ON THE FITTING HEIGHT OF SOLUBLE GROUPS

by

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Abstract

We consider five separate problems in finite group theory which cover a range of topics including properties of 2-generated subgroups, permutation groups, fixed-point-free automorphisms and the study of Sylow structure. The treatments of these problems are largely self-contained, but they all share an underlying theme which is to study finite soluble groups in terms of their Fitting height.

Firstly, we prove that if A is a maximal subgroup of a group G subject to being 2-generated, and $V \leq G$ is a nilpotent subgroup normalised by A, then $F^*(A)V$ is quasinilpotent. Secondly, we investigate the structure of soluble primitive permutation groups generated by two p^n-cycles and find upper bounds for their Fitting height in terms of p and n. Thirdly, we extend a recent result regarding fixed-point-free automorphisms. Namely, let $R \cong \mathbb{Z}_r$ be cyclic of prime order act on the extraspecial group $F \cong s^{1+2n}$ such that $F = [F, R]$, and suppose that RF acts on a group G such that $C_G(F) = 1$ and $(r, |G|) = 1$. Then we show that $F(C_G(R)) \subseteq F(G)$. In particular, when $r \neq s^n+1$, then $f(C_G(R)) = f(G)$. Fourthly, we show that there is no absolute bound on the Fitting height of a group with two Sylow numbers. Lastly, we characterise partial HNE-groups as precisely those groups which split over their nilpotent residual, which itself is cyclic of square-free order.

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Contents

Introduction

In mathematics, we often find series of inclusions of types of mathematical objects. For example, in module theory we have

$$\text{free modules} \subset \text{projective modules} \subset \text{flat modules} \subset \text{modules},$$

and in analysis we have

$$\text{Hilbert spaces} \subset \text{Banach spaces} \subset \text{metric spaces} \subset \text{topological spaces}.$$

Let A be a mathematical object of type X and B a mathematical object of type Y. Suppose further that A is not of type Y but B, by virtue of being type Y, is automatically of type X. So

$$\text{mathematical objects of type } Y \subset \text{mathematical objects of type } X.$$

A natural question to ask is: is there a well defined way of measuring how far A is from being a mathematical object of type Y? So for example, let R be a ring and M an R-module which is not flat. Is there a well defined way of measuring how far M is from being a flat R-module? Indeed there is, and the answer lies in the concept of flat dimension. For any R-module, say X, there exist flat R-modules X_i and an exact sequence (possibly of infinite length)

$$\dots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} \dots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X \to 0.$$

Such a sequence is called a flat resolution of X. If there exists an n such that $X_{n+1} = 0$, then X has finite flat dimension, and the smallest such n is called the flat dimension of X. The flat dimension of a module can be seen as a measure of how far a module is from being flat. We can also define projective and free resolutions by replacing the X_i with projective and free R-modules respectively. Furthermore, for any R-module X, we can always form projective and free resolutions of X, and thus have well defined concepts of both projective and free dimension. These can be seen as measures of how far an R-module is from being projective or free respectively.

So what about in finite group theory? If we have an inclusion of groups

$$\text{groups of type } Y \subset \text{groups of type } X,$$

then how do we measure how far any given group G of type X is from being a group of type Y. The most common approach is as follows: look for subgroups G_i such that

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_n = G$$

where G_i/G_{i-1} are groups of type Y for $1 \leq i \leq n$. If we are guaranteed the existence of such G_i, then we could search for the shortest chain and take this to be a 'measure' of how far G is being a group of type Y. Of course, this may not be possible. If we take an arbitrary finite group and wish to know how far it is from being abelian, then we might be at a loss by adopting this approach. Indeed, we would fail if we tried this with any nontrivial perfect group. However, there are many instances where this approach is successful. Consider the following inclusion

$$\text{nilpotent groups} \subset \text{soluble groups}.$$

For any soluble group G, it is possible to find subgroups G_i such that

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

where G_i/G_{i-1} is a nilpotent group for all $1 \leq i \leq n$. The smallest such n such that G possesses a series as above is called the Fitting height of G. As such, we can think of Fitting height as a measure of how far a soluble group is from being nilpotent.

Being able to break groups up in this way can often be very useful. It can allow us to break problems down about certain classes of groups to questions about other classes of groups, which may be easier to understand, and how these groups can be 'put together'. This is quite a generic idea in finite group theory and has often been the motivation behind research projects. The Jordan-Hölder theorem is one of the most fundamental results in group theory and states that: given any finite group G and two series for G

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = G, \; 1 = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_m = G$$

such that H_i/H_{i-1} and K_j/K_{j-1} are simple for all $1 \leq i \leq n$ and $1 \leq j \leq m$, then $n = m$ and $\{H_i/H_{i-1} \mid 1 \leq i \leq n\} = \{K_j/K_{j-1} \mid 1 \leq j \leq m\}$. This leads to the following questions: what are the finite simple groups; and, given any finite groups H and N, what groups G have a normal subgroup $M \trianglelefteq G$ such that $M \cong N$ and $G/M \cong H$? (Such a group G is said to be an extension of N by H.) In light of the Jordan-Hölder theorem, answering these two questions should allow us to construct and classify all finite groups. The finite simple groups have been classified but there is yet more work to be done before we fully understand how, given two arbitrary finite groups H and N, what groups are an extension of N by H.

In this thesis, we explore five separate problems in finite group theory, considered in turn from chapters 2 to 6. These problems cover a range of topics within finite group theory including properties of 2-generated subgroups, permutation groups, fixed-point-free automorphisms and the study of Sylow structure. We give more specific introductions to these problems at the beginning of their respective chapters. As such, chapters 2 to 6 are all largely self-contained and can be read in any order. However, all of the problems considered share an underlying theme, which is to study finite soluble groups in terms of their Fitting height. We begin each of chapters 2 to 6 by stating the problem which is to be considered and explaining where this fits in with current literature. At the end of each of these chapters we suggest further questions that naturally arise from our investigation.

In Chapter 1 we recall definitions and results which will be relevant throughout this thesis. Further background material which is specific to a particular result or problem is included only in the chapter where it is used.

In Chapter 2 we begin by studying groups in terms of their soluble 2-generated subgroups. In particular, we prove that if A is a maximal subgroup of a group G subject to being soluble and 2-generated, and $V \leq G$ is a nilpotent subgroup normalised by A, then $F(A)V$ is nilpotent. As an application of this, we offer new proofs of two well-known results, namely: every soluble group G possesses a 2-generated subgroup with the same Fitting height as G; and, that a group is soluble if and only if every three elements generate a soluble subgroup. We then attempt to find analogous results for insoluble groups. In particular, we take a subgroup $A \leq G$ which is maximal subject to being 2-generated, a quasinilpotent subgroup $V \leq G$ which is normalised by A, and ask whether $F^*(A)V$ is quasinilpotent. We obtain positive results when $V = F(V)$ and partial results when $V = E(V)$.

In Chapter 3 we investigate the structure of soluble primitive permutation groups generated by two p^n-cycles where p is a prime and $n \in \mathbb{N}$. We conclude this chapter by showing that the Fitting height of such a group is bounded in terms of p and n. As an application of this bound, we suggest a method of studying soluble subgroups of $\mathrm{Sym}(m)$ in terms of a particular class of 2-generated subgroups. This relates to some of the results presented in Chapter 2.

In Chapter 4 we study fixed-point-free automorphisms. It was recently proved by E. Khukhro in [28] that if a Frobenius group with complement R and kernel F acts on a group G such that $C_G(F) = 1$, then the Fitting heights of G and $C_G(R)$ are equal. Related to this result we prove the following: let $R \cong \mathbb{Z}_r$ for some prime r, act on an extraspecial group $F \cong s^{1+2n}$ such that $F = [F, R]$, and suppose RF acts on a group G such that $C_G(F) = 1$ and $(r, |G|) = 1$, then $F(C_G(R)) \leq F(G)$. As a corollary, we find that when $r \neq s^n + 1$, the Fitting heights of G and $C_G(R)$ are equal.

In Chapter 5 we answer a question recently asked in a paper by A. Moretó. He recently proved that a group with two Sylow numbers is the product of two nilpotent Hall subgroups. Due to a result of Kegel and Wielandt, such a group is soluble. As we will see, it is quite easy to construct a group whose order is divisible by two distinct primes (and thus has at most two Sylow numbers) and whose Fitting height is arbitrarily large. However, A. Moretó asked: if a group has two Sylow numbers and $n > 2$ distinct primes dividing its order, can its Fitting height be bounded in terms of n? We show by constructing examples that this is not the case. In particular, we show how to construct a group with two Sylow numbers whose order is divisible by an arbitrarily large number of distinct primes and whose Fitting height is also arbitrarily large.

Finally, in Chapter 6, we consider a question regarding Hall normally embedded subgroups. In [32] the following was asked: let G be a group such that for every subgroup $B \leq G$ there exists a Hall normally embedded subgroup $H \leq G$ such that $|B| = |H|$; is G soluble? In this chapter we find a characterisation for such groups and show that not only are they soluble but they have Fitting height no greater than two. Indeed, we find that such groups are precisely those which split over their nilpotent residual, which itself is cyclic of square-free order.

Chapter 1

Background material

We begin by recalling some results which will be required throughout this thesis. These results may be used without reference in later chapters. Most of the group theoretic results included in this chapter can be found with detailed proofs in either [25] or [2]. Similarly, most of the representation theoretic results can be found in either [20] or [24]. Specific references for some results are given; usually where the proofs tend to be lengthy or of greater difficulty. Where results may not be found in the above, the author either provides a proof or clearly indicates where one can be found. No great originality is claimed by the author in this first chapter.

1.1 General group theoretic results

Throughout this thesis, the word 'group' will mean 'finite group'.

Definition 1.1.1. Let G be a group. We define the *commutator subgroup* G' to be

$$G' = \langle [g,h] = g^{-1}h^{-1}gh \mid g,h \in G \rangle.$$

We define the n^{th} commutator subgroup $G^{(n)}$ inductively as follows. We set $G^{(0)} = G$, $G^{(1)} = G'$ and for $n > 1$, $G^{(n)} = (G^{(n-1)})'$. If $G = G'$, then G is said to be a *perfect* group.

Lemma 1.1.2. *Let G be a group and $g,h,k \in G$. Then the following identities hold:*

1. $[g,h][h,g] = 1$;
2. $[gh,k] = [g,k]^h[h,k]$;
3. $[g,hk] = [g,k][g,h]^k$.

Definition 1.1.3. Let G be a group and $H, K \leq G$ be subgroups. Then the commutator of H and K, denoted $[H, K]$, is defined to be

$$[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle.$$

By noting that the generators of $[H, K]$ are the inverses of the generators of $[K, H]$, we can see that $[H, K] = [K, H]$.

Lemma 1.1.4. *Let G be a group and $H, K \leq G$ be subgroups. Then*

$$[H, K] \trianglelefteq \langle H, K \rangle.$$

Lemma 1.1.5. *Let G be a group and $H, K \leq G$ be subgroups. Then $K \leq N_G(H)$ if and only if $[H, K] \subseteq H$.*

We can define higher commutators as follows.

Definition 1.1.6. Let G be a group and $g_1, \ldots, g_n \in G$. The triple commutator $[g_1, g_2, g_3]$ is computed using left association, so

$$[g_1, g_2, g_3] = [[g_1, g_2], g_3].$$

We inductively define higher commutators, so

$$[g_1, g_2, \ldots, g_n] = [[g_1, g_2, \ldots, g_{n-1}], g_n].$$

This allows us to define higher commutator groups.

Definition 1.1.7. Let G be a group and H, K, $L \leq G$ be subgroups. Then

$$[H, K, L] = \langle [[h, k], l] \mid h \in H,\ k \in K,\ l \in L \rangle.$$

Lemma 1.1.8. (Three subgroups lemma) *Let G be a group and H, K, $L \leq G$ be subgroups. If $[H, K, L] = 1$ and $[K, L, H] = 1$, then $[L, H, K] = 1$.*

Theorem 1.1.9. (Dedekind's modular law) *Let H, K and L be subgroups of a group G, and suppose $H \subseteq L \subseteq G$. Then*

$$HK \cap L = H(K \cap L).$$

Lemma 1.1.10. (The Frattini argument) *Let G be a group and $N \trianglelefteq G$, where N is finite. Suppose that $P \in \mathrm{Syl}_p(N)$. Then $G = N_G(P)N$.*

Lemma 1.1.11. *Let G be a group and $g, h \in G$ be two involutions. Then $\langle g, h \rangle$ is a dihedral subgroup of G.*

Proof. See [25, Lemma 2.14(b)]. □

Proposition 1.1.12. *A minimal normal subgroup of a dihedral group is cyclic.*

Proof. Let
$$G = \langle r, s \mid r^n = s^2 = 1,\ srs = r^{-1} \rangle,$$
and $N \trianglelefteq G$ be a minimal normal subgroup. Suppose there is an element in N of the form r^k where $1 \le k \le n$. Set $R = \langle r^k \rangle$. Since R is a subgroup of G, $r^{-k} \in R$. Hence $R = \langle r^k, r^{-k} \rangle$. So R is normal in G since $(r^k)^s = r^{-k}$ and both r^k and r^{-k} commute with r. Since N is minimal normal and $R \subseteq N$, we must have $R = N$. Hence N is cyclic.

Otherwise, $N^\#$ consists only of elements of the form $r^i s$ where $1 \le i \le n$. Let $r^i s, r^j s \in N$ where $1 \le i, j \le n$ and $i \ne j$. Then

$$r^i s r^j s = r^i r^{-j} = r^{i-j} \ne 1.$$

However, we are assuming that N does not contain any such elements. Thus $i = j$ and $N \cong \mathbb{Z}_2$. □

Definition 1.1.13. Let G be a group, $n \in \mathbb{N}$ and $H \le \mathrm{Sym}(n)$. Let G^n denote the direct product of n copies of G. Let $\pi \in H$ and define an action of H on G^n by
$$\pi^{-1} : (g_1, \ldots, g_n) \longmapsto (g_{1^\pi}, \ldots, g_{n^\pi}).$$
The semidirect product of G^n by H formed in this way is called the *wreath product* of G by H. This is often denoted $G \mathrm{wr} H$.

Definition 1.1.14. Let G be a group and $\{G_i \mid 1 \le i \le n\}$ be a set of subgroups of G. If $G = \langle G_i \mid 1 \le i \le n \rangle$ and $[G_i, G_j] = 1$ for $i \ne j$, then G is said to be a *central product* of the subgroups G_i, $1 \le i \le n$. This is often denoted
$$G = G_1 * \ldots * G_n.$$

The subgroups G_i in Definition 1.1.14 have the property that

$$\bigcap_{i=1}^{n} G_i \subseteq Z(G).$$

Definition 1.1.15. Let G be a group and $H \le G$ a subgroup. The *normal closure* of H in G is the smallest normal subgroup $N \trianglelefteq G$ such that $H \subseteq N$.

We find that

$$\langle H^G \rangle = \langle H^g \mid g \in G \rangle$$

is the normal closure of H in G. Since the intersection of normal subgroups is normal, we also find that $\langle H^G \rangle$ is the intersection of all normal subgroups in G which contain H.

Definition 1.1.16. Let G be a group. A subgroup $S \leq G$ is said to be *subnormal* in G if there exist subgroups H_i of G such that

$$S = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_r = G.$$

We write $S \trianglelefteq\trianglelefteq G$ to denote that S is subnormal in G.

Definition 1.1.17. Let G be a group and $H \leq G$ a subgroup. The *subnormal closure* of H in G is the smallest subnormal subgroup $N \trianglelefteq\trianglelefteq G$ such that $H \subseteq N$.

Since the intersection of subnormal subgroups is again subnormal (see [25, Corollary 2.4]) we find that the subnormal closure of H in G is the intersection of all subnormal subgroups in G which contain H.

1.2 Soluble, nilpotent and p-groups

1.2.1 p-groups and π-groups

Lemma 1.2.1. *Let G be a finite p-group and $1 \neq N \trianglelefteq G$. Then*

$$N \cap Z(G) \neq 1.$$

Corollary 1.2.2. *The centre of a nontrivial finite p-group G is nontrivial.*

Proposition 1.2.3. *Let π be a set of primes, G a group, and $M, N \trianglelefteq G$ be normal π-subgroups of G. Then MN is a normal π-subgroup of G.*

Proof. Since M and N are both normal in G, MN is a normal subgroup of G. Also,

$$|MN| = \frac{|M||N|}{|M \cap N|},$$

and thus MN is a π-subgroup of G. □

Definition 1.2.4. Let G be a group and π a set of primes. The subgroup generated by all normal π-subgroups of G is denoted $\mathcal{O}_\pi(G)$.

Note that $\mathcal{O}_\pi(G)$ is the largest normal π-subgroup of a group G and is well defined by Proposition 1.2.3. When $\pi = \{p\}$, we write $\mathcal{O}_p(G)$ to denote the subgroup generated by all normal p-subgroups of G (as opposed to $\mathcal{O}_{\{p\}}(G)$). This is the largest normal p-subgroup of G.

Lemma 1.2.5. *Let G be a group and p a prime such that*

$$G = G_1 \times \ldots \times G_n$$

for subgroups G_i of G, $1 \leq i \leq n$. Set $\overline{G} = G/\mathcal{O}_p(G)$. Then:

1. $\mathcal{O}_p(G) = \mathcal{O}_p(G_1) \times \ldots \times \mathcal{O}_p(G_n)$; *and*
2. $\overline{G} = \overline{G_1} \times \ldots \times \overline{G_n}$.

Proof. Since G is a direct product, G_i commutes with G_j for all $i \neq j$. Hence

$$\mathcal{O}_p(G_1) \times \ldots \times \mathcal{O}_p(G_n) \trianglelefteq G.$$

Of course this direct product is a p-group and so

$$\mathcal{O}_p(G_1) \times \ldots \times \mathcal{O}_p(G_n) \subseteq \mathcal{O}_p(G).$$

Observe also that

$$\mathcal{O}_p(G) \subseteq (\mathcal{O}_p(G) \cap G_1) \times \ldots \times (\mathcal{O}_p(G) \cap G_n),$$

and so

$$\mathcal{O}_p(G) \subseteq \mathcal{O}_p(G_1) \times \ldots \times \mathcal{O}_p(G_n).$$

Since we have inclusion in both directions, it follows that

$$\mathcal{O}_p(G) = \mathcal{O}_p(G_1) \times \ldots \times \mathcal{O}_p(G_n).$$

For the second part of this lemma we must prove that each element $\overline{g} \in \overline{G}$ can be written uniquely in the form

$$\overline{g} = \overline{g_1} \cdots \overline{g_n}$$

with $\overline{g_i} \in \overline{G_i}$. We first consider the existence of such an expression. Since G is a direct product, each $g \in G$ can be written in the form

$$g = g_1 \cdots g_n$$

with $g_i \in G_i$. Since the map $G \longrightarrow \overline{G}$ is a homomorphism,

$$\overline{g} = \overline{g_1} \cdots \overline{g_n}$$

where $g_i \longmapsto \overline{g_i}$ for each i. In order to prove the uniqueness of this expression, it suffices to show that the identity has a unique such expression. Write

$$1 = \overline{g_1} \cdots \overline{g_n}$$

with $g_i \in G_i$. Then

$$g_1 \cdots g_n \in \mathcal{O}_p(G) = \mathcal{O}_p(G_1) \times \ldots \times \mathcal{O}_p(G_n)$$

so

$$g_1 \cdots g_n = h_1 \cdots h_n$$

for $h_i \in \mathcal{O}_p(G_i)$. Using the fact that the subgroups G_i commute, then

$$(h_1^{-1} g_1) \cdots (h_n^{-1} g_n) = 1$$

and $(h_i^{-1} g_i) \in G_i$ for all i. As $G = G_1 \times \ldots \times G_n$, we get $(h_i^{-1} g_i) = 1$, and so $g_i = h_i$. Therefore $\overline{g_i} = \overline{h_i} = 1$. □

Proposition 1.2.6. *Let G be a finite group and π be any set of primes. Then there exists a unique smallest normal subgroup N such that G/N is a π-group.*

Proof. It suffices to show that for normal subgroups $H, K \trianglelefteq G$ such that G/H and G/K are π-groups, that $G/(H \cap K)$ is a π-group. Let H and K be two such subgroups. The map $\phi : G \longrightarrow G/H \times G/K$ defined by

$$g \longmapsto (gH, gK),$$

is a homomorphism with kernel $H \cap K$. Hence $G/(H \cap K)$ is isomorphic to a subgroup of $G/H \times G/K$, which is a π-group. □

Let G and N be as in Proposition 1.2.6. Then N is denoted $\mathcal{O}^{\pi}(G)$, and if $\pi = \{p\}$ for some prime p, we write $\mathcal{O}^p(G)$.

Definition 1.2.7. Let G be a group and p and q be primes. Then $\mathcal{O}_{p,q}(G)$ is defined to be the full inverse image of $\mathcal{O}_q(G/\mathcal{O}_p(G))$ in G.

Definition 1.2.8. Let G be a p-group. Then $\Omega_1(G) = \langle g \mid g^p = 1 \rangle$.

Note that if G is an abelian p-group, then $\Omega_1(G)$ is elementary abelian.

1.2.2 Nilpotent groups

Definition 1.2.9. Let G be a group. Let $Z_0 = 1$ and $Z_1 = Z(G)$ and define Z_i inductively as the unique subgroup such that $Z_i/Z_{i-1} = Z(G/Z_{i-1})$. The following chain of normal subgroups in G

$$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \dots$$

is called the *upper central series* of G. If for some $n \in \mathbb{N}$ we have $G = Z_n$, then G is said to be *nilpotent*. If $Z_{n-1} < G$ and $Z_n = G$, then G is said to be of *nilpotence class* n.

Note that the nontrivial abelian groups are precisely the nilpotent groups of nilpotence class one (we define the nilpotence class of the trivial group to be zero). Hence

$$\text{abelian groups} \subset \text{nilpotent groups}.$$

We can think of nilpotence class as a 'measure' of how far a nilpotent group is from being abelian. This is strongly related to the ideas outlined in the introduction.

Theorem 1.2.10. *Let G be a finite group. Then the following are equivalent:*

1. *G is nilpotent;*
2. *$H < N_G(H)$ for every proper subgroup $H < G$;*
3. *Every maximal subgroup of G is normal;*
4. *Every Sylow p-subgroup of G is normal;*
5. *Every nontrivial homomorphic image of G has a nontrivial centre.*

Proof. Note that the equivalence of 1, 2, 3 and 4 is proved in [25, Theorem 1.26]. The equivalence of 1 and 5 is proved in [25, Lemma 1.20]. □

Corollary 1.2.11. *Let G be a finite group and $\pi(G) = \{p_1, \dots, p_n\}$. Then G is nilpotent if and only if*

$$G = \mathcal{O}_{p_1}(G) \times \dots \times \mathcal{O}_{p_n}(G).$$

Proof. Suppose

$$G = \mathcal{O}_{p_1}(G) \times \dots \times \mathcal{O}_{p_n}(G).$$

Then $|G|_{p_i} = |\mathcal{O}_{p_i}(G)|$ for all i, $1 \leq i \leq n$. Since $\mathcal{O}_{p_i}(G) \trianglelefteq G$, $\mathcal{O}_{p_i}(G)$ is the unique Sylow p_i-subgroup of G for any i. Thus all Sylow subgroups of G are normal and so G is nilpotent by Theorem 1.2.10.

If G is nilpotent, then every Sylow p-subgroup of G is normal in G. Since $\mathcal{O}_{p_i}(G)$ is maximal among normal p_i-subgroups of G, $\mathcal{O}_{p_i}(G) \in \mathrm{Syl}_{p_i}(G)$ for all i. Therefore
$$G = \mathcal{O}_{p_1}(G) \cdots \mathcal{O}_{p_n}(G).$$
Note that since any Sylow subgroup of G is normal, that the product of any Sylow subgroup with any other subgroup of G, is a subgroup of G. Now

$$\mathcal{O}_{p_i}(G) \text{ and } \prod_{p_i \neq p_j} \mathcal{O}_{p_j}(G)$$

have coprime order, so

$$\mathcal{O}_{p_i}(G) \cap \prod_{p_i \neq p_j} \mathcal{O}_{p_j}(G) = 1$$

for all $1 \leq i \leq n$. Therefore

$$G = \mathcal{O}_{p_1}(G) \times \ldots \times \mathcal{O}_{p_n}(G).$$

□

Corollary 1.2.12. *A group G is nilpotent if and only if elements of coprime order commute.*

Proof. If G is nilpotent, then

$$G = \mathcal{O}_{p_1}(G) \times \ldots \times \mathcal{O}_{p_n}(G),$$

where $\{p_1, \ldots, p_n\} = \pi(G)$. Let $g, h \in G$ be elements of coprime order. We can write $g = g_1 \cdots g_n$ and $h = h_1 \cdots h_n$ where $g_i, h_i \in \mathcal{O}_{p_i}(G)$ for all i. Note that the subgroups $\mathcal{O}_{p_i}(G)$ commute and elements in distinct $\mathcal{O}_{p_i}(G)$ have coprime order. Hence, if we denote by $o(x)$ the order of the element $x \in G$, then $o(g) = \prod_{i=1}^{n} o(g_i)$ and $o(h) = \prod_{i=1}^{n} o(h_i)$. Since g and h have coprime order, at least one of g_i or h_i is trivial for each i. Since g_i commutes with h_j for all $i \neq j$, it follows that g and h commute.

If all coprime elements commute, then Sylow subgroups of different orders commute. Hence all Sylow subgroups of G are normal, and so G is nilpotent by Theorem 1.2.10. □

Proposition 1.2.13. *Let G be a group and $H, K \trianglelefteq G$ be normal nilpotent subgroups. Then $HK \trianglelefteq G$ is normal and nilpotent.*

Definition 1.2.14. Let G be a group and define the *Fitting subgroup*, denoted $F(G)$, by

$$F(G) = \langle N \mid N \trianglelefteq G, N \text{ is nilpotent} \rangle.$$

It is clear from Proposition 1.2.13 that $F(G)$ is maximal among normal nilpotent subgroups of an arbitrary group G.

Definition 1.2.15. Let G be a group and set $F_0(G) = 1$. Then we define *higher Fitting subgroups* of G inductively by letting $F_i(G)$ be the full inverse image of $F(G/F_{i-1}(G))$ in G for all $i \geq 1$.

1.2.3 Soluble groups

Lemma 1.2.16. *Let G be a group and N a minimal normal subgroup. If N is soluble, then it is elementary abelian.*

Proof. See [25, Lemma 3.11]. □

Corollary 1.2.17. *Let $G \neq 1$ be a soluble group, then $F(G) \neq 1$.*

Proof. By Lemma 1.2.16, a minimal normal subgroup $N \trianglelefteq G$ is elementary abelian. Abelian groups are nilpotent and minimal normal subgroups are nontrivial by definition. □

Theorem 1.2.18. *Let G be a soluble group. Then $C_G(F(G)) \subseteq F(G)$.*

Proof. See [20, Page 218] □

Corollary 1.2.19. *Let G be a soluble group. Suppose that every abelian normal subgroup $N \trianglelefteq G$ is contained in $Z(G)$. If $F(G)$ is abelian, then G is abelian.*

Proof. By hypothesis, $F(G) \subseteq Z(G)$ since $F(G) \trianglelefteq G$ and is abelian. Now, $C_G(F(G)) \subseteq F(G)$ by Theorem 1.2.18. Hence,

$$G \subseteq C_G(F(G)) \subseteq F(G)$$

where the inclusion on the left follows since $F(G) \subseteq Z(G)$. Thus $G = F(G)$, and so G is abelian. □

Theorem 1.2.20. (Feit–Thompson odd order theorem) *Let G be a finite group of odd order. Then G is soluble.*

Proof. See [15]. □

Definition 1.2.21. Let G be a finite group and π a set of primes. Then a *Hall π-subgroup* of G is a π-subgroup with index involving no prime in π.

Definition 1.2.22. Let G be a group and p a prime. A *normal p-complement* in G is a normal Hall p'-subgroup H of G.

Theorem 1.2.23. (Hall's theorem) *Let G be a finite soluble group and π a set of primes. Then:*

1. *There exists a Hall π-subgroup H of G;*
2. *All Hall π-subgroups of G are conjugate;*
3. *Every π-subgroup K of G is contained in some Hall π-subgroup of G.*

Proof. See [21]. □

Note that when π consists of a single prime p, a Hall π-subgroup is a Sylow p-subgroup of G. Also, if the set of primes π in Hall's theorem consists of a single prime p, then the statement coincides with Sylow's theorem restricted to soluble groups. Hence Hall's theorem can be seen as a generalisation of Sylow's theorem for soluble groups.

1.2.4 Frattini subgroup

We now define another very important characteristic subgroup, namely, the Frattini subgroup. For particular groups G, the Frattini quotient $G/\Phi(G)$ has nice properties. Several proofs in this thesis proceed by minimal counterexample, and in those, it is very common at some stage to make a reduction after the consideration of a group's Frattini quotient.

Definition 1.2.24. Let G be a group. Let $\Phi(G) \leq G$ be the intersection of all maximal subgroups of G. We call this subgroup the *Frattini subgroup* of G.

Lemma 1.2.25. *Let G be a p-group for some prime p. Then $G/\Phi(G)$ is elementary abelian.*

Proof. See [20, Page 174]. □

Lemma 1.2.26. *Let G be a p-group. Then $\Phi(G)$ is the smallest normal subgroup N of G such that G/N is elementary abelian.*

Proof. See [30, 5.2.8]. □

Lemma 1.2.27. *Let G be a group. If $G = \Phi(G)H$ for some subgroup $H \leq G$, then $G = H$.*

Proof. See [20, Page 173]. □

Lemma 1.2.28. *Let G be a group and $N \trianglelefteq G$ a normal subgroup. Then $\Phi(N) \subseteq \Phi(G)$.*

Proof. Suppose not, then there exists a maximal subgroup $M \leq G$ such that $\Phi(N) \nsubseteq M$. Hence, since $\Phi(N) \trianglelefteq G$, we can form the subgroup $\Phi(N)M$, and we have that $G = \Phi(N)M$. Since $\Phi(N) \subseteq N$, we have $N = \Phi(N)(N \cap M)$ by Dedekind's modular law. Then $N = N \cap M$ by Lemma 1.2.27. However, this forces $N \subseteq M$, which is a contradiction since $\Phi(N) \nsubseteq M$ □

Proposition 1.2.29. *Let G be a group. Then $\Phi(G)$ is nilpotent and hence $\Phi(G) \subseteq F(G)$.*

Proof. Let $P \in \mathrm{Syl}_p(\Phi(G))$. By the Frattini argument we have that

$$G = N_G(P)\Phi(G).$$

Thus $G = N_G(P)$ by Lemma 1.2.27 and so $P \trianglelefteq G$. In particular, $P \trianglelefteq \Phi(G)$. Since P is an arbitrary Sylow subgroup of $\Phi(G)$, we see that all Sylow subgroups of $\Phi(G)$ are normal and hence $\Phi(G)$ is nilpotent by Theorem 1.2.10. Since $\Phi(G) \trianglelefteq G$, it follows that $\Phi(G) \subseteq F(G)$. □

Lemma 1.2.30. *Let G be a finite nilpotent group. Then $G/\Phi(G)$ is abelian.*

Proof. Let M be a maximal subgroup of G. Then $M \trianglelefteq G$ by Theorem 1.2.10. Hence G/M is simple by the correspondence theorem. Since G is nilpotent, G/M must be an abelian simple group (nilpotent groups have nontrivial centres). Hence $G' \subseteq M$. Since M was chosen arbitrarily, G' is contained in every maximal subgroup of G. Hence it is contained in their intersection and so $G' \subseteq \Phi(G)$. □

Lemma 1.2.31. *Let G be a p-group with $\Phi(G) \leq Z(G)$. Then G' is elementary abelian.*

Proof. Note that $G/\Phi(G)$ is elementary abelian by Lemma 1.2.25, so $G' \leq \Phi(G) \leq Z(G)$. Thus G' is abelian. Let $g, h \in G$. Then

$$[g, h]^p = [g, h^p] = 1.$$

The second equality follows since $h^p \in \Phi(G) \leq Z(G)$. Thus for arbitrary $g, h \in G$, $[g, h]$ has order 1 or p. Since G' is abelian and is generated by commutators, the result follows. □

Lemma 1.2.32. *Let G be a p-group such that $Z(\Phi(G)) \leq Z(G)$. Then $\Phi(G) \leq Z(G)$.*

Proof. See [16, Lemma 3.2]. □

Corollary 1.2.33. *Let G be a p-group. Suppose that every abelian normal subgroup $N \trianglelefteq G$ is contained in $Z(G)$. Then*

$$G' \subseteq \Phi(G) \subseteq Z(G).$$

Proof. The inclusion on the left holds since $G/\Phi(G)$ is abelian. Now $Z(\Phi(G))$ is an abelian normal subgroup of G, hence $Z(\Phi(G)) \subseteq Z(G)$ by hypothesis. By Lemma 1.2.32, we have that $\Phi(G) \subseteq Z(G)$. □

1.3 Fitting height

As outlined in the introduction, the notion of Fitting height is central to this thesis. We will be studying soluble groups in many different settings but quite often the aim will be able to say something about their Fitting height. It can be thought of in some sense as a 'measure' of how far a soluble group is from being nilpotent.

Definition 1.3.1. Let G be a soluble group and let G_i be subgroups of G such that

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G \tag{1.3.1}$$

where G_{i+1}/G_i nilpotent for all $0 \leq i \leq n-1$. Then 1.3.1 is called a *Fitting series* for G of length n.

Note that for any finite soluble group G, there exists a Fitting series for G. This can be seen since the Fitting subgroup of a nontrivial soluble group is nontrivial.

Definition 1.3.2. Let G be a soluble group. The smallest integer n such that G possesses a Fitting series of length n is called the *Fitting height* of G. We denote this by $f(G)$.

Lemma 1.3.3. *Let G be a soluble group and $H \leq G$. Then $f(H) \leq f(G)$.*

Proof. Let G have Fitting height n and

$$1 = F_0 \trianglelefteq F_1 \trianglelefteq \ldots \trianglelefteq F_n = G$$

be a Fitting series for G. Consider the normal series for H given by

$$1 = F_0 \cap H \trianglelefteq F_1 \cap H \trianglelefteq \ldots \trianglelefteq F_n \cap H = H.$$

We claim that this is a Fitting series for H. Since this series has length n, the result will follow. For each i, $1 \leq i \leq n$, let

$$\phi_i : F_i \cap H \longrightarrow F_i/F_{i-1}$$

be defined by

$$f\phi_i = fF_{i-1}.$$

Then ϕ_i is a homomorphism with $\ker(\phi) = F_{i-1} \cap H$. Hence $(F_i \cap H)/(F_{i-1} \cap H)$ is isomorphic to a subgroup of F_i/F_{i-1} for each i, $1 \le i \le n$. Since F_i/F_{i-1} is nilpotent for each i, $1 \le i \le n$, the claim follows. □

Lemma 1.3.4. *Let G be a soluble group and $K \trianglelefteq G$. Then*

$$f(G) \le f(K) + f(G/K).$$

Proof. Let G/K have Fitting height n and

$$1 = \overline{F_0} \trianglelefteq \overline{F_1} \trianglelefteq \ldots \trianglelefteq \overline{F_n} = G/K$$

be a Fitting series for G/K. Let F_i be the inverse image of $\overline{F_i}$ in G for $1 \le i \le n$. Then

$$1 = K_0 \trianglelefteq K \trianglelefteq F_1 \trianglelefteq \ldots \trianglelefteq F_n = G$$

is a normal series for G. Let K have Fitting height m and

$$1 = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_m = K$$

be a Fitting series for K. Then

$$1 = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_m = K \trianglelefteq F_1 \trianglelefteq \ldots \trianglelefteq F_n = G$$

is a Fitting series of G of length $n + m$. Thus the claim follows. □

Lemma 1.3.5. *Let G be a nontrivial soluble group. Then*

$$f(G/F(G)) = f(G) - 1.$$

Proof. Set $\overline{G} = G/F(G)$. Let

$$1 = \overline{F_0} \trianglelefteq \ldots \overline{F_n} = \overline{G}$$

be a Fitting series for $\overline{G}$ of minimal length. For each i, $0 \le i \le n$, let F_i be the inverse image of $\overline{F_i}$ in G. Then we have the following normal series for G

$$F(G) = F_0 \trianglelefteq F_1 \trianglelefteq \ldots \trianglelefteq F_n = G. \qquad (1.3.2)$$

The third isomorphism theorem tells us

$$F_{i+1}/F_i \cong \overline{F_{i+1}}/\overline{F_i}.$$

Since $\overline{F_{i+1}}/\overline{F_i}$ is nilpotent for each i, $0 \le i \le n-1$, and $F(G)$ is nilpotent, 1.3.2 becomes a Fitting series for G of length $n + 1$ by appending 1 at the beginning. This is clearly a Fitting series of minimal length, otherwise we could construct a Fitting series for $\overline{G}$ of length less than n. □

Lemma 1.3.6. *Let $G = H \times K$ be a soluble group. Then*

$$f(G) = \max\{f(H), f(K)\}.$$

Proof. Let

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_n = H$$

be a Fitting series for H and

$$1 = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_m = K$$

a Fitting series for K, both of minimal length, for some $n, m \in \mathbb{N}$. Assume without loss of generality that $m \leq n$ and consider the following normal series for $H \times K$

$$1 = H_0 \times K_0 \trianglelefteq \ldots \trianglelefteq H_{n-m} \times K_0 \trianglelefteq H_{n-m+1} \times K_1 \trianglelefteq \ldots \trianglelefteq H_n \times K_m = H \times K.$$

Since

$$(H \times K)/(A \times B) \cong (H/A) \times (K/B)$$

for all $A \trianglelefteq H$ and $B \trianglelefteq K$, we see that each factor of the normal series for $H \times K$ above is nilpotent, and so it is a Fitting series. Thus $f(H \times K) \leq \max\{f(H), f(K)\}$.

Consider the subgroup $H \times 1 \leq H \times K$. Now $H \times 1 \cong H$ and so $f(H) = f(H \times 1)$. By Lemma 1.3.3, $f(H) \leq f(H \times K)$. Thus $\max\{f(H), f(K)\} \leq f(H \times K)$, and the result follows. □

Definition 1.3.7. Let G be a nontrivial soluble group. Then we let

$$\psi(G) = \bigcap\{H \trianglelefteq G \mid f(G/H) < f(G)\}.$$

If $G = 1$, then we let $\psi(G) = 1$.

Lemma 1.3.8. *Let G be a nontrivial soluble group. Then $\psi(G)$ is the unique smallest normal subgroup N of G such that $f(G/N) < f(G)$. We also have that $1 \neq \psi(G) \leq F(G)$.*

Proof. It suffices to show that for any two subgroups $H, K \trianglelefteq G$ such that

$$f(G/H) < f(G) \text{ and } f(G/K) < f(G),$$

that

$$f(G/H \cap K) < f(G).$$

Consider the map

$$\phi : G \longrightarrow G/H \times G/K$$

defined by

$$g \longmapsto (gH, gK).$$

Then ϕ is a homomorphism with kernel $H \cap K$. Therefore $G/H \cap K$ is isomorphic to a subgroup of $(G/H) \times (G/K)$. So

$$f(G/(H \cap K)) \leq f((G/H) \times (G/K))$$

by Lemma 1.3.3. However, by Lemma 1.3.6

$$f((G/H) \times (G/K)) = \max\{f(G/H), f(G/K)\} < f(G).$$

Therefore, $f(G/(H \cap K)) < f(G)$.

The second claim follows since $\psi(G) \leq F(G)$ by Lemma 1.3.5. Also, since $f(G/1) = f(G)$, it follows that $1 \neq \psi(G)$. □

Lemma 1.3.9. *Let $H \leq G$ with G soluble and $f(H) = f(G)$. Then*

$$\psi(H) \leq \psi(G) \leq F(G).$$

Proof. See [17, Lemma 1.1]. □

Lemma 1.3.10. *Let G be soluble and $N \trianglelefteq G$. Let $\overline{G} = G/N$ and suppose that $\overline{G} \neq 1$. Then the following are equivalent:*

1. $\overline{\psi(G)} \neq 1$;
2. $f(\overline{G}) = f(G)$;
3. $\psi(\overline{G}) = \overline{\psi(G)}$.

Proof. See [17, Lemma 1.2]. □

Definition 1.3.11. Let $G \neq 1$ be a soluble group and set $\psi_1(G) = \psi(G)$. Then for $i > 2$ we define $\psi_i(G)$ inductively as the full inverse image of $\psi(G/\psi_{i-1}(G))$ in G.

Proposition 1.3.12. *Let G be soluble group of Fitting height greater than 2. Then $\varphi(G) = [\varphi_2(G), \varphi(G)]$.*

Proof. Let $[\varphi_2(G), \varphi(G)] = N$, and suppose $\varphi(G) \neq N$. Set $\overline{G} = G/N$. Now $N \subseteq \varphi(G)$ and so since $N \neq \varphi(G)$, $f(G) = f(\overline{G})$. Hence

$$\varphi(\overline{G}) = \overline{\varphi(G)}.$$

Now

$$[\overline{\varphi(G)}, \overline{\varphi_2(G)}] = \overline{N} = 1$$

and so $\overline{\varphi(G)} \subseteq Z(\overline{\varphi_2(G)})$. Note that $\overline{\varphi_2(G)}/\overline{\varphi(G)}$ is nilpotent and so $\overline{\varphi_2(G)} \subseteq F(\overline{G})$. Hence

$$f(\overline{G}/F(\overline{G})) \leq f(G/\varphi_2(G)) = f(G) - 2 = f(\overline{G}) - 2.$$

This is a contradiction, hence $\varphi(G) = N$. □

1.4 Generalised Fitting subgroup

In soluble groups we find that $C_G(F(G)) \leq F(G)$. This gives us a representation of G as a subgroup of $\mathrm{Aut}(F(G))$ with kernel $Z(F(G))$. Hence, the structure of a soluble group G is often heavily restricted by properties of $F(G)$. In Chapter 2, we generalise some results about soluble groups to arbitrary groups. Quite naturally this involves the generalised Fitting subgroup $F^*(G)$. As the name suggests, $F^*(G)$ is an analogue of $F(G)$ for arbitrary groups. We find that $C_G(F^*(G)) \leq F^*(G)$, and so again we have a representation of G and can use $F^*(G)$ as a means of studying the structure of a group G. Further to the action of G on $F(G)$, we have a permutation representation on something called the components of G. Thus $F^*(G)$ also has much control over the structure of G.

Definition 1.4.1. A group G is said to be *quasisimple* if it is perfect and $G/Z(G)$ is a nonabelian simple group.

Lemma 1.4.2. *Let G be a quasisimple group. Every proper normal subgroup of G is contained in its centre and every nontrivial homomorphic image of G is quasisimple.*

Proof. See [25, Lemma 9.2]. □

Corollary 1.4.3. *Let G be a quasisimple group and p a prime. Then $G = \mathcal{O}^p(G)$.*

Proof. Suppose $\mathcal{O}^p(G) < G$. Then since $\mathcal{O}^p(G) \trianglelefteq G$, $\mathcal{O}^p(G) \subseteq Z(G)$ by Lemma 1.4.2. Then $G/Z(G)$ is a p-group by the definition of $\mathcal{O}^p(G)$. This is a contradiction since $G/Z(G)$ is nonabelian and simple. □

Definition 1.4.4. Let G be a group and $H \leq G$ a subnormal quasisimple subgroup. Then we call H a *component* of G. We denote by $\mathrm{Comp}(G)$, the set of components of G.

Theorem 1.4.5. *Let H and K be distinct components of a finite group G. Then $[H, K] = 1$.*

Proof. See [25, Theorem 9.4]. □

Definition 1.4.6. Let G be a group. The subgroup generated by the components of G is called the *layer* of G and is denoted $E(G)$.

By Theorem 1.4.5 it follows that the layer $E(G)$ of a group G is a central product of its components.

Definition 1.4.7. A group G is said to be *semisimple* if is a direct product of nonabelian simple groups.

Lemma 1.4.8. *Let G be a finite group and $N \trianglelefteq G$ an insoluble minimal normal subgroup. Then N is semisimple.*

Proof. See [25, Lemma 9.6]. □

Theorem 1.4.9. *Let G be a finite group, $E = E(G)$ the layer of G and $Z = Z(E)$. Then the following hold:*

1. *$E' = E$;*
2. *E/Z is semisimple;*
3. *If N is a soluble normal subgroup of G, then $[E, N] = 1$.*

Proof. See [25, Theorem 9.7]. □

Definition 1.4.10. Let G be a group. Then we define the *generalised Fitting subgroup* of G to be $F^*(G) = E(G)F(G)$. If $G = F^*(G)$, then G is said to be *quasinilpotent*.

Lemma 1.4.11. *Let G be a finite group and p a prime. Then $F^*(G) = \mathcal{O}_p(G) * \mathcal{O}^p(F^*(G))$.*

Proof. Note that $F^*(G) = \mathcal{O}_p(G)\mathcal{O}^p(F^*(G))$. We can see this since

$$F^*(G)/\mathcal{O}^p(F^*(G))$$

is a p-group by the definition of $\mathcal{O}^p(F^*(G))$. The full inverse image will be the product of a normal p-group and $\mathcal{O}^p(F^*(G))$. So $F^*(G) \subseteq \mathcal{O}_p(G)\mathcal{O}^p(F^*(G))$. The reverse inclusion is obvious since both $\mathcal{O}_p(G)$ and $\mathcal{O}^p(F^*(G))$ are subgroups of $F^*(G)$. Now by definition

$$F^*(G) = \mathcal{O}_p(G)\mathcal{O}_{p'}(G)E(G).$$

Therefore the factor group

$$F^*(G)/\mathcal{O}_{p'}(G)E(G)$$

is a p-group. So $\mathcal{O}^p(F^*(G)) \subseteq \mathcal{O}_{p'}(G)E(G)$ by Proposition 1.2.6. Now

$$[\mathcal{O}_p(G), \mathcal{O}_{p'}(G)] = 1 \text{ and } [\mathcal{O}_p(G), E(G)] = 1$$

by Corollary 1.2.12 applied to $F(G)$ and part 3 of Theorem 1.4.9 respectively. Thus

$$[\mathcal{O}_p(G), \mathcal{O}^p(F^*(G))] = 1,$$

and so

$$F^*(G) = \mathcal{O}_p(G) * \mathcal{O}^p(F^*(G))$$

by the definition of a central product (see Definition 1.1.14). □

Definition 1.4.12. A *central extension* of a group G is a pair (H, π) where H is a group and $\pi : H \longrightarrow G$ is a surjective homomorphism with $\ker(\pi) \leq Z(H)$. If H is perfect, then (H, π) is said to be a *perfect central extension* of G.

Quasisimple groups are precisely the perfect central extensions of simple groups.

1.5 Coprime action

Definition 1.5.1. Let G and A be groups. Then A is said to *act via automorphisms* on G if A acts on G as a set and

$$(gh) \cdot a = (g \cdot a)(h \cdot a)$$

for all $g, h \in G$ and $a \in A$. If furthermore $(|G|, |A|) = 1$, then A is said to act *coprimely* on G.

Theorem 1.5.2. *Let A be a group which acts coprimely on the group G and $N \trianglelefteq G$ be an A-invariant normal subgroup. Set $\overline{G} = G/N$. Then:*

1. $C_{\overline{G}}(A) = \overline{C_G(A)}$*;*
2. $G = C_G(A)[G, A]$ *and* $[G, A] = [G, A, A]$*;*
3. *If G is abelian, then* $G = C_G(A) \times [G, A]$*;*
4. *If* $[G', A] = 1$*, then* $G = C_G(A) * [G, A]$.

Proof. For part 1 see [25, Page 132]. For parts 2 and 3 see [25, Lemmas 4.28 and 4.29 and Theorem 4.34].

We now prove part 4. By part 2, G is generated by $C_G(A)$, and $[G, A]$. By hypothesis $[G', A] = [A, G'] = 1$. So by the three subgroups lemma we have $[G, A, G] = 1$. Thus $[G, A] \subseteq Z(G)$. Therefore $[C_G(A), [G, A]] = 1$, and so the claim follows from the definition of a central product (see Definition 1.1.14). □

Note that part 3 of Theorem 1.5.2 is often referred to as Fitting's theorem.

Theorem 1.5.3. (Thompson's $P \times Q$ lemma) *Let A be a finite group that acts on the p-group G, and suppose that $A = P \times Q$ where P is a p-group and Q a p'-group. Suppose $[Q, C_G(P)] = 1$. Then $[Q, G] = 1$.*

Proof. See [2, 24.2]. □

Theorem 1.5.4. *Let A act on G where both A and G are finite groups such that $(|A|, |G|) = 1$. Then for each prime p we have the following:*

1. *There exists an A-invariant Sylow p-subgroup of G;*
2. *If S and T are A-invariant Sylow p-subgroups of G, then $S^c = T$ for some element $c \in C_G(A)$;*
3. *Let P be an A-invariant p-subgroup, then P is contained in some A-invariant Sylow p-subgroup of G.*

Proof. See [2, 18.7]. □

If $A = 1$ in Theorem 1.5.4, then it becomes the statement of Sylow's theorem. Also, Theorem 1.5.4 should also require that at least one of A or G is soluble. However, since $(|A|, |G|) = 1$, at least one of A and G has odd order, and so this is guaranteed by the Feit–Thompson odd order theorem.

1.6 Special groups and symplectic forms

Proposition 1.6.1. *An elementary abelian p-group of order p^n is isomorphic to a vector space of dimension n over $\mathbb{F}_p$.*

Proof. See [20, Page 10]. □

Definition 1.6.2. Let V be a vector space and β a nondegenerate alternating form on V. Then we call β a *symplectic form* on V.

Lemma 1.6.3. *Let V be a vector space. If V admits a symplectic form β, then V has even dimension.*

Proof. See [49, §3.4.4]. □

Definition 1.6.4. Let G be a p-group. Then G is called *special* if

$$G' = \Phi(G) = Z(G).$$

Such a group is called *extraspecial* if furthermore $G' \cong \mathbb{Z}_p$.

Lemma 1.6.5. *Let G be a p-group such that*

$$\mathbb{Z}_p \cong G' \leq \Phi(G) \leq Z(G),$$

and set $\overline{G} = G/Z(G)$. Let k be a generator for G'. Then $\overline{G}$ is a vector space over $\mathbb{F}_p$, and the map

$$f : \overline{G} \times \overline{G} \longrightarrow Z(G) \text{ defined by } [g,h] = k^{f(\overline{g},\overline{h})}$$

is a symplectic form on $\overline{G}$.

Proof. Since $\Phi(G) \leq Z(G)$, $\overline{G}$ is elementary abelian and is thus an $\mathbb{F}_p$-vector space by Proposition 1.6.1. Let $g, h \in G$. Then by Lemma 1.1.2 we have $[g,h] = [h,g]^{-1}$ and $[g,g] = 1$. Thus $f(\overline{g},\overline{h}) = -f(\overline{g},\overline{h})$ and $f(\overline{g},\overline{g}) = 0$. Hence f is well defined. Let $g_1, g_2, h \in G$. Then

$$[g_1 g_2, h] = [g_1, h]^{g_2}[g_2, h] = [g_1, h][g_2, h]$$

where the first equality follows by Lemma 1.1.2, and the second since $G' \subseteq Z(G)$. A similar calculation shows

$$[h, g_1 g_2] = [h, g_1][h, g_2].$$

So f defines an alternating bilinear form.

Let $\overline{g} \in \overline{G}$ and suppose $f(\overline{g},\overline{h}) = 0$ for all $\overline{h} \in \overline{G}$. Then $[g,h] = 1$ for all $h \in G$, and so $g \in Z(G)$. Thus f is nondegenerate, and so defines a symplectic form on $\overline{G}$. □

Note that p-groups which satisfy the hypotheses of Lemma 1.6.5 have the property that $|G/Z(G)| = p^{2l}$ for some $l \in \mathbb{N}$. This follows from Lemma 1.6.3. Extraspecial groups are an example of such a group. If G is extraspecial, then $|Z(G)| = p$ and thus $|G| = p^{1+2l}$ for some $l \in \mathbb{N}$. We sometimes write $G \cong p^{1+2l}$ to denote that G is an extraspecial group of order p^{1+2l}.

1.7 Permutation and Frobenius groups

Definition 1.7.1. Let G act transitively on a set Ω. If for $\alpha, \beta \in \Omega$ there exists a unique $g \in G$ such that $\alpha \cdot g = \beta$, then G is said to act *regularly* on Ω.

When a group G acts regularly on a set Ω, it follows that all point stabilisers are trivial. Thus the following definition is a generalisation of this notion.

Definition 1.7.2. Let G act on a set Ω. If the stabiliser of any given point $\alpha \in \Omega$ is trivial, then G is said to act *semiregularly* on Ω.

In particular, a group G acts regularly on a set Ω if it acts both transitively and semiregularly.

Definition 1.7.3. Let G be a group which acts on the set Ω. Then we define:

1. $\mathrm{Mov}_\Omega(G) = \{\alpha \in \Omega \mid \alpha^g \neq \alpha \text{ for some } g \in G\}$; and

2. $\mathrm{Fix}_\Omega(G) = \{\alpha \in \Omega \mid \alpha^g = \alpha \text{ for all } g \in G\}$.

We sometimes omit the subscript and write $\mathrm{Mov}(G)$ and $\mathrm{Fix}(G)$ when it is clear what set G is acting on.

Definition 1.7.4. Let A be a group which acts on the group G such that A acts semiregularly on the nonidentity elements of G. Then we say that the action of A on G is *Frobenius*.

Theorem 1.7.5. *Let G be a group which is the split extension of a normal subgroup $N \trianglelefteq G$ by $A \leq G$. Then the following are equivalent:*

1. *The conjugation action of A on N is Frobenius;*

2. *$A \cap A^g = 1$ for all $g \in G - A$;*

3. *$C_G(a) \subseteq A$ for all $a \in A^\#$;*

4. *$C_G(n) \subseteq N$ for all $n \in N^\#$.*

Proof. See [25, Theorem 6.4]. □

Definition 1.7.6. Let A, N and G be as in Theorem 1.7.5. Then G is a *Frobenius group* with *Frobenius kernel* N and *Frobenius complement* A

When the context is clear we sometimes refer to the Frobenius kernel as the 'kernel'.

Theorem 1.7.7. (Thompson) *Frobenius kernels are nilpotent.*

Proof. See [25, Theorem 6.24]. □

Proposition 1.7.8. *Let G be a group that acts transitively on a set Ω containing more than one element and let $H \leq G$ be the stabiliser of $\alpha \in \Omega$. Then G acts primitively on Ω if and only if H is maximal.*

Proof. See [49, Proposition 2.1]. □

Lemma 1.7.9. *Let G be a group that acts transitively on a set Ω and let $N \trianglelefteq G$. Then the orbits in the action of N on G form a system of imprimitivity for the action of G on Ω.*

Corollary 1.7.10. *Let G act primitively on a set Ω and let $N \trianglelefteq G$. Then N either acts trivially or transitively on Ω.*

Lemma 1.7.11. *Let G be a group which acts transitively on some set Ω and let H be the stabiliser of $\alpha \in \Omega$. Suppose that there exists a normal subgroup $N \trianglelefteq G$ such that N acts regularly on Ω, $G = HN$ and $H \cap N = 1$. Then the conjugation action of H on the nonidentity elements of N is permutation isomorphic to the action of H on $\Omega - \{\alpha\}$.*

Proof. See [25, Lemma 8.5]. □

1.8 Representation theory

We now consider the representation theory results that we will require throughout this thesis before moving on to looking at free modules and the first cohomology group. Throughout the rest of this chapter we let $\mathbb{F}$ be a field.

Definition 1.8.1. Let G be a group and V a G-module. If we can write

$$V = V_1 \oplus \ldots \oplus V_n$$

where for each i there exists a field $\mathbb{F}_i$ such that V_i is an irreducible $\mathbb{F}_i[G]$-module, we say that V is a *completely reducible G-module possibly of mixed characteristic.*

Definition 1.8.2. Let G be a group, ϕ a representation of G on $V/\mathbb{F}$ and $\mathbb{F} \leq \mathbb{K}$ a field extension. Then we can extend the representation from ϕ to $\phi_{\mathbb{K}}$ on

$$V^{\mathbb{K}} = V \otimes_{\mathbb{F}} \mathbb{K}.$$

We call ϕ *absolutely irreducible* provided the extended representation $\phi_{\mathbb{K}}$ of G on $V^{\mathbb{K}}/\mathbb{K}$ is irreducible for every extension $\mathbb{K}$ of $\mathbb{F}$. If a field $\mathbb{F}$ has the property that every irreducible representation ϕ of G on a vector space $V/\mathbb{F}$ is absolutely irreducible, then $\mathbb{F}$ is called a *splitting field* for G.

Lemma 1.8.3. *Let G be a group and V an $\mathbb{F}[G]$-module. Let $\mathbb{F} \leq \mathbb{K}$ be a field extension, then:*

1. $C_{V^{\mathbb{K}}}(G) = C_V(G) \otimes_{\mathbb{F}} \mathbb{K}$;
2. $C_{V^{\mathbb{K}}}(G) = 0$ *if and only if* $C_V(G) = 0$;
3. *Suppose V is faithful and irreducible. Then every irreducible submodule of $V^{\mathbb{K}}$ is faithful.*

Proof. See [16, Lemma 2.2]. □

Theorem 1.8.4. *Let G be a group and V a faithful completely reducible $\mathbb{F}[G]$-module. Let $\mathbb{F} \leq \mathbb{K}$ be a field extension where $\mathbb{K}$ is a splitting field for G, then*

$$V^{\mathbb{K}} = V \otimes_{\mathbb{F}} \mathbb{K}$$

is a faithful completely reducible $\mathbb{K}[G]$-module.

Proof. Since V is faithful, every irreducible G-submodule of $V^{\mathbb{K}}$ is faithful by Lemma 1.8.3. Now the result follows from [24, Theorem 9.21]. □

Lemma 1.8.5. *Let G be a group and V an $\mathbb{F}[G]$-module. For any normal subgroup $N \trianglelefteq G$, $C_V(N)$ is an $\mathbb{F}[G]$-submodule of V.*

Proof. So we need to show that $C_V(N)$ is G-invariant. Let $v \in C_V(N)$, $g \in G$ and $n \in N$. Then $n^{g^{-1}} \in N$ since $N \trianglelefteq G$. So

$$v^g = v^{n^{g^{-1}}g} = v^{gn}.$$

Therefore $v^g \in C_V(n)$. Since n was chosen arbitrarily, it follows that $v^g \in C_V(N)$. Then since v and g were chosen arbitrarily in $C_V(N)$ and G respectively, the claim follows. □

Lemma 1.8.6. *Let G be a group and V a faithful irreducible module for G over a field $\mathbb{F}$ of characteristic p. Then $\mathcal{O}_p(G) = 1$.*

Proposition 1.8.7. *Let G be a group which possesses a faithful irreducible representation. Then $Z(G)$ is cyclic.*

Lemma 1.8.8. *Let G be an abelian group of order n and $\mathbb{F}$ a field which contains a primitive n^{th} root of unity. Then every irreducible representation of G over $\mathbb{F}$ is 1-dimensional.*

Proposition 1.8.9. *Let ϕ be a degree 1 representation of a group G and let K be the kernel of this representation. Then G/K is cyclic.*

Corollary 1.8.10. *Let G be a noncyclic group. Then G does not possess a faithful representation of degree 1.*

Theorem 1.8.11. (Maschke's theorem) *Let G be a group and V an $\mathbb{F}[G]$-module for some field $\mathbb{F}$. Suppose*

$$(\mathrm{char}(\mathbb{F}), |G|) = 1.$$

Then V is completely reducible.

Definition 1.8.12. Let G be a group and V an $\mathbb{F}[G]$-module. Let $\Omega = \{V_1, \dots, V_n\}$ be a collection of nonzero subspaces of V such that

$$V = V_1 \oplus \dots \oplus V_n$$

and $V_i g \in \Omega$ for all i and $g \in G$. This gives a permutation representation for G on Ω and we call Ω a *G-system of imprimitivity* for V. We say that V is *primitive* if $\{V\}$ is the only G-system of imprimitivity for V. Otherwise, we say that V is an *imprimitive* G-module.

Definition 1.8.13. Let G be a group, V a completely reducible $\mathbb{F}[G]$-module and $U \leq V$ an irreducible $\mathbb{F}[G]$-submodule. If for every irreducible submodule $U' \leq V$ we have that

$$U \cong_{\mathbb{F}[G]} U',$$

then V is said to be *homogeneous* as an $\mathbb{F}[G]$-module.

Theorem 1.8.14. (Clifford's theorem) *Let V be an irreducible $\mathbb{F}[G]$-module and $N \trianglelefteq G$ a normal subgroup. Then V is the direct sum of N-invariant $\mathbb{F}$-subspaces V_i, $1 \leq i \leq k$, which satisfy the following conditions:*

1. *$V_i = V_{i1} \oplus \dots \oplus V_{il}$, where each V_{ij} is an irreducible $\mathbb{F}[N]$-submodule. The number l is independent of i and V_{ij}, $V_{i'j'}$ are isomorphic as $\mathbb{F}[N]$-modules if and only if $i = i'$;*

2. *For any $\mathbb{F}[N]$-submodule $U \leq V$, we have*

$$U = U_1 \oplus \dots \oplus U_k$$

   where $U_i = U \cap V_i$ for each i. In particular, any irreducible $\mathbb{F}[N]$-submodule of V is contained in one of the V_i;

3. *Let $g \in G$ and define the map $\pi_g : V_i \longrightarrow V_i g$. Then π_g is a permutation of the set $S = \{V_1, \dots, V_k\}$ and the mapping $g \longmapsto \pi_g$ defines a permutation representation on the set S. Furthermore, $NC_G(N)$ is contained in the kernel of this representation.*

Proof. See [20, Pages 70-72]. □

Corollary 1.8.15. *Let V be a primitive irreducible $\mathbb{F}[G]$-module and $N \trianglelefteq G$ a normal subgroup. Then V is homogeneous as an $\mathbb{F}[N]$-module.*

Proof. By Clifford's Theorem, V is a direct sum of N-invariant subspaces $\{V_1, \dots, V_k\}$ which are transitively permuted by G. This constitutes a system of imprimitivity for the action of G on V. However, since V is a primitive module for G, this forces $k = 1$. Thus V is a direct sum of N-isomorphic irreducible N-submodules. □

Corollary 1.8.16. *Let G be a group and V a primitive $\mathbb{F}[G]$-module for some algebraically closed field $\mathbb{F}$. Then all abelian normal subgroups of G are cyclic and contained in $Z(G)$.*

Proof. Let $N \trianglelefteq G$ be an abelian normal subgroup. Then V is homogeneous as an $\mathbb{F}[N]$-module by Corollary 1.8.15. Thus V_N is a direct sum of $\mathbb{F}[N]$-isomorphic irreducible modules. Since N is abelian and $\mathbb{F}$ is algebraically closed, N acts on each of these direct summands by some scalar $\lambda \in \mathbb{F}$. Hence $[G, N]$ acts trivially on V. Since V is a faithful $\mathbb{F}[G]$-module, $[G, N] = 1$. Hence $N \subseteq Z(G)$. Also, since N embeds into $\mathrm{Aut}(V)$ and acts on V by scalar multiplication, N is cyclic. □

Let G be a group, V an $\mathbb{F}[G]$-module and $H \leq G$ a subgroup of G. We denote by V_H the $\mathbb{F}_q[H]$-module V restricted to H.

Theorem 1.8.17. *Let $A = \langle a \rangle$ be a cyclic group which acts semiregularly on the abelian group N. Let V be a faithful $\mathbb{F}[AN]$-module where AN is the split extension of N by A. Assume that* $\mathrm{char}(\mathbb{F})$ *and $|N|$ are coprime and $C_V(N) = 0$. Then V_A is free.*

Proof. See [16, Theorem 2.9]. □

Theorem 1.8.18. (Hall–Higman) *Let A be a cyclic group that acts on the extraspecial p-group $P \cong p^{1+2n}$. Assume that A is semiregular on P/P' and trivial on P'. Let V be an irreducible $\mathbb{F}[AP]$-module with $V_{P'}$ faithful for some field $\mathbb{F}$. Assume further that at least one of the following holds:*

1. *$\mathbb{F}$ is algebraically closed;*
2. *$\mathbb{F}$ is a splitting field for P and* $\mathrm{End}_{[AP]}(V) = \mathbb{F} \cdot 1$*;*
3. *$\mathbb{F}$ is a splitting field for P and V_P is irreducible.*

Then V is faithful, V_P is irreducible, $\dim V = p^n$ *and there exists a 1-dimensional $\mathbb{F}[A]$-module U such that at least one of the following holds;*

- *$|A|$ divides $p^n + 1$ and*
$$V_A \cong \left(\frac{p^n + 1}{|A|} - 1\right) \times \mathbb{F}[A] \oplus \mathbb{F}[A]/U;$$
- *$|A|$ divides $p^n - 1$, A does not act irreducibly on P/P' and*
$$V_A \cong \left(\frac{p^n - 1}{|A|}\right) \times \mathbb{F}[A] \oplus U.$$

Proof. See [16, Theorem 4.1]. □

1.9 Free modules

If a group A acts on a soluble group G, then a normal subgroup $N \trianglelefteq G$ which is minimal with respect to being A-invariant is elementary abelian. Thus we can consider N to be a module for A. We will encounter this situation quite a lot and often we find that N is a free module for some cyclic subgroup of A. We review here the necessary properties of free modules.

Definition 1.9.1. Let R be a ring. An R-module V is a *free* R-module if there exists $B \subseteq V$ such that for all R-modules W and all maps $\theta : B \longrightarrow W$, θ extends uniquely to an R-module homomorphism $\hat{\theta} : V \longrightarrow W$. For such a B and V, we call B a set of *free generators* or a *free generating set* for V.

Proposition 1.9.2. *Let R be a ring, V an R-module and B a subset of V. Then the following are equivalent:*

1. *For every R-module W, every mapping $\theta : B \longrightarrow W$ extends uniquely to an R-module homomorphism $\hat{\theta} : V \longrightarrow W$;*
2. *B is an R-linearly independent spanning set of V.*

Proof. See [7, Page 49, Theorem 2.1.22]. □

In what follows, for an arbitrary ring R, let R^n denote the direct sum of n copies of R.

Corollary 1.9.3. *Let R be a ring. Then for any $n \in \mathbb{N}$, $V = R^n$ is a free R-module.*

Proof. Let $b_i = (0, \dots, 1, \dots, 0) \in R^n$ be the element with 0s in each column except for the i^{th} column where the entry is 1. Then $B = \{b_i \mid 1 \leq i \leq n\}$ is an R-linearly independent spanning set for V. Thus V is free by Proposition 1.9.2. □

Lemma 1.9.4. *Let R be a ring and V and W be free R-modules. Then $V \cong W$ if and only if their free generating sets are of equal cardinality.*

Proof. See [40, Page 59, Proposition 2.37]. □

Proposition 1.9.5. *Let R be a ring and V an R-module. Then V is a free R-module if and only if $V \cong R^n$ for some $n \in \mathbb{N}$.*

Proof. We know that R^n is free for any $n \in \mathbb{N}$ by Corollary 1.9.3, so if $V \cong R^n$ for some $n \in \mathbb{N}$, then V is clearly a free R-module.

Now let V be a free R-module with free generating set B. Suppose $|B| = n$. Then $V \cong R^n$ by Lemma 1.9.4. □

Theorem 1.9.6. *Let G be a group and V an $\mathbb{F}[G]$-module. Then the following are equivalent:*

1. *V is a free $\mathbb{F}[G]$-module;*

2. *$V \cong \mathbb{F}[G]^n$ for some $n \in \mathbb{N}$;*

3. *V possesses a G-invariant basis on which G acts semiregularly;*

4. *V possesses a system of imprimitivity on which G acts semiregularly.*

Proof. The equivalence of 1 and 2 is a direct consequence of Proposition 1.9.5. We first prove the equivalence on 2 and 3 and then of 3 and 4.

Let V be an $\mathbb{F}[G]$-module that possesses a basis B on which G acts semiregularly. Let $\mathcal{O}_i$ denote the orbits in the action of G on B, then

$$V = \bigoplus_i \langle \mathcal{O}_i \rangle.$$

Thus it will suffice to show that $\langle \mathcal{O}_i \rangle \cong \mathbb{F}[G]$ for any i. Let $W = \langle \mathcal{O}_i \rangle$ for some i. We know that $\mathbb{F}[G]$ has basis $B' = \{1\}$ as a module over $\mathbb{F}[G]$. Consider the map

$$\phi : B' \longrightarrow W \text{ given by } 1\phi = b_i$$

where b_i is some basis vector of V contained in the orbit $\mathcal{O}_i$. Since $\mathbb{F}[G]$ is free, ϕ extends uniquely to an $\mathbb{F}[G]$-homomorphism $\hat{\phi} : \mathbb{F}[G] \longrightarrow W$. By the definition of W, $\hat{\phi}$ is onto. However, since $\mathbb{F}[G]$ and W have the same dimension, $\hat{\phi}$ defines an $\mathbb{F}[G]$-isomorphism between them as desired.

Now suppose that V is a free $\mathbb{F}[G]$-module. Then

$$V \cong \mathbb{F}[G] \oplus \ldots \oplus \mathbb{F}[G].$$

We know that G acts semiregularly on the basis of $\mathbb{F}[G]$ (the action of a group on itself by right multiplication is regular) and so it acts semiregularly on the basis of n copies of $\mathbb{F}[G]$.

We now prove the equivalence of 3 and 4. Let V be an $\mathbb{F}[G]$-module with basis B on which G acts semiregularly. Let $B = \{b_1, \ldots, b_n\}$ and $V_i = \langle b_i \rangle$. Then

$$V = \bigoplus_{i=1}^{n} V_i$$

defines a system of imprimitivity on which G acts semiregularly.

Now suppose that V is an $\mathbb{F}[G]$-module which possesses a system of imprimitivity on which G acts semiregularly. So

$$V = \bigoplus_{i=1}^{n} V_i,$$

and for any i, $V_i g = V_i$ if and only if $g = 1$. Let B_i be a basis for V_i. Then

$$B = \coprod B_i$$

is a basis for V. Let $b \in B$. Then $b \in B_i$ for some i. Suppose $bg = b$ for some $g \neq 1$. Then g must fix V_i. This is a contradiction since g acts semiregularly on the subspaces V_i. □

Lemma 1.9.7. *Let G be a group and $V \cong \mathbb{F}[G]$. Then*

$$\dim C_V(G) = 1.$$

Proof. The action of G on $\mathbb{F}[G]$ is assumed to be from the right. Consider $G = \{g_1, \ldots, g_n\}$ as a set. Then $\{g_1, \ldots, g_n\}$ is a basis of $\mathbb{F}[G]$. Let $0 \neq v \in C_{\mathbb{F}[G]}(G)$. Then

$$v = \sum_{i=1}^{n} \lambda_i g_i$$

where $\lambda_i \in \mathbb{F}$. Since $v \neq 0$, $\lambda_i \neq 0$ for some i. Suppose $\lambda_j = 0$ for some $i \neq j$. Then v cannot be fixed by $g_i^{-1} g_j$. Thus $\lambda_j \neq 0$ for all $i \neq j$.

Now suppose that $\lambda_i \neq \lambda_j$. Then again we see that v is not fixed by $g_i^{-1} g_j$. Thus

$$v = \lambda \sum_{i=1}^{n} g_i \text{ for some } \lambda \in \mathbb{F},$$

and so $C_{\mathbb{F}[G]}(G) = \langle v \rangle$. Since $\dim\langle v \rangle = 1$, the result follows. □

Immediately we obtain the following corollary.

Corollary 1.9.8. *Let G be a group and V an $\mathbb{F}[G]$-module for some field $\mathbb{F}$. Suppose V has a free direct summand. Then $\dim C_V(G) > 0$.*

Proof. This follows trivially from Lemma 1.9.7. □

Theorem 1.9.9. *Let A be a cyclic group and V an $\mathbb{F}[A]$-module for some field $\mathbb{F}$. Let $\mathbb{F} \leq \mathbb{K}$ be a field extension, then:*

1. *V is free if and only if $V^{\mathbb{K}}$ is free;*

2. *$\mathbb{F}[A]$ is a direct summand of V if and only if $\mathbb{K}[A]$ is a direct summand of $V^{\mathbb{K}}$.*

Proof. See [16, Corollary 2.8]. □

1.10 First cohomology group

Much of what appears in this short section can be found in [2, Section 17]. Results which have been taken from elsewhere are clearly indicated.

Throughout let G be a finite group and V an $\mathbb{F}_p[G]$-module. Then we can form the semidirect product $V \rtimes G = VG$ where $V \trianglelefteq VG$ and G is a complement to V in VG.

Definition 1.10.1. Let $\gamma : G \longrightarrow V$ be such that

$$(gh)\gamma = (g\gamma)^h + h\gamma \quad \text{for all } g, h \in G.$$

Then γ is called a *derivation* from G to V. The set of derivations from G into V is denoted $\mathrm{Der}(G, V)$.

Lemma 1.10.2. *The set* $\mathrm{Der}(G, V)$ *is an* $\mathbb{F}_p$*-vector space where addition and scalar multiplication are defined as follows*

$$g(\gamma + \delta) = g\gamma + g\delta \quad \gamma, \delta \in \mathrm{Der}(G, V), g \in G$$

and

$$g(\lambda\gamma) = \lambda(g\gamma) \quad \gamma \in \mathrm{Der}(G, V), g \in G, \lambda \in F.$$

Lemma 1.10.3. *Let* $\gamma \in \mathrm{Der}(G, V)$ *and define* $S(\gamma) = \{g(g\gamma) \mid g \in G\}$*. The map*

$$S : \gamma \longrightarrow S(\gamma)$$

defines a bijection between $\mathrm{Der}(G, V)$ *and the set of complements to* V *in* VG*.*

Corollary 1.10.4. *The number of complements to* V *in* VG *is* $|\mathrm{Der}(G, V)|$*.*

Lemma 1.10.5. *Let* $\gamma \in \mathrm{Der}(G, V)$ *and define an action of* G *on* $\mathrm{Der}(G, V)$ *by*

$$h(\gamma^g) = [(h^{g^{-1}})\gamma]^g \text{ where } g, h \in G.$$

Then the map π *given by*

$$g\pi : \gamma \longmapsto \gamma^g$$

is an $\mathbb{F}_p$*-representation of* G *on* $\mathrm{Der}(G, V)$*.*

Lemma 1.10.6. *Let* $v \in V$ *and define* $\alpha_v : G \longrightarrow V$ *by*

$$g\alpha_v = [g, v] = v - v^g.$$

The map $\alpha : V \longrightarrow \mathrm{Der}(G, V)$ *defined by*

$$\alpha : v \longmapsto \alpha_v$$

is an $\mathbb{F}_p[G]$*-homomorphism with kernel* $C_V(G)$*.*

Definition 1.10.7. Let $\gamma \in \mathrm{Der}(G, V)$. If $\gamma = \alpha_v$ for some $v \in V$ as above, then we call γ an *inner derivation*. We denote the set of inner derivations from G to V by $\mathrm{IDer}(G, V)$.

Note that under the addition and multiplication that we defined in Lemma 1.10.2, the set $\mathrm{IDer}(G, V)$ is an $\mathbb{F}_p$-subspace of $\mathrm{Der}(G, V)$. If we have α_u, $\alpha_v \in \mathrm{IDer}(G, V)$ and $\lambda \in \mathbb{F}_p$, then

$$\alpha_u + \alpha_v = \alpha_{u+v} \text{ and } \lambda\alpha_u = \alpha_{\lambda u}.$$

Definition 1.10.8. The *first cohomology group* of G on V, denoted $H^1(G, V)$, is given by

$$H^1(G, V) \cong \mathrm{Der}(G, V)/\mathrm{IDer}(G, V)$$

where the right hand side is a quotient of $\mathbb{F}_p$-vector spaces.

Theorem 1.10.9. *There are $|H^1(G, V)|$ conjugacy classes of complements to V in VG.*

Theorem 1.10.10. (Hochschild–Serre) *Let π be a representation of a group G on the $\mathbb{F}$-vector space V, $K \trianglelefteq G$ and V^K the fixed point subspace of K on V. Then there exists an exact sequence as follows:*

$$0 \longrightarrow H^1(G/K, V^K) \xrightarrow{\alpha} H^1(G, V) \xrightarrow{\beta} H^1(K, V)^G.$$

Proof. See [43, Pages 213-216]. □

Note that in Theorem 1.10.10, if $H^1(K, V)^G = 0$, then α is an isomorphism. Also, the action of G on $H^1(K, V)$ is defined as in Lemma 1.10.5.

Theorem 1.10.11. (Aschbacher–Guralnick) *Let G be a finite group and V a faithful irreducible $\mathbb{F}_p[G]$-module. Then $|H^1(G, V)| < |V|$.*

Proof. See [3, Theorem A]. □

Theorem 1.10.12. (Aschbacher–Guralnick) *Let $G = \langle g_1, \ldots, g_d \rangle$ be a finite group and V a nontrivial irreducible $\mathbb{F}_p[G]$-module, then VG can be generated by d elements if and only if $|H^1(G, V)| < |V|^{d-1}$.*

Proof. See [3, (2.5)]. □

Chapter 2

Properties of 2-generated subgroups

In [18] the following theorem is proved:

Theorem 2.0.1. (Flavell) *Let G be a finite group and suppose that P is a soluble $\{2,3\}'$-subgroup of G. Define*

$$\Sigma_G(P) = \{A \leq G \mid A \text{ is soluble and } A = \langle P, P^a \rangle \text{ for some } a \in A\}.$$

Let A be a maximal member of $\Sigma_G(P)$ with respect to inclusion. Then

$$F(A)V$$

is nilpotent for every nilpotent subgroup V of G that is normalised by A.

The aim of this chapter is to prove results of a similar nature to that of Theorem 2.0.1. However, we will consider the case where the subgroup P (as in Theorem 2.0.1) is cyclic and will remove the condition that A is soluble. Hence A will be a 2-generated subgroup. A nice analogue of Theorem 2.0.1 might be the following:

> Let G be a finite group and suppose that P is a soluble $\{2,3\}'$-subgroup of G. Define
>
> $$\Sigma_G(P) = \{A \leq G \mid A = \langle P, P^a \rangle \text{ for some } a \in A\}.$$
>
> Let A be a maximal member of $\Sigma_G(P)$ with respect to inclusion. Then
>
> $$F^*(A)V$$
>
> is quasinilpotent for every quasinilpotent subgroup V of G that is normalised by A.

However, the following example shows that this statement is not true.

Example 2.0.2. Let $A \cong SL(2,5)$ and let V be the natural module for A. Let $G = AV$ be the semidirect product of V with A formed in the obvious way. A $\{2,3\}'$-subgroup of G must be a 5-group. Let $P \in \mathrm{Syl}_5(A)$ and set

$$\Sigma_G(P) = \{B \leq G \mid B = \langle P, P^b \rangle \text{ for some } b \in B\}.$$

We claim that A is a maximal member of $\Sigma_G(P)$ with respect to inclusion. Now V is a quasinilpotent subgroup of G normalised by A. We claim further that AV is not quasinilpotent and thus since $A = F^*(A)$, we will have that $F^*(A)V$ is not quasinilpotent.

It is clear that $A \in \Sigma_G(P)$ but it is not so clear whether it is a maximal member with respect to inclusion. Since V is an irreducible module for A, this will follow if we can show that $G \neq \langle P, P^g \rangle$ for all $g \in G$.

We first ask if there are any more complements to V in G which contain P. Notice that for all $v \in C_V(P)$, A^v is a complement to V which contains P. Now both P and V are 5-groups. Thus $C_V(P) \neq 1$. Since A is faithful on V, and $\dim V = 2$, we have $|C_V(P)| = 5$. How many distinct complements to V in AV does this give us? Suppose $A^{v_1} = A^{v_2}$ with $v_1, v_2 \in C_V(P)$. Let $u = v_1 v_2^{-1}$, so $A^u = A$. Then for $b \in A$ we have

$$[u, b] \in A \cap V = 1.$$

Therefore $u \in C_V(A) = 1$ and so $v_1 = v_2$. So we get five complements to V in G of this form.

How many conjugates of P do all of these complements contain? Any pair only have P in common otherwise they would generate the same complement. Thus the number of distinct conjugates that these complements contain, is the number of complements multiplied by the number of distinct conjugates of P contained in A. Note that there are six Sylow 5-subgroups in A. Thus these complements contain 25 conjugates of P which are not equal to P.

How many conjugates are there of P in G? This is given by

$$|G : N_G(P)| = \frac{|V||A|}{|N_G(P)|}.$$

Now

$$N_G(P) = C_V(P)N_A(P).$$

So

$$|G : N_G(P)| = \frac{|V|}{|C_V(P)|}\frac{|A|}{|N_A(P)|} = 5 \cdot 6.$$

Therefore, the number of G-conjugates of P which are not equal to P, is 29.

Consider $\langle P, P^v \rangle$. If $v \in V$ and $P^v \neq P$, then P^v is not contained in any complement to V which contains P. We get four conjugates to P like this. This accounts for the other four conjugates of P, but we see here that $\langle P, P^v \rangle$ does not generate G. This is because the factor group $\langle P, P^v \rangle / V$ is cyclic. If $\langle P, P^v \rangle = G$, then this factor group would be isomorphic to A. Thus A is indeed a maximal member of $\Sigma_G(P)$ with respect to inclusion.

Suppose AV is quasinilpotent, so $AV = F^*(AV)$. Then by Lemma 1.4.11,

$$AV = \mathcal{O}_p(AV) * \mathcal{O}^p(AV)$$

for any prime p. Now $V \subseteq \mathcal{O}_5(AV)$. If AV is quasinilpotent, then V must commute with all $5'$-elements of AV. However, this implies that the action of A on V is not faithful. However, this action is faithful and so AV cannot be quasinilpotent.

How about if we only restrict A to being maximal among 2-generated subgroups? It turns out that we can obtain some analogous results to Theorem 2.0.1 in this case, and as such, arbitrary 2-generated subgroups will be the object of study in this chapter. The main result that we will prove is the following:

Theorem 2.0.3. *Let G be a finite group, $A \leq G$ a subgroup which is maximal subject to being 2-generated and $V \leq G$ a quasinilpotent subgroup which is normalised by A. Then $F^*(A)F(V)$ is quasinilpotent.*

However, we begin this chapter by looking at some results regarding soluble 2-generated subgroups before extending these to arbitrary 2-generated subgroups. In particular, we prove:

Theorem 2.0.4. *Let G be a finite group, $A \leq G$ a subgroup which is maximal subject to being soluble and 2-generated and $V \leq G$ a nilpotent subgroup which is normalised by A. Then $F(A)V$ is nilpotent.*

As a result of Theorem 2.0.4, we provide new proofs of a couple of well-known results. Namely, that: any finite soluble group G possesses a 2-generated subgroup A with $f(A) = f(G)$; and, a finite group is soluble if and only if every three elements generate a soluble subgroup. The former of these results was first proved by R. Carter, B. Fischer and T. Hawkes in [9] and the latter by M. Powell (an account of his work can be found in [8, pages 473-476]).

We first recall some preliminary results that will be required in this chapter.

2.1 Preliminary results

The results presented in this section are well-known. References for most of the results are given, but the author provides proofs of results which are difficult to find. No originality is claimed by the author in this section.

Theorem 2.1.1. (Baer–Suzuki) *Let G be a finite group and $H \leq G$. Then $H \subseteq F(G)$ if and only if $\langle H, H^g \rangle$ is nilpotent for all $g \in G$.*

Proof. See [25, Theorem 2.12]. □

Definition 2.1.2. Let P be a p-group and $\epsilon(P)$ be the set of all elementary abelian subgroups of P that have the maximum possible order. Then the *Thompson subgroup*, denoted $J(P)$, is defined to be the subgroup generated by all of the members of $\epsilon(P)$.

Theorem 2.1.3. (Thompson's normal p-complement theorem) *Let G be a finite group, p an odd prime and $P \in \mathrm{Syl}_p(G)$. Suppose that $C_G(Z(P))$ and $N_G(J(P))$ have normal p-complements. Then G has a normal p-complement.*

Proof. See [25, Chapter 7]. □

Theorem 2.1.4. (Dade) *Suppose $H \trianglelefteq G$ and that $G = C_G(P)H$ whenever P is a Sylow subgroup of H. Then $G/C_G(H)H$ is soluble.*

Proof. See [11]. □

In the latter parts of this chapter it will be necessary for us to know about the subgroup structure of $PSL(2,17)\mathrm{wr}\mathbb{Z}_2$. We devote the rest of this section to recalling the necessary facts.

Lemma 2.1.5. *Let $G \cong PSL(2,q)$ for some prime q. Then*

$$|G| = \frac{1}{(2,q-1)} q(q^2-1).$$

Proof. See [49, Section 3.3.1]. □

Theorem 2.1.6. (Dickson) *Let q be a power of the prime p. Then a subgroup of $PSL(2,q)$ is isomorphic to one of the following groups:*

1. *The dihedral groups of order $2(q \pm 1)/d$ and their subgroups where $d = (2, q-1)$;*

2. *A group H of order $q(q-1)/d$ and its subgroups. A Sylow p-subgroup P of H is elementary abelian, $P \trianglelefteq H$ and the factor group H/P is a cyclic group of order $(q-1)/d$;*

3. Sym(4) *or* Alt(5) *and their subgroups;*

4. $PSL(2,r)$ *or* $PGL(2,r)$ *where* r *is a power of* p *such that* $r^m = q$ *for some* $m \in \mathbb{N}$.

Proof. See [43, Pages 412-413]. □

Corollary 2.1.7. *Let* M *be a maximal subgroup of* $G \cong PSL(2,17)$. *Then* M *is isomorphic to one of the following:* Sym(4), Dih(16), Dih(18) *or* $\mathbb{Z}_{17} \rtimes \mathbb{Z}_8$.

Proof. Note that $|G| = 2^4 3^2 17$ by Lemma 2.1.5. We check the list of possible subgroups as outlined in Theorem 2.1.6. This gives us the following possibilities.

The dihedral groups of order $2(17 \pm 1)/d$ where $d = (2, 17-1)$. Since $d = 2$, this gives us Dih(16) and Dih(18).

A group H of order $17(17-1)/d$ where $d = (2, 17-1)$. The Sylow 17-subgroup P of H is elementary abelian, $P \trianglelefteq H$ and the factor group H/P is cyclic of order $(17-1)/2 = 8$. Since $|G|_{17} = 17$, P must be cyclic. Thus $H \cong \mathbb{Z}_{17} \rtimes \mathbb{Z}_8$.

By considering the order of G, part 3 of Theorem 2.1.6 gives us Sym(4).

Since M is maximal and 17 is prime, part 4 of Theorem 2.1.6 does not give us any extra possibilities. □

Lemma 2.1.8. *The maximal soluble subgroups of* $PSL(2,17)\text{wr}\mathbb{Z}_2$ *are as follows:*

1. $H = H_1 \times H_2$ *where* H_1 *and* H_2 *are isomorphic to maximal soluble subgroups of* $PSL(2,17)$ *and* $H_1 \ncong H_2$;

2. $H = (H_1 \times H_2)\langle h \rangle$ *where* $h \in H - (H_1 \times H_2)$, $|H : H_1 \times H_2| = 2$ *and* H_1 *and* H_2 *are isomorphic to maximal soluble subgroups of* $PSL(2,17)$ *such that* $H_1 \cong H_2$.

Proof. Let H be a maximal soluble subgroup of $G = PSL(2,17)\text{wr}\mathbb{Z}_2$. For notational purposes write this as

$$G \cong (K_1 \times K_2) \rtimes \langle t \rangle$$

where $K_i \cong PSL(2,17)$ for $i \in \{1,2\}$ and $\langle t \rangle \cong \mathbb{Z}_2$.

Suppose $H \subseteq K_1 \times K_2$. Then $H \subseteq H\pi_1 \times H\pi_2 = H_0$ where π_i is the projection map of $(K_1 \times K_2)$ onto K_i for $i \in \{1,2\}$. Now $H\pi_i$ is soluble as a homomorphic image of a soluble group. So H_0 is soluble. Also, H_0 is strictly contained in G, and so the maximality of H forces $H = H_0$. We see that H must be a direct product of subgroups H_1 and H_2, which are isomorphic to maximal soluble subgroups of $PSL(2,17)$. However, if $H_1 \cong H_2$, then $H < (H_1 \times H_2)\langle t \rangle < G$, contradicting the maximality of H.

Now suppose that $H \nsubseteq K_1 \times K_2$. Let $h \in H - (K_1 \times K_2)$. By Dedekind's modular law we get

$$H = (H \cap (K_1 \times K_2))\langle h \rangle.$$

By considering the projection maps π_1 and π_2 and the maximality of H, we find that

$$H \cap (K_1 \times K_2) = (H \cap K_1) \times (H \cap K_2).$$

Now $H \cap K_i$ must be a maximal soluble subgroup of K_i, otherwise we can find a soluble subgroup of G which strictly contains H. Now h interchanges the subgroups $H \cap K_i$ between the direct factors K_1 and K_2, and so

$$H \cap K_1 \cong H \cap K_2.$$

By considering the canonical epimorphism $G \longrightarrow \overline{G} = G/(K_1 \times K_2)$, we see that $|H : H_1 \times H_2| = 2$. □

2.2 Properties of 2-generated soluble subgroups

Throughout this section, G will denote a finite group and Σ_G the set of 2-generated soluble subgroups of G. This set is partially ordered by inclusion. We denote by Σ_G^* the subset of maximal members of Σ_G. As outlined at the beginning of this chapter, we look to prove results of a similar flavour to that of Theorem 2.0.1. To that end we have the following:

Theorem 2.2.1. *Let $A \in \Sigma_G^*$ and $V \leq G$ be a nilpotent subgroup which is normalised by A. Then $F(A)V$ is nilpotent.*

Before proving Theorem 2.2.1 we need the following lemma.

Lemma 2.2.2. *Suppose the group A acts on the group N where both A and N are nilpotent. If for all primes $q \neq r$ we have*

$$[\mathcal{O}_q(A), \mathcal{O}_r(N)] = 1,$$

then AN is nilpotent.

Proof. Let q be any prime. We work to show that AN has a normal Sylow q-subgroup. Since both A and N are nilpotent, it follows that $\mathcal{O}_q(A)\mathcal{O}_q(N) \in \mathrm{Syl}_q(AN)$. Set $Q = \mathcal{O}_q(A)\mathcal{O}_q(N)$. Note that A normalises Q since $\mathcal{O}_q(A) \trianglelefteq A$

and $\mathcal{O}_q(N) \trianglelefteq AN$. (The latter follows from the fact that since A normalises N, it normalises any characteristic subgroup of N, and thus it normalises $\mathcal{O}_q(N)$.) Next notice that $\mathcal{O}_q(N)$ normalises Q since $\mathcal{O}_q(N) \leq Q$. Further $\mathcal{O}_{q'}(N)$ centralises Q since

$$[\mathcal{O}_q(A), \mathcal{O}_{q'}(N)] = 1$$

by hypothesis, and since

$$[\mathcal{O}_q(N), \mathcal{O}_{q'}(N)] = 1.$$

Therefore

$$Q \trianglelefteq A\mathcal{O}_q(N)\mathcal{O}_{q'}(N) = AN.$$

Since the prime q was chosen arbitrarily, we see that all Sylow subgroups of AN are normal and thus AN is nilpotent. □

Proof. (of Theorem 2.2.1) Assume the statement of Theorem 2.2.1 is false and consider a counterexample with $|G| + |V|$ minimal. Then $G = AV$, and by applying Lemma 2.2.2 to the action of $F(A)$ on V, we see that there exist distinct primes r and q such that $[\mathcal{O}_r(V), \mathcal{O}_q(A)] \neq 1$.

Note that $V = \mathcal{O}_r(V)$, since if

$$V = \mathcal{O}_r(V) \times \mathcal{O}_{r'}(V)$$

with $\mathcal{O}_{r'}(V) \neq 1$, we could take $U = \mathcal{O}_r(V)$ and find a smaller counterexample, contradicting the minimality of G. The minimality of V also forces $V = [\mathcal{O}_r(V), \mathcal{O}_q(A)]$. If $V \neq U = [\mathcal{O}_r(V), \mathcal{O}_q(A)]$, then this forces

$$[\mathcal{O}_r(V), \mathcal{O}_q(A), \mathcal{O}_q(A)] = 1,$$

otherwise we can take $G = AU$ and again find a smaller counterexample. However, $\mathcal{O}_q(A)$ acts coprimely on $\mathcal{O}_r(V)$, so by part 2 of Theorem 1.5.2

$$[\mathcal{O}_r(V), \mathcal{O}_q(A), \mathcal{O}_q(A)] = [\mathcal{O}_r(V), \mathcal{O}_q(A)].$$

However, this cannot be the case here since $[\mathcal{O}_r(V), \mathcal{O}_q(A)] \neq 1$. Thus we indeed have that $V = [\mathcal{O}_r(V), \mathcal{O}_q(A)]$.

We now work towards showing that V is elementary abelian. Suppose V is not elementary abelian, so $\Phi(V) \neq 1$. Set $\overline{G} = G/\Phi(V)$.

We first show that $\overline{V}$ is minimal normal in $\overline{G}$. Now

$$\overline{V} = \overline{[V, \mathcal{O}_q(A)]} = [\overline{V}, \mathcal{O}_q(A)].$$

Here we are using that $\Phi(V)$ is an A-invariant normal subgroup of V, so A acts via the rule $\overline{v}^a = \overline{v^a}$. Since $\overline{V}$ is abelian, we have

$$\overline{V} = C_{\overline{V}}(\mathcal{O}_q(A)) \times [\overline{V}, \mathcal{O}_q(A)]$$

by Fitting's theorem. Together with the above it follows that

$$C_{\overline{V}}(\mathcal{O}_q(A)) = 1.$$

Let $\overline{U} \leq \overline{V}$ be a minimal normal subgroup of $\overline{G}$ and suppose $\overline{U} \subset \overline{V}$. If U is the inverse image of $\overline{U}$ in G, then the minimality of V implies $[\mathcal{O}_q(A), U] = 1$, whence

$$\overline{U} \leq C_{\overline{V}}(\overline{\mathcal{O}_q(A)}) = 1.$$

This is a contradiction, and so $\overline{U} = \overline{V}$.

We now claim that $\overline{A}$ is maximal in $\overline{G}$ and $\overline{A} \cap \overline{V} = 1$. Note that $\overline{A} \cap \overline{V} \trianglelefteq \overline{G}$ since $\overline{V}$ is abelian and normal in $\overline{G}$. Since $\overline{V}$ is minimal normal in $\overline{G}$, this implies that $\overline{A} \cap \overline{V} = 1$ or $\overline{V}$. If $\overline{A} \cap \overline{V} = \overline{V}$, then $\overline{V} \subseteq \overline{A}$ and so $\overline{G} = \overline{A}$. Thus $G = A\Phi(V)$ and $V = (V \cap A)\Phi(V)$. Hence, by Lemma 1.2.27, we have $V = V \cap A$, and so $G = A$. However, G cannot be 2-generated and so we must have $\overline{A} \cap \overline{V} = 1$. Suppose $\overline{A}$ is not maximal, then there exists $\overline{B} \leq \overline{G}$ such that $\overline{A} < \overline{B} < \overline{G}$. Then $\overline{G} = \overline{BV}$ and $\overline{B} \cap \overline{V} \neq 1$. Now $\overline{V}$ is minimal normal in $\overline{G}$ and so this forces $\overline{V} \subseteq \overline{B}$ and $\overline{G} = \overline{B}$. This contradicts our choice of $\overline{B}$, so $\overline{A}$ must be maximal in $\overline{G}$.

Since $A \in \Sigma_G$, we have that $\overline{A} \in \Sigma_{\overline{G}}$. However, we do not know whether $\overline{A} \in \Sigma^*_{\overline{G}}$. Suppose $\overline{A} \notin \Sigma^*_{\overline{G}}$. Then since $\overline{A}$ is a maximal subgroup of $\overline{G}$, we have $\overline{G} = \langle \overline{a}, \overline{b} \rangle$ for some $a, b \in G$. Then $G = \Phi(V)\langle a, b \rangle$, and so using the modular law

$$V = \Phi(V)(V \cap \langle a, b \rangle).$$

Thus by Lemma 1.2.27 we have $V = V \cap \langle a, b \rangle$. Therefore $V \leq \langle a, b \rangle$, and so $\Phi(V) \leq \langle a, b \rangle$. This forces $G = \langle a, b \rangle$, but G is not 2-generated. Hence $\overline{A} \in \Sigma^*_{\overline{G}}$. Now since $\overline{F(A)V} = F(\overline{A})\overline{V}$, we have that the conditions of the theorem are satisfied by $\overline{A}, \overline{V}$; but $F(\overline{A})\overline{V}$ is not nilpotent. If $F(\overline{A})\overline{V}$ is nilpotent, then $[\overline{V}, \mathcal{O}_q(\overline{A})] = 1$. However,

$$[\overline{V}, \mathcal{O}_q(\overline{A})] = \overline{V} \neq 1.$$

Hence, by the minimality of G, we have $\Phi(V) = 1$, and so V is elementary abelian.

Also $A \cap V = 1$. If $A \cap V \neq 1$, then $\mathcal{O}_q(A)$ would centralise $V \cap A$ since it is a normal r-subgroup of A and $q \neq r$. However, since V is abelian and

$V = [V, \mathcal{O}_q(A)]$, $\mathcal{O}_q(A)$ has no fixed-points on V. Hence A is a complement to V in G.

We next claim that any complement to V in G is conjugate to A. Consider $K = \mathcal{O}_q(A)V \trianglelefteq G$ and suppose B is a complement to V in G. Then $G = BV$ and $K = (B \cap K)V$, so $B \cap K$ is a complement to V in K. Since V is an r-group, $B \cap K \in \mathrm{Syl}_q(K)$ and similarly $\mathcal{O}_q(A) \in \mathrm{Syl}_q(K)$. By Sylow's theorem, $B \cap K$ and $\mathcal{O}_q(A)$ are conjugate in K. But since $K = \mathcal{O}_q(A)V$, we may choose $v \in V$ such that

$$(B \cap K)^v = \mathcal{O}_q(A).$$

Note that $A = N_G(\mathcal{O}_q(A))$. If $A < N = N_G(\mathcal{O}_q(A))$, then $N \cap V \neq 1$, and we can consider $R = N \cap V \trianglelefteq N$. Now

$$[R, \mathcal{O}_q(A)] \subseteq R \cap \mathcal{O}_q(A) = 1$$

since R and $\mathcal{O}_q(A)$ have coprime orders. Thus $R \subseteq C_V(\mathcal{O}_q(A)) = 1$, which is a contradiction. Now note

$$\mathcal{O}_q(A) = (B \cap V)^v \trianglelefteq B^v,$$

and so $B^v \subseteq N_G(\mathcal{O}_q(A)) = A$. However, $|B^v| = |A|$ and so $B^v = A$ for some $v \in V$ as claimed.

Now $A = \langle a, b \rangle$, for some $a, b \in G$. Let $v \in V$ and set $B = \langle a, bv \rangle$. Then B is a complement to V in G. Note that since $A \cap V = 1$, for each choice of $v \in V$ we obtain a distinct complement to V in AV. Now all of these complements are conjugate, so there exists $u \in V$ such that

$$B^u = A.$$

Therefore $A = \langle a^u, (bv)^u \rangle$, and so

$$[a, u] = a^{-1}a^u \in A \cap V = 1.$$

We get that $u \in C_V(a)$. Also, $(bv)^u = b^u v \in A$ (using the fact here that V is abelian). Therefore $b^{-1}b^u v \in A$ and

$$[b, u]v \in A \cap V = 1.$$

Thus $v = [u, b]$. For each choice of $v \in V$, we find a distinct $u \in C_V(a)$, and so

$$|V| \leq |C_V(a)|.$$

Therefore $V = C_V(a)$. Similarly, $V = C_V(b)$, and so

$$[V, A] = 1.$$

However

$$1 < [V, \mathcal{O}_q(A)] \subseteq [V, A] = 1,$$

which is a contradiction. □

The following result is well-known and was first proved using the theory of formations by R. Carter, B. Fischer and T. Hawkes in [9]. Here we present a more direct proof.

Theorem 2.2.3. *Let G be a soluble group. Then G possesses a 2-generated subgroup with the same Fitting height as G.*

Proof. So we want to show that there exists $A \in \Sigma_G$ such that $f(A) = f(G)$. Assume the statement of Theorem 2.2.3 is false and let G be a minimal counterexample.

Choose $q \in \pi(F(G))$ and set $\overline{G} = G/\mathcal{O}_q(G)$. We claim that $F(G) = \mathcal{O}_q(G)$, and thus it will follow from Lemma 1.3.5 that $f(\overline{G}) = f(G)-1$. Since $|\overline{G}| < |G|$, the minimality of $|G|$ forces there to be a 2-generated subgroup $\overline{A}$ of $\overline{G}$ such that $f(\overline{A}) = f(\overline{G})$. If $f(\overline{G}) = f(G)$, then $f(\overline{A}) = f(G)$. Now there exists a 2-generated subgroup B of G which maps onto $\overline{A}$. Since $f(B) \geq f(\overline{A})$, $f(B) = f(G)$, contradicting our choice of G. So

$$f(\overline{G}) < f(G).$$

If $\mathcal{O}_{q'}(G) \neq 1$, then by the same argument we have $f(G/\mathcal{O}_{q'}(G)) < f(G)$. Consider the following map

$$\phi : G \longrightarrow G/\mathcal{O}_q(G) \times G/\mathcal{O}_{q'}(G)$$

given by

$$g\phi = (g\mathcal{O}_q(G), g\mathcal{O}_{q'}(G)).$$

This is a homomorphism with kernel $\mathcal{O}_q(G) \cap \mathcal{O}_{q'}(G) = 1$, and so G is isomorphic to a subgroup of $G/\mathcal{O}_q(G) \times G/\mathcal{O}_{q'}(G)$. This forces $F(G) = \mathcal{O}_q(G)$, otherwise G embeds into a direct product of groups both with Fitting height $f(G) - 1$. So we must have $\mathcal{O}_{q'}(G) = 1$ and the claim follows.

Let $A \leq G$ be such that A maps onto $\overline{A}$ under the canonical epimorphism $G \longrightarrow \overline{G}$. Then there exists $A^* \in \Sigma_G^*$ such that $A \leq A^*$. Now

$$f(G) - 1 = f(\overline{G}) = f(\overline{A}) \leq f(A) \leq f(A^*).$$

If $f(\overline{A}) < f(A)$, then

$$f(A) = f(A^*) = f(G)$$

which is a contradiction. So

$$f(\overline{A}) = f(A) = f(A^*).$$

Since A^* is soluble (as G is soluble) and $f(A) = f(A^*)$, we have

$$\psi(A) \leq F(A^*)$$

by Lemma 1.3.9. By Theorem 2.2.1, $F(A^*)\mathcal{O}_q(G)$ is nilpotent and so $\psi(A)\mathcal{O}_q(G)$ is nilpotent since

$$\psi(A)\mathcal{O}_q(A) \leq F(A^*)\mathcal{O}_q(G).$$

Now $F(G) = \mathcal{O}_q(G)$ and G is soluble, hence $C_G(\mathcal{O}_q(G)) \leq \mathcal{O}_q(G)$. We claim that $\psi(A)$ is a q-group. Let $x \in \psi(A)$ be a q'-element, then

$$[x, \mathcal{O}_q(A)] = 1$$

since x has coprime order to all elements in $\mathcal{O}_q(G)$, and by Corollary 1.2.12 coprime elements in nilpotent groups commute. Then $x \in C_G(\mathcal{O}_q(G)) \leq \mathcal{O}_q(G)$, so x is a q'-element of $\mathcal{O}_q(G)$. Therefore $x = 1$ and so $\psi(A)$ is a q-group.

Since $f(\overline{A}) = f(A)$, we have that $\psi(\overline{A}) = \overline{\psi(A)}$ by Lemma 1.3.10. Now $\overline{\psi(A)}$ is a q-group as the homomorphic image of a q-group. Thus $\psi(\overline{A})$ is also a q-group. Now $f(\overline{A}) = f(\overline{G})$, so $\psi(\overline{A}) \leq F(\overline{G})$ by Lemma 1.3.9. However, $F(G/\mathcal{O}_q(G)) = F(\overline{G})$ is a q'-group and so $\psi(\overline{A})$ is a q'-group. This forces $\psi(\overline{A}) = 1$. Then $\overline{A} = 1$ by Lemma 1.3.8 and so $f(\overline{A}) = 0$. Therefore $f(\overline{G}) = 0$. Hence $\overline{G} = 1$ and $G = \mathcal{O}_q(G)$. Then G is nilpotent and so any nontrivial 2-generated subgroup of G has the same Fitting height as G. Therefore no such minimal counterexample exists. □

Theorem 2.2.3 tells us that, given a soluble group G, we can obtain some global information about G just from studying Σ_G. Namely, we can bound the Fitting height of G just by bounding the Fitting height of its 2-generated subgroups. Hence, if we are trying to study the structure of an arbitrary soluble group, we can perhaps reduce the problem to a question regarding Σ_G. This idea forms a large part of the motivation behind much of what will be studied in Chapter 3. We conclude this section with another well-known result; but one which can be obtained as a corollary to Theorem 2.2.3. This was first proved by M. Powell (an account of his work can be found in [8, pages 473-476]). Here we present a different proof.

Corollary 2.2.4. *A group G is soluble if and only if every three of its elements generate a soluble subgroup.*

Proof. Of course if G is soluble, then every subgroup of G is soluble, so in particular every 3-generated subgroup is soluble.

Let G be minimal with the property that it is an insoluble group and every three elements generate a soluble subgroup. Let $A = \langle a, b \rangle$, where $a, b \in G$, be a subgroup of maximal Fitting height among 2-generated subgroups of G. Further let $H = \langle a, b, c \rangle$ for some $c \in G$. Then H is soluble by hypothesis since it is a 3-generated subgroup of G. By Theorem 2.2.3, every soluble subgroup

K of G contains a 2-generated subgroup with Fitting height $f(K)$. By our choice of A then this forces $f(A) = f(H)$. Now since H is soluble, we have

$$\psi(A) \leq F(H)$$

by Lemma 1.3.9. Consider

$$A^c = \langle a^c, b^c \rangle \leq H.$$

Now $\psi(A^c) = \psi(A)^c$ so

$$\psi(A)^c \leq F(H).$$

Thus

$$\langle \psi(A), \psi(A)^c \rangle \leq F(H).$$

Since c was chosen arbitrarily in G, we see that $\langle \psi(A), \psi(A)^c \rangle$ is nilpotent for all $c \in G$. By the Baer–Suzuki theorem we have $\psi(A) \leq F(G)$. In particular $F(G) \neq 1$, and so $|G/F(G)| < |G|$. Now since $G/F(G)$ is a homomorphic image of G, every three elements of $G/F(G)$ generate a soluble subgroup. Hence, by the minimality of G, we must have that $G/F(G)$ is soluble. However, this forces G to be soluble, which is a contradiction. □

2.3 Properties of 2-generated subgroups

Further to the notation defined in Section 2.2, let Σ_G^f denote the set of soluble 2-generated subgroups of G of maximal Fitting height. Also, let Γ_G denote the set of 2-generated subgroups of G and Γ_G^* the subset of maximal members of Γ_G. Of course, $\Sigma_G \subseteq \Gamma_G$. The aim of this chapter is to investigate the following. Let G be a group, $A \in \Gamma_G^*$ and V a quasinilpotent subgroup which is normalised by A: what can be said about the action of $F^*(A)$ on V? We may sometimes impose the extra hypothesis that G be nearly soluble. Let's first make clear what we mean by a nearly soluble group.

Definition 2.3.1. A group G is said to be *nearly soluble* if the composition factors of G are either cyclic of prime order or isomorphic to $\mathrm{Alt}(5)$.

In answering the question posed above, it is natural to break the problem up by considering the action of A on $F(V)$ and $E(V)$. Since, if A normalises V, it will normalise $F(V)$ and $E(V)$ as they are both characteristic in V. If we can say something about the action of A on $F(V)$ and $E(V)$, then given that V is quasinilpotent, we are in a good position to say something about its action on V. First we consider the action of A on $F(V)$.

Theorem 2.3.2. *Let G be a group, $A \in \Gamma_G^*$ and $V \leq G$ be a quasinilpotent subgroup which is normalised by A. Then $F^*(A)F(V)$ is quasinilpotent.*

Proof. We begin by considering a counterexample with $|G|+|V|$ minimal. Now A normalises $F(V)$ and so $V = F(V)$, otherwise $F^*(A)F(V)$ is quasinilpotent by induction. Furthermore, $G = AV$ since otherwise we can find a smaller counterexample $H < G$, namely $H = AV$. The following lemma allows us to assume that V is a p-group.

Lemma 2.3.3. *Let $G = AV$ where $A \in \Gamma_G^*$ and $V \leq G$ is a nilpotent subgroup which is normalised by A . If $F^*(A)\mathcal{O}_p(V)$is quasinilpotent for all $p \in \pi(V)$, then $F^*(A)V$ is quasinilpotent.*

Proof. Let $K = F^*(A)V$. Since $F^*(A)\mathcal{O}_p(V)$ is quasinilpotent for all $p \in \pi(V)$, and $\mathcal{O}_p(V) \subseteq \mathcal{O}_p\big(F^*(A)\mathcal{O}_p(V)\big)$, we have that $\mathcal{O}_p(V)$ commutes with all p'-elements of $F^*(A)\mathcal{O}_p(V)$ by Lemma 1.4.11. By Corollary 1.4.3, $E(A)$ is generated by p'-elements for any prime p and so

$$[E(A), \mathcal{O}_p(V)] = 1$$

for all p. Then since V is nilpotent, we have

$$[E(A), V] = 1.$$

It then follows that $E(A) \trianglelefteq K$ and so $E(A) \subseteq E(K)$. We now work to show that $F(A)V$ is nilpotent. Then since $F(A)V \trianglelefteq K$, it will follow that $F(A)V \subseteq F(K)$. Let p be a prime, then

$$\mathcal{O}_p(A)\mathcal{O}_p(V) \in \mathrm{Syl}_p(F(A)V).$$

Similarly for a prime $q \neq p$

$$\mathcal{O}_q(A)\mathcal{O}_q(V) \in \mathrm{Syl}_q(F(A)V).$$

Recall that if a group G is quasinilpotent, then $\mathcal{O}_p(G)$ commutes with all p'-elements of G by Lemma 1.4.11. By hypothesis, $F^*(A)\mathcal{O}_p(V)$ is quasinilpotent for all $p \in \pi(V)$. Since $\mathcal{O}_p(V) \subseteq \mathcal{O}_p\big(F^*(A)\mathcal{O}_p(V)\big)$, it follows that $\mathcal{O}_p(V)$ commutes with all p'-elements of $F^*(A)\mathcal{O}_p(V)$. So

$$[\mathcal{O}_p(V), \mathcal{O}_q(A)] = 1$$

for all primes $p \neq q$. Similarly

$$[\mathcal{O}_q(V), \mathcal{O}_p(A)] = 1$$

and so

$$[\mathcal{O}_p(A)\mathcal{O}_p(V), \mathcal{O}_q(A)\mathcal{O}_q(V)] = 1.$$

Therefore, all Sylow subgroups of $F(A)V$ commute and thus $F(A)V$ is nilpotent. Then K is quasinilpotent as

$$K = F^*(A)V = E(A)F(A)V \subseteq E(K)F(K) = F^*(K).$$

□

Since we are working in a minimal counterexample, we can deduce that V is a p-group. We now show that in fact, V must be elementary abelian.

Lemma 2.3.4. *Let $G = AV$ be as above. Then V is elementary abelian.*

Proof. Assume V is not elementary abelian so that $\Phi(V) \neq 1$, and set $\overline{G} = AV/\Phi(V)$. Suppose $\overline{A} \in \Gamma^*_{\overline{G}}$. Then $F^*(\overline{A})\overline{V}$ is quasinilpotent by induction. Now $\overline{F^*(A)} \subseteq F^*(\overline{A})$ so $\overline{F^*(A)}\overline{V}$ is quasinilpotent but this is a contradiction for the following reason. We have that $\overline{V}$ is a normal p-subgroup of $\overline{F^*(A)}\overline{V}$ and so it is contained in $\mathcal{O}_p\big(\overline{F^*(A)}\overline{V}\big)$. Since $\overline{F^*(A)}\overline{V}$ is quasinilpotent, $\overline{V}$ commutes with all p'-elements of $\overline{F^*(A)}\overline{V}$. However, $F^*(A)V$ is not quasinilpotent and so there exists a p'-element of $F^*(A)V$ that does not centralise V. Now, $F^*(A)\Phi(V)$ is quasinilpotent by induction. So this p'-element had better be in the kernel of the map $G \longrightarrow \overline{G}$. However, $\Phi(V)$ is a p-group so there are no p'-elements in the kernel of this map.

So $\overline{A} \notin \Gamma^*_{\overline{G}}$. Therefore we can choose $\overline{B} \in \Gamma^*_{\overline{G}}$ such that $\overline{A} \subseteq \overline{B}$. Let $\widetilde{B}$ be the full inverse image of $\overline{B}$ in G. So $A < \widetilde{B} = A(\widetilde{B} \cap V)$. Certainly $F^*(\overline{B})\overline{V}$ is quasinilpotent by induction. Suppose $\widetilde{B} \neq G$. Then $\widetilde{B} \cap V < V$, otherwise $V \subseteq \widetilde{B}$ and thus $G \subseteq \widetilde{B}$. Now $A \subseteq \widetilde{B}$ and $\widetilde{B} \cap V \trianglelefteq \widetilde{B}$, and so A normalises $\widetilde{B} \cap V$. Hence $F^*(A)(\widetilde{B} \cap V)$ is quasinilpotent by induction. Now

$$F^*(A)(\widetilde{B} \cap V) \trianglelefteq A(\widetilde{B} \cap V) = \widetilde{B}.$$

Note that $F^*(\widetilde{B})$ is the largest normal quasinilpotent subgroup of $\widetilde{B}$ so $F^*(A)(\widetilde{B} \cap V) \subseteq F^*(\widetilde{B})$, and therefore

$$F^*(A) \subseteq F^*(\widetilde{B}).$$

We conclude that $\overline{F^*(A)}\overline{V}$ is quasinilpotent which can be deduced from the following series of inclusions

$$\overline{F^*(A)}\overline{V} \leq \overline{F^*(\widetilde{B})}\overline{V} \leq F^*(\overline{\widetilde{B}})\overline{V} = F^*(\overline{B})\overline{V}$$

and noting that $F^*(\overline{B})\overline{V}$ is quasinilpotent. Again, by noting that there are no p'-elements in the kernel of the map $G \longrightarrow \overline{G}$, we obtain a contradiction and thus conclude that $\widetilde{B} = G$.

Now choose $b_1, b_2 \in G$ such that $\overline{B} = \langle \overline{b_1}, \overline{b_2} \rangle$. Then

$$\widetilde{B} = \langle b_1, b_2 \rangle \Phi(V),$$

and so by the modular law

$$V = (V \cap \langle b_1, b_2 \rangle)\Phi(V).$$

Then

$$V = V \cap \langle b_1, b_2 \rangle \subseteq \langle b_1, b_2 \rangle$$

by Lemma 1.2.27, and so this forces

$$G = \widetilde{B} = \langle b_1, b_2 \rangle.$$

This gives us our final contradiction since G is not 2-generated. We conclude that $\Phi(V) = 1$ and so V is elementary abelian. $\square$

Thus we can think of V as an $\mathbb{F}_p[A]$-module.

Lemma 2.3.5. *V is a completely reducible module for A.*

Proof. Let $U \subset V$ be an irreducible submodule for A and set $\overline{G} = G/U$ ($U \trianglelefteq G$ since V is abelian). By a previous argument we again find that $A \notin \Gamma^*_{\overline{G}}$. So we can choose $\overline{B}$ with $\overline{A} < \overline{B} \in \Gamma^*_{\overline{G}}$. Again we find that $\widetilde{B} = G$ where $\widetilde{B}$ is the full inverse image of $\overline{B}$ in G. Therefore $\overline{G}$ is 2-generated, so there exists $b_1, b_2 \in G$ with $G = U\langle b_1, b_2 \rangle$. Then $U \cap \langle b_1, b_2 \rangle = 1$ since U is a minimal normal subgroup and G is not 2-generated. Therefore

$$V = U \times \big(V \cap \langle b_1, b_2 \rangle\big).$$

The result now follows by induction. $\square$

Lemma 2.3.6. *V is an irreducible module for A.*

Proof. Suppose not, then by Lemma 2.3.5 we may write

$$V = \bigoplus_i U_i$$

where $i > 1$ and each U_i is an irreducible A-submodule of V. By induction, $F^*(A)U_i$ is quasinilpotent for all i. So

$$[E(A), U_i] = 1$$

for all i and thus

$$E(A) \subseteq E(F^*(A)V).$$

We now show that $F(A)V$ is nilpotent. For all i, U_i commutes with all p'-elements of $F^*(A)$ and so commutes with $\mathcal{O}_{p'}(A)$. Now

$$\mathcal{O}_p(A)U_i \in \mathrm{Syl}_p(F(A)U_i)$$

and $\mathcal{O}_q(A) \in \mathrm{Syl}_q(F(A)U_i)$ for all $q \neq p$. So since

$$[\mathcal{O}_p(A), \mathcal{O}_q(A)] = 1$$

and

$$[U_i, \mathcal{O}_q(A)] = 1,$$

we see that all Sylow subgroups of $F(A)V$ commute, and thus it is nilpotent. Now $F(A)V \trianglelefteq F^*(A)V$ so $F(A)V \subseteq F(K)$. Therefore $F^*(A)V$ is quasinilpotent. □

By the minimality of our counterexample, we have that V is an irreducible $\mathbb{F}_p[A]$-module. It follows that $A \cap V = 1$ since V is minimal normal in G.

Lemma 2.3.7. *Let $C = C_A(V)$. Then $H^1(A/C, V) \cong H^1(A, V)$.*

Proof. By Theorem 1.10.10, there exists an exact sequence

$$0 \longrightarrow H^1(A/C, V) \longrightarrow H^1(A, V) \longrightarrow H^1(C, V)^A.$$

We work to show that $H^1(C, V)^A = 0$. First we claim that

$$H^1(C, V)^A \cong \mathrm{Hom}_A(C, V).$$

Let $\gamma \in \mathrm{Der}(C, V)$ and $g, h \in C$. Then

$$\begin{aligned} (gh)\gamma &= (g\gamma)^h + h\gamma \\ &= g\gamma + h\gamma \end{aligned}$$

where the second equality follows since $g\gamma \in V$ and $h \in C$. Let $\alpha \in \mathrm{IDer}(C, V)$. So there exists $v \in V$ such that for $g \in C$

$$g\alpha = v - v^g = v - v = 0.$$

So $\mathrm{IDer}(C, V) = 0$. From this we conclude that $H^1(C, V) \cong \mathrm{Hom}(C, V)$. Let $\gamma \in H^1(C, V)^A$ so $g\gamma = g\gamma^h$ for all $g \in C$ and $h \in A$. Then

$$g\gamma = (g^{h^{-1}}\gamma)^h$$

and so

$$(g\gamma)^{h^{-1}} = g^{h^{-1}}\gamma.$$

Thus γ is an A-homomorphism from C into V.

Let $\varphi \in \mathrm{Hom}_A(C,V)$ be nontrivial. Then φ is onto since V is irreducible. Let $q \neq p$, then

$$[\mathcal{O}_q(A), C] \subseteq \mathcal{O}_q(C) \subseteq \ker(\varphi)$$

where the second inclusion follows because $\mathcal{O}_q(V) = 1$. Therefore

$$[g,c]\varphi = 1$$

for all $g \in \mathcal{O}_q(A)$, $c \in C$. Now let $v \in V$ and $c \in C$ be such that $c\varphi = v$. Let $g \in \mathcal{O}_q(A)$, then

$$v^g = (c\varphi)^g = (c^g)\varphi$$

where the second equality follows since φ is an A-homomorphism. So

$$v^g = (c\varphi)(c^{-1}c^g)\varphi = v$$

where the second equality follows since $(c^{-1}c^g)\varphi = 1$. So $\mathcal{O}_q(A)$ is trivial on V for all $q \neq p$.

Similarly $E(A)$ acts trivially on V. We have

$$[E(A), C] \subseteq E(A) \cap C \subseteq \ker(\varphi).$$

The inclusion on the right follows since $E(V) = 1$. Thus $E(A)\varphi = 1$ since every nontrivial homomorphic image of a quasisimple group is quasisimple. From this we see that V commutes with all p'-elements of $F^*(A)V$, which implies $F^*(A)V$ is quasinilpotent. Thus there are no nontrivial A-homomorphisms from C into V. We conclude that $H^1(C,V)^A = 0$ and thus the result follows. □

Lemma 2.3.8. $|\mathrm{IDer}(A,V)| = |V|$

Proof. Now $|\mathrm{IDer}(A,V)| \leq |V|$ since an inner derivation γ_v is determined by our choice of $v \in V$. Suppose

$$\gamma_u = \gamma_v$$

for some $u, v \in V$, $u \neq v$. Then for all $a \in A$

$$v^a - v = u^a - u$$

and so

$$(v-u)^a = v - u.$$

Since A acts nontrivially and irreducibly on V, this forces $u = v$. However, $u \neq v$ and so $\gamma_u \neq \gamma_v$ for all $u \neq v$ and the result follows. □

Lemma 2.3.9. *There are $|V|^2$ complements to V in G.*

Proof. Let $A = \langle a, b\rangle$. Then $A_{v_1,v_2} = \langle av_1, bv_2\rangle$ is a complement to V in G for all $v_1, v_2 \in V$. Suppose there exists v_1, v_2 such that $A = \langle av_1, bv_2\rangle$. Then $a^{-1}av_1 \in A$ and so $v_1 \in A$. But $A \cap V = 1$ so $v_1 = 1$. Similarly $v_2 = 1$ and thus we can form at least $|V|^2$ complements to V in G.

Consider the map

$$\xi : \mathrm{Der}(A, V) \longrightarrow V \oplus V$$

defined by

$$\gamma \longmapsto (a\gamma, b\gamma).$$

Then ξ is an injection since elements of $\mathrm{Der}(A, V)$ are uniquely determined by their action on the generators of A. So since there are $|\mathrm{Der}(A, V)|$ complements to V in AV, we see that there are at most $|V|^2$ complements to V in G and thus the claim follows. □

Lemmas 2.3.8 and 2.3.9 together imply that $|H^1(A, V)| = |V|$ and so

$$|H^1(A/C, V)| = |V|$$

by Lemma 2.3.7. However, since A/C is a finite 2-generated group and V is faithful and irreducible for A/C, $|H^1(A/C, V)| < |V|$ by Theorem 1.10.11. This contradiction completes the proof of Theorem 2.3.2. □

Remark 2.3.10. We might wonder at this point why the example we discussed earlier concerning $G \cong V \rtimes SL(2, 5)$ is not a counterexample to Theorem 2.3.2. The reason is because this group is 2-generated itself. This can be seen since $SL(2, 5)$ is 2-generated and is faithfully and irreducibly represented on V. Thus by Theorems 1.10.11 and 1.10.12, the semidirect product $VSL(2, 5)$ is also 2-generated.

We now turn our attention to the action of A on $E(V)$. Of course, what we would like to prove is: if G is a group, $A \in \Gamma^*_G$ and $V \leq G$ a quasinilpotent subgroup normalised by A, then $F^*(A)E(V)$ is quasinilpotent. Now if $F^*(A)E(V)$ is quasinilpotent, then the components in V will be normal in $F^*(A)E(V)$. We can see this as follows. Since $E(V) \trianglelefteq F^*(A)E(V)$, we have $E(V) \subseteq E(F^*(A)E(V))$. Now components in quasinilpotent groups are normal. Hence, if $F^*(A)E(V)$ is quasinilpotent, then the components of V will be normal in $F^*(A)E(V)$. Unfortunately, we do not prove that $F^*(A)E(V)$ is quasinilpotent but we do outline results which suggest that $F^*(A)$ acts trivially on $\mathrm{Comp}(V)$. Note that if a group A acts on a group V such that A is trivial on $\mathrm{Comp}(V)$, we do not necessarily have that $F^*(A)E(V)$ is quasinilpotent.

Again, we split up the problem and consider the action of $E(A)$ and $F(A)$ on V. This in turn will allow us to say something about the action of $F^*(A)$ on V; or more precisely on $\mathrm{Comp}(V)$.

Theorem 2.3.11. *Let G be a group, $A \in \Gamma_G$ of maximal order and $V \leq G$ be a quasinilpotent subgroup which is normalised by A. Then either $E(A)$ acts trivially on* $\mathrm{Comp}(V)$ *or $A \cap V \neq 1$.*

Proof. If $A \cap V \neq 1$, then we are done. Hence, let G be a minimal counterexample with $A \cap V = 1$. So we see that $G = AV$ where $V = E(V)$ and A is transitive on $\mathrm{Comp}(V)$. Let's first assume that $Z(V) = 1$ so $V = V_1 \times \ldots \times V_n$ where each V_i is a nonabelian simple group.

Consider the following subgroup

$$L = \langle K \in \mathrm{Comp}(A) \mid K \text{ is nontrivial on } \mathrm{Comp}(V)\rangle \leq A.$$

Since we are considering a counterexample, we have that $L \neq 1$. Now $L \trianglelefteq A$ for the following reason. We know A permutes its components around. If $K \leq L$, then K^a is nontrivial on $\mathrm{Comp}(V)$ for all $a \in A$, so it is clear that L is fixed under conjugation from A.

Let $p \in \pi(V)$ and $P \in \mathrm{Syl}_p(V)$, then by the Frattini argument

$$G = N_G(P)V = AV.$$

Therefore,

$$N_G(P)/N_V(P) \cong A.$$

Since A is 2-generated, there exists $n_1, n_2 \in N_G(P)$ such that

$$N_G(P) = \langle n_1, n_2\rangle N_V(P).$$

Then $G = \langle n_1, n_2\rangle V$. Let $B = \langle n_1, n_2\rangle$. Since

$$A \cong B/B \cap V,$$

it follows that $|A| \leq |B|$. However, since A is of maximal order in Γ_G, we have that $|A| = |B|$, and so $B \cap V = 1$. Thus B is of maximal order in Γ_G. So $B \in \Gamma_G^*$ and since B normalises P, it follows from Theorem 2.3.2 that $F^*(B)P$ is quasinilpotent. Let $X = LV \trianglelefteq G$. By the modular law we have that $X = (B \cap X)V$. Also, $B \cap X \cap V = 1$ since $B \cap V = 1$. We now claim that $B \cap X$ centralises P. By Corollary 1.4.3, $E(B) \subseteq \mathcal{O}^p(F^*(B)P)$ and since $F^*(B)P$ is quasinilpotent, $E(B)$ centralises P by Lemma 1.4.11. Now $L \cong B \cap X \trianglelefteq B$ and so $B \cap X \subseteq E(B)$. So $B \cap X$ centralises P and thus $X = C_X(P)V$. Now $p \in \pi(V)$ was chosen arbitrarily, and so it follows that

$X = C_X(Q)V$ whenever Q is a Sylow subgroup of V. We now appeal to Theorem 2.1.4 for a contradiction. We get

$$X/C_X(V)V \cong L/C_X(V)$$

is soluble. However, $L = E(L)$ and so we must have $L = C_X(V)$. By the definition of L, this cannot be true. Thus $L = 1$ and so $E(A)$ normalises every component of V.

Now suppose that $Z(V) \neq 1$. Clearly $Z(V)$ is an A-invariant subgroup of V. Now A acts on $\overline{V} = V/Z(V)$ via

$$(vZ(V))^a = v^a Z(V).$$

Hence we can form the group $A\overline{V}$. Let $\varphi : AV \longrightarrow A\overline{V}$ be the canonical epimorphism.

Note that A is a member of $\Gamma_{A\overline{V}}$ of maximal order. This follows since A injects into $A\overline{V}$ and for every 2-generated subgroup R of $A\overline{V}$ there exists a 2-generated subgroup S of AV with $S\varphi = R$. Hence, by induction $E(A)$ acts trivially on $\mathrm{Comp}(\overline{V})$. Now every component of V maps onto a component of $\overline{V}$. Let $K \in \mathrm{Comp}(\overline{V})$ such that $K \in \mathrm{Comp}(V)$ maps onto $\overline{K}$. Let I be the full inverse image of $\overline{K}$ in V, so $I = KZ(V)$. Let $a \in E(A)$ and suppose there exists $k \in K - Z(V)$ such that $k^a \notin K - Z(V)$, so k^a lies in a different component. Then $\overline{k}^a$ must be in a different component of $\overline{V}$ than $\overline{k}$ since $\overline{k^a} = \overline{k}^a$. However this contradicts that $E(A)$ acts trivially on $\mathrm{Comp}(\overline{V})$. □

At this point we would like to say something about the action of $F(A)$ on $\mathrm{Comp}(V)$ for some $A \in \Gamma_G^*$. We digress for a moment to prove some results which will help us with this.

Lemma 2.3.12. *Let A be a soluble group which acts on a group $G \neq 1$. Then there exists a nontrivial A-invariant soluble subgroup of G.*

Proof. Consider a counterexample with $|A| + |G|$ minimal. Let $p \in \pi(F(A))$. We first argue that $C_G(\mathcal{O}_p(A)) = 1$. Suppose $C_G(\mathcal{O}_p(A)) \neq 1$ and let $g \in C_G(\mathcal{O}_p(A))$, $a \in A$ and $c \in \mathcal{O}_p(A)$. Then

$$(g^a)^c = g^{ac} = g^{aca^{-1}a} = g^a,$$

where the final equality since $\mathcal{O}_p(A) \trianglelefteq A$. So $C_G(\mathcal{O}_p(A))$ is A-invariant. If $C_G(\mathcal{O}_p(A)) \neq G$, then we can apply induction to find an A-invariant soluble subgroup of $C_G(\mathcal{O}_p(A))$. Since we are working in a counterexample, this forces $C_G(\mathcal{O}_p(A)) = G$. Consider $\overline{A} = A/\mathcal{O}_p(A)$. Now $\overline{A}$ acts on G and by induction there is a nontrivial soluble $\overline{A}$-invariant subgroup $H < G$. This is a contradiction so we find that $C_G(\mathcal{O}_p(A)) = 1$.

Suppose $p \in \pi(G)$ and choose $P \in \mathrm{Syl}_p(A)$. By Sylow's theorem $P \leq S \in \mathrm{Syl}_p(PG)$, which we can write as

$$S = P(S \cap G)$$

using the modular law. Since $G \trianglelefteq PG$, we have that $S \cap G \trianglelefteq S$. Now $S \cap G \neq 1$, and so

$$1 \neq Z(S) \cap S \cap G \subseteq C_G(\mathcal{O}_p(A)) = 1$$

by Lemma 1.2.1, which is a contradiction.

Let $q \in \pi(G)$ such that $q \neq p$. By coprime action, there is an $\mathcal{O}_p(A)$-invariant Sylow q-subgroup Q of G. Since $C_G(\mathcal{O}_p(A))$ is transitive on the set of $\mathcal{O}_p(A)$-invariant Sylow q-subgroups of G, it follows that Q is uniquely determined. Both A and $\mathcal{O}_p(A)$ act on the set Ω of Sylow q-subgroups of G. Now $\mathrm{Fix}_\Omega(\mathcal{O}_p(A))$ is A-invariant and so we see that Q is A-invariant, which is a contradiction. □

Proposition 2.3.13. *Let G be a group, $A \leq G$ a soluble subgroup and $P \leq A$ a p-group such that $P \leq \mathcal{O}_p(H)$ whenever $A \leq H \leq G$ with H soluble. Suppose $2 \neq p \in \pi(F(A))$ and*

$$V = V_1 \times \ldots \times V_n \leq G$$

with $V_1, \ldots, V_n$ permuted by A and $\mathcal{O}^p(V_i) \neq 1$ for all i. Then P normalises each V_i.

Proof. Consider a counterexample with $|G| + |V|$ minimal. Then $G = AV$ and $\{V_1, \ldots, V_n\}$ is a set upon which A acts transitively and P acts nontrivially. We start by arguing that $\mathcal{O}_p(V) = 1$. If $\mathcal{O}_p(V) \neq 1$, then we can look at the action of A on $\overline{V} = V/\mathcal{O}_p(V)$. First we claim that $\overline{V}$ is a direct product. Note that by Lemma 1.2.5 we have

$$\mathcal{O}_p(V) = \mathcal{O}_p(V_1) \times \ldots \times \mathcal{O}_p(V_n).$$

Now we can rewrite $\overline{V}$ as

$$\overline{V} = (V_1 \times \ldots \times V_n)/(\mathcal{O}_p(V_1) \times \ldots \times \mathcal{O}_p(V_n)) \cong (V_1/\mathcal{O}_p(V_1)) \times \ldots \times (V_n/\mathcal{O}_p(V_n))$$

since $\mathcal{O}_p(V_i) \trianglelefteq V_i$ for all i. Therefore $\overline{V}$ can be written as the following direct product

$$\overline{V} = \overline{V_1} \times \ldots \times \overline{V_n}.$$

Now P acts on V, and since $\mathcal{O}_p(V)$ is a P-invariant subgroup, P acts on $\overline{V}$ as follows

$$(v\mathcal{O}_p(V))^g = v^g \mathcal{O}_p(V)$$

where $g \in P$. Consider the group $\overline{G} = \overline{AV}$. Then since we are assuming $\mathcal{O}_p(V) \neq 1$, we have that $|\overline{A}| + |\overline{V}| < |A| + |V|$. Also, $\mathcal{O}^p(V_i) \neq 1$ for all i, and since $\mathcal{O}_p(V_i)$ is a p-group, there will be p'-elements in $\overline{V_i}$ for all i. Hence $\mathcal{O}^p(\overline{V_i}) \neq 1$ for all i. Suppose we have the following inclusions

$$\overline{P} \subseteq \overline{A} \subseteq \overline{H}$$

where $\overline{P}$ is a p-group and $\overline{H}$ a soluble group. Let H be the full inverse image of $\overline{H}$ in G. Then H is soluble and we have the following inclusions

$$P \subseteq A \subseteq H.$$

By hypothesis, $P \subseteq \mathcal{O}_p(H)$ and so

$$\overline{P} \subseteq \overline{\mathcal{O}_p(H)} \subseteq \mathcal{O}_p(\overline{H}).$$

Thus we see that all of the hypotheses are satisfied by $\overline{G}$, but since G is a minimal counterexample, we can conclude $\overline{P}$ normalises $\overline{V_i}$. Since V is not a p-group, this contradicts the fact that P is nontrivial on $\{V_1, \ldots, V_n\}$. So $\mathcal{O}_p(V) = 1$.

By Lemma 2.3.12, there exists a nontrivial soluble A-invariant subgroup $W \leq V$. Without loss of generality we can assume that

$$W = (W \cap V_1) \times \ldots \times (W \cap V_n).$$

We can assume this since each $W_i = W \cap V_i$ is soluble, and so the direct product of the W_i, which is normalised by A, is also soluble. We could thus take

$$\prod_i W_i$$

as the soluble subgroup of V normalised by A and see in this case the group chosen is equal to its direct product of projections onto the V_i. Thus it is no loss to assume that this is the case to begin with. Suppose that $W = V$. Then G is soluble, and so by hypothesis $P \subseteq \mathcal{O}_p(G)$. Now

$$[P, V_i] \subseteq \mathcal{O}_p(G) \cap V \subseteq \mathcal{O}_p(V) = 1,$$

and so P centralises V_i for any i.

Thus $W \neq V$ and W must be a p-group. Otherwise $\mathcal{O}^p(W_i) \neq 1$ for all i and we could apply induction to conclude that P normalises each W_i, and thus it would normalise each V_i. Consider the normaliser $N = N_V(W)$. We claim that

$$N = N_{V_1}(W \cap V_1) \times \ldots \times N_{V_n}(W \cap V_n).$$

Since V_i commutes V_j for all $i \neq j$, we clearly have

$$N_{V_1}(W \cap V_1) \times \ldots \times N_{V_n}(W \cap V_n) \subseteq N.$$

Now let $v \in N$, so

$$v = v_1 \cdots v_n$$

where $v_i \in V_i$. Then

$$W_i^v = W_i^{v_i} \subseteq V_i$$

where the first equality holds since the subgroups V_i commute. Since $v \in N$, we need $v_i \in N_{V_i}(W_i)$ and therefore

$$N \subseteq N_{V_1}(W \cap V_1) \times \ldots \times N_{V_n}(W \cap V_n).$$

Since we have inclusion in both directions, this forces equality here. Since $\mathcal{O}_p(V) = 1$, we have $N < V$, and so as before we find that N is a p-group. Using the fact that 'normalisers grow' in p-groups, we have that A normalises a Sylow p-subgroup P of V. Thus it normalises the characteristic subgroups $Z(P)$ and $J(P)$. By the same argument, $C_V(Z(P))$ and $N_V(J(P))$ are both p-groups and so trivially have normal p-complements. Thus by Thompson's normal p-complement theorem, V also has a normal p-complement, say C, where

$$C = \mathcal{O}_{p'}(V_1) \times \ldots \times \mathcal{O}_{p'}(V_n).$$

Since V is not a p-group, this is nontrivial, and so we can apply induction to the action of A on C to deduce the result. □

Recall that we are considering, for some group G, the action of $A \in \Gamma_G^*$ on some quasinilpotent subgroup V which it normalises. In particular we want to say something about the action of $F(A)$. Consider a soluble group G and $H \leq G$. It is not necessarily true that $F(H) \leq F(G)$. However, if we take H such that $f(H) = f(G)$, then we get $\psi(H) \leq F(G)$ by Lemma 1.3.9. Thus any p-subgroup of $\psi(H)$ will be contained in $\mathcal{O}_p(G)$. Proposition 2.3.13 becomes very useful to us when we consider this property of $\psi(H)$. This leads us to the next result, where we consider the action of $\psi(A)$ on V rather than $F(A)$.

Corollary 2.3.14. *Let G be a group, $A \in \Sigma_G^f$ and V be a semisimple subgroup which is normalised by A. Then $\mathcal{O}_{2'}(\psi(A))$ acts trivially on* $\mathrm{Comp}(V)$.

Proof. Since V is semisimple, we can write

$$V = V_1 \times \ldots \times V_n$$

where each of the subgroups V_i is a nonabelian simple component of V. Thus for an arbitrary prime p, $\mathcal{O}^p(V_i) \neq 1$ for any i. Let $H \leq G$ be a soluble subgroup such that $A \leq H$. By Theorem 2.2.3, H has a 2-generated subgroup with Fitting height $f(H)$. Since A was chosen with maximal Fitting height in Σ_G, we have that $f(A) = f(H)$. Thus by Lemma 1.3.9 we have

$$\psi(A) \leq \psi(H) \leq F(H).$$

So

$$\mathcal{O}_p(\psi(A)) \leq \mathcal{O}_p(H)$$

for all primes $p \in \pi(F(A))$. By Proposition 2.3.13, $\mathcal{O}_p(\psi(A))$ normalises each V_i for all primes $2 \neq p \in (F(A))$. Thus $\mathcal{O}_{2'}(\psi(A))$ acts trivially on $\mathrm{Comp}(V)$. □

Immediately, it is of interest to know whether Corollary 2.3.14 can be extended to say that $\psi(A)$ acts trivially on $\mathrm{Comp}(V)$, rather than restricting ourselves to $\mathcal{O}_{2'}(\psi(A))$. It is due to the use of Thompson's normal p-complement theorem in the proof of Proposition 2.3.13 that Corollary 2.3.14 only concerns the action of $\mathcal{O}_{2'}(\psi(A))$. It turns out that when G is nearly soluble, we can conclude that $\psi(A)$ acts trivially on $\mathrm{Comp}(V)$. The proof of this hints at a weaker hypothesis that may still allow us to extend Corollary 2.3.14. We first prove a few lemmas necessary for this result.

Lemma 2.3.15. *Let G be a group and $N \trianglelefteq G$ such that G/N is 2 generated and soluble. Then there exists a subgroup $H \leq G$ which is 2 generated and soluble such that $G = HN$.*

Proof. If $N = 1$, then the claim is trivial since we can just take $H = G$.

So we can assume that $N \neq 1$. Let $p \in \pi(N)$ and $P \in \mathrm{Syl}_p(N)$. By the Frattini argument we have $G = N_G(P)N$. Also

$$G/N \cong N_G(P)/(N_G(P) \cap N),$$

and so $N_G(P)/N_G(P) \cap N$ is 2-generated and soluble. Suppose $N_G(P) \neq G$. Then since $N_G(P) \cap N \trianglelefteq N_G(P)$, we can apply induction to conclude that there exists $H \leq N_G(P)$ which is 2-generated and soluble such that $N_G(P) = H(N_G(P) \cap N)$. Therefore

$$G = H(N_G(P) \cap N)N = HN,$$

and the claim follows in this case.

Now suppose that $N_G(P) = G$. Then $P \trianglelefteq G$ whenever P is a Sylow subgroup of N. Since the prime p was chosen arbitrarily, all Sylow subgroups of

N are normal in G and thus normal in N. Then N is nilpotent by Theorem 1.2.10. Since G/N is soluble, it follows that G is soluble and thus any subgroup of G is also soluble. Choose $g, h \in G$ such that $G/N = \langle gN, hN \rangle$, and set $H = \langle g, h \rangle$. Then H has the desired properties. □

Lemma 2.3.16. *Let G be a group and $A \in \Sigma_G^f$ be of maximal order. Then $A \in \Sigma_G^*$.*

Proof. Let $A \in \Sigma_G^f$. Then $A \leq B \in \Sigma_G^*$ and so $f(A) \leq f(B)$ by Lemma 1.3.3. However, since $A \in \Sigma_G^f$, this forces $f(A) = f(B)$. If we choose A to be of maximal order among elements of Σ_G^f, then this forces $A = B$ and so $A \in \Sigma_G^*$. □

Lemma 2.3.17. *The only perfect central extensions of* $\mathrm{Alt}(5)$ *are* $\mathrm{Alt}(5)$ *and $SL(2,5)$.*

Proof. Note that the pair $(SL(2,5), \pi)$ where

$$\pi : SL(2,5) \longrightarrow SL(2,5)/Z(SL(2,5)) = PSL(2,5)$$

is the canonical epimorphism, is a perfect central extension of $PSL(2,5)$. The proof now follows from [49, page 51] and [2, 33.15]. □

Theorem 2.3.18. *Let G be a nearly soluble group. Let $A \in \Sigma_G^f$ and $V \leq G$ be a quasinilpotent subgroup which is normalised by A. Then $\psi(A)$ acts trivially on* $\mathrm{Comp}(V)$*.*

Proof. Again we work inside a minimal counterexample. So $G = AV$ where $V = E(V)$ and we can assume that A acts transitively on $\mathrm{Comp}(V)$. We can also assume that $A \in \Sigma_G^*$. If not, then $A < A^* \in \Sigma_G^*$. Then since $A \in \Sigma_G^f$, we have that $f(A) = f(A^*)$ and so $\psi(A) \leq \psi(A^*)$ by Lemma 1.3.9. So if $\psi(A^*)$ acts trivially on $\mathrm{Comp}(V)$, it follows that $\psi(A)$ does as well. Thus it is no loss to assume that $A \in \Sigma_G^*$ to begin with. Let's first assume that $Z(V) = 1$ so that $V = V_1 \times \ldots \times V_n$ where each V_i is isomorphic to $\mathrm{Alt}(5)$. We first argue that $A \cap V \neq 1$.

Since we are working in a counterexample, we can choose a prime p such that $\mathcal{O}_p(\psi(A))$ is nontrivial on $\mathrm{Comp}(V)$. Assume $A \cap V = 1$. Let $q \in \pi(V)$ such that $p \neq q$ and choose $Q \in \mathrm{Syl}_q(V)$. Then $G \cong N_G(Q)V$ by the Frattini argument, and since $A \cap V = 1$, we have that

$$N_G(Q)/N_V(Q) \cong G/V \cong A.$$

Since A is soluble and 2-generated, there exists a soluble 2-generated subgroup $B \leq N_G(Q)$ with $N_G(Q) = BN_V(Q)$ by Lemma 2.3.15. So G may be written as $G = BV$. Now $B/N_V(Q) \cong A$ and so $f(A) \leq f(B)$ and $|A| \leq |B|$. Since $A \in \Sigma_G^f$ of maximal order, $f(A) = f(B)$ and $|A| = |B|$. So $B \in \Sigma_G^f$ and is of maximal order. Thus by Lemma 2.3.16, $B \in \Sigma_G^*$. Now $B \subseteq N_G(Q)$ and so B normalises Q. Consider the soluble group $H = BQ$. Then since $B \in \Sigma_G^*$ and $B \subseteq H$, we certainly have that $B \in \Sigma_H^*$. By Theorem 2.2.1, $F(B)Q$ is nilpotent, and so in particular, every p-element of $F(B)Q$ commutes with Q. Since $V \leq \mathcal{O}_p(\psi(A))V$, we can rewrite this as

$$\mathcal{O}_p(\psi(A))V = (\mathcal{O}_p(\psi(A))V \cap B)V$$

by the modular law. Since $\mathcal{O}_p(\psi(A))V \trianglelefteq G$ and $B \cap V = 1$, it follows that $\mathcal{O}_p(\psi(A))V \cap B \subseteq \mathcal{O}_p(B)$, so

$$\mathcal{O}_p(\psi(A))V \subseteq \mathcal{O}_p(B)V.$$

We know $\mathcal{O}_p(B)$ centralises Q and so it follows that $\mathcal{O}_p(\psi(A))$ does as well. Now

$$Q \subseteq (Q \cap V)\pi_1 \times \ldots \times (Q \cap V)\pi_n$$

where π_i is the projection map of V onto its i^{th} component. Since Q is nontrivial, this contradicts the fact that $\mathcal{O}_p(\psi(A))$ is nontrivial on $\text{Comp}(V)$, and so we conclude that $A \cap V \neq 1$.

Note that since $A \cap V \neq 1$, $(A \cap V)\pi_i \neq 1$ for some $i \in \{1, \ldots, n\}$. Since A is transitive on $\text{Comp}(V)$, $(A \cap V)\pi_i \cong (A \cap V)\pi_j$ for all $i, j \in \{1, \ldots, n\}$.

Assume $(A \cap V)\pi_i \in \text{Syl}_p(V_i)$ where $p = 2, 3$ or 5. We have

$$A \cap V \subseteq (A \cap V)\pi_1 \times \ldots \times (A \cap V)\pi_n = V_0.$$

Therefore A normalises

$$N_V(V_0) = N_{V_1}((A \cap V)\pi_1) \times \ldots \times N_{V_n}((A \cap V)\pi_n).$$

From now on let $N = N_V(V_0)$ and $N_i = N_{V_i}((A \cap V)\pi_i)$. Since A is soluble, $A \cap V$ is soluble and so $(A \cap V)\pi_i < V_i$ for each i. Since $V_i \cong \text{Alt}(5)$ for each i, it follows that N_i is soluble for each i and thus N is soluble. Let $H = AN$. Then H is soluble and since $A \in \Sigma_G^f$, it follows from Theorem 2.2.3 that $f(A) = f(H)$. Thus $\psi(A) \subseteq \psi(H)$ by Lemma 1.3.9 and so $\mathcal{O}_p(\psi(A)) \subseteq \mathcal{O}_p(H)$. Therefore

$$[\mathcal{O}_p(\psi(A)), N] \subseteq [\mathcal{O}_p(H), N] \subseteq \mathcal{O}_p(H) \cap N = \mathcal{O}_p(N).$$

Set $\overline{H} = H/\mathcal{O}_p(H)$. The following sequence of inclusions tells us that $\mathcal{O}_p(\psi(A))$ normalises $N_i\mathcal{O}_p(N)$ for all i

$$[\mathcal{O}_p(\psi(A)), N_i\mathcal{O}_p(N)] \subseteq [\mathcal{O}_p(\psi(A)), N] \subseteq \mathcal{O}_p(N) \subseteq N_i\mathcal{O}_p(N).$$

Therefore $\mathcal{O}_p(\psi(A))$ normalises $\mathcal{O}^p(N_i\mathcal{O}_p(N))$. Now $N_i \trianglelefteq N_i\mathcal{O}_p(N)$ and since $N_i\mathcal{O}_p(N)/N_i$ is a p-group, $\mathcal{O}^p(N_i\mathcal{O}_p(N)) \subseteq N_i$. Clearly

$$\mathcal{O}^p(N_i\mathcal{O}_p(N)) \trianglelefteq N_i$$

and $N_i/\mathcal{O}^p(N_i\mathcal{O}_p(N))$ is a p-group. Now $N_i = \mathcal{O}^p(N_i)$ since we are working inside $\mathrm{Alt}(5)$, so we get the following

$$N_i = \mathcal{O}^p(N_i) \subseteq \mathcal{O}^p(N_i\mathcal{O}_p(N)) \subseteq N_i.$$

Of course the inclusions are in fact equalities and thus $\mathcal{O}_p(\psi(A))$ normalises N_i. So

$$[\mathcal{O}_p(\psi(A)), N_i] \subseteq N_i \cap \mathcal{O}_p(N) = \mathcal{O}_p(N_i).$$

Therefore

$$[\mathcal{O}_p(\psi(A)), \overline{N_i}] \subseteq \overline{\mathcal{O}_p(N_i)} = 1.$$

Since i was chosen arbitrarily, $[\mathcal{O}_p(\psi(A)), \overline{N}] = 1$. If $\mathcal{O}_p(\psi(A))$ acts nontrivially on $\mathrm{Comp}(V)$, then it cannot centralise $\overline{N}$ unless $\overline{N}$ is trivial. However, $\overline{N} \neq 1$. Note that since $\psi(A) \subseteq \psi(H)$, $\mathcal{O}_{p'}(\psi(A)) \subseteq \mathcal{O}_{p'}(H)$. So

$$[\mathcal{O}_{p'}(\psi(A)), N] \subseteq [\mathcal{O}_{p'}(H), N] \subseteq \mathcal{O}_{p'}(H) \cap N = \mathcal{O}_{p'}(N) = 1.$$

This clearly implies that $\mathcal{O}_{p'}(\psi(A))$ acts trivially on $\mathrm{Comp}(V)$ and so $\psi(A)$ acts trivially on $\mathrm{Comp}(V)$ in this case.

Now consider the case where $(A \cap V)\pi_i \cong \mathbb{Z}_2$ for all i. Note that any two distinct Sylow 2-subgroups of $\mathrm{Alt}(5)$ intersect trivially. So each involution a_i, where $(A \cap V)\pi_i = \langle a_i \rangle$, is contained in exactly one $S \in \mathrm{Syl}_2((A \cap V)\pi_i)$, call this S_i. Since A permutes the V_i, it permutes the S_i, and thus normalises the direct product

$$S = S_1 \times \ldots \times S_n.$$

Now the previous argument can be repeated with $N = S$ to show

$$[\mathcal{O}_2(\psi(A)), \overline{S_i}] = 1$$

for all i.

Finally we need to consider the cases when $(A \cap V)\pi_i \cong S_3, A_4$ or D_{10}. In this case we just take

$$N = (A \cap V)\pi_1 \times \ldots \times (A \cap V)\pi_n.$$

The argument again goes through unchanged since A normalises this direct product.

Now suppose that $Z(V) \neq 1$. Clearly $Z(V)$ is an A-invariant subgroup of V and so $Z(V) \trianglelefteq AV$. Set $\overline{AV} = AV/Z(V)$.

We now claim that $A \in \Sigma^f_{\overline{AV}}$. Note that if $f(\overline{A}) = f(A)$, then the claim will follow since the set Σ_G maps onto $\Sigma_{\overline{G}}$. If $A \cap Z(V) = 1$, then this claim is trivial, so assume that $A \cap Z(V) \neq 1$. By Theorem 1.4.9 and Lemma 2.3.17 we have that $Z(V) \cong \mathbb{Z}_2$. Hence $Z(V) \subseteq Z(A)$. If $f(\overline{A}) < f(A)$, then $Z(V) = \varphi(A)$. However this forces A to be nilpotent by Proposition 1.3.12. Then $A = \psi(A) \subseteq V$, which is a contradiction since A acts nontrivially on $\mathrm{Comp}(V)$. Thus $A \in \Sigma^f_{\overline{AV}}$.

So by induction, $\psi(\overline{A})$ acts trivially on $\mathrm{Comp}(\overline{V})$. Now every component of V maps onto a component of $\overline{V}$. Let $\overline{K} \in \mathrm{Comp}(\overline{V})$ such that $K \in \mathrm{Comp}(V)$ maps onto $\overline{K}$. Let I be the full inverse image of $\overline{K}$ in V so $I = KZ(V)$. Let $a \in \psi(A)$ and $k \in K - Z(V)$ such that $k^a \notin K - Z(V)$, so k^a lies in a different component. We know such a and k exist since $\psi(A)$ acts nontrivially on $\mathrm{Comp}(V)$. Also note that $a \notin Z(V)$, hence $\overline{a} \neq 1$. Then $\overline{k}^{\overline{a}}$ must be in a different component of $\overline{V}$ than $\overline{k}$ since $\overline{k^a} = \overline{k}^{\overline{a}}$. However this contradicts that $\psi(\overline{A})$ acts trivially on $\mathrm{Comp}(\overline{V})$. □

Crucial to the proof of Theorem 2.3.18 is the fact that the Sylow subgroups of Alt(5) are not maximal. We discussed earlier that there may be particular circumstances where Corollary 2.3.14 may be extended to say that $\psi(A)$ acts trivially on $\mathrm{Comp}(V)$. Maybe the extra hypothesis that is needed is that the composition factors of G be cyclic of prime order or isomorphic to a nonabelian simple group whose Sylow subgroups are not maximal. It turns out that no nonabelian simple group possesses a maximal Sylow subgroup of odd order (this is a corollary to [20, Theorem 3.2, page 340]). So this extra hypothesis is not as strong as it initially appears.

We conclude this section with a counterexample to the following statement:

> Let G be a group, $A \in \Sigma^*_G$ and V a quasinilpotent subgroup which is normalised by A. Then $\psi(A)$ acts trivially on $\mathrm{Comp}(V)$.

Example 2.3.19. Let $G \cong PSL(2,17)\mathrm{wr}\mathbb{Z}_2$. For notational purposes, write this as

$$G \cong (K_1 \times K_2) \rtimes \langle t \rangle$$

where $K_i \cong PSL(2,17)$ for $i \in \{1,2\}$ and $\langle t \rangle \cong \mathbb{Z}_2$. Also, let $V = K_1 \times K_2 \leq G$. The method behind this counterexample is to find a maximal soluble 2-generated subgroup A which is a 2-group and which acts nontrivially on $\mathrm{Comp}(V)$. A 2-group is nilpotent and so $A = \psi(A)$. Thus if A acts nontrivially on $\mathrm{Comp}(V)$, then $\psi(A)$ acts nontrivially on $\mathrm{Comp}(V)$.

Let $D_i \in \mathrm{Syl}_2(K_i)$ for $i \in \{1,2\}$. By considering the order of $PSL(2,17)$, it follows from Corollary 2.1.7 that $D_i \cong \mathrm{Dih}(16)$ for $i \in \{1,2\}$. Let $D_1 = \langle a, b \mid a^2 = b^2 = (ab)^8 = 1\rangle$. We can assume without loss of generality that $D_2 = \langle a^t, b^t\rangle$. Let $A = \langle ta, b\rangle$. Then A clearly acts nontrivially on $\mathrm{Comp}(V)$. We claim that the following projection maps

$$\phi_1 : (K_1 \times K_2) \cap A \longrightarrow D_1 \text{ and } \phi_2 : (K_1 \times K_2) \cap A \longrightarrow D_2$$

are surjective. First note that

$$(ta)^2 = tata = a^t a$$

where the second equality follows since t is an involution. So

$$a^t a \in (K_1 \times K_2) \cap A.$$

Now

$$\phi_1(a^t a) = a \text{ and } \phi_1(b) = b$$

and so ϕ_1 is certainly a surjection. Now

$$[ta, b] = (b^{-1})^t b$$

and so $b^t \in A$. Similarly, we see that ϕ_2 is a surjection since

$$\phi_2(a^t a) = a^t \text{ and } \phi_2(b^t) = b^t.$$

We now claim that 2^8 divides $|A|$. Note that since $G = (K_1 \times K_2)\langle ta\rangle$ and $\langle ta\rangle \le A$, it follows from Dedekind's modular law that

$$A = (A \cap (K_1 \times K_2))\langle ta\rangle.$$

So

$$|A| = \frac{|\langle ta\rangle||(K_1 \times K_2) \cap A|}{|\langle ta\rangle \cap ((K_1 \times K_2) \cap A)|}.$$

Now $(ta)^4 = (a^t a)^2 = 1$ and so $|\langle ta\rangle| = 4$. Also, since $ta \notin (K_1 \times K_2) \cap A$ and $(ta)^2 \in (K_1 \times K_2) \cap A$, it follows that

$$|\langle ta\rangle \cap ((K_1 \times K_2) \cap A)| = 2.$$

So $|A| = 2|(K_1 \times K_2) \cap A|$, and thus it will suffice to show that 2^7 divides $|(K_1 \times K_2) \cap A|$. Earlier we saw that $a^t, b^t \in A$ and so

$$a^t, b^t \in A \cap (K_1 \times K_2).$$

So the following projection map

$$\phi(A \cap (K_1 \times K_2)) \longrightarrow D_2$$

is a surjection. Note that

$$[(ta)^2, b] = [a^t a, b] = [a, b] = (ab)^2$$

where the final equality follows since both a and b are involutions. So

$$\langle (ab)^2, b\rangle \subseteq A \cap K_1 \subseteq \ker(\phi).$$

We claim that $\langle (ab)^2, b\rangle \cong \mathrm{Dih}(8)$. This follows since $((ab)^2)^4 = b^2 = 1$ and $((ab)^2)^b = ((ab)^2)^{-1}$. Thus 2^3 divides $|\ker(\phi)|$. Thus by the first isomorphism theorem, it follows that 2^7 divides $|(K_1 \times K_2) \cap A|$, and so 2^8 divides $|A|$. Now by considering the maximal soluble subgroups of G as outlined in Lemma 2.1.8, it follows that any 2-generated subgroup of G which contains A and is maximal subject to being soluble is a 2-group. Thus we indeed have a counterexample as outlined earlier.

2.4 Concluding remarks

At the beginning of Section 2.3 we asked the following question: let G be a group, $A \in \Gamma_G^*$ and V a quasinilpotent subgroup which is normalised by A; what can be said about the action of $F^*(A)$ on V? Theorem 2.3.2 provides us with a very nice answer to this question in terms of how A acts on $F(V)$. Namely, $F^*(A)F(V)$ is quasinilpotent. However, when we considered the action of A on $E(V)$, the results we have obtained have regarded the action of $F^*(A)$ on $\mathrm{Comp}(V)$ as opposed to properties of $F^*(A)E(V)$; and in some cases we have taken $A \in \Sigma_G^f$ as opposed to $A \in \Gamma_G^*$. Of course, some members of Σ_G^f may well be members of Γ_G^* but we cannot guarantee that this is the case. The final example in Section 2.3 is testimony to that. However, these latter results along with our example concerning $PSL(2,17)\mathrm{wr}\mathbb{Z}_2$ hint that maybe the following is true:

> Let G be a group whose composition factors are either cyclic or nonabelian containing no maximal Sylow 2-subgroups. Let $A \in \Gamma_G^*$ and $V \leq G$ a quasinilpotent subgroup of G normalised by A. Then $F^*(A)F(V)$ is quasinilpotent and $F^*(A)$ acts trivially on $\mathrm{Comp}(V)$.

Of course, we have already shown in this setup that $F^*(A)F(V)$ is quasinilpotent, so the real question is: does $F^*(A)$ act trivially on $\mathrm{Comp}(V)$?

In Section 2.2 we proved the well-known result that every soluble group G possesses a 2-generated subgroup with Fitting height $f(G)$. Thus we can obtain global information about the structure of a soluble group by studying its 2-generated subgroups. Crucial to our proof of this result was Theorem 2.2.1. With this in mind, it is natural to ask whether the results obtained in Section 2.3 can give us similar results relating the structure of arbitrary groups to that of their 2-generated subgroups. If there are analogous results to be found, then surely we need some analogous notion of Fitting height for insoluble groups. It seems that a natural way to proceed is with the following definition.

Definition 2.4.1. Let G be a group. Then the *generalised Fitting height* of G, denoted $f^*(G)$, is the smallest $n \in \mathbb{N}$ such that G possesses a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

where each G_i/G_{i-1} is quasinilpotent for $1 \leq i \leq n$.

Then we might suppose that every group G has a 2-generated subgroup with generalised Fitting height $f^*(G)$. However, we run into problems quite quickly with this definition. Indeed, the notion of generalised Fitting height as above is not respected by subgroups. By which we mean, for a group G and subgroup $H \leq G$, we do not necessarily have that $f^*(H) \leq f^*(G)$. We can think of Fitting height as a measure of how 'complex' the structure of a soluble group is. We would like to think of generalised Fitting height in the same way for arbitrary groups. However, it seems absurd to suggest that the structure of a group G is less 'complex' than that of one of its subgroups H, as might be purported by $f^*(G) < f^*(H)$ for some subgroup $H \leq G$. So what would be a 'sensible' definition of generalised Fitting height? Ideally, we want $f^*(G) = f(G)$ for all soluble groups G and $f^*(H) \leq f^*(G)$ for all subgroups $H \leq G$ of arbitrary groups. If such a notion exists, then a reasonable question to ask might be: do the results obtained in Section 2.3 give us a way of relating the structure of a group to that of its 2-generated subgroups in terms of generalised Fitting height?

Chapter 3

Soluble primitive permutation groups

The study of permutation groups is one of the oldest in group theory. C. Jordan laid the foundations for the subject in 1870 in [26], building on earlier work by Cauchy, Galois and Lagrange. Influential texts since then include H. Wielandt's *Finite permutation groups* [48] and much more recently *Permutation groups* [12] by Dixon and Mortimer. Despite its antiquity, the theory of permutation groups continues to be a very active area of research within group theory. Much recent work in this area has come as a result of the O'Nan-Scott theorem which allows many problems in finite permutation groups to be reduced to questions about maximal subgroups of almost simple groups (the O'Nan-Scott theorem was first stated in [1] but the reader is referred to [33] for a self-contained and complete proof).

In this chapter we investigate the structure of soluble primitive permutation groups generated by two p^n-cycles (by 'p^n-cycle' we mean an element which can be written as the product of n disjoint cycles of length p) for a prime p and $n \in \mathbb{N}$. As we shall see, the study of these groups is interesting in its own right, but we also give a potential application of the results of this chapter which relate to the work in Chapter 2. There we proved the well-known result that, any finite soluble group G possesses a 2-generated subgroup A with $f(A) = f(G)$. Thus we can obtain information about a soluble group from its 2-generated subgroups. We apply a similar idea to soluble subgroups of $\mathrm{Sym}(m)$ by studying a particular class of 2-generated subgroups of $\mathrm{Sym}(m)$.

Let G be a soluble primitive permutation group generated by two p^n-cycles. Suppose that G acts faithfully on some set Ω. A standard argument shows that G has an elementary abelian minimal normal subgroup $N \trianglelefteq G$ which acts regularly on Ω and thus $|\Omega| = q^x$ for some prime q and $x \in \mathbb{N}$. We will prove that

$$f(G) \leq 4 + \frac{3}{2}\log_2(1 + \log_q np).$$

First we recall some preliminary results that will be required throughout this chapter.

3.1 Preliminary results

The results presented in this section are well-known. References for most of the results are given, but the author provides proofs of results which are difficult to find. No originality is claimed by the author in this section.

We first work towards determining the dimensions of the faithful irreducible $\mathbb{F}_p[A]$-modules where $A \cong \mathbb{Z}_n$.

Lemma 3.1.1. *Let G be a group and V an irreducible $\mathbb{F}[G]$-module for some field $\mathbb{F}$. Then $\mathrm{Hom}_G(V, V)$ is a division algebra with $\mathbb{F}$ in its centre.*

Proof. See [20, Theorem 5.2, page 76]. □

Theorem 3.1.2. (Wedderburn's Little Theorem) *A finite division algebra is a field.*

Proof. See [46]. □

Proposition 3.1.3. *Let $A \cong \mathbb{Z}_n$ and V be a faithful irreducible $\mathbb{F}_p[A]$-module. Then $\dim_{\mathbb{F}_p} V = m$ where m is the order of p modulo n.*

Proof. Note that since A acts faithfully on V, it embeds into $\mathrm{End}_{\mathbb{F}_p}(V)$. Hence we can identify $A \subseteq \mathrm{End}_{\mathbb{F}_p}(V)$. Denote by $\mathbb{K}$ the subring of $\mathrm{End}_{\mathbb{F}_p}(V)$ generated by A. By Lemma 3.1.1, $\mathbb{K}$ is a division algebra with $\mathbb{F}_p$ in its centre. Since $\mathbb{K}$ is finite, it is a field by Wedderburn's Little theorem. Hence $\mathbb{K} \cong \mathbb{F}_{p^m}$ for some $m \in \mathbb{N}$. Now V is a vector space over $\mathbb{K}$ of dimension 1. Thus

$$\dim_{\mathbb{F}_p} V = m = \dim_{\mathbb{F}_p} \mathbb{K}.$$

Now A embeds into $\mathbb{K}^* \cong \mathbb{Z}_{p^m-1}$ since V is a $\mathbb{K}[A]$-module of dimension 1. So n divides $p^m - 1$. Therefore the order of p modulo n divides m. Suppose that $p^{m'} \equiv 1$ modulo n for some m' dividing m. Then since m' divides m, $\mathbb{K}$ contains a subfield $\mathbb{L} \cong \mathbb{F}_{p^{m'}}$. Now the multiplicative group of a finite field is cyclic. Hence $\mathbb{K}^*$ has a unique subgroup of order d for each divisor d of $|\mathbb{K}^*|$. Therefore $\mathbb{K}^*$ has a unique subgroup of order n. This subgroup must be A. Now since n also divides $|\mathbb{L}^*|$, we have that $\mathbb{L}^*$ contains A by the same argument. By the definition of $\mathbb{K}$, we have that $\mathbb{L} = \mathbb{K}$ and so $m' = m$. □

Note that if a group G possesses a faithful irreducible $\mathbb{F}[G]$-module, then $Z(G)$ is cyclic by Proposition 1.8.7. Furthermore, if V is primitive and $\mathbb{F}$ is algebraically closed, then all abelian normal subgroups of G are cyclic and contained in $Z(G)$ by Corollary 1.8.16. When this situation arises for G soluble, it is natural to consider whether $F(G)$ is abelian or not. If $F(G)$ is abelian, then G is abelian by Corollary 1.2.19. When $F(G)$ is nonabelian, G possesses a nonabelian normal p-subgroup P. This subgroup will satisfy

$$1 \neq P' \subseteq \Phi(P) \subseteq Z(P) \tag{3.1.1}$$

by Corollary 1.2.33. We will encounter groups G and P satisfying these conditions quite often during this chapter. Hence we devote the remainder of this section to results regarding such groups. In particular, we consider the faithful irreducible modules of p-groups satisfying 3.1.1. Note that such p-groups have nilpotence class two.

Lemma 3.1.4. *Let G be a group, p a prime and $P \trianglelefteq G$ a nonabelian p-group. Assume that every abelian subgroup of P that is normal in G is cyclic and contained in $Z(G)$. Then $P/Z(P)$ is a completely reducible $\mathbb{F}_p[G]$-module. Each irreducible summand possesses a G-invariant symplectic form.*

Proof. See [16, Corollary 3.7]. □

Lemma 3.1.5. *Let G be a group, p a prime and $P \trianglelefteq G$ a nonabelian p-group. Assume that every abelian subgroup of P that is normal in G is cyclic and contained in $Z(G)$. If T is a p'-subgroup of G with $[P,T] \neq 1$, then $[P,T]$ is extraspecial with*

$$[P,T]' = C_{[P,T]}(T) = P' = Z(P) \cap [P,T].$$

Proof. See [16, Lemma 3.5(b)]. □

Lemma 3.1.6. *Let G be a p-group of nilpotence class no greater than two and p odd. If G is generated by elements of order p, then G has exponent p.*

Proof. See [20, Lemma 3.9(i), page 183]. □

Lemma 3.1.7. *Let G be a group and $g \in G$ such that $[h,g] \in Z(G)$ for all $h \in G$. Then the map from G into $Z(G)$ defined by*

$$h \longmapsto [h,g]$$

is a homomorphism.

Proof. Let h, $k \in G$. Then

$$\begin{array}{rcl} hk & \longmapsto & [hk, g] \\ & = & k^{-1}h^{-1}g^{-1}hkg \\ & = & k^{-1}h^{-1}g^{-1}hgg^{-1}kg \\ & = & k^{-1}[h, g]g^{-1}kg \\ & = & [h, g][k, g] \end{array}$$

where the final equality follows because $[h, g] \in Z(G)$. □

Lemma 3.1.8. *Let G be a p-group of nilpotence class no greater than two with the property that all abelian normal subgroups are cyclic. Then G is either cyclic or is isomorphic to Q_8, the quaternion group of order 8.*

Proof. Follows from [20, Theorems 4.10(i) and 4.3(ii)(a) on pages 199 and 191 respectively]. □

Lemma 3.1.9. *Let $G \cong Q_8$, the quarternion group of order 8, and V a faithful irreducible $\mathbb{F}[G]$-module for some algebraically closed field $\mathbb{F}$. Then* $\dim V = 2$.

Proof. See [20, Theorem 5.4, page 206]. □

Proposition 3.1.10. *Let P be a p-group such that $1 \neq P' \leq \Phi(P) \leq Z(P)$ and V a faithful irreducible $\mathbb{F}[P]$-module where $\mathbb{F}$ is algebraically closed. Then $|P/Z(P)| = p^{2k}$ for some $k \in \mathbb{N}$ and $\dim V = p^k$.*

Proof. Since P possesses a faithful irreducible representation, it follows from Proposition 1.8.7 that $Z(P)$ is cyclic. Now $\Phi(P) \subseteq Z(P)$ and so P' is elementary abelian by Lemma 1.2.31 . Thus $P' \cong \mathbb{Z}_p$, and so the first claim follows from Lemmas 1.6.3 and 1.6.5.

Suppose that every abelian normal subgroup of P is cyclic. Then by Lemma 3.1.8, P is either cyclic or is isomorphic to Q_8 (the quaternion group of order 8). Since $P' \neq 1$, P cannot be cyclic and so $P \cong Q_8$. Since V is a faithful irreducible representation of P over an algebraically closed field, the claim follows from Lemma 3.1.9. Hence we must consider what happens when P has a noncyclic abelian normal subgroup.

Let $N \trianglelefteq P$ be a noncyclic abelian normal subgroup. We now show that there exists a normal subgroup $A \trianglelefteq P$ such that $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Set $M = \Omega_1(N)$. Then M is elementary abelian and noncyclic since N is abelian and noncyclic. Also, $M \trianglelefteq P$ since $M \operatorname{char} N \trianglelefteq P$. Set $Z = Z(P) \cap M$. Then $Z \neq 1$ by Lemma 1.2.1 since $1 \neq M \trianglelefteq P$. In fact, we must have $Z \cong \mathbb{Z}_p$ since $Z(P)$ is cyclic.

Consider the factor group M/Z which is nontrivial since M is noncyclic. Now P acts on M/Z and has a nontrivial fixed point since both P and M/Z are p-groups. Let $1 \neq g \in C_{M/Z}(P)$ and A be the full inverse image of $\langle g \rangle \leq M/Z$ in M. Then $A \trianglelefteq P$ and $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

There are $p+1$ hyperplanes in A of which Z is one. Thus there are p hyperplanes of A which intersect $Z(P)$ trivially. Let these hyperplanes be denoted $A_1, \ldots A_p$ and set $V_i = C_V(A_i)$. We now claim that

$$V = V_1 \oplus \ldots \oplus V_p.$$

Let $U \leq V$ be an irreducible A-submodule. Since A is elementary abelian, it follows from Proposition 1.8.9 that $C_A(U)$ is a hyperplane of A. Now $C_V(Z(P)) = 0$ since V is faithful and irreducible for P, so $C_A(U) = A_i$ for some i. Thus $V_i \neq 0$ for some i. Now note that P acts on $\{V_i \mid 1 \leq i \leq p\}$ and so the irreducibility of P on V forces

$$V = \prod_i V_i.$$

Note that if $i \neq j$, then $V_i \cap V_j \leq C_A(V) = 0$ since $A = \langle A_i, A_j \rangle$ whenever $i \neq j$. Let $i < p$ be maximal such that

$$V_1 + \ldots + V_i = V_1 \oplus \ldots \oplus V_i.$$

Since A is abelian, $A_i \trianglelefteq A$ for all i and so A normalises each V_i by Lemma 1.8.5. We have

$$V_{i+1} \cap (V_1 \oplus \ldots \oplus V_i) = (V_{i+1} \cap (V_1)) \oplus \ldots \oplus (V_{i+1} \cap (V_i)) = 0.$$

This contradicts the maximality of i and hence

$$V = V_1 \oplus \ldots \oplus V_p.$$

Since V is irreducible for P, we have that P must transitively permute the V_i. Let $Q = N_P(V_1)$. (Note at this stage that since V is a faithful irreducible $\mathbb{F}[P]$-module and $\mathbb{F}$ is algebraically closed that $Z(P)$ acts by scalars on V and hence $Z(P) \subseteq Q$.) So Q is a point stabiliser in the permutation action of P on the V_i. Since $i = p$, this action is primitive and thus Q is maximal in P by Proposition 1.7.8. Since P is a p-group, $Q \trianglelefteq P$ and so Q must normalise each V_i. Also, $|P : Q| = p$. Choose $g \in P - Q$, so that $P = Q\langle g \rangle$. We claim that V_i is an irreducible Q-module for each i. Let $U_i \leq V_i$ be an irreducible Q-submodule for each i. Now $g^p \in Q$ so

$$U = U_1 \oplus \ldots \oplus U_p$$

is a P-module. The irreducibility of V forces $V = U$ and so $U_i = V_i$ for all i.

Let $K_i = C_Q(V_i)$. Notice that $A_i \subseteq K_i \trianglelefteq Q$. Since $P' \subseteq Z(P)$, the map $x \longmapsto [x, g]$ is a homomorphism from K_i to P' with kernel $C_{K_i}(g)$. Note that

$$[K_i, Q] \subseteq P' \cap K_i = 1$$

where the equality on the right holds since P' has no nontrivial fixed points on V. Hence $K_i \subseteq Z(Q)$. Now $C_{K_i}(g) \subseteq Z(P) \cap K_i = 1$. The equality holds here since $Z(P)$ cannot have fixed points in V. Hence K_i injects into P' and has order 1 or p. However, since $\mathbb{Z}_p \cong A_i \subseteq K_i$, it follows that $K_i \cong \mathbb{Z}_p$.

Set $\overline{Q} = Q/K_1$. Then

$$\overline{Q}' \leq \Phi(\overline{Q}) \leq Z(\overline{Q}).$$

We can see this as follows. Let W be the inverse image of $Z(\overline{Q})$ in Q. Then

$$[W, Q] \subseteq P' \cap K_1 = 1$$

where the equality holds since P' cannot have fixed points in V. Thus $Z(\overline{Q}) = \overline{Z(Q)}$. Also

$$\overline{Q}/Z(\overline{Q}) \cong \overline{Q}/\overline{Z(Q)}$$

is elementary abelian as a homomorphic image of $Q/Z(Q)$. Hence $\Phi(\overline{Q}) \leq Z(\overline{Q})$. The inclusion $\overline{Q}' \leq \Phi(\overline{Q})$ follows since $\overline{Q}$ is nilpotent. We work to show that $\overline{Q}/Z(\overline{Q})$ has order $p^{2(k-1)}$. Then $\dim V_1 = p^{k-1}$ by induction and the result will follow since $\dim V = p \dim V_1$.

Suppose that $\overline{Q}$ is nonabelian. The map $z \longmapsto [z, g]$ is a homomorphism from $Z(Q)$ into P' with kernel $C_{Z(Q)}(g) = Z(P)$. Now $K_1 \subseteq Z(Q)$ and so it follows that $Z(Q) = K_1 \times Z(P)$. Also $|Q/Z(Q)| = |\overline{Q}/Z(\overline{Q})|$. Now $|Q| = \frac{1}{p}|P|$ and $|Z(Q)| = p|Z(P)|$ and so

$$|\overline{Q}/Z(\overline{Q})| = |P : Z(P)| \Big/ p^2 = p^{2(k-1)}.$$

As outlined earlier we have by induction that $\dim V_1 = p^{k-1}$. Since $\dim V = p \dim V_1$, the claim follows.

Now suppose that $\overline{Q}$ is abelian. Since $\overline{Q}$ is faithful and irreducible on V_1, $\dim V_1 = 1$ and $\dim V = p$. By Proposition 1.8.9, $\overline{Q}$ is cyclic. Since $K_1 \subseteq Z(Q)$, $Q/Z(Q)$ is cyclic and so Q is abelian. Consider the homomorphism from Q into P' defined by

$$x \longmapsto [x, g].$$

This has kernel $C_Q(g) = Z(P)$. Since P is nonabelian, it follows that $|Q : C_Q(g)| = p$ and so $|Q : Z(P)| = p$. Now $|P : Q| = p$ and so $|P : Z(P)| = p^2$. $\square$

Corollary 3.1.11. *Let P be an extraspecial p-group of order p^{1+2k} and let $\mathbb{F}$ be a field of characteristic prime to p. Then the faithful irreducible representations of P over $\mathbb{F}$ are all of degree cp^k for some $c \in \mathbb{N}$.*

Proof. Let V be a faithful irreducible $\mathbb{F}[P]$-module and $\mathbb{K}$ be the algebraic closure of $\mathbb{F}$. Let

$$V^{\mathbb{K}} = V \otimes_{\mathbb{F}} \mathbb{K}.$$

Then by Maschke's theorem, $V^{\mathbb{K}}$ is completely reducible

$$V^{\mathbb{K}} = V_1 \oplus \ldots \oplus V_c.$$

Since V is faithful, every irreducible submodule of $V^{\mathbb{K}}$ is faithful by Lemma 1.8.3. Thus each V_i is a faithful irreducible $\mathbb{K}[P]$-module. By Proposition 3.1.10, $\dim_{\mathbb{K}} V_i = p^k$ for all i. Thus $\dim_{\mathbb{K}} V^{\mathbb{K}} = cp^k$. Since

$$\dim_{\mathbb{F}} V = \dim_{\mathbb{K}} V^{\mathbb{K}},$$

the result follows. □

3.2 Basic structure

Before investigating the structure of soluble primitive permutation groups generated by two p^n-cycles, we first outline the basic structure of soluble primitive permutation groups in general. The results stated in this section are either well-known or follow readily from well-known results. As such, no originality is claimed by the author in this section. However, we include proofs to give the reader a feel for the groups that will be the object of study throughout this chapter. It will also give us the opportunity to set up some notation.

Lemma 3.2.1. *Let G be a soluble primitive permutation group which acts faithfully on the set Ω. Then we can write G as the split extension of a regular normal subgroup V by a point stabiliser H.*

Proof. Let $H \le G$ be the stabiliser of a point in Ω. Then since G is primitive on Ω, we have that H is maximal in G by Proposition 1.7.8. Now let $V \trianglelefteq G$ be a minimal normal subgroup. Then since G is soluble, we see that V is a soluble minimal normal subgroup, and thus by Lemma 1.2.16, it is elementary q-abelian for some prime q. Then since V is normal in G, we can form the subgroup $VH \le G$. By the maximality of H, we have that either $VH = H$ or $VH = G$. Suppose that $VH = H$. Then $V \subseteq H$ and so V fixes a point in Ω. However, G is transitive on Ω and thus it acts transitively by conjugation on the subgroups which stabilise a point in Ω. Since $V \trianglelefteq G$, V is contained in the point stabiliser of every point in Ω and so V acts trivially on Ω. However, $V \neq 1$, so this contradicts the fact that G is faithful on Ω. Therefore, $VH = G$ and V must act transitively on Ω.

Now consider the subgroup $V \cap H \leq G$. We certainly have that $V \cap H \trianglelefteq H$, but since V is abelian, it also follows that $V \cap H \trianglelefteq V$, and thus $V \cap H \trianglelefteq G$. The minimality of V forces $V \cap H = 1$ or $V \cap H = V$ but $V \cap H \neq V$ because then $V \subseteq H$ which we have already seen cannot happen. Thus $V \cap H = 1$ and so we see that G is the split extension of V by H.

Notice here that H has been chosen arbitrarily in the sense that we did not specify a particular point in Ω for which it is the point stabiliser. So we see that V intersects trivially with any given point stabiliser in G. Combined with the fact that V is transitive on Ω then we have that V is in fact regular on Ω. □

Throughout the remainder of this section, we will assume the following:

Hypothesis 3.2.2.

- *G is a soluble primitive permutation group which acts faithfully on Ω.*

By Lemma 3.2.1, G is the split extension regular minimal subgroup by a point stabiliser. We now set some notation for these subgroups.

- *V is a minimal normal subgroup of G which acts regularly on Ω;*
- *H is the stabiliser of a point in Ω;*
- *$G = V \rtimes H$.*

Note that by Lemma 1.7.11, the action of H on V is always faithful.

Lemma 3.2.3. *The set Ω has prime power order.*

Proof. Since V acts regularly on Ω, we have that $|\Omega| = |V|$. Now $|V| = q^m$ for some $m \in \mathbb{N}$ since it is elementary q-abelian. Thus the size of Ω is prime power order. □

Lemma 3.2.4. *The subgroup $V \trianglelefteq G$ can be viewed as a faithful irreducible module for H over $\mathbb{F}_q$.*

Proof. We have just seen that V is an elementary q-abelian subgroup of G and thus by Proposition 1.6.1 we can identify V with a vector space over $\mathbb{F}_q$. The conjugation action of H on V realises V as an $\mathbb{F}_q[H]$-module, and since V is minimal normal in G, it must be irreducible as a H-module. By assumption, the action of G on Ω is faithful and thus by Lemma 1.7.11 the action of H on V is also faithful. □

Corollary 3.2.5. *The subgroup* $\mathcal{O}_q(H) \leq H$ *is trivial.*

Proof. By Lemma 3.2.4, H possesses a faithful irreducible module over $\mathbb{F}_q$. Hence, $\mathcal{O}_q(H) = 1$ by Lemma 1.8.6. □

Corollary 3.2.6. *The centre of* H *is cyclic.*

Proof. By Lemma 3.2.4, H possesses a faithful irreducible module. Hence, $Z(H)$ is cyclic by Proposition 1.8.7. □

Lemma 3.2.7. *Let* $\alpha \in \Omega$, H_α *the point stabiliser of* α *in* G *and* $h \in H_\alpha$. *Then*

$$|\mathrm{Fix}_\Omega(h)| = |C_V(h)|.$$

Proof. Note first that since $h \in H_\alpha$, we have that $\mathrm{Fix}_\Omega(h)$ is nonempty. We work to show that $C_V(h)$ acts regularly on $\mathrm{Fix}_\Omega(h)$. First we must show that $C_V(h)$ acts on $\mathrm{Fix}_\Omega(h)$. Let $v \in C_V(h)$ and $\beta \in \mathrm{Fix}_\Omega(h)$, then

$$\beta^v = \beta^{hv} = \beta^{vh}.$$

The first equality follows since $\beta \in \mathrm{Fix}_\Omega(h)$ and the second equality since $v \in C_V(h)$. From this we deduce that $\beta^v \in \mathrm{Fix}_\Omega(h)$ and thus $C_V(h)$ does indeed act on $\mathrm{Fix}_\Omega(h)$.

Now V is regular on Ω and so there exists a unique $v \in V$ such that $\alpha \cdot v = \beta$. Let H_β denote the point stabiliser of β in G and observe that

$$H_\beta = H_{\alpha\cdot v} = H_\alpha^v.$$

So if we look at the following commutator

$$[v, h] = v^{-1}h^{-1}vh \in V \cap H_\beta = 1,$$

we find that $v \in C_V(h)$. We can see this as follows. Since $V \trianglelefteq G$, we have $h^{-1}vh \in V$ and thus $[v, h] \in V$. Also note that $h^{-1} \in H_\alpha$ because $h \in H_\alpha$ and so $v^{-1}h^{-1}v \in H_\beta$ since $H_\beta = H_\alpha^v$. Thus $C_V(h)$ acts regularly on $\mathrm{Fix}_\Omega(h)$. □

Corollary 3.2.8. *Let* $h \in H$. *Then* $|\mathrm{Fix}_\Omega(h)|$ *divides* $|\Omega|$ *and thus the number of points fixed by* h *is of prime power order.*

Proof. Since $C_V(h) \leq V$, $|C_V(h)|$ divides $|V|$ and thus since $|V| = |\Omega|$ and $|\mathrm{Fix}_\Omega(h)| = |C_V(h)|$, the result follows. □

Note further that since G is faithful on Ω, the number of fixed points of a nonidentity element $h \in H$ cannot be more than $\frac{1}{2}|\Omega|$.

3.3 Soluble primitive permutation groups generated by two p^n-cycles

We now focus our attention on soluble primitive permutation groups which can be generated by two p^n-cycles. By 'p^n-cycle', we mean an element which can be written as the product of n disjoint cycles of length p. Throughout the remainder of this section, we will consider the following:

Hypothesis 3.3.1.

- *Hypothesis 3.2.2 holds;*
- *G is generated by two p^n-cycles where p is a prime and $n \in \mathbb{N}$;*

By Lemma 3.2.3, Ω has prime power order. Also, H is 2-generated as a homomorphic image of G. We now set some notation.

- *$|\Omega| = q^m$ where q is a prime and $m \in \mathbb{N}$;*
- *$H = \langle a, b \rangle$, $A = \langle a \rangle$ and $B = \langle b \rangle$.*

Straightaway this places further constraints (than those outlined in Section 3.2) on the order of Ω, namely

$$np \leq |\Omega| < 2np.$$

The first inequality comes from the fact that it does not make sense to consider the action of a p^n-cycle on a set of order less than np. The second inequality follows since if $|\Omega| \geq 2np$, either both of our generating elements would fix some $\alpha \in \Omega$, and thus the whole group would fix α, or we would have $2n$ orbits of length p. In either case, G would not be transitive on Ω and thus it would certainly not be primitive. Before we really get into investigating the structure of these groups, we first quickly dispose of a couple of special cases. Firstly, we will look at the case when $p = 2$. Then we will consider the case when $n = 1$. In these instances, it is quite straightforward to describe the structure of G, but furthermore, having knowledge of these cases will help us later on.

Proposition 3.3.2. *Assume Hypothesis 3.3.1 and that $p = 2$. Then q is odd, $m = 1$ and $G \cong \mathrm{Dih}(2q)$.*

Proof. Let $G = \langle x, y \rangle$ where x and y are 2^n-cycles. Since x and y are both involutions, G is dihedral by Lemma 1.1.11. Since V is minimal normal in G, V is cyclic by Proposition 1.1.12 and so $m = 1$. Now V is a faithful irreducible

$\mathbb{F}_q[H]$-module and since V has dimension 1, H is cyclic by Proposition 1.8.9. Since H is generated by involutions, we have $H \cong \mathbb{Z}_2$. Also, since H is nontrivial on V, $G \cong \mathrm{Dih}(2q)$. Lastly, since H is a 2-group, $q \neq 2$ because V is a nontrivial irreducible $\mathbb{F}_q[H]$-module. □

Proposition 3.3.3. *Assume Hypothesis 3.3.1 and that $n = 1$. Then one of the following holds:*

1. $G \cong \mathbb{Z}_p$;
2. $G \cong \mathbb{Z}_2^m \rtimes \mathbb{Z}_p$ *where* $p = 2^m - 1$;
3. $G \cong \mathrm{Sym}(3)$.

Proof. It is clear that

$$p \leq |\Omega| < 2p$$

for reasons stated in Section 3.2.

Suppose $|\Omega| = p$. Then $|V| = p$ and $G \leq \mathrm{Sym}(p)$. Thus $V \in \mathrm{Syl}_p(G)$ and so both of our generating elements are contained in V. Hence $V = G$ and $G \cong \mathbb{Z}_p$.

Now suppose $|\Omega| > p$. Then we can assume one of the generating elements of G, call it x, is in the point stabiliser H. By Corollary 3.2.8 the number of points fixed by x divides $|\Omega|$. Therefore

$$(|\Omega| - p) \big| |\Omega|,$$

which forces $|\Omega| = p + 1$ since $|\Omega| < 2p$ and the prime divisors of p are 1 and p. Since x is a p-cycle, H is transitive on $V - \{1\}$. A group action which is transitive on a set of prime order is in fact primitive. So we can view H as a soluble primitive permutation group generated by two p-cycles which acts faithfully on a set of order p. Thus $H \cong \mathbb{Z}_p$ by our argument above. Since p and q have different parity, we have that either $p = 2$ or $q = 2$. If $p = 2$, then $q = 3$, so $N \cong \mathbb{Z}_3$ and $H \cong \mathbb{Z}_2$. Since H is nontrivial on V, it follows that $G \cong \mathrm{Sym}(3)$. If $q = 2$, then $p = 2^m - 1$ and $G \cong \mathbb{Z}_2^m \rtimes \mathbb{Z}_p$. □

We have already remarked on how the structure of G places constraints on $|\Omega|$, however, we make this more precise with the following.

Proposition 3.3.4. *Assume Hypothesis 3.3.1. Then*

$$|\Omega| = q^m = np + cq^k$$

where $c \in \{0, 1\}$ *and* $k \in \mathbb{N} \cup \{0\}$.

At this stage, it is tempting to split the problem up into considering the following cases for the order of Ω in terms of n and p:

1. $|\Omega| = np = q^m$;
2. $|\Omega| = np + 1 = q^m$;
3. $|\Omega| = np + q^k = q^m$ where $k > 0$.

The second and third cases arise for the following reason. If we suppose $|\Omega| > np$, then we can assume without loss of generality that one of our generating elements, say x, lies in H. We know that the number of fixed points of x divides the order of Ω and thus $|\Omega| - np$ must be a power of q. Note also in case 1 that $p = q$.

However, we will not split the problem up in this way since the structure of G is much more strongly related to other factors. For example, whether or not V is a primitive module for H. Recall that $H = \langle a, b \rangle$, $A = \langle a \rangle$ and $B = \langle b \rangle$. In the case where V is a primitive module for H, the structure of G is further heavily restricted by whether $\mathbb{F}_q[A]$ or $\mathbb{F}_q[B]$ appear as a direct summand of V_A or V_B respectively. As we shall see, when V is an imprimitive module for H, both $\mathbb{F}_q[A]$ and $\mathbb{F}_q[B]$ necessarily appear as direct summands of V_A and V_B respectively. In what follows, we will see how these factors govern the structure of G. We will make clear the importance of the order of Ω and how this is related to the structure of G as it arises. As such, we will realise the importance of $|\Omega|$, and how this may be expressed in terms of n, p and q. For instance, when $|\Omega| = np + 1$, V cannot be imprimitive for H. In fact, neither $\mathbb{F}_q[A]$ nor $\mathbb{F}_q[B]$ can appear as a direct summand of V_A and V_B respectively.

We begin with a few results regarding the generators of H. Note that when $|\Omega| = np$, our two p^n-cycles cannot fix any points in Ω and so they are certainly not contained in H. In this case, all we know is that H is generated by two elements of order dividing p since H is a homomorphic image of G. However, when $|\Omega| > np$ we have the following very useful result.

Lemma 3.3.5. *Let $|\Omega| > np$. Then H is generated by two p^n-cycles.*

Proof. Let $G = \langle x, y \rangle$ where x and y are p^n-cycles. Denote by $\overline{G}$ the factor group G/V. Then we see that

$$\overline{\langle x, y \rangle} \cong H.$$

Since $|\Omega| > np$, we can assume without loss of generality that $x \in H$. Of course $y \notin H$, otherwise G would fix a point in Ω. However, since all point stabilisers are conjugate, there exists $g \in G$ such that $y^g \in H$. Now since G is the split extension of V by an arbitrary point stabiliser, these point stabilisers are all conjugate by elements of V. Hence, there in fact exists $v \in V$ such that $y^v \in H$. Now

$$H \cong \overline{\langle x, y \rangle} \cong \langle \overline{x}, \overline{y} \rangle = \langle \overline{x}, \overline{y}^{\overline{v}} \rangle$$

where the final equality follows since $\overline{v} = 1$. Now

$$\langle \overline{x}, \overline{y}^{\overline{v}} \rangle = \overline{\langle x, y^v \rangle} \leq \overline{H}.$$

By considering orders,

$$\overline{\langle x, y^v \rangle} = \overline{H}.$$

Since $\langle x, y^v \rangle \leq H$ and $H \cap V = 1$, it follows that $\langle x, y^v \rangle = H$. Thus H is also generated by two p^n-cycles. □

In the case where $|\Omega| > np$, it will henceforth be assumed that the generators a and b of H are both p^n-cycles. The next couple of results place constraints on the dimensions of $C_V(a)$ and $C_V(b)$.

Lemma 3.3.6. *Suppose $|\Omega| > np$. Then both $\dim C_V(a)$ and $\dim C_V(b)$ are bounded above by $\frac{1}{2} \dim V$.*

Proof. Since $|\Omega| > np$, both $\dim C_V(a)$ and $\dim C_V(b)$ are given by

$$\log_q(q^m - np)$$

where $|\Omega| = q^m$. Suppose that $\dim C_V(a)$ and $\dim C_V(b)$ are both greater than $\frac{1}{2} \dim V$. Then

$$C_V(a) \cap C_V(b) \neq 0$$

otherwise

$$\dim\Big(C_V(a) + C_V(b)\Big) > m.$$

Since H is generated by a and b,

$$C_V(a) \cap C_V(b) = C_V(H).$$

However, $C_V(H) = 0$. □

Lemma 3.3.7. *Suppose $|\Omega| = np + 1$. Then $\dim C_V(a) = 0 = \dim C_V(b)$.*

Proof. Since $|\Omega| = q^m = np+1$ for some $m \in \mathbb{N}$, both $\dim C_V(a)$ and $\dim C_V(b)$ are given by

$$\log_q(q^m - np) = \log_q 1 = 0.$$

□

Corollary 3.3.8. *Assume Hypothesis 3.3.1 and suppose $|\Omega| = np + 1$. Then neither $\mathbb{F}_q[A]$ nor $\mathbb{F}_q[B]$ can appear as direct summands of V_A or V_B respectively.*

Proof. By Lemma 3.3.7 we have that $\dim(C_V(A)) = 0$ and so it is clear from Lemma 1.9.7 that $\mathbb{F}_q[A]$ cannot appear as a direct summand of V_A. Similarly, $\mathbb{F}_q[B]$ cannot appear as a direct summand of V_B. □

Lemma 3.3.7 and Corollary 3.3.8 turn out to be very powerful in determining the structure of G when $|\Omega| = np + 1$. Indeed, we now work towards showing that every abelian normal subgroup of H is contained in $Z(H)$ and that V is a primitive module for H when $|\Omega| = np + 1$. We do this by showing that if either V is imprimitive for H or if H contains an abelian normal subgroup N such that $N \nsubseteq Z(H)$, then at least one of $\mathbb{F}_q[A]$ or $\mathbb{F}_q[B]$ appear as a direct summand of V_A or V_B respectively. The latter of these results is much easier to prove when p and $|N|$ are coprime and so we look at this case before proving the result in generality.

Lemma 3.3.9. *Suppose that there exists an abelian normal subgroup $N \trianglelefteq H$ such that $N \nsubseteq Z(H)$ and that p and $|N|$ are coprime. Then either $\mathbb{F}_q[A]$ or $\mathbb{F}_q[B]$ appears as a direct summand of V_A or V_B respectively.*

Proof. Since $N \nsubseteq Z(H)$, at least one of $\langle a \rangle$ and $\langle b \rangle$ acts nontrivially on N. Assume without loss of generality that $[N, \langle a \rangle] \neq 1$. Since N is abelian and $\langle a \rangle$ acts coprimely on N, we have

$$N = C_N(\langle a \rangle) \times [N, \langle a \rangle]$$

by Fitting's theorem. Let $N_0 = [N, \langle a \rangle]$. Then $\langle a \rangle$ acts semiregularly on N_0 since $C_N(\langle a \rangle) \cap N_0 = 1$. Now

$$V = C_V(N_0) \oplus [V, N_0].$$

Note that $[V, N_0] \neq 0$ since $N_0 \neq 1$ and H acts faithfully on V. Since $\langle a \rangle$ acts on both V and N_0, it acts on $[V, N_0]$. Thus $[V, N_0]$ is an $\mathbb{F}[AN_0]$-module (where AN_0 is the spilt extension of N_0 by A). Now $C_{[V,N_0]}(N_0) = 0$ since $C_V(N_0) \cap [V, N_0] = 0$. So the hypotheses of Theorem 1.8.17 are satisfied with A, N_0 and $[V, N_0]$ in place of A, N and V respectively, hence $[V, N_0]$ is a free $\mathbb{F}_q[A]$-module. Since

$$V = C_V(N_0) \oplus [V, N_0],$$

the result follows. □

Corollary 3.3.10. *Assume the hypotheses of Lemma 3.3.9. If either* $[N, \langle a \rangle]$ *or* $[N, \langle b \rangle]$ *is nontrivial and normal in* H*, then* $\dim V = lp$ *for some* $l \in \mathbb{N}$*.*

Proof. Assume without loss of generality that $1 \neq [N, \langle a \rangle] \trianglelefteq H$. Let $N_0 = [N, \langle a \rangle]$. Since H acts faithfully and irreducibly on V, $C_V(N_0) = 0$. Thus $V = [V, N_0]$ and the hypotheses of Theorem 1.8.17 are satisfied with A, N_0 and V in place of A, N and V respectively. Hence, V is free as an A-module and has dimension $l|A| = lp$ for some $l \in \mathbb{N}$. □

If either $[N, \langle a \rangle]$ or $[N, \langle b \rangle]$ in Corollary 3.3.10 is trivial, then this forces the other to be nontrivial and normal in H. They cannot both be trivial since $N \nsubseteq Z(H)$. If for example $[N, \langle a \rangle] \neq 1$ and $[N, \langle b \rangle] = 1$, then $[N, \langle a \rangle] \trianglelefteq H$ since $[N, \langle a \rangle]$ is normalised by a and centralised by b.

Corollary 3.3.11. *Assume Hypothesis 3.3.1 and suppose* $|\Omega| = np + 1$*. Then all abelian normal* p'*-subgroups* $N \trianglelefteq H$ *are cyclic and contained in* $Z(H)$*.*

Proof. Suppose there exists an abelian p'-subgroup $N \trianglelefteq H$ such that $N \nsubseteq Z(H)$. Then at least one of $\mathbb{F}_q[A]$ or $\mathbb{F}_q[B]$ appear as a direct summand of V_A or V_B respectively by Lemma 3.3.9. However, this contradicts Corollary 3.3.8 and so $N \subseteq Z(H)$. Then N must be cyclic since $Z(H)$ is cyclic. □

Lemma 3.3.12. *Let* W *be an imprimitive irreducible* $\mathbb{K}[H]$*-module for some field* $\mathbb{K}$*. Then at least one of* $\mathbb{K}[A]$ *or* $\mathbb{K}[B]$ *appears as a direct summand of* W_A *or* W_B *respectively.*

Proof. Since W is imprimitive for H, there exists a nontrivial system of imprimitivity

$$W = W_1 \oplus \ldots \oplus W_k.$$

Let $\Gamma = \{W_1, \ldots, W_k\}$. Since W is an irreducible module for H, at least one of a and b acts nontrivially on Γ. So we can assume without loss of generality that $W_1^a \neq W_1$. Then W_1 is in an orbit of length p under the action of a. We can reorder such that this corresponds to the blocks 1 to p. If we look at the subspace $U = W_1 \oplus \ldots \oplus W_p$, then this is a $\mathbb{K}[A]$-module. This is an imprimitive module for A and we see that A acts semiregularly on this system of imprimitivity. Thus it follows from Theorem 1.9.6 that U is free as an A-module. Since

$$W = U \oplus W_{p+1} \oplus \ldots \oplus W_k,$$

the result follows. □

Proposition 3.3.13. *Let G be a group and suppose there exists an abelian normal subgroup $N \trianglelefteq G$ such that $N \nsubseteq Z(G)$. Let V be a faithful irreducible $\mathbb{F}_q[G]$-module and $\mathbb{F}_q \leq \mathbb{K}$ be a field extension which contains a primitive $|N|^{\text{th}}$ root of unity. Then every irreducible G-submodule of*

$$V^{\mathbb{K}} = V \otimes_{\mathbb{F}} \mathbb{K}$$

is imprimitive.

Proof. Let $U \leq V^{\mathbb{K}}$ be an irreducible G-submodule. Suppose U is primitive for G. Then by Corollary 1.8.16, all abelian normal subgroups of G are cyclic and contained in $Z(G)$. This is a contradiction since $N \nsubseteq Z(G)$. □

Proposition 3.3.14. *Assume Hypothesis 3.3.1 and suppose $|\Omega| = np + 1$. Then all abelian normal subgroups $N \trianglelefteq H$ are cyclic and contained in $Z(H)$.*

Proof. Suppose that there exists an abelian normal subgroup $N \trianglelefteq H$ such that $N \nsubseteq Z(H)$. Let $\mathbb{F}_q \leq \mathbb{K}$ be a field extension which contains a primitive $|N|^{\text{th}}$ root of unity and set $V^{\mathbb{K}} = V \otimes_{\mathbb{F}_q} \mathbb{K}$. Then by Proposition 3.3.13, every irreducible $\mathbb{K}[H]$-submodule of $V^{\mathbb{K}}$ is imprimitive. Thus by Lemma 3.3.12, at least one of $\mathbb{K}[A]$ or $\mathbb{K}[B]$ appear as a direct summand of $V_A^{\mathbb{K}}$ or $V_B^{\mathbb{K}}$ respectively. However, by Theorem 1.9.9 this can only occur if one of $\mathbb{F}_q[A]$ or $\mathbb{F}_q[B]$ appear as a direct summand of V_A or V_B respectively. This is a contradiction to Corollary 3.3.8 since $|\Omega| = np + 1$. Hence $N \subseteq Z(H)$. Now $Z(H)$ is cyclic by Corollary 3.2.6 and so N must also be cyclic. □

Lemma 3.3.15. *Suppose $|\Omega| = np + 1$. Then V is a primitive module for H.*

Proof. This is a direct consequence of Lemma 3.3.12 and Corollary 3.3.8. □

Proposition 3.3.14 and Lemma 3.3.15 are prime examples of how the structure of G is largely controlled by $|\Omega|$ and how this can be expressed in terms of p and n. They essentially follow from the fact that once we have a system of imprimitivity Γ for H on V, at least one of the generators of H must act nontrivially Γ. We will soon show that in fact both generators of H must act nontrivially of Γ. Firstly though, we have the following.

Lemma 3.3.16. *Suppose that there exists an abelian normal subgroup $N \trianglelefteq H$ such that $N \nsubseteq Z(H)$. Suppose further that* $\dim V = p$*. Then either $|\Omega| = q^p = np + q^x$ where $x \in \mathbb{N}$, or $|\Omega| = p^p = np$. In either case, $H \cong K \rtimes L$ where $1 \neq K$ is a direct product of cyclic groups and $L \cong \mathbb{Z}_p$.*

Proof. Note that $|\Omega| \neq np+1$ by Proposition 3.3.14 . Also, since $\dim V = p$, $|\Omega| = q^p = np + q^x$ where $x \in \mathbb{N}$ or $|\Omega| = p^p = np$.

Let $\mathbb{K}$ denote the algebraic closure of $\mathbb{F}_q$. We claim that

$$V^{\mathbb{K}} = V \otimes_{\mathbb{F}_q} \mathbb{K}$$

is irreducible. Suppose $V^{\mathbb{K}}$ is not an irreducible H-module. Then there exists a proper submodule $U \leq V^{\mathbb{K}}$ which is irreducible. Since

$$\dim_{\mathbb{F}} V = \dim_{\mathbb{K}} V^{\mathbb{K}},$$

$\dim_{\mathbb{K}} U < p$ and so neither $\mathbb{K}[A]$ nor $\mathbb{K}[B]$ can appear as a direct summand of U_A or U_B respectively. If U is an imprimitive module for H, then by Lemma 3.3.12, at least one of $\mathbb{K}[A]$ or $\mathbb{K}[B]$ will appear as a direct summand of U_A or U_B respectively (note that Lemma 3.3.12 is independent of the field over which V is defined). So U is a primitive module for H. Since $\mathbb{K}$ is algebraically closed, this forces all abelian normal subgroups to be cyclic and contained in $Z(H)$. Thus $V^{\mathbb{K}}$ is an irreducible and imprimitive H-module.

By Clifford's Theorem $V^{\mathbb{K}}_N$ is a direct sum of irreducible N-submodules. Since $\mathbb{K}$ contains a primitive $|N|^{\text{th}}$ root of unity, V_N is a direct sum of p 1-dimensional N-submodules. If V is homogeneous with respect to N, then all of these submodules will be N-isomorphic and thus N acts by the same scalar on each of them. However, this would mean $N \subseteq Z(H)$. Thus each of these N-submodules are pairwise non N-isomorphic and so N acts on each of them by a different scalar. This collection of N-submodules constitutes a nontrivial system of imprimitivity for the action of H on V, call it Γ. Let K be the full kernel of the action of H on Γ. Of course $N \subseteq K$ and since $N \neq 1$, we have that $K \neq 1$. Now H/K is a primitive permutation group generated by two p-cycles which acts faithfully on a set of order p. Thus by Proposition 3.3.3, $H/K \cong \mathbb{Z}_p$. □

Lemma 3.3.17. *Suppose that V is an imprimitive module for H and let $\Gamma = \{V_1, \ldots, V_k\}$ be a nontrivial system of imprimitivity. Then both a and b act nontrivially on Γ.*

Proof. Assume for a contradiction that this is not the case, so at least one of a and b act trivially on Γ. Of course we cannot have both a and b acting trivially on Γ since V is an irreducible module for H and Γ is a nontrivial system of imprimitivity. Assume without loss of generality that only a acts trivially on Γ. Now b must act regularly on Γ because V is irreducible for H. Since b has order p, this forces $k = p$.

Suppose $|\Omega| = q^m = np + q^x$ where $x \in \mathbb{N}$. Then $|\mathrm{Fix}_\Omega(b)| = q^x$ and so $\dim C_V(b) = x$. Note also that

$$C_V(b) = \{v_1 v_1^b v_1^{b^2} \cdots v_1^{b^{p-1}} \mid v_1 \in V_1\}$$

and thus

$$|C_V(b)| = |V_1|.$$

From this it follows that $\dim V_1 = x$ and thus $m = px$.

Now b is a p^n-cycle by Lemma 3.3.5 and so it moves np elements of V. Since b is regular on Γ, we have that $n = q^x - 1$. Now

$$q^m = np + q^x$$

and so

$$q^{px} = (q^x - 1)p + q^x.$$

Suppose $q = p$. Then

$$q^{qx} = (q^x - 1)q + q^x.$$

If $x = 1$, then this forces $q = 2$ and thus $p = 2$. However, when $p = 2$, V has dimension 1 by Proposition 3.3.2, but V here has dimension at least p. Thus $x > 1$ and so

$$q^{qx-1} + 1 = q^{x-1}(q + 1).$$

However, this has no solutions for q prime and $x \in \mathbb{N}$, $x > 1$. We can see this by reducing modulo q.

Now suppose $q \neq p$, so

$$q^{px} = (q^x - 1)p + q^x.$$

Noting that $x > 0$ and reducing modulo q implies $p \equiv 0$ modulo q, which is a contradiction.

Now suppose $|\Omega| = p^m = np$. Again we assume that a acts trivially on Γ and b nontrivially. Let b be a p^l-cycle where $l \in \mathbb{N}$, $l < n$. By the same argument we find that $m = p \dim V_1$. Let $\dim V_1 = x$ so $n = p^{px-1}$. Now b fixes $np - lp$ elements in V, so by Corollary 3.2.8

$$np - lp = p^c$$

for some $c \in \mathbb{N}$. Arguing as before

$$l = p^{\dim V_1} - 1 = p^x - 1$$

and so

$$p^{px} - p(p^x - 1) = p^c.$$

If $c = 1$, then

$$p^{px-1} - p^x + 1 = 1$$

and so

$$px - 1 = x.$$

Since p is prime and $x \in \mathbb{N}$, we must have that $p = 2$ and $x = 1$. However, we have already seen that $p \neq 2$. Therefore $c > 1$ and so

$$p^{px-1} - p^x + 1 = p^{c-1}.$$

By noting that $x > 0$ and reducing modulo p we see that this has no solutions. □

Corollary 3.3.18. *Assume Hypothesis 3.3.1. If V is imprimitive for H, then both $\mathbb{F}_q[A]$ and $\mathbb{F}_q[B]$ appear as direct summands of V_A and V_B respectively.*

Proof. Since V is imprimitive for H, there exists a system of imprimitivity

$$V = V_1 \oplus \ldots \oplus V_k.$$

Let $\Gamma = \{V_1, \ldots, V_k\}$. By Lemma 3.3.17, both a and b act nontrivially on Γ. The proof is now identical to that of Lemma 3.3.12. □

Lemma 3.3.19. *Suppose that V is an imprimitive module for H of dimension p. Then either $H \cong L$ or $H \cong K \rtimes L$ where $L \cong \mathbb{Z}_p$ and K is a direct product of p subgroups, each of which is a subgroup of $\mathbb{Z}_{q-1}$. In the latter case, the direct factors of K are cyclically permuted by L.*

Proof. Since V is imprimitive for H, there exists a system of imprimitivity

$$\Gamma = \{V_1, \ldots, V_k\}$$

and we can consider the permutation action of H on Γ. Since $\dim V = p$, $\dim V_i = 1$ for each subspace V_i. Thus the generating elements of H act as p-cycles on Γ. Let K be the kernel of the action of H on Γ. Then H/K satisfies all of the hypotheses of Proposition 3.3.3. Since $|\Gamma| = p$, $H/K \cong \mathbb{Z}_p$. Thus if $K = 1$, then $H \cong \mathbb{Z}_p$.

Now assume that $K \neq 1$. Since V is a direct sum of 1-dimensional subspaces, K acts by scalar multiplication on each component. Since $K \trianglelefteq H$, we see by Clifford's theorem that K either acts by the same scalar on each component or it does not act by the same scalar on any two distinct V_i. If K acts by the same scalar on each component, then $K \subseteq Z(H)$. So $H/Z(H)$ is cyclic and thus H is abelian. Since H is faithfully and irreducibly represented, $Z(H)$ is cyclic and thus if H is abelian, then $H \cong \mathbb{Z}_p$. So if $K \neq 1$, then K cannot act on each subspace V_i by the same scalar. □

Consider the following hypothesis.

Hypothesis 3.3.20.

- *Hypothesis 3.3.1 holds;*
- *All abelian normal subgroups of H are cyclic and contained in $Z(H)$.*

We recall that when $|\Omega| = np + 1$, all abelian normal subgroups of H are necessarily contained in $Z(H)$. As such they are necessarily cyclic since $Z(H)$ is cyclic. Hence the remainder of this section has particular relevance to the case where $|\Omega| = np + 1$.

Theorem 3.3.21. *Assume Hypothesis 3.3.20. If $F(H)$ is abelian, then $H \cong \mathbb{Z}_p$ and $G \cong \mathbb{Z}_q^m \rtimes \mathbb{Z}_p$ where m is the order of q modulo p. Furthermore, if $|\Omega| = np + 1$, then G is a Frobenius group.*

Proof. By Corollary 1.2.19, we have that H is abelian. Hence H is cyclic by Corollary 3.2.6. Since H is generated by elements of order p, $H \cong \mathbb{Z}_p$. We already know that V is a faithful irreducible $\mathbb{F}_q[H]$-module and so since $H \cong \mathbb{Z}_p$, the second claim follows by Proposition 3.1.3.

Suppose $|\Omega| = np + 1$. Since H is generated by a p^n-cycle and $|V| = np + 1$, H acts by inducing n orbits of length p on V. Every nonidentity element of H only fixes $1 \in V$ and so G is a Frobenius group by part 4 of Theorem 1.7.5 with kernel V and complement H. □

The following corollary has relevance when $|\Omega| \neq np + 1$.

Corollary 3.3.22. *Assume hypothesis 3.3.20 and suppose V is a primitive module for H and either $\mathbb{F}_q[A]$ or $\mathbb{F}_q[B]$ is a direct summand of V_A or V_B respectively. Then $F(H)$ is nonabelian.*

Proof. Suppose for a contradiction that $F(H)$ is abelian. Then $H \cong \mathbb{Z}_p$ by Theorem 3.3.21. This rules out the case where $|\Omega| = np$ and $|\Omega| = np + p^x$ since a group G such that $\mathcal{O}_p(G) \neq 1$ cannot be faithfully and irreducibly represented over a field of characteristic p. Thus $|\Omega| = np + q^x = q^m$ where $p \neq q$ and $x \in \mathbb{N}$. Now m is the order of q modulo p by Theorem 3.3.21. However, $m \geq p$ by hypothesis. This is a contradiction since every element in $\mathbb{Z}_p^*$ has order dividing $p - 1$. □

We now consider the case where $F(H)$ is nonabelian. In particular, we will consider the structure of G when H is nilpotent and when H is soluble nonnilpotent.

Proposition 3.3.23. *Assume Hypothesis 3.3.20. If H is nilpotent and nonabelian, then it is an extraspecial p-group of exponent p and order p^3 where p is odd. Furthermore, V is an irreducible H-module of dimension cp where $c \in \mathbb{N}$.*

Proof. If H is nilpotent, then both generating elements of H are contained inside $\mathcal{O}_p(H)$. Thus they will generate a p-group. If $p = 2$, then G will be a dihedral group by Proposition 3.3.2. As we saw in the proof of Proposition 3.3.2, $H \cong \mathbb{Z}_2$ and so H is abelian in this case. So if H is nilpotent and nonabelian, then p must be odd. Recall that we are assuming that every abelian normal subgroup of H is cyclic and contained in $Z(H)$. Thus

$$H' \leq \Phi(H) \leq Z(H)$$

by Corollary 1.2.33 since H is a p-group. Since H is nonabelian, it has nilpotence class 2. So H has exponent p by Lemma 3.1.6 and we have that $Z(H) \cong \mathbb{Z}_p$ since $Z(H)$ is cyclic. Now $H' \neq 1$, and so the above inclusions become equalities. Thus H is extraspecial.

Now H is 2-generated as a homomorphic image of G. Hence its Frattini quotient is an $\mathbb{F}_p$-vector space of dimension 2. Hence, H has order p^3. We know that V is a faithful irreducible module for H over characteristic coprime to p and so by Corollary 3.1.11 it must have dimension cp for some $c \in \mathbb{N}$. □

When $|\Omega| = np$, $n > 1$, we have that H cannot be nilpotent. If H is nilpotent, then it will be a p-group since it is generated by two elements of order p. If $|\Omega| = np$, then H is faithfully and irreducibly represented over a field of characteristic p, so $\mathcal{O}_p(H) = 1$ by Lemma 1.8.6. Then since H is maximal in G, it forces G to be cyclic of prime order. This cannot happen if $n > 1$ since G will contain a noncyclic elementary abelian subgroup.

We now investigate what happens when H is nonnilpotent.

Proposition 3.3.24. *Assume Hypothesis 3.3.20 and suppose that H is soluble nonnilpotent, $\mathcal{O}_{p'}(H)$ is nonabelian and neither $\mathbb{F}_q[A]$ nor $\mathbb{F}_q[B]$ appear as a direct summand of V_A or V_B respectively. Then $p = 2^x + 1$ for some $x \in \mathbb{N}$ and $q \neq 2$.*

Proof. Since $\mathcal{O}_{p'}(H)$ is nonabelian, there exists a prime $s \neq p$ such that $\mathcal{O}_s(H)$ is nonabelian. We can assume without loss of generality that A acts nontrivially on $\mathcal{O}_s(H)$. If $\mathcal{O}_s(H)$ were to commute with both of the generators of H, then $\mathcal{O}_s(H) \subseteq Z(H)$ which would contradict the assumption that $\mathcal{O}_s(H)$ is nonabelian. Let $S = [\mathcal{O}_s(H), A]$. By Lemma 3.1.5, S is extraspecial and $C_S(A) = S'$. Also, since A is cyclic, it is semiregular on S/S'.

Let $\mathbb{K}$ denote the algebraic closure of $\mathbb{F}_q$ and consider

$$V^{\mathbb{K}} = V \otimes_{\mathbb{F}_q} \mathbb{K}.$$

Since $\mathbb{F}_q[A]$ does not appear as a direct summand of V_A, $\mathbb{K}[A]$ does not appear as a direct summand of $V_A^{\mathbb{K}}$ by Theorem 1.9.9. Also, since V is faithful, any irreducible submodule of $V^{\mathbb{K}}$ is faithful by part 3 of Lemma 1.8.3. Let $U \leq V^{\mathbb{K}}$ be an irreducible H-submodule. Then the hypotheses of Theorem 1.8.18 are satisfied with A and S in place of A and P. Since $\mathbb{K}[A]$ does not appear as a direct summand of V_A, this forces $p = s^x + 1$ where $x \in \mathbb{N}$. Since p and s are primes of different parity and $p \neq 2$, we have that $s = 2$ and $p = 2^x + 1$. Remembering that V is a faithful irreducible module for H over $\mathbb{F}_q$ we conclude by Lemma 1.8.6 that $q \neq 2$ since $\mathcal{O}_2(H) \neq 1$. □

Corollary 3.3.25. *Assume Hypothesis 3.3.20. If H is soluble nonnilpotent and neither $\mathbb{F}_q[A]$ nor $\mathbb{F}_q[B]$ appear as a direct summand of V_A or V_B respectively, then $\mathcal{O}_{\{2,p\}'}(H)$ is abelian.*

Proof. Suppose H is soluble nonnilpotent and s is a prime such that $s \neq 2, p$ and $\mathcal{O}_s(H)$ is nonabelian. Again we can assume without loss of generality that $S = [\mathcal{O}_s(H), A] \neq 1$. Arguing exactly as in the proof of Proposition 3.3.24 we find that $s = 2$ which is a contradiction. □

Proposition 3.3.26. *Assume Hypothesis 3.3.20. Suppose H is soluble nonnilpotent and that $\mathcal{O}_{p'}(H)$ is abelian. Then H contains an extraspecial p-group and $m = lcp^k$ where $l, c, k \in \mathbb{N}$.*

Proof. Since $\mathcal{O}_{p'}(H)$ is abelian and $F(H)$ is nonabelian, we certainly have that $\mathcal{O}_p(H) \neq 1$. Set $\overline{H} = H/\mathcal{O}_p(H)$. Then since H is soluble nonnilpotent, there exists a prime s such that $\mathcal{O}_s(\overline{H}) \neq 1$. Set $K = \mathcal{O}_{p,s}(H) \trianglelefteq H$ and take $S \in \mathrm{Syl}_s(K)$. So $K = \mathcal{O}_p(H)S$ and by the Frattini argument we have $H = KN_H(S)$. So

$$\begin{aligned}
H &= KN_H(S) \\
H &= \mathcal{O}_p(H)SN_H(S) \\
H &= [\mathcal{O}_p(H), S]C_{\mathcal{O}_p(H)}(S)N_H(S) \\
H &= [\mathcal{O}_p(H), S]N_H(S)
\end{aligned}$$

where the third equality follows by part 2 of Theorem 1.5.2 and the inclusion $S \subseteq N_H(S)$, and the fourth equality since $C_{\mathcal{O}_p(H)}(S) \leq N_H(S)$. We have that $P = [\mathcal{O}_p(H), S] \trianglelefteq H$ since both $\mathcal{O}_p(H)$ and S are normalised by $N_H(S)$. The hypotheses of Lemma 3.1.5 are satisfied with $H, \mathcal{O}_p(H)$ and S in place of G, P and T respectively and so P is extraspecial. Let P have order p^{1+2k} for some $k \in \mathbb{N}$. Then by Clifford's theorem, V_P is homogeneous and is the direct sum of faithful irreducible $\mathbb{F}_q[P]$-submodules each of which has dimension cp^k for some $c \in \mathbb{N}$ by Corollary 3.1.11. This forces $m = lcp^k$ since there will be some constant $l \in \mathbb{N}$ copies of this representation in V_P. □

This last result cannot occur when $|\Omega| = np$ or when $|\Omega| = np+q^x$, for $x > 1$ and $p = q$. In both of these cases, V is a faithful irreducible $\mathbb{F}_p[H]$-module and thus by Lemma 1.8.6, $\mathcal{O}_p(H) = 1$. This will force $F(H)$ to be abelian since $\mathcal{O}_{p'}(H)$ is abelian. Then $H \cong \mathbb{Z}_p$ by Theorem 3.3.21. This is a contradiction since H is soluble nonnilpotent.

Corollary 3.3.27. *Assume Hypothesis 3.3.20. Suppose H is soluble nonnilpotent. Then for every prime r such that $\mathcal{O}_r(H)$ is nonabelian, H contains an extraspecial subgroup $R \cong r^{1+2k}$ for some $k \in \mathbb{N}$, and r^k divides* $\dim V$.

Proof. Let r be a prime such that $\mathcal{O}_r(H)$ is nonabelian. The proof proceeds exactly as in the proof of Proposition 3.3.26 with $\mathcal{O}_r(H)$ in place of $\mathcal{O}_p(H)$. □

3.4 Bounding the Fitting height in terms of n and p

The previous sections of this chapter indicate that in certain circumstances, it can be very difficult to determine the structure of a soluble primitive permutation group G generated by two p^n-cycles. In this final section, we try to measure the complexity of G in some way and relate this to p and n. To be more precise, we give a bound on the Fitting height of G in terms of p and n. Theorems 3.4.1 and 3.4.3 will prove extremely useful in achieving this. They can both be found in [50, Theorem 2]. We offer a slightly different proof of the latter result although some of the ideas used are the same.

Theorem 3.4.1. (Zhang) *Let G be a soluble subgroup of* $\mathrm{Sym}(n)$. *Then $f(G) \leq \frac{3}{2}\log_2 n$.*

Proof. See [50, Theorem 2]. □

Lemma 3.4.2. *Let G be a soluble group. If G is a subgroup of one of $GL(2,p)$ or $GL(4,2)$, then $f(G) \leq 3$. If G is a subgroup of $GL(pq,2)$ where p and q are two primes, then $f(G) \leq 4$.*

Proof. Follows from [34, Theorems 2.11, 2.14 and 2.15]. □

Theorem 3.4.3. (Zhang) *Let G be a soluble group and V a faithful completely reducible $\mathbb{F}[G]$-module of dimension n for some field $\mathbb{F}$. Then $f(G) \leq 3 + \frac{3}{2}\log_2 n$.*

Proof. Let G and V be a counterexample such that $|G| + n$ is minimal, so

$$f(G) > 3 + \frac{3}{2}\log_2 n.$$

Note that G is clearly nonnilpotent. Also $F(G)$ is a p-group for some prime p, otherwise, by the minimality of $|G|$, G embeds into a direct product of groups both with Fitting height less than $f(G)$.

Let $\mathbb{F} \leq \mathbb{K}$ be a field extension where $\mathbb{K}$ is a splitting field for G. Then by Theorem 1.8.4

$$V^{\mathbb{K}} = V \otimes_{\mathbb{F}} \mathbb{K}$$

is a faithful completely reducible $\mathbb{K}[G]$-module of dimension n. So we may assume that $\mathbb{F}$ is algebraically closed. We now claim that V must be irreducible for G. Suppose V is not irreducible, then we may write

$$V = V_1 \oplus V_2$$

where each subspace V_i is an $\mathbb{F}[G]$-module of dimension $n_i > 0$. Let

$$\phi : G \longrightarrow G/C_G(V_1) \times G/C_G(V_2)$$

be defined by

$$g \longmapsto \big(gC_G(V_1), gC_G(V_2)\big).$$

Then ϕ is a homomorphism with kernel $C_G(V_1) \cap C_G(V_2) = 1$, and so G is isomorphic to a subgroup of $G/C_G(V_1) \times G/C_G(V_2)$. By Lemmas 1.3.3 and 1.3.6 we have that

$$f(G) \leq \max\{f\big(G/C_G(V_1)\big), f\big(G/C_G(V_2)\big)\}.$$

Since

$$|C_G(V_i)| \leq |G| \text{ and } \dim V_i < \dim V$$

for $i = 1, 2$,

$$f\big(G/C_G(V_i)\big) \leq 3 + \frac{3}{2}\log_2 \dim V_i$$

by induction. This is a contradiction since

$$3 + \frac{3}{2}\log_2 \dim V_i < 3 + \frac{3}{2}\log_2 n$$

for $i = 1, 2$. Thus we have that V is irreducible for G.

Suppose V is imprimitive for G so

$$V = V_1 \oplus \ldots \oplus V_m$$

for some nontrivial system of imprimitivity $\Gamma = \{V_1, \dots, V_m\}$. Let K be the kernel in the permutation action of G on Γ. Then $f(G/K) \le \frac{3}{2}\log_2 m$ by Theorem 3.4.1. Since G is faithful on V, K embeds into a direct product of m copies of $GL\big(\frac{n}{m}, \mathbb{F}\big)$. As before we can apply induction to conclude

$$f(K) \le 3 + \frac{3}{2}\log_2\Big(\frac{n}{m}\Big).$$

Now

$$\begin{aligned} f(G) &\le f(K) + f(G/K) \\ &\le 3 + \frac{3}{2}\log_2\Big(\frac{n}{m}\Big) + \frac{3}{2}\log_2 m \\ &= 3 + \frac{3}{2}\log_2 n \end{aligned}$$

contrary to our assumption. Thus V is a primitive module for G.

Since V is primitive and irreducible for G over an algebraically closed field, it follows from Corollary 1.8.16 that every abelian normal subgroup of G is cyclic and contained in $Z(G)$. We must have that $F(G)$ is nonabelian, since if $F(G)$ is abelian, then G is abelian by Corollary 1.2.19. However, G is nonnilpotent. So $F(G) = P$ where P is a nonabelian p-group for some prime p. This subgroup satisfies the following

$$1 \ne P' \le \Phi(P) \le Z(P)$$

by Corollary 1.2.33. Also, since $F(G) = P$, we have that $G/F(G)$ acts faithfully on $P/Z(P)$ by Theorem 1.2.18. This follows since if there is an element $g \in G/F(G)$ which acts trivially on $P/Z(P)$, it must act trivially on P since $Z(P) \subseteq Z(G)$ (all abelian normal subgroups of G are cyclic and contained in $Z(G)$). In combination with Lemma 3.1.4, we see that $\overline{P} = P/Z(P)$ is a faithful completely reducible $\mathbb{F}_p[G/F(G)]$-module and each irreducible direct summand possesses a G-invariant symplectic form. Let $\overline{Q}$ be an irreducible direct summand of order p^{2k} and let Q be the inverse image of $\overline{Q}$ in P. Then $Q \trianglelefteq G$ and again we have

$$1 \ne Q' \le \Phi(Q) \le Z(Q).$$

Now V_Q is homogeneous by Clifford's theorem, and so $\dim V = lp^k$ by Proposition 3.1.10 (using here that since G is faithful on V, Q is also faithful on V). Now $\dim \overline{Q} = 2k \le q^k$ for all primes q and $k \in \mathbb{N}$. Now since $G/F(G)$ is faithful on $\overline{P}$, it embeds into a direct product of symplectic groups. Thus we can apply induction to conclude that

$$f(G/F(G)) \le 3 + \frac{3}{2}\log_2 2k.$$

By Lemma 1.3.5 we have the following inequality

$$3 + \frac{3}{2}\log_2 n < 4 + \frac{3}{2}\log_2 2k.$$

After some manipulation this becomes

$$n < 2^{\frac{5}{3}}k.$$

Given that q^k divides n, the only possibilities for k, n and q are as follows:

1. $k = 1$ and $n = q$ where $q = 2$ or 3;
2. $k = 2$ and $n = q^2 = 4$; or
3. $k = 3$ and $n = q^3 = 8$.

Recall

$$3 + \frac{3}{2}\log_2 n < f(G).$$

So if we can show that the soluble subgroups of $\mathrm{Sp}_{2k}(q)$ have Fitting height no greater than

$$3 + \frac{3}{2}\log_2 n$$

for the above values of n, k and q, then we are done. However, using the fact that $\mathrm{Sp}_{2k}(q) \leq GL(2k, q)$, this follows from Lemma 3.4.2. □

Recall that Ω denotes the set upon which G acts and either $|\Omega| = np$ or $|\Omega| = np + q^x$ for some prime q and $x \in \mathbb{N} \cup \{0\}$.

Theorem 3.4.4. *Let G be a soluble primitive permutation group generated by two p^n-cycles which acts faithfully on some set Ω.*

1. *If $|\Omega| = np$, then*
$$f(G) \leq 4 + \frac{3}{2}\log_2(1 + \log_p n).$$
2. *If $|\Omega| = np + q^x$, then*
$$f(G) \leq 4 + \frac{3}{2}\log_2(1 + \log_q np).$$
In particular, if $p = q$, then
$$f(G) \leq 4 + \frac{3}{2}\log_2(2 + \log_p n).$$

Proof. Recall that $G = V \rtimes H$ where V is a faithful irreducible module for H. Also, by Lemma 3.2.1, $|V| = |\Omega|$. Since V is elementary abelian and $V \trianglelefteq G$, we have that $V \subseteq F(G)$. Now $f(H) = f(G/V)$ so either

$$f(H) = f(G) \text{ or } f(H) = f(G) - 1.$$

So $f(G) \leq f(H) + 1$.

Suppose $|\Omega| = np = p^m$. By Theorem 3.4.3,

$$f(H) \leq 3 + \frac{3}{2}\log_2 m.$$

Now $np = p^m$ so $m = \log_p np$. Therefore

$$f(G) \leq f(H) + 1 \leq 4 + \frac{3}{2}\log_2(1 + \log_p n).$$

Now suppose $|\Omega| = np + q^x = q^m$. Recall from Section 3.2 that $|\Omega| < 2np$ so $m < \log_q 2np \leq 1 + \log_q np$. Thus

$$f(G) \leq f(H) + 1 \leq 4 + \frac{3}{2}\log_2(1 + \log_q np).$$

□

3.5 A further application for this bound

In [18] the following theorem is proved:

Theorem 3.5.1. (Flavell) *Let G be a soluble finite group and let C be a conjugacy class of $\{2,3\}'$-subgroups of G. If G is generated by C, then there exist two members of C that generate a subgroup with the same Fitting height as G. Moreover, the two members of C may be chosen to be conjugate in the subgroup that they generate.*

Suppose G is a soluble subgroup of $\mathrm{Sym}(m)$ generated by a conjugacy class C of $\{2,3\}'$-elements. Then by Theorem 3.5.1, there exist two elements of C which generate a subgroup A with $f(A) = f(G)$. If we take C to be a class of p^n-cycles for some prime $p > 3$, then we could use the results of this chapter to further restrict the Fitting height of G. Namely, if A acts faithfully and primitively on some set Ω, then

$$f(G) \leq 4 + \frac{3}{2}\log_2(1 + \log_q np)$$

where $|\Omega| = q^x$ for some prime q and $x \in \mathbb{N}$.

If p and n happened to be much smaller than m, then this could prove to be a very strong bound for the Fitting height of G. In particular, it could be a vast improvement on the bound given by Theorem 3.4.1, namely

$$f(G) \leq \frac{3}{2} \log_2 m.$$

3.6 Concluding remarks

In this chapter we have investigated the structure of soluble primitive permutation groups generated by two p^n-cycles. We have obtained many structural results but what we have is far from a classification of such groups. As such, there is much more work to be done before we fully understand these groups.

In Section 3.5 we outlined a very nice application for the bounds on the Fitting height of these groups. By considering Theorem 3.5.1, it would be interesting to see if similar bounds could be found for the Fitting height of soluble primitive permutation groups generated by two elements of the same cycle type. This would give us further information when studying certain classes of soluble subgroups of $\mathrm{Sym}(m)$.

Another direction to take might be to investigate soluble *transitive* permutation groups generated by two p^n-cycles. What sort of bounds can we obtain for the Fitting height of these groups and how do they compare to those bounds found in Section 3.4?

Chapter 4

On fixed-point-free automorphisms

If a group G admits a fixed-point-free automorphism ϕ, then one can often say something about the structure of G given certain properties of ϕ. Results of this kind can be traced back over a century but the study of fixed-point-free automorphisms continues to receive attention and remains central to the study of finite groups. In his book, *Theory of Groups of Finite Order* of 1911, W. Burnside proved that groups which admit fixed-point-free automorphisms of order 2 are abelian and groups which admit fixed-point-free automorphisms of order 3 are nilpotent of class at most 2 (proofs of these results by B. H. Neumann also appear in [37] and [38] respectively). These results were later generalised by J. G. Thompson and G. Higman who showed that groups which admit a fixed-point-free automorphism of prime order p are nilpotent [44] and that their nilpotency class is bounded in terms of p [22]. More than thirty years later it was shown by P. Rowley, using the classification of finite simple groups, that groups which admit a fixed-point-free automorphism are soluble [41]; thus generalising earlier results to the case where the automorphism has composite order. It is of great interest also to consider the case where the group acting on G is noncyclic. Indeed, it was conjectured that if a group A acts fixed-point-freely on the soluble group G (by which we mean $C_G(A) = 1$) then the Fitting height of G is bounded above by the length of the longest chain of subgroups in A. This is known as the Fitting height conjecture and has been largely settled in the case where A acts coprimely on G in a series of papers by T. Berger, A. Turull and many others (most of these results are collected in [45]). If we do not assume that A acts coprimely on G however, then things are much different. For instance, it was shown in [5] by S. D. Bell and B. Hartley that if A is a nonnilpotent group, then there is a group G of arbitrarily large Fitting height on which A

acts such that $C_G(A) = 1$. As such, there has been attention towards solving the Fitting height conjecture when A is nilpotent. For example K. Cheng [10] and much more recently by G. Ercan and İ. Güloğlu in [13] and [14].

In this chapter we follow the slightly different approach recently taken by E. Khukhro. In 2012 he proved the following:

Theorem 4.0.1. (Khukhro) *Suppose that a finite group G admits a Frobenius group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then:*

1. *$F_i(C_G(H)) = F_i(G) \cap C_G(H)$ for all i;*
2. *the Fitting height of G is equal to the Fitting height of $C_G(H)$.*

Proof. See [28, Theorem 2.1]. □

Since Frobenius kernels are nilpotent, the condition that $C_G(F) = 1$ forces G to be soluble due to a result by V. V. Belyaev and B. Hartley [6, Theorem 0.11]. Thus the Fitting height of G is well defined. So E. Khukhro is still considering the situation where a nilpotent group acts fixed-point-freely on a group G but there is also an 'additional' action which comes from the complement H; and indeed it is in terms of the action of this complement that we obtain structural information about G. Namely, that its Fitting height is equal to that of the fixed-point subgroup of H. This work by E. Khukhro led the author to ask the following question:

> Let $R \cong \mathbb{Z}_r$ for some prime r and let F be a nilpotent group on which R acts such that $F = [F, R]$. Suppose RF acts on a group G such that $C_G(F) = 1$. Then do we necessarily have that $F(C_G(R)) \subseteq F(G)$?

To make clear how this is related to the work by E. Khukhro we note that if RF is a Frobenius group with kernel F and complement R, then $F = [F, R]$. Also, if $F(C_G(R)) \subseteq F(G)$, then $F_i(C_G(R)) = F_i(G) \cap C_G(R)$ for all i and $f(G) = f(C_G(R))$ provided R has nontrivial fixed points on any RF-invariant section of G. In this chapter we consider the case when F is extraspecial and $(r, |G|) = 1$. In particular we prove the following:

Theorem 4.0.2. *Let $R \cong \mathbb{Z}_r$ for some prime r and $F \cong s^{1+2l}$. Suppose that R acts on F such that $F = [F, R]$. Suppose further that RF acts on a group G such that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $F(C_G(R)) \leq F(G)$.*

Before we prove this however, we require some preliminary results and set some notation.

4.1 Preliminary results

The results presented in this section are well-known. References for some of the results are given, but the author provides proofs of results which are difficult to find. No originality is claimed by the author in this section.

Lemma 4.1.1. *Let X be a group and $G \trianglelefteq X$. Then*

$$F(G/\Phi(X) \cap G) = F(G)/(\Phi(X) \cap G).$$

Proof. Set $\overline{G} = G/(\Phi(X) \cap G)$. The image of $F(G)$ in $\overline{G}$ is normal and nilpotent and thus

$$\overline{F(G)} \subseteq F(\overline{G}).$$

We work to show the reverse inclusion.

Let K be the full inverse image of $F(\overline{G})$ in G. Let $P \in \mathrm{Syl}_p(K)$. Now

$$\overline{P} = \mathcal{O}_p(\overline{G}) \trianglelefteq \overline{X},$$

which follows since $\mathcal{O}_p(\overline{G})\mathrm{char}\overline{G} \trianglelefteq \overline{X}$. Therefore

$$P(\Phi(X) \cap G) \trianglelefteq X,$$

and so by the Frattini argument

$$X = N_X(P)P(\Phi(X) \cap G).$$

Thus

$$X = N_X(P)\Phi(X)$$

and so $X = N_X(P)$. Therefore K is a normal nilpotent subgroup of G and so $K \subseteq F(G)$. So

$$F(\overline{G}) = \overline{K} \subseteq \overline{F(G)}.$$

□

Corollary 4.1.2. *Let $G \neq 1$ be a soluble group. Then $\Phi(G) \neq F(G)$.*

Proof. Let G be a counterexample. By Lemma 4.1.1 with $X = G$ we have

$$F(G/\Phi(G)) = F(G)/\Phi(G).$$

Now $F(G)/\Phi(G) = 1$ since $F(G) = \Phi(G)$. Thus $F(G/\Phi(G)) = 1$. However, since $G/\Phi(G)$ is soluble, we must have $G/\Phi(G) = 1$. However, this forces $G = \Phi(G)$, which is a contradiction since $G \neq 1$. □

Lemma 4.1.3. *Let G be a soluble group. Then $f(G) = f(G/\Phi(G))$.*

Proof. First note that $\Phi(G) \subseteq F(G)$ by Proposition 1.2.29. Also, $\Phi(G) \neq F(G)$. Set $\overline{G} = G/\Phi(G)$ and suppose $f(\overline{G}) \neq f(G)$. Then $f(\overline{G}) = f(G) - 1$. Now $F(\overline{G}) = \overline{F(G)}$ by Lemma 4.1.1 with $X = G$. Hence

$$f(G/F(G)) = f(\overline{G}/\overline{F(G)}) = f(G) - 2.$$

This is a contradiction, hence $f(G) = f(\overline{G})$. □

Proposition 4.1.4. *Let X be a group and $G \trianglelefteq X$. Then*

$$F(G)/(\Phi(X) \cap G)$$

is a completely reducible module for X possibly of mixed characteristic.

Proof. Set $\overline{X} = X/(\Phi(X) \cap G)$ and $\overline{F} = \overline{F(G)}$. Note that for all primes p that $\mathcal{O}_p(G) \trianglelefteq X$ and hence $\Phi(\mathcal{O}_p(G)) \subseteq \Phi(X)$ by Lemma 1.2.28. So $\overline{F}$ is a direct product of elementary abelian subgroups, each of which is an $\mathbb{F}_p[X]$-module for some prime p.

Suppose $\overline{V} \leq \overline{F}$ is a minimal normal subgroup in $\overline{X}$. Let V be the full inverse image of $\overline{V}$ in X. Then $V \nsubseteq \Phi(X) \cap G$. Note that $V \subseteq G$ by Lemma 4.1.1. Hence $V \nsubseteq \Phi(X)$, and so there exists a maximal subgroup $M \leq X$ such that $X = MV$ (note that MV is indeed a subgroup of X since $V \trianglelefteq X$). Now $\overline{X} = \overline{MV}$. Suppose $\overline{M} \cap \overline{V} \neq 1$. Then since $\overline{V}$ is minimal normal in $\overline{X}$, $\overline{V} \subseteq \overline{M}$, and so $\overline{X} = \overline{M}$. Hence

$$X = M(\Phi(X) \cap G) \subseteq M\Phi(X) = \Phi(X),$$

which is a contradiction. Thus $\overline{M} \cap \overline{V} = 1$. Now $\overline{V} \subseteq \overline{F}$ and so

$$\overline{F} = (\overline{F} \cap \overline{M})\overline{V}$$

by Dedekind's modular law. It is clear that $\overline{F} \cap \overline{M}$ is a complement to $\overline{V}$ in $\overline{F}$. We now claim that it is also X-invariant. It is $\overline{V}$-invariant since $\overline{F}$ is abelian and it is $\overline{M}$-invariant since it is normal in $\overline{M}$. Hence it is $\overline{X}$-invariant. By considering how X acts on $\overline{X}$, we see that $\overline{F} \cap \overline{M}$ is also X-invariant.

This shows that any minimal normal subgroup $\overline{V} \leq \overline{F}$ has an X-invariant complement in $\overline{F}$ and hence the claim follows. □

Lemma 4.1.5. *Suppose that a finite group G admits a group RF of automorphisms where RF is the split extension of the nilpotent group F by R. Suppose further that $C_G(F) = 1$. Then there is a unique RF-invariant Sylow p-subgroup of G for each prime $p \in \pi(G)$.*

Proof. See [29, Lemma 2.6]. This also uses [39, Theorem 9.5.9]. □

Theorem 4.1.6. (Belyeav–Hartley) *Let A be a finite nilpotent group which acts on a finite group G such that $C_G(A) = 1$. Then G is soluble.*

Proof. See [6, Theorem 0.11]. □

Definition 4.1.7. A *Carter subgroup* of a soluble group G is a nilpotent subgroup C such that $C = N_G(C)$.

Lemma 4.1.8. *Let G be a finite group admitting a nilpotent group F of automorphisms such that $C_G(F) = 1$. If N is a normal F-invariant subgroup of G, then $C_{G/N}(F) = 1$.*

Proof. This proof expands that of [29, Lemma 2.2].

We first claim that F is a Carter subgroup of GF. Suppose not, so $F \subset N_{GF}(F)$. Then

$$N_{GF}(F) \cap G \neq 1.$$

Let $1 \neq g \in N_{GF}(F) \cap G$. Then

$$[g, F] \subseteq F \cap G = 1.$$

Thus $1 \neq g \in C_G(F) = 1$. This is a contradiction, so F is indeed a Carter subgroup of GF.

By the correspondence theorem, the image of F under the canonical epimorphism

$$GF \longrightarrow GF/N$$

is a Carter subgroup of G/N. So $C_{G/N}(F) = 1$. □

Lemma 4.1.9. *Let G be a transitive permutation group on a set Ω, G_α the stabiliser of a point $\alpha \in \Omega$ and $P \in \mathrm{Syl}_p(G_\alpha)$. Then $N_G(P)$ is transitive on $\mathrm{Fix}_\Omega(P)$.*

Proof. Let $g \in N_G(P)$, $p \in P$ and $\beta \in \mathrm{Fix}_\Omega(P)$. Then

$$\beta gp = \beta p^{g^{-1}} g = \beta g.$$

So βg is fixed by P and therefore $N_G(P)$ acts on $\mathrm{Fix}_\Omega(P)$.

Let α, $\beta \in \mathrm{Fix}_\Omega(P)$. Since G acts transitively on Ω, there exists $g \in G$ such that $\beta g = \alpha$. Now α is fixed by P^g and so $P^g \leq G_\alpha$. By Sylow's theorem there exists a $h \in G_\alpha$ such that $P^g = P^h$. Thus $hg^{-1} \in N_G(P)$. Now

$$\alpha h g^{-1} = \alpha g^{-1} = \beta.$$

Therefore for all α, $\beta \in \mathrm{Fix}_\Omega(P)$, there exists $g \in N_G(P)$ such that $\alpha g = \beta$. □

If we have a vector space V such that

$$V = V_0 \oplus \ldots \oplus V_n,$$

then we write

$$V_0 \oplus \ldots \oplus \widehat{V_i} \oplus \ldots \oplus V_n$$

to denote the subspace

$$V_0 \oplus \ldots \oplus V_{i-1} \oplus V_{i+1} \oplus \ldots \oplus V_n.$$

Lemma 4.1.10. *Let RG be a group and V an irreducible RG-module on which G acts faithfully. Suppose*

$$V = V_0 \oplus \ldots \oplus V_n$$

where each V_i is a G-submodule of V. Let $H \leq RG$ be such that

$$H \subseteq C_G(\widehat{V_0} \oplus V_1 \ldots \oplus \ldots \oplus V_n)$$

and $G = \langle H^{RG} \rangle$. Then $G = G_0 \times \ldots \times G_n$ where $G_i = C_G(V_0 \oplus \ldots \oplus \widehat{V_i} \oplus \ldots \oplus V_n)$.

Proof. We work to show that $G \subseteq G_0 \ldots G_n$ since the reverse inclusion is clear.

First note that $H \subseteq G_0$. Let $x \in RG$ and suppose $V_0^x = V_i$. Let $h \in H$ and $v \in V_j \neq V_i$. Then $v^{x^{-1}} \notin V_0$ and so

$$[v^{x^{-1}}, h] = 1.$$

So $v^{h^x} = v$. Therefore $h^x \in G_i$, and so

$$G = \langle H^{RG} \rangle \subseteq G_0 \ldots G_n.$$

Note that each G_i is normal in G as the kernel of an action. Suppose there exists an i such that

$$G_i \bigcap \prod_{j \neq i} G_j \neq 1.$$

Let $1 \neq g \in G_i \bigcap \prod_{j \neq i} G_j$. Then g centralises $V_0 \oplus \ldots \widehat{V_i} \oplus \ldots \oplus V_n$ since $g \in G_i$, and it centralises V_i since $g \in \prod_{j \neq i} G_j$. Thus g is a nontrivial element of G which centralises V. This is a contradiction since V is a faithful G-module. Thus

$$G = G_i \times \prod_{j \neq i} G_j.$$

By induction it follows that

$$G = G_0 \times \ldots \times G_n.$$

□

Lemma 4.1.11. *Let G be a group which acts on a group $H = H_0 \times \ldots \times H_n$ such that $C_H(G) = 1$ and for each $H_i \in \{H_0, \ldots, H_n\}$ and $g \in G$ we have $H_i^g \in \{H_0, \ldots, H_n\}$. Let $G_0 = N_G(H_0)$. Then $C_{H_0}(G_0) = 1$.*

Proof. Note that by induction, we may assume that G acts transitively on $\{H_0, \ldots, H_n\}$. Now let $T = \{1 = g_0, g_1, \ldots, g_n\}$ be a set of representatives for the right cosets of G_0 in G. Suppose $C_{H_0}(G_0) \neq 1$ and choose $1 \neq h \in C_{H_0}(G_0)$. Let

$$\hat{h} = \prod_{i=0}^{n} h^{g_i}.$$

We claim that $\hat{h}$ is fixed by G.

First note that elements in a common coset of G_0 in G act in the same way on h. Let $g_i' \in G_0 g_i$ so $g_i' = g_0 g_i$ for some $g_0 \in G_0$. Then

$$h^{g_i'} = h^{g_0 g_i} = h^{g_i}.$$

Now notice that the h^{g_i} commute. This follows since for distinct g_i, $g_j \in T$, h^{g_i} and h^{g_j} lie in distinct H_k. Suppose this is not the case so there exist $g_i \neq g_j$ such that h^{g_i}, $h^{g_j} \in H_k$ for some $k \in \{0, \ldots, n\}$. Then

$$h^{g_i g_j^{-1}} \in H_0.$$

So $g_i g_j^{-1} \in G_0$ and thus $g_i \in G_0 g_j$. However this contradicts the fact that g_i and g_j are distinct representatives for the cosets of G_0 in G.

Since for each $g_i \in T$ the set

$$Tg_i = \{tg_i \mid t \in T\}$$

is another set of representatives for the cosets of G_0 in G, it follows that each member of T centralises $\hat{h}$. Therefore every element of G must also centralise $\hat{h}$ by the argument above. This of course forces $C_H(G) \neq 1$ since $1 \neq \hat{h}$, which is a contradiction since $C_H(G) = 1$. □

Theorem 4.1.12. (Flavell) *Let r be a prime, $R \cong \mathbb{Z}_r$ and P an r'-group on which R acts. Let V be a faithful irreducible RP-module over a field of characteristic p such that $C_V(R) = 0$. Then either:*

1. $[R, P] = 1$*; or*
2. $[R, P]$ *is a nonabelian special* 2*-group and* $r = 2^n + 1$ *for some* $n \in \mathbb{N}$.

Proof. See [19, Theorem A]. □

4.2 Proof of the main theorem

Let $R \cong \mathbb{Z}_r$ for some prime r, act on the extraspecial group $F \cong s^{1+2l}$ such that $F = [F, R]$. Then we must have that both $r \neq s$ and $C_F(R) \subseteq \Phi(F)$. The former holds by [25, Lemma 4.32]. The latter can be seen as follows. Set $\overline{F} = F/\Phi(F)$. Then $\overline{F} = [\overline{F}, R]$. However, $\overline{F}$ is abelian since F is nilpotent and thus since R acts coprimely on $\overline{F}$, it follows that $C_{\overline{F}}(R) = 1$. However, we also have that $\overline{C_F(R)} = C_{\overline{F}}(R)$ by coprime action and thus $C_F(R) \subseteq \Phi(F)$. In what follows we will show that if RF acts on a group G such that $C_G(F) = 1$, then $F(C_G(R)) \subseteq F(G)$. The proof will proceed by considering a minimal counterexample RFG. Thus we must have that $C_F(R) = Z(F)$. Otherwise $C_F(R) = 1$; but we know that a counterexample does not exist in this case by Theorem 4.0.1. Hence, we will establish Theorem 4.0.2 by proving the following:

Theorem 4.2.1. *Let $R \cong \mathbb{Z}_r$ for some prime r and $F \cong s^{1+2l}$. Suppose that R acts on F such that $[R, Z(F)] = 1$ and $RF/Z(F)$ is a Frobenius group. Suppose further that RF acts on a group G such that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $F(C_G(R)) \leq F(G)$.*

Proof. First note that F is nilpotent since it is extraspecial. Thus the condition that $C_G(F) = 1$ forces G to be soluble by Theorem 4.1.6. We begin by considering a counterexample with $|RFG|$ minimal. So $F(C_G(R)) \notin F(G)$. For notational purposes set $X = RFG$, so $G \trianglelefteq X$.

Lemma 4.2.2. *With G and X as above we have that $F(G)$ is a completely reducible X-module.*

Proof. We know by Proposition 4.1.4 that $F(G)/(\Phi(X) \cap G)$ is a completely reducible module for X. We work to show that $\Phi(X) \cap G = 1$.

Suppose that $\Phi(X) \cap G \neq 1$ and set $\overline{G} = G/(\Phi(X) \cap G)$. By minimality we have that

$$F(C_{\overline{G}}(R)) \leq F(\overline{G}).$$

We also have by Lemma 4.1.1 that

$$\overline{F(G)} = F(\overline{G}).$$

Now

$$\overline{F(C_G(R))} \leq F(C_{\overline{G}}(R))$$

and so

$$\overline{F(C_G(R))} \leq \overline{F(G)}.$$

If we consider the full inverse image of both $\overline{F(C_G(R))}$ and $\overline{F(G)}$ in G, then

$$F(C_G(R))\,(\Phi(X) \cap G) \leq F(G)\,(\Phi(X) \cap G) = F(G)$$

where the equality on the right follows since $\Phi(X)$ is a normal nilpotent subgroup of X, and so $\Phi(X) \cap G$ is a normal nilpotent subgroup of G and thus contained in $F(G)$. However, this is a contradiction since $F(C_G(R)) \nsubseteq F(G)$. Thus $\Phi(X) \cap G = 1$ and the result follows. □

Lemma 4.2.3. *There exists a prime p such that $F(G) = \mathcal{O}_p(G)$ is an irreducible X-module.*

Proof. We know from Lemma 4.2.2 that $F(G)$ is a completely reducible X-module. An irreducible constituent of $F(G)$ will be minimal normal in G with respect to being X-invariant.

Suppose that $F(G)$ is not an irreducible X-module and let U and V denote two distinct irreducible X-submodules. Consider the map

$$\phi : G \longrightarrow G/U \times G/V$$

given by

$$g \longmapsto (gU, gV).$$

Then ϕ is a homomorphism with kernel $U \cap V$. Now $U \cap V$ is X-invariant and contained in both U and V. Thus since both U and V are irreducible and are distinct, $U \cap V = 1$. Thus ϕ is injective and so G embeds into $G/U \times G/V$.

Now let $\overline{G} = G/U$. Then

$$\overline{F(C_G(R))} \leq F(C_{\overline{G}}(R)) \leq F(\overline{G})$$

where the inclusion on the right both follows by minimality. Thus it follows that

$$\overline{\langle F(C_G(R))^G \rangle} \leq F(\overline{G}).$$

Similarly, if we set $\overline{G} = G/V$, then $\overline{\langle F(C_G(R))^G \rangle} \leq F(\overline{G})$. So the image of $\langle F(C_G(R))^G \rangle$ under ϕ is nilpotent. However, since ϕ is injective, we have that $\langle F(C_G(R))^G \rangle$ must also be nilpotent. So

$$\langle F(C_G(R))^G \rangle \subseteq F(G)$$

since $\langle F(C_G(R))^G \rangle \trianglelefteq G$. This is a contradiction since $F(C_G(R)) \nsubseteq F(G)$. Thus $F(G)$ contains only one irreducible X-submodule and so the result follows since $F(G)$ is a completely reducible X-module. □

For notational purposes set $F(G) = V$.

Lemma 4.2.4. *There exists a nontrivial RF-invariant Sylow q-subgroup Q of G such that $G = QV$ for some prime $q \neq p$.*

Proof. Set $\overline{G} = G/V$. By minimality we have

$$\overline{F(C_G(R))} \leq F(\overline{G}).$$

Now $F(C_G(R)) \nsubseteq F(G)$ and so there exists a prime $q \neq p$ such that

$$\overline{\mathcal{O}_q(C_G(R))} \neq 1.$$

By the above we have that

$$\overline{\mathcal{O}_q(C_G(R))} \leq \mathcal{O}_q(\overline{G}).$$

Let K denote the full inverse image of $\mathcal{O}_q(\overline{G})$ in G. So

$$\mathcal{O}_q(C_G(R)) \subseteq K \trianglelefteq RFG.$$

Now K is certainly RF-invariant and $C_K(F) = 1$, and so by Lemma 4.1.5 there exists a unique RF-invariant Sylow q-subgroup Q of K. So $K = QV$. However, $F(K) = V$ and so by minimality it follows that $G = K$. □

Lemma 4.2.5. *Let $1 \neq H \leq \mathcal{O}_q(C_G(R))$. Then $Q = \langle H^F \rangle$.*

Proof. By Lemma 4.2.4, $G = QV$ where Q is an RF-invariant Sylow q-subgroup of G. In particular, Q is an R-invariant Sylow q-subgroup of G. Now $\mathcal{O}_q(C_G(R))$ is an R invariant q-subgroup of G and so by coprime action it is contained in some R-invariant Sylow q-subgroup of G. We know that $C_G(R)$ acts transitively by conjugation on R-invariant Sylow q-subgroups of G and so since $\mathcal{O}_q(C_G(R)) \trianglelefteq C_G(R)$, it follows that $\mathcal{O}_q(C_G(R))$ must be contained in every R-invariant Sylow q-subgroup of G. Thus $\mathcal{O}_q(C_G(R)) \leq Q$.

Let $Q_0 = \langle H^{RF} \rangle$. Then $Q_0 = \langle H^F \rangle$ since H is centralised by R. Suppose $Q_0 < Q$ and set $G_0 = Q_0V$. Now $C_G(V) = V$ and so $\mathcal{O}_q(G_0) = 1$. By minimality

$$F(C_{G_0}(R)) \leq F(G_0) = V.$$

However,

$$1 \neq H \subseteq F(C_{G_0}(R)).$$

This contradiction forces

$$Q_0V = G_0 = G = QV.$$

and so $Q_0 = Q$. □

We may consider V as an irreducible $\mathbb{F}_p[RFQ]$-module. We now extend the ground field to a splitting field k for RFQ and consider the following $k[RFQ]$-module

$$W = V \otimes_{\mathbb{F}_p} k.$$

Let $\overline{V}$ be an irreducible $k[RFQ]$-submodule of W.

Lemma 4.2.6. *Q acts faithfully on $\overline{V}$ and $C_{\overline{V}}(F) = 0$.*

Proof. Suppose Q does not act faithfully on $\overline{V}$. Then there exists $1 \neq K \subseteq Q$ with $K \trianglelefteq RFQ$ such that $C_{\overline{V}}(K) = \overline{V}$. Now

$$C_{\overline{V}}(K) \subseteq C_W(K) = C_V(K) \otimes_{\mathbb{F}_p} k,$$

and so $C_V(K) \neq 0$. Since $K \trianglelefteq RFQ$ then $C_V(K)$ is normalised by RFQ. By the irreducibility of V we have $C_V(K) = V$. However, Q acts faithfully on V. This is a contradiction and thus Q acts faithfully on $\overline{V}$.

The second claim follows directly from part 2 of Lemma 1.8.3 with F in place of G. □

Lemma 4.2.7. $[C_{\overline{V}}(R), \mathcal{O}_q(C_G(R))] = 0.$

Proof. Now

$$C_G(R) = C_V(R)C_Q(R) \text{ and } C_V(R) \trianglelefteq C_G(R).$$

Thus

$$[C_V(R), \mathcal{O}_q(C_G(R))] = C_V(R) \cap \mathcal{O}_q(C_G(R)) = 1.$$

By considering $C_V(R)$ as an $\mathbb{F}_p[\mathcal{O}_q(C_G(R))]$-module, we have that

$$[C_W(R), \mathcal{O}_q(C_G(R))] = 0$$

by Lemma 1.8.3. Since $C_{\overline{V}}(R) \subseteq C_W(R)$, we have that

$$[C_{\overline{V}}(R), \mathcal{O}_q(C_G(R))] = 0.$$

□

Lemma 4.2.8. *Suppose $\overline{V}$ is an imprimitive module for RFQ. Then $\mathcal{O}_q(C_G(R))$ centralises any block which is not normalised by R.*

Proof. Let

$$\overline{V} = U_0 \oplus \ldots \oplus U_n$$

where the U_i are blocks of imprimitivity in the action of RFQ on $\overline{V}$, and set $\Omega = \{U_0, \ldots, U_n\}$. Let $R = \langle a \rangle$. We want to show that $\mathcal{O}_q(C_G(R))$ centralises

$$U = \bigoplus_{U_i \in \mathrm{Mov}_\Omega(R)} U_i.$$

We must first show that $\mathcal{O}_q(C_G(R))$ acts on $\mathrm{Mov}_\Omega(R)$. Let $U_i \in \mathrm{Mov}_\Omega(R)$ and $g \in \mathcal{O}_q(C_G(R))$. Then

$$U_i^{ga} = U_i^{ag} = U_j^g$$

where $U_j \in \mathrm{Mov}_\Omega(R)$. So since $U_i^g \neq U_j^g$, we have that $U_i^g \in \mathrm{Mov}_\Omega(R)$, and so $\mathcal{O}_q(C_G(R))$ indeed acts on $\mathrm{Mov}_\Omega(R)$.

Let $U_i \in \mathrm{Mov}_\Omega(R)$ and consider

$$U' = \bigoplus_{j=1}^{r} U_i^{a^j}.$$

Then for $u \in V_i$,

$$w = u + u^a + \ldots + u^{a^{r-1}}$$

is centralised by R. Thus it is also centralised by $\mathcal{O}_q(C_G(R))$. Hence U' is normalised by $\mathcal{O}_q(C_G(R))$. The orbit of U_i under the action of $R \times \mathcal{O}_q(C_G(R))$ has length r and thus $R \times \mathcal{O}_q(C_G(R))$ acts primitively on the orbit of U_i since r is prime. Now $\mathcal{O}_q(C_G(R)) \trianglelefteq R \times \mathcal{O}_q(C_G(R))$ and so its action is either trivial or transitive on the orbit of U_i. Since $\mathcal{O}_q(C_G(R))$ is a q-group, we see that this action must be trivial, hence, $\mathcal{O}_q(C_G(R))$ normalises any block in the orbit of U_i under the action of $R \times \mathcal{O}_q(C_G(R))$. Since w is centralised by $\mathcal{O}_q(C_G(R))$ and $\mathcal{O}_q(C_G(R))$ normalises U_i, it must centralise u. Since u was chosen arbitrarily in U_i, which was chosen arbitrarily in $\mathrm{Mov}_\Omega(R)$, it follows that $\mathcal{O}_q(C_G(R))$ centralises all $U_i \in \mathrm{Mov}_\Omega(R)$. □

Henceforth, we will write

$$\overline{V} = V_0 \oplus \ldots \oplus V_n$$

where the V_i are the homogeneous components with respect to $Z(Q)$. Set $\Gamma = \{V_0, \ldots, V_n\}$.

Our next major goal is to prove that $[Z(Q), Z(F)] \neq 1$. We thus proceed with the assumption that this is not the case and work to obtain a contradiction. We first need a few lemmas.

Lemma 4.2.9. *Assume $[Z(Q), Z(F)] = 1$. Then R has only one fixed point on Γ.*

Proof. Let $R = \langle a \rangle$. By Lemma 4.2.8 we have that $\mathcal{O}_q(C_G(R))$ centralises all of the subspaces $V_i \in \mathrm{Mov}_\Gamma(R)$. Now $C_Q(\overline{V}) = 1$ by Lemma 4.2.6 and so we have the strict inclusion $\mathrm{Mov}_\Gamma(R) \subset \Gamma$. Now $\mathrm{Fix}_\Gamma(R) \neq \varnothing$, hence R is in the stabiliser of a point in the action of RFQ on Γ. Since $R \in \mathrm{Syl}_r(RFQ)$, R is a Sylow r-subgroup of this stabiliser. Thus $N_{RFQ}(R)$ acts transitively on $\mathrm{Fix}_\Gamma(R)$ by Lemma 4.1.9. Now $N_{RFQ}(R) = RZ(F)C_Q(R)$. Clearly R acts trivially on $\mathrm{Fix}_\Gamma(R)$. Now $[C_Q(R), Z(Q)] = 1$ and by hypothesis we have $[Z(Q), Z(F)] = 1$. Therefore $Z(F)C_Q(R) \subseteq C_{RFQ}(Z(Q))$. Thus by Clifford's Theorem, $Z(F)C_Q(R)$ acts trivially on Γ. In particular, $Z(F)C_Q(R)$ acts trivially on $\mathrm{Fix}_\Gamma(R)$ and so $|\mathrm{Fix}_\Gamma(R)| = 1$. □

Recall that if we have a vector space

$$U = U_0 \oplus \ldots \oplus U_n,$$

then we write

$$U_0 \oplus \ldots \oplus \widehat{U_i} \oplus \ldots \oplus U_n$$

to denote the subspace

$$U_0 \oplus \ldots \oplus U_{i-1} \oplus U_{i+1} \oplus \ldots \oplus U_n.$$

In the following lemma let $Q_i = C_Q(V_0 \oplus \ldots \oplus \widehat{V_i} \oplus \ldots \oplus V_n)$.

Lemma 4.2.10. *If* $[Z(Q), Z(F)] = 1$, *then* $Q = Q_0 \times \ldots \times Q_n$.

Proof. We can assume without loss of generality that $\mathrm{Fix}_\Gamma(R) = \{V_0\}$. Since R has no fixed points on $\Gamma - \{V_0\}$, it follows by Lemma 4.2.8 that $V_1 \oplus \ldots \oplus V_n$ is centralised by $\mathcal{O}_q(C_G(R))$. Thus $\mathcal{O}_q(C_G(R)) \subseteq Q_0$. The result now follows from Lemma 4.1.10 with Q and $\overline{V}$ in place of G and V respectively. □

Let $F_0 = N_F(V_0)$. Then $F_0 \neq 1$, otherwise V_0 would be in a regular orbit under the action of F on Γ. Thus $\langle V_0^F \rangle$ would be a free F-module and so $C_{\overline{V}}(F) \neq 1$, contrary to Lemma 4.2.6.

Lemma 4.2.11. *If* $[Z(Q), Z(F)] = 1$ *then* $Q_0 = \langle \mathcal{O}_q(C_G(R))^{F_0} \rangle$.

Proof. Let $f \in F$ and suppose $V_0^f = V_i$. Let $g \in \mathcal{O}_q(C_G(R))$ and $v \in V_j \neq V_i$. Then $v^{f^{-1}} \notin V_0$ and so

$$[v^{f^{-1}}, g] = 1.$$

So $v^{g^f} = v$. Therefore $g^f \in Q_i$ and so $\mathcal{O}_q(C_G(R))^f \subseteq Q_0$ if and only if $f \in F_0$.

Now

$$Q/(Q_1 \times \ldots \times Q_n) \cong Q_0$$

and

$$\langle \mathcal{O}_q(C_G(R))^{F-F_0}\rangle \subseteq Q_1 \times \ldots \times Q_n.$$

So if we consider the canonical epimorphism

$$\phi : Q \longrightarrow Q/(Q_1 \times \ldots \times Q_n),$$

it follows that $\langle \mathcal{O}_q(C_G(R))^{F_0}\rangle$ maps onto $Q/Q_1 \times \ldots \times Q_n$ under ϕ. So

$$Q = \langle \mathcal{O}_q(C_G(R))^{F_0}\rangle \times Q_1 \times \ldots \times Q_n.$$

By considering orders it follows that $|Q_0| = |\langle \mathcal{O}_q(C_G(R))^{F_0}\rangle|$ and so $Q_0 = \langle \mathcal{O}_q(C_G(R))^{F_0}\rangle$. □

Lemma 4.2.12. *If* $[Z(Q), Z(F)] = 1$, *then* $C_{Q_0}(F_0) = 1$.

Proof. By noting that $C_Q(F) = 1$, then this follows by Lemma 4.1.11 with F and Q in place of G and H respectively. □

Lemma 4.2.13. $[Z(Q), Z(F)] \neq 1$

Proof. Assume that this is not the case so $[Z(Q), Z(F)] = 1$. Then

$$Q \cong Q_0 \times \ldots \times Q_n$$

where the Q_i are defined as in Lemma 4.2.10, and $C_{Q_0}(F_0) = 1$ as in Lemma 4.2.12.

Now since the V_i are homogeneous components for $Z(Q)$ and k is a splitting field for $Z(Q)$, $Z(Q)$ acts on V_0 by a scalar $\lambda \in k$. However,

$$Z(Q) = Z(Q_0) \times \ldots \times Z(Q_n)$$

and

$$Z(Q_1) \times \ldots \times Z(Q_n)$$

acts trivially on V_0. So $Z(Q_0)$ acts on V_0 by λ and we must have that $\lambda \neq 1$ otherwise $C_Q(\overline{V}) \neq 1$ as $Z(Q_0) \neq 1$. This follows since

$$1 \neq \mathcal{O}_q(C_G(R)) \subseteq Q_0.$$

Now F_0 normalises Q_0 by Lemma 4.2.11 and so we can think about the action of $[F_0, Z(Q_0)]$ on V_0. We know that $Z(Q_0)$ acts by scalars on V_0 and so every element in $[F_0, Z(Q_0)]$ acts on V_0 trivially. However, since Q_0 acts faithfully on V_0 and $[F_0, Z(Q_0)] \subseteq Z(Q_0)$, F_0 must centralise $Z(Q_0)$ and thus

$$1 \neq C_{Q_0}(F_0).$$

This is a contradiction to Lemma 4.2.12. □

Corollary 4.2.14. *$Z(F)$ acts semiregularly Γ.*

Proof. Suppose $Z(F)$ normalises some $V_j \in \Gamma$. Then since RF is transitive on Γ and $Z(F) = Z(RF)$, we have that $Z(F)$ acts trivially on Γ. Now $Z(Q)$ acts on each $V_i \in \Gamma$ by scalars and so $[Z(F), Z(Q)]$ must act trivially on each $V_i \in \Gamma$. This forces $[Z(F), Z(Q)] = 1$ since $C_Q(\overline{V}) = 1$. This is a contradiction to Lemma 4.2.13. □

Lemma 4.2.15. *Q acts trivially on any system of imprimitivity in the action of RFQ on $\overline{V}$.*

Proof. Let

$$\overline{V} = U_0 \oplus \ldots \oplus U_n$$

where the U_i are blocks of imprimitivity in the action of RFQ on $\overline{V}$, and set $\Omega = \{U_0, \ldots, U_n\}$. We work to show that $\mathcal{O}_q(C_G(R))$ acts trivially on Ω. Then the normal closure of $\mathcal{O}_q(C_G(R))$ in RFQ will also act trivially Ω. Since $\langle \mathcal{O}_q(C_G(R))^{RFQ} \rangle = Q$, the claim will follow.

Let $R = \langle a \rangle$. Then $\mathcal{O}_q(C_G(R))$ centralises any $U_i \in \mathrm{Mov}_\Omega(R)$ by Lemma 4.2.8. Also, as in the proof of Lemma 4.2.9, we get that $\mathrm{Fix}_\Omega(R) \neq \varnothing$ and $N_{RFQ}(R)$ is transitive on $\mathrm{Fix}_\Omega(R)$. Now

$$N_{RFQ}(R) = RZ(F)C_Q(R).$$

Clearly, R acts trivially on $\mathrm{Fix}_\Omega(R)$. Let $U_j \in \mathrm{Fix}_\Omega(R)$ and suppose $Z(F) \notin N_F(U_j)$. Then

$$N_F(U_j) \cap Z(F) = 1$$

since $Z(F)$ is cyclic of prime order. In particular, R acts semiregularly on $N_F(U_j)$ because $C_F(R) = Z(F)$. Note that $N_F(U_j) \neq 1$, since otherwise F would have a regular orbit on Ω and thus a nontrivial fixed point on $\overline{V}$, contrary to Lemma 4.2.6. Therefore $N_F(U_j)$ must be elementary abelian since it is isomorphic to its image under the canonical epimorphism

$$\varphi : F \longrightarrow F/Z(F).$$

Also $C_{U_j}(N_F(U_j)) = 0$ by Lemma 4.1.11. Hence $C_{U_j}(R) \neq 0$ by Theorem 1.8.17. Thus by Lemma 4.2.7, $\mathcal{O}_q(C_G(R))$ normalises U_j.

Suppose that $\mathcal{O}_q(C_G(R))$ does not normalise $U_j \in \mathrm{Fix}_\Omega(R)$. Then reasoning as above we must have $Z(F) \subseteq N_F(U_j)$ and $C_{U_j}(R) = 0$. Thus $C_Q(R)$ can only map U_j to a subspace $U_i \in \mathrm{Fix}_\Omega(R)$ which itself is normalised by $Z(F)$. So $Z(F)$ must act trivially on $\mathrm{Fix}_\Omega(R)$, otherwise we get two distinct orbits in the action of $N_{RFQ}(R)$ on $\mathrm{Fix}_\Omega(R)$. Since $Z(F)$ is trivial on $\mathrm{Fix}_\Omega(R)$,

$$Z = [\mathcal{O}_q(C_G(R)), Z(F)]$$

is also trivial on $\mathrm{Fix}_{\Omega}(R)$. Also, Z centralises each subspace $U_j \in \mathrm{Mov}_{\Omega}(R)$ since $Z \subseteq \mathcal{O}_q(C_G(R))$, and so Z acts trivially on Ω. Note that $Z \neq 1$ since $[Z(Q), Z(F)] \neq 1$ and so by Lemma 4.2.5 we have that $Q = \langle Z^F \rangle$. Thus it follows that Q also acts trivially on Ω. This is a contradiction since U_j is not normalised by $\mathcal{O}_q(C_G(R))$. □

Corollary 4.2.16. *Every characteristic abelian subgroup of Q is contained in $Z(Q)$.*

Proof. Let A be a characteristic abelian subgroup of Q. Let

$$\overline{V} = U_0 \oplus \ldots \oplus U_n$$

where the U_i are homogeneous components with respect to A. Then $\Omega = \{U_0, \ldots, U_n\}$ is a system of imprimitivity for RFQ on $\overline{V}$, and so Q is trivial on Ω. Since A acts by scalars on any given $U_i \in \Omega$, $[Q, A]$ centralises $\overline{V}$. This forces $[Q, A] = 1$ and thus $A \subseteq Z(Q)$. □

Corollary 4.2.17. *Q has nilpotence class at most two.*

Proof. Since every characteristic abelian subgroup of Q is contained in $Z(Q)$, $Z(\Phi(Q)) \subseteq Z(Q)$. Thus $\Phi(Q) \subseteq Z(Q)$ by Lemma 1.2.32 and so $Q/Z(Q)$ is abelian. □

Recall that Γ is the set of $Z(Q)$-homogeneous components in $\overline{V}$. We know that the subset of components in Γ which are normalised by R is nonempty and that $N_{RF}(R) = R \times Z(F)$ acts transitively on this set. We also know, since $Z(F) \trianglelefteq RF$, that the orbits of the action of $Z(F)$ on Γ forms a system of imprimitivity

$$\overline{V} = W_0 \oplus \ldots \oplus W_m$$

for the action of RF on Γ. We can assume without loss of generality that V_0 is normalised by R and that W_0 is the direct sum of components in the orbit of V_0 under the action of $Z(F)$ on Γ. We also set

$$Q_i = C_{\overline{V}}(W_0 \oplus \ldots \oplus \widehat{W_i} \oplus \ldots \oplus W_m)$$

and find that $Q = Q_0 \times \ldots \times Q_m$, which follows exactly as in Lemma 4.2.10.

Lemma 4.2.18. $Q_0 = \langle \mathcal{O}_q(C_G(R))^{N_F(V_0)} \rangle$.

Proof. We first show that $N_F(W_0) = Z(F) \times N_F(V_0)$. We can assume without loss of generality that

$$W_0 = V_0 \oplus \ldots \oplus V_{s-1}$$

and set $\Delta = \{V_0, \dots, V_{s-1}\}$. By definition, $Z(F)$ is contained in $N_F(W_0)$ and is transitive on Δ. In particular, since $|\Delta| = s$, $N_F(W_0)$ is primitive on Δ. Since $N_F(W_0)$ is an s-group and $|\Delta| = s$, $N_F(V_0)$ must be the full kernel in the action of $N_F(W_0)$ on Δ. We find that $N_F(W_0)/N_F(V_0)$ is regular on Δ and so $N_F(W_0)/N_F(V_0) \cong \mathbb{Z}_s$. Thus it follows

$$N_F(W_0) = Z(F) \times N_F(V_0).$$

Arguing exactly as in the proof of Lemma 4.2.11 we find

$$Q_0 = \langle \mathcal{O}_q(C_G(R))^{N_F(W_0)} \rangle.$$

Now $[R, Z(F)] = 1$ and so $\mathcal{O}_q(C_G(R))^{Z(F)} = \mathcal{O}_q(C_G(R))$. Thus

$$Q_0 = \langle \mathcal{O}_q(C_G(R))^{Z(F) \times N_F(V_0)} \rangle = \langle \mathcal{O}_q(C_G(R))^{N_F(V_0)} \rangle.$$

□

Lemma 4.2.19. $[Z(Q_0), R] = 1$.

Proof. Since the subspaces $V_i \subseteq \overline{V}$ are homogeneous components for $Z(Q)$, $Z(Q_0)$ acts on them by scalars. Now W_0 is the direct sum of components which are normalised by R. Since $Z(Q_0)$ acts by scalars on any given $V_i \subseteq W_0$, $[Z(Q_0), R]$ acts trivially on W_0. However, Q is faithful on $\overline{V}$ and since Q_0 centralises $W_1 \oplus \dots \oplus W_m$, this forces $[Z(Q_0), R] = 1$. □

Lemma 4.2.20. *Q is abelian.*

Proof. Note that

$$Q' \cap \mathcal{O}_q(C_G(R)) = 1.$$

If this is not the case, then

$$Q = \langle (Q' \cap \mathcal{O}_q(C_G(R)))^{RF} \rangle \subseteq Q'$$

where the equality on the left follows by Lemma 4.2.5 and the inclusion on the right since Q' is characteristic in Q. However, this is a contradiction since Q is nontrivial and nilpotent.

It follows that

$$[\mathcal{O}_q(C_G(R)), C_Q(R)] \subseteq Q' \cap \mathcal{O}_q(C_G(R)) = 1.$$

Thus

$$\mathcal{O}_q(C_G(R)) \subseteq Z(C_Q(R)).$$

By Lemma 4.2.19, we have $[Z(Q_0), R] = 1$ and so since Q has nilpotence class at most two,

$$Q_0 = [Q_0, R] * C_{Q_0}(R).$$

However, since $\mathcal{O}_q(C_G(R)) \subseteq Z(C_Q(R))$, we have that

$$\mathcal{O}_q(C_G(R)) \subseteq Z(C_{Q_0}(R)),$$

and so

$$\mathcal{O}_q(C_G(R)) \subseteq Z(Q_0) \subseteq Z(Q).$$

Set $G_0 = Z(Q)V$. If $G_0 < G$, then by induction

$$F(C_{G_0}(R)) \subseteq F(G_0) = V.$$

However, since $\mathcal{O}_q(C_G(R)) \subseteq Z(Q)$, there are clearly q-elements contained in $F(C_{G_0}(R))$. Thus $G_0 = G$ and so $Z(Q) = Q$. □

It follows from Corollary 4.2.14 that $Z(F) \nsubseteq N_F(V_i)$ for any $V_i \in \Gamma$. Thus $Z(F) \cap N_F(V_0) = 1$ since $Z(F)$ is cyclic of prime order. Hence

$$N_F(V_0) = [R, N_F(V_0)].$$

Since Q is abelian, Lemma 4.2.19 now says that $[Q_0, R] = 1$, and so

$$[Q_0, N_F(V_0)] = 1.$$

Thus it follows

$$Q_0 = \langle \mathcal{O}_q(C_G(R))^{N_F(V_0)} \rangle = \mathcal{O}_q(C_G(R)).$$

Now $N_F(V_0)$ is abelian and $C_{V_0}(N_F(V_0)) = 0$ by Lemma 4.1.11. Hence $C_{V_0}(R) \neq 0$ by Theorem 1.8.17. Then since $[\mathcal{O}_q(C_G(R)), C_V(R)] = 1$, we must have that Q_0 acts trivially on $C_{V_0}(R)$. However, V_0 is a homogeneous component for Q_0 and so Q_0 must act trivially on V_0. It follows then that Q_0 acts trivially on W_0 and thus Q_0 acts trivially on $\overline{V}$. This is a contradiction since $C_Q(\overline{V}) = 1$. This completes the proof of Theorem 4.2.1. □

Corollary 4.2.21. *Let $R \cong \mathbb{Z}_r$ for some prime r and $F \cong s^{1+2l}$. Suppose that R acts on F such that $[R, Z(F)] = 1$ and $RF/Z(F)$ is a Frobenius group. Suppose further that RF acts on a group G such that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $F_i(C_G(R)) = F_i(G) \cap C_G(R)$ for all i.*

Proof. Let $i \in \mathbb{N}$ be the least such that

$$F_i(C_G(R)) \nsubseteq F_i(G).$$

We know that $F(C_G(R)) \leq F(G)$ by Theorem 4.2.1 and so $i > 1$. Let $\overline{G} = G/F_{i-1}(G)$ and ϕ be the canonical epimorphism from G onto $\overline{G}$. Now

$$\overline{F_i(C_G(R))} \trianglelefteq \overline{C_G(R)}$$

and $\overline{F_i(C_G(R))}$ is nilpotent since $F_{i-1}(C_G(R)) \subseteq \ker(\phi)$. Now

$$\overline{F_i(C_G(R))} \leq F(\overline{C_G(R)}) = F(C_{\overline{G}}(R)) \leq F(\overline{G}).$$

By the definition of $F_i(G)$ we have that

$$F(\overline{G}) = \overline{F_i(G)}.$$

Therefore

$$\overline{F_i(C_G(R))} \subseteq \overline{F_i(G)}.$$

However, this is a contradiction since $F_i(C_G(R)) \nsubseteq F_i(G)$. □

Corollary 4.2.22. *Let $R \cong \mathbb{Z}_r$ for some prime r and $F \cong s^{1+2l}$ such that $r \neq s^l + 1$. Suppose that R acts on F such that $[R, Z(F)] = 1$ and $RF/Z(F)$ is a Frobenius group. Suppose further that RF acts on a group G such that $C_G(F) = 1$ and $(r, |G|) = 1$. Then $f(C_G(R)) = f(G)$.*

Proof. Since $C_G(R) \leq G$, $f(C_G(R)) \leq f(G)$. So it will suffice to show that $f(G) \leq f(C_G(R))$.

Let $n \in \mathbb{N}$ be the Fitting height of $C_G(R)$ so $F_n(C_G(R)) = C_G(R)$. Now we know that

$$F_n(C_G(R)) = F_n(G) \cap C_G(R)$$

and so we have that $C_G(R) \leq F_n(G)$. We work to show that $F_n(G) = G$. Suppose this is not the case so $F_n(G) < G$. Let S be an RF-invariant section of $G/F_n(G)$ which has no proper RF-invariant subgroups. Since $F_n(G)\text{char}G$, $F_n(G)$ is a normal F-invariant subgroup of G. Thus $C_{G/F_n(G)}(F) = 1$ by Lemma 4.1.8. We can consider S as an irreducible $\mathbb{F}_p[RF]$-module for some prime p. Now R acts coprimely on G and so since $C_G(R) \subseteq F_n(G)$, it follows that R acts fixed point freely on $G/F_n(G)$.

Suppose $C_F(S) \neq 1$. Now, F acts nontrivially on S and so $F/C_F(S) \neq 1$. Also, since $C_F(S) \trianglelefteq F$ and $Z(F)$ is cyclic of prime order, then $Z(F) \subseteq C_F(S)$. It follows by coprime action that R acts semiregularly on $F/C_F(S)$. By Theorem 1.8.17, S is free as an R-module. However, this is a contradiction since R acts fixed-point-freely on S.

Thus $C_F(S) = 1$. By Theorem 1.8.18, R can only act fixed-point-freely on S if $r = s^l + 1$. However, this does not hold by hypothesis. □

It should be noted that E. Khukhro first proved Theorem 4.0.1 with the added hypothesis that $(|H|,|G|) = 1$. This hypothesis was later removed by use of a cohomological argument which cannot be used to the same effect to remove the hypothesis that $(r,|G|) = 1$ in Theorem 4.2.1. In particular, he used the fact that when a Frobenius group FH acts on a group G with $C_G(F) = 1$, any section V of G which is minimal with respect to being RF-invariant is free as a module for R. Thus $\mathrm{H}^1(R,V) = 0$, a result which Khukhro uses to show that for any RF-invariant normal subgroup N of G that $\overline{C_G(R)} = C_{\overline{G}}(R)$ (see [28, Section 1]). Now under the hypotheses of Theorem 4.2.1, we do not necessarily have that any normal subgroup N of G which is minimal with respect to being RF-invariant is free as a module for R. Indeed, let N be minimal normal with respect to being RF-invariant. Then N is an irreducible module for RF over some field $\mathbb{F}$. Suppose F acts faithfully on N, then RF must also act faithfully on N. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ and set $V = N \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Let $\overline{V}$ be an irreducible $\overline{\mathbb{F}}[RF]$-submodule of V. Then $\overline{V}$ is faithful $\overline{\mathbb{F}}[RF]$-module by Lemma 1.8.3. Thus by Theorem 1.8.18, we have that $\overline{V}_R$ is not free as an $\overline{\mathbb{F}}[R]$-module. Thus we certainly cannot conclude that V_R is a free $\overline{\mathbb{F}}[R]$-module, and in particular, we cannot conclude that N_R is a free $\mathbb{F}[R]$-module. However, it must be mentioned that it may not be necessary to require that for any RF-invariant normal subgroup N of G that $\overline{C_G(R)} = C_{\overline{G}}(R)$ for the arguments in Lemmas 4.2.2, 4.2.3 and 4.2.4 to extend to when $(r,|G|) = r$. Indeed, having $\overline{C_G(R)} = C_{\overline{G}}(R)$ allowed us to conclude that $\overline{F(C_G(R))} \leq F(C_{\overline{G}}(R))$ since $F(\overline{C_G(R)}) = F(C_{\overline{G}}(R))$. However, if for any RF-invariant normal subgroup N of G we have that $\overline{F(C_G(R))} \leq F(C_{\overline{G}}(R))$ where $\overline{G} = G/N$ (even when $\overline{C_G(R)} \neq C_{\overline{G}}(R)$), then we could at least begin to make the same reductions as in the proof of Theorem 4.2.1. Of course there are other parts of the proof which rely on the coprimeness of r and $|G|$, such as Lemma 4.2.5, and this would also need to be overcome in order to remove the hypothesis that $(r,|G|) = 1$ from Theorem 4.2.1.

We finish this section with an example of a group which shows that we cannot remove the hypothesis that $r \neq s^l + 1$ from Corollary 4.2.22. In particular, we exhibit a group RFG such that:

1. $R \cong \mathbb{Z}_r$ where r is a prime;

2. $F \cong s^{1+2l}$ is an extraspecial s-group upon which R acts such that $[R, Z(F)] = 1$, $RF/Z(F)$ is a Frobenius group and $r = s^l + 1$; and

3. G is a group upon which RF acts such that $C_G(F) = 1$ and $f(C_G(R)) \neq f(G)$.

Example 4.2.23. Let $R \cong \mathbb{Z}_r$ where r is a prime of the form $r = 2^n+1$ for some $n \in \mathbb{N}$ and F be an extraspecial group of order 2^{2n+1}. Let R act on F such that $[R, Z(F)] = 1$ and $RF/Z(F)$ is Frobenius. Let $\mathbb{F}$ be an algebraically closed field. We first claim that there exists a faithful irreducible $\mathbb{F}[RF]$-module. First note that the only $\mathbb{F}[RF]$-module on which R acts trivially is the trivial $\mathbb{F}[RF]$-module since $F = [F, R]$. Let V be an irreducible $\mathbb{F}[RF]$-module and suppose that RF does not act faithfully on V. If R acts nontrivially on V, then $Z(RF) = Z(F) \subseteq C_{RF}(V)$ since $Z(F)$ is cyclic of prime order. Hence, if there does not exist a faithful irreducible $\mathbb{F}[RF]$-module, then $Z(RF)$ acts trivially on the regular $\mathbb{F}[RF]$-module. However, this is a contradiction since RF acts faithfully on the regular $\mathbb{F}[RF]$-module.

Now let V be a faithful irreducible $\mathbb{F}[RF]$-module where $\mathbb{F}$ is an algebraically closed field. Then by Theorem 1.8.18

$$V_R \cong \mathbb{F}[R]/U$$

where U is a 1-dimensional $\mathbb{F}[R]$-module. Note that since $\mathbb{F}$ is algebraically closed, we can write

$$\mathbb{F}[R] = M_1 \oplus \ldots \oplus M_{|R|}$$

where each M_i is a 1-dimensional $\mathbb{F}[R]$-module and $M_i \not\cong_{\mathbb{F}[R]} M_j$ for $i \neq j$. Assume without loss of generality that M_1 is the trivial $\mathbb{F}[R]$-module.

Suppose that $U \cong_{\mathbb{F}[R]} M_1$ so that

$$V_R \cong M_2 \oplus \ldots \oplus M_{|R|}.$$

Then $C_{V_R}(R) = 0$ and hence $f(C_{V_R}(R)) = 0$. However, $f(V_R) = 1$ and so we have constructed a group as outlined earlier.

So what happens if $U \not\cong_{\mathbb{F}[R]} M_1$? It turns out that we can construct an $\mathbb{F}[RF]$-module W from V such that $W \cong_{\mathbb{F}[R]} \mathbb{F}[R]/M_1$. We can assume without loss of generality that $U \cong_{\mathbb{F}[R]} M_{|R|}$.

Let N be a 1-dimensional $\mathbb{F}[R]$-module. Then N is also an $\mathbb{F}[RF]$-module if we define the action of F on N to be trivial. Now consider the $\mathbb{F}[R]$-module $T = M_i \otimes_{\mathbb{F}[R]} N$. We know that R acts on M_i by some scalar $\lambda \in \mathbb{F}^*$ and on N by some scalar $\mu \in \mathbb{F}^*$. Hence R acts on T by the scalar $\lambda\mu$.

Now consider $W = V \otimes_{\mathbb{F}[R]} N$. We have

$$W = M_1 \otimes_{\mathbb{F}[R]} N \oplus \ldots \oplus M_{|R|-1} \otimes_{\mathbb{F}[R]} N.$$

Note that we can choose N such that none of the $M_i \otimes_{\mathbb{F}[R]} N$ is the trivial $\mathbb{F}[R]$-module. Now N itself cannot be the trivial $\mathbb{F}[R]$-module since by assumption we have that one of the M_i, where $1 \leq i \leq |R| - 1$, is the trivial $\mathbb{F}[R]$-module. Hence R acts on N by some scalar $\lambda \neq 1$. There are $|R| - 1$

choices for λ. If the trivial $\mathbb{F}[R]$-module is a summand of W, then R must act by λ^{-1} on one of the M_i. However, only $|R| - 2$ of the summands of W are nontrivial as an $\mathbb{F}[R]$-module. Hence we can choose λ such that R does not act by λ^{-1} on any of the M_i. Then the trivial $\mathbb{F}[R]$-module will not be a direct summand of W and $C_W(R) = 0$. Hence $W \cong_{\mathbb{F}[R]} \mathbb{F}[R]/M_1$ as desired. Note that since N is trivial as a module for F that $V \cong_{\mathbb{F}[F]} W$.

4.3 Further work

At the beginning of this chapter we asked the following question:

> Let $R \cong \mathbb{Z}_r$ for some prime r and let F be a nilpotent group on which R acts such that $F = [F, R]$. Suppose RF acts on a group G such that $C_G(F) = 1$. Then do we necessarily have that $F(C_G(R)) \subseteq F(G)$?

We managed to answer this question positively in the case where F is extraspecial and $(r, |G|) = 1$. An obvious extension of this work is to investigate whether $F(C_G(R)) \subseteq F(G)$ when F is an arbitrary nilpotent group. It may be that the result holds only for certain classes of nilpotent groups but not for nilpotent groups in general. In which case it would be very interesting to know where this boundary is and what properties of F really affect the outcome of this question. If we supposed that F is a nilpotent group, then an obvious approach would be to try to reduce to the case where F is abelian or extraspecial and then invoke the results obtained in this chapter and by E. Khukhro. However, it is not at all clear how to do this. Even reducing to the case where F is a p-group presents difficulty. Another approach may be as follows. Under what circumstances do we have that $f(C_G(R)) = f(G)$? If, when F satisfies certain conditions, it arises that $f(C_G(R))$ is not necessarily equal to $f(G)$, then this may hint at conditions that F must satisfy in order for us to conclude that $F(C_G(R)) \subseteq F(G)$. In order to see why this is we consider the following theorem:

Theorem 4.3.1. *Let $R \cong \mathbb{Z}_r$ for some non-Fermat prime r, act on the nilpotent group F such that $F = [F, R]$. Suppose that RF acts on a group G such that $C_G(F) = 1$, $(r, |G|) = 1$ and $F(C_G(R)) \subseteq F(G)$. Then $f(C_G(R)) = f(G)$.*

Proof. Since $C_G(R) \leq G$, $f(C_G(R)) \leq f(G)$. So it will suffice to show that $f(G) \leq f(C_G(R))$.

Let $n \in \mathbb{N}$ be the Fitting height of $C_G(R)$ so $F_n(C_G(R)) = C_G(R)$. Now we know that

$$F_n(C_G(R)) = F_n(G) \cap C_G(R)$$

and so we have that $C_G(R) \leq F_n(G)$. We work to show that $F_n(G) = G$. Suppose this is not the case so $F_n(G) < G$. Let S be an RF-invariant section of $G/F_n(G)$ which has no proper RF-invariant subgroups. Since $F_n(G)\mathrm{char}G$, $F_n(G)$ is a normal F-invariant subgroup of G. Thus $C_{G/F_n(G)}(F) = 1$ by Lemma 4.1.8. We can consider S as an irreducible $\mathbb{F}_p[RF]$-module for some prime p. Now R acts coprimely on G and so since $C_G(R) \subseteq F_n(G)$, it follows that R acts fixed point freely on $G/F_n(G)$.

Now F acts nontrivially on S and so after factoring out the kernel of F on S we see by Theorem 4.1.12 that either $[R, F/C_F(S)] = 1$ or $[R, F/C_F(S)]$ is a nonabelian special 2-group and $r = 2^m + 1$ for some $m \in \mathbb{N}$. By hypothesis, the former must hold, but then since $C_F(R) = \Phi(F)$, we have that $F = C_F(S)\Phi(F)$. However, this implies that $F = C_F(S)$, which is a contradiction. □

If we found that for all nilpotent F that $f(C_G(R)) = f(G)$, then the approach above would in some sense fail, in that, it would not give us any insight into when $F(C_G(R))$ may not necessarily be contained in $F(G)$. However, it would not be a complete loss since we would be in receipt of a result which is satisfactory in its own right. Related to this, it would be interesting to consider if, for F nilpotent such that $f(C_G(R)) \neq f(G)$, whether there is an absolute bound on $f(G) - f(C_G(R))$. Maybe this is related to the nilpotence class of F?

Of course the other obvious direction to take would be to completely remove the hypothesis that $(r, |G|) = 1$.

4.4 The dihedral group as a group of automorphisms

In this final section we prove another result which relates to the main idea behind the work in this chapter. Namely, if a group A acts on a group G such that there is a nilpotent subgroup N of A which acts fixed-point-freely, then we can find a relationship between $f(G)$ and the Fitting height of subgroups which are centralised by elements of A not contained in N. In particular, we consider a situation where the group acting is dihedral.

In [42] the following theorem is proved:

Theorem 4.4.1. (Shumyatsky) *Suppose that $D = \langle \alpha, \beta \rangle$ is a dihedral group where α and β are both involutions. Let D act on a finite group G in such a manner that $C_G(\alpha\beta) = 1$ and both $C_G(\alpha)$ and $C_G(\beta)$ are nilpotent. Then G is nilpotent.*

Proof. See [42, Theorem 2.11]. □

We generalise this result by proving the following:

Theorem 4.4.2. *Suppose that* $D = \langle \alpha, \beta \rangle$ *is a dihedral group where* α *and* β *are both involutions. Let* D *act on a finite group* G *in such a manner that* $C_G(\alpha\beta) = 1$ *and* $\max\{f(C_G(\alpha)), f(C_G(\beta))\} = n$*. Then* $f(G) = n$*.*

Before proving Theorem 4.4.2 we state the following preparatory lemmas. These lemmas are well-known and we offer references for their proofs.

Lemma 4.4.3. *Let* G *be a group of odd order which admits an automorphism* ϕ *of order* 2*. Set* $F = C_G(\phi)$ *and let* $I = \{g \in G \mid g\phi = g^{-1}\}$*. Then* $G = FI = IF$ *and* $F \cap I = 1$*.*

Proof. See [20, Lemma 10.4.1]. □

Lemma 4.4.4. *Let* D *and* G *be as in Theorem 4.4.2. Then there exists a unique* D*-invariant Sylow* p*-subgroup* $P \in \mathrm{Syl}_p(G)$ *for each prime* $p \in \pi(G)$*.*

Proof. Follows directly from Lemma 4.1.5. □

Lemma 4.4.5. *Let* D *and* G *be as in Theorem 4.4.2 and let* N *be a* D*-invariant normal subgroup of* G*. Set* $\overline{G} = G/N$*. Then* $C_{\overline{G}}(\alpha\beta) = 1$*,* $C_{\overline{G}}(\alpha) = \overline{C_G(\alpha)}$ *and* $C_{\overline{G}}(\beta) = \overline{C_G(\beta)}$

Proof. See [42, Lemma 2.7]. □

Proof. (of Theorem 4.4.2) Let G be a counterexample of minimal order. Let $F = \langle \alpha\beta \rangle$. First note that G cannot be nilpotent. Suppose G is nilpotent then we must have that both $C_G(\alpha) = 1 = C_G(\beta)$. Hence G must be a group of odd order. Then both α and β act on each element of G by inversion by Lemma 4.4.3. Then F must act trivially on G forcing $G = 1$.

We first claim that for any nontrivial D-invariant normal subgroup N of G we have $f(G/N) < f(G)$. Let $1 \neq N \trianglelefteq G$ be D-invariant and set $\overline{G} = G/N$. Note that $C_{\overline{G}}(F) = 1$, $C_{\overline{G}}(\alpha) = \overline{C_G(\alpha)}$ and $C_{\overline{G}}(\beta) = \overline{C_G(\beta)}$ by Lemma 4.4.5. Then since $|\overline{G}| < |G|$, we have

$$\max\{f(C_{\overline{G}}(\alpha)), f(C_{\overline{G}}(\beta))\} = f(\overline{G})$$

by minimality. Thus our claim follows since

$$\max\{f(C_{\overline{G}}(\alpha)), f(C_{\overline{G}}(\beta))\} \leq \max\{f(C_G(\alpha)), f(C_G(\beta))\} < f(G).$$

We now claim that $F(G)$ is a completely reducible module for DG. By Proposition 4.1.4, it will suffice to show that $\Phi(DG) \cap G = 1$. Suppose

$\Phi(DG) \cap G \neq 1$ and set $\overline{G} = G/(\Phi(DG) \cap G)$. Since $\Phi(DG) \cap G$ is a D-invariant normal subgroup and $1 \neq \Phi(DG) \cap G \subseteq F(G)$,

$$f(\overline{G}) = f(G) - 1.$$

If $F(\overline{G}) \neq 1$, then

$$f(G/F(G)) = f(\overline{G}/\overline{F(G)}) = f(G) - 2.$$

This cannot be and so $G = \Phi(DG) \cap G$. However, G is not nilpotent and so we must have $\Phi(DG) \cap G = 1$.

We now show that $F(G)$ is minimal with respect to being D-invariant. Let U and V be minimal normal in G with respect to being D-invariant and suppose $U \neq V$. Now $U \cap V$ is normal in G and is D-invariant and so by the minimality of U and V we have that $U \cap V = 1$. Thus G embeds into $G/U \times G/V$. However, by an earlier claim we have that $f(G/U)$, $f(G/V) < f(G)$. This is a contradiction, hence we must have $U = V$.

Let $q \in \pi(F_2(G))$. Now by Lemma 4.4.4 there exists a D-invariant Sylow q-subgroup Q of $F_2(G)$. We choose q here such that $Q \nsubseteq F(G)$, which is possible since G is not nilpotent. By the Frattini argument we have

$$G = N_G(Q)F_2(G) = N_G(Q)F(G).$$

Now $N_G(Q) \cap F(G) = 1$. If not, then $1 \neq M = N_G(Q) \cap F(G)$ is normal in G and is D-invariant. Thus $F(G) = M$ and so $F(G) \subseteq N_G(Q)$. However, this implies that $Q \trianglelefteq G$, which is a contradiction since $Q \nsubseteq F(G)$

Now F has no fixed points on Q and thus FQ is a Frobenius group since F is cyclic. We can consider $F(G)$ as a module for FQ. Now $F(G)$ is self-centralising in G and so $[Q, F(G)] \neq 1$. Since $F(G)$ is abelian, Q acts fixed-point-freely on $[Q, F(G)]$, and so $[Q, F(G)]$ is free as an F-module. However, F acts fixed-point-freely on G. □

Chapter 5

Groups with two Sylow numbers

In [36], A. Moretó proved that a group with two Sylow numbers is the product of two nilpotent Hall subgroups. In particular, due to a result by Kegel and Wielandt, such a group is soluble (see [27] and [47]). He also mentioned that it is possible to construct groups having arbitrarily large Fitting height whose order is only divisible by two distinct primes. It is then suggested that it may be possible to obtain an absolute bound on the Fitting height of groups with two Sylow numbers whose order is divisible by more than two distinct primes. In this chapter, we show that an absolute bound does not exist, even for an arbitrarily large number of distinct primes dividing the group order. In particular, we construct a group with only two Sylow numbers but whose Fitting height, and number of distinct primes dividing the group order, are both arbitrarily large. We will also see an example of a group with only two distinct prime divisors and arbitrarily large Fitting height.

5.1 Preliminaries

The results presented in this section are well-known. References for most of the results are given, but the author provides proofs of results which are difficult to find. No originality is claimed by the author in this section.

Throughout this chapter, G will be a finite group and $\pi(G)$ the set of prime divisors of $|G|$. We write $n_p(G)$ to denote the number of Sylow p-subgroups of G.

Definition 5.1.1. Let G be a group. Then $m \in \mathbb{N}$ is a *Sylow number* of G if there exists a prime $p \in \pi(G)$ such that $n_p(G) = m$.

Proposition 5.1.2. *Let G be a finite group. Then G has only one Sylow number if and only if it is nilpotent.*

Proof. If G is nilpotent, then every Sylow subgroup is normal and so the only Sylow number is one.

Now suppose that G has only one Sylow number n. We have that $\pi(n) \subseteq \pi(G)$. Let $p \in \pi(G)$. Then $p \notin \pi(n)$ since $n \equiv 1 \mod p$. Since this is true for all primes in $\pi(G)$, we must have $n = 1$. Therefore, every Sylow subgroup of G is normal and hence G is nilpotent. □

Theorem 5.1.3. (Moretó) *A group with two Sylow numbers is the product of two nilpotent Hall subgroups.*

Proof. See [36]. □

Theorem 5.1.4. (Kegel–Wielandt) *A finite group G is soluble if it contains nilpotent subgroups $G_1, \ldots, G_k$ such that $G = G_1 \cdots G_k$ and $G_iG_j = G_jG_i$ for all $1 \leq i, j \leq k$. In particular, G is soluble if $G = AB$ where A and B are nilpotent subgroups of G.*

Proof. See [27] and [47]. □

Definition 5.1.5. Let G be a group and $H \leq G$ be a subgroup of finite index. Let M be a right $\mathbb{F}[H]$-module for some field $\mathbb{F}$. Set

$$M^G = M \otimes_{\mathbb{F}[H]} \mathbb{F}[G]$$

where $\mathbb{F}[G]$ is a left $\mathbb{F}[H]$-module. Then M^G is a right $\mathbb{F}[G]$-module via the rule

$$(m \otimes r)s = m \otimes (rs)$$

where $m \in M$ and r, $s \in \mathbb{F}[G]$. The right module M^G is called the *induced module* of M.

Lemma 5.1.6. *Let G be a group and $H \leq G$ be a subgroup of finite index. Let M be a right $\mathbb{F}[H]$-module for some field $\mathbb{F}$. Then*

$$\dim_{\mathbb{F}} M^G = [G : H] \cdot \dim_{\mathbb{F}} M.$$

Proof. See [39, Page 231]. □

Proposition 5.1.7. *Let G be a group and $H \leq G$. Let M be an $\mathbb{F}[H]$-module for some field $\mathbb{F}$ and $N \trianglelefteq G$ such that $N \subseteq H$. If $C_M(N) = 0$, then $C_{M^G}(N) = 0$.*

Proof. Let T be a right transversal for H in G. Since $N \trianglelefteq G$, we can write

$$M^G = \bigoplus_{t \in T} Mt$$

as an $\mathbb{F}[N]$-module. So if there exists $0 \neq m \in C_{M^G}(N)$, then for some $t \in T$ there will exist $0 \neq m' \in Mt$ which is fixed by N. Now N is normalised by G, so for each $t \in T$, t maps the fixed point subspace of N in M to the fixed point subspace of N in Mt. Since $C_M(N) = 0$, no such $m' \neq 0$ can exist. Hence $C_{M^G}(N) = 0$. □

Proposition 5.1.8. *For any $n \in \mathbb{N}$ there exists a prime p such that $p-1$ has at least n distinct prime divisors.*

Proof. A theorem of Euler states that every arithmetic progression beginning with 1 contains an infinite number of primes. Choose $a \in \mathbb{N}$ such that a has n distinct prime divisors and consider the arithmetic progression with first term 1 and common difference a. By Euler's theorem there exists a prime p in this sequence and so

$$p = 1 + ka$$

for some $k \in \mathbb{N}$. Thus a divides $p-1$ and so $p-1$ has at least n distinct prime divisors. □

5.2 Constructing examples

We will now look at some examples of groups which have two Sylow numbers and arbitrarily large Fitting height. It is quite simple to construct such a group G where $|G|$ is divisible by only two distinct primes as we shall see in the first example of this section. Indeed we only need to worry about increasing the Fitting height since such a group can only have one or two Sylow numbers. However, if we stipulate that $|G|$ is divisible by more than two distinct primes, it becomes much more difficult to increase $f(G)$ whilst ensuring that G still only has two Sylow numbers. At the end of this section we will exhibit a construction which achieves this goal.

Example 5.2.1. Let p and q be distinct primes. We will construct a sequence of groups G_n, $n \in \mathbb{N}$, such that for all n, G_n has Fitting height n and $F(G_n) = \mathcal{O}_q(G_n)$ if n is odd and $F(G_n) = \mathcal{O}_p(G_n)$ if n is even.

We let $G_1 \cong \mathbb{Z}_q$; this clearly satisfies the conditions above. We now define G_{n+1} in terms of G_n. Suppose n is odd, so $F(G_n) = \mathcal{O}_q(G_n)$ and let $V = \mathbb{F}_p[G_n]$. Then V is an $\mathbb{F}_p$-module on which G_n acts faithfully. Set

$$G_{n+1} = V \rtimes G_n.$$

Note that

$$F(G_{n+1}) = (F(G_{n+1}) \cap G_n)V$$

by the modular law. Now V is a p-group, so since $F(G_{n+1})$ is nilpotent, any q-element of $F(G_{n+1}) \cap G_n$ centralises V. Since G_n acts faithfully on V, it follows that $F(G_{n+1}) \cap G_n$ is a p-group. Now

$$F(G_{n+1}) \cap G_n \trianglelefteq G_n \text{ and } \mathcal{O}_p(G_n) = 1.$$

Therefore

$$F(G_{n+1}) \cap G_n = 1$$

and so $F(G_{n+1}) = V$. Then

$$F(G_{n+1}) = \mathcal{O}_p(G_{n+1}) \text{ and } f(G_{n+1}) = f(G_n) + 1 = n + 1.$$

If n is even, so that $F(G_n) = \mathcal{O}_p(G_n)$, let $V = \mathbb{F}_q[G_n]$ and repeat the construction in the obvious way.

Example 5.2.2. We now show how to construct a group with two Sylow numbers whose Fitting height is arbitrarily large and whose order is divisible by at least three distinct primes. We will first construct such a group with order divisible by exactly three distinct prime divisors p, q and r, after which it will be clear how to adapt the construction so that the group order has any number $n \geq 3$ distinct prime divisors. Note that we will require both q and r to divide $p - 1$, but we can easily choose primes to satisfy this. The idea is to ensure

$$G = H \times K$$

such that:

1. $|H|_p = |K|_p$;
2. $\pi(H) = \{p, q\}$ and $\pi(K) = \{p, r\}$;
3. H contains a Sylow q-subgroup of G which is self-normalising in H; and
4. K contains a Sylow r-subgroup of G which is self-normalising in K.

This will ensure that $n_q = n_r$ and so G will have no more than two Sylow numbers. Since for any soluble group $A = B \times C$, we have

$$f(A) = \max\{f(B), f(C)\},$$

we increase the Fitting height of G by increasing the Fitting height of the factors H and K.

We now construct a sequence of groups H_n, $n \in \mathbb{N}$ satisfying the following properties:

1. $f(H_n) = n$;
2. $\pi(H_n) = \{p, q\}$ if $n \geq 2$;
3. If $Q \in \mathrm{Syl}_q(H_n)$, then $N_{H_n}(Q) = Q$;
4. $|H_n|_p$ is independent of q.

The group H_n will be our candidate for H when we eventually construct G. In the following, at *Step i* we will construct H_i.

Step 1: Set $H_1 = Q_1 \cong \mathbb{Z}_q$. Then H_1 clearly satisfies the conditions above.

Step 2: Let P_2 be a nontrivial irreducible $\mathbb{F}_p[H_1]$-module. Then P_2 is 1-dimensional since $q \mid (p-1)$. Set $H_2 = H_1 \ltimes P_2$. We claim that $F(H_2) = P_2$. Certainly $P_2 \subseteq F(H_2)$. If $P_2 \neq F(H_2)$, then $F(H_2) = H_2$. Then $H_2 = H_1 \times P_2$, which is a contradiction since H_1 acts nontrivially on P_2, and so $F(H_2) = P_2$. Hence $f(H_2) = 2$ and thus the first condition above is satisfied.

Now $Q_1 \in \mathrm{Syl}_q(H_2)$. If $Q_1 \neq N_{H_2}(Q_1)$, then $Q_1 \trianglelefteq H_2$. This implies that $H_2 = H_1 \times P_2$ which again we know is not the case. Hence the third condition above is satisfied. Also, $\pi(H_2) = \{p, q\}$ and $|H_2|_p = p$, so conditions two and four are satisfied.

Step 3: Let M be a nontrivial irreducible $\mathbb{F}_q[P_2]$-module. Let Q_3 be an irreducible submodule of the $\mathbb{F}_q[H_2]$-module induced from M. Set $H_3 = H_2 \ltimes Q_3$. We claim that $F(H_3) = Q_3$. Certainly $Q_3 \subseteq F(H_3)$. Suppose $Q_3 \neq F(H_3)$. If there are any p-elements in $F(H_3)$, then $P_2 \subseteq F(H_3)$. Thus P_2 would centralise Q_3. This is not the case since $1 \neq M \leq Q_3$ and $C_M(P) = 1$. Thus $F(H_3)$ is a q-group. Now $Q_1 \nsubseteq F(H_3)$ otherwise there would be nontrivial q-elements in $F(H_2)$. Thus $F(H_3) = Q_3$ and $f(H_3) = 3$, so the first condition above is satisfied.

Now $Q_1 Q_3 \in \mathrm{Syl}_q(H_3)$. If $Q_1 Q_3 \neq N_{H_3}(Q_1 Q_3)$, then $Q_1 Q_3 \trianglelefteq H_3$ and so $Q_1 \trianglelefteq H_2$. Again this is not the case, so $Q_1 Q_3 = N_{H_3}(Q_1 Q_3)$ and condition three is satisfied. Also, $\pi(H_3) = \{p, q\}$ and $|H_3|_p = p$, so conditions two and four are satisfied.

Step 4: Now Q_1 acts on the q-group Q_3 and so normalises a hyperplane $H \trianglelefteq Q_3$. Set

$$\overline{Q_1 Q_3} = Q_1 Q_3 / H.$$

Then

$$\overline{Q_1 Q_3} \cong \overline{Q_1} \times \overline{Q_3}$$

since $\overline{Q_3} \cong \mathbb{Z}_q$. Therefore $Q_1H \trianglelefteq Q_1Q_3$ and

$$Q_1Q_3/Q_1H \cong \mathbb{Z}_q.$$

Let M be an irreducible nontrivial $\mathbb{F}_p[Q_1Q_3/Q_1H]$-module. Then M is a nontrivial 1-dimensional module for Q_1Q_3 where $C_{Q_1Q_3}(M) = Q_1H$. Now let P_4 be the $\mathbb{F}_p[H_3]$-module obtained by inducing from M. Set $H_4 = H_3 \ltimes P_4$. We claim that $F(H_4) = P_4$. If $|F(H_4)|_p > |P_4|$, then $P_2 \subseteq F(H_4)$. Thus $P_2 \subseteq F(H_3)$ which is not the case. Hence $|F(H_4)|_p = |P_4|$. Note that $F(H_4) \cap Q_3 \trianglelefteq Q_3$. Now Q_3 is an irreducible $\mathbb{F}_q[H_2]$-module and since Q_3 is nontrivial on P_4, $F(H_4) \cap Q_3 = 1$. Now $C_{H_3}(P_4)$ is a normal q-subgroup of H_3 and is thus contained in Q_3. This shows that $C_{H_3}(P_4) = 1$ and so $F(H_4) = P_4$. Thus $f(H_4) = 4$ and the first condition is satisfied.

Now $Q_1Q_3 \in \mathrm{Syl}_q(H_4)$. If $Q_1Q_3 \neq N_{H_4}(Q_1Q_3)$, then $N_{P_2P_4}(Q_1Q_3) \neq 1$. Let $g \in N_{P_2P_4}(Q_1Q_3)$. Set $\overline{H_4} = H_4/P_4$. Then

$$\overline{g} \in \overline{N_{P_2P_4}(Q_1Q_3)} \subseteq N_{\overline{P_2P_4}}(\overline{Q_1Q_3}) = 1$$

since the Sylow q-subgroup of H_3 is self-normalising. So $g \in P_4$. Therefore

$$[Q_1Q_3, g] \subseteq Q_1Q_3 \cap P_4 = 1$$

and so $g \in C_{P_4}(Q_1Q_3)$. However,

$$C_{P_4}(Q_1Q_3) \subseteq C_{P_4}(Q_3) = 1$$

where the equality on the right follows by Proposition 5.1.7. Thus $g = 1$, which is a contradiction, and so $Q_1Q_3 = N_{H_4}(Q_1Q_3)$.

Also, $\pi(H_4) = \{p, q\}$ and $|H_4|_p = p^{p+1}$, so conditions two and four are satisfied.

At this stage we note that we are alternately building up groups

$$H_i = \begin{cases} H_{i-1} \ltimes P_i & \text{when } i \text{ is even} \\ H_{i-1} \ltimes Q_i & \text{when } i \text{ is odd.} \end{cases}$$

We now describe '*Step i*' when i is odd and even. Each of the former will be similar to *Step 3* and each of the latter to *Step 4*.

Step i=2j+1: Note that since i is odd

$$H_{i-1} = H_{i-2} \ltimes P_{i-1}.$$

Let M be a nontrivial irreducible $\mathbb{F}_q[P_{i-1}]$-module. Let Q_i be an irreducible submodule of the $\mathbb{F}_q[H_{i-1}]$-module induced from M. Set $H_i = H_{i-1} \ltimes Q_i$.

Step i=2j: In what follows $i \geq 4$. Note that since i is even

$$H_{i-1} = H_{i-2} \ltimes Q_{i-1}.$$

Now $Q = Q_1 \dots Q_{i-3}$ acts on the q-group Q_{i-1} and so normalises a hyperplane $H \trianglelefteq Q_{i-1}$. Set

$$\overline{QQ_{i-1}} = QQ_{i-1}/H.$$

Then

$$\overline{QQ_{i-1}} \cong \overline{Q} \times \overline{Q_{i-1}}$$

since $\overline{Q_{i-1}} \cong \mathbb{Z}_q$. Therefore $QH \trianglelefteq QQ_{i-1}$ and

$$QQ_{i-1}/QH \cong \mathbb{Z}_q.$$

Let M be an irreducible nontrivial $\mathbb{F}_p[QQ_{i-1}/QH]$-module. Then since q divides $p-1$, M is a nontrivial 1-dimensional module for QQ_{i-1} where

$$C_{QQ_{i-1}}(M) = QH.$$

Now let P_i be the $\mathbb{F}_p[H_{i-1}]$-module obtained by inducing from M. Set $H_i = H_{i-1} \ltimes P_i$.

We claim that the groups H_i satisfy the conditions above. Note that by Lemma 5.1.6

$$|P_i| = p^{|P_2 \dots P_{i-2}|}$$

for $i > 2$ and so since $|P_2| = p$, $|H_i|_p$ is certainly independent of q. Thus the fourth condition is always satisfied. Also, $\pi(H_i) = \{p, q\}$ for $n \geq 2$ and so the second condition is also always satisfied.

Lemma 5.2.3. *Let $i \in \mathbb{N}$ be odd. Then $C_{P_{i+1}}(Q_i) = 1$.*

Proof. This follows directly from Proposition 5.1.7. □

Corollary 5.2.4. *Let $i \in \mathbb{N}$ be even. Then $F(H_i)$ is a p-group.*

Proof. Note that $F(H_2) = P_2$ and so the result is clear when $i = 2$. Hence we can assume $i > 2$.

For $i > 2$ the result will follow if we can show $F(H_i) \cap Q_{i-1} = 1$. Now $P_i \subseteq F(H_i)$ for all i since it is a normal p-subgroup of H_i. If $F(H_i) \cap Q_{i-1} \neq 1$, then we must have $F(H_i) \cap Q_{i-1} = Q_{i-1}$ since Q_{i-1} is an irreducible $\mathbb{F}_q[H_{i-2}]$-module by construction. Thus Q_{i-1} must act trivially on P_i. However, $C_{P_i}(Q_{i-1}) = 1$ by Lemma 5.2.3, so this cannot happen. □

Lemma 5.2.5. *Let $Q \in \mathrm{Syl}_q(H_i)$. Then $Q = N_{H_i}(Q)$.*

Proof. Note that we have already proved this when $i \in \{1, 2, 3, 4\}$. Let i be even so that

$$H_i = H_{i-1} \ltimes P_i.$$

We claim that if $Q_1 \dots Q_{i-1}$ is self-normalising in H_i, then $Q_1 \dots Q_{i+1}$ is self-normalising in both H_{i+1} and H_{i+2}.

Now $Q_{i+1} \trianglelefteq H_{i+1}$, so if there are any p-elements in $N_{H_{i+1}}(Q_1 \dots Q_{i+1})$, then there must be nontrivial p-elements in $N_{H_i}(Q_1 \dots Q_{i-1})$. However, $Q_1 \dots Q_{i-1}$ is self-normalising in H_i by hypothesis. So $Q_1 \dots Q_{i+1}$ is self-normalising in H_{i+1}. Furthermore, if there is a nontrivial p-element

$$g \in N_{H_{i+2}}(Q_1 \dots Q_{i+1}),$$

then $g \in P_{i+2}$. Otherwise the Sylow q-subgroup of $H_{i+2}/Q_{i+1}P_{i+2}$ would not be self-normalising. In particular, $g \in C_{P_{i+2}}(Q_1 \dots Q_{i+1})$ since $P_{i+2} \trianglelefteq H_{i+2}$. However

$$C_{P_{i+2}}(Q_1 \dots Q_{i+1}) \subseteq C_{P_{i+2}}(Q_{i+1}) = 1$$

where the equality on the right follows by Lemma 5.2.3. Thus $Q_1 \dots Q_{i+1}$ is self-normalising in H_{i+2}. □

Lemma 5.2.6. *Let $i \in \mathbb{N}$ be even. If $\varphi(H_i) = P_i$, then:*

1. *$\varphi(H_{i+1}) = Q_{i+1}$; and*
2. *$\varphi(H_{i+2}) = P_{i+2}$.*

Proof. Note that if $\varphi(H_{i+1}) \subseteq Q_{i+1}$, then $\varphi(H_{i+1}) = Q_{i+1}$ since Q_{i+1} is a minimal normal subgroup of H_{i+1}. Suppose $\varphi(H_{i+1}) \neq Q_{i+1}$ then

$$f(H_{i+1}/Q_{i+1}) = f(H_{i+1}).$$

In particular, $f(H_i) = f(H_{i+1})$. Since $H_i \leq H_{i+1}$, $\varphi(H_i) \subseteq \varphi(H_{i+1})$. By hypothesis $P_i = \varphi(H_i)$. So $Q_{i+1} \nsubseteq \varphi(H_{i+1})$, otherwise P_i would centralise Q_{i+1}. Indeed, Q_{i+1} contains an $\mathbb{F}_q[P_i]$-module M such that $C_{P_i}(M) \neq P_i$. Hence $Q_{i+1} \cap \varphi(H_{i+1}) = 1$ since Q_{i+1} is minimal normal in H_{i+1}. Now

$$f(H_{i+1}/P_iQ_{i+1}) < f(H_{i+1})$$

and so $\varphi(H_{i+1}) \subseteq P_iQ_{i+1}$. If $\mathcal{O}_q(\varphi(H_{i+1})) \neq 1$, then $Q_{i+1} \cap \varphi(H_{i+1}) \neq 1$. Thus $\mathcal{O}_q(\varphi(H_{i+1})) = 1$ and so $\varphi(H_{i+1}) = P_i$. In particular $P_i \trianglelefteq H_{i+1}$. Thus

$$[P_i, Q_{i+1}] = 1$$

and so $C_{P_i}(Q_{i+1}) = P_i$. This is a contradiction, hence

$$\varphi(H_{i+1}) = Q_{i+1}.$$

By Corollary 5.2.4, $\varphi(H_{i+2})$ is a p-group. Suppose $f(H_{i+1}) = f(H_{i+2})$. Then since $H_{i+1} \leq H_{i+2}$, we have $\varphi(H_{i+1}) \subseteq \varphi(H_{i+2})$. However, $\varphi(H_{i+1})$ is a nontrivial q-group. Thus $f(H_{i+1}) < f(H_{i+2})$ and so $\varphi(H_{i+2}) \subseteq P_{i+2}$.

Suppose $\varphi(H_{i+2}) \neq P_{i+2}$. Set $\overline{H_{i+2}} = H_{i+2}/\varphi(H_{i+2})$. Then $\overline{P_{i+2}} \neq 1$. Now $f(\overline{H_{i+2}}) = f(\overline{H_{i+1}})$. So since $\overline{H_{i+1}} \leq \overline{H_{i+2}}$,

$$\overline{Q_{i+1}} = \varphi(\overline{H_{i+1}}) \subseteq \varphi(\overline{H_{i+2}}) \subseteq F(\overline{H_{i+2}}).$$

Hence $\overline{Q_{i+1}}$ centralises $\overline{P_{i+2}}$ since $\overline{P_{i+2}} \subseteq F(\overline{H_{i+2}})$. This is a contradiction since $C_{P_{i+2}}(Q_{i+1}) = 1$. Thus

$$\varphi(H_{i+2}) = P_{i+2}.$$

□

Corollary 5.2.7. *$\varphi(H_i) = P_i$ when i is even and $\varphi(H_i) = Q_i$ when i is odd.*

Proof. By construction $H_1 = Q_1 \cong \mathbb{Z}_q$ and so trivially $\varphi(H_1) = Q_1$. Also, $F(H_2) = P_2 \cong \mathbb{Z}_p$ and so $\varphi(H_2) = P_2$. By Lemma 5.2.6 the result now follows by induction. □

Corollary 5.2.8. *$f(H_i) = i$ for all $i \in \mathbb{N}$.*

Proof. This follows trivially from Corollary 5.2.7. □

Now, if we want to construct a group G with two Sylow numbers which has Fitting height n and three distinct prime divisors, we do the following. Construct H_n as outlined above. Construct another group K_n using the same process but with a prime r (not equal to p or q) in place of q. Let

$$G = H_n \times K_n.$$

Then G has the desired properties. If we wish to increase the number of distinct prime divisors of $|G|$, then we construct another group, say L_n, using the process above but with another prime s not equal q or r such that s also divides $p - 1$, and take the direct product

$$G = H_n \times K_n \times L_n.$$

Note that by Proposition 5.1.8, this process can go on indefinitely, and so we can construct a group with two Sylow numbers with arbitrarily large Fitting height and any number $n \geq 3$ distinct primes dividing its order.

Note how crucial it is in Example 5.2.2 that the order of the Sylow p-subgroup in H_n is independent of q and that the Sylow q-subgroup of H_n is self-normalising. Indeed, it is due to this that we are able to make sure that G has only two Sylow numbers. Now consider the groups G_n that we constructed in Example 5.2.1. These groups have two Sylow numbers $n_p(G_n)$ and $n_q(G_n)$, but we do not have much control over the values that either of these take as n grows. For $n \geq 2$, $|G_n|_p$ is dependent on q. Hence, even though the Sylow q-subgroups of G_n are self-normalising, we do not have much control over $n_q(G_n)$. Also, for $n \geq 3$, $|G_n|_q$ is dependent on p and the Sylow p-subgroups are not self-normalising. In particular, we cannot use these constructions, taking another prime r in place of p or q and form direct products to construct a group with the properties that G has in Example 5.2.2. If we did so, it would be almost impossible to tell how many Sylow numbers such a group would have.

Chapter 6

On Hall normally embedded subgroups

Characterising groups which possess subgroups of a prescribed order, perhaps with further properties such as being normal, has long been a source of strong and useful results within group theory. A classic example of such a result is P. Hall's generalisation of Sylow's theorem to soluble groups. This characterises finite soluble groups as precisely those which, for an arbitrary set of primes π, possess a π-subgroup whose order is coprime to its index. Another example of such a result, due to C. V. Holmes [23], is that a finite group G is nilpotent if and only if G possesses a normal subgroup of order d for each divisor d of $|G|$. These results have become virtually indispensable in the study of soluble and nilpotent groups. In this chapter we obtain another characterisation result of this nature.

A subgroup H of a finite group G is said to be Hall normally embedded in G if H is a Hall subgroup of its normal closure $\langle H^G \rangle$. We classify groups G which possess a Hall normally embedded subgroup H of order $|B|$ for each subgroup $B \leq G$. Such groups will be referred to as partial HNE-groups. In particular, we prove the following:

Theorem 6.0.1. *A finite group G is a partial HNE-group if and only if $G = H \ltimes N$ where H is nilpotent and N is a cyclic normal subgroup of square-free order.*

This result has application to the following problem posed in [32] by S. Li and J. Liu:

> **Problem** 1. Study groups G in which there exists a Hall normally embedded subgroup H of order $|B|$ for each $B \leq G$. In particular, is G soluble?

We see that Theorem 6.0.1 answers this question in the affirmative. Our results also have application to recent work of Li, He, Nong and Zhou in [31]. Here the structure of a partial HNE-group G is described under the assumption that G is a CLT-group, which we recall is a group which satisfies the converse to Lagrange's theorem. Hence they necessarily assume that G is soluble since all CLT-groups are soluble (see [35]). We show that this hypothesis is in fact redundant. In particular, we prove the following.

Theorem 6.0.2. *Let G be a partial HNE-group. Then G is a CLT-group.*

So a group for which there exists a Hall normally embedded subgroup H of order $|B|$ for each $B \leq G$ is necessarily a CLT-group.

The approach that we take in proving Theorem 6.0.1 allows us to describe the normal closure of the Sylow subgroups of partial HNE-groups and to describe how these subgroups can be used to construct the nilpotent residual. This will be important since we find that the nilpotent residual of such a group is cyclic. As such, these groups are not only soluble but have Fitting height no greater than two. We then move on to showing that partial HNE-groups are CLT-groups. Once this is identified, some of the results regarding their structure can be found in [31]. We finish by giving necessary and sufficient conditions for a group to be a partial HNE-group, thus completing the characterisation. In particular, we prove the assertions of Theorem 6.0.1

6.1 Preliminaries

The results presented in this section are well-known. References for most of the results are given, but the author provides proofs of results which are difficult to find. No originality is claimed by the author in this section.

Definition 6.1.1. Let G be a group which satisfies the converse to Lagrange's theorem, so that for each divisor d of $|G|$ there exists a subgroup $H \leq G$ such that $|H| = d$. Then we call G a *CLT-group.*

Definition 6.1.2. Let G be a group. A subgroup $H \leq G$ is said to be *Hall normally embedded* in G if it is a Hall subgroup of its normal closure $\langle H^G \rangle$.

Definition 6.1.3. Let G be a group. If for each divisor d of $|G|$ there exists a Hall normally embedded subgroup H of order d, then G is said to be a *HNE-group.*

Definition 6.1.4. Let G be a group. If for each subgroup $B \leq G$ there exists a Hall normally embedded subgroup $H \leq G$ such that $|H| = |B|$, then G is said to be a *partial HNE-group.*

It is clear that every HNE-group is a partial HNE-group. However, it is not clear a priori that all partial HNE-groups are HNE-groups. We will later show that partial HNE-groups are CLT-groups, thus proving that all partial HNE-groups are indeed HNE-groups.

Definition 6.1.5. Let G be a group. A subgroup $H \leq G$ is said to be *slim* if $|H|_p \leq p$ for all primes p.

Proposition 6.1.6. *Let G be a finite group. Then there exists a unique smallest normal subgroup N such that G/N is nilpotent.*

Proof. It suffices to show that for normal subgroups $H, K \trianglelefteq G$ such that G/H and G/K are nilpotent, we have that $G/(H \cap K)$ is nilpotent. Let H and K be two such subgroups. The map $\phi : G \longrightarrow G/H \times G/K$ defined by

$$g \longmapsto (gH, gK)$$

is a homomorphism with kernel $H \cap K$. So $G/(H \cap K)$ is isomorphic to a subgroup of $G/H \times G/K$, which is nilpotent since it is a direct product of nilpotent groups. □

Definition 6.1.7. Let G be a finite group. The *nilpotent residual* of G, denoted $G^{\mathcal{N}}$, is the smallest normal subgroup N of G such that G/N is nilpotent.

Note that by Proposition 6.1.6, the nilpotent residual of a finite group G is well defined. It is the intersection of all normal subgroups $N \trianglelefteq G$ such that G/N is nilpotent.

Proposition 6.1.8. *Let G be a finite group and p the smallest prime divisor of $|G|$. Let $P \in \mathrm{Syl}_p(G)$ and assume that P is cyclic. Then G has a normal p-complement.*

Proof. This is a corollary to Burnside's normal p-complement theorem. See [25, Corollary 5.14]. □

Theorem 6.1.9. (Holmes) *A finite group G is nilpotent if and only if there exists a normal subgroup of order d for each divisor d of $|G|$.*

Proof. See [23]. □

6.2 Proofs of the main results

The first main result that we will prove in this section is the following:

Theorem 6.2.1. *Let G be a partial HNE-group. Then G is soluble.*

We provide two proofs of this result, one which appeals to the Feit–Thompson odd order theorem and one which does not. As expected the former proof is easier to follow. Before providing either of these proofs, we require a few preparatory lemmas.

Lemma 6.2.2. *Let G be a partial HNE-group and p and q be distinct primes in $\pi(G)$. If G does not possess a subgroup of order pq, then $|G|_p = p$ and $|G|_q = q$.*

Proof. Since G is a partial HNE-group, there exists $\mathbb{Z}_p \cong P \leq G$ with $P \in \mathrm{Syl}_p(\langle P^G \rangle)$ and $\mathbb{Z}_q \cong Q \leq G$ with $Q \in \mathrm{Syl}_q(\langle Q^G \rangle)$. By the Frattini argument we may write

$$G = \langle P^G \rangle N_G(P).$$

Since G does not possess a subgroup of order pq, $N_G(P)$ must be a q'-group. Hence all q-elements are contained in $\langle P^G \rangle$. In particular $\langle Q^G \rangle \subseteq \langle P^G \rangle$. Similarly we find that $\langle P^G \rangle \subseteq \langle Q^G \rangle$ and so $\langle P^G \rangle = \langle Q^G \rangle$. Hence

$$G = \langle Q^G \rangle N_G(P),$$

and so $Q \in \mathrm{Syl}_q(G)$. Thus $|G|_q = q$. Similarly $P \in \mathrm{Syl}_p(G)$ and so $|G|_p = p$. □

The next couple of results help us describe the normal closure of the Sylow subgroups of partial HNE-groups. This will prove crucial to both proofs of Theorem 6.2.1.

Lemma 6.2.3. *Let G be a partial HNE-group, $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$. If $r \in \pi(G)$ such that $r \neq p$, then $|\langle P^G \rangle|_r = 1$ or r.*

Proof. Let $r \in \pi(G)$ such that $r \neq p$. Note that if $|G|_r \leq r$, then the result follows trivially, so we may assume that $|G|_r > r$. Then by Lemma 6.2.2, there exists a subgroup $B \leq G$ of order pr. Since G is a partial HNE-group, there is a Hall normally embedded subgroup $H \leq G$ of order pr. Then $P\langle H^G \rangle$ is a subgroup of G with $|P\langle H^G \rangle|_p = |G|_p$ and $|P\langle H^G \rangle|_r = r$. By hypothesis, there is a Hall normally embedded subgroup $K \leq G$ with $|K| = |P\langle H^G \rangle|$. Now $P \subseteq \langle K^G \rangle$ and so $\langle P^G \rangle \subseteq \langle K^G \rangle$. Thus $|\langle P^G \rangle|_r \leq r$. □

Corollary 6.2.4. *Let G be a partial HNE-group, $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$. If $r \in \pi(G)$ such that $r < p$, then $|\langle P^G \rangle|_r = 1$*

Proof. Let s be the smallest prime in $\pi(\langle P^G \rangle)$ and suppose $s \neq p$. Then $|\langle P^G \rangle|_s = s$ by Lemma 6.2.3, and so in particular, $\langle P^G \rangle$ has a cyclic Sylow s-subgroup. By Proposition 6.1.8, $\langle P^G \rangle$ has a normal s-complement S. Now S is characteristic in $\langle P^G \rangle$ and so $S \trianglelefteq G$. Thus $\langle P^G \rangle \subseteq S$ since S is a normal subgroup of G which contains P. However, $\langle P^G \rangle \nsubseteq S$ and so $s = p$. □

Corollary 6.2.5. *Let G be a partial HNE-group. Then G has a normal 2-complement.*

Proof. Let N be the product of the normal closures of the Sylow subgroups of G of odd order, so

$$N = \prod_{\substack{P \in \mathrm{Syl}_p(G) \\ p \neq 2}} \langle P^G \rangle.$$

Then N is a normal 2-complement in G. □

Proof. (of Theorem 6.2.1) By Corollary 6.2.5, G has a normal 2-complement N. By the Feit–Thompson odd order theorem, N is soluble. Also, G/N is soluble since it is a 2-group. Thus G is soluble. □

Proof. (of Theorem 6.2.1 without an appeal to the Feit–Thompson odd order theorem) Let $\pi(G) = \{p_1, \ldots, p_n\}$ where p_i is the i^{th} largest prime in $\pi(G)$. Let $P_i \in \mathrm{Syl}_{p_i}(G)$ for $1 \leq i \leq n$. By Corollary 6.2.4 we have the following normal series

$$1 \trianglelefteq \langle P_n^G \rangle \trianglelefteq \langle P_n^G \rangle \langle P_{n-1}^G \rangle \trianglelefteq \ldots \trianglelefteq \langle P_n^G \rangle \ldots \langle P_2^G \rangle \trianglelefteq \langle P_n^G \rangle \ldots \langle P_1^G \rangle = G$$

where $\langle P_n^G \rangle \ldots \langle P_i^G \rangle / \langle P_n^G \rangle \ldots \langle P_{i+1}^G \rangle \cong P_i$ for all $1 \leq i \leq n-1$ and $\langle P_n^G \rangle = P_n$. So G possesses a normal series where each section is a p-group for some prime p. Hence G is soluble. □

Corollary 6.2.6. *Let G be a partial HNE-group, $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$. Then $\langle P^G \rangle = PN$ where N is a slim subgroup whose order is divisible only by primes greater than p.*

Proof. Note that G is soluble by Theorem 6.2.1, and so $\langle P^G \rangle$ is also soluble. Thus there exists a Hall p'-subgroup N of $\langle P^G \rangle$ by Hall's theorem. Now N is slim by Lemma 6.2.3 and has order divisible only by primes greater than p by Corollary 6.2.4. □

Corollary 6.2.7. *Let G be a partial HNE-group. Then $f(G)$ is bounded above by $|\pi(G)|$.*

Proof. This is clear from the second proof of Theorem 6.2.1. □

Corollary 6.2.7 gives us a nice bound on the Fitting height of partial HNE-groups. However, when the order of the group has more than two distinct prime divisors, we can do better.

Theorem 6.2.8. *Let G be a partial HNE-group. Then $f(G) \leq 2$.*

We require a few lemmas before proving Theorem 6.2.8.

Lemma 6.2.9. *Let G be a slim group. Then both $F(G)$ and $G/F(G)$ are cyclic.*

Proof. Since G is slim, it follows that $F(G)$ is also slim. So $F(G)$ may be written

$$F(G) \cong C_1 \times \ldots \times C_n$$

where each C_i is cyclic of prime order and $C_i \cong C_j$ if and only if $i = j$. Thus $F(G)$ is cyclic.

Now $C_G(F(G)) \subseteq F(G)$ since G is soluble, and so $G/F(G)$ acts faithfully on $F(G)$. Thus $G/F(G)$ embeds into $\mathrm{Aut}(F(G))$. Since $F(G)$ is cyclic, $\mathrm{Aut}(F(G))$ is abelian, and so $G/F(G)$ is abelian. Therefore, since $G/F(G)$ is slim, it may also be written

$$G/F(G) \cong D_1 \times \ldots \times D_m$$

where each D_i is cyclic of prime order and $D_i \cong D_j$ if and only if $i = j$. Thus $G/F(G)$ is also cyclic. □

Lemma 6.2.10. *Let G be a group. Suppose there exists a slim normal subgroup $N \trianglelefteq G$ such that $|G : N| = p^n$ for some prime p. Then $f(G) \leq 2$.*

Proof. By Lemma 6.2.9 we have that $F(N)$ is cyclic. In particular, $\mathrm{Aut}(F(N))$ is abelian. Thus

$$[G, N] \leq C_N(F(N)) = F(N).$$

Set $\overline{G} = G/F(N)$. Then $\overline{N} \subseteq Z(\overline{G})$ since

$$[\overline{G}, \overline{N}] = \overline{[G, N]} \subseteq \overline{F(N)} = 1.$$

Thus

$$\overline{G} = \overline{P} * \overline{N}$$

where $P \in \mathrm{Syl}_p(G)$. Since both $\overline{P}$ and $\overline{N}$ are nilpotent, $\overline{G}$ is nilpotent. Thus $f(G) \leq 2$. □

Henceforth, when we specify that G is a partial HNE-group we will assume the following:

1. $\pi(G) = \{p_1, \ldots, p_n\}$ where $p_{i+1} > p_i$ for all $1 \le i \le n-1$;
2. $P_i \in \mathrm{Syl}_{p_i}(G)$;
3. $\langle P_i^G \rangle = P_i N_i$ where N_i is a slim subgroup whose order is only divisible by primes larger the p_i.

Lemma 6.2.11. *Let G be a partial HNE-group. Then for each i we have that the subgroup N_i is normal in $\langle P_i^G \rangle$.*

Proof. This is immediate since for each i we have

$$N_i = \langle P_i^G \rangle \bigcap \prod_{\substack{P \in \mathrm{Syl}_p(G) \\ p > p_i}} \langle P^G \rangle.$$

$\square$

Corollary 6.2.12. *Let G be a partial HNE-group. Then for each i we have $f(\langle P_i^G \rangle) \le 2$ and $\langle P_i^G \rangle / F(\langle P_i^G \rangle)$ is a p_i-group.*

Proof. The first claim is immediate from Lemmas 6.2.10 and 6.2.11.

Set $\overline{G} = G/F(\langle P_i^G \rangle)$. Now $\langle P_i^G \rangle / F(\langle P_i^G \rangle)$ is nilpotent and so

$$\overline{\langle P_i^G \rangle} = \overline{P_i} \times \overline{N_i}.$$

However,

$$\langle \overline{P_i}^{\overline{G}} \rangle = \overline{\langle P_i^G \rangle}$$

and so this forces $\overline{N_i} = 1$. Hence $N_i \subseteq F(\langle P_i^G \rangle)$ and so $\langle P_i^G \rangle / F(\langle P_i^G \rangle)$ is indeed a p_i-group. $\square$

Corollary 6.2.13. *Let G be a partial HNE-group. Then the subgroups N_i are cyclic.*

Proof. By Corollary 6.2.12 we have $N_i \subseteq F(\langle P_i^G \rangle)$ for all i. Hence the subgroups N_i are cyclic since they are nilpotent and slim. $\square$

So for any partial HNE-group G and $P \in \mathrm{Syl}_p(G)$ we have that

$$\langle P^G \rangle = PN$$

where N is a normal cyclic slim subgroup of G whose order is only divisible by primes greater than p. So given that the normal closures of the Sylow subgroups of partial HNE-groups are described in this way, we see in some sense that such groups are not far from being nilpotent, as is the assertion of Theorem 6.2.8.

Proof. (of Theorem 6.2.8) For each i we have $F(\langle P_i^G \rangle) \subseteq F(G)$ and

$$\langle P_i^G \rangle / F(\langle P_i^G \rangle)$$

is nilpotent. Set $\overline{G} = G/F(G)$. Then $\overline{G}$ is generated the subgroups $\overline{\langle P_i^G \rangle}$. Since each of these subgroups is normal and nilpotent in $\overline{G}$, we have that $\overline{G}$ is nilpotent. Hence $f(G) \leq 2$. □

We now move on to showing that partial HNE-groups are CLT-groups. This will show that a group is a partial HNE-group if and only if it is a HNE-group. As such, these notions are equivalent, and the running hypothesis in [31] that G be a CLT-group is redundant. We finish by giving necessary and sufficient conditions for a group to be a HNE-group.

Theorem 6.2.14. *Let G be a partial HNE-group. Then G is a CLT-group.*

Proof. We may write

$$|G| = p_1^{r_1} \cdots p_n^{r_n}$$

where $r_i \in \mathbb{N}$. We now construct a subgroup $H \leq G$ such that

$$|H| = p_1^{s_1} \cdots p_n^{s_n}$$

where $s_i \in \mathbb{N} \cup \{0\}$. Let P_1 be a subgroup of order $p_1^{s_1}$. Let P_2 be a Hall normally embedded subgroup of order $p_2^{s_2}$. Consider the following subgroup

$$P_1 \langle P_2^G \rangle.$$

Then $P_1 \langle P_2^G \rangle = P_1 P_2 N$ where $N \trianglelefteq G$ is a slim subgroup whose order is divisible only by primes greater than p_2. Thus $P_1 \langle P_2^G \rangle$ contains a Hall subgroup of order $p_1^{s_1} p_2^{s_2}$. Call this subgroup K. Now let P_3 be a Hall normally embedded subgroup of G of order $p_3^{s_3}$ and consider $K \langle P_3^G \rangle$. By the same argument $K \langle P_3^G \rangle$ contains a subgroup of order $p_1^{s_1} p_2^{s_2} p_3^{s_3}$. We can repeat this process until we have a subgroup of order $|H|$. □

Theorem 6.2.14 tells us that partial HNE-groups are precisely the HNE-groups. Henceforth, we will refer to such groups only as HNE-groups. However, the notation that we set out earlier regarding such groups will remain the same.

Proposition 6.2.15. *Let G be a HNE-group. The subgroup $N \trianglelefteq G$ generated by the subgroups N_i is slim and cyclic.*

Proof. Suppose that N is not slim. Then there exists primes p_i, p_j and p_k with $p_i < p_j < p_k$ and Sylow subgroups $P_i \in \mathrm{Syl}_{p_i}(G)$, $P_j \in \mathrm{Syl}_{p_j}(G)$ such that p_k divides both $|\langle P_i^G \rangle|$ and $|\langle P_j^G \rangle|$ but not $|\langle P_i^G \rangle \cap \langle P_j^G \rangle|$. By Theorem 6.2.14, G is a CLT-group. Thus there exists a subgroup of order $|P_i||P_j|p_k$. In particular, since G is a HNE-group, there exists a Hall normally embedded subgroup H of order $|P_i||P_j|p_k$. Now $\langle H^G \rangle$ contains both P_i^G and P_j^G and thus contains the subgroup $\langle P_i^G \rangle \langle P_j^G \rangle$. However, $|\langle P_i^G \rangle \langle P_j^G \rangle|_{p_k} = p_k^2$. This is a contradiction since $|\langle H^G \rangle|_{p_k} = p_k$.

Now N is nilpotent since it is generated by cyclic normal subgroups. Thus since N is slim, it must be cyclic. □

Now that we have established that HNE-groups are CLT-groups, some of the results hereon are similar to those found in [31]. Particular attention is drawn to [31, Theorems 10 and 11]. We also make clear how to construct the nilpotent residual of HNE-groups.

Corollary 6.2.16. *Let G be a HNE-group. Then G has a normal cyclic slim subgroup whose corresponding factor group G/N is nilpotent.*

Proof. By Proposition 6.2.15 the subgroup

$$N = \prod_i N_i$$

is normal, cyclic and slim. Since $N_i \subseteq N$ for all i, the Sylow subgroups of G/N are normal and hence G/N is nilpotent. □

We now show that the converse to Corollary 6.2.16 holds.

Theorem 6.2.17. *Let G be a group. Suppose G has a normal cyclic slim subgroup N such that G/N is nilpotent. Then G is a HNE-group.*

Proof. First note that every subgroup of N is characteristic since N is cyclic. Thus every subgroup of N is normal in G since $N \trianglelefteq G$.

Let $\pi(G) = \{p_1, \ldots, p_n\}$ and

$$|G| = p_1^{r_1} \cdots p_n^{r_n}$$

where $p_{i+1} > p_i$ for all $1 \leq i \leq n-1$. Let $H \leq G$ such that

$$|H| = p_1^{h_1} \cdots p_n^{h_n}$$

where $h_i \geq 0$ for all $1 \leq i \leq n$. Now G/N is nilpotent and so there exists a normal subgroup for every order dividing $|G/N|$ by Theorem 6.1.9. Let $K \trianglelefteq G/N$ be a normal subgroup of order

$$|K| = p_1^{k_1} \cdots p_n^{k_n}$$

where

$$k_i = \begin{cases} h_i & \text{if } h_i \geq 1 \text{ and } p_i \notin \pi(N) \\ h_i - 1 & \text{if } h_i \geq 1 \text{ and } p_i \in \pi(N) \\ 0 & \text{if } h_i = 0. \end{cases}$$

Let L be the inverse image of K in G. So

$$|L| = p_1^{l_1} \cdots p_n^{l_n}$$

where $l_i = 0$ or 1 if $h_i = 0$ and $l_i = h_i$ if $h_i \geq 1$. Suppose there exists $p_i \in \pi(L)$ such that $p_i \notin \pi(H)$. Then $|L|_{p_i} = p_i$. Let $P_i \in \mathrm{Syl}_{p_i}(L)$. Then $P_i \subseteq N$ and so $P_i \trianglelefteq G$. Therefore, L has a normal Hall subgroup A divisible by all primes in $\pi(L)$ which are not in $\pi(H)$. This subgroup A has a complement C in L of order $|H|$. Since $|C|$ is coprime to its index in L and $L \trianglelefteq G$, C is a Hall normally embedded subgroup of G of order $|H|$. If $|H| = |L|$, then L is a normal subgroup of order $|H|$ and hence a Hall normally embedded subgroup of order $|H|$. □

Proposition 6.2.18. *Let G be a HNE-group. The subgroup N of G is the nilpotent residual of G.*

Proof. Since G/N is nilpotent, we have that $G^{\mathcal{N}} \subseteq N$. Suppose $N \neq G^{\mathcal{N}}$. Then since N is slim, there exists $p \in \pi(N)$ such that $p \notin \pi(G^{\mathcal{N}})$. Let $P \in \mathrm{Syl}_p(N)$. Then since N is nilpotent and $P \cap G^{\mathcal{N}} = 1$, every p'-element of G commutes with P. Also, since $P \trianglelefteq G$ and $P \cong \mathbb{Z}_p$, $P \subseteq Z(P_1)$ where $P_1 \in \mathrm{Syl}_p(G)$. Thus $P \subseteq Z(G)$. Now $P \subseteq N$ and so there exists a prime $q < p$ and $Q \in \mathrm{Syl}_q(G)$ such that $P \subseteq \langle Q^G \rangle$. Now $\langle Q^G \rangle$ contains a Hall p'-subgroup H. This is a p-complement, but since $P \subseteq Z(G)$, $H \trianglelefteq \langle Q^G \rangle$ and so it is in fact a normal p-complement. Now $H \trianglelefteq G$ since H is characteristic in $\langle Q^G \rangle$. This is a contradiction since $Q \subseteq H \trianglelefteq G$ and $\langle Q^G \rangle \nsubseteq H$. Thus $N/G^{\mathcal{N}} = 1$ and so $N = G^{\mathcal{N}}$. □

Lemma 6.2.19. *Let G be a HNE-group. Let H be a Hall π-subgroup for some set of primes π. Then H is also a HNE-group.*

Proof. Since G is a HNE-group, it is soluble by Theorem 6.2.1. Hence, any normal π-subgroup of G is contained in H. Let $K \leq H$ be a subgroup of H. Then since G is a HNE-group, there exists a subgroup L with $|K| = |L|$ which is Hall normally embedded in G. Note that every G-conjugate of L is also Hall normally embedded in G and that some conjugate of L is contained in H by Hall's theorem. We can assume without loss of generality that $L \subseteq H$. Then

$$L \subseteq \langle L^G \rangle \cap H \trianglelefteq H.$$

Now L is a Hall subgroup of $\langle L^G \rangle$, so it is certainly a Hall subgroup of $\langle L^G \rangle \cap H$. Hence L is Hall normally embedded in H and $|K| = |L|$. Thus H is a HNE-group. □

Theorem 6.2.20. *Let G be a HNE-group. Then G is the split extension of its nilpotent residual.*

Proof. Let G be a minimal counterexample and N its nilpotent residual. First note that $|\pi(G)| > 1$, otherwise G is nilpotent and thus splits over its nilpotent residual. Also, since N is cyclic, any subgroup of N is characteristic, and so is normal in G since $N \trianglelefteq G$.

Let p be the largest prime in $\pi(G)$ and $P \in \mathrm{Syl}_p(G)$. Then $P \trianglelefteq G$ by Corollary 6.2.6. Let H be a Hall p'-subgroup of G. If $[H, P] = 1$, then $H^{\mathcal{N}} = N$. Now H is a HNE-group by Lemma 6.2.19 and so H splits over N by induction. Hence

$$H = K \ltimes N \text{ and } G = (K \ltimes N) \ltimes P.$$

Then KP is a complement to N in G.

Hence $[H, P] \neq 1$. By coprime action we have

$$P = C_P(H)[H, P].$$

Set $\overline{G} = G/N$. Now $[\overline{H}, \overline{P}] = 1$ and so $[H, P] \subseteq P \cap N$. Hence $[H, P] = P \cap N$ since $[H, P] \neq 1$ and $|P \cap N| = p$. Thus $|P : C_P(H)| = p$ and so $C_P(H) \trianglelefteq P$. Also $(P \cap N) \trianglelefteq P$ and $C_P(H) \cap (P \cap N) = 1$ and so

$$P = C_P(H) \times (P \cap N).$$

Let $M = H^{\mathcal{N}}$. Then $M \subseteq N \cap H$. In particular $M \trianglelefteq G$. Suppose $M \neq N \cap H$. Then $M(P \cap N) \neq N$ but $G/(M(P \cap N))$ is nilpotent. This is a contradiction, hence $M = N \cap H$.

Now H is a Hall subgroup of G and so is a HNE-group by Lemma 6.2.19. Thus H splits over M by induction. So

$$H = K \ltimes M \text{ and } G = (K \ltimes M) \ltimes (C_P(H) \times (P \cap N)).$$

Now $M(P \cap N) = N$ and $KC_P(H)$ is a complement to N in G. □

Note that Theorems 6.2.17 and 6.2.20, Proposition 6.2.18 and Corollary 6.2.16 confirm Theorem 6.0.1. Since proving Theorem 6.0.1 the author has been informed that the same result has also been proved by A. Ballester-Bolinches and S. Qiao and is due to be published in Archiv der Mathematik [4]. Their proof is by induction and is thus very different to the one presented in this chapter. In particular, since they work with a minimal counterexample, many of the structural properties of HNE-groups are not presented.

We close this chapter by further putting into context the results which have been established. We also outline some potential further research problems.

6.3 Related work

We mentioned at the beginning of this chapter the result which states that a finite group G is nilpotent if and only if there exists a normal subgroup H of order $|B|$ for each $B \leq G$. We relaxed the normality condition and characterised HNE-groups as precisely those which are the split extension of a normal cyclic slim subgroup by a nilpotent group. We now mention other types of groups whose characterisations mirror those of the aforementioned groups.

A group D is called a *Dedekind* group if and only if *every* subgroup of D is normal. For example all abelian groups are Dedekind groups (a nonabelian Dedekind group is called a *Hamiltonian* group). Such a group D is either abelian or may be written

$$D \cong Q_8 \times R \times S$$

where Q_8 is the quarternion group of order 8, R an elementary abelian 2-group and S an abelian group with all its elements of odd order [39, Theorem 5.3.7, page 139]. Now a group G such that *every* subgroup of G is Hall normally embedded is the split extension of a normal cyclic slim subgroup by a Dedekind group [32, Theorem 3.4]. So if we begin with a group where *every* subgroup is normal, we obtain Dedekind groups. Then replacing 'normal' by 'Hall normally embedded' we get precisely those groups which are the split extension of a normal cyclic slim subgroup by a Dedekind group. However, if we begin with a group where *there exists* a normal subgroup of each subgroup order, we obtain nilpotent groups. Then replacing 'normal' by 'Hall normally embedded' we obtain precisely HNE-groups which are the split extension of normal cyclic slim groups by nilpotent groups.

It is interesting to consider what happens when we replace normal by subnormal. In [32, Theorem 3.3] groups in which *every* subgroup is Hall subnormally embedded (one which is a Hall subgroup of its subnormal closure) are characterised. One thing to notice from this characterisation by Li and Liu is that such groups are the split extension of their nilpotent residual $G^{\mathcal{N}}$. In particular, once we factor out $G^{\mathcal{N}}$ we obtain a nilpotent group. This observation becomes particularly interesting when we realise that nilpotent groups may be characterised as groups in which *every* subgroup is subnormal [25, Lemma 2.1].

Now finite nilpotent groups may also be characterised as those which, for every subgroup, *there exists* a subnormal subgroup of the same order. We can see this as follows. Note that in a finite group G, a subgroup H is contained in $F(G)$ if and only if H is nilpotent and subnormal in G (see [25, Theorem 2.2]). Let G be a finite group such that for each $B \leq G$ there exists a subnormal subgroup $H \leq G$ such that $|B| = |H|$. Then in particular, for each $p \in \pi(G)$, there exists $P \in \mathrm{Syl}_p(G)$ such that P is subnormal in G. Then since P is nilpotent, we must have $P \subseteq F(G)$. Hence G is contained in $F(G)$ and so it must be nilpotent. To prove the converse we note that a finite group is nilpotent if and only if every subgroup is subnormal (see [25, Lemma 2.1]). Hence a finite nilpotent group certainly has a subnormal subgroup for each subgroup order.

It would be interesting to characterise groups where *there exists* a Hall subnormally embedded subgroup of each subgroup order. Do these groups also split over their nilpotent residual? Or to put it another way: are these groups also the split extension of a normal subgroup N by a group in which *there exists* a subnormal subgroup of each subgroup order? (Thus mirroring the characterisations above.) If this is the case, then are there also comparisons to be made between the nilpotent residuals of: groups in which *every* subgroup is Hall subnormally embedded; and, groups in which *there exists* a Hall subnormally embedded subgroup of each subgroup order? If there are comparisons to be made, then this would also reflect the characterisations that we have just considered. Namely, that the nilpotent residuals of both: groups in which *every* subgroup is Hall normally embedded; and, groups in which *there exists* a Hall normally embedded subgroup of each subgroup order; are normal, cyclic and slim.

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