CLASSICAL DIFFUSION OF A PARTICLE IN A ONE DIMENSIONAL RANDOM POTENTIAL

by

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Abstract

This thesis examines the topic of classical diffusion of a particle in the presence of disorder. The presence of disorder has the effect of subjecting the classical particle to an additional random potential and it is the form of this random potential that is of interest. We consider two forms of the random potential and calculate several disorder averaged quantities including the particles probability distribution which is described by the Fokker-Planck equation [1, 2] and the transport properties of the particle, including the mean-square displacement and the velocity and diffusion coefficients.

The first part of the thesis deals with a random potential that is characterized by short-ranged correlations and some constant term known as drift. This is a problem that was first formulated some thirty years ago by Sinai [3], who showed that for a particle with zero drift the mean-square displacement had the form $\langle x^2(t)\rangle \approx \ln^4(t)$. We employ a combination of Green's functions, distribution functions and asymptotic matching to not only analytically re-produce this result, but also the expectation value of the probability distribution and all transport properties for an arbitrary value of drift, which is an original result.

For the second half of the thesis we consider essentially the same problem again but with a random potential that has long-ranged logarithmic correlations. To solve the problem we use the renormalization and functional renormalization group techniques in an attempt to re-create known results in an effort to find a general method that can deal with such one-dimensional systems. We calculate the particles distribution function using a functional renormalization group approach, which we use to partially re-derive the phase transition in the first-passage time distribution.

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To my parents...

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Chapter 1

INTRODUCTION - OVERVIEW OF THESIS

As the title suggests the topic of this thesis is that of classical diffusion of a particle in a one-dimensional random potential. The starting point of all classical diffusion problems of the type we consider is the Fokker-Planck equation [1, 2] which describes the evolution of the probability distribution of our classical particle. The Fokker-Planck equation is characterized by a random stationary potential and in the course of this thesis we consider two different realizations of this random potential. The thesis is therefore divided into two main chapters where we solve the Fokker-Planck equation for the two different random potentials using completely different techniques united by their origins in quantum mechanics. The first potential we consider is one with short-ranged delta function correlations with drift, the so called Sinai model [3], where we use the techniques developed for the seemingly different problem of the Anderson model [4] to solve for the probability distribution. For the second potential where we consider long-ranged logarithmic correlations [5] we use the methods of renormalization [6] and functional renormalization [7] group theory to re-derive known results in an attempt to find a method of dealing with such one-dimensional problems. The aim of this chapter is to give a complete overview of the thesis indicating the important steps we take and the methods we use.

For the first chapter of the thesis we build up all of the foundations required to understand the topic of classical diffusion of a particle in a one-dimensional random potential. We begin by considering a collection of particles dissolved in a fluid. We use the fact that the number density of these particles is conserved to find they follow a conservation law which leads to Fick's law. We will then derive the standard Einstein's relation [8] which surprisingly relates

the diffusion coefficient to temperature using the idea that we can ignore interactions between the particles in the fluid in the dilute limit. In the dilute limit we assume the particles are far enough away from each other so there are no interactions between them, which means we can just consider the motion of an individual particle. Doing so we find the particles motion is random, a phenomena known as Brownian motion [9, 10] which is described by two forces, random thermal fluctuations which drive the particles motion and viscous drag slowing it. These random thermal fluctuations are known as stochastic forces and we will explain how such a force arises. We will describe the particles motion using Langevin theory which we will use to derive the Diffusion equation and show the usual $\langle x^2(t)\rangle = 2Dt$ behaviour for diffusion. We call this the pure case as there are no effects of disorder. After a brief general discussion of the effects of disorder we consider a particle which is subject to both thermal fluctuations and disorder which creates an additional random force on the particle. For this situation the probability distribution is given by the Fokker-Planck equation, which as we have previously stated is the central equation to the whole thesis. The Fokker-Planck equation is characterized by a random potential and it is at this point where we introduce the general form of the correlation function

$$\langle (U(x) - U(x'))^2 \rangle = K(x - x') = \gamma \left[(x - x')^{\alpha} + \ln \left(\frac{|x - x'|}{a} \right) (1 - \delta^{(a)}(x - x')) \right], \quad (1.0.1)$$

which is written in terms of a general parameter α [11] which we will be used to separate the correlation function into cases, γ is the strength of disorder, a is the ultraviolet cut-off and $\delta^{(a)}(x - x') = 1$ for $|x - x'| < a$ and 0 for $|x - x'| > a$. A full explanation of this correlation function will be given in chapter two. We give a brief discussion into the two limits of interest for this correlation function, but with all of the main details presented in the chapter dedicated to that particular case. For the choice of $\alpha = 0$ we have logarithmic correlations which is an excellent model for relaxation in glasses [12], transport in solids [13] and vortex glass dynamical scaling [11]. For the case of $\alpha = 1$ we have Sinai's diffusion which is an excellent model for the dynamics of random field magnets [14], dislocation dynamics [14] and glass dynamics [11]. All of the work considered upto this point will be for the continuous version of the problem. For the last section of the chapter we consider the Master

equation and the Lattice Hopping model [13], which is a completely equivalent approach to the continuous problem when we take the lattice spacing to zero. As we will come to see the majority of the work done on classical diffusion has been based on the Lattice Hopping model. We will also give a discussion on what has been achieved using the method and use the description to illustrate the physical processes occurring in the continuous problem.

For the following chapter we consider our first disorder averaged quantity, the first-passage time distribution [15], which gives a description of the time spent in a system as a function of its initial starting point. As we will see the first-passage time distribution will provide a very effective formalism for the calculation of various characteristics of the particle traveling in some random potential. The equation which describes this first-passage time distribution is the Pontryagin equation, which we will derive from the Fokker-Planck equation. The Pontryagin equation is a simple second order differential equation which can be solved with an integration factor to find the n^{th} moment of the distribution. We follow this approach, but also include an alternative method where we follow the method of Lerner and Yurkevich [16] and use a Green's function written in terms of stochastic functions with their boundary conditions to the left or right of some source. The aim of this is to illustrate the method we will be using in the following chapter to solve the Fokker-Planck equation for short-ranged delta function correlations.

With the solution to the n^{th} moment of the first-passage time distribution we follow the arguments of [15] and derive the tails of the probability in terms of the general correlation function. From this we calculate the mean-square displacement and give the first of many derivations of Sinai's diffusion $\langle x^2(t)\rangle \approx \ln^4(t)$ [3] result. We also follow further arguments of [15] to derive the mean velocity of the particle from the solution to the Pontryagin equation. We first solve for the mean velocity in terms of the general correlation function $K(x-x')$ and then follow the arguments of [11] to derive its form for four specific cases. We first consider the two cases of Sinai's diffusion and logarithmic correlations, which will be followed by a consideration of the intermediate case of $0 < \alpha < 1$ which is known as creep motion and $\alpha < 0$ known as the ohmic regime [11]. We will find for all $\alpha > 1$ that the integral which describes the mean velocity diverges and its value is always zero. For Sinai's diffusion we find that the model exhibits algebraic distributions of waiting times giving rise to aging

phenomena [17], which as we will see mimics essential aspects of spin glass behaviour. For the case of logarithmic correlations we find that it resembles the vortex glass transition which separates the creep and ohmic behaviours.

For the next chapter of the thesis we will be considering a particle diffusing in a random environment with short ranged delta function correlations and drift, the so called Sinai model [3]. This is a well formulated problem which has been in existence for the past thirty years. There has been much interest in the problem, but very few exact analytical solutions have been found. In fact there is only an exact analytical solution for zero drift [18] which agrees with the scaling results of Kesten [19, 20]. For the case of finite drift there has been no exact analytical solution up until now, only simple scaling arguments [14, 21] based on the discrete model and a renormalization group analysis [22]. We will begin by following the scaling arguments of [14] to show that we expect a variety of diffusive behaviours and to see that it is the ratio of drift velocity to the strength of disorder that controls the physics of the problem. We calculate two disordered averages the return probability and the probability of finding the particle at some arbitrary position in space. The approach we take follows similar methods to the ones we used in solving for the moments of first-passage time, where we set up a Green's function and then introduce stochastic functions with their boundary conditions to the left and right. We begin by giving a general formalism to the problem and introduce two distribution functions each described by their own Fokker-Planck equation. The aim is to solve for each of these distribution functions with their form depending on the disordered average under consideration. For the return probability, which calculates the probability of the particle returning to its origin we find the simplification that makes the distribution functions solvable is to assume stationary solutions and we find a variety of processes depending on the value of drift velocity. We find the return probability description paints a very useful picture of the effects of drift and disorder. In addition to this we also calculate the moments of the return probability and show that it is not a self-averaging quantity.

For the probability of finding the particle at some arbitrary position we solve the Fokker-Planck equations for the distribution functions by writing them in the form of a Schrödinger equation. We will then use the method of asymptotic matching to find the eigen-values

of these equations, which requires us to find two asymptotic regions of the Schrödinger potential and match them in the region where they overlap. For the case of zero drift we find that all the states live in a quasi-continuous region and we re-create the known result of Sinai diffusion [14, 18]. In contrast, for constant drift we find the emergence of a gap in the potential of the Schrödinger equation which leads to regions with discrete states and it is in these discrete states which we need to solve the Schrödinger equation in. It is this emergence of a gap in the potential which can not be treated correctly by previous techniques [18] and is the reason why there has, until now, not been an exact analytical solution for the case with drift. Solving in these regions of the potential leads to several different diffusive behaviour including Lévy flight, sub-diffusive and diffusive with re-normalized diffusion and velocity coefficients.

Chapter five will provide the background to the techniques that we will use in the final chapter of the thesis where we solve the Fokker-Planck equation with logarithmic correlations using the renormalization [23, 24] and functional renormalization [7] group techniques. We begin by considering Grassmann variables as an alternative to fermionic operators with their complicated anti-commutation relations [25]. This leads naturally to fermionic coherent states and the resolution of the identity. This will allow us to derive the fermionic functional integral. With the fermionic path integral and Grassmann numbers understood we will finally give an introduction to renormalization group theory and the three step recipe [26] for deriving the Gell-Mann Low equations which describe how the parameter of interest changes with time[27]. The three step process, as we will see, involves the sub-division of fields into fast and slow modes, where the definition of fast and slow is given by some cut-off in momentum space. The second step is where the renormalization procedure takes place and we average over fast modes. This averaging procedure will lead to a new action defined on a different kinematic scale to what it was originally. Therefore the final step is to re-scale the new action so it is on the same kinematic scale as the old action. From this we can compare the new and old actions and see how the parameter of interest changes under the whole process.

The final chapter excluding the conclusion is where we use the techniques introduced in the previous chapter to solve the Fokker-Planck equation with logarithmic correlations.

We begin by introducing the first-passage time distribution and notice that for logarithmic correlations there is a phase transition between a low temperature glass state and a high temperature phase as a function of the moments [5]. The aim of this chapter is to re-create this transition using the renormalization and functional renormalization group approach as a test to see if it is an effective method for dealing with such one-dimensional problems. We will also discuss why the method we used to calculate the transport properties of a particle in short-ranged correlations can not be applied to logarithmic correlations. We begin by considering why a system with logarithmic correlations describes such a phase transition and why short or longer ranged correlations only have one state or the other [28]. To do this we consider the competition between the entropy of the typical sites and the low energy states [28]. We will also give a description of the two-dimensional model and show a variety of different diffusive behaviour depending on certain natural vector constraints [6, 29] *i.e.* a solenoidal or potential field. The two-dimensional problem as we will see is an excellent model of the random motion of a traveller in a liquid with stationary random streams [6]. With the background understood we introduce the model which we will be considering and perform what is known as tree-level scaling to ensure the problem is scale invariant, a condition required to perform the renormalization group procedure. We then find the one-loop corrections to the diffusion and strength of disorder coefficients which will allow us to find how the mean-square displacement scales. We then consider the higher cumulants by allowing fluctuations in the diffusion coefficient [30, 31]. We will find that the higher cumulants only give a correction to themselves. We therefore consider what will be known as linear corrections to the higher cumulants and use the results and a functional renormalization group approach to find the whole distribution function which will be used to re-create the phase transition in the first-passage time distribution. We will find that this functional renormalization group equation only partially re-creates the phase transition in the first-passage time distribution. In an attempt to fix this we consider non-linear corrections to the higher cumulants and derive a new functional renormalization group equation.

Chapter 2

INTRODUCTION TO CLASSICAL DIFFUSION

For this chapter of the thesis we will be building up all of the physical ideas and mathematical concepts that make up classical diffusion. We begin with the idea that classical diffusion is simply the process of particles moving from a high concentration to a low concentration. When we consider the number of particles in such a situation we find their number density follows a conservation law which leads to Fick's law. We then use Fick's law to derive the standard Einstein's relation [8] which surprisingly relates the diffusion coefficient to temperature using the fact that in the dilute limit the interactions between the particles in the fluid can be ignored. As we are in the dilute limit we can consider the motion of a single particle to find that it is completely random, a phenomena known as Brownian motion. If we consider the forces acting on the particle we find for the pure case, *i.e.* the case of no disorder, the only forces acting on the particle are thermal fluctuations which are described by a random function and viscous drag which is proportional to the particles velocity. Using the standard Newton's law which relates force and acceleration we derive a Langevin equation for the particles motion. A Langevin equation is simply an equation which describes a stochastic variable. This Langevin equation will be used to derive the probability distribution for a single diffusing particle and we find that the equation that describes the probability distribution is the usual Diffusion equation. We will then solve the Diffusion equation using standard techniques and show the usual $\langle x^2(t)\rangle = 2Dt$ behaviour of diffusion. The main topic of this thesis though is to consider diffusion in the presence of disorder. We find that with the addition of disorder the particle is subject to an additional random force and the probability distribution for such a system is described by the Fokker-

Planck equation. The Fokker-Planck equation is characterized by a random force, and it is the form of this random force which is of interest. We therefore introduce the general correlation function for this random force as a function of some parameter α which separates several cases of interest. We conclude the chapter by describing an alternative formulation of the problem, the Lattice Hopping model, which is described by the Master equation.

2.1 Fick's Law

If we begin by considering normal diffusion we know that a collection of particles suspended in some fluid will spread out from a region of high concentration to that of a low concentration. If we consider a number of particles dissolved or suspended in this fluid we notice that their number does not vary with time. We therefore say the number density $n(x,t)$ obeys a conservation law

$$\frac{\partial n(x,t)}{\partial t} + \nabla . j(x,t) = 0, \tag{2.1.1}$$

where $j(x,t)$ is the current density which is defined by $j(x,t) = \sum_\alpha v_\alpha(t)\delta(x - x_\alpha(t))$, where $v_\alpha(t)$ is the velocity of the α^{th} particle. Now in thermal equilibrium the particles are distributed uniformly throughout the fluid and the thermal average of the number density $\langle n(x,t)\rangle$ is independent of both position x and time t. If we consider the problem when there is a spatially non-uniform density caused by some external potential, then we find it must eventually move back to its uniform state once the potential is turned off. This is due to the motion of the particles. We can therefore conclude that a spatially non-uniform density leads to a non-zero current and we would expect the current to be proportional to the gradient of the density, *i.e.* $j = -D\nabla n$, to find

$$\frac{\partial n(x,t)}{\partial t} = D\nabla^2 n(x,t), \tag{2.1.2}$$

where D is a constant, the diffusion coefficient. This is known as Fick's Law and it states that a spatially non-uniform density leads to currents in directions opposite to the direction of changes in densities, *i.e.* to currents to re-establish spatial uniformity of $n(x,t)$. Fick's law can be solved using standard mathematical techniques to show that

$n(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp[-\frac{x^2}{4Dt}]$, which is a function that spreads out over time from its initial position. We can use this expression for the conservation of number density to derive the Einstein's relation by considering the dilute limit, which is when we ignore interactions between particles.

2.1.1 External Potentials and the Einstein Relation

Fick's Law is only appropriate in situations when there is no external potential. If we consider densities of diffusing particles dilute enough so that interactions between them can be ignored then we can only consider the external force acting on a given particle and from molecules comprising the fluid for which it collides with. These collisions with molecules has the effect of introducing a friction force proportional to its velocity. Now in steady state this force must be equal to any external force

$$v_D = \frac{1}{\gamma} f_{ext} = -\frac{1}{\gamma} \nabla U(x), \tag{2.1.3}$$

where γ is the friction coefficient, $U(x)$ is some external potential and v_D is the drift velocity. This drift velocity gives rise to a drift current $j_D = n v_D$ in addition to the diffusion current. We therefore say the total current j_{tot} becomes

$$j_{tot} = -D\nabla n + j_D = -D\nabla n - \frac{n}{\gamma} \nabla U(x), \tag{2.1.4}$$

and we see the equation for the number density becomes

$$\frac{\partial n(x,t)}{\partial t} = D\nabla^2 n(x,t) + \frac{1}{\gamma} \nabla n(x,t) \nabla U(x) + \frac{1}{\gamma} n(x,t) \nabla^2 U(x). \tag{2.1.5}$$

In the thermodynamic equilibrium the total particle current must be zero and the density must satisfy the Boltzman relation $n_{eq} = \exp[-U(x)/T]$ leading to

$$j_{tot} \;=\; \frac{Dn}{T} \nabla U(x) - \frac{n}{\gamma} \nabla U(x) = 0, \tag{2.1.6}$$

and to satisfy the condition of zero current we find $D = \frac{T}{\gamma}$ which is the standard Einstein's relation, a result first derived by Einstein [8, 9].

For this section we have simply considered a collection of particles diffusing and have noticed that they follow a conservation known as Fick's law. We then considered the limit that the interactions between particles can be ignored to derive the standard Einstein's relation, this is known as the dilute limit. The dilute limit assumes that the particles are far enough away from each other so that the interactions between them can be ignored. This means we can just consider a single diffusing particle and the forces acting on it. On doing so we find the particles motion is random, a phenomena known as Brownian motion named after Robert Brown who first discovered it in 1827 whilst using a light microscope to make observations of pollen grains suspended in water.

2.2 Brownian Motion

Before we consider Brownian motion we should consider a particle immersed in a fluid under the influence of just a friction force, which is caused by the particle colliding with the molecules of the fluid. The equation that describes the particles motion is given by

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} = 0, \tag{2.2.1}$$

which can be solved to show $v(t) = v(0)\exp[-\gamma t/m]$, which shows that the momentum of the particle is transfered to the molecules in the fluid leading to a velocity that decays to zero. This differential equation (2.2.1) is a deterministic equation as the velocity $v(t)$ at a time t is completely determined by its initial value. This picture of the particle only subject to a friction force is only valid if the mass of the particle is large enough so that its velocity due to thermal fluctuations is negligible. From the equipartition law, the mean energy of the particle in one-dimension is

$$\frac{1}{2}m\langle v^2\rangle = \frac{1}{2}kT, \tag{2.2.2}$$

where k is the Boltzmanns constant and T is the temperature.

If we now return to the picture of a Brownian particle where we don't assume its mass is

much larger than those in the fluid then we must include thermal fluctuations. For this case when deriving the equation of motion for the particle we must consider thermal fluctuations and viscous drag which leads to a stochastic differential equation of the form

$$m\frac{d^2x}{dt^2} = \eta(t) - \gamma\frac{dx}{dt}, \tag{2.2.3}$$

where $\eta(t)$ describes the random thermal fluctuations which are assumed to have the form $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = G(t-t')$, which is the time correlation of the thermally fluctuating force. We will consider shortly why a stochastic force occurs. We choose $\langle\eta(t)\rangle = 0$ because we know the equation of motion of the average velocity $\langle v(t)\rangle$ should be given by (2.2.1). To find the form of the correlation function $G(t-t')$ we find that when we multiply two Langevin forces at different times we assume that their average value is zero for times $t-t'$ which are larger than the duration time of the collision time τ_0 *i.e.* $G(t-t') = 0$ for $|t-t'| \geq \tau_0$. This assumption would seem to be reasonable because the collisions of different molecules of the fluid with the particle are approximately independent. Usually the duration time τ_0 of a collision is much smaller than the relaxation time $\tau = 1/\gamma$ of the particles velocity. We therefore take the limit as $\tau_0 \to 0$ as a reasonable approximation leading to $\langle\eta(t)\eta(t')\rangle = G_0\delta(t-t')$. The delta function appears because otherwise the average energy of the particle would not be finite as it should be according to the equipartition law. This will be the case we are considering, which is known as Gaussian white noise. The term white noise is so called because its spectral density $S(\omega)$ which is the Fourier transform of the correlation function $G(t-t')$ is independent of the frequency ω [32] *i.e.*

$$S(\omega) = 2G_0\int_{-\infty}^{\infty}\exp[-i\omega\tau]\delta(\tau) = 2G_0. \tag{2.2.4}$$

In general, if the spectral density had some frequency dependence, then the noise would be termed coloured.

A differential equation of the kind (2.2.3), *i.e.* one that describes a stochastic variable is known as a Langevin equation [33], named after the French physicist Paul Langevin. The Langevin equation is a powerful tool for the study of dynamical properties of many inter-

esting systems in chemistry, physics and engineering [34, 35]. The success of the Langevin equation rests on the description of macroscopic quantities starting from microscopic dynamics, where the effect of fast degrees of freedom, for example a heat bath in statistical physics, can be taken into account by noise [36]. In the problem we are considering the stochastic force occurs because we can not treat the situation exactly. If we were to treat the problem exactly we would have to solve the coupled equations of motion for all the molecules of the fluid and for the particle and we would have no stochastic force. Due to the large number of particles in the fluid $\sim 10^{23}$ it is not generally possible to solve these coupled equations. Furthermore we do not know the initial values of all the molecules. As usual in thermodynamic problems we consider an ensemble of such particle and fluid systems, a Gibbs ensemble, then the stochastic force varies from system to system and the only thing we can do is to consider averages of the force for the ensemble. For example, we will show in this section that Gaussian noise leads to normal diffusion. Equivalently, such phenomena can also be described by the Fokker-Planck equation which describes the probability distribution of the particle and this is the approach we will be taking. For now though we will return to the Langevin description.

We can solve equation (2.2.3) via an integration factor to find the velocity $v(x)$ in terms of the random function $\eta(t)$

$$\frac{dx}{dt} = v(x) = v(0)\exp[-\frac{\gamma}{m}(t-t_0)] + \frac{1}{m}\int_{t_0}^{t}\eta(t')\exp[-\frac{\gamma}{m}(t-t')]dt', \tag{2.2.5}$$

and to find the thermally averaged velocity we employ the correlation function of $\eta(t)$ to find

$$\begin{aligned} \langle v(t)\rangle &= v(0)\exp[-\gamma(t-t_0)], \\ \langle v^2(t)\rangle - \langle v(t)\rangle^2 &= \frac{G_0}{2\gamma m}\Big[1-\exp[-\frac{\gamma}{m}(t-t_0)]\Big]. \end{aligned} \tag{2.2.6}$$

In the equilibrium limit as $t_0 \to -\infty$, $\langle v(t)\rangle = 0$ and

$$\langle v^2(t)\rangle = \frac{G_0}{2\gamma m} = \frac{kT}{m}, \tag{2.2.7}$$

which comes from the equipartition law, which states that the mean energy of the particle $\frac{1}{2}m\langle v^2\rangle = \frac{1}{2}kT$. This leads to a value $G_0 = 2D$ using the standard Einstein's relation. Using this result we define the final expression for the correlation function of $\eta(t)$, as $\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$. To proceed the usual assumption that is made for Brownian motion is consider the over-damped limit, which is to say the mass of the particle $m \to 0$. We therefore find the Langevin equation (2.2.3) becomes

$$\frac{dx}{dt} = \eta(t), \tag{2.2.8}$$

where we have written the equation in terms of dimensionless quantities. This equation can be solved exactly to show that for normal diffusion $\langle x(t)\rangle = 0$ and $\langle x^2(t)\rangle = 2Dt$, by simply using integration and the averages of $\eta(t)$. As we have stated previously, a completely equivalent approach we can take is to consider the probability distribution of the particle which is given by the Diffusion equation.

If we define the probability distribution for such a particle as $P(x,t) = \langle \delta(x(t)-x)\rangle$ where the average is taken over thermal fluctuations we can derive an equation for its evolution by simply differentiating it with respect to time t and using the Langevin equation (2.2.8) to find

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\langle \delta(x(t)-x)\frac{dx}{dt}\rangle = -\frac{\partial}{\partial x}\langle \delta(x(t)-x)\eta(t)\rangle, \tag{2.2.9}$$

where the term $\langle \delta(x(t)-x)\eta(t)\rangle$ can be found using the result of Furutsu-Novikov. The result of Furutsu-Novikov is simply a generalization of the splitting property of Gaussian averages and it states that for a zero-mean Gaussian process $u(t)$ with the correlation function $K(t,t')$ that

$$\langle u(t)R[u]\rangle = \int dt' K(t,t')\left\langle \frac{\delta R[u]}{\delta u(t')}\right\rangle, \tag{2.2.10}$$

where $R[u]$ is some functional of the Gaussian process $u(t)$. As this formula is crucial to this thesis we will sketch its proof in Appendix A. By considering the term $\langle \delta(x(t)-x)\eta(t)\rangle$ we find

$$\langle \delta(x(t)-x)\eta(t)\rangle \;=\; \int dt \langle \eta(t)\eta(t')\rangle \left\langle \frac{\delta}{\delta\eta(t')}\delta(x(t)-x)\right\rangle$$

$$= \quad 2D \lim_{t' \to t} -\frac{\partial}{\partial x} \langle \delta(x(t) - x) \frac{\delta x(t)}{\delta \eta(t')} \rangle = -D \frac{\partial P(x,t)}{\partial x}. \qquad (2.2.11)$$

To calculate the form of $\delta x(t)/\delta \eta(t')$ we simply integrated equation (2.2.8) and took the functional derivative to find $\frac{\delta x(t)}{\delta \eta(t)} = \frac{1}{2}$. Bringing this all together we find the equation that describes the evolution of the probability distribution $P(x,t)$

$$\frac{\partial P(x,t)}{\partial t} = D \frac{\partial^2 P(x,t)}{\partial x^2}, \qquad (2.2.12)$$

which is the Diffusion equation. This situation will be known as the pure case as we have not included any effects of disorder. We will now derive several results for the pure case including the probability distribution $P(x,t)$, the return probability, the mean-square displacement and the velocity of the particle and show that the relevant results agree with those we found using the Langevin equation approach.

2.2.1 The Pure Case

The solution to the Diffusion equation with the initial condition $P(x,0) = \delta(x - x')$ has the form

$$P(x, x', t) = \frac{1}{\sqrt{4\pi D t}} \exp\left[- \frac{(x - x')^2}{4Dt} \right], \qquad (2.2.13)$$

where x' is the starting point of the random walk. This result can be found using a variety of techniques including a separation of variables and Fourier and Laplace transforms. This is the probability of finding the particle at some point in space at some arbitrary time and it shows that over time the particle spreads out from its origin as expected for a diffusive particle. We can use the result to calculate another quantity, the return probability. The return probability is simply the probability of the particle returning to the origin of its random walk and can be found by simply setting $x = x'$. For the pure case we find a return probability of the form $P(t) = \frac{1}{\sqrt{4\pi t}}$. The return probability will be an important quantity when we solve the Fokker-Planck equation for short-ranged correlations.

The probability distribution $P(x,t)$ can also be used to calculate all of the moments, cumulants and subsequently the transport properties of the particle. The general form of

the moments can be found from the expression

$$\langle x^n(t)\rangle = \int_{-\infty}^{\infty} x^n P(x,t)dx. \tag{2.2.14}$$

The mean-square displacement which calculates the total distance traveled by the particle comes from the case of $n = 2$. If we calculate $\langle x(t)\rangle$ for the pure case we find $\langle x(t)\rangle = 0$. If we calculate the mean-square displacement we find the defining result of diffusion $\langle x^2(t)\rangle = 2Dt$, and we find the two results are in agreement with those we calculated from the Langevin equation approach. These results lead naturally to the definition of the diffusion and velocity coefficients

$$\begin{aligned} D &= \frac{1}{2}\lim_{t\to\infty}\frac{\langle x^2(t)\rangle - \langle x(t)\rangle^2}{t}, \\ V &= \lim_{t\to\infty}\frac{\langle x(t)\rangle}{t}, \end{aligned} \tag{2.2.15}$$

where the limit of $t \to \infty$ is chosen as we want the equilibrium behaviour of the particle. These results also define the first two cumulants Ξ_n of the distribution, $\Xi_1 = V$ and $\Xi_2 = D$ and we find the velocity of a particle in the pure case if zero. In later sections we will consider further cumulants of the distribution which are described by the cumulant expansion.

The Diffusion equation and its solution can be applied to more than just the problem of diffusion. If we consider Fourier's law and the conservation of energy we find the flow rate of heat through a surface is proportional to the negative temperature gradient across the surface, $Q = -k\partial_x u$ where u is the temperature and k is the thermal conductivity. In the absence of work done a change in the internal energy per unit volume in the material δQ is proportional to the change in temperature, $\delta Q = c_p\rho\partial_t u$, where c_p is the specific heat capacity and ρ is the density of the material. If we now consider an increase in the energy in a small spatial region in a small time period we find it is entirely accounted for by the flux of heat across the boundaries. By conservation of energy we find

$$\frac{\partial u}{\partial t} = \frac{k}{c_p\rho}\frac{\partial^2 u}{\partial x^2}, \tag{2.2.16}$$

which as we can see is the Diffusion equation with $D = k/C_p\rho$. The solution of this equation has the form of (2.2.9) which is to say that heat spreads out from warmer to colder areas, much like a high concentration of particles diffusing to a low concentration.

As we have already stated this is known as the pure case as we have not included the effect of disorder. The main topic of this thesis is to consider classical diffusion of a particle in a one-dimensional random environment which is caused by disorder. We will see that the effect of disorder is to subject the particle to some additional random force. At no point in the discussion will we give a description of what causes the disorder, only the effect it has on the particles motion. As the effect of disorder is completely random, we can not say anything about its functional dependence, we can only describe its correlation function as we did with the thermal fluctuations. The idea is again that we can't solve a system with impurities exactly as we would need to solve the coupled equations of motion for all the molecules of the fluid, for disorder and with the particle. The motion of the molecules is described by thermal fluctuations as usual, and the motion of disorder is given by the random term and we simply have a Gibbs ensemble of molecule, disorder and particle systems. The addition of this new random function introduces a new term to the Langevin equation that describes the particles motion. This means the equation that describes the probability distribution also changes. To find the new equation we follow the same steps as we did previously to derive the Diffusion equation to find the Fokker-Planck equation.

2.2.2 The Fokker-Planck Equation

As we have previously stated the addition of disorder into the system has the effect of subjecting the particle to an additional random potential $U(x)$ or random velocity potential $V(x)$. We therefore find the Langevin equation that describes the particles motion is given by

$$\frac{dx}{dt} = V(x(t)) + \eta(t), \tag{2.2.17}$$

where we are still considering the over-damped limit. If we follow the same steps as we did previously, *i.e.* differentiate with respect to time and use the Langevin equation (2.2.17) we

find

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial x}\langle\delta(x(t)-x)\eta(t)\rangle - \frac{\partial}{\partial x}\langle\delta(x(t)-x)V(x)\rangle \\ &= -\frac{\partial}{\partial x}\langle\delta(x(t)-x)\eta(t)\rangle - \frac{\partial}{\partial x}\Big(V(x)P(x,t)\Big). \end{aligned} \tag{2.2.18}$$

If we use the result of Furutsu-Novikov to perform the thermal averages, as we did for the pure case, we derive the Fokker-Planck equation which describes the evolution of the probability distribution $P(x,t)$

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\Big(V(x)P(x,t)\Big), \tag{2.2.19}$$

where $D\frac{\partial^2 P}{\partial x^2}$ is known as the diffusive term and $\frac{\partial}{\partial x}\Big(V(x)P\Big)$ is the advection term. We are allowed to perform the averaging over thermal fluctuations and disorder independently as they occur on different time scales. For example thermal fluctuations occur on a much faster time scale than those caused by disorder. This is the usual Fokker-Planck equation with some arbitrary random velocity potential which describes the equation of motion for the transition probability $P(x,t|x',t')$. A completely equivalent formalism is that of the backwards Fokker-Planck equation which is found by differentiating with respect to x' and t', *i.e.* with respect to the stochastic variable $\eta(t')$ at the earlier time of $t' < t$. We will derive the backwards Fokker-Planck equation in chapter three where we consider the first-passage time distribution, which describes the time it takes a diffusive particle to leave a pre-defined region. It is obvious from (2.2.19) that the Fokker-Planck equation is a non-Hermitian operator [37, 38]. This will be an important point to consider when we use the renormalization group approach to solve for long-ranged logarithmic correlations. As we will see the non-Hermiticity of the Fokker-Planck equation leads to its functional integral being divergent.

Equation (2.2.19) is a Fokker-Planck equation that describes the motion of a general Brownian particle under the influence of disorder. This is a special form of the Fokker-

Planck equation. The most general Fokker-Planck equation for one variable x has the form

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}(D^{(2)}(x)P(x,t)) - \frac{\partial}{\partial x}(D^{(1)}(x)P(x,t)), \tag{2.2.20}$$

where $D^{(2)}(x) > 0$ is called the diffusion coefficient and $D^{(1)}(x)$ is the drift coefficient. If we solve this Fokker-Planck equation we obtain a distribution function from which we can calculate any averaged macroscopic variable by integration. As the Fokker-Planck equation is not restricted to systems near equilibrium it can be applied to systems far from thermal equilibrium, for example the statistics of laser light or an ion in a superionic conductor under the influence of an additional string external field [32].

From this section we have derived the Diffusion equation which describes the evolution of the probability distribution of the particle. We have seen that the typical behaviour of such a particle is to spread out from its original position. We have also derived the mean-square displacement $\langle x^2(t)\rangle$ which gave the definition of the diffusion coefficient. This was known as the pure case as we did include any effects of disorder. This lead naturally to a discussion of disorder and we found that it has the effect of subjecting the particle to an additional random force $U(x)$. We found that a particle subject to this random force is described by the Fokker-Planck equation which is made up of a diffusive term and an advection term. We have seen that the Fokker-Planck equation is characterized by this random velocity potential which we will discuss in the next section.

2.3 General Correlation Function

The motion of an over-damped classical particle in a random potential proves an effective description of a variety of phenomena. The behaviour of the particle and the phenomena which it is a model of is determined by the form of the random potential $U(x)$ which it is diffusing in. Throughout this thesis we will be considering a correlation function for $U(x)$ of the form

$$\langle (U(x) - U(x'))^2 \rangle = K(x - x') = \gamma \left[(x - x')^\alpha + \ln\left(\frac{|x - x'|}{a}\right)(1 - \delta^{(a)}(x - x')) \right], \tag{2.3.1}$$

where γ is a measure of the strength of disorder, $\delta^{(a)}(x-x')$ is zero for $|x-x'|>a$ and one otherwise and the choice of alpha determines the model we are interested in. The first term which dominates at large distances describes long-ranged correlations in the random potential which generalizes the Sinai model. We find that the second term only describes the behaviour at $\alpha=0$. It should also be noted the relationship between the random potential $U(x)$ and the random velocity potential $V(x)$ is given by $V(x)=dU(x)/dx$. We consider two cases of interest in this thesis $\alpha=0$ which is the case of logarithmic correlations and $\alpha=1$ which is known as Sinai's diffusion, and each case will be solved using completely different mathematical techniques. We will discuss each model briefly in this introductory section with the main discussions taking place in the appropriate chapter. It should be noted that in chapter three where we consider the first-passage time distribution we also consider two additional cases of $0<\alpha<1$ and $\alpha<0$ which are known as the creep and ohmic regimes respectively.

2.3.1 $\alpha=0$ - Logarithmic Correlations

For logarithmic correlations we have $\alpha=0$ and a correlation function of the form $\langle(U(x)-U(x'))^2\rangle = \gamma\ln\left(\frac{|x-x'|}{a}\right)(1-\delta^{(a)}(x-x'))$. It is a model that precisely describes several problems including a single vortex in a XY spin model with Gaussian random gauge disorder [39], relaxation in glasses [11] and non Hermitian quantum mechanics [37, 38]. The interest we have in such a problem is that we want to attempt to re-derive known results using a renormalization and function renormalization group approach in an attempt to find a general method that can deal with such one-dimensional systems. We will first attempt to re-derive the mean-square displacement and the form of the diffusion coefficient. As we will come to see logarithmic correlations are themselves very interesting as they exhibit a phase transition in the first-passage time distribution between a low temperature glass phase and a high temperature phase as a function of its moments. This will be another result we try to re-create using the renormalization and functional renormalization group approaches by first calculating the particles distribution function.

2.3.2 $\alpha = 1$ - Sinai Diffusion

As we have previously stated the correlation function for Sinai diffusion has the form $\langle (U(x) - U(x'))^2 \rangle = \gamma |x - x'|$, which is a long-ranged correlation function for the random potential. In this thesis we are more interested in Sinai's diffusion in terms of a random velocity potential $V(x)$which has a correlation function of the form $\langle V(x)V(x') \rangle = 2\gamma\delta(x - x')$, which will be known as short ranged correlations in the velocity potential. A correlation function of this form simply states that each impurity does not feel the effect of any other impurity. We can also introduce the concept of drift which subjects the particle to an additional non-random constant velocity term V_0, and we write the random velocity potential as $V(x) = V_0 + v(x)$ with $v(x)$ satisfying $\langle v(x) \rangle = 0$ and $\langle v(x)v(x') \rangle = 2\gamma\delta(x - x')$. The model with zero drift is known as Sinai's diffusion and was first solved by Sinai to show the $\langle x^2(t) \rangle \approx \ln^4(t)$ result which we will derive three times throughout the thesis. We will see that Sinai's diffusion with and without drift will be an exceptional model for glass dynamics [11], dynamics of interfaces [40, 41] and dislocations and dynamics of charge density waves [42]. The majority of the work done on Sinai's diffusion has been based on the Lattice Hopping model and the Master equation which we will discuss next.

We have now completely introduced the continuous problem of classical diffusion in the presence of disorder. An equivalent formalism would have been to consider the Lattice Hopping model and take the limit of the lattice spacing $a \to 0$, something which we will now discuss.

2.4 The Lattice Hopping Model

As already stated a completely equivalent approach to the problem is to consider the Lattice Hopping model [13]. The Lattice Hopping model is equivalent to a random walk [43] on a random lattice with the transfer rates from one site to another described by some random hopping rate. The equation that describes the Lattice Hopping model is the Master equation and to derive it we need to consider how the probability of finding the particle on the n^{th} site of the lattice changes with time. We see that the change in the probability is given by the probability of hopping from the $(n-1)^{th}$ or the $(n+1)^{th}$ site to the n^{th} site minus the

probability of hopping from the n^{th} site to either the $(n-1)^{th}$ or the $(n+1)^{th}$ site. This leads to a Master equation of the form

$$\frac{dP_n(t)}{dt} = W_{n,n+1}P_{n+1}(t) + W_{n,n-1}P_{n-1}(t) - (W_{n+1,n} + W_{n-1,n})P_n(t), \tag{2.4.1}$$

where $W_{n,n+1}$ are the random hopping rates from the $(n+1)^{th}$ site to the n^{th} site. These random transfer rates are assumed to be asymmetric which in other words means $W_{n,n+1}$ is not equal to $W_{n+1,n}$. In addition to this the pairs $W_{n,n+1}$ and $W_{n+1,n}$ are assumed to be independent from one link to the other. It is also assumed that the initial condition $p_n(0) = \delta_{n,0}$, which is to say that at a time $t = 0$ the particle is localized on the site $n = 0$. This model is sometimes described as the *random random walk* [44]. The most generic case for these hopping rates is given by

$$W_{n,n+1} = \frac{D}{a^2}\exp\left[-a\frac{V_{n+1}}{2D}\right], \tag{2.4.2}$$

$$W_{n+1,n} = \frac{D}{a^2}\exp\left[a\frac{V_{n+1}}{2D}\right], \tag{2.4.3}$$

where V_n is the associated random potential at that point on the lattice and a is the lattice spacing [14]. We can use the Lattice Hopping model to illustrate the physical processes going on in the continuous case. As we can see from figure (2.4) which comes from [14] we find that for a particle sat on the lattice there is a time scale associated with the particle hopping to the next site. We have seen that this hopping rate is determined by the potential at that point. This view of the discrete case can be applied to the continuous case. For the continuous case we find disorder has the effect of creating traps in the diffusion landscape which the particle has to escape from with the size of the trap determined by the strength of disorder.

We can see the equivalence between the Master and Fokker-Planck equations if we expand the right-hand side of the Master equation and take the limit as the lattice spacing a gets taken to zero. By expanding the hopping rates in terms of a in the limit as the spacing goes

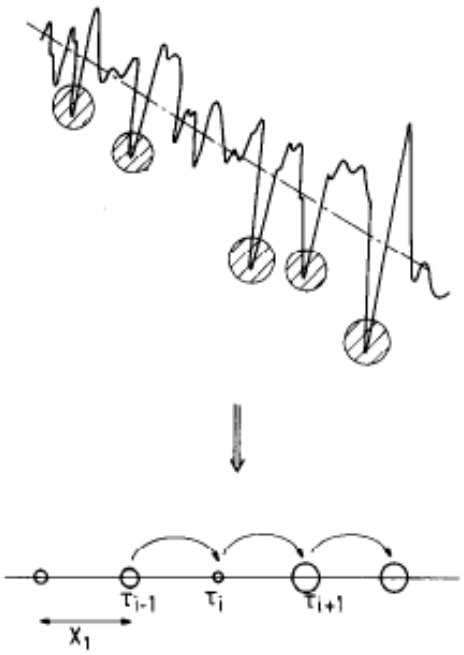

Figure 2.1: Figure to show the relation between the Lattice Hopping model and the continuous model. For each site on the lattice there is a waiting time till the particle can hop to the next site.

to zero we find

$$\frac{dP_n}{dt} = \frac{D}{a^2}[P_{n+1} + P_{n-1} - 2P_n] + \frac{1}{2a}[V_n P_{n-1} + V_n P_n - V_{n+1} P_{n+1} - V_{n+1} P_n], \qquad (2.4.4)$$

and we return to the continuous Fokker-Planck equation by taking the limit as a tends to zero, with the probability density $P(x,t)$ given by $\frac{1}{a} P_{n=\frac{x}{a}}(t)$.

The majority of the work done on classical diffusion in a random one-dimensional environment has been based around the Lattice Hopping model. The original work done by Sinai some thirty years ago was based on the discrete hopping model [3] where he showed the $\langle x^2(t) \rangle \approx \ln^4(t)$ result. For the Sinai model with drift the work done on [14] where they derive several results for the transport properties of the particle the arguments were based around this discrete model. The effect of bias or drift as it will come to be known appears quite naturally with models with random hopping rates, as it simply means the particle has some preferential direction when hopping.

The lattice model as a problem itself, has been solved using various techniques depending

on the form of the hopping rates. For example the model we introduced here, where the hopping rates are assumed to be asymmetric was solved using a recursion method first proposed by Bernasconi and Schneider [45, 46] for the study of drift properties in a particular asymmetric model. They used the method to calculate the form of the diffusion coefficient for a constant finite value of drift. This method was generalized to any random asymmetric model with non-zero hopping rates by [44] which allows for the calculation of the drift velocity. The recursion method of [47] can also be used in the calculation of the diffusion coefficient. In addition to this the one-dimensional hopping model of arbitrary period N has also been considered by Derrida [47]. Derrida directly calculated the properties of the steady state and obtained exact expressions for the velocity and diffusion coefficients by taking the limit of an infinite period. It has also been shown that the return probability $\langle P_0(\epsilon)\rangle$ is an important quantity to consider as it can be expanded to give the transport properties of the particle [44], *i.e.*

$$\langle P_0(\epsilon)\rangle \approx \frac{1}{V} - \frac{2D}{V^3}\epsilon + \ldots, \tag{2.4.5}$$

where the return probability is written in terms of the Laplace parameter ϵ, which has been used to eliminate the time derivative of the Master equation. They have shown for the Sinai model that depending on the value of drift μ, which as we will see in chapter four is the ratio of drift V_0 to the strength of disorder γ, they expect a mixture of diffusive behaviours. For the finite velocity regime of $\mu > 1$ the diffusion coefficient is found to be infinite for $1 < \mu < 2$ and finite for $\mu > 2$. This is something we find for the continuous model. They also show that for the zero velocity regime *i.e.* $0 < \mu < 1$ the diffusion coefficient is zero for $\mu < \frac{1}{2}$ and infinite for $\frac{1}{2} < \mu < 1$. This is again something we show for the continuous model. There has also been a comparison of the statistical and dynamical properties of the discrete Sinai model at finite times [48]. In [48] it was shown that the statistical distribution which they calculated explicitly and the dynamical distribution of the position which they calculated numerically matched in the high temperature limit in the large asymptotic regime. They found however at low temperatures and at finite times that there is a clear correspondence between the statistical and dynamical distributions.

From this section on the Lattice Hopping model we have seen that it provides an alter-

native formulation of the continuous problem. We have seen that if the lattice spacing gets taken to zero we arrive at the Fokker-Planck equation. The Lattice Hopping model also paints an intuitive picture of the diffusion process in the presence of disorder.

2.5 Summary

For this chapter of the thesis we have built up all of the physical and mathematical concepts of classical diffusion. We began by considering a number of particles dissolved or suspended in a fluid where we found that their number obeys a conservation law. This conservation law lead to Fick's Law which states that a spatially non-uniform density leads to currents in directions opposite to the direction of changes in densities. We have also seen that Fick's law is only appropriate in situations where there are no external potentials. We therefore considered the dilute limit of the problem which meant we could just consider the motion of a single particle which is random and is known as Brownian motion. We then considered the forces acting on such a particle and derived a Langevin equation for its motion. We considered the over-damped limit of the Langevin equation which is the limit as the mass of the particle gets taken to zero. We then used this Langevin equation to derive the probability distribution for a Brownian particle and derived the Diffusion equation. We found the solution to the Diffusion equation and used it to calculate the moments and cumulants of the distribution. We have shown that the first two cumulants Ξ_1 and Ξ_2 give the velocity and diffusion coefficients respectively. The work we presented upto this point was for the case of zero disorder and is known as the pure case. We then gave a general introduction to disorder. For the case of Brownian motion we found that disorder has the effect of subjecting the particle to an additional random force $U(x)$ and we found the probability distribution for such a particle is described by the Fokker-Planck equation. We then gave a discussion into the general correlation function of the random potential and found there are two cases of interest, the case of logarithmic correlations and Sinai diffusion. We then concluded the chapter by considering the Master equation and the Lattice Hopping model where we found that it a is completely equivalent formulation of the problem when we send the lattice spacing a to zero.

Chapter 3

FIRST-PASSAGE TIME DISTRIBUTION

For this chapter we will be considering the problem of first-passage time which gives a description of the time spent in a system by a particle as a function of its initial starting point and as we will find it is a very effective formalism for the calculation of various characteristics of the random motion. We find the moments of the first-passage time distribution are described by the Pontryagin equation which we derive from the usual Fokker-Planck equation using properties of conditional probability, Markovian and non-Markovian processes, which forms the basis of the first section of this chapter. With the properties of Markovian processes and conditional probability understood we derive the Pontryagin equation. As the Pontryagin equation is a simple second order differential equation we solve for the n^{th} moment via an integration factor. The aim of the following section is to find the solution to the equation using techniques from Lerner and Yurkevich [16] as an illustration of the method we will be using in the following chapter to solve the Fokker-Planck equation for short ranged delta function correlations [3]. The solution to the Pontryagin equation will be found in terms of the general correlation function $K(x-x')$ and by following the arguments of [15] we use the result to calculate the tails of the probability distribution of the moments. We use this result to find a general expression for the mean-square displacement showing that for the case of Sinai's diffusion the known result of $\langle x^2(t)\rangle \sim \ln^4(t)$[3]. We then follow the arguments of [11] and use the first-passage time formalism to derive the form of the mean velocity for a general correlation function and then solve for the two specific cases we discussed in chapter two. We will also consider the additional case of $\alpha < 0$ which is known as the ohmic regime [11] and $0 < \alpha < 1$ which is known as the creep regime [11]. We begin

the discussion by considering conditional probability and Markovian processes which leads to a full derivation of the Chapman-Kolmogorov and Master equations.

3.1 Conditional Probability, Markovian and Non Markovian Processes

A Markov process is a mathematical model for the random evolution of a memoryless system. That is to say any future state, at any moment in time, depends only on its present state and not on any states in the past. If there are a series of Markov processes then they form what is known as a Markov chain which states that the system may change from its current state to another, or remain in the same state according to a certain probability distribution. It should be noted that the opposite case of a system whose future state is determined by past events is known as a non-Markovian process. To describe a Markov chain in a formal mathematical language requires the use of conditional probability. The conditional probability $P_{1|1}(y_2, t_2 \mid y_1, t_1)$ is given by

$$P_2(y_1, t_1; y_2, t_2) = P_{1|1}(y_2, t_2 \mid y_1, t_1) P_1(y_1, t_1), \tag{3.1.1}$$

which has the meaning that the joint probability of finding the particle at y_1 at t_1 and y_2 at t_2 equals the product of the probability of being at y_1 at t_1 and the conditional probability of finding y_2 at t_2 given y_1 at t_1. A Markov process can be written as the product of several conditional probabilities

$$P_{1|n-1}(y_n, t_n \mid y_{n-1}, t_{n-1}; \ldots; y_1, t_1) = P_{1|1}(y_n, t_n \mid y_{n-1}, t_{n-1}), \tag{3.1.2}$$

with $t_1 < t_2 < ... < t_n$, a relation known as the Markov property. This relation simply states that the probability of a transition at a time t_{n-1} from y_{n-1} to a value y_n at a time t_n only depends on the step before it and not on the past history of the problem. By taking the case of $n = 3$ we derive the Chapman-Kolmogorov equation which leads to a derivation of the Master equation.

3.1.1 The Chapman-Kolmogorov and Master Equations

For the case of $n = 3$ we find the conditional probability becomes

$$\begin{aligned} P_3(y_3, t_3 \mid y_1, t_1) &= P_2(y_2, t_2|y_1, t_1)P_{1|2}(y_3, t_3|y_2, t_2; y_1, t_1) \\ &= P_1(y_1, t_1)P_{1|1}(y_2, t_2 \mid y_1, t_1)P_{1|1}(y_3, t_3 \mid y_2, t_2). \end{aligned} \tag{3.1.3}$$

We can derive the Chapman-Kolmogorov equation by first integrating with respect to y_2 to find

$$\int P_3(y_3, t_3; y_2, t_2 : y_1, t_1)dy_2 = \int P_1(y_1, t_1)P_{1|1}(y_2, t_2|y_1, t_1)P_{1|1}(y_3, t_3|y_2, t_2)dy_2, \tag{3.1.4}$$

which becomes

$$P_2(y_3, t_3; y_1, t_1) = P_1(y_1, t_1) \int P_{1|1}(y_2, t_2|y_1, t_1)P_{1|1}(y_3, t_3|y_2, t_2)dy_2, \tag{3.1.5}$$

and by using the Markov property and dividing through by $P(y_1, t_1)$ we find

$$P_{1|1}(y_3, t_3 \mid y_1, t_1) = \int P_{1|1}(y_3, t_3 \mid y_2, t_2)P_{1|1}(y_2, t_2 \mid y_1, t_1)dy_2, \tag{3.1.6}$$

which is the Chapman-Kolmogorov equation, with $t_1 < t_2 < t_3$. The Chapman-Kolmogorov equation can be interpreted by saying that at a time t_1 at the point y_1 we reach the point y_3 at t_3 via any one of the possible values of y_2 at the intermediate time t_2. We can use the Chapman-Kolmogorov equation to derive the Master equation (2.4.1) by denoting the transition probability as $P_{1|1}(y_2, t_2|y_1, t_1) = T_\tau(y_2|y_1)$ with $T_{\tau+\tau'}(y_3|y_1) = \int T_{\tau'}(y_3|y_2)T_\tau(y_2|y_1)dy_2$ and expanding the transition probability as a Taylor series in the limit of short time

$$T_{\tau'}(y_3|y_2) = (1 - \alpha_0\tau')\delta(y_2 - y_3) + \tau' W(y_3|y_2) + O(\tau'^2), \tag{3.1.7}$$

where $(1 - \alpha_0\tau')$ takes into account the probability for no transition to take place and the delta function ensures that the probability to stay at the same state after no time time has elapsed is one. The final term, which is the random hopping rate is simply the time

derivative of the transition probability at $\tau' = 0$. If we put this definition into (3.1.6) we find

$$\frac{T_{\tau+\tau'}(y_3|y_1) - T_\tau(y_3|y_1)}{\tau'} = \int [W(y_3|y_2)T_\tau(y_2|y_1) - W(y_2|y_3)T_\tau(y_3|y_1)]dy_2, \tag{3.1.8}$$

and in the limit as $\tau' \to 0$ the left hand side becomes the derivative of $T_\tau(y_3|y_1)$

$$\frac{\partial T_\tau(y_3|y_1)}{\partial \tau} = \int [W(y_3|y_2)T_\tau(y_2|y_1) - W(y_2|y_3)T_\tau(y_3|y_1)]dy_2. \tag{3.1.9}$$

If we re-write this in terms of the probability distribution we find for the continuous case that

$$\frac{\partial P(y,t)}{\partial t} = \int [W(y|y')P(y',t) - W(y'|y)P(y,t)]dy', \tag{3.1.10}$$

and for the discrete case we find the usual Master equation

$$\begin{aligned} \frac{dP_n}{dt} &= \sum_{n'} [W_{nn'}P_{n'}(t) - W_{n'n}P_n(t)] \\ &= W_{n,n+1}P_{n+1}(t) + W_{n,n-1}P_{n-1}(t) - (W_{n+1,n} + W_{n-1,n})P_n(t), \end{aligned} \tag{3.1.11}$$

where we just consider the two nearest neighbours, $n' = n + 1$ and $n' = n - 1$, which is agreement with (2.4.1).

Throughout this section we have discussed the properties of Markovian and non-Markovian processes and have seen that they are described by conditional probability. We have seen by considering a chain of Markov processes that we can derive the Chapman-Kolmogorov equation which can be interpreted by saying that the particle can take any path from y_1 to y_3. We have also used the Chapman-Kolmogorov equation to derive the Master equation. We would like to proceed to derive the Pontryagin equation from the Fokker-Planck equation using the methods described in this section.

3.2 From the Fokker-Planck to Pontryagin Equation

Defining some stochastic variable $x(t)$ with a probability distribution $P(x,t|x',t_0)$ and $x' \in [a;b]$ where a and b are absorbing boundaries, *i.e.* $P(x=a,t) = P(x=b,t) = 0$ which states that as soon as the particle reaches a or b it is absorbed. We can define two time scales $T(a)$ and $T(b)$ which are called the Fokker-Planck times which are simply the times for the particle to reach the absorbing boundaries at a or b. If we begin with the usual Fokker-Planck equation

$$\frac{\partial P(x,t|x',t_0)}{\partial t} = D\frac{\partial^2 P(x,t|x',t_0)}{\partial x^2} - \frac{\partial}{\partial x}\bigg(V(x)P(x,t|x',t_0)\bigg), \tag{3.2.1}$$

we want the quantity $g(t,b) = \text{Prob}(T(b)=t)$ or its integral $G(b,t) = \text{Prob}(T(b)\geq t)$ which is simply the probability of the time t being at the Fokker-Planck time $T(b)$ which as we know is the time the particle leaves the system. If we now use the fact that for a Markovian process the probability $P(x,t|x',t_0)$ is given by (3.1.6) for any $t' \in [t_0,t]$ and by taking the derivative with respect to time t' we find

$$\begin{aligned} 0 &= \int dy\bigg[\frac{\partial P(x,t|y,t')}{\partial t'}P(y,t'|x',t_0) + P(x,t|y,t')\frac{\partial P(y,t'|x',t_0)}{\partial t'}\bigg] \\ &= \int dy\bigg[\frac{\partial P(x,t|y,t')}{\partial t'}P(y,t'|x',t_0) + P(x,t|y,t')\bigg(D\frac{\partial^2 P(y,t'|x',t_0)}{\partial y^2} \\ &\qquad - \frac{\partial}{\partial y}[V(y)P(y,t|x',t_0)\bigg)\bigg], \end{aligned}$$

and by integrating by parts

$$\frac{\partial P(x,t|y,t')}{\partial t'} = V(y)\frac{\partial P(x,t|y,t')}{\partial y} - D\frac{\partial P(x,t|y,t')}{\partial y^2}, \tag{3.2.2}$$

which is the backwards Fokker-Planck equation [32]. The aim now is to find the quantity $G(b,t) = \text{Prob}(T(b)\geq t)$ and we set $a=-\infty$ for simplicity. We know that $G(b,t) = \int_{-\infty}^{b} P(x,t|x't_0)dx$ with the initial condition $P(x,t_0|x',t_0) = \delta(x-x')$ and the equation that

describes its evolution is given by

$$\frac{\partial G(b,t)}{\partial t} = \int_{-\infty}^{b} \frac{\partial P(x,t|x',t_0)}{\partial t} dx = \int_{-\infty}^{b} \frac{\partial P(x,0|x',t_0-t)}{\partial t} dx, \tag{3.2.3}$$

and using the backwards Fokker-Planck equation (3.2.2) we find

$$\begin{aligned} \frac{\partial G(b,t)}{\partial t} &= \int_{-\infty}^{b} dx \Big[-V(x') \frac{\partial P(x,t|x',t_0)}{\partial x'} + D \frac{\partial^2 P(x,t|x't_0)}{\partial x'^2} \Big] \\ &= -V(x') \frac{\partial G(b,t)}{\partial x'} + D \frac{\partial^2 G(b,t)}{\partial x'^2}. \end{aligned} \tag{3.2.4}$$

As we want $t(x') = \int_0^\infty g(b,t) t dt = \int_0^\infty dt G(b,t)$, which comes from integration by parts, we integrate the backwards Fokker-Planck equation (3.2.2) to find

$$D \frac{d^2 t}{dx'^2} - V(x') \frac{dt}{dx'} = -1, \tag{3.2.5}$$

where the right-hand side comes from the boundary conditions of $G(b,t)$. This method can be generalized to the n^{th} moment

$$D \frac{d^2 t_n}{dx'^2} - V(x') \frac{dt_n}{dx'} = -n t_{n-1}, \tag{3.2.6}$$

where the right-hand side simply comes from integrating by parts n times, with $t_0 = 1$. This is the Pontryagin equation [15] and it describes the moments of the first-passage time distribution.

The aim of the following section is to solve the Pontryagin equation using two very different methods. As we have seen the Pontryagin equation is a simple second order differential equation that can be solved with an integration factor [15]. For the second method of Lerner and Yurkevich [16] we re-derive the result using a Green's function approach so to illustrate the method used in the next chapter where we solve the Fokker-Planck equation for short ranged correlations.

3.3 The Pontryagin Equation

As we have already stated the aim of this section is to derive the solution to the Pontryagin equation using two different methods. The first as used in [15] is an integration factor and the second is the method from [16]. The idea behind [16] is to set up a Green's function for the equation by writing it in terms of functions acting to the left and right side of the source. The form of the Green's function will be found by writing it in terms of a transfer matrix and the boundary conditions of the problem using left and right acting functions. We solve the Pontryagin equation for the general correlation function (2.3.1) and follow the arguments of [15] to calculate the tails of the probability distribution and the form of the mean-square displacement.

We therefore begin with the Pontryagin equation written in terms of the random field $U(x)$

$$D\frac{d^2t_n}{dx'^2} + \frac{dU}{dx'}\frac{dt_n}{dx'} = -nt_{n-1}(x'). \tag{3.3.1}$$

We must first state the boundary conditions for the problem which are given by $t(L) = 0$ and $\frac{dt}{dx} \mid_{x=0} = 0$, which is to say we are considering a reflective barrier at $x = 0$ and an absorbing barrier at $x = L$. The aim of this section is to calculate the time it takes for the particle to arrive at the absorbing barrier at $x = L$ knowing that the particle starts its random walk at $x = x'$.

3.3.1 Solution Via an Integration Factor

The first method we use to solve the Pontryagin equation is that of an integration factor. If we solve for the n^{th} moment we find

$$t_n(x') = \frac{n}{T}\int_{x'}^{L} dx'_1 \exp\left[\frac{U(x')}{T}\right]\int_0^{x'_1} \exp\left[-\frac{U(y_0)}{T}\right]t_{n-1}(y_0)dy_0, \tag{3.3.2}$$

where we have re-written the diffusion coefficient in terms of the temperature via the usual Einstein's relation (2.1.6) with the mobility constant set to unity. We do this to make our work consistent with that done previously. By expressing the solution for $t_{n-1}(x')$ in terms of $t_{n-2}(x')$ and substituting in the equation for $t_n(x')$ and proceeding iteratively we obtain

the result

$$t_n(x') = \frac{n!}{T^n}\int_{x'}^{L} dx'_n \int_0^{x'_n} y_{n-1}\ldots\int_{y_1}^{L} dx'_1\int_0^{x'_1} dy_0 \exp\left[\frac{1}{T}\Big(\sum_{i=1}^{n} U(x'_i) - \sum_{i=0}^{n-1} U(y_i)\Big)\right],$$

and after averaging over Gaussian disorder we find

$$\begin{aligned}\langle t_n(x')\rangle &= (-1)^n \frac{n!}{T^n}\int_{x'}^{L} dx'_n \int_0^{x'_n} y_{n-1}\ldots\int_{y_1}^{L} dx'_1\int_0^{x'_1} dy_0 \\ &\times \exp\left[\frac{1}{2T^2}\sum_{i,j} K(y_i - x'_j) - K(y_i - y_j) - K(x'_i - x'_j)\right], \end{aligned} \tag{3.3.3}$$

which will be the result we are trying to re-create using a Green's function.

3.3.2 First Moment of the Mean First-Passage Time

We would now like to re-create the result of (3.3.3) using the method from [16] as an illustration of the method we will be using in the next chapter to derive the probability distribution of a particle in a one-dimensional random potential with short ranged correlations and drift. The method we will follow is to consider the first moment of first-passage time and re-write it in terms of a Green's function with functions acting to the left and right. We then notice that these left and right acting functions can be written as a Langevin equation which can be reduced to a vector equation. The vector equation will be solved to find the form of these left and right acting functions and hence the solution to the first-passage time.

We begin with the first moment of mean first-passage time

$$\frac{d^2t}{dx'^2} - \frac{1}{T}\frac{dU}{dx'}\frac{dt}{dx'} = -\frac{1}{T}. \tag{3.3.4}$$

It can be shown that this equation has a solution of the form $t(x') = -\frac{1}{T}\int_0^L G(x', y)dy$, with the Green's function $G(x', y)$ of the form

$$G(x', y) = \frac{1}{R'_+(y)R_-(y) - R_+(y)R'_-(y)} \begin{cases} R_+(x')R_-(y) & x' \geq y, \\ R_-(x')R_+(y) & x' \leq y, \end{cases} \tag{3.3.5}$$

with the random functions $R_{\pm}(y)$ satisfying the homogeneous Langevin equation

$$\frac{d^2R_{\pm}}{dx'^2} - \frac{1}{T}\frac{dU}{dx'}\frac{dR_{\pm}}{dx'} = 0, \tag{3.3.6}$$

with their boundary conditions on the right and left boundary respectively. The first step we should make to solve the problem is to re-write (3.3.6) in terms of the vector $\Psi_{\pm}(y) = \begin{pmatrix} R_{\pm}(y) \\ R'_{\pm}(y) \end{pmatrix}$ so to reduce it to a first order vector equation

$$\frac{d\Psi_{\pm}}{dy} = \begin{pmatrix} 0 & 1 \\ 0 & \frac{1}{T}U(y) \end{pmatrix} \Psi_{\pm}(y) = \mu(y)\Psi_{\pm}(y). \tag{3.3.7}$$

The next step we would like to make is to introduce the transfer matrix $m(y, x)$. The transfer matrix just takes a function at y and transfers it to the value at x. It has the form $\Psi_{\pm}(y) = m(y, x)\Psi_{\pm}(x)$. To write the Green's function in terms of the transfer matrix we should consider the properties of the transfer matrix $m(y, x)$ itself. It is easy to show that to transfer to the same point in space the transfer matrix is just the identity, *i.e.* $m(x, x) = 1$. It can also be shown that a transformation to an intermediate step and then to the final step is just the transformation matrix of the initial to the final point, *i.e.* $m(x, y)m(y, z) = m(x, z)$. The final identity which will be used is a result for its inverse. The inverse of the transfer matrix is the transfer matrix with its arguments swapped round, *i.e.* $m^{-1}(x, y) = m(y, x)$. Using equation (3.3.7) and the transfer matrix we can derive two differential equations

$$\frac{dm(y, x)}{dy} = \mu(y)m(y, x), \qquad \frac{dm(y, x)}{dx} = -m(y, x)\mu(x).$$

If we consider the case when $x' > y$ and using the above transformations, the Green's function can be written in terms of the transfer matrix and Pauli matrices σ_x and σ_y

$$G(x', y) = -\frac{i}{2}\frac{\Psi_-^T(0)\sigma_y m(0, y)(\sigma_y + i\sigma_x)m(x', L)\Psi_+(L)}{\Psi_-^T(0)\sigma_y m(0, L)\Psi_+(L)}, \tag{3.3.8}$$

which as we can see is written in terms of the boundary conditions of $t(x')$. The boundary conditions as already stated are $t(L) = 0$ and $\frac{dt}{dx'} \mid_{x'=0} = 0$. Therefore the Green's function becomes

$$G(x', y) = \frac{m_{12}(x', L)}{m_{22}(y, L)}, \tag{3.3.9}$$

where both $m_{12}(x', y)$ and $m_{22}(x', y)$ satisfy the differential equations

$$\frac{dm_{12}(x', y)}{dy} = m_{22}(x', y), \qquad \frac{dm_{22}(x', y)}{dx'} = m_{22}(x', y)\frac{U(x')}{T}. \tag{3.3.10}$$

All that is required now is to solve the above equations. Doing this and substituting the solutions into equation (3.3.9) we arrive at the solution for the first moment of the first-passage time distribution averaged over disorder

$$\langle t(x') \rangle = \frac{1}{T} \int_{x'}^{L} dx_1' \int_{0}^{x_1'} dy_0 \exp\left[\frac{1}{2T^2} K(x_1' - y_0)\right], \tag{3.3.11}$$

where we have employed the correlation function (2.3.1), which is in agreement with (3.3.3) for $n = 1$. We would now like to generalize the result to find the n^{th} moment of the first-passage time distribution.

3.3.3 n-th Moment of First Passage Time

To generalize the method to the n^{th} moment of first-passage time we simply need to solve for the n^{th} moment which can be written as a Green's function of the form (3.3.9) and the $(n-1)^{th}$ moment. We then need to write down the solution to the $(n-1)^{th}$ solution in terms of this Green's function and substitute in for the solution of the n^{th} moment. Treating this iteratively we should arrive at the required result.

We begin with the Pontryagin equation for the n^{th} moment

$$\frac{d^2 t_n}{dx''^2} - \frac{1}{T}\frac{dU}{dx'}\frac{dt_n}{dx'} = -\frac{n}{T} t_{n-1}, \tag{3.3.12}$$

and using the same Green's function argument as for the first moment we find the solution to the n^{th} moment is given by

$$t_n(x') = -n \int_0^L G(x', y) t_{n-1}(y) dy, \tag{3.3.13}$$

where $G(x', y)$ satisfies equation (3.3.5). If we iteratively substitute in each subsequent moment we find the n^{th} moment is just the product of n Green's functions given by (3.3.9). If we substitute in for $G(x', y)$ and take care with the limits of the integrals we find we arrive at the intended result of

$$\begin{aligned} \langle t_n(x') \rangle &= (-1)^n \frac{n!}{T^n} \int_{x'}^{L} dx'_n \int_0^{x'_n} y_{n-1} \ldots \int_{y_1}^{L} dx'_1 \int_0^{x'_1} dy_0 \\ &\times \exp\left[\frac{1}{2T^2} \sum_{i,j} K(y_i - x'_j) - K(y_i - y_j) - K(x'_i - x'_j) \right]. \end{aligned} \tag{3.3.14}$$

We will now use this result to calculate the tails of the probability distribution and to make an estimation of the mean-square displacement.

3.4 Application of the First-Passage Time Distribution

Equation (3.3.14) has the form of a partition sum of $2n$ particles, with n particles of the type y and n of the type x [15]. Particles of the same type attract via a potential $K(r)/2T$ whilst those of the opposite type repel via a potential $-K(r)/T$. We can see that this equation imposes the $2n$ restrictions of $0 \leq x_{i-1} \leq y_i$ and $x_i \leq y_i \leq L$ for $i = 1, ..., n$. In general this integral can not be solved exactly. However if we follow the arguments of [15] and consider the large L limit we can solve the integral via the method of steepest descents which describes the large time limit. We can see that the integrand reaches its maximum value at the point $(x_i, y_i) = (0, L)$ and using the fact that $K(0) = K'(0) = 0$ we obtain

$$\langle t_n(0) \rangle = \frac{n!}{T} \left(\frac{2T^2}{K'(L)n} \right)^{2n} \exp\left[\frac{n^2 K(L)}{2T^2} \right], \tag{3.4.1}$$

where the constant has been determined by the functional dependence close to the saddle point. This approximation is obviously only true for the case when $\alpha > 0$ for the correlation function (2.3.1). The approximation can not be used for logarithmic correlations or when $\alpha < 0$ as $K(0) = \infty$.

We can attempt to re-construct the tails of the probability distribution for the first-passage time if we assume a solution of the form $P(t) \sim \exp[-A \ln^{\beta}(t/t_0)]$ and comparing with the result for $\langle t_n(0) \rangle$ we find

$$P(t) \sim \exp\left(- \frac{T^2}{2K(L)} \ln^2(t/t_0) \right). \tag{3.4.2}$$

We can use this result to make an approximation concerning the large time behaviour of the mean-square displacement. The characteristic value of $x(t)$ can be found from the equation $[T^2/K(x)] \ln^2(t/t_0) \sim 1$ and with $K(x) \sim |x|^{\alpha}$ for $\alpha > 0$ we find

$$\langle x^2(t) \rangle \sim \left(\frac{T^2}{C} \right)^{\frac{2}{\alpha}} \ln^{\frac{4}{\alpha}}(t/t_0), \tag{3.4.3}$$

and we see that if $\alpha = 1$ we arrive at Sinai's known result of $\langle x^2(t) \rangle \sim \ln^4(t)$ [3]. It is at this point where we check the conjecture that the n^{th} moment can be found from the first moment, *i.e.* $\langle t_n(0) \rangle = \langle t_1(0) \rangle^n$. By simply considering (3.3.11) and (3.3.14) we see that this conjecture is false and $\langle t_n(0) \rangle \neq \langle t_1(0) \rangle^n$.

We have seen from this section that there are two methods to solve for the moments of first-passage time. We have seen that the solution to the first-passage time distribution can be used to derive the mean-square displacement which is in complete agreement with known results. We would like to proceed by following the arguments of [11] to calculate the mean velocity of a particle under the influence of some constant external force $V_0 x$.

3.5 The Mean Velocity

If we now consider the particle in the presence of a constant external force $V_0 x$, we can calculate the mean velocity as a function of this external force $V_0 x$ and the general correlation

function $K(x-x')$[15]. If we initially consider the first moment of the first-passage time distribution with the potential $\tilde{U}(x) = U(x) - V_0 x$, where the correlation function for $U(x)$ is defined as before we find

$$t_1(x) = \frac{1}{T}\int_{x'}^{L} dz \exp\left[-\frac{V_0 z}{T}\right] \exp\left[\frac{U(z)}{T}\right] \int_{-\infty}^{z} dx \exp\left[\frac{V_0 x}{T}\right] \exp\left[-\frac{U(x)}{T}\right], \qquad (3.5.1)$$

where we assume that the point at L is still an absorbing barrier, and the reflecting barrier at zero has been taken to minus infinity. In this limit the particle does not feel the effect of the left boundary as the external force is assumed to be positive. If we now average the first moment of the first-passage time distribution over disorder we find

$$\langle t_1(x)\rangle = \frac{1}{T}\int_{x'}^{L} dz \exp\left[-\frac{V_0 z}{T}\right] \int_{-\infty}^{z} dx \exp\left[\frac{V_0 x}{T}\right] \left\langle \exp\left[\frac{U(z)-U(x)}{T}\right]\right\rangle, \qquad (3.5.2)$$

and by introducing the new variable $y = x - z$ we find

$$\langle t_1(x)\rangle = \frac{L-x}{T}\int_{0}^{\infty} dy \exp\left[-\frac{V_0 y}{T}\right] \left\langle \exp\left[\frac{U(x+y)-U(x)}{T}\right]\right\rangle. \qquad (3.5.3)$$

In the limit that $L - x'$ tends to infinity we can apply the central limit theorem which implies that $\frac{1}{\langle t_1(x')\rangle} = \langle 1/t_1(x')\rangle$. Therefore the mean velocity V is simply $\langle (L-x')/t_1(x')\rangle$ and we find for a Gaussian random potential that

$$\begin{aligned} \frac{1}{V} &= \frac{1}{T}\int_{0}^{\infty} dy \exp\left[-\frac{V_0 y}{T}\right] \left\langle \exp\left[\frac{U(x+y)-U(x)}{T}\right]\right\rangle \\ &= \frac{1}{T}\int_{0}^{\infty} dy \exp\left[-\frac{V_0 y}{T} + \frac{K(y)}{2T^2}\right]. \end{aligned} \qquad (3.5.4)$$

The physical interpretation of this result in terms of an Arrhenius waiting time is clear. The average waiting time $1/V$ is a sum of Boltzmann weights associated with the barriers the particle must overcome to move in the direction of the driving force. Therefore the barriers $U(x+y) - U(x)$ with $y > 0$ produce the largest waiting times [15]. We also see that this expression shows some general features [11]. For large V_0 we see that $V \approx V_0$. We also see that for small V_0 the response will be linear only if the barriers saturate which is to say they don't grow at large distances. In addition, if the potentials are uncorrelated at large

distances *i.e.* $y \to \infty$ then we separate the potentials as $\langle \exp[(U(x+y) - U(x))/T] \rangle = \langle \exp[U/T] \rangle \langle \exp[-U/T] \rangle$ and the velocity V is given by

$$V \propto \frac{D}{D_0} V_0 = \frac{V_0}{\langle \exp[U/T] \rangle \langle \exp[-U/T] \rangle}, \tag{3.5.5}$$

where D is the diffusion coefficient in the presence of disorder and D_0 is the diffusion coefficient in the absence of disorder. We would like to derive the form of the mean velocity for four cases, the first two being the case of Sinai diffusion and logarithmic correlations as discussed in chapter two. We would also like to follow [11] and introduce two further cases, the first of which is $0 < \alpha < 1$ which describes creep motion and $\alpha < 0$ which is known as the ohmic regime. We will see that the logarithmic correlations with $\alpha = 0$ separates the creep and ohmic regimes. We notice that we should only calculate the velocity for systems with $\alpha \leq 1$ as the integral (3.5.4) is divergent for $\alpha > 1$ leading to a zero value of the velocity.

3.5.1 $\alpha = 1$ - Sinai's Diffusion

For the case of $\alpha = 1$ in (2.3.1) we find that $K(x) = \gamma |x|$ and the velocity of the particle is given by

$$\frac{1}{V} = \frac{1}{T} \int_0^\infty dx \exp\left[- \left(\frac{V_0}{T} - \frac{\gamma}{2T^2} \right) x \right], \tag{3.5.6}$$

and we find that if $V_0 < V_0^{th} = \frac{\gamma}{2T}$ then the integral diverges and the velocity is zero. For the case of $V_0 > V_0^{th}$ then the integral is convergent and we find the velocity of the particle is given by $V = V_0 - V_0^{th}$. We therefore see that the Sinai model exhibits algebraic distributions of waiting times giving rise to aging phenomena [17] and we see that Sinai's model mimics essential aspects of the spin glass behaviour [11]. We will find when we solve the Fokker-Planck equation for short-ranged correlations with drift, the velocity of the particle is given by the exact same expression found here, as expected.

3.5.2 $0 < \alpha < 1$ - Creep Motion

If we consider the case of $0 < \alpha < 1$ in equation (2.3.1) we find the creep dynamic regime. Creep motion is defined as a sequence of thermally activated jumps from one state to the

next as favored by the applied force. If we define the dynamical exponent as $z = 2 + \frac{\gamma}{2T^2}$ and the characteristic force $V_0^c = (\gamma/2T^2)^{\frac{1}{\alpha}}T$ we find

$$\frac{T}{V} = \frac{1}{V_0^{cz-1}} \int_0^\infty dv v^{z-2} \exp\left[\frac{V_0}{V_0^c} v + v^\alpha\right], \tag{3.5.7}$$

and we notice that when $\alpha \to 1$ the characteristic force V_0^c reduces to the threshold force V_0^{th}. For $V_0 \ll V_0^c$ using the method of steepest descents we find a velocity of the form

$$V = \sqrt{\frac{\alpha(1-\alpha)}{2\pi}} T V_0^{cz-1} \left(\frac{V_0}{V_0^c}\right)^{\frac{z-1-\alpha/2}{1-\alpha}} \exp\left[-(1-\alpha)\left(\frac{\alpha V_0^c}{f}\right)^{\frac{\alpha}{1-\alpha}}\right] \approx \exp[-\text{const}/V_0^\mu], \tag{3.5.8}$$

and we notice that the linear response as $V_0 \to 0$ is absent and the characteristic barriers that control the dynamics diverge as $1/V_0^\mu$ [11]. This unlimited growth of the creep barriers is now taken as the characteristic feature of glassy dynamics.

3.5.3 $\alpha = 0$ - Logarithmic Correlations

For the case of $\alpha = 0$ in (2.3.1) we arrive at a correlation function that is just logarithmic. We therefore find the velocity becomes

$$\frac{1}{V} = \frac{1}{T}\int_0^\infty dx \exp\left[-\frac{V_0 x}{T}\right] \exp\left[\frac{\gamma \ln|x|}{2T^2}\right] = \frac{1}{T}\int_0^\infty x^{\frac{\gamma}{2T^2}} \exp\left[-\frac{V_0 x}{T}\right] dx, \tag{3.5.9}$$

which leads to velocity of the form

$$V = \frac{V_0^{z-1}}{T^{z-2}} \frac{1}{\Gamma[z-1]}, \qquad \text{with} \qquad z = 2 + \frac{\gamma}{2T^2}, \tag{3.5.10}$$

resembling power law critical behaviour proposed at the vortex-glass transition [11]. This transition separates the creep dynamics $V \approx \exp[-\text{const}/V_0^\mu]$ in the vortex glass state from the ohmic behaviour above the transition.

3.5.4 $\alpha < 0$ - Ohmic Regime

For the ohmic regime we consider a correlation function of the form $\langle U(x)U(0)\rangle = \gamma C(x)$ with $C(0) = 1$ we find the mean velocity is given by

$$\frac{1}{V} = \frac{1}{T}\int_0^\infty dz \exp\left[-\frac{V_0 z}{T} + \frac{\gamma}{T^2}(1 - C(z))\right], \tag{3.5.11}$$

and we find for the case of large V_0 the velocity is proportional to V_0 *i.e.* $V \approx V_0$ and for small V_0 we find $V \approx \exp[-\gamma/T^2]V_0$ [11]. We find for high temperatures that these two regimes match. We also find that for low temperatures there is a sharp crossover with a de-pinning temperature $T_p = \sqrt{\gamma}$ which separates two different behaviours which are known as unpinned and pinned [11].

3.6 Summary

To summarize this chapter we have considered the first-passage time distribution which gives a description of the time the particle spends in the system as a function of its initial starting point. We first introduced the concepts of Markov and non-Markov processes which describe a system whose future state is either independent or dependent of its past states. We showed that a series of Markovian processes is described by a Markov chain and that such a chain is is described by conditional probability. This allowed us to derive the Chapman-Kolmogorov equation which states that an object which has some initial position can reach its future state via many intermediate states at any time. Using the Chapman-Kolmogorov equation we were able to derive the Master equation from first principles. These results allowed us to derive the Pontryagin equation which describes the moments of the first-passage time distribution by considering the Fokker-Planck equation. As we have already seen the Pontryagin is a simple second order differential equation which can be solved with a simple integration factor. The aim of this chapter though was to introduce the techniques used in the next chapter to derive the Fokker-Planck equation for short-ranged delta function correlations. The method we used was to write the equation in terms of a Green's function with left and right acting functions. With the solution to the first-passage time distribution understood

we then found the form of the tails of the probability distribution $P(t)$ which we used to estimate the mean-square displacement in terms of the general correlation function $K(x)$. We then used the first-passage time distribution to calculate the average velocity first of all for a general correlation function and then for four specific cases of interest. For the case of Sinai's diffusion we have seen that the velocity is zero for $V_0 < V_0^{th}$ and $V = V_0 - V_0^{th}$ for $V_0 > V_0^{th}$, which as we have seen gives rise to aging phenomena mimicking essential aspects of spin glass behaviour. We have seen for the creep regime that the linear response as the applied force $V_0 \to 0$ is absent and the characterized barriers which control the dynamics diverge as $1/V_0^{\mu}$. We have also shown there is a transition between this creep motion and the ohmic regime which is modeled by a particle traveling in a random environment with logarithmic correlations. This logarithmic regime is characterized by a power law critical behaviour at the transition.

Chapter 4

FOKKER-PLANCK EQUATION FOR SHORT-RANGED CORRELATIONS

In this chapter we will be solving the continuous Fokker-Planck equation for short ranged delta function correlations with drift, the so called Sinai model [3]. We begin the discussion by considering the model, what has already been done and why the model is of interest, for this we consider dynamics of field magnets and dislocation dynamics [14]. We then perform a statistical analysis to predict the behaviour of the particle as certain parameters are changed. From this we show that the single parameter that controls the physics of the problem is the ratio of the drift velocity to the strength of disorder μ. These scaling arguments also show that there is a time scale for which the particle is insensitive to the disordered nature of the environment. We will then solve the Fokker-Planck equation for short-ranged correlations using an approach built on the methods developed for the seemingly different problem of a quantum particle moving in random quenched disorder, the Anderson model [4]. We present a generic approach and then calculate two disorder averaged quantities, the return probability and the probability of finding the particle at some arbitrary distance away from the origin. For both problems we solve for an arbitrary value of drift velocity [21] and show that it does indeed control the main physical properties of the system, as predicted by the scaling arguments. For the return probability we find several regions of interest showing the effect of drift and disorder. For the case of zero drift we find disorder has the effect of localizing the particle around its origin. We find that with the introduction of drift competition emerges between drift driving the particle away and disorder localizing

the particle. We also find, in agreement with scaling arguments, that there is a time scale where the diffusion process follows that of normal diffusion and the particle is insensitive to the disordered nature of the environment. We also calculate the moments of the return probability to see that $\langle P_n[0,t]\rangle \neq \langle P(0,t)\rangle^n$ which means the return probability is not a self averaging quantity. For the probability of finding the particle at some arbitrary distance away from the origin we find a variety of diffusive behaviour including Lévy flight, sub-diffusive and diffusive motion depending on the value of drift velocity. For the case of zero drift we find a result which is in agreement with the scaling arguments of Kesten [19, 20] and the exact results of [14, 18]. We then find several other distributions depending on the range of drift velocity considered. For each case we will also consider the moments and cumulants of the distribution which will give us the form of the transport properties of the particle including those of the velocity and diffusion coefficients.

4.1 Introduction to the Model

As we have previously stated we are considering the Sinai model with drift. As we have seen from chapter two the Sinai model corresponds to a random potential with a correlation function of the form $\langle [U(x) - U(x')]^2\rangle = \gamma|x - x'|$. The formulation we are interested in is written in terms of a random velocity potential $V(x) = V_0 + v(x)$, where we remind the reader that V_0 is the drift velocity and $v(x)$ has the properties

$$\langle v(x)\rangle = 0, \qquad \text{and} \qquad \langle v(x)v(x')\rangle = 2\gamma\delta(x - x'). \tag{4.1.1}$$

This model was formulated some thirty years ago by Sinai [3] who showed rigorously by considering the discrete time model we described in chapter two, that the mean-square displacement for the case of zero drift has the form $\langle x^2(t)\rangle \approx \ln^4(t)$. The majority of the work done since that of Sinai has been based around scaling arguments with the first major contribution coming from Kesten [19, 20] where he found the form of the scaling function for zero drift. This result was later confirmed in [14] using the replica trick. The first full analytic result for the expectation value of the probability distribution for zero drift was

found in [18], where they considered a technique very similar to the one we considered.

The model with drift has been a far more challenging problem. The main contributions up to now have again been based around scaling arguments [14] which predicts a mixture of diffusive processes depending on the value of drift velocity. To find such results they considered an approach based on the Lattice Hopping model in the continuum limit. Using approximate techniques they found that we would expect to find a mixture of sub-diffusive, Lévy flight and diffusive behaviours, something that we prove with our analytic solution. There has also been a real space renormalization group investigation into the problem [22] where they obtained the scaling form of the distribution of the position $x(t)$ of the particle, the probability of it not returning to the origin and the distributions of first passage times in an infinite sample as well as in the presence of a boundary and in a finite but large sample. One of the main successes of this paper was to re-derive the result obtained by Kesten using this real space renormalization group method. There has also been an investigation into the persistence properties of the model [21] which measures the probability of return to the origin and the associated decay exponents. Therefore the success of this chapter is that we find an exact analytical method to derive the probability distribution and transport properties for a particle in the presence of drift. Coincidentally the method we originally used to calculate the expectation of the probability distribution was done in [18] which as we will explain will not give the result for finite drift, only that of zero drift. We will now consider some physical realizations of the model.

4.2 Physical Applications

We have previously seen that the model we are considering can be applied to many physical situations for example roughening of domain walls [49, 50], dynamics of interfaces and dislocations [40, 41] and dynamics of charge density waves [42]. We have also seen from the discussion on first-passage time that the model also mimics essential aspects of spin glass behaviour [11] due to algebraic waiting times leading to aging phenomena. For this short section though we will consider two specific physical problems, the dynamics of random field magnets and dislocation dynamics. We will introduce the problems and explain how the

model applies to them.

4.2.1 Dynamics of Random Field Magnets

Consider domain-wall dynamics in the one-dimensional random field Ising model

$$H = -J \sum_i S_i S_{i+1} - \sum_i h_i S_i. \tag{4.2.1}$$

Consider now a domain wall separating up spins to the left of site i from down spins to the right. The energy required to flip the spin S_i and move the domain wall from $(i, i+1)$ to $(i+1, i+2)$ is $\Delta H = 2h_i$ which corresponds to an activated transition rate of the form

$$W_{i+1,i} = W_0 \exp[\Delta H/2kT] = W_0 \exp[\beta h_i], \tag{4.2.2}$$

$$W_{i,i+1} = W_0 \exp[-\Delta H/2kT] = W_0 \exp[-\beta h_i], \tag{4.2.3}$$

and the thermally activated dynamics of the domain wall precisely follow this model [14].

4.2.2 Dislocation Dynamics

If we consider the motion of a dislocation in a perfect crystal we see it can be modeled by the dynamics of a string in the Peierls potential, the so called washboard potential. The downhill progression of the string is described by the following. The string nucleates a kink anti-kink pair which gets torn apart by the external stress until annihilated with the neighbour nucleated pair which results in the translation of the full dislocation. We therefore see that the motion of the kink is one-dimensional and its progression results in an energy gain from the dislocation proportional to the traveled distance. That is to say that the kinks position satisfies the Langevin equation (2.2.3) with $V(x) = V_0$ [14].

We would now like to perform an analysis using scaling arguments and a statistical analysis to make several predictions about the model to show that it is the ratio of drift velocity to the strength of disorder that controls the main physics of the problem.

4.3 Scaling Arguments and Statistical Analysis

We can make predictions about the model by noticing that there are some invariance properties under the re-scaling of time and position as shown in [14]. It can be seen that the Langevin (2.2.17) and Master (2.4.1) equations remain unchanged under the transformations $x = \lambda\tilde{x}$ and $t = \rho\tilde{t}$, provided the random force and thermal noise scale as $\eta(t) = \frac{\lambda}{\rho}\tilde{\eta}(t)$ and $V = \frac{\lambda}{\rho}\tilde{V}$. Under these re-scalings the probability distribution scales as

$$P(x,t;D;V_0;\gamma) = \frac{1}{\lambda}P(\tilde{x},\tilde{t};\tilde{D};\tilde{V_0};\tilde{\gamma}), \tag{4.3.1}$$

where the new parameters $\tilde{D} = \frac{\rho}{\lambda^2}D$, $\tilde{V_0} = \frac{\rho}{\lambda}V_0$ and $\tilde{\gamma} = \frac{\rho^2}{\lambda^3}\gamma$. We can see by expressing the length and time in natural units $\lambda = \frac{4D^2}{\gamma} \equiv x_1$, $\rho = \frac{8D^3}{\gamma^2} \equiv \tau_1$ and the parameter $\tilde{V_0} = \frac{2V_0D}{\gamma} \equiv \mu$ the probability distribution becomes

$$P(x,t;D;V_0;\gamma) = \frac{1}{x_1}P\left(\frac{x}{x_1},\frac{t}{\tau_1};\frac{1}{2};\mu;1\right). \tag{4.3.2}$$

We therefore see that the single parameter that controls all of the physics is μ, which is the ratio of the mean bias and the diffusion coefficient to the strength of the disorder. Throughout this chapter we call the value μ the drift velocity, which some may say is slightly misleading terminology.

We would now like to make a statistical analysis of the model and to do so we consider the typical energy barrier encountered by the particle when moving a distance x in the random potential $U(x)$

$$\Delta U(x) \approx -V_0x \pm \sqrt{\gamma x}, \tag{4.3.3}$$

which is a sum of the mean bias V_0 and random fluctuating part. This defines another length scale $x_0 \sim \frac{\gamma}{V_0^2}$, which comes from comparing the strength of each term in the random potential (4.3.3). We have two cases to consider, the first being when $x \ll x_0$ and secondly when $x \gg x_0$. For the case of $x \ll x_0$ we find that the random fluctuations dominate and the particle does not feel the effect of the mean bias and its motion is slowed down with a typical energy barrier growing as $\sqrt{\gamma x}$. For the opposite situation of $x \gg x_0$, where the

mean bias $-V_0 x$ becomes dominant we expect to find the particles motion slowed down only by very large fluctuations of the random potential. It can be seen that the typical energy barrier to be overcome in order to reach the scale x_0 is of order $\Delta U(x_0) \sim \frac{\gamma}{V_0}$. Therefore the dimensionless parameter μ turns out to be the ratio between $\Delta U(x_0)$ and the characteristic thermal energy scale $2kT$, $\mu \equiv \frac{2V_0 D}{\gamma} = \frac{2kT}{\Delta U(x_0)}$, which compares the degree of disorder to the strength of thermal activation and as previously seen this is the control parameter for the whole problem. The length scale $x_1 = \frac{4D^2}{\gamma}$ appears when comparing the fluctuations of the potential $\sqrt{\gamma x}$ to the thermal energy $2kT$. It is the length scale below which the particle is insensitive to the disordered character of the potential and τ_1 is the corresponding diffusion time. It should also be noted that the ratio of the length scales $\frac{x_1}{x_0} = \mu^2$.

We should now consider three cases for the value of μ. For $\mu = 0$ we have, as we know Sinai's diffusion [3], which is the case when there is no bias on the average *i.e.* $V_0 = 0$ and the length scale x_0 diverges. In this case the typical potential barrier which is of order $\sqrt{\gamma x}$ controls the diffusion process and the time required to cross this barrier by thermal activation is of order

$$t \approx \tau_1 \exp\left[\frac{\sqrt{\gamma x}}{2D}\right], \tag{4.3.4}$$

which suggests that diffusion follows a logarithmic law $\langle x^2(t)\rangle \sim \ln^4(t)$, which comes from the Arrhenius argument. This is in agreement with the known result of Sinai and with the result we derived from the first-passage time distribution (3.4.3). For the case of $\mu \neq 0$ there are two further cases we should consider. For the case of $\mu \leq 1$ we have $x_1 \leq x_0$ as long as $x \leq x_1$ and diffusion remains normal. This remains true up to times of order $\tau_1 = \frac{x_1^2}{2D} = \frac{8D^3}{\gamma^2}$. For $x_1 \leq x \leq x_0$ the typical behaviour $\sqrt{\gamma x}$ dominates over the mean bias and the diffusion follows the usual logarithmic law, as if μ were zero. This regime holds true until times of order $\tau_0 = \tau_1 e^{\frac{1}{\mu}}$, which becomes large for small μ.

For the case when $\mu \geq 1$ the length scales x_0 and x_1 are reversed and there is no logarithmic regime and the particle feels only the mean bias and is only slowed by large fluctuations of the potential. In both of these regimes the problem is characterized by the competition between the convective term and the large fluctuations of the potential.

What we can conclude from these quite simple scaling arguments is that the parameter

that controls all of the physics is the ratio of the mean bias to the strength of disorder, what we call drift velocity. We also expect to find a series of physical processes as a function of drift velocity. For the next section we derive a general formalism to the problem and then solve the Fokker-Planck equation to show the different regions of behaviour.

4.4 General Formulation

As we have already stated we are interested in two disorder averaged quantities; the return probability and the probability of finding the particle at some arbitrary position. For this section we present a general formulation to the problem upto the point where we consider the two disordered averages. The approach we take is to take the Laplace transform of the Fokker-Planck equation and notice that it takes the form of a Green's function of a similar form to the one we derived for the first-passage time distribution. We perform the average over disorder by introducing functions that have boundary conditions to the left and right and the aim is to solve for each of these functions.

We begin with the usual Fokker-Planck equation written in terms of the dimensionless quantities we introduced in the previous section

$$\frac{\partial P(x,x',t)}{\partial t} = \frac{\partial^2 P(x,x',t)}{\partial x^2} - \frac{\partial}{\partial x}\bigg(V(x)P(x,x',t)\bigg), \tag{4.4.1}$$

with $V(x) = \mu + v(x)$, which as we know is the sum of some constant drift term and a randomly distributed term with short ranged delta function correlations, *i.e.* $\langle v(x)v(x')\rangle = 2\delta(x-x')$. In addition to this we should define appropriate boundary and initial conditions. For this problem we define the initial condition as $P(x,x';t=0) = \delta(x-x')$, which is to say the particle starts the diffusion process at $x = x'$. The two boundary conditions we have are at $x = \pm L$, which will eventually get taken to infinity. The first step to take is to eliminate the time derivative of the probability distribution via a simple Laplace transformation

$$\epsilon P(x,x'\epsilon) - \frac{\partial^2 P(x,x';\epsilon)}{\partial x^2} - \frac{\partial}{\partial x}\bigg(V(x)P(x,x';\epsilon)\bigg) = \delta(x-x'), \tag{4.4.2}$$

where ϵ are the eigen-energies of the system and $P(x,x',\epsilon) = \int_0^\infty \exp[-\epsilon t]P(x,x',t)$. The solution to this equation has the form of a Green's function and we therefore parameterise as such

$$P(x,x';\epsilon) = -\frac{1}{R'_+(x')R_-(x') - R'_-(x')R_+(x')} \begin{cases} R_+(x)R_-(x'), & x \geq x' \\ R_-(x)R_+(x'), & x \leq x' \end{cases} \tag{4.4.3}$$

which has been derived using the fact that there is a source at $x = x'$, along with the fact that there is a discontinuity at this point in the first derivative and that the function should be continuous through the whole domain. The functions $R_\sigma(x)$ ($\sigma = \pm$) satisfy the equation

$$\frac{\partial^2 R_\sigma(x)}{\partial x^2} - \frac{\partial}{\partial x}\bigg(V(x)R_\sigma(x)\bigg) = \epsilon R_\sigma(x), \tag{4.4.4}$$

with the single boundary condition $(\partial_x - V(x))R_\sigma(x) \mid_{x=\sigma L} = 0$. To proceed we make the substitution $\frac{R'_\sigma(x)}{R_\sigma(x)} = -\sigma\sqrt{\epsilon}\xi_\sigma(x) + V(x)$ and the probability distribution becomes

$$P(x,x';\epsilon) = \frac{1}{\sqrt{\epsilon}}\frac{1}{\xi_+(x') + \xi_-(x')} \begin{cases} \exp[\int_{x'}^x dx_1(-\sqrt{\epsilon}\xi_+(x_1) + V(x_1))], & x \geq x' \\ \exp[\int_{x'}^x dx_1(\sqrt{\epsilon}\xi_-(x_1) + V(x_1))]. & x \leq x' \end{cases} \tag{4.4.5}$$

We can re-write this probability distribution in terms of two delta functions and two integrals over ξ_+ and ξ_-

$$\begin{aligned} P(x,x';\epsilon) &= \frac{1}{\sqrt{\epsilon}}\int_0^\infty d\xi_+ \int_0^\infty d\xi_- \frac{\delta(\xi_+(x) - \xi_+)\delta(\xi_-(x) - \xi_-)}{\xi_+ + \xi_-} \\ &\times \begin{cases} \exp[\int_{x'}^x dx_1(-\sqrt{\epsilon}\xi_+ + V(x_1))], & x \geq x' \\ \exp[\int_{x'}^x dx_1(\sqrt{\epsilon}\xi_- + V(x_1))]. & x \leq x' \end{cases} \end{aligned} \tag{4.4.6}$$

The stochastic functions $\xi_\sigma(x)$ satisfy the Langevin equation

$$\frac{d\xi_\sigma}{dx} = \sigma\sqrt{\epsilon}(\xi_\sigma^2(x) - 1) - (\mu + v(x))\xi_\sigma(x). \tag{4.4.7}$$

We can use this Langevin equation to determine some properties of the functions $\xi_\sigma(x)$. By simply differentiating (4.4.7) with respect to ϵ we find that both $\xi_\sigma(x)$ are ever increasing

functions, which we will use when we introduce the two distribution functions. Another property that will be required is the functional derivative $\frac{\delta\xi_\sigma(x)}{\delta V(x')}$, which can be found by simply taking the functional derivative of (4.4.7) and solving for $\frac{\delta\xi_\sigma(x)}{\delta V(x')}$, *i.e.*

$$\frac{\delta\xi_\sigma(x)}{\delta v(x')} = \sigma\theta(\sigma x' - \sigma x)\xi_\sigma(x')\exp\left[-\int_{x'}^{x} dx_1(2\sqrt{\epsilon}\xi_\sigma(x_1) + v(x_1))\right], \tag{4.4.8}$$

where we have used the fact that $\xi_-(x)$ and $\xi_+(x)$ have boundary conditions to the left and right respectively. This will be of use when we are required to perform the average over disorder where we use the result of Furutsu-Novikov to perform the averaging, as we did to derive the Fokker-Planck and Diffusion equations in chapter two. We are now in a position to start solving for the expectation value of the probability distribution for the return probability and the probability of the finding the particle at some arbitrary position in space. To actually proceed the method we will follow is to re-write the expectation of the probability distribution as a pair of integrals over two distribution functions [16]. We therefore write the expectation value of the probability distribution as

$$\begin{aligned}\langle P(x,x';\epsilon)\rangle &= \frac{1}{\sqrt{\epsilon}}\int_0^\infty d\xi_+\int_0^\infty d\xi_-\frac{1}{\xi_+ + \xi_-}p_+(\xi_+)p_-(\xi_-) \\ &= \frac{1}{\sqrt{\epsilon}}\int_0^\infty d\tau\int_0^\infty d\xi_+ p_+(\xi_+)\exp[-\xi_+\tau]\int_0^\infty d\xi_- p_-(\xi_-)\exp[-\xi_-\tau],\end{aligned} \tag{4.4.9}$$

where the distribution functions $p_\sigma(\xi_\sigma) = \langle\delta(\xi_\sigma(x') - \xi_\sigma)\exp[\int_{x'}^{x} dx_1(-\sigma\sqrt{\epsilon}\xi_\sigma(x_1) + \mu + v(x_1))]\rangle$ and $p_{-\sigma}(\xi_{-\sigma}) = \langle\delta(\xi_{-\sigma}(x) - \xi_{-\sigma})\rangle$ with the value of σ depending on whether we are considering $x > x'$ or $x' > x$. We can separate the two quantities on averaging as they occur on either side of the source x' without any overlap. We can therefore split the averaging into two separate parts. We have also introduced a third integral over τ, $\frac{1}{\xi_+ + \xi_-} = \int_0^\infty d\tau\exp[-(\xi_+ + \xi_-)\tau]$, which makes the integrals separable. For each of these distribution functions we derive a Fokker-Planck equation as we did when we first derived the original Fokker-Planck equation in chapter two. Once we have this Fokker-Planck equation we will solve it using various techniques to find the form of the expectation value of the probability distribution.

4.5 The Return Probability

The return probability, as we know, is simply the probability of the particle returning to the origin of its random walk. There has been much work done on the return probability with full analytic results derived in [14, 21] for zero and non-zero drift. We use the result primarily to check the method successfully re-creates known results and it also provides an excellent illustration of the effect of drift and disorder. In addition to this, we also calculate an analytic result for the moments of the return probability and show that it is not a self averaging quantity in agreement with [14], even though at no point in this paper do they analytically solve for the result.

To calculate the return probability we set $x = x'$ and the two distribution functions become $p_+(x) = \langle\delta(\xi_+(x) - \xi_+)\rangle$ and $p_-(x) = \langle\delta(\xi_-(x) - \xi_-)\rangle$. We find the form of these distribution functions by deriving a Fokker-Planck equation for their evolution by simply differentiating with respect to x and using the result of Furutsu-Novikov

$$\frac{\partial p_\sigma}{\partial x} = -\frac{\partial}{\partial \xi_\sigma}\left[\sigma\sqrt{\epsilon}(\xi_\sigma^2 - 1)p_\sigma - \mu\xi_\sigma p_\sigma - \xi_\sigma\langle\delta(\xi_\sigma(x) - \xi_\sigma)v(x)\rangle\right]. \tag{4.5.1}$$

We use the result of Furutsu-Novikov to find the form of the average $\langle\delta(\xi_\sigma(x) - \xi_\sigma)v(x)\rangle$ by taking the integral over the product of the correlation function of the potential we are averaging over and the average of the functional derivative of the function that is being averaged, *i.e.*

$$\langle\delta(\xi_\sigma(x') - \xi_\sigma)v(x')\rangle \;=\; \int\langle v(x')v(x'')\rangle\langle\frac{\delta}{\delta v(x'')}\delta(\xi_\sigma(x') - \xi_\sigma)\rangle dx''. \tag{4.5.2}$$

This result can be found by using the correlation function of $v(x)$ and the functional derivative $\frac{\delta\xi_\sigma(x)}{\delta v(x)}$ to find

$$\langle\delta(\xi_\sigma(x')\xi_\sigma)v(x')\rangle = -\sigma\frac{\partial}{\partial\xi_\sigma}(\xi_\sigma p_\sigma). \tag{4.5.3}$$

Using this result we arrive at a Fokker-Planck equation for the evolution of the distribution functions $p_\sigma(\xi_\sigma)$

$$\frac{\partial}{\partial x}p_\sigma(\xi_\sigma) = -\frac{\partial}{\partial \xi_\sigma}\left[\sigma\sqrt{\epsilon}(\xi_\sigma^2 - 1)p_\sigma - \mu\xi_\sigma p_\sigma + \sigma\xi_\sigma\frac{\partial}{\partial \xi_\sigma}(\xi_\sigma p_\sigma)\right]. \tag{4.5.4}$$

The problem is now formally solved as it has been reduced to a set of non-random equations, although it is still not easy to deal with. The simplification which makes the problem solvable comes from the thermodynamic limit $L \to \infty$, since we are not interested in finite system sizes. The variables $\xi_\sigma(x)$ take non-random values at the boundaries which dictate that the initial conditions for $p_\sigma(\xi_\sigma, x = L) = \delta(\xi_\sigma(L) - \xi_\sigma)$. If we send $L \to \infty$, we can ignore the true boundary conditions and search for stationary solutions, thus we find two normalized solutions of the form

$$p_+(\xi_+) = \frac{1}{2K_\mu(2\sqrt{\epsilon})}\frac{1}{\xi_+^{(1-\mu)}}\exp[-\sqrt{\epsilon}(\xi_+ + \frac{1}{\xi_+})], \tag{4.5.5}$$

$$p_-(\xi_-) = \frac{1}{2K_\mu(2\sqrt{\epsilon})}\frac{1}{\xi_-^{(1+\mu)}}\exp[-\sqrt{\epsilon}(\xi_- + \frac{1}{\xi_-})], \tag{4.5.6}$$

where $K_\nu(z)$ are modified Bessel functions of the second kind [51]. To solve for the return probability we simply need to perform the three integrals over ξ_+, ξ_- and τ. On doing so we find the expectation value of the return probability distribution becomes

$$\langle P(\epsilon)\rangle = \frac{K_{\mu-1}(2\sqrt{\epsilon})K_{\mu+1}(2\sqrt{\epsilon})}{K_\mu^2(2\sqrt{\epsilon})} - 1, \tag{4.5.7}$$

where the integrals have been done with the aid of [51, 52]. This is the most general result for the return probability with drift. The result is in agreement with that of [14] and it would seem that the method we use does indeed re-create known results. We would like to write this as a function of time, but to do so we should consider its asymptotic behaviour to separate it into short and long time scales. For the short time scale as $t \to 0$ we need to consider the large epsilon limit $\epsilon \to \infty$ which leads to

$$\langle P(\epsilon)\rangle = \frac{1}{2\sqrt{\epsilon}} - \frac{\mu^2}{16}\frac{1}{\epsilon^{\frac{3}{2}}} + \ldots, \tag{4.5.8}$$

where we have simply used the asymptotics of the Bessel functions [51]. For the long time equilibrium behaviour which is equivalent to $\epsilon \to 0$ we find several regions of interest. For the case of zero drift we find the return probability becomes

$$\langle P(\epsilon)\rangle = \frac{K_1^2(2\sqrt{\epsilon})}{K_0^2(2\sqrt{\epsilon})} - 1 = \frac{1}{4\epsilon}\frac{1}{\ln^2(2\sqrt{\epsilon})}. \tag{4.5.9}$$

For finite drift we find a series of solutions depending on the value of drift velocity

$$P(\epsilon) = \begin{cases} \frac{\Gamma[1+\mu]\Gamma[1-\mu]}{\Gamma^2[\mu]}\epsilon^{-1+\mu}, & \text{for} \quad 0<\mu<1 \\ \frac{1}{\mu-1} + \frac{\Gamma[1+\mu]\Gamma[1-\mu]}{\Gamma^2[\mu]}\epsilon^{-1+\mu}, & \text{for} \quad 1<\mu<2 \\ \frac{1}{\mu-1} - \frac{2\epsilon}{(\mu-2)(\mu-1)^2} + \frac{\Gamma[1+\mu]\Gamma[1-\mu]}{\Gamma^2[\mu]}\epsilon^{-1+\mu}. & \text{for} \quad 2<\mu<3 \end{cases} \tag{4.5.10}$$

We can generalize this result to any arbitrary value of μ to find

$$\langle P(\epsilon)\rangle = \sum_{m=1}^{[\mu]} c_m \epsilon^{m-1} + c_\mu \epsilon^{\mu-1}, \tag{4.5.11}$$

where $[\mu]$ means to take the integer value of μ in the region $m < \mu < m+1$. The coefficients c_m can be found by expanding equation (4.5.7) to the $(m-1)^{th}$ order in ϵ in the limit $\epsilon \to 0$. To find the time dependence of the return probability we must perform the inverse Laplace transformation to find

$$P(t) = \begin{cases} \frac{1}{\ln^2(t)}, & \text{for} \quad \mu = 0 \\ \frac{\Gamma[1+\mu]}{\Gamma[\mu]^2}t^{-\mu}, & \text{for} \quad 0<\mu<2 \\ \frac{2t^{-2}}{(\mu-1)^2(\mu-2)}, & \text{for} \quad \mu>2 \end{cases} \tag{4.5.12}$$

for the long time equilibrium behaviour. For short time scales we see that the return probability becomes

$$\langle P(t)\rangle = \frac{1}{\sqrt{4\pi t}}\exp\left[-\frac{\mu^2}{4}t\right], \tag{4.5.13}$$

which has the form of normal diffusion as we found for the pure case.

These results are simply for the first moment of the return probability. We would now like to generalize the method to find the n^{th} moment. We will follow the exact same

steps as we did previously with the only difference arising when we introduce the third integral over τ, where instead of using $\frac{1}{\xi_+ + \xi_-} = \int_0^\infty \exp[-(\xi_+ + \xi_-)\tau]$ we use $\frac{1}{(\xi_+ + \xi_-)^n} = \frac{1}{\Gamma[n]} \int_0^\infty t^{n-1} \exp[-(\xi_+ + \xi_-)\tau]$.

4.5.1 Moments of the Return Probability

As far as we know the only work done on the moments of the return probability has been in [14] where they show that $\langle P_n(t) \rangle \neq \langle P(t) \rangle^n$, *i.e.* $\langle P(t) \rangle$ is not a self averaging quantity, which they show without an explicit result for $\langle P_n(t) \rangle$. For this section we generalize the method for the first moment to find an analytical expression for $\langle P_n(t) \rangle$, which as we have already stated is an original result as far as we know. The result we find can be shown to be consistent with those of the first moment we calculated in the previous section. It should also be noted that the method we use can be used to generalize to any arbitrary time scale, but we only consider long time scales as the only relevant scale.

To calculate the moments of the return probability we follow the same procedure as we did previously to find

$$P_n(\epsilon) = \frac{1}{\sqrt{\epsilon}^n} \frac{1}{(\xi_+(x) + \xi_-(x))^n}, \tag{4.5.14}$$

with $\xi_\sigma(x)$ still satisfying the Langevin equation (4.4.7). We find by introducing the usual distribution functions (4.4.9) whose solution is still given by (4.5.5) the n^{th} moment of the expectation of the return probability becomes

$$\langle P_n(\epsilon) \rangle = \frac{1}{\sqrt{\epsilon}^n} \frac{1}{\Gamma(n)} \frac{1}{K_\mu^2(2\sqrt{\epsilon})} \int_0^\infty d\tau \tau^{n-1} K_\mu^2 \Big[2\sqrt{\sqrt{\epsilon}(\sqrt{\epsilon} + \tau)} \Big]. \tag{4.5.15}$$

By making the substitution $u^2 = 4(\epsilon + \sqrt{\epsilon}\tau)$ we find the expectation value of the probability distribution becomes

$$\begin{aligned} \langle P_n(\epsilon) \rangle &= \frac{1}{2^{2n-1}\epsilon^n} \frac{1}{\Gamma(n)} \frac{1}{K_\mu^2(2\sqrt{\epsilon})} \int_{2\sqrt{\epsilon}}^\infty u(u^2 - 4\epsilon)^{n-1} K_\mu^2(u) du \\ &= \frac{1}{\Gamma[n]} \frac{1}{2^{2n-1}} \frac{1}{K_\mu^2(2\sqrt{\epsilon})} \frac{1}{\epsilon^n} \sum_{k=0}^{n-1} \binom{n-1}{k} (-4\epsilon)^k \int_{2\sqrt{\epsilon}}^\infty u^{2n-1-2k} K_\mu^2(u), \end{aligned}$$

where we have simply expanded the term $(u^2 - 4\epsilon)^{n-1}$ using the binomial expansion. If we perform this integral using [52] we find the moments of the expectation value of the return probability are given by

$$\langle P_n(\epsilon)\rangle = \sum_{k=0}^{n-1}(-1)^k \binom{n-1}{k} \left[\frac{4^{n-k}\epsilon^{n-\mu}\Gamma^2[\mu]}{k+\mu-n} + \epsilon^k \frac{2\sqrt{\pi}\Gamma[n-k-\mu]\Gamma[n-k+\mu]}{\Gamma[\frac{1}{2}-k+n]}\right] \times \frac{1}{\Gamma[n]}\frac{1}{2^{2n+2}}\frac{1}{K_\mu^2(2\sqrt{\epsilon})}\frac{1}{\epsilon^n}, \tag{4.5.16}$$

where we have considered the limit $\epsilon \to 0$ in the solution to the integral. We could have derived a general result for any value of epsilon to find a series of generalized hypergeometric functions, but it would have been somewhat unnecessary as we are only interested in the long time equilibrium limit. This is the most general result for the moments of the return probability and as usual the next step is to consider the result for different values of drift. For zero drift we find the moments are given simply by $k = 0$ as all subsequent terms are epsilon times smaller. We therefore find

$$\langle P_n(\epsilon)\rangle = \frac{\sqrt{\pi}}{2^{2n+1}}\frac{1}{\epsilon^n}\frac{\Gamma^2[n]}{\Gamma[n+\frac{1}{2}]}\frac{1}{\ln^2(2\sqrt{\epsilon})}, \tag{4.5.17}$$

which when we set $n = 1$ is consistent with the first moment. For the case of finite drift we find that for $0 < \mu < 1$ the return probability is again given by the case of $k = 0$ and we find

$$\langle P_n(\epsilon)\rangle = \frac{\sqrt{\pi}}{2^{2n-1}}\frac{\Gamma[n-\mu]\Gamma[n+\mu]}{\Gamma[\frac{1}{2}+n]\Gamma^2[\mu]}\epsilon^{-n+\mu}, \tag{4.5.18}$$

which is again consistent with the first moment. It can also be shown that for any arbitrary value of drift velocity the result is consistent with the first moment we derived in the previous section. To find the result for any other moment or drift requires us to consider the dominant regions of (4.5.16). To check the result of [14] that the return probability is not a self-averaging quantity we should write it as a function of time. If we perform the inverse Laplace transforms to find the time dependence of (4.5.17) and (4.5.18) we find that for zero

drift

$$\langle P_n(t)\rangle = \frac{\sqrt{\pi}}{2^{2n+1}}\frac{\Gamma[n]}{\Gamma[n+\frac{1}{2}]}\frac{t^{-1+n}}{\ln^2(t)}, \tag{4.5.19}$$

and for $0 < \mu < 1$

$$\langle P_n(t)\rangle = \frac{\sqrt{\pi}}{2^{2n-1}}\frac{\Gamma[n+\mu]}{\Gamma[\frac{1}{2}+n]\Gamma^2[\mu]}t^{-1+n-\mu}, \tag{4.5.20}$$

and we see that $\langle P_n(t)\rangle \neq \langle P(t)\rangle^n$. We can therefore conclude that the return probability is not a self-averaging quantity.

4.5.2 Summary

To summarize this section on the return probability we notice that we have derived several results for its distribution function. The first result we found was that for the short time behaviour of a randomly diffusing particle which we found follows that of normal diffusion, *i.e.* the particle diffuses freely with $\langle x^2(t)\rangle = 2Dt$. The particle does not feel any effect of disorder indicating that there is a finite time scale which needs to have passed before disorder has an effect. The more significant case is that of the long time equilibrium distribution which has several distinct regions of interest, all determined by the value of drift velocity. For the case of Sinai's diffusion we find the probability distribution decays logarithmically indicating that disorder has the effect of localizing the particle around the origin. This seems quite natural when we consider the physical process going on. As the strength of disorder increases we find it takes longer for the particle to overcome its trapping potential. As a consequence we expect it to take longer to jump to the next trap and we find its motion is slowed, hence we say it is sub-diffusive. We see that with the introduction of the drift velocity there are further regions of interest. For non-zero drift we find the probability distribution becomes a power law and we see the drift velocity drives the particle away from the origin but there is now competition between the two opposing effects. We also found the result for the return probability can be written in the form $\langle P(\epsilon)\rangle = \sum_{m=1}^{[\mu]} c_m\epsilon^{m-1} + c_\mu\epsilon^\mu$ a result which will be of interest for the next section. In addition to just the return probability

we have also calculated a result for the moments of the distribution. We have seen that $\langle P_n(t)\rangle \neq \langle P(t)\rangle^n$ meaning $\langle P(t)\rangle$ is not a self averaging quantity. We were also able to give a physical meaning to $\langle P(t)\rangle$ which was that it represents the fraction of particles that are left at their initial site after some time t. We would now like to generalize the method to calculate the probability of finding the particle at some arbitrary point.

4.6 Arbitrary Probability Distribution

In this section of the thesis we will be solving for the probability distribution of finding the particle at some arbitrary position. We do this by following the same procedure as we did previously by first deriving a Fokker-Planck equation for each distribution function. The difference now is that we can't assume stationary solutions and we need to expand over eigen-states. On assuming an expansion over eigen-states we will be required to solve the Fokker-Planck equation for the distribution function using the method of asymptotic matching. This asymptotic matching method requires the equation to be solved in two overlapping asymptotic regions and then matched in the region of overlap. This will lead to the energy spectrum of the problem and the form of the distribution function will be essentially solved. It is this method of asymptotic matching that makes the problem solvable for the case of finite drift.

If we begin with the usual definition of the expectation value of the probability distribution

$$\langle P(x,x';\epsilon)\rangle = \frac{1}{\sqrt{\epsilon}}\int_0^\infty d\xi_+ \int_0^\infty d\xi_- \int_0^\infty d\tau \exp[-(\xi_+ + \xi_-)\tau] p_+(\xi_+) p_-(\xi_-),$$

we can immediately reduce the problem down to two integrals over ξ_σ and τ by using the result previously found for the return probability

$$\begin{aligned}\langle P(x,x';\epsilon)\rangle &= \frac{1}{\sqrt{\epsilon}}\frac{1}{K_\mu(2\sqrt{\epsilon})}\int_0^\infty d\tau \overbrace{\int_0^\infty d\xi_{-\sigma}\exp[-\xi_{-\sigma}\tau]p(\xi_{-\sigma})}^{p(\tau)} \\ &\times (1+\frac{\tau}{\sqrt{\epsilon}})^{-\frac{\sigma\mu}{2}} K_\mu(2\sqrt{\epsilon(1+\frac{\tau}{\sqrt{\epsilon}})}). \qquad (4.6.1)\end{aligned}$$

To find the form of the other probability distribution

$$p_\sigma(\xi_\sigma) = \langle \delta(\xi_\sigma(x') - \xi_\sigma) \exp[\int_{x'}^{x} dx_1(-\sigma\sqrt{\epsilon}\xi_\sigma(x_1) + +\mu + v(x_1))]\rangle, \tag{4.6.2}$$

we need to follow the same steps as we did for the return probability *i.e.* differentiate with respect to x and use the result of Furutsu-Novikov to calculate the averages. Following these steps we find a Fokker-Planck equation of the form

$$\sigma\frac{\partial p_\sigma}{\partial x} = \xi_\sigma^2\frac{\partial^2 p_\sigma}{\partial \xi_\sigma^2} + (-\sigma\mu\xi_\sigma + \xi_\sigma + \xi_\sigma^2\sqrt{\epsilon} - \sqrt{\epsilon})\frac{\partial p_\sigma}{\partial \xi_\sigma} + \sqrt{\epsilon}\xi_\sigma p_\sigma. \tag{4.6.3}$$

We notice from equation (4.6.1) that we would like to find the Laplace transform of the solution to this probability distribution. We can do this now by performing the Laplace transform of our Fokker-Planck equation for $p_\sigma(\xi_\sigma)$ to find an equation for $p(\tau)$ $(= \int_0^\infty d\xi_\sigma \exp[-\xi_\sigma\tau]p_\sigma(\xi_\sigma))$. We can then reduce our expression for the expectation value of the probability distribution to a single integral over τ

$$\langle P(x, x'; \epsilon)\rangle = \frac{1}{\sqrt{\epsilon}}\frac{1}{K_\mu(2\sqrt{\epsilon})}\int_0^\infty d\tau p(\tau)(1 + \frac{\tau}{\sqrt{\epsilon}})^{-\frac{\sigma\mu}{2}} K_\mu(2\sqrt{\epsilon(1 + \frac{\tau}{\sqrt{\epsilon}})}). \tag{4.6.4}$$

By making the substitution $z = 1 + \frac{\tau}{\sqrt{\epsilon}}$ we find the expectation of the probability distribution becomes

$$\langle P(x, x'; \epsilon)\rangle = \int_1^\infty dz z^{-\frac{\sigma\mu}{2}}\frac{K_\mu(2\sqrt{\epsilon z})}{K_\mu(2\sqrt{\epsilon})}p_{-\sigma}(z), \tag{4.6.5}$$

and the equation for $p_\sigma(z)$ has the form

$$\sigma\frac{\partial p_\sigma(z)}{\partial \hat{x}} = (z^2 - z)\frac{\partial^2 p_\sigma(z)}{\partial z^2} + ((3 + \sigma\mu)z - 2 - \sigma\mu)\frac{\partial p_\sigma(z)}{\partial z} + (1 + \sigma\mu - \epsilon z + \epsilon)p_\sigma(z), \tag{4.6.6}$$

where x has been re-scaled in terms of x', *i.e.* $\hat{x} = x - x'$. We would now like to solve this equation and we do so by assuming an expansion over eigen-states *i.e.* $p(z) = \sum_{n=0}\psi_n(z)\psi_n(z')\exp[-E|x - x'|]$, where E are the eigen-energies. To find the correct asymptotic limits of this equation we should write it as a Schrödinger equation by assuming

$\psi(z) = \psi_0(z)\psi_1(z)$ where $\psi_0(z)$ is chosen so to eliminate the first derivative of $\psi_1(z)$

$$-\psi_1''(s)+\left(-\frac{1}{16(\cosh^2[\frac{s}{2}]-1)}-\frac{(1+2\sigma\mu)(3+2\sigma\mu)}{16\cosh^2[\frac{s}{2}]}+\frac{\mu^2}{4}+\epsilon(\cosh^2[\frac{s}{2}]-1)\right)\psi_1(s) = E\psi_1(s), \tag{4.6.7}$$

where we used the substitution $z = \cosh^2(\frac{s}{2})$ to eliminate the term in front of the second derivative. Now if we plot this potential we see that there are two regions of interest. The first when the eigen-energy E is greater than $\frac{\mu^2}{4}$ which puts the particle in the quasi-

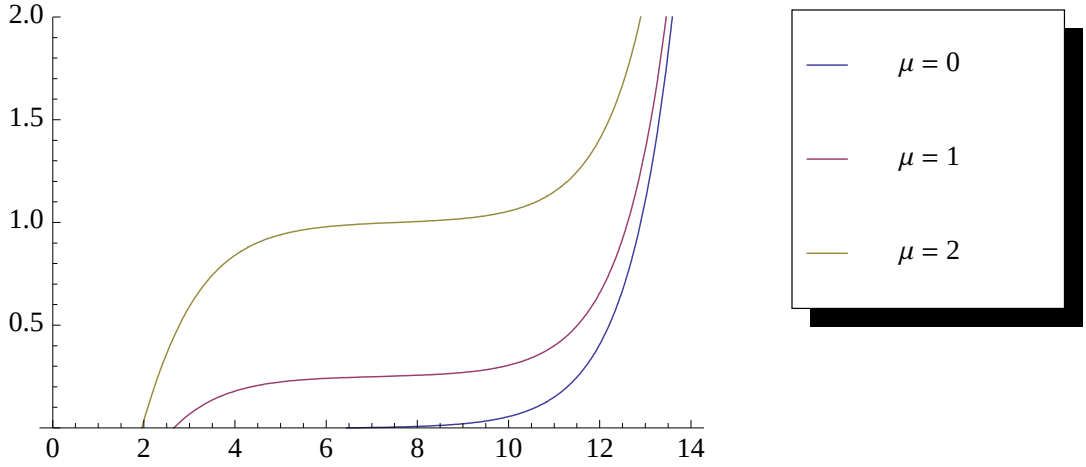

Figure 4.1: Typical shape of the potential showing the emergence of a gap in the potential, which separates discrete states and quasi continuous states.

continuous region and the second when the eigen-energy is less than $\frac{\mu^2}{4}$ and the particle sits in a bound state with discrete energy levels. From this we can see that on considering the zero drift case we will only have states in the quasi-continuous region. In contrast for the finite drift case we must consider only the bound states to find the energy spectrum if we only consider the large separation limit. It is at this point where the difference between the method we have used and previous attempts arises [18]. The equivalent equation for this distribution function used in [18] has the form

$$-E\psi(u) = -\frac{\partial^2\psi}{\partial u^2}+\left(\frac{\epsilon}{4}\sinh^2(u)-\frac{\sqrt{\epsilon}}{2}(1\pm\mu)\sinh(u)+\frac{\mu^2}{4}\right)\psi(u), \tag{4.6.8}$$

which is equivalent to the potential of (4.6.7) if we make the appropriate transformations. For zero drift this equation can be approximated to an infinite well potential and the energy

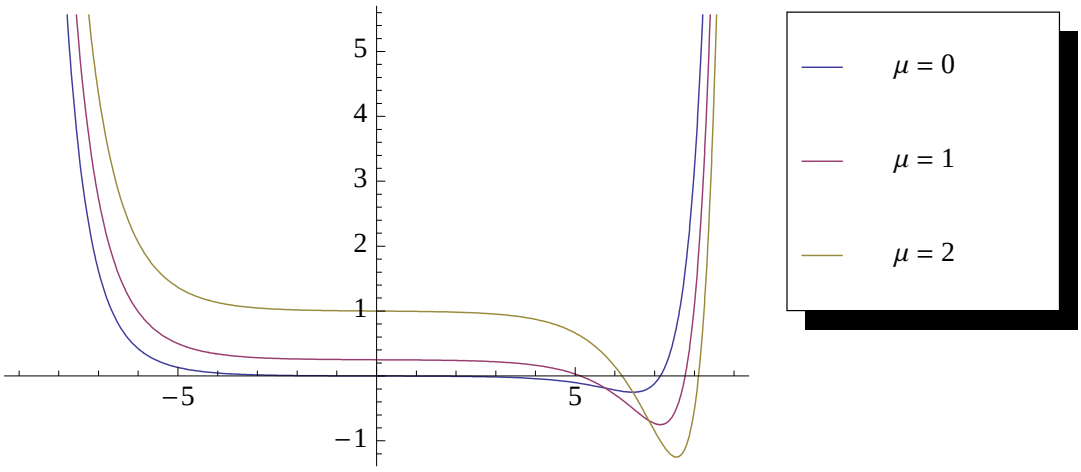

Figure 4.2: Typical shape of the potential of equation (4.6.8), which is usually approximated to an infinite well potential.

E is found using standard methods. The problem with this approach is that it provides two values for the energy as it is a symmetrical well and only one of them follows the known scaling argument of Kesten [19, 20] and this value is chosen without any justification. As we will see the method of asymptotic matching will only provide one value for E, the correct one. In addition to this, when we look at figure (4.6.8) with drift turned on there is this region with bound states. This can not be treated correctly when approximating the potential to an infinite well and hence the reason why there has been no previous analytical solution to this problem with finite drift.

If we move back to our own formulation we see by considering equation (4.6.7) there are two natural asymptotic limits to consider. The first being $z \gg 1$ and the second $z \ll \frac{1}{\epsilon}$. If we consider such limits we find two asymptotic equations

$$\begin{aligned} -E\psi(z) &= z^2\frac{\partial^2\psi(z)}{\partial z^2} + (3+\sigma\mu)z\frac{\partial\psi(z)}{\partial z} + (1+\sigma\mu-\epsilon z+\epsilon)\psi(z), \\ -E\psi(z) &= (z^2-z)\frac{\partial^2\psi(z)}{\partial z^2} + ((3+\sigma\mu)z-2-\sigma\mu)\frac{\partial\psi(z)}{\partial z} + (1+\sigma\mu)\psi(z), \end{aligned}$$

for $z \gg 1$ and $z \ll \frac{1}{\epsilon}$ respectively. The aim now is to solve these equations and then asymptotically match them in the region $1 \ll z \ll \frac{1}{\epsilon}$ to find the form of the energy spectrum E. We have two cases to consider, the first being for zero drift and the second for finite

drift. We treat each case separately as the methods we use to asymptotically match the solutions are completely different. For the case of zero drift velocity we can simply match the two solutions as they have the form of plane waves. For the finite drift case we have to use a perturbation solution and then asymptotically match the correct functions.

4.6.1 Zero Drift Velocity

For the case of zero drift velocity we see that there are no bound states and we sit in the state with $E = -q^2$ in the quasi-continuous region. We find two asymptotic solutions, the first for $z \gg 1$

$$\psi(z) = Az^{-1}K_{2iq}(2\sqrt{\epsilon z}), \tag{4.6.9}$$

where as we know from the return probability, $K_\nu(z)$ are modified Bessel functions of the second kind and A is simply a constant of integration. If we now consider the other asymptotic limit of $z \ll \frac{1}{\epsilon}$ then we find a solution of the form

$$\psi(z) \quad = \quad BA_{iq}z^{-1-iq}F[1+iq, iq, 1+2iq; \frac{1}{z}] - BA_{-iq}z^{-1+iq}F[1-iq, -iq, 1-2iq; \frac{1}{z}],$$

where $F[a, b, c, z]$ is a Hypergeometric function [51] with

$$A_{iq} = \frac{\Gamma[1-2iq]}{\Gamma[1-iq]\Gamma[-iq]}, \qquad \text{and} \qquad A_{-iq} = \frac{\Gamma[1+2iq]}{\Gamma[1+iq]\Gamma[iq]}, \tag{4.6.10}$$

with the choice of integration constants chosen so to avoid the logarithmic divergence as $z \to 1^+$ [51]. As these two solutions are asymptotic limits of the same solution they should be equivalent in the region $1 \ll z \ll \frac{1}{\epsilon}$. In this overlapping region the two asymptotic solutions are plane waves and we can safely match them. Doing so we find the energy spectrum is given by

$$\epsilon^{2iq} = \frac{\Gamma[1-iq]\Gamma[-iq]\Gamma^2[2iq]}{\Gamma[1+iq]\Gamma[iq]\Gamma^2[-2iq]}, \tag{4.6.11}$$

and as we are considering the small q, due to the fact we are in the large separation limit, we find the energy spectrum has the form

$$q = \frac{(2n+1)\pi}{2\ln(\epsilon)}, \tag{4.6.12}$$

which is in complete agreement with known result of Kesten [19]. We now only need to normalize our two solutions to find the form of the integration constants. Doing this we find two normalized solutions

$$\begin{aligned}
\psi(z) &= \frac{2\sqrt{2}q}{\ln^{\frac{1}{2}}(\frac{1}{\epsilon})} z^{-1} K_{2iq}(2\sqrt{\epsilon z}), \\
\psi(z) &= \frac{1}{\sqrt{2}} \frac{1}{\ln^{\frac{1}{2}}(\frac{1}{\epsilon})} \frac{\epsilon^{-iq}}{\Gamma[1-2iq]} z^{-1+iq} F[1-iq, -iq, 1-2iq, \frac{1}{z}] \\
&- \frac{1}{\sqrt{2}} \frac{1}{\ln^{\frac{1}{2}}(\frac{1}{\epsilon})} \frac{\epsilon^{iq}}{\Gamma[1+2iq]} z^{-1-iq} F[1+iq, iq, 1+2iq, \frac{1}{z}],
\end{aligned}$$

for $z \gg 1$ and $z \ll \frac{1}{\epsilon}$ respectively. We are now in a position to find the expression for the expectation value of the probability distribution

$$\langle P(x, x'; \epsilon) \rangle = \frac{1}{\ln^3(2\sqrt{\epsilon})} \sum_{n=0}^{\infty} a_q b_q q^2 \exp\left[-\frac{(2n+1)^2\pi^2}{4\ln^2(\epsilon)} \mid x - x' \mid \right], \tag{4.6.13}$$

where the terms a_q and b_q are integrals of the form

$$a_q = \int_1^{\infty} dz K_0(2\sqrt{\epsilon z}) K_{2iq}(2\sqrt{\epsilon z}) = \frac{1}{2\epsilon} \frac{\pi^2 q^2}{\cosh(2\pi q) - 1} = \frac{1}{8\epsilon}, \tag{4.6.14}$$

and

$$b_q = \int_1^{\infty} \frac{dz}{z} K_0(2\sqrt{\epsilon z}) K_{2iq}(2\sqrt{\epsilon z}) = \frac{(-1)^n}{q^3}, \tag{4.6.15}$$

which have been performed using standard integrals [52]. It should be noted that we have ignored the contribution from the Hypergeometric functions. This is because the contribution is only significant as $z \to 1^+$ so we say the distribution is essentially given by just the Bessel function $K_{2iq}(2\sqrt{\epsilon z})$ with the Hypergeometric function just giving a correction to the distribution. Substituting in the values of a_q and b_q we find the expectation of the

probability distribution becomes

$$\langle P(x,x';\epsilon)\rangle = \frac{4}{\pi\epsilon\ln^2(\epsilon)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\exp\left[-\frac{(2n+1)^2\pi^2}{4\ln^2(\epsilon)} \mid x-x' \mid \right]. \tag{4.6.16}$$

We can now check whether this result is consistent with our earlier result of the return probability by simply setting $x = 0$

$$\langle P(x=0;\epsilon)\rangle = \frac{4}{\pi\epsilon\ln^2(\epsilon)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)} = \frac{4}{\pi\epsilon\ln^2(\epsilon)}\frac{\pi}{4} = \frac{1}{\epsilon\ln^2(\epsilon)},$$

which is indeed the result of the return probability we found previously. We should now perform the inverse Laplace transform to find the time dependence of the probability distribution

$$\langle P(x,x';t)\rangle = \frac{4}{\pi\ln^2(t)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\exp\left[-\frac{(2n+1)^2\pi^2}{4\ln^2(t)} \mid x-x' \mid \right], \tag{4.6.17}$$

which is in agreement with the result of [14, 18]. We have now re-created all known results using the method of asymptotic matching. We would now like to calculate an analytic expression for the expectation value of the probability distribution with drift.

4.6.2 Finite Drift Velocity

As we have already seen, we have calculated the the form of the probability distribution for zero drift, but we would like to use the method to calculate the probability distribution for finite drift, a result which has not previously been found analytically. As we have seen the addition of drift brings the emergence of a gap in the potential and it is in this region which we need to solve in. If we consider the equation

$$(z^2-z)\psi''(z) + ((3+\sigma\mu)z - 2 - \sigma\mu)\psi'(z) + (1+\sigma\mu)\psi(z) = (\epsilon z - \epsilon - E)\psi(z), \tag{4.6.18}$$

but now in the region with discrete states, *i.e.* $0 < E < \frac{\mu^2}{4}$, then we find in the region $z \gg 1$ a solution of the form

$$\psi(z) = Az^{-(1+\frac{\sigma\mu}{2})}K_{2r}(2\sqrt{\epsilon z}), \tag{4.6.19}$$

where $r = \sqrt{\frac{\mu^2}{4} - E}$ and A is again a constant of integration. We can also solve this equation in the region $z \ll \frac{1}{\epsilon}$ to find a solution of the form

$$\begin{aligned} \psi(z) &= Bz^{-(1+\frac{\sigma\mu}{2})-r}F[1+\frac{\sigma\mu}{2}+r, -\frac{\sigma\mu}{2}+r, 1+2r; \frac{1}{z}] \\ &- Cz^{-(1-\frac{\sigma\mu}{2})-r}F[1+\frac{\sigma\mu}{2}-r, -\frac{\sigma\mu}{2}-r, 1-2r; \frac{1}{z}], \end{aligned} \tag{4.6.20}$$

where B and C are constants of integration and to eliminate the logarithmic divergence as $z \to 1^+$ we must set each parameter of the Hypergeometric function equal to a negative integer [51]. Doing this we find for $\sigma = +1$ the energy E is zero and $r = \frac{\mu}{2}$. For the opposite case when $\sigma = -1$ we find that it is $n = -1$ that minimizes the energy. We therefore find the probability distribution becomes

$$\langle P(x, x'; \epsilon)\rangle = \begin{cases} \exp[-a|\hat{x}|], & \hat{x} \leq 0 \\ \text{Const.} & \hat{x} \geq 0 \end{cases} \tag{4.6.21}$$

This analysis shows the effect of drift velocity; when it is turned on we find it has the effect of pushing the particle to the region of positive x. We therefore need to solve for the energy E in this region to the right with $\sigma = 1$ in the limits $E \to 0$ and $\epsilon \to 0$. It should be noted that this is only true for positive μ. If we had a negative value of drift the whole distribution would be mirrored along the line $x = 0$. Using this idea we will be able to expand the results for positive drift to those for negative drift. We know in this limit of $E \to 0$ and $\epsilon \to 0$ that $r = \sqrt{\frac{\mu^2}{4} - E} = \frac{\mu}{2} - \frac{E}{\mu}$ and in the region $z \gg 1$

$$\begin{aligned} \psi(z) &= Az^{-(1+\frac{\mu}{2})}K_{\mu-\frac{2}{\mu}E}(2\sqrt{\epsilon z}) \\ &= A\left[\frac{\epsilon^{-\frac{\mu}{2}+\frac{E}{\mu}}z^{-(1+\mu)+\frac{E}{\mu}}}{\Gamma[1-\mu]} - \frac{\epsilon^{\frac{\mu}{2}-\frac{E}{\mu}}z^{-1-\frac{E}{\mu}}}{\Gamma[1+\mu]}\right]. \end{aligned} \tag{4.6.22}$$

For the case of $z \ll \frac{1}{\epsilon}$ we need to solve equation (4.6.18) by making a perturbation expansion in ϵ and E

$$\psi(z) = \psi_0(z) + (E-\epsilon)\psi_1(z) + (E-\epsilon)^2\psi_2(z) + \ldots, \tag{4.6.23}$$

and we need to solve to each order of ϵ and E. Upon substitution of this perturbation expansion we find that the equation for the n^{th} term of the perturbation expansion is given by

$$(z^2-z)\psi_n''(z) + ((3+\mu)z - 2 - \mu)\psi_n'(z) + (1+\mu)\psi_n(z) = \frac{(\epsilon z - \epsilon - E)}{E-\epsilon}\psi_{n-1}(z). \tag{4.6.24}$$

The aim is to find a solution to this equation and then asymptotically match it in the region $1 \ll z \ll \frac{1}{\epsilon}$ to find the form of the energy spectrum. We should therefore look for two solutions to this equation which are proportional to either $z^{-(1+\mu)}$ or z^{-1}. If we now consider the zeroth order of perturbation we find

$$(z^2-z)\psi_0''(z) + ((3+\mu)z - 2 - \mu)\psi_0'(z) + (1+\mu)\psi_0(z) = 0, \tag{4.6.25}$$

which has a solution of the form $\psi_0(z) = z^{-(1+\mu)}$. If we now consider the first order expansion in ϵ and E we are required to solve an equation of the form

$$\begin{aligned}(z^2-z)\psi_1''(z) + ((3+\mu)z - 2 - \mu)\psi_1'(z) + (1+\mu)\psi_1(z) &= \frac{(\epsilon z - \epsilon - E)}{E-\epsilon}\psi_0(z) \\ &= \frac{(\epsilon z - \epsilon - E)}{E-\epsilon}z^{-(1+\mu)},\end{aligned}$$

and we find a solution which is proportional to z^{-1}

$$\psi_1(z) = \frac{1}{E-\epsilon}\frac{1}{\mu^2}\left(\frac{\epsilon}{\mu-1} - E\right)z^{-1}. \tag{4.6.26}$$

Now if we match the two solutions in the region $1 \ll z \ll \frac{1}{\epsilon}$ we find the energy is given by

$$E = \frac{\Gamma[1+\mu]\Gamma[1-\mu]}{\Gamma[\mu]^2}\epsilon^\mu + \frac{\epsilon}{\mu-1}. \tag{4.6.27}$$

Now this is enough for the region $0 < \mu < 2$, but if we want to consider the higher values of μ we find we must go further in the perturbation expansion. For second order in the perturbation expansion we must solve

$$(z^2 - z)\psi_2''(z) + ((3+\mu)z - 2 - \mu)\psi_2'(z) + (1+\mu)\psi_2(z) = \frac{(\epsilon z - \epsilon - E)}{E - \epsilon}\psi_1(z), \quad (4.6.28)$$

and we find a term which is proportional to z^{-1} of the form

$$\psi_2(z) = -\frac{1}{(E-\epsilon)^2}\frac{2\epsilon^2}{(\mu-2)(\mu-1)^2\mu^2}z^{-1}, \quad (4.6.29)$$

and on matching the energy becomes

$$E = \frac{\epsilon}{\mu - 1} - \frac{2\epsilon^2}{(\mu-2)(\mu-1)^2} + \epsilon^\mu \frac{\Gamma[\mu-1]\Gamma[1+\mu]}{\Gamma[\mu]^2}. \quad (4.6.30)$$

We notice by looking at this result that the most general form of the energy E is given by the sum $E = \sum_{m=1}^{[\mu]} c_m \epsilon^m + c_\mu \epsilon^\mu$. A result of this kind was suggested in [14] but there was no proof until now. This is exactly what we found for the return probability in the previous section. We therefore make the conjecture that the energy spectrum E is given by $E = \epsilon \times \langle P(\epsilon)\rangle$. It should be noted that we have neglected the contribution from the quasi-continous region as we are only interested in the large separation limit.

Using this general expression for the eigen-energy E we write the solution to the expectation of the probability distribution with drift velocity as

$$\langle P(x,x';\epsilon)\rangle = \left(\frac{K_{\mu-1}(2\sqrt{\epsilon})K_{\mu+1}(2\sqrt{\epsilon})}{K_\mu^2(2\sqrt{\epsilon})} - 1\right)\exp\left[-\epsilon\left(\frac{K_{\mu-1}(2\sqrt{\epsilon})K_{\mu+1}(2\sqrt{\epsilon})}{K_\mu^2(2\sqrt{\epsilon})} - 1\right)|x - x'|\right].$$

This expression comes from the fact that $\langle P(x,x';\epsilon)\rangle$ should be normalized to $\frac{1}{\epsilon}$ and give the expectation value of the return probability when $x = x'$. It should be noted that we have not been able to prove this result exactly. We have only been able to show that the two expansions coincide with each other, although with our knowledge of the problem we know this result should be expected. If we consider different regions of drift velocity we find

that for $0 < \mu < 1$ the probability distribution becomes

$$\langle P(x,x';\epsilon)\rangle = \frac{\Gamma[1-\mu]\Gamma[1+\mu]}{\Gamma^2[\mu]}\epsilon^{-1+\mu}\exp\left[-\frac{\Gamma[1-\mu]\Gamma[1+\mu]}{\Gamma^2[\mu]}\epsilon^{\mu}|x-x'|\right],$$

which shows a Lévy flight distribution. For the marginal case of $\mu = 1$ we find the probability distribution becomes

$$\langle P(x,x';\epsilon)\rangle = -\ln(2\sqrt{\epsilon})\exp\left[\epsilon\ln(2\sqrt{\epsilon})|x-x'|\right], \tag{4.6.31}$$

where the logarithmic term comes from the asymptotics of the Bessel function when $\mu = 0$. For the case of $1 < \mu < 2$ we find the expectation value of the probability distribution becomes

$$\langle P(x,x';\epsilon)\rangle = \left(\frac{1}{\mu-1}+\frac{\Gamma[1-\mu]\Gamma[1+\mu]}{\Gamma^2[\mu]}\epsilon^{-1+\mu}\right)\exp\left[-\left(\frac{\epsilon}{\mu-1}+\frac{\Gamma[1-\mu]\Gamma[1+\mu]}{\Gamma^2[\mu]}\epsilon^{\mu}\right)|x-x'|\right],$$

and for $2 < \mu < 3$ we find

$$\begin{aligned}\langle P(x,x';\epsilon)\rangle &= \left(\frac{1}{\mu-1}-\frac{2}{(\mu-2)(\mu-1)^2}\epsilon+\frac{\Gamma[1-\mu]\Gamma[1+\mu]}{\Gamma^2[\mu]}\epsilon^{-1+\mu}\right)\\ &\times \exp\left[-\left(\frac{\epsilon}{\mu-1}-\frac{2}{(\mu-2)(\mu-1)^2}\epsilon^2\frac{\Gamma[1-\mu]\Gamma[1+\mu]}{\Gamma^2[\mu]}\epsilon^{\mu}\right)|x-x'|\right].\end{aligned}$$

This result can be generalized to any order of drift velocity so to find a general solution of the form

$$\langle P(x,x';\epsilon)\rangle = \left(\sum_{m=1}^{[\mu]}c_m\epsilon^{m-1}+c_\mu\epsilon^{\mu-1}\right)\exp\left[-\left(\sum_{m=1}^{[\mu]}c_m\epsilon^{m}+c_\mu\epsilon^{\mu}\right)|x-x'|\right], \tag{4.6.32}$$

and the time dependence can be found from performing the inverse Laplace transformation, but this will not be the path we will follow. Instead we use these expressions for the probability distributions to calculate the transport properties of the particle. For each case we will calculate a general expression for the quantity $\langle x^n(t)\rangle$ from which we can calculate the transport properties of the particle and other relevant quantities including the mean-square displacement. We can also use this expression to calculate further properties of the

particle including the velocity and diffusion coefficients. For each case we will consider the general expression for all of the cumulants of the distribution.

4.7 Transport Properties

As we have previously stated we can use the probability distributions we derived in the previous section to calculate a general expression for the moments of the distribution $\langle x^n(t)\rangle$. To calculate the quantity $\langle x^n(t)\rangle$ we should first consider $\langle x^n(\epsilon)\rangle$, which we can calculate from

$$\langle x^n(\epsilon)\rangle = \int_0^\infty x^n P(x, x'; \epsilon) dx, \tag{4.7.1}$$

and to find the time dependence we simply need to perform the inverse Laplace transformation. Doing this for each case of drift we find a variety of diffusive behaviours. We begin by considering the case of zero drift where we re-derive the known result of the mean-square displacement $\langle x^2(t)\rangle$ and show that disorder has the effect of destroying the diffusion process. We then calculate further regions of drift including $0 < \mu < 1$ which is characterized by a Lévy distribution, $1 < \mu < 2$ which is again Lévy flight and $\mu > 2$ where we find the particle returns to normal diffusive behaviour. We find that the transports properties for each case are in agreement with those found in [14]. We do though find several differences between the results we calculate and those in [14] for the mean-square displacement.

4.7.1 $\mu = 0$

For the zero drift case we find the general expression for the moments is given by

$$\langle x^n(t)\rangle = \frac{4^{n+2}}{\pi^{2n+3}} \ln^{2n}(t) \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^{2n+3}}, \tag{4.7.2}$$

and we find for the third and final time in this thesis the known result of Sinai's diffusion $\langle x^2(t)\rangle = \frac{61}{360} \ln^4(t)$. If we calculate the quantity $\lim_{t\to\infty} \frac{\langle x^n(t)\rangle}{t}$ we find that it is zero for all moments and we say that all cumulants are zero. This therefore means that the velocity and diffusion coefficients are both zero and we see that the effect of disorder is to slow the

particles motion and destroy the diffusion process. We call this sub-diffusive behaviour, *i.e.* slower than diffusion. The fact that even the smallest amount of disorder destroys the diffusion process means we are in the strong disorder regime. The slowing of the particles motion is due to the fact that the random potential controls the transport of the particle. This was not unexpected as when we considered the scaling arguments we showed that the length scale x_0, the region when the random fluctuations dominate, diverges and we find that disorder controls the transport process.

4.7.2 $0 < \mu < 1$

For the case of $0 < \mu < 1$ we find the general expression for the moments is given by

$$\langle x^n(t)\rangle = \frac{1}{\Gamma[1+n\mu]}\frac{\Gamma^{2n}[\mu]}{\Gamma^n[1+\mu]\Gamma^n[1-\mu]}t^{n\mu}, \tag{4.7.3}$$

and we find a mean-square displacement of the form $\langle x^2(t)\rangle = \frac{1}{\Gamma[1+2\mu]}\frac{\Gamma^4[\mu]}{\Gamma^2[1+\mu]\Gamma^2[1-\mu]}t^{2\mu}$, which is in agreement with [14]. We find that the first cumulant $\Xi_1 = V = \lim_{t\to\infty}\frac{\langle x(t)\rangle}{t} = 0$ and we see the particles velocity is zero. We see that on calculating the other cumulants we get a mixture of behaviours depending on the value of drift velocity. For example if we calculate the second cumulant we find

$$\Xi_2 = D = \frac{1}{2}\lim_{t\to\infty}\frac{\langle x^2(t)\rangle - \langle x(t)\rangle^2}{t} = \frac{1}{2}\lim_{t\to\infty}\frac{\Gamma^4[\mu]}{\Gamma[1+2\mu]\Gamma^2[1+\mu]\Gamma^2[-\mu]}t^{2\mu-1}, \tag{4.7.4}$$

and we see that for $\mu < \frac{1}{2}$ that $\Xi_2 = 0$ and for $\frac{1}{2} < \mu < 1$ the cumulant $\Xi_2 = \infty$. This value of $\mu = \frac{1}{2}$ is important because the distribution function at this point is Gaussian when we move to the time description. This behaviour of the cumulants being either zero or infinite continues for all moments and cumulants of the distribution with the separation between cases given by $\mu < \frac{1}{n}$ where n is the moment being considered ($n \geq 2$). These results for the diffusion coefficient have also been shown in [14, 44]. For this case the particles motion is known as anomalous drift [14].

If we consider the first moment of the distribution written in terms of dimensional quan-

tities we find

$$\frac{\langle x(t)\rangle}{x_1} = \frac{1}{\Gamma[1+2\mu]}\frac{\Gamma^2[\mu]}{\Gamma[1+\mu]\Gamma[1-\mu]}\left(\frac{t}{\tau_1}\right)^{\mu}, \tag{4.7.5}$$

and if we consider the small drift limit, *i.e.* $\mu \to 0$ we find $\frac{\langle x(t)\rangle}{x_1} = \frac{1}{\mu^2}\left(\frac{t}{\tau_1}\right)^{\mu}$. The $\frac{1}{\mu^2}$ dependence of the pre-factor for small μ is very important. As we have seen from scaling arguments for times $t < \tau_0 = \tau_1 e^{\frac{1}{\mu}}$ the behaviour of the particle is insensitive to the drift and it follows the usual logarithmic regime. The point is that this expression correctly crosses $\frac{1}{\mu^2}(\frac{t}{\tau_1})^{\mu}$ for $t = \tau_0$ and $x = x_0 = \frac{x_1}{\mu}$ re-creating the result we found via a statistical analysis for the separation of disorder and drift controlled regions.

4.7.3 $\mu = 1$

For the case of $\mu = 1$ we find the general expression for the moments of the distribution is given by

$$\langle x^n(t)\rangle \approx \frac{\Gamma[1+n]t^n}{\ln^n(t)}, \tag{4.7.6}$$

and we find the first two moments are given by $\langle x(t)\rangle = \frac{t}{\ln^2(t)}$ in agreement with [14] and $\langle x^2(t)\rangle = 2\frac{t^2}{\ln^2(t)}$. On calculating the cumulants of the distribution we find the first cumulant $\Xi_1 = 0$, *i.e.* the particles velocity is zero, with all other cumulants being infinite.

4.7.4 $1 < \mu < 2$

For the case of $1 < \mu < 2$ we find

$$\langle x^n(t)\rangle = (\mu-1)^n t^n - \frac{n\Gamma[1-\mu]\Gamma[1+\mu](\mu-1)^{n+1}\Gamma[1+n]}{\Gamma^2[\mu]\Gamma[2+n-\mu]}t^{1+n-\mu}, \tag{4.7.7}$$

and the first cumulant is given by $\Xi_1 = V = \mu - 1$ leading to the value $\frac{V}{V_0} = 1 - \frac{1}{\mu}$ in agreement with [14]. If we compare this result with what we calculated from the mean first-passage time distribution we find they are indeed identical as expected. To see this we notice that the result from the first-passage time has the form $V = 0$ for $V_0 < V_0^{th}$ and $V = V_0 - V_0^{th}$ otherwise. We find that when we write $\mu = \frac{2V_0 D}{\gamma}$ we see that when $V_0 < \frac{\gamma}{2T}$ which is when $\mu < 1$ we do indeed find the velocity is zero, as expected. For $\mu > 1$, when

we write everything in dimensional quantities we find $V = V_0 - \frac{\gamma}{2D}$ which is equivalent to $V = V_0 - V_0^{th}$ when we write the diffusion coefficient in terms of the temperature using the standard Einstein's relation. If we calculate the mean-square displacement we find

$$\langle x^2(t)\rangle = (\mu-1)^2t^2 - \frac{2\Gamma[1-\mu]\Gamma[1+\mu](\mu-1)^3}{\Gamma^2[\mu]\Gamma[4-\mu]}t^{3-\mu}, \tag{4.7.8}$$

and we find the quantity

$$\begin{aligned}\langle x^2(t)\rangle - (\langle x(t)\rangle)^2 &= \frac{\Gamma[1-\mu]\Gamma[1+\mu](\mu-1)^3}{\Gamma^2[\mu]}\left[\frac{1}{\Gamma[3-\mu]} - \frac{1}{\Gamma[4-\mu]}\right]t^{3-\mu} \\ &- \frac{\Gamma^2[1-\mu]\Gamma^2[1+\mu](\mu-1)^4}{\Gamma^4[\mu]\Gamma^2[3-\mu]}t^{4-2\mu},\end{aligned} \tag{4.7.9}$$

which we find is in disagreement with the result in [14]. If we calculate all other moments of the distribution we find that they are all infinite, including the diffusion coefficient. We therefore only have one finite moment with all the others infinite and the particles motion is known as anomalous diffusion [14].

4.7.5 $2 < \mu < 3$

For the case of $2 < \mu < 3$ we find

$$\langle x^n(t)\rangle = (\mu-1)^nt^n - \frac{n\Gamma[1-\mu]\Gamma[1+\mu](\mu-1)^{n+1}\Gamma[1+n]}{\Gamma^2[\mu]\Gamma[2+n-\mu]}t^{1+n-\mu} + 2\frac{(\mu-1)^{n-1}}{(\mu-2)}t^{n-1}. \tag{4.7.10}$$

If we calculate the cumulants we find the first two are non-zero with $\Xi_1 = V = \mu - 1$ and $\Xi_2 \to \frac{D}{D_0} = \frac{\mu-1}{\mu-2}$ with all the rest being infinite. We now see the particles motion is diffusive with a positive non-zero velocity and we say that the particle has overcome the effect of disorder. This emergence of diffusive behaviour should not be surprising. We have seen from the definition of μ *i.e.* $\mu = \frac{2V_0D_0}{\gamma}$ that drift and thermal activation overcome disorder when $\mu > 2$. We now say the particle only feels the effect of the mean bias and is only slowed by large fluctuations in the potential.

We can use these results to derive a velocity-diffusion relation of the form

$$\frac{D}{V} = \frac{D_0}{V_0}\frac{\mu}{\mu - 2}. \tag{4.7.11}$$

We notice that we arrive back at the case of pure diffusion and drift only when $\mu \to \infty$, which from the definition only occurs when we have an infinite drift velocity or zero disorder as expected. For the case of infinite drift we would find all cumulants and moments are finite, something that would be expected with what is known about the case of pure diffusion.

These results for the transport properties are completely consistent with that of [14] and we can in fact re-create diagram (4.7.5) which comes from the scaling arguments of [14]. We

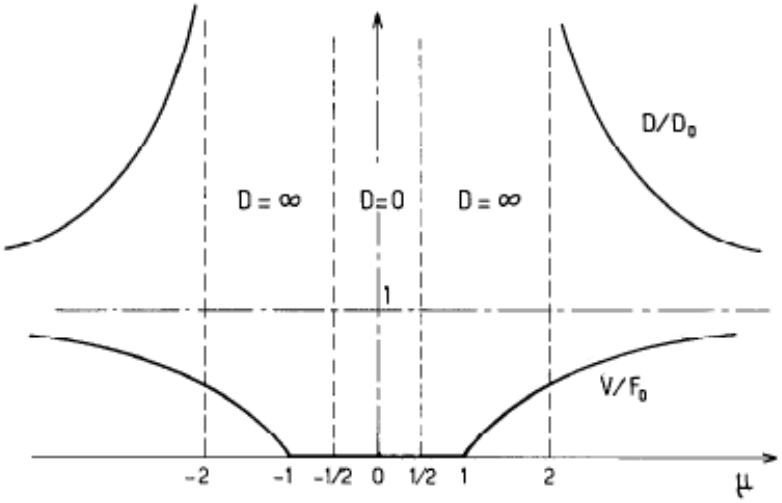

Figure 4.3: Figure to show the velocity and diffusion coefficients as a function of drift velocity.

have extended the results to the region of negative μ by simply mirroring them in the line $x = 0$. We also notice that there is a pattern in the results for cumulants. It should be noted by calculating all of the cumulants in the range $m < \mu < m + 1$ that the first m cumulants are finite with all the rest infinite. This was first suggested in [14] without proof, but we have shown that it is indeed true. We can also compare the results we have found to those of calculated by Pottier and Saint-James [44] where they found the expansion of the return probability to linear order in ϵ was enough to derive the velocity and diffusion coefficients. We find that our results agree with those of [44] and we can take the assumption further by

saying that if our conjecture is true, although we can't prove it, we believe it to be, then just knowing the return probability is enough to calculate all of the transport properties of the particles. The is would be useful result to prove and in our opinion, future work on this problem should be based around proving this conjecture.

4.8 Summary

For this chapter we have solved the Fokker-Planck equation for short-ranged delta function correlations with drift. We began the discussion by considering the model we are interested in with a brief overview of what has been done previously. We have seen that there has been some success in finding a solution to the problem. We have seen that there has only been an analytic solution for the case of zero drift and scaling and renormalization group arguments for the case with drift. We have also given a few physical situations which is modeled by a particle traveling in an environment with short ranged correlations including dynamics of random field magnets and dislocation dynamics. The first step we made was to consider scaling arguments where we found that the drift velocity controls all of the physics in the problem, something we have also shown with our analytic solution. Scaling arguments also showed a time scale for which the particle is insensitive to the disordered nature of the system which again our analytical solution shows.

The first step we made in the calculation was to describe the general approach to solving the problem. To do this we wrote the expectation value of the probability distribution as a Green's function in terms of stochastic functions. This Green's function was then written in terms of two distributions which we derived a Fokker-Planck equation for. We then calculated two disorder averaged quantities; the return probability and the probability of finding the particle at some arbitrary position and the form of the distribution function we considered was governed by which disordered average we were interested in. The return probability as we have seen gives the probability of the particle returning to the origin of its random walk. We have seen that for short time scales the probability distribution follows that of normal diffusion indicating that there is a time scale for which the particle is insensitive to the disordered nature of the environment as we predicted by scaling arguments. The

result also showed a mixture of diffusive behaviour depending on the value of drift velocity, again as predicted by scaling arguments. In this section on the return probability we also calculated its moments. The result we found was consistent with that of the first moment. We used the result to show that the return probability is not a self averaging quantity, in agreement with [14].

The more general case we considered was that of the probability of finding the particle at some arbitrary distance away from the origin. For zero drift we re-derived the known result for the probability distribution which was in agreement with Kesten's scaling argument [19, 20] and the exact analytical results of [14, 18]. The advantage of the method we used to calculate this quantity was that we arrived at a single value for the energy spectrum whereas previous attempts found two values and the choice was made so to agree with the known result. Calculating the transport properties for such a particle we find that all of the moments and cumulants are zero and we see the effect of disorder is to slow the particles motion and destroy the diffusion process, *i.e.* the particle becomes sub-diffusive. The real advantage of the method we used was for calculating the probability distribution for the case of non-zero drift velocity. With the introduction of drift we found the emergence of a gap in the potential of our Schrödinger equation. It is inside this gap where we needed to solve the Schrödinger equation so to find the value for the discrete energy spectrum. We found for the energy spectrum that its solution was of the form $E = \sum_{n=1}^{[\mu]} c_n \epsilon^n + c_\mu \epsilon^\mu$ which leads to several diffusion processes. We calculated the first three terms of this expansion and have shown that they follow the definition of the return probability, as expected. Unfortunately we have not been able to generalize the result to show that every term follows that of the return probability, something that any future work should consider attempting. For the case of $0 < \mu < 1$ we found Lévy flight which is characterized by heavy tails of the probability distribution. We found for this case that depending on the value of drift velocity the moments and cumulants were either zero or infinite, with the separation between the two cases given by $\mu = \frac{1}{n}$, where n is the moment being considered. For the case of $1 < \mu < 2$ we also found Lévy flight, but we also found that the particle now had some finite velocity given by $\frac{V}{V_0} = 1 - \frac{1}{\mu}$. We also showed that the result agreed with the one we found using the first-passage time distribution as expected. We found that on calculation of the second

cumulant we found a result which scaled as $t^{3-\mu}$ compared to $t^{\frac{2}{\mu}}$ as previously found in [14]. We found when calculating all other cumulants they were all infinite in the long time limit. For $\mu > 2$ we found the particle now has some finite velocity and a finite non-zero value for the diffusion coefficient $\frac{D}{D_0} = \frac{\mu-1}{\mu-2}$ in agreement with [14] indicating that we now have normal diffusion but with a renormalized value for the diffusion coefficient. In general it can be shown that the first m cumulants in the region $m < \mu < m + 1$ are non-zero with the rest being infinite.

Chapter 5

FUNCTIONAL INTEGRATION AND RENORMALISATION GROUP THEORY

In this chapter we give an introduction to the concepts of functional integration and renormalization group theory. These are the tools that we will be using in the following chapter to derive the long time behaviour of the transport coefficients of a particle in an environment with long-range disorder. The discussion will begin by motivating the use of Grassmann algebra [53] as an alternative to the complicated anti-commutation relations of fermion operators, which leads to fermionic coherent states and the resolution of the identity. This will give us the techniques required to derive the fermionic functional integral. Finally we give an introduction to renormalization group theory and describe the three step recipe for deriving the Gell-Mann Low equations [26] which describes the evolution of some parameter under the renormalization group process.

5.1 Grassmann Variables, Fermion Coherent States and the Resolution of Unity

The idea of functional integration is to re-write a Green's function or a partition function (or whatever quantity is of interest) in a form where instead of using fermion operators with their complicated anti-commutation relations $\{\hat{a}_k, \hat{a}_k^\dagger\} = \delta_{kk'}$ we use something else which has simpler anti-commutation rules, *i.e.* $\{\psi_k, \psi_{k'}\} = 0$. In order to do this we need to introduce Grassmann algebra, named after the mathematician Hermann Grassmann. If we

begin by considering a system of bosons, where $\hat{b}_k^\dagger$ creates a boson in the state $|k\rangle$ and $\hat{b}_k$ annihilates a boson in the same state. The question is whether we can build a basis out of the eigenfunctions of the operators $\hat{b}_k^\dagger$ or $\hat{b}_k$. It should be clear from just counting arguments that we will not be able to find any eigenfunction for $\hat{b}_k^\dagger$ apart from the vacuum state $|0\rangle$. If we suppose that such a function exists, then it will always be possible to represent it in a form of a superposition where each term has a well-defined number of bosons. As this number cannot be less than zero we can always find a term which corresponds to the minimum number of bosons in the superposition. The action of $\hat{b}_k^\dagger$ will always increase this minimum number by one and thus there is no chance of getting the original function back. The situation is completely different if we want to find an eigenfunction for $\hat{b}_k$. This is due to the fact that there is nothing wrong in having an infinite number of bosons in a state. We can see by denoting the vacuum state as $|0\rangle$, then $\hat{b}_k e^{\lambda \hat{b}_k^\dagger}|0\rangle = \lambda e^{\lambda \hat{b}_k^\dagger}|0\rangle$ and $e^{\lambda \hat{b}_k^\dagger}|0\rangle$ is an eigen-state for the annihilation operator $\hat{b}_k$. Of course, if it were possible to have a state with a negative infinity number of bosons we would be able to build an eigenfunction for $\hat{b}_k^\dagger$ following similar arguments to above, but a system with such properties would be difficult to comprehend.

If we apply the same counting arguments to the fermionic case we arrive at problems since we are not allowed to have an infinite number of fermions in one state. If we define an arbitrary superposition as $\psi = \alpha|0\rangle + \beta\hat{a}_k^\dagger|0\rangle$ where we only consider the k-state, the operator $\hat{a}_k^\dagger$ creates a fermion in this state, where α and β are some constants. Acting an annihilation operator on this state we have $\hat{a}_k\psi = \beta|0\rangle$ and it is clear that there are no such coefficients α, β which would make ψ an eigenfunction of $\hat{a}_k$. It is possible though to build the eigenfunctions of both $\hat{a}_k$ and $\hat{a}_k^\dagger$, if the coefficients in the superposition are not complex but Grassmann numbers.

Grassmann algebra is defined by a set of generators $\{\psi_\alpha\}$, $\alpha = 1, ..., n$ which anti-commute with each other

$$\psi_\alpha\psi_\beta + \psi_\beta\psi_\alpha = 0, \tag{5.1.1}$$

which clearly states that when $\alpha = \beta$

$$\psi_\alpha^2 = 0, \tag{5.1.2}$$

i.e. the square of any Grassmann number is zero. Using this property we see that any analytic function f defined on this algebra can only be a linear function

$$f(\psi) = f_0 + f_1\psi, \tag{5.1.3}$$

as all other terms of the expansion are zero by definition. We can also define the following properties for conjugation in Grassmann algebra

$$\overline{(\psi_\alpha)} = \bar{\psi}_\alpha, \qquad \text{and} \qquad \overline{(\bar{\psi}_\alpha)} = \psi_\alpha, \tag{5.1.4}$$

and if λ is some complex number then

$$\overline{(\lambda\psi_\alpha)} = \lambda^*\bar{\psi}_\alpha. \tag{5.1.5}$$

As for ordinary complex functions, a derivative can be defined for functions of Grassmann variables. It is defined identically to that of the complex derivative, except for the derivative operator $\frac{\partial}{\partial\psi}$ to act on ψ, the variable ψ must be anti-commuted through until it is adjacent to $\frac{\partial}{\partial\psi}$ *i.e.*

$$\frac{\partial}{\partial\psi}(\bar{\psi}\psi) = -\frac{\partial}{\partial\psi}(\psi\bar{\psi}) = -\bar{\psi}. \tag{5.1.6}$$

In contrast to the definition of a derivative, integration of Grassmann numbers has no analog to that of integration over ordinary variables. The properties of integration are such that the integral of one is zero where one is the derivative of ψ with respect to ψ. The only term which does not vanish is that of ψ, hence the definite integral over the Grassmann algebra is given by

$$\int d\psi = 0, \qquad \text{and} \qquad \int d\psi\psi = 1, \tag{5.1.7}$$

and as in the case of the derivative, in order to apply these rules we must first anti-commute the variable ψ so to bring it next to $d\psi$. For example, consider the integral

$$\int d\psi_2\psi_1\psi_2\psi_3 = -\int d\psi_2\psi_2\psi_1\psi_3 = -\psi_1\psi_3. \tag{5.1.8}$$

The integration of the conjugate variable is defined in the exact same manner

$$\int d\bar{\psi} = 0, \qquad \text{and} \qquad \int d\bar{\psi}\bar{\psi} = 1. \tag{5.1.9}$$

As we now understand the rules of Grassmann numbers and algebra we can now build the fermionic coherent states. To do this we must first define a Grassmann algebra by associating a generator ψ_α with each annihilation operator $\hat{a}_\alpha$ and a generator $\bar{\psi}_\alpha$ with each creation operator $\hat{a}^\dagger_\alpha$. We now define the fermion coherent state $\mid \psi\rangle$ as

$$\mid \psi\rangle = \exp[-\sum_\alpha \psi_\alpha \hat{a}^\dagger_\alpha] \mid 0\rangle = \prod_\alpha (1 - \psi_\alpha \hat{a}^\dagger_\alpha) \mid 0\rangle, \tag{5.1.10}$$

where the exponential of a sum can be be written as a product of many exponentials and all higher order expansions of the exponential are zero by definition. We are now required to verify that the coherent states are indeed eigen-states of the annihilation operator, *i.e*

$$\hat{a}_\alpha(1 - \psi_\alpha \hat{a}^\dagger_\alpha) \mid 0\rangle = -\hat{a}_\alpha \psi_\alpha \hat{a}^\dagger_\alpha \mid 0\rangle = \psi_\alpha \mid 0\rangle = \psi_\alpha(1 - \psi_\alpha \hat{a}^\dagger_\alpha) \mid 0\rangle, \tag{5.1.11}$$

we therefore find

$$\hat{a}_\alpha \mid \psi\rangle = \hat{a}_\alpha \prod_\beta (1 - \psi_\beta \hat{a}^\dagger_\beta) \mid 0\rangle = \psi_\alpha \prod_\beta (1 - \psi_\beta \hat{a}^\dagger_\beta) \mid 0\rangle = \psi_\alpha \mid \psi\rangle, \tag{5.1.12}$$

and we have shown that the coherent fermionic state is an eigen-state of the annihilation operator. Similarly, the adjoint of the coherent state is given by

$$\langle \psi \mid = \langle 0 \mid \exp[\sum_\alpha \bar{\psi}_\alpha \hat{a}_\alpha], \tag{5.1.13}$$

which is a left eigenfunction of the creation operator $\hat{a}^\dagger_\alpha$, *i.e.*

$$\langle \psi \mid \hat{a}^\dagger_\alpha = \langle \psi \mid \bar{\psi}_\alpha. \tag{5.1.14}$$

The action of the creation operator $\hat{a}^\dagger_\alpha$ on a coherent state is given by

$$\hat{a}^\dagger_\alpha \mid \psi\rangle = \hat{a}^\dagger_\alpha(1-\psi_\alpha\hat{a}^\dagger_\alpha)\prod_{\beta\neq\alpha}(1-\psi_\beta\hat{a}^\dagger_\beta)\mid 0\rangle = \hat{a}^\dagger_\alpha\prod_{\beta\neq\alpha}(1-\psi_\beta\hat{a}^\dagger_\beta)\mid 0\rangle = -\frac{\partial}{\partial\psi_\alpha}\mid\psi\rangle.$$

Similarly it can be shown that

$$\langle\psi \mid \hat{a}_\alpha = \frac{\partial}{\partial\psi}\langle\psi \mid . \tag{5.1.15}$$

The overlap of two coherent states can easily shown to be

$$\langle\psi \mid \psi'\rangle = \langle 0 \mid \prod_\alpha(1-\bar{\psi}_\alpha\hat{a}_\alpha)(1-\psi'_\alpha\hat{a}^\dagger_\alpha)\mid 0\rangle = \prod_\alpha(1+\bar{\psi}_\alpha\psi'_\alpha) = \exp[\sum_\alpha\bar{\psi}_\alpha\psi'_\alpha]. \tag{5.1.16}$$

Using integration with respect to Grassmann variables it is possible to build the unity operator. Instead of doing this constructively we will first declare the answer and then prove that the operator is built in such a way that is possesses all the properties of unity. If we consider the operator

$$\hat{A} = \int\prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_\alpha\bar{\psi}_\alpha\psi_\alpha]\mid\psi\rangle\langle\bar{\psi}\mid, \tag{5.1.17}$$

where $\mid\psi\rangle$ and $\langle\bar{\psi}\mid$ are defined by (5.1.10) and (5.1.13). The first step is to prove that the operator $\hat{A}$ commutes with any annihilation operator, *i.e.* $\hat{a}_k\hat{A} = \hat{A}\hat{a}_k$. We can do this by acting $\hat{a}_k$ to the left of $\hat{A}$

$$\begin{aligned}\hat{a}_k\hat{A} &= \hat{a}_k\int\prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha\exp[-\sum_\alpha\bar{\psi}_\alpha\psi_\alpha]\mid\psi\rangle\langle\bar{\psi}\mid \qquad (5.1.18)\\ &= \int\prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha\exp[-\sum_\alpha\bar{\psi}_\alpha\psi_\alpha]\psi_k\exp[-\sum_\alpha\psi_\alpha\hat{a}^\dagger_\alpha]\mid 0\rangle\langle 0\mid\exp[-\sum_\alpha\hat{a}_\alpha\bar{\psi}_\alpha].\end{aligned}$$

Now if we remove the k dependence from the the exponents in the integrals

$$\hat{a}_k\hat{A} = \int\prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha\exp[-\sum_{\alpha\neq k}\bar{\psi}_\alpha\psi_\alpha]\psi_k\exp[-\sum_{\alpha\neq k}\psi_\alpha\hat{a}^\dagger_\alpha]\mid 0\rangle\langle 0\mid\exp[-\sum_{\alpha\neq k}\hat{a}_\alpha\bar{\psi}_\alpha](1-\hat{a}_k\bar{\psi}_k), \tag{5.1.19}$$

and by integrating with respect to ψ_k and $\bar{\psi}_k$

$$\hat{a}_k\hat{A} = \int \prod_{\alpha\neq k} d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_{\alpha\neq k}\bar{\psi}_\alpha\psi_\alpha]\psi_k \exp[-\sum_{\alpha\neq k}\psi_\alpha\hat{a}^\dagger_\alpha] \mid 0\rangle\langle 0 \mid \exp[-\sum_{\alpha\neq k}\hat{a}_\alpha\bar{\psi}_\alpha]\hat{a}_k, \tag{5.1.20}$$

and using $\int d\psi^*_k d\psi_k \exp[-\psi^*_k\psi_k]\exp[-\psi_k\hat{a}^\dagger_k] = 1$ we remove the restrictions on the summation

$$\hat{a}_k\hat{A} = \int \prod_{\alpha} d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_{\alpha}\bar{\psi}_\alpha\psi_\alpha]\exp[-\sum_{\alpha}\psi_\alpha\hat{a}^\dagger_\alpha] \mid 0\rangle\langle 0 \mid \exp[-\sum_{\alpha\neq k}\hat{a}_\alpha\bar{\psi}_\alpha]\hat{a}_k. \tag{5.1.21}$$

Now by using the fact the $\hat{a}_k\hat{a}_k = 0$ the full operator $\hat{A}$ can be restored meaning $\hat{a}_k\hat{A} = \hat{A}\hat{a}_k$. It can also be shown that $\hat{A}$ commutes with all creation operators $\hat{a}^\dagger_k$. Therefore $\hat{A}$ must just be a number and by calculating $\langle 0 \mid \hat{A} \mid 0\rangle$ we find

$$\begin{aligned}
\langle 0 \mid \hat{A} \mid 0\rangle &= \int \prod_{\alpha} d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_{\alpha}\bar{\psi}_\alpha\psi_\alpha]\langle 0 \mid \exp[-\sum_{\alpha}\psi_\alpha\hat{a}^\dagger_\alpha] \mid 0\rangle \\
&\times \langle 0 \mid \exp[-\sum_{\alpha}\hat{a}_\alpha\bar{\psi}_\alpha] \mid 0\rangle \\
&= \int \prod_{\alpha} d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_{\alpha}\bar{\psi}_\alpha\psi_\alpha] = \hat{1},
\end{aligned} \tag{5.1.22}$$

from which we can conclude that $\hat{A}$ is a unity operator

$$\hat{1} = \int \prod_{\alpha} d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_{\alpha}\bar{\psi}_\alpha\psi_\alpha] \mid \psi\rangle\langle\bar{\psi} \mid . \tag{5.1.23}$$

We will use this result extensively in the following section where we use it to build the functional integral formalism.

To summarize this section we have motivated the use of Grassmann numbers and used them as an alternative to the complicated anti-commutation properties of fermionic operators. We have in addition to this derived several results for Grassmann algebra including differentiation and integration. We have also introduced the fermionic coherent states and the resolution of the identity which we will now use to derive the fermionic field integral.

5.2 Fermionic Field Integral

We now have all the tools required to derive the fermionic functional integral. Beginning with the usual definition of a quantum mechanical partition function

$$\begin{aligned} \mathcal{Z} &= \mathrm{Tr}\exp[-\beta H] \\ &= \sum_n \langle n \mid \exp[-\beta(\hat{H}-\mu\hat{N})] \mid n\rangle, \end{aligned} \tag{5.2.1}$$

where $\beta = \frac{1}{T}$, the inverse of temperature, μ is some chemical potential and the sum is over the complete set of Fock states. Now by inserting the resolution of the identity (5.1.23) the partition function becomes

$$\mathcal{Z} = \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_\alpha \bar{\psi}_\alpha \psi] \sum_n \langle n \mid \psi\rangle\langle\psi \mid \exp[-\beta(\hat{H}-\mu\hat{N})] \mid n\rangle, \tag{5.2.2}$$

and to remove the sum over Fock space we must commute $\langle n \mid \psi\rangle$ to the right-hand side. We must be careful though with the minus signs as fermionic coherent states change signs upon permutation, $\langle n \mid \psi\rangle\langle\psi \mid n\rangle = \langle -\psi \mid n\rangle\langle n \mid \psi\rangle$. Therefore the partition function becomes

$$\begin{aligned} \mathcal{Z} &= -\int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha] \sum_n \langle \psi \mid \exp[-\beta(\hat{H}-\mu\hat{N})] \mid n\rangle\langle n \mid \psi\rangle \\ &= -\int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha \exp[-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha] \langle \psi \mid \exp[-\beta(\hat{H}-\mu\hat{N})] \mid \psi\rangle, \end{aligned} \tag{5.2.3}$$

where we have used the identity $\sum_n \mid n\rangle\langle n \mid = 1$. If we now divide the "time interval" β into N segments and insert coherent state resolutions of identity we can re-write the partition function $\mathcal{Z}$ as

$$\mathcal{Z} \int_{\substack{\bar{\psi}=-\bar{\psi}^N \\ \psi^0=-\psi^N}} \prod_{n=0}^{N} \prod_\alpha d\bar{\psi}^n_\alpha d\psi^n_\alpha \exp[-\delta \sum_{n=0}^{N-1} \delta^{-1}(\bar{\psi}^n - \bar{\psi}^{n+1})\psi^n + H(\bar{\psi}^{n+1}, \psi^n) - \mu N(\bar{\psi}^{n+1}, \psi^n)],$$

where $\delta = \frac{\beta}{N}$ and

$$\frac{\langle \psi \mid \hat{H}[\hat{a}^\dagger, \hat{a}] \mid \psi'\rangle}{\langle \psi \mid \psi'\rangle} = H(\bar{\psi}, \psi'), \tag{5.2.4}$$

and similarly for $N(\bar{\psi}, \psi)$. Finally by taking the limit as $N \to \infty$ we obtain the continuum version of the path integral

$$\mathcal{Z} = \int \mathcal{D}(\bar{\psi}, \psi) \exp[-S[\bar{\psi}, \psi]], \tag{5.2.5}$$

with the action $S[\bar{\psi}, \psi]$ defined by

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau [\bar{\psi}\partial_\tau \psi + H(\bar{\psi}, \psi) - \mu N(\bar{\psi}, \psi)], \tag{5.2.6}$$

where $D(\bar{\psi}, \psi) = \lim_{N\to\infty} \prod_{n=1}^{N} \prod_\alpha d\bar{\psi}_\alpha^n d\psi_\alpha^n$.

From this section we have derived the form of the fermionic field integral. We have done so via the usual quantum mechanical partition function and the repeated use of the identity of unity. We now have all of the tools required to consider renormalization group theory.

5.3 The Renormalisation Group Theory

The formulation of the renormalization group procedure was first introduced by Stuckelberg and Petermann in 1953 [23] as a method for dealing with high energy particle physics. Its application to quantum electrodynamics was explored in a paper by Gell-Mann and Low in 1954 [27], but it was not until the late 60's, early 70's that the idea of recursively generating flows of coupled equations arose [54]. However, it took the insight of Kenneth G. Wilson [24] to actually develop the full potential of the approach and develop it into a widely accepted and useful technique.

The renormalization group process is a powerful and efficient tool to explore interacting theories for problems when perturbation theory fails, such as the ϕ^4 model [55]. There are many models which exhibit both short-range (of order some microscopic cut-off a) and long-range fluctuations (of order L, the system size), where it is the long-range fluctuations which are of interest. The aim of the renormalization group process is to find an effective theory of long-range fluctuations by integrating over all short-range fluctuations.

There is a clear problem with the "separation" of short to long-range fluctuations or what will now be known as separation into fast and slow modes, as all types of fluctuations

are treated equally [56, 57, 58]. To implement an integration over fast modes to generate an effective action of slow degrees of freedom, we declare an artificial length scale $a^{(1)} \equiv b > a$ as the scale which separates the short-wavelength fluctuations on scales $[a, a^{(1)}]$ from the long-wavelength fluctuations on $[a^{(1)}, L^{-1}]$. Having done this we may then integrate out the short-ranged fluctuations changing the action of the long-ranged degrees of freedom. This leads to an effective action of the slow degrees of freedom which is structurally similar to that of the original. The integration therefore has the effect of changing a set of coupling constants, and we arrive at a theory identical to the original but for a renormalised set of coupling constants and an increased short-distance cut-off $a \to a^{(1)} = ba$. Now if this procedure is continued iteratively via $a^{(2)} = ba^{(1)} = ba^2$ and the short-range fluctuations are integrated out in the region $[a^{(2)}, a^{(1)}]$ then we find a new set of coupling constants. This iterative process continues until the cut-off $a^{(n)} \sim L^{-1}$, the length scale we are interested in and the new coupling constants encode all of the long-range behaviour.

We should now have a vague idea about the renormalization group process so it would be useful to give a general outline to the procedure. If we begin with a field theory of the form [56]

$$S[\bar{\psi}, \psi] = \sum_{\alpha=1}^{N} g_\alpha \int d^d x (\nabla\bar{\psi})^n (\nabla\psi)^m, \tag{5.3.1}$$

then the aim of the process is to derive a set of Gell-Mann Low equations [27] for each of the parameters g_α for the above action. The Gell-Mann Low equations describe the flow of the coupling constants under an infinitesimal change of the control parameter λ, *i.e.*

$$\frac{dD}{d\ln(\lambda^{-1})} = \beta(D), \tag{5.3.2}$$

where the right-hand side is known, for historical reasons, as the Beta function. To find the Gell-Mann Low equation there are three stages to follow [26], the first being the sub-division of fields into fast and slow modes, secondly the integration over fast modes and finally re-scaling.

5.3.1 Sub-Division of the Fields

A completely equivalent description of the renormalization group procedure can be performed in momentum space via a Fourier transform

$$\psi(x) = \sum_{q_1=0}^{\Lambda} \psi(q_1) \exp[iq_1 x], \qquad \bar{\psi}(x) = \sum_{q_1=0}^{\Lambda} \bar{\psi}(q_1) \exp[-iq_1 x], \tag{5.3.3}$$

where Λ is the ultraviolet cut-off in momentum space. If we are interested in the physics at long distances in real space then this translates to small q in momentum space. We can imagine that we are interested only in correlations which lie within a tiny ball of radius $\lambda\Lambda$ centered at the origin, where $0 < \lambda < 1$ defines the region for slow modes and all other regions define the fast modes.

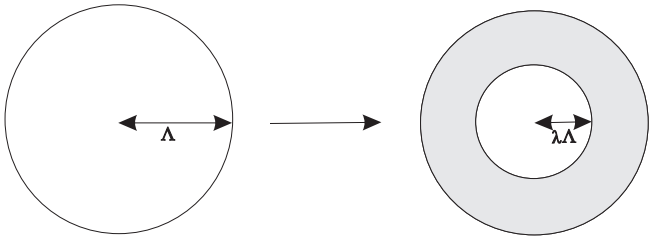

Figure 5.1: Separation of momentum space into fast and slow modes. Λ describes the ultraviolet cut off in momentum space, and λ is a scaling parameter that describes the separation between fast and slow modes.

This allows us to define two sets of fields

$$\begin{aligned} \psi_< &= \psi(q), \qquad \text{for} \qquad 0 < q < \lambda\Lambda \\ \psi_> &= \psi(q), \qquad \text{for} \qquad \lambda\Lambda < q < \Lambda \end{aligned} \tag{5.3.4}$$

where $\psi_<$ defines the slow modes and $\psi_>$ defines the fast modes. Now if we write down our action, which has the form of some Gaussian term $S_0[\bar{\psi}, \psi]$ and some interaction term $S_{int}[\bar{\psi}, \psi]$, *i.e.* we just consider the first two terms of the sum (5.3.1) we find

$$S[\bar{\psi}, \psi] = S_0[\bar{\psi}, \psi] + S_{int}[\bar{\psi}, \psi], \tag{5.3.5}$$

where

$$S_0[\bar{\psi},\psi] = \int dx(\nabla\bar{\psi}(x))(\nabla\psi(x)), \qquad S_{int}[\bar{\psi},\psi] = \int dx(\nabla\bar{\psi}(x))^2(\nabla\psi(x))^2. \tag{5.3.6}$$

Now if we write the action in terms of fast and slow modes we find

$$S[\bar{\psi},\psi] = S_0[\psi_<] + S_0[\psi_>] + S_{int}[\psi_<,\psi_>], \tag{5.3.7}$$

which is made up of three terms. We see that one term $S_0[\psi_<]$ is composed entirely of slow modes, a term which is known as tree-level scaling. Tree level scaling allows us to find the dimensionality of the fields and the coupling constants, which allows us to find the conditions for the action to be scale invariant. This is a vital requirement for the renormalization group procedure to work. The term scaling invariant simply means the action remains unchanged under re-scaling of momentum. If we make a re-scaling of the momentum $q' = \lambda q$, then we can force the dimensionality of the fields or coupling constants such that the action remains invariant under the re-scaling. If the coupling constants change under tree-level scaling then this gives the first term of the Gell-Mann Low equation. The other terms $S_0[\psi_>]$ and $S_{int}[\psi_<,\psi_>]$ are functions of both fast and slow modes which we perform the renormalization group process on.

5.3.2 The RG Procedure

Once we have defined the dimensionality of the system and separated the system into its fast and slow contributions we begin the process of integrating out the fast momentum modes. The basic principle is given by the following [26]

$$\begin{aligned} Z &= \int [d\psi_<] \int [d\psi_>] \exp(S_0[\bar{\psi}_<,\psi_<]) \exp(S_0[\bar{\psi}_>,\psi_>]) \exp(S_{int}[\psi_<,\psi_>]) \\ &= \int [d\psi_<] \exp(S_0[\bar{\psi}_<,\psi_<]) \int [d\psi_>] \exp(S_0[\bar{\psi}_>,\psi_>]) \exp(S_{int}[\psi_<,\psi_>]) \\ &= \int [d\psi_<] \exp(S'[\bar{\psi}_<,\psi_<]), \end{aligned} \tag{5.3.8}$$

where we have defined the effective action $S'[\psi_<]$ as

$$\begin{aligned}
\exp(S'[\bar{\psi},\psi_<]) &= \exp(S_0[\bar{\psi}_<,\psi_<])\int[d\psi_>]\exp(S_0[\bar{\psi}_>,\psi_>])\exp(S_{int}[\psi_<,\psi_>]) \\
&= \exp(S_0[\bar{\psi}_<,\psi_<])\int[d\psi_>]\exp(S_0[\bar{\psi}_>,\psi_>])\exp(S_{int}[\psi_<,\psi_>]) \\
&\times \frac{\int[d\psi_>]\exp(S_0[\bar{\psi}_>,\psi_>])}{\int[d\psi_>]\exp(S_0[\bar{\psi}_>,\psi_>])} \\
&= \exp(S_0[\bar{\psi}_<,\psi_<])\langle\exp(S_{int}[\psi_<,\psi_>])\rangle_0, \qquad (5.3.9)
\end{aligned}$$

where $\langle \quad \rangle_0$ is the average over the fast modes. It should also be noted that the term $\int[d\psi_>]\exp[S_0(\psi_>) = Z_0$, is just a constant and will be ignored. The idea is to separate the problem into fast and slow modes and then integrate over the fast modes. Now to actually perform these averages requires some approximations. There are many approximations that can be used, but we will consider the so called loop approximation. The loop approximation requires the integration over the fast modes to be organised according to the number of independent momentum integrals.

If we consider a term containing four fields, which for the current example is given by the interaction term $S_{int}[\bar{\psi},\psi]$, then we represent the vertex as a dot with four legs, *i.e.* consider diagram (5.3.2), for which there are three possibilities for the arrangements of the

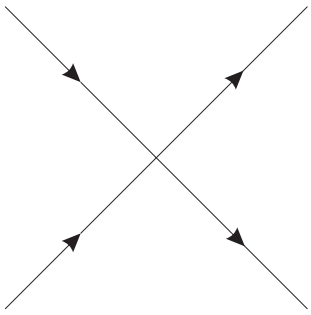

Figure 5.2: Representation of a typical vertex. If all fields are slow we have tree level scaling.

fields. If all four fields are slow, then we have, as we already know the tree-level diagram. If two of the fields are slow and two are fast, we can perform the contractions between the fast modes. This is known as a one-loop correction as there is only one independent momentum integral. This process leads to a vertex which has two legs, which gives a correction to the

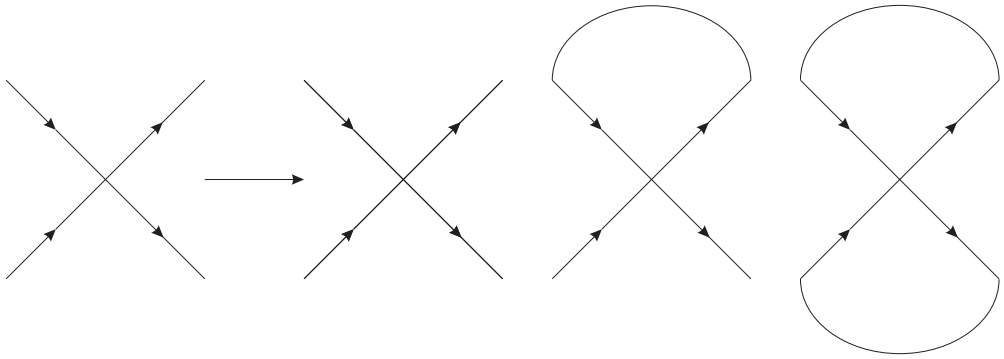

Figure 5.3: Averaging of a term with four legs has three possible contributions, the first being all slow, which is tree level, the second which gives a correction to the Gaussian term and the last which is just a constant.

Gaussian term. The final possibility is when all four fields are fast, and the contractions lead to a two-loop correction with no slow fields. This is known as a two loop diagram because there are now two independent momentum integrals. This term is just a constant and can be ignored. The small parameter which justifies the loop-expansion comes from the number of loops. The rules for calculating diagrams of this type are quite simple. The first rule is

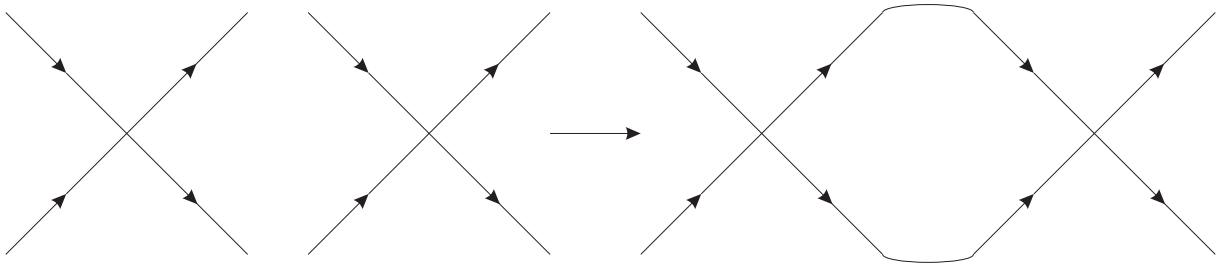

Figure 5.4: The only possible one-loop correction that can come from averaging two vertices. All disconnected diagrams are cancelled.

that momentum should be conserved in each diagram, and this is enforced by the x integral, and the delta function which comes from it. The other rule is that contractions can only be performed between terms involving $\bar{\psi}$ and ψ, as contractions between $\langle\bar{\psi}\bar{\psi}\rangle_0$ and $\langle\psi\psi\rangle_0$ will be zero by definition. From Gaussian averaging of Grassmann variables we find

$$\langle\bar{\psi}(q_1)\psi(q_2)\rangle_0 = \frac{\int d\bar{\psi}(q_1)d\psi(q_2)\bar{\psi}(q_1)\psi(q_2)\exp[-\bar{\psi}A\psi]}{\int d\bar{\psi}(q_1)d\psi(q_2)\exp[-\bar{\psi}A\psi]} = \frac{1}{A}\delta(q_1 - q_2). \tag{5.3.10}$$

On performing these steps to the action (5.3.1) we arrive at a new renormalized action

$$S'[\bar{\psi},\psi] = \sum_{\alpha=1}^{N} g'_\alpha \int d^d x (\nabla\bar{\psi})^n (\nabla\psi)^m, \tag{5.3.11}$$

where the coupling constants of the remaining slow fields have been altered. It should be noted that this process can lead to a new set of operators which were not present in the original action. The way to treat these operators is to investigate whether they are "relevant" *i.e.* they increase under the renormalization group procedure. If the new operators are indeed relevant then they must be included in the original action with some appropriate coupling constant. If on the other hand the operators are irrelevant, *i.e.* they decrease under the renormalization group procedure we can safely ignore them. A more detailed discussion of the meaning of relevant and irrelevant operators will be given in the next section where we consider the final step of the renormalization group process, re-scaling.

5.3.3 Re-Scaling

After performing the renormalization group procedure we have a new action $S'[\bar{\psi},\psi]$ defined by (5.3.11) which has experienced mode elimination, such that it is defined completely differently to the original action $S[\bar{\psi},\psi]$. We would like to compare the two equations to find how the coupling constants flow, but the problem is the two actions are defined on different kinematic scales. For example, $S[\bar{\psi},\psi]$ is defined on $0 < k < \Lambda$, whereas $S'[\bar{\psi},\psi]$ is defined on $0 < k < \lambda\Lambda$ (this of course would not be a problem if Λ was sent to infinity, as it sometimes is for high energy physics problems). To correct the problem we should define a new momenta after mode elimination, $k' = \frac{k}{\lambda}$, which runs over the same range as k did before the mode elimination.

If we follow these three steps we have a mapping which takes an action defined in a certain phase space to an action in the same space. If we represent the initial action as a point in a coupling constant space, the point will flow under the renormalization group process to another point in the same space. Once we have derived our evolution equation for the coupling constants, *i.e.* the Gell-Mann Low equation, we should consider how the coupling constants flow under the renormalization group process. If the Beta function is

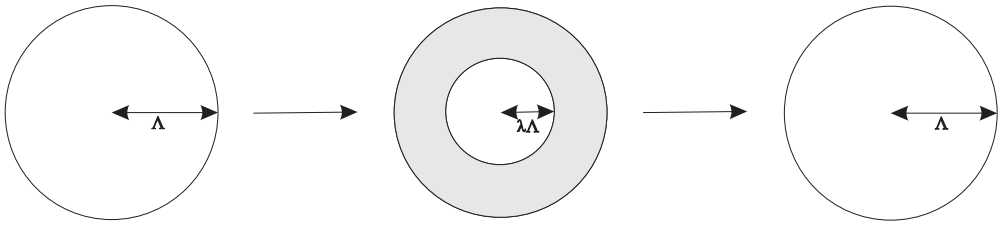

Figure 5.5: Illustration of the three step recipe. Beginning with an action with a defined cut-off we average over fast modes to define a new action. This action needs to be re-scaled so to compare with the previous action.

a positive quantity the action grows under the renormalization group process making it a relevant process. If the Beta function was exactly zero then the action remains fixed under the renormalization group process, this is known as the marginal case. For the final case when the Beta function is a negative quantity, then we say the action has been renormalized to zero, or we say it is irrelevant. These Gell-Mann Low equations represent the main result of the renormalization group process. If we treat $\ln(\lambda^{-1})$ as some time variable then we can imagine the equation to be some dynamical system, *i.e.* the system describing the evolution of the effective coupling constants on changing time scales.

5.4 Summary

For this chapter we have derived all of the tools required to calculate the transport properties of a particle in an environment with logarithmic correlations. We began the discussion by considering Grassmann algebra as an alternative to fermionic operators with their complex anti-commutation relations. We derived and explained several results for Grassmann algebra including differentiation and integration. We then introduced the fermionic coherent state and used it to derive the resolution of unity. The repeated use of this resolution of identity allowed us to derive the form of the path integral which gave us all the tools required to introduce the renormalization group approach. The renormalization group process we have found is a very useful tool for exploring interacting theories where perturbation theory fails and the aim of the process is to find an effective theory for long ranged fluctuations by integrating over all short-ranged fluctuations. We have also seen that the renormalization group theory follows a three step recipe. We have seen that the first step is to define some

cut-off in momentum space. We then need to divide the momentum space into two regions which we defined as fast and slow modes. The second step we had to follow is to average over the fast modes to derive a new action. We have seen that this new action is defined on a different kinematic scale to that of the old action and so we need to re-scale the new one. Upon re-scaling we can compare the two actions and see how the parameter of interest has changed which is given by the Gell-Mann Low equations. We have seen that if the Gell-Mann Low equation is a positive function then it grows under the renormalization group procedure and we say that it is a relevant process. If it is zero then we say it is a marginal case and finally if the Gell-Mann Low equation is negative we say that it is irrelevant under the renormalization process and we can ignore it. Finally it should be noted that the original action should be scale invariant to perform the renormalization group procedure.

Chapter 6

FOKKER-PLANCK EQUATION FOR LONG-RANGED LOGARITHMIC CORRELATIONS

For this chapter of the thesis we use the method of renormalization and functional renormalization group theory to re-derive the transport properties and first-passage time distribution of a particle in a one-dimensional random environment with logarithmic correlations. As we have previously seen this is an excellent model for relaxation in glasses [12], transport in solids [13] and vortex glass dynamics [11] as discussed in the first-passage time chapter. We begin the chapter with an introductory section showing a transition between a low temperature glass phase and high temperature phase in the first-passage time distribution [5] and it is this phase transition which we are trying to re-create. We then consider why only logarithmic correlations show this transition [28] and then give a discussion of the equivalent two-dimensional problem [29]. We then introduce the model and use the one-loop approximation to calculate the transport properties of the particle, including the diffusion coefficient and the mean-square displacement. We then proceed to consider how the higher cumulants [31] of the distribution act under the renormalization process which we use to derive a functional renormalization group equation for the distribution function. We solve this equation to find that it describes a particle diffusing with drift. Using this solution we derive the first-passage time distribution and show that whilst it does find the existence of a transition it does not reproduce the correct exponent. In an attempt to correct this we consider non-linear corrections to the higher cumulants and then calculate a new functional renormalization group equation.

6.1 Introduction and Background

In this introductory section we consider the first-passage time distribution and show the existence of a phase transition between a low temperature glass phase and a high temperature state. The existence of this phase transition can be seen by considering the high and low temperature limits of equation (3.3.14). We will see that the high temperature state is dominated by an entropic saddle point and the low temperature state is dominated by an energetic saddle point. This behaviour leads to a phase transition between states as a function of its moments. We then proceed to explain why this phase transition can only be described by logarithmic correlations and not by short or long-ranged correlations. For this discussion we follow the arguments of [28] and consider whether the entropy of typical sites can overcome the energy of low-energy states. As we will see correlations that are shorter will only have a high temperature state whereas longer correlations will have a low temperature glass phase. We conclude the section by considering the equivalent two-dimensional problem.

6.1.1 The First-Passage Time Distribution

For this section we consider the first-passage distribution and we show the existence of a phase transition between a low temperature glass phase and a high temperature state when the correlations are logarithmic. As we have seen in chapter three the moments of the first-passage time distribution are described by the Pontryagin equation

$$\frac{d^2t_n}{dx'^2} - \frac{1}{T}\frac{dU}{dx'}\frac{dt_n}{dx'} = -\frac{n}{T}t_{n-1}, \tag{6.1.1}$$

which as we have seen has a solution of the form

$$\begin{aligned}\langle t_n(x')\rangle &= (-1)^n\frac{n!}{T^n}\int_{x'}^{L}dx'_n\int_0^{x'_n}y_{n-1}\ldots\int_{y_1}^{L}dx'_1\int_0^{x'_1}dy_0 \\ &\times \exp\left[\frac{1}{2T^2}\sum_{i,j}K(y_i-x'_j)-K(y_i-y_j)-K(x'_i-x'_j)\right],\end{aligned} \tag{6.1.2}$$

in terms of the general correlation function $K(x-x')$. As we know the correlation function we are considering is given by $K(x-x') = \langle(U(x)-U(x'))^2\rangle = \gamma\ln|x-x'|(1-\delta^{(1)}(x-x'))$,

where γ is a measure of the strength of disorder, and we have set the ultraviolet cut-off $a = 1$. If we follow the arguments of [28] we find that at high temperatures there is an entropic variational saddle point that dominates *i.e.* all the y's and x's are far away $O(L)$ from each other. At low temperatures there is an energetic saddle point which dominates *i.e.* all y's and x's are close together $O(1)$ and x's and y's far away. In the large L limit the moments show the transition

$$\langle t(x')\rangle^n \sim \begin{cases} L^{2n(1+g)}, & g < \frac{1}{2n} \\ L^{2+2n^2 g}, & g > \frac{1}{2n} \end{cases} \tag{6.1.3}$$

which shows a sequence of transition temperatures $T_{c,p} = \sqrt{\gamma} T_c$. This will be the result we are trying to re-create using the renormalization and functional renormalization group procedures. We will now explain why this transition can only be modeled by logarithmic correlations and not by longer or shorter ranged correlations.

6.1.2 Why Only Logarithmic Correlations?

The most general model we can consider is that of a particle in a Gaussian random potential $U(x)$ with spatial correlations $\langle [U(x) - U(x')]^2 \rangle = K(x - x')$, which are invariant by translation. As stated we follow the arguments of [59] and consider three forms of this correlation function, short-ranged, long-ranged and the intermediate logarithmic regime. The arguments of [59] are based around the competition between the entropy of the typical sites and the low energy states. The first case we consider is that of short-ranged correlations of the form $K(x) = \text{Const}$, secondly the case of long-ranged correlations, $K(x) = |x|^\delta$ and finally the intermediate case of logarithmic correlations $K(x) = \gamma \ln |x|$. For short-ranged correlations we find that there is only a high-temperature phase with no phase transition. The entropy of typical sites is $T \ln(L)$ which always overcomes the energy of optimal sites $V_{min} \sim \sqrt{\frac{\gamma}{2} \ln(L)}$ in the large L limit. For long-range correlations we find that there is only a low-temperature phase and no transition. This can be seen by considering the typical energy $V_{min} \sim L^{\frac{\delta}{2}}$ which can completely overcome the entropy $\sim T \ln(L)$. For logarithmic correlations, which provides the intermediate state between the short and long-range corre-

lations we see that the typical minimum energy is given by $\sim \sqrt{\gamma}\ln(L)$ which is enough to overcome the entropy of the sites $\sim T\ln(L)$ depending on the temperature. This leads to a transition temperature T_c between the high temperature phase and low-temperature frozen phase $T_c = \sqrt{\gamma}$. Before we proceed to introduce the model we are considering we will give a description of the equivalent two-dimensional problem.

6.1.3 The Two-Dimensional Problem

So far we have seen the existence of a phase transition between a low-temperature glass state and a high-temperature phase and we have explained why only logarithmic correlations describe such a phase transition. We would now like to proceed with a discussion of the equivalent problem in two-dimensions. Following the work from [6, 29] we notice there are two models of the random potential that have been considered each with some natural vector constraint. If we impose the natural constraint that $(\partial_\alpha v_\alpha(r) = 0)$, *i.e.* a solenoidal field and $(\partial_\alpha v_\beta(r) - \partial_\beta v_\alpha(r) = 0)$ *i.e.* a potential field we can define two models for the correlation function of the vector potential $\langle v_\alpha(r) v_\beta(r')\rangle = \gamma_0 F_{\alpha\beta}(r - r')$

$$F_{\alpha\beta}(k) = \begin{cases} \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}, & \text{Transverse Disorder} \\ \frac{k_\alpha k_\beta}{k^2}. & \text{Longitudinal Disorder} \end{cases} \tag{6.1.4}$$

Now the physical realizations of these models are that of the random motion of a traveller in a liquid with stationary random streams [6, 29]. More specifically the model with transverse

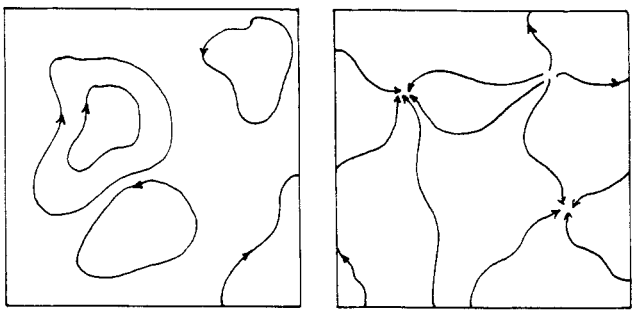

Figure 6.1: Typical realizations of the random potential $V(r)$ - Figure on the left shows transverse disorder showing lines of drift. Figure on the right shows longitudinal disorder creating a series of sources and sinks slowing the particles motion.

disorder represents a particle traveling in an incompressible liquid, an analogy of a bottle in an ocean. In contrast the model of longitudinal disorder is one of a particle traveling in a compressible liquid with random non circulatory flows.

To derive the transport properties for each model we perform a one-loop renormalization group analysis to find

$$\begin{aligned}
\frac{d\ln D}{d\ln(\lambda^{-1})} &= \kappa g_0, \\
\frac{d\ln \gamma}{d\ln(\lambda^{-1})} &= -(1-\kappa)g_0, \\
\frac{d\ln g_0}{d\ln(\lambda^{-1})} &= -(1+\kappa)g_0,
\end{aligned}$$

where g_0 is the effective coupling given by $g_0 = \frac{\gamma}{4\pi D^2}$ with $\kappa = 1$ for transverse disorder and $\kappa = -1$ for longitudinal disorder. For this we show no details of the calculation as we have followed the arguments of [29]. If we now solve these equations we find the the diffusion coefficient has the form

$$D(t) = \begin{cases} \ln^{\frac{1}{2}}(t), & \text{Transverse Disorder} \\ t^{-\frac{\gamma_0}{8\pi D_0^2}}, & \text{Longitudinal Disorder} \end{cases} \tag{6.1.5}$$

and we see that for transverse disorder the particles motion is super-diffusive and for longitudinal disorder the motion is sub-diffusive. The interpretation of these results are that for transverse disorder the super-diffusive behaviour is due to the ballistic motion along lines of drift. On the contrary, for longitudinal disorder the random potential is characterized by a set of sources and sinks, and these sinks act as traps. This trapping turns out to dominate the tendency to ballistic motion leading to sub-diffusive behaviour. An illustration of these random potentials is given by (6.1.3) from [29] showing the lines of drift for transverse disorder and the source and sinks created by longitudinal disorder.

To summarize this section, we have shown the existence of a phase transition in the first-passage time distribution and seen that our aim is to re-create this result using the renormalization and functional renormalization group procedures. We have also explained why only logarithmic correlations describe this transition and that correlations which are

longer or shorter would only have one state or another. We then concluded the discussion by considering the equivalent two-dimensional problem where we considered two models each with natural vector constraints which lead to a variety of diffusive behaviours. In the following section we introduce the model which we will be considering and give an explanation why we can't use the methods we have used previously to solve the problem.

6.2 The Model

As previously stated we are solving the model of an over-damped particle in a random environment with long-ranged logarithmic correlations using the method of renormalization group theory. We choose to use the renormalization group method instead of the one we used in chapter four as we don't know how to perform the averaging when the correlations are logarithmic. For short ranged correlations when we introduced the distribution functions and derived a Fokker-Planck equation to describe their evolution we had to use the result of Furutsu-Novikov to perform the averages which is only applicable when we have delta function short ranged correlation. If we wanted to perform a similar analysis for logarithmic correlations we would need to find an alternative averaging method to Furutsu-Novikov, something which we are not currently aware of. We therefore needed to find an alternative method and we chose the renormalization method as it is proven to work well in two dimensions as we have already discussed. A renormalization group procedure for the one-dimensional model has already been used to calculate the form of the diffusion coefficient and mean-square displacement [5]. We also calculate these quantities using the renormalization group procedure but we also attempt to calculate the distribution function of the particle using a combination of renormalization ad functional renormalization group techniques. This is an original method for solving for the particles distribution function and as a test to see whether this provides the correct result we use it to attempt to re-derive the phase transition in the first-passage time distribution.

We begin with the usual Fokker-Planck equation written in terms of the random potential $U(x)$

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial P(x,t)}{\partial x} - \frac{\partial}{\partial x}\left(\frac{dU(x)}{dx}P(x,t)\right) \tag{6.2.1}$$

with the initial condition that $P(x, t = 0) = \delta(x - x')$. The correlation function for the random potential $U(x)$ is given by equation (2.3.1) where we consider $\alpha = 0$ and the large separation limit *i.e.*

$$\langle [U(x) - U(x')]^2 \rangle = \gamma \ln |x - x'|. \tag{6.2.2}$$

The functional integral representing the Green's function of the Fokker-Planck equation can be written in terms of conjugate complex Grassmann fields $\psi(x)$ and $\bar{\psi}(x)$ [6]

$$G(x, x', \omega) = \frac{\int \bar{\psi}(x)\psi(x') \exp(iS[\bar{\psi}, \psi])\mathcal{D}\bar{\psi}\mathcal{D}\psi}{\int \exp(iS[\bar{\psi}, \psi])\mathcal{D}\bar{\psi}\mathcal{D}\psi}, \tag{6.2.3}$$

where the effective action functional is given by

$$S[\bar{\psi}, \psi] = \int dx(i\bar{\psi}\omega\psi + \partial_x V(x)(\partial_x\bar{\psi})\psi - D(\partial_x\bar{\psi})(\partial_x\psi)). \tag{6.2.4}$$

As we discussed in a chapter two the functional integral representation of the Green's function of the Fokker-Planck is divergent. To overcome this we need to write the functional integral as a perturbation series and for this model we consider the weak disorder limit so we can treat disorder as the perturbation parameter [37, 38]. We would now like to average over disorder and to do so we employ the standard replica trick: the fields $\psi(x)$ and $\bar{\psi}(x)$ and the functional (6.2.3) are N-replicated and the independent averaging of the numerator and denominator is justified in the replica limit as $N \to 0$. Taking into account the disorder in the random potential $U(x)$ we write our effective action as

$$S[\bar{\psi}, \psi] = \overbrace{\int dx\bar{\psi}^a D\partial_x^2\psi^a}^{S_0[\bar{\psi},\psi]} + \overbrace{\frac{1}{2}\int dxdx'(\partial_x\bar{\psi}^b\psi^b)_x\partial_x\partial_{x'}F(x - x')(\partial_{x'}\bar{\psi}^c\psi^c)_{x'}}^{S_{int}[\bar{\psi},\psi]}, \tag{6.2.5}$$

where $\omega = 0$ due to conservation of probability and $a, b, c...$ are replica indices with an implied sum from 1 to N. The term $S_0[\bar{\psi}, \psi]$ is simply the non-interacting Gaussian term, and $S_{int}[\bar{\psi}, \psi]$ is the interaction term. We now have the required action and from the previous chapter we know that the first step in any renormalization group calculation is to ensure the action is scale invariant using tree level scaling.

To perform tree level scaling we re-scale momentum via $q' = \lambda q$ where $0 < \lambda < 1$ is

the scaling parameter. To ensure the action is scale invariant we find the fields must scale as $\psi(q') = \lambda^{-\frac{3}{2}}\psi(q'/\lambda)$ which leaves both the diffusion coefficient D and the strength of disorder γ as dimensionless quantities. We therefore find the Gell-Mann Low equation for D and γ is given by

$$\frac{dD}{d\ln(\lambda^{-1})} = 0, \qquad \text{and} \qquad \frac{d\gamma}{d\ln(\lambda^{-1})} = 0, \tag{6.2.6}$$

and we say that as the Beta function is zero they are both marginal under the renormalization process. To find the next term of the Gell-Mann Low equation we need to consider one-loop corrections.

6.3 One-Loop Corrections to the Transport Coefficients

To calculate the next term of the Gell-Mann Low equations for the diffusion and strength of disorder coefficients we consider one-loop corrections. We first calculate the result for the diffusion coefficient and then the strength of disorder. To perform a one-loop correction the standard procedure to follow is to expand $\exp(iS_{int}[\bar{\psi}, \psi])$ as a power series and integrate with the weight $\exp(iS_0[\bar{\psi}, \psi])$ over the fast components of the fields and exponentiating the result of the integration. This is otherwise known as the cumulant expansion.

6.3.1 One-Loop Correction to the Diffusion Coefficient

To calculate the correction to the diffusion coefficient we should consider averaging the interaction term of the action. Doing so we find that there are only three possible loop diagrams, the first being when there are no fast modes and we simply have tree level scaling. The second when there are two fast fields which is classed as a one-loop diagram. This leaves a term with two slow fields which would provide a one-loop correction to the Gaussian term. The third and final diagram, where the average is between every field we find a two-loop diagram which is simply a constant. We therefore need to calculate a diagram of the form (6.3.1) to find the correction to the diffusion coefficient is given by

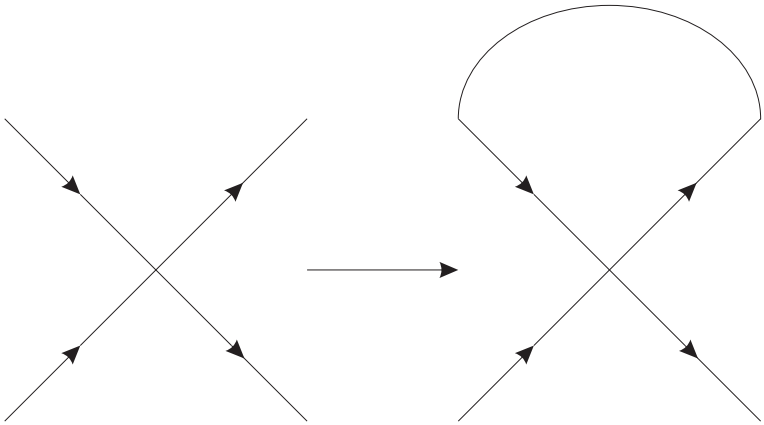

Figure 6.2: One-loop correction to the diffusion coefficient.

$$
\begin{aligned}
\delta D &= -\Big\langle (2\pi)^2 \frac{\gamma}{4} \sum_{\substack{q_1,q_2,q_3=0 \\ q_4,k=0}} \bar{\psi}^b(q_1) q_1 \psi^b(q_2) |k| \bar{\psi}^c(q_3) q_3 \psi^c(q_4) \\
&\times \delta(-q_1+q_2+k)\delta(-q_3+q_4-k) \Big\rangle_0 \\
&= (2\pi)^2 \frac{\gamma}{4} \sum_{q_1,q_4=0} \bar{\psi}^b(q_1) q_1 \psi^c(q_4) \sum_{q_2,q_3,k=\lambda\Lambda} \langle \bar{\psi}^c(q_3) \psi^b(q_2) \rangle_0 q_3 |k| \\
&\times \delta(-q_1+q_2+k)\delta(-q_3+q_4-k) \\
&= (2\pi) \frac{\gamma}{4D} \sum_{q_1=0} \bar{\psi}^b(q_1) q_1^2 \psi^b(q_1) \sum_{k=\lambda\Lambda} \frac{1}{|k|} \\
&= (2\pi) \frac{\gamma}{4\pi D^2} \sum_{q_1=0} \bar{\psi}^b(q_1) q_1^2 D \psi^b(q_1) \ln(\lambda^{-1}).
\end{aligned} \tag{6.3.1}
$$

To find the exact correction to the diffusion coefficient we should count the number of diagrams that contribute. We notice that there are six possible combinations for there to be two fast and two slow fields. This can be immediately reduced to four as contractions between $\langle \bar{\psi}\bar{\psi} \rangle_0$ and $\langle \psi\psi \rangle_0$ are zero by definition. Finally we can eliminate two more combinations by noticing that if we want k to be fast we are not permitted to have both q_1, q_2 or q_3, q_4 to be fast. This therefore leaves us with only two contributing diagrams and the correction to the diffusion coefficient is given by

$$
D = D - 2g_0 \ln(\lambda^{-1}) D, \tag{6.3.2}
$$

where $g_0 = \frac{\gamma}{4\pi D^2}$ is a dimensionless quantity which represents the effective coupling constant. As a consequence we find the Gell-Mann Low equation becomes

$$\frac{d\ln(D)}{d\ln(\lambda^{-1})} = -2g_0, \tag{6.3.3}$$

which decreases under the renormalization process. We now follow similar arguments to calculate the one-loop contribution to the strength of disorder which we will use to calculate the Gell-Mann Low equation for the effective coupling constant g_0.

6.3.2 One-Loop Correction to the Strength of Disorder

To calculate the one-loop correction to the strength of disorder γ we must consider the second term of the cumulant expansion. Doing so we find the only term that gives a correction to it comes from $\langle S_{int}^2[\bar{\psi},\psi]\rangle$. There are several ways to perform this averaging, but the only relevant one-loop diagram is given by diagram (6.3.2). Calculating this diagram we find the one-loop correction to the strength of disorder is

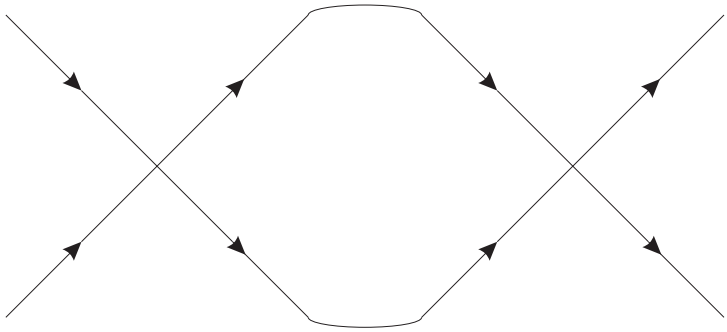

Figure 6.3: One-loop correction to the strength of disorder γ.

$$\begin{aligned}
\delta\gamma &= \Bigg\langle (2\pi)^4\frac{\gamma^2}{32}\sum_{\substack{q_1,q_2,q_3=0\\ q_4,k=0}} \bar{\psi}^b(q_1)q_1\psi^b(q_2)|k|\bar{\psi}^c(q_3)q_3\psi^c(q_4)\delta(-q_1+q_2+k) \\
&\times \delta(-q_3+q_4-k)\sum_{\substack{l_1,l_2,l_3=0\\ l_4,k'=0}} \bar{\psi}^d(l_1)l_1\psi^d(l_2)|k'|\bar{\psi}^e(l_3)l_3\psi^e(l_4) \\
&\times \delta(-l_1+l_2+k')\delta(-l_3+l_4-k')\Bigg\rangle_0
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^2 \frac{\gamma^2}{32D^2} \sum_{\substack{q_3,q_4,l_2=0 \\ l_3,k=0}} \bar{\psi}^c(q_3) q_3 \psi^c(q_4) |k| \bar{\psi}^b(l_3) l_3 \psi^b(l_2) \\
&\times \quad \delta(l_2 - l_3 + k)\delta(-q_4 + q_3 + k) \sum_{k'=\lambda\Lambda} \frac{1}{|k'|} \\
&= (2\pi) \frac{\gamma^2}{16D^2} \ln(\lambda^{-1}) \sum_{q_3,l_3,k=0} \bar{\psi}^b(l_3) l_3 \psi^b(l_3 - k) |k| \bar{\psi}^c(q_3) q_3 \psi^c(q_3 + k).
\end{aligned}
$$

To find the exact correction to the strength of disorder we should again count the number of diagrams that contribute to the correction by following the same arguments we used for the correction to the diffusion coefficient *i.e.* contractions can only be between $\langle \bar{\psi}\psi \rangle$ and the restrictions that come from momentum conservation. Following these rules we find the correction to the strength of disorder is given by

$$
\gamma = \gamma - 4g_0\gamma, \tag{6.3.4}
$$

and the Gell-Mann Low equation has the form

$$
\frac{d\ln(\gamma)}{d\ln(\lambda^{-1})} = -4g_0, \tag{6.3.5}
$$

which clearly decreases under the renormalization group process. We would like to use these equations to calculate the form of the diffusion coefficient. The problem is they are a function of g_0 so we should calculate a Gell-Mann Low equation for g_0. We can do this using the fact that $g_0 \approx \frac{\gamma}{D^2}$ to find

$$
\frac{d\ln(g_0)}{d\ln(\lambda^{-1})} = \frac{d\ln(\gamma)}{d\ln(\lambda^{-1})} - 2\frac{d\ln(D)}{d\ln(\lambda^{-1})} = 0. \tag{6.3.6}
$$

It is clear from this that the coupling term is just a constant and is not renormalized under the renormalization group process. Using this we can solve the Gell-Mann Low equations for D and γ to find

$$
\gamma = \gamma_0 \lambda^{4g_0}, \qquad \text{and} \qquad D = D_0 \lambda^{2g_0}, \tag{6.3.7}
$$

where γ_0 and D_0 are constants of integration. If we write this in time representation *i.e.* $\lambda = (t/\tau)^{-\frac{1}{2}}$ we find the long time behaviour of the diffusion coefficient goes as

$$D(t) = D_0 \left(\frac{t}{\tau}\right)^{-g_0}, \tag{6.3.8}$$

and we can conclude that the particles motion is sub-diffusive as $t \to \infty$. This is caused by the presence of sources and sinks which act as traps which dominates the particles tendency to ballistic motion. We can also calculate the form of the strength of disorder to find

$$\gamma(t) = \gamma_0 \left(\frac{t}{\tau}\right)^{-2g_0}. \tag{6.3.9}$$

The diffusion coefficient can be used to calculate the mean-square displacement $\langle x^2(t)\rangle$. Doing so we find the mean-square displacement scales as $\langle x^2(t)\rangle \sim t^{\frac{2}{z}}$ with $z = 2 + 2g_0$ in agreement with [5].

To summarize this section we have derived the transport coefficients of a particle in a one-dimensional random potential with logarithmic correlations. We have shown that such a system has sub-diffusive behaviour and a mean-square displacement in agreement with what is already know. We would now like to proceed to calculate the whole distribution function by allowing the diffusion coefficient to fluctuate and seeing how the higher cumulants of the distribution act under the renormalization group procedure [31]. This is the main original result of the chapter.

6.4 Introduction of Higher Cumulants

The aim of this section is to use the renormalization and functional renormalization group techniques to derive the particles full distribution function. To fulfill this aim we introduce the higher cumulants of the distribution by allowing fluctuations in the diffusion coefficient. We would then like to investigate how these higher cumulants act under the renormalization group procedure and then relate them back to their distribution function. The idea is to derive a functional renormalization group equation for the distribution function using the fact that we know how the cumulants relate to the distribution function and how they act

under the renormalization procedure.

To introduce the higher cumulants we define the fluctuations in the diffusion coefficient by writing them as the sum of some bare term $\bar{D}$ and some fluctuating part $\delta D(x)$ to find $D = \bar{D} + 2\delta D(x)$. Using this expression for the diffusion coefficient we find the action becomes

$$\begin{aligned} S[\bar{\psi},\psi] &= \int dx[\partial_x V(x)(\partial_x(\bar{\psi})\psi - D(\partial_x\bar{\psi})(\partial_x\psi)] \\ &= \int dx[\partial_x V(x)(\partial_x\bar{\psi})\psi - \bar{D}(\partial_x\bar{\psi})(\partial_x\psi) - 2\delta D(x)(\partial_x\bar{\psi})(\partial_x\psi)], \end{aligned}$$

with the fluctuations in the diffusion coefficient having the form [31]

$$\begin{aligned} \langle\langle \delta D(x_1)\delta D(x_2)\rangle\rangle &= 2! g^{(2)}\delta(x_1 - x_2), \\ \langle\langle \delta D(x_1)...\delta D(x_s)\rangle\rangle &= s! g^{(s)}\delta(x_1 - x_2)...\delta(x_1 - x_s). \end{aligned} \tag{6.4.1}$$

If we now take this new action and average over disorder and the fluctuations in the diffusion coefficient we find

$$\begin{aligned} S[\bar{\psi},\psi] &= \overbrace{\int dx\bar{\psi}^a \bar{D}\partial_x^2\psi^a}^{S_0[\bar{\psi},\psi]} + \overbrace{\frac{1}{2}\int dxdx'(\partial_x\bar{\psi}^b\psi^b)_x\partial_x\partial_{x'}\langle V(x)V(x')\rangle(\partial_{x'}\bar{\psi}^c\psi^c)_{x'}}^{S_{int}[\bar{\psi},\psi]} \\ &+ \underbrace{\sum_{s=2}^{\infty} g^{(s)}\int dx\prod_{i=1}^{s}(\partial_{x_i}\bar{\psi}^{d_i}\partial_{x_i}\psi^{d_i})}_{S_{cum}[\bar{\psi},\psi]}, \end{aligned} \tag{6.4.2}$$

where $S_{cum}[\bar{\psi},\psi]$ is the cumulant term where $g^{(s)}$ is the coupling term to each cumulant and d_i are the replica indices. This extra cumulant term becomes

$$S_{cum}[\bar{\psi},\psi] = (2\pi)\sum_{s=2}^{\infty} g^{(s)}\prod_{i=1}^{s}\sum_{q_i,q_i'=0}\bar{\psi}^{d_i}(q_i)\psi^{d_i}(q_i')q_iq_i'\delta(\sum_{m=1}^{s}(q_m - q_m')), \tag{6.4.3}$$

when we move to momentum space. The aim now is to derive the Gell-Mann Low equation for the coupling term $g^{(s)}$. As we know the first term of the Gell-Mann equation comes from tree level scaling. We have previously found that the fields scale as $\psi(q') = \lambda^{-\frac{3}{2}}\psi(q'/\lambda)$ so

to leave the cumulant part of the action invariant under the renormalization group process the coupling constant $g^{(s)}$ must scale as $g^{(s)} = \lambda^{(s-1)} g^{(s)}$and the first term of the Gell-Mann Low equation becomes

$$\frac{dg^{(s)}}{d\ln(\lambda^{-1})} = -(s-1)g^{(s)}. \tag{6.4.4}$$

From this we can see that all $g^{(s)}$ are irrelevant under the renormalization group procedure except for $s = 1$ which is marginal. To calculate the next term of the Gell-Mann Low equation we again need to consider one-loop corrections. It is clear from the construction of this term that it will not have a correction to the diffusion coefficient or to the strength of disorder due to the number of gradients, but it will give corrections to itself.

To calculate the one-loop linear corrections to the higher cumulants we should consider the term $\langle S_{int}[\bar{\psi}, \psi] S_{cum}[\bar{\psi}, \psi] \rangle_0$ which comes from the second term of the cumulant expansion. To calculate the correction that comes from this term we need to know the number of diagrams that contribute to it. We notice that the cumulant term comes with a total of $2s$ fields and gradients, and the problem is to consider how many combinations can arise from this. To find the corrections to the couplings it should be noted that the averaging over fast modes $\langle \bar{\psi}\psi \rangle_0$ must involve $\bar{\psi}$ coming from the interaction term and ψ from the cumulant term. Using this idea to calculate the number of possible diagrams we find the first $\bar{\psi}$ from

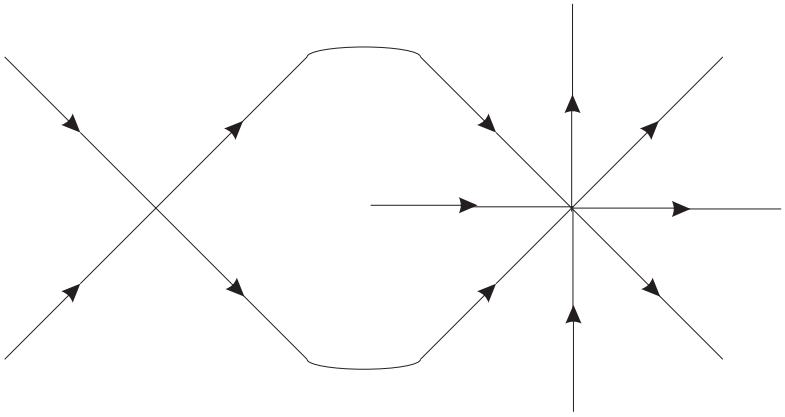

Figure 6.4: One-loop corrections to the higher moments. Contractions can only occur between $\bar{\psi}$ coming from the interaction term and ψ from the cumulant term.

the interaction term has a choice of s ψ fields from the cumulant term, and the second $\bar{\psi}$ has a choice of $(s-1)$ terms to choose from. Therefore the total number of diagrams to

each order is given by $s(s-1)$. If we now perform the averaging over fast modes we find

$$\begin{aligned}\delta g^{(s)} &= -\Big\langle (2\pi)^3 \frac{\gamma}{4} \sum_{\substack{l_1,l_2,l_3=0 \\ l_4,k=0}} \bar{\psi}^a(l_1) l_1 \psi^a(l_2) |k| \bar{\psi}^b(l_3) l_3 \psi^b(l_4) \delta(-l_1+l_2+k)\delta(-l_3+l_4-k) \\ &\times \sum_{s=2}^{\infty} g^{(s)} \prod_{i=1}^{s} \sum_{q_i,q_i'=0} \bar{\psi}^{d_i}(q_i)\psi^{d_i}(q_i') q_i q_i' \delta(\sum_{m=1}^{s}(q_m - q_m'))\Big\rangle \\ &= g_0 s(s-1) \sum_{s=2}^{\infty} g^{(s)} [\prod_{i=1}^{s} \sum_{q_i,q_i'=0} \bar{\psi}^{d_i}(q_i)\psi^{d_i}(q_i') q_i q_i' \delta(\sum_{m=1}^{s}(q_m - q_m'))] \ln(\lambda^{-1}), \end{aligned} \tag{6.4.5}$$

which leads to a correction to the s^{th} cumulant of the form

$$\delta g^{(s)} = s(s-1) g_0 g^{(s)}, \tag{6.4.6}$$

which increases under the renormalization group process. We therefore find the Gell-Mann Low equation becomes

$$\frac{d\ln(g^{(s)})}{d\ln(\lambda^{-1})} = -(s-1) + g_0 s(s-1), \tag{6.4.7}$$

where $-(s-1)g^{(s)}$ comes from tree level scaling and $g_0 s(s-1)g^{(s)}$ comes from the one-loop linear corrections. We can consider two limits of this equation, first, $sg_0 \ll 1$ where the fluctuations are small and we find a Gaussian distribution. For the opposite case of $sg_0 \gg 1$ we find the Gell-Mann Low equation becomes

$$\frac{d\ln(g^{(s)})}{d\ln(\lambda^{-1})} = s(s-1) g_0, \tag{6.4.8}$$

and we have fluctuations that lead to a lognormal asymptotic tail for the distribution function. The form of this distribution function can be found by writing the cumulants in the form $g^{(s)} \sim v^s \exp[us^2]$ [30] to find the asymptotic tails go as

$$f(g^{(s)}) \approx \frac{1}{g^{(s)}} \exp\Big[-\frac{1}{4g_0 \ln(\lambda^{-1})} \ln^2(g^{(s)} \lambda^{-g_0}) \Big], \tag{6.4.9}$$

which in time representation has the form

$$f(g^s) \approx \frac{1}{g^{(s)}} \exp\left[-\frac{1}{4g_0 \ln(\frac{t}{\tau})^{\frac{1}{2}}} \ln^2\left(g^{(s)} \left(\frac{t}{\tau}\right)^{\frac{g_0}{2}} \right) \right], \tag{6.4.10}$$

for the long-time distribution.

To summarize this section we have calculated the one-loop linear corrections to the higher cumulants. We have introduced the higher cumulants by allowing the diffusion coefficient to fluctuate. The purpose of this was to allow us to derive an equation that describes how the cumulants change under the renormalization process. We have shown that the distribution of these cumulants are Gaussian when sg_0 is small. We have also shown that the tails of the distribution are log normal. We will use these results in the following section to derive a functional renormalization group equation to solve for the distribution function.

6.5 Calculation of the Distribution Function

In the previous section we were interested in the higher cumulants of the distribution which were introduced by allowing the diffusion coefficient to fluctuate. We then derived an equation that described the evolution of the coupling constant to each cumulant. We will now use this equation to derive a functional renormalization group equation for the distribution function by relating the distribution function to its cumulants. We begin by considering the distribution of the cumulants

$$\int dD P_l(D) \exp[-zD] = \exp[-F[z]], \tag{6.5.1}$$

and the cumulant generating function [60] which is defined by

$$F[\lambda, z] = \sum_{s=0} g^{(s)} z^s. \tag{6.5.2}$$

We now have an expression for the distribution function in terms of the cumulants and we can derive a functional renormalization group equation by simply differentiating equation (6.5.2) by $\ln(\lambda^{-1})$. Doing so, and using equation (6.4.7) for the evolution of cumulants we

find

$$\begin{aligned}\frac{dF[\lambda,z]}{d\ln(\lambda^{-1})} &= \sum_{s=0}^{\infty}\frac{dg^{(s)}}{d\ln(\lambda^{-1})}z^s \\ &= -\sum_{s=0}^{\infty}sg^{(s)}z^s+\sum_{s=0}^{\infty}g^{(s)}z^s+g_0\sum_{s=0}^{\infty}s(s-1)g^{(s)}z^s \\ &= F[z]-zF'[z]+g_0z^2F''[z], \end{aligned} \tag{6.5.3}$$

where we have used $\sum_{s=0}^{\infty}sg^{(s)}z^s=-zF'[z]$ and $\sum_{s=0}^{\infty}s(s-1)g^{(s)}z^s=z^2F''[z]$. If we make the substitution $z=e^v$ then our functional renormalization group equation becomes

$$\partial_l G[v]=G[v]-(1+g_0)G'[v]+g_0G''[v], \tag{6.5.4}$$

which has a solution of the form

$$G[l,v]=\exp[l]\int\frac{dv'}{\sqrt{4g_0\pi l}}\exp\left[-\frac{[v-(1+g_0)l-v']^2}{4g_0l}\right]G(0,v'), \tag{6.5.5}$$

which describes a diffusion packet with some drift velocity $(1+g_0)$ being smeared by l, with $G(0,v)$ the initial condition. If we write equation (6.5.1) in terms of this new variable $z=e^v$ then we find

$$\int dDP_l(D)\exp[-e^vD]=\exp[-G(v)], \tag{6.5.6}$$

and in the limit as $v\to-\infty$ then equation (6.5.6) can be expanded to give

$$\begin{aligned}\exp[-G(v)] &= \int dDP_l(D)\exp[-e^vD]=\int dDP_l(D)(1-e^vD+\frac{1}{2}e^{2v}D^2+\dots) \\ &= 1-\langle D\rangle e^v+\frac{1}{2}\langle D^2\rangle e^{2v}+\dots, \end{aligned} \tag{6.5.7}$$

and we find $G(v)$ is given by

$$G(v)=\langle D\rangle e^v-\frac{1}{2}[\langle D^2\rangle-\langle D\rangle^2]e^{2v}+\dots, \tag{6.5.8}$$

which is simply the cumulant expansion on re-exponentiating. We can find another possible solution to the functional renormalization equation (6.5.4) by assuming a perfect exponential

solution of the form

$$G(v) = \exp[\alpha(v + c_\alpha l)] = \exp[l(1-\alpha)(1-\alpha g)]\exp[\alpha v], \tag{6.5.9}$$

which corresponds to a front with a velocity $-c_\alpha$

$$c_\alpha = \frac{(1-\alpha)(1-\alpha g)}{\alpha}, \tag{6.5.10}$$

where the form of c_α comes from simply substituting the perfect exponential solution into the equation for $G(v)$. We would now like to use this solution to calculate the first-passage time distribution in an attempt to find the phase transition between a low temperature glass phase and a high temperature state.

To summarize this section we have calculated a functional renormalization group equation for the particles distribution function using the Gell-Mann Low equation for the coupling constants we derived in the previous section. We found the solution to this equation had the form of a particle diffusing with some drift velocity. We also found that as $v \to -\infty$ or $z \to 0$ we arrive at the cumulant expansion for the diffusion coefficient. We found another solution to this functional renormalization group equation by assuming a perfect exponential solution which we will now use to derive the first-passage time distribution in an attempt to re-create the phase transition.

6.6 Comparison to First-Passage Time

We have already seen that the renormalization group procedure can be used to calculate the results for the mean-square displacement and diffusion coefficient. We have also seen that the first-passage time distribution for a particle in an environment with logarithmic correlations undergoes a transition between a low temperature glass phase and a high temperature phase as a function of its moments. We would now like to re-derive this result using this renormalization and functional renormalization group procedures.

If we define $P(x, x', t)$ as the solution of the Fokker-Planck equation with the initial condition $P(x, x', t = 0) = \delta(x - x')$ with absorbing boundary conditions $P(x = -\frac{L}{2}, x', t) =$

$P(x = \frac{L}{2}, x', t) = 0$, then the mean first-passage time for a particle at x' is given by

$$\begin{aligned}
\langle t(x')\rangle &= -\int_0^\infty dtt\partial t \int_{-\frac{L}{2}}^{\frac{L}{2}} dx P(x, x', t) \\
&= \int_0^\infty dt \int_{-\frac{L}{2}}^{\frac{L}{2}} dx P(x, x', t) \\
&= \int_{-\frac{L}{2}}^{\frac{L}{2}} dx G(x, x', i\omega = 0) \\
&= \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \langle \psi(x)\bar{\psi}(x')\rangle, \qquad (6.6.1)
\end{aligned}$$

where the path integral should be subject to the boundary condition $\psi(\frac{L}{2}) = \psi(-\frac{L}{2}) = 0$, *i.e.* the total probability decays. We can extend this result to find the n^{th} moment of the first-passage time distribution

$$\langle t(x')^n\rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx_1 dx_2 \dots dx_n \langle \bar{\psi}_{a_1}(x') \dots \bar{\psi}_{a_n}(x')\psi_{a_1}(x_1) \dots \psi_{a_n}(x_n)\rangle. \qquad (6.6.2)$$

We perform the averaging by making a tree with only a log disorder vertex with $2n$ external legs and propagators, $n-2$ internal propagators and $n-1$ vertices. If we perform this estimate with a vertex

$$\gamma \int (\partial_x \bar{\psi}\psi)_x \partial_x \partial_{x'} \ln \mid x - x' \mid)(\partial_{x'} \bar{\psi}\psi)_{x'} dx dx', \qquad (6.6.3)$$

we find it yields a factor $\gamma L^{-2} = g\bar{D}^2 L^{-2}$ per vertex to find

$$\langle t(x')^n\rangle \sim L^n (g\bar{D}^2 L^{-2})^{n-1} (L\bar{D})^{n-2+2n} = g^{n-1} L^{2n(1+g)}, \qquad (6.6.4)$$

which is exactly the result we found earlier for the lower limit of the phase transition. We find the upper limit of the phase transition by making a direct calculation with the $\bar{D}^n$ vertex we calculated earlier, where

$$\bar{D}^n \sim L^{(1-n)(1-ng)}, \qquad (6.6.5)$$

which leads to an estimate of first-passage time

$$\langle t(x')\rangle^n \sim L^{n+1}\bar{D}^n\bar{D}^{-2n} = L^{2+3ng+gn^2}. \tag{6.6.6}$$

If we compare these with the known results of [5] and equation (6.1.3) we see that we do in part re-create the phase transition in the first-passage time distribution. We see for the lower limit of $g < \frac{1}{(2n)}$ we find exactly the result of [5]. Unfortunately the method does not exactly re-create the upper limit of $\langle t(x')\rangle^n$, but it does show the existence of an upper limit. Therefore the aim of the next section is to describe a possible method that may find the correct upper-limit. For this we consider non-linear corrects to the higher cumulants.

6.7 Non-Linear Corrections to the Higher Cumulants

As we have seen, by only considering the linear corrections to the higher cumulants we find the transition between the low temperature glass state and high temperature phase but we don't find the correct exponent. In an attempt to correct this we now consider the non-linear corrections to the higher cumulants and to do so we consider the third order of the cumulant expansion. Out of the four terms that arise from the cumulant expansion only one of them will contribute to the non linear corrections, the term $\langle S^3_{int}[\bar{\psi},\psi]\rangle_0$ will not as it does not have the correct number of gradients, neither does $\langle S^2_{int}[\bar{\psi},\psi]S_{cum}[\bar{\psi},\psi]\rangle_0$. Finally $\langle S^3_{cum}[\bar{\psi},\psi]\rangle_0$ will not have a logarithmic divergence and we find the only term that contributes to the cumulants is given by $\frac{1}{2}\langle S_{int}[\bar{\psi},\psi]S^2_{cum}[\bar{\psi},\psi]\rangle_0$. To calculate the non-linear correction to the higher cumulants we need to calculate a diagram of the form (6.7). This leads to a correction of the form

$$\begin{aligned}
\delta g^{(s)} &= -(2\pi)^4\frac{\gamma}{8}\Bigg\langle \sum_{\substack{q_1,q_2,q_3=0\\ q_4,k=0}} \bar{\psi}^a(q_1)q_1\psi^a(q_2)|k|\bar{\psi}^b(q_3)q_3\psi^b(q_4)\delta(-q_1+q_2+k)\\
&\times \delta(-q_3+q_4-k)\sum_{s=2}^{\infty} g^{(s)}\prod_{i=1}^{s}\sum_{l_i,l_i'=0}\bar{\psi}^{c_i}(l_i)\psi^{c_i}(l_i')l_il_i'\delta(\sum_{m=1}^{s}(l_m-l_m'))\\
&\times \sum_{t=2}^{\infty} g^{(t)}\prod_{j=1}^{t}\sum_{n_j,n_j'=0}\bar{\psi}^{d_j}(n_j)\psi^{d_j}(n_j')n_jn_j'\delta(\sum_{p=1}^{t}(n_p-n_p'))\Bigg\rangle_0 .
\end{aligned} \tag{6.7.1}$$

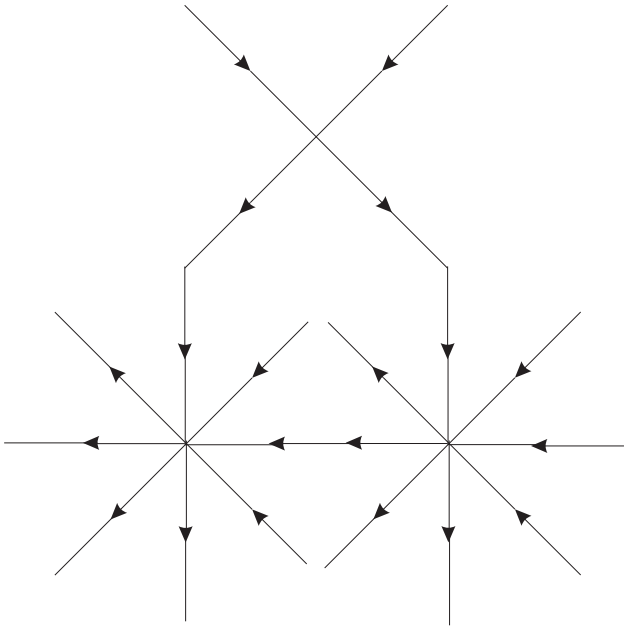

Figure 6.5: One-loop corrections to the non-linear corrections of the higher cumulants. Two contributions to the diagram depending on how the averaging is performed.

As usual we need to count the number of diagrams which contribute to the correction. It is clear that the number of diagrams is given by the sum of two terms, the first when the averaging between the two higher gradient terms has the form $\langle\bar{\psi}(l)\psi(n)\rangle_0$ and the other when the averaging has the form $\langle\bar{\psi}(n)\psi(l)\rangle_0$. The number of diagrams from $\langle\bar{\psi}(l)\psi(n)\rangle_0$ is given by $s^2t(t-1)$ and the number of diagrams from $\langle\bar{\psi}(n)\psi(l)\rangle_0$ is given by $t^2s(s-1)$ leading to a correction of the form

$$\begin{aligned}\delta g^{(s+t-1)} &= -g_0 g^{(s)} g^{(t)} \ln(\lambda^{-1})[s^2t(t-1)+t^2(s-1)s] \\ &\times \prod_{i=1}^{s+t-1} \sum_{q_i,q_i'=0} \bar{\psi}^{a_i}(q_i)\psi^{a_i}(q_i')\delta(\sum_{m=1}^{s+t-1}(q_m-q_m')),\end{aligned} \tag{6.7.2}$$

which is a correction to the $(s+t-1)^{th}$ term. It is clear therefore that the general correction to the s^{th} term is given by

$$\delta g^{(s)} = -g_0 \ln(\lambda^{-1}) \sum_{t=2}^{m-1}[t^2(s-t+1)(s-t)+(s-t+1)^2t(t-1)]g^{(s-t+1)}g^{(t)},$$

and we have a Gell-Mann Low equation of the form

$$\begin{aligned}\frac{dg^{(s)}}{d\ln(\lambda^{-1})} &= -(s-1)g^{(s)} + g_0 s(s-1)g^{(s)} \\ &- g_0 \sum_{t=2}^{m-1}[t^2(s-t+1)(s-t) + (s-t+1)^2 t(t-1)]g^{(s-t+1)}g^{(t)},\end{aligned} \tag{6.7.3}$$

where $-(s-1)g^{(s)}$ comes from tree level scaling and $g_0 s(s-1)g^{(s)}$ comes from the linear corrections to the higher cumulants. We will now use this flow equation to derive a functional renormalization group equation for the distribution of the higher cumulants as we did for the case of linear corrections. Following the same procedure as we did for the linear case we find

$$\begin{aligned}\frac{dF[\lambda,z]}{d\ln(\lambda^{-1})} &= F[z] - zF'[z] + g_0 z^2 F''[z] \\ &- g_0 \sum_{s=0}^{\infty}\sum_{t=2}^{s-1}[t^2(s-t+1)(s-t) + (s-t+1)^2 t(t-1)]g^{(s-t+1)}g^{(t)}z^s,\end{aligned}$$

and if we make the change $m = s-t+1$ we find a functional renormalization group equation of the form

$$\frac{\partial F[\lambda,z]}{\partial\ln(\lambda^{-1})} = F[\lambda,z] - zF'[\lambda,z] + g_0 z^2 F''[\lambda,z] - 2g_0 z^2 F''[\lambda,z]F'[\lambda,z] - 2g_0 z^3 F''[\lambda,z]^2.$$

We now have an equation that describes the cumulants of the distribution which has been derived using linear and non-linear corrections. Unfortunately we have not been able to solve this equation, therefore the aim of any future work should be based around solving this equation and to follow similar arguments to those we presented previously to hopefully find the correct exponent for the upper limit of the phase transition. If it were possible to show that this method gives the correct exponent for the phase transition we would have found a powerful technique for calculating the properties of such one-dimensional systems.

6.8 Summary

For this chapter we have considered the problem of an over-damped particle traveling in an environment with logarithmic correlations. We first considered the first-passage time distribution and found the existence of a phase transition between a low temperature glass state and a high temperature phase and it was this phase transition which we were trying to re-create using the renormalization and function renormalization group approach. We then gave an explanation why only logarithmic correlations show this phase transition whereas shorter or longer range correlations would only have one state or the other. We then considered the equivalent two-dimensional problem where we followed the arguments of [6, 29] to derive one-loop corrections to the diffusion coefficient. On doing so we found a mixture of diffusive behaviours depending on the particular vector constraint we were considering. For the case of a solenoidal vector potential we found super-diffusive behaviour due to the existence of drift lines. For the case of a potential field we found the particles motion is sub-diffusive caused by disorder inducing a series of sources and sinks destroying the diffusive behaviour. We then re-derived a series of results for the one-dimensional problem using renormalization group theory.

The first result we have found was that of the mean-square displacement which was in agreement with the known result of $\langle x^2(t)\rangle \sim t^{\frac{2}{z}}$ with $z = 2+2g_0$. We also found the diffusion coefficient had the form $D(t) = D_0 t^{-g_0}$ meaning the particles motion is sub-diffusive. We found these results by first calculating the one-loop correction to the diffusion and strength of disorder coefficients and then finding how the coupling constant g_0 changes. In addition to this we have also shown the existence of the phase transition between a frozen state and high temperature state, but with an incorrect value for the exponent. To do this we considered the higher cumulants of the distribution by allowing fluctuations in the diffusion coefficient. On averaging over these fluctuations we found an additional term in the action which described the cumulants. We found that under the renormalization group process these cumulants only have corrections to themselves. The one-loop corrections showed their distribution was Gaussian for $sg_0 \ll 1$ *i.e.* small fluctuations. For $sg_0 \gg 1$, where the fluctuations are large we found the distribution had logarithmic tails. We then used the Gell-Mann Low equation

for the higher cumulants to calculate the particles distribution function. We first derived a functional renormalization group equation by only considering linear corrections to the flow equations. Solving this equation we found the distribution function had the form of a particle diffusing with some drift velocity. As we have seen by only considering these linear corrections we found the transition but not the correct exponent. In an attempt to fix this we have calculated the non-linear corrections and derived a new functional renormalization group equation. We were not able to solve this new functional renormalization group equation, but if we were able to we would have followed the same arguments as those previously to calculate the phase transition in the first-passage time distribution. Therefore we believe any future work should be based around solving this functional renormalization group equation.

Chapter 7

CONCLUSIONS AND FUTURE WORK

For this thesis we have considered the problem of classical diffusion of a particle in a variety of one-dimensional random potentials. We have considered two main projects in this thesis involving the calculation of certain transport properties of our particle. The two projects have been successful to varying degrees which will be discussed in this conclusion. We will also give a summary of each chapter including a discussion of our successes and failures and any possible future paths for the work.

The thesis began by giving a general introduction to classical diffusion and the Fokker-Planck equation. We first considered the conservation of the number density of particles dissolved or suspended in a fluid. This conservation law lead to Fick's law which states that a spatially non-uniform density leads to currents in directions opposite to the direction of changes in densities. We then considered the dilute limit of this situation which allowed us to just concentrate on the motion of a single particle which is found to be random, *i.e.* Brownian motion. A particle of this type is simply exposed to a combination of thermal fluctuations and viscous drag and it is these random thermal fluctuations that leads to the diffusive behaviour. This diffusive behaviour is governed by the usual Diffusion equation which in this thesis is known as the pure case as it is not subject to the effect of disorder. This provided us with a result $\langle x^2(t)\rangle = 2Dt$ to compare to for the disordered case. We then introduced disorder and found that its presence subjects the particle to an additional random potential. We found the probability distribution for such a particle is given by the Fokker-Planck equation and it is this equation that formed the basis of the work presented in this thesis. It was at this point where we introduced the general form of the correlation

function $K(x - x')$ in terms of a parameter α. We have seen that for this parameter that when it is zero we have logarithmic correlations which was discussed in chapter six. For the case of $\alpha = 1$ we found Sinai's diffusion which we discussed in chapter four. We also considered two further cases of this correlation function in the chapter on first-passage time. The first case known as creep motion had $0 < \alpha < 1$ and the second has $\alpha < 0$ which is known as the ohmic regime. We also presented a completely equivalent formalism by considering the one-dimensional Lattice Hopping model [13]. We have shown by considering a single site on a lattice that the change in the probability distribution is given by the probability of the particle hopping to the right in addition to the probability of a particle hopping to the left. This equation is known as the Master equation. We have seen that the Lattice Hopping model is equivalent to the continuous Fokker-Planck model by taking the limit of the lattice spacing going to zero.

The first disordered quantity we investigated was that of the first-passage time distribution which as we have seen gives a description of the time spent in a system as a function of its initial starting point. We have seen that the moments of the first-passage time distribution are described by the Pontryagin equation, which is just a simple second order differential equation. We derived this equation by considering properties of Markov processes which are described by conditional probability. We also used the properties of conditional probability to derive the Chapman-Kolmogorov equation which states that the probability of hopping from one point to another can occur via any intermediate state. This Chapman-Kolmogorov equation was then used to derive the Master equation. As the Pontryagin equation is a second order differential equation it was solved with a simple integration factor. We then used an alternative method [16] using a Green's function to illustrate the method we used in the proceeding chapter where we solved the Fokker-Planck equation for Sinai's diffusion with drift. The method we used was to write the solution in terms of a Green's function with functions acting to the left and right of some source. We then re-wrote these left and right acting functions in terms of a transfer matrix and then derived a differential equation for each component of the matrix. By solving said differential equation we were able to find the form of the first moment of the first-passage time distribution in terms of a general correlation function. We were then able to generalize the problem to the n^{th} moment by

simply using a recursive relation between the n^{th} and $(n-1)^{th}$ moment. We found that using this method of introducing distribution functions was long winded and some what unnecessary but it did re-create the known result of the disordered averaged first-passage time and is a good illustration of the techniques used in the proceeding chapter. We then followed the arguments of [15] and derived the form of the probability distribution for the general correlation function. We used this to calculate the mean-square displacement in terms of the general correlation function and then more specifically for the case of $\alpha > 0$. We showed that for $\alpha = 1$ we arrive at the first of many derivations of the known result of Sinai diffusion $\langle x^2(t) \rangle \approx \ln^4(t)$. In addition to this we followed the arguments of [11, 15] and derived the mean velocity for the general form of the correlation function and then for four separate cases. We showed that for Sinai's diffusion the model exhibits algebraic distributions of waiting time times giving rise to aging phenomena mimicking essential aspects of the spin glass behaviour, a result we showed again when we solved the Fokker-Planck equation for short-ranged delta function correlations. For creep motion we noticed that the characteristic barriers that control the dynamics diverge as $1/f^{\mu}$ which is the characteristic feature of glassy dynamics. We have found for logarithmic correlations that they act as the transition between creep dynamics and the ohmic state.

The most successful work in this thesis came in chapter four where we solved the Fokker-Planck equation for short ranged delta function correlations with drift, the Sinai model. We used the methods developed for the seemingly different problem of Anderson localization to develop results for two disordered quantities. We began by considering a statistical analysis of the problem to make predictions about various properties of the system. By introducing dimensionless quantities we found that the single parameter that controlled the physics of the problem was μ, the ratio of drift velocity to the strength of disorder. These scaling arguments showed several length and time scales that separate regions of diffusive behaviour, for example we have shown that there is a time scale for which the particle in insensitive to the disordered nature of the environment something which our analytic results have shown.

The first disorder averaged quantity we calculated was that of the return probability, which as we have seen is simply the probability of the particle returning to the origin of its random walk. The method we used to calculate this quantity was to solve the Fokker-Planck

equation with a Green's function that has random functions acting to the left and right. We then wrote these random functions in terms of a distribution function and then derived a Fokker-Planck equation for each one. To actually find a solution to these Fokker-Planck equations we had to assume stationary solutions as we were only interested in infinite sized systems. This led to two normalized solutions which we used to solve for the expectation value of the return probability. We have seen in the short time limit of this result that the probability distribution follows that of pure diffusion, showing that there is a time scale where the particle is insensitive to the disordered nature of the environment as predicted by scaling arguments. For the long time equilibrium behaviour we found several more interesting regions depending on the value of drift velocity. For the case of zero drift we found that the probability distribution decays as the square of the logarithm. This is much slower than pure diffusion and we see that the effect of disorder is to destroy the diffusive motion and slow the particles motion. The reason for this is clear when we consider the process going on. We have seen that a particle diffusing in some random potential is like a Lattice Hopping model, with the particle hopping from from site to another. We have also seen that each site acts like a well trapping the particle and the time it takes for the particle to escape is dependent on the strength of disorder. We have seen that the introduction of drift drives the particle away and this is seen by the return probability becoming a power law. We have also shown that the return probability can be written as a general sum of the form $\langle P(\epsilon)\rangle = \sum_{m=1}^{[\mu]} c_m \epsilon^{m-1} + c_\mu \epsilon^{\mu-1}$ a result which turned out to be of great importance when it came to calculate the probability of finding the particle at some arbitrary distance away from its origin. We have also considered the moments of the return probability and showed that $\langle [P(0,t)]^n\rangle \neq \langle P(0,t)\rangle^n$ which is to say say that $P(0,t)$ is not a self-averaging quantity in the usual sense in agreement with [14].

The second disordered quantity of interest was the probability of finding the particle at some arbitrary distance away from the origin. We solved for this quantity following similar arguments to those we used for the return probability, *i.e.* introduce two distribution functions and then derive a Fokker-Planck equation for each one. On doing this we arrived at a Schrödinger equation which showed two regions of interest depending on the value of the drift velocity. For the case of zero drift velocity we found the Schrödinger potential had

only quasi-continuous states, whereas with the introduction of drift came the emergence of a localized solution with discrete states something that was missed by previous attempts to solve the problem. For the case of zero drift velocity we solved in the quasi-continuous region of the potential to find two asymptotic solutions in overlapping regions. By asymptotically matching the two solutions in this region of overlap we were able to derive the eigen-energy of the Fokker-Planck equation of the distribution function of interest. The result for the eigen-energy was in agreement with the known result of Kesten. We used this result and the the asymptotic solutions of the distribution functions to derive the form of the expectation value of the probability distribution to show that it is agreement with known results of [14, 18]. Using this result we were able to derive the mean-square displacement which gave a result of the form $\langle x^2(t)\rangle = \frac{61}{360}\ln^4(t)$, in agreement with Sinai's result [3]. Calculating the diffusion coefficient for this case gave $D = 0$ which as we have seen shows the effect of disorder is destroy the diffusion process and the particles motion becomes sub-diffusive *i.e.* slower than normal diffusion. We also found that all higher cumulants of the distribution were zero. With the introduction of drift we found the emergence of a gap in the potential with discrete states. The difference this time was that we were not simply allowed to asymptotically match the two solutions in the region of overlap because the two solutions are of different orders to each order and this was the reason why we had to introduce a perturbation series. By solving to each order in the perturbation series and then matching the two solutions we were able to find the eigen-energy of the Fokker-Planck equation describing the distribution functions which we found had the general form $E = \sum_{m=1}^{[\mu]} c_m\epsilon^m + c_\mu\epsilon^\mu$ which was identical to what we found for the return probability and we therefore made the conjecture that $E = \epsilon\langle P(\epsilon)\rangle$. This result was predicted in [14] but was not proved. This then allowed us to solve for the expectation value of the probability distribution and we found a mixture of diffusive behaviours. For example we found for $0 < \mu < 1$ the distribution function had the form of Lévy flight, as did the distribution in the region $1 < \mu < 2$. The most interesting results came about when we considered the transport properties of the particle.

On calculating the cumulants and the moments of the distribution we found several results of interest. For drift velocity in the range $0 < \mu < 1$ we found that the velocity coefficient was zero for all values of μ. We found that the behaviour of the higher moments was

slightly more unusual. We found that the higher cumulants of the distribution were either zero or infinite depending on the moment being considered. We found that the separation between zero and infinity occurs when $\mu = \frac{1}{n}$, where n is the n^{th} moment being considered. This behaviour only occurs when we consider $0 < \mu < 1$. These results, whilst unusual, were in agreement with those found by scaling arguments in [14] as was the result we found for the mean-square displacement. We were also able to re-create the region showing the separation between drift and disorder controlled diffusion. For this case the particles motion is known as anomalous drift. We found when considering $1 < \mu < 2$, the particle acquires some finite velocity, something we expected from the work done on the first-passage time distribution. On calculating all other moments we found that they were all infinite, including that of the diffusion coefficient and we say the particle is undergoing anomalous diffusion. We have also seen that the particles motion has gone from being sub-diffusive for $0 < \mu < \frac{1}{2}$ to super-diffusive or ballistic for $\frac{1}{2} < \mu < 2$, which is somewhat surprising, but in agreement with the results of [14]. We did find one result for the second cumulant $\Xi_2 = \langle x^2(t)\rangle - \langle x(t)\rangle^2$ which was in disagreement with that of [14]. We have shown that the second cumulant Ξ_2 should scale as $t^{3-\mu}$ compared to the result $t^{\frac{2}{\mu}}$ which was calculated from scaling arguments. The result which we have found would seem to be the more physically appropriate result as drift is only a linear effect and we would expect a linear response as we have seen with previous results. When we considered the region $2 < \mu < 3$ we found the first two cumulants of the distribution were non-zero *i.e.* the particle has some velocity and a finite non-zero value for the diffusion coefficient. We therefore see the particle has become diffusive again. On calculating the moments for further values of drift we found that for drift in the region $m < \mu < m + 1$ the first m moments are non-zero, with the rest being infinite. It should be noted that the conjecture we made regarding the energy E being given by the return probability has not been rigorously proven and any future work should be directed along the path of showing this result. If this could be done we would be able to conclude that just knowing the return probability is enough to know all of the transport properties of the particle. This would generalize the result of [44] where they showed that the form of the diffusion and velocity coefficients could be found from the expansion of the return probability upto first order in epsilon.

In the following chapter we introduced the idea of Grassmann numbers, functional integration and the renormalization group technique, the tools that were required to calculate the transport properties of an over-damped particle in a random environment with logarithmic correlations. We began by motivating the use of Grassmann numbers which as we have seen are useful if we want to re-write a Green's function in a form where instead of using fermionic operators with their complicated anti-commutation relations we use something else that has simpler anti-commutation rules. We derived several properties of Grassmann algebra including the resolution of the identity, integration and differentiation. The resolution of the identity allowed us to derive the form of the functional integral. With Grassmann numbers and functional integration understood we were able to consider the renormalization group theory. For renormalization group theory we see that the action must be scale invariant, which means the action remains unchanged under the re-scaling of momentum. We follow a three step recipe to find the Gell-Mann Low equation which as we know describes how the different parameters act under the renormalization procedure. The first step, as we have seen, is the sub-division of fields into slow and fast modes, with the second step performing the averaging over the fast modes. This averaging over fast modes leads to a new action defined on different kinematic scales with a set of modified parameters. This new action can only be compared to the old action if they are both on the same kinematic scales and to achieve this the new action needs to be re-scaled. This re-scaling is the final step of the renormalization recipe and it allows us to find how the parameters change as the cut-off λ changes, the Gell-Mann Low equations.

For the final chapter of the thesis we solved the Fokker-Planck equation with logarithmic correlations using the renormalization and functional renormalization group approach we described in the previous chapter. Although the results for such a problem have previously been calculated the aim of the work was to re-create all known results using the functional renormalization group approach in the hope that we would find a general method of dealing with such one-dimensional problems. The discussion began by considering the first-passage time distribution with logarithmic correlations and we showed the existence of a freezing point transition using a simple saddle point approximation. We then argued that a transition of this form could only be described by logarithmic correlations and we

did this by considering the competition between the entropy of the typical sites and the low energy states. A discussion of this form showed that short ranged correlations would only describe a high temperature phase with no phase transition and long ranged correlations would only have a low temperature phase, again without a phase transition. We then gave a description of the equivalent model in two-dimensions, where we considered two specific vector constraints on its potential. We have seen by considering transverse disorder which describes hopping conduction in a two-dimensional medium with charged impurities the particle shows super-diffusive behaviour due to its ballistic motion along lines of drift. For the second model where we considered longitudinal disorder we found it described a particle traveling in a compressible liquid with random non circulatory flows and it showed sub-diffusive behaviour due to presence of sources and sinks, where the sinks act as traps.

The first result we attempted to derive was that of the mean-square displacement. We did this using the one-loop approximation where we had to consider how both the diffusion and strength of disorder coefficient scaled under the renormalization process. This re-scaling showed that both coefficients were irrelevant processes as their Beta function was negative. On solving the Gell-Mann Low equation for the diffusion coefficient we found that the particles motion was sub-diffusive, as to be expected for a one-dimensional diffusion problem with disorder. Using this result we were able to calculate the form of the mean-square displacement, $\langle x^2(t)\rangle = t^{\frac{2}{z}}$ where $z = 2 + 2g_0$, which was in agreement with the known results of [5]. In an attempt to find the freezing transition in the first-passage time distribution we were required to consider fluctuations in the diffusion coefficient, what we called the higher cumulants. The higher cumulants were found to only provide corrections to themselves under the renormalization process and not to the diffusion coefficient or the strength of disorder. We used the Gell-Mann Low equation of the higher cumulants to derive the form of a functional renormalization group equation which described the cumulants of the distribution. This functional renormalization group equation was then solved to find that the particles distribution had the form of a particle diffusing with some drift whilst being smeared over time. This picture was shown to be misleading when we used its solution to re-derive the first-passage time distribution. On calculating the first-passage time distribution we found the existence of a freezing transition and we even found the correct exponent for the lower

limit, but unfortunately the method did not re-create exactly the correct exponent for the higher limit. In an attempt to fix the incorrect exponent we derived an equation for the non-linear contributions to the higher cumulants, from which we derived a new functional renormalization group equation. Unfortunately we have not been able to solve this equation and thus can not say if it will give the correct cumulant. Solving this equation and then possibly finding the correct exponent will provide the basis for future work based on this problem. Alternatively if the method does not return the correct exponent, then further work should be planned to find a method to deal with such one-dimensional problems.

Appendix A

THE FURUTSU-NOVIKOV FORMULISM

Throughout this thesis we use an extremely useful Gaussian averaging theorem which is know as the Furutsu-Novikov formula or "Gaussian-integration". As we have previously stated, when we consider a zero mean Gaussian process $u(t)$ with the correlation function $K(t,t')$ the result of Furutsu-Novikov states that

$$\langle u(t)R[u]\rangle = \int [du]u(t)R[u]\exp\left[-\frac{1}{2}\int dt'dt''u(t')K(t',t'')u(t'')\right] = \int dt'K(t,t')\left\langle\frac{\delta R[u]}{\delta u(t')}\right\rangle.$$

In this appendix we will sketch a derivation of this result. We begin the derivation by considering the more simple example of a Gaussian distribution with a delta function distribution, *i.e.* $K(t',t'') = \Omega^{-1}\delta(t'-t'')$. For this example the averaging becomes

$$\langle u(t)R[u]\rangle = \int [du]u(t)R[u]\exp\left[-\frac{1}{2\Omega}\int_{-\infty}^{\infty}u^2(t')dt'\right]. \tag{A.0.1}$$

We notice that we can write

$$\Omega^{-1}u(t)\exp\left[-\frac{1}{2\Omega}\int_{-\infty}^{\infty}u^2(t')dt'\right] = -\frac{\delta}{\delta u(t)}\exp\left[-\frac{1}{2\Omega}\int_{-\infty}^{\infty}u^2(t')dt'\right], \tag{A.0.2}$$

and we find the average becomes

$$\langle u(t)R[u]\rangle = -\Omega \int [du]R[u]\frac{\delta}{\delta u(t)}\exp\left[-\frac{1}{2\Omega}\int_{-\infty}^{\infty}u^2(t')dt'\right], \tag{A.0.3}$$

and by integrating by parts once we find

$$\langle u(t)R[u]\rangle = \Omega\left\langle \frac{\delta R[u]}{\delta u(t)}\right\rangle. \tag{A.0.4}$$

We can generalize this result to an arbitrary Gaussian distribution with a correlation function of the form $K(t,t')$. For this arbitrary case the average becomes

$$\langle u(t)R[u]\rangle = \int [du]u(t)R[u]\exp\left[-\frac{1}{2}\int dt'dt''u(t')K(t',t'')u(t'')\right], \tag{A.0.5}$$

and by following the same arguments as above we find

$$\langle u(t)R[u]\rangle = \int dt'K(t,t')\int [du]\exp\left[-\frac{1}{2}\int dt'dt''u(t')K(t',t'')u(t'')\right]\frac{\delta R[u]}{\delta u(t')}, \tag{A.0.6}$$

and hence we find the required result of

$$\langle u(t)R[u]\rangle = \int dt'K(t,t')\left\langle \frac{\delta R[u]}{\delta u(t')}\right\rangle. \tag{A.0.7}$$

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