be true in other types. We also expect that some of the material in this section is known by experts.

## 6.1 Normalizer of $\mathfrak{h}$ in $G$

Let $G = \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}[t, t^{-1}])$ and $\mathfrak{h}$ be the subalgebra of diagonal matrices in $\mathfrak{gl}_n(\mathbb{C})$. We will determine $N_G(\mathfrak{h})$ and explain its structure.

Let $U(\mathbb{C}([t, t^{-1}]) = \{u \in \mathbb{C}[t, t^{-1}] \mid uv = 1 \text{ for some } v \in \mathbb{C}[t, t^{-1}]\}$ be the group of units of $\mathbb{C}[t, t^{-1}]$. Then $U(\mathbb{C}([t, t^{-1}]) = \{at^i \mid a \in \mathbb{C}^*, i \in \mathbb{Z}\}$. To show this, let $p(t) = a_m t^m + a_{m+1} t^{m+1} + \ldots + a_n t^n$ and $q(t) = b_r t^r + b_{r+1} t^{r+1} + \ldots + b_s t^s$ be in $U(\mathbb{C}([t, t^{-1}])$ where $a_m, a_n, b_r, b_s \neq 0$. Suppose that $p(t)q(t) = 1$. Then $p(t)q(t) = a_m b_r t^{m+r} +$ higher powers of $t$. Since $a_m, b_r \neq 0$, we have $m + r = 0$ and $a_m b_r = 1$. Also $p(t)q(t) = a_n b_s t^{n+s} +$ lower powers of $t$. Since $a_n, b_s \neq 0$, we have $n + s = 0$ and $a_n b_s = 1$. But $m = -r$ and $n = -s$ implies $m = n$ and $r = s$ and $p(t) = a_m t^m$ and $q(t) = {a_m}^{-1} t^{-m}$.

**Lemma 6.1.1.** *$N_G(\mathfrak{h})$ is the group of monomial matrices in $G$ with entries in $U(\mathbb{C}([t, t^{-1}])$ where monomial matrices are square matrices with exactly one nonzero entry in each row and column.*

*Proof.* Consider $V = \mathbb{C}[t, t^{-1}]^n$, as a module for $\mathfrak{h}$ and take

$$\mathfrak{B} = \left\{ \begin{pmatrix} t^{i_1} & 0 & \ldots & 0 \end{pmatrix}^{\mathrm{tr}}, \begin{pmatrix} 0 & t^{i_2} & \ldots & 0 \end{pmatrix}^{\mathrm{tr}}, \ldots, \begin{pmatrix} 0 & 0 & \ldots & t^{i_n} \end{pmatrix}^{\mathrm{tr}} \mid i_j \in \mathbb{Z}, 0 \leq j \leq n \right\}$$

as a $\mathbb{C}$-basis for $V$. Then we have $V = \bigoplus_{i=1}^n V_{\epsilon_i}$, where $\epsilon_i : \mathfrak{h} \to \mathbb{C}$ is a map defined by $\epsilon_i(\mathrm{diag}(a_1, \ldots, a_n)) = a_i$ and $V_{\epsilon_i} = \{v \in V \mid xv = \epsilon_i(x)v \text{ for all } x \in \mathfrak{h}\}$. Then $V_{\epsilon_j} = \langle t^i e_j \mid i \in \mathbb{Z}, \rangle$ where $e_j$ with nonzero $j$th position and $0 \leq j \leq n$. Let $g \in N_G(\mathfrak{h})$ and $x \in \mathfrak{h}$. Then $(gxg^{-1}) \cdot v = \epsilon_i(gxg^{-1})v$ for all $1 \leq i \leq n$ and $v \in V_{\epsilon_i}$. So $x(g^{-1}v) = \epsilon_i(gxg^{-1})(g^{-1}v)$. This is true for all $x \in \mathfrak{h}$. So $g^{-1}v$ is a simultaneous eigenvector for all $x \in \mathfrak{h}$. Hence $g^{-1}v \in V_{\epsilon_j}$ for some $j$. Therefore if we view $e_1, \ldots, e_n$ as a basis of $\mathbb{C}[t, t^{-1}]^n$ as a free $\mathbb{C}[t, t^{-1}]$-module then any element $N_G(\mathfrak{h})$ sends $e_i$ to $p_i(t)e_{\sigma(i)}$ where

$p_i(t) \in \mathbb{C}[t, t^{-1}]$ and $\sigma(i) \in [1, n]$. Any element $g \in N_G(\mathfrak{h})$ is determined by $g(e_i)$ for all $i = 1, \ldots, n$. So given $g \in N_G(\mathfrak{h})$ the $i$th column of $g$ is $\begin{pmatrix} 0 & \ldots & 0 & p_i(t) & 0 & \ldots & 0 \end{pmatrix}^{\mathrm{tr}}$ with nonzero $\sigma(i)$th position. Recall that $\mathrm{GL}_n(\mathbb{C}[t, t^{-1}]) = \{x \in \mathrm{Mat}_n(\mathbb{C}[t, t^{-1}]) \mid xy = 1 = yx \text{ for some } y \in \mathrm{Mat}_n(\mathbb{C}[t, t^{-1}])\} = \{x \in \mathrm{Mat}_n(\mathbb{C}[t, t^{-1}]) \mid \det(x) \in U(\mathbb{C}[t, t^{-1}])\}$. As $\pm \prod_{i=1}^{n} p_i(t) = \det(g) \in U(\mathbb{C}[t, t^{-1}])$ for any $g \in N_G(\mathfrak{h})$. So each $p_i(t) \in U(\mathbb{C}[t, t^{-1}])$ and $p_i(t) = a_i t^{m_i}$ for some $a_i \in \mathbb{C}$ and $m_i \in \mathbb{Z}$. So $g \in G$ is a monomial matrix with entries in $U(\mathbb{C}[t, t^{-1}])$. $\square$

Now we give notation, $g \in N_G(\mathfrak{h})$ is a matrix of the form $g = P(\sigma, \underline{a}, \underline{m})$ where $\sigma \in \mathrm{Sym}([1, n])$, $\underline{a} = (a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$, $\underline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and the $(i, j)$th entry of $g \in N_G(\mathfrak{h})$ is $\delta_{i,\sigma(j)} a_j t^{m_j}$. Then we have

$$N_G(\mathfrak{h}) = \{P(\sigma, \underline{a}, \underline{m}) \mid \sigma \in \mathrm{Sym}([1, n]), \underline{a} \in (\mathbb{C}^*)^n, \underline{m} \in \mathbb{Z}^n\}.$$

and $p(\sigma, \underline{a}, \underline{m}) = p(\sigma', \underline{a}', \underline{m}')$ if and only if $\sigma = \sigma'$, $\underline{a} = \underline{a}'$ and $\underline{m} = \underline{m}'$. Now view $\underline{m}$ as a function $\underline{m} : [1, n] \to \mathbb{Z}$ defined by $\underline{m}(i) = m_i$. Then we can define a right action of $\mathrm{Sym}([1, n])$ on $\mathbb{Z}^n$ by $\underline{m}^{\sigma}(i) = \underline{m}(\sigma(i))$. We have $(\underline{m}^{\sigma\sigma'})(i) = \underline{m}((\sigma\sigma')(i))$. Also $((\underline{m}^{\sigma})^{\sigma'})(i) = \underline{m}^{\sigma}(\sigma'(i)) = \underline{m}(\sigma\sigma'(i))$. Then $\underline{m}^{(\sigma\sigma')} = (\underline{m}^{\sigma})^{\sigma'}$ where $(m_1, \ldots, m_n)^{\sigma} = (m_{\sigma(1)}, \ldots, m_{\sigma(n)})$. Similarly view $\underline{a}$ as a map $\underline{a} : [1, n] \to \mathbb{C}^*$ defined by $\underline{a}(i) = a_i$. Now we can define a right action of $\mathrm{Sym}([1, n])$ on $\mathbb{C}^*$ by $\underline{a}^{\sigma}(i) = \underline{a}(\sigma(i))$.

Let $P(\sigma, \underline{a}, \underline{m})$ and $P(\delta, \underline{b}, \underline{n})$ be in $N_G(\mathfrak{h})$. Then we can check that

$$P(\sigma, \underline{a}, \underline{m}) P(\delta, \underline{b}, \underline{n}) = P(\sigma\delta, \underline{a}^{\delta}\underline{b}, \underline{m}^{\delta} + \underline{n})$$

where $\underline{a}^{\delta}\underline{b}$ and $\underline{m}^{\delta} + \underline{n}$ mean pointwise multiplication and addition respectively and $(m_1, \ldots, m_n)^{\delta} = (m_{\delta(1)}, \ldots, m_{\delta(n))})$ and $(a_1, \ldots, a_n)^{\delta} = (a_{\delta(1)}, \ldots, a_{\delta(n))})$as mentioned above.

**Example 6.1.2.** Let $P = \begin{pmatrix} 0 & a_2t^{m_2} & 0 \\ 0 & 0 & a_3t^{m_3} \\ a_1t^{m_1} & 0 & 0 \end{pmatrix}$ and $P' = \begin{pmatrix} 0 & {a_2}'t^{{m_2}'} & 0 \\ {a_1}'t^{{m_1}'} & 0 & 0 \\ 0 & 0 & {a_3}'t^{{m_3}'} \end{pmatrix}$.
Then $P = P((132), (a_1, a_2, a_3), (m_1, m_2, m_3))$ and $P' = P((12), ({a_1}', {a_2}', {a_3}'), ({m_1}', {m_2}', {m_3}')$.
Also

$$PP' = \begin{pmatrix} a_2{a_1}'t^{m_2+{m_1}'} & 0 & 0 \\ 0 & 0 & a_3{a_3}'t^{m_3+{m_3}'} \\ 0 & a_1{a_2}'t^{m_1+{m_2}'} & 0 \end{pmatrix}.$$

So $PP' = P((23), (a_2{a_1}', a_1{a_2}', a_3{a_3}'), (m_2+{m_1}', m_1+{m_2}', m_3+{m_3}')) = P(\sigma\sigma', \underline{a}^{\sigma'}\underline{a}', \underline{m}^{\sigma'} + \underline{m}')$ where $\sigma = (132)$, $\sigma' = (12)$, $\underline{a} = (a_1, a_2, a_3)$ and $\underline{a}' = ({a_1}', {a_2}', {a_3}')$.

We have

$$N_G(\mathfrak{h}) = N_G(H) = \{P(\sigma, \underline{a}, \underline{m}) \mid \sigma \in \mathrm{Sym}([1, n]), \underline{a} \in (\mathbb{C}^*)^n, \underline{m} \in \mathbb{Z}^n\}$$

where $H$ is the subgroup of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$ and therefore

$$H = \{P(1, \underline{a}, \underline{0}) \mid \underline{a} \in (\mathbb{C}^*)^n\}.$$

Now we work out a complement $K$ of $H$ in $N_G(H)$ such that $N_G(H) = K \rtimes H$. So $N_G(H)/H \cong K$. Let $K = \{P(\theta, \underline{1}, \underline{m}) \mid \theta \in \mathrm{Sym}([1, n]), \underline{m} \in \mathbb{Z}^n\}$ then we have $N_G(H) = KH$ and $H \cap K = 1$.

**Proposition 6.1.3.** *$K$ is generated by* $\{\tilde{S}_i = P(s_i, \underline{1}, \underline{0}) \mid s_i = (i\, i+1), i = 1, \ldots, n-1\} \cup \{\tilde{T} = P(1, \underline{1}, (1, 0, 0, \ldots, 0))\}$ .

*Proof.* We know that $K$ is monomial matrices with entries $t^{m_i}$ for $m_i \in \mathbb{Z}$. Let $\alpha = P(\theta, \underline{1}, \underline{m}) \in K$ where $\theta \in \mathrm{Sym}([1, n])$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Then as $\mathrm{Sym}([1, n])$ is generated by transpositions $s_1, \ldots, s_{n-1}$, we can write $\theta = s_{i_1}, s_{i_2}, \ldots, s_{i_r}$ where $1 \leq i_j \leq$

$n-1$ and $1 \leq r$ and we have $P(\theta, \underline{1}, \underline{0}) = \tilde{S}_{i_1}\tilde{S}_{i_2}\ldots\tilde{S}_{i_r}$. Also $P(1, \underline{1}, (0, \ldots, m_j, \ldots, 0)) = \operatorname{diag}(1, \ldots, t^{m_j}, \ldots, 1) = P((1\, j), \underline{1}, \underline{0})P(1, \underline{1}, (1, 0, \ldots, 0))^{m_j}P((1\ j), \underline{1}, \underline{0})$ for $j > 2$. So $P(1, \underline{1}, (0, \ldots, m_i, \ldots, 0))$ is in the group generated by $\tilde{S}_1, \ldots, \tilde{S}_{n-1}, \tilde{T}$. Hence, we have $\alpha = P(\theta, \underline{1}, \underline{0}) \prod_{j=1}^{n} P(1, \underline{1}, (0, \ldots, m_j, \ldots, 0))$ and $\alpha$ is generated by $\tilde{S}_1, \ldots, \tilde{S}_{n-1}, \tilde{T}$. □

Now we want to show $K \cong \operatorname{Sym}(\mathbb{Z})^{T_n}$. To do this recall that $\operatorname{Sym}(\mathbb{Z})^{T_n}$ is generated by $T = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ n+1 & 2 & \ldots & n-1 & n \end{pmatrix}$ and $S_i = \begin{pmatrix} 1 & 2 & \ldots & i & i+1 & \ldots & n \\ 1 & 2 & \ldots & i+1 & i & \ldots & n \end{pmatrix}$ for $i = 1, \ldots, n-1$. Also we know that $K$ is generated by $\tilde{T} = P(1, \underline{1}, (1, 0, \ldots, 0))$ and $\tilde{S}_i = P(s_i, \underline{1}, \underline{0})$ for $s_i = (i\, i+1)$ and $i = 1, \ldots, n-1$. We try to find defining relations for $K$ when it is generated by $\tilde{S}_1, \ldots, \tilde{S}_{n-1}$ and $\tilde{T}$. These include $\tilde{S}_i^2 = 1$, $(\tilde{S}_i\, \tilde{S}_j)^{m_{i\,j}} = 1$ where $m_{i\,i+1} = 3$ and $m_{i\,j} = 2$ if $|i-j| > 1$. Also $\tilde{S}_i\tilde{T} = \tilde{T}\tilde{S}_i$ if $i > 1$ and moreover $(\tilde{S}_1\tilde{T}\tilde{S}_1)\tilde{T} = \tilde{T}(\tilde{S}_1\tilde{T}\tilde{S}_1)$. We need to check these are all relations. Consider

$$\Gamma = \langle s_1, \ldots, s_n, t \mid s_1ts_1t = ts_1ts_1, s_it = ts_i \text{ for } i > 1, (s_is_j)^{m_{i\,j}} = 1\rangle$$

where $m_{i\,i} = 1$, $m_{i\,i+1} = 3$, and $m_{i\,j} = 2$ if $|i-j| > 1$.

**Lemma 6.1.4.** *Each element of* $\Gamma$ *can be written in the form*

$$w(s_1, \ldots, s_{n-1})t^{m_1}(s_1ts_1)^{m_2}(s_2s_1ts_1s_2)^{m_3}\ldots(s_{n-1}\ldots s_1ts_1\ldots s_{n-1})^{m_n},$$

*where* $w(s_1, \ldots, s_{n-1})$ *is a word in* $s_1, \ldots, s_{n-1}$.

*Proof.* We prove this by induction on the number of times that a power of $t$ occurs in a word in $s_1, \ldots, s_n$ and $t$. Suppose our claim is true for any word containing a power of $t$ at most $k$ times and we prove this is true for any word containing a power of $t$ occurring $k+1$ times. Let $t^m$ be the first power of $t$ occurring from the right side in a word $w(s_1, \ldots, s_n, t)$. Then we have $w(s_1, \ldots, s_n, t) = w'(s_1, \ldots, s_n, t)t^m s_{i_1}s_{i_2}\ldots s_{i_{p-1}}s_{i_p}$ where $i_j \in \{1, \ldots n\}$

and $j \in \{1, \dots, p\}$. If $s_{i_1} \neq s_1$, then $w(s_1, \dots, s_n, t) = w'(s_1, \dots, s_n, t)s_{i_1}t^m s_{i_2} \dots s_{i_{p-1}} s_{i_p}$ and $t^m$ moves right until it might meet $s_1$. In general where $w(s_1, \dots, s_n, t) = w'(s_1, \dots, s_n, t)t^m$ $s_1 s_2 \dots s_r s_{j_1} s_{j_2} \dots s_{j_{n-1}} s_{j_n}$ such that $j_1 \neq r+1$ for some $r \in \{1, \dots, n\}$, we have $w(s_1, \dots, s_n, t) =$ $w'(s_1, \dots, s_n, t)t^{m-1} s_1 s_2 \dots s_r (s_r \dots s_2 s_1 t s_1 s_2 \dots s_r) s_{j_1} s_{j_2} \dots s_{j_{n-1}} s_{j_n} = w'(s_1, \dots, s_n, t)t^{m-2} s_1 s_2$ $\dots s_r (s_r \dots s_2 s_1 t s_1 s_2 \dots s_r)(s_r \dots s_2 s_1 t s_1 s_2 \dots s_r) s_{j_1} s_{j_2} \dots s_{j_{n-1}} s_{j_n} = \dots = w'(s_1, \dots, s_n, t) s_1 s_2$ $\dots s_r (s_r \dots s_2 s_1 t s_1 s_2 \dots s_r)^m s_{j_1} s_{j_2} \dots s_{j_{n-1}} s_{j_n} = w''(s_1, \dots, s_n, t)(s_r \dots s_2 s_1 t s_1 s_2 \dots s_r)^m s_{j_1} s_{j_2}$ $\dots s_{j_{n-1}} s_{j_n}$. Since $j_1 \neq r+1$ then $(s_r \dots s_2 s_1 t s_1 s_2 \dots s_r)^m$ will commute with all $s_{j_i}$ until it meets the first $s_{q+1}$. In this case if we have $w(s_1, \dots, s_n, t) = w''(s_1, \dots, s_n, t)(s_r \dots s_2 s_1 t s_1 s_2 \dots s_r)^m$ $s_{r+1} \dots s_{r+u} s_{j_1} \dots s_{j_n}$ for some $u \in \{1, \dots, n\}$ such that $r + u \in \{1, \dots, n\}$, then we have $w(s_1, \dots, s_n, t) = w''(s_1, \dots, s_n, t)(s_r \dots s_2 s_1 t s_1 s_2 \dots s_r)^{m_1} s_{r+1} \dots s_{r+u} s_{q_1} \dots s_{q_n} =$ $w''(s_1, \dots, s_n, t)(s_{r+1} \dots s_{r+u})(s_{r+u} \dots s_{r+1}(s_r \dots s_2 s_1 t s_1 s_2 \dots s_r)^m s_{r+1} \dots s_{r+u}) s_{q_1} \dots s_{q_n}$. Then we have $w(s_1, \dots, s_n, t) = w'''(s_1, \dots, s_n, t)(s_{r+u} \dots s_{r+1} s_r \dots s_2 s_1 t s_1 s_2 \dots s_r s_{r+1} \dots s_{r+u})^m$ $s_{q_1} \dots s_{q_n}$. We will continue this procedure until we have $w(s_1, \dots, s_n, t) = \lambda(s_1, \dots, s_n, t)$ $(s_v \dots s_1 t s_1 \dots s_v)^m$ for some $v \in \{1, \dots, n\}$. Now let $t_{r+1} = s_r \dots s_1 t s_1 \dots s_r$. Hence we have $w(s_1, \dots, s_n, t) = \lambda(s_1, \dots, s_n, t) {t_{v+1}}^{m_{v+1}}$ where $m_{v+1} = m$. Since $\lambda(s_1, \dots, s_n, t)$ is a word with a power of $t$ occurring $k$ times then by induction we see that all elements of $\Gamma$ can be written in the form $w(s_1, \dots, s_{n-1})v(t_1, \dots, t_n)$. It is easy to check that $t_1, \dots, t_n$ pointwise commute in order to write $v(t_1, \dots, t_n) = t_1^{m_1} t_2^{m_2} \dots t_n^{m_n}$. ☐

**Proposition 6.1.5.** $K \cong \Gamma$

*Proof.* We define a map $\Phi : \Gamma \to K$ by $s_i \mapsto \tilde{S}_i$, $t \mapsto \tilde{T}$ and we prove that $\Phi$ is an isomorphism. Obviously, $\Phi$ is surjective and to show $\Phi$ is injective let $g \in \Gamma$. Then by Lemma 6.1.4 we can write $g = w(s_1, \dots, s_{n-1}) t_1^{m_1} \dots t_m^{m_n}$ and we have $\Phi(g) =$ $w(\tilde{S}_1, \dots, \tilde{S}_{n-1}) {T_1}^{m_1} \dots T_n^{m_n}$ where $T_{i+1} = \tilde{S}_i \tilde{S_{i-1}} \dots \tilde{T} \dots \tilde{S_{i-1}} \tilde{S}_i$. If $\Phi(g) = 1$ then $m_1, \dots, m_n =$ $0$ and $w(\tilde{S}_1, \dots, \tilde{S}_{n-1}) = 1$. We know $w(\tilde{S}_1, \dots, \tilde{S}_{n-1}) = 1$ implies $w(s_1, \dots, s_{n-1}) = 1$. Therefore $\Phi$ is injective. ☐

**Proposition 6.1.6.** $\Gamma \cong \mathrm{Sym}(\mathbb{Z})^{T_n}$

*Proof.* We prove that $\Psi : \Gamma \to \mathrm{Sym}(\mathbb{Z})^{T_n}$ defined by $\Gamma(s_i) = S_i$ and $\Gamma(t) = T$ is injective. This is well defined because $T$ and $S_i$ satisfy relations of $\Gamma$. We need to show every element of $\mathrm{Sym}(\mathbb{Z})^{T_n}$ can be written in the form $w(S_1, \dots, S_{n-1}){S'_1}^{m_1} \dots {S'_m}^{m_n}$ with $m_1, \dots, m_n \in \mathbb{Z}$ where $S'_i = S_{i-1}S_{i-2} \dots T \dots S_{i-2}S_{i-1}$ so that $S'_i(i) = i + n$ and $S'_i(j) = j$ for $j \neq i$. Also it is clear that $w(S_1, \dots, S_{n-1}){S'_1}^{m_1} \dots {S'_m}^{m_n} = 1$ if and only if $m_1, \dots, m_n = 0$ and $w(S_1, \dots, S_{n-1}) = 1$ because every element of $\mathrm{Sym}(\mathbb{Z})^{T_n}$ can be obtained first by translating $1, \dots, n$ individually and then permuting $1, \dots, n$. □

**Corollary 6.1.7.** $K \cong \mathrm{Sym}(\mathbb{Z})^{T_n}$.

*Proof.* This follows from Propositions 6.1.5 and 6.1.6. □

Therefore we have $N_G(\mathfrak{h}) \cong \mathrm{Sym}(\mathbb{Z})^{T_n} \ltimes (\mathbb{C}^*)^n$.

## 6.2 Normalizer of $\mathfrak{g}_Y$ in $G$

In this section let $G = \mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$ and $Y \subseteq [1, n-1]$. Then we have $G_Y \subseteq \mathrm{GL}_n(\mathbb{C})$ which is the Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$ corresponding to $Y$. Let $\mathfrak{g}_Y$ be the Levi subalgebra corresponding to $Y$. In Chapter 5 we determined a partition $L$ of $[1, n]$ from $Y$. The general rule is that $i$ and $i + 1$ lie in the same part of the partition if and only if $i \in Y$. For consistency from now on we denote $G_Y$ by $G_L$ and $\mathfrak{g}_Y$ by $\mathfrak{g}_L$. Therefore $\mathfrak{g}_L$ is generated by $e_{ij}$ such that $i$ and $j$ lie in the same part of $L$.

**Lemma 6.2.1.** $N_G(\mathfrak{g}_L) = G_L(N_G(\mathfrak{g}_L) \cap N_G(\mathfrak{h}))$.

*Proof.* Let $g \in N_G(\mathfrak{g}_L)$. Then $g \cdot t \in \mathfrak{g}_L$ for all $t \in \mathfrak{h}$ where $\cdot$ is the adjoint action of $G$ on $\mathfrak{g}$. So $g \cdot \mathfrak{h} \subseteq \mathfrak{g}_L$ and $g \cdot \mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}_L$. All maximal toral subalgebras of $\mathfrak{g}_L$ are conjugate by an element of $G_L$. So there exists $x \in G_L$ such that $x \cdot g \cdot \mathfrak{h} = \mathfrak{h}$. Therefore $xg \in N_G(\mathfrak{h})$ and we have $N_G(\mathfrak{g}_L) = G_L(N_G(\mathfrak{g}_L) \cap N_G(\mathfrak{h}))$. □

To work out $N_G(\mathfrak{g}_L)$ we need to find out all elements of $N_G(\mathfrak{g}_L) \cap N_G(\mathfrak{h})$. So we look for elements $P(\sigma, \underline{a}, \underline{m}) \in N_G(\mathfrak{h})$ that are in $N_G(\mathfrak{g}_L)$.

We can check that the element $P(\sigma, \underline{a}, \underline{m})$ acts on $e_{ij}t^k$ by conjugation such that

$$P(\sigma, \underline{a}, \underline{m}) e_{ij} P(\sigma, \underline{a}, \underline{m})^{-1} = a_i a_j^{-1} t^{m_i - m_j} e_{\sigma(i)\sigma(j)}.$$

We know that $\Phi_Y = \{\alpha_{ij} \mid i, j \in [1, n-1], i \neq j\}$.

**Proposition 6.2.2.** *$P(\sigma, \underline{a}, \underline{m}) \in N_G(\mathfrak{g}_L)$ if and only if $\sigma(L) = L$ i.e if $i$ and $j$ lie in the same part of $L$, then so do $\sigma(i)$and $\sigma(j)$ and moreover for any $r$ and $s$ in $L_i$ for some $i$ we have $m_r = m_s$.*

*Proof.* This is true because $P(\sigma, \underline{a}, \underline{m}) \in N_G(\mathfrak{g}_L)$ if and only if for all $i, j$ such that $e_{ij} \in \mathfrak{g}_L$ we have

$$P(\sigma, \underline{a}, \underline{m}) e_{ij} P(\sigma, \underline{a}, \underline{m})^{-1} = a_i a_j^{-1} t^{m_i - m_j} e_{\sigma(i)\sigma(j)} = b e_{kl}$$

where $\sigma(i) = k$, $\sigma(j) = l$, $a_i a_j^{-1} = b \in \mathbb{C}$ and $m_i = m_j$ . □

Therefore $K \cap N_G(\mathfrak{g}_L)$ is the set of all $P(\sigma, \underline{1}, \underline{m})$ such that $\sigma(L) = L$ and if $i$ and $j$ are in $L_k$ for some $k$ then $\sigma(i)$ and $\sigma(j)$ are in $L_r$ for some $r$ and for any $s$ and $t$ in $L_i$ for some $i$ we have $m_s = m_t$.

## 6.3 Levi type Weyl groups

For any $Y \subseteq [1, n-1]$ we get a partition $L = \{L_1, \ldots, L_m\}$ of $[1, n]$. In this section we prove that $W^L$ known as Levi type Weyl group is isomorphic to $N_G(\mathfrak{g}_L)/G_L$.

First we define

$$K_L = K \cap G_L = \{P(\sigma, \underline{1}, \underline{0}) \mid \sigma \in \mathrm{Sym}([1, n]) \text{ and } \sigma L_i = L_i \text{ for all } i = 1, \ldots, m\}.$$

Now let

$$N_{K,L} = K \cap N_G(\mathfrak{g}_L) = \{P(\sigma, \underline{1}, \underline{m}) \mid \sigma(L) = L \text{ and } m_r = m_s \text{ whenever } r, s \in L_i \text{ for some } i\}.$$

We can show $N_{K,L} = N_K(K_L)$.

**Lemma 6.3.1.** *The map $\Omega : N_{K,L} \to N_G(\mathfrak{g}_L)/G_L$ defined by $\Omega(x) = xG_L$ is a surjective map with kernel $K_L$.*

*Proof.* Let $P(\sigma, \underline{1}, \underline{m}) \in \ker(\Omega)$ then $P(\sigma, \underline{1}, \underline{m})G_L = G_L$ so $P(\sigma, \underline{1}, \underline{m}) \in G_L$ therefore $\underline{m} = 0$ and $\sigma(L_i) = L_i$ for all $i$ and $\ker(\Omega) = K_L$. To show it is surjective we know that any element of $N_G(\mathfrak{g}_L)/G_L$ is of the form $xG_L$ where $x \in N_G(G_L) \cap N_G(H)$. As $N_G(H) = K \ltimes H$ then we can write $x = kh$ where $k \in K$ and $h \in H$. Then $xG_L = khG_L = kG_L$. Also it is clear that $k \in N_G(G_L)$ because $kh \in N_G(G_L)$ and $h \in N_G(G_L)$. Therefore $\Omega(k) = xG_L$ and $\Omega$ is surjective and we have $N_{K,L}/K_L \cong N_G(\mathfrak{g}_L)/G_L$. □

For any $Y \subseteq [1, n-1]$ and the corresponding partition $L = \{L_1, \ldots, L_m\}$ we have

$$N_{G_L}(\mathfrak{h}) = \{P(\sigma, \underline{a}, \underline{0}) \mid \sigma \in \mathrm{Sym}([1, n]) \text{ and } \sigma(L_i) = L_i \text{ for all } i = 1, \ldots, m\}.$$

and $N_{G_L}(\mathfrak{h}) = K_L \ltimes H$ where

$$K_L = \{P(\sigma, \underline{1}, \underline{0}) \mid \sigma \in \mathrm{Sym}([1, n]) \text{ and } \sigma(L_i) = L_i,\ i = 1, \ldots, m\}.$$

We have $N_{K,L} = N_K(K_L)$ as a subset of $K = \{P(\sigma, \underline{1}, \underline{m}) \mid \sigma \in \mathrm{Sym}(n) \text{ and } \underline{m} \in \mathbb{Z}^n\}$ and we try to determine a complement $K^L$ of $K_L$ in $N_{K,L}$.

Let $L$ be a partition of $[1, n]$ corresponding to the subset $Y \subseteq [1, n-1]$ as before. Let $\bar{L}$ be the partition of $\mathbb{Z}$ stable under $T_n$ obtained from $L$. By Corollary 6.1.7 we have

$\widehat{W} = \text{Sym}(\mathbb{Z})^{T_n} \cong K$ under the map $\tau : K \to \text{Sym}(\mathbb{Z})^{T_n}$ defined by

$$P(\sigma, \underline{1}, \underline{m}) \mapsto [\sigma]{S'}_1^{m_1} \ldots {S'}_n^{m_n}$$

where $\underline{m} = (m_1, \ldots, m_n)$, and $S'_i$ is defined on $[1, n]$ by $S'_i(i) = i + n$ and $S'_i(j) = j$ if $i \neq j$ and $[\sigma]{S'}_1^{m_1} \ldots {S'}_n^{m_n}(i) = [\sigma](i + m_i n) = \sigma(i) + m_i n$.

Recall that

$$\widehat{W}_{\bar{L}} = \{\sigma \in \widehat{W} \mid \sigma(L_i) = L_i \text{ for all } i\},$$

$$N_{\widehat{W}}(\widehat{W}_{\bar{L}}) = N_{\widehat{W},\bar{L}} = \{\sigma \in \widehat{W} \mid \sigma(\bar{L}) = \bar{L}\}$$

and

$$\widehat{W}^{\bar{L}} = \{\sigma \in \bar{W} \mid \sigma(\bar{L}) = \bar{L} \text{ and } \text{ if } r, s \in \bar{L}_i \text{ for some } i \text{ with } r < s \text{ then } \sigma(r) < \sigma(s)\}$$

such that $N_{\widehat{W},\bar{L}} \cong \widehat{W}^{\bar{L}} \ltimes \widehat{W}_{\bar{L}}$ where $L$ is a partition of $[1, n]$ corresponding to the subset $Y \subseteq [1, n-1]$ and $\bar{L}$ is the partition of $\mathbb{Z}$ stable under $T_n$ obtained from $L$.

**Proposition 6.3.2.** $\tau(N_{K,L}) = N_{\widehat{W},\bar{L}}$.

*Proof.* To prove our claim that $\tau(N_{K,L}) = N_{\widehat{W},\bar{L}}$, let $P(\sigma, \underline{1}, \underline{m}) \in N_{K,L}$ and let $M = T_n^j(L_i) \in \bar{L}$ where $\bar{L} = \bigcup_{j \in \mathbb{Z}} T_n^j(L) = \{T_n^j(L_i) \mid j \in \mathbb{Z}, i \in [1, m]\}$ where $T_n$ is translation $n$ steps to right. Then let $\delta = \tau(P(\sigma, \underline{1}, \underline{m}))$. We show that $\delta(M) \in \bar{L}$ so that $\delta \in N_{\widehat{W},\bar{L}}$. Let $r, s \in M$, then $T_n^{-j}(r), T_n^{-j}(s) \in L_i$. So we have $\delta(r) = T_n^j(\delta(T_n^{-j}(r))) = T_n^j(\sigma(r - jn) + m_{r-jn}n) = \sigma(r - nj) + m_{r-jn}n + nj$. Similarly $\delta(s) = \sigma(s - jn) + m_{s-jn}n + nj$. Since $L_h = \sigma(L_i) \in L$ for some $h$, so $\sigma(r - jn), \sigma(s - jn) \in L_h$. Note that $m_{r-jn} = m_{s-jn}$ because $r - jn, s - jn \in L_i$. So $\delta(r), \delta(s) \in {T_n}^{m_{r-jn}+j}(L_h)$. This implies that $\delta(M) = {T_n}^{m_{r-jn}+j}(L_h) \in \bar{L}$. Therefore, $\tau(P(\sigma, \underline{1}, \underline{m})) \in N_{\widehat{W},L}$, so $\tau(N_{K,L}) \subseteq N_{\widehat{W},\bar{L}}$. The proof of reverse inclusion is similar. □

**Corollary 6.3.3.** $N_G(\mathfrak{g}_L)/G_L \cong N_{\widehat{W}}(\widehat{W}_{\bar{L}})/\widehat{W}_{\bar{L}}$.

*Proof.* Under the isomorphism in Corollary 6.1.7 we can check that $K_L \cong \widehat{W}_{\bar{L}}$. We also have $N_{K,L} \cong N_{\widehat{W},\bar{L}}$ by Proposition 6.3.2. So there exists a subgroup of $N_{K,L}$ denoted by $K^L$ which is isomorphic to $\widehat{W}^{\bar{L}}$ and $N_{K,L} = K^L \ltimes K_L$. Hence by Lemma 6.3.1 we have $N_G(\mathfrak{g}_L)/G_L \cong N_{\widehat{W}}(\widehat{W}_{\bar{L}})/\widehat{W}_{\bar{L}}$. □

We also expect $\widehat{W}^{\bar{L}} \cong N_{G^e}(\mathfrak{h}^e)/C_{G^e}(\mathfrak{h}^e)$. For the finite case, see [3, Lemma 14].

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