M-Axial Algebras Related to 4-Transposition Groups

by

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Abstract

The main result of this thesis concerns the classification of 3-generated M-axial algebras A such that every 2-generated subalgebra of A is a Sakuma algebra of type NX, where $N \in \{2,3,4\}$ and $X \in \{A,B,C\}$. This goal requires the classification of all groups G which are quotients of the groups $T^{(s_1,s_2,s_3)} = \langle x, y, z \mid x^2, y^2, z^2, (xy)^{s_1}, (xz)^{s_2}, (yz)^{s_3} \rangle$ for $s_1, s_2, s_3 \in \{3,4\}$ and the set of all conjugates of x, y and z satisfies the 4-transposition condition. We show that those groups are quotients of eight groups. We show which of these eight groups can be generated by Miyamoto involutions. This can be done by classifying all possible M-axial algebras for them. In addition, we discuss the embedding of Fisher spaces into a vector space over GF(2) in Chapter 3.

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Contents

Chapter 1

Introduction

The Monster group, denoted by M, is a finite simple group which has the highest order among the sporadic simple groups. It was constructed by Robert Griess [12] as the group of automorphisms of a commutative non-associative real algebra V of dimension 196,884, called the Griess algebra (aka the Monster algebra). The algebra V is equipped with an inner product $\langle , \rangle$ satisfying $\langle u \cdot v, w\rangle = \langle u, v \cdot w\rangle$ for all $u, v, w \in V$. The group M is a 6-transposition group. This means that M is generated by an invariant set of involutions (called $2A$-involutions in [3]) and the order of the product of any pair of these involutions a and b is less than or equal to 6 [2]. More precisely, the conjugacy class of ab is one of the eight classes $2A,\ 2B,\ 3A,\ 3C,\ 4A,\ 4B,\ 5A$, and $6A$ [2, 3].

In the Griess algebra, every $2A$-involution a determines a unique idempotent e_a, called the axis of a, and for a second $2A$-involution b, the conjugacy class of ab determines the value of the inner product of e_a and e_b [3]. From the viewpoint of the vertex operator algebras (VOAs), the Monster group can be viewed as the group of automorphisms of the Moonshine VOA $V^\natural = \oplus_{n=0}^{\infty} V_n^\natural$. This VOA has been constructed in [10, 28].

In [30], for a VOA $V = \oplus_{n=0}^{\infty} V_n$ over the real numbers $\mathbb{R}$, satisfying the extra assumption that $V_0 = \mathbb{R}1$ and $V_1 = 0$, special vectors in V_2, called Ising vectors, were defined and it was shown that every Ising vector u leads to an automorphism τ_u of V, called the *Miyamoto involution* [27]. When $V = V^\natural$, V_2 coincides with the Griess algebra and the Ising vectors are multiples of the $2A$-axes while the corresponding Miyamoto involutions are the $2A$-involutions of M [10]. The main result of [30] is that the 6-transposition property of M holds for Miyamoto involutions of an arbitrary VOA. To prove this he completely classified subalgebras in V_2 generated by two Ising vectors. The structure of such a subalgebra is determined uniquely by the inner product of the two Ising vectors.

Starting from Sakuma's proof, Ivanov [19] extracted the relevant properties of V_2 and he made them the axioms of a new class of algebras, *Majorana algebras*. Every Majorana algebra has a finite group of automorphisms (generated by Miyamoto involutions) associated with it. Sakuma's theorem [30] is equivalent to a classification of all Majorana algebras for the dihedral groups. He showed that there are exactly eight such algebras and they are all subalgebras of the Monster algebra. This allow to label the eight 2-generated algebras with the classes $2A$, $2B$, $3A$, $3C$, $4A$, $4B$, $5A$, and $6A$ as above. For a general 6-transposition group G, a corresponding Majorana algebra may or may not exists. However, if G is a subgroup of M and G is generated by $2A$-involutions, then the algebra exists. Such an algebra of G is said to be *based on an embedding of G in the Monster.*

All previous work on Majorana algebras (see [20, 21, 22, 23, 24, 31]) has been done under the additional assumption, the so-called $(2A)$-condition. It states that *if T is the set of Miyamoto involutions in G, t_0 and t_1 are in T, and product $t_0 \cdot t_1$ is also contained in T, then the corresponding idempotents a_0 and a_1 generate a subalgebra of type $2A$ and $a_\rho = a_0 + a_1 - 8a_0 \cdot a_1$ is an Ising vector corresponding to $t_0 \cdot t_1$.*

For instance, we take the symmetric group S_4 of degree 4 and try to classify all Majorana algebras of it without $(2A)$-condition. The group S_4 has two conjugacy classes of involutions, the six transpositions and the three double transpositions. Since S_4 must be generated by the Miyamoto involutions, we always take the six transpositions. The three double transpositions may or may not be chosen to be Miyamoto involutions.

At this point, we need to discuss the so-called *shapes*. First of all, Ising vectors correspond to Miyamoto involutions. Hence pairs of Ising vectors correspond to pairs of involutions. According to Sakuma [30], two Ising vectors generate one of the eight particular algebras, and the order of the product of the involutions corresponding to the Ising vectors limits the type of the 2-generated algebra. For example, if the order of the product of Miyamoto involutions is 3 then the two corresponding Ising vectors generate a subalgebra of type $3A$ or $3C$; and so on. If two pairs of Ising vectors are conjugate then they generate a subalgebras of the same type. So, for each orbit of pairs, the shape prescribes the type of the algebras those pairs generate.

For the group S_4, there are two classes of involutions. Hence we have two cases. The first one is the case where we just take the six transpositions. Then we have only two orbits on pairs and the order of the product of any two involutions is either 2 or 3. Therefore, the corresponding Ising vectors generate a subalgebra either of type $2X$ or $3Y$, where $X \in \{A, B\}$ and $Y \in \{A, C\}$. Therefore, we have four possible shapes $(2X, 3Y)$. Otherwise, we have all nine involutions and five orbits on pairs with order of the product in the orbits being 2, 2, 2, 3, or 4. So, we have the shape $(2X, 2Y, 2Z, 3W, 4U)$. The fifth entry in the shape, $4U$, corresponding to the orbit of pair of involutions whose product has order 4, determine the type of the subalgebras $2X$ and $2Y$ because both of them are subalgebras in $4U$, and then X and Y must be the same. Altogether, in the two cases for S_4 we obtain twelve possible shapes (see Table 1.2). In each case, the shape may or may not lead to an algebra.

For a number of groups, the corresponding Majorana algebras have been determined for all or almost all shapes. The paper [22] deals with the symmetric group S_4 of degree 4 and only four shapes are covered in. The Master of Research thesis [25] determined the remaining cases of the group S_4. The Majorana algebras of the groups A_5 [23], A_6 and A_7 [21, 20], and $L_3(2)$ [24] were classified. Ákos Seress [31] computed Majorana algebras for a list of groups, such as $S_5, S_6, 3.A_6, 3.S_6, (S_4 \times S_3) \cap A_7, 3.A_7, S_7, 3.S_7, L_2(11), L_3(3)$ and M_{11}, by using computer algebra system GAP [11]. Table 2.5 gives the shapes and dimensions of the known Majorana algebras for the groups mentioned above.

As a generalisation of Majorana algebras as well as commutative associative algebras, axial algebras have been defined in [18]. They are not necessarily associative commutative algebras generated by primitive axes, that is, semisimple idempotents in which every axis spans its own 1-eigenspace.

In this thesis, 3-generated M-axial algebras A such that every 2-generated subalgebra of A is a Sakuma algebra of type NX, where $N \in \{2, 3, 4\}$ and $X \in \{A, B, C\}$ has been studied. To achieve this, we require the classification of all groups G which are quotients of the group $T^{(s_1,s_2,s_3)} = \langle x, y, z \mid x^2, y^2, z^2, (xy)^{s_1}, (xz)^{s_2}, (yz)^{s_3} \rangle$ for $s_1, s_2, s_3 \in \{3, 4\}$ and the set of all conjugates of x, y and z is 4-transposition.

This thesis consists of five main chapters. In Chapter 2, the necessary background of axial algebras has been given. At the end of the chapter, we define the concept of axial representations in order to find the axial algebras for the groups that are determined in Chapter 4.

In Chapter 3, we study a typical type of M-axial algebras involving only subalgebras of type $2A$ and $2B$. In this case, the corresponding involutions to axes are σ-involutions and a group generated by the set of such σ-involutions is a 3-transposition group. For each 3-transposition group there is a Fischer space on a set of σ-involutions associated with it.

In the last section of the chapter, we calculate the dimension of the embedding of such Fischer space into a $GF(2)$ vector space.

Chapter 4 of this thesis is purely group theoretical. All groups G generated by three Miyamoto involutions has been classified. This requires the classification of all groups satisfy the following property:

Property (Δ). *A group G satisfies property (Δ) if and only if the following hold:*

1. *G is generated by three involutions a, b and c.*
2. *The order of the product of any two distinct elements in $T := a^G \cup b^G \cup c^G$ is at most 4.*

The main result in Chapter 4 is the following theorem. Note that in the second column of the Table 1.1, $B(2, 4)$ refers to the Burnside group of rank 2 and exponent 4.

Theorem 1.0.1. *A group satisfies property (Δ) if it is a quotient of at least one of the groups in Table 1.1.*

Groups	Isomorphism Type	(s_1, s_2, s_3)	Group Order
T_1	$(4 \times 2^2) : 2$	$(4, 4, 4)$	32
T_2	$3^2 : S_3$	$(3, 3, 3)$	54
T_3	$4^2 : S_3$	$(3, 3, 3)$	96
T_4	$2 \times L_3(2)$	$(3, 3, 4)$	336
T_5	$(((2 \times D_8) : 2) : 3).2) : 2$	$(3, 4, 4)$	384
	$= (2.(((2^4) : 3) : 2)$		
T_6	$(S_4 \times S_4) : 2$	$(3, 4, 4)$	1152
T_7	$(((3 \times ((3^2) : 3)) : 3) : Q_8) : 2$	$(4, 4, 4)$	3888
T_8	$B(2, 4) : 2$	$(4, 4, 4)$	8192

Table 1.1: Largest 3-generated 4-transposition groups

In Chapter 5, we describe many of the M-axial algebras for the groups in Chapter 4. In Chapter 6, we revisit the main results of this thesis and we discuss some possible future work in this direction.

$(2B, 3A)$	$(2B, 2B, 2B, 3A, 4A)$	$(2A, 2A, 2B, 3A, 4B)$
$(2B, 3C)$	$(2B, 2B, 2B, 3C, 4A)$	$(2A, 2A, 2B, 3C, 4B)$
$(2A, 3A)$	$(2B, 2B, 2A, 3A, 4A)$	$(2A, 2A, 2A, 3A, 4B)$
$(2A, 3C)$	$(2B, 2B, 2A, 3C, 4A)$	$(2A, 2A, 2A, 3C, 4B)$

Table 1.2: S_4-Shapes

Chapter 2

Axial algebras

Most of the contents of this chapter can be found in [22, 18] and [17].

The notion of an axial algebra first was introduced in [18]. In order to present this definition, we need first to review some related concepts.

2.1 Fusion rules, axes and axial algebras

Definition 2.1.1. *A fusion table over a field k is a finite set $\mathfrak{F}$ of elements of k and a map $* : \mathfrak{F} \times \mathfrak{F} \to 2^{\mathfrak{F}}$.*

Note that elements of a fusion table defined above can be arranged in a square symmetric table. Each entry in a fusion table can be viewed as a rule. So sometimes we refer to fusion tables as fusion rules.

We give two examples of fusion tables. These are in fact the fusion rules that will feature prominently in this thesis. First, take $J_\alpha = \{1, 0, \alpha\}$ with $0 \neq \alpha \neq 1$, and let the fusion rules be given by the table below.

	1	0	α
1	1	0	α
0	0	0	α
α	α	α	1,0

Table 2.1: Fusion rules J_α

The second example is where $M = \{1, 0, \frac{1}{4}, \frac{1}{32}\}$ and the fusion rules are given by the table below.

	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	0	$\frac{1}{4}$	$\frac{1}{32}$
0	0	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1,0	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	1,0,$\frac{1}{4}$

Table 2.2: Fusion rules M

Assume that A is a not necessarily associative commutative algebra over a field k. The adjoint of $a \in A$, denoted by ad$(a)\in$ End(A), is the mapping $b \to ab$. The *eigenvector* of ad(a) with respect to an eigenvalue $\lambda \in k$ is a vector $b \in A$ such that ad$(a)b = ab = \lambda b$.

By the λ-*eigenspace* of ad(a) we mean the set of all eigenvectors of ad(a) corresponding to the eigenvalue λ. We denote the λ-eigenspace by A^a_λ, that is, $A^a_\lambda = \{b \in A \mid ab = \lambda b\}$. We write $A^a_\lambda = 0$ if $\lambda \in \mathfrak{F}$ is not an eigenvalue of ad(a).

Let $\mathfrak{F}$ be the set of eigenvalues of a. We say that a is *semisimple* if A decomposes into a direct sum of the eigenspaces of ad(a), that is, $A = \bigoplus_{\lambda\in\mathfrak{F}} A^a_\lambda$.

If a is an idempotent, that is, $a^2 = a$, then we have that $1 \in \mathfrak{F}$ and $a \in A^a_1$. Indeed, this is true since $aa = a = 1a$.

When A is associative, if $a \in A$ is an idempotent, then one can deduce that $A = A_1^a \oplus A_0^a$, hence $\{1\} \subseteq \mathfrak{F} = \{1, 0\}$ (it can be that $\mathfrak{F} = \{1\}$). The more interesting case is where the algebra is not associative, which means that $\mathfrak{F}$ can be arbitrary, containing 1.

Definition 2.1.2. *Let a be an element of the algebra A. Then a is said to be an $\mathfrak{F}$-axis if the following hold:*

(A1) *a is an idempotent;*

(A2) *a is semisimple;*

(A3) *the fusion rules $\mathfrak{F}$ are satisfied, that is, $A_\lambda^a A_\mu^a \subseteq A_{\lambda * \mu}^a$ for $\lambda, \mu \in \mathfrak{F}$ under the algebra product.*

Definition 2.1.3. *For an $\mathfrak{F}$-axis a of an algebra A, if $A_1^a = \langle a \rangle$ then a is called* primitive*.*

Definition 2.1.4. *A nonassociative commutative algebra A is an* $\mathfrak{F}$-axial algebra *if it is generated by a set of primitive $\mathfrak{F}$-axes.*

Sometimes there is an extra structure on an $\mathfrak{F}$-axial algebra, a bilinear form, which is an important feature of the algebra.

Definition 2.1.5. *An $\mathfrak{F}$-axial algebra A is called* Frobenius *if there is a nonzero bilinear form $\langle , \rangle : A \times A \to k$ such that for all $a, b, c \in A$, $\langle a, bc \rangle = \langle ab, c \rangle$. Additionally, for any $\mathfrak{F}$-axis a, we require that $\langle a, a \rangle \neq 0$.*

Lemma 2.1.6. *The form $\langle , \rangle$ is symmetric.*

Proof. The algebra A is generated by $\mathfrak{F}$-axes. Furthermore, A is spanned by monomials, each of $\mathfrak{F}$-axis is a product of two other elements. Let x, y be arbitrary monomial products of axes. Then $x = x_1 x_2$ and $\langle x, y \rangle = \langle x_1 x_2, y \rangle = \langle x_1, x_2 y \rangle = \langle x_1 y, x_2 \rangle = \langle y, x_1 x_2 \rangle = \langle y, x \rangle$. □

Lemma 2.1.7. *For any $\mathfrak{F}$-axis a in the Frobenius algebra A, the eigenspaces A^a_λ and A^a_μ are perpendicular whenever $\lambda \neq \mu$.*

Proof. Let $v \in A^a_\lambda$ and $w \in A^a_\mu$. Then $\lambda\langle v, w\rangle = \langle \lambda v, w\rangle = \langle av, w\rangle = \langle v, aw\rangle = \langle v, \mu w\rangle = \mu\langle v, w\rangle$. This implies $(\lambda - \mu)\langle v, w\rangle = 0$. Since $\lambda \neq \mu$, we have $\langle v, w\rangle = 0$. □

The following two lemmas are the useful tools used to calculate all or almost all unknown algebra products in Chapter 5 of this thesis.

Lemma 2.1.8. *Every $\mathfrak{F}$-axis associates with each element of its 0-eigenspace.*

Proof. Let a be any axis. Since the eigenvectors of a generate A, it is enough to show that the equality hold for any v in the eigenspaces of a. Assume that $v \in A^a_\lambda$ and $b \in A^a_0$. There are two cases, the first case is, if $\lambda = 1$, then $v = sa$ for some $s \in R$, which implies that $a(vb) = a(sab) = a(s0) = 0$. On the other hand $(av)b = (a(sa))b = s(aa)b = s(ab) = s0 = 0$. Therefore $a(vb) = (av)b$. For the second case, that is if $\lambda \neq 1$, then by fusion rules, $vb \in A^a_{\lambda * 0} = A^a_\lambda$ and hence $a(vb) = \lambda(vb) = (\lambda v)b = (av)b$, the claim yields. □

The following lemma, called the *resurrection principle lemma*, is the second useful tool used in Chapter 5 where $\mathfrak{F} = M = \{1, 0, \frac{1}{4}, \frac{1}{32}\}$.

Lemma 2.1.9. *Let α, β be 0-eigenvectors and γ is $\frac{1}{4}$-eigenvector of the axis a_0. Then $t = \alpha \cdot \beta$ is a 0-eigenvector of a_0, $s = \alpha \cdot \gamma$ is a $\frac{1}{4}$-eigenvector of a_0 and $4a_0(s - t) = s$.*

Proof. From the fusion rules, it can be seen that t and s are 0- and $\frac{1}{4}$-eigenvectors of a_0, respectively . It means that $a_0 \cdot t = 0$ and $a_0 \cdot s = \frac{1}{4} \cdot s$. Therefore $4a_0(s - t) = s$. □

Now we turn to automorphisms of an $\mathfrak{F}$-axial algebra A. By an automorphism we mean an invertible linear transformation of A that preserves the algebra product. For Frobenius algebras we also require the automorphism to preserve the form. Next, we give the following definition.

Definition 2.1.10. *For an abelian group G, we say that the fusion table $\mathfrak{F}$ is G-graded if $\mathfrak{F}$ can be partitioned into parts $\{\mathfrak{F}_g\}_{g \in G}$ such that for every $g, h \in G$, if $x \in \mathfrak{F}_g$ and $y \in \mathfrak{F}_h$, then $x * y \subseteq \mathfrak{F}_{gh}$*

For example, if we take $G = \mathbb{Z}/2\mathbb{Z} = \{+, -\}$, then the partition consists of two subsets $\mathfrak{F}_+$ and $\mathfrak{F}_-$ and the fusion table should satisfy the following: if we take $a, b \in \mathfrak{F}_+$ and $c, d \in \mathfrak{F}_-$, then $a * b \in \mathfrak{F}_+$, $a * c \in \mathfrak{F}_-$ and $c * d \in \mathfrak{F}_+$. To be more precise, we review the Tables 2.1 and 2.2. The double lines in the tables separated the plus and minus parts in such way that the column on the right and under the double lines refers to the minus part.

	1	0	α
1	1	0	α
0	0	0	α
α	α	α	1,0

The $\mathbb{Z}/2\mathbb{Z}$-grading is: $J_{\alpha+} = \{1, 0\}$ and $J_{\alpha-} = \{\alpha\}$.

	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	0	$\frac{1}{4}$	$\frac{1}{32}$
0	0	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1,0	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	1,0,$\frac{1}{4}$

Here the $\mathbb{Z}/2\mathbb{Z}$-grading is: $M_+ = \{1, 0, \frac{1}{4}\}$ and $M_- = \{\frac{1}{32}\}$.

Remark 2.1.11. *If a is an $\mathfrak{F}$-axis of the algebra A, then $A^a_X := \bigoplus_{\lambda \in X} A^a_\lambda$ for a subset $X \subseteq \mathfrak{F}$.*

For an abelian group G, a linear character χ of G is a homomorphism $\chi : G \to k^\times$, where $k^\times$ is the multiplicative group of the field k.

Proposition 2.1.12. *Let χ be a linear character of the abelian group G and $\mathfrak{F}$ be G-graded. Suppose that a is an $\mathfrak{F}$-axis of the algebra A. Then the linear transformation $\tau = \tau_\chi(a)$ of A defined by*

$$\tau|_{A^a_{\mathfrak{F}_g}} := \chi(g)\mathrm{Id}_{\mathrm{A}^{\mathrm{a}}_{\mathfrak{F}_{\mathrm{g}}}}$$

for all $g \in G$, is an automorphism. Furthermore, if A is Frobenius, then τ also preserves the form.

Proof. From the fusion rules, we have that $A_{\mathfrak{F}_g} A_{\mathfrak{F}_h} \subseteq A_{\mathfrak{F}_{gh}}$ for any $g, h \in G$. That is, for any two eigenvectors x and y of $\mathrm{ad}(a)$, we have $x^\tau y^\tau = (\chi(g)x)(\chi(h)y) = \chi(g)\chi(h)xy = \chi(gh)xy = (xy)^\tau$. Since A is spanned by eigenvectors, then τ preserves the algebra product. Thus, τ is an automorphism of the algebra A. If A is Frobenius, then by Lemma 2.1.7 we have that τ preserves the form. □

Example 2.1.13.

If we consider $\mathbb{Z}/2\mathbb{Z}$-graded fusion rules $\mathfrak{F}$, then $\mathfrak{F} = \mathfrak{F}_- \cup \mathfrak{F}_+$. So the automorphism τ in Proposition 2.1.12 is as follows

$$\tau = \begin{cases} \mathrm{id} & \text{on } A^a_{\mathfrak{F}_+}, \\ -\mathrm{id} & \text{on } A^a_{\mathfrak{F}_-}, \end{cases}$$

and has order at most 2.

The automorphisms τ of order two as in Example 2.1.13 are called the *Miyamoto involu-*

tions.

2.2 Monster and Majorana algebras

The largest sporadic simple group, denoted by M, was first constructed by Robert Griess [12] as the group of automorphisms of a commutative non-associative real algebra V_M of dimension 196,884. M and V_M are known as the *Monster group* and the *Griess algebra* (sometimes called the *Monster algebra*), respectively.

In [2] J. H. Conway defined a particular idempotent in the Monster algebra V_M called $2A$-*axis* associating to the so called $2A$-*involution* in the Monster group M. S. Norton [29] classified all subalgebras of V_M generated by any two $2A$-axes, called *Norton-Sakuma algebras*. He proved that any subalgebra U of V_M generated by the $2A$-axes a_s and a_t corresponding to the $2A$-involutions s and t, respectively, is determined completely by the conjugacy class of the product st in M. Furthermore, the conjugacy class of st is one of the nine classes $1A, 2A, 3A, 4A, 5A, 6A, 2B, 4B$ and $3C$.

A Vertex Operator Algebra (VOA) $V^\natural$ was constructed by Frenkel, Lepowsky and Meurman [10], called the *Moonshine module*. From this point of view, M is the automorphism group of $V^\natural$.

Consider a real VOA $V = \oplus_{n=o}^{\infty} V_n$ such that $V_0 = \mathbb{R}1$ and $V_1 = 0$. Then the weight 2 subspace V_2 of V, called *Griess algebra*, coincides with V_M and has a structure of a commutative nonassociative algebra. In [27] M. Miyamoto showed that the automorphisms τ_a of V corresponding to the generators $a \in V_2$, called *Ising vectors*, are involutions. Note that the Ising vectors are multiples of the $2A$-axes while the corresponding Miyamoto involutions are the $2A$-involutions. It is remarked by S. Sakuma [30] that the order of the product of any two such involutions does not exceed six and he also noticed that any subalgebra of V_2 generated by two Ising vectors is isomorphic to a Norton-Sakuma algebra.

In 2009, A. A. Ivanov introduced the concept of Majorana algebras (see [19] and a refined version in [22]) as a Frobenius axial algebras A over the field $\mathbb{R}$ of real numbers such that the generators have length one, the bilinear form on A is positive definite and the Norton inequality is satisfied, that is, $\langle a \cdot a, b \cdot b \rangle \geq \langle a \cdot b, a \cdot b \rangle$ for any $a, b \in A$. From now on by M-axial algebras we mean Majorana algebras generated by a set of M-axes.

The definition of Majorana algebras was derived from the properties used in the Sakuma theorem, which classifies the subalgebras generated by two M-axes. All such subalgebras are based on the embedding into the Monster algebra. In [22] Ivanov *et al* proved that Sakuma's theorem also hold for Majorana algebras.

The following theorem is a version of Sakuma's theorem in terms of M-axial algebras as in [22].

Theorem 2.2.1. *There are exactly nine M-axial algebras generated by two M-axes.*

The above theorem is also true in a more general case, that is, the theorem is hold for any Frobenius algebras. This has been proved in [18].

The structure of the nine M-axial algebras mentioned in Theorem 2.2.1 and their dimensions are given in the following theorem in the language of M-axial algebras.

Theorem 2.2.2 (Sakuma)**.** *Let V be an M-axial algebra and U be a subalgebra of V generated by two M-axes a_0 and a_1. Then U is isomorphic to one the subalgebras as described in Table 2.3. Moreover, dim(U)≤ 8.*

Type	Basis	Products and angles
$2A$	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_\rho)$, $a_0 \cdot a_\rho = \frac{1}{2^3}(a_0 + a_\rho - a_1)$ $\langle a_0, a_1\rangle = \langle a_0, a_\rho\rangle = \langle a_1, a_\rho\rangle = \frac{1}{2^3}$
$2B$	a_0, a_1	$a_0 \cdot a_1 = 0$, $\langle a_0, a_1\rangle = 0$
$3A$	$a_{-1}, a_0, a_1,$ u_ρ	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_\rho$ $a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$ $u_\rho \cdot u_\rho = u_\rho$ $\langle a_0, a_1\rangle = \frac{13}{2^8}, \langle a_0, u_\rho\rangle = \frac{1}{4}, \langle u_\rho, u_\rho\rangle = \frac{8}{5}$
$3C$	a_{-1}, a_0, a_1	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1})$, $\langle a_0, a_1\rangle = \frac{1}{2^6}$
$4A$	$a_{-1}, a_0, a_1,$ a_2, v_ρ	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 + a_2 + a_{-1} - 3v\rho)$ $a_0 \cdot v_\rho = \frac{1}{2^4}(5a_0 - 2a_1 - a_2 - 2a_{-1} + 3v\rho)$ $v_\rho \cdot v_\rho = v_\rho$, $a_0 \cdot a_2 = 0$ $\langle a_0, a_1\rangle = \frac{1}{2^5}$, $\langle a_0, a_2\rangle = 0$, $\langle a_0, v_\rho\rangle = \frac{3}{2^3}$, $\langle v_\rho, v_\rho\rangle = 2$
$4B$	$a_{-1}, a_0, a_1,$ a_2, a_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$ $a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho^2})$ $\langle a_0, a_1\rangle = \frac{1}{2^6}$, $\langle a_0, a_2\rangle = \langle a_0, a_\rho\rangle = \frac{1}{2^3}$
$5A$	$a_{-2}, a_{-1}, a_0,$ a_1, a_2, w_ρ	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$ $a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$ $a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$ $w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$ $\langle a_0, a_1\rangle = \frac{3}{2^7}$, $\langle a_0, w_\rho\rangle = 0$, $\langle w_\rho, w_\rho\rangle = \frac{5^3 \cdot 7}{2^{19}}$
$6A$	$a_{-2}, a_{-1}, a_0,$ $a_1, a_2, a_3,$ a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho^2}$ $a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3}), a_{\rho^3} \cdot u_{\rho^2} = 0$ $\langle a_0, a_1\rangle = \frac{5}{2^8}$, $\langle a_0, a_2\rangle = \frac{13}{2^8}$, $\langle a_0, a_3\rangle = \frac{1}{2^3}$, $\langle a_{\rho^3}, u_{\rho^2}\rangle = 0$

Table 2.3: Norton-Sakuma algebras

In the following table, we present the λ-eigenvectors of the M-axis a_0, where $\lambda \in \{0, \frac{1}{4}, \frac{1}{32}\}$.

Type	0	$\frac{1}{4}$	$\frac{1}{32}$
$2A$	$a_1 + a_\rho - \frac{1}{2^2}a_0$	$a_1 - a_\rho$	
$2B$	a_1		
$3A$	$u_\rho - \frac{2\cdot5}{3^3}a_0 + \frac{2^5}{3^3}(a_1 + a_{-1})$	$u_\rho - \frac{2^3}{3^2\cdot5}a_0 - \frac{2^5}{3^2\cdot5}(a_1 + a_{-1})$	$a_1 - a_{-1}$
$3C$	$a_1 + a_{-1} - \frac{1}{2^5}a_0$		$a_1 - a_{-1}$
$4A$	$v_\rho - \frac{1}{2}a_0 + 2(a_1 + a_{-1}) + a_2,\ a_2$	$v_\rho - \frac{1}{3}a_0 - \frac{2}{3}(a_1 + a_{-1}) - \frac{1}{3}a_2$	$a_1 - a_{-1}$
$4B$	$a_1 + a_{-1} - \frac{1}{2^5}a_0 - \frac{1}{2^3}(a_{\rho^2} - a_2)$, $a_2 + a_{\rho^2} - \frac{1}{2^2}a_0$	$a_2 - a_{\rho^2}$	$a_1 - a_{-1}$
$5A$	$w_\rho + \frac{3}{2^9}a_0 - \frac{3\cdot5}{2^7}(a_1 + a_{-1}) - \frac{1}{2^7}(a_2 + a_{-2})$, $w_\rho - \frac{3}{2^9}a_0 + \frac{1}{2^7}(a_1 + a_{-1}) + \frac{3\cdot5}{2^7}(a_2 + a_{-2})$	$w_\rho + \frac{1}{2^7}(a_1 + a_{-1} - a_2 - a_{-2})$	$a_1 - a_{-1}$, $a_2 - a_{-2}$
$6A$	$u_{\rho^2} + \frac{2}{3^2\cdot5}a_0 - \frac{2^8}{3^2\cdot5}(a_1 + a_{-1}) - \frac{2^5}{3^2\cdot5}(a_2 + a_{-2} + a_3 - a_{\rho^3})$, $a_3 + a_{\rho^3} - \frac{1}{2^2}a_0$, $u_{\rho^2} - \frac{2\cdot5}{3^3}a_0 + \frac{2^5}{3^3}(a_2 + a_{-2})$	$u_{\rho^2} - \frac{2^3}{3^2\cdot5}a_0 - \frac{2^5}{3^3\cdot5}(a_2 + a_{-2} + a_3 - a_{\rho^3})$, $a_3 - a_{\rho^3}$	$a_1 - a_{-1}$, $a_2 - a_{-2}$

Table 2.4: Eigenvectors of a_0

By an M-axial algebra A of a finite group G generated by a normal set T of involutions is a map that sends each $t \in T$ to an M-axis a_t such that $A = \langle\langle a_t \mid t \in T \rangle\rangle$ and G acts on the indices of the axes a_t by conjugation.

The *shape* of an M-axial algebra A of a finite group G is a rule which prescribes the type of Norton-Sakuma algebras generate by any two M-axes and respect the inclusion between the algebras.

For the groups $S_4, S_5, S_6, S_7, 3.S_6, 3.S_7, A_5, A_6, A_7, 3.A_6, 3.A_7, L_2(11), L_3(2), L_3(3), (S_4 \times S_3) \cap A_7$ and M_{11}, the M-axial algebras have been determined for all or almost all shapes (see Table 2.5). The computer algebra system GAP [11] had been used for this purpose.

Note that in Table 2.5, the shapes with the removed line in between for the group S_4 lead to the same algebra.

Group	$\|T\|$	Shape	Dimension	Reference(s)
S_4	6	$(2B, 3A)$	13	[22]
S_4	6	$(2B, 3C)$	6	[22]
S_4	6	$(2A, 3A)$	13	[25]
	6+3	$(2A, 2A, 2A, 3A, 4B)$		[22]
S_4	6	$(2A, 3C)$	9	[31, 25]
	6+3	$(2A, 2A, 2A, 3C, 4B)$		[22]
S_4	6+3	$(2B, 2B, 2B, 3C, 4A)$	12	[31, 25]
S_4	6+3	$(2B, 2B, 2B, 3A, 4A)$	25	[25]
S_4	6+3	$(2A, 2A, 2B, 3C, 4B)$	12	[25]
S_4	6+3	$(2A, 2A, 2B, 3A, 4B)$	16	[25]
S_4	6+3	$(2B, 2B, 2A, 3A, 4A)$	0	[25]
S_4	6+3	$(2B, 2B, 2A, 3C, 4A)$	0	[25]
$L_3(2)$	21	$(2A, 3C, 4B)$	21	[24]
$L_3(2)$	21	$(2A, 3A, 4B)$	49	[24]
$L_3(3)$	117	$(2A, 3AC, 4B)$	144	[31]
M_{11}	165	$(2A, 3A, 4B)$	286	[31]
$L_2(11)$	55	$(2A, 3A)$	101	[6, 31]
A_5	15	$(2A, 3C)$	20	[23]
A_5	15	$(2A, 3A)$	26	[23]
A_5	15	$(2B, 3C)$	21	[31]
S_5	15+10	$(2A, 3A, 4B)$	36	[31]
S_6	45+15	$(2AB, 3A, 4B)$	91	[31]
A_6	45	$(2A, 3A, 4B)$	76	[21, 31]
A_6	45	$(2A, 3C, 4B)$	70	[20, 31]
$3.A_6$	45	$(2A, 3A, 4B)$	76	[31]
$3.A_6$	45	$(2A, 3C, 4B)$	70	[31]
$3.A_6$	45	$(2A, 3AC, 4B)$	105	[31]
$3.S_6$	45+15	$(2A, 3AC, 4B)$	136	[31]
A_7	105	$(2A, 3A, 4B)$	196	[21, 31]
S_7	105+21	$(2AB, 3A, 4B)$	217	[31]
$3.A_7$	105	$(2A, 3A, 4B)$	196	[31]
$3.A_7$	105	$(2A, 3AC, 4B)$	211	[31]
$3.S_7$	105+21	$(2AB, 3AC, 4B)$	254	[31]
$(S_4 \times S_3) \cap A_7$	18+3	$(2A, 3A, 4B)$	30	[31]

Table 2.5: Known M-axial algebras

2.3 Axial algebras of Jordan type

Let A be an J-axial algebra, where $J = \{1, 0, \alpha\}$ with $0 \neq \alpha \neq 1$, with the fusion rules as described in Table 2.1. This kind of algebra known as a *primitive axial algebra of Jordan type* α and it has been studied in [17]. For a special value of α, namely $\alpha = \frac{1}{2}$, it is called a *Jordan algebra*. The most interesting feature of axial algebras A of Jordan type $\alpha \neq \frac{1}{2}$ over a field $\mathbb{F}$ of characteristic not equal to two is that the set of Miyamoto involutions corresponding to the axes form a normal set of 3-transpositions in the group they generate. As a consequence of this, if the algebra A is finitely generated, then it is finite dimensional.

Also in [17], the automorphism groups of axial algebras of Jordan type α that are generated by Miyamoto involutions have been discussed in order to examine the type of the dihedral subgroups generated by two Miyamoto involutions. The following is the main result.

Proposition 2.3.1. *Let a, b be two axes in the axial algebra A of Jordan type α. Suppose $\langle\langle a, b\rangle\rangle$ be the subalgebra of type NX, where $N \in \{1, 2, 3\}, X \in \{A, B, C\}$, with τ_a and τ_b the corresponding Miyamoto involutions to a and b, respectively. Then*

1. $|\tau_a\tau_b| = 1$ *if* $NX = 1A$;
2. $|\tau_a\tau_b| = 2$ *if* $NX = 2B$;
3. $|\tau_a\tau_b| = 3$ *if* $NX = 3C$.

2.4 Automorphisms of M-axial algebras

In this section we consider the fusion rules M. We try to understand the automorphism groups of M-axial algebras generated by Miyamoto involutions. More precisely, we describe the dihedral subgroups generated by two Miyamoto involutions.

In the more general situation of M-axial algebras, the order of the product of any two Miyamoto involutions still does not exceed six. There is detail in the next theorem.

Theorem 2.4.1. *Suppose A is an M-axial algebra and a and b are axes in A. If $B := \langle\langle a, b\rangle\rangle$ is the Sakuma algebra of type NX, then the order of the element $\tau_a\tau_b$ in $Aut(A)$ is either N or $N/2$.*

Proof. Suppose that D is a subgroup of $Aut(A)$ generated by τ_a and τ_b. Then D acts on B. If one of the τ_a and τ_b is trivial, say τ_a, then $|\tau_a\tau_b| = |\tau_b|$ which is at most 2. If τ_a and τ_b are different and commute, then $|\tau_a\tau_b| = 2$, and D is an abelian dihedral subgroup. We next assume that τ_a and τ_b are different nontrivial involutions such that $\tau_a\tau_b \neq \tau_b\tau_a$. Consider the homomorphism $\varphi : D \to Aut(\langle\langle a, b\rangle\rangle)$.

Let $x \in Ker\varphi$. Then x fixes a and b and hence it centralizes τ_a and τ_b. Therefore $x \in Z(D)$. Then $Ker\varphi$ lies in the center of D, so it has size at most 2.

Here we have two cases to consider. If N is odd, then D acts transitively on the axes in B. Hence τ_a and τ_b are conjugate in D, which means that $|Ker\varphi| = 1$, so $|\tau_a\tau_b| = N$. Thus D is the dihedral group of order $2N$. If N is even, then $|\tau_a\tau_b| = N/2$ and D is the dihedral group of order N . □

We denote the automorphism group of the algebras of type NX by G_{NX}. With the information in the above theorem we have the following.

Corollary 2.4.2. *The group G_{NX} is one of the following:*

1. $G_{2Y} \cong C_2$ *for* $Y \in \{A, B\}$,
2. $G_{3Z} \cong S_3$ *for* $Z \in \{A, C\}$,
3. $G_{4W} \cong C_2 \times C_2$ *for* $W \in \{A, B\}$,
4. $G_{5A} \cong D_{10}$,
5. $G_{6A} \cong S_3$.

2.5 $\mathfrak{F}$-Axial representations

In this section we introduce the notion of axial representations for a finite group G which aims to describe $\mathfrak{F}$-axial algebras A invariant under G. One of the mysteries behind it is the classification of subalgebras of A invariant under subgroups of G, which helps us to perform calculations more precisely on subgroups rather than the whole of G.

Definition 2.5.1. *Let G be a finite group generated by an invariant set T of involutions which is the union of some conjugacy classes of G. Let A be an $\mathfrak{F}$-axial algebra generated by a set X of $\mathfrak{F}$-axes in which G acts on. Then (A, X) is called an* axial representation *of (G, T) if there is a linear representation $\varphi : G \to GL(A)$ and a map $\tau : X \to T$ such that $\tau(x^g) = (\tau(x))^{\varphi(g)}$ for all $x \in X$ and $g \in G$.*

Note that the map τ is not required to be injective.

2.6 Some Well Known Results of Group Theory

In order to find the group structure of some groups discussed in Chapter 4, we make use of two well known theorems of group theory, which are Burnside's $p^a \cdot q^b$ Theorem and Schur-Zassenhaus Theorem. The main reference of this section is [1].

Theorem 2.6.1 (Burnside's $p^a \cdot q^b$ Theorem)**.**

Let p, q be two distinct primes and $a, b \in \mathbb{N}$. Any group of order $p^a \cdot q^b$ is solvable.

Theorem 2.6.2 (Schur-Zassenhaus Theorem)**.**

Let G be a finite group, let $H \trianglelefteq G$ and assume

(1) $(|H|, |G/H|) = 1$*, and*

(2) *either H or G/H is solvable.*

Then

(1) *G splits over H, and*

(2) *G is transitive on the complements to H in G.*

Chapter 3

On Abelian Subgroups of 6-transposition Groups

3.1 General setup

Recall that a pair (G, D) is called a 6-transposition group if G is a finite group and D is a normal generating set of involutions in G such that for any $d, e \in D$ the order of de is at most 6.

Suppose that A is an M-axial algebra corresponding to (G, D). We start with a subgroup E of G which is an elementary abelian 2-group, such that $E = \langle E \cap D \rangle$, and we denote the corresponding subalgebra by $B := A_E$.

Assume that there is a bijection between $R := E \cap D = \{r_1, ..., r_n\}$ and the set I of M-axes generating B. Since the elements of R commute, then by Sakuma's theorem any two of these M-axes generate a subalgebra of type $2A$ or $2B$.

From this point on we assume the $(2A)$-condition which states that: *For distinct com-*

muting involutions $t_0, t_1 \in D$, the subalgebra generated by the corresponding M-axes a_0 and a_1 is of type $2B$ if $t_0t_1 \notin D$, and it is of type $2A$ otherwise. Furthermore, in the latter case, the vector, a_2, corresponding to $t_2 = t_0t_1$ lies in the subalgebra generated by a_0 and a_1.

3.2 M-Axial Algebras for E

Let $I_B = \{b_1, ..., b_n\}$ be the set of M-axes corresponding to $R = \{r_1, ..., r_n\}$. Then the subalgebra B is spanned by I_B. From Sakuma's theorem and the $(2A)$-condition we can find the product $b_i \cdot b_j$ and the value of $\langle b_i, b_j\rangle$, which are

$$b_i \cdot b_j = \begin{cases} b_i & \text{if } i{=}j, \\ \frac{1}{8}(b_i + b_j - b_k) & \text{if } r_ir_j = r_k \in R, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\langle b_i, b_j\rangle = \begin{cases} 1 & \text{if } i{=}j, \\ \frac{1}{8} & \text{if } r_ir_j \in R, \\ 0 & \text{otherwise.} \end{cases}$$

Here we can see that for any M-axis b in I_B, the eigenspace $B^b_{\frac{1}{32}}$ is equal to zero. This means that the corresponding involution r to b acts trivially on B. Therefore, we get σ-involutions, where

$$\sigma_b = \begin{cases} id & \text{on } B_1^b \oplus B_0^b, \\ -id & \text{on } B_{\frac{1}{4}}^b, \end{cases}$$

rather than τ-involutions, and σ_b only acts on B and it does not extend to act on the algebra A.

In [27], Miyamoto showed that the set of such σ involutions generates a 3-transposition group. So, we get the group $(H, \hat{R})$ of 3-transpositions, where $\hat{R}$ is the set of all $\sigma-$ *involutions* and $H = \langle \hat{R} \rangle$.

3.3 Fischer Space on $\hat{R}$

For the group $(H, \hat{R})$, we have the Fischer space Π on the set $\hat{R}$ associated with it. In Π, two involutions σ_{b_i} and σ_{b_j} are collinear if they do not commute and the line through them is $\{\sigma_{b_i}, \sigma_{b_j}, \sigma_{b_k}\}$, where $\sigma_{b_k} = \sigma_{b_i}^{\sigma_{b_j}} = \sigma_{b_j}^{\sigma_{b_i}}$.

Also, in the Fischer space if two lines intersect in a point, then they lie in a plane and the planes are either affine planes of order 3 or dual affine planes of order 2.

Now corresponding to the group $(H, \hat{R})$, there is a free algebra $\tilde{B}$ with the basis $\{\tilde{b}_r\}_{r\in\hat{R}}$ such that B is the quotient of $\tilde{B}$ if $\tilde{B}$ has an ideal, otherwise they will be isomorphic.

We define the product and the form on $\tilde{B}$ as follows

$$\tilde{b_{\sigma_i}} \cdot \tilde{b_{\sigma_j}} = \begin{cases} \tilde{b_{\sigma_i}} & \text{if } i{=}j, \\ \frac{1}{8}(\tilde{b_{\sigma_i}} + \tilde{b_{\sigma_j}} - \tilde{b_{\sigma_k}}) & \text{if } \{\sigma_i,\, \sigma_j,\, \sigma_k\} \text{ is a line,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\langle \tilde{b_{\sigma_i}}, \tilde{b_{\sigma_j}} \rangle = \begin{cases} 1 & \text{if } i{=}j, \\ \frac{1}{8} & \text{if } \sigma_i \text{ and } \sigma_j \text{ are collinear,} \\ 0 & \text{otherwise.} \end{cases}$$

Now we have the following theorem

Theorem 3.3.1. *$\tilde{B}$ is an M-axial algebra of shape only involving $2A$ and $2B$. Furthermore, B is a factor algebra of $\tilde{B}$.*

From $(2A)$-condition we can see that the map $\varphi : \hat{R} \to E$ which send σ_{b_i} to r_i defines an embedding of Π into E, viewed as a vector space over $GF(2)$.

Before we find the embedding of Π into E, we require to give the definition of the embedding of the Fischer space into the vector space over $GF(2)$ as introduced in [13].

Definition 3.3.2. *Suppose the Fischer space $\Pi = (D, L(D))$, the embedding of Π into a vector space E over $GF(2)$ is the map $\varphi : D \to E$ such that φ maps D injectively into $E - \{0\}$ and $\varphi(a) + \varphi(b) + \varphi(c) = 0$ whenever $\{a, b, c\}$ is a line in $L(D)$.*

From now we denote the affine plane of order 3 by Σ and the dual affine plane of order 2 by Δ.

The following lemma gives the description of the Fischer spaces that have an embedding in E. Note that in the proof of Lemma 3.3.3, the notation $a \bullet b$ means the line through a and b.

Lemma 3.3.3. *The affine plane of order 3, Σ, does not have an embedding.*

Proof. Suppose by contradiction that Σ has an embedding φ into T. Recall that any two points of Σ are collinear. Select distinct points $a, b, c \in \Sigma$ such that $c \notin a \bullet b$. Then the lines $a \bullet b$ and $a \bullet c$ are distinct. Let x be the third point on $a \bullet b$ and, similarly, y be the third point on $a \bullet c$. We define the subspace $T_0 := \langle \varphi(a), \varphi(b), \varphi(c) \rangle$. There are two cases to consider. First, if $T_0 = T$, then in T_0, we have $\varphi(a) + \varphi(b) + \varphi(x) = 0$ and $\varphi(a) + \varphi(c) + \varphi(y) = 0$. Since Σ has nine points and T_0 has seven points, then by pigeonhole principle, $\varphi(t) = \varphi(s)$ for distinct points t and s. Let z be the third point on $t \bullet s$. Then $\varphi(z) = \varphi(t) + \varphi(s) = 0$; a contradiction.

The second case is where $T_0 \lneqq T$. T_0 is spanned by the image two intersecting lines. The dimension of the intersection of these two lines can not be 2 because b and c are distinct and so it is 1-dimensional. Thus, T_0 is 3-dimensional. Since T_0 is a subspace over $GF(2)$, then it has seven points.

To show that each point of T_0 is the image. Let d be a point such that $d \notin a \bullet b$ and $d \notin a \bullet c$. Through d, we have one line $a \bullet d$, one line parallel to $a \bullet b$, and one line parallel to $a \bullet c$. Since altogether there are four lines through d, we conclude that there is a line through d that does not contain a, meets with $a \bullet b$, and meets with $a \bullet c$. Suppose this line meets with $a \bullet b$ in the point s, and it meets $a \bullet c$ in the point t. Since $\varphi(a) + \varphi(b) = \varphi(s)$ and $\varphi(a) + \varphi(c) = \varphi(t)$, then T_0 contains the image of s and t and so also contains the image of d. By the same argument for each line, we conclude that every element in T_0 is the image.

Take another line through a, say $a \bullet e$, distinct from $a \bullet b$, $a \bullet c$ and $a \bullet d$. Let the subspace $T_1 := \langle \varphi(a), \varphi(b), \varphi(e) \rangle$. By the same argument for T_0, T_1 also has seven points. Since T_0 and T_1 share in a line, then the number of points of T_0 and T_1 in T is $7 + 7 - 3 = 11$ and each of them is the image of a point in Σ which has nine points, this contradicts the injectivety of φ. $\square$

The direct result from above lemma is the following.

Corollary 3.3.4. *Every plane of* Π *is a dual affine plane of order 2.*

Such Fischer spaces and 3-transposition groups are said to be of *symplectic type*. Examples are S_n for $n \geq 3$, $O^+_{2n}(2)$ for $n \geq 4$, $O^-_{2n}(2)$ for $n \geq 3$ and $Sp_{2n}(2)$ for $n \geq 3$ [26].

Suppose that (G, D) is a 3-transposition group. For any $d \in D$, denote $D_d = \{e \in D \mid ed = de, e \neq d\}$, $A_d = \{e \in D \mid ed \neq de\}$, $d\tau e$ if and only if $A_d = A_e$, $d\theta e$ if and only if $D_d = D_e$, $\tau(G) = [O_2(G), G]$, $\theta(G) = [O_3(G), G]$ (see [9]) and $\rho(G) = \tau(G)\theta(G)$. The 3-transposition group (G, D) is called *irreducible* if $\rho(G) = \tau(G) = \theta(G) = 1$ [5].

It has been shown in [5] that any finite irreducible 3-transposition group of symplectic type is isomorphic to one of the following: $O^\epsilon_{2n}(2)$ for $\epsilon = \pm$ and $n \geq$ but $(n, \epsilon) \neq (2, +)$, $Sp_{2n}(2)$ for $n \geq 3$, or S_n for $n \geq 5$.

Hall in [13] described the full embedding of these spaces into $GF(2)$-vector space V.

Our remaining results describe the embedding of the reducible symplectic spaces.

3.4 Embedding of reducible symplectic spaces into a $GF(2)$-vector spaces

In this section, we find the embedding of the reducible symplectic spaces into a $GF(2)$-vector spaces. First, we list some examples of 3-transposition groups of symplectic type and the dimension of their embeddings in the following table.

G	$\|D\|$	Dimension
S_n	$\frac{n(n-1)}{2}$	$n-1$
$2^4 : S_5$	20	5
$2^6 : S_7$	42	7
$2^6 : S_8$	56	8
$2^8 : S_9$	72	9
$2^8 : S_{10}$	90	10
$O_6^-(2)$	36	6
$O_{10}^+(2)$	496	10
$2^6 : O_6^-(2)$	72	7
$2^8 : O_8^-(2)$	272	9
$2^8 : O_8^+(2)$	240	9
$Sp_6(2)$	63	7
$Sp_8(2)$	255	9
$Sp_{10}(2)$	1023	11
$2^6 : Sp_6(2)$	126	8
$2^8 : Sp_8(2)$	510	10

In [13], Hall proved that if φ is the embedding of the Fischer space $\Delta = (D, L(D))$, then Δ has irreducible subspace $\Delta^* = (D^*, L(D)^*)$ and V has a subspace W intersecting the span of $\varphi(D^*)$ trivially such that $\varphi(\Delta)$ can be constructed from W and Δ^*. W called the radical part of φ. This will help us to prove Theorem 3.4.2.

Hall in [14] and [15] showed that an indecomposable 3-transposition group with trivial center of symplectic type is isomorphic to the extension of one of the groups $O_{2n}^{\epsilon}(2)$ for $\epsilon = \pm$ and $n \geq$ but $(n, \epsilon) \neq (2, +)$, $Sp_{2n}(2)$ for $n \geq 3$, and S_n for $n \geq 3$ by the direct sum of copies of the natural module.

In this situation, the natural module is isomorphic to 2^{2n}. For the group S_n we exclude the case $n = 4$ because $S_4 = 2^2 : S_3$.

For all of these cases, $\rho(G) \neq 1$. Then one of $\tau(G)$ and $\theta(G)$ is not 1 [16]. The next lemma will show the case that $\theta(G) = 1$.

Lemma 3.4.1. $\theta(G) = 1$.

Proof. By contradiction, suppose that $\theta(G) \neq 1$. Then there exist $a, b \in D$ such that $D_a = D_b$. Since $a \notin D_b$ and $b \notin D_a$, then $|ab| = 3$. Thus, there is a line $a \bullet b$ through a and b with $c \in a \bullet b$. Since $D_a = D_b = D_c$, then c is also θ-equivalent. Assume that there is another line through a, say $a \bullet d$. So $D_a = D_d$. Then d has to be collinear with both b and c because $D_d = D_a = D_b = D_c$. Thus any two distinct points are collinear. Therefore, we have an affine plane of order 3, which contradicts the Lemma 3.3.3. $\square$

The consequence for the above lemma is, for a reducible cases we have $\rho(G) = \tau(G)$.

We need to keep in mind the following isomorphic groups: $O_2^+(2) \cong Z_2$, $O_2^-(2) \cong Sp_2(2) \cong S_3$, $O_4^-(2) \cong S_5$, $O_4^+(2) \cong S_3 \times S_3$, $S_6 \cong Sp_4(2)$ and $O_6^+(2) \cong S_8$.

Theorem 3.4.2. *Assume that $G = E : O_{2n}^{\epsilon}(2)$, where E is a direct sum of k copies of the $2n$-dimensional natural module over $GF(2)$ for $O_{2n}^{\epsilon}(2)$, $\epsilon = \pm$, and $n \geq 2$ but $(n, \epsilon) \neq (2, +)$ or $(3, +)$. Then Δ has an embedding of dimension $2n + k$.*

Proof. By induction on k. Suppose that $k = 1$, then $G = 2^{2n} : O_{2n}^{\epsilon}(2) = \langle D \rangle$, where D is the set of 3-transpositions. Let Δ be the associated Fisher space to G and φ be the embedding of Δ in a $GF(2)$-space V. Here $\tau(G) = 2^{2n}$, and then $\bar{G} = G/\tau(G) \cong O_{2n}^{\epsilon}(2)$. Assume that $\bar{\Delta}$ is the associated Fischer space to $\bar{G}$. Then $\bar{\Delta}$ has an embedding $\bar{\varphi}$ of dimension $2n$ [13]. Let $\bar{G} = \langle \bar{D} \rangle$ and define the natural homomorphism $f : G \to \bar{G}$. Let $a \in \bar{D}$. Then $[2^{2n} : C_{2^{2n}}(a)] = 2$, that is, the fiber of a is of size 2 in D. Let $f^{-1}(a) = \{r, s\}$.

Then r and s are τ-equivalent. It means that all τ-classes have size 2. Thus, the size of D is precisely twice as $\bar{D}$. Let $W = \{r + s \mid r\tau s; r, s \in D\}$. Then W is a subspace of V by Lemma 1.3 in [13]. Thus, the point set D of Δ is $\{p + w \mid p \in \bar{D}, w \in W\}$ and the line set $L(D)$ is $\{\{a, b, c\}, \{a + w, b + w, c\}, \{a + w, b, c + w\}, \{a, b + w, c + w\} \mid w \in W, \{a, b, c\} \in L(\bar{D})\}$. In the Fischer space Δ, $x\tau y$ if and only if $x \in y + W$ and all τ-classes have size 2. Then Δ/τ is isomorphic to $\bar{\Delta}$ by the natural projection of τ-classes. This means that by adding only one point we obtain the full embedding of Δ. Therefore, the dimension of the embedding φ is $2n + 1$.

Suppose that the theorem is true for $k - 1$, that is, if E_{-1} is the direct sum of $k - 1$ copies of $2n$-dimensional natural model for $O^{\epsilon}_{2n}(2)$, then Δ has an embedding of dimension $2n + k - 1$. Let $H = E_{-1} : O^{\epsilon}_{2n}(2)$. Then by the hypothesis, the corresponding Fischer space of H has an embedding of dimension $2n + k - 1$. Suppose that G is the extension of H by a natural module 2^{2n}, that is $G = 2^{2n} : H$. By part 1 of the proof we obtain that one extension of H implies one extra dimension of the embedding in $GF(2)$-space. Therefore Δ has an embedding of dimension $2n + k$ in $GF(2)$-space V. □

By the same argument, the following can be proved.

Theorem 3.4.3. *Assume that $G = E : Sp_{2n}(2)$, where E is a direct sum of k copies of the $2n$-dimensional natural module over $GF(2)$ for $Sp_{2n}(2)$ for $n \geq 2$. Then Δ has two embeddings of dimension $2n + k$ and the other of dimension $2n + 1 + k$ for $O_{2n+1}(2)$.*

Theorem 3.4.4. *Assume that $G = E : S_m$, where E is a direct sum of k copies of the $2n$-dimensional natural module over $GF(2)$ for S_m for $m \geq 3$ and $m = 2n+1$ or $m = 2n+2$. Then Δ has an embedding of dimension $m - 1 + k$.*

Chapter 4

The 3-generated 4-transposition Groups

4.1 Motivation

As we mentioned in the introduction that the main aim of this thesis is to classify all 3-generated M-axial algebras A such that every 2-generated subalgebra of A is a Sakuma algebra of type NX, where $N \in \{2, 3, 4\}$ and $X \in \{A, B, C\}$.

The algebras A are invariant under the group G generated by three Miyamoto involutions corresponding to the generators of the algebra A. By Theorem 2.4.1 the order of the product of any pair of these Miyamoto involutions does not exceed 4. Hence, to accomplish the main goal of the thesis, which is the classification of all 3-generated M-axial algebras not including the Sakuma subalgebras $5A$ and $6A$, we require to classify all groups satisfying the following property.

Property (Δ):

1. G is generated by three involutions a, b and c.

2. The order of the product of any two distinct elements in $T := a^G \cup b^G \cup c^G$ is at most 4.

Remark 4.1.1. *We do not assume that the three generators a, b and c of the group G in the above definition are distinct.*

4.2 Main result

In this section, we present the main result of the chapter which is the classification of all groups satisfying (Δ). We notice from condition 2 of (Δ) that the order of the product of any two of generators of G be either 1, 2, 3 or 4. The group G is a factor group of the group $T^{(s_1,s_2,s_3)}$ given by the presentation $\langle a, b, c \mid a^2, b^2, c^2, (ab)^{s_1},\ (ac)^{s_2}, (bc)^{s_3}\rangle$, where $s_i \in \{2, 3, 4\}$ for all i. If one of the s_i's is equal to 1, then we are in a situation which Sakuma considered, which is hence not of interest to us, and so we skip this case.

In principle, the value $s_i = 2$, for all i, also can be skipped because this subcase is covered by the case $s_i = 4$. However, we keep it in this chapter because they lead to groups which appears as subgroups of bigger groups, so we get an idea about what the bigger groups look like.

It is well known which of the groups $T^{(s_1,s_2,s_3)}$ are finite. In Section 4.3, we summarise all the finite groups $T^{(s_1,s_2,s_3)}$ and show that they all satisfy property (Δ). Which means also that all their quotients are of interest of us. For the infinite cases, we introduce further four relations $R_i^{r_i} = 1$ for $i = 1, 2, 3, 4$, where

$$R_1 := a \cdot b^c, R_2 := a \cdot c^b, R_3 := b \cdot c^a \text{ and } R_4 := a^c \cdot a^b.$$

They came from the generators of G and their conjugates. In view of (Δ), we must have that $r_i \in \{1, 2, 3, 4\}$ for all i. In Sections 4.4 and 4.5, we find the list of finite groups which are quotients of infinite groups $T^{(s_1,s_2,s_3)}$ by some or all of the four relations $R_i^{r_i} = 1$ and we determine which of them satisfy property (Δ). In some of the cases we have to introduce further relations to make the group finite. The main result of this chapter is Theorem 4.2.1 and it will be proved in Sections 4.4, 4.5 and 4.6.

Recall that the notion $B(2, 4)$ in Table 4.1 refers to the Burnside group of rank 2 and exponent 4.

Theorem 4.2.1. *A group satisfies property* (Δ) *if and only if it is a quotient of at least one of the groups in Table 4.1.*

Groups	Isomorphism Type	(s_1, s_2, s_3)	$(r_1, r_2, r_3, r_4; r_5, r_6)$	Group Order
T_1	$(4 \times 2^2) : 2$	$(4, 4, 4)$	$(4, 4, 4, 4; 3, 4)$	32
T_2	$3_+^{1+2} : 2$	$(3, 3, 3)$	$(3, -, -, -; -, -)$	54
T_3	$4^2 : S_3$	$(3, 3, 3)$	$(4, -, -, -; -, -)$	96
T_4	$2 \times L_3(2)$	$(3, 3, 4)$	$(4, -, -, -; -, -)$	336
T_5	$(((2 \times D_8) : 2) : 3).2) : 2$	$(3, 4, 4)$	$(3, 4, -, -; -, -)$	384
	$= (2.(((2^4) : 3) : 2)$			
T_6	$(S_4 \times S_4) : 2$	$(3, 4, 4)$	$(4, 4, -, -; 3, -)$	1152
T_7	$(((3 \times ((3^2) : 3)) : 3) : Q_8) : 2$	$(4, 4, 4)$	$(4, 4, 4, 3; 3, -)$	3888
T_8	$B(2, 4) : 2$	$(4, 4, 4)$	$(4, 4, 4, 4; 4, 4)$	8192

Table 4.1: Largest 3-generated 4-transposition groups

Let us give a brief explanation of how to read Table 4.1. The group T_4 is a quotient of $T^{(3,3,4)}$ by the relation $R_1^4 = (a \cdot b^c)^4 = 1$. It has order 336 and is isomorphic to $2 \times L_3(2)$.

4.3 The finite 4-transposition groups $T^{(s_1,s_2,s_3)}$

Our groups $T^{(s_1,s_2,s_3)}$ belong to the class of triangle groups, which in turn are a subclass of Coxeter groups. In particular, $T^{(s_1,s_2,s_3)}$ can be realized as a group generated by reflections in the sides of the triangle with angles $\frac{\pi}{s_1}, \frac{\pi}{s_2}, \frac{\pi}{s_3}$. There are three classes of triangle group, **Euclidean, Spherical and Hyperbolic**, depending on whether $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3}$ is equal to, greater than, or less than 1, respectively. Since the presentation for $T^{(s_1,s_2,s_3)}$ is symmetric in a, b and c, the isomorphism type of $T^{(s_1,s_2,s_3)}$ does not change for any permutation of $\{s_1, s_2, s_3\}$. Thus, we further assume $s_1 \leq s_2 \leq s_3$.

The standard criterion for the group $T^{(s_1,s_2,s_3)}$ to be finite is given in Theorem 4.3.1. The orders of some of the finite groups $T^{(s_1,s_2,s_3)}$ can be found in [4] and [8].

Theorem 4.3.1. *The group $T^{(s_1,s_2,s_3)}$ is finite if and only if it is of spherical type, that is,* $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} > 1$.

As $2 \leq s_1, s_2, s_3 \leq 4$, only the triples in the set $\Gamma := \{(2,2,2), (2,2,3), (2,2,4), (2,3,3), (2,3,4)\}$ lead to finite groups $T^{(s_1,s_2,s_3)}$.

Proposition 4.3.2. *The orders and isomorphism types of the groups $T^{(s_1,s_2,s_3)}$ for $(s_1, s_2, s_3) \in \Gamma$ are as described in Table 4.2.*

(s_1, s_2, s_3)	Group Order	Isomorphism Type	$\|cc\|$
$(2,2,2)$	8	2^3	1+1+1
$(2,2,3)$	12	D_{12}	1+3
$(2,2,4)$	16	$2 \times D_8$	1+2+2
$(2,3,3)$	24	S_4	6
$(2,3,4)$	48	$2 \times S_4$	6+3

Table 4.2: The finite 4-transposition groups $T^{(s_1,s_2,s_3)}$

Let us give a brief explanation of how to read Table 4.2. For example, the group $T^{(2,2,2)}$ is isomorphic to the elementary abelian group 2^3. The fourth column in the table, titled $|cc|$, represents the sizes of the conjugacy classes $\{a^{T^{(s_1,s_2,s_3)}}\} \cup \{b^{T^{(s_1,s_2,s_3)}}\} \cup \{c^{T^{(s_1,s_2,s_3)}}\}$. If the order of the product of two generators is odd, then they are conjugate and hence belong to the same conjugacy class. Because of this reason, sometimes we see only one number or two numbers in the fourth column.

Proof. The proof is based on Tietze transformations, we derive a relation from the existing relations to get an isomorphic presentation. We then use the information in Table 1 of [8] to get the orders and isomorphism types of the presentations.

We prove each case separately.

Case $(s_1, s_2, s_3) = (2, 2, 2)$

Since $(ab)^2 = (ac)^2 = (bc)^2 = 1$, then a, b and c are commute. Hence $T^{(2,2,2)} \cong 2^3$.

Case $(s_1, s_2, s_3) = (2, 2, 3)$

The group $T^{(2,2,3)}$ has order 12. Clearly a commutes with b and c and hence $T^{(2,2,3)} \cong \langle a \rangle \times \langle b, c \rangle \cong 2 \times D_6 \cong D_{12}$.

Case $(s_1, s_2, s_3) = (2, 2, 4)$

Similar to the case $(2, 2, 3)$, $T^{(2,2,4)} \cong \langle a \rangle \times \langle b, c \rangle \cong 2 \times D_8$.

Case $(s_1, s_2, s_3) = (2, 3, 3)$

It is clear from the structure of the presentation that $T^{(2,3,3)}$ is the Weyl group of the root system of type A_3, which is S_4.

Case $(s_1, s_2, s_3) = (2, 3, 4)$

The group $T^{(2,3,4)}$ is isomorphic to the group $G^{(3,4,6)} := \langle r, s, t \mid r^2, s^2, t^2, (rs)^2, (rt)^3,$

$(ts)^4, (rst)^6\rangle \cong 2\times S_4$ as in Table 1 in [8] via the isomorphism $a \mapsto t(rt)^2$, $b \mapsto s((st)^4)^{rt(st)^2}$, $c \mapsto t$. This was checked by GAP. □

Proposition 4.3.3. *The groups in Table 4.2 are all satisfy property* (Δ).

Proof. It can be seen from the structure of the groups that the order of the product of any two involutions does not exceed 4 except for the group D_{12}. In D_{12}, there are pairs of involutions their products have order 6. Let $\{a, b, c\}$ be the generating set of involutions in a presentation as that of $T^{(2,2,3)}$. Since a commutes with b and c, then the product of a with b and c or any conjugates to b and c has order 2. As the product bc has order three, then b and c are conjugate. So the product of b or any conjugate of b with c or any conjugate of c has order 3. Therefore, the group D_{12} satisfy (Δ). □

4.4 The 4-transposition quotients of the infinite groups $T^{(s_1,s_2,s_3)}$

In this section we try to find the largest finite quotients of the infinite groups $T^{(s_1,s_2,s_3)}$, that satisfy (Δ) for $(s_1, s_2, s_3) \notin \Gamma$. Recall that $2 \le s_1 \le s_2 \le s_3 \le 4$. Then the groups $T^{(s_1,s_2,s_3)}$ are infinite if $(s_1, s_2, s_3) \in \Lambda := \{(2, 4, 4), (3, 3, 3), (3, 3, 4), (3, 4, 4), (4, 4, 4)\}$. We can exclude $(2, 4, 4)$ because $T^{(2,4,4)}$ is a factor group of $T^{(4,4,4)}$. To achieve our goal, we need to introduce some extra relations on the generators of $T^{(s_1,s_2,s_3)}$ to reduce it to a finite group. First, let us recall the relation $R_1^{r_1} = 1$, where $R_1 = a \cdot b^c$ and $r_1 \in \{1, 2, 3, 4\}$.

Denote the group given by the presentation $\langle a, b, c \mid a^2, b^2, c^2, (ab)^{s_1}, (ac)^{s_2}, (bc)^{s_3}, R_1^{r_1}\rangle$ by $T^{(s_1,s_2,s_3;r_1)}$. So $T^{(s_1,s_2,s_3;r_1)}$ is the quotient of $T^{(s_1,s_2,s_3)}$ by the normal closure of $R_1^{r_1}$.

In the next section we consider some quotients of $T^{(s_1,s_2,s_3;r_1)}$ by some additional relations because some of the groups $T^{(s_1,s_2,s_3;r_1)}$ remain infinite. We set the following words:

$$R_2 := a \cdot c^b, R_3 := b \cdot c^a, R_4 := a^c \cdot a^b,$$

and we denote the quotient of $T^{(s_1,s_2,s_3;r_1)}$ by the normal closure of $R_2^{r_2}, R_3^{r_3}$ and $R_4^{r_4}$ by

$T^{(s_1,s_2,s_3;r_1,r_2,r_3,r_4)} := \langle a, b, c \mid a^2, b^2, c^2, (ab)^{s_1}, (ac)^{s_2}, (bc)^{s_3}, (a \cdot b^c)^{r_1}, (a \cdot c^b)^{r_2}, (b \cdot c^a)^{r_3}, (a^c \cdot a^b)^{r_4} \rangle$, where $r_2, r_3, r_4 \in \{1, 2, 3, 4\}$ to avoid contradicting (Δ).

Note that we use the dashes $-$ in tables to indicate that the relation is not applicable to that group. In this section, we use $T^{(s_1,s_2,s_3;r_1)}$ for $T^{(s_1,s_2,s_3;r_1,-,-,-)}$ for simplicity.

Each triple $(s_1, s_2, s_3) \in \Lambda$ will be treated in a separate proposition and we show which groups $T^{(s_1,s_2,s_3;r_1)}$ are finite for $r_1 \in \{1, 2, 3, 4\}$. We present the information in tables. In each table, the first column gives the value of r_1 in the relation $R_1^{r_1} = 1$, the second column gives the order of the group $T^{(s_1,s_2,s_3;r_1)}$ computed with GAP, the third column gives the isomorphism type of the group. The fourth column gives the sum of the sizes of conjugacy classes of the generators of the group and the last column tells whether the group satisfies property (Δ) or not.

Before we start to prove our results in this chapter we give the following easy but important lemma which comes from the symmetry in $\{b, b^c\}$ of the presentation for $T^{(s_1,s_2,s_3;r_1)}$.

Lemma 4.4.1. *The permutation (b, b^c) swaps s_1 and r_1 and gives the isomorphism $T^{(s_1,s_2,s_3;r_1)} \cong T^{(r_1,s_2,s_3;s_1)}$.*

Proof. The permutation (b, b^c) permutes ab with ab^c. Suppose that G is a group generated by a, b^c and c. By a simple calculation in G, we can compute the following: $o(ab^c) = r_1, o(ac) = s_2, o(b^c c) = o(bc) = s_3$ and $o(a(b^c)^c) = o(ab) = s_1$. □

Proposition 4.4.2. *The groups $T^{(3,3,3;r_1)}$ are all finite for $r_1 \in \{1, 2, 3, 4\}$. Orders and the isomorphism types are as described in the following table :*

r_1	Group Order	Isomorphism Type	$\lvert cc \rvert$	(Δ)
1	6	S_3	3	Y
2	24	S_4	6	Y
3	54	$3^{1+2}_+ : 2$	9	Y
4	96	$4^2 : S_3$	12	Y

Proof. We discuss each case separately.

For $r_1 = 1$ the group $T^{(3,3,3;1)}$ is of order 6. Since $R_1 = ab^c = 1$, we have that $a = b^c$. By substituting this in the other relations, we obtain the presentation $\langle b, c \mid b^2, c^2, (bc)^3 \rangle \cong S_3$.

Let $T = T^{(3,3,3;2)}$. The group T has order 24. Note that $T = \langle a, b, c \rangle = \langle a, c, b^c \rangle$ by Lemma 4.4.1. Since $o(ac) = 3, o(ab^c) = 2$ and $o(cb^c) = o(bc) = 3$, we have that T is a factor group of $T^{(2,3,3)}$. By Proposition 4.3.2, $T^{(2,3,3)} \cong S_4$. Therefore, $T^{(3,3,3;2)} \cong S_4$.

For $r_1 = 3$, the group $G := T^{(3,3,3;3)}$ has order $54 = 2 \cdot 3^3$. Hence, G is solvable by Burnside's $p^a \cdot q^b$ Theorem. We find in GAP that $N = \langle ab, ac \rangle$ is of order 3^3 and hence it is a Sylow 3-subgroup. Since N has index 2 in G, we have $N \triangleleft G$. Then the hypotheses of the Schur-Zassenhaus Theorem are satisfied, so $G \cong N \rtimes K$, where $|K| = 2$ and we can take K be any subgroup of G of order 2, say $K \cong \langle a \rangle$.

Since G is generated by involutions, G has no factor group of order 3. So 2 acts by inversion on 3^2 and fixes $Z(G)$. If G has a factor group of order 3^2, then 2 acts on $Z(G)$ and one of 3^2 by inversion, in this case G will not be generated by involutions. Thus, $N \cong 3^{1+2}_+$, an extraspecial group of exponent 3 "+ "type, and $G \cong 3^{1+2}_+ : 2$.

For $r_1 = 4$, the group $G := T^{(3,3,3;4)}$ has order 96. First, define N to be the normal closure of the element $x := ab^c$ in G. Then $N = \langle ab^c, cb^a \rangle \cong 4^2$, as checked in GAP. The group G has trivial center. Take the element ac of order three so that it acts fixed point freely on N.

Let $H := \langle a, c \rangle \cong S_3$. One can see that $G = NH$ and $N \cap H = 1$. Hence the group G is isomorphic to $4^2 : S_3$. □

Proposition 4.4.3. *The groups $T^{(3,3,4;r_1)}$ are all finite for $r_1 \in \{1, 2, 3, 4\}$ and the orders and the isomorphism types for them are as described in the following table:*

r_1	Group Order	Isomorphism type	$\lvert cc \rvert$	(Δ)
1	2	2	1	Y
2	2	2	1	Y
3	96	$4^2 : S_3$	12	Y
4	336	$2 \times L_3(2)$	21	Y

Proof. Similar to the previous cases, we prove each case separately.

If $r_1 = 1$, then $a = b^c$. We have $(ab)^3 = 1$ and b and b^c commute, then $(ab)^3 = (b^c b)^2 (b^c b) = (b^c b) = 1$ so that $b = b^c = a$. Also $(ac)^3 = 1$ so that $1 = (b^c c)^3 = (cb)^3 = bc$ and then $b = c$. Therefore, the presentation for $T^{(3,3,4;1)}$ is equivalent to $\langle a \mid a^2 \rangle \cong 2$.

For $r_1 = 2$, the group $G := T^{(3,3,4;2)}$ is of order 2. It is a quotient of the group $T^{(3,3,4;4)}$.

If $r_1 = 3$, then by Lemma 4.4.1 $T^{(3,3,4;3)} \cong 4^2 : S_3$.

For $r_1 = 4$, the group $G := T^{(3,3,4;4)}$ has order 336. GAP was used for calculating the orders of elements of G. The element abc has order 14. Consider the presentation $F := \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (ac)^3, (bc)^4, (a \cdot b^c)^4, (abc)^7 \rangle$, so that F is a factor group of G. By [7], F is isomorphic to $L_3(2)$. Suppose that N be the normal closure of the involution $n := (abc)^7 = a(bac)^2 ac^b ab^c$ in G. Since G/N has order 168, then N must has size two.

Thus $G \cong SL_2(7)$ or $2 \times L_2(7)$. The group G contains at least 21 involutions whereas the group $SL_2(7)$ contains only one involution. Thus $G \cong 2 \times L_2(7)$, our claim. $\square$

Proposition 4.4.4. *The groups $T^{(3,4,4;r_1)}$ are all finite for $r_1 \in \{1, 2\}$. The orders and the isomorphism types for them are as described in the following table:*

r_1	Group Order	Isomorphism type	$\|cc\|$	(Δ)
1	4	2^2	1+1	Y
2	72	$(S_3 \times S_3) : 2$	6+6	Y

Proof. If $r_1 = 1$, then $a = b^c$. Thus, $ab = (bc)^2$ so that $(ab)^2 = 1 = (ab)^3$ and hence $a = b$. Therefore, the presentation for $T^{(3,4,4;1)}$ is $\langle b, c \mid b^2, c^2, (bc)^2 \rangle \cong 2^2$.

For $r_1 = 2$, the group $G := T^{(3,4,4;2)}$ is of order 72. Let the subgroups $H_1 = \langle a, b \rangle$ and $H_2 = \langle a^c, b^c \rangle$ of G. It is clear from the structure of the presentation that the elements aa^c, ab^c and bb^c have order 2. The element ba^c is the conjugate of ab^c, so it has also order 2. The involution c swaps H_1 and H_2. Therefore, $G \cong (S_3 \times S_3) : 2$. $\square$

To prove Proposition 4.4.6, we require the following lemma.

Lemma 4.4.5. *The group $T^{(2,4,4;1)}$ is isomorphic to the group D_8.*

Proof. Since $r_1 = 1$ then $a = b^c$. By substituting it in the other relations we see that $ab = b^c b = (cb)^2$ and $ac = b^c c = cb$ so that the group $T^{(2,4,4;1)}$ has presentation $\langle b, c \mid b^2, c^2, (bc)^4 \rangle$, which is isomorphic to D_8. $\square$

Proposition 4.4.6. *The groups $T^{(4,4,4;r_1)}$ are all finite for $r_1 \in \{1, 2\}$ and the orders and the isomorphism types for them are as described in the following table:*

r_1	Group Order	Isomorphism type	$\|cc\|$	(Δ)
1	8	D_8	2+2	Y
2	128	$(D_8 \times D_8) : 2$	4+4+8	Y

Proof. For $r_1 = 1$ the group $T^{(4,4,4;1)}$ has order 8. We have $(ab)^4 = (aa^b)^2 = 1$. Now $aa^b = abab = b^c bb^c b = (b^c b)^2 = 1$ because $(bc)^4 = 1$. Then a and b commute so that ab has order 2. Therefore $T^{(4,4,4;1)}$ is isomorphic to $T^{(2,4,4;1)}$, which is isomorphic to D_8 by Lemma 4.4.5.

If $r_1 = 2$ then the group $G := T^{(4,4,4;2)}$ is of order 128. Similar to the case of $(3, 4, 4; 2)$, we prove by construction that G is isomorphic to $(D_8 \times D_8) : 2$. Let $H_1 := \langle a, b\rangle, H_2 := \langle a^c, b^c\rangle$ be two subgroups of G. It clear that the element c interchanging H_1 and H_2. Since $[a, a^c] = [a, b^c] = [b, b^c] = 1$ and we have $[a, b^c] = 1$ then $[a^c, b] = 1$. So that $G \cong (H_1 \times H_2) : \langle c\rangle$. □

4.5 Further cases

In the last section we were not able to compute the order of some groups with GAP which probably means that these groups are infinite. To resolve those cases, we introduce three extra relations $R_i^{r_i} = 1$, where $R_2 = a \cdot c^b$, $R_3 = b \cdot c^a$, $R_4 = a^c \cdot a^b$ and $r_2, r_3, r_4 \in \{1, 2, 3, 4\}$, and we recall that $T^{(s_1,s_2,s_3;r_1,r_2,r_3,r_4)}$ is the group with the presentation

$T^{(s_1,s_2,s_3;r_1,r_2,r_3,r_4)} := \langle a, b, c \mid a^2, b^2, c^2, (ab)^{s_1}, (ac)^{s_2}, (bc)^{s_3}, (a \cdot b^c)^{r_1}, (a \cdot c^b)^{r_2}, (b \cdot c^a)^{r_3}, (a^c \cdot a^b)^{r_4}\rangle$,

Note that we use the dashes − in tables in this section to indicate that the relation is omitted in that group.

Proposition 4.5.1. *The groups $T^{(3,4,4;3,r_2,-,-)}$ are all finite for $r_2 \in \{1,2,3,4\}$ and the orders and the isomorphism types for them are as described in the following table:*

r_2	Group Order	Isomorphism type	$\|cc\|$	(Δ)
1	2	2	1	Y
2	48	$2 \times S_4$	3+6	Y
3	2	2	1	Y
4	384	$(((2 \times D_8) : 2) : 3).2) : 2 =$ $(2.(((2^4) : 3) : 2)$	12+12	Y

Proof. If $r_2 = 1$, then $a = c^b$ so that $ab^c = c^b b^c = (bc)^3 = cb$. Since ab^c has order 3, then so is cb. But $1 = (cb)^3 = bc$, then $b = c$ and hence $a = b = c$. Therefore $T^{(3,4,4;3,1,-,-)} \cong 2$.

For $r_2 = 2$ the group $G := T^{(3,4,4;3,2,-,-)}$ has order 48. Let H be the subgroup $\langle a, c^b, b\rangle$. We notice that H is a quotient of $T^{(2,3,4)}$, which from Proposition 4.3.2 is a group isomorphic to $2 \times S_4$.

If $r_2 = 4$, then the group $G := T^{(3,4,4;3,4,-,-)}$ has order 384. Let N be the normal closure of a in G. Then $N = \langle a, b, a^c, b^c\rangle \cong (((2 \times D_8) : 2) : 3) : 2$ is of order 192, as checked in GAP. Let the subgroup $H := \langle c\rangle$ so that $N \cap H = 1$ and c interchanges the generators of N. Then $G = N : H$. □

Proposition 4.5.2. *The groups $T^{(3,4,4;4,r_2,-,-)}$ are all finite for $r_2 \in \{1,2,3,4\}$ and the orders and the isomorphism types for them are as described in the following table:*

r_2	Group Order	Isomorphism type	$\|cc\|$	(Δ)
1	2	2	1	Y
2	4	2^2	1+1	Y
3	336	$2 \times L_3(2)$	21	Y
4	2304	$(((((2 \times D_8) : 2) : 3) : 3) : 4) : 2$	24+24	N

Proof. If $r_2 = 3$ then the group $G := T^{(3,4,4;4,3,-,-)}$ has order 336. Take the subgroup $H := \langle a, b, c^b \rangle$ of G. We see that $o(ab) = 3, o(ac^b) = 3, o(bc^b) = o(bc) = 4$ and $o(a(b)^{c^b}) = o(a(b^{bcb})) = o(a(bcbcb)) = o(abb^cb) = o(ab^c) = 4$. Thus H is a quotient of $T^{(3,3,4;4)}$, which from Proposition 4.4.3 is a group of order 336 isomorphic to $2 \times L_3(2)$.

For $r_2 = 4$, the group $T := T^{(3,4,4;4,4,-,-)}$ has order $2304 = 2^8 \cdot 3^2$. GAP gives the structure of T, which is $(((((2 \times D_8) : 2) : 3) : 3) : 4) : 2$. The group T does not satisfy (Δ) because there exists pairs of involutions whose product has order greater than 4, for example the product $c \cdot c^{ab}$ has order 6. □

Lemma 4.5.3. *The groups $T^{(3,4,4;4,r_2,-,-)}$ and $T^{(4,4,4;3,r_2,-,-)}$ are isomorphic.*

Proof. The group $T^{(3,4,4;4,r_2,-,-)}$ is the quotient of the group $T^{(3,4,4;4)}$ by the normal closure of $R_2^{r_2}$. From Lemma 4.4.1, we have that $T^{(3,4,4;4)} \cong T^{(4,4,4;3)}$, and so the result follows. □

Proposition 4.5.4. *The groups $T^{(4,4,4;4,r_2,-,-)}$ are all finite for $r_2 \in \{1, 2, 3\}$ and the orders and the isomorphism types for them are as described in the following table:*

r_2	Group Order	Isomorphism type	$\|cc\|$	(Δ)
1	8	D_8	2+2	Y
2	128	$(D_8 \times D_8) : 2$	4+8+4	Y
3	2304	$(((((2 \times D_8) : 2) : 3) : 3) : 4) : 2$	24+24	N

Proof. For $r_2 = 1$ the group $G := T^{(4,4,4;4,1,-,-)}$ has order 8. Let $H := \langle a, c^b, b \rangle$ be the subgroup of G. We notice that $o(ac^b) = 1, o(ab) = 4, o(c^b b) = o(bc) = 4$ and $o(a(c^b)^b) = o(ac) = 4$. Thus H is a quotient of $T^{(1,4,4;4)}$. By Lemma 4.4.1, $T^{(1,4,4;4)} \cong T^{(4,4,4;1)}$. By Proposition 4.4.6, $T^{(4,4,4;1)}$ is isomorphic to D_8.

For $r_2 = 2$, the proof is very similar to the case when $r_2 = 1$.

If $r_2 = 3$, then the group $G := T^{(4,4,4;4,3,-,-)}$ has order 2304. Let H be the subgroup of G as above. Since $o(ab^{bcb}) = o(abcbcb) = o(abb^c b) = o(ab^c) = 4$, hence H is a quotient of $T^{(3,4,4;4,4,-,-)}$. By Proposition 4.5.2, it is isomorphic to $(((((2 \times D_8) : 2) : 3) : 3) : 4) : 2$ of order 2304. □

Proposition 4.5.5. *The groups $T^{(4,4,4;4,4,4,r_4)}$ are all finite for $r_4 \in \{1, 2, 3, 4\}$ and the orders and the isomorphism types for them are as described in the following table:*

r_4	Group Order	Isomorphism type	$\lvert cc \rvert$	(Δ)
1	32	$(4 \times 2^2) : 2$	2+4+4	Y
2	1024	$(((2 \times ((4 \times 2^2) : 4)) : 2) : 2) : 2$	8+16+16	Y
3	7776	$((3 \times ((3^2) : 3)) : 3) : ((4 \times 2^2) : 2)$	18+36+36	N
3	32768	?	32+64+64	N

Proof. Only in this proof we write G_{r_4} for $T^{(4,4,4;4,4,4,r_4)}$ for simplicity.

The group G_3 has order $7776 = 2^5 \cdot 3^5$. G_3 is solvable by Burnside' $p^a \cdot q^b$ Theorem. We find by inspection the Sylow 3-subgroup $N := \langle a^{cb}a, a^c a^b, a^{bc}a, ba^{bc}ab, b^c acbac \rangle$ of order 3^5. Relabelling the generators $a^{cb}a, a^c a^b, a^{bc}a, ba^{bc}ab, b^c acbac$ of N by v_1, v_2, v_3, v_4, v_5 and using the relations $R_1^4 = 1$ and $R_2^4 = 1$, we notice that $v_1^a = v_1^{-1}, v_2^a = v_2, v_3^a = v_3^{-1}, v_4^a = v_4, v_5^a = v_5, v_1^b = v_2, v_2^b = v_1, v_3^b = v_4, v_4^b = v_3, v_5^b = v_5^{ab}, v_1^c = v_5, v_2^c = v_3^{-1}, v_3^c = v_2^{-1}, v_4^c = v_4^{ac}, v_5^c = v_1$. Thus $N \lhd G$. The group G_3 satisfies the hypotheses of Schur-Zassenhaus Theorem so that $G_3 \cong N \rtimes H$, where $|H| = 2^5$. Also by inspection we find a Sylow 2-subgroup $H := \langle a^c, c^{ba}, b^{acb} \rangle$ which is not elementary abelian because a^c and c^{ba} are not commute. The group G_3 does not satisfy (Δ) as there exists pairs of involution with product of order greater than 4, for instance the element $b \cdot b^{ca}$ has order 6.

For the group G_4, GAP gives $|G_4| = 32768 = 2^{15}$. By inspection, we find that G_4 has elements of order 8 such as $b \cdot b^{ac}$ and $c \cdot c^{ab}$, so that G_4 does not satisfy (Δ). $\square$

4.6 Quotients of $T^{(s_1,s_2,s_3;r_1,r_2,r_3,r_4)}$ satisfying (Δ)

As we saw earlier that generally $T^{(s_1,s_2,s_3;r_1,r_2,r_3,r_4)}$ does not satisfy (Δ). In this section we find the largest quotient of groups $T^{(s_1,s_2,s_3;r_1,r_2,r_3,r_4)}$ satisfying (Δ). Here we fix $(s_1, s_2, s_3; r_1, r_2, r_3, r_4)$ to be one of $(3, 4, 4; 4, 4, -, -)$, $(4, 4, 4; 4, 4, 4, 3)$ or $(4, 4, 4; 4, 4, 4, 4)$. We will discuss each case separately in a subsection.

4.6.1 The largest 4-transposition quotients of the group $T^{(3,4,4;4,4,-,-)}$

Let $T_1 := T^{(3,4,4;4,4,-,-)}$. In this subsection we will prove the following.

Theorem 4.6.1. *The unique largest 4-transposition quotient of* T_1 *is* $H := T_1/\langle (c \cdot c^{ab})^3 \rangle$ *which is isomorphic to* $(S_4 \times S_4) : 2$.

In Proposition 4.5.2, we showed that the group T_1 does not satisfy (Δ) as it contains the product $c \cdot c^{ab}$ of order 6. We start by quotienting T_1 by the relation $(c \cdot c^{ab})^3 = 1$ and resulting in a group isomorphic to $(S_4 \times S_4) : 2$ satisfying (Δ).

We denote the group given by the presentation

$\langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (ac)^4, (bc)^4, (a \cdot b^c)^4, (a \cdot c^b)^4, -, -, (c \cdot c^{ab})^{r_5} \rangle$ by $T^{(3,4,4;4,4,-,-;r_5)}$

Proposition 4.6.2. *Let* $T := T^{(3,4,4;4,4,-,-;r_5)}$ *where* $r_5 \in \{1, 2, 3, 4\}$. *Then*

(i) *if* $r_5 \in \{1, 2, 4\}$ *then* $T \cong D_8$,

(ii) *if* $r_5 \in \{3\}$ *then* $T \cong (S_4 \times S_4) : 2$.

Proof. (i) If $r_5 = 1$ the $c = c^{ab}$ so that $c^a = c^b$. By the same argument as in proof of Proposition 4.5.4 we can see that T has a subgroup H where $H := \langle c, c^a, b \rangle$. By a straightforward computation we see that $o(c \cdot c^a) = 2, o(c \cdot b) = 4, o(c^a \cdot b) = o(c^b \cdot b) = o(bc) = 4$ and $o(c \cdot (c^a)^b) = o(c \cdot c^{ab}) = 1$. Thus H is a quotient of $T^{(2,4,4;1)}$, which is isomorphic to D_8 by Lemma 4.4.5.

(ii) If $r_4 = 3$ then $|T| = 1152 = 2^7 \cdot 3^2$. Similar to the proof of the case $(4, 4, 4; 2)$ we let the subgroups $H_1 := \langle a, b, b^{cac} \rangle$ and $H_2 := \langle a^c, b^c, b^{ca} \rangle$ of T. It is clear that $o(ab) = 3, o(ab^{cac}) = o(acacbcac) = o(acab^c cacaca) = o(acacbacaca) = o((ba)^{caca}) = 3$ and $o(bb^{cac}) = o(bcacbcac) = o((ba^c)^2) = o(((ab^c)^{ac})^2) = 2$. Then H_1 is a quotient of $T^{(3,3,2)}$. By Proposition 4.3.2, H_1 is isomorphic to S_4. The element c swaps H_1 with H_2 so that the group T is isomorphic to $(H_1 \times H_2) : \langle c \rangle$. □

4.6.2 The largest 4-transposition quotients of the group $T^{(4,4,4;4,4,4,3)}$

Let $T_2 := T^{(4,4,4;4,4,4,3)}$. The main result in this subsection is the following theorem.

Theorem 4.6.3. *There are two largest 4-transposition quotients of* T_2 *which are* $H_1 := T_2/\langle (c \cdot c^{ab})^3 \rangle$ *and* $H_2 := T_2/\langle (c \cdot c^{ab})^4 \rangle$ *such that* $H_1 \cong (((3 \times ((3^2) : 3)) : 3) : Q_8) : 2$ *and* $H_2 \cong (4 \times 2^2) : 2$.

In Proposition 4.5.5, we showed that the group T_2 does not satisfy (Δ) as it contains the product $c \cdot c^{ab}$ of order 6. We start by quotienting T_2 by the relation $(c \cdot c^{ab})^3 = 1$ and resulting in a group isomorphic to $(((3 \times ((3^2) : 3)) : 3) : Q_8) : 2$ satisfying (Δ).

We denote the group given by the presentation

$\langle a, b, c \mid a^2, b^2, c^2, (ab)^4, (ac)^4, (bc)^4, (a{\cdot}b^c)^4, (a{\cdot}c^b)^4, (b{\cdot}c^a)^4, (a^c{\cdot}a^b)^3, (c{\cdot}c^{ab})^{r_5} \rangle$ by $T^{(4,4,4;4,4,4,3;r_5)}$

Proposition 4.6.4. *Let* $T := T^{(4,4,4;4,4,4,3;r_5)}$ *where* $r_5 \in \{1,2,3,4\}$. *Then*

(i) *if* $r_5 \in \{1\}$ *then* $T \cong (4 \times 2) : 2$,

(ii) *if* $r_5 \in \{2,4\}$ *then* $T \cong (4 \times 2^2) : 2$,

(iii) *if* $r_5 \in \{3\}$ *then* $T \cong (((3 \times ((3^2) : 3)) : 3) : Q_8) : 2$.

Proof. (iii) For $r_5 = 3$ the group $T := T^{(4,4,4;4,4,4,3;3)}$ is of order $3888 = 2^4 \cdot 3^5$. The proof is quite similar to the proof in the case of $T^{(4,4,4;4,4,4,3)}$. The group T is solvable by Burnside's $p^a \cdot q^b$ Theorem. We take a Sylow 3-subgroup $N := \langle bb^{ca}, a^{cb}a, c^bc^a, cc^{ba}\rangle$. It was checked by GAP that N is normal in T. Then the hypotheses of Schur-Zassenhaus Theorem are satisfied so that $T \cong N \rtimes H$, where $|H| = 2^4$. By inspection we find the Sylow 2-subgroup $H := \langle a^{cba}, b^{aca}, c^{(ba)^2}\rangle \cong (4 \times 2) : 2$. □

4.6.3 The largest 4-transposition quotients of the group $T^{(4,4,4;4,4,4,4)}$

In this subsection we aim to find the largest quotient of the group $T^{(4,4,4;4,4,4,4)}$ satisfying (Δ). We notice from Proposition 4.5.5 that the element $c \cdot c^{ab}$ has order 8. First we quotient the group $T^{(4,4,4;4,4,4,4)}$ by the relation $(c \cdot c^{ab})^4 = 1$ and we denote the following presentation

$\langle a, b, c \mid a^2, b^2, c^2, (ab)^4, (ac)^4, (bc)^4, (a{\cdot}b^c)^4, (a{\cdot}c^b)^4, (b{\cdot}c^a)^4, (a^c{\cdot}a^b)^4, (c{\cdot}c^{ab})^4\rangle$ by $T^{(4,4,4;4,4,4,4;r_5)}$.

The following lemma is obtained with GAP.

Lemma 4.6.5. *Let* $K := T^{(4,4,4;4,4,4,4;r_5)}$ *where* $r_5 \in \{1,2,3,4\}$. *Then*

(i) *If* $r_5 \in \{1,3\}$ *then* $K \cong (4 \times 2^2) : 2$,

(ii) *If* $r_5 \in \{2\}$ *then* $K \cong ((2 \times ((4 \times 2^2) : 4)) : 4) : 2$,

(iii) *If* $r_5 \in \{4\}$ *then* $|K| = 16384$.

Proof. (iii) For $r_5 = 4$, the group K has order 16384 and does not satisfy (Δ) as there exists a pair of involutions, for example b and b^{ac}, with their product of order 8. $\square$

Again we quotient the group $T^{(4,4,4;4,4,4,4;4)}$ by the relation $(b \cdot b^{ac})^4 = 1$ and we denote by $T^{(4,4,4;4,4,4,4;4,r_6)}$ the group given with the presentation

$T^{(4,4,4;4,4,4,4;4,r_6)} := \langle a, b, c \mid a^2, b^2, c^2, (ab)^4, (ac)^4, (bc)^4, (a \cdot b^c)^4, (a \cdot c^b)^4, (b \cdot c^a)^4, (a^c \cdot a^b)^4, (c \cdot c^{ab})^4, (b \cdot b^{ca})^{r_6} \rangle$.

Proposition 4.6.6. *Let* $T := T^{(4,4,4;4,4,4,4;4,r_6)}$ *where* $r_6 \in \{1, 2, 3, 4\}$*. Then*

(i) *if* $r_6 \in \{1, 3\}$ *then* $T \cong (4 \times 2^2) : 2$*,*

(ii) *if* $r_6 \in \{2\}$ *then* $T \cong ((2 \times ((4 \times 2^2) : 4)) : 4) : 2$*,*

(iii) *if* $r_6 \in \{4\}$ *then* $T \cong B(2, 4) : 2$*.*

Proof. (iii) If $r_6 = 4$ then $|T| = 2^{13}$. Let $H := \langle x, y \rangle$, where $x = ab, y = ac$, be the subgroup of T. We noticed that H is the Burnside group $B(2, 4)$ of order 2^{12}. The involution a acts by inversion on both x and y so that $T \cong B(2, 4) : 2$. $\square$

Chapter 5

M-Axial Algebras for the 4-transposition Groups

In this chapter, we study 3-generated M-axial algebras A such that every 2-generated subalgebra of A is a Sakuma algebra of type NX, where $N \in \{2, 3, 4\}$ and $X \in \{A, B, C\}$. For this aim, we classified all 3-generated 4-transposition groups in Chapter 4. By studying their M-axial representations we can achieve our goal. Since our list of 4-transposition groups is too big, we only focus on some cases where it is possible to do a proof "by hand". All others can be found in the appendix part of the thesis. They were done by computer. We need to keep in mind that we do not assume that there is a bijection between the selected set of involutions and the set of M-axes and the $(2A)$-condition is not considered.

The group $(S_3 \times S_3) : 2$ appears as a subgroup of the groups $(((3 \times ((3^2) : 3)) : 3) : Q_8) : 2$ and $(S_4 \times S_4) : 2$. Also, the group D_{12}, the dihedral group of order 12, appears as a subgroup of the group $(S_3 \times S_3) : 2$. So it is desirable to start with D_{12} to find its M-axial algebras. Throughout this chapter, the symbol $\langle\langle S \rangle\rangle$ refers to the subalgebra generated by $S \subseteq A$.

5.1 M-Axial algebras for the group D_{12}

In the current section, the group $G := D_{12}$ is considered. Let $G = \langle a, b, c\rangle$, where $a = (i, j)$, $b = (j, k)$ and $c = (l, m)$. It is clear that the group G satisfies the presentation $\langle a,\ b,\ c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^2 = 1\rangle$, which is the same as the group $T^{(3,2,2)} = T^{(2,2,3)}$ given in Proposition 4.3.2. Since the product of a and b has order 3, they belong to the same conjugacy class. Thus, the sizes of the conjugacy classes with representatives a and c are 3 and 1, respectively. Thus, $a^G = \{(j, k), (i, j), (i, k)\}$ and $c^G = \{(l, m)\}$. It is well known that the group D_{12} has three classes of involutions with sizes 1, 3 and 3. Here we only take the cases where the sizes are 1 and 3 because they only appear in the classes of the group $(S_3 \times S_3) : 2$. So, the cases 1, 3, 3 and 3, 3 are not of our interest.

Relabeling the involutions in $T := a^G \cup c^G$ by s, t, v and r. It can be seen that there are two orbits on pairs of elements in T with representatives $\{t, s\}$ and $\{t, r\}$.

Let a_s, a_t, a_v and a_r be the corresponding M-axes to the involutions s, t, v and r, respectively. So G has the following possible shapes

$(2A, 3A)$
$(2A, 3C)$
$(2B, 3A)$
$(2B, 3C)$

Table 5.1: D_{12}-shapes

We treat each case in a separate subsection.

5.1.1 The shape $(2A, 3A)$

In this case, we consider the generating set of the algebra A consisting of eight elements. Which are four axes denoted by a_s, a_t, a_v and a_r, three vectors denoted by $b_{r,s}, b_{r,t}$ and $b_{r,v}$ which are vectors in the subalgebras $\langle\langle a_r, a_s\rangle\rangle$, $\langle\langle a_r, a_t\rangle\rangle$ and $\langle\langle a_r, a_v\rangle\rangle$ of type $2A$, respectively, and the vector u_1 in the subalgebra $\langle\langle a_s, a_t\rangle\rangle$ of type $3A$. The following are the known products of the vectors in the spanning set according to the Table 2.3.

$$a_s \cdot a_s = a_s;\ a_r \cdot a_r = a_r;$$

$$b_{r,s} \cdot b_{r,s} = b_{r,s};$$

$$a_s \cdot a_r = \frac{1}{8}(a_s + a_r - b_{r,s});$$

$$a_s \cdot b_{r,s} = \frac{1}{8}(a_s + b_{r,s} - a_r);$$

$$a_r \cdot b_{r,s} = \frac{1}{8}(a_r + b_{r,s} - a_s).$$

$$a_s \cdot a_t = \frac{1}{2^5}(2a_s + 2a_t + a_v) - \frac{3^3 \cdot 5}{2^{11}}u_1;$$

$$a_s \cdot u_1 = \frac{1}{3^2}(2a_s - a_t - a_v) + \frac{5}{2^5}u_1.$$

By Table 2.4, we present the known eigenvectors of the axes a_s and a_r in the following tables.

Type	0-eigenvectors	$\frac{1}{4}$-eigenvectors	$\frac{1}{32}$-eigenvectors
$2A$	$a_r + b_{r,s} - \frac{1}{4}a_s$	$a_r - b_{r,s}$	
$3A$	$u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)$	$u_1 - \frac{8}{45}a_s - \frac{32}{45}(a_t + a_v)$	$a_t - a_v$

Table 5.2: Eigenvectors of a_s

Type	0-eigenvectors	$\frac{1}{4}$-eigenvectors
$2A$	$a_s + b_{r,s} - \frac{1}{4}a_r$	$a_s - b_{r,s}$
	$a_t + b_{r,t} - \frac{1}{4}a_r$	$a_t - b_{r,t}$
	$a_v + b_{r,v} - \frac{1}{4}a_r$	$a_v - b_{r,v}$

Table 5.3: Eigenvectors of a_r

Our main aim in this section is to show that the M-axial algebra A of the shape $(2A, 3A)$ is of dimension 8.

Lemma 5.1.1.

$b_{r,v} - b_{r,t}$ is $\frac{1}{32}$-eigenvector of a_s.

Proof. By fusion rules, the product $w_1 := (a_r + b_{r,s} - \frac{1}{4}a_s)(a_t - a_v)$ is $\frac{1}{32}$-eigenvector of a_s. Then $w_1 = \frac{1}{8}(a_r + a_t - b_{r,t}) - \frac{1}{8}(a_r + a_v - b_{r,v}) + a_t b_{r,s} - a_v b_{r,s} - \frac{1}{4}(\frac{1}{64}(a_s + a_t - a_v - a_s - a_v + a_t)) = a_t b_{r,s} - a_v b_{r,s} + \frac{15}{128}a_t - \frac{15}{128}a_v - \frac{1}{8}b_{r,t} + \frac{1}{8}b_{r,v}$.

Also, $w_2 := (a_r - b_{r,s})(a_t - a_v)$ is $\frac{1}{32}$-eigenvector of a_s and so $w_2 = a_v b_{r,s} - a_t b_{r,s} + \frac{1}{8}(a_r + a_t - b_{r,t}) - \frac{1}{8}(a_r + a_v - b_{r,v}) = a_v b_{r,s} - a_t b_{r,s} + \frac{1}{8}a_t - \frac{1}{8}a_v - \frac{1}{8}b_{r,t} + \frac{1}{8}b_{r,v}$. It is clear that $w_1 + w_2 = \frac{31}{128}(a_t - a_v) + \frac{1}{4}(b_{r,v} - b_{r,t})$ is $\frac{1}{32}$-eigenvector of a_s. From Table 5.2, we have also that $a_t - a_v$ is $\frac{1}{32}$-eigenvector of a_s and so $b_{r,v} - b_{r,t}$ is $\frac{1}{32}$-eigenvector of a_s. □

By using the action of G, we have the following.

Corollary 5.1.2.

$b_{r,v} - b_{r,s}$ and $b_{r,t} - b_{r,s}$ are $\frac{1}{32}$-eigenvectors of a_t and a_v, respectively.

Before we find all remaining products, we give the following definition, which can be found in [31].

Definition 5.1.3. *Let A be an $\mathfrak{F}$-axial algebra and $X \subseteq A$. Then we say that the subalgebra $\langle\langle X \rangle\rangle$ is k-closed if it is the linear span of the k-long products $\{x_1 \cdot x_2 \cdot ... \cdot x_k | x_i \in X\}$.*

At this point, the algebra A is 3-closed, we find all possible products involving terms of a product of three vectors.

Lemma 5.1.4. $a_s \cdot (a_r u_1) = -\frac{1}{108}a_s + \frac{1}{216}a_t + \frac{1}{216}a_v + \frac{1}{27}b_{r,s} - \frac{5}{216}b_{r,t} - \frac{1}{72}b_{r,v} + \frac{5}{256}u_1 + \frac{8}{27}a_s b_{r,t} - \frac{4}{27}a_t b_{r,s} - \frac{4}{27}a_v b_{r,s} + \frac{1}{8}a_r u_1 - \frac{1}{8}b_{r,s}u_1$.

Proof. From Table 5.2 and Lemma 2.1.8, we see that the axis a_s associates with $u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)$ in the sense that $(a_s \cdot x)\cdot(u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)) = a_s \cdot (x \cdot (u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)))$ for all $x \in A$. In particular, $(a_s \cdot a_r)\cdot(u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)) = a_s \cdot (a_r \cdot (u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)))$. Thus, $\frac{1}{54}a_t + \frac{1}{54}a_v + \frac{11}{432}a_r + \frac{5}{432}b_{r,s} - \frac{1}{54}b_{r,t} - \frac{1}{54}b_{r,v} - \frac{4}{27}a_t b_{r,s} - \frac{4}{27}a_v b_{r,s} + \frac{1}{8}a_r u_1 - \frac{1}{8}b_{r,s}u_1 = \frac{1}{108}a_s + \frac{1}{72}a_t + \frac{1}{72}a_v + \frac{11}{432}a_r - \frac{11}{432}b_{r,s} + \frac{1}{216}b_{r,t} - \frac{1}{216}b_{r,v} - \frac{5}{256}u_1 - \frac{8}{27}a_s b_{r,t} + a_s \cdot (a_r u_1)$ and the result follows. □

Lemma 5.1.5. $a_s \cdot (a_t b_{r,s}) = -\frac{1}{128}a_s - \frac{1}{256}a_r + \frac{1}{64}b_{r,s} - \frac{1}{128}b_{r,t} - \frac{1}{256}b_{r,v} + \frac{135}{16384}u_1 + \frac{1}{8}a_s b_{r,t} + \frac{1}{16}a_t b_{r,s} + \frac{1}{32}a_v b_{r,s} - \frac{135}{2048}a_r u_1 - \frac{135}{2048}b_{r,s}u_1$.

Proof. By Lemma 2.1.8, we have that the axis a_s associates with the 0-eigenvector $a_r + b_{r,s} - \frac{1}{4}a_s$ and so $(a_s \cdot a_t)\cdot(a_r + b_{r,s} - \frac{1}{4}a_s) = a_s \cdot (a_t \cdot (a_r + b_{r,s} - \frac{1}{4}a_s))$. Thus, $\frac{9}{4096}a_s + \frac{39}{8192}a_t + \frac{9}{8192}a_v + \frac{3}{256}a_r - \frac{1}{128}b_{r,t} - \frac{1}{256}b_{r,v} + \frac{135}{32768}u_1 + \frac{1}{16}a_t b_{r,s} + \frac{1}{32}a_v b_{r,s} - \frac{135}{2048}a_r u_1 - \frac{135}{2048}b_{r,s}u_1 = \frac{41}{4096}a_s + \frac{39}{8192}a_t + \frac{9}{8192}a_v + \frac{1}{64}a_r - \frac{1}{64}b_{r,s} - \frac{135}{32768}u_1 - \frac{1}{8}a_s b_{r,t} + a_s \cdot (a_t b_{r,s})$ and the result follows. □

A direct consequence of Lemmas 5.1.4 and 5.1.5 is the following corollary.

Corollary 5.1.6. (i) $a_s \cdot (a_v b_{r,s}) = -\frac{1}{128}a_s - \frac{1}{256}a_r + \frac{1}{64}b_{r,s} - \frac{1}{128}b_{r,t} - \frac{1}{256}b_{r,v} + \frac{135}{16384}u_1 + \frac{1}{8}a_s b_{r,t} + \frac{1}{32}a_t b_{r,s} + \frac{1}{16}a_v b_{r,s} - \frac{135}{2048}a_r u_1 - \frac{135}{2048}b_{r,s}u_1$,

(ii) $a_t \cdot (a_s b_{r,t}) = -\frac{1}{128}a_t - \frac{1}{256}a_r - \frac{9}{1024}b_{r,s} + \frac{17}{1024}b_{r,t} - \frac{1}{256}b_{r,v} + \frac{135}{16384}u_1 + \frac{1}{16}a_s b_{r,t} - \frac{135}{2048}a_r u_1 - \frac{135}{2048}b_{r,t}u_1 + \frac{1}{8}a_t b_{r,s} + \frac{1}{32}a_v b_{r,s}$,

(iii) $a_t \cdot (a_v b_{r,s}) = -\frac{3}{256}a_t - \frac{5}{512}b_{r,s} + \frac{7}{512}b_{r,t} - \frac{1}{256}b_{r,v} + \frac{135}{16384}u_1 + \frac{1}{32}a_s b_{r,t} - \frac{135}{2048}a_r u_1 - \frac{135}{2048}b_{r,t}u_1 + \frac{5}{32}a_t b_{r,s} + \frac{1}{16}a_v b_{r,s}$,

(iv) $a_v \cdot (a_s b_{r,t}) = -\frac{3}{256}a_v - \frac{5}{512}b_{r,s} - \frac{5}{1024}b_{r,t} + \frac{15}{1024}b_{r,v} + \frac{135}{16384}u_1 + \frac{1}{16}a_s b_{r,t} - \frac{135}{2048}a_r u_1 - \frac{135}{2048}b_{r,v}u_1 + \frac{1}{32}a_t b_{r,s} + \frac{5}{32}a_v b_{r,s}$,

(v) $a_v \cdot (a_t b_{r,s}) = -\frac{3}{256}a_v - \frac{5}{512}b_{r,s} - \frac{5}{1024}b_{r,t} + \frac{15}{1024}b_{r,v} + \frac{135}{16384}u_1 + \frac{1}{32}a_s b_{r,t} - \frac{135}{2048}a_r u_1 - \frac{135}{2048}b_{r,v}u_1 + \frac{1}{16}a_t b_{r,s} + \frac{5}{32}a_v b_{r,s}$.

Lemma 5.1.7. $a_s \cdot (b_{r,s}u_1) = \frac{7}{108}a_s - \frac{1}{54}a_t - \frac{1}{54}a_v - \frac{1}{36}a_r - \frac{1}{27}b_{r,s} + \frac{1}{27}b_{r,t} + \frac{1}{36}b_{r,v} - \frac{5}{256}u_1 - \frac{8}{27}a_s b_{r,t} + \frac{1}{32}a_r u_1 + \frac{9}{32}b_{r,s}u_1 + \frac{1}{27}a_t b_{r,s} + \frac{1}{27}a_v b_{r,s}$.

Proof. By the same argument as in the previous lemmas, the axis a_s associates with the 0-eigenvector $u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)$ and then $(a_s \cdot b_{r,s}) \cdot (u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)) = a_s \cdot (b_{r,s} \cdot (u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)))$. So

$-\frac{1}{54}a_t - \frac{1}{54}a_v - \frac{11}{432}a_r - \frac{5}{432}b_{r,s} + \frac{1}{54}b_{r,t} + \frac{1}{54}b_{r,v} - \frac{1}{8}a_r u_1 + \frac{1}{8}b_{r,s}u_1 + \frac{4}{27}a_t b_{r,s} + \frac{4}{27}a_v b_{r,s} = -\frac{7}{108}a_s + \frac{1}{432}a_r + \frac{11}{432}b_{r,s} - \frac{1}{54}b_{r,t} - \frac{1}{108}b_{r,v} + \frac{5}{256}u_1 + \frac{8}{27}a_s b_{r,t} - \frac{5}{32}a_r u_1 - \frac{5}{32}b_{r,s}u_1 + \frac{1}{9}a_t b_{r,s} + \frac{1}{9}a_v b_{r,s} + a_s \cdot (b_{r,s}u_1)$ and the result follows. □

Corollary 5.1.8. (i) $a_t \cdot (b_{r,t}u_1) = -\frac{1}{54}a_s + \frac{7}{108}a_t - \frac{1}{54}a_v - \frac{1}{36}a_r + \frac{31}{864}b_{r,s} - \frac{31}{864}b_{r,t} + \frac{1}{36}b_{r,v} - \frac{5}{256}u_1 + \frac{1}{27}a_s b_{r,t} + \frac{1}{32}a_r u_1 + \frac{9}{32}b_{r,t}u_1 - \frac{8}{27}a_t b_{r,s} + \frac{1}{27}a_v b_{r,s}$,

(ii) $a_v \cdot (b_{r,v}u_1) = -\frac{1}{54}a_s - \frac{1}{54}a_t + \frac{7}{108}a_v - \frac{1}{36}a_r + \frac{31}{864}b_{r,s} + \frac{23}{864}b_{r,t} - \frac{5}{144}b_{r,v} - \frac{5}{256}u_1 + \frac{1}{27}a_s b_{r,t} + \frac{1}{32}a_r u_1 + \frac{9}{32}b_{r,v}u_1 + \frac{1}{27}a_t b_{r,s} - \frac{8}{27}a_v b_{r,s}$.

From now we try to find all possible products between a_r and the other vectors, we start with the following lemma.

Lemma 5.1.9. $a_r \cdot (a_s b_{r,t}) = \frac{3}{256}a_r + \frac{1}{128}b_{r,s} + \frac{1}{128}b_{r,t} + \frac{1}{256}b_{r,v} - \frac{135}{16384}u_1 + \frac{1}{8}a_s b_{r,t} + \frac{135}{2048}a_r u_1 - \frac{1}{8}b_{r,s}b_{r,t} - \frac{1}{8}a_t b_{r,s}$.

Proof. From Table 5.3 and Lemma 2.1.8, the axis a_r associates with the 0-eigenvector $a_t + b_{r,t} - \frac{1}{4}a_r$ in the sense that $(a_r \cdot a_s) \cdot (a_t + b_{r,t} - \frac{1}{4}a_r) = a_r \cdot (a_s \cdot (a_t + b_{r,t} - \frac{1}{4}a_r))$. Thus, $\frac{1}{128}a_t + \frac{1}{256}a_v + \frac{1}{128}b_{r,s} - \frac{135}{16384}u_1 + \frac{1}{8}a_s b_{r,t} - \frac{1}{8}b_{r,s}b_{r,t} - \frac{1}{8}a_t b_{r,s} = \frac{1}{128}a_t + \frac{1}{256}a_v - \frac{3}{256}a_r -$

$\frac{1}{128}b_{r,t} - \frac{1}{256}b_{r,v} - \frac{135}{2048}a_r u_1 + a_r \cdot (a_s b_{r,t})$. Therefore, $a_r \cdot (a_s b_{r,t}) = \frac{3}{256}a_r + \frac{1}{128}b_{r,s} + \frac{1}{128}b_{r,t} + \frac{1}{256}b_{r,v} - \frac{135}{16384}u_1 + \frac{1}{8}a_s b_{r,t} + \frac{135}{2048}a_r u_1 - \frac{1}{8}b_{r,s}b_{r,t} - \frac{1}{8}a_t b_{r,s}$. □

Corollary 5.1.10. (i) $a_r \cdot (a_t b_{r,s}) = \frac{3}{256}a_r + \frac{1}{128}b_{r,s} + \frac{1}{128}b_{r,t} + \frac{1}{256}b_{r,v} - \frac{135}{16384}u_1 - \frac{1}{8}a_s b_{r,t} + \frac{135}{2048}a_r u_1 - \frac{1}{8}b_{r,s}b_{r,t} + \frac{1}{8}a_t b_{r,s}$,

(ii) $a_r \cdot (a_v b_{r,s}) = \frac{3}{256}a_r + \frac{1}{128}b_{r,s} + \frac{1}{128}b_{r,t} + \frac{1}{256}b_{r,v} - \frac{135}{16384}u_1 - \frac{1}{8}a_s b_{r,t} + \frac{135}{2048}a_r u_1 + \frac{1}{8}a_v b_{r,s} - \frac{1}{8}b_{r,s}b_{r,v}$.

In the following lemma we find the product between a_r and $b_{r,s}b_{r,t}$.

Lemma 5.1.11. $a_r \cdot (b_{r,s}b_{r,t}) = -\frac{1}{128}a_s - \frac{1}{128}a_t - \frac{1}{256}a_v + \frac{5}{256}a_r - \frac{1}{128}b_{r,s} - \frac{1}{128}b_{r,t} - \frac{1}{256}b_{r,v} + \frac{135}{8192}u_1 - \frac{135}{2048}a_r u_1 + \frac{1}{4}b_{r,s}b_{r,t}$.

Proof. From fusion rules and Table 5.3, we see that the product $u := (a_s + b_{r,s} - \frac{1}{4}a_r)(a_t + b_{r,t} - \frac{1}{4}a_r) = \frac{1}{16}a_s + \frac{1}{16}a_t + \frac{1}{32}a_v - \frac{1}{16}a_r - \frac{135}{2048}u_1 + a_s b_{r,t} + b_{r,s}b_{r,t} + a_t b_{r,s}$ is 0-eigenvector of a_r. Since $a_r \cdot u = 0$ and by applying Lemma 5.1.9 and Corollary 5.1.10, the result follows. □

Corollary 5.1.12. $a_r \cdot (b_{r,s}b_{r,v}) = -\frac{1}{128}a_s - \frac{1}{128}a_t - \frac{1}{256}a_v + \frac{5}{256}a_r - \frac{1}{128}b_{r,s} - \frac{1}{128}b_{r,t} - \frac{1}{256}b_{r,v} + \frac{135}{8192}u_1 - \frac{135}{2048}a_r u_1 + \frac{1}{8}b_{r,s}b_{r,t} + \frac{1}{8}a_t b_{r,s} - \frac{1}{8}a_v b_{r,s} + \frac{1}{8}b_{r,s}b_{r,v}$.

In the following lemma we try to find the products $b_{r,s} \cdot b_{r,v}$ and $b_{r,t} \cdot b_{r,v}$ by using Lemma 2.1.9.

Lemma 5.1.13. (i) $b_{r,s} \cdot b_{r,v} = -\frac{1}{32}a_t + \frac{1}{32}a_v - \frac{1}{32}b_{r,t} + \frac{1}{32}b_{r,v} + b_{r,s}b_{r,t} + a_t b_{r,s} - a_v b_{r,s}$,

(ii) $b_{r,t} \cdot b_{r,v} = -\frac{1}{32}a_s + \frac{1}{32}a_v - \frac{1}{32}b_{r,t} + \frac{1}{32}b_{r,v} + a_s b_{r,t} + b_{r,s}b_{r,t} - a_v b_{r,s}$.

Proof. We apply Lemma 2.1.9 to find the product $b_{r,t} \cdot b_{r,v}$. From Table 5.3 and fusion rules, the vectors $u := (b_{r,t} + a_t - \frac{1}{4}a_r)(b_{r,v} + a_v - \frac{1}{4}a_r)$ and $w := (b_{r,t} + a_t - \frac{1}{4}a_r)(b_{r,v} - a_v)$ are 0- and $\frac{1}{4}$-eigenvectors of a_r, respectively. So, $w = b_{r,t} \cdot b_{r,v} - \frac{1}{32}a_s - \frac{1}{16}a_t - \frac{1}{32}b_{r,t} - \frac{1}{32}b_{r,v} + \frac{135}{2048}u_1 + a_t b_{r,s} - a_v b_{r,s}$. We see that $w - u = (b_{r,t} + a_t - \frac{1}{4}a_r)(-2a_v + \frac{1}{4}a_4) = -\frac{1}{16}a_s - \frac{1}{8}a_t -$

$\frac{1}{16}a_v + \frac{1}{16}a_r + \frac{1}{16}b_{r,s} - \frac{1}{16}b_{r,t} - \frac{1}{16}b_{r,v} + \frac{135}{1024}u_1 - 2a_vb_{r,s}$. By Corollary 5.1.10 and the fact that $w = 4a_r(w-u)$, then $4a_r(w-u) = -\frac{1}{16}a_s - \frac{1}{32}a_t - \frac{1}{32}b_{r,t} - \frac{1}{32}b_{r,v} + \frac{135}{2048}u_1 + a_sb_{r,t} - a_vb_{r,s} + b_{r,s}b_{r,v}$ and so

$$b_{r,t} \cdot b_{r,v} = -\frac{1}{32}a_s + \frac{1}{32}a_t + a_sb_{r,t} - a_tb_{r,s} + b_{r,s}b_{r,v}. \tag{1}$$

The action of τ_{a_s} on equation (1) gives

$$b_{r,t} \cdot b_{r,v} = -\frac{1}{32}a_s + \frac{1}{32}a_v - \frac{1}{32}b_{r,t} + \frac{1}{32}b_{r,v} + a_sb_{r,t} - a_vb_{r,s} + b_{r,s}b_{r,t} \tag{2}$$

and so part (ii) is proved. By Subtracting equation (2) from equation (1), we get that

$$b_{r,s} \cdot b_{r,v} = -\frac{1}{32}a_t + \frac{1}{32}a_v - \frac{1}{32}b_{r,t} + \frac{1}{32}b_{r,v} + b_{r,s}b_{r,t} + a_tb_{r,s} - a_vb_{r,s} \tag{3}$$

and then part (i) is also proved. □

In the following lemma, we will find the products $a_tb_{r,s}$ and $a_vb_{r,s}$ in terms of a's, b's and $a_sb_{r,t}$.

Lemma 5.1.14. (i) $a_tb_{r,s} = -\frac{1}{32}a_s + \frac{1}{32}a_t + \frac{1}{32}b_{r,s} - \frac{1}{32}b_{r,t} + a_sb_{r,t}$,

(ii) $a_vb_{r,s} = -\frac{1}{32}a_s + \frac{1}{32}a_v + \frac{1}{32}b_{r,s} - \frac{1}{32}b_{r,t} + a_sb_{r,t}$.

Proof. By fusion rules and Table 5.3, the vector $v_1 := (a_s - b_{r,s})(a_v - b_{r,v}) = \frac{1}{16}a_s + \frac{3}{32}a_v - \frac{135}{2048}u_1 - a_sb_{r,t} + b_{r,s}b_{r,t} + a_tb_{r,s} - 2a_vb_{r,s}$ lies in $\{1, 0\}$-eigenspace of the axis a_r, that is, the product $a_r \cdot v_1$ must equal to the scalar multiple of a_r, say λa_r, for a real number λ. By Lemmas 5.1.9 and 5.1.11 and Corollary 5.1.10, this implies that

$$a_r \cdot v_1 = -\frac{1}{64}a_t + \frac{1}{64}a_v + \frac{1}{64}a_r - \frac{1}{32}b_{r,s} - \frac{1}{32}b_{r,t} - \frac{1}{64}b_{r,v} + \frac{135}{4096}u_1 - \frac{135}{512}a_ru_1 + \frac{1}{2}b_{r,s}b_{r,t} + \frac{1}{2}a_tb_{r,s} - \frac{1}{2}a_vb_{r,s} = \lambda a_r \tag{4}$$

By applying τ_{a_v}, we get another equation, which is

$$-\frac{1}{64}a_s+\frac{1}{64}a_v+\frac{1}{64}a_r-\frac{1}{32}b_{r,t}-\frac{1}{32}b_{r,s}-\frac{1}{64}b_{r,v}+\frac{135}{4096}u_1- \frac{135}{512}a_ru_1+\frac{1}{2}b_{r,t}b_{r,s}+\frac{1}{2}a_sb_{r,t}-\frac{1}{2}a_vb_{r,t}=\lambda a_r. \quad (5)$$

By subtracting equation (4) from equation (5), we get the following

$$-\frac{1}{64}a_s+\frac{1}{64}a_t+\frac{1}{2}(a_sb_{r,t}-a_tb_{r,s})-\frac{1}{2}a_v(b_{r,t}-b_{r,s})=0. \quad (6)$$

By Corollary 5.1.2, equation (6) becomes

$$-\frac{1}{64}a_s+\frac{1}{64}a_t+\frac{1}{2}(a_sb_{r,t}-a_tb_{r,s})-\frac{1}{64}(b_{r,t}-b_{r,s})=0.$$

and so $a_tb_{r,s}=-\frac{1}{32}a_s+\frac{1}{32}a_t+\frac{1}{32}b_{r,s}-\frac{1}{32}b_{r,t}+a_sb_{r,t}$. Thus, part (i) is proved. Also by applying τ_{a_s} on part (i) and making use of Lemma 5.1.1, we get part (ii). □

Corollary 5.1.15. (i) $a_tb_{r,v}=-\frac{1}{32}a_s+\frac{1}{32}a_t-\frac{1}{32}b_{r,t}+\frac{1}{32}b_{r,v}+a_sb_{r,t}$,

(ii) $a_vb_{r,t}=-\frac{1}{32}a_s+\frac{1}{32}a_v+a_sb_{r,t}$.

In the following lemma, we rewrite the products in Lemmas 5.1.4 and 5.1.7.

Lemma 5.1.16. (i) $a_s\cdot(a_ru_1)=\frac{1}{36}b_{r,s}-\frac{1}{72}b_{r,t}-\frac{1}{72}b_{r,v}+\frac{5}{256}u_1+\frac{1}{8}a_ru_1-\frac{1}{8}b_{r,s}u_1$,

(ii) $a_s\cdot(b_{r,s}u_1)=\frac{1}{16}a_s-\frac{5}{288}a_t-\frac{5}{288}a_v-\frac{1}{36}a_r-\frac{5}{144}b_{r,s}+\frac{5}{144}b_{r,t}+\frac{1}{36}b_{r,v}-\frac{5}{256}u_1-\frac{2}{9}a_sb_{r,t}+\frac{1}{32}a_ru_1+\frac{9}{32}b_{r,s}u_1$.

Proof. The results come after substituting the products in Lemma 5.1.14 in the products of Lemmas 5.1.4 and 5.1.7. □

Lemma 5.1.17. $b_{r,s}u_1=\frac{2}{9}b_{r,s}-\frac{1}{9}b_{r,t}-\frac{1}{9}b_{r,v}+\frac{5}{32}u_1-a_ru_1$.

Proof. By Lemma 5.1.16, it is easy to check that the vector $b_{r,s} - \frac{1}{2}b_{r,t} - \frac{1}{2}b_{r,v} + \frac{45}{64}u_1 - \frac{9}{2}a_r u_1 - \frac{9}{2}b_{r,s}u_1$ is $\frac{5}{32}$-eigenvector of the axis a_s. Since our algebra A only decomposes into a direct sum of 1-, 0-, $\frac{1}{4}$- and $\frac{1}{32}$-eigenspaces, then any other eigenspace must vanish, and so the result follows. □

Corollary 5.1.18. (i) $b_{r,t}u_1 = -\frac{1}{9}b_{r,s} + \frac{2}{9}b_{r,t} - \frac{1}{9}b_{r,v} + \frac{5}{32}u_1 - a_r u_1$,

(ii) $b_{r,v}u_1 = -\frac{1}{9}b_{r,s} - \frac{1}{9}b_{r,t} + \frac{2}{9}b_{r,v} + \frac{5}{32}u_1 - a_r u_1$.

In the next lemma we will find the product $a_s \cdot (a_s b_{r,t})$.

Lemma 5.1.19. $a_s \cdot (a_s b_{r,t}) = \frac{15}{1024}a_s - \frac{7}{2048}b_{r,t} + \frac{7}{2048}b_{r,v} + \frac{1}{4}a_s b_{r,t}$.

Proof. By fusion rules, the product $u := (a_r + b_{r,s} - \frac{1}{4}a_s)(u_1 - \frac{10}{27}a_s + \frac{32}{27}(a_t + a_v)) = -\frac{1}{6}a_s + \frac{5}{27}a_t + \frac{5}{27}a_v + \frac{8}{27}a_r + \frac{8}{27}b_{r,s} - \frac{1}{3}b_{r,t} - \frac{7}{27}b_{r,v} + \frac{5}{32}u_1 + \frac{64}{27}a_s b_{r,t}$ is a 0-eigenvector of a_s. Since $a_s \cdot u = 0$, we have that $-\frac{5}{144}a_s + \frac{7}{864}b_{r,t} - \frac{7}{864}b_{r,v} - \frac{16}{27}a_s b_{r,t} + \frac{64}{27}a_s(a_s b_{r,t}) = 0$. □

By the same argument as in Lemma 5.1.14, we obtain the following.

Lemma 5.1.20. $a_r \cdot u_1 = \frac{1}{45}a_s - \frac{1}{45}a_t - \frac{1}{45}a_v + \frac{1}{45}a_r - \frac{1}{45}b_{r,s} + \frac{1}{45}b_{r,t} - \frac{1}{45}b_{r,v} + \frac{1}{32}u_1 - \frac{64}{45}a_s b_{r,t}$.

Proof. By fusion rules, the product $w_1 := (a_r - b_{r,s})(u_1 - \frac{8}{45}a_s - \frac{32}{45}(a_t + a_v)) = -\frac{2}{45}a_s - \frac{1}{15}a_t - \frac{1}{15}a_v - \frac{2}{9}a_r - \frac{2}{15}b_{r,s} + \frac{7}{45}b_{r,t} + \frac{1}{5}b_{r,v} - \frac{5}{32}u_1 + \frac{64}{45}a_s b_{r,t} + 2a_r u_1$ is in $\{1, 0\}$-eigenspace of a_s. The product $a_s \cdot w_1$ must equal to a multiple of a_s, say λa_s, for a real number λ. That is, $-\frac{1}{9}a_s + \frac{1}{90}a_t + \frac{1}{90}a_v - \frac{1}{90}a_r + \frac{1}{90}b_{r,s} - \frac{1}{90}b_{r,t} + \frac{1}{90}b_{r,v} - \frac{1}{64}u_1 + \frac{32}{45}a_s b_{r,t} + \frac{1}{2}a_r u_1 =$ [1] λa_s, and so

$$-\frac{1}{90}a_s + \frac{1}{90}a_t + \frac{1}{90}a_v - \frac{1}{90}a_r + \frac{1}{90}b_{r,s} - \frac{1}{90}b_{r,t} + \frac{1}{90}b_{r,v} - \frac{1}{64}u_1 + \frac{32}{45}a_s b_{r,t} + \frac{1}{2}a_r u_1 = 0 \quad (7)$$

[1] In this case, the value of $\lambda \neq -\frac{1}{9}$. Here we choose $\lambda = -\frac{1}{10}$ and this matches our computer program, and then our computations lead to an algebra of dimension 8.

and then

$$a_r \cdot u_1 = \frac{1}{45}a_s - \frac{1}{45}a_t - \frac{1}{45}a_v + \frac{1}{45}a_r - \frac{1}{45}b_{r,s} + \frac{1}{45}b_{r,t} - \frac{1}{45}b_{r,v} + \frac{1}{32}u_1 - \frac{64}{45}a_s b_{r,t}. \quad (8)$$

□

Lemma 5.1.21. $b_{r,s} \cdot b_{r,t} = \frac{5}{128}a_s - \frac{5}{128}a_t - \frac{5}{128}a_v + \frac{5}{128}a_r + \frac{3}{128}b_{r,s} + \frac{13}{128}b_{r,t} - \frac{1}{128}b_{r,v} - \frac{45}{4096}u_1 - \frac{5}{2}a_s b_{r,t}$.

Proof. By computation, the vector $v_1 := a_s - a_t - a_v + a_r + \frac{3}{5}b_{r,s} + \frac{13}{5}b_{r,t} - \frac{1}{5}b_{r,v} - \frac{9}{32}u_1 - 64a_s b_{r,t} - \frac{128}{5}b_{r,s} \cdot b_{r,t}$ is $-\frac{1}{16}$-eigenvector of a_r. This can be checked by multiplying a_r and v_1. The algebra A only decomposes to 1-, 0, $\frac{1}{4}$- and $\frac{1}{32}$-eigenspaces, any other eigenspace should vanish and then $v_1 = 0$ so that the product $b_{r,s} \cdot b_{r,t}$ can be obtained. □

The remaining products in this subsection are the products of $a_s b_{r,t}$ with u_1, b's and itself. We start with the following lemma.

Lemma 5.1.22. $(a_s b_{r,t}) \cdot u_1 = \frac{1}{720}a_s + \frac{1}{480}a_t + \frac{1}{480}a_v - \frac{1}{180}a_r + \frac{1}{480}b_{r,s} + \frac{1}{720}b_{r,t} + \frac{1}{480}b_{r,v} + \frac{19}{2048}u_1 + \frac{16}{45}a_s b_{r,t}$.

Proof. By Lemma 2.1.8, the axis a_s associates with its 0-eigenvector $u_1 - \frac{10}{27}a_1 + \frac{32}{27}(a_t + a_v)$ in the sense that $(a_s \cdot x) \cdot (u_1 - \frac{10}{27}a_1 + \frac{32}{27}(a_t + a_v)) = a_s \cdot (x \cdot (u_1 - \frac{10}{27}a_1 + \frac{32}{27}(a_t + a_v)))$ for all $x \in A$. In particular, put $x = b_{r,t}$, then $(a_s b_{r,t}) \cdot u_1 - \frac{251}{13824}a_s - \frac{1}{288}a_t - \frac{1}{144}a_v - \frac{1}{216}a_r + \frac{7}{864}b_{r,s} - \frac{173}{27648}b_{r,t} + \frac{77}{27648}b_{r,v} - \frac{5}{1024}u_1 + \frac{25}{54}a_s b_{r,t} = -\frac{1159}{69120}a_s - \frac{1}{720}a_t - \frac{7}{1440}a_v - \frac{11}{1080}a_r + \frac{11}{1080}b_{r,s} - \frac{673}{138240}b_{r,t} + \frac{673}{138240}b_{r,v} + \frac{9}{2048}u_1 + \frac{221}{270}a_s b_{r,t}$. Thus, the required product can be obtained. □

Lemma 5.1.23. $(a_s b_{r,t}) \cdot b_{r,s} = \frac{21}{2048}a_s - \frac{5}{512}a_t - \frac{5}{512}a_v + \frac{19}{2048}a_r - \frac{35}{2048}b_{r,s} + \frac{17}{2048}b_{r,t} - \frac{25}{2048}b_{r,v} + \frac{135}{8192}u_1 - \frac{5}{8}a_s b_{r,t}$.

Proof. The same procedure can be taken as in the previous lemma that the axis a_s associates with its 0-eigenvector $a_r + b_{r,s} - \frac{1}{4}a_s$ in which $(a_s \cdot b_{r,t}) \cdot (a_r + b_{r,s} - \frac{1}{4}a_s) = a_s \cdot (b_{r,t} \cdot (a_r + b_{r,s} - \frac{1}{4}a_s))$. This implies that

$(a_sb_{r,t})\cdot b_{r,s} - \frac{13}{4096}a_s - \frac{1}{2048}a_t + \frac{7}{2048}a_v + \frac{17}{2048}a_r - \frac{1}{2048}b_{r,s} + \frac{11}{8192}b_{r,t} + \frac{21}{8192}b_{r,v} - \frac{315}{65536}u_1 + \frac{5}{32}a_sb_{r,t} = \frac{29}{4096}a_s - \frac{21}{2048}a_t - \frac{13}{2048}a_v + \frac{9}{512}a_r - \frac{9}{512}b_{r,s} + \frac{79}{8192}b_{r,t} - \frac{79}{8192}b_{r,v} + \frac{765}{65536}u_1 - \frac{15}{32}a_sb_{r,t}$ and the result follows. □

By the action of the group G on the algebra A, we can find the products $(a_sb_{r,t}) \cdot b_{r,t}$ and $(a_sb_{r,t}) \cdot b_{r,v}$.

To find the product of $a_sb_{r,t}$ with itself, we require to have the following lemma.

Lemma 5.1.24. *Suppose that* $\alpha_1 := a_sb_{r,t} - \frac{15}{1024}a_s - \frac{9}{64}b_{r,t} - \frac{7}{64}b_{r,v}$, $\alpha_2 := a_sb_{r,t} - \frac{15}{256}a_r - \frac{15}{256}b_{r,s} - \frac{9}{64}b_{r,t} - \frac{7}{64}b_{r,v}$ *and* $\beta := a_sb_{r,t} + \frac{5}{64}(a_t + a_v) - \frac{1}{64}b_{r,t} + \frac{1}{64}b_{r,v} - \frac{225}{2048}u_1$. *Then* α_1 *and* α_2 *are 0-eigenvectors and* β *is* $\frac{1}{4}$*-eigenvector of* a_s*, respectively.*

Proof. The proof is a straightforward computation by applying a_s. □

We use Lemma 2.1.9 to find the product $(a_sb_{r,t}) \cdot (a_sb_{r,t})$.

Lemma 5.1.25. $(a_sb_{r,t}) \cdot (a_sb_{r,t}) = \frac{1527}{262144}a_s - \frac{1407}{262144}a_t - \frac{1351}{262144}a_v + \frac{1435}{262144}a_r - \frac{1407}{262144}b_{r,s} + \frac{1527}{262144}b_{r,t} - \frac{1351}{262144}b_{r,v} + \frac{65205}{8388608}u_1 - \frac{1383}{4096}a_sb_{r,t}$.

Proof. We make use of Lemma 2.1.9 and vectors in Lemma 5.1.24. Let $u := \alpha_1 \cdot \alpha_2$ and $w := \alpha_1 \cdot \beta$, where α_1, α_2 and β are the same as defined in Lemma 5.1.24. Then u and w are 0- and $\frac{1}{4}$-eigenvectors of a_s, respectively. So $w = (a_sb_{r,t}) \cdot (a_sb_{r,t}) - \frac{855}{262144}a_s + \frac{135}{131072}a_t + \frac{107}{131072}a_v - \frac{259}{262144}a_r + \frac{231}{262144}b_{r,s} - \frac{381}{131072}b_{r,t} + \frac{293}{131072}b_{r,v} - \frac{1755}{1048576}u_1 + \frac{309}{2048}a_sb_{r,t}$. Now, $w - u = \frac{35}{262144}a_s - \frac{1189}{524288}a_t - \frac{1189}{524288}a_v + \frac{11}{4096}a_r - \frac{103}{16384}b_{r,s} - \frac{9711}{524288}b_{r,t} - \frac{10273}{524288}b_{r,v} + \frac{131625}{16777216}u_1 - \frac{281}{8192}a_sb_{r,t}$. On the other hand, $w = 4a_s(w - u) = \frac{21}{8192}a_s - \frac{1137}{262144}a_t - \frac{1137}{262144}a_v + \frac{147}{32768}a_r - \frac{147}{32768}b_{r,s} +$

$\frac{765}{262144}b_{r,t} - \frac{765}{262144}b_{r,v} + \frac{51165}{8388608}u_1 - \frac{765}{4096}a_s b_{r,t}$. Therefore, $(a_s b_{r,t})\cdot(a_s b_{r,t}) = \frac{1527}{262144}a_s - \frac{1407}{262144}a_t - \frac{1351}{262144}a_v + \frac{1435}{262144}a_r - \frac{1407}{262144}b_{r,s} + \frac{1527}{262144}b_{r,t} - \frac{1351}{262144}b_{r,v} + \frac{65205}{8388608}u_1 - \frac{1383}{4096}a_s b_{r,t}$. □

At this point, we can rewrite all products in terms of a's, b's, u_1 and $a_s \cdot b_{r,t}$. In the next lemma, we try to find the product $a_s \cdot b_{r,t}$ in terms of a's, b's and u_1.

Lemma 5.1.26. $a_s \cdot b_{r,t} = \frac{1}{64}a_s - \frac{1}{64}a_t - \frac{1}{64}a_v + \frac{1}{64}a_r - \frac{1}{64}b_{r,s} + \frac{1}{64}b_{r,t} - \frac{1}{64}b_{r,v} + \frac{45}{2048}u_1$.

Proof. It is easy to check that $u := a_r - \frac{8}{3}b_{r,s} + \frac{8}{3}b_{r,t} - \frac{105}{8}u_1 - \frac{256}{3}a_s b_{r,s}$ is a 0-eigenvector of a_r. By Lemmas 5.1.9, 5.1.20, 5.1.21, 5.1.22, 5.1.23 and 5.1.25, we can compute that $u \cdot u = \frac{183}{4}a_s - \frac{1183}{36}a_t - \frac{1183}{36}a_v + \frac{887}{36}a_r - \frac{239}{36}b_{r,s} + \frac{703}{36}b_{r,t} - \frac{1183}{36}b_{r,v} + \frac{31955}{128}u_1 - \frac{15088}{9}a_s b_{r,t}$. By fusion rules, $u \cdot u$ is 0-eigenvector of a_r, that is $a_r \cdot (u \cdot u) = 0$. However, $0 = a_r \cdot (u \cdot u) = \frac{203}{18}a_s - \frac{203}{18}a_t - \frac{203}{18}a_v + \frac{203}{18}a_r - \frac{203}{18}b_{r,s} + \frac{203}{18}b_{r,t} - \frac{203}{18}b_{r,v} + \frac{1015}{64}u_1 - \frac{6496}{9}a_s b_{r,t}$. Therefore, $a_s \cdot b_{r,t} = \frac{1}{64}a_s - \frac{1}{64}a_t - \frac{1}{64}a_v + \frac{1}{64}a_r - \frac{1}{64}b_{r,s} + \frac{1}{64}b_{r,t} - \frac{1}{64}b_{r,v} + \frac{45}{2048}u_1$. □

The summary of the products in this subsection are as follows:

$$a_s \cdot a_s = a_s;\ a_r \cdot a_r = a_r;$$

$$b_{r,s} \cdot b_{r,s} = b_{r,s};$$

$$a_s \cdot a_r = \frac{1}{8}(a_s + a_r - b_{r,s});$$

$$a_s \cdot b_{r,s} = \frac{1}{8}(a_s + b_{r,s} - a_r);$$

$$u_1 \cdot u_1 = u_1;$$

$$a_s \cdot a_t = \frac{1}{32}(2a_s + 2a_t + a_v) - \frac{135}{2^{11}}u_1;$$

$$a_s \cdot u_1 = \frac{1}{9}(2a_s - a_t - a_v) + \frac{5}{32}u_1;$$

$$b_{r,s}\cdot b_{r,t}=\frac{1}{32}(2b_{r,s}+2b_{r,t}+b_{r,v})-\frac{135}{2^{11}}u_1;$$

$$b_{r,s}\cdot u_1=\frac{1}{9}(2b_{r,s}-b_{r,t}-b_{r,v})+\frac{5}{32}u_1;$$

$$a_r\cdot u_1=0;$$

$$a_s\cdot b_{r,t}=\frac{1}{64}a_s-\frac{1}{64}a_t-\frac{1}{64}a_v+\frac{1}{64}a_r-\frac{1}{64}b_{r,s}+\frac{1}{64}b_{r,t}-\frac{1}{64}b_{r,v}+\frac{45}{2048}u_1;$$

Before we conclude this subsection, the author suggests the reader to save the appendices B, C, D, E and F in a G files under the names "D12-2a3a.g", "D12-2b3a.g", "D12-2b3c.g", "algebraaxioms.g" and "fusionrules.g", respectively.

We conclude this subsection with the following proposition.

Proposition 5.1.27. *The M-axial algebra of the shape $(2A,3A)$ for the group D_{12} has dimension 8.*

Proof. The proof can be done with GAP as follows:

gap> Read("D12-2a3a.g");

gap> Read("algebraaxioms.g");

gap> Read("fusionrules.g");

if nothing appeared, then the shape $(2A,3A)$ leads to an algebra. □

5.1.2 The shape $(2A,3C)$

In this subsection, we assume that the algebra A is generated by a set of four axes denoted by a_s, a_t, a_v and a_r and the three vectors denoted by $b_{r,s}, b_{r,t}$ and $b_{r,v}$ which are vectors in the subalgebras $\langle\langle a_r, a_s\rangle\rangle$, $\langle\langle a_r, a_t\rangle\rangle$ and $\langle\langle a_r, a_v\rangle\rangle$ of type $2A$, respectively. By Table 2.3, the following products are known.

$$a_s \cdot a_s = a_s;\ a_r \cdot a_r = a_r;$$

$$b_{r,s} \cdot b_{r,s} = b_{r,s};$$

$$a_s \cdot a_t = \frac{1}{64}(a_s + a_t - a_v);$$

$$a_s \cdot a_r = \frac{1}{8}(a_s + a_r - b_{r,s});$$

$$a_s \cdot b_{r,s} = \frac{1}{8}(a_s + b_{r,s} - a_r);$$

$$a_r \cdot b_{r,s} = \frac{1}{8}(a_r + b_{r,s} - a_s).$$

The main aim in this subsection is to prove the following proposition.

Proposition 5.1.28. *The M-axial algebra of the shape $(2A, 3C)$ is trivial.*

We will prove Proposition 5.1.28 in several lemmas. First, we give all known eigenvectors of the axes a_s and a_r in the following tables.

Type	0-eigenvectors	$\frac{1}{4}$-eigenvectors	$\frac{1}{32}$-eigenvectors
$2A$	$a_r + b_{r,s} - \frac{1}{4}a_s$	$a_r - b_{r,s}$	
$3C$	$a_t + a_v - \frac{1}{32}a_s$		$a_t - a_v$

Table 5.4: Eigenvectors of a_s

Type	0-eigenvectors	$\frac{1}{4}$-eigenvectors
$2A$	$a_s + b_{r,s} - \frac{1}{4}a_r$	$a_s - b_{r,s}$
	$a_t + b_{r,t} - \frac{1}{4}a_r$	$a_t - b_{r,t}$
	$a_v + b_{r,v} - \frac{1}{4}a_r$	$a_v - b_{r,v}$

Table 5.5: Eigenvectors of a_r

Lemma 5.1.29.

$b_{r,v} - b_{r,t}$ and $a_t b_{r,s} - a_v b_{r,s}$ are $\frac{1}{32}$-eigenvectors of a_s.

Proof. By fusion rules, the product $w_1 := (a_r + b_{r,s} - \frac{1}{4}a_s)(a_t - a_v)$ is $\frac{1}{32}$-eigenvector of a_s. Then $w_1 = \frac{1}{8}(a_r + a_t - b_{r,t}) - \frac{1}{8}(a_r + a_v - b_{r,v}) + a_t b_{r,s} - a_v b_{r,s} - \frac{1}{4}(\frac{1}{64}(a_s + a_t - a_v - a_s - a_v + a_t)) = a_t b_{r,s} - a_v b_{r,s} + \frac{15}{128}a_t - \frac{15}{128}a_v - \frac{1}{8}b_{r,t} + \frac{1}{8}b_{r,v}$.

Also, $w_2 := (a_r - b_{r,s})(a_t - a_v)$ is $\frac{1}{32}$-eigenvector of a_s and so $w_2 = a_v b_{r,s} - a_t b_{r,s} + \frac{1}{8}(a_r + a_t - b_{r,t}) - \frac{1}{8}(a_r + a_v - b_{r,v}) = a_v b_{r,s} - a_t b_{r,s} + \frac{1}{8}a_t - \frac{1}{8}a_v - \frac{1}{8}b_{r,t} + \frac{1}{8}b_{r,v}$. It is clear that $w_1 + w_2 = \frac{31}{128}(a_t - a_v) + \frac{1}{4}(b_{r,v} - b_{r,t})$ and $w_1 - w_2 = 2(a_t b_{r,s} - a_v b_{r,s}) - \frac{1}{128}(a_t - a_v)$ are $\frac{1}{32}$-eigenvectors of a_s. From Table 5.4, we have also that $a_t - a_v$ is $\frac{1}{32}$-eigenvector of a_s and so $b_{r,v} - b_{r,t}$ and $a_t b_{r,s} - a_v b_{r,s}$ are $\frac{1}{32}$-eigenvectors of a_s. □

By using the action of G, we have the following.

Corollary 5.1.30.

$b_{r,v} - b_{r,s}$ and $b_{r,t} - b_{r,s}$ are $\frac{1}{32}$-eigenvectors of a_t and a_v, respectively.

From now, we try to find all possible unknown products in the algebra A.

Lemma 5.1.31. (i) $a_s \cdot (a_t \cdot b_{r,s}) = -\frac{7}{512}a_s - \frac{1}{64}a_r + \frac{1}{64}b_{r,s} - \frac{1}{512}b_{r,t} + \frac{1}{512}b_{r,v} + \frac{1}{8}a_s b_{r,t} + \frac{1}{64}a_t b_{r,s} - \frac{1}{64}a_v b_{r,s}$,

(ii) $a_s \cdot (a_v \cdot b_{r,s}) = -\frac{7}{512}a_s - \frac{1}{64}a_r + \frac{1}{64}b_{r,s} - \frac{1}{512}b_{r,t} + \frac{1}{512}b_{r,v} + \frac{1}{8}a_s b_{r,t} - \frac{1}{64}a_t b_{r,s} + \frac{1}{64}a_v b_{r,s}$.

Proof. From fusion rules and Table 5.4, the product $u := (a_r + b_{r,s} - \frac{1}{4}a_s)(a_t + a_v - \frac{1}{32}a_s)$ is 0-eigenvector of a_s and so $u = -\frac{1}{128}a_s + \frac{1}{8}a_t + \frac{1}{8}a_v + \frac{1}{4}a_r - \frac{1}{8}b_{r,t} - \frac{1}{8}b_{r,v} + a_t b_{r,s} + a_v b_{r,s}$. Since $a_s \cdot u = 0$, we have

$$a_s \cdot (a_t b_{r,s}) + a_s \cdot (a_v b_{r,s}) = -\frac{7}{256}a_s - \frac{1}{32}a_r + \frac{1}{32}b_{r,s} - \frac{1}{256}b_{r,t} + \frac{1}{256}b_{r,v} + \frac{1}{4}a_s b_{r,t}. \quad (9)$$

By Lemma 5.1.29, we have that

$$a_s \cdot (a_t b_{r,s}) - a_s \cdot (a_v b_{r,s}) = \frac{1}{32}(a_t b_{r,s} - a_v b_{r,s}). \qquad (10)$$

By solving equations (9) and (10) the result follows. □

Lemma 5.1.32. $a_r \cdot (a_s b_{r,t}) = \frac{15}{2^9}a_r + \frac{1}{2^9}b_{r,s} + \frac{1}{2^9}b_{r,t} - \frac{1}{2^9}b_{r,v} + \frac{1}{2^3}a_s b_{r,t} - \frac{1}{2^3}a_t b_{r,s} - \frac{1}{2^3}b_{r,s}b_{r,t}$.

Proof. From Table 5.5, $u := a_t + b_{r,t} - \frac{1}{4}a_r$ is 0-eigenvector of a_r and by Lemma 2.1.8, a_r associates with u in the sense that $(a_4 \cdot v) \cdot u = a_4 \cdot (v \cdot u)$ for all $v \in A$. Choose $u = a_s$, then $\frac{1}{8}(a_s + a_r - b_{r,s})(a_t + b_{r,t} - \frac{1}{4}a_r) = a_r \cdot (-\frac{1}{64}a_s + \frac{1}{64}a_t - \frac{1}{64}a_v - \frac{1}{32}a_r + \frac{1}{32}b_{r,s} + a_s b_{r,t})$, which implies that $-\frac{3}{512}a_s + \frac{1}{512}a_t - \frac{1}{512}a_v + \frac{1}{128}b_{r,s} + \frac{1}{8}a_s b_{r,t} - \frac{1}{8}a_t b_{r,s} - \frac{1}{8}b_{r,s}b_{r,t} = -\frac{3}{512}a_s + \frac{1}{512}a_t - \frac{1}{512}a_v - \frac{15}{512}a_r + \frac{3}{512}b_{r,s} - \frac{1}{512}b_{r,t} + \frac{1}{512}b_{r,v} + a_r \cdot (a_s b_{r,t})$, so the result follows. □

Since the group G acts on the algebra A, we can have the following.

Corollary 5.1.33. (i) $a_r \cdot (a_t b_{r,s}) = \frac{15}{512}a_r + \frac{1}{512}b_{r,s} + \frac{1}{512}b_{r,t} - \frac{1}{512}b_{r,v} - \frac{1}{8}a_s b_{r,t} + \frac{1}{8}a_t b_{r,s} - \frac{1}{8}b_{r,s}b_{r,t}$,

(ii) $a_r \cdot (a_v b_{r,s}) = \frac{15}{512}a_r + \frac{1}{512}b_{r,s} + \frac{1}{512}b_{r,t} - \frac{1}{512}b_{r,v} - \frac{1}{8}a_s b_{r,t} + \frac{1}{8}a_v b_{r,s} - \frac{1}{8}b_{r,s}b_{r,v}$.

In the next lemma we try to find the product between a_r and $b_{r,s}b_{r,t}$.

Lemma 5.1.34. $a_r \cdot (b_{r,s}b_{r,t}) = -\frac{1}{512}a_s - \frac{1}{512}a_t + \frac{1}{512}a_v + \frac{1}{512}a_r - \frac{1}{512}b_{r,s} - \frac{1}{512}b_{r,t} + \frac{1}{512}b_{r,v} + \frac{1}{4}b_{r,s}b_{r,t}$.

Proof. By Lemma 2.1.8, the 0-eigenvector $u := a_s + b_{r,s} - \frac{1}{4}a_r$ of a_r associates with a_r. Thus, $(a_r \cdot b_{r,t}) \cdot u = a_r \cdot (b_{r,t} \cdot u)$ and so $-\frac{1}{512}a_s + \frac{3}{512}a_t + \frac{1}{512}a_v - \frac{1}{128}b_{r,t} + \frac{1}{8}a_s b_{r,t} - \frac{1}{8}a_t b_{r,s} + \frac{1}{8}b_{r,s}b_{r,t} = a_r \cdot (\frac{1}{32}a_t - \frac{1}{32}a_r - \frac{1}{32}b_{r,t} + a_s b_{r,t} + b_{r,s}b_{r,t})$. By Lemma 5.1.32, $-\frac{1}{512}a_s + \frac{3}{512}a_t + \frac{1}{512}a_v - \frac{1}{128}b_{r,t} + \frac{1}{8}a_s b_{r,t} - \frac{1}{8}a_t b_{r,s} + \frac{1}{8}b_{r,s}b_{r,t} = \frac{1}{128}a_t - \frac{1}{512}a_r + \frac{1}{512}b_{r,s} - \frac{3}{512}b_{r,t} - \frac{1}{512}b_{r,v} + \frac{1}{8}a_s b_{r,t} - \frac{1}{8}a_t b_{r,s} - \frac{1}{8}b_{r,s}b_{r,t} + a_r \cdot (b_{r,s}b_{r,t})$, and the result follows. □

The following corollary is a direct consequence of the above lemma.

Corollary 5.1.35. $a_r \cdot (b_{r,s}b_{r,v}) = -\frac{1}{512}a_s + \frac{1}{512}a_t - \frac{1}{512}a_v + \frac{1}{512}a_r - \frac{1}{512}b_{r,s} + \frac{1}{512}b_{r,t} - \frac{1}{512}b_{r,v} + \frac{1}{4}b_{r,s}b_{r,v}$.

The following lemma gives a useful relation which help us to show that most of the axes are vanish.

Lemma 5.1.36. $b_{r,s}b_{r,v} = -\frac{1}{32}a_t + \frac{1}{32}a_v - \frac{1}{32}b_{r,t} + \frac{1}{32}b_{r,v} + a_tb_{r,s} - a_vb_{r,s} + b_{r,s}b_{r,t}$.

Proof. It is easy to check that the vector $-\frac{1}{32}a_t + \frac{1}{32}a_v - \frac{1}{32}b_{r,t} + \frac{1}{32}b_{r,v} + a_tb_{r,s} - a_vb_{r,s} + b_{r,s}b_{r,t} - b_{r,s}b_{r,v}$ is $\frac{1}{2^3}$-eigenvector of the axis a_r. Since our algebra A is only decomposes into a direct sum of 1-, 0-, $\frac{1}{4}$- and $\frac{1}{32}$-eigenspaces, then any other eigenspace should vanish, so the result follows. □

In the next lemma, we try to find the product between a_t and $a_sb_{r,t}$.

Lemma 5.1.37. $a_t \cdot (a_sb_{r,t}) = -\frac{7}{512}a_t - \frac{1}{64}a_r - \frac{3}{2048}b_{r,s} + \frac{31}{2048}b_{r,t} + \frac{1}{512}b_{r,v} + \frac{1}{64}a_sb_{r,t} + \frac{1}{8}a_tb_{r,s} - \frac{1}{64}a_vb_{r,s}$.

Proof. From Table 5.4 and Lemma 2.1.8, the 0-eigenvector $u := a_r + b_{r,t} - \frac{1}{4}a_t$ of a_t associates with a_t and so $(a_t \cdot a_s) \cdot u = a_t(\cdot a_s \cdot u)$. Thus, $\frac{15}{8192}a_s - \frac{15}{8192}a_v - \frac{3}{2048}b_{r,s} - \frac{1}{2048}b_{r,t} + \frac{1}{512}b_{r,v} + \frac{1}{64}a_sb_{r,t} - \frac{1}{64}a_vb_{r,s} = \frac{15}{8192}a_s + \frac{7}{512}a_t - \frac{15}{8192}a_v + \frac{1}{64}a_r - \frac{1}{64}b_{r,t} - \frac{1}{8}a_tb_{r,s} + a_t \cdot (a_sb_{r,t})$. Therefore, $a_t \cdot (a_sb_{r,t}) = -\frac{7}{512}a_t - \frac{1}{64}a_r - \frac{3}{2048}b_{r,s} + \frac{31}{2048}b_{r,t} + \frac{1}{512}b_{r,v} + \frac{1}{64}a_sb_{r,t} + \frac{1}{8}a_tb_{r,s} - \frac{1}{64}a_vb_{r,s}$. □

Corollary 5.1.38. $a_v \cdot (a_sb_{r,t}) = -\frac{9}{512}a_v - \frac{3}{256}a_r - \frac{5}{2048}b_{r,s} + \frac{5}{2048}b_{r,t} + \frac{3}{256}b_{r,v} + \frac{1}{64}a_sb_{r,t} - \frac{1}{64}a_tb_{r,s} + \frac{5}{32}a_vb_{r,s}$.

Lemma 5.1.39. (i) $a_tb_{r,s} = -\frac{1}{8}a_s + \frac{1}{8}a_t + \frac{1}{8}b_{r,s} - \frac{13}{96}b_{r,t} + \frac{1}{96}b_{r,v} + a_sb_{r,t}$,

(ii) $a_vb_{r,s} = -\frac{1}{8}a_s + \frac{1}{8}a_v + \frac{1}{8}b_{r,s} - \frac{1}{48}b_{r,t} - \frac{5}{48}b_{r,v} + a_sb_{r,t}$.

Proof. From Table 5.4 and Lemma 2.1.8 we have

$$(a_s \cdot b_{r,t}) \cdot (a_t + a_v - \frac{1}{32}a_s) = (a_s \cdot (b_{r,t} \cdot (a_t + a_v - \frac{1}{32}a_s)). \tag{11}$$

By Lemma 5.1.37 and Corollary 5.1.38, left-hand side of equation (11) is equal to

$$-\frac{7}{512}a_t - \frac{9}{512}a_v - \frac{7}{256}a_r - \frac{1}{256}b_{r,s} + \frac{9}{512}b_{r,t} + \frac{7}{512}b_{r,v} + \frac{1}{32}a_s b_{r,t} + \frac{7}{64}a_t b_{r,s} + \frac{9}{64}a_v b_{r,s} - \frac{1}{32}(a_s b_{r,t}) \cdot a_s, \tag{12}$$

and from Corollary 5.1.30 and Lemma 5.1.31, right-hand side of equation (11) is equal to

$$-\frac{1}{32}a_s + \frac{1}{512}a_t - \frac{1}{512}a_v - \frac{7}{256}a_r + \frac{7}{256}b_{r,s} - \frac{1}{512}b_{r,t} + \frac{1}{512}b_{r,v} + \frac{9}{32}a_s b_{r,t} - \frac{1}{64}a_t b_{r,s} + \frac{1}{64}a_v b_{r,s} - \frac{1}{32}a_s \cdot (a_s b_{r,t}). \tag{13}$$

The equality of (12) and (13) gives the following

$$\frac{1}{32}a_s - \frac{1}{64}a_t - \frac{1}{64}a_v - \frac{1}{32}b_{r,s} + \frac{5}{256}b_{r,t} + \frac{3}{256}b_{r,v} - \frac{1}{4}a_s b_{r,t} + \frac{1}{8}a_t b_{r,s} + \frac{1}{8}a_v b_{r,s} = 0. \tag{14}$$

The action of τ_{a_t} on equation (14) gives another equation, which is

$$-\frac{1}{64}a_s - \frac{1}{64}a_t + \frac{1}{32}a_v + \frac{1}{64}b_{r,s} + \frac{3}{256}b_{r,t} - \frac{7}{256}b_{r,v} + \frac{1}{8}a_s b_{r,t} + \frac{1}{8}a_t b_{r,s} - \frac{1}{4}a_v b_{r,s} = 0. \tag{15}$$

From equations (14) and (15), we get the following

$$a_t b_{r,s} = -\frac{1}{8}a_s + \frac{1}{8}a_t + \frac{1}{8}b_{r,s} - \frac{13}{96}b_{r,t} + \frac{1}{96}b_{r,v} + a_s b_{r,t}. \tag{16}$$

Substituting equation (16) in equation (15) we obtain

$$a_v b_{r,s} = -\frac{1}{8}a_s + \frac{1}{8}a_v + \frac{1}{8}b_{r,s} - \frac{1}{48}b_{r,t} - \frac{5}{48}b_{r,v} + a_s b_{r,t}. \tag{17}$$

□

We can rewrite Lemma 5.1.37 and Corollary 5.1.38 as the following.

Lemma 5.1.40. (i) $a_t \cdot (a_s b_{r,t}) = -\frac{7}{512}a_s + \frac{1}{512}a_t - \frac{1}{512}a_v - \frac{1}{64}a_r + \frac{25}{2048}b_{r,s} - \frac{3}{2048}b_{r,t} + \frac{5}{1024}b_{r,v} + \frac{1}{8}a_s b_{r,t}$,

(ii) $a_v \cdot (a_s b_{r,t}) = -\frac{9}{512}a_s - \frac{1}{512}a_t + \frac{1}{512}a_v - \frac{3}{256}a_r + \frac{31}{2048}b_{r,s} + \frac{1}{768}b_{r,t} - \frac{29}{6144}b_{r,v} + \frac{5}{32}a_s b_{r,t}$.

From now we try to find relations in order to show that the spanning set of the algebra A contains only zero.

Lemma 5.1.41. $b_{r,s} = b_{r,t} = b_{r,v}$.

Proof. From information in Table 5.4, the vectors $u := (a_v + b_{r,t} - \frac{1}{4}a_t)(a_s + a_v - \frac{1}{32}a_t) = -\frac{1}{128}a_t + \frac{1}{4}a_v + \frac{1}{4}a_r - \frac{1}{32}b_{r,s} + \frac{1}{96}b_{r,t} - \frac{11}{48}b_{r,v} + 2a_s b_{r,t}$ and $w := (a_r - b_{r,t})(a_s + a_v - \frac{1}{32}a_t) = \frac{1}{4}a_s + \frac{31}{128}a_r - \frac{7}{32}b_{r,s} - \frac{1}{384}b_{r,t} - \frac{1}{48}b_{r,v} - 2a_s b_{r,t}$ are 0- and $\frac{1}{4}$-eigenvectors of a_t, respectively. Since $a_t \cdot u = 0$ and $a_t \cdot w - \frac{1}{4}w = 0$, we have that

$$\frac{1}{768}a_s - \frac{1}{768}a_r - \frac{1}{1024}b_{r,s} + \frac{11}{4608}b_{r,t} - \frac{1}{9216}b_{r,v} - \frac{1}{96}a_s b_{r,t} = 0 \tag{18}$$

and

$$-\frac{1}{768}a_s + \frac{1}{768}a_r + \frac{1}{1024}b_{r,s} + \frac{25}{4608}b_{r,t} - \frac{71}{9216}b_{r,v} + \frac{1}{96}a_s b_{r,t} = 0, \tag{19}$$

respectively. From equations (18) and (19), we see that $\frac{1}{128}b_{r,t} - \frac{1}{128}b_{r,v} = 0$, which implies $b_{r,t} = b_{r,v}$ and by the action of the group G on A, we get that $b_{r,s} = b_{r,t} = b_{r,v}$. □

Lemma 5.1.42. *All axes vanish.*

Proof. From Table 5.5 and Lemma 5.1.41, we can see that $(a_s+b_{r,s}-\frac{1}{4}a_r)-(a_t+b_{r,t}-\frac{1}{4}a_r) = a_s-a_t$ and $(a_s-b_{r,s})-(a_t-b_{r,t}) = a_s-a_t$ are 0- and $\frac{1}{4}$-eigenvectors of a_r, respectively. Since the only vector can be an eigenvector for two distinct eigenvalues is zero, then $a_s-a_t = 0$, that is, $a_s = a_t$ and hence $a_s = a_t = a_v$. Back to Table 5.4, $a_t + a_v - \frac{1}{32}a_s = \frac{63}{32}a_s$ is 0-eigenvetor of a_s, that is, $a_s \cdot a_s = 0$, but $a_s \cdot a_s = a_s$, then $a_s = 0$ and so $a_s = a_t = a_v = 0$. Also the vector $v := b_{r,s} - \frac{1}{4}a_r$ is 0-eigenvector of a_r, then $a_r \cdot v = -\frac{1}{8}a_r + \frac{1}{8}b_{r,s} = 0$. Thus, $a_r = b_{r,s}$ and by the same argument as before $a_r = 0$. Therefore, the algebra A is trivial. □

The Proposition 5.1.28 is now proved.

5.1.3 The shape $(2B, 3A)$

In this case, we assume that the algebra A is generated by five elements, which are the four axes denoted by a_s, a_t, a_v and a_r and the vector u_1 in the subalgebra $\langle\langle a_s, a_t\rangle\rangle$ of type $3A$. According to the Table 2.3, The following are the known products of the vectors in the generating set of A.

$$a_s \cdot a_s = a_s;\ a_r \cdot a_r = a_r;$$

$$a_s \cdot a_r = 0;$$

$$a_s \cdot a_t = \frac{1}{2^5}(2a_s + 2a_t + a_v) - \frac{3^3 \cdot 5}{2^{11}}u_1;$$

$$a_s \cdot u_1 = \frac{1}{3^2}(2a_s - a_t - a_v) + \frac{5}{2^5}u_1.$$

The only unknown product in this case is $a_r \cdot u_1$, which can be done in the following lemma.

Lemma 5.1.43. $a_r \cdot u_1 = 0$.

Proof. Since the vectors a_s and a_t are 0-eigenvectors of a_r, by fusion rules, we have that $u := a_s \cdot a_t$ is also 0-eigenvector of a_r. So $u = a_s \cdot a_t = \frac{1}{2^5}(2a_s + 2a_t + a_v) - \frac{3^3 \cdot 5}{2^{11}} u_1$. Since $a_r \cdot u = 0$ and $a_r \cdot a_s = a_r \cdot a_t = a_r \cdot a_v = 0$, then the result follows. □

Proposition 5.1.44. *The M-axial algebra of the shape* $(2B, 3A)$ *is of dimension 5.*

Proof. The proof can be done with GAP as follows:

gap> Read("D12-2b3a.g");

gap> Read("algebraaxioms.g");

gap> Read("fusionrules.g");

if nothing appeared, then the shape $(2B, 3A)$ leads to an algebra. □

5.1.4 The shape $(2B, 3C)$

In this case, we consider the algebra A is generated by the set of four M-axes denoted by a_s, a_t, a_v and a_r such that the subalgebras $\langle\langle a_r, a_s\rangle\rangle$, $\langle\langle a_r, a_t\rangle\rangle$ and $\langle\langle a_r, a_v\rangle\rangle$ are all of type $2B$ and the subalgebra $\langle\langle a_s, a_t\rangle\rangle$ is of type $3C$. The algebra A is 1-closed, that is, all products between M-axes are known from Table 2.3 as below

$$a_s \cdot a_s = a_s;\ a_r \cdot a_r = a_r;$$

$$a_s \cdot a_r = 0;$$

$$a_s \cdot a_t = \frac{1}{64}(a_s + a_t - a_v).$$

So, we can only have the following proposition.

Proposition 5.1.45. *The M-axial algebra of the shape* $(2B, 3C)$ *is of dimension 4.*

Proof. The proof can be done with GAP as follows:

gap> Read("D12-2b3c.g");

gap> Read("algebraaxioms.g");

gap> Read("fusionrules.g");

if nothing appeared, then the shape $(2B, 3C)$ leads to an algebra. □

5.2 M-Axial algebras for the group $(S_3 \times S_3) : 2$

In this section we consider the group $G :=(S_3 \times S_3) : 2$. Let $G = (\langle x, s\rangle \times \langle y, t\rangle) : \langle u\rangle$, where $x = (1, 2, 3), s = (2, 3), y = (4, 5, 6), t = (5, 6)$ and $u = (1, 4)(2, 5)(3, 6)$. It is clear that $x^3 = y^3 = s^2 = t^2 = u^2 = 1$, $x^s = x^{-1}, y^t = y^{-1}, x^t = x, y^s = y, x^u = y$ and $s^u = t$.

Let $a = s = (2, 3), b = sx = (1, 2)$ and $c = u^{yt} = (1, 6)(2, 5)(3, 4)$. Then the group G satisfies the presentation $\langle a,\ b,\ c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^4 = (bc)^4 = (ab^c)^2 = 1\rangle$, which is the same as the group $T^{(3,4,4;2)}$ given in Proposition 4.4.4. Since the product ab has order 3, the elements a and b belong to the same conjugacy class, say a^G, and also a and t are conjugate, then t and its conjugates belong to a^G. From here we can say that the sizes of the conjugacy classes with representatives a and c are 6 and 6, respectively. Thus, $a^G = \{(5, 6), (4, 5), (4, 6), (2, 3), (1, 2), (1, 3)\}$ and $c^G = \{(1, 4)(2, 5)(3, 6), (1, 5)(2, 6)(3, 4), (1, 6)(2, 4)(3, 5), (1, 4)(2, 6)(3, 5), (1, 5)(2, 4)(3, 6), (1, 6)(2, 5)(3, 4)\}$.

Relabeling the involutions in $T := a^G \cup c^G$ by $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}$ and b_{12} and determining the number of orbits on pairs of elements in T and the type of the Sakuma subalgebras generated by any pair of idempotents correspond to a pair of involutions in T (see Figure 5.1).

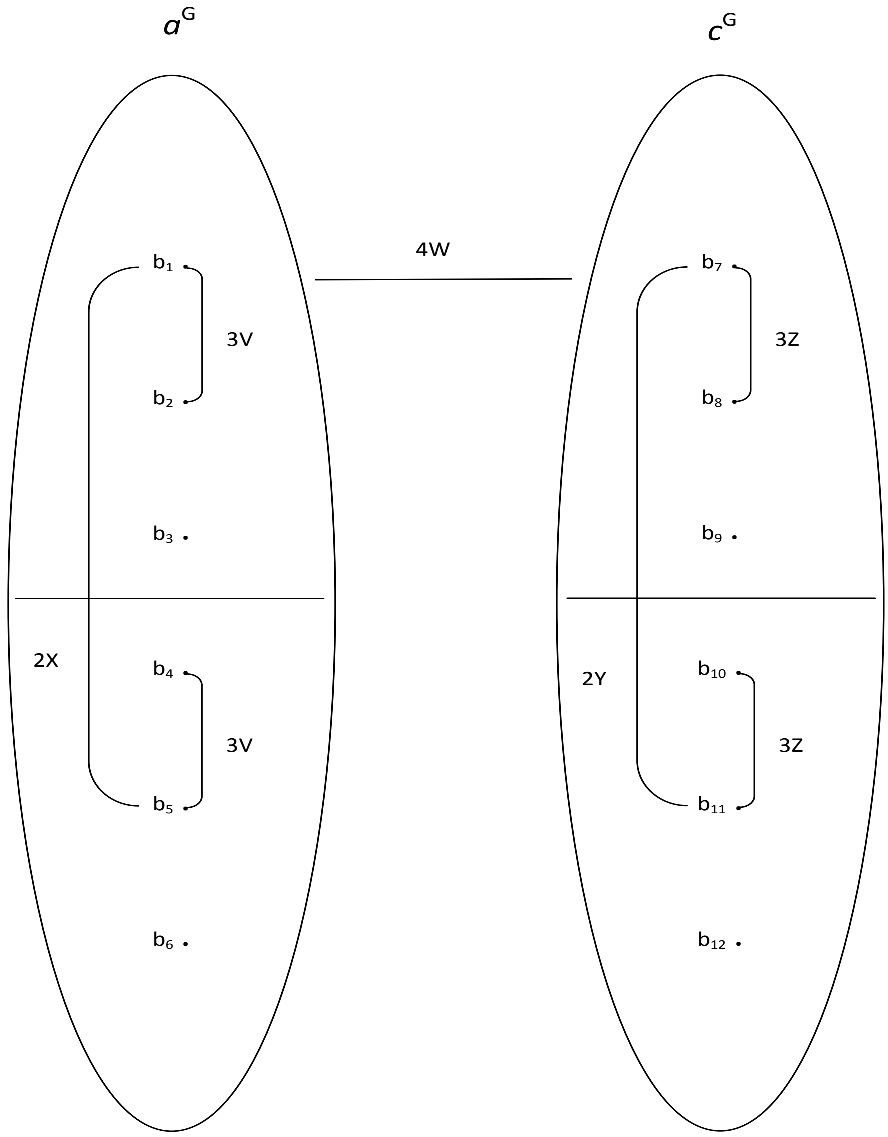

Figure 5.1: Orbits on pairs for the group $(S_3 \times S_3) : 2$

In Figure 5.1, the lines correspond to orbits on pairs of involutions from T and the letters nN, where $n \in \{2, 3, 4\}$, $N \in \{V, W, X, Y, Z\}$, correspond to the type of the Sakuma subalgebras generated by a pair of axes. The subalgebras $2X$ and $2Y$ corresponding to the orbit of pair of involutions whose product has order 2, determine the type of the subalgebra $4W$ because both of them are subalgebras of it, and then X and Y must be equal. The subalgebras $3V$ and $3Z$ are independent. So all possible shapes for the group G are listed in the following table.

$(3V, 3Z, 4W)$
$(3A, 3A, 4A)$
$(3C, 3A, 4A)$
$(3C, 3C, 4A)$
$(3A, 3A, 4B)$
$(3C, 3A, 4B)$
$(3C, 3C, 4B)$

Table 5.6: G-shapes

Each case will be treated separately in a subsection.

5.2.1 The shape $(3C, 3C, 4A)$

In this case, the generating set of the algebra A consists of twelve M-axes, denoted by $a_1, ..., a_{12}$ and nine vectors, denoted by $v_1, ..., v_9$ which are vectors in the subalgebras $\langle\langle a_i, a_j \rangle\rangle$, for $i = 1, ..., 6$ and $j = 7, ..., 12$, of type $4A$. The following are the known products in algebra by the information in Table 2.3.

$$a_1 \cdot a_1 = a_1; \;\; a_7 \cdot a_7 = a_7;$$

$$a_1 \cdot a_4 = 0;\ a_7 \cdot a_{10} = 0;$$

$$a_1 \cdot a_2 = \frac{1}{2^6}(a_1 + a_2 - a_3);\ a_7 \cdot a_8 = \frac{1}{2^6}(a_7 + a_8 - a_9);$$

$$a_1 \cdot a_7 = \frac{1}{2^6}(3a_1 + 3a_7 + a_4 + a_{10} - 3v_5);$$

$$a_1 \cdot v_5 = \frac{1}{2^4}(5a_1 - 2a_7 - a_4 - 2a_{10} + 3v_5);$$

$$v_5 \cdot v_5 = v_5;\ a_7 \cdot v_5 = \frac{1}{2^4}(5a_7 - 2a_1 - a_{10} - 2a_4 + 3v_5);$$

Our aim in this subsection is to prove the following proposition, which can be done in several lemmas.

Proposition 5.2.1. *The M-axial algebra of the shape $(3C, 3C, 4A)$ is trivial.*

Here we start by the following lemma.

Lemma 5.2.2.

$v_2 - v_1$ is $\frac{1}{32}$-eigenvector of the M-axis a_5.

Proof. First, we list some of the known eigenvectors of the axis a_5 in Table 5.7.

Type	0-eigenvectors	$\frac{1}{4}$-eigenvectors	$\frac{1}{32}$-eigenvectors
$2B$	a_1, a_2, a_3		
$3C$	$a_4 + a_6 - \frac{1}{32}a_5$		$a_4 - a_6$
$4A$	$v_3 - \frac{1}{2}a_5 + 2(a_7 + a_{11})$	$v_3 - \frac{1}{3}a_5 - \frac{2}{3}(a_7 + a_{11}) - \frac{1}{3}a_2$	$a_7 - a_{11}$
	$v_8 - \frac{1}{2}a_5 + 2(a_9 + a_{10})$	$v_8 - \frac{1}{3}a_5 - \frac{2}{3}(a_9 + a_{10}) - \frac{1}{3}a_3$	$a_9 - a_{10}$
	$v_9 - \frac{1}{2}a_5 + 2(a_8 + a_{12})$	$v_9 - \frac{1}{3}a_5 - \frac{2}{3}(a_8 + a_{12}) - \frac{1}{3}a_1$	$a_8 - a_{12}$

Table 5.7: Eigenvectors of a_5

From Table 5.7 and fusion rules, we see that the vector $v := a_2(a_9 - a_{10}) = \frac{1}{64}a_4 - \frac{1}{64}a_6 - \frac{1}{64}a_8 + \frac{3}{64}a_9 - \frac{3}{64}a_{10} + \frac{1}{64}a_{12} - \frac{3}{64}v_1 + \frac{3}{64}v_2$ is $\frac{1}{32}$-eigenvector of a_5. We can compute that

$a_5v = \frac{1}{2048}a_4 - \frac{1}{2048}a_6 - \frac{1}{2048}a_8 + \frac{3}{2048}a_9 - \frac{3}{2048}a_{10} + \frac{1}{2048}a_{12} - \frac{3}{64}a_5v_1 + \frac{3}{64}a_5v_2$. Since $a_5v = \frac{1}{32}v$, we have that

$$a_5(v_2 - v_1) = \frac{1}{32}(v_2 - v_1).$$

□

By the action of the group G on the algebra A, we have the following

$$a_5(v_6 - v_4) = \frac{1}{32}(v_6 - v_4). \tag{20}$$

and

$$a_5(v_7 - v_5) = \frac{1}{32}(v_7 - v_5). \tag{21}$$

Corollary 5.2.3.

The vector $a_2(v_6 - v_4)$ is $\frac{1}{32}$-eigenvector of a_5

Lemma 5.2.4.

$a_5v_1 = \frac{1}{96}a_1 - \frac{13}{192}a_2 + \frac{1}{96}a_3 - \frac{1}{96}a_4 + \frac{13}{192}a_5 - \frac{1}{96}a_6 + \frac{3}{64}v_1 + \frac{1}{64}v_2 - \frac{3}{64}v_8 - \frac{1}{64}v_9 + a_2v_8$.

Proof. Again from Table 5.7 and fusion rules, the vectors $u := a_2(v_8 - \frac{1}{2}a_5 + 2(a_9 + a_{10}))$ and $w := a_2(v_8 - \frac{1}{3}a_5 - \frac{2}{3}(a_9 + a_{10}) - \frac{1}{3}a_3)$ are 0- and $\frac{1}{4}$-eigenvectors of a_5, respectively. So $w = a_2v_8 + \frac{1}{192}a_1 - \frac{13}{192}a_2 - \frac{1}{192}a_3 - \frac{1}{96}a_4 - \frac{1}{96}a_6 - \frac{1}{96}a_8 - \frac{1}{32}a_9 - \frac{1}{32}a_{10} - \frac{1}{96}a_{12} + \frac{1}{32}(v_1 + v_2)$ and $w - u = \frac{1}{192}a_1 - \frac{49}{192}a_2 - \frac{1}{192}a_3 - \frac{1}{24}a_4 - \frac{1}{24}a_6 - \frac{1}{24}a_8 - \frac{1}{8}a_9 - \frac{1}{8}a_{10} - \frac{1}{24}a_{12} + \frac{1}{8}(v_1 + v_2)$.

On the other hand, $w = 4a_5(w - u) = \frac{1}{2}(a_5v_2 + a_5v_1) - \frac{1}{192}a_1 - \frac{1}{64}a_3 - \frac{13}{192}a_5 - \frac{1}{96}a_8 - \frac{1}{32}a_9 - \frac{1}{32}a_{10} - \frac{1}{96}a_{12} + \frac{3}{64}v_8 + \frac{1}{64}v_9$. Then

$$a_2v_8 - \frac{1}{2}(a_5v_2 + a_5v_1) = -\frac{1}{96}a_1 + \frac{13}{192}a_2 - \frac{1}{96}a_3 + \frac{1}{96}a_4 - \frac{13}{192}a_5 + \frac{1}{96}a_6 - \frac{1}{32}v_1 - \frac{1}{32}v_2 + \frac{3}{64}v_8 + \frac{1}{64}v_9. \quad (22)$$

From Lemma 5.2.2 and equation (22), we get the following

$$a_2v_8 = -\frac{1}{96}a_1 + \frac{13}{192}a_2 - \frac{1}{96}a_3 + \frac{1}{96}a_4 - \frac{13}{192}a_5 + \frac{1}{96}a_6 - \frac{3}{64}v_1 - \frac{1}{64}v_2 + \frac{3}{64}v_8 + \frac{1}{64}v_9 + a_5v_1. \quad (23)$$

Rewrite the equation (23) as the following

$$a_5v_1 = \frac{1}{96}a_1 - \frac{13}{192}a_2 + \frac{1}{96}a_3 - \frac{1}{96}a_4 + \frac{13}{192}a_5 - \frac{1}{96}a_6 + \frac{3}{64}v_1 + \frac{1}{64}v_2 - \frac{3}{64}v_8 - \frac{1}{64}v_9 + a_2v_8. \quad (24)$$

□

Since the group G acts on the algebra A, the products of all a_i's with all v_j's, for $i = 1, .., 6$ and $j = 1, .., 9$, can be computed except nine of them, namely a_1v_1, a_1v_2, a_1v_3, a_2v_4, a_2v_6, a_2v_8, a_3v_1, a_3v_2 and a_3v_3. For simplicity, we let $t_1 = a_1v_1$, $t_2 = a_1v_2$, $t_3 = a_1v_3$, $t_4 = a_2v_4$, $t_5 = a_2v_6$, $t_6 = a_2v_8$, $t_7 = a_3v_1$, $t_8 = a_3v_2$ and $t_9 = a_3v_3$.

Lemma 5.2.5.

$a_5 \cdot t_3 = -\frac{13}{1024}a_1 - \frac{1}{1024}a_2 + \frac{1}{1024}a_3 - \frac{1}{512}a_4 - \frac{1}{512}a_6 - \frac{3}{512}a_7 - \frac{1}{512}a_9 - \frac{1}{512}a_{10} - \frac{3}{512}a_{11} + \frac{3}{512}v_3 +$

$\frac{3}{512}v_5 + \frac{3}{512}v_7 - \frac{3}{512}v_8 + \frac{3}{16}t_3$.

Proof. By fusion rules and information in Table 5.7, the vector $u := a_1(v_3 - \frac{1}{2}a_5 + 2(a_7 + a_{11}))$ is a 0-eigenvector of a_5. So $u = t_3 + \frac{3}{16}a_1 + \frac{1}{32}a_4 + \frac{1}{32}a_6 + \frac{3}{32}a_7 + \frac{1}{32}a_9 + \frac{1}{32}a_{10} + \frac{3}{32}a_{11} - \frac{3}{32}v_5 - \frac{3}{32}v_7$. By Lemma 5.2.4 and the fact that $a_5 \cdot u = 0$, we have that

$$a_5 \cdot t_3 + \frac{13}{1024}a_1 + \frac{1}{1024}a_2 - \frac{1}{1024}a_3 + \frac{1}{512}a_4 + \frac{1}{512}a_6 + \frac{3}{512}a_7 + \frac{1}{512}a_9 + \frac{1}{512}a_{10} + \frac{3}{512}a_{11} - \frac{3}{512}v_3 - \frac{3}{512}v_5 - \frac{3}{512}v_7 + \frac{3}{512}v_8 - \frac{3}{16}t_3 = 0.$$

□

Lemma 5.2.6.
$a_5 \cdot t_4 = -\frac{27}{4096}a_1 + \frac{15}{4096}a_2 - \frac{37}{4096}a_3 - \frac{5}{2048}a_4 + \frac{1}{256}a_5 - \frac{5}{2048}a_6 - \frac{1}{1024}a_8 - \frac{1}{1024}a_9 - \frac{1}{1024}a_{10} - \frac{1}{1024}a_{12} - \frac{1}{2048}v_1 - \frac{1}{2048}v_2 - \frac{17}{2048}v_3 + \frac{9}{2048}v_4 + \frac{7}{2048}v_5 + \frac{9}{2048}v_6 + \frac{7}{2048}v_7 - \frac{3}{1024}v_8 - \frac{1}{2048}v_9 + \frac{7}{64}t_3 + \frac{1}{64}t_4 - \frac{1}{64}t_5 - \frac{1}{64}t_6 + \frac{9}{64}t_9$.

Proof. From Table 5.7, we see that $u := (a_4 + a_6 - \frac{1}{32}a_5)(v_3 - \frac{1}{2}a_5 + 2(a_7 + a_{11})) = \frac{1}{12}a_1 - \frac{13}{96}a_2 + \frac{1}{12}a_3 + \frac{47}{192}a_4 - \frac{7}{192}a_5 + \frac{47}{192}a_6 + \frac{3}{16}a_7 + \frac{1}{32}a_8 + \frac{1}{32}a_9 + \frac{1}{32}a_{10} + \frac{3}{16}a_{11} + \frac{1}{32}a_{12} + \frac{1}{64}v_1 + \frac{1}{64}v_2 + \frac{3}{32}v_3 - \frac{9}{64}v_4 - \frac{7}{64}v_5 - \frac{9}{64}v_6 - \frac{7}{64}v_7 + t_4 + t_5$ is a 0-eigenvector of a_5. Since $a_5 \cdot u = 0$, we have that

$$a_5 \cdot t_4 + a_5 \cdot t_5 = -\frac{27}{2048}a_1 + \frac{15}{2048}a_2 - \frac{37}{2048}a_3 - \frac{5}{1024}a_4 + \frac{1}{128}a_5 - \frac{5}{1024}a_6 - \frac{1}{512}a_8 - \frac{1}{512}a_9 - \frac{1}{512}a_{10} - \frac{1}{512}a_{12} - \frac{1}{1024}v_1 - \frac{1}{1024}v_2 - \frac{17}{1024}v_3 + \frac{9}{1024}v_4 + \frac{7}{1024}v_5 + \frac{9}{1024}v_6 + \frac{7}{1024}v_7 - \frac{3}{512}v_8 - \frac{1}{1024}v_9 + \frac{7}{32}t_3 - \frac{1}{32}t_6 + \frac{9}{32}t_9. \quad (25)$$

But in Corollary 5.2.3, we have

$$a_5 \cdot t_5 - a_5 \cdot t_4 = \frac{1}{32}(t_5 - t_4), \tag{26}$$

then by solving equations (25) and (26), the result follows. □

Lemma 5.2.7.

$v_8 - v_3 = 0.$

Proof. Back to Table 5.7 and by fusion rules, $w := a_1(v_8 - \frac{1}{3}a_5 - \frac{2}{3}(a_9 + a_{10}) - \frac{1}{3}a_3) = \frac{13}{2^6}a_1 - \frac{1}{2^6}a_2 + \frac{1}{2^6}a_3 + \frac{1}{2^5}a_4 + \frac{1}{2^5}a_6 + \frac{1}{2^5}a_7 + \frac{3}{2^5}a_9 + \frac{3}{2^5}a_{10} + \frac{1}{2^5}a_{11} + \frac{3}{2^5}v_3 - \frac{3}{2^5}v_5 - \frac{3}{2^5}v_7 - \frac{3}{2^5}v_8 - 3t_3$ is $\frac{1}{4}$-eigenvector of a_5. By Lemma 5.2.5, $a_5 \cdot t_3$ is know, then $a_5 \cdot w = \frac{13}{2^8}a_1 - \frac{1}{2^8}a_2 + \frac{1}{2^8}a_3 + \frac{1}{2^7}a_4 + \frac{1}{2^7}a_6 + \frac{1}{2^7}a_7 + \frac{3}{2^7}a_9 + \frac{3}{2^7}a_{10} + \frac{1}{2^7}a_{11} - \frac{3}{128}v_5 - \frac{3}{2^7}v_7 - \frac{3}{4}t_3$. Since $a_5 \cdot w - \frac{1}{4}w = 0$, we have that $-\frac{3}{2^7}v_3 + \frac{3}{2^7}v_8 = 0$. □

From Lemma 5.2.7, we can compute that $a_2v_8 = a_2v_3 = \frac{1}{16}(5a_2 - 2a_7 - a_5 - 2a_{11} + 3v_3)$ and $a_7v_8 = a_7v_3 = \frac{1}{16}(5a_7 - 2a_2 - a_{11} - 2a_5 + 3v_3)$. For now, the product of all a_i's with v_j's can be written in terms of a_i's and v_j's, for $i = 1, ..., 12$ and $j = 1, ..., 9$.

Lemma 5.2.8.

$a_5 = 0.$

Proof. Let $u := v_8 - \frac{1}{2}a_5 + 2(a_9 + a_{10})$. By fusion rules, $\alpha_1 := a_1 \cdot u$ and $\alpha_2 := a_2 \cdot u$ are 0-eigenvectors of a_5. Then

$$\alpha_1 = \frac{1}{2}a_1 + \frac{1}{32}a_4 - \frac{1}{16}a_5 + \frac{1}{32}a_6 + \frac{1}{32}a_7 - \frac{1}{8}a_8 + \frac{3}{32}a_9 + \frac{3}{32}a_{10} + \frac{1}{32}a_{11} - \frac{1}{8}a_{12} - \frac{3}{32}v_5 - \frac{3}{32}v_7 + \frac{3}{16}$$

and

$$\alpha_2 = \frac{1}{2}a_2 + \frac{1}{32}a_4 - \frac{1}{16}a_5 + \frac{1}{32}a_6 - \frac{1}{8}a_7 + \frac{1}{32}a_8 + \frac{3}{32}a_9 + \frac{3}{32}a_{10} - \frac{1}{8}a_{11} + \frac{1}{32}a_{12} - \frac{3}{32}v_1 - \frac{3}{32}v_2 + \frac{3}{16}v_3.$$

Since $a_5 \cdot \alpha_1 = 0$ and $a_5 \cdot \alpha_2 = 0$, we have that

$$-\frac{1}{256}a_1 + \frac{1}{1024}a_2 + \frac{3}{1024}a_3 - \frac{63}{1024}a_5 + \frac{1}{512}a_7 - \frac{1}{128}a_8 + \frac{3}{512}a_9 + \frac{3}{512}a_{10} + \frac{1}{512}a_{11} - \frac{1}{128}a_{12} - \frac{3}{1024}v_3 - \frac{9}{1024}v_8 + \frac{3}{256}v_9 = 0 \quad (27)$$

and

$$\frac{1}{1024}a_1 - \frac{1}{256}a_2 + \frac{3}{1024}a_3 - \frac{63}{1024}a_5 - \frac{1}{128}a_7 + \frac{1}{512}a_8 + \frac{3}{512}a_9 + \frac{3}{512}a_{10} - \frac{1}{128}a_{11} + \frac{1}{512}a_{12} + \frac{3}{256}v_3 - \frac{9}{1024}v_8 - \frac{3}{1024}v_9 = 0, \quad (28)$$

respectively. The vector $\alpha_3 := a_7 - a_9 - a_{10} + a_{11}$ is 0-eigenvector of a_5, then

$$a_5 \cdot \alpha_3 = \frac{1}{32}a_2 - \frac{1}{32}a_3 + \frac{1}{16}a_7 - \frac{1}{16}a_9 - \frac{1}{16}a_{10} + \frac{1}{16}a_{11} - \frac{3}{32}v_3 + \frac{3}{32}v_8 = 0. \quad (29)$$

From equations (27) and (28), we get

$$-\frac{1}{3}a_2 + \frac{1}{3}a_3 - 7a_5 - \frac{2}{3}a_7 + \frac{2}{3}a_9 + \frac{2}{3}a_{10} - \frac{2}{3}a_{11} + v_3 - v_8 = 0 \quad (30)$$

and from equations (29) and (30), we see that $a_5 = 0$. □

By the action of the group G, we get that $a_i = 0$, for $i = 1, ..., 6$. By symmetry, we can see that all a_i's are equal to zero, for $i = 1, ..., 12$. Thus, the algebra A is trivial and then the Proposition 5.2.1 is proved.

Corollary 5.2.9. *The M-axial algebra of the shape* $(3C, 3A, 4A)$ *is trivial.*

5.2.2 The shapes $(3C, 3C, 4B)$ and $(3C, 3A, 4B)$

The two cases are considered in this subsection, which are $(3C, 3C, 4B)$ and $(3C, 3A, 4B)$, as they contains the same subalgebra of the shape $(2A, 3C)$ for the subgroup D_{12} of the group G. By Proposition 5.1.28, the subalgebra of the shape $(2A, 3C)$ for the subgroup D_{12} is trivial, then the algebras for the both cases $(3C, 3C, 4B)$ and $(3C, 3A, 4B)$ are trivial and we can state the following proposition.

Proposition 5.2.10. *The M-axial algebras of the shapes* $(3C, 3C, 4B)$ *and* $(3C, 3A, 4B)$ *are trivial.*

5.2.3 The shapes $(3A, 3A, 4A)$ and $(3A, 3A, 4B)$

These two cases, $(3A, 3A, 4A)$ and $(3A, 3A, 4B)$, seem too big and the computer was not able to calculate them. So, the author of this thesis was not able to verify their algebras at the moment. It is his future work to find a good methodology to find algebras for any different shape of any arbitrary group.

At this point, we can say that most of the M-axial algebras of the groups $(((3 \times ((3^2) : 3)) : 3) : Q_8) : 2$ and $(S_4 \times S_4) : 2$ are trivial except those marked dashes $-$ in the fifth column of the table in Appendix A, which seems too big and computer was not able to compute them.

Chapter 6

Conclusion

In this thesis, we concentrated on the study of 3-generated M-axial algebras A such that every 2-generated subalgebra of A is a Sakuma algebra of type NX, where $N \in \{2, 3, 4\}$ and $X \in \{A, B, C\}$. For this purpose, we found all 3-generated 4-transposition groups such that the order of the product of any pair of generators does not exceed four. This has been done in Chapter 4. So the main result in Chapter 4 is the following

Theorem 6.0.11. *A group satisfies property* (Δ) *if it is a quotient of at least one of the groups in Table 4.1.*

For a particular case, in Chapter 3, we studied the M-axial algebras only involving $2A$ and $2B$ subalgebras without restriction to the number of generators. We noticed that the group of automorphisms of M-axial algebras is a 3-transposition group. Hence, we have a Fischer space associated with it. In the last section, we found the dimension of the embedding of such Fischer spaces into a $GF(2)$ vector space.

In Chapter 5, we classified M-axial algebras for many of the groups found in Chapter 4. This has been done by calculating subalgebras for subgroups of groups in above theorem. We saw that most of them lead to the trivial algebra. However, some cases were not computed because they are too big and the computer was not able to calculate them. We left those cases open and we might be able to find them in our future work.

Appendix A

M-Axial Algebras and Their Dimensions

Note the following:

1. GF(p) refers to a finite field with p elements.
2. Groups in the following table which do not appear in Chapter 4 are factor groups of the group $B(2,4):2$.
3. The dashes $-$ in the following table indicate that the M-axial algebra has not been determined yet.

$G = \langle a,b,c \rangle$	$\|G\|$	$\|a^G \cup b^G \cap c^G\|$	Shape	Dimension
2^3	8	1+1+1	$(2A, 2B^2)$	4
			$(2B^3)$	3
			$(2A^2, 2B)$	6
			$(2A^3)$	-
D_8	8	2+2	$(2A^2, 4B)$	5
			$(2B^2, 4A)$	5
D_8	8	2+2+1	$(2A, 2B^3, 4A)$	10
			$(2B^4, 4A)$	6
			$(2A^2, 2B^2, 4A)$	14
			$(2A^3, 2B, 4B)$	8

			$(2A^2, 2B^2, 4B)$	6
			$(2A^4, 4B)$	5
D_{12}	12	1+3	$(2B, 3A)$	5
			$(2B, 3C)$	4
			$(2A, 3A)$	8
			$(2A, 3C)$	0
$2 \times D_8$	16	2+1+2	$(2A, 2B^3, 4A)$	10
			$(2B^4, 4A)$	6
			$(2A^2, 2B^2, 4A)$	14
			$(2A^3, 2B, 4B)$	8
			$(2A^2, 2B^2, 4B)$	6
			$(2A^4, 4B)$	5
$2 \times D_8$	16	2+2+2	$(2A, 2B^4, 4A^2)$	0
			$(2B^5, 4A^2)$	9
			$(2A^4, 2B, 4B^2)$	8
			$(2A^3, 2B^2, 4B^2)$	7
			$(2A^2, 2B^3, 4A^2)$	-
			$(2A^5, 4B^2)$	11
$(4 \times 2) : 2$	16	2+2+2	$(2A^3, 4B^3)$	7
			$(2B^3, 4A^3)$	-
$2^4 : 2$	32	2+2+4	$(2A, 2B^6, 4A^2)$	0
			$(2B^7, 4A^2)$	13
			$(2A^3, 2B^4, 4A, 4B)$	12
			$(2A^2, 2B^5, 4A, 4B)$	0
			$(2A^2, 2B^5, 4A^2)$	0

			$(2A, 2B^6, 4A^2)$	-
			$(2A^4, 2B^3, 4A, 4B)$	15
			$(2A^3, 2B^4, 4A, 4B)$	0
			$(2A^5, 2B^2, 4B^2)$	6
			$(2A^4, 2B^3, 4B^2)$	10
			$(2A^6, 2B, 4B^2)$	0
			$(2A^5, 2B^2, 4B^2)$	0
			$(2A^3, 2B^4, 4A^2)$	0
			$(2A^2, 2B^5, 4A^2)$	0 Over GF(p)
			$(2A^5, 2B^2, 4A, 4B)$	0 Over GF(p)
			$(2A^4, 2B^3, 4A, 4B)$	0
			$(2A^7, 4B^2)$	0
			$(2A^6, 2B, 4B^2)$	-
T_1	32	4+2+4	$(2A, 2B^6, 4A^4)$	0
			$(2B^7, 4A^4)$	0
			$(2A^2, 2B^5, 4A^4)$	0
			$(2A^3, 2B^4, 4A^2, 4B^2)$	0
			$(2A^2, 2B^5, 4A^2, 4B^2)$	0
			$(2A^4, 2B^3, 4A^2, 4B^2)$	0
			$(2A^4, 2B^3, 4A^2, 4B^2)$	0
			$(2A^3, 2B^4, 4A^2, 4B^2)$	0
			$(2A^5, 2B^2, 4A^2, 4B^2)$	13
			$(2A^6, 2B, 4B^4)$	8
			$(2A^5, 2B^2, 4B^4)$	12
			$(2A^7, 4B^4)$	6

$(4 \times 4) : 2$	32	4+4+4	$(2A, 2B^8, 4A^6)$	0
			$(2B^9, 4A^6)$	0
			$(2A^2, 2B^7, 4A^6)$	0
			$(2A^3, 2B^6, 4A^4, 4B^2)$	0
			$(2A^2, 2B^7, 4A^4, 4B^2)$	0
			$(2A^4, 2B^5, 4A^4, 4B^2)$	0
			$(2A^3, 2B^6, 4A^6)$	0
			$(2A^4, 2B^5, 4A^4, 4B^2)$	0
			$(2A^3, 2B^6, 4A^4, 4B^2)$	0
			$(2A^5, 2B^4, 4A^4, 4B^2)$	0
			$(2A^5, 2B^4, 4A^2, 4B^4)$	0
			$(2A^4, 2B^5, 4A^2, 4B^4)$	0
			$(2A^6, 2B^3, 4A^2, 4B^4)$	0
			$(2A^5, 2B^4, 4A^2, 4B^4)$	0
			$(2A^7, 2B^2, 4A^2, 4B^4)$	0
			$(2A^6, 2B^3, 4A^2, 4B^4)$	0
			$(2A^7, 2B^2, 4B^6)$	0
			$(2A^6, 2B^3, 4B^6)$	15
			$(2A^8, 2B, 4B^6)$	0
			$(2A^9, 4B^6)$	0
$2 \times S_4$	48	6+3	$(2B^3, 3A, 4A)$	25
			$(2B^3, 3C, 4A)$	12
			$(2A, 2B^2, 3A, 4A)$	0
			$(2A, 2B^2, 3C, 4A)$	0
			$(2A^2, 2B, 3A, 4B)$	16

			$(2A^2, 2B, 3C, 4B)$	12
			$(2A^3, 3A, 4B)$	13
			$(2A^3, 3C, 4B)$	9
T_2	54	9	$(3A, 3C^3)$	12
			$(3C^4)$	9
			$(3A^2, 3C^2)$	-
			$(3A^3, 3C)$	-
			$(3A^4)$	-
$(2^2 \times D_8) : 2$	64	4+4+4	$(2A, 2B^8, 4A^6)$	0
			$(2B^9, 4A^6)$	-
			$(2A^2, 2B^7, 4A^6)$	0
			$(2A^3, 2B^6, 4A^4, 4B^2)$	0
			$(2A^2, 2B^7, 4A^4, 4B^2)$	0
			$(2A^4, 2B^5, 4A^4, 4B^2)$	-
			$(2A^3, 2B^6, 4A^6)$	0
			$(2A^4, 2B^5, 4A^4, 4B^2)$	0
			$(2A^3, 2B^6, 4A^4, 4B^2)$	0
			$(2A^5, 2B^4, 4A^4, 4B^2)$	0
			$(2A^5, 2B^4, 4A^2, 4B^4)$	0
			$(2A^4, 2B^5, 4A^2, 4B^4)$	0
			$(2A^6, 2B^3, 4A^2, 4B^4)$	0
			$(2A^5, 2B^4, 4A^2, 4B^4)$	0
			$(2A^7, 2B^2, 4A^2, 4B^4)$	0
			$(2A^6, 2B^3, 4A^2, 4B^4)$	-
			$(2A^7, 2B^2, 4B^6)$	11

			$(2A^6, 2B^3, 4B^6)$	15
			$(2A^8, 2B, 4B^6)$	8
			$(2A^9, 4B^6)$	-
$(((4 \times 2) : 2) : 2) : 2$	64	4+4+4	$(2A, 2B^7, 4A^3)$	-
			$(2B^8, 4A^3)$	0
			$(2A^3, 2B^5, 4A^2, 4B)$	0
			$(2A^2, 2B^6, 4A^2, 4B)$	-
			$(2A^2, 2B^6, 4A^3)$	-
			$(2A, 2B^7, 4A^3)$	0
			$(2A^4, 2B^4, 4A^2, 4B)$	0
			$(2A^3, 2B^5, 4A^2, 4B)$	-
			$(2A^5, 2B^3, 4A, 4B^2)$	-
			$(2A^4, 2B^4, 4A, 4B^2)$	0
			$(2A^6, 2B^2, 4A, 4B^2)$	-
			$(2A^5, 2B^3, 4A, 4B^2)$	0
			$(2A^3, 2B^5, 4A^2, 4B)$	0
			$(2A^2, 2B^6, 4A^2, 4B)$	0
			$(2A^5, 2B^3, 4A, 4B^2)$	0
			$(2A^4, 2B^4, 4A, 4B^2)$	0
			$(2A^4, 2B^4, 4A^2, 4B)$	0
			$(2A^3, 2B^5, 4A^2, 4B)$	0
			$(2A^6, 2B^2, 4A, 4B^2)$	0
			$(2A^5, 2B^3, 4A, 4B^2)$	0
			$(2A^7, 2B, 4B^3)$	0
			$(2A^6, 2B^2, 4B^3)$	0

			$(2A^8, 4B^3)$	0
			$(2A^7, 2B, 4B^3)$	0
$(S_3 \times S_3) : 2$	72	6+6	$(2B^2, 3A, 3C, 4A)$	0
			$(2B^2, 3C^2, 4A)$	0
			$(2B^2, 3A^2, 4A)$	-
			$(2A^2, 3A, 3C, 4B)$	0
			$(2A^2, 3C^2, 4B)$	0
			$(2A^2, 3A^2, 4B)$	-
T_3	96	12	$(2A, 3C, 4B)$	15
			$(2B, 3C, 4A)$	15
			$(2A, 3A, 4B)$	0
			$(2B, 3A, 4A)$	-
$(D_8 \times D_8) : 2$	128	4+4+8	$(2A, 2B^7, 4A^3, 4B)$	0
			$(2B^8, 4A^4)$	0
			$(2A^3, 2B^5, 4A^2, 4B^2)$	0
			$(2A^2, 2B^6, 4A^3, 4B)$	0
			$(2A^2, 2B^6, 4A^3, 4B)$	0
			$(2A, 2B^7, 4A^4)$	0
			$(2A^4, 2B^4, 4A^2, 4B^2)$	0
			$(2A^3, 2B^5, 4A^3, 4B)$	0
			$(2A^5, 2B^3, 4A, 4B^3)$	0
			$(2A^4, 2B^4, 4A^2, 4B^2)$	0
			$(2A^6, 2B^2, 4A, 4B^3)$	0
			$(2A^5, 2B^3, 4A^2, 4B^2)$	0
			$(2A^3, 2B^5, 4A^2, 4B^2)$	0

			$(2A^2, 2B^6, 4A^3, 4B)$	-
			$(2A^5, 2B^3, 4A, 4B^3)$	0
			$(2A^4, 2B^4, 4A^2, 4B^2)$	-
			$(2A^4, 2B^4, 4A^2, 4B^2)$	0
			$(2A^3, 2B^5, 4A^3, 4B)$	0
			$(2A^6, 2B^2, 4A, 4B^3)$	0
			$(2A^5, 2B^3, 4A^2, 4B^2)$	0
			$(2A^7, 2B, 4B^4)$	0
			$(2A^6, 2B^2, 4A, 4B^3)$	-
			$(2A^8, 4B^4)$	0
			$(2A^7, 2B, 4A, 4B^3)$	0
T_4	336	21	$(2B^2, 3A, 4A)$	-
			$(2B^2, 3C, 4A)$	57
			$(2A^2, 3A, 4B)$	49
			$(2A^2, 3C, 4B)$	21
T_5	384	12+12	$(2A, 2B^6, 3C, 4A^3, 4B)$	0
			$(2B^7, 3C, 4A^4)$	0
			$(2A^3, 2B^4, 3C, 4A^2, 4B^2)$	0
			$(2A^2, 2B^5, 3C, 4A^3, 4B)$	60
			$(2A^5, 2B^2, 3C, 4A, 4B^3)$	0
			$(2A^4, 2B^3, 3C, 4A^2, 4B^2)$	0
			$(2A, 2B^6, 3A, 4A^3, 4B)$	0
			$(2B^7, 3A, 4A^4)$	0
			$(2A^3, 2B^4, 3A, 4A^2, 4B^2)$	0
			$(2A^2, 2B^5, 3A, 4A^3, 4B)$	0

			$(2A^5, 2B^2, 3A, 4A, 4B^3)$	0
			$(2A^4, 2B^3, 3A, 4A^2, 4B^2)$	0
			$(2A^7, 3C, 4B^4)$	0
			$(2A^6, 2B, 3C, 4A, 4B^3)$	42 Over GF(11)
			$(2A^7, 3A, 4B^4)$	0
			$(2A^6, 2B, 3A, 4A, 4B^3)$	59 Over GF(11)
T_6	1152	24+12	$(2A, 2B^3, 3C^2, 4A^2)$	0
			$(2B^4, 3C^2, 4A^2)$	0
			$(2A, 2B^3, 3A, 3C, 4A^2)$	0
			$(2B^4, 3A, 3C, 4A^2)$	0
			$(2A^2, 2B^2, 3C^2, 4A, 4B)$	0
			$(2A, 2B^3, 3C^2, 4A, 4B)$	0
			$(2A^2, 2B^2, 3A, 3C, 4A, 4B)$	0
			$(2A, 2B^3, 3A, 3C, 4A, 4B)$	0
			$(2A^3, 2B, 3C^2, 4A, 4B)$	0
			$(2A^2, 2B^2, 3C^2, 4A, 4B)$	0
			$(2A^3, 2B, 3A, 3C, 4A, 4B)$	0
			$(2A^2, 2B^2, 3A, 3C, 4A, 4B)$	0
			$(2A^4, 3C^2, 4B^2)$	0
			$(2A^3, 2B, 3C^2, 4B^2)$	0
			$(2A^4, 3A, 3C, 4B^2)$	0
			$(2A^3, 2B, 3A, 3C, 4B^2)$	0
			$(2A, 2B^3, 3A, 3C, 4A^2)$	0
			$(2B^4, 3A, 3C, 4A^2)$	0
			$(2A, 2B^3, 3A^2, 4A^2)$	-

			$(2B^4, 3A^2, 4A^2)$	0
			$(2A^2, 2B^2, 3A, 3C, 4A, 4B)$	0
			$(2A, 2B^3, 3A, 3C, 4A, 4B)$	0
			$(2A^2, 2B^2, 3A^2, 4A, 4B)$	0
			$(2A, 2B^3, 3A^2, 4A, 4B)$	0
			$(2A^3, 2B, 3A, 3C, 4A, 4B)$	0
			$(2A^2, 2B^2, 3A, 3C, 4A, 4B)$	0
			$(2A^3, 2B, 3A^2, 4A, 4B)$	-
			$(2A^2, 2B^2, 3A^2, 4A, 4B)$	0
			$(2A^4, 3A, 3C, 4B^2)$	0
			$(2A^3, 2B, 3A, 3C, 4B^2)$	0
			$(2A^4, 3A^2, 4B^2)$	0
			$(2A^3, 2B, 3A^2, 4B^2)$	0
T_7	3888	18+18+18	$(2B^3, 3A, 3C^5, 4A^3)$	0
			$(2B^3, 3C^6, 4A^3)$	0
			$(2B^3, 3A^2, 3C^4, 4A^3)$	0
			$(2B^3, 3A^2, 3C^4, 4A^3)$	0
			$(2B^3, 3A^3, 3C^3, 4A^3)$	0
			$(2B^3, 3A^2, 3C^4, 4A^3)$	0
			$(2B^3, 3A^4, 3C^2, 4A^3)$	0 Over GF(11)
			$(2B^3, 3A^3, 3C^3, 4A^3)$	0
			$(2B^3, 3A^4, 3C^2, 4A^3)$	0
			$(2B^3, 3A^4, 3C^2, 4A^3)$	0 Over GF(11)
			$(2B^3, 3A^3, 3C^3, 4A^3)$	0 Over GF(11)
			$(2B^3, 3A^5, 3C, 4A^3)$	0 Over GF(11)

				$(2B^3, 3A^6, 4A^3)$	-
				$(2A^3, 3A, 3C^5, 4B^3)$	0
				$(2A^3, 3C^6, 4B^3)$	0
				$(2A^3, 3A^2, 3C^4, 4B^3)$	0
				$(2A^3, 3A^2, 3C^4, 4B^3)$	0
				$(2A^3, 3A^3, 3C^3, 4B^3)$	0
				$(2A^3, 3A^2, 3C^4, 4B^3)$	0
				$(2A^3, 3A^4, 3C^2, 4B^3)$	0
				$(2A^3, 3A^3, 3C^3, 4B^3)$	0
				$(2A^3, 3A^4, 3C^2, 4B^3)$	0
				$(2A^3, 3A^4, 3C^2, 4B^3)$	0
				$(2A^3, 3A^3, 3C^3, 4B^3)$	0
				$(2A^3, 3A^5, 3C, 4B^3)$	0
				$(2A^3, 3A^6, 4B^3)$	-

Appendix B

M-Axial Algebra of the Shape $(2A, 3A)$ for the Group D_{12}

```
g:=Group((1,2),(2,3),(4,5));

cc2:=Concatenation(Elements(ConjugacyClass(g,(1,2))),
                   Elements(ConjugacyClass(g,(4,5))));

n2:=Length(cc2);

cc22:=Elements(ConjugacyClass(g,(1,2)(4,5)));
n22:=Length(cc22);

cc3:=[(1,2,3)];
n3:=Length(cc3);

n:=n2+n22+n3;

D12Action:=function(p,e)
 local z;
 if p<=n2 then
  return Position(cc2,cc2[p]^e);
 elif p<=n2+n22 then
  z:=cc22[p-n2]^e;
  return Position(cc22,z)+n2;
 else
  z:=cc3[p-n2-n22]^e;
  if z in cc3 then
   return Position(cc3,z)+n2+n22;
  else
```

```
   return Position(cc3,z^-1)+n2+n22;
  fi;
 fi;
end;

phi:=ActionHomomorphism(g,[1..n],D12Action);

V:=Rationals^n;
e:=Basis(V);

Mult:=List([1..n],i->[]);

for i in [1..n2] do
 for j in [1..n2] do
  x:=cc2[i]*cc2[j];
  if Order(x)=1 then
   Mult[i][j]:=e[i];
  elif Order(x)=2 then
   k:=Position(cc22,x)+n2;
   Mult[i][j]:=(e[i]+e[j]-e[k])/8;
   Mult[j][i]:=(e[i]+e[j]-e[k])/8;
   Mult[i][k]:=(e[i]+e[k]-e[j])/8;
   Mult[k][i]:=(e[i]+e[k]-e[j])/8;
   Mult[j][k]:=(e[j]+e[k]-e[i])/8;
   Mult[k][j]:=(e[j]+e[k]-e[i])/8;
   Mult[k][k]:=e[k];
  else
   k:=Position(cc2,x*cc2[i]);
   if x in cc3 then
    s:=Position(cc3,x)+n2+n22;
   else
    s:=Position(cc3,x^-1)+n2+n22;
   fi;
   Mult[i][j]:=(2*e[i]+2*e[j]+e[k])/32-(3^3*5/2^11)*e[s];;
   Mult[j][i]:=(2*e[i]+2*e[j]+e[k])/32-(3^3*5/2^11)*e[s];;
   Mult[i][s]:=(2*e[i]-e[j]-e[k])/3^2+(5/2^5)*e[s];;
   Mult[s][i]:=(2*e[i]-e[j]-e[k])/3^2+(5/2^5)*e[s];;
   Mult[j][s]:=(-e[i]+2*e[j]-e[k])/3^2+(5/2^5)*e[s];;
   Mult[s][j]:=(-e[i]+2*e[j]-e[k])/3^2+(5/2^5)*e[s];;
   Mult[s][s]:=e[s];;
  fi;
 od;
od;
```

```
Times:=function(u,v)
 local p,i,j;
 p:=ShallowCopy(Zero(V));
 for i in [1..n] do
  for j in [1..n] do
   p:=p+u[i]*v[j]*Mult[i][j];
  od;
 od;
 return p;
end;

gg:=Image(phi);

# products between a's and b's

orb1:=Orbit(gg,[1,6], OnPairs);
 for p in orb1 do
  x:=RepresentativeAction(gg,[1,6],p,OnPairs);
  Mult[p[2]][p[1]]:=
  Permuted([ 1/64, -1/64, -1/64, 1/64, -1/64, 1/64, -1/64, 45/2048 ],x);
  Mult[p[1]][p[2]]:=
  Permuted([ 1/64, -1/64, -1/64, 1/64, -1/64, 1/64, -1/64, 45/2048 ],x);
 od;

# product between a_r and u_1

 orb2:=Orbit(gg,[4,8], OnPairs);
 for p in orb2 do
  Mult[p[1]][p[2]]:=Zero(V);
  Mult[p[2]][p[1]]:=Zero(V);
 od;

# products between b's and u_1

orb3:=Orbit(gg,[5,8], OnPairs);
 for p in orb3 do
  x:=RepresentativeAction(gg,[5,8],p,OnPairs);
  Mult[p[2]][p[1]]:=Permuted([ 0, 0, 0, 0, 2/9, -1/9, -1/9, 5/32 ],x);
  Mult[p[1]][p[2]]:=Permuted([ 0, 0, 0, 0, 2/9, -1/9, -1/9, 5/32 ],x);
 od;

# products among b's
```

```
orb4:=Orbit(gg,[5,6], OnPairs);
 for p in orb4 do
  x:=RepresentativeAction(gg,[5,6],p,OnPairs);
  Mult[p[2]][p[1]]:=Permuted([ 0, 0, 0, 0, 1/16, 1/16, 1/32, -135/2048 ],x);
  Mult[p[1]][p[2]]:=Permuted([ 0, 0, 0, 0, 1/16, 1/16, 1/32, -135/2048 ],x);
 od;
```

Appendix C

M-Axial Algebra of the Shape $(2B, 3A)$ for the Group D_{12}

```
g:=Group((1,2),(2,3),(4,5));

cc2:=Concatenation(Elements(ConjugacyClass(g,(1,2))),
                   Elements(ConjugacyClass(g,(4,5))));

n2:=Length(cc2);

cc3:=[(1,2,3)];
n3:=Length(cc3);

n:=n2+n3;

D12Action:=function(p,e)
 local z;
 if p<=n2 then
  return Position(cc2,cc2[p]^e);
 else
  z:=cc3[p-n2]^e;
  if z in cc3 then
   return Position(cc3,z)+n2;
  else
   return Position(cc3,z^-1)+n2;
  fi;
 fi;
end;

phi:=ActionHomomorphism(g,[1..n],D12Action);
```

```
V:=Rationals^n;
e:=Basis(V);

Mult:=List([1..n],i->[]);

for i in [1..n2] do
 for j in [1..n2] do
  x:=cc2[i]*cc2[j];
  if Order(x)=1 then
   Mult[i][j]:=e[i];
  elif Order(x)=2 then
   Mult[i][j]:=Zero(V);
   Mult[j][i]:=Zero(V);
  else
   k:=Position(cc2,x*cc2[i]);
   if x in cc3 then
    s:=Position(cc3,x)+n2;
   else
    s:=Position(cc3,x^-1)+n2;
   fi;
   Mult[i][j]:=(2*e[i]+2*e[j]+e[k])/32-(3^3*5/2^11)*e[s];;
   Mult[j][i]:=(2*e[i]+2*e[j]+e[k])/32-(3^3*5/2^11)*e[s];;
   Mult[i][s]:=(2*e[i]-e[j]-e[k])/3^2+(5/2^5)*e[s];;
   Mult[s][i]:=(2*e[i]-e[j]-e[k])/3^2+(5/2^5)*e[s];;
   Mult[j][s]:=(-e[i]+2*e[j]-e[k])/3^2+(5/2^5)*e[s];;
   Mult[s][j]:=(-e[i]+2*e[j]-e[k])/3^2+(5/2^5)*e[s];;
   Mult[s][s]:=e[s];;
  fi;
 od;
od;

Times:=function(u,v)
 local p,i,j;
 p:=ShallowCopy(Zero(V));
 for i in [1..n] do
  for j in [1..n] do
   p:=p+u[i]*v[j]*Mult[i][j];
  od;
 od;
 return p;
end;
```

```
gg:=Image(phi);

# product between a_r and u_1

 orb1:=Orbit(gg,[4,5], OnPairs);
 for p in orb1 do
  Mult[p[1]][p[2]]:=Zero(V);
  Mult[p[2]][p[1]]:=Zero(V);
 od;
```

Appendix D

M-Axial Algebra of the Shape $(2B, 3C)$ for the Group D_{12}

```
g:=Group((1,2),(2,3),(4,5));

cc2:=Concatenation(Elements(ConjugacyClass(g,(1,2))),
                   Elements(ConjugacyClass(g,(4,5))));

n2:=Length(cc2);

n:=n2;

D12Action:=function(p,e)
 local z;
 if p<=n2 then
  return Position(cc2,cc2[p]^e);
 fi;
end;

phi:=ActionHomomorphism(g,[1..n],D12Action);

V:=Rationals^n;
e:=Basis(V);

Mult:=List([1..n],i->[]);

for i in [1..n2] do
 for j in [1..n2] do
  x:=cc2[i]*cc2[j];
  if Order(x)=1 then
```

```
   Mult[i][j]:=e[i];
  elif Order(x)=2 then
   Mult[i][j]:=Zero(V);
   Mult[j][i]:=Zero(V);
  else
   k:=Position(cc2,x*cc2[i]);
   Mult[i][j]:=(e[i]+e[j]-e[k])/64;
   Mult[j][i]:=(e[i]+e[j]-e[k])/64;
  fi;
 od;
od;

Times:=function(u,v)
 local p,i,j;
 p:=ShallowCopy(Zero(V));
 for i in [1..n] do
  for j in [1..n] do
   p:=p+u[i]*v[j]*Mult[i][j];
  od;
 od;
 return p;
end;

gg:=Image(phi);
```

Appendix E

M-Axial Algebra Axioms Code

```
# e is the basis of V
# Times is the product function
# n2 is the number of M-axes
# Mult[i] is the adjoint matrix of i's axis

#Checking Condition 1
for i in [1..n2] do
 a:=e[i];
 if Times(a,a)<>a then
  Print("Fail Condition 1",a,i,"\n");
 fi;
od;

#Checking Condition 2
for i in [1..n2] do
 a:=Mult[i];
  Eigen:=Eigenvalues(Rationals,a);
  if Eigen=[1,1/4,1/32,0] then
   Es:=Eigenspaces(Rationals,a);
   x:=0;
   for j in [1..4] do
    x:=x+Dimension(Es[j]);
   od;
   if Dimension(V)<>x then
    Print("Fail Condition 2",i,"\n");
   fi;
  elif Eigen=[1,1/32,0] then
   Es:=Eigenspaces(Rationals,a);
   x:=0;
```

```
   for j in [1..3] do
    x:=x+Dimension(Es[j]);
   od;
   if Dimension(V)<>x then
    Print("Fail Condition 2",i,"\n");
   fi;
  elif Eigen=[1,1/4,0] then
   Es:=Eigenspaces(Rationals,a);
   x:=0;
   for j in [1..3] do
    x:=x+Dimension(Es[j]);
   od;
   if Dimension(V)<>x then
    Print("Fail Condition 2",i,"\n");
   fi;
  else
  # Eigen=[1,0];
   Es:=Eigenspaces(Rationals,a);
   x:=0;
   for j in [1..2] do
    x:=x+Dimension(Es[j]);
   od;
   if Dimension(V)<>x then
    Print("Fail Condition 2",i,"\n");
   fi;
  fi;
  if (Dimension(Es[1]) <> 1) or not (e[i] in Es[1]) then
   Print("Fail in primitivity",i,"\n");
  fi;
od;
```

Appendix F

Fusion Rules Code

```
# This code checking condition 3 of the definition of axial algebras.
# e is the basis of V
# Times is the product function
# n2 is the number of M-axes
# Mult[i] is the adjoint matrix of i's axis

  for i in [1..n2] do
     A:=Mult[i];
      Eigen:=Eigenvectors(Rationals, A);
       zz:=Filtered(Eigen,u->Times(e[i],u)=Zero(V));
       Vz:=Subspace(V,zz);
       qq:=Filtered(Eigen,u->Times(e[i],u)=u/4);
       Vq:=Subspace(V,qq);
       th:=Filtered(Eigen,u->Times(e[i],u)=u/32);
       Vth:=Subspace(V,th);
     for u in [e[i]] do
      for v in [e[i]] do
       if not Times(u,v) in Subspace(V,[e[i]]) then
        Print("Fail in One","\n");
       fi;
      od;
     od;
     for u in [e[i]] do
      for v in zz do
       if not Times(u,v) in Vz then
        Print("Fail in One and Zero","\n");
       fi;
      od;
     od;
```

```
for u in [e[i]] do
 for v in qq do
  if not Times(u,v) in Vq then
   Print("Fail in One and Quarter","\n");
  fi;
 od;
od;
for u in [e[i]] do
 for v in th do
  if not Times(u,v) in Vth then
   Print("Fail in One and Thirty Two","\n");
  fi;
 od;
od;
for u in zz do
 for v in zz do
  if not Times(u,v) in Vz then
   Print("Fail in Zero","\n");
  fi;
 od;
od;
for u in zz do
 for v in qq do
  if not Times(u,v) in Vq then
   Print("Fail in Zero and Quarter","\n");
  fi;
 od;
od;
for u in zz do
 for v in th do
  if not Times(u,v) in Vth then
   Print("Fail in Zero and Thirty Two","\n");
  fi;
 od;
od;
for u in qq do
 for v in qq do
  if not Times(u,v) in Subspace(V,Concatenation([e[i]],zz)) then
   Print("Fail in Quarter and Quarter","\n");
  fi;
 od;
od;
for u in qq do
```

```
   for v in th do
    if not Times(u,v) in Vth then
     Print("Fail in Quarter and Thirty Two","\n");
    fi;
   od;
  od;
  for u in th do
   for v in th do
    if not Times(u,v) in
     Subspace(V,Concatenation([e[i]],zz,qq)) then
     Print("Fail in One, Zero and Quarter","\n");
    fi;
   od;
  od;
od;
```

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