SYMMETRIC AND SEMISYMMETRIC GRAPHS

by

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Abstract

This thesis is an investigation into symmetric and semisymmetric graphs of prime valency. Our approach is via amalgams of groups and coverings of such graphs by trees. We develop theoretical and computational methods to inform this problem. In the case of symmetric graphs of valency five we find that there are twenty five finite faithful amalgams, in the case of semisymmetric graphs of valency five we find there are one hundred and five finite faithful amalgams. We determine presentations for the universal completions of such amalgams and find completions in finite groups.

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CONTENTS

INTRODUCTION

Algebraic graph theory is the study of graphs with tools from algebra. In the context of this thesis, those tools are group theoretic. Far from being an application of group theory however, this also presents us with an ability to study groups by their action on graphs. The benefit of the latter is that graphs are (or at least appear to be) simple objects, whilst groups are inherently complex. The sharing of knowledge between these two areas has influenced the development of both, and perhaps more importantly, has provided many deep problems for study. An example is to classify the pairs (Γ, G) where Γ is a graph with certain properties and G is a group acting on Γ in a certain fashion. In this thesis our aim is to investigate such pairs when Γ has prime valency and the action of G is edge-transitive.

If a graph Γ (with no isolated vertices) admits a group G acting transitively on edges, then there are at most two orbits on the vertices of Γ. In the case that there are exactly two, we say that Γ is (G-)*semisymmetric*. If there is one orbit on the vertices, then we consider the action on the arcs of Γ. An *arc* is an ordered pair of adjacent vertices. If G acts transitively on the arcs, then we say that Γ is (G-)*symmetric*, otherwise, Γ is called $\frac{1}{2}$-*arc transitive*. A result of Tutte says that if Γ is $\frac{1}{2}$-arc transitive, then Γ (which must be regular) has even valency (see Proposition 2.1.1). Since the components of regular graphs of valency one are just edges and the components of regular graphs of valency two are circuits, it makes sense to concentrate on symmetric graphs with valency at least three. More generally, for $s \geq 2$, an s-arc is an ordered sequence of vertices such that every successive pair is an arc and every three successive vertices are all distinct. We say that a graph Γ is (G, s)-*transitive* (for $G \leq \mathrm{Aut}(\Gamma)$) if s is the largest integer such that G acts transitively on the set of s-arcs of Γ. Thus G-symmetric graphs are (G, s)-transitive for some $s \geq 1$, and every (G, s)-transitive graph with no vertices of valency 1 is G-symmetric. For a semisymmetric graph Γ, Γ is bi-regular with valencies k and l say. If $l = 1$ then the components of Γ are k-stars. If $k = l = 2$ then the components of Γ are circuits. Hence for semisymmetric graphs we usually assume that $k \geq 3$ and $l \geq 2$. A graph Γ is called *locally* (G, s)-*transitive* if s is the largest integer such that for each vertex $x \in \Gamma$ the stabiliser in G of x acts transitively on the set of s-arcs with initial vertex x. We will see (Lemma 2.1.2) that locally (G, s)-transitive graphs with $s \geq 1$ are G-semisymmetric.

Finite semisymmetric graphs for which $k = 3 = l$ were studied by Goldschmidt in the seminal paper [17]. In this paper Goldschmidt developed a method to study this problem using amalgams of groups and Serre's [34] covering theory of graphs by trees. This proceeds as follows (a full discussion can be found in Chapter 2). To a finite G-semisymmetric graph Γ we associate an *amalgam* $\mathcal{A}$ formed by a pair of vertex stabilisers G_x and G_y such that $\{x, y\}$ is an edge of the graph. The degree of the amalgam is the pair of indices $(|G_x : G_{xy}|, |G_y : G_{xy}|)$ where $G_{xy} = G_x \cap G_y$. As we shall see in Lemma 2.1.2, this is equal to (k, l), the bi-valency of the graph. Furthermore, we see $G = \langle G_x, G_y \rangle$ so G is a completion for $\mathcal{A}$ (see Definition 1.6.2). For a completion X of $\mathcal{A}$ we can consider the coset graph $\Gamma(\mathcal{A}, X)$ of $\mathcal{A}$ with respect to X (see Definition 1.6.4). This has vertex set $X/G_x \cup X/G_y$ and edges between two cosets $G_x g$ and $G_y h$ if and only if $G_x g \cap G_y h \neq \emptyset$. Lemma 2.1.6 shows that $\Gamma \cong \Gamma(\mathcal{A}, G)$ and therefore implies that every semisymmetric graph of bi-valency (k, l) is the coset graph of an amalgam of degree (k, l) with respect to some completion. Thus our focus will be upon first classifying amalgams of degree (k, l) and then to try and understand their completions. Our method involves considering the coset graph $\widetilde{\Gamma} = \Gamma(\mathcal{A}, \mathcal{G}(\mathcal{A}))$, where $\mathcal{G}(\mathcal{A})$ is the universal completion of the amalgam (see Section 1.6). The graph $\widetilde{\Gamma}$ is a tree, moreover the subgroup $\mathcal{G}(\mathcal{A})$ fixing a vertex of $\widetilde{\Gamma}$ is equal to either G_x or G_y, so $\mathcal{G}(\mathcal{A})$ is a *locally finite* subgroup of $\mathrm{Aut}(\widetilde{\Gamma})$. By studying the action of $\mathcal{G}(\mathcal{A})$ on $\widetilde{\Gamma}$, properties and structural information of the groups G_x and G_y can be found. This is "local" information which, when combined with knowledge of

$\mathcal{G}(\mathcal{A})$, enables us to recover G and therefore Γ. In this approach the "local actions" $G_x^{\Delta(x)}$ and $G_y^{\Delta(y)}$ (the permutation groups induced by G_x and G_y on the neighbours of x and y respectively) has considerable influence. We will see in Chapter 2 these are transitive groups and for the particular valencies that we are interested in we understand their structure very well (see Section 1.4). We gain information about the graph Γ in this way since the local action on the graph Γ and on the tree are the same. For $k = 3 = l$ we have $|\Delta(x)| = 3$ for every $x \in \Gamma$ and so the group $G_x^{\Delta(x)}$ is either cyclic of order three or the full symmetric group of degree 3.

One of the many interesting results from Goldschmidt's paper is that the automorphism group of the amalgam $\mathcal{A}$ arising from a semisymmetric graph Γ embeds into the full automorphism group of the tree $\Gamma(\mathcal{A}, \mathcal{G}(\mathcal{A}))$. In fact, the image is the full normaliser of $\mathcal{G}(\mathcal{A})$ in the automorphism group of the tree, a group which is considerably more complex than $\mathcal{G}(\mathcal{A})$ (for example, it is uncountable). We offer a constructive proof of this result in Section 2.2 in which we explicitly show how this embedding arises. Moreover our approach allows us to make the process computational and we have designed computer programs in MAGMA to facilitate this (see Sections A.1 and A.2). In particular, we are able to calculate the possible extensions of the group G inside the automorphism group of the tree using only local information, this is the Extension Theorem 2.2.25. Of course this can only be done in this way, since we cannot compute with the automorphism group of the tree itself. We also extend a result of Goldschmidt's from the valency three case to arbitrary primes. Theorem 2.4.2 shows that in the semisymmetric case there exists a certain minimal simple amalgam. Together with the Extension Theorem, this would allow us to compute all amalgams of degree (p, p) after finding the simple amalgams. We use this in our investigation into semisymmetric graphs of valency five.

For semisymmetric graphs of valency three the main result of [17] is that, up to isomorphism, there are precisely fifteen amalgams which arise from finite semisymmetric graphs of valency three. Such an amalgam is called *faithful* and is finite in the sense that it is formed of finite groups. The degree of the amalgam is the pair (k, l). Results prior to Goldschmidt's were of a more combinatorial flavour. Examples of this are results of Tutte [49, 50] for amalgams of degree $(3, 2)$ (which arise from symmetric graphs of valency three). In particular, here we have that the stabiliser in G of an edge has order dividing 16. Using Tutte's results, Djoković and Miller [12] classified the possible amalgams that arise and found that there are precisely seven. Perhaps motivated by these two results, Goldschmidt made the following conjecture (for the terminology see Section 2.2).

Conjecture (Goldschmidt, 1980)**.** *Suppose that k and l are primes. Then (up to isomorphism) there are finitely many finite faithful amalgams of degree (k, l).*

The graph theoretic version of this conjecture is the following.

Conjecture. *Up to conjugacy in $\mathrm{Aut}(\Gamma)$, there are finitely many locally finite edge-transitive subgroups of $\mathrm{Aut}(\Gamma)$ where Γ is a bi-regular tree with prime valencies.*

Results of Djokovic [11] and Bass, Kulkarni [3] show that there are infinitely many isomorphism classes of amalgams with composite degree (that is, at least one of k or l composite). Conversely, evidence for the validity of above conjecture was provided first by Rowley [33] and then by Fan [13]. Both of these results were proved under an assumption which implies that the local action is either dihedral or Frobenius and that the edge stabiliser is a p-group for some prime p with $k \neq p \neq l$. Using a different approach Djoković [11] showed that for certain classes of primes there exists a bound on the number of finite faithful amalgams (the methods involved permutation group theory). In the same paper, he proposes the harder problem of classifying the actual isomorphism types of the amalgams. In Section 1.4 we use the Aschbacher-O'Nan-Scott theorem together with theorems of Burnside and Cameron to gain some control over the local action that can occur in this situation. We find that the local action is either affine or on a list which is delivered by Lemma 1.4.5. This list shows that in general the local action will be affine or linear. Three exceptions occur, which are valency five, valency eleven and valency twenty-three. In this thesis we consider the valency five symmetric and semisymmetric cases. Briefly, our main theorems are below, consult Theorem 3.0.5, Theorem 4.0.1 and

Theorem 4.0.2 for more technical statements. The following theorem is connected to the symmetric case.

Theorem. *Suppose that $\mathcal{A}$ is a finite faithful amalgam of degree $(5, 2)$. Then $\mathcal{A}$ is isomorphic to one of twenty-five such amalgams, the types of which are listed in Table 3.1.*

The next theorem applies to the semisymmetric case.

Theorem. *Suppose that $\mathcal{A}$ is a finite faithful amalgam of degree $(5, 5)$. Then $\mathcal{A}$ isomorphic to one of one hundred and five such amalgams, the types of which are listed in Table 4.1 and Tables 4.3-4.7.*

Our theorems are informed by two results which we now describe. The first concerns (G, s)-transitive graphs. The earlier mentioned result of Tutte [49] shows that $s \leq 5$ when Γ is regular of valency three. Weiss extended this result to regular graphs of arbitrary valency and showed $s \leq 7$ [52] (although this result depends on the Classification of Finite Simple Groups (CFSG) by using Cameron's classification of 2-transitive groups). For graphs of small valency Weiss determined a presentation for the group G. The following theorem is contained in Theorem 2 of [54] (and because of the restriction on the valency, is independent of the CFSG).

Theorem (Weiss)**.** *Suppose that Γ is a graph of valency five and which is (G, s)-transitive for $s \in \{4, 5\}$. Then G admits one of the presentations given in Table 3.3.*

In Chapter 3 we explain the contribution of the above theorem to our investigation. It remains to classify the groups which occur for $s \leq 3$ which we do in the remainder of Chapter 3. A result of Gardiner [16] helps to show that the groups involved in the amalgam must be "small". After some careful analysis of the possible configurations, we arrive at the list in Table 3.1. Having compiled the list, we derive presentations for the universal completions of these amalgams which we present in Tables 3.2 and 3.3.

For the semisymmetric case we invoke the Thompson-Wielandt theorem (proved in Section 2.5) to make a case division. In the first case we make use of a deep theorem of

Stellmacher and Delgado, the proof of which forms [10]. This technical result requires several definitions, which we make in Section 1.6. Roughly speaking, the result tells us that the amalgam resembles an amalgam which comes from a generalised n-gon. A generalised n-gon is a graph with diameter n and girth $2n$; a theorem of Feit and Higman [15] shows that finite generalised n-gons exist only for $n \in \{2, 3, 4, 6, 8\}$. Generalised n-gons can be constructed from vector spaces. For an example let W be a 3-dimensional vector space over the field with four elements. Let $\mathcal{V}$ be the set of proper non-trivial subspaces of W. Let $\mathcal{E}$ consist of sets of the form $\{V_1, V_2\}$ where V_i has dimension i and $V_1 \leq V_2$. Then the graph $\Gamma = (\mathcal{V}, \mathcal{E})$ has 42 vertices and each vertex has 5 neighbours. The group $G = \mathrm{PSL}_3(4)$ acts edge-transitively on Γ, so Γ is a semisymmetric graph of valency five. The diameter of Γ is 3 and the girth of Γ is 6, so Γ is an example of a generalised 3-gon. The generalised n-gons that occur as examples of semisymmetric graphs of valency five have $n \in \{3, 4, 6\}$. The amalgam to which $\mathrm{PSL}_3(4)$ gives rise is the amalgam we denote by $\mathcal{S}_{13}$ in Table 4.1.

The second case is similar to some of our work on the symmetric case in that the groups involved in the amalgam are small, but there are more configurations here. The third case shows how the amalgams arising from semisymmetric graphs differ from those that come from symmetric graphs - the vertex stabilisers G_x and G_y have rather different structures and actions. The group G_x acts faithfully on vertices at distance two from x, whereas the group G_y does not act faithfully on the vertices at distance two from y. To make progress on this case we concentrate on the simple amalgam which we obtain from Theorem 2.4.2. Then using information about GF(2)-modules for the group $G_x^{\Delta(x)}$ we gain control over G_x. This allows us to determine G_y. Having found the amalgams, we also determine presentations for their universal completions. We expect these presentations will be useful to anyone wanting to compute with semisymmetric graphs of valency five or with the amalgams.

The tools that we have developed to deal with the case of valency five are certainly applicable to the valency eleven and valency twenty-three cases. It is expected that the amalgams which will appear will be similar to the amalgams $\mathcal{S}_1$-$\mathcal{S}_{12}$, but that there will be no analogue of the amalgams $\mathcal{S}_{13}$-$\mathcal{S}_{15}$ since there are no groups of Lie type which produce a valency eleven or valency twenty-three graph in the way that the generalised n-gons arise for valency five. For the semisymmetric case of valency p with $p \notin \{5, 11, 23\}$ we expect that this is a rather more difficult problem. The presence of local action which involves a linear group considerably increases the complexity of the problem. Firstly, this will force us to consider the pushing up problem for a large (rank greater than 3) linear group (a problem unsolved in full generality). Secondly, the examples of generalised 3-gons of valency $p = r^d + \cdots + r + 1$ show that one will eventually have to identify a group such as $\mathrm{PSL}_{d+1}(r)$. On the other hand, Stellmacher announced in Siena in 1996 that $s \leq 9$ for locally (G, s)-transitive graphs. The proof of this bound will contribute to our understanding of the situation. Considering the conjecture for semisymmetric graphs of valency p and q with $p \neq q$ is again a more difficult problem, there is no minimal simple amalgam present and the two local actions could be quite different. We expect amalgams with similar properties to $\mathcal{S}_5$-$\mathcal{S}_{12}$ appear (indeed, considering these amalgams one can already write down examples).

A generalisation of Goldschmidt's conjecture is to replace the assumption of prime valency with locally primitive action. For the symmetric case, this is the Weiss Conjecture [53] which is still open. See [31] for recent progress.

Conjecture (Weiss, 1979)**.** *Let Γ be a regular graph of valency k and $G \leq \mathrm{Aut}(\Gamma)$ be vertex-transitive. Suppose that for every vertex $x \in \Gamma$ the group $G_x^{\Delta(x)}$ is primitive. Then there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $|G_x| \leq f(k)$ for all $x \in \Gamma$.*

A graph satisfying the hypothesis of the above conjecture is (G, s)-transitive for some $s \geq 1$. Under the assumption that $s \geq 2$, the conjecture has been shown to be true. We observe that a (G, s)-transitive graph Γ with $s \geq 2$ has the following property: for every $x \in \Gamma$ the group $G_x^{\Delta(x)}$ is 2-transitive. Therefore Cameron's list of 2-transitive groups has an impact on this problem, and draws attention to the locally linear case. Under this assumption, a deep theorem of Trofimov shows that the conjecture holds. The proof of this theorem begins with [38] and [39], and is then continued in the two series [40, 41] and [42, 43, 44, 45] and the summary [46]. For the general case with $s \geq 2$, Trofimov and Weiss [47, 48] show that the above conjecture holds. It remains to see what can be said in the $s = 1$ case, and indeed what can be said in the more general situation of semisymmetric graphs. Therefore there are several interesting avenues for further research here, it would be extremely interesting to see the development of this theory.

Notation. We write C_n (or just n) for the cyclic group of order n and if $m > 1$ we write n^m for the direct product of m cyclic groups each of order n. We write $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ for the symmetric and alternating groups of degree n. We write $\mathrm{Dih}(2n)$ for the dihedral group of order $2n$ and $\mathrm{Frob}(nk)$ for the Frobenius group with Frobenius kernel of order n and complement of order k (e.g. $\mathrm{Frob}(20)$ where it is clear that the Frobenius kernel has order 5). For the finite simple groups we prefer to give them longer names, for example we write $\mathrm{PSL}_n(q)$ instead of $\mathrm{L}_n(q)$. We write $G \sim A.B$ for the *shape* of G, so G has a normal subgroup A such that G/A is isomorphic to B, this does not describe the isomorphism type of G. We write the semidirect product of A and B via the homomorphism $\pi : B \to \mathrm{Aut}(A)$ as $A \rtimes_\pi B$ (or just $A : B$ if the homomorphism is clear). Direct products we write as $A \times B$ and central products as $A \circ B$. If H and K are two subgroups with unique index two subgroups H_1 and K_1 respectively, then by $H \curlywedge K$ we denote the index two subgroup of $H \times K$ which contains $H_1 \times K_1$ but neither H nor K. We also follow standard notation as used in [9, 18].

CHAPTER 1

PRELIMINARIES

1.1 Group actions

This thesis is concerned with actions of groups on graphs. We begin therefore with the definitions of group actions. Throughout this section G is an arbitrary finite group with subgroups H and K and normal subgroups N and M. By 1 we denote the identity element in G.

Definition 1.1.1. An *action* of G on a set Ω is a map $\pi : \Omega \times G \to \Omega$ such that for all $a \in \Omega$ we have

(i) $\pi(a, 1) = a$ and

(ii) $\pi(a, gh) = \pi(\pi(a, g), h)$ for all $g, h \in G$.

We will suppress the map π and write a^g for the element $\pi(a, g)$. An action of G on a graph Γ is an action on the vertex set of Γ which preserves the edge set of Γ. An action of G on a group A is an action on the elements of A such that for all $g \in G$ and all $a, b \in A$ we have $(ab)^g = a^g b^g$.

There is an equivalence between actions of a group G on a set Ω, a graph Γ, a group A and homomorphisms from G to $\mathrm{Sym}(\Omega)$, $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(A)$ respectively. We adopt whichever viewpoint is convenient and if R is the kernel of the mentioned homomorphism, we say that G acts with kernel R. If G acts with kernel G we say the action is *trivial* and if G acts with trivial kernel the action is called *faithful*. If G acts faithfully on a set Ω, we say that G is a *permutation group* on Ω. For $\omega \in \Omega$ we write

$$G_w = \mathrm{Stab}_G(\omega) = \{g \in G \mid \omega^g = \omega\}.$$

In the particular case of a group G acting on another group A, we can always view this as occurring in the semidirect product $A : G$.

Example 1.1.2. *The following are examples of actions of G,*

(i) on the left/right cosets of H by left/right multiplication,

(ii) on the elements of N by conjugation,

(iii) on the elements/subgroups of G in a conjugacy class of elements/subgroups of G by conjugation.

Included in (ii) of the above is the example of G acting on G by conjugation. The *orbits* of the action of G are the equivalence classes of Ω under the relation $a \sim b$ if and only if there is $g \in G$ such that $a^g = b$. The action of G is *transitive* if there is only one orbit. The idea of the following proof is known as the Frattini argument and it will be applied frequently in this thesis.

Proposition 1.1.3. *Suppose that G acts transitively on Ω. Then H is transitive on Ω if and only if $G = HG_\omega$ for some $\omega \in \Omega$.*

Proof. Suppose that H is transitive on Ω and let $g \in G$, $\omega \in \Omega$. Since H is transitive, there exists $h \in H$ such that $\omega^g = \omega^h$, whence $hg^{-1} \in G_\omega$ and $g \in HG_\omega$.

Now suppose that $G = HG_\omega$. For each $\beta \in \Omega$ there exists $g \in G$ such that $\omega^g = \beta$. Writing $g = xh$ for some $h \in H$ and $x \in G_\omega$ we have $\omega^g = \omega^h = \beta$, hence H is transitive on Ω. □

We refer to Proposition 1.1.3 as the Frattini Argument. Note that the finiteness of G is not used in the proof and no assumption is made on the size of Ω, thus we may use the statement for infinite groups acting on infinite sets (and do in Theorem 2.4.2). Since both G and N act transitively on the Sylow p-subgroups of N, if S is a non-trivial Sylow p-subgroup of N for some prime p we obtain the factorisation $G = \mathrm{N}_G(S)N$ (which is the usual formulation of the Frattini argument). From now on we also fix a finite group A on which G acts (perhaps $A = G$ with the conjugation action). For a subgroup B of A the *normaliser* in G of B is $\mathrm{N}_G(B) = \{g \in G \mid B^g = B\}$ and the *centraliser* in G of B is $\mathrm{C}_G(B) = \{g \in G \mid b^g = b \text{ for all } b \in B\}$. Note that $\mathrm{N}_G(B)$ acts on B by conjugation with kernel $\mathrm{C}_G(B)$.

For elements x, y of G the commutator $[x, y]$ of x and y is $x^{-1}y^{-1}xy$. The group $[H, K]$ is the subgroup generated by the commutators $[h, k]$ for $h \in H$ and $k \in K$.

Proposition 1.1.4. *Let B be a subgroup of A. The following hold,*

(i) G normalises B if and only if $[B, G] \leq B$,

(ii) if A and G normalise B then A and G normalise $[B, G]$.

Proof. This follows from commutator relations (found in [1, pg.27] for example). □

This is an appropriate time to mention a consequence concerning commutators. The following lemma is usually employed with $N = 1$.

Lemma 1.1.5 (Three subgroups lemma)**.** *Let X, Y and Z be subgroups of G. Suppose that $[X, Y, Z] \leq N$ and $[Y, Z, X] \leq N$. Then $[Z, X, Y] = N$.*

Proof. See [36, pg.6]. □

A *section* of G is a quotient H/L where L is a normal subgroup of H. The action of G on A induces an action of G on the set of sections of A (naturally $(B/C)^g = B^g/C^g$). A section B/C is G-invariant if G normalises B and C and in this case we can make G act on B/C by defining $(bC)^g = b^gC$ for $b \in B$, $g \in G$. This action is well defined: if $bC = b'C$ then $(b^g)^{-1}(b'^g) = (b^{-1}b')^g \in C^g = C$, so $(bC)^g = (b'C)^g$.

Proposition 1.1.6. *Let B be a normal G-invariant subgroup of A. The following hold.*

(i) G acts trivially on A/B if and only if $[A, G] \leq B$.

(ii) If G acts trivially on B then G acts trivially on $A/C_A(B)$.

(iii) If G acts trivially on B and A/B then $[A, G] \leq \mathrm{Z}(B)$ and G' centralises A.

Proof. For (i), we have $(aB)^g = aB$ if and only if $a^{-1}a^g = [a, g] \in B$.

For part (ii), we have $[B, G] = 1$, and since B is normal in A, $[B, A] \leq B$ so we have

$$[A, B, G] = 1 = [B, G, A]$$

and the three subgroups lemma implies that $[G, A, B] = 1$, i.e. that $[G, A] \leq \mathrm{C}_A(B)$. Now part (i) implies that G centralises $A/\mathrm{C}_A(B)$.

Suppose now that G acts trivially on B and on A/B. By part (i), we get $[A, G] \leq B$. Part (ii) implies that G acts trivially on $A/\mathrm{C}_A(B)$. Since $\mathrm{C}_A(B)$ is an G-invariant normal subgroup of A, we may apply part (i) to obtain $[A, G] \leq \mathrm{C}_A(B)$. Hence

$$[A, G] \leq B \cap \mathrm{C}_A(B) = \mathrm{Z}(B).$$

Thus $[A, G, G] = [G, A, G] = 1$, and so $[G, G, A] = [G', A] = 1$ which implies $G' \leq \mathrm{C}_G(A)$. $\square$

We say that the action of G on A is coprime (or G acts coprimely on A) if $(|G|, |A|) = 1$.

Lemma 1.1.7 (Coprime action)**.** *Suppose that G acts coprimely on A. The following hold,*

(i) if B is a G-invariant normal subgroup of A, then $C_{A/B}(G) = C_A(G)B/B$,

(ii) $A = C_A(G)[A, G]$ and if A is abelian then $A = C_A(G) \times [A, G]$,

(iii) $[A, G, G] = [A, G]$,

(iv) if B is a G-invariant normal subgroup of A and G acts trivially on B and A/B then G acts trivially on A.

Proof. For (i), see [23, pg.184]. For (ii), set $B = [A, G]$ and then G acts trivially on A/B so that $A/B = \mathrm{C}_{A/B}(G)$. Now (i) implies that $A/B = \mathrm{C}_A(G)B/B$ which gives $A = \mathrm{C}_A(G)B = \mathrm{C}_A(G)[A, G]$ as required.

For part (iii), we apply (ii) to get $[A, G] = [\mathrm{C}_A(G)[A, G], G]$. For $c \in \mathrm{C}_A(G)$, $a \in [A, G]$, $g \in G$ we have $[ca, g] = [c, g]^a[a, g] = [a, g]$ so that $[A, G] = [A, G, G]$ as required. For part (iv) we see that $[A, G, G] \leq [B, G] = 1$ and then (iii) gives the result. $\square$

At the opposite end of the spectrum from coprime action, we have the following.

Proposition 1.1.8. *Suppose that G and $1 \neq A$ are non-trivial p-groups. Then $A > [A, G] > [A, G, G] > \cdots > 1$ and $C_A(G) \neq 1$.*

Proof. Repeated applications of Proposition 1.1.4 (ii) shows that the subgroups $[A, G, \ldots, G]$ are normalised by G and A. It suffices therefore to show that $[A, G] < A$ (note that the final non-trivial term of this series is contained in $\mathrm{C}_A(G)$). We assume the result is false. Considering the semidirect product $X = AG$, we see that $[X, A] = [A, A][G, A] = A$, and so $[A, X, X] = [A, X] = A$. This contradicts the nilpotency of X. $\square$

Definition 1.1.9. We say that G acts *quadratically* on A if $[A, G, G] = 1$ and $[A, G] \neq 1$.

Thus quadratic action of G on A means that G acts trivially on both $[A, G]$ and $A/[A, G]$. Quadratic action will show up in the proof of Theorem 1.5.6.

If we do not fully understand the action of G on A it is best to examine the action of G on sections of A. The following concept provides the "best" set of sections to look at.

Definition 1.1.10. A normal series $1 = A_0 \lhd A_1 \lhd \ldots \lhd A_n = A$ is a G-*chief series of* A if A_{i-1} is a maximal G-invariant subgroup of A_i for $1 \leq i \leq n$. The factor groups A_i/A_{i-1} are called G-chief factors of A.

If A_i/A_{i-1} is a G-chief factor, we see that either $[A_i, G] \leq A_{i-1}$ or $[A_i, G]A_{i-1} = A_i$. We call a chief factor *central* in the former case and *non-central* in the latter.

Lemma 1.1.11. *Let $1 = A_0, \dots, A_n = A$ be a G-chief series for A. For $i = 1, \dots, n$ set $\overline{A_i} = A_i/A_{i-1}$, then*

$$|A/C_A(G)| \geq \prod_{i=1}^{n} |\overline{A_i}/C_{\overline{A_i}}(G)|.$$

Proof. See [30, pg.27]. □

With repeated applications of Lemma 1.1.7 (iv) we obtain the following.

Lemma 1.1.12. *Suppose that G acts coprimely on A and every G-chief factor of A is central. Then G centralises A.*

1.2 Characteristic subgroups

In this section we recall some facts about certain characteristic subgroups that are related to the structure of p-groups. Our aim is to show how the structure of normal p-subgroups of G influences the structure of G itself.

Definition 1.2.1. If G is a p-group for $i \in \mathbb{Z}$, we define $\Omega_i(G) = \langle x \in G \mid x^{p^i} = 1 \rangle$ and $\mho^i(G) = \langle x \in G \mid x \text{ is a } p^i\text{-th power} \rangle$.

We usually abbreviate $\Omega_1(G)$ and $\mho^1(G)$ to $\Omega(G)$ and $\mho(G)$ respectively. If G is abelian then $\Omega(G)$ and $G/\mho(G)$ are elementary abelian. Note that the subgroups defined above are characteristic.

Definition 1.2.2. Let G be a group and let $\Phi(G)$ be the intersection of all the maximal subgroups of G. The group $\Phi(G)$ is called the *Frattini subgroup* of G.

Clearly the Frattini subgroup is characteristic. A useful property of the Frattini subgroup is the following.

Lemma 1.2.3. *Suppose that $H\Phi(G) = G$. Then $G = H$.*

Proof. Let H be as in the hypothesis and assume that $H < G$. Then we may choose M maximal such that $H \leq M$. But $\Phi(G) \leq M$, so $G = H\Phi(G) \leq M < G$, a contradiction. □

The following was the first application of the Frattini argument and hence the reason for its name.

Proposition 1.2.4. *The Frattini subgroup is nilpotent.*

Proof. Let P be a Sylow subgroup of $\Phi(G)$. The Frattini Argument gives $G = \Phi(G)\mathrm{N}_G(P)$ and the previous lemma implies $G = \mathrm{N}_G(P)$. Thus P is normal in $\Phi(G)$. $\square$

If G is a p-group then another characterisation of $\Phi(G)$ is the following: $\Phi(G)$ is the smallest normal subgroup of G such that $G/\Phi(G)$ is elementary abelian. This implies that $\Phi(G) = [G, G]\mho(G)$. Together with a factorisation of G we can calculate $\Phi(G)$ in the following way.

Proposition 1.2.5. *If $G = PQ$ then $\Phi(G) = \Phi(P)\Phi(Q)[P, Q]$.*

The following two subgroups are central to the subject of "local" group theory.

Definition 1.2.6. Let π be a set of primes. A group H is a π-group if H is finite and every prime divisor of $|H|$ belongs to π. A π-subgroup is a subgroup which is a π-group. By $\mathrm{O}_\pi(G)$ we denote the largest normal π-subgroup of G. By $\mathrm{O}^\pi(G)$ we denote the smallest normal subgroup of G such that $G/\mathrm{O}^\pi(G)$ is a π-group.

Observe that $\mathrm{O}_\pi(G)$ is the product of all the normal π-subgroups of G and (if G is finite) $\mathrm{O}^\pi(G)$ is generated by all the π'-subgroups of G. Recall that a subgroup of G is called p-local if it is the normaliser of some non-identity p-subgroup of G, so with the above terminology if M is p-local then $\mathrm{O}_p(M) \neq 1$. Note that for any group G we have $\mathrm{O}_p(G) = G$ if and only if $\mathrm{O}^p(G) = 1$.

Proposition 1.2.7. *The following hold,*

(i) $\mathrm{O}_\pi(G) \cap H \leq \mathrm{O}_\pi(H)$ *with equality if* $H \lhd G$*,*

(ii) *if* $\mathrm{O}_\pi(G) \leq H$ *and* $H \lhd G$ *then* $\mathrm{O}_\pi(G) = \mathrm{O}_\pi(H)$*,*

(iii) $\mathrm{O}^\pi(H) \leq H \cap \mathrm{O}^\pi(G)$.

Note that (iii) still holds if replace the assumption that G is finite with the assumption that H has *finite index* in G. We use this in Lemma 1.3.5.

Remark 1.2.8. The above proposition is most useful when the conclusion of (ii) holds. Observe that in part (i), if H is not normal in G then the containment can be strict. Take $G = \mathrm{Sym}(3)$, $H = \langle (1,2) \rangle$ and $\pi = \{2\}$. Then $1 = \mathrm{O}_\pi(G) \cap H < \mathrm{O}_\pi(H) = H$.

Also, in the third part, even if H is a normal subgroup of G then we can have $\mathrm{O}^\pi(H) < \mathrm{O}^\pi(G) \cap H$. For example, take $G = \mathrm{Sym}(3)$, $H = \langle (1,2,3) \rangle$ and $\pi = \{3\}$. Then $\mathrm{O}^\pi(H) = 1$ and $\mathrm{O}^\pi(G) = G$ which gives $\mathrm{O}^\pi(H) < \mathrm{O}^\pi(G) \cap H = H$.

We write $\pi(G)$ for the set of prime divisors of $|G|$.

Proposition 1.2.9. *Let G be a group and let p, $q \in \pi(G)$ be distinct primes. Then* $[\mathrm{O}_p(G), \mathrm{O}_q(G)] = 1$.

Proof. Note that $\mathrm{O}_p(G) \cap \mathrm{O}_q(G) = 1$ since $p \neq q$, and the order of the intersection must be both a p- and q-group. Now since both $\mathrm{O}_p(G)$ and $\mathrm{O}_q(G)$ are normal in G, we have $[\mathrm{O}_p(G), \mathrm{O}_q(G)] \leq \mathrm{O}_p(G) \cap \mathrm{O}_q(G) = 1$. □

Definition 1.2.10. Let G be a group, the Fitting subgroup of G is defined to be

$$\mathbf{F}(G) = \langle \mathrm{O}_p(G) \mid p \in \pi(G) \rangle.$$

Lemma 1.2.11. *Suppose that G is a group. Then $\boldsymbol{F}(G)$ is the largest (by containment) normal nilpotent subgroup of G.*

Proof. By Proposition 1.2.9, $\mathrm{O}_p(G) \in \mathrm{Syl}_p(\mathbf{F}(G))$ for a prime $p \in \pi(\mathbf{F}(G))$. Since $\mathrm{O}_p(G)$ is normal in G, it is normal in $\mathbf{F}(G)$, so $\mathbf{F}(G)$ is nilpotent.

Suppose now that N is a normal nilpotent subgroup of G. Then $N = \langle \mathrm{O}_p(N) \mid p \in \pi(N) \rangle$ and the normality of N in G together with Proposition 1.2.7 shows that $N \leq \mathbf{F}(G)$. □

The importance of the Fitting subgroup is indicated by the following theorem. It says that if G is soluble, $G/Z(\mathbf{F}(G))$ is a faithful group of automorphisms of $\mathbf{F}(G)$.

Theorem 1.2.12. *Let G be a soluble group. Then $C_G(\boldsymbol{F}(G)) \leq \boldsymbol{F}(G)$.*

Instead of proving the theorem here, we will later deduce it as a corollary to Theorem 1.3.20. The following two theorems deliver results on the structure of G related to structure of p-subgroups. We say that a group G splits over a normal subgroup N if there is $H \leq G$ such that $G = HN$ and $H \cap N = 1$. The subgroup H is referred to as a *complement* to N (in G), and G is isomorphic to the semidirect product $N : H$.

Theorem 1.2.13 (Gaschütz's Theorem)**.** *Let $P \in \mathrm{Syl}_p(G)$ and suppose that V is a normal abelian p-subgroup of G. Then G splits over V if and only if P splits over V.*

Proof. See [1, pg.31]. □

Theorem 1.2.14 (Schur-Zassenhaus Theorem)**.** *Suppose that K is a normal subgroup of G and $(|G/K|, |K|) = 1$. Then G splits over K. In addition, if one of G/K or K is soluble then all complements to K in G are conjugate.*

Proof. See [23, pg.125]. □

1.3 Subnormal subgroups

In the previous section we defined a characteristic subgroup of G which (when G is soluble) "controls" the structure of G. The result of this section is the analogue for insoluble groups.

Definition 1.3.1. Let G be a group. A subgroup H of G is called *subnormal* if there exists subgroups H_i of G such that

$$H = H_l \lhd H_{l-1} \lhd \cdots \lhd H_1 \lhd H_0 = G$$

and we write $H \lhd\lhd G$. The number of proper subgroups in the shortest possible chain of subgroups is called the *subnormal depth* of H (in G).

For example, if we take $G \cong D_8 = \langle (1,2,3,4),(2,4) \rangle$, $H_1 = \langle (1,3)(2,4),(2,4) \rangle$, $H_2 = \langle (2,4) \rangle$. Then $H_1 \lhd G$, so H_1 has subnormal depth 1, $H_2 \lhd H_1$ but H_2 is not normal, so H_2 has subnormal depth 2 and (as always) G has subnormal depth 0.

Subnormality is a transitive relation on the set of subgroups of a group G. To see this, let $K \lhd\lhd H$ and $H \lhd\lhd G$, then there are subnormal series from K to H and from H to G and by "gluing" one series to the other we obtain a subnormal series for K in G.

Proposition 1.3.2. *Let G be a group with subgroups K and N.*

(i) Suppose that $K \lhd\lhd G$ and $N \leq G$. Then $K \cap N \lhd\lhd N$.

(ii) Suppose that $K \lhd\lhd G$ and $K \leq N \leq G$. Then $K \lhd\lhd N$.

(iii) If $K, N \lhd\lhd G$ then $K \cap N \lhd\lhd G$.

Proof. For (i), we let $K = K_0 \lhd K_1 \lhd \cdots \lhd K_r = G$ be a subnormal series for K in G. Then

$$K \cap N = K_0 \cap N \lhd K_1 \cap N \lhd \cdots \lhd K_r \cap N = G \cap N = N$$

is a subnormal series from $K \cap N$ to N.

Now (ii) follows immediately from (i) since $K = K \cap N \lhd\lhd N$.

For (iii) we first apply (i) to obtain $K \cap N \lhd\lhd N$. But $N \lhd\lhd G$ and subnormality is a transitive relation, thus $K \cap N \lhd\lhd G$. □

Above we showed that the set of subnormal subgroups of G is closed under intersection. One can show more, the set is also closed under products.

Theorem 1.3.3. *Let G be a group with subnormal subgroups K and N. Then*

$$\langle K, N \rangle \lhd\lhd G.$$

Proof. See [22, Theorem 2.5, pg.48]. □

The next three results intertwine subnormality and the subgroups $\mathrm{O}_\pi(G)$ and $\mathrm{O}^\pi(G)$ defined in the previous sections.

Lemma 1.3.4. *Suppose that K is a π-subgroup of G and K is subnormal in G. Then $K \leq \mathrm{O}_\pi(G)$. In particular, the subgroup generated by two subnormal π-subgroups of G is contained in $\mathrm{O}_\pi(G)$.*

Proof. Let K be as in the statement, we apply induction on the subnormal depth l of K in G. If $l = 0$, then $K = G$ and $G = \mathrm{O}_\pi(G)$, so we are done trivially. If $l = 1$ then $K \lhd G$ and so $K \leq \mathrm{O}_\pi(G)$ by definition.

Assume now that $l > 1$ and write $K = U_l \lhd U_{l-1} \lhd\lhd G$. Then since U_l is a π-group, $K \leq \mathrm{O}_\pi(U_{l-1})$ and $\mathrm{O}_\pi(U_{l-1}) \lhd U_{l-2}$ since it is characteristic in U_{l-1}. Therefore $\mathrm{O}_\pi(U_{l-1})$ has subnormal depth at most $l-1$. By induction, $\mathrm{O}_\pi(U_{l-1}) \leq \mathrm{O}_\pi(G)$, and hence $K \leq \mathrm{O}_\pi(G)$.

Suppose now that K and H are subnormal π-subgroups of G. Then $K, H \leq \mathrm{O}_\pi(G)$ by the above, which implies $\langle K, H\rangle \leq \mathrm{O}_\pi(G)$. $\square$

The following lemma still holds if we drop the assumption that G is finite, we will use this in Theorem 2.4.2.

Lemma 1.3.5. *Suppose that K is subnormal in G and $|G : K|$ is a π-number, then $\mathrm{O}^\pi(G) = \mathrm{O}^\pi(K)$. In particular, $\mathrm{O}^\pi(G) = \mathrm{O}^\pi(\mathrm{O}^\pi(G))$.*

Proof. Suppose that K is as in the statement of the lemma, by induction on the subnormal depth of K it suffices to assume that $K \lhd G$. Also, by Proposition 1.2.7 (iii) it suffices to show that $\mathrm{O}^\pi(G) \leq \mathrm{O}^\pi(K)$. Since $\mathrm{O}^\pi(K)$ is characteristic in K, we have that $\mathrm{O}^\pi(K) \lhd G$, and

$$G/K \cong (G/\mathrm{O}^\pi(K))/(K/\mathrm{O}^\pi(K)),$$

thus $G/\mathrm{O}^\pi(K)$ is a π-group which implies $\mathrm{O}^\pi(G) \leq \mathrm{O}^\pi(K)$ as required. $\square$

Lemma 1.3.6. *Suppose that $H \lhd\lhd G$. Then $\mathrm{O}_\pi(G)$ normalises $\mathrm{O}^\pi(H)$.*

Proof. Let $X = H\mathrm{O}_\pi(G)$ and note that $H \lhd\lhd X$ by Proposition 1.3.2(ii). Now $|X : \mathrm{O}^\pi(H)| = |X : H||H : \mathrm{O}^\pi(H)|$ is a π-number, so Lemma 1.3.5 implies that $\mathrm{O}^\pi(X) = \mathrm{O}^\pi(H)$. Since this is a normal subgroup of X and $\mathrm{O}_\pi(G) \leq X$, we are done. $\square$

Recall that a group K is said to be *perfect* if $K = K'$. We will call K *quasisimple* if K is perfect and $K/Z(K)$ is a non-abelian simple group. Note that the only proper normal subgroups of quasisimple groups are contained in the centre.

Definition 1.3.7. A *component* of a group G is a subgroup K such that K is subnormal in G and K is quasisimple.

The corollary to the next theorem shows that a pair of components of G relate to each other almost in the same way as the groups $\mathrm{O}_p(G)$ and $\mathrm{O}_q(G)$ for distinct primes p and q.

Theorem 1.3.8. *Suppose that K is a component of G and $U \lhd\lhd G$. Then either $K \leq U$ or $[K, U] = 1$.*

Proof. We assume that the theorem is false, and amongst counterexamples choose G such that $|G|$ is minimal, and then such that $|G : U|$ is minimal. Thus $K \nleq U$ and $[K, U] \neq 1$. Note that $K \nleq U$ forbids $U = G$, also $[K, U] \neq 1$ forbids $K = G$ since this would imply $U \leq \mathrm{Z}(K)$.

Since U is subnormal in G, we may choose a maximal normal subgroup U_1 of G containing U. Now $K \leq U_1$ would imply that $[K, U] = 1$ since $|U_1| < |G|$, so U_1 is not a counterexample, but this contradicts our assumption. Thus $K \nleq U_1$ and $1 \neq [K, U] \leq [K, U_1]$. The minimality of $|G : U|$ now forces $U = U_1$, so $U \lhd G$. This implies that $[K, U] \lhd U$.

Pick a maximal normal subgroup K_1 of G containing K (which exists since $K < G$). Now $[K, U] \leq [K_1, U] \leq K_1$. By the conclusion of the above paragraph therefore, $[K, U]$ is subnormal in G, and so is subnormal in K_1. But $K \nleq U$ so $K \nleq [K, U]$. Since $|K_1| < |G|$ therefore, $[K, U, K] = 1$. But $1 = [K, U, K] = [U, K, K]$ and so the three subgroups lemma implies that $[K, K, U] = 1$. But K is a component, so $1 = [K, K, U] = [K, U]$, a contradiction which delivers the result. $\square$

Corollary 1.3.9. *If H and K are distinct components of G, then $[H, K] = 1$.*

Observe that the set of components of G is invariant under automorphisms of G. This leads us to the following characteristic subgroup.

Definition 1.3.10. Let G be a group, we define the *layer of* G, $\mathbf{E}(G)$, to be the subgroup generated by all components of G.

The following is a second corollary to Theorem 1.3.8. It shows us that any subnormal subgroup we have to hand is normalised by the layer. Thus for a component K of G we have that $K \lhd \mathbf{E}(G) \lhd G$, so remarkably, components of G have subnormal depth at most 2.

Corollary 1.3.11. *Let $U \lhd\lhd G$ for some group G. Then $\boldsymbol{E}(G)$ normalises U.*

Proof. Let K be an arbitrary component of G, then either $K \leq U$ or $[K, U] = 1$, so certainly $K \leq \mathrm{N}_G(U)$. Since K was arbitrary, $\mathbf{E}(G) \leq \mathrm{N}_G(U)$. □

Note that K being a component of G tells us two things. On one hand, K is a quasisimple group, a property which is intrinsic to K, and on the other hand, K is subnormal in G which tells us something about the subgroup structure of G. Thus the two properties may seem to be considered independent from one another, and we exploit this below.

Proposition 1.3.12. *Suppose that N is a subnormal subgroup of G. Then the components of N are components of G. In particular, if $\boldsymbol{E}(G) \leq N$ then $\boldsymbol{E}(G) = \boldsymbol{E}(N)$.*

Proof. Let K be a component of N. Since N is subnormal in G, K is also subnormal in G, and since K is quasisimple, K is therefore a component of G.

Now suppose that $\mathbf{E}(G) \leq N$. If K is a component of G then $K \leq N$, and so by Proposition 1.3.2 (ii) we have $K \lhd\lhd N$. Thus $K \leq \mathbf{E}(N)$, which gives $\mathbf{E}(G) \leq \mathbf{E}(N)$. But the reverse equality holds also, so we have $\mathbf{E}(G) = \mathbf{E}(N)$. □

Lemma 1.3.13. *Suppose that K and L are distinct components of G. Then $K \cap L = Z(K) \cap Z(L)$. Moreover, $Z(K) = Z(E) \cap K$ where $E = \boldsymbol{E}(G)$.*

Proof. One inclusion holds trivially. By Corollary 1.3.9, $[k, l] = 1$ for all $k \in K$ and all $l \in L$. Therefore any element $m \in K \cap L$ commutes with all of L since it lies in K, but on the other hand m commutes with all of K since m lies in L. Thus $m \in Z(K) \cap Z(L)$.

For the second part, if $k \in Z(K)$, then k commutes with all of K, and since $[K, L] = 1$ for distinct components, k commutes with every element in the layer, so $k \in Z(E) \cap K$. Since the reverse inclusion also holds, we are done. $\square$

The following lemma shows us that if G is a group with a soluble subnormal subgroup U such that $\mathrm{C}_G(U) \leq U$, then the layer of G is trivial. In particular, the statement $\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$ gives this.

Lemma 1.3.14. *Suppose $U \lhd\lhd G$ and $C_G(U) \leq U$. Then $\boldsymbol{E}(G) \leq U$.*

Proof. Let K be a component of G and assume that $K \nleq U$. Then by Corollary 1.3.11 we have $[U, K] = 1$ so that $K \leq \mathrm{C}_G(U) \leq U$, a contradiction. Thus $K \leq U$ and so $\mathbf{E}(G)$, the product of all components of G, is contained in U. $\square$

Definition 1.3.15. We say that M is a *minimal normal subgroup* of G if the only non-trivial normal subgroup of G contained in M is M itself.

Proposition 1.3.16. *Suppose that G is a group and M, N are minimal normal subgroups. Either $M = N$ or $[M, N] = 1$.*

Proof. Since M and N are normal in G, $[M, N] \leq M \cap N$. Thus if $M \neq N$, then $M \cap N$ is a normal subgroup of G properly contained in one of M or N, so $M \cap N = 1$ which gives $[M, N] = 1$. $\square$

If G is a group with a normal subgroup S such that S is simple, then clearly S is a minimal normal subgroup. We give the class of groups which are generated by such minimal normal subgroups a name.

Definition 1.3.17. A group is *semisimple* if it is a product of non-abelian simple normal subgroups.

Combining the above remark and Proposition 1.3.16, we see that a semisimple group is isomorphic to a direct product of non-abelian simple groups. For any group G we are able to show that there is a normal subgroup which is either an elementary abelian p-group or semisimple.

Lemma 1.3.18. *Let G be any group and let A be a minimal normal subgroup of G. Then A is either an elementary abelian p-group for some prime p or A is semisimple.*

Proof. Let G and A be as in the statement and pick a minimal normal subgroup S of A. Let $\mathcal{S}$ be the subgroup of A generated by the minimal normal subgroups of A which are isomorphic to S. Then $\mathcal{S} \cong S \times \cdots \times S$ by Proposition 1.3.16.

We claim that $\mathcal{S}$ is characteristic in A. Indeed, if ϕ is an automorphism of A and T is a minimal normal subgroup of A isomorphic to S, then T^ϕ is again a minimal normal subgroup and $T^\phi \cong T \cong S$, so $\mathcal{S}^\phi = \mathcal{S}$. But this gives $\mathcal{S} \lhd G$ so $A = \mathcal{S}$. We claim that S is simple. Otherwise, there is a proper non-trivial subgroup N of A normal in S. Since $A = \mathcal{S}$ then, we see that $N \lhd A$. But S was a minimal normal subgroup, so $N = S$.

If S is abelian then S is cyclic of order p for some prime p, and A is an elementary abelian p-group. Otherwise, S is non-abelian and A is semisimple. $\square$

The following subgroup is the analogue of the Fitting subgroup for an insoluble group. We shall see that it controls the structure of G in the same way as $\mathbf{F}(G)$ when G is soluble.

Definition 1.3.19. The generalised Fitting subgroup of a group G is defined to be

$$\mathbf{F}^*(G) = \mathbf{E}(G)\mathbf{F}(G)$$

where $\mathbf{F}(G)$ is the Fitting subgroup of G. The generalised Fitting subgroup is a characteristic subgroup of G.

Theorem 1.3.20. *Let G be any finite group. Then $C_G(\boldsymbol{F}^*(G)) \leq \boldsymbol{F}^*(G)$.*

This result can be found in [22, pg.276] for example. Before giving a proof, we prove two lemmas which allow us to recognise when subgroups of G are contained in either $\mathbf{F}(G)$ or $\mathbf{E}(G)$. We state these two lemmas for an arbitrary finite group A.

Lemma 1.3.21. *Let A be a finite group and let $Z \leq \mathrm{Z}(A)$. Then A is nilpotent if and only if A/Z is nilpotent.*

Proof. Let p be a prime divisor of $|A|$ and let $P \in \mathrm{Syl}_p(A)$. It follows that PZ/Z is normal in A/Z if and only if PZ is normal in A. Thus if A is nilpotent, certainly A/Z is. On the other hand, if A/Z is nilpotent, then $PZ \lhd A$, and since $P \lhd PZ$ (and $P \in \mathrm{Syl}_p(PZ)$), we have $P = \mathrm{O}_p(PZ) \lhd A$, so A is nilpotent. □

Lemma 1.3.22. *Suppose that A is a finite group such that $A/\mathrm{Z}(A)$ is a non-abelian simple group. Then A' is perfect and $A'/\mathrm{Z}(A') \cong A/\mathrm{Z}(A)$ is non-abelian simple.*

Proof. Suppose first that $\overline{A} := A/\mathrm{Z}(A)$ is abelian. Then $\overline{A}$ is cyclic of prime order, in particular, A is abelian which gives $A = \mathrm{Z}(A)$, a contradiction. Hence $\overline{A}$ is a non-abelian simple group. Now $A' \nleq \mathrm{Z}(A)$, but $\overline{A}$ is simple so we have $\overline{A'} = \overline{A}$ which implies $A = A'\mathrm{Z}(A)$. Now $A' = [A'\mathrm{Z}(A), A'\mathrm{Z}(A)] = [A', A'] = A''$ so A' is perfect. Moreover, $\mathrm{Z}(A')$ commutes with $\mathrm{Z}(A)$ and with A' so $[\mathrm{Z}(A'), A] = [\mathrm{Z}(A'), A'\mathrm{Z}(A)] = 1$. Hence $\mathrm{Z}(A') \leq A' \cap \mathrm{Z}(A)$, but the reverse inclusion obviously holds, so we have $\mathrm{Z}(A') = A' \cap \mathrm{Z}(A)$. Via an isomorphism theorem therefore, $\overline{A} = A'\mathrm{Z}(A)/\mathrm{Z}(A) \cong A'/\mathrm{Z}(A) \cap A' = A'/\mathrm{Z}(A')$. □

Proof of Theorem 1.3.20. Set $F = \mathbf{F}^*(G)$ and $C = \mathrm{C}_G(\mathbf{F}^*(G))$ and suppose that $C \nleq F$. Using the bar notation we set $\overline{G} = G/(C \cap F)$. Choose a normal subgroup A of G such that A is minimal with respect to $C \cap F \leq A \leq C$ but $A \nleq F$ (note that A exists since C satisfies this property). We claim that $\overline{A}$ is a minimal normal subgroup of $\overline{G}$. Indeed, if $\overline{B} \lhd \overline{G}$ and $B \leq A$ then by our minimal choice of A we either have $B = A$ so that $\overline{B} = \overline{A}$ or $B \leq F$ which implies $\overline{B} = 1$. Thus we may apply Lemma 1.3.18 to see that $\overline{A}$ is either abelian or semisimple.

In both cases we observe that $C \cap F \leq \mathrm{Z}(C)$ and so $C \cap F \leq \mathrm{Z}(A)$. In the first case then Lemma 1.3.21 implies that A is nilpotent and so by Lemma 1.2.11, $A \leq \mathbf{F}(G) \leq F$, a contradiction.

Hence we may assume we are in the second case. Let $T \leq A$ be such that $\overline{T}$ is a minimal normal subgroup of $\overline{A}$. Similar to above we see that $C \cap F \leq \mathrm{Z}(T)$, but we chose T such that $\overline{T}$ is a minimal normal subgroup of $\overline{A}$. This implies that $\mathrm{Z}(T) = C \cap F$. Thus $\overline{T} = T/\mathrm{Z}(T)$ is simple, so T' is perfect by Lemma 1.3.22 and $T'/\mathrm{Z}(T') \cong T/\mathrm{Z}(T)$ is a non-abelian simple group. Since $T' \lhd\lhd G$ this implies that T' is a component of G which gives $T' \leq \mathbf{E}(G) \leq F$. Hence $T' \leq C \cap F$ and so $\overline{T}$ is abelian, a contradiction. $\square$

As promised, we now give the proof of Theorem 1.2.12.

Corollary 1.3.23. *Suppose that G is a soluble group, then $C_G(\boldsymbol{F}(G)) \leq \boldsymbol{F}(G)$.*

Proof. Since G is soluble $\mathbf{E}(G) = 1$ and so $\mathbf{F}^*(G) = \mathbf{F}(G)$. Hence Theorem 1.3.20 delivers the result. $\square$

Remark 1.3.24. If $G \neq 1$ then by examining a minimal normal subgroup of G we conclude that $\mathbf{F}^*(G) \neq 1$.

1.4 Permutation groups of prime degree

In this section we let G be a (non-trivial) transitive permutation group on a finite set Ω. A *block* is a non-empty subset B of Ω such that for all $g \in G$ we have $B^g \cap B = \emptyset$ or $B^g = B$. We call G *imprimitive* if there exists a block B such that $1 \neq |B| \neq |\Omega|$, and otherwise, we call G *primitive.* Note that the orbits of G on Ω are blocks, so a primitive group must be transitive. Note that G_B is transitive on the elements of B, thus $|G_B : G_\omega| = |B|$ for each $\omega \in B$ and so $|B|$ divides $|\Omega|$ for any block B.

Proposition 1.4.1. *Suppose that $|\Omega|$ is prime. Then G is primitive or acts trivially on Ω.*

Primitive actions are characterised by the following property.

Lemma 1.4.2. *A transitive action of G on Ω is primitive if and only if G_α is a maximal subgroup for any $\alpha \in \Omega$.*

If G is transitive but imprimitive on Ω, with a block B, then G_B (the setwise stabiliser of B) is transitive on B and G is transitive on the set $\{B^g \mid g \in G\}$. When one is trying to understand the action of G on Ω, this directs our attention towards primitive actions. By the previous lemma, this is equivalent to G acting on the cosets of a maximal subgroup of G, so a description of maximal subgroups is required. This was delivered (independently) by O'Nan and Scott. However there was a hole in both proofs and a case was missed. A corrected version of the theorem was given by Aschbacher and Scott [2].

Theorem 1.4.3 (Aschbacher-O'Nan-Scott)**.** *Suppose that the action of G on Ω is primitive. Then exactly one of the following hold,*

(i) $\boldsymbol{F}^*(G) = \boldsymbol{F}(G)$ *is the unique minimal normal subgroup of G,*

(ii) $\boldsymbol{F}(G) = 1$ *and* $\boldsymbol{F}^*(G)$ *is the direct product of the only two minimal normal subgroups of G (which are isomorphic),*

(iii) $\boldsymbol{F}(G) = 1$ *and* $\boldsymbol{F}^*(G)$ *is the unique minimal normal subgroup of G.*

Proof. This follows from [23, 6.6.12] once we see that if $N \lhd G_\omega$ for any $\omega \in \Omega$, then $G_\omega = \mathrm{N}_G(N)$, so G_ω is a *primitive maximal subgroup* of G (as defined in [23]). $\square$

The following theorem is more applicable to our situation, although it can now be seen to be a consequence of the above.

Theorem 1.4.4 (Burnside)**.** *Suppose that $|\Omega|$ is a prime. Then either G is 2-transitive and the unique minimal normal subgroup is non-abelian, or G is permutational isomorphic to a subgroup of* $\mathrm{AGL}_1(p)$ *acting on* $\mathbb{F}_p$*.*

Proof. Burnside's original proof used complex character theory [6, Theorem VII, Chapter XVI]. Recently a short and elementary proof has been given by P. Müller [27]. $\square$

Note that if the second conclusion of the above theorem holds then G is 2-transitive if and only if $G = \mathrm{AGL}_1(p)$. The following lemma is a consequence of the classification of 2-transitive groups due to Cameron, and therefore depends upon the GFSG. It is not required in any result in this thesis, we merely state it to direct our attention towards certain interesting situations.

Lemma 1.4.5. *Suppose that $|\Omega| = p$ is a prime and G acts 2-transitively on Ω. Then $\boldsymbol{F}^*(G)$ and p appear in Table 1.1.*

Proof. We apply the previous result to see that either the situation is as in line one of Table 1.1, or $\mathbf{F}^*(G)$ is a non-abelian simple group. The list of 2-transitive finite permutation groups is contained in [8, (5.3)]. Assuming that $\mathbf{F}^*(G)$ is on this list, we require that the degree is prime. The only options are those appearing in Table 1.1. □

p	$\mathbf{F}^*(G)$	Point stabiliser in $\mathbf{F}^*(G)$
all	C_p	Trivial
all	$\mathrm{Alt}(p)$	$\mathrm{Alt}(p-1)$
$q^{d-1} + q^{d-2} + \cdots + 1$, q a prime power	$\mathrm{PSL}_d(q)$, $d \geq 2$	$q^{d-1} : \mathrm{PGL}_{d-1}(q)$
11	$\mathrm{PSL}_2(11)$	$\mathrm{Alt}(5)$
11	M_{11}	M_{10}
23	M_{23}	M_{22}

Table 1.1: Generalised Fitting subgroups of 2-transitive permutation groups of prime degree

Remark 1.4.6. In Table 1.1 we have only given the isomorphism shape of $\mathbf{F}^*(G)$, we also need to give the set Ω on which $\mathbf{F}^*(G)$ is acting and how this action arises. For the cyclic group, the alternating group and the Mathieu groups, the action is the natural action of degree p. For $G = \mathrm{PSL}_2(11)$ there are two inequivalent actions on 11 points, arising from the two conjugacy classes of subgroups isomorphic to $\mathrm{Alt}(5)$ in G. For $G = \mathrm{PSL}_d(q)$, there are also two actions on p points (where $p = q^{d-1} + \cdots + 1$), these are on the sets of points and hyperplanes of the natural d-dimensional module for G over $\mathbb{F}_q$, which are inequivalent if $d > 2$.

1.5 A pushing-up result

The main result of this section is Theorem 1.5.6. We expect that the result is well known, but we were unable to find a reference. A pushing up problem is the following: We have a group G, a p-subgroup Q and $R = \mathrm{N}_G(Q)$. We say that R can be *pushed up* if there is a p-subgroup P of G such that $R < \mathrm{N}_G(P)$. Determining why R cannot be pushed up is equally valuable and the general pushing up problem is to describe the obstructions to pushing up. For a description of some of the important results in this area see [30, (24.2)]. We will need the following result.

Theorem 1.5.1. *Let X be a finite group such that $C_X(\mathrm{O}_2(X)) \leq \mathrm{O}_2(X)$. Let $S \in \mathrm{Syl}_2(X)$ and set $Z = \langle \Omega_1(\mathrm{Z}(S))^X \rangle$. Suppose that $X/\mathrm{O}_2(X) \cong \mathrm{PSL}_2(2^n)$ for some $n \in \mathbb{N}$, that no non-trivial characteristic subgroup of S is normal in X and that there exists a subgroup $H \leq \mathrm{Aut}(S)$ with $|H : N_H(\mathrm{O}_2(X))|$ odd. Then $\langle Z^H \rangle$ is a normal subgroup of X which is contained in $\mathrm{O}_2(X)$.*

Proof. This is Corollary 3.14 in [29]. □

Definition 1.5.2. Let S be a p-group. Then $\mathcal{A}(S)$ is the set of abelian subgroups of maximal order. We define

$$J(S) = \langle A \mid A \in \mathcal{A}(S) \rangle,$$

the *Thompson* subgroup (of S).

The set $\mathcal{A}(S)$ is invariant under automorphisms of S, thus $J(S)$ is a characteristic subgroup of S. Here is one property of the Thompson subgroup.

Proposition 1.5.3. *Let R be a group, let $S \in \mathrm{Syl}_p(R)$ and suppose Q is a subgroup of S such that $J(S) \leq Q$. Then $J(S) = J(Q)$.*

Proof. We will show that $\mathcal{A}(S) = \mathcal{A}(Q)$. Indeed, let $A \in \mathcal{A}(S)$ and $B \in \mathcal{A}(Q)$. Then B is an abelian subgroup of maximal order in Q, but $B \leq Q \leq S$, so $|B| \leq |A|$. Now $A \leq J(S) \leq Q$ so A is an abelian subgroup of Q, and so $|A| \leq |B|$ as B has the maximal order of an abelian subgroup of Q. Thus $|A| = |B|$ which gives $B \in \mathcal{A}(S)$ and $A \in \mathcal{A}(Q)$. $\square$

Another property of the elements of $\mathcal{A}(S)$ is the following.

Proposition 1.5.4. *Let $A \in \mathcal{A}(S)$. Then $C_S(A) = A$.*

Proof. Let $x \in \mathrm{C}_S(A)$. Then $\langle x, A\rangle$ is abelian, and since $|A|$ is maximal amongst abelian subgroups of S, we have $|\langle x, A\rangle| = |A|$ which implies $x \in A$. Hence $\mathrm{C}_S(A) \leq A$, but A is abelian, so $A \leq \mathrm{C}_S(A)$ and we are done. $\square$

Theorem 1.5.5 (Thompson Replacement Theorem)**.** *Let S be a p-group, $A \in \mathcal{A}(S)$ and let Z be an abelian p-subgroup of S. Assume A normalises Z but Z does not normalise A. Then there exists an element $A^* \in \mathcal{A}(S)$ such that:*

i) $A \cap Z < A^* \cap Z$,

ii) A^* normalises A.

Proof. See [18, Thm. 8.2.5, pg.273]. $\square$

Let $R = \mathrm{Sym}(4)$ and let $Q = \mathrm{O}_2(R) \cong 2^2$. Observe $R/Q \cong \mathrm{Sym}(3) \cong \mathrm{Dih}(6) \cong \mathrm{AGL}_1(3)$ and note that for any Sylow 2-subgroup S of R, we have $S \cong \mathrm{Dih}(8)$ and the only characteristic subgroup of S which is normal in R is the trivial subgroup. This shows that the conclusion $p = 2$ and $r = 3$ cannot be removed from the next theorem.

Theorem 1.5.6. *Suppose that p and r are primes with $r > 2$. Let R be a group with $Q = \mathrm{O}_p(R) = \boldsymbol{F}^*(R)$ and let $S \in \mathrm{Syl}_p(R)$. Suppose that R/Q is a normal subgroup of $\mathrm{AGL}_1(r)$. Then either $p = 2$ and $r = 3$, or there exists a non-trivial characteristic subgroup C of S which is normal in R.*

Proof. We assume that no non-trivial characteristic subgroup of S is normal in R (in particular, $S < R$). Set $Z = \Omega_1(\mathrm{Z}(Q))$ and $X = \Omega_1(\mathrm{Z}(S))$. Since $1 \neq X$ char S, we see that X is not a normal subgroup of R.

(1) $X \leq Z$ and $\mathrm{C}_R(Z) = Q$.

Since $\mathrm{C}_R(Q) = \mathrm{Z}(Q)$ and $[X, Z] \leq [\mathrm{Z}(S), S] = 1$ we get $X \leq \mathrm{Z}(Q)$ and since X is elementary abelian, we get $X \leq Z$. We have $Q \leq \mathrm{C}_R(Z)$ and if the containment is proper, then since $\mathrm{C}_R(Z) \lhd R$, we see that $1 \neq \mathrm{C}_R(Z)/Q \lhd R/Q$ which implies that $r \mid |\mathrm{C}_R(Z)|$. Then $r \mid |\mathrm{C}_R(X)|$ also as $\mathrm{C}_R(Z) \leq \mathrm{C}_R(X)$, but already $S \leq \mathrm{C}_R(Z)$ and for every other prime l dividing $|R/Q|$ we can find a Sylow l-subgroup which normalises S and therefore X, hence $R = \mathrm{N}_R(X)$, a contradiction.

By (1) we may consider a R/Q-chief series of Z.

(2) There are non-central R/Q-chief factors in Z.

Otherwise coprime action implies $\mathrm{O}^p(R/Q)$ centralises Z. But then $1 \neq \mathrm{O}^p(R/Q) \leq \mathrm{C}_{R/Q}(Z) = 1$, a contradiction.

(3) $J(S) \nleq Q$.

Otherwise Proposition 1.5.3 implies $J(S) = J(Q)$ and provides a contradiction.

(4) There is $A \in \mathcal{A}(S)$ with $[Z, A, A] = 1$ and $A \nleq Q$.

By the previous claim we may choose $A \in \mathcal{A}(S)$ such that $A \nleq Q$. Amongst such A we choose A with $|A \cap Z|$ as large as possible and we claim that Z normalises A. If this is not the case, Thompson's Replacement Theorem gives $A^* \in \mathcal{A}(S)$ with $A \cap Z < A^* \cap Z$ and A^* normalises A. By the choice of A we must have $A^* \leq Q$ which gives $[Z, A^*] = 1$ and so $Z \leq A^* \leq \mathrm{N}_R(A)$, a contradiction proving that Z normalises A. Hence $[Z, A] \leq A$ and so $[Z, A, A] = 1$.

(5) $p = 2$.

Otherwise, the previous claim implies that AQ/Q and a conjugate act quadratically on Z (since $A \nleq Q$ we cannot have $A \lhd R$), and so R/Q contains a subgroup isomorphic to $\mathrm{SL}_2(p)$ by [18, Theorem 3.8.1] (after considering an irreducible submodule and tensoring with $\overline{\mathbb{F}_p}$). Since R/Q is soluble we have $p = 3$, but the Sylow 2-subgroups of $\mathrm{SL}_2(3)$ are non-cyclic as opposed to the Sylow 2-subgroups of $\mathrm{AGL}_1(r)$ and this contradiction proves the claim.

(6) For all $A \in \mathcal{A}(S)$ with $A \nleq Q$ we have $|Z/\mathrm{C}_Z(A)| \leq |A/\mathrm{C}_A(Z)| = 2$.

Since $Z\mathrm{C}_A(Z)$ is elementary abelian, we have $|Z\mathrm{C}_A(Z)| \leq |A|$ for all $A \in \mathcal{A}(S)$. Now $|Z\mathrm{C}_A(Z)| = |Z||\mathrm{C}_A(Z)|/|Z \cap \mathrm{C}_A(Z)|$ and $Z \cap \mathrm{C}_A(Z) \leq \mathrm{C}_Z(A)$, so $|Z/\mathrm{C}_Z(A)| \leq |Z/Z \cap \mathrm{C}_A(Z)|$. Hence $|Z/\mathrm{C}_Z(A)| \leq |A/\mathrm{C}_A(Z)|$. By (1) $\mathrm{C}_A(Z) = A \cap Q$ and $|AQ/Q| = 2$ since A is elementary abelian and the Sylow 2-subgroups of R/Q are cyclic.

(7) We have $r = 3$.

By (2) there is a non-central R/Q-chief-factor, W say, contained in Z. By (3) we may choose $A \in \mathcal{A}(S)$ such that $A \nleq Q$ and we have $\mathrm{N}_R(A) \neq R$ (since AQ/Q is not normal in R/Q). Since $|W/\mathrm{C}_W(A)| \leq |Z/\mathrm{C}_Z(A)| \leq 2$, we see that AQ/Q centralises a hyperplane of W. Let $B = A^g$ for some $g \notin \mathrm{N}_R(A)$ and note that $R/Q = \langle AQ/Q, BQ/Q \rangle$. Now $|W/\mathrm{C}_W(B)| \leq 2$ also and combined with $\mathrm{C}_W(B) \cap \mathrm{C}_W(A) = \mathrm{C}_W(R/Q) = 1$ gives $|W| \leq 4$. Since R/Q acts faithfully on W, we must have $r = 3$.

Statements (5) and (7) now give the conclusion of the theorem. $\square$

1.6 Amalgams of groups

Definition 1.6.1. An *amalgam* $\mathcal{A}$ is a 5-tuple $(P_1, P_2, B, \pi_1, \pi_2)$ of three groups P_1, P_2 and B and two monomorphisms $\pi_i : B \to P_i$ $(i = 1, 2)$.

We say that two amalgams $\mathcal{A}$ and $\mathcal{B} = (R_1, R_2, D, \phi_1, \phi_2)$ have the same *type* if there is a triple of isomorphisms $\beta : D \to B$ and $\alpha_i : R_i \to P_i$ $(i = 1, 2)$ such that for $i = 1, 2$ we have $\alpha_i(\phi_i(D)) = \pi_i(\beta(D))$.

The *degree* of an amalgam is the pair $(|P_1 : \pi_1(B)|, |P_2 : \pi_2(B)|)$. We say the amalgam is finite if B is finite and the degree is a pair of integers.

Note that the amalgam $(P_1, P_2, B, \alpha\pi_1, \beta\pi_2)$ has the same type as $\mathcal{A}$ for every pair of automorphisms α and β of P_1 and P_2 respectively. Hence the type of an amalgam amounts to a choice of an $\mathrm{Aut}(P_1)$-conjugacy class of subgroups of P_1 and an $\mathrm{Aut}(P_2)$-conjugacy class of subgroups of P_2.

Amalgams of the same type can have different properties. For example, let

$$\begin{aligned} P_1 &= \langle (1,2),(3,4,5),(4,5)\rangle, \\ P_2 &= \langle (6,7),(8,9,10),(9,10)\rangle, \\ B &= \langle (11,12),(13,14)\rangle, \end{aligned}$$

(so $P_1 \cong P_2 \cong \mathrm{C}_2 \times \mathrm{Sym}(3)$ and $B \cong 2^2$). Define $\pi_1 : B \to P_1$ by

$$\pi_1((11,12)) = (1,2) \text{ and } \pi_1((13,14)) = (4,5).$$

Define $\pi_2 : B \to P_2$ by

$$\pi_2((11,12)) = (6,7) \text{ and } \pi_2((13,14)) = (9,10).$$

Set $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$. Notice that $\pi_i((11,12)) \in \mathrm{Z}(P_i)$ for $i = 1,2$. Let γ be the automorphism of B which swaps $(11,12)$ and $(13,14)$ and set $\mathcal{B} = (P_1, P_2, B, \pi_1\gamma, \pi_2)$. Then $\mathcal{A}$ and $\mathcal{B}$ have the same type, but $\mathcal{B}$ has the following property which $\mathcal{A}$ does not,

$$(\pi_1\gamma)^{-1}(\mathrm{Z}(P_1)) \cap \pi_2^{-1}(\mathrm{Z}(P_2)) = 1.$$

Thus amalgams of the same type can encode rather different behaviour.

Definition 1.6.2. A *completion* of $\mathcal{A}$ is a triple (G, ρ_1, ρ_2) of a group G and two homomorphisms $\rho_i : P_i \to G$ so that $G = \langle \rho_1(P_1), \rho_2(P_2)\rangle$ and the subdiagram of Figure 1.1 consisting of G, P_1, P_2, B and the maps between them commutes.

A *universal completion* of $\mathcal{A}$ is a completion (R, ϕ_1, ϕ_2) such that if (G, ρ_1, ρ_2) is any other completion then there exists a unique homomorphism κ such that Figure 1.1 commutes.

If (G, ρ_1, ρ_2) is a completion of $\mathcal{A}$ (as above) we say that the completion is *faithful* if the maps ρ_i are monomorphisms. By an abuse of language, we will also refer to G as

$$
\begin{tikzcd}
 & P_1 \arrow[dr, "\phi_1"] \arrow[drr, bend left, "\rho_1"] & & \\
B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & R \arrow[r, dashed, "\kappa"] & G \\
 & P_2 \arrow[ur, "\phi_2"'] \arrow[urr, bend right, "\rho_2"'] & &
\end{tikzcd}
$$

Figure 1.1: An amalgam with universal completion and completion

the completion, and as such, we call a completion *finite* if G is finite. Note that finite completions of amalgams always exist, we can simply take $G = 1$ and ρ_1, ρ_2 to be the trivial maps. We remark that if the groups involved in an amalgam are finite, then a faithful finite completion of the amalgam exists, see [28].

Every amalgam has a unique faithful completion. The uniqueness follows from the uniqueness of the map κ in Definition 1.6.2. We construct a universal completion in the following way. Let $X = P_1 * P_2$ (the free product of P_1 and P_2) and write α_i for the natural monomorphism from P_i to X, we set N to be the normal closure in X of the set $\{\alpha_1(\pi_1(b))\alpha_2(\pi_2(b^{-1})) \mid b \in B\}$. Then $Y = X/N$ is a faithful universal completion of $\mathcal{A}$. In Appendix A.3 we give a program written in Magma that creates, as a finitely presented group, the universal completion of an amalgam.

Given an amalgam $\mathcal{A}$ with a faithful completion G we may identify B, P_1 and P_2 with their images in G (and then $P_1 \cap P_2 \geq B$). When we do this we may drop reference to the maps, and in that case, we will write the amalgam as the triple (P_1, P_2, B).

Notation 1.6.3. We write $\mathcal{G}(\mathcal{A})$ for the universal completion of the amalgam $\mathcal{A}$.

We have mentioned earlier the connection between amalgams and semisymmetric graphs. Here we make this connection explicit.

Definition 1.6.4. Let $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_1)$ be an amalgam and suppose that G is a faithful completion of $\mathcal{A}$. The *coset graph* $\Gamma(\mathcal{A}, G)$ of $\mathcal{A}$ with respect to G is the graph with vertex set

$$G/P_1\dot{\cup}G/P_2$$

and edge set $\{\{P_1g, P_2h\} \mid P_1g \cap P_2h \neq \emptyset\}$.

The next proposition shows that amalgams give rise to semisymmetric graphs.

Proposition 1.6.5. *Let $\mathcal{A}$ be as above and let G be a faithful completion. Then $\Gamma = \Gamma(\mathcal{A}, G)$ is a connected bi-regular graph of bi-valency the degree of $\mathcal{A}$. G acts edge-transitively on Γ and has two orbits θ_1 and θ_2 on the vertices of Γ. A stabiliser of a vertex in θ_i is conjugate in G to P_i and the stabiliser of an edge is conjugate in G to $P_1 \cap P_2$. The kernel of this action is the core in G of $P_1 \cap P_2$.*

Since we are only aiming to deal with groups which act faithfully on graphs, in light of the above proposition, we make the following definition.

Definition 1.6.6. An amalgam $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ is called *faithful* if whenever $K \leq B$ and $\pi_i(K) \lhd P_i$ for $i = 1, 2$ we have $K = 1$.

Later in the thesis we will encounter a certain class of amalgams which are defined in [10]. Here we recall the definition and mention the relevant results which we will later call upon.

Definition 1.6.7. Let p be a prime and let G be a group with a pair of finite subgroups (P_1, P_2) such that $G = \langle P_1, P_2 \rangle$ and no non-trivial normal subgroup is contained in $P_1 \cap P_2$. Then (P_1, P_2) is a *weak (B, N)-pair of characteristic p* (with respect to G) if there exists normal subgroups P_1^* and P_2^* of P_1 and P_2 respectively such that for $\{i, j\} = \{1, 2\}$ the following hold

(i) $O_p(P_i) \leq P_i^*$ and $P_i = P_i^*(P_1 \cap P_2)$,

(ii) $C_{P_i}(O_p(P_i)) \leq O_p(P_i)$,

(iii) $P_i^* \cap P_j = N_{P_i^*}(S)$ for some $S \in \mathrm{Syl}_p(P_i^*)$,

(iv) $P_i^*/\mathrm{O}_p(P_i) \cong \mathrm{PSL}_2(p^{n_i})$, $\mathrm{SL}_2(p^{n_i})$, $\mathrm{PSU}_3(p^{n_i})$, $\mathrm{SU}_3(p^{n_i})$, $\mathrm{Sz}(2^{n_i})$ or $\mathrm{Dih}(10)$ (and $p = 2$), $\mathrm{Ree}(3^{n_i})$ or $\mathrm{Ree}(3)'$ (and $p = 3$) for some $n_1, n_2 \in \mathbb{N}$.

Recall that $\mathrm{Ree}(3)' \cong \mathrm{PSL}_2(8)$ and $\mathrm{Sz}(2) \cong \mathrm{Frob}(20)$. Note that if (P_1, P_2) is a weak (B, N)-pair for G and N is a normal subgroup of G with $P_1 \cap N = 1 = N \cap P_2$, then setting $\overline{G} = G/N$ we have that $(\overline{P_1}, \overline{P_2})$ is a weak (B, N)-pair of $\overline{G}$. From [10] we derive the following theorem.

Theorem 1.6.8. *Suppose that $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ is an amalgam, $G = \mathcal{G}(\mathcal{A})$ and that (P_1, P_2) is a weak (B, N)-pair for G of characteristic 2 with $|P_1 : B| = 5 = |P_2 : B|$. Then there is a free normal subgroup N of G such that $H = G/N$ is finite and $\boldsymbol{F}^*(H) \cong \mathrm{PSL}_3(4)$, $\mathrm{Sp}_4(4)$ or $\mathrm{G}_2(4)$.*

Proof. We apply [10, Theorem A, pg.100]. The index of B in P_1 and P_2 restricts to the list above. □

The following was noted in [10, pg.97]. Let X be isomorphic to $\mathrm{PSL}_3(4)$, $\mathrm{Sp}_4(4)$ or $\mathrm{G}_2(4)$ and let $S \in \mathrm{Syl}_2(X)$. Setting $Y = \mathrm{N}_X(S)$ there is a unique pair of subgroups R_1 and R_2 such that $Y = R_1 \cap R_2$ and (R_1, R_2) is a weak (B, N)-pair for X.

Notation 1.6.9. We write $\mathcal{S}_{13}$, $\mathcal{S}_{14}$ and $\mathcal{S}_{15}$ for the amalgams (R_1, R_2, Y, i_1, i_2) which come from the groups $\mathrm{PSL}_3(4)$, $\mathrm{Sp}_4(4)$ and $\mathrm{G}_2(4)$ so that (R_1, R_2) is a weak (B, N)-pair.

From the theorem we obtain the following.

Corollary 1.6.10. *If the hypothesis of Theorem 1.6.8 holds then, for $i = 1, 2$, $P_i \cong R_i$ and $B \cong Y$ where (R_1, R_2, Y, i_1, i_2) is one of the amalgams $\mathcal{S}_{13}$, $\mathcal{S}_{14}$ or $\mathcal{S}_{15}$.*

CHAPTER 2

GROUPS AND GRAPHS

Recall that a graph $\Gamma = (V, E)$ is a pair of vertices V, and edges E, where E is a subset of $\{\{x, y\} \mid x, y \in V, x \neq y\}$. Therefore our graphs contain no loops and no multiple edges. We also assume that all the graphs are connected and we use $\mathrm{d}(-,-)$ to denote the usual distance metric. For a vertex x and an integer i we define

$$\Delta^{[i]}(x) = \{y \in \Gamma \mid \mathrm{d}(x, y) = i\}$$

and we write $\Delta(x)$ for $\Delta^{[1]}(x)$, the *neighbourhood* of x. We say Γ is a G-graph if G is a group acting faithfully on Γ. For $x \in \Gamma$ we set

$$G_x^{[i]} = \{g \in G \mid y^g = y \text{ for all } y \in \Delta^{[i]}(x)\}$$

and observe that for $j \geq i$ we have $G_x^{[j]} \lhd G_x^{[i]}$. We write G_x for $G_x^{[0]}$. Moreover, if $(x, y, z, \dots)$ is a path then we write $G_{xyz\dots}^{[i]}$ for $G_x^{[i]} \cap G_y^{[i]} \cap G_z^{[i]} \cap \dots$ and $G_{xyz\dots}$ for $G_{xyz\dots}^{[0]}$.

2.1 Edge-transitive groups of automorphisms

We distinguish three different actions of a group on a graph. We say that the action of G is *semisymmetric* if G acts edge-transitively, but not vertex-transitively. We say that the action of G is *symmetric* if G acts vertex-transitively, edge-transitively and

$|G_{\{\alpha,\beta\}} : G_{\alpha\beta}| = 2$ for some (and therefore every) edge $\{\alpha, \beta\}$. We say that the action of G is $\frac{1}{2}$-*arc transitive* if the action is not symmetric but is both vertex- and edge-transitive.

Proposition 2.1.1. *Suppose that G is vertex- and edge-transitive on Γ and that Γ has odd valency. Then G acts symmetrically.*

Proof. Let k be the valency of Γ, which is odd by assumption and let $\{\alpha, \beta\}$ be an edge of Γ. Suppose that $G_{\{\alpha,\beta\}} = G_{\alpha\beta}$ and let m be the number of edges. By edge-transitivity $m = |G : G_{\alpha\beta}|$. On the other hand, $m = \frac{k|G:G_\alpha|}{2}$ by the Hand Shaking Lemma. It follows that $k = 2|G_\alpha : G_{\alpha\beta}|$, a contradiction. $\square$

It follows from the proposition that groups acting edge-transitively on graphs of odd valency act either symmetrically or semisymmetrically. There is a unique smallest $\frac{1}{2}$-arc transitive graph, Holt's graph [20] with 27 vertices, 54 edges and valency four. All vertex stabilisers have order two and all edge stabilisers are trivial.

Lemma 2.1.2. *Suppose that Γ is either G-symmetric or G-semisymmetric. Then G_α acts transitively on $\Delta(\alpha)$. Moreover, $|G_\alpha : G_{\alpha\beta}| = |\Delta(\alpha)|$ for each $\beta \in \Delta(\alpha)$.*

Proof. Let β and δ be neighbours of α and let $g \in G$ be such that $\{\alpha, \beta\}^g = \{\alpha, \delta\}$. If Γ is G-semisymmetric, then $\Gamma = \alpha^G \cup \beta^G$, and so we have $\beta^g = \delta$ and $g \in G_\alpha$ as required. If Γ is symmetric then after acting with $G_{\{\alpha,\delta\}}$ (if necessary) we have $\beta^g = \delta$ and $g \in G_\alpha$ as above. The final statement of the lemma follows by the Orbit-Stabiliser theorem. $\square$

The following follows from the connectivity of Γ.

Lemma 2.1.3. *Suppose that $A \leq G_\alpha$ and $B \leq G_\beta$ are transitive on $\Delta(\alpha)$ and $\Delta(\beta)$ respectively. Then $X = \langle A, B \rangle$ is edge-transitive. Moreover, if $A \cap G_{\alpha\beta} \not\leq G_{\alpha\beta} \cap B$ then $B < X_\beta$.*

With the notation of the above lemma, we may consider the question of when $A = X_\alpha$ and $B = X_\beta$. The lemma suggests we might hope that it is enough to require $A \cap G_{\alpha\beta} = G_{\alpha\beta} \cap B$. However, this is not the case. For example, let Γ be the graph obtained from the incidence geometry associated to the vector space of three dimensions over GF(2). For an edge $\{\alpha, \beta\}$ we have $G_\alpha \cong \mathrm{Sym}(4) \cong G_\beta$. A judicious choice of $A \leq G_\alpha$ and $B \leq G_\beta$ with $A \cong B \cong \mathrm{C}_3$ gives $X = G$, and so both $A < X_\alpha$ and $B < X_\beta$ hold. Note that $A \cap G_{\alpha\beta} = G_{\alpha\beta} \cap B$ however. In the next section, we consider the case that Γ is a tree and we shall see that the condition $A \cap G_{\alpha\beta} = G_{\alpha\beta} \cap B$ ensures that $A = X_\alpha$ and $B = X_\beta$.

The following result is one of the most powerful tools which we have at our disposal. It shall be applied frequently in our investigation, so if it's use is sufficiently clear we shall omit reference.

Lemma 2.1.4. *Suppose that $\{u, v\}$ is an edge of Γ and $K \leq G_{uv}$ is such that $N_{G_u}(K)$ and $N_{G_v}(K)$ are transitive on $\Delta(u)$ and $\Delta(v)$ respectively. Then $K = 1$.*

Proof. Lemma 2.1.3 implies that $X = \langle \mathrm{N}_{G_u}(K), \mathrm{N}_{G_v}(K) \rangle \leq \mathrm{N}_G(K)$ is edge-transitive. It follows that for any $z \in \Gamma$ there is $g \in X$ such that either $u^g = z$ or $v^g = z$. Then $K = K^g \leq (G_{uv})^g \leq G_z$. Hence K fixes every vertex of Γ, so $K = 1$. $\square$

We omit the proof of the following lemma as it merely serves to highlight the local and global properties which motivate definitions in the next section.

Lemma 2.1.5. *Suppose that G acts edge-transitively on Γ and that $H \leq G$ also acts edge-transitively. For each edge $\{\alpha, \beta\}$ of Γ the following hold*

(1) $G = HG_{\alpha\beta}$ and $H = \langle H_\alpha, H_\beta \rangle$,

(2) $H_\alpha = H \cap G_\alpha$,

(3) $H_\alpha G_{\alpha\beta} = G_\alpha$,

(4) $H_\alpha \cap H_\beta = H_\alpha \cap G_{\alpha\beta}$,

(5) if H is normal in G then $H_\alpha \lhd G_\alpha$.

The following lemma shows that we may reduce the study of semisymmetric graphs of bi-valency (k, l) to the problem of classifying faithful amalgams of degree (k, l) and their completions.

Lemma 2.1.6. *Suppose that Γ is G-semisymmetric and let $\mathcal{A}$ be the amalgam formed by the stabilisers of adjacent vertices. Then the graphs Γ and $\Gamma(\mathcal{A}, G)$ are isomorphic as G-graphs.*

Proof. Write $\mathcal{A} = (G_\alpha, G_\beta, G_{\alpha\beta}, \pi_\alpha, \pi_\beta)$. Observe that every vertex of Γ is conjugate to precisely one of α or β. We define $\theta : \Gamma \to \Gamma(\mathcal{A}, G)$ by

$$\theta : \gamma \mapsto \begin{cases} G_\alpha g & \text{if } \gamma = \alpha^g \text{ for some } g \in G \\ G_\beta h & \text{if } \gamma = \beta^h \text{ for some } h \in G \end{cases}$$

and we claim that θ is the required isomorphism. First we check that θ commutes with the action of G. Let $\gamma \in \Gamma$ and assume $\gamma = \alpha^g$ for some $g \in G$ (the other case being similar). Let $k \in G$, then $\theta(\gamma^k) = G_\alpha gk = (G_\alpha g)k = \theta(\gamma)^k$ as required. We now check that θ is a graph homomorphism. If $\{\gamma, \delta\}$ is an edge of Γ, then by edge-transitivity there is $g \in G$ such that $\gamma = \alpha^g$ and $\delta = \beta^g$ (after relabelling if necessary). Then $\theta(\gamma) = G_\alpha g$ and $\theta(\delta) = G_\beta g$, so $\theta(\gamma) \sim \theta(\delta)$ as required.

It is clear that θ is surjective, so we need only check that θ is well defined and injective. If $\gamma = \alpha^g = \alpha^k$ then $gk^{-1} \in G_\alpha$ so that $G_\alpha g = G_\alpha k$, similarly for vertices in the orbit of β. Now if $G_\alpha k = \theta(\gamma) = \theta(\mu) = G_\alpha g$ say, then there are $g, k \in G$ such that $\gamma = \alpha^k$ and $\mu = \alpha^g$. Since $kg^{-1} \in G_\alpha$ we have $\gamma^{g^{-1}} = \alpha$ so that $\mu = \alpha^g = \gamma^{g^{-1}g} = \gamma$ which completes the proof. $\square$

For the symmetric case we follow a similar procedure to that for semisymmetric graphs. We show that classifying symmetric graphs of valency k is equivalent to classifying faithful amalgams of degree $(k, 2)$ and their completions.

Definition 2.1.7. Suppose that the action of G on Γ is symmetric and let $e = \{\alpha, \beta\}$ be an edge of Γ. Set $\mathcal{A} = (G_\alpha, G_e, G_{\alpha\beta}, \pi_\alpha, \pi_e)$. We define $\Pi(\mathcal{A}, G)$ to be the graph with vertex set $\{G_\alpha g \mid g \in G\}$. Two vertices $G_\alpha g$ and $G_\alpha h$ are adjacent if and only if $gh^{-1} \in G_\alpha a G_\alpha$ for some $a \in G_e - G_{\alpha\beta}$.

In the above definition, we should check that the definition of the graph does not depend on the choice of $a \in G_{\{\alpha,\beta\}} - G_{\alpha\beta}$. To our knowledge, this construction is due to [24]. Note that the amalgam $\mathcal{A}$ has degree $(k, 2)$ where k is the valency of Γ.

Lemma 2.1.8. *Suppose that* Γ *is* G*-symmetric. Let* $e = \{\alpha, \beta\}$ *be an edge and let* $\mathcal{A}$ *be the amalgam formed by* G_α *and* G_e*. Then the graphs* Γ *and* $\Pi(\mathcal{A}, G)$ *are isomorphic as* G*-graphs. Furthermore, the amalgam* $\mathcal{A}$ *is faithful.*

Proof. The required map is $\theta : \Gamma \to \Pi(\mathcal{A}, G)$ given by $\theta : \gamma \mapsto G_\alpha g$ where g is such that $\gamma = \alpha^g$. The details are similar to above. The amalgam is faithful since G acts faithfully on Γ. □

If Γ is a graph and B is a partition of the vertices of Γ, we can define the *quotient graph* Γ_B of Γ (with respect to B) as follows. The vertices of Γ_B are the parts comprising B, and two vertices of Γ_B are adjacent if there is an edge between the respective parts of Γ. There is a canonical graph homomorphism $\pi_B : \Gamma \to \Gamma_B$ given by taking a vertex x to the part to which it belongs. We are interested in quotient graphs because of the following.

Proposition 2.1.9. *Suppose that* B *is a* G*-invariant partition of* Γ*. Then* G *acts on* Γ_B *and the action commutes with the map* π_B*.*

Note that the orbits of a normal subgroup form a partition of B which is G-invariant. As an example of this, take Γ to be the circuit of length six and $G = \mathrm{Aut}(\Gamma) \cong \langle (1, 2, 3, 4, 5, 6), (1, 2)(3, 6)(4, 5)\rangle$. Then letting B be the orbits of $N = \langle (1, 4)(2, 5)(3, 6)\rangle$ we see that Γ_B is a circuit of length three, and $G/N \cong \mathrm{Dih}(6)$ acts faithfully on Γ_B.

The so-called global approach to analysing symmetric or semisymmetric graphs is to reveal the structure of G together with Γ by considering quotients determined by orbits of normal subgroups. More explicitly, suppose that N is a normal subgroup of G and let B be the partition of Γ induced by the orbits of N on Γ. Then the group G/N acts on Γ_B and various properties of Γ_B are inherited from G and Γ.

2.2 The local viewpoint, amalgams and extensions

We now develop a local approach to analysing symmetric and semisymmetric graphs. We defined amalgams in Section 1.6, we now develop a theory. Motivated by our need to compute with amalgams, we seek to understand the foundations of the topic. The goal is Theorem 2.2.25 which may be viewed as a constructive version of [17, (2.8)]. Indeed, it is by studying the proof of this result and how it is applied in [17] that informed our understanding of this topic. In particular, one consequence of our approach is a computational implementation of Theorem 2.2.25 which can be found in Section A.2. Throughout we fix an amalgam $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ and set $G = \mathcal{G}(\mathcal{A})$.

To develop the local viewpoint further, we introduce some terminology which aims to mirror the terminology associated to the global approach.

Definition 2.2.1. Let $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ be an amalgam and suppose that R_1, R_2 and D are subgroups of P_1, P_2 and B respectively. If properties (i) and (ii) below hold, we say that $\mathcal{B} = (R_1, R_2, D, \pi_1|_D, \pi_2|_D)$ is a *subamalgam* of $\mathcal{A}$.

(i) For $i = 1, 2$ we have $P_i = R_i\pi_i(B)$.

(ii) For $i = 1, 2$ we have $R_i \cap \pi_i(B) = \pi_i(D)$.

Lemma 2.2.2. *Suppose that $\mathcal{B}$ is a subamalgam of $\mathcal{A}$. Then the degree of $\mathcal{A}$ is equal to the degree of $\mathcal{B}$.*

Proof. We adopt the notation of Definition 2.2.1. For $i = 1, 2$ since $P_i = R_i\pi_i(B)$ it follows that $|P_i : \pi_i(B)| = |R_i : R_i \cap \pi_i(B)|$. Now $R_i \cap \pi_i(B) = \pi_i(D)$ and the maps involved are monomorphisms, so we obtain $|P_i : \pi_i(B)| = |R_i : \pi_i|_D(D)|$. □

Definition 2.2.3. With $\mathcal{A}$ and $\mathcal{B}$ as in Definition 2.2.1 we say that $\mathcal{B}$ is a *normal subamalgam* of $\mathcal{A}$ if R_i is a normal subgroup of P_i for $i = 1, 2$. An amalgam is *simple* if it has no proper normal subamalgams.

An immediate consequence of the definition of a normal subamalgam is that if $\mathcal{A}$ and $\mathcal{B}$ are as above and $\mathcal{B}$ is a normal subamalgam of $\mathcal{A}$, then D is a normal subgroup of B.

Lemma 2.2.4. *Suppose that* $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ *is a faithful amalgam and that* $\mathcal{N} = (N_1, N_2, D, \pi_1|_D, \pi_2|_D)$ *is a normal subamalgam. Then* $\mathcal{N}$ *is a faithful amalgam.*

Proof. Suppose that $1 \neq K \leq D$ and $\pi_i|_D(K) \lhd N_i$ for $i = 1, 2$. Set $L = \langle K^B \rangle$. Since π_i is a homomorphism we have $\pi_i(L) = \langle \pi_i(K)^{\pi_i(B)} \rangle$. Now $P_i = N_i\pi_i(B)$, N_i is normalised by $\pi_i(B)$ and normalises $\pi_i(K)$, hence $\pi_i(L) = \langle \pi_i(K)^{P_i} \rangle$ is a normal subgroup of P_i. It follows then that $1 \neq \pi_i(L) \lhd P_i$ and $L \leq B$, which contradicts the hypothesis that $\mathcal{A}$ is a faithful amalgam. $\square$

It will be important for us to find normal subamalgams of a given amalgam. The next proposition tells us how to go about this.

Proposition 2.2.5. *Let* $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ *be an amalgam. For* $i = 1, 2$ *suppose there are normal subgroups* U_i *of* P_i *such that* $U_i\pi_i(B) = P_i$*. Set*

$$C = \pi_1^{-1}(U_1 \cap \pi_1(B))\pi_2^{-1}(\pi_2(B) \cap U_2)$$

and $N_i = U_i\pi_i(C)$*. Then* $\mathcal{N} = (N_1, N_2, C, \pi_1|_C, \pi_2|_C)$ *is a normal subamalgam of* $\mathcal{A}$*.*

Proof. We have $N_i\pi_i(B) \geq U_i\pi_i(B) = P_i$ for $i = 1, 2$ and the Dedekind identity confirms that $N_i \cap \pi_i(B) = \pi_i(C)$. We also need to verify that N_i is normal in P_i, this follows from the factorisation $U_i\pi_i(B) = P_i$ and that C is a normal subgroup of B. $\square$

Definition 2.2.6. If (U_1, U_2) is a pair of subgroups satisfying the hypothesis of Proposition 2.2.5, we call the normal subamalgam constructed in Proposition 2.2.5 the *normal subamalgam generated by* (U_1, U_2).

Having defined normal subamalgams we also want to find a notion of a characteristic subamalgam. For this, we need to define isomorphisms between amalgams. Once we have established this, we can define the automorphism group of an amalgam, and the characteristic subamalgams are those that are invariant under the automorphism group.

Definition 2.2.7. An *amalgam homomorphism* from $\mathcal{B} = (R_1, R_2, D, \phi_1, \phi_2)$ to $\mathcal{A}$ is a triple of homomorphisms $\Theta = (\alpha, \beta, \gamma)$ such that $\{i, j\} = \{1, 2\}$ and the diagram in

Figure 2.1 commutes. The set of amalgam homomorphisms from $\mathcal{B}$ to $\mathcal{A}$ is denoted by

$$\begin{tikzcd}
 & & R_1 \arrow[rr, "\alpha"] & & P_i & \\
\mathcal{B} & D \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] \arrow[rr, "\gamma"] & & B \arrow[ur, "\pi_i"'] \arrow[dr, "\pi_j"] & & \mathcal{A} \\
 & & R_2 \arrow[rr, "\beta"] & & P_j &
\end{tikzcd}$$

Figure 2.1: An amalgam homomorphism

$\mathrm{Hom}(\mathcal{B}, \mathcal{A})$. The amalgam homomorphism Θ is an *amalgam monomorphism*, respectively *amalgam isomorphism*, if it is a triple of monomorphisms, respectively, isomorphisms. We write $\mathrm{Aut}(\mathcal{A})$ for automorphism group of $\mathcal{A}$, that is, the set of amalgam isomorphisms $\Theta : \mathcal{A} \to \mathcal{A}$ (which forms a group under composition).

Note that by our definition, the amalgams $\mathcal{A}$ and $(P_2, P_1, B, \pi_2, \pi_1)$ are isomorphic. Observe that $\mathrm{Aut}(\mathcal{A})$ has a subgroup of index at most two which normalises P_1 and P_2. We denote this subgroup by $\mathrm{Aut}^\circ(\mathcal{A})$. We now introduce our approach to working with $\mathrm{Aut}(\mathcal{A})$; we shall see shortly that $\mathrm{Aut}^\circ(\mathcal{A})$ is easier to deal with.

Example 2.2.8. Amalgams of type $\mathcal{Q}_3^1$. *Let $G = \mathrm{Sym}(9)$ and choose elements $a = (1,2,3,4,5)$, $b = (2,3,5,4)$ and $c = (2,6)(3,7)(5,8)(4,9)$ of G. Define $A = \langle a, b, b^c\rangle$ and $B = \langle b, c\rangle$ and $C = A \cap B$. Let π_1 and π_2 be the identifications of C as a subgroup of A and B respectively. Note that the core in A and B of $\pi_i(C)$ is trivial for $i = 1, 2$.*

Let γ and δ be the automorphisms of C induced by the maps $\gamma : b \mapsto b$, $\gamma : b^c \mapsto bb^c$ and $\delta : b \mapsto b$, $\delta : b^c \mapsto b^3b^c$. Let $\mathcal{A} = (A, B, C, \pi_1, \pi_2)$, $\mathcal{A}_1 = (A, B, C, \pi_1, \pi_2\gamma)$ and $\mathcal{A}_2 = (A, B, C, \pi_1, \pi_2\delta)$ and set $X = \langle b^c\rangle$. Now the three amalgams are all of the same type, but the following commutators show that the amalgams are pairwise non-isomorphic

$$[\pi_1(X), A] = [\pi_2\gamma(X), B] = 1,\ [\pi_2\delta(X), B] = \langle (bb^c)^2\rangle,\ [\pi_2(X), B] = \langle bb^c\rangle.$$

Notation 2.2.9. We write each $\Theta \in \mathrm{Aut}(\mathcal{A})$ as a triple $\Theta = (\alpha_1, \alpha_2, \gamma)$ where the domain of α_i is P_i and of γ is B. Since Θ acts on the set $\{1,2\}$ via the map $\mathrm{Aut}(\mathcal{A}) \mapsto \mathrm{Aut}(\mathcal{A})/\mathrm{Aut}^\circ(\mathcal{A})$, we may calculate $\Theta(1)$ and $\Theta(2)$ (and observe that these values determine whether Θ swaps P_1 and P_2 or not).

With the notation for elements of $\mathrm{Aut}(\mathcal{A})$ introduced above, we have to understand how composition works. For future reference, we summarise the rules below (these are determined by drawing diagrams and evaluating the maps).

Proposition 2.2.10. *The following hold for* $\Theta = (\alpha_1, \alpha_2, \gamma_1)$, $\Omega = (\beta_1, \beta_2, \gamma_2) \in \mathrm{Aut}(\mathcal{A})$.

(i) $\Theta^{-1} = (\alpha_{\Theta(1)}^{-1}, \alpha_{\Theta(2)}^{-1}, \gamma_1)$.

(ii) $\Theta\Omega = (\alpha_{\Omega(1)}\beta_1, \alpha_{\Omega(2)}\beta_2, \gamma_1\gamma_2)$.

(iii) For $\Omega \in \mathrm{Aut}^\circ(\mathcal{A})$,

$$\Omega^\Theta = (\alpha_1^{-1}\beta_{\Theta(1)}\alpha_1, \alpha_2^{-1}\beta_{\Theta(2)}\alpha_2, \gamma_1^{-1}\gamma_2\gamma_1).$$

Determining $\mathrm{Aut}(\mathcal{A})$ for a given amalgam requires knowledge of the automorphism groups of P_1, P_2 and B and information on $\mathrm{Hom}(P_1, P_2)$. The group $\mathrm{Aut}^\circ(\mathcal{A})$ is easier to deal with then, since we do not need information on the latter set. We now explain how we can calculate $\mathrm{Aut}^\circ(\mathcal{A})$ via another approach. Let $\Theta = (\alpha, \beta, \gamma) \in \mathrm{Aut}^\circ(\mathcal{A})$, so the diagram in Figure 2.2 commutes. We have $\alpha(\pi_1(B)) = \pi_1(\gamma(B))$ which implies $\alpha \in \mathrm{N}_{\mathrm{Aut}(P_1)}(\pi_1(B))$. Similarly, we have $\beta \in \mathrm{N}_{\mathrm{Aut}(P_2)}(\pi_2(B))$ and since the diagram commutes $\pi_1^{-1}\alpha\pi_1 = \gamma = \pi_2^{-1}\beta\pi_2$. For any pair $(\alpha, \beta) \in \mathrm{N}_{\mathrm{Aut}(P_1)}(\pi_1(B)) \times \mathrm{N}_{\mathrm{Aut}(P_2)}(\pi_2(B))$ such that $\pi_1^{-1}\alpha\pi_1 = \pi_2^{-1}\beta\pi_2$ we define an automorphism $\Theta = (\alpha, \beta, \pi_1^{-1}\alpha\pi_1)$ of $\mathcal{A}$. In other words, we have established an isomorphism:

$$\mathrm{Aut}^\circ(\mathcal{A}) \cong \{(\alpha, \beta) \in \mathrm{N}_{\mathrm{Aut}(P_1)}(\pi_1(B)) \times \mathrm{N}_{\mathrm{Aut}(P_2)}(\pi_2(B)) \mid \pi_1^{-1}\alpha\pi_1 = \pi_2^{-1}\beta\pi_2\}.$$

For the purpose of calculating $\mathrm{Aut}^\circ(\mathcal{A})$ it is beneficial to identify $\mathrm{Aut}^\circ(\mathcal{A})$ as above. For theoretical purposes it may be convenient to revert to the notation given in 2.2.9.

$$\begin{tikzcd}
 & & P_1 \arrow[rr, "\alpha"] & & P_1 & \\
\mathcal{A} & B \arrow[ur, "\pi_1"'] \arrow[rr, "\gamma"] \arrow[dr, "\pi_2"] & & B \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & & \mathcal{A} \\
 & & P_2 \arrow[rr, "\beta"] & & P_2 &
\end{tikzcd}$$

Figure 2.2: The automorphism of $\mathcal{A}$ determined by Θ

Definition 2.2.11. A subamalgam of $\mathcal{A}$ is *characteristic* if it is $\mathrm{Aut}(\mathcal{A})$-invariant. We say that an amalgam $\mathcal{B}$ is an *extension of $\mathcal{A}$ via* Θ if $\Theta : \mathcal{A} \to \mathcal{B}$ is an amalgam monomorphism such that $\Theta(\mathcal{A})$ is a normal subamalgam of $\mathcal{B}$.

Given $\mathcal{A}$ as above we observe that there are infinitely many extensions of $\mathcal{A}$. Indeed, set $R_i = P_i \times \mathrm{C}_n$, $D = B \times \mathrm{C}_n$ and define ϕ_1, ϕ_2 in the obvious way, then $\mathcal{B} = (R_1, R_2, D, \phi_1, \phi_2)$ is an extension of $\mathcal{A}$ (which is *not* faithful). On the other hand, Theorem 2.2.25 at the end of this section says that there are finitely many extensions which are faithful and a (unique) largest such extension exists. Lemma 2.2.4 shows that every faithful amalgam which is not simple is an extension of a faithful amalgam. Theorem 2.2.25 will allow us to recover such an amalgam from the normal subamalgam.

Given a group G and only the knowledge that G has a normal subgroup N of which one knows the isomorphism type, it is usually a difficult task to determine the isomorphism type of G, that is, to classify the possible extensions of N. We would like to know how many of these extensions are central, split, non-split and so forth. If we are given the additional information that $N = F^*(G)$ and that N is a non-abelian simple group, we then know that G is isomorphic to a subgroup of $\mathrm{Aut}(N)$, so we just have to determine $\mathrm{Aut}(N)$ and amongst the subgroups of $\mathrm{Aut}(N)$ which contain $\mathrm{Inn}(N)$ we find the possibilities for G (and we only have to consider these subgroups up to conjugacy). We aim to obtain a similar recipe for amalgams. In generality this is not possible of course, but in the class of faithful amalgams, a similar trick works. First we need to set up some machinery.

For the remainder of this section we fix an amalgam $\mathcal{N} = (N_1, N_2, D, \phi_1, \phi_2)$. We transport the automorphisms of N_1 and N_2 which induce automorphisms of D into $\mathrm{Aut}(\mathcal{N})$ as follows. Let $H_1 = \mathrm{N}_{\mathrm{Aut}(N_1)}(\phi_1(D))$ and $H_2 = \mathrm{N}_{\mathrm{Aut}(N_2)}(\phi_2(D))$ and set $H = H_1 \times H_2$. For $\alpha \in H_i$ we define a map $\theta_i(\alpha) : D \to D$ by $\theta_i(\alpha) : d \mapsto \phi_i^{-1}(\alpha(\phi_i(d)))$. Then $\theta_i : H_i \to \mathrm{Aut}(D)$ defined by $\theta_i : \alpha \mapsto \theta_i(\alpha)$ is a homomorphism. We set $H_i^* = \theta_i(H_i)$. With this notation we state the amalgam counting lemma of Goldschmidt. This tells us how many amalgams of a certain type there are, up to isomorphism.

Lemma 2.2.12 (Goldschmidt)**.** *The number of isomorphism classes of amalgams of the same type as $\mathcal{N}$ is the number of (H_1^*, H_2^*)-double cosets in $\mathrm{Aut}(D)$. Moreover if $\gamma_1, \ldots, \gamma_n$ are representatives for these double cosets then*

$$\{(N_1, N_2, D, \phi_1, \phi_2\gamma_i) \mid i \in [1, n]\}$$

is a complete set of representatives for the isomorphism classes of amalgams of this type and contains no repetitions.

Proof. See [17, (2.7)]. □

Example 2.2.13. *Let $a = (1, 2, 3, 4, 5)$, $b = (2, 5)(3, 4)$, $c = (6, 7, 8, 9, 10)$ and $d = (7, 10)(8, 9)$ and let $P_1 = \langle a, b, d\rangle$, $P_2 = \langle b, c, d\rangle$ and $B = \langle b, d\rangle$ so that $P_1 \cong \mathrm{Dih}(20) \cong P_2$ and $B \cong 2^2$. Set $\mathcal{A} = (P_1, P_2, B, \mathrm{id}_{P_1}, \mathrm{id}_{P_2})$. Write $x_1 = b$, $x_2 = d$ and $x_3 = bd$ and view $\mathrm{Aut}(B)$ as $\mathrm{Sym}(3)$ acting on subscripts. We see that $H_1^* = \langle (1, 3)\rangle$ and $H_2^* = \langle (2, 3)\rangle$. The three cosets in $\mathrm{Aut}(B)$ of H_1^* are H_1^*, $H_1^*(2, 3)$ and $H_1^*(1, 2)$ so H_2^* has two orbits on the cosets of H_1^* in $\mathrm{Aut}(B)$. Hence there are exactly two amalgams of the same type as $\mathcal{A}$. The first we have written down above is faithful, the second is given by interchanging b and d in the embedding of B in P_2. The effect of this is that d is central in both P_1 and P_2, so the second amalgam is not faithful.*

We have identified $\mathrm{Aut}^\circ(\mathcal{N})$ with $\{(\alpha, \beta) \in H \mid \theta_1(\alpha) = \theta_2(\beta)\}$. This gives the following presentation of $\mathrm{Aut}^\circ(\mathcal{N})$

$$\mathrm{Aut}^\circ(\mathcal{N}) = \left\langle \begin{array}{c|c} (\theta_1^{-1}(\mu), \theta_2^{-1}(\mu)) & \alpha \in C_{H_1}(\phi_1(D)),\ \beta \in C_{H_2}(\phi_2(D)), \\ (\alpha, 1),\ (1, \beta) & \mu \in H_1^* \cap H_2^* \end{array} \right\rangle.$$

Recall that to determine $\mathrm{Aut}(\mathcal{N})$ we require knowledge of the isomorphisms between N_1 and N_2. We give the following example which shows how one can determine $\mathrm{Aut}(\mathcal{N})$ after finding $\mathrm{Aut}^\circ(\mathcal{N})$.

Example 2.2.14. *Take* $N_1 = \langle (1, 2, 3) \rangle$*,* $N_2 = \langle (4, 5, 6) \rangle$ *and* $D = 1$*. Let* $\phi_i : D \to N_i$ *be the inclusions and set* $\mathcal{N} = (N_1, N_2, D, \phi_1, \phi_2)$*. We will write automorphisms of* N_1 *and* N_2 *and maps between* N_1 *and* N_2 *as permutations. We have* $\mathrm{Aut}(N_1) = \langle (1, 2) \rangle$, $\mathrm{Aut}(N_2) = \langle (4, 5) \rangle$ *and, since* $D = 1$, $\mathrm{Aut}(N_i) = N_{\mathrm{Aut}(N_i)}(\phi_i(D))$*. We define* $\Theta_1 = ((1, 2), 1, 1)$ *and* $\Theta_2 = (1, (4, 5), 1)$*, which are clearly automorphisms of* $\mathcal{N}$*. Moreover* $\mathrm{Aut}^\circ(\mathcal{N}) = \langle \Theta_1, \Theta_2 \rangle$ *(since it cannot be larger). Now the permutation* $x = (1, 4)(2, 5)(3, 6)$ *induces the automorphism* $\Theta_3 = (x, x, 1) \notin \mathrm{Aut}^\circ(\mathcal{N})$ *and* $(\Theta_3)^2 = (1, 1, 1)$*. Hence* $\mathrm{Aut}(\mathcal{N}) = \langle \Theta_1, \Theta_2, \Theta_3 \rangle \cong \mathrm{Dih}(8)$*.*

From now on we assume that $\mathcal{N}$ is a faithful amalgam.

For $i = 1, 2$ we regard H_i as a subgroup of H. We define $\theta : D \to H$ by

$$\theta : d \mapsto (\mathrm{c}_1(\phi_1(d)), \mathrm{c}_2(\phi_2(d)))$$

where $\mathrm{c}_i(b)$ is the automorphism of N_i defined by $a \mapsto a^b$ for a, $b \in N_i$.

Lemma 2.2.15. *The map* θ *defined above is a monomorphism. Moreover* $\theta(D)$ *is a normal subgroup of* $\mathrm{Aut}(\mathcal{N})$ *which is contained in* $\mathrm{Aut}^\circ(\mathcal{N})$*.*

Proof. We first check that θ is a homomorphism. Let $a, b \in D$ and consider $\theta(a)\theta(b)$. Let $a_i = \phi_i(a)$ and $b_i = \phi_i(b)$ for $i = 1, 2$ (so that $a_1 b_1 = \phi_1(ab)$ and $a_2 b_2 = \phi_2(ab)$). Then we find

$$\theta(a)\theta(b) = (c_1(a_1), c_2(a_2))(c_1(b_1), c_2(b_2)) = (c_1(a_1)c_1(b_1), c_2(a_2)c_2(b_2))$$

and since c_i and ϕ_i are monomorphisms for $i = 1, 2$, we have $\theta(a)\theta(b) = \theta(ab)$. Now observe that $[\phi_i(\ker\theta), N_i] = 1$ for $i = 1, 2$. Hence $\ker\theta = 1$ since $\mathcal{N}$ is faithful. It is clear that $\theta(D) \leq \mathrm{Aut}^\circ(\mathcal{N})$.

Now let $\Theta \in \mathrm{Aut}(\mathcal{N})$ and (recalling 2.2.9) write $\Theta = (\alpha_1, \alpha_2, \beta)$. We need to show that $\theta(a)^\Theta \in \theta(D)$. Writing $\theta(a)$ as a triple we have $\theta(a) = (c_1(a_1), c_2(a_2), c_D(a))$ (where $c_D : D \to \mathrm{Inn}(D)$ is the obvious map). Then Proposition 2.2.10 gives

$$\begin{aligned} \theta(a)^\Theta &= (\alpha_1^{-1} c_{\Theta(1)}(a_{\Theta(1)})\alpha_1, \alpha_2^{-1} c_{\Theta(2)}(a_{\Theta(2)})\alpha_2, \beta^{-1} c_D(a))\beta) \\ &= (c_1(\alpha_1^{-1}(a_{\Theta(1)})), c_2(\alpha_2^{-1}(a_{\Theta(2)})), c_D(\gamma^{-1}(a))) \end{aligned}$$

Since $\Theta^{-1} \in \mathrm{Aut}(\mathcal{N})$ the following diagram commutes

$$\begin{tikzcd} & N_{\Theta(1)} \arrow[rr, "\alpha_1^{-1}"] & & N_1 \\ D \arrow[ur, "\phi_{\Theta(1)}"] \arrow[rr, "\gamma^{-1}"] & & D \arrow[ur, "\phi_1"] & \end{tikzcd}$$

hence $\alpha_1^{-1}(a_{\Theta(1)}) = \phi_1(\gamma^{-1}(a))$. Similarly, we obtain $\alpha_2^{-1}(a_{\Theta(2)}) = \phi_2(\gamma^{-1}(a))$ so setting $b = \gamma^{-1}(a)$ and $b_i = \phi_i(b)$ for $i = 1, 2$ we obtain $\theta(a)^\Theta = (c_1(b_1), c_2(b_2), c_D(b))$ as required.

□

Notation 2.2.16. The *inner automorphism group* $\mathrm{Inn}(\mathcal{N})$ is the image of D under θ. The outer automorphism $\mathrm{Out}(\mathcal{N})$ is the quotient $\mathrm{Aut}(\mathcal{N})/\mathrm{Inn}(\mathcal{N})$ and we write $\mathrm{Out}^\circ(\mathcal{N})$ for the quotient $\mathrm{Aut}^\circ(\mathcal{N})/\mathrm{Inn}(\mathcal{N})$.

Example 2.2.17. *Continuing with Example 2.2.14, the image of D under θ is the trivial subgroup, so trivially is a normal subgroup of* $\mathrm{Aut}(\mathcal{N})$ *and all automorphisms are outer.*

Working in a bigger amalgam gives us a better example. Set $P_1 = \langle (1,2,3), (1,2)\rangle$ and $P_2 = \langle (4,5,6), (4,5)\rangle$ and let $B = \langle (7,8)\rangle$. Define π_1 by $\pi_1((7,8)) = (1,2)$ and π_2 by $\pi_2((7,8)) = (4,5)$ and set $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$. Then identifying $\mathrm{Aut}(P_i)$ *with P_i we have that* $\mathrm{Inn}(\mathcal{A}) = \langle ((1,2),(4,5),1)\rangle$ *and (the same as in Example 2.2.14)* $\mathrm{Aut}(\mathcal{A}) = \langle \Theta_1, \Theta_2, \Theta_3\rangle$ *we see that* $\mathrm{Inn}(\mathcal{A}) = \langle \Theta_1\Theta_2\rangle$ *which is central in* $\mathrm{Aut}(\mathcal{A})$.

By the Correspondence Theorem, there is a bijection between the conjugacy classes of subgroups of $\mathrm{Aut}(\mathcal{N})$ which contain $\mathrm{Inn}(\mathcal{N})$ and are contained in $\mathrm{Aut}^\circ(\mathcal{N})$ and the conjugacy classes of subgroups of $\mathrm{Out}(\mathcal{N})$ which are contained in $\mathrm{Out}^\circ(\mathcal{N})$. We now proceed to show how an extension of $\mathcal{N}$ may be constructed from a subgroup of $\mathrm{Aut}^\circ(\mathcal{N})$.

Let R be a subgroup of $\mathrm{Aut}^\circ(\mathcal{N})$ which contains $\theta(D)$. Writing elements of $\mathrm{Aut}^\circ(\mathcal{N})$ as pairs (α_1, α_2) where $\alpha_i \in H_i$ (identified as a subgroup of H) we have homomorphisms $\xi_i : H \to \mathrm{Aut}(N_i)$ defined by $\xi_i : (\alpha_1, \alpha_2) \mapsto \alpha_i$. Restricting these maps to R we set $U_i = N_i \rtimes_{\xi_i} R$. We identify N_i and R with the subgroups $\{(n, 1) \mid n \in N_i\}$ and $\{(1, r) \mid r \in R\}$ of U_i respectively. Define $\mu_i : D \to U_i$ by $d \mapsto (\phi_i(d^{-1}), \theta(d))$ and set $C_i = \mu_i(D)$ for $i = 1, 2$.

Lemma 2.2.18. *The set C_i is a normal subgroup of U_i. Moreover, $C_i \cap N_i = 1 = C_i \cap R$.*

Proof. We first claim that the map μ_i is an isomorphism, since ϕ_i is a monomorphism this will follow once we show that μ_i is a homomorphism. Let a, $b \in D$ and consider $\mu_i(a)\mu_i(b)$. We have

$$\begin{aligned} \mu_i(a)\mu_i(b) &= (\phi_i(a^{-1}), \theta(a))(\phi_i(b^{-1}), \theta(b)) \\ &= (\phi_i(a^{-1})(\phi_i(b^{-1}))^{\xi_i(\theta(a))^{-1}}, \theta(a)\theta(b)) \end{aligned}$$

and $\phi_i(b^{-1})^{\xi_i(\theta(a))^{-1}} = \phi_i(b^{-1})^{\phi_i(a^{-1})}$ which gives

$$\phi_i(a^{-1})(\phi_i(b^{-1}))^{\xi_i(\theta(a))^{-1}} = \phi_i(a^{-1})\phi_i(a)\phi_i(b^{-1})\phi_i(a)^{-1}.$$

Since ϕ_i and θ are homomorphisms then, we see that μ_i is a homomorphism. In particular, C_i is a subgroup of U_i.

Now let $(x, y) \in U_i$ and $b \in D$. We need to show $\mu_i(b)^{(x,y)} \in \mu_i(D)$. Observe that

$$\mu_i(b)^{(x,y)} = ((x^{-1}\phi_i(b^{-1})x^{\phi_i(b^{-1})})^{\xi_i(y)}, \theta(b)^y) = (\phi_i(b^{-1})^{\xi_i(y)}, \theta(b)^y).$$

Now let $\alpha \in H_1$ and $\beta \in H_2$ be such that $y = (\alpha, \beta)$. Since $(\alpha, \beta) \in \mathrm{Aut}^\circ(\mathcal{N})$ there is $b' \in$

D such that $\phi_1(b') = b^\alpha$ and $\phi_2(b') = b^\beta$ we have $\theta(b)^y = (c_1(b)^\alpha, c_2(b)^\beta) = (c_1(b'), c_2(b'))$ and $\phi_i(b^{-1})^{\xi_i(y)} = \phi_i((b')^{-1})$. Thus $\mu_i(b)^{(x,y)} = \mu_i(b') \in \mu_i(D)$ as required.

For the assertion about intersections, note that every element of N_i is written as $(n, 1)$ and so if $\mu(d) \in N_i \cap \mu_i(D)$ we have $\theta(d) = 1$ which gives $d = 1$. Similarly, if $\mu(d) \in \mu_i(D) \cap R$ then $\phi_i(d) = 1$ whence $d = 1$. $\square$

By Lemma 2.2.18 we may define the quotient $V_i = U_i/C_i$. Set $\rho_i : N_i \to V_i$ by $n \mapsto (n, 1)C_i$. We let $\mathcal{E} = (V_1, V_2, R, \beta_1, \beta_2)$ where $\beta_i : R \to V_i$ is defined by $r \mapsto (1, r)C_i$. Lemma 2.2.18 implies that the maps β_i are monomorphisms so $\mathcal{E}$ is an amalgam.

Lemma 2.2.19. *The amalgam $\mathcal{E}$ is faithful and is an extension of $\mathcal{N}$.*

Proof. We claim that the triple $\Theta = (\rho_1, \rho_2, \theta)$ is an amalgam monomorphism $\Theta : \mathcal{N} \to \mathcal{E}$ with $\Theta(\mathcal{N})$ a normal subamalgam of $\mathcal{E}$. It follows from the definition of V_i as a quotient of U_i that the diagram in Figure 2.3 commutes. Set $M_i = \rho_i(N_i)$ and $F = \theta(D)$. We need to show that the triple (M_1, M_2, F) defines a subamalgam of $\mathcal{E}$.

$$\begin{tikzcd}
 & & N_1 \arrow[rr, "\rho_1"] & & V_1 & \\
\mathcal{N} & D \arrow[ur, "\phi_1"] \arrow[rr, "\theta"] \arrow[dr, "\phi_2"'] & & R \arrow[ur, "\beta_1"'] \arrow[dr, "\beta_2"] & & \mathcal{E} \\
 & & N_2 \arrow[rr, "\rho_2"] & & V_2 &
\end{tikzcd}$$

Figure 2.3: The amalgam monomorphism $\Theta : \mathcal{N} \to \mathcal{E}$

By construction we have $M_i\beta_i(R) = V_i$ and $M_i \cap \beta_i(R) = \rho_i\phi_i(D) = \beta_i\theta(D) = \beta_i(F)$. Also M_i is a normal subgroup of V_i, so $\Theta(\mathcal{N})$ is indeed a normal subamalgam of $\mathcal{E}$.

Let $K \leq R$ and assume that $\beta_i(K)$ is a normal subgroup of V_i for $i = 1, 2$. Since $\beta_i(K) \cap M_i \leq \beta_i(R) \cap M_i = \beta_i\theta(D)$, setting $K_0 = K \cap \theta(D)$ we see that $\phi_i\theta^{-1}(K_0)$ is a normal subgroup of N_i for $i = 1, 2$. Now $\mathcal{N}$ is a faithful amalgam so we obtain $K_0 = 1$, whence $\beta_i(K)$ commutes with M_i. Letting $(\alpha_1, \alpha_2) \in K$ it follows that $n_i^{\alpha_i} = n_i$ for all $n_i \in N_i$, and so $\alpha_i = 1$ for $i = 1, 2$. Hence $K = 1$ and we are done. $\square$

Definition 2.2.20. With the notation as above, we write $\mathcal{E}(\mathcal{N}, R)$ for the amalgam $\mathcal{E}$ constructed in Lemma 2.2.19.

We now show that to find the isomorphism type of an extension such as $\mathcal{E}(\mathcal{N}, R)$ we only need to consider the $\mathrm{Aut}(\mathcal{N})$-conjugacy class of subgroups of $\mathrm{Aut}^\circ(\mathcal{N})$ to which R belongs.

Lemma 2.2.21. *Let* $\Theta \in \mathrm{Aut}(\mathcal{N})$ *and* $R \leq \mathrm{Aut}^\circ(\mathcal{N})$. *Then* $\mathcal{E}(\mathcal{N}, R)$ *is isomorphic to* $\mathcal{E}(\mathcal{N}, R^\Theta)$.

Proof. We view both extensions as subamalgams of $\mathcal{E}(\mathcal{N}, \mathrm{Aut}^\circ(\mathcal{N}))$. Then the result is obvious. $\square$

Example 2.2.22. *Continuing with Example 2.2.14 we have* $\mathrm{Aut}(\mathcal{N}) = \langle \Theta_1, \Theta_2, \Theta_3 \rangle \cong \mathrm{Dih}(8)$ *and* $\mathrm{Aut}^\circ(\mathcal{N}) = \langle \Theta_1, \Theta_2 \rangle$.

Let $R_1 = \langle \Theta_1 \rangle$, $R_2 = \langle \Theta_2 \rangle$, $R_3 = \langle \Theta_1\Theta_2 \rangle$, $R_4 = \mathrm{Aut}^\circ(\mathcal{A})$ *and let* $\mathcal{E}_i = \mathcal{E}(\mathcal{A}, R_i)$. *Then we see that the types of* $\mathcal{E}_i$ *are* $(\mathrm{Sym}(3), \mathrm{C}_6, \mathrm{C}_2)$, $(\mathrm{C}_6, \mathrm{Sym}(3), \mathrm{C}_2)$, $(\mathrm{Sym}(3), \mathrm{Sym}(3), \mathrm{C}_2)$ *and* $(\mathrm{C}_2 \times \mathrm{Sym}(3), \mathrm{C}_2 \times \mathrm{Sym}(3), 2^2)$. *Note that the amalgams* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are isomorphic, and that* R_1 *and* R_2 *are conjugate in* $\mathrm{Aut}(\mathcal{A})$. *So in this case,* $\mathrm{Aut}(\mathcal{N})$ *conjugacy just reminds us that the amalgams* $(N_1, N_2, D, \phi_1, \phi_2)$ *and* $(N_2, N_1, D, \phi_2, \phi_1)$ *are isomorphic.*

Fix now a faithful amalgam $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ and assume that $\mathcal{N}$ is a normal subamalgam of $\mathcal{A}$ (so that $N_i \lhd P_i$, $D \lhd B$ and $\phi_i = \pi_i|_D$ for $i = 1, 2$). We define $\widetilde{\theta} : B \to \mathrm{Aut}(N_1) \times \mathrm{Aut}(N_2)$ by

$$\widetilde{\theta} : b \mapsto (c_1(\pi_1(b)), c_2(\pi_2(b)))$$

where the map $c_i(x) : N_i \to N_i$ is conjugation induced by $x \in P_i$.

Lemma 2.2.23. *The map* $\widetilde{\theta}$ *is a monomorphism and* $\widetilde{\theta}|_D = \theta$. *Moreover* $\widetilde{\theta}(B)$ *is contained in* $\mathrm{Aut}^\circ(\mathcal{N})$.

Proof. Since $\widetilde{\theta}$ is a composition of homomorphisms it is itself a homomorphism. Let $b \in \ker \widetilde{\theta}$, then $\pi_i(b)$ centralises N_i and so $\pi_i(\ker \widetilde{\theta})$ is a normal subgroup of P_i for $i \in \{1, 2\}$, whence $\ker \widetilde{\theta} = 1$. Let $d \in D$, then $\widetilde{\theta}(d) = (c_1(\pi_1(d)), c_2(\pi_2(d))) = (c_1(\phi_1(d)), c_2(\phi_2(d))) = \theta(d)$ as required.

Finally, we need to verify that $\widetilde{\theta}(B)$ is contained in $\mathrm{Aut}^\circ(\mathcal{N})$. This requires

$$(c_1(\pi_1(b)), c_2(\pi_2(b))) \in \mathrm{Aut}^\circ(\mathcal{N})$$

for all $b \in B$, so we need to verify that the following equality holds:

$$\phi_1^{-1} c_1(\pi_1(b))\phi_1 = \phi_2^{-1} c_2(\pi_2(b))\phi_2$$

for all $b \in B$. But these maps are both the automorphism of D induced by conjugation by b, so the result follows. $\square$

Lemma 2.2.23 shows that $\theta(D) = \widetilde{\theta}(D) \leq \widetilde{\theta}(B) \leq \mathrm{Aut}^\circ(\mathcal{N})$. Hence if $\mathcal{N}$ is a normal subamalgam of $\mathcal{A}$ then $\widetilde{\theta}$ defines a subgroup of $\mathrm{Aut}^\circ(\mathcal{N})$ which contains $\theta(D)$. Let us now drop the assumption that $\mathcal{N}$ is a normal subamalgam of $\mathcal{A}$, and assume that $\mathcal{A}$ is an extension of $\mathcal{N}$. Therefore, there is an amalgam monomorphism $\Theta : \mathcal{N} \to \mathcal{A}$ such that $\mathcal{M} := \Theta(\mathcal{N})$ is a normal subamalgam of $\mathcal{A}$. Then (abusing the language we have introduced above) $\mathcal{A}$ determines a subgroup $\widetilde{\theta}(B)$ of $\mathrm{Aut}^\circ(\mathcal{M})$. Since $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{N})$ are isomorphic (via Θ) the extension $\mathcal{A}$ of $\mathcal{N}$ determines a subgroup of $\mathrm{Aut}(\mathcal{N})$, but note the dependence on Θ. The notation we introduce below highlights this dependence.

Notation 2.2.24. We write $E(\mathcal{N}, \mathcal{A}, \Theta)$ for the subgroup of $\mathrm{Aut}^\circ(\mathcal{N})$ defined by a faithful amalgam $\mathcal{A}$ which is an extension of $\mathcal{N}$ via Θ.

By Lemma 2.2.19 we may construct the amalgam $\mathcal{E}(\mathcal{N}, E(\mathcal{N}, \mathcal{A}, \Theta))$, and naturally, we ask if this differs from $\mathcal{A}$. The following theorem answers this question "No", and so we conclude that all faithful extensions of $\mathcal{N}$ can be "seen" inside $\mathrm{Aut}(\mathcal{N})$.

Theorem 2.2.25 (Extension theorem)**.** *The amalgams $\mathcal{A}$ and $\mathcal{E}(\mathcal{N}, E(\mathcal{N}, \mathcal{A}, \Theta))$ are isomorphic.*

Proof. Let $\Theta = (\alpha_1, \alpha_2, \beta)$. We make two simplifications to the notation so that we will not need to explicitly refer to Θ. First we identify $\mathcal{N}$ with $\Theta(\mathcal{N})$, so that N_1, N_2 and D are identified with their images in P_1, P_2 and B respectively. Then the map $\widetilde{\theta} : B \to \mathrm{Aut}(\mathcal{N})$ is defined. Note that our identifications mean that we may write $\widetilde{\theta}(b) = (c_1(\pi_1(b)), c_2(\pi_2(b)))$ rather than having to write $\widetilde{\theta}(b) = (\alpha_1^{-1}(c_{\alpha_1(N_1)}(\pi_1(b))\alpha_1, \alpha_2^{-1}(c_{\alpha_2(N_2)}(\pi_2(b))\alpha_2))$. We expect therefore that this identification will simplify the following exposition. Now let $R = \widetilde{\theta}(B)$, $\mathcal{E} = \mathcal{E}(\mathcal{N}, R)$ and then matching notation with Lemma 2.2.19 we have $\mathcal{E} = (V_1, V_2, R, \beta_1, \beta_2)$. We now identify N_1, N_2 and D with their images in V_1, V_2 and R respectively, that is N_i is identified with the subgroup $\{(n, 1)C_i \mid n \in N_i\}$ of V_i and D is identified with $\widetilde{\theta}(D) = \theta(D)$. We need to find isomorphisms which make the following diagram commute.

$$\begin{tikzcd}
 & N_1 \arrow[rr] \arrow[rrrr, bend left] & & P_1 \arrow[rr, dashed] & & V_1 \\
D \arrow[ur, "\phi_1"] \arrow[rr] \arrow[dr, "\phi_2"'] & & B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] \arrow[rr, dashed] & & \widetilde{\theta}(B) = R \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"] & \\
 & N_1 \arrow[rr] \arrow[rrrr, bend right] & & P_1 \arrow[rr, dashed] & & V_2
\end{tikzcd}$$

There is an obvious choice for these maps. We have an isomorphism $\widetilde{\theta} : B \to R$ and we need to find maps from P_i to V_i for $i = 1, 2$. Using that P_i admits the factorisation $P_i = N_i \pi_i(B)$ we define $\gamma_i : P_i \to V_i$ by

$$\gamma_i : n_i \pi_i(b) \mapsto (n_i, 1)(1, \widetilde{\theta}(b))C_i.$$

Note that for $b \in B$ we have $\gamma_i(\pi_i(b)) = (1, \widetilde{\theta}(b))C_i = \beta_i(\widetilde{\theta}(b))$. So with $\Delta = (\gamma_1, \gamma_2, \widetilde{\theta})$ the diagram above will commute. To ensure that Δ is an amalgam isomorphism then we need to check that the maps γ_1 and γ_2 are group isomorphisms.

First we will show that γ_i is well-defined (since possibly $N_i \cap \pi_i(B) = \pi_i(D)$ is non-trivial). So for $i \in \{1, 2\}$ suppose there are $m, n \in N_i$ and $a, b \in B$ such that $n\pi_i(a) = m\pi_i(b)$. Then $m^{-1}n = \pi_i(ba^{-1}) \in N_i \cap \pi_i(B) = \phi_i(D) = \pi_i(D)$. So there is $d \in D$ such that $n = m\phi_i(d)$ and $a = d^{-1}b$. Hence

$$\begin{aligned}
\gamma_i(n\pi_i(a)) &= (m\phi_i(d), \widetilde{\theta}(d^{-1}b))C_i \\
&= (m\phi_i(d), \widetilde{\theta}(d^{-1})\widetilde{\theta}(b))C_i \\
&= (m, 1)C_i(\phi_i(d), \widetilde{\theta}(d^{-1})C_i(1, \widetilde{\theta}(b))C_i \\
&= (m, 1)C_i(\phi_i((d^{-1})^{-1}), \widetilde{\theta}(d^{-1}))C_i(1, \widetilde{\theta}(b))C_i
\end{aligned}$$

By the definition of C_i we have $(\phi_i((d^{-1})^{-1}), \widetilde{\theta}(d^{-1}))C_i = C_i$, so we see the last equation is equal to $(m, 1)C_i(1, \widetilde{\theta}(b))C_i = (m, \widetilde{\theta}(b))C_i = \gamma_i(m\pi_i(b))$.

It is clear that γ_i is surjective, so it remains to see that γ_i is a monomorphism. Let $x, y \in P_i$ and pick m, n and a, b so that $x = n\pi_i(a)$, $y = m\pi_i(b)$. Then $xy = nm^{\pi_i(a^{-1})}\pi_i(ab)$. Let $m' \in N_i$ be such that $m' = m^{\pi_i(a^{-1})}$ so that $xy = nm'\pi_i(ab)$ and $\gamma_i(xy) = (nm', \widetilde{\theta}(ab))C_i$. Now it follows from the definition of the semidirect product U_i that

$$\begin{aligned}
\gamma_i(x)\gamma_i(y) &= (n, \widetilde{\theta}(a))C_i(m, \widetilde{\theta}(b))C_i \\
&= (nm^{\xi_i(\widetilde{\theta}(a^{-1}))}, \widetilde{\theta}(a)\widetilde{\theta}(b))C_i \\
&= (nm^{\pi_i(a^{-1})}, \widetilde{\theta}(ab))C_i \\
&= (nm', \widetilde{\theta}(ab))C_i \\
&= \gamma_i(xy)
\end{aligned}$$

so γ_i is a homomorphism. Note that γ_i is a monomorphism (this follows from the definition of the subgroup C_i). We now see Δ is an amalgam isomorphism $\mathcal{A} \cong \mathcal{E}$ since we observed above that the diagram below commutes.

$$\begin{tikzcd}
 & N_1 \arrow[rr] \arrow[rrrr, bend left] & & P_1 \arrow[rr, "\gamma_1"] & & V_1 \\
D \arrow[ur, "\phi_1"] \arrow[rr] \arrow[dr, "\phi_2"'] & & B \arrow[ur, "\pi_1"] \arrow[rr, "\widetilde{\theta}"] \arrow[dr, "\pi_2"] & & R \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"] & \\
 & N_1 \arrow[rr] \arrow[rrrr, bend right] & & P_1 \arrow[rr, "\gamma_2"] & & V_2
\end{tikzcd}$$

$\square$

A question arises from considering Theorem 2.2.25 and Lemma 2.2.21 together. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are extensions of $\mathcal{N}$, via Θ_1 and Θ_2 respectively, such that the subgroups $E(\mathcal{N}, \mathcal{A}, \Theta_1)$ and $E(\mathcal{N}, \mathcal{B}, \Theta_2)$ are conjugate in $\mathrm{Aut}^\circ(\mathcal{N})$, then Lemma 2.2.21 implies that $\mathcal{A} \cong \mathcal{B}$. On the other hand, if $\mathcal{A} \cong \mathcal{B}$, we have been unable to show in general that $E(\mathcal{N}, \mathcal{A}, \Theta_1)$ and $E(\mathcal{N}, \mathcal{B}, \Theta_2)$ are $\mathrm{Aut}(\mathcal{N})$-conjugate. Under an assumption on the degree of the amalgam, we show in the next section that a converse to Lemma 2.2.21 holds.

A further question is the following. Given an amalgam $\mathcal{S}_0$, for $i \in \mathbb{N}$ define $\mathcal{S}_i = \mathcal{E}(\mathcal{S}_{i-1}, \mathrm{Aut}^\circ(\mathcal{S}_{i-1}))$. Does there exist an $n \in \mathbb{N}$ for which $\mathcal{S}_n \cong \mathcal{S}_{n+1}$? Or is this true for a certain class of amalgams? This question is motivated by the related problem for finite groups: If G_0 is a finite group with $\mathrm{Z}(G_0) = 1$, for $i \in \mathbb{N}$ define $G_i = \mathrm{Aut}(G_{i-1})$. Then Wielandt's Automorphism Tower Theorem says that (up to isomorphism) there are finitely many groups in the sequence $(G_0, G_1, G_2, \dots)$. Although we expect that the answer to this question for general amalgams is "no", we shall prove in the next section that this holds for amalgams of certain degrees.

2.3 Edge-transitive groups of automorphisms of trees

The material in this section follows work of Serre [34] and Goldschmidt [17]. We establish the connection between amalgams and the action of groups on trees.

Definition 2.3.1. A *tree* is a connected locally finite graph without circuits.

Recall that a graph is locally finite if and only if every vertex has finitely many neighbours.

Example 2.3.2. *In Figure 2.4 we have three trees. The first is non-regular, the second is regular of valency two (and the dashed lines indicate it carries on) and the third is bi-regular of valency* $(5, 2)$.

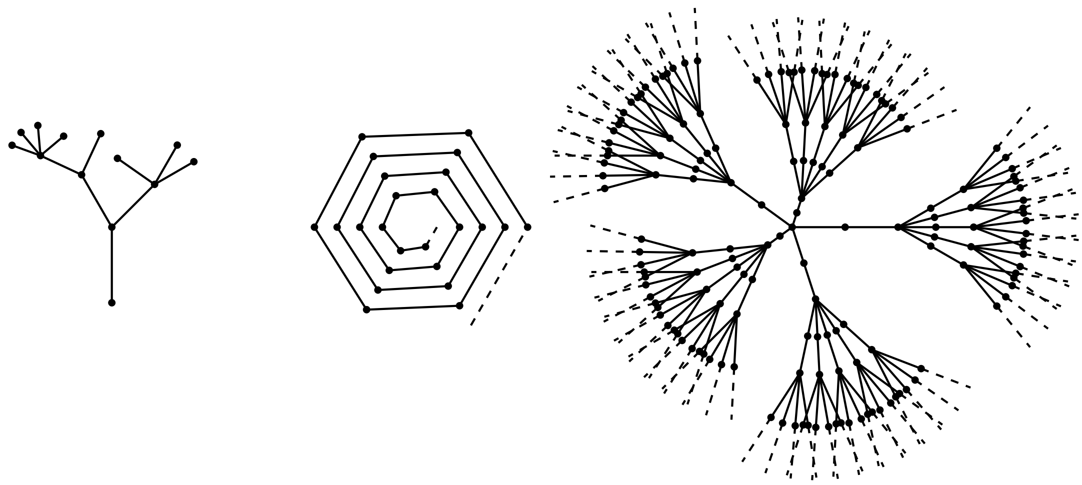


Figure 2.4: Examples of trees

We will focus on edge-transitive groups of automorphisms of a tree Γ. It follows that Γ is bi-regular, if the orbits of such a group are $\mathcal{O}_1$ and $\mathcal{O}_2$ there are integers k_1 and k_2 so that every vertex in $\mathcal{O}_i$ has valency k_i $(i = 1, 2)$. As remarked in the introduction, we may assume that $k_1 \geq 3$ and $k_2 \geq 2$. In particular, Γ has infinitely many vertices. We write $\Gamma = \Gamma_{k_1,k_2}$ to indicate the valencies of Γ (and just Γ_{k_1} if $k_1 = k_2$). The connection between trees and amalgams is given by the following theorem of Serre.

Theorem 2.3.3. *Let* X *be a graph,* $\{x, y\}$ *an edge of* X *and* G *an edge-transitive subgroup of* $\mathrm{Aut}(X)$*. The following are equivalent.*

(i) X is a tree.

(ii) $G = \mathcal{G}(\mathcal{A})$ where $\mathcal{A} = (G_x, G_y, G_{xy}, \pi_x, \pi_y)$.

Proof. This is Theorem 6 in [34, pg.32]. $\square$

With $G = \mathcal{G}(\mathcal{A})$ the theorem tells us that $\Gamma(\mathcal{A}, \mathcal{G}(\mathcal{A})) \cong \Gamma_{k_1,k_2}$ where (k_1, k_2) is the degree of the amalgam $\mathcal{A}$. Our motivation for considering trees is applications to finite graphs, so we identify the following class of subgroups of $\mathrm{Aut}(\Gamma)$.

Definition 2.3.4. A subgroup G of $\mathrm{Aut}(\Gamma)$ is *locally finite* if for each vertex x of Γ we have $|G_x| < \infty$.

Note that the group G being locally finite is equivalent to the finiteness of the amalgam $(G_x, G_y, G_{xy}, \pi_x, \pi_y)$.

Lemma 2.3.5. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic amalgams. Then $\mathcal{G}(\mathcal{A}) \cong \mathcal{G}(\mathcal{B})$.*

Proof. This follows from the universality property of the completions of the amalgams. $\square$

Lemma 2.3.6. *Suppose that $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ and $\mathcal{B} = (R_1, R_2, D, \phi_1, \phi_2)$ are finite amalgams and let $G = \mathcal{G}(\mathcal{A})$ and $H = \mathcal{G}(\mathcal{B})$. If $\alpha : G \to H$ is an isomorphism then there is $g \in H$ such that $(\alpha(P_1))^g = R_1$ and $(\alpha(P_2))^g = R_2$. In particular, $\mathcal{A} \cong \mathcal{B}$ and $\mathrm{Aut}(G) = \mathrm{Inn}(G)\mathrm{Stab}_{\mathrm{Aut}(G)}(\{P_1, P_2\})$.*

Proof. This follows from [5, Theorem 2.4.4]. $\square$

From now on we work under the following hypothesis.

Hypothesis: The group G is an edge-transitive locally finite subgroup of $A = \mathrm{Aut}(\Gamma)$. We set $\mathcal{A} = (G_x, G_y, G_{xy}, \pi_x, \pi_y)$ where $\{x, y\}$ is an edge of Γ.

We wish to establish a dictionary between the local viewpoint and the global approach. The next two lemmas show that we can detect edge-transitive subgroups locally.

Lemma 2.3.7. *Let $\mathcal{B}$ be a (normal) subamalgam of $\mathcal{A}$. Then $\mathcal{G}(\mathcal{B})$ is a (normal) edge-transitive subgroup of G.*

Proof. See [34, Proposition 3, page 6]. $\square$

Lemma 2.3.8. *Let H be a (normal) edge-transitive subgroup of G. Then*

$$\mathcal{B} = (H_x, H_y, H_{xy}, \pi_x, \pi_y)$$

is a (normal) subamalgam of $\mathcal{A}$.

Proof. See page 8 of [34]. $\square$

Lemma 2.3.9. *Suppose that Θ is an amalgam isomorphism $\Theta : \mathcal{A} \to \mathcal{B}$ and let θ be the induced isomorphism $\theta : \mathcal{G}(\mathcal{A}) \to \mathcal{G}(\mathcal{B})$. Then there is $g \in \mathrm{Aut}(\Gamma)$ such that $\theta(h) = h^g$ for all $h \in \mathcal{G}(\mathcal{A})$.*

Proof. Write $\Theta = (\alpha_1, \alpha_2, \alpha)$, $\mathcal{B} = (R_1, R_2, D)$ and $\mathcal{G}(\mathcal{B}) = H$. We have an isomorphism $\Gamma \cong \Gamma(\mathcal{A}, G)$ by Lemma 2.1.6. We define a graph homomorphism

$$\mu : \Gamma(\mathcal{A}, G) \to \Gamma(\mathcal{B}, H)$$

by

$$\mu : G_x g \mapsto H_u \theta(g),\ \ \mu : G_y h \mapsto H_v \theta(h)$$

where $\alpha_1(G_x) = \theta(G_x) = H_u$ and $\alpha_2(G_y) = \theta(G_y) = H_v$. It is easy to check that μ is an isomorphism, thus $\Gamma(\mathcal{B}, H)$ is isomorphic to Γ also. Let $g \in \mathrm{Aut}(\Gamma)$ be the composition of these isomorphisms.

Now let $h \in \mathcal{G}(\mathcal{A})$ and let z be a vertex of Γ. Identifying z with $H_w k$ for some $k \in H$ and $w \in \{u, v\}$ we have the following for some $r \in \{x, y\}$

$$z^{h^g} = (H_w k)^{g^{-1}hg} = (G_r \theta^{-1}(k))^{hg} = (G_r \theta^{-1}(k) h)^g = H_w k \theta(h) = z^{\theta(h)}.$$

It follows that $h^g(\theta(h))^{-1}$ fixes every vertex of Γ. Since $\mathrm{Aut}(\Gamma)$ acts faithfully on Γ we have $h^g = \theta(h)$ and $\mathcal{G}(\mathcal{A})^g = \mathcal{G}(\mathcal{B})$. $\square$

The following theorem captures the equivalence between the local and global viewpoints.

Theorem 2.3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite amalgams of degree (k_1, k_2). The following are equivalent,*

(i) $\mathcal{B}$ is isomorphic to $\mathcal{A}$,

(ii) $\mathcal{G}(\mathcal{B})$ is isomorphic to $\mathcal{G}(\mathcal{A})$,

(iii) $\mathcal{G}(\mathcal{B})$ and $\mathcal{G}(\mathcal{A})$ are $\mathrm{Aut}(\Gamma_{k_1,k_2})$-conjugate.

Proof. Lemmas 2.3.5 and 2.3.6 show the equivalence of (i) and (ii). Lemma 2.3.9 shows that (ii) implies (iii) and the reverse implication is immediate. □

The statement of the following lemma can be found in the proof of [17, (2.8)], we provide more details.

Lemma 2.3.11. *Let $N = N_{\mathrm{Aut}(\Gamma)}(G)$. There is a canonical isomorphism $N_{\{x,y\}} \to \mathrm{Aut}(\mathcal{A})$.*

Proof. Let $g \in N_{\{x,y\}}$ and let $\alpha_1^g : G_x \to (G_x)^g$, $\alpha_2^g : G_y \mapsto (G_y)^g$ and $\alpha^g : G_{xy} \to (G_{xy})^g = G_{xy}$ be the monomorphisms induced by conjugation by g. Clearly the diagram in Figure 2.5 commutes so $\Theta^g := (\alpha_1^g, \alpha_2^g, \alpha^g) \in \mathrm{Aut}(\mathcal{A})$.

$$\begin{tikzcd}
 & G_x \arrow[rr, "g"] & & G_{x^g} \\
G_{xy} \arrow[ur, "\pi_x"] \arrow[dr, "\pi_y"'] \arrow[rr, "g"] & & G_{xy} \arrow[ur, "\pi_{x^g}"] \arrow[dr, "\pi_{y^g}"'] & \\
 & G_y \arrow[rr, "g"] & & G_{y^g}
\end{tikzcd}$$

Figure 2.5: The automorphism Θ_g

Conversely, if $\Theta \in \mathrm{Aut}(\mathcal{A})$ then Lemma 2.3.9 gives $g^\Theta \in \mathrm{Aut}(\Gamma)$ such that $G^\Theta = \mathcal{G}(\mathcal{A})^{g^\Theta} = \mathcal{G}(\Theta(\mathcal{A})) = \mathcal{G}(\mathcal{A})$, so $g^\Theta \in \mathrm{N}_A(G)$. Moreover $\{x^{g^\Theta}, y^{g^\Theta}\} = \{x, y\}$ so $g^\Theta \in N_{\{x,y\}}$.

Let $\alpha : \mathrm{Aut}(\mathcal{A}) \to N_{\{x,y\}}$ be given by $\alpha : \Theta \mapsto g^\Theta$ and $\beta : N_{\{x,y\}} \to \mathrm{Aut}(\mathcal{A})$ be given by $\beta : g \mapsto \Theta^g$. Then these maps (which are clearly homomorphisms) are inverses of each other, so we have the desired isomorphism. □

2.4 Trees of prime valency

In this section, we suppose that G is an edge-transitive locally finite group of automorphisms of the tree $\Gamma = \Gamma_r$ for some odd prime r. By Theorem 2.3.3, $G = \mathcal{G}(\mathcal{A})$ where $\mathcal{A}$ is the amalgam formed by a pair (G_x, G_y) where $\{x, y\}$ is an edge of Γ. Our aim is to generalise some of the results from [17] which hold under the assumption $r = 3$ to the case where $r \geq 3$. We will show that every such group is contained in the normaliser of an edge-transitive subgroup of $\mathrm{Aut}(\Gamma_r)$ which has no proper normal edge-transitive subgroups. We call these groups *simple edge-transitive groups*. Then changing to the local perspective, we will show precisely how to describe the normaliser and the conjugacy classes of such groups. Set

$$\pi = \{p \in \mathbb{N} \mid p < r \text{ and } p \text{ a prime}\}.$$

If K is a group which is not necessarily finite, by $\mathrm{O}^\pi(K)$ we mean the smallest normal subgroup of *finite index* so that $K/\mathrm{O}^\pi(K)$ is a π-group.

Lemma 2.4.1. *The group G_{xy} is a π-group.*

Proof. Let $g \in G_{xy}$ and suppose that g has prime order q with $q \geq r$. Since g fixes x and y, g has acts trivially on $\Delta(x)$ and $\Delta(y)$, so $g \in G_x^{[1]} \cap G_y^{[1]}$. By connectivity and induction we see that g fixes every vertex of Γ, so $g = 1$, a contradiction. □

It follows from the degree of $\mathcal{A}$ that $|G_x| = r|G_{xy}|$ and so $\mathrm{O}^{r'}(G_x) = \mathrm{O}^\pi(G_x)$. Similarly for G_y. In particular, notice that $\mathrm{O}^{r'}(G_x)$ is generated by the r-subgroups of G_x, so it acts transitively on $\Delta(x)$.

Theorem 2.4.2. *Let $K = \mathrm{O}^{\pi}(G)$. Then K is a simple edge-transitive subgroup of G.*

Proof. First note that K is an edge-transitive subgroup of G since K_x and K_y contain r-elements which act transitively on $\Delta(x)$ and $\Delta(y)$ respectively. Now assume that H is a normal edge-transitive subgroup of K. Then $G = HG_{xy}$ and the previous lemma shows that $|G : H|$ is a π-number. Since H is subnormal in G we may apply Lemma 1.3.5 which yields $\mathrm{O}^{\pi}(H) = \mathrm{O}^{\pi}(G) = K$ and therefore $H = K$. $\square$

We have the following description of $\mathrm{O}^{\pi}(G)$.

Lemma 2.4.3. *Let $K = \mathrm{O}^{\pi}(G)$. Then $K = \langle \mathrm{O}^{r'}(G_x), \mathrm{O}^{r'}(G_y)\rangle$. Moreover $K = \mathcal{G}(\mathcal{N})$ where $\mathcal{N}$ is the normal subamalgam of $\mathcal{A}$ generated by the pair $(\mathrm{O}^{r'}(G_x), \mathrm{O}^{r'}(G_y))$.*

Proof. Set $Y = \langle \mathrm{O}^{r'}(G_x), \mathrm{O}^{r'}(G_y)\rangle$. Since $\mathrm{O}^{r'}(G_x)$ and $\mathrm{O}^{r'}(G_y)$ contain r-elements and G_{xy} is an r'-group by Lemma 2.4.1, these groups are transitive on $\Delta(x)$ and $\Delta(y)$ respectively. Hence for each $z \in \Gamma$ there is $g \in Y$ such that (without loss of generality) $x^g = z$. Now $Y \geq (\mathrm{O}^{r'}(G_x))^g = \mathrm{O}^{r'}(G_{x^g}) = \mathrm{O}^{r'}(G_z)$, thus $Y \lhd G$. We have $Y \leq K$ by Lemma 1.2.7 and now the previous theorem gives $Y = K$ as required.

Let $\mathcal{N}$ be the normal subamalgam of $\mathcal{A}$ generated by the pair $(\mathrm{O}^{r'}(G_x), \mathrm{O}^{r'}(G_y))$ (recall Definition 2.2.6). Then $\mathcal{G}(\mathcal{N})$ is a normal subgroup of $\mathcal{G}(\mathcal{A}) = G$ and is edge-transitive (reasoning as above). By the previous paragraph, we have $\mathcal{G}(\mathcal{N}) \leq K$ and so Theorem 2.4.2 implies that $\mathcal{G}(\mathcal{N}) = K$. $\square$

Notation 2.4.4. We write $\mathrm{O}^{r'}(\mathcal{A})$ for the normal subamalgam generated by the pair $(\mathrm{O}^{r'}(G_x), \mathrm{O}^{r'}(G_y))$.

Recalling Proposition 2.2.5, if $\mathcal{A} = (G_x, G_y, G_{xy}, \pi_x, \pi_y)$ then letting $C = (\mathrm{O}^{r'}(G_x) \cap G_{xy})(\mathrm{O}^{r'}(G_y) \cap G_{xy})$ we have

$$\mathrm{O}^{r'}(\mathcal{A}) = (\mathrm{O}^{r'}(G_x)C, \mathrm{O}^{r'}(G_y)C, C, \pi_x|_C, \pi_y|_C).$$

We can now establish the converse to Lemma 2.2.21 for amalgams of degree (r, r).

Theorem 2.4.5. *The amalgam $\mathcal{A}$ is an extension of $N := \mathrm{O}^{r'}(\mathcal{A})$. Moreover, up to isomorphism, the number of extensions of $\mathcal{N}$ is the number of* $\mathrm{Aut}(\mathcal{N})$*-conjugacy classes of subgroups of* $\mathrm{Aut}^{\circ}(\mathcal{N})$ *which contain* $\mathrm{Inn}(\mathcal{N})$*.*

Proof. The first part follows from the previous lemma. For the second part, let $\mathcal{B}$ and $\mathcal{C}$ be extensions of $\mathcal{N}$ by Θ_B and Θ_C respectively. Lemma 2.2.21 shows that $\mathcal{B} \cong \mathcal{C}$ if $E_B := E(\mathcal{N}, \mathcal{B}, \Theta_B)$ and $E_C := E(\mathcal{N}, \mathcal{C}, \Theta_C)$ are conjugate in $E := \mathrm{Aut}(\mathcal{N})$.

We now assume that $\mathcal{B} \cong \mathcal{C}$ and we need to show that E_B and E_C are conjugate in E. Set $G_B = \mathcal{G}(\mathcal{B})$, $G_C = \mathcal{G}(\mathcal{C})$ and $G_N = \mathcal{G}(\mathcal{N})$. After conjugation and using Theorem 2.4.2 we have $G_N = \mathrm{O}^{r'}(G_B) = \mathrm{O}^{r'}(H)$, $G_B = G_N(G_B)_{xy}$ and $G_C = G_N(G_C)_{xy}$. Since $\mathcal{B}$ is isomorphic to $\mathcal{C}$ there is $g \in \mathrm{Aut}(\Gamma)$ such that $G_B^g = G_C$, and after composition with an element of N we may assume that g fixes $\{x, y\}$ so that $((G_B)_{xy})^g = (G_C)_{xy}$. Now $G_N^g = \mathrm{O}^{r'}(G_B)^g = \mathrm{O}^{r'}(G_B^g) = \mathrm{O}^{r'}(G_C) = G_N$ so that $g \in \mathrm{N}_{\mathrm{Aut}(\Gamma)}(G_N)_{xy}$. Identifying g with its image under the canonical isomorphism of Lemma 2.3.11 we obtain $E_B^g = E_C$, as required. $\square$

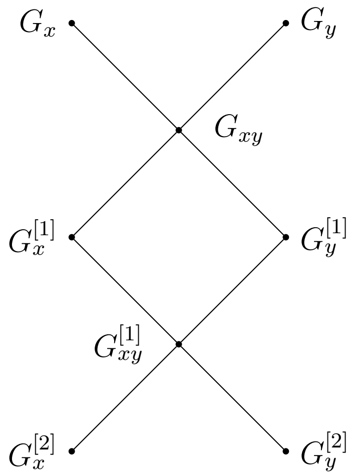
2.5 Thompson-Wielandt style theorems

In this section we prove a version of the Thompson-Wielandt theorem. Thompson's original theorem [37] implies that for a primitive permutation group G acting on a set Ω, $|G_\alpha : \mathrm{O}_p(G_\alpha)|$ is bounded where $\alpha \in \Omega$. The intended application for the theorem was the Sims Conjecture (see [7]). A completely reworked proof was given by Wielandt using subnormality methods, and the theorem is now attributed to both. Many variations of the theorem have appeared and the focus of these theorems is mostly upon the existence of a prime p such that $\mathbf{F}^*(G_\alpha) = \mathrm{O}_p(G_\alpha)$. For graph theoretic problems it has been widely applied, see [16, (2.3)] and more recently [35]. The proof we give is a combination of ideas from Fan's proof [14] and from van Bon's [51].

We work under the following hypothesis.

Hypothesis: Γ is a graph on which G acts faithfully and edge-transitively.

We fix a fundamental edge $\{x, y\}$. The diagram below shows the various inclusions of the subgroups relevant to the statement of Theorem 2.5.1.



Theorem 2.5.1 (Thompson-Wielandt)**.** *Suppose that the local action at x and y is primitive. Then, up to interchanging x and y, one of the following holds.*

(i) There exists a prime p such that all of $\boldsymbol{F}^*(G_x)$, $\boldsymbol{F}^*(G_y)$, $\boldsymbol{F}^*(G_{xy})$ are p-groups.

(ii) $G_{xy}^{[1]} = G_y^{[2]}$ and $G_x^{[2]} = 1$.

Recall that the *local action* at a vertex x by a subgroup H of G_x is the permutation group induced by H acting on $\Delta(x)$, for which we write $H^{\Delta(x)}$. Clearly the proof of the above theorem will involve considerations of the subgroup $G_{xy}^{[1]}$. Note that whenever the hypothesis of the following lemma holds we can conclude that statement (ii) of the Thompson-Wielandt theorem holds. Hence in our proof of the Thompson-Wielandt theorem, we are done whenever we can apply the following lemma.

Lemma 2.5.2. *Let $z \in \Gamma$ and let $\{z, w\}$ be an edge of Γ. Suppose that the local action at z is primitive and $C_{G_z}(G_z^{[1]}) \nleq G_z^{[1]}$. Then the following hold.*

(i) $C_{G_z}(G_z^{[1]})$ is transitive on $\Delta(z)$.

(ii) $G_{zw}^{[1]} = G_z^{[2]}$ and $G_w^{[2]} = 1$.

Proof. Since $\mathrm{C}_{G_z}(G_z^{[1]})$ is not contained in $G_z^{[1]}$, $\mathrm{C}_{G_z}(G_z^{[1]})^{\Delta(z)}$ is a non-trivial normal subgroup of $G_z^{\Delta(z)}$. Since $G_z^{\Delta(z)}$ is primitive we have that $\mathrm{C}_{G_z}(G_z^{[1]})$ is transitive on $\Delta(z)$ which gives part (i).

By (i) for each $y \in \Delta(z)$ there is $g \in \mathrm{C}_{G_z}(G_z^{[1]})$ such that $w^g = y$. Now $G_{zw}^{[1]} = (G_{zw}^{[1]})^g = G_{zy}^{[1]}$, so we obtain $G_{zw}^{[1]} \leq G_z^{[2]}$. It follows that $G_z^{[2]} = G_{zw}^{[1]}$. Since $G_w^{[2]} \leq G_z^{[1]}$ we see that $G_w^{[2]}$ is normalised by $\langle G_w, \mathrm{C}_{G_z}(G_z^{[1]})\rangle$ which is an edge-transitive subgroup of G, hence $G_w^{[2]} = 1$. □

Proof of 2.5.1. We may assume that $G_{xy}^{[1]}$ is non-trivial, otherwise (ii) holds. Our first claim reduces the problem to considering Fitting subgroups.

(1) $\mathbf{E}(G_x) = \mathbf{E}(G_y) = \mathbf{E}(G_{xy}) = 1$.

Set $E_x = \mathbf{E}(G_x)$, $E_y = \mathbf{E}(G_y)$. Suppose that there is a component K of G_x not contained in $G_x^{[1]}$. Since a component centralises a normal subgroup in which it is not contained, we have $\mathrm{C}_{G_x}(G_x^{[1]}) \not\leq G_x^{[1]}$ and applying Lemma 2.5.2 with $z = x$ and $y = w$ shows that (ii) holds. A similar argument applies to G_y so we may assume that $\mathbf{E}(G_x^{[1]}) = E_x$ and $\mathbf{E}(G_y^{[1]}) = E_y$ (and note that both are contained in G_{xy}). Suppose that $E_x \not\leq E_y$, so that there is a component K of G_x with $K \not\leq G_y^{[1]}$. Now Theorem 1.3.8 implies that $[K, G_y^{[1]}] = 1$, so we can apply Lemma 2.5.2 to see that (ii) holds. A symmetric argument applies if $E_y \not\leq E_x$ so we have that $E_x = E_y = 1$. If there is a component K of G_{xy}, then $K \not\leq G_x^{[1]}$ since $E_x = 1$, so $[K, G_x^{[1]}] = 1$ and again Lemma 2.5.2 shows that (ii) holds. We may assume therefore that $\mathbf{E}(G_{xy}) = 1$. Hence the claim.

Set $\pi = \pi(\mathbf{F}^*(G_x^{[1]})) = \pi(\mathbf{F}(G_x^{[1]}))$, where the last equality follows from (1).

(2) $\pi(\mathbf{F}(G_x)) = \pi(\mathbf{F}(G_x^{[1]})) = \pi(\mathbf{F}(G_{xy})) = \pi(\mathbf{F}(G_y^{[1]})) = \pi(\mathbf{F}(G_y))$.

Let $X \in \{G_x, G_{xy}, G_y^{[1]}\}$ and suppose that $q \in \pi$. If $\mathrm{O}_q(X) = 1$ then we must have $X = G_y^{[1]}$ and $\mathrm{O}_q(G_x^{[1]}) \cap X = 1$ and $[\mathrm{O}_q(G_x^{[1]}), X] = 1$ which allows us to apply Lemma 2.5.2 with $z = y$ and $w = x$. So we obtain $\pi \subseteq \pi(\mathbf{F}(X))$. Now if $q \in \pi(\mathbf{F}(X))$ then $\mathrm{O}_q(X) \cap G_x^{[1]} = 1$ implies $\mathrm{O}_q(X)$ centralises $G_x^{[1]}$, and Lemma 2.5.2 shows then that (ii) holds. Otherwise, $\mathrm{O}_q(X) \cap G_x^{[1]} \neq 1$, so $q \in \pi$ and $\pi = \pi(\mathbf{F}(X))$. This argument now applies to y to deliver the claim.

By (2) we see that (i) holds if $|\pi| = 1$. Assuming this is not the case then, we pick distinct primes $p, q \in \pi$ and set $P_x = \mathrm{O}_p(G_x^{[1]})$, $P_y = \mathrm{O}_p(G_y^{[1]})$, $Q_x = \mathrm{O}_q(G_x^{[1]})$ and $Q_y = \mathrm{O}_q(G_y^{[1]})$. Our assumption implies that these are all non-trivial groups.

(3) $P_x \cap P_y \neq 1 \neq Q_x \cap Q_y$.

Observe $[P_x, G_y^{[1]}] \leq P_x \cap G_y^{[1]} = P_x \cap P_y$. Now P_x is non-trivial, therefore $P_x \cap P_y = 1$ would allow us to apply Lemma 2.5.2 with $z = y$ and $w = x$, hence the claim.

(4) After swapping x and y if necessary, $\mathrm{O}_q(G_{xy}) = Q_y$ and $\mathrm{O}_p(G_{xy}) = P_x$. Moreover $Q_y \not\leq G_x^{[1]}$ and $P_x \not\leq G_y^{[1]}$.

Note that $Q_x Q_y \leq \mathrm{O}_q(G_{xy})$, hence if $\mathrm{O}_q(G_{xy}) \leq G_x^{[1]}$ and $\mathrm{O}_q(G_{xy}) \leq G_y^{[1]}$ then we have $Q_x = Q_y$ and therefore $Q_x = 1$, which is against our assumption. Hence we may assume that $\mathrm{O}_q(G_{xy}) \not\leq G_x^{[1]}$. If $\mathrm{O}_q(G_{xy}) \not\leq G_y^{[1]}$ also, then we would see that $P_x \cap P_y$ is centralised by $\langle \langle \mathrm{O}_q(G_{xy})^{G_x} \rangle, \langle \mathrm{O}_q(G_{xy})^{G_y} \rangle \rangle$, which is against (3). Hence the claim holds for q. Arguing similarly, we have that at most one of $\mathrm{O}_p(G_{xy}) \not\leq G_x^{[1]}$ or $\mathrm{O}_p(G_{xy}) \not\leq G_y^{[1]}$ holds. Assume the former and set $L_1 = \langle \mathrm{O}_q(G_{xy})^{G_x} \rangle$, $L_2 = \langle \mathrm{O}_p(G_{xy})^{G_x} \rangle$, then L_1 normalises $\mathrm{O}^q(G_{xy}^{[1]})$ and $\mathrm{O}^q(G_y^{[2]})$ since $G_{xy}^{[1]}$ is subnormal in the conjugates of $\mathrm{O}_q(G_{xy})$, and similarly L_2 normalises $\mathrm{O}^p(G_{xy}^{[1]})$ and $\mathrm{O}^p(G_y^{[2]})$. It follows that $\mathrm{O}^q(G_y^{[2]}) = 1$ and $\mathrm{O}^p(G_y^{[2]}) = 1$ whence $G_y^{[2]} = 1$. Also we have that $\mathrm{O}^q(G_{xy}^{[1]}) \leq G_x^{[2]}$ and $\mathrm{O}^p(G_{xy}^{[1]}) \leq G_x^{[2]}$ which implies $G_{xy}^{[1]}/G_x^{[2]}$ is both a p- and a q-group, and therefore $G_{xy}^{[1]} = G_x^{[2]}$. Thus (ii) holds. We can therefore conclude that $\mathrm{O}_p(G_{xy}) \not\leq G_y^{[1]}$ and this proves the claim.

Now set $L = \langle \mathrm{O}_q(G_{xy})^{G_x} \rangle$ and note that L is transitive on $\Delta(x)$. Now L normalises $\mathrm{O}^q(G_y^{[2]})$ so this implies $G_y^{[2]}$ is a q-group. On the other hand, (4) implies that $P_y \leq P_x$ so $P_y \leq G_{xy}^{[1]}$. Since P_y is normal in G_y we have $P_y \leq G_y^{[2]}$ and therefore $P_y = 1$, a contradiction which completes the proof. $\square$

Remark 2.5.3. The prime p asserted to exist by the Thompson-Wielandt theorem cannot be guessed without further knowledge. We will see that the shape of $G_x/G_x^{[1]}$ and knowledge of the prime divisors of $|G_x|$ allows us to determine p in some applications.

Note that if one of $G_x^{[2]}$ or $G_y^{[2]}$ is trivial then we have succeeded in bounding the order of G_x and G_y. In some applications this is sufficient, for example, verifying the Sims Conjecture (see [7]). If one wants to determine the structure of G_x and G_y (as we will need to do in the later sections) we shall require more information in the $G_x^{[2]} = 1$ case.

For the arc-transitive case we have the following corollary.

Corollary 2.5.4. *Suppose the hypothesis of Theorem 2.5.1 holds and that G acts arc-transitively on Γ. Then one of the following holds:*

(i) there exists a prime p such that $\boldsymbol{F}^*(G_x)$, $\boldsymbol{F}^*(G_{\{x,y\}})$ and $\boldsymbol{F}^*(G_{xy})$ are all p-groups,

(ii) $G_{xy}^{[1]} = 1$.

Proof. If conclusion (ii) of Theorem 2.5.1 holds and G is arc-transitive, then we have $|G_x^{[2]}| = |G_y^{[2]}| = 1$ and so $G_{xy}^{[1]} = 1$. Suppose that conclusion (i) of Theorem 2.5.1 holds and note that since $|G_{\{x,y\}} : G_{xy}| = 2$, every component of $G_{\{x,y\}}$ lies in G_{xy} and is therefore trivial. Let q be a prime distinct from p. If $\mathrm{O}_q(G_{\{x,y\}}) \leq G_{xy}$ we are done, so we may assume $q = 2$. Now p is odd and $\mathrm{O}_2(G_{\{x,y\}})$ centralises $\mathbf{F}^*(G_{xy})$, so $\mathrm{O}_2(G_{\{x,y\}}) \cap G_{xy} = 1$ and $\mathrm{O}_2(G_{\{x,y\}})$ centralises G_{xy}. It follows that $G_x^{[1]} = G_y^{[1]} = 1$ and so (ii) holds. $\square$

CHAPTER 3

SYMMETRIC GRAPHS OF VALENCY FIVE

We begin by setting $\Gamma = \Gamma_5$ (the regular tree of valency five) and we let G be a locally finite subgroup of $\mathrm{Aut}(\Gamma)$ such that Γ is G-symmetric. We fix an edge $\{x, y\}$ of Γ and we aim to determine the isomorphism type of the finite faithful amalgam $\mathcal{A} = (G_x, G_{\{x,y\}}, G_{xy}, \pi_1, \pi_2)$ which has degree $(5, 2)$ (recall that $G = \mathcal{G}(\mathcal{A})$. We prove the following theorem.

Theorem 3.0.5. *Up to isomorphism, there are exactly twenty-five finite faithful amalgams of degree* $(5, 2)$*. Each is the unique faithful amalgam of its type. The types of the amalgams are listed in Table 3.1.*

The work of Weiss [54] and Zhou, Feng [55] is a contribution to the proof of Theorem 3.0.5 which was included in the author's MPhil(Qual) thesis [26]. Since that work was completed [19] has appeared. Thus the isomorphism type of the group G_x in the amalgam $\mathcal{A}$ is known. The isomorphism types of the group $G_{\{x,y\}}$ and the amalgam $\mathcal{A}$ have not yet been determined, therefore we concentrate on the identification of the edge stabilisers and the classification of the amalgam $\mathcal{A}$.

Remark 3.0.6 (On Table 3.1)*.* Let (P_1, P_2) be one of the pairs in Table 3.1. One can easily check there is a unique conjugacy class of index five subgroups in P_1, let B be one of these subgroups. In P_2 there is a unique $\mathrm{Aut}(P_2)$-conjugacy class of subgroups isomorphic to B. Thus the pair (P_1, P_2) does indeed determine the *type* of an amalgam. For each amalgam we have provided a sample finite completion G. In Section 3.3 we prove that the group

Amalgam	P_1	P_2	G	s
$\mathcal{Q}_1^1$	C_5	C_2	$\mathrm{C}_5 \wr \mathrm{C}_2$	1
$\mathcal{Q}_1^2$	Dih(10)	2^2	Alt(5)	1
$\mathcal{Q}_1^3$	Dih(10)	C_4	Alt(6)	1
$\mathcal{Q}_1^4$	Dih(20)	Dih(8)	$\mathrm{PSL}_2(11):2$	1
$\mathcal{Q}_2^1$	Frob(20)	$\mathrm{C}_4 \times \mathrm{C}_2$	Sym(6)	2
$\mathcal{Q}_2^2$	Frob(20)	C_8	M_{11}	2
$\mathcal{Q}_2^3$	Frob(20)	Dih(8)	Sym(5)	2
$\mathcal{Q}_2^4$	Frob(20)	Q_8	M_{10}	2
$\mathcal{Q}_2^5$	$\mathrm{Frob}(20) \times \mathrm{C}_2$	N_{16}	Sym(9)	2
$\mathcal{Q}_2^6$	$\mathrm{Frob}(20) \times \mathrm{C}_2$	M_{16}	Aut(Alt(6))	2
$\mathcal{Q}_2^7$	Alt(5)	Sym(4)	Alt(6)	2
$\mathcal{Q}_2^8$	Alt(5)	$\mathrm{Alt}(4) \times \mathrm{C}_2$	Sym(6)	2
$\mathcal{Q}_2^9$	Sym(5)	$\mathrm{Sym}(4) \times \mathrm{C}_2$	Sym(6)	2
$\mathcal{Q}_3^1$	$\mathrm{Frob}(20) \times \mathrm{C}_4$	$\mathrm{C}_4 \wr \mathrm{C}_2$	Sym(9)	3
$\mathcal{Q}_3^2$	$\mathrm{Alt}(5) \times \mathrm{Alt}(4)$	$\mathrm{Alt}(4) \wr \mathrm{C}_2$	Alt(9)	3
$\mathcal{Q}_3^3$	$\mathrm{Sym}(5) \curlywedge \mathrm{Sym}(4)$	L_1	Alt(9)	3
$\mathcal{Q}_3^4$	$\mathrm{Sym}(5) \curlywedge \mathrm{Sym}(4)$	L_2	Sym(9)	3
$\mathcal{Q}_3^5$	$\mathrm{Sym}(5) \times \mathrm{Sym}(4)$	$\mathrm{Sym}(4) \wr \mathrm{C}_2$	Sym(9)	3
$\mathcal{Q}_4^1$	$2^4 : \mathrm{Alt}(5)$	$2^{4+2} : \mathrm{Sym}(3)$	$\mathrm{PSL}_3(4).\langle g \rangle$	4
$\mathcal{Q}_4^2$	$2^4 : \mathrm{Alt}(5)$	$2^{4+2} : \mathrm{C}_6$	$\mathrm{PSL}_3(4).\langle gf \rangle$	4
$\mathcal{Q}_4^3$	$2^4 : \mathrm{Sym}(5)$	$2^{4+2+1} : \mathrm{Sym}(3)$	$\mathrm{PSL}_3(4).\langle f, g \rangle$	4
$\mathcal{Q}_4^4$	$2^4 : (\mathrm{Alt}(5) \times \mathrm{C}_3)$	$(2^{4+2} : \mathrm{C}_3) : \mathrm{Sym}(3)$	$\mathrm{PGL}_3(4).\langle g \rangle$	4
$\mathcal{Q}_4^5$	$2^4 : (\mathrm{Alt}(5) \times \mathrm{C}_3)$	$(2^{4+2} : \mathrm{C}_3) : \mathrm{C}_6$	$\mathrm{PGL}_3(4).\langle gf \rangle$	4
$\mathcal{Q}_4^6$	$2^4 : \mathrm{Sym}(5) \curlywedge \mathrm{Sym}(3)$	$2^{4+2} : \mathrm{Sym}(3)^2$	$\mathrm{Aut}(\mathrm{PSL}_4(4))$	4
$\mathcal{Q}_5^1$	$2^6 : \mathrm{Sym}(5) \curlywedge \mathrm{Sym}(3)$	$(2^6 : (\mathrm{Alt}(4) \times \mathrm{C}_3)) : \mathrm{C}_4$	$\mathrm{Aut}(\mathrm{Sp}_4(4))$	5

Table 3.1: The types of finite, faithful amalgams of degree (5,2)

G is indeed a finite completion of the amalgam. The group M_{16} is the *modular group* of order sixteen with presentation $\langle u, v \mid u^8 = 1, v^2 = 1, u^v = u^5 \rangle$. The group N_{16} is the subgroup $\langle (1,2,3,4)(5,6,7,8),\ (5,7)(6,8),\ (1,5)(2,6)(3,7)(4,8) \rangle$ of Sym(8). By L_1 and L_2 we denote the groups

$$\mathrm{L}_1 = \langle (1,2,3), (2,3,4), (5,6,7), (6,7,8), (1,2)(5,6), (1,5)(2,6)(3,7)(4,8) \rangle,$$

$$\mathrm{L}_2 = \langle (1,2,3), (2,3,4), (5,6,7), (6,7,8), (1,6,2,5)(3,7)(4,8) \rangle,$$

which are both extensions of $\mathrm{Alt}(4) \times \mathrm{Alt}(4)$.

3.1 Edge stabilisers

In this section we aim to classify the isomorphism type of the group $G_{\{x,y\}}$ and the amalgam $\mathcal{A}$. As remarked in the introduction, Γ is (G,s)-transitive for some $s \leq 5$. If $s \geq 4$ then [54] determines $\mathcal{A}$ up to isomorphism and we have $\mathcal{A} \in \{\mathcal{Q}_4^1, \ldots, \mathcal{Q}_4^6, \mathcal{Q}_5^1\}$. The amalgams $\mathcal{Q}_4^1$-$\mathcal{Q}_4^6$ are visible in the group $\mathrm{Aut}(\mathrm{PSL}_3(4))$ which acts on the generalised triangle associated to $\mathrm{PSL}_3(4)$. In Table 3.1 we have indicated which extension each amalgam is visible in, where g and f are graph and field automorphisms respectively. The amalgam $\mathcal{Q}_5^1$ is visible in $\mathrm{Aut}(\mathrm{Sp}_4(4))$ which acts on the generalised quadrangle associated to $\mathrm{Sp}_4(4)$. We now turn to the case $1 \leq s \leq 3$. The following two theorems are [55, Theorem 4.1] and [19, Theorem 1.1] respectively.

Theorem 3.1.1. *Suppose that G_x is soluble and $1 \leq s \leq 3$. Then G_x is isomorphic to one of* C_5, $\mathrm{Dih}(10)$, $\mathrm{Dih}(20)$ *if* $s = 1$, *one of* $\mathrm{Frob}(20)$, $\mathrm{Frob}(20) \times \mathrm{C}_2$ *if* $s = 2$ *or* $\mathrm{Frob}(20) \times \mathrm{C}_4$ *if* $s = 3$.

Theorem 3.1.2. *Suppose that G_x is insoluble and $1 \leq s \leq 3$. Then $s \geq 2$ and G_x is isomorphic to one of* $\mathrm{Alt}(5)$ *or* $\mathrm{Sym}(5)$ *if* $s = 2$ *and one of* $\mathrm{Alt}(5) \times \mathrm{Alt}(4)$, $\mathrm{Sym}(4) \curlywedge \mathrm{Sym}(5)$ *or* $\mathrm{Sym}(5) \times \mathrm{Sym}(4)$ *if* $s = 3$.

Recalling Corollary 2.5.4 we know that the group $G_{xy}^{[1]}$ is a p-group for some prime p (the first proof of this fact seems to be [16, (2.3)]). For the small values of s under current consideration, it is in fact trivial.

Proposition 3.1.3. *Suppose that $1 \leq s \leq 3$. Then $G_{xy}^{[1]} = 1$, $G_x^{[1]} \cong G_x^{[1]}G_y^{[1]}/G_y^{[1]} \lhd G_{xy}/G_y^{[1]}$ and $[G_x^{[1]}, G_y^{[1]}] = 1$.*

Proof. The first assertion is contained in the proofs of [55, Theorem 4.1] and [19, Theorem 1.1]. The remaining assertions follow by a homomorphism theorem, the normality of $G_x^{[1]}$ and $G_y^{[1]}$ in G_{xy} and the definition of $G_{xy}^{[1]}$. □

Recall that the kernel of the action of G_x on the vertices adjacent to x is the core in G_x of G_{xy}. Since there is a unique class of index five subgroups, the group $G_x^{[1]}$ is uniquely determined by Theorems 3.1.1 and 3.1.2.

Lemma 3.1.4. *Suppose that* $G_x^{[1]} = 1$. *Then* $\mathcal{A}$ *has the same type as one of* $\mathcal{Q}_1^1$*-* $\mathcal{Q}_1^3$*,* $\mathcal{Q}_2^1$*-*$\mathcal{Q}_2^4$ *or* $\mathcal{Q}_2^7$*-*$\mathcal{Q}_2^9$*.*

Proof. We use the fact that G_{xy} is uniquely determined by G_x, and we consider each of the groups of order $2|G_{xy}|$ which has a subgroup isomorphic to G_{xy}. Each turns out to be a candidate. $\square$

From now on we may assume that $G_x^{[1]} \neq 1$.

Lemma 3.1.5. *Suppose that* $G_x \cong \mathrm{Dih}(20)$*. Then* $G_{\{x,y\}} \cong \mathrm{Dih}(8)$ *and* $\mathcal{A}$ *has the same type as* $\mathcal{Q}_1^4$*.*

Proof. As $G_x \cong \mathrm{Dih}(20)$ we see $G_x^{[1]}$ has order 2 and $G_{xy} \cong 2^2$. Then $G_{\{x,y\}}$ is a non-abelian group of order 8 with an elementary abelian subgroup of order 4. It follows that $G_{\{x,y\}} \cong \mathrm{Dih}(8)$. $\square$

In the next lemma we find the relevant edge stabilisers have order sixteen. Recall the definitions of M_{16} and N_{16} from Remark 3.0.6. Observe that N_{16} has a central cyclic subgroup of order 4, modulo which it is elementary abelian of order 4.

Lemma 3.1.6. *Suppose that* $G_x \cong \mathrm{Frob}(20) \times \mathrm{C}_2$*. Then* $G_{\{x,y\}} \cong \mathrm{M}_{16}$ *or* $G_{\{x,y\}} \cong \mathrm{N}_{16}$ *and* $\mathcal{A}$ *has the same type as* $\mathcal{Q}_2^5$ *or* $\mathcal{Q}_2^6$*.*

Proof. We have $G_{xy} \cong \mathrm{C}_4 \times \mathrm{C}_2$, fix notation $G_{xy} = \langle h, j \rangle$ where h has order 4 and j has order 2. Additionally, we may assume that $\langle j \rangle = G_x^{[1]}$ and $\langle h^2 j \rangle = G_y^{[1]}$ since j is not a square in G_{xy}. We know there is $t \in G_{\{x,y\}}$ such that $j^t = h^2 j$, and we choose such a t with order as small as possible. If t has order 2, then we find that $G_{\{x,y\}} \cong N_{16}$, otherwise t has order 4 or 8. If t has order 8, then (after changing notation if necessary) we have $t^2 = h$ and so $j^t = h^2 j = t^4 j$ implies that $t^j = t^5$ and we see $G_{\{x,y\}} \cong M_{16}$. It remains to see that t cannot have order 4.

There are exactly two cyclic subgroups of order 4 in G_{xy} and these are generated by h and hj respectively. We claim that t centralises one of these subgroups. First assume that $h^t = hj$ or $h^t = h^3j$. Then $t^2 \in G_{xy}$, so $h^{t^2} = h$. On the other hand, both of $h^t = hj$ and $h^t = h^3j$ imply that $h^{t^2} = h^3$, whence $h = h^3$, a contradiction. Hence either $h^t = h$, in which case t centralises $\langle h \rangle$ or $h^t = h^3$. Then we find that $(hj)^t = h^3h^2j = hj$, so t centralises $\langle hj \rangle$. In both cases, we find an element of order 4, k say, in G_{xy} which is centralised by t. Hence $t^2 = k^2$ and so $(tk)^2 = 1$, but $tk \notin G_{xy}$, and this contradicts our choice of t with minimal order. □

Lemma 3.1.7. *Suppose that $G_x \cong \mathrm{Frob}(20) \times \mathrm{C}_4$. Then $G_{\{x,y\}} \cong \mathrm{C}_4 \wr \mathrm{C}_2$ and $\mathcal{A}$ has the same type as $\mathcal{Q}_3^1$.*

Proof. Since $G_x^{[1]} \cong \mathrm{C}_4$, we have $G_{xy} = G_x^{[1]}G_y^{[1]} \cong \mathrm{C}_4 \times \mathrm{C}_4$. Choose $q \in G_{\{x,y\}}$ of least order such that $q \notin G_{xy}$, we claim q has order 2. Writing $G_x^{[1]} = \langle a \rangle$, set $b = a^q$, then $G_y^{[1]} = \langle b \rangle$ and $(a^i)^q = b^i$ for $i \in \mathbb{N}$. Since $G_{\{x,y\}}$ is non-abelian, it follows that $\mathrm{Z}(G_{\{x,y\}}) = \langle ab \rangle$. Now $q^2 \in G_{xy}$ which is abelian, so $q^2 \in \mathrm{Z}(G_{\{x,y\}})$. If $q^2 = 1$ we are done. Suppose first that $q^2 = a^2b^2$. Then $(qab)^2 = 1$, and $qab \notin G_{xy}$ since $q \notin G_{xy}$, this contradicts our choice of q. Similarly, if $q^2 = ab$ or $q^2 = a^3b^3$, we find that qb^3, respectively, qb, are involutions, and do not lie in G_{xy}. Thus we may assume q is an involution, and therefore $G_{\{x,y\}} \cong \mathrm{C}_4 \wr \mathrm{C}_2$. □

From now on we may assume $G_x/G_x^{[1]}$ is insoluble.

Lemma 3.1.8. *We have $C_{G_{\{x,y\}}}(G_x^{[1]}G_y^{[1]}) = C_{G_{xy}}(G_x^{[1]}G_y^{[1]}) = C_{G_x}(G_x^{[1]}G_y^{[1]}) = \mathrm{Z}(G_x^{[1]}G_y^{[1]})$. Moreover, $G_{\{x,y\}}$ acts faithfully on $G_x^{[1]}G_y^{[1]}$ by conjugation.*

Proof. Set $C_e = \mathrm{C}_{G_{\{x,y\}}}(G_x^{[1]}G_y^{[1]})$ and $C_x = \mathrm{C}_{G_x}(G_x^{[1]}G_y^{[1]})$. The first equality will follow once we have shown $C_e \leq G_{xy}$. If this were not the case, then C_e acts transitively on $\{x, y\}$. Also we see that $[C_e, G_x^{[1]}] \leq [C_e, G_x^{[1]}G_y^{[1]}] = 1$, hence $G_x^{[1]}$ is a normal subgroup of $\langle G_x, C_e \rangle$, which acts transitively on $V(\Gamma)$. It follows that $G_x^{[1]} = 1$, a contradiction. Now $\mathrm{Z}(G_x^{[1]}G_y^{[1]}) \leq C_e \leq C_x$, so it remains to see that the latter subgroup is contained in $G_x^{[1]}G_y^{[1]}$. Since $G_x/G_x^{[1]} \cong \mathrm{Alt}(5)$ or $G_x/G_x^{[1]} \cong \mathrm{Sym}(5)$, we see that normal subgroups of $G_{xy}/G_x^{[1]}$ contain their centralisers in $G_x/G_x^{[1]}$, therefore

$$C_xG_x^{[1]}/G_x^{[1]} \leq \mathrm{C}_{G_x/G_x^{[1]}}(G_x^{[1]}G_y^{[1]}/G_x^{[1]}) \leq G_x^{[1]}G_y^{[1]}/G_x^{[1]}$$

and so $C_x \leq C_x G_x^{[1]} \leq G_x^{[1]} G_y^{[1]}$ as required. The second assertion of the lemma now follows from the isomorphisms $G_x^{[1]} \cong \mathrm{Alt}(4)$ or $G_x^{[1]} \cong \mathrm{Sym}(4)$ which follow from Theorem 3.1.2. □

We define $A = \mathrm{Aut}(G_x^{[1]} G_y^{[1]}) \cong \mathrm{Sym}(4) \wr \mathrm{C}_2$. This isomorphism follows from the observation that there are exactly two normal subgroups isomorphic to Alt(4) in $G_x^{[1]} G_y^{[1]}$. Lemma 3.1.8 allows us to identify $G_{\{x,y\}}$ with a subgroup of A. Note that $\mathrm{O}^2(A) \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4)$ and by Lemma 3.1.8 $\mathrm{O}^2(A) \leq G_x^{[1]} G_y^{[1]}$. Thus we see $G_{\{x,y\}}/\mathrm{O}^2(A)$ in the quotient $A/\mathrm{O}^2(A) \cong \mathrm{Dih}(8)$. We use these observations below.

Lemma 3.1.9. *Suppose that $G_x \cong \mathrm{Alt}(5) \times \mathrm{Alt}(4)$. Then $G_{\{x,y\}} \cong \mathrm{Alt}(4) \wr \mathrm{C}_2$ and $\mathcal{A}$ has the same type as $\mathcal{Q}_3^2$.*

Proof. Theorem 3.1.2 gives $G_x^{[1]} \cong \mathrm{Alt}(4)$. Now $G_{xy} = G_x^{[1]} G_y^{[1]} \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4)$. Since $G_x^{[1]}$ and $G_y^{[1]}$ are conjugate in $G_{\{x,y\}}$, inspecting the possibilities for $G_{\{x,y\}}$ in A we must have $G_{\{x,y\}} \cong \mathrm{Alt}(4) \wr \mathrm{C}_2$. □

Lemma 3.1.10. *Suppose that $G_x \cong \mathrm{Sym}(5) \times \mathrm{Sym}(4)$. Then $G_{\{x,y\}} \cong \mathrm{Sym}(4) \wr \mathrm{C}_2$ and $\mathcal{A}$ has the same type as $\mathcal{Q}_3^5$.*

Proof. We have $G_{xy} = G_x^{[1]} G_y^{[1]} \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$ and so $G_{\{x,y\}} \cong \mathrm{Aut}(G_{xy})$. □

Finally, we have to deal with the possibility that $G_x/G_x^{[1]} \cong \mathrm{Sym}(5)$ and $G_x^{[1]} \cong \mathrm{Alt}(4)$. There are two types of amalgam which have this property.

For the final possibility for the shape of G_x there are two different possibilities for $G_{\{x,y\}}$. These groups differ in the isomorphism type of $G_{\{x,y\}}/G_x^{[1]} G_y^{[1]}$, which has order 4, but is either cyclic or elementary abelian. Recall the definitions of the groups L_1 and L_2 from Remark 3.0.6.

Lemma 3.1.11. *Suppose that $G_x/G_x^{[1]} \cong \mathrm{Sym}(5)$ and $G_x^{[1]} \cong \mathrm{Alt}(4)$. Then $G_{\{x,y\}}$ is isomorphic to one of L_1 or L_2 and $\mathcal{A}$ has the same type as one of $\mathcal{Q}_3^3$ or $\mathcal{Q}_3^4$.*

Proof. Comparing orders, we see that $|G_{\{x,y\}} : G_x^{[1]} G_y^{[1]}| = 4$. By Lemma 3.1.8 and the remarks following, $G_{\{x,y\}}$ can be identified with a subgroup of index two in A which contains the characteristic subgroup $\mathrm{O}^2(A)$. There are precisely three of these, L_1 and L_2 above and $\mathrm{L}_3 \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$. Identifying $G_x^{[1]}$ with its image in A we see $G_x^{[1]} \lhd \mathrm{L}_3$, so we must have $G_{\{x,y\}} \cong \mathrm{L}_1$ or $G_{\{x,y\}} \cong \mathrm{L}_2$. □

Note that so far, even though we have shown that the amalgam $\mathcal{A}$ has the same type as one of the amalgams in Table 3.1, we have not determined how many isomorphism classes of amalgam of each type there are. This problem is addressed in the next section.

3.2 Uniqueness and presentations

The aim of this section is to verify that each of the finite faithful amalgams of degree $(5, 2)$ is unique. We will use the notation established in Section 2.2. For an amalgam $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ we have $H_i = \mathrm{N}_{\mathrm{Aut}(P_i)}(\pi_i(B))$ and H_i^* is the image of this group in $\mathrm{Aut}(B)$ (under the image $H_i \mapsto H_i/\mathrm{C}_{\mathrm{Aut}(P_i)}(\pi_i(B))$). Goldschmidt's amalgam counting lemma (Lemma 2.2.12) then says that the number of amalgams with the same type as $\mathcal{A}$ is the number of (H_1^*, H_2^*)-double cosets in $\mathrm{Aut}(B)$.

Lemma 3.2.1. *There is a unique class of amalgams of type $\mathcal{Q}_i^j$ for (i, j) in the set* $\{(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(2, 4)\}$.

Proof. If $i = 1$ then there is nothing to prove since $\mathrm{Aut}(B) = 1$. For the remaining amalgams $\mathrm{Aut}(B) \cong \mathrm{C}_2$. Inspecting $\mathrm{Aut}(P_2)$ we find an element which inverts B in all cases, so we are done. □

Lemma 3.2.2. *There are two isomorphism classes of amalgams of type $\mathcal{Q}_1^4$ and precisely one is faithful.*

Proof. We see that $\mathrm{Aut}(B) \cong \mathrm{Sym}(3)$. After choosing a labelling, one finds that H_1^* is the subgroup $\langle (1, 2) \rangle$ and that $H_2^* = \langle (2, 3) \rangle$. Hence there are two (H_1^*, H_2^*) double cosets in $\mathrm{Aut}(B)$. For both of these amalgams we have $\mathrm{Z}(P_1)\mathrm{Z}(P_2) \leq B$, but the faithful amalgam has $\mathrm{Z}(P_1) \cap \mathrm{Z}(P_2) = 1$, and the non-faithful amalgam has $\mathrm{Z}(P_1) = \mathrm{Z}(P_2)$. □

Lemma 3.2.3. *There is a unique class of amalgams of types $\mathcal{Q}_2^5$ and $\mathcal{Q}_2^6$ respectively.*

Proof. We write $B = \langle x, y\rangle$ where x has order 4 and y has order 2, and consider the action of the groups $\mathrm{Aut}(B)$, $\mathrm{N}_{\mathrm{Aut}(P_1)}(B)$ and $\mathrm{N}_{\mathrm{Aut}(P_2)}(B)$ on $\Omega = \{x, x^{-1}, xy, x^{-1}y\}$, the elements of order 4 in B. Since $\mathrm{Aut}(B) \cong \mathrm{Dih}(8)$ acts faithfully on Ω we may write elements of H_1^* and H_2^* as permutations of $\{1, 2, 3, 4\}$ (acting on subscripts after labelling $x_1 = x$, $x_2 = x^{-1}$, $x_3 = xy$, $x_4 = x^{-1}y$). In both cases we see $H_1^* = \langle (1, 3)(2, 4)\rangle$ and H_2^* contains the subgroup $\langle (1, 2)(3, 4), (3, 4)\rangle$. Hence $\mathrm{Aut}(B) = H_1^* H_2^*$, so by the Goldschmidt Lemma there is a unique class of amalgams. $\square$

Lemma 3.2.4. *There is a unique class of amalgams of type $\mathcal{Q}_2^j$ with $j \in \{7, 8, 9\}$.*

Proof. Observe that $\mathrm{Aut}(B) \cong \mathrm{Sym}(4)$ for each of these amalgams. Now $\mathrm{C}_{\mathrm{Aut}(P_1)}(B) = 1$ and $\mathrm{N}_{\mathrm{Aut}(P_1)}(B) \cong \mathrm{Sym}(4)$, thus we find $\mathrm{Aut}(B) = H_1^*$. It follows that there is a unique class of amalgams. $\square$

Lemma 3.2.5. *There are three isomorphism classes of amalgams of type $\mathcal{Q}_3^1$ and precisely one is faithful.*

Proof. We identify $\mathrm{Aut}(B)$ with the group $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$. Using generators for the group $\mathrm{Aut}(P_1) \cong \mathrm{Frob}(20) \times \mathrm{Dih}(8)$ and $\mathrm{Aut}(P_2) \cong \mathrm{Dih}(8) : 2^2$, we find that $H_1^* \cong \mathrm{Dih}(8)$ and $H_2^* \cong 2^3$ and these groups are generated by the matrices

$$H_1^* = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle, \; H_2^* = \left\langle \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\rangle.$$

Either by hand or with the aid of Magma or Gap one can verify that there are three (H_1^*, H_2^*) double cosets in $\mathrm{Aut}(B)$, and so there are three isomorphism classes of amalgams with this type. In Example 2.2.8 we constructed three pairwise non-isomorphic amalgams of this type and precisely one is faithful. $\square$

Lemma 3.2.6. *Let $\mathcal{A}$ be an amalgam of type $\mathcal{Q}_i^j$ with $i = 3$ and $j = 2, 3, 4, 5$. Then $\mathcal{A}$ is the unique amalgam of this type.*

Proof. It is clear that $\mathrm{Aut}(P_1) \cong \mathrm{Sym}(5) \times \mathrm{Sym}(4)$ and $\mathrm{Aut}(P_2) \cong \mathrm{Sym}(4) \wr \mathrm{C}_2$ for each of the amalgams. Then the image of $\mathrm{N}_{\mathrm{Aut}(P_1)}(B)$ in $\mathrm{Aut}(B)$ is the subgroup isomorphic to $\mathrm{Sym}(4) \times \mathrm{Sym}(4)$. Since there is an inner automorphism of P_2 which normalises B and swaps the factors, the image of this element in $\mathrm{Aut}(B)$ lies outside the $\mathrm{Sym}(4) \times \mathrm{Sym}(4)$ subgroup. Hence $\mathrm{Aut}(B) = H_1^* H_2^*$, so there is a unique amalgam of these types by the Goldschmidt Lemma. □

In Table 3.2 we give presentations of the universal completions of the finite faithful amalgams of degree (5,2). These presentations have the advantage that vertex and edge stabilisers are relatively easy to identify. The presentations are perhaps not the most efficient for computational purposes, however using the **Simplify** command in MAGMA on these presentations returns the presentation unchanged, so the presentations are satisfactory. The presentations will be available for download at [25].

For the final seven completions, we know that the universal completion satisfies one of the nine presentations $R_{5,4+}$, $R_{5,4-}$, $R_{5,4+}^{\{g\}}$, $R_{5,4-}^{\{g\}}$, $R_{5,4+}^{\{f\}}$, $R_{5,4-}^{\{f\}}$, $R_{5,4+}^{\{f,g\}}$, $R_{5,4-}^{\{f,g\}}$, $R_{5,5}$ due to Weiss [54, Theorem (1.1)]. We have presented them here in an "uncompressed" form so that the number of amalgams is clear, and to underline the point that the two pairs of amalgams defined by the presentations $R_{5,4\pm}^{\{g\}}$ and $R_{5,4\pm}^{\{f,g\}}$ are isomorphic (so to abuse notation and language, we are saying $R_{5,4+}^{\{g\}} \cong R_{5,4-}^{\{g\}}$ and $R_{5,4+}^{\{f,g\}} \cong R_{5,4-}^{\{f,g\}}$). We prove this below.

Lemma 3.2.7. *Suppose that X and X' are groups defined by the presentations $R_{5,4+}^{\{g\}}$ and $R_{5,4-}^{\{g\}}$ respectively. Then $X \cong X'$.*

Suppose that Y and Y' are groups defined by the presentations $R_{5,4+}^{\{f,g\}}$ and $R_{5,4-}^{\{f,g\}}$ respectively. Then $Y \cong Y'$.

Proof. Since all the groups are defined by their presentations, we will show that X and X' admit the same presentation, and similarly for Y and Y'. From [54] we have that $X = \langle a, e, c, g \rangle$. Since $g^2 = 1$, $\langle ag, e, c, g \rangle = X$. It is easy to check that the relations in the subgroup $\langle ag, e, c, g \rangle$ are those that hold in X'.

Type	Generators	Relations
$\mathcal{Q}_1^1$	a, b	a^5, b^2
$\mathcal{Q}_1^2$	a, b, c	a^5, b^2, c^2, $(ac)^2$, $(bc)^2$
$\mathcal{Q}_1^3$	a, b	a^5, b^4, $(b^2a)^2$
$\mathcal{Q}_1^4$	a, b, c	a^5, b^4, c^2, $(bc)^2$, $(ab^2)^2$, $[a,c]$
$\mathcal{Q}_2^1$	a, b, c	a^5, b^2, c^4, a^ca^3, $[b,c]$
$\mathcal{Q}_2^2$	a, b	a^5, b^8, $a^{b^2}a^3$
$\mathcal{Q}_2^3$	a, b, c	a^5, b^2, c^4, a^ca^3, $(cb)^2$
$\mathcal{Q}_2^4$	a, b, c	a^5, b^4, c^4 a^ca^3, c^bc
$\mathcal{Q}_2^5$	a, b, c, d	a^5, b^2, c^4, d^2, a^ca^3, $[a,d]$, $[b,c]$, $[c,d]$, d^bc^2d
$\mathcal{Q}_2^6$	a, b, c	a^5, b^8, c^2, $a^{b^2}a^3$, b^cb^3, $[a,c]$
$\mathcal{Q}_2^7$	a, b, c, d	a^3, b^2, c^3, d^3, $(dc)^2$, $(da)^2$, c^ac^2d, $(bc)^2$, b^dbc
$\mathcal{Q}_2^8$	a, b, c, d	a^3, b^2, c^3, d^3, $(dc)^2$, $(da)^2$, c^ac^2d, $[b,c]$, $[b,d]$
$\mathcal{Q}_2^9$	a, b, c, d	a^5, b^2, c^4, d^2, $(cd)^3$, $[b,c]$, $[b,d]$, a^3cad
$\mathcal{Q}_3^1$	a, b, c	a^5, b^2, c^4, a^ca^3, $[a,c^b]$, $[c,c^b]$
$\mathcal{Q}_3^2$	a, b, c, d, e, f	a^3, b^2, c^3, d^3, e^3, f^3, $(fe)^2$, $[e,c]$, $[f,c]$, $[e,d]$, $[f,d]$, $(dc)^2$, $[e,a]$, $[f,a]$, $(ad)^2$, c^ac^2d, e^bc, f^bd
$\mathcal{Q}_3^3$	a, b, c, d, e, f, g	c^3, d^3, e^2, f^3, g^3, $(gf)^2$, $[f,c]$, $[g,c]$, $[f,d]$, $[g,d]$, $(dc)^2$, $(ef)^2$, $(ec)^2$, e^gf^2e, e^dc^2e, a^3, $[f,a]$, $[g,a]$, $(ad)^2$, ee^a, b^2, f^2c^b, g^2d^b, $(eb)^2$
$\mathcal{Q}_3^4$	a, b, c, d, e, f	c^3, d^3, e^3, f^3, $(dc)^2$, $[c,e]$, $[d,e]$, $[c,f]$, $[d,f]$, $(fe)^2$, b^4, c^2e^b, d^2f^b, ec^b, d^bef^2, a^3, $[c,a]$, $[d,a]$, $(af)^2$, $[b^2,a]$
$\mathcal{Q}_3^5$	a, b, c, d, e, f	c^4, d^2, e^4, f^2, $(cd)^3$, $(ef)^3$, $[c,e]$, $[c,f]$, $[d,e]$, $[d,f]$, a^5, a^3cad, $[a,e]$, $[a,f]$, b^2, c^be^3, d^bf

Table 3.2: Presentations for the universal completions of finite faithful (5,2) amalgams with $s \leq 3$.

Similarly we have $Y = \langle a, e, c, g, f\rangle$. Then $Y = \langle ag, e, c, g, f^{-1}\rangle$ and it is again a routine exercise to verify that the relations in the subgroup just constructed are those that hold in Y'. $\square$

The universal completions for the amalgams $\mathcal{Q}_4^1$-$\mathcal{Q}_4^6$ are generated by elements a, e_0, c, f and g. For $i \in \mathbb{Z}$ we define $e_i := a^ie_0a^{-i}$ and $t = e_0e_3e_0$. The universal completion of the amalgam $\mathcal{Q}_5^1$ is generated by elements a, e_0 and c, and as before set $e_i := a^ie_0a^{-i}$. We also define $t := e_0e_4e_0$, $f := aca^{-1}$ and $g = (ta)^2$.

Type	Generators	Relations
$\mathcal{Q}_4^1$	a, e_0, c	e_0^2,c^3,$(e_0e_3)^3$, $tct^{-1}c$, $(e_0c)^3$, $(ce_0e_3)^5$, $tat^{-1}a$, $[e_0,e_1]$, $[e_0,ce_1c^{-1}]$, $[e_0,e_2]e_1$, $[e_0,ce_2c^{-1}]c^{-1}e_1c$, $aca^{-1}c$
$\mathcal{Q}_4^2$	a, e_0, c	e_0^2, c^3, $(e_0e_3)^3$, $tct^{-1}c$, $(e_0c)^3$, $(ce_0e_3)^5$, $tat^{-1}a$, $[e_0,e_1]$, $[e_0,ce_1c^{-1}]$, $[e_0,e_2]e_1$, $[e_0,ce_2c^{-1}]c^{-1}e_1c$, $[a,c]$
$\mathcal{Q}_4^3$	a, e_0, c, f	e_0^2, c^3, f^3, $(e_0e_3)^3$, $tct^{-1}c$, $(e_0c)^3$, $(ce_0e_3)^5$, $tat^{-1}a$, $[e_0,e_1]$, $[e_0,ce_1c^{-1}]$, $[e_0,e_2]e_1$, $[e_0,ce_2c^{-1}]c^{-1}e_1c$, $[c,a]$, $[c,f]$, $[e,f]$, $af(cfa)^{-1}$
$\mathcal{Q}_4^4$	a, e_0, c, f	e_0^2, c^3, f^3, $(e_0e_3)^3$, $tct^{-1}c$, $(e_0c)^3$, $(ce_0e_3)^5$, $tat^{-1}a$, $[e_0,e_1]$, $[e_0,ce_1c^{-1}]$, $[e_0,e_2]e_1$, $[e_0,ce_2c^{-1}]c^{-1}e_1c$, $aca^{-1}c$, $[c,f]$, $[e,f]$, $afa^{-1}fc^{-1}$
$\mathcal{Q}_4^5$	a, e_0, c, g	e_0^2, c^3, g^2, $(e_0e_3)^3$, $tct^{-1}c$, $(e_0c)^3$, $(ce_0e_3)^5$, $tat^{-1}a$, $[e_0,e_1]$, $[e_0,ce_1c^{-1}]$, $[e_0,e_2]e_1$, $[e_0,ce_2c^{-1}]c^{-1}e_1c$, $[a,c]$, $[e_0,g]$, $[a,g]$, $gcgc$
$\mathcal{Q}_4^6$	a, e_0, c, f, g	e_0^2, c^3, g^2, f^3, $(e_0e_3)^3$, $tct^{-1}c$, $(e_0c)^3$, $(ce_0e_3)^5$, $tat^{-1}a$, $[e_0,e_1]$, $[e_0,ce_1c^{-1}]$, $[e_0,e_2]e_1$, $[e_0,ce_2c^{-1}]c^{-1}e_1c$, $[c,a]$, $[e_0,g]$, $[a,g]$, $gcgc$, $gfgf$, $[c,f]$, $[e,f]$, $af(cfa)^{-1}$
$\mathcal{Q}_5^1$	a, e_0, c	c^3, e_0^2, $(e_0e_4)^3$, $tct^{-1}c$, g^2, $[e_0,g]$, $[a,g]$, c^gc, $(e_0c)^3$, $[e_2,c]$, $(ce_0e_4)^5$, $[c,f]$, $af(cfa)^{-1}$, $[e_0,e_1]$, $[e_0,e_2]$, $[e_0,e_3]e_2e_1$

Table 3.3: Presentations for the universal completions of finite faithful (5,2) amalgams with $s \geq 4$.

3.3 Finite Completions

In this section we provide examples of finite faithful completions for finite faithful amalgams of degree (5,2). We deal with the cases where $s \leq 3$. For the bigger values of s, we have indicated in Table 3.1 how completions can be obtained in the groups $\mathrm{Aut}(\mathrm{PSL}_3(4))$ and $\mathrm{Aut}(\mathrm{Sp}_4(4))$. For the remaining amalgams, our target for completions are also almost simple groups. Our reasoning for this is the following. If K is an almost simple group with two proper subgroups M and N such that $K = \langle M, N\rangle$ and $M \cap N < \mathbf{F}^*(K)$, then the amalgam $\mathcal{A} = (M, N, M \cap N, i_M, i_N)$ is faithful since $\mathbf{F}^*(K)$ has no non-trivial proper normal subgroups and K is a faithful completion of $\mathcal{A}$. Since there is a unique faithful amalgam of each type $\mathcal{Q}_i^j$ if we can show that an almost simple group exhibits an amalgam of type $\mathcal{Q}_i^j$, then this amalgam is the faithful one, and hence the group K is a completion.

The amalgam $\mathcal{Q}_1^1$ has as a finite completion any group generated by an involution and an element of order five. Thus $G = \mathrm{C}_5 \wr \mathrm{C}_2$ is indeed a finite completion of $\mathcal{Q}_1^1$. The graph

$\Gamma(\mathcal{A}, G)$ is isomorphic to the complete bipartite graph $\mathrm{K}_{5,5}$. For the amalgam $\mathcal{Q}_2^3$, consider $P_1 = \langle (1,2,3,4,5), (2,3,5,4) \rangle \cong \mathrm{Frob}(20)$ and $\langle (2,3,5,4), (3,4) \rangle \cong \mathrm{Dih}(8)$. Then $\langle P_1, P_2 \rangle$ contains a 5-cycle and a transposition, so $\langle P_1, P_2 \rangle = \mathrm{Sym}(5)$ is a finite completion for the amalgam $\mathcal{Q}_2^3$. Now P_1 contains a subgroup of index 2 isomorphic to $\mathrm{Dih}(10)$. In P_2 there is a subgroup of index 2 which is isomorphic to 2^2 and does not normalise $\mathrm{O}_5(P_1)$. The intersection of these two subgroups has order two, thus we see that $\mathrm{Alt}(5)$ is a completion for the amalgam of type $\mathcal{Q}_1^2$.

The group $\mathrm{Aut}(\mathrm{Alt}(6))$ provides us with completions for seven of our amalgams. For the amalgams $\mathcal{Q}_1^3$ and $\mathcal{Q}_2^1$, this is readily seen by considering the subgroups $P_1 = \langle (1,2,3,4,5),\ (1,2)(3,5) \rangle$ and $P_2 = \langle (1,3,2,5)(4,6) \rangle$, and then in $\mathrm{Sym}(6)$ considering $P_1^* = \langle (1,2,3,4,5),\ (1,3,2,5) \rangle$ and $P_2^* = \langle (1,3,2,5),\ (4,6) \rangle$. We see then that only a point stabiliser contains P_1 and P_1^* in $\mathrm{Alt}(6)$ and $\mathrm{Sym}(6)$ respectively, so $\mathrm{Alt}(6) = \langle P_1, P_2 \rangle$ and $\mathrm{Sym}(6) = \langle P_1^*, P_2^* \rangle$. Appealing now to the non-split extension of $\mathrm{Alt}(6)$ by a cyclic group of order 2, which is isomorphic to M_{10}, we see two maximal subgroups isomorphic to $\mathrm{Frob}(20)$ and $3^2 : \mathrm{Q}_8$. Choosing appropriate conjugacy class representatives, we obtain groups isomorphic to $\mathrm{Frob}(20)$ and Q_8 which intersect in a subgroup isomorphic to C_4. It follows that M_{10} is a faithful completion of our amalgam $\mathcal{Q}_2^4$. Finally, we consider the full automorphism group of $\mathrm{Alt}(6)$ which is isomorphic to $\mathrm{P\Gamma L}_2(9)$ and is a non-split extension of $\mathrm{Alt}(6)$ by 2^2. Here, our subgroup isomorphic to $\mathrm{Frob}(20)$ becomes a group isomorphic to $\mathrm{Frob}(20) \times 2$, and in the maximal subgroup of size 2^5, we find an index two subgroup which we denote by M_{16} with isomorphism shape $(\mathrm{C}_4 \times \mathrm{C}_2).\mathrm{C}_2$. Again, we may choose representatives of the conjugacy classes so that these groups intersect in a group isomorphic to $\mathrm{C}_4 \times \mathrm{C}_2$, which gives us $\mathrm{P\Gamma L}_2(9)$ as a faithful completion of our amalgam $\mathcal{Q}_2^6$.

Turning now to the amalgams $\mathcal{Q}_2^7$-$\mathcal{Q}_2^9$ we choose $P_1 = \mathrm{Stab}_{\mathrm{Alt}(6)}(1) \cong \mathrm{Alt}(5)$ and $P_2 = \mathrm{Stab}_{\mathrm{Alt}(6)}(\{1,2\}) \cong \mathrm{Sym}(4)$. Then $P_1 \cap P_2 \cong \mathrm{Alt}(4)$ and $\mathrm{Alt}(6) = \langle P_1, P_2 \rangle$. Hence $\mathrm{Alt}(6)$ is a faithful completion of the amalgam $\mathcal{Q}_2^7$. We see that $\mathrm{Sym}(6) = \langle \mathrm{N}_{\mathrm{Sym}(6)}(P_1), \mathrm{N}_{\mathrm{Sym}(6)}(P_2) \rangle$, $\mathrm{N}_{\mathrm{Sym}(6)}(P_1) \cong \mathrm{Sym}(5)$ and $\mathrm{N}_{\mathrm{Sym}(6)}(P_2) \cong \mathrm{Sym}(4) \times \mathrm{C}_2$. Hence

Sym(6) is a completion of the $\mathcal{Q}_2^9$ amalgam. There is a unique subgroup $P_2 \leq P_2^* \leq \mathrm{N}_{\mathrm{Sym}(6)}(P_2)$ such that $P_2^* \cong \mathrm{Alt}(4) \times \mathrm{C}_2$. Setting $P_1^* = P_1(\mathrm{N}_{\mathrm{Sym}(6)}(P_1) \cap P_2^*)$ we have $P_1^* \cong \mathrm{Alt}(5)$ and $P_1^* \cap P_2^* \cong \mathrm{Alt}(4)$. Since P_2^* contains transpositions, we have that $\langle P_1^*, P_2^* \rangle = \mathrm{Sym}(6)$ so that Sym(6) is a completion for the amalgam $\mathcal{Q}_2^8$. For the amalgam $\mathcal{Q}_1^4$, we consult the Atlas [9, pg.7] to see that $\mathrm{PSL}_2(11) : 2$ has a maximal subgroup isomorphic to Dih(20). We may choose representatives H and K of this conjugacy class which are interchanged by the outer automorphism of order 2, and thus obtain a subgroup L generated by $H \cap K$ and the outer automorphism. Then $L \cong \mathrm{Dih}(8)$, and since H is maximal, $\mathrm{PSL}_2(11) : 2$ is generated by H and L. Thus $\mathrm{PSL}_2(11) : 2$ is a finite faithful completion of the amalgam $\mathcal{Q}_1^4$.

The Mathieu group M_{11} appears for us as a completion of our amalgam $\mathcal{Q}_2^2$. Inside the maximal subgroups isomorphic to either Sym(5) or M_{10}, we see a subgroup which we will call A_1 isomorphic to Frob(20). Choosing an element $x \in A_1$ of order 4, we let $A_2 = \mathrm{C}_{\mathrm{M}_{11}}(x)$. The character table of M_{11} shows us that $|A_2| = 8$, and since all elements of order 4 in M_{11} are conjugate, and there are elements of order 8, A_2 is cyclic of order 8 (and $A_1 \cap A_2 \cong \mathrm{C}_4$). Suppose now that $G = \langle A_1, A_2 \rangle \neq \mathrm{M}_{11}$ and let N be a maximal subgroup containing G. Then $40 \mid |N|$, and so $N \cong \mathrm{M}_{10}$ or $N \cong \mathrm{Sym}(5)$. The second of these is clearly impossible, and so possibly $N \cong \mathrm{M}_{10}$. Now the derived subgroup of M_{10} has index 2 and is isomorphic to Alt(6), thus the unique subgroup in A_2 of index 2 lies in Alt(6). But the element of order 5 in A_1 must also lie in the derived subgroup, and so this implies that Alt(6) contains a subgroup isomorphic to Frob(20), which is not the case. Hence $G = \mathrm{M}_{11}$, and G is a completion of an amalgam of type $\mathcal{Q}_2^2$.

For the amalgams $\mathcal{Q}_2^5$, $\mathcal{Q}_3^1$-$\mathcal{Q}_3^5$ we claim that either Alt(9) or Sym(9) is a completion. In $G = \mathrm{Sym}(9)$, let A_1 be the natural embedding of $\mathrm{Sym}(5) \times \mathrm{Sym}(4)$ viewed as the stabiliser of the partition $\{\{1,2,3,4,5\},\{6,7,8,9\}\}$. Now take A_2 to be the normaliser in G of the stabiliser in A_1 of the point 1, then $A_2 \cong \mathrm{Sym}(4) \wr \mathrm{C}_2$, and $B := A_1 \cap A_2 \cong \mathrm{Sym}(4)^2$ and $|A_1/B| = 5$, $|A_2/B| = 2$. Since A_1 is a maximal subgroup of G, we see that $G = \langle A_1, A_2 \rangle$, and since Alt(9) is simple (and $\mathrm{Alt}(9) \not\leq B$) we see that the triple (A_1, A_2, B), together with the embeddings of B in A_1 and A_2, is a faithful amalgam of degree $(5, 2)$ which is our amalgam $\mathcal{Q}_3^5$. Now A_1 and A_2 contain some obvious subgroups which give us our amalgams listed at the beginning of this paragraph, and aside from the cases $\mathcal{Q}_3^2$ and $\mathcal{Q}_3^4$ where we dip into Alt(9), Sym(9) is a completion for these amalgams too. This is easy to check, and simply requires knowledge of the maximal subgroups of Sym(9) and Alt(9), which are delivered by the ATLAS [9].

CHAPTER 4

SEMISYMMETRIC GRAPHS OF VALENCY FIVE

In this chapter we focus on semisymmetric graphs of valency five and we aim to describe the resulting amalgams. As we have seen, this is equivalent to classifying the finite faithful amalgams of degree (5,5). Using Theorem 2.2.25 it is enough to classify the simple amalgams, this we do in Sections 4.4-4.6. In Section 4.2 we show how the extensions of the simple amalgams arise. We prove the following two theorems.

Theorem 4.0.1. *Suppose that $\mathcal{A}$ is a simple finite faithful amalgam of degree $(5,5)$. Then the type of $\mathcal{A}$ is in Table 4.1. Moreover, $\mathcal{A}$ is the unique faithful amalgam of this type.*

Theorem 4.0.2. *Suppose that $\mathcal{E}$ is a faithful extension of one of the faithful amalgams $\mathcal{S}_i$ for $i \in [1, 15]$. Then $\mathcal{E}$ has the same type as one of the amalgams in Tables 4.3-4.7. In particular, there are ninety non-trivial extensions of the amalgams $\mathcal{S}_1$-$\mathcal{S}_{15}$.*

Remark 4.0.3 (On Table 4.1)*.* We have given a sample finite completion G for each of the amalgams. We will justify this in Section 4.7.

For the amalgams $\mathcal{S}_7$, $\mathcal{S}_8$, $\mathcal{S}_{11}$ and $\mathcal{S}_{12}$ we have only given the shape of G_y which does not identify the isomorphism type of the group. We will remedy this in Section 4.7 where we give a permutation representation of the groups appearing in the amalgam.

Amalgam	G_x	G_y	G
$\mathcal{S}_1$	C_5	C_5	5^2
$\mathcal{S}_2$	$\mathrm{Alt}(5)$	$\mathrm{Alt}(5)$	$\mathrm{Alt}(6)$
$\mathcal{S}_3$	$\mathrm{Alt}(5)$	$\mathrm{Alt}(4) \times \mathrm{C}_5$	$\mathrm{Alt}(9)$
$\mathcal{S}_4$	$\mathrm{Alt}(5) \times \mathrm{Alt}(4)$	$\mathrm{Alt}(5) \times \mathrm{Alt}(4)$	$\mathrm{Alt}(9)$
$\mathcal{S}_5$	$2^4 : \mathrm{C}_5$	$2^3 \times \mathrm{Dih}(10)$	$\mathrm{Alt}(21)$
$\mathcal{S}_6$	${\mathrm{C}_4}^4 : \mathrm{C}_5$	${\mathrm{C}_4}^3 \times \mathrm{Frob}(20)$	$\mathrm{Alt}(21)$
$\mathcal{S}_7$	$2^4 : \mathrm{Alt}(5)$	$(\mathrm{C}_5 \times 2^3).(\mathrm{Alt}(4) \times \mathrm{C}_2)$	$\mathrm{Alt}(21)$
$\mathcal{S}_8$	${\mathrm{C}_4}^4 : \mathrm{Alt}(5)$	$(\mathrm{C}_5 \times {\mathrm{C}_4}^3).(\mathrm{Alt}(4) \times \mathrm{C}_4)$	$\mathrm{Alt}(21)$
$\mathcal{S}_9$	$\mathrm{Alt}(4) \wr \mathrm{C}_5$	$\mathrm{Alt}(4)^4 \times \mathrm{Alt}(5)$	$\mathrm{Alt}(21)$
$\mathcal{S}_{10}$	$\mathrm{Alt}(4) \wr \mathrm{Alt}(4)$	$\mathrm{Alt}(4) \wr \mathrm{Alt}(4) \times \mathrm{Alt}(5)$	$\mathrm{Alt}(21)$
$\mathcal{S}_{11}$	$\mathrm{O}^2(\mathrm{Sym}(4) \wr \mathrm{C}_5)$	$\mathrm{Alt}(5).\mathrm{Sym}(4)^4$	$\mathrm{Alt}(21)$
$\mathcal{S}_{12}$	$\mathrm{O}^2(\mathrm{Sym}(4) \wr \mathrm{Alt}(5))$	$\mathrm{Alt}(5).\mathrm{Sym}(4) \wr \mathrm{Alt}(4)$	$\mathrm{Alt}(21)$
$\mathcal{S}_{13}$	$2^4 : \mathrm{PSL}_2(4)$	$2^4 : \mathrm{PSL}_2(4)$	$\mathrm{PSL}_3(4)$
$\mathcal{S}_{14}$	$2^{2+4} : \mathrm{GL}_2(4)$	$2^{2+4} : \mathrm{GL}_2(4)$	$\mathrm{Sp}_4(4)$
$\mathcal{S}_{15}$	$2^{2+8} : (\mathrm{PSL}_2(4) \times \mathrm{C}_3)$	$2^{4+6} : (\mathrm{PSL}_2(4) \times \mathrm{C}_3)$	$\mathrm{G}_2(4)$

Table 4.1: The types of faithful finite simple amalgams of degree (5,5)

4.1 Uniqueness of simple amalgams

In this short section we remove any ambiguity concerning the amalgams introduced in Table 4.1, that is, we prove that there is a unique faithful amalgam of each type. After this section then we may refer to "the faithful amalgam $\mathcal{S}_i$".

Theorem 4.1.1. *Let $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ be a faithful amalgam of type $\mathcal{S}_i$ for $i \in [1, 15]$. Then there is a unique faithful amalgam of this type.*

Proof. We use Lemma 2.2.12 and the notation introduced there. For $\mathcal{S}_1$ there is nothing to prove. For both $\mathcal{S}_2$ and $\mathcal{S}_3$ we have that $\mathrm{Aut}(B) \cong \mathrm{Sym}(4)$ and $\mathrm{N}_{\mathrm{Aut}(P_1)}(\pi_1(B)) \cong \mathrm{Sym}(4)$ which means there is a unique amalgam of this type. For $\mathcal{S}_4$ we see that $\mathrm{Aut}(B) \cong \mathrm{Sym}(4) \wr \mathrm{C}_2$. Now $H_1^* = H_2^* \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$ has index two in $\mathrm{Aut}(B)$, so there are two isomorphism classes of amalgams of this type. Moreover $\mathrm{Aut}(B)$ shows how the two classes differ; in one the image of an $\mathrm{Alt}(4)$ factor of B is normal in both P_1 and P_2, whereas the other class of amalgams is faithful.

For the amalgams $\mathcal{S}_{13}$-$\mathcal{S}_{15}$ Theorem 1.6.8 shows that any amalgam has the same type as an amalgam over a Sylow 2-subgroup of one of $\mathrm{PSL}_3(4)$, $\mathrm{Sp}_4(4)$ or $\mathrm{G}_2(4)$ (see the

Amalgam	No. of extensions
$\mathcal{S}_1$	11
$\mathcal{S}_2$	2
$\mathcal{S}_3$	8
$\mathcal{S}_4$	4
$\mathcal{S}_5$	15
$\mathcal{S}_6$	15
$\mathcal{S}_7$	8
$\mathcal{S}_8$	8
$\mathcal{S}_9$	8
$\mathcal{S}_{10}$	8
$\mathcal{S}_{11}$	5
$\mathcal{S}_{12}$	5
$\mathcal{S}_{13}$	4
$\mathcal{S}_{14}$	2
$\mathcal{S}_{15}$	2

Table 4.2: The number of extensions of faithful finite simple amalgams of degree (5,5)

remark after Theorem 1.6.8). Since the Sylow 2-subgroups are conjugate, the conjugation map defines an isomorphism between any two such amalgams. Hence there is a unique faithful amalgam of this type.

For the amalgams $\mathcal{S}_5$-$\mathcal{S}_{12}$ we employ the computer program given in Section A.1 which shows there is a unique faithful amalgam of these types. $\square$

4.2 Extensions

Here we calculate the extensions of the simple amalgams using Theorem 2.2.25. We prove the following theorem.

Theorem 4.2.1. *For $i \in [1, 15]$ the number of extensions of the faithful amalgam $\mathcal{S}_i$ is given in Table 4.2.*

We begin with the smallest, recall Definition 2.2.7 of the automorphism group of an amalgam.

Lemma 4.2.2. *The group $\mathrm{Aut}(\mathcal{S}_1)$ is isomorphic to $\mathrm{C}_4 \wr \mathrm{C}_2$.*

Amalgam	P_1	P_2	B
$\mathcal{E}_1^1$	Dih(10)	C_{10}	C_2
$\mathcal{E}_1^2$	Dih(10)	Dih(10)	C_2
$\mathcal{E}_1^3$	Dih(20)	Dih(20)	2^2
$\mathcal{E}_1^4$	Frob(20)	C_{20}	C_4
$\mathcal{E}_1^5$	Frob(20)	Frob(20)	C_4
$\mathcal{E}_1^6$	Frob(20)	Frob(20)	C_4
$\mathcal{E}_1^7$	Frob(20)	Q_{20}	C_4
$\mathcal{E}_1^8$	Dih(10) $\times \mathrm{C}_4$	Frob(20) $\times \mathrm{C}_2$	$\mathrm{C}_2 \times \mathrm{C}_4$
$\mathcal{E}_1^9$	Frob(20) $\times \mathrm{C}_2$	Frob(20) $\times \mathrm{C}_2$	$\mathrm{C}_4 \times \mathrm{C}_2$
$\mathcal{E}_1^{10}$	Frob(20) $\times \mathrm{C}_4$	Frob(20) $\times \mathrm{C}_4$	$\mathrm{C}_4 \times \mathrm{C}_4$

Table 4.3: The types of the extensions of the faithful amalgam $\mathcal{S}_1$

Proof. Write $\mathcal{S}_1 = (P_1, P_2, B, \pi_1, \pi_2)$ so that $P_1 \cong P_2 \cong \mathrm{C}_5$ and $B \cong 1$. We have that $\mathrm{Aut}(P_1) \cong \mathrm{Aut}(P_2) \cong \mathrm{C}_4$. Since every automorphism of P_1 and P_2 normalises B we have $H_1^* \times H_2^* = H$. Since $\mathrm{Aut}(B) = 1$, every automorphism of P_1 and P_2 has the same action on B. Hence $\mathrm{Aut}^\circ(\mathcal{A}) = H$. Clearly there is an automorphism of order two swapping P_1 and P_2, so we have $\mathrm{Aut}(\mathcal{A}) \cong \mathrm{C}_4 \wr \mathrm{C}_2$. □

We may now calculate the extensions, recall Definition 2.2.20.

Lemma 4.2.3. *There are ten non-trivial extensions of $\mathcal{S}_1$. The extensions have one of the types (P_1, P_2, B) given in Table 4.3.*

Proof. Write $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ with $P_1 \cong P_2 \cong \mathrm{C}_5$ and $B \cong 1$. First note that up to $\mathrm{Aut}(\mathcal{A})$-conjugacy there are eleven subgroups of $\mathrm{Aut}^\circ(\mathcal{A})$. We make the identifications $\mathrm{Aut}^\circ(\mathcal{A}) = \langle \alpha, \beta \rangle$ with $\langle \alpha \rangle = H_1^*$ and $\langle \beta \rangle = H_2^*$. Then representatives for these conjugacy classes are $R_1 = \langle \alpha^2 \rangle$, $R_2 = \langle \alpha^2\beta^2 \rangle$, $R_3 = \langle \alpha^2, \beta^2 \rangle$, $R_4 = \langle \alpha \rangle$, $R_5 = \langle \alpha\beta \rangle$, $R_6 = \langle \alpha\beta^3 \rangle$, $R_7 = \langle \alpha\beta^2 \rangle$, $R_8 = \langle \alpha^2, \beta \rangle$, $R_9 = \langle \alpha\beta, \alpha^2 \rangle$ and $R_{10} = \langle \alpha, \beta \rangle$.

For $i \in \{1, \ldots, 10\}$ the extension $\mathcal{E}_1^i := \mathcal{E}(\mathcal{S}_1, R_i)$ has the same type as the amalgam described in Table 4.3. The extensions $\mathcal{E}(\mathcal{S}_1, R_5)$ and $\mathcal{E}(\mathcal{S}_1, R_6)$ have the same type, but the amalgams are non-isomorphic (see Example 4.2.4). □

In the proof of the above lemma we saw that two extensions of $\mathcal{S}_1$ have the same type, but are non-isomorphic. We construct an explicit example to demonstrate this.

Example 4.2.4. The amalgams $\mathcal{E}_1^5$ and $\mathcal{E}_1^6$. *Let $P_1 = \langle x, w\rangle$, $P_2 = \langle y, z\rangle$ be so that x and y have order 5, w and z have order 4 and $x^w = x^2$, $y^z = y^2$. Let $B = \langle u\rangle$ be cyclic of order 4. Define $\pi_1 : B \to P_1$ by $u \mapsto w$ and $\pi_2 : B \to P_2$ by $u \mapsto z$. Let $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$. Then $\mathcal{A}$ has the same type as $\mathcal{E}_1^5$. Applying Lemma 2.2.12 we see that $N_{\mathrm{Aut}(P_i)}(\pi_i(B)) = C_{\mathrm{Aut}(P_i)}(\pi_i(B))$ for $i = 1, 2$. Therefore $H_1^* = H_2^* = 1$ and so there are two (H_1^*, H_2^*) double cosets in $\mathrm{Aut}(B) \cong \mathrm{C}_2$. A representative of the second isomorphism class is given by setting $\pi_3 : u \mapsto z^3$. This second amalgam is the extension $\mathcal{E}_1^6$.*

Lemma 4.2.5. *Let $\mathcal{E}_2^1 = (\mathrm{Sym}(5), \mathrm{Sym}(5), \mathrm{Sym}(4))$ be the (unique) faithful amalgam of this type. Then $\mathcal{E}_2^1$ is the unique non-trivial extension of $\mathcal{S}_2$.*

Proof. Note that $H_1 \times H_2 \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$ and $\mathrm{Inn}(\mathcal{S}_2)$ is the diagonal $\mathrm{Alt}(4)$ subgroup. Then we find that $\mathrm{Aut}^\circ(\mathcal{S}_2)$ is the diagonal $\mathrm{Sym}(4)$ subgroup and $\mathrm{Aut}(\mathcal{S}_2) \cong \mathrm{Sym}(4) \times \mathrm{C}_2$. Thus $\mathrm{Out}^\circ(\mathcal{S}_2) \cong \mathrm{C}_2$ so there is a unique non-trivial extension. □

Lemma 4.2.6. *We have* $\mathrm{Aut}(\mathcal{S}_3) = \mathrm{Aut}^\circ(\mathcal{S}_3) \cong \mathrm{Sym}(4) \times \mathrm{C}_4$ *and* $\mathrm{Out}^\circ(\mathcal{S}_3) \cong \mathrm{C}_2 \times \mathrm{C}_4$.

Proof. Write $\mathcal{S}_3 = (P_1, P_2, B)$ and note that $\mathrm{Aut}(\mathcal{S}_3) = \mathrm{Aut}^\circ(\mathcal{S}_3)$. We have $\mathrm{Aut}(P_1) \cong \mathrm{Sym}(5)$ and $\mathrm{N}_{\mathrm{Aut}(P_1)}(B) \cong \mathrm{Sym}(4)$ and for P_2 we find that $\mathrm{Aut}(P_2) \cong \mathrm{Sym}(4) \times \mathrm{C}_4 = \mathrm{N}_{\mathrm{Aut}(P_2)}(B)$. Hence $H_1^* \times H_2^* \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4) \times \mathrm{C}_4$. Note that $(1, x) \in \mathrm{Aut}^\circ(\mathcal{S}_3)$ for all x in the normal cyclic subgroup of order four in H_2^* since B is centralised by these elements. The rest of the automorphism group $\mathrm{Aut}(\mathcal{S}_3)$ is generated by the "diagonal" $\mathrm{Sym}(4)$ subgroup. Hence we see $\mathrm{Aut}^\circ(\mathcal{S}_3)/\mathrm{Inn}(\mathcal{S}_3) \cong \mathrm{C}_2 \times \mathrm{C}_4$. □

Lemma 4.2.7. *There are seven non-trivial extensions of $\mathcal{S}_3$. The types of these amalgams are given in Table 4.4.*

Proof. Using the previous lemma we know that the number of extensions is the number of subgroups of $\mathrm{C}_2 \times \mathrm{C}_4$. Writing $\mathrm{Aut}^\circ(\mathcal{S}_3) = \langle \mathrm{Inn}(\mathcal{S}_3), \alpha, \beta\rangle$ where α is an involution so that $\langle \mathrm{Inn}(\mathcal{S}_3), \alpha\rangle \cong \mathrm{Sym}(4)$ and β is an element of order four which commutes with the

Amalgam	P_1	P_2	B
$\mathcal{E}_3^1$	Sym(5)	Sym(4) $\times$ C_5	Sym(4)
$\mathcal{E}_3^2$	Sym(5)	Sym(4) $\curlywedge$ Dih(10)	Sym(4)
$\mathcal{E}_3^3$	Alt(5) $\times$ C_2	Alt(4) $\times$ Dih(10)	Alt(4) $\times$ C_2
$\mathcal{E}_3^4$	Sym(5) $\times$ C_2	Sym(4) $\times$ Dih(10)	Sym(4) $\times$ C_2
$\mathcal{E}_3^5$	Alt(5) $\times$ C_4	Alt(4) $\times$ Frob(20)	Alt(4) $\times$ C_4
$\mathcal{E}_3^6$	Alt(5) : 4	Sym(4) $\curlywedge$ Frob(20)	Alt(4) : 4
$\mathcal{E}_3^7$	Sym(5) $\times$ C_4	Sym(4) $\times$ Frob(20)	Sym(4) $\times$ C_4

Table 4.4: The types of the extensions of the faithful amalgam $\mathcal{S}_3$

Sym(4) subgroup just constructed. The extensions by subgroups of order two in $\mathrm{Out}^\circ(\mathcal{S}_3)$ are given by

$$\begin{aligned}
\mathcal{E}_3^1 &:= \mathcal{E}(\mathcal{S}_3, \langle \mathrm{Inn}(\mathcal{S}_3), \alpha \rangle), \\
\mathcal{E}_3^2 &:= \mathcal{E}(\mathcal{S}_3, \langle \mathrm{Inn}(\mathcal{S}_3), \alpha\beta^2 \rangle), \\
\mathcal{E}_3^3 &:= \mathcal{E}(\mathcal{S}_3, \langle \mathrm{Inn}(\mathcal{S}_3), \beta^2 \rangle).
\end{aligned}$$

The three extensions corresponding to subgroups of order four in $\mathrm{Out}^\circ(\mathcal{S}_3)$ are given by

$$\begin{aligned}
\mathcal{E}_3^4 &:= \mathcal{E}(\mathcal{S}_3, \langle \mathrm{Inn}(\mathcal{S}_3), \alpha, \beta^2 \rangle), \\
\mathcal{E}_3^5 &:= \mathcal{E}(\mathcal{S}_3, \langle \mathrm{Inn}(\mathcal{S}_3), \beta \rangle), \\
\mathcal{E}_3^6 &:= \mathcal{E}(\mathcal{S}_3, \langle \mathrm{Inn}(\mathcal{S}_3), \alpha\beta \rangle).
\end{aligned}$$

The final extension is $\mathcal{E}_3^7 := \mathcal{E}(\mathcal{S}_3, \mathrm{Aut}^\circ(\mathcal{A}))$. It is easy to verify that these amalgams have types given in Table 4.4. $\square$

Lemma 4.2.8. *We have* $\mathrm{Aut}(\mathcal{S}_4) \cong \mathrm{Sym}(4) \wr \mathrm{C}_2$, $\mathrm{Aut}^\circ(\mathcal{S}_4) \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4)$ *and* $\mathrm{Inn}(\mathcal{S}_4) \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4)$. *There are 3 non-trivial extensions of* $\mathcal{S}_4$*; their types are given in Table 4.5.*

Proof. Let $\mathcal{S}_4 = (P_1, P_2, B, \pi_1, \pi_2)$. We first determine the group $H_1 \times H_2$. Note that $\mathrm{Aut}(P_1) \cong \mathrm{Sym}(5) \times \mathrm{Sym}(4) \cong \mathrm{Aut}(P_2)$ and $H_1 \cong \mathrm{Sym}(4) \times \mathrm{Sym}(4) \cong H_2$. Hence $H_1 \times H_2$ is the direct product of four copies of Sym(4). Make the identifications $H_1 =$

Amalgam	P_1	P_2	B
$\mathcal{E}_4^1$	$\mathrm{Alt}(5) \times \mathrm{Sym}(4)$	$\mathrm{Alt}(4) \times \mathrm{Sym}(5)$	$\mathrm{Alt}(4) \times \mathrm{Sym}(4)$
$\mathcal{E}_4^2$	$\mathrm{Sym}(5) \curlywedge \mathrm{Sym}(4)$	$\mathrm{Sym}(5) \curlywedge \mathrm{Sym}(4)$	$\mathrm{Sym}(4) \curlywedge \mathrm{Sym}(4)$
$\mathcal{E}_4^3$	$\mathrm{Sym}(5) \times \mathrm{Sym}(4)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(5)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(4)$

Table 4.5: The types of the extensions of the faithful amalgam $\mathcal{S}_4$

$\langle (1,2),(1,2,3,4),(5,6),(5,6,7,8)\rangle$ and $H_2 = \langle (9,10),(9,10,11,12),(13,14),(13,14,15,16)\rangle$ and regard $H_1 \times H_2$ as a subgroup of $\mathrm{Sym}(16)$. We find that

$$\mathrm{Inn}(\mathcal{S}_4) = \langle (1,2,3)(9,10,11),(2,3,4)(10,11,12),(5,6,7)(13,14,15),(6,7,8)(14,15,16)\rangle$$

and $\mathrm{Aut}^\circ(\mathcal{S}_4) = \langle (1,2)(9,10),\ (5,6)(13,14), \mathrm{Inn}(\mathcal{S}_4)\rangle$. Since $\mathrm{Out}(\mathcal{S}_4) \cong \mathrm{Dih}(8)$ there are three non-trivial extensions of $\mathcal{S}_4$. Here $\mathrm{Out}(\mathcal{S}_4)$ conjugacy implies the extensions defined by the groups $\langle \mathrm{Inn}(\mathcal{S}_4),(1,2)(9,10)\rangle$ and $\langle \mathrm{Inn}(\mathcal{S}_4),(5,6)(13,14)\rangle$ are isomorphic. We define

$$\begin{aligned}
\mathcal{E}_4^1 &:= \mathcal{E}(\mathcal{S}_4, \langle \mathrm{Inn}(\mathcal{S}_4),(5,6)(13,14)\rangle),\\
\mathcal{E}_4^2 &:= \mathcal{E}(\mathcal{S}_4, \langle \mathrm{Inn}(\mathcal{S}_4),(1,2)(9,10)(5,6)(13,14)\rangle),\\
\mathcal{E}_4^3 &:= \mathcal{E}(\mathcal{S}_4, \langle \mathrm{Inn}(\mathcal{S}_4),(1,2)(9,10),(5,6)(13,14)\rangle),
\end{aligned}$$

then it is easy to see that the amalgams $\mathcal{E}_4^1$-$\mathcal{E}_4^3$ have the types given in Table 4.5. □

Lemma 4.2.9. *For $i \in [5,12]$ we have $\mathrm{Aut}(\mathcal{S}_i) = \mathrm{Aut}^\circ(\mathcal{S}_i)$, and $\mathrm{Out}(\mathcal{S}_i)$ is described in Table 4.6.*

Proof. Since $P_1 \ncong P_2$ for all of these amalgams we have $\mathrm{Aut}(\mathcal{S}_i) = \mathrm{Aut}^\circ(\mathcal{S}_i)$. Now we use the computer program **Ext** given in Section A.2. □

Remark 4.2.10. We do not attempt to describe the shape of the groups appearing in the extensions of the amalgams $\mathcal{S}_5$-$\mathcal{S}_{12}$ at this stage, since they are extensions by 2-groups this would not provide much information. In Section 4.7 we will work with a fixed completion of these amalgams, and then the extensions will become transparent.

Amalgam	Outer automorphism group
$\mathcal{S}_5$	$C_4 \times C_4$
$\mathcal{S}_6$	$C_4 \times C_4$
$\mathcal{S}_7$	$C_2 \times C_4$
$\mathcal{S}_8$	$C_2 \times C_4$
$\mathcal{S}_9$	$C_2 \times C_4$
$\mathcal{S}_{10}$	$C_2 \times C_4$
$\mathcal{S}_{11}$	2^2
$\mathcal{S}_{12}$	2^2

Table 4.6: The outer automorphism groups of the faithful amalgams $\mathcal{S}_5$-$\mathcal{S}_{12}$

Amalgam	P_1	P_2	G
$\mathcal{E}_{13}^1$	$2^4 : \mathrm{GL}_2(4)$	$2^4 : \mathrm{GL}_2(4)$	$\mathrm{PSL}_3(4).\langle i\rangle$
$\mathcal{E}_{13}^2$	$2^4 : \Gamma\mathrm{L}_2(4)$	$2^4 : \Gamma\mathrm{L}_2(4)$	$\mathrm{PSL}_3(4) : \langle f\rangle$
$\mathcal{E}_{13}^3$	$2^4 : (\mathrm{GL}_2(4) : C_2)$	$2^4 : (\mathrm{GL}_2(4) : C_2)$	$\mathrm{PSL}_3(4) : \langle f, i\rangle$
$\mathcal{E}_{14}^1$	$2^{2+4} : \Gamma\mathrm{L}_2(4)$	$2^{2+4} : \Gamma\mathrm{L}_2(4)$	$\mathrm{Sp}_4(4) : \langle f\rangle$
$\mathcal{E}_{15}^1$	$2^{2+8} : (\mathrm{PSL}_2(4) \times C_3) : C_2$	$2^{4+6} : (\mathrm{PSL}_2(4) \times C_3) : C_2$	$\mathrm{Aut}(G_2(4))$

Table 4.7: The types of the extensions of the faithful amalgam $\mathcal{S}_{13}$-$\mathcal{S}_{15}$

Lemma 4.2.11. *There are 3 non-trivial extensions of the amalgam $\mathcal{S}_{13}$. All are visible in the group* $\mathrm{Aut}(\mathrm{PSL}_3(4))$. *There is one non-trivial extension of the amalgam $\mathcal{S}_{14}$, visible in* $\mathrm{Aut}(\mathrm{Sp}_4(4))$. *There is one non-trivial extension of the amalgam $\mathcal{S}_{15}$, visible in* $\mathrm{Aut}(G_2(4))$.

Proof. This follows from Theorem 1.6.8. □

We use similar notation for the extensions of the amalgams $\mathcal{S}_{13}$-$\mathcal{S}_{15}$ as introduced for the amalgams $\mathcal{S}_1$-$\mathcal{S}_4$. The types of the amalgams can be seen inside the groups described in the above lemma, we record them in Table 4.7 and indicate which extension the amalgam is present in (where f denotes a field automorphism and i is a diagonal automorphism of order three).

Remark 4.2.12. We calculate that $\mathrm{Aut}(\mathcal{S}_{13}) \cong 2 \times \mathrm{Sym}(3)$ and $\mathrm{Aut}^\circ(\mathcal{S}_{13}) \cong \mathrm{Sym}(3)$. We have $\mathrm{Aut}(\mathcal{S}_{14}) \cong C_4$ and $\mathrm{Aut}^\circ(\mathcal{S}_{14}) \cong C_2$. Furthermore $\mathrm{Aut}(\mathcal{S}_{15}) = \mathrm{Aut}^\circ(\mathcal{S}_{15}) \cong C_2$. These calculations are in agreement with the lemma above.

Proof of Theorem 4.2.1. This is a combination of Lemmas 4.2.3, 4.2.5, 4.2.7, 4.2.8, 4.2.9

and 4.2.11. $\square$

4.3 Simple amalgams of degree (5,5)

Let us now fix $\mathcal{S}$ a simple finite faithful amalgam of degree $(5,5)$. Let $G = \mathcal{G}(\mathcal{S})$ and $\Gamma = \Gamma(\mathcal{S}, G) \cong \Gamma_{5,5}$, a tree of valency five. Fix an edge $\{x, y\}$ of Γ and set $\mathcal{A} = (G_x, G_y, G_{xy}, \pi_x, \pi_y)$. Now $\mathcal{S} \cong \mathcal{A}$ and the aim is thus to determine $\mathcal{A}$. Furthermore, by Theorem 2.4.5 we have that $\mathcal{A} = \mathrm{O}^{5'}(\mathcal{A})$, the normal subamalgam of $\mathcal{A}$ generated by the pair $(\mathrm{O}^{5'}(G_x), \mathrm{O}^{5'}(G_y))$. By Proposition 2.2.5, setting

$$B = (\mathrm{O}^{5'}(G_x) \cap G_{xy})(\mathrm{O}^{5'}(G_y) \cap G_{xy}),$$

we have that

$$\begin{aligned} G_x &= \mathrm{O}^{5'}(G_x)B, \\ G_y &= \mathrm{O}^{5'}(G_y)B, \\ G_{xy} &= B. \end{aligned}$$

Our first result concerns the edge stabiliser. It follows immediately from Lemma 2.4.1 and Burnside's $p^a q^b$-Theorem. We may use this lemma without reference.

Lemma 4.3.1. *The group G_{xy} is a $\{2,3\}$-group. In particular, G_{xy} is soluble.*

Next we identify the local action.

Lemma 4.3.2. *For $z \in \Gamma$ the group $G_z^{\Delta(z)}$ is isomorphic to one of* C_5, $\mathrm{Dih}(10)$*,* $\mathrm{Frob}(20)$*,* $\mathrm{Alt}(5)$ *or* $\mathrm{Sym}(5)$*.*

Proof. By Lemma 2.1.2, $G_z^{\Delta(z)}$ is a transitive subgroup of $\mathrm{Sym}(\Delta(z)) \cong \mathrm{Sym}(5)$. The result is then an easy calculation in $\mathrm{Sym}(5)$. $\square$

Our second result shows that $|\mathbf{F}(G_x)|$ and $|\mathbf{F}(G_y)|$ have restricted prime divisors.

Lemma 4.3.3. *Suppose that $z \in \{x, y\}$ and p is a prime such that $\mathrm{O}_p(G_z) \neq 1$. Then $p \in \{2, 5\}$. Moreover, $\mathrm{O}_2(G_z) = \mathrm{O}_2(G_z^{[1]})$.*

Proof. We prove the statement for $z = x$, a symmetric argument yields the assertion for $z = y$. Let p be as in the statement and set $Q = \mathrm{O}_p(G_x)$. If $Q \not\leq G_x^{[1]}$, then $G_x^{\Delta(x)}$ is p-local, and so $p = 5$ by Lemma 4.3.2. We may assume then that $Q \leq G_x^{[1]}$. If $Q \not\leq G_y^{[1]}$ then we see that $QG_y^{[1]}/G_y^{[1]}$ is a normal subgroup of $G_{xy}/G_y^{[1]}$, so $p = 2$ (again by the previous lemma). Since $Q = \mathrm{O}_p(G_y^{[1]})$ allows us to apply Lemma 2.1.4 and obtain $Q = 1$, we may assume that $Q < \mathrm{O}_p(G_y^{[1]})$. Now we have $\mathrm{O}_p(G_y^{[1]}) \not\leq G_x^{[1]}$, and so now by considering $G_{xy}/G_x^{[1]}$ we see $p = 2$. The final statement follows since $G_x^{\Delta(x)}$ has no normal 2-subgroup. $\square$

We now make a case division according to Theorem 2.5.1 which delivers two cases. The second of these with $G_{xy}^{[1]} = G_x^{[2]}$ is more easily considered as two cases, where $G_{xy}^{[1]}$ is trivial or not. For $z \in \Gamma$ we define

$$\begin{aligned} Q_z &= \mathrm{O}_2(G_z^{[1]}), \\ F_z &= \mathrm{O}_5(G_z). \end{aligned}$$

With this notation Lemma 4.3.3 says that for $z \in \Gamma$ we have

$$\begin{aligned} \mathbf{F}^*(G_z) &= F_z Q_z \mathbf{E}(G_z) \text{ and} \\ \mathbf{F}^*(G_z^{[1]}) &= Q_z. \end{aligned}$$

Note that Lemma 4.3.1 shows that 5^2 does not divide $|G_z|$ so that either $F_z = 1$ or $\mathbf{E}(G_z) = 1$.

Theorem 4.3.4. *Exactly one of the following holds.*

(i) The group $G_{xy}^{[1]}$ is trivial.

(ii) The group $G_{xy}^{[1]} = G_y^{[2]}$ is non-trivial, $G_x^{[2]} = 1$, $\boldsymbol{F}^*(G_x) = Q_x$ and $\boldsymbol{F}^*(G_y) \neq Q_y$.

(iii) For $z = x, y$ we have $\boldsymbol{F}^*(G_z) = \boldsymbol{F}^*(G_z^{[1]}) = Q_z$ and $\boldsymbol{F}^*(G_{xy}) = \mathrm{O}_2(G_{xy})$.

Proof. Since $G_x^{\Delta(x)}$ and $G_y^{\Delta(y)}$ are primitive, we may apply Theorem 2.5.1. This shows that either $G_{xy}^{[1]} = G_y^{[2]}$ and $G_x^{[2]} = 1$ or there is a prime p such that $\mathbf{F}^*(G_z) = \mathrm{O}_p(G_z)$ for $z = x, y$. If the latter holds then the previous lemma shows that $p \in \{2, 5\}$. If $p = 2$ then (iii) holds and if $p = 5$ then for $\mathbf{F}^*(G_{xy}) = \mathrm{O}_p(G_{xy})$ to hold, Lemma 4.3.1 shows that $\mathbf{F}^*(G_{xy}) = 1$, hence (i) holds. Suppose now that $G_{xy}^{[1]} = G_y^{[2]}$ and $G_x^{[2]} = 1$ hold. We may assume that (i) doesn't hold so that $G_{xy}^{[1]}$ is a non-trivial subnormal subgroup of G_x. If $F_x \neq 1$ or $\mathbf{E}(G_x) \neq 1$ then Lemma 4.3.1 implies that $F_x \cap G_x^{[1]} = 1$ or $\mathbf{E}(G_x) \nleq G_x^{[1]}$ holds. It follows that $[F_x, G_{xy}^{[1]}] = 1$ or $[\mathbf{E}(G_x), G_{xy}^{[1]}] = 1$. Since $G_{xy}^{[1]} = G_y^{[2]}$ is a normal subgroup of G_y, this would imply $G_{xy}^{[1]} = 1$, which is against our assumption. Thus we may conclude $\mathbf{F}^*(G_x) = Q_x$. Now $\mathrm{O}_3(G_{xy})$ centralises Q_x, so $\mathrm{O}_3(G_{xy}) = 1$ and $\mathbf{F}^*(G_{xy}) = \mathrm{O}_2(G_{xy})$. Now either (iii) holds, or $\mathbf{F}^*(G_y) \neq Q_y$ and (ii) holds. □

In the next three sections we consider the cases delivered by Theorem 4.3.4 in turn. We establish some notation which will be used in the next three sections.

Notation 4.3.5. We will use the bar notation to denote subgroups of $G_x^{\Delta(x)}$ and $G_y^{\Delta(y)}$ in the following way. Recall that $G_x^{\Delta(x)} = G_x/G_x^{[1]}$. For a subgroup H of G_x we write $\overline{H}$ for the group $HG_x^{[1]}/G_x^{[1]} = H^{\Delta(x)}$. Similarly $G_y^{\Delta(y)} = G_y/G_y^{[1]}$ and for a subgroup K of G_y we write $\widetilde{K}$ for $KG_y^{[1]}/G_y^{[1]} = K^{\Delta(y)}$.

4.4 $G_{xy}^{[1]} = 1$

In this section we assume that $G_{xy}^{[1]} = 1$, the case delivered by Theorem 4.3.4(i). It follows that $[Q_x, Q_y] = 1$ and so $\mathrm{C}_{G_z}(Q_z) \nleq Q_z$ for $z = x, y$. Since $Q_z = \mathbf{F}^*(G_z^{[1]})$ Theorem 1.3.20 gives $\mathbf{F}^*(G_z) \neq \mathbf{F}^*(G_z^{[1]})$ and we draw two conclusions from this. The first is that G_z either has components or $F_z \neq 1$. The second is that $\mathrm{C}_{G_z}(Q_z)$ is transitive on $\Delta(z)$ (since $G_z^{\Delta(z)}$ is primitive and $\mathrm{C}_{G_z}(Q_z)^{\Delta(z)}$ is a normal subgroup).

Lemma 4.4.1. *Suppose that K is a component of G_z for $z \in \{x, y\}$. Then $K \cong \mathrm{Alt}(5)$ and $K = \boldsymbol{E}(G_z)$.*

Proof. Without loss, suppose that K is a component of G_x and let $L = K \cap G_{xy}$ and $Z = Z(K) = Z(L)$. By Lemma 4.3.2 we see that $KG_x^{[1]}/G_x^{[1]}$ must be isomorphic to $\mathrm{Alt}(5)$ and so it follows that $K \cap G_x^{[1]} = Z$ and $K/Z \cong \mathrm{Alt}(5)$. Looking at the projection of L over $G_y^{[1]}$ we see that $Z \leq G_y^{[1]}$. Hence Z is a subnormal 2-subgroup of G_y and so is contained in Q_y. Since $\mathrm{C}_{G_y}(Q_y)$ is transitive on $\Delta(y)$ we have $Z \lhd \langle K, \mathrm{C}_{G_y}(Q_y)\rangle$ which gives $Z = 1$. We now have that $\mathbf{E}(G_x)$ is a direct product of $\mathrm{Alt}(5)$ subgroups and so $\mathbf{E}(G_x) \cap G_x^{[1]} = 1$. It follows that there can be at most one component. $\square$

For $z \in \Gamma$ we set

$$H_z = \mathrm{O}^{5'}(G_z).$$

As remarked above we now have $H_z \cong \mathrm{C}_5$ or $\mathrm{Alt}(5)$. Our assumption from the beginning of Section 4.3 is that

$$\mathcal{A} = (H_x(H_y \cap G_{xy}), H_y(H_x \cap G_{xy}), (H_x \cap G_{xy})(H_y \cap G_{xy}))$$

and we may now proceed to identify $\mathcal{A}$. We may assume we have labelled so that $|H_x| \leq |H_y|$ and we have three cases to consider.

Lemma 4.4.2. *Suppose that $H_x \cong \mathrm{C}_5$. Then $\mathcal{A} = (\mathrm{C}_5, \mathrm{C}_5, 1)$ or*

$$\mathcal{A} = (\mathrm{Alt}(4) \times \mathrm{C}_5, \mathrm{Alt}(5), \mathrm{Alt}(4)).$$

Proof. We arrive at the first conclusion if $H_y \cong \mathrm{C}_5$. This is clear since in this case $H_x \cap G_{xy} = 1 = H_y \cap G_{xy}$. Suppose now that $H_y \cong \mathrm{Alt}(5)$. As $H_x \cap G_{xy} = 1$ it remains to determine the isomorphism type of $H_x(H_y \cap G_{xy})$. Since $H_y \cap G_{xy} \cong \mathrm{Alt}(4)$ has no cyclic quotients of order 2 or 4, we see that $[H_x, (H_y \cap G_{xy})] = 1$. $\square$

Lemma 4.4.3. *Suppose that $H_x \cong H_y \cong \mathrm{Alt}(5)$. Then $\mathcal{A} = (\mathrm{Alt}(5), \mathrm{Alt}(5), \mathrm{Alt}(4))$ or*

$$\mathcal{A} = (\mathrm{Alt}(5) \times \mathrm{Alt}(4), \mathrm{Alt}(4) \times \mathrm{Alt}(5), \mathrm{Alt}(4) \times \mathrm{Alt}(4)).$$

Proof. It follows from the isomorphism types of H_x and H_y that $H_x \cap G_{xy} = H_y \cap G_{xy}$, or $(H_x \cap G_{xy}) \cap (H_y \cap G_{xy})$ is elementary abelian of order four or trivial. The first and last possibilities give the first and second conclusions of the lemma respectively, so we may assume that $(H_x \cap G_{xy}) \cap (H_y \cap G_{xy}) \cong 2^2$. Then we find $(H_x \cap G_{xy})(H_y \cap G_{xy}) \cong \mathrm{C}_3 \times \mathrm{Alt}(4)$ and we can choose t to be a central element of order 3. Considering the projection of this subgroup over $G_x^{[1]}$ and $G_y^{[1]}$ respectively, we find that $t \in G_x^{[1]}$ and $t \in G_y^{[1]}$. Hence $t \in G_{xy}^{[1]}$ which implies $\langle t \rangle = 1$, a contradiction which completes the lemma. □

4.5 Mixed Type

In this section we work under the following hypothesis.

Hypothesis (M): Conclusion (ii) of Theorem 4.3.4 holds.

First we recall some of the statements from Theorem 4.3.4 and add some details concerning the structure of G_x and G_y.

Lemma 4.5.1. *The following hold:*

(i) $G_{xy}^{[1]} = G_y^{[2]} \neq 1$, $\boldsymbol{F}^*(G_x) = Q_x$ *and* $G_x^{[2]}$ *is trivial,*

(ii) $C_{G_y}(Q_y)$ *is transitive on* $\Delta(y)$. *Moreover, if* $U \leq Q_x \cap Q_y$ *is normal in* G_x *then* $U = 1$,

(iii) Q_x *is abelian. Moreover* Q_x *is elementary abelian if* $\widetilde{G_y} \ncong \mathrm{Frob}(20)$.

Proof. Part (i) is part of (ii) of Theorem 4.3.4.

By part (ii) of Theorem 4.3.4 we have $\mathbf{F}^*(G_y) \neq Q_y$ so that $\mathrm{C}_{G_y}(Q_y) \nleq Q_y$. Now $\mathrm{C}_{G_y}(Q_y) \nleq G_y^{[1]}$ since $\mathbf{F}^*(G_y^{[1]}) = Q_y$ which gives $\mathrm{C}_{G_y^{[1]}}(Q_y) \leq Q_y$. Hence the primitivity of $G_y^{\Delta(y)}$ implies that $\mathrm{C}_{G_y}(Q_y)$ is transitive on $\Delta(y)$. Now assume that $U \leq Q_x \cap Q_y$ and $U \lhd G_x$. Then $U \lhd \langle G_x, \mathrm{C}_{G_y}(Q_y) \rangle$ which acts transitively on Γ and $U \leq G_{xy}$, hence $U = 1$.

For (iii), first we observe that (i) and (ii) imply $Q_x \nleq Q_y$. Thus $Q_x \cap Q_y < Q_x$ and $Q_x \cap Q_y = Q_x \cap G_y^{[1]}$. Hence $Q_x/(Q_x \cap Q_y) = Q_x/(Q_x \cap G_y^{[1]}) \cong \widetilde{Q_x}$ gives $1 \neq \widetilde{Q_x} \lhd \widetilde{G_{xy}}$ and by considering the various isomorphism shapes of $\widetilde{G_{xy}}$ we see that $\widetilde{Q_x}$ is abelian. Hence $Q_x' \leq Q_x \cap Q_y$ and again using (ii) we have $Q_x' = 1$, so that Q_x is abelian. Suppose now that $\widetilde{G_y} \not\cong \mathrm{Frob}(20)$. Then $\widetilde{Q_x}$ is elementary abelian, hence $\Phi(Q_x) \leq Q_x \cap Q_y$ and again (ii) gives $\Phi(Q_x) = 1$ so Q_x is elementary abelian. This completes the lemma. □

We require some knowledge of the structure of G_y to make further progress.

Lemma 4.5.2. *Suppose that K is a component of G_y. Then $K \cong \mathrm{Alt}(5)$ and $K = \boldsymbol{E}(G_y)$.*

Proof. Note that $5 \mid |K|$ since $\pi(G_y) = \{2,3,5\}$ and $\{2,3\}$-groups are soluble. If L is another component and $K \neq L$, then $|KL| = |K||L|/|K \cap L|$. Since $K \cap L \leq \mathrm{Z}(K)$ which has order at most 2, we see that 5^2 divides $|KL|$ which divides $|G_y|$. This is incompatible with the order of G_y. Thus K is the unique component and, since $KG_y^{[1]}/G_y^{[1]}$ is a subgroup of $\mathrm{Sym}(5)$, we know that K is a component of type $\mathrm{Alt}(5)$, so we may assume for a contradiction that $K \cong \mathrm{SL}_2(5)$. Now $\mathrm{SL}_2(3) \cong U := K \cap G_{xy} \lhd G_{xy}$, and so $U \cap Q_x \lhd U$. Since Q_x is abelian and there is a unique abelian normal 2-subgroup of U, we have $U \cap Q_x = \mathrm{Z}(U)$. Then for $T \in \mathrm{Syl}_3(U)$ we see $[Q_x, T] \leq [Q_x, U] \leq Q_x \cap U = \mathrm{Z}(U)$ and so $[Q_x, T, T] \leq [\mathrm{Z}(U), T] = 1$. By coprime action (Lemma 1.1.7) we obtain $[Q_x, T] = 1$, which contradicts $Q_x = \mathbf{F}^*(G_x)$. Hence $K \cong \mathrm{Alt}(5)$. □

Theorem 4.5.3. *The following hold:*

(i) $\widetilde{G_y} \not\cong \mathrm{C}_5$,

(ii) $\boldsymbol{F}^*(G_y) \cong \mathrm{C}_5 \times Q_y$ *if and only if* $\widetilde{G_y}$ *is isomorphic to* $\mathrm{Dih}(10)$ *or* $\mathrm{Frob}(20)$, *and*

(iii) $\boldsymbol{F}^*(G_y) \cong \mathrm{Alt}(5) \times Q_y$ *if and only if* $\widetilde{G_y}$ *is isomorphic to* $\mathrm{Alt}(5)$ *or* $\mathrm{Sym}(5)$.

Proof. If $\widetilde{G_y} \cong \mathrm{C}_5$ we would have $G_{xy} = G_y^{[1]}$ which gives $Q_x \leq Q_y$ and this implies $Q_x = 1$, a contradiction to Lemma 4.5.1(i). To see (ii), assume that $\widetilde{G_y}$ has the required shape, then G_y is soluble, and so $\mathbf{E}(G_y) = 1$. Since $\mathrm{O}_p(G_y) = 1$ for $p > 5$ and $p = 3$, if $\mathrm{O}_5(G_y) = 1$ also, then $\mathbf{F}^*(G_y) = Q_y$ and since $\mathrm{C}_{G_y}(\mathbf{F}^*(G_y)) \leq \mathbf{F}^*(G_y)$ we would have $\mathrm{C}_{G_y}(Q_y) \leq Q_y$, contradicting our assumption. Hence $\mathrm{O}_5(G_y) \neq 1$, and so $\mathrm{O}_5(G_y) \cong \mathrm{C}_5$. The reverse direction is immediate with $\mathrm{O}_5(G_y) \neq 1$.

For (iii), first assume that $\widetilde{G_y}$ has the required shape. Then $\mathrm{O}_5(G_y) = 1$ and so we must have $\mathbf{E}(G_y) \neq 1$ (similarly to above). Hence there is a component K. By Lemma 4.5.2 there is a unique component and it is isomorphic to $\mathrm{Alt}(5)$. Then $K \cap Q_y = 1$ and so $\mathbf{F}^*(G_y) = \mathbf{E}(G_y)\mathbf{F}(G_y) = KQ_y \cong \mathrm{Alt}(5) \times Q_y$. The reverse direction is obvious. □

The previous theorem shows that $\mathrm{O}^{5'}(G_y) \cong \mathrm{C}_5$ or $\mathrm{Alt}(5)$.

4.5.1 Mixed type where G_y is soluble

In this section we assume that G_y is soluble. We prove the following theorem.

Theorem 4.5.4. *Assume Hypothesis* **(M)** *holds and that G_y is soluble. Then $\mathcal{A}$ is isomorphic to one of $\mathcal{S}_i$ for $i \in \{5, 6, 7, 8\}$.*

As in the previous section we set

$$\begin{aligned} H_x &= \mathrm{O}^{5'}(G_x), \\ H_y &= \mathrm{O}^{5'}(G_y). \end{aligned}$$

Then $H_y = F_y = \mathrm{O}^{5'}(G_y)$ by Theorem 4.5.3 and this group is isomorphic to C_5.

Proposition 4.5.5. *We have $G_x = H_x$ and $Q_x = G_x^{[1]}$.*

Proof. Our assumption on the simplicity of $\mathcal{A}$ implies that $G_x = H_x(H_y \cap G_{xy})$ and clearly $H_y \cap G_{xy} = 1$.

For $S \in \mathrm{Syl}_3(G_x^{[1]})$ the Frattini argument gives $G_x = G_x^{[1]}\mathrm{N}_{G_x}(S)$ so that $\mathrm{N}_{G_x}(S)$ is transitive on $\Delta(x)$. Now $G_{xy}/G_y^{[1]}$ is a 2-group, so we see that $S \leq G_{xy}^{[1]} = G_y^{[2]}$. It follows that $S \in \mathrm{Syl}_3(G_y^{[2]})$ and so $G_y = G_y^{[2]}\mathrm{N}_{G_y}(S)$ implies that $\mathrm{N}_{G_y}(S)$ is transitive on $\Delta(y)$. Hence $S = 1$ by Lemma 2.1.4. □

Since $G_y = H_y(H_x \cap G_{xy})$ we have $|H_x \cap G_{xy} : H_x \cap G_y^{[1]}| \leq 4$. Thus the majority of the work remaining is to determine the group H_x.

Proposition 4.5.6. *The group G_x/Q_x is isomorphic to* C_5 *or* $\mathrm{Alt}(5)$.

Proof. For all sets of primes π we have $\mathrm{O}^{\pi}(G) = \mathrm{O}^{\pi}(\mathrm{O}^{\pi}(G))$. Since $G_x = H_x = \mathrm{O}^{5'}(G_x)$ by Proposition 4.5.5 it follows that $\mathrm{O}^{5'}(G_x) = G_x$ (taking $\pi = \{2, 3\}$). In particular, this tells us that G_x has no non-trivial images in a 2-group, and so $G_x/Q_x = G_x/G_x^{[1]}$ has no normal subgroup of index a power of 2. Inspecting the possibilities of Lemma 4.3.2 we see $G_x/Q_x \cong \mathrm{C}_5$ or $G_x/Q_x \cong \mathrm{Alt}(5)$. □

The subgroup $R_x := [Q_x, G_x]$ will be important. We determine some properties in the following proposition.

Proposition 4.5.7. *The following hold,*

(i) $[R_x, N] = R_x$ *for any* $N \lhd G_x$ *such that* $N \nleq Q_x$*,*

(ii) if $U \lhd G_x$ *and then* $[R_x/U, G_x] = R_x/U$*.*

Proof. Since G_x/Q_x is simple, if N is a normal subgroup not contained in Q_x we have $Q_xN = G_x$. Hence $[R_x, N] = [R_x, Q_xN] = [R_x, G_x]$. Now $G_x = \mathrm{O}^{5'}(G_x)$, so coprime action gives $[Q_x, G_x, G_x] = [Q_x, G_x]$, that is $[R_x, G_x] = R_x$. This is (i).

Part (ii) follows from the properties of commutators. □

Lemma 4.5.8. *One of* $G_x/R_x \cong \mathrm{C}_5$*,* $G_x/R_x \cong \mathrm{Alt}(5)$ *or* $G_x/R_x \cong \mathrm{SL}_2(5)$ *holds.*

Proof. Set $L = G_x/R_x$, $Z = Q_x/R_x$ and $K = G_x/Q_x \cong L/Z$. Using Proposition 4.5.6 we have $\mathrm{O}^{5'}(L) = L$. By the definition of R_x we have $Z \leq \mathrm{Z}(L)$. In particular if $K \cong \mathrm{C}_5$, then L is the direct product of a group of order 5 and a 2-group which therefore gives $L \cong \mathrm{C}_5$. We may assume therefore that $K \cong \mathrm{Alt}(5)$ and $Z \neq 1$.

We claim that L is perfect. It suffices to prove that G_x is perfect. Since G_x/Q_x is non-abelian, $G_x' \nleq Q_x$, and so Proposition 4.5.6 gives $G_x = G_x'Q_x$. Then G_x/G_x' is a 2-group, and so it follows that $G_x = G_x'$. Now L is a perfect group such that $L/\mathrm{Z}(L) \cong \mathrm{Alt}(5)$ and $\mathrm{Z}(L) = \mathrm{O}_2(L)$. A well known result now implies $L \cong \mathrm{SL}_2(5)$ (for a reference see [21, Satz 5.25.7]). □

In the final stages of the identification of G_x we will use the following lemma to rule out the third possibility in the lemma above.

Lemma 4.5.9. *G_x does not contain a subgroup isomorphic to* $\mathrm{SL}_2(5)$*.*

Proof. Suppose that $L \leq G_x$ is such a subgroup, set $Z := Z(L)$ and pick $z \in Z$ so that $\langle z \rangle = Z$ (and note $Z \cong \mathrm{C}_2$). Then $L \cap Q_x = \langle z \rangle$ (since otherwise $LQ_x/Q_x \cong \mathrm{SL}_2(5)$, which cannot hold). We can choose $P \in \mathrm{Syl}_2(L)$ such that $P \leq G_{xy}$ and pick $r, s \in P$ such that $[r, s] = z$. Now $G_{xy}/G_y^{[1]}$ is abelian, so $z \in G_y^{[1]}$. Hence $z \in Q_x \cap G_y^{[1]} \leq Q_y$. In particular $Z \lhd \langle L, H_y \rangle$ which forces Z to be trivial. This is a contradiction which proves that such a subgroup does not occur. □

We now aim to find the isomorphism type of R_x. The subgroup $P_x = \Omega_1(R_x)$ of R_x is useful. By W we will denote the 4-dimensional GF(2)-module for Alt(5) obtained from the isomorphism $\mathrm{Alt}(5) \cong \mathrm{P}\Omega_4^-(2)$ (sometimes referred to as the deleted permutation module).

Proposition 4.5.10. *Viewed as a module for G_x/Q_x, $P_x \cong 2 \oplus W$ or W.*

Proof. Since P_x is elementary abelian, $P_xQ_y/Q_y \cong 2$. Thus $P_x \cap Q_y$ is a hyperplane of P_x. The stabiliser in G_x of $P_x \cap Q_y$ is G_{xy}, and so there are five G_x conjugates of $P_x \cap Q_y$, let them be $P_1, \ldots, P_5$. Note that $\bigcap_{i\in[1,5]} P_i$ is normal in G_x and is contained in Q_y, thus $\bigcap_{i\in[1,5]} P_i = 1$. Let V be the dual of P_x and consider the action of G_x/Q_x on V. Then V is generated by five 1-spaces which form an orbit under the action of G_x/Q_x. Hence V is a quotient of the permutation module. By [1, (24.3)], an element of order 5 acts faithfully on P_x, so $2^4 \leq |P_x|$. Hence V is isomorphic to W or $2 \oplus W$. Since the orthogonal module is self-dual, the result holds for P_x. □

Lemma 4.5.11. *Suppose that $\Phi(R_x) = 1$. Then $R_x \cong W$ as a G_x/Q_x-module.*

Proof. We have $R_x = P_x$ and by the previous proposition, $P_x \cong 2 \oplus W$ or $P_x \cong W$ as a G_x/Q_x module. By Proposition 4.5.7 (ii), $[R_x, G_x] = R_x$, and so $P_x \cong W$. □

We may now assume that $\Phi(R_x) \neq 1$. Since R_x is abelian, $\Phi(R_x)$ is contained in P_x. The subgroups are also related in the following way.

Proposition 4.5.12. *Suppose $\Phi(R_x) \neq 1$. Then as G_x-modules, R_x/P_x is isomorphic to $\Phi(R_x)$.*

Proof. Define a map $\phi : R_x \to R_x$ by $\phi : r \mapsto r^2$. Since R_x is abelian, ϕ is a homomorphism and moreover, ϕ commutes with the action of G_x. Now $(\ker \phi)^\#$ consists of the involutions in R_x, so $\ker \phi = P_x$. Also, $\mathrm{im}\phi$ consists of the squares in R_x, so $\mathrm{im}\phi = \Phi(R_x)$. Hence $R_x/P_x \cong \Phi(R_x)$. □

Proposition 4.5.13. *Suppose that $\Phi(R_x) \neq 1$. Then there are G_x-module isomorphisms $R_x/P_x \cong \Phi(R_x) \cong W$.*

Proof. The first isomorphism is the previous proposition. By Proposition 4.5.10 we see that R_x/P_x is one of the modules 2, $2 \oplus W$, W. On the other hand, Proposition 4.5.7 (ii) gives $[R_x/P_x, G_x] = R_x/P_x$. Hence $R_x/P_x \cong W$. □

Lemma 4.5.14. *Suppose that $\Phi(R_x) \neq 1$. Then $P_x = \Phi(R_x)$.*

Proof. By Proposition 4.5.13, $R_x/P_x \cong W$ and $\Phi(R_x) \cong W$. Since $P_x \cong W$ or $P_x \cong 2 \oplus W$, we are done unless $P_x \cong 2 \oplus W$. In which case, $R_x/\Phi(R_x) \cong 2 \oplus W$ (since the orthogonal module is projective), but Proposition 4.5.7 (ii) implies $[R_x/\Phi(R_x), G_x] = R_x/\Phi(R_x)$, a contradiction. □

Theorem 4.5.15. *$R_x \cong 2^4$ or $R_x \cong {\mathrm{C}_4}^4$.*

Proof. If $\Phi(R_x) = 1$ then we are done by Lemma 4.5.11. When $\Phi(R_x) \neq 1$, Lemma 4.5.14 implies $\Phi(R_x) = P_x$. Since R_x is abelian and has exponent 4, R_x is the direct product of $\log_2(|P_x|) = 4$ copies of a cyclic group of order 4. □

In the following lemma we see how the structure of R_x influences the structure of G_x.

Lemma 4.5.16. *G_x splits over Q_x and $R_x = Q_x$.*

Proof. By Lemma 4.5.8 the result is trivial if $G_x/Q_x \cong 5$ so we may assume $G_x/Q_x \cong \mathrm{Alt}(5)$. If $G_x/R_x \cong \mathrm{Alt}(5)$ then we use the following argument from [32, Lemma 4.1]. For the duration of this proof only, we write $\overline{G_x} = G_x/R_x$ and let t and s be elements such that $\bar{t}$ is an involution, $\overline{s}$ an element of order three and $\overline{ts}$ has order five. We know there is an element of order five $\overline{u} \in \overline{G_x}$ such that $\bar{t}$ inverts $\overline{u}$, and we choose a pre-image $u \in G_x$ of order five. Now $\mathrm{C}_{G_x}(u) = \langle u \rangle$ and $\mathrm{N}_{G_x}(\langle u \rangle) \cong \mathrm{Dih}(10)$. So we may choose t to have order two and we choose s to have order three. Then $\langle t, s \rangle \cong \mathrm{Alt}(5)$, so the extension splits. Now we may assume that $G_x/R_x \cong \mathrm{SL}_2(5)$ (and $Q_x/R_x \cong \mathrm{C}_2$). If G_x does split over Q_x, then a complement is isomorphic to $\mathrm{SL}_2(5)$, which is a contradiction to Lemma 4.5.9. So below we just need to prove the splitting occurs.

First assume that $\Phi(R_x) = 1$ so that $R_x = P_x$. We claim that Q_x is elementary abelian. Otherwise, there is an element of order four in Q_x, q say. Then $q^2 \in P_x$. Since $Q_x = P_x \cup P_x q$ and Q_x is abelian, for each $r \in Q_x$ we have $r^2 = 1$ or $r^2 = q^2$. It follows that q^2 is the unique square in Q_x, and is therefore G_x-invariant. Now $1 \neq \langle q^2 \rangle < P_x$, and P_x is irreducible by Lemma 4.5.11, a contradiction. Hence $\Phi(Q_x) = 1$.

Since $[Q_x, G_x] = P_x$ is the orthogonal module for $\mathrm{Alt}(5)$, we see that $Q_x = 2 \oplus W$. Let Z be the unique normal subgroup of G_x of order two. Considering G_x/Z we see that this is an extension of W by $\mathrm{Alt}(5)$, and therefore splits. It follows that G_x contains a subgroup isomorphic to $\mathrm{SL}_2(5)$, a contradiction.

Now assume that $\Phi(R_x) \neq 1$ so that $\Phi(R_x) = P_x$ and we have $1 < P_x < R_x < Q_x$ with successive quotients 2^4, 2^4, C_2. First we consider G_x/P_x and apply the same argument as in the previous paragraph to see a subgroup L of G_x such that $L/P_x \cong \mathrm{SL}_2(5)$. Then considering L we apply the argument from the previous paragraph (with L in place of G_x) to see that L contains a subgroup isomorphic to $\mathrm{SL}_2(5)$, the final contradiction. $\square$

We summarise the results of this section in the theorem below. Note that after specifying the action of G_x on $\mathrm{O}_2(G_x)$, the groups below are uniquely determined. The group for which we write ${\mathrm{C}_4}^4 : \mathrm{Alt}(5)$ is isomorphic to $\mathrm{O}^2(\mathrm{C}_4 \wr \mathrm{Alt}(5))$.

Theorem 4.5.17. *G_x is one of the four groups, $2^4 : 5$, ${\mathrm{C}_4}^4 : 5$, $2^4 : \mathrm{Alt}(5)$, ${\mathrm{C}_4}^4 : \mathrm{Alt}(5)$ (where the action on the sections of $\mathrm{O}_2(G_x)$ is the action of $G_x/\mathrm{O}_2(G_x)$ on W).*

We now proceed to identify the isomorphism type of G_y. Recall that $G_y = F_y G_{xy}$, and essentially it suffices to determine the centraliser in G_{xy} of F_y.

Proposition 4.5.18. *Suppose that* $G_x/R_x \cong \mathrm{C}_5$. *Then* $G_y \cong 2^3 \times \mathrm{Dih}(10)$ *or* $G_y \cong {\mathrm{C}_4}^3 \times \mathrm{Frob}(20)$.

Proof. Note $\mathrm{C}_{G_{xy}}(F_y) = G_y^{[1]}$ (by considering $\widetilde{G_y}$). Now $G_{xy} \cong 2^4$ or ${\mathrm{C}_4}^4$, and $R_x/R_x \cap Q_y \cong 2$ or 4. So in the first case an element of order 2 inverts F_y, and we see $G_y \cong \mathrm{Dih}(10) \times \mathrm{Frob}(20)$. In the second case, an element of order 4 acts as the square map on F_y, and so $G_y \cong \mathrm{Frob}(20) \times {\mathrm{C}_4}^3$. $\square$

Proposition 4.5.19. *Suppose that* $G_x \cong 2^4 : \mathrm{Alt}(5)$. *Then* $G_y^{[1]} = \mathrm{O}^2(G_{xy}) \cong 2^3 : \mathrm{Alt}(4)$ *and* G_y *has isomorphism shape* $2^3.\mathrm{Alt}(4).\mathrm{Dih}(10)$.

Proof. As $G_{xy}/G_y^{[1]} \cong 2$ we have that $\mathrm{O}^2(G_{xy}) \leq G_y^{[1]}$. Examining the isomorphism type of G_{xy} we see that $\mathrm{O}^2(G_{xy})$ has index two, so equality holds. Note that G_{xy} splits over $\mathrm{O}^2(G_{xy})$ since R_x is elementary abelian, thus G_y contains a $\mathrm{Dih}(10)$ subgroup which complements $G_y^{[1]}$. $\square$

Proposition 4.5.20. *Suppose that* $G_x \cong {\mathrm{C}_4}^4 : \mathrm{Alt}(5)$. *Then* $G_y^{[1]} = \mathrm{O}^2(G_{xy}) \cong {\mathrm{C}_4}^3 : \mathrm{Alt}(4)$ *and* G_y *has isomorphism shape* $(\mathrm{C}_5 \times {\mathrm{C}_4}^3).(\mathrm{Alt}(4) \times \mathrm{C}_4)$.

Proof. As in the proof of the previous proposition we see that $G_{xy}/G_y^{[1]} \cong \mathrm{C}_4$ so that $\mathrm{O}^2(G_{xy}) \leq G_y^{[1]}$ and considering the isomorphism type of G_{xy}, we see that $|G_{xy} : \mathrm{O}^2(G_{xy})| = 4$ so we have equality. Thus $F_yG_y^{[1]} \cong \mathrm{C}_5 \times {\mathrm{C}_4}^3$ and $G_y/(F_yG_y^{[1]}) = G_{xy}/G_y^{[1]} \cong \mathrm{C}_4 \times \mathrm{Alt}(4)$. $\square$

We can now prove the theorem stated at the beginning of this section.

Proof of Theorem 4.5.4. We have seen that $\mathcal{A}$ has the same type as the amalgams listed in the statement. Since each $\mathcal{S}_i$ is the unique faithful amalgam of its type by Theorem 4.1.1, the amalgams are isomorphic. $\square$

4.5.2 Mixed type where G_y non-soluble

In this section we assume that G_y is insoluble. We prove the following theorem.

Theorem 4.5.21. *Assume Hypothesis* **(M)** *holds and that G_y is non-soluble. Then $\mathcal{A}$ is isomorphic to one of $\mathcal{S}_i$ for $i \in \{9, 10, 11, 12\}$.*

Since G_y is non-soluble, we have $\mathbf{F}^*(G_y) = \mathbf{E}(G_y)Q_y$ by Theorem 4.5.3. We set

$$\begin{aligned} E_y &= \mathbf{E}(G_y), \\ E_{xy} &= E_y \cap G_x, \end{aligned}$$

and similarly for all vertices which are in the same orbit as y. We let

$$\begin{aligned} H_x &= \mathrm{O}^2(G_x), \\ H_y &= E_y(H_x \cap G_{xy}). \end{aligned}$$

Both H_x and H_y are transitive on $\Delta(x)$, so $|H_x : H_x \cap G_{xy}| = 5 = |H_y : H_y \cap G_{xy}|$.

The first goal of this section is to show that $\mathcal{B} = (H_x, H_y, H_x \cap H_y)$ is a normal subamalgam of $\mathcal{A}$. It then follows that $\mathcal{A} = \mathcal{B}$ so we will classify $\mathcal{B}$.

Lemma 4.5.22. *We have $E_{xy} \leq H_x$ and $H_x \cap G_{xy} = H_x \cap H_y = G_{xy} \cap H_y$.*

Proof. Since $E_{xy} \cong \mathrm{Alt}(4)$ we have $E_{xy} = \langle S, T \rangle$ for some cyclic subgroups S and T of order 3. Then $E_{xy} \leq \mathrm{O}^2(G_x) = H_x$. Since $H_x \cap H_y$ fixes the edge $\{x, y\}$ we have $H_x \cap H_y \leq G_{xy}$. Also it follows from the definition of H_y that $H_x \cap G_{xy} \leq H_y$, so it remains to see that $G_{xy} \cap H_y \leq H_x$. By the Dedekind identity, $H_y \cap G_{xy} = (H_x \cap G_{xy})E_{xy} = H_x \cap G_{xy}$, as required. □

Now without any ambiguity we may set

$$H_{xy} = H_x \cap H_y.$$

By the previous lemma, the amalgam $\mathcal{B} = (H_x, H_y, H_{xy})$ is a faithful (5,5) amalgam.

Lemma 4.5.23. *The amalgam $\mathcal{B}$ is a normal subamalgam of $\mathcal{A}$. In particular, $\mathcal{B} = \mathcal{A}$.*

Proof. By definition H_x is a normal subgroup of G_x. It follows that H_{xy} is a normal subgroup of G_{xy} and so $H_y = E_y H_{xy}$ is a normal subgroup of $E_y G_{xy} = G_y$. Since G_{xy} is maximal in both G_y and G_x, it is immediate that $G_x = H_x G_{xy}$ and $G_y = H_y G_{xy}$. Finally we need to check that $H_x \cap G_{xy} = H_x \cap H_y = G_{xy} \cap H_y$, which is the content of Lemma 4.5.22. Our assumption on the simplicity of $\mathcal{A}$ gives the second part. □

We now continue to work with $\mathcal{B}$.

Proposition 4.5.24. *We have $H_x/H_x \cap G_x^{[1]} \cong \mathrm{C}_5$ or $\mathrm{Alt}(5)$.*

Proof. This follows from the isomorphism $H_x/H_x \cap G_x^{[1]} \cong H_x G_x^{[1]}/G_x^{[1]}$, the structure of $G_x/G_x^{[1]}$ and that $\mathrm{O}^2(H_x) = H_x$. □

The following lemma is useful.

Lemma 4.5.25. *We have $C_{G_y}(E_{xy}) = G_y^{[1]} = C_{G_y}(E_y)$.*

Proof. First we see that $[E_y, G_y^{[1]}] \leq E_y \cap G_y^{[1]} = 1$ so that $G_y^{[1]} \leq \mathrm{C}_{G_y}(E_{xy}) \cap \mathrm{C}_{G_y}(E_y)$. Now we consider the quotient $G_y/G_y^{[1]}$ and see that the images of E_y and E_{xy} (which are $\mathrm{Alt}(5)$ and $\mathrm{Alt}(4)$ subgroups respectively) have trivial centralisers. This gives the reverse inclusion and completes the proof. □

We now set $L = \langle E_{xy}^{H_x} \rangle$. We will see that this subgroup controls the structure of H_x.

Theorem 4.5.26. *The group L is isomorphic to $\mathrm{Alt}(4) \times \mathrm{Alt}(4) \times \mathrm{Alt}(4) \times \mathrm{Alt}(4) \times \mathrm{Alt}(4)$.*

Proof. We proceed with a number of claims.

(1) Let $t \in E_{xy}$ have order 3, then $Q_x = \mathrm{C}_{Q_x}(t) \times [t, Q_x]$ and $[Q_x, E_{xy}] = [t, Q_x] \cong 2^2$.

First note that Q_x and E_{xy} normalise each other since both are normal in G_{xy}. The first assertion is immediate using coprime action (recall that Q_x is elementary abelian by Lemma 4.5.1). For the second assertion, observe that $E_{xy} \cap Q_x = 1$ would give $E_{xy} \leq \mathrm{C}_{G_x}(Q_x) = Q_x$, a contradiction, so $E_{xy} \cap Q_x = \mathrm{O}_2(E_{xy}) \cong 2^2$, and $1 \neq [E_{xy}, Q_x] \lhd E_{xy}$, so $[E_{xy}, Q_x] = E_{xy} \cap Q_x \cong 2^2$. Now let $T \in \mathrm{Syl}_3(E_{xy})$ and let $t \in T^\#$. If $[Q_x, t] \cong \mathrm{C}_2$ then $[Q_x, t, t] = 1$ and coprime action gives $[Q_x, t] = 1$, contradiction. Thus $[Q_x, t] \cong \mathrm{C}_2$ and so $[Q_x, t] \leq [Q_x, E_{xy}]$ implies the result.

(2) $E_{xy} \leq G_x^{[1]}$.

Otherwise, $E_{xy} \cap G_x^{[1]} = E_{xy} \cap Q_x \cong 2^2$ and so $\mathrm{C}_3 \cong \overline{E_{xy}} \lhd \overline{G_{xy}} \cong \mathrm{Alt}(4)$, a contradiction.

We now set $\Delta(x) = \{y, y_1, y_2, y_3, y_4\}$ so that E_{y_i} and E_{xy_i} are the conjugates of E_y and E_{xy} under G_x.

(3) $E_{xy} \cap E_{xy_1} = 1$.

By (2), $E_{xy_1} \leq G_{xy}$ so that E_{xy_1} normalises E_{xy} and vice versa. Thus (3) holds unless $E_{xy} = E_{xy_1}$ or $E_{xy} \cap E_{xy_1} \cong 2^2$. Assume the former first, then $E_{xy} \lhd \langle G_{xy}, G_{xy_1} \rangle = G_x$. By the Frattini Argument, for $T \in \mathrm{Syl}_3(E_{xy})$ we get $G_x = \mathrm{N}_{G_x}(T) E_{xy}$, hence $\mathrm{N}_{G_x}(T)$ is transitive on $\Delta(x)$. Note that $\mathrm{C}_{Q_x}(T) \lhd \mathrm{N}_{G_x}(T)$ and $\mathrm{C}_{Q_x}(T) \leq \mathrm{C}_{G_{xy}}(T)$ which we see is equal to $T\mathrm{C}_{G_y^{[1]}}(T) = TG_y^{[1]}$ since $\widetilde{\mathrm{C}_{G_{xy}}(T)} \leq \mathrm{C}_{\widetilde{G_{xy}}}(\widetilde{T}) = \widetilde{T}$. Thus $\mathrm{C}_{Q_x}(T) \leq G_y^{[1]}$ and so $[E_y, \mathrm{C}_{Q_x}(T)] = 1$. But now $\mathrm{C}_{Q_x}(T) \lhd \langle \mathrm{N}_{G_x}(T), E_y \rangle$ which is transitive on Γ, and so $\mathrm{C}_{G_x}(T) = 1$. Now (1) gives $Q_x = [t, Q_x] \cong 2^2$, which is impossible. Hence $E_{xy} \neq E_{xy_1}$.

We may now assume $E_{xy} \cap E_{xy_1} = \mathrm{O}_2(E_{xy}) = \mathrm{O}_2(E_{xy_1})$. By (1) we have $2^2 \cong [Q_x, t] \leq E_{xy}$, so that $[Q_x, t] = \mathrm{O}_2(E_{xy}) = \mathrm{O}_2(E_{xy_1})$ for any t of order 3 in E_{xy}. Let $R = E_{xy}E_{xy_1}$ and observe $|R| = 2^2 3^2$, and a Sylow 3-subgroup S is elementary abelian (there are two elements of order 3 in E_{xy} which are not in E_{xy_1}, so $S \not\cong \mathrm{C}_9$). Hence $\mathrm{C}_{[Q_x,t]}(S) \neq 1$, from which we deduce $R \cong \mathrm{C}_3 \times \mathrm{Alt}(4)$ (and $\mathrm{Z}(R) \cong \mathrm{C}_3$). Now $[Q_x, R] = [Q_x, E_{xy}][Q_x, E_{xy_1}] \leq R$. Choose $1 \neq z \in \mathrm{Z}(R)$, then $[Q_x, z, z] \leq [Q_x, R, z] \leq [R, z] = 1$. But then coprime action implies $[Q_x, z] = 1$ which implies $z = 1$, a contradiction which proves (3).

(4) $E_{xy} \leq G_{y_1}^{[1]}$.

Since $[E_{xy}, E_{xy_1}] = 1$ by the previous claim the result follows by Lemma 4.5.25.

We can now complete the proof. We have $L = E_{xy}E_{xy_1}E_{xy_2}E_{xy_3}E_{xy_4}$. Since $E_{xy} \leq G_{y_1}^{[1]}$ we have $E_{xy}E_{xy_1} \cong E_{xy} \times E_{xy_1}$. Similarly $E_{xy}E_{xy_1} \cap E_{xy_2} = 1$ and $[E_{xy}E_{xy_1}, E_{xy_2}] = 1$. Repeating this process we have the result. $\square$

Lemma 4.5.27. *H_x acts faithfully by conjugation on L.*

Proof. Let $C = \mathrm{C}_{H_x}(L)$ and note C is a normal subgroup of H_x since L is normal in H_x. Suppose that C is not contained in $G_x^{[1]}$. Then C is transitive on $\Delta(x)$ and, since E_{y_1} centralises E_{xy} and acts transitively on $\Delta(y)$, we have $E_{xy} = 1$ by Lemma 2.1.4, a contradiction. Thus $C \leq G_x^{[1]}$. Now $C \leq G_y$ so Lemma 4.5.25 implies $C \leq G_y^{[1]}$. By the same reasoning applied to the vertices $y_1, \ldots, y_4$ we get $C \leq G_x^{[2]} = 1$ as required. □

With the previous lemma, H_x has the following normal series, $L \lhd H_x \cap G_x^{[1]} \lhd H_x$. We have identified the subgroup L and the quotient $H_x/H_x \cap G_x^{[1]}$. We now need to determine the structure of $H_x \cap G_x^{[1]}/L$ and how the group fits together. To achieve both of these goals we use the following.

Proposition 4.5.28. *L embeds into* $\mathrm{Aut}(L) \cong \mathrm{Sym}(4) \wr \mathrm{Sym}(5)$.

Proof. The embedding follows from Lemma 4.5.27. The isomorphism type of $\mathrm{Aut}(L)$ follows from [4, Theorem 3.1]. □

Let $A = \mathrm{Aut}(L)$ and under the embedding of Proposition 4.5.28 we identify H_x and its subgroups with their images in A. Let $B = \bigcap_{u \in \Delta(x)} \mathrm{N}_A(E_{xu})$ which is isomorphic to $\mathrm{Sym}(4) \times \mathrm{Sym}(4) \times \mathrm{Sym}(4) \times \mathrm{Sym}(4) \times \mathrm{Sym}(4)$. We see that $L = B' = \mathrm{Inn}(A)$ and B/L is elementary abelian of order 2^5.

In the next theorem we identify H_x. Note that $\mathrm{Sym}(4) \wr \mathrm{C}_5$ has isomorphism shape $\mathrm{Alt}(4)^5.2^5.\mathrm{C}_5$, and modulo the $\mathrm{Alt}(4)^5$ subgroup has isomorphism shape $2 \times 2^4 : \mathrm{C}_5$. Thus $\mathrm{O}^2(\mathrm{Sym}(4) \wr \mathrm{C}_5)$ is a subgroup of index 2 and has isomorphism shape $\mathrm{Alt}(4)^4.2^4.\mathrm{C}_5$. The group $\mathrm{O}^2(\mathrm{Sym}(4) \wr \mathrm{Alt}(5))$ can be described similarly.

Theorem 4.5.29. *The group H_x is isomorphic to one of the following groups*

(i) $\mathrm{Alt}(4) \wr \mathrm{C}_5$,

(ii) $\mathrm{Alt}(4) \wr \mathrm{Alt}(5)$,

(iii) $\mathrm{O}^2(\mathrm{Sym}(4) \wr \mathrm{C}_5)$,

(iv) $\mathrm{O}^2(\mathrm{Sym}(4) \wr \mathrm{Alt}(5))$.

Proof. By the preceding discussion it suffices to work in $A/L \cong 2 \wr \mathrm{Sym}(5)$ which we may take to be generated by the permutations $(1,2)$, $(1,3,5,7,9)(2,4,6,8,10)$, and $(1,3)(2,4)$. Since $L \leq H_x \cap G_x^{[1]} \leq B$ we have $H_x \cap G_x^{[1]} \lhd BH_x$. In particular, the image in A/L of $H_x \cap G_x^{[1]}$ is a subgroup normalised by an element of order 5 and so is either the trivial subgroup, $\mathrm{Z}(A/L) = \langle (1,2)(3,4)(5,6)(7,8)(9,10) \rangle$, $[A/L, B/L] = \langle (1,2)(3,4)$, $(3,4)(5,6)$, $(5,6)(7,8)$, $(7,8)(9,10) \rangle$ or B/L.

We claim that $|H_x \cap G_x^{[1]}/L| \in \{1, 2^4\}$. Otherwise we have $|H_x \cap G_x^{[1]}/L| = 2$ or $|H_x \cap G_x^{[1]}/L| = 2^5$. In both cases, we consider $S = (A/L)/[A/L, B/L] \cong 2 \times \mathrm{Sym}(5)$ and we see that $T = (H_x/L)[A/L, B/L]/[A/L, B/L]$ is a homomorphic image of H_x. By our assumption, $|\mathrm{Z}(T)| = 2$ and $T/\mathrm{Z}(T) \cong \mathrm{C}_5$ or $T/\mathrm{Z}(T) \cong \mathrm{Alt}(5)$ holds. Thus T splits over $\mathrm{Z}(T)$ and so T has a 2-quotient, which contradicts the assumption that $\mathrm{O}^2(H_x) = H_x$.

Assume first that $|H_x \cap G_x^{[1]}/L| = 1$. Then $H_x/L \cong \mathrm{C}_5$ or $H_x/L \cong \mathrm{Alt}(5)$. In the first case we see that H_x is conjugate to $LF \cong \mathrm{Alt}(4) \wr \mathrm{C}_5$ where $F \in \mathrm{Syl}_5(A)$. In the second case, we have $H_x/L \leq 2 \wr \mathrm{Alt}(5)$ (by considering the quotient A/B for example) and so H_x/L is a complement to B/L in this subgroup. Since there is a unique conjugacy class of complements, we find that H_x is conjugate to a subgroup isomorphic to $\mathrm{Alt}(4) \wr \mathrm{Alt}(5)$. Hence the isomorphism of (i) or (ii) holds.

Now we assume that $|H_x \cap G_x^{[1]}/L| = 2^4$ and so $H_x \cap G_x^{[1]}/L = [A/L, B/L]$. We now consider the quotient $(A/L)/[A/L, B/L] \cong 2 \times \mathrm{Sym}(5)$ and we see there is a unique conjugacy class of subgroups isomorphic to C_5 or isomorphic to $\mathrm{Alt}(5)$. Hence H_x/L is conjugate to a member of one of these classes, and therefore H_x is isomorphic to one of (iii) or (iv). □

We may now identify H_y based upon the four cases specified in Theorem 4.5.29

Theorem 4.5.30. *With respect to the cases in Theorem 4.5.29, H_y is isomorphic to*

(i) $\mathrm{Alt}(5) \times \mathrm{Alt}(4) \times \mathrm{Alt}(4) \times \mathrm{Alt}(4) \times \mathrm{Alt}(4)$,

(ii) $\mathrm{Alt}(5) \times \mathrm{Alt}(4) \wr \mathrm{Alt}(4)$,

(iii) $\mathrm{Alt}(5) : (\mathrm{Sym}(4) \times \mathrm{Sym}(4) \times \mathrm{Sym}(4) \times \mathrm{Sym}(4))$ *or*

(iv) $\mathrm{Alt}(5) : \mathrm{Sym}(4) \wr \mathrm{Alt}(4)$.

Proof. Recall that $H_y = E_y H_{xy}$. We proceed by analysing the cases delivered by Theorem 4.5.29. Lemma 4.5.25 gives $\mathrm{C}_{H_{xy}}(E_{xy}) = H_{xy} \cap G_y^{[1]}$, which allows us in all cases to determine the struture of H_y.

In case (i) we have $H_{xy} = E_{xy}E_{xy_1}E_{xy_2}E_{xy_3}E_{xy_4}$ and $[E_{xy_1}E_{xy_2}E_{xy_3}E_{xy_4}, E_{xy}] = 1$. Thus $H_y = E_y E_{xy_1}E_{xy_2}E_{xy_3}E_{xy_4}$ and by Lemma 4.5.25 we have $\mathrm{C}_{H_y}(E_y) = \mathrm{C}_{H_y}(E_{xy})$ so that $H_y \cong E_y \times E_{xy_1}E_{xy_2}E_{xy_3}E_{xy_4}$ which gives the result in this case.

In case (ii) we have $H_{xy} \cong \mathrm{Alt}(4) \times \mathrm{Alt}(4) \wr \mathrm{Alt}(4)$. Thus $H_{xy} = E_{xy}\mathrm{C}_{H_{xy}}(E_{xy})$ and so $H_y = E_y\mathrm{C}_{H_y}(E_y) \cong \mathrm{Alt}(5) \times \mathrm{Alt}(4) \wr \mathrm{Alt}(4)$.

For (iii) we see that $\mathrm{C}_{H_{xy}}(E_{xy})$ has isomorphism shape $\mathrm{Alt}(4)^4 : 2^3$, and has index 24 in H_{xy}. Note that H_{xy} splits over E_{xy} and a complement is isomorphic to $\mathrm{Sym}(4)^4$. Thus $H_y = E_y H_{xy} \cong \mathrm{Alt}(5) : \mathrm{Sym}(4)^4$. Note that there are multiple split extensions which may be described as such, but having specified the subgroup $\mathrm{C}_{H_y}(E_y)$ we have fixed the isomorphism class to which this group belongs.

In case (iv) we again see that H_{xy} splits over E_{xy} and that a complement, C say, is isomorphic to $\mathrm{Sym}(4) \wr \mathrm{Alt}(4)$. Thus $\mathrm{C}_{H_{xy}}(E_{xy}) = \mathrm{C}_C(E_{xy})$ has isomorphism shape $\mathrm{Alt}(4)^4.2^3.\mathrm{Alt}(4)$. Hence $\mathrm{C}_{H_{xy}}(E_{xy})$ is the unique index 2 subgroup in C, and so $H_y = E_y H_{xy} = H_y C \cong \mathrm{Alt}(5) : \mathrm{Sym}(4) \wr \mathrm{Alt}(4)$ is the unique split extension which is not a direct product. □

We can now prove Theorem 4.5.21.

Proof of Theorem 4.5.21. We have seen that $\mathcal{A} = \mathcal{B}$ and Theorems 4.5.29 and 4.5.30 show that $\mathcal{B}$ has the same type as $\mathcal{S}_i$ for some $i \in [9, 12]$. The result now follows from our assumption that $\mathcal{A}$ is faithful and Theorem 4.1.1. □

4.6 $\mathbf{C}_{G_z}(Q_z) \leq Q_z$ for $z = x, y$

In this section we work towards proving Theorem 4.6.8. This is the final case that is delivered by Theorem 4.3.4 and it's proof will complete the classification of simple finite faithful amalgams of degree $(5, 5)$. We will show that (G_x, G_y) is a weak (B, N)-pair for G. Recalling Definition 1.6.7, we need to show that for $z = x, y$ there exists a normal subgroup G_z^* of G_z containing Q_z such that $G_z^*/Q_z \in \Lambda$ where

$$\Lambda := \{\mathrm{Dih}(10), \mathrm{Frob}(20), \mathrm{Alt}(5)\},$$

and $G_z^* \cap G_{xy} = \mathrm{N}_{G_z^*}(P)$ for some $P \in \mathrm{Syl}_2(G_z^*)$. If this holds, then Corollary 1.6.10 identifies the amalgam as one of $\mathcal{S}_{13}$, $\mathcal{S}_{14}$, $\mathcal{S}_{15}$. We first prove the following theorem.

Theorem 4.6.1. *For $z = x, y$ we have $Q_x \cap Q_y \neq Q_z$.*

Proof. We will establish the proof of the theorem after a number of steps.

We assume for a contradiction that $Q_x \leq Q_y$ and set $L = \langle Q_y^{G_x} \rangle$. Note that $L \not\leq G_x^{[1]}$ or this would give $Q_y \leq Q_x$, hence $G_x/G_x^{[1]} \cong \mathrm{C}_5$ cannot occur in the following.

(1) For any $J \leq G_x$ such that J is transitive on $\Delta(x)$ we have $L = \langle Q_y^J \rangle$.

We see that $L = Q_y[Q_y, G_x] = Q_y[Q_y, JG_{xy}] \leq [Q_y, G_{xy}][Q_y, J] \leq Q_y[Q_y, J] = \langle Q_y^J \rangle$, but the reverse containment is automatic.

(2) $L \cap G_x^{[1]} = Q_x$

Set $M = L \cap G_x^{[1]}$ and choose $T \in \mathrm{Syl}_2(G_{xy})$ such that $Q_y \leq T$. Then $T_0 := T \cap M \in \mathrm{Syl}_2(M)$ and $T_0 \lhd T$ since $M \lhd G_{xy}$. Then $Q_y \leq \mathrm{N}_L(T_0)$ and by the Frattini argument $L = \mathrm{N}_L(T_0)M$, so we see that $\mathrm{N}_L(T_0)$ is transitive on $\Delta(x)$. Hence $L \geq \mathrm{N}_L(T_0) \geq \langle Q_y^{\mathrm{N}_L(T_0)} \rangle = \langle Q_y^{G_x} \rangle = L$, so $T_0 \lhd L$ which implies $T_0 = Q_x$ (as $Q_x \leq M$). For the duration of this claim set $\overline{G_x} = G_x/Q_x$. Then $\overline{M}$ is a $2'$-group. Also, $[Q_y, G_x^{[1]}] \leq Q_y \cap G_x^{[1]} = Q_x$, so $[L, G_x^{[1]}] = [\langle Q_y^{G_x} \rangle, G_x^{[1]}] = \langle [Q_y, G_x^{[1]}]^{G_x} \rangle \leq Q_x$, that is $[\overline{L}, \overline{G_x^{[1]}}] = 1$. Then $[\overline{L}, \overline{M}] = 1$, however $\overline{L}/\overline{M}$ is isomorphic to one of $\mathrm{Dih}(10), \mathrm{Frob}(20)$, $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$ and none of these groups has a non-trivial extension by a 3-group, so we see that $\overline{L} \cong \overline{K} \times \overline{M}$ where

$\overline{K}$ is one of the previous mentioned groups. Then $\overline{Q_y} \leq \overline{K}$ which implies $L = \langle Q_y^L \rangle \leq K$, hence $L = K$ and $\overline{M} = 1$. Hence $L \cap G_x^{[1]} = Q_x$ as required.

(3) After replacing L with one of its subgroups, the following hold,

(a) $L/\mathrm{O}_2(L) \cong \mathrm{Dih}(10)$, $\mathrm{Frob}(20)$, $\mathrm{Alt}(5)$,

(b) L is transitive on $\Delta(x)$,

(c) $Q_y \in \mathrm{Syl}_2(L)$,

(d) No non-trivial characteristic subgroup of Q_y is normal in L.

Note that once we have that L which satisfies (a), (b) and (c) then (d) is immediate. Indeed, if $1 \neq C \lhd L$ and C char Q_y, then $C \lhd G_y$ also, and so $C \lhd \langle L, G_y \rangle$ which is transitive on Γ. This would imply $C = 1$, a contradiction.

For the duration of this claim we set $\overline{G_x} = G_x/\mathrm{O}_2(L)$. Note that $\mathrm{O}_2(L) = Q_x = L \cap G_x^{[1]}$ so that $\overline{L}$ is isomorphic to one of $\mathrm{Dih}(10)$, $\mathrm{Frob}(20)$, $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$. Thus $5 \nmid |G_x : L|$ so $|L : L \cap G_{xy}| = 5$ and $|\overline{L} : \overline{L \cap G_{xy}}| = 5$ also.

Firstly, if $\overline{L} \cong \mathrm{Dih}(10)$ we see $\overline{Q_y} \cong \mathrm{C}_2$ and so $Q_y \in \mathrm{Syl}_2(L)$ and L itself is our desired subgroup. If $\overline{L} \cong \mathrm{Frob}(20)$, either $|\overline{Q_y}| = 4$ or $|\overline{Q_y}| = 2$. In the second case we would see a unique index two normal subgroup K of L containing Q_y, but then we would obtain $L = \langle Q_y^{G_x} \rangle \leq K$, a contradiction. Hence L itself satisfies (a)-(c).

When $\overline{L} \cong \mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$, the normality of $\overline{Q_y}$ in $\overline{L \cap G_{xy}}$ implies that $Q_y \cong 2^2$ and $\overline{Q_y} \leq \overline{L}' \cong \mathrm{Alt}(5)$. Thus we obtain the desired conclusion after replacing L with the pre-image of $\overline{L}'$ here.

We now assemble some pushing up results which deliver our conclusion.

(4) If $L/\mathrm{O}_2(L) \cong \mathrm{Dih}(10)$ or $\mathrm{Frob}(20)$, $\mathrm{O}_2(L) = \mathbf{F}^*(L)$ and $S \in \mathrm{Syl}_2(L)$ then there exists C char S such that $1 \neq C \lhd L$.

To see this apply Theorem 1.5.6 with $r = 5$ and $p = 2$.

We are now ready for the contradiction.

(5) $Q_x \not\leq Q_y$.

Let L be the subgroup of G_x delivered by (3). In the instance $L/\mathrm{O}_2(L) \cong \mathrm{Dih}(10)$ or $\mathrm{Frob}(20)$ we apply (4) with $Q_y = S$ and obtain a non-trivial characteristic subgroup C of Q_y. But this contradicts (3) (d). In the second instance, we need (3) (d) to invoke Theorem 1.5.1 and we set $\widetilde{H} = Q_y P$ where $P \in \mathrm{Syl}_5(G_y)$. Then $Q_x \lhd \widetilde{H}$ implies $Q_x = 1$ since $\widetilde{H}$ is transitive on $\Delta(y)$. Thus since $Q_x \lhd Q_y$, we see $Q_y = \mathrm{N}_{\widetilde{H}}(Q_x)$. Now let $H = \widetilde{H}/\mathrm{C}_{\widetilde{H}}(Q_y)$ be viewed as a subgroup of $\mathrm{Aut}(Q_y)$. Now we apply Theorem 1.5.1 with $Q_y = S$, $Z = \Omega_1(\mathrm{Z}(S))$ and the subgroup H just constructed, so that $|H : \mathrm{N}_H(Q_x)| = 5$. This implies $\langle Z^H \rangle \leq Q_x$ and $\langle Z^H \rangle \lhd L$. But $\langle Z^H \rangle$ is normalised by H, that is, it is invariant under all the conjugation maps induced by $\widetilde{H}$, so certainly $\langle Z^H \rangle \lhd \widetilde{H}$. Then $1 \neq Z \leq \langle Z^H \rangle \lhd \langle L, \widetilde{H} \rangle$ which is transitive on Γ, a contradiction. $\square$

With the proof of Theorem 4.6.1 completed, we may assume the following hypothesis for the rest of this section.

Hypothesis (A): The group $Q_x \cap Q_y$ is a proper subgroup of both Q_x and Q_y.

Note that **(A)** immediately implies that $G_x/G_x^{[1]} \in \{\mathrm{Dih}(10), \mathrm{Frob}(20), \mathrm{Alt}(5), \mathrm{Sym}(5)\}$.

Lemma 4.6.2. *Let $u, v \in \Delta(x)$ with $u \neq v$. Then the group $L = \langle G_x^{[1]}, Q_u, Q_v \rangle$ is transitive on $\Delta(x)$.*

Proof. Set $\overline{G_x} = G_x/G_x^{[1]}$ and observe that $\overline{Q_u}$ and $\overline{Q_v}$ are distinct 2-subgroups of $\overline{G_x}$. It is easy to check from the list above that $\overline{L}$ contains a subgroup isomorphic to $\mathrm{Dih}(10)$, as required. $\square$

Recall that $G_{xy}^{[1]} = G_x^{[1]} \cap G_y^{[1]}$.

Lemma 4.6.3. *The group $G_{xy}^{[1]}$ is a 2-group.*

Proof. Let $u \neq y$ be a vertex adjacent to x and let $v \neq x$ be a vertex adjacent to y. Note that $G_{xy}^{[1]}$ is a subnormal subgroup of G_{xu} and so Lemma 1.3.6 implies that $\mathrm{O}_2(G_{xu})$ normalises $\mathrm{O}^2(G_{xy}^{[1]})$, hence Q_u normalises $G_{xy}^{[1]}$. The same argument shows that $Q_v \leq \mathrm{N}_{G_y}(\mathrm{O}^2(G_{xy}^{[1]}))$. Thus $\mathrm{O}^2(G_{xy}^{[1]}) \lhd \langle G_x^{[1]}, Q_u, Q_y \rangle$ and $\mathrm{O}^2(G_{xy}^{[1]}) \lhd \langle G_y^{[1]}, Q_v, Q_x \rangle$. Since these two groups are transitive on $\Delta(x)$ and $\Delta(y)$ by the previous lemma, we find that $\mathrm{O}^2(G_{xy}^{[1]}) = 1$. $\square$

With the previous lemma we are able to limit the structure of G_x as follows. Since $G_x^{[1]} \lhd G_{xy}$ we find that $G_x^{[1]}/G_{xy}^{[1]} \cong G_x^{[1]}G_y^{[1]}/G_y^{[1]} \lhd G_{xy}/G_y^{[1]}$. The latter subgroup being isomorphic to one of $\{\mathrm{C}_2, \mathrm{C}_4, \mathrm{Alt}(4), \mathrm{Sym}(4)\}$ we see that either $G_x^{[1]}$ is a 2-group and $G_x^{[1]} = Q_x$, or possibly $G_x^{[1]}/Q_x \in \{\mathrm{C}_3, \mathrm{Sym}(3)\}$. Before considering these cases, we determine how much of a problem the Sylow 3-subgroups of G_x could pose.

Proposition 4.6.4. *Let $T \in \mathrm{Syl}_3(G_x)$. Then $|T| \leq 3^2$ and T is elementary abelian.*

Proof. We may assume that $T \leq G_{xy}$. The previous lemma gives that $G_{xy}^{[1]}$ is a 2-group, and so $T \cong TG_{xy}^{[1]}/G_{xy}^{[1]} \leq G_{xy}/G_{xy}^{[1]}$. Since this group is a permutation group acting on the set $(\Delta(x) \cup \Delta(y)) \setminus \{x, y\}$ preserving the partition, it is a subgroup $\mathrm{Sym}(4) \times \mathrm{Sym}(4)$ and the assertion is immediate. $\square$

Proposition 4.6.5. *Suppose that $G_x^{[1]} = Q_x$. Then there exists $G_x^* \lhd G_x$ such that $G_x^*/Q_x \in \Lambda$.*

Proof. Let $\overline{G_x} = G_x/G_x^{[1]} = G_x/Q_x$. We have already seen that $\overline{G_x} \in \{\mathrm{Dih}(10), \mathrm{Frob}(20), \mathrm{Alt}(5), \mathrm{Sym}(5)\}$ and so we may take $G_x^* = G_x$ unless the final isomorphism holds, in which case we take G_x^* to be the preimage of the derived subgroup of $\overline{G_x}$. $\square$

Lemma 4.6.6. *Suppose that $G_x^{[1]}/Q_x \cong \mathrm{Sym}(3)$. Then there exists $G_x^* \lhd G_x$ such that $G_x^*/Q_x \in \Lambda$.*

Proof. Set $\overline{G_x} = G_x/Q_x$. We first observe that this situation can only arise if $G_y/G_y^{[1]} \cong \mathrm{Sym}(5)$ and $G_x^{[1]}G_y^{[1]} = G_{xy}$. Thus $G_{xy}/G_{xy}^{[1]} \cong G_x^{[1]}/G_{xy}^{[1]} \times G_y^{[1]}/G_{xy}^{[1]}$ where $G_x^{[1]}/G_{xy}^{[1]} \cong \mathrm{Sym}(4)$ and $G_y^{[1]}/G_{xy}^{[1]} \cong G_{xy}/G_x^{[1]}$. Hence $G_{xy}/Q_x \cong \mathrm{Sym}(3) \times G_{xy}/G_x^{[1]}$. In particular, any Sylow 2-subgroup P of $\overline{G_{xy}}$ splits over $P \cap \overline{G_x^{[1]}}$.

Now the assumption implies that $\overline{G_x}$ has a normal subgroup T of order 3. Proposition 4.6.4 on the Sylow 3-subgroups of G_x shows that either $T \in \mathrm{Syl}_3(\overline{G_x})$ or the Sylow 3-subgroups of $\overline{G_x}$ split over T. In the former case we may apply the Schur-Zassenhaus Theorem and in the latter case we apply Gaschütz's Theorem to obtain K, a complement to T in $\overline{G_x}$. Observe that K has a normal, and therefore central, subgroup S of order 2.

We claim that the Sylow 2-subgroups of $\overline{G_x}$ split over S. To see this, we choose a subgroup $S^0 \in \mathrm{Syl}_2(\overline{G_{xy}})$ which contains S (note that $S^0 \in \mathrm{Syl}_2(\overline{G_x})$). But now $S^0 \cap \overline{G_x^{[1]}} = S$, and so the previous paragraph implies that S^0 does indeed split over S. Invoking Gaschütz's Theorem again, we find L, a complement in K to S, and since $S \leq \mathrm{Z}(K)$ we see that $K \cong L \times S$.

We claim now that $[L, T] = 1$. Of course, L normalises T so $L/\mathrm{C}_L(T) \hookrightarrow \mathrm{Aut}(T) \cong \mathrm{C}_2$. Thus it suffices to show that a Sylow 2-subgroup of L centralises T. We may take $S^0 \cap L$ to be this subgroup, and since $S^0 \cong S \times S^0 \cap L$, looking in G_{xy}/Q_x we indeed see that $S^0 \cap L$ centralises T. Thus $L \lhd \overline{G_x}$ and we have shown that $\overline{G_x} \cong L \times \overline{G_x^{[1]}}$.

Choosing G_x^* as the preimage of L (or L' if $L \cong \mathrm{Sym}(5)$), we see that $G_x^*/Q_x \in \Lambda$ as required. $\square$

Lemma 4.6.7. *Suppose that $G_x^{[1]}/Q_x \cong \mathrm{C}_3$. Then there exists $G_x^* \lhd G_x$ such that $G_x^*/Q_x \in \Lambda$.*

Proof. Again we set $\overline{G_x} = G_x/Q_x$ and observe that there exists a normal subgroup T isomorphic to C_3. Hence Gaschütz's Theorem or the theorem of Schur-Zassenhaus gives a complement K to T in $\overline{G_x}$. As before, we see that $|K/\mathrm{C}_K(T)| \leq 2$ and $K \cong G_x/G_x^{[1]}$.

Let us first consider the case that $K \cong \mathrm{Dih}(10)$ and $K/\mathrm{C}_K(T) \cong \mathrm{C}_2$. Then we find that $|G_{xy}/G_x^{[1]}| = 2$. If $G_y/G_y^{[1]} \cong \mathrm{Sym}(5)$, then we would have $|G_{xy}/G_x^{[1]}G_y^{[1]}| = 2$ which forces $G_y^{[1]}G_x^{[1]} = G_x^{[1]}$, and so $Q_y \leq Q_x$, a contradiction to (A). Hence we have $G_y/G_y^{[1]} \cong \mathrm{Alt}(5)$ and $G_{xy} = G_x^{[1]}G_y^{[1]}$. But now we see that $G_{xy}/G_{xy}^{[1]} \cong \mathrm{Alt}(4) \times \mathrm{Dih}(10)$ and so $\overline{G_{xy}} \cong \mathrm{C}_3 \times \mathrm{Dih}(10)$, a contradiction to $\mathrm{C}_{\overline{G_x}}(T) \cong \mathrm{C}_5$.

We return now to the general situation where $|K/\mathrm{C}_K(T)| \leq 2$. If $K = \mathrm{C}_K(T)$ then $K \lhd \overline{G_x}$ and we let G_x^* be the preimage of K (or K' if $K \cong \mathrm{Sym}(5)$) in G_x. Otherwise, $K/\mathrm{C}_K(T) \cong \mathrm{C}_2$ and $\mathrm{C}_K(T)$ is isomorphic to one of $\{\mathrm{Dih}(10), \mathrm{Frob}(20), \mathrm{Alt}(5)\}$ (since the case $K \cong \mathrm{Dih}(10)$ and $\mathrm{C}_K(T) \cong \mathrm{C}_5$ was ruled out above) so here we let G_x^* be the preimage of $\mathrm{C}_K(T)$. Thus $G_x^*/Q_x \in \Lambda$ as required. $\square$

Theorem 4.6.8. *The pair (G_x, G_y) is a weak (B, N)-pair of characteristic two for G.*

Proof. The previous results guarantee the existence of the normal subgroup G_x^* required. Note that $|G_x^*/G_x^* \cap G_{xy}| = 5$ in all cases, so we may choose a Sylow 2-subgroup P of G_x^* contained in G_{xy}. But now $P/Q_x \in \mathrm{Syl}_2(G_{xy}/Q_x)$ and this group is isomorphic to one of $\{\mathrm{C}_2, \mathrm{C}_4, \mathrm{Alt}(4)\}$. Hence $P \lhd G_{xy} \cap G_x^*$. Since $P \neq Q_x$, we conclude $G_{xy} \cap G_x^* = \mathrm{N}_{G_x^*}(P)$ as required. After relabelling, the results in this section hold for G_y also (since all we have used to prove the statement for G_x is that $5 \mid |G_y|$), so we are done. □

4.7 Finite completions

Here we construct finite faithful completions for the simple amalgams $\mathcal{S}_1$-$\mathcal{S}_{12}$. Recall that the amalgams $\mathcal{S}_{13}$-$\mathcal{S}_{15}$ and their extensions have completions inside $\mathrm{Aut}(\mathrm{PSL}_3(4))$, $\mathrm{Aut}(\mathrm{Sp}_4(4))$ and $\mathrm{Aut}(\mathrm{G}_2(4))$, respectively. We will construct the completions in certain groups which make the extensions visible. In particular, this is beneficial for the amalgams $\mathcal{S}_5$-$\mathcal{S}_{12}$ as we have not concretely constructed the extensions of these amalgams yet.

Lemma 4.7.1. *Let $G = \mathrm{Sym}(9)$ and set $P_1 = \langle (1,2,3,4,5) \rangle$, $P_2 = \langle (5,6,7,8,9) \rangle$ and $B = P_1 \cap P_2$. Then the amalgam (P_1, P_2, B) is isomorphic to $\mathcal{S}_1$ and G' is a completion of $\mathcal{S}_1$. Moreover, $N_G(P_1) \cap N_G(P_2) \cong \mathrm{C}_4 \times \mathrm{C}_4$ and each of the extensions $\mathcal{E}_1^1$-$\mathcal{E}_1^{10}$ is visible in G.*

Proof. It is clear that the amalgam (P_1, P_2, B) is isomorphic to $\mathcal{S}_1$. Inspecting the maximal subgroups of $\mathrm{Alt}(9)$ we see that $G' = \langle P_1, P_2 \rangle$, so G' is a faithful completion of $\mathcal{S}_1$. We have that $\mathrm{N}_G(P_1) \cap \mathrm{N}_G(P_2) = \langle (1,2,4,3), (6,7,9,8) \rangle$, writing $\alpha = (6,7,9,8)$ and $\beta = (1,2,4,3)$ we set $R_1 = \langle \alpha^2 \rangle$, $R_2 = \langle \alpha^2\beta^2 \rangle$, $R_3 = \langle \alpha^2, \beta^2 \rangle$, $R_4 = \langle \alpha \rangle$, $R_5 = \langle \alpha\beta \rangle$, $R_6 = \langle \alpha\beta^3 \rangle$, $R_7 = \langle \alpha\beta^2 \rangle$, $R_8 = \langle \alpha^2, \beta \rangle$, $R_9 = \langle \alpha\beta, \alpha^2 \rangle$ and $R_{10} = \langle \alpha, \beta \rangle$.

Set $\mathcal{A}_i = (P_1R_i, P_2R_i, R_i)$ and we claim that $\mathcal{A}_i$ is isomorphic to $\mathcal{E}_1^i$. Since $\mathcal{A}_i$ is a faithful amalgam that is an extension of $\mathcal{S}_1$, it is isomorphic to $\mathcal{E}_1^j$ for some j. Considering the isomorphism type of P_1R_i our claim holds unless $i = 5$ or $i = 6$. The amalgams $\mathcal{A}_5$ and $\mathcal{A}_6$ both have type $(\mathrm{Frob}(20), \mathrm{Frob}(20), \mathrm{C}_4)$. Suppose for a contradiction that $\Theta = (f_1, f_2, f_3)$ is an amalgam isomorphism between $\mathcal{A}_5$ and $\mathcal{A}_6$ (recall Definition 2.2.7 of an amalgam isomorphism and Proposition 2.2.10). Observe that $f_1(\alpha\beta) = f_3(\alpha\beta) = \alpha\beta^3$ or $f_1(\alpha\beta) = \alpha^3\beta = f_2(\alpha\beta)$. There are four cases to consider in total, depending on the value of $\Theta(1)$.

Assume that $\Theta(1) = 2$ and let i be an integer such that $f_1(x) = y^i$ where $x = (1, 2, 3, 4, 5)$ and $y = (5, 6, 7, 8, 9)$. Assume also that $f_1(\alpha\beta) = \alpha\beta^3$, then $f_1(x^{\alpha\beta}) = f_1(x^2) = y^{2i}$. Since f_1 is a homomorphism, we have $f_1(x^{\alpha\beta}) = f_1(x)^{f_1(\alpha\beta)} = (y^i)^{\alpha\beta^3} = y^{-2i}$. This implies $y^{2i} = y^{-2i}$, a clear contradiction. The other cases yield contradictions with similar arguments. This shows that no isomorphism Θ can exist, hence $\mathcal{A}_5$ and $\mathcal{A}_6$ belong to distinct isomorphism classes of amalgams. With the notation we've set up, we see $\mathcal{A}_i$ is isomorphic to $\mathcal{E}_1^i$ for $i \in [1, 10]$.

Also note that $G' \leq \langle P_1 R_i, P_2 R_i \rangle \leq G$, so either $\mathrm{Alt}(9)$ or $\mathrm{Sym}(9)$ is a faithful completion for the amalgam $\mathcal{E}_1^i$ where $i \in [1, 10]$. □

Lemma 4.7.2. *Let $G = \mathrm{Alt}(6)$ and let $P_1 = \mathrm{Stab}_G(1)$, $P_2 = \mathrm{Stab}_G(2)$ (in the usual action on six points). Then $\langle P_1, P_2 \rangle = G$ is a faithful completion of the amalgam $\mathcal{S}_2$. Moreover the extension $\mathcal{E}_2^1$ is visible in $\mathrm{Sym}(6)$.*

Proof. We have that $P_1 \cap P_2 \cong \mathrm{Alt}(4)$ and since $|G : P_1| = 5$ we have $G = \langle P_1, P_2 \rangle$. Since G is simple, the amalgam formed by P_1, P_2 and $P_1 \cap P_2$ is faithful, hence is isomorphic to $\mathcal{S}_2$. The rest of the lemma follows immediately. □

Lemma 4.7.3. *Let $G = \mathrm{Sym}(9)$, $P_1 = \langle (1, 2, 3), (3, 4, 5) \rangle$ and $P_2 = \langle (1, 2, 3), (2, 3, 4), (5, 6, 7, 8, 9) \rangle$. Then $\mathrm{Alt}(9)$ is a faithful completion of $\mathcal{S}_3$ and all extensions are visible in G.*

Proof. It is easy to check that $\mathrm{Alt}(9) = \langle P_1, P_2 \rangle$. Set $B = P_1 \cap P_2$. Since $\mathrm{Alt}(9)$ is simple, the amalgam formed by P_1, P_2 and B is faithful and is therefore isomorphic to $\mathcal{S}_3$. We find that $\mathrm{N}_G(P_1) \cap \mathrm{N}_G(P_2) = \langle B, \alpha, \beta \rangle$ where $\alpha = (3,4)$ and $\beta = (6,7,9,8)$. Set

$$\begin{aligned}
R_1 &:= \langle B, \alpha \rangle \\
R_2 &:= \langle B, \alpha\beta^2 \rangle \\
R_3 &:= \langle B, \beta^2 \rangle \\
R_4 &:= \langle B, \alpha, \beta^2 \rangle \\
R_5 &:= \langle B, \beta \rangle \\
R_6 &:= \langle B, \alpha\beta \rangle \\
R_7 &:= \langle B, \alpha, \beta \rangle
\end{aligned}$$

then the group $\langle P_1R_i, P_2R_i \rangle$ is a faithful completion for $\mathcal{E}_3^i$ where $i \in [1,7]$. Note that $\langle P_1R_i, P_2R_i \rangle = \mathrm{Alt}(9)$ for $i = 3,6$ only, and $\langle P_1R_i, P_2R_i \rangle = \mathrm{Sym}(9)$ in the remaining cases. $\square$

Lemma 4.7.4. *Let $G = \mathrm{Sym}(9)$, $P_1 = \langle (1,2,3),\ (3,4,5),\ (6,7,8),\ (7,8,9) \rangle$ and $P_2 = \langle (1,2,3),\ (2,3,4),\ (6,7,8),\ (8,9,10) \rangle$. Then $\mathrm{Alt}(9)$ is a faithful completion of $\mathcal{S}_4$ and all extensions are visible in G.*

Proof. Here we compute that $\mathrm{N}_G(P_1) \cap \mathrm{N}_G(P_2) = \langle P_1 \cap P_2, (1,2), (5,6) \rangle$. The calculations are similar to the previous lemma. $\square$

For the remaining amalgams we set $G = \mathrm{Sym}(21)$ and show that either G or $G' = \mathrm{Alt}(21)$ is a faithful completion for the amalgams $\mathcal{S}_5$-$\mathcal{S}_{12}$ and their extensions. We define

$$\begin{aligned}
X_1 &:= \langle (1,3)(2,4)(5,7)(6,8), (5,7)(6,8)(9,11)(10,12), \\
&\quad\ (9,11)(10,12)(13,15)(14,16), (14,16)(13,15)(17,19)(18,20) \rangle, \\
X_2 &:= \langle (1,2,3,4)(5,8,7,6), (5,6,7,8)(9,12,11,10), \\
&\quad\ (9,10,11,12)(13,16,15,14), (13,14,15,16)(17,20,19,18) \rangle,
\end{aligned}$$

and note that $X_2 \cong \mathrm{C}_4{}^4$ and $\Omega_1(X_2) = X_1 \cong 2^4$. We also need some permutations which act on the orbits of X_1 and X_2;

$$\begin{aligned}
f &:= (1,5,9,13,17)(2,6,10,14,18)(3,7,11,15,19)(4,8,12,16,20),\\
s &:= (1,5,9)(2,6,10)(3,7,11)(4,8,12),\\
t &:= (9,13,17)(10,14,18)(11,15,19)(12,16,20),\\
r &:= (17,20,18,19,21)
\end{aligned}$$

Observe that $\langle f\rangle \cong \langle r\rangle \cong \mathrm{C}_5$. Also note that s and t have order three and $\langle s,t\rangle$ is isomorphic to $\mathrm{Alt}(5)$ acting primitively on the set $[1,20]$ with block system $[1,4]\cup[5,8]\cup[9,12]\cup[13,16]\cup[17,20]$. We define the following subgroups of G (recall the shape notation from the notation introduced in the introduction),

$$\begin{aligned}
P_{1,5} &= \langle X_1, f\rangle \sim 2^5.\mathrm{C}_5,\\
P_{1,6} &= \langle X_2, f\rangle \sim 4^4.\mathrm{C}_5,\\
P_{1,7} &= \langle X_1, s, t\rangle \sim 2^4.\mathrm{Alt}(5),\\
P_{1,8} &= \langle X_2, s, t\rangle \sim 4^4.\mathrm{Alt}(5),
\end{aligned}$$

and, for $i = 5,6,7,8$, we set $B_i = \mathrm{Stab}_{P_{1,i}}(\{17,18,19,20\})$ and $P_{2,i} = \langle B_i, r\rangle$. Now let $\mathcal{A}_i$ be the amalgam formed by $(P_{1,i}, P_{2,i}, B_i)$ for $i = 5,6,7,8$.

Lemma 4.7.5. *For $i = 5,6,7,8$ the amalgams $\mathcal{S}_i$ and $\mathcal{A}_i$ are isomorphic. Moreover, G' is a faithful completion of $\mathcal{S}_i$.*

Proof. Let $i \in \{5,6,7,8\}$ and observe that $P_{1,i}$ acts transitively on the blocks $[1,4]\cup\cdots\cup[17,20]$, so that $|P_{1,i} : B_i| = 5$. Also we have $B_5 = X_1$, $B_6 = X_2$, $B_7 = \langle X_1, s, t'\rangle$ and $B_8 = \langle X_2, s, t'\rangle$ where

$$t' = (5,9,13)(6,10,14)(7,11,15)(8,12,16).$$

Thus B_i normalises $\langle r\rangle$ which gives $|P_{2,i} : B_i| = 5$ and we see that the amalgam $\mathcal{A}_i$ is of degree $(5,5)$. We have to show that $G' = \langle P_{1,i}, P_{2,i} \rangle$ for $i = 5,6,7,8$. Note the inclusions $P_{1,5} \leq P_{1,6} \leq P_{1,8}$ and $P_{1,7} \leq P_{1,8}$. Similarly, $P_{2,5} \leq P_{2,6} \leq P_{2,8}$ and $P_{2,7} \leq P_{2,8}$. Hence we simply need to observe that $G' = \langle P_{1,5}, P_{2,5} \rangle$ (this can be shown using MAGMA for example). Also, since G' is simple, we have shown that the amalgam $\mathcal{A}_i$ is a faithful amalgam of degree $(5,5)$. Using Theorems 4.0.1 and 4.0.2 and the isomorphism type of $P_{1,i}$, we conclude that $\mathcal{A}_i$ is isomorphic to $\mathcal{S}_i$. □

Lemma 4.7.6. *For $i \in \{5,6\}$ the extensions of $\mathcal{S}_i$ are visible in G.*

Proof. Let $i \in \{5,6,7,8\}$ and set $N = \mathrm{N}_G(P_{1,i}) \cap \mathrm{N}_G(P_{2,i})$. Now let R be a subgroup of N which contains B_i. Observe that

$$G' = \langle P_{1,i}, P_{2,i} \rangle \leq \langle P_{1,i}, P_{2,i} \rangle R = \langle P_{1,i}R, P_{2,i}R \rangle \leq G$$

and therefore G' or G is a completion of the amalgam $\mathcal{A}_i := (P_{1,i}R, P_{2,i}R, R)$. Since $P_{1,i} \cap R = B_i = P_{2,i} \cap R$ this amalgam has degree (5,5) and is therefore an extension of $\mathcal{S}_i$. Moreover N is a split extension of B by a subgroup isomorphic to $\mathrm{C}_4 \times \mathrm{C}_4$, therefore there are 15 amalgams which are extensions of $\mathcal{S}_i$ visible in G. We use MAGMA to see that if S is a subgroup of N containing B_i such that $S \neq R$, then $P_{2,i}R$ and $P_{2,i}S$ are non-isomorphic, except in two specific cases which we discuss below.

Let $R = \langle B_i, x \rangle$ and $S = \langle B_i, y \rangle$ where

$$\begin{aligned} x &:= (1,12,15,6)(2,9,16,7)(3,10,13,8)(4,11,14,5)(17,20,19,18), \\ y &:= (1,8,15,10)(2,5,16,11)(3,6,13,12)(4,7,14,9)(17,20,19,18). \end{aligned}$$

Then $P_{1,i}R \cong P_{1,i}S$ and $P_{2,i}R \cong P_{2,i}S$, which implies (because of the degree) that the amalgams $\mathcal{A} := (P_{1,i}R, P_{2,i}R, R)$ and $\mathcal{B} := (P_{1,i}S, P_{2,i}S, S)$ are of the same type. We claim they are non-isomorphic. Suppose for a contradiction that $\Theta = (f_1, f_2, f_3)$ is an amalgam isomorphism from $\mathcal{A}$ to $\mathcal{B}$ (recall Definition 2.2.7 of an amalgam isomorphism). The following argument is similar to the argument showing that the amalgams $\mathcal{A}_5$ and $\mathcal{A}_6$

are non-isomorphic in the proof of Lemma 4.7.1, but requires slightly more work. Note that $\Theta(1) = 1$ and $\Theta(2) = 2$ since P_1R and P_2S are non-isomorphic.

Now $f_1(f) = f^b$ for some $b \in B_i$ (since $P_1 = \langle f \rangle B_i$ is a normal subgroup of P_1S). Now $\langle r \rangle$ is a normal Sylow 5-subgroup of both $P_{2,i}R$ and $P_{2,i}S$, therefore $f_2(r) = r^i$ for some i. Note that neither x nor y centralise r, so we have that $f_3(x) \notin \mathrm{C}_S(r)$. Now $S/\mathrm{C}_S(r)$ is cyclic of order four and $f_3(x)$ is an element of order four which does not square into $\mathrm{C}_S(r)$, so we see that $f_3(x) = yz$ or y^3z for some $z \in \mathrm{C}_S(r)$. Since f_2 is a homomorphism we have that

$$r^{2i} = f_2(r^2) = f_2(r^x) = f_2(r)^{f_2(x)} = (r^i)^{f_3(x)}$$

since $r^{y^3z} = r^{-2}$ we find that $f_3(x) = yz$ must hold. Now we consider

$$f_1(f^x) = f_1(f^3) = f_1(f)^3 = (f^3)^b.$$

On the other hand, this is equal to $(f^b)^{yz} = f^{byz} = (f^y)^{b^yz}$. Now $f^y = f^2$ and so $b(b^yz)^{-1} \in \mathrm{N}_{B_i}(\langle f \rangle) = \mathrm{C}_{B_i}(f) = 1$, whence $b = b^yz$ which implies $(f^2)^{b^yz} = (f^3)^{b^yz}$ and therefore $f^2 = f^3$, a clear contradiction.

It follows then that the amalgams arising in G as described above are all distinct and therefore all extensions of $\mathcal{S}_i$ are visible in G. □

Lemma 4.7.7. *For $i \in \{7, 8\}$ the extensions of $\mathcal{S}_i$ are visible in G.*

Proof. Let $i \in \{7, 8\}$. Then $N := \mathrm{N}_G(P_{1,i}) \cap \mathrm{N}_G(P_{2,i})$ is a split extension of B_i by a group isomorphic to $\mathrm{C}_2 \times \mathrm{C}_4$. There are therefore 8 extensions of $\mathcal{S}_i$ visible in G, and considering the isomorphism types of $P_{2,i}R$ for $B_i \leq R \leq N$ we see these extensions are all distinct. Hence every extension of $\mathcal{S}_i$ is visible in G. □

To construct completions for the amalgams $\mathcal{S}_9$ - $\mathcal{S}_{12}$ we define the following permutations,

$$\begin{aligned} a &:= (1,2,3), \\ b &:= (2,3,4), \\ c &:= (1,2)(5,6), \end{aligned}$$

and note that $\langle a,b\rangle \cong \mathrm{Alt}(4)$ and $\langle a,b,c\rangle \cong \mathrm{Sym}(4)$. We define the following subgroups of G',

$$\begin{aligned} P_{1,9} &= \langle a,b,f\rangle \cong \mathrm{Alt}(4) \wr \mathrm{C}_5, \\ P_{1,10} &= \langle a,b,c,f\rangle \cong \mathrm{O}^2(\mathrm{Alt}(4) \wr \mathrm{C}_5), \\ P_{1,11} &= \langle a,b,s,t\rangle \cong \mathrm{Alt}(4) \wr \mathrm{Alt}(5), \\ P_{1,12} &= \langle a,b,c,s,t\rangle \cong \mathrm{O}^2(\mathrm{Sym}(4) \wr \mathrm{Alt}(5)), \end{aligned}$$

and as above, for $i = 9, 10, 11, 12$, we set $B_i = \mathrm{Stab}_{P_{1,i}}(\{17, 18, 19, 20\})$ and $P_{2,i} = \langle B_i, r\rangle$ and define $\mathcal{A}_i$ to be the amalgam formed by $P_{1,i}$ and $P_{2,i}$.

Lemma 4.7.8. *For $i = 9, 10, 11, 12$ the amalgams $\mathcal{A}_i$ and $\mathcal{S}_i$ are isomorphic. Moreover, G' is a faithful completion of $\mathcal{S}_i$.*

Proof. Let $i \in \{9, 10, 11, 12\}$. First observe that $P_{1,5} \leq P_{1,i}$ and $P_{2,5} \leq P_{2,i}$ hence $G' = \langle P_{1,i}, P_{2,i}\rangle$. It follows that the amalgam $\mathcal{A}_i$ is faithful. For the same reason as in the proof of Lemma 4.7.5 we have $|P_{1,i} : B_i| = 5$. For $j \in \{0, 1, 2, 3, 4\}$ we set

$$a_j = a^{f^j},\ b_j = b^{f^j},\ c_j = c^{f^j}.$$

Observe that $1 = [a_j, r] = [b_j, r] = [c_j, r]$ for $j \neq 4$. Now $\langle a_4, b_4, r\rangle \cong \mathrm{Alt}(5)$ and $\langle a_4, b_4, c_4, r\rangle \cong \mathrm{Sym}(5)$. It follows that $B_i = \mathrm{Stab}_{P_{2,i}}(21)$ and $|P_{2,i} : B_i| = 5$. Hence $\mathcal{A}_i$ is a faithful (5,5) amalgam. Considering the isomorphism type of $P_{1,i}$ and Theorems 4.0.1 and 4.0.2 we conclude that $\mathcal{A}_i$ is isomorphic to $\mathcal{S}_i$. $\square$

Lemma 4.7.9. *For $i = 9, 10, 11, 12$ the extensions of $\mathcal{S}_i$ are visible in G.*

Proof. Let $i \in \{9, 10, 11, 12\}$ and let $N = \mathrm{N}_G(P_{1,i}) \cap \mathrm{N}_G(P_{2,i})$. We calculate that N splits over B_i with a complement isomorphic to $\mathrm{C}_2 \times \mathrm{C}_4$, for $i \in \{9, 10\}$, and a complement is isomorphic to 2^2, for $i \in \{11, 12\}$. Let R be a subgroup of G such that $B_i \leq R \leq N$. Then

$$G' = \langle P_{1,i}, P_{2,i} \rangle \leq \langle P_{1,i}, P_{2,i} \rangle R = \langle P_{1,i}R, P_{2,i}R \rangle \leq G$$

so either G' or G is a completion of the amalgam $(P_{1,i}R, P_{2,i}R, R)$, which is therefore faithful. A Magma calculation shows that if $S \leq N$ is such that $B \leq S$ and $R \neq S$ then the groups $P_{2,i}S$ and $P_{2,i}R$ are non-isomorphic. Hence for each choice of R we obtain a distinct isomorphism class of amalgam. For $i \in \{9, 10\}$ then, all eight extensions of $\mathcal{S}_i$ are visible in G and, for $i \in \{11, 12\}$, all five extensions of $\mathcal{S}_i$ are visible in G. $\square$

4.8 Presentations

In this section we give presentations for the universal completions of the simple finite faithful amalgams of degree $(5, 5)$, with the exception of $\mathcal{S}_{12}$. We have provided a presentation for the universal completions of the amalgam $\mathcal{S}_{12}$ for download at [25]. The forbidding length of this presentation (which has six generators and 143 relations) means that it is unsuitable for display. To obtain these presentations, we employ the function **AP** described in Section A.3 and use the finite completions from Section 4.7.

Type	Generators	Relations
$\mathcal{S}_1$	a, b	a^5, b^5
$\mathcal{S}_2$	a, b, c	a^2, b^3, $(ba)^2$, c^5, $(c^{-1}b)^3$, $(c^{-1}b^{-1}c^{-1})^2$, $bc^2b^{-1}c^2bc^{-1}$, $(cb^{-1}c^{-1}a)^3$
$\mathcal{S}_3$	a, b, c, d	a^3, b^3,c^3, d^5, $(cb^{-1})^2$, $[b,d]$, $(ab^{-1})^2$, $[c,d]$, $(ac)^2$, $(bc)^3$
$\mathcal{S}_4$	a, b, c, d, e, f	e^3, c^3, a^3, b^3, f^3, d^3, $(bd)^2$, $[b,f]$, $[b,e]$, $[d,e]$, $[a,c]$, $[d,f]$, $[b,c]$, $ab^{-1}a^{-1}bd^2$, $(ef^{-1}ef)^2$, $(ef^{-1}e^{-1}f^{-1})^2$, $c^{-1}d^{-1}bd^{-1}cd^{-1}bd^{-1}$, $a^{-1}b^{-1}ad^{-1}bd^{-1}ba^{-1}b$, $cf^{-1}efe^{-1}cef^{-1}ef$, $(c^{-1}f^{-1}efe^{-1})^2$, $a^{-1}f^{-1}efe^{-1}aef^{-1}e^{-1}f$, $([f,e^{-1}])^3$, $f^{-1}ed^{-1}fe^{-1}bd^{-1}ef^{-1}e^{-1}d^{-1}fbd^{-1}$

Table 4.8: Presentations for simple finite faithful (5,5) amalgams.

Type	Generators	Relations
$\mathcal{S}_5$	a, b, c	a^2, $ac^{-1}ac$, b^5, c^5, $(bab^{-1}a)^2$, $bab^{-1}c^{-1}bab^{-1}c$, $(b^{-1}abc^{-1})^2$, $(ba)^5$, $(ab^2ab^{-2})^2$, $b^2ab^{-2}c^{-1}b^2ab^{-2}c$
$\mathcal{S}_6$	a, b, c	a^4, $[a,c]$, b^5, c^5, $a^{-1}b^{-1}a^{-1}bab^{-1}ab$, $b^{-1}a^{-1}bc^{-1}b^{-1}abc$, $(a^{-1}b)^5$, $a^{-1}b^2a^{-1}b^{-2}ab^2ab^{-2}$, $b^{-2}a^{-1}b^2c^{-1}b^{-2}ab^2c$, $ab^{-2}c^{-1}b^2a^{-1}b^{-2}c^{-2}b^2$
$\mathcal{S}_7$	a, b, c, d	b^3, a^4, $[b,d]$, $[a,d]$, c^5, d^5, $(b^{-1}c^{-2})^2$, $(a^{-1}b^{-1})^3$, $a^{-1}c^{-1}bab^{-1}c$, $c^{-1}b^{-1}cd^{-1}c^{-1}bcd$, $a^{-1}c^{-1}a^{-1}bab^{-1}ac$, $a^{-1}ca^{-1}c^{-1}abab^{-1}$, $c^{-1}a^{-1}cd^{-1}c^{-1}acd$, $(b^{-1}c^{-1}b^{-1}c)^2$, $b^{-1}c^{-1}a^{-1}ca^{-1}bc^{-1}ac$, $a^{-1}c^{-2}a^{-1}cbab^{-1}c^{-1}ac^2$, $c^{-3}b^{-1}a^{-1}c^{-1}a^{-1}c^{-1}b^{-1}cbab^{-1}$, $c^{-2}ac^2d^{-1}c^{-2}a^{-1}c^2d^{-2}$
$\mathcal{S}_8$	a, b, c, d	a^2, b^3, $[b,d]$, $ad^{-1}ad$, c^5, d^5, $(b^{-1}a)^3$, $b^{-1}abc^{-1}ac$, $(b^{-1}c^{-2})^2$, $cac^{-1}d^{-1}cac^{-1}d$, $cb^{-1}c^{-1}d^{-1}cbc^{-1}d$, $(cb^{-1}c^{-1}b^{-1})^2$, $bcac^{-1}ab^{-1}cac^{-1}$, $(c^{-1}babcb^{-1})^2$, $(c^{-1}b^{-1}abcd^{-1})^2$, $(ac^{-1}b^{-1}abc)^2$, $cac^{-2}b^{-1}abc^2abc^2abc$

Table 4.9: Presentations for simple finite faithful (5,5) amalgams.

Type	Generators	Relations
$\mathcal{S}_9$	$a,\ b,\ c$	b^3, $(cb^{-1})^2$, c^5, a^5, $b^{-1}aca^{-1}bac^{-1}a^{-1}$, $[[b,a^{-1}],b^{-1}]$, $c^{-1}aba^{-1}cab^{-1}a^{-1}$, $a^{-2}b^{-1}a^2ba^{-2}ba^2b^{-1}$, $a^2ba^{-2}ca^2b^{-1}a^{-2}c^{-1}$, $b^{-1}a^2ca^{-2}ba^2c^{-1}a^{-2}$, $ac^2bca^{-1}cac^{-1}b^{-1}c^{-2}a^{-1}c^{-1}$, $c^2bcbc^2b^{-1}c^{-1}b^{-1}cbc^{-2}b^{-1}$, $a^{-1}c^2b^{-1}c^{-1}ac^{-1}a^{-1}bc^{-1}b^{-1}c^{-2}ac$, $(a^{-1}cbc^{-2}abc^{-1})^2$, $c^2b^{-1}c^{-1}a^{-2}c^{-1}a^2cbc^{-2}a^{-2}ca^2$, $c^2b^{-1}cb^{-1}a^{-1}c^2b^{-1}c^{-1}ac^{-1}a^{-1}cbc^{-2}a$, $a^2c^{-1}a^{-2}cbc^{-2}a^2ca^{-2}cbc^{-2}$, $abc^2b^{-1}c^{-1}b^{-1}a^{-1}c^2bcabcbc^{-2}b^{-1}a^{-1}c^{-1}b^{-1}c^{-2}$, $bc^2a^2bc^2b^{-1}c^{-1}a^{-2}c^{-2}b^{-1}a^2cbc^{-2}b^{-1}a^{-2}$, $a^{-1}c^{-1}b^{-1}c^{-2}a^2c^{-1}b^{-1}c^{-2}a^{-2}c^2bca^2c^2bca^{-1}$, $bcbc^{-2}a^{-1}bcbc^{-2}b^{-1}ac^2bcba^{-1}bcbc^{-2}b^{-1}a$, $a^2bc^2b^{-1}c^{-1}b^{-1}a^{-2}b^{-1}c^{-1}b^{-1}a^2bcbc^{-2}b^{-1}a^{-2}b^{-1}c^{-1}b^{-1}$

Table 4.10: Presentations for simple finite faithful (5,5) amalgams.

Type	Generators	Relations
$\mathcal{S}_{10}$	$a,\ b,\ c,\ d$	$b^2,\ c^5,\ a^5,\ bcbd^{-1},\ (dc^{-1})^2,\ (a^{-1}bab)^2,\ caba^{-1}c^{-1}aba^{-1},\ (a^{-1}b)^5,\ (acbc^{-1}a^{-1}b)^2,\ (ba^{-2}ba^2)^2,\ (cb)^6,\ (bc^{-2})^4,$ $a^{-2}ba^2ca^{-2}ba^2c^{-1},\ (ba^{-1}dc^{-1}a)^2,\ a^{-1}cba^2ba^{-2}c^{-1}ba^2ba^{-1},\ c^2bca^{-1}bac^{-1}bc^{-2}a^{-1}ba,\ cbc^{-1}a^2c^{-1}a^{-2}cbc^{-1}a^2ca^{-2},$ $cac^{-1}bc^{-1}bc^{-1}a^{-1}c^{-1}acdca^{-1},\ (c^{-1}acbc^{-1}a^{-1}cb)^2,\ cbca^{-1}cbc^{-1}ac^{-1}bc^{-2}a^{-1}cbc^{-1}ac,\ c^{-1}a^{-2}c^{-1}bc^{-2}a^2ca^{-2}c^2bca^2,$ $(acbc^{-1}aba^2ba)^2,\ cbcac^2bca^{-1}c^{-1}bc^{-2}ac^{-1}bc^{-2}a^{-1}c,\ (a^{-1}cdcababa^{-1})^2,\ abc^{-1}a^{-2}badc^{-1}acba^{-2}badc^{-1},$ $ca^2c^{-2}bc^2a^{-2}bc^{-1}a^2c^{-2}bc^2a^{-2}b,\ ac^{-1}bc^{-2}ba^{-1}bc^2bcabc^2bca^{-1}c^{-1}bc^{-2}b,\ cbac^{-1}bc^{-1}bc^{-1}a^{-1}bc^{-1}a^{-2}ba^{-2}c^{-1}bc^{-1}bc^{-1}a^2ba^2,$ $(bc^2badc^{-1}a^{-1}bc^{-2})^2,\ cbc^{-1}ba^{-2}c^{-1}bc^{-1}bc^{-1}a^2cbc^{-1}ba^{-2}cdca^2,\ a^2cbc^{-1}a^{-2}cbc^{-1}ba^{-1}ba^{-2}cbc^{-1}a^2bacbc^{-1}b,$ $cbc^{-1}a^{-2}bacdca^{-1}ba^2cbc^{-1}a^{-1}c^{-1}bc^{-1}bc^{-1}a,\ a^{-2}cbc^{-1}abac^{-1}bc^{-1}bc^{-1}a^{-2}cbc^{-1}abacdc,$ $a^{-1}dc^{-1}a^{-1}ba^2c^{-2}bc^2a^{-1}cbc^{-1}ba^{-1}ba^2c^{-2}bc^2,\ cbc^{-1}a^{-1}c^{-1}bc^{-2}ba^{-2}ba^{-2}cbc^{-1}a^2ba^2bc^2bca,$ $(a^2ba^2cbc^{-1}ac^{-1}bc^{-1}bc^{-1})^2,\ bc^{-1}bc^{-2}a^2cbc^{-1}ba^2bac^2da^2cbc^{-1}ba^2ba,\ c^{-1}bc^{-1}bc^{-1}acbc^{-1}ba^2ba^2cdcacbc^{-1}ba^2ba^2,$ $ca^{-1}ba^{-1}cbc^{-1}a^{-1}cbc^{-1}ba^{-1}ba^{-1}c^{-1}babadc^{-1}acbc^{-1}abab,\ ba^{-1}bc^{-2}bc^2ba^{-1}cbc^{-1}a^2bc^{-1}a^{-1}bc^{-2}bc^2ba^{-1}cbc^{-1}a^2c,$ $ba^{-2}c^{-1}bc^{-1}bc^{-1}a^{-2}c^{-1}bc^{-1}bc^{-1}a^{-1}ba^{-2}c^{-1}bc^{-1}bc^{-1}a^{-2}cdca^{-1},\ c^2dac^{-2}bc^2a^{-1}ba^{-1}bac^{-1}bc^{-2}ac^{-2}ababa^{-2}c^2a^{-1}b,$ $c^{-2}bc^2bababc^{-2}bc^2a^{-2}bc^{-2}bc^2a^2c^{-2}bc^2ba^{-1}ba^{-1},\ cbca^{-1}ba^{-1}cbc^{-1}a^{-1}dc^{-1}a^{-1}ba^{-1}c^{-1}bc^{-1}a^{-1}ba^{-1}cbc^{-1}a^{-1}dc^{-1}a^{-1}ba^{-1},$ $aba^{-1}cda^{-1}c^{-1}bc^{-1}bc^{-1}a^{-1}ba^2bc^{-1}bc^{-1}aba^{-1}c^{-1}a^{-1}c^{-1}bc^{-1}bc^{-1}a^{-1}ba^2c,\ babc^{-2}bc^2a^2dcbca^{-2}bc^{-2}bc^2a^{-1}ba^{-2}bc^{-1}bc^{-1}bc^{-1}a^2,$ $c^{-1}a^2bc^{-1}babc^{-2}bc^2a^2dc^{-1}a^2dac^{-2}bc^2ba^2d,\ ba^2dc^2bc^{-1}acdca^2ba^2bc^2dcacdca^2,\ (c^{-2}bc^2babacbc^{-1}acbc^{-1}aba)^2,$ $(abc^{-2}bc^2a^2ba^2bc^{-2}bc^2b)^2,\ (cdcacbc^{-1}bacbc^{-1}baba^2b)^2,\ (abc^{-2}bc^2ba^{-1}ba^2cbc^{-1}ba^2ba)^2,$ $abaca^2ba^{-2}c^{-2}ba^{-1}ba^{-2}cbc^{-1}acbc^{-1}babac^2bc^{-1}babacbc^{-1}acbc^{-1},$ $bc^{-1}a^{-1}c^{-2}bc^2badc^{-1}a^2ba^{-1}cbc^{-1}a^{-4}ba^2cbc^{-2}bc^{-2}a^2cbc^{-1}a^{-1}bc,$ $c^{-1}a^{-1}ba^{-2}ba^{-2}c^{-1}bc^{-1}bc^{-1}a^{-1}bc^{-2}bc^2acbc^{-1}bc^{-1}a^2ba^{-2}c^{-2}a^{-1}c^{-2}bc^2abc^{-1}bc^{-1},$ $c^2ba^{-2}cbc^{-1}ba^2ca^2c^{-2}bc^2a^2babc^{-1}bc^{-2}ba^{-2}cbc^{-1}a^{-1}bc^{-2}bc^2a^{-1}ba^{-1}b,$ $cba^{-1}cbc^{-1}a^2bacdca^{-2}bc^{-2}bc^2ba^{-1}ba^{-2}bc^{-1}bc^{-1}bc^{-1}abacbc^{-1}ac^{-2}bc,$ $bc^2bca^2ba^2ca^2dacbca^2ba^{-2}ca^{-1}bc^{-1}ba^{-1}c^{-1}bc^{-2}ba^{-1}bc^{-1}a^{-2}cdca^{-2},$ $ba^{-1}ba^2bc^{-2}bc^2bac^{-1}bc^{-1}bc^{-1}a^{-2}dcbcacacdca^2ba^2cdcbc^{-1}a^{-1}c^{-1}bc^{-1}bc^{-1},$ $abc^{-2}bc^2a^{-1}c^{-1}ba^2bc^{-2}bc^2ba^{-1}c^{-2}bc^2bac^{-1}ba^{-2}ba^{-2}c^{-1}bc^{-1}bc^{-1}a^{-1}cbc^{-1}bc^{-1}bc^2bacdca^2c,$ $(dc^2bc^{-1}a^{-2}ba^{-2}c^{-1}bc^{-1}bc^{-1}a^{-1}c^{-1}bc^{-1}bac^{-1}bc^{-1}bc^{-1}a^{-1})^2,$ $c^{-2}bc^2baba^2bc^{-2}bc^2ba^{-2}ba^{-1}bc^{-2}bc^2a^2bacdcadcbca^2ba^2bc^2bcacdcab,$ $a^{-1}ba^2c^2bca^2ba^2cbc^2dacbcbca^{-1}bcacdca^2ba^2cbc^{-2}bc^{-1}bc^{-1}acda^{-2}ba^2cba^{-1}c^2bcabcbc^{-1}a^{-1}c^{-1}bc^{-2}b,$ $ba^{-2}bc^{-1}bc^{-1}bc^{-1}a^{-1}bc^{-1}bc^{-2}babc^{-1}bc^{-1}bc^{-1}a^{-1}ba^{-1}c^{-1}bc^{-1}bc^{-1}ac^2dabc^{-1}bc^{-1}bc^{-1}a^{-2}bc^2da^{-1}ba^{-2}c^{-1}\ldots$ $\ldots bc^{-1}bc^{-1}ba^{-2}c^{-2}bc^2a^{-1},$ $bc^{-1}ac^2bca^{-2}ba^{-2}bc^2bca^2ba^2bc^2dabc^{-1}bc^{-2}ba^2ba^2cdc^2bc^{-2}acaba^{-1}dca^{-1}c^{-2}a^{-1}c^{-1}bc^{-2}a^2c^{-1}bc^{-1}bc^{-1}ba^2\ldots$ $\ldots cacbc^{-1}ba^2ba^2c^{-1}ac^{-1}bc^{-1}$

Table 4.11: Presentations for simple finite faithful (5,5) amalgams.

Type	Generators	Relations
$\mathcal{S}_{11}$	$a,\ b,\ c,\ d,\ e$	$d^3,\ c^3,\ a^3,\ b^3,\ [b^{-1},e^{-1}],\ [b^{-1},a],\ [e^{-1},d],\ ca^{-1}c^{-1}b^{-1}cac^{-1}b,\ bdbd^{-1}b^{-1}db^{-1}d^{-1},$ $c^{-1}b^{-1}cd^{-1}c^{-1}bcd,\ cb^{-1}c^{-1}e^{-1}cbc^{-1}e,\ (cac^{-1}a)^2,\ [[b,c^{-1}],b],\ (c^{-1}ac^{-1}a^{-1})^2,$ $ec^{-1}b^{-1}ce^{-1}cdb^{-1}d^{-1}c^{-1},\ adb^{-1}d^{-1}a^{-1}dcbc^{-1}d^{-1},\ e^{-2}c^{-1}bcecdb^{-1}d^{-1}c^{-1},$ $d^{-1}a^{-1}d^{-1}b^{-1}dacb^{-1}c^{-1}db^{-1},\ bc^{-1}dbd^{-1}cd^{-1}cbc^{-1}d,\ dbd^{-1}aca^{-1}db^{-1}d^{-1}ac^{-1}a^{-1},$ $bd^{-1}cbc^{-1}db^{-1}c^{-1}db^{-1}d^{-1}c,\ cadb^{-1}d^{-1}a^{-1}c^{-1}adbd^{-1}a^{-1},\ ad^{-1}cbc^{-1}da^{-1}d^{-1}cb^{-1}c^{-1}d,$ $cbc^{-1}ada^{-1}cb^{-1}c^{-1}ad^{-1}a^{-1},\ eacb^{-1}c^{-1}a^{-1}e^{-1}acbc^{-1}a^{-1},\ c^{-1}dbd^{-1}ce^{-1}c^{-1}db^{-1}d^{-1}ce,$ $a^{-1}dbd^{-1}ad^{-1}a^{-1}db^{-1}d^{-1}ad,\ ea^{-1}cbc^{-1}ae^{-1}a^{-1}cb^{-1}c^{-1}a,\ (c^{-1}bce^{-1})^3,$ $c^{-1}dbd^{-1}cac^{-1}db^{-1}d^{-1}ca^{-1},\ ea^{-1}dbd^{-1}ae^{-1}a^{-1}db^{-1}d^{-1}a,\ ae^2c^{-1}b^{-1}cea^{-1}e^2c^{-1}b^{-1}ce,$ $d^{-1}e^2c^{-1}b^{-1}cede^2c^{-1}b^{-1}ce,\ dbd^{-1}acae^{-1}c^{-1}adb^{-1}d^{-1}caec^{-1},$ $bc^{-1}db^{-1}d^{-1}c^{-1}ac^{-1}bc^{-1}db^{-1}d^{-1}c^{-1}a^{-1}c^{-1},\ cb^{-1}d^{-1}cbc^{-1}dc^{-1}dbc^{-1}db^{-1}d^{-1}cd^{-1},$ $cd^{-1}b^{-1}d^{-1}cb^{-1}c^{-1}d^{-1}c^{-1}dcbc^{-1}dbd,\ cbc^{-1}acbc^{-1}a^{-1}cb^{-1}c^{-1}acb^{-1}c^{-1}a^{-1},$ $c^{-1}a^{-1}c^{-1}bc^{-1}dcbc^{-1}bda^{-1}d^{-1}bda^{-1}b^{-1},\ acbc^{-1}a^{-1}db^{-1}d^{-1}cacb^{-1}c^{-1}a^{-1}c^{-1}dbd^{-1},$ $b^{-1}d^{-1}cbcdb^{-1}d^{-1}cb^{-1}db^{-1}d^{-1}cbc^{-1}d^{-1}cb^{-1}c^{-1}d^{-1},$ $c^{-1}dbd^{-1}c^{-1}b^{-1}c^{-1}dbd^{-1}b^{-1}c^{-1}dbd^{-1}c^{-1}b^{-1}c^{-1}b^{-1}dbd^{-1},$ $bc^{-1}dbd^{-1}bc^{-1}db^{-1}d^{-1}cdb^{-1}d^{-1}cb^{-1}c^{-1}bc^{-1}db^{-1}d^{-1}c^{-1},$ $bc^{-1}db^{-1}d^{-1}a^{-1}cabca^{-1}bcacdb^{-1}d^{-1}ca^{-1}cb^{-1}c^{-1}a,$ $acbc^{-1}b^{-1}dbd^{-1}c^{-1}bc^{-1}b^{-1}dbd^{-1}c^{-1}a^{-1}bc^{-1}d^{-1}cbc^{-1}b^{-1}dc,$ $a^{-1}c^{-1}bc^{-1}d^{-1}cb^{-1}c^{-1}dacb^{-1}c^{-1}ac^{-1}db^{-1}d^{-1}bcab^{-1}cb^{-1}cacbc^{-1}b^{-1},$ $cbc^{-1}dbd^{-1}ad^{-1}c^{-1}db^{-1}d^{-1}acadcbc^{-1}b^{-1}d^{-1}cb^{-1}c^{-1}a^{-1}c^{-1}a^{-1}cda^{-1},$ $acbc^{-1}a^{-1}cdb^{-1}dcbc^{-1}dca^{-1}bc^{-1}db^{-1}d^{-1}bca^{-1}cb^{-1}c^{-1}a^{-1}bcdbd^{-1},$ $c^{-1}dbd^{-1}ae^2cdb^{-1}d^{-1}cbcec^{-1}b^{-1}c^{-1}dbd^{-1}c^{-1}a^{-1}db^{-1}d^{-1}ca^{-1}d^{-1}bda,$ $a^{-1}c^{-1}b^{-1}dbd^{-1}a^{-1}d^{-1}b^{-1}dcebec^{-1}b^{-1}cebc^{-1}b^{-1}dbd^{-1}cb^{-1}a^{-1}c^{-1}bdbd^{-1}a^{-1}d^{-1}cbc^{-1}d^{-1}b^{-1}d^{-1}cb,$ $acadb^{-1}d^{-1}cb^{-1}ac^{-1}a^{-1}bc^{-1}dbd^{-1}a^{-1}b^{-1}c^{-1}a^{-1}dcbc^{-1}b^{-1}d^{-1}cbd^{-1}a^{-1}b^{-1}c^{-1}dbd^{-1}a^{-1}cbd^{-1}cb^{-1}c$

Table 4.12: Presentations for simple finite faithful (5,5) amalgams.

Type	Generators	Relations
$\mathcal{S}_{13}$	$a,\ b,\ c$	$b^3,\ a^4,\ c^4,\ c^{-1}b^{-1}c^2b^{-1}c^{-2}b^{-1}c^{-1},\ (c^{-1}b^{-1})^5,\ (c^{-1}b)^5,\ a^{-2}bc^2b^{-1}a^{-2}bc^{-1}bc^2b^{-1}c^{-1}b^{-1},$ $ac^2bc^{-1}bc^{-1}bcbcb^2cb^{-1}ca,$ $b^{-1}c^{-1}b^{-1}cb^{-1}c^{-1}b^{-1}c^{-1}bcbc^{-2}b^{-1}cbc,$ $(c^{-1}bcb^{-1}cb)^3,$ $b^{-1}c^{-1}bcb^{-1}cb^{-1}a^{-1}cb^{-1}c^{-1}bc^{-1}bca^{-1}bc^{-2}b^{-1}a,$ $a^{-1}bc^{-1}bc^{-1}b^{-1}cba^{-2}b^{-1}c^{-1}bcb^{-1}cb^{-1}ac^{-1}b^{-1}c^{-1}bc^{-1}bcb,$ $(a^{-2}bc^{-1}bc^{-1}b^{-1}cb)^3,$ $cb^{-1}c^{-1}b^{-1}cb^{-1}cb^{-1}c^{-1}b^{-1}a^{-2}bc^{-1}bc^{-1}b^{-1}c^{-2}b^{-1}cb^{-1}c^{-2}b^{-1}c^{-1}bc^{-1}a^{-2},$ $c^{-1}bcb^{-1}cb^{-1}abc^{-1}bc^{-1}b^{-1}cbab^{-1}c^{-1}bcb^{-1}c^{-1}b^{-1}ac^{-1}b^{-1}cb^{-1}cbcbc^{-1}bc^2b^{-1}c^{-1}b,$ $(bc^{-1}bc^{-1}b^{-1}cba)^5,\ (b^{-1}c^{-1}bcb^{-1}cb^{-1}a)^5$
$\mathcal{S}_{14}$	$a,\ b,\ c$	$b^2,\quad (bc)^6,\quad a^{15},\quad c^{15},\quad cbc^{-1}bcbcbc^2bc^{-2}bc^{-1}bc^{-1}b,\quad cbc^{-1}bc^{-1}bc^4bc^{-4}bcb,$ $(c^{-1}bc^2bc^{-1})^3,\ c^{-1}bc^{-2}bc^{-5}bc^{-2}bcbc^{-3},\ a^{-4}bc^4bc^{-1}bc^{-6}a^{-2}c^{-5}bc^{-1}bc^{-1}bc^4a^2c^{-5}bcbcbc,$ $bc^{-1}bc^4ba^{-5}c^{-5}bc^{-1}bc^{-1}bc^4a^5c^{-1}bc^6bcbc^6,\ a^{-1}c^{-5}bc^{-1}bc^{-1}bc^4a^{-7}c^{-1}bc^6bcbc^{-3}a^4cbc^4bc^{-2}bc,$ $a^{-2}c^{-5}bc^{-1}bc^{-1}bc^4a^{-1}c^{-5}bc^{-1}bc^{-1}bc^4a^2cbc^4bc^{-2}bca^6c^{-4}bc^2bcbc^{-2}bc,$ $a^{-1}c^{-5}bc^{-1}bc^{-1}bc^4a^{-3}c^{-5}bc^{-1}bc^{-1}bc^4a^{-3}c^{-5}bc^{-1}bc^{-1}bc^4a^4c^{-1}bc^3bcbcbc^{-2}bc,$ $a^{-4}c^{-5}bc^{-1}bc^{-1}bc^4a^{-2}c^3bc^{-1}bc^{-6}bca^{-7}c^{-5}bc^{-1}bc^{-1}bc^4ac^{-5}bc^{-1}bc^{-1}bc^4,$ $ac^{-5}bc^{-1}bc^{-1}bc^4a^3c^6bcbc^{-4}ba^4bc^5bac^{-5}bc^{-1}bc^{-1}bc^4ac^6bcbc^{-4}b,\quad (c^4bc^6bcbc^{-4}ba^4bc)^3,$ $c^{-5}bc^{-1}bc^{-1}bc^4a^{-1}c^{-5}bc^{-1}bc^{-1}bc^4a^4c^{-5}bc^{-1}bc^{-1}bc^4a^{-3}c^{-1}bc^2bc^{-4}bc^{-1}a^{-2}c^{-5}bc^{-1}bc^{-1}bc^4a^3,$ $bc^{-4}bcbc^5bc^2bc^{-4}bc^{-1}a^{-2}c^{-6}bc^{-1}bc^4ba^6c^{-5}bc^{-1}bc^{-1}bc^3bc^2bc^{-4}bc^{-1}a^{-2}c^{-6}bc^{-1}bc^4ba^6,$ $(a^{-1}c^{-5}bc^{-1}bc^{-1}bc^4)^6$

Table 4.13: Presentations for simple finite faithful (5,5) amalgams.

Type	Generators	Relations
$\mathcal{S}_{15}$	a, b, c, d	$a^2,\quad c^2,\quad b^3,\quad (cd^{-2}cd^{-1})^3,\quad d^{15},\quad (dcd)^6,\quad (dcd^{-2}cd^{-1}cdcd)^2,$ $d^2cd^{-1}cd^{-3}cd^3cdcd^{-2}cdcd^{-1}c,\quad (dcd^{-5}cd^4)^2,\quad (dcd^{-3}cdcd^{-1}cd^2)^2,$ $(dcd^3cdc)^3,\quad (ab^{-1})^{15},\quad bab^{-1}ab^{-1}ababab^{-1}ad^4cdcd^{-1}cdcd^{-1}cd^{-4}c,$ $(bab^{-1}ab^{-1}a)^6,\quad (bab^{-1}a)^{10},\quad (ab^{-1}ab^{-1}ab^{-1}abababab^{-1})^3,$ $b^{-1}ab^{-1}abab^{-1}ab^{-1}ababab^{-1}ab^{-1}ab^{-1}abababab^{-1}ab^{-1}ad^3cd^{-5}cd^{-3},$ $ab^{-1}abababab^{-1}ababab^{-1}abab^{-1}abab^{-1}ab^{-1}d^6cdcd^3cdcd^{-1}c,$ $abab^{-1}ab^{-1}ababab^{-1}abab^{-1}ab^{-1}ababab^{-1}d^{-1}cdcd^2cd^{-1}cd^{-1}cd^{-1}cd^5cd^{-1},$ $(bab^{-1}abab^{-1}abab^{-1}abab^{-1}ab^{-1}ab^{-1}ab^{-1}a)^2,$ $ab^{-1}abab^{-1}ab^{-1}abab^{-1}ab^{-1}ab^{-1}abab^{-1}abababab^{-1}d^{-3}cd^{-3}cd^{-6}cdc,$ $abab^{-1}ab^{-1}ab^{-1}ababab^{-1}ab^{-1}ababab^{-1}ab^{-1}abab^{-1}ab^{-1}dcdcd^{-1}cd^4cd^{-1}cdcd,$ $abab^{-1}abababab^{-1}ababab^{-1}abab^{-1}ab^{-1}ab^{-1}abab^{-1}ab^{-1}d^{-4}cd^3cdcd^3c,$ $ababab^{-1}ababababab^{-1}ab^{-1}abababab^{-1}ad^2cd^{-1}cd^{-2}cd^3cdcd^{-3}c,$ $bab^{-1}abababab^{-1}ab^{-1}ab^{-1}abab^{-1}abab^{-1}ab^{-1}ababab^{-1}d^{-1}cd^2cd^{-5}cd^3cd,$ $b^{-1}abab^{-1}ab^{-1}abababab^{-1}ab^{-1}ab^{-1}ab^{-1}ab^{-1}ababababab^{-1}ab^{-1}ab^{-1}ababab^{-1}aba,$ $b^{-1}ababab^{-1}abababab^{-1}abab^{-1}ab^{-1}ab^{-1}ababdcd^{-1}cdcd^{-4}cd^{-3}cd^{-2}cd^{-1}c$

Table 4.14: Presentations for simple finite faithful (5,5) amalgams.

APPENDIX A

COMPUTER PROGRAMS

A.1 Counting isomorphism classes of amalgams of a certain type

The following function is a computer implementation of Goldschmidt's counting lemma (Lemma 2.2.12) in MAGMA. If $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ then the function **AC** accepts as input the tuple $(P_1, P_2, \pi_1(B), \pi_2\pi_1^{-1})$. The function returns a list **dc**=$(\gamma_1, \ldots, \gamma_n)$ of double coset representatives of H_1^* and H_2^* in $\mathrm{Aut}(B)$ (see Lemma 2.2.12 for notation). A complete set of representatives for the isomorphism classes of amalgams of the same type as $\mathcal{A}$ is then $\{(P_1, P_2, B, \pi_1, \pi_2\gamma_i) \mid i \in [1, \ldots, n]\}$.

```
function AC(p1,p2,b1,isom);


// The first part of the function computes the holomorphs
// of p1, p2 and b1


ab1:=AutomorphismGroup(b1);
phi,P:=PermutationRepresentation(ab1);
phii:=Inverse(phi);


a1:=AutomorphismGroup(p1);
```

```
phi1,perm1:=PermutationRepresentation(a1);
phi1i:=Inverse(phi1);

e1,p1toe1,perm1toe1:=SemidirectProduct(p1,perm1,phi1i);
c1:=sub<e1|{x@perm1toe1 : x in Generators(perm1)}>;


a2:=AutomorphismGroup(p2);
phi2,perm2:=PermutationRepresentation(a2);
phi2i:=Inverse(phi2);


e2,p2toe2,perm2toe2:=SemidirectProduct(p2,perm2,phi2i);
c2:=sub<e2|{x @perm2toe2 : x in Generators(perm2)} >;


// This part finds the normalisers of b1 in the automorphism
// groups of p1 and p2

n1:=Normaliser(e1,b1@p1toe1) meet c1;
n2:=Normaliser(e2,(b1@isom)@p2toe2) meet c2;

// This pulls n1 and n2 back into the automorphism groups
// of p1 and p2 respectively

a1n1:={ ( ( x@@perm1toe1 ) @phi1i ) : x in Generators(n1)};
a2n2:={ ( ( x@@perm2toe2 ) @phi2i ) : x in Generators(n2)};
```

```
// Next we define images in aut(b1)

astar:= { ab1!hom<b1->b1 | x:-> x@y   > : y in a1n1 };
cstar:={ ab1!hom<b1->b1 |
x:-> ((x@isom)@y)@@isom  > : y in a2n2 };

// Now we move astar and cstar into the permutation
// representation of aut(b1)

astarP:=sub<P|{ x@ phi : x in astar}>;
cstarP:=sub<P|{ x@ phi : x in cstar}>;

// In the permutation representation of aut(b1) we can
// compute the double coset reps quickly
dblc:=DoubleCosetRepresentatives(P,astarP ,cstarP );
dc:=[x@phii : x in dblc ];

// the function returns a list of these double coset
// representatives
return dc;

end function;
```

A.2 Computing extensions of an amalgam

The MAGMA function **Ext** computes extensions of amalgams. If $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$ is an amalgam, the function accepts as input the tuple $(P_1, P_2, \pi_1(B), \pi_2\pi_1^{-1})$.

```
function Ext(p1,p2,b1,isom);

// The first part of the program constructs the holomorphs
// of p1, p2 and b1.

ab1:=AutomorphismGroup(b1);
phi,P:=PermutationRepresentation(ab1);
phii:=Inverse(phi);

a1:=AutomorphismGroup(p1);
phi1,perm1:=PermutationRepresentation(a1);
phi1i:=Inverse(phi1);

e1,p1toe1,perm1toe1:=SemidirectProduct(p1,perm1,phi1i);
c1:=sub<e1|{x@perm1toe1 : x in Generators(perm1)}>;

perm1toe1i:=Inverse(perm1toe1);

a2:=AutomorphismGroup(p2);
phi2,perm2:=PermutationRepresentation(a2);
phi2i:=Inverse(phi2);

e2,p2toe2,perm2toe2:=SemidirectProduct(p2,perm2,phi2i);
c2:=sub<e2|{x @perm2toe2 : x in Generators(perm2)} >;

perm2toe2i:=Inverse(perm2toe2);

// Now we calculate the normalisers of b1 in the automorphism
```

```
// groups of p1 and p2

n1:=Normaliser(e1,b1@p1toe1) meet c1;
n2:=Normaliser(e2,(b1@isom)@p2toe2) meet c2;

// The following part finds the centralisers of b1 in the
// automorphism groups of p1 and p2 and defines the image
// of the normalisers in the automorphism group of b1

mu1:=hom<n1-> P | { x->
(ab1!hom<b1->b1 | y:->y@((x@@perm1toe1)@phi1i)>)@phi
 : x in Generators(n1) }>;
mu2:=hom<n2-> P | {x-> (ab1!
 hom<b1->b1 | y:->((y@isom)@((x@@perm2toe2)@phi2i))@@isom>
 )@phi : x in Generators(n2) }>;
k1:=Kernel(mu1);
k2:=Kernel(mu2);
gensk1:=Generators(k1);
gensk2:=Generators(k2);

// The group A below is just the direct product of the
// normalisers of b1 in the automorphism groups of p1 and p2

A,injs,projs:=DirectProduct(n1,n2);

// By taking the intersection of the images in the automorphism
// group of b1 we locate the elements of aut(p1) and aut(p2)
// which induce the same automorphisms on b1
```

```
ast:=n1@mu1;
cst:=n2@mu2;
acst:=ast meet cst;
gensacst:=Generators(acst);

// Here nstar is the automorphism group of the amalgam

nstar:=sub<A | {x@injs[1] : x in gensk1},
{x@injs[2] : x in gensk2},
{ (x@@mu1)@injs[1] * (x@@mu2)@injs[2] : x in gensacst } >;

nu1:=hom<b1->n1 | {x-> ( (a1!
 hom<p1->p1 | {y->y^x : y in Generators(p1) }>
 )@phi1)@perm1toe1 : x in Generators(b1) } >;
nu2:=hom<b1->n2 | {x-> ( (a2!
hom<p2->p2 | {y->y^(x@isom) : y in Generators(p2) }>
)@phi2)@perm2toe2 : x in Generators(b1) }>;
nu:=hom<b1->A | {x-> ((x@nu1)@injs[1])*( (x@nu2)@injs[2])
 : x in Generators(b1) } >;

// Here dstar is the image of b1 in the automorphism group
// of the amalgam

dstar:=b1@nu;

// We need to find all the subgroups of the automorphism
// group which contain dstar
```

```
q,f:=quo<nstar|dstar>;
subs:=Subgroups(q);
subs2:=[ sub<nstar | dstar,subs[i]`subgroup@@f> :
 i in [1..#subs] ];

// We can now define the extensions by the subgroups in subs2

gx:=[];gy:=[];gxy:=[];pi1:=[];pi2:=[];

// The following code creates a list of the extensions of the
// amalgam by the subgroups in subs2

for i in [1..#subs2] do
gxy:=gxy cat [subs2[i]];
l1,f1,f2:=SemidirectProduct(
p1, subs2[i], projs[1]*perm1toe1i*phi1i );
k:=sub<l1| { x@f1*((x^-1)@nu@f2) : x in Generators(b1) } >;
l2,map:=quo<l1|k>;
gx:=gx cat [l2];
b2:=sub<l2|{ x@(f2*map) : x in Generators(subs2[i])}>;
pi1:=pi1 cat [ hom<subs2[i]->b2 |
 {x ->x@(f2*map) : x in Generators(subs2[i])} > ];
l1,f1,f2:=SemidirectProduct(
 p2, subs2[i], projs[2]*perm2toe2i*phi2i );
k:=sub<l1| {(x@isom)@f1*((x^-1)@nu@f2) :
 x in Generators(b1) } >;
l2,map:=quo<l1|k>;
```

```
gy:=gy cat [l2];
b2:=sub<l2|{ x@(f2*map) : x in Generators(subs2[i])} >;
pi2:=pi2 cat [hom<subs2[i]->b2|
 {x -> x@(f2*map) : x in Generators(subs2[i])} > ];
end for;

return gx,gy,gxy,pi1,pi2;

end function;
```

A.3 Amalgamated products

Finally we provide details of the Magma function **AP** which computes the universal completion of an amalgam $\mathcal{A} = (P_1, P_2, B, \pi_1, \pi_2)$. The function accepts as input the tuple $(P_1, P_2, \pi_1(B), \pi_2\pi_1^{-1})$. This function is built around the existing Magma function **FreeProduct**. The function **FPGroupStrong** is used to convert the groups P_1 and P_2 into finitely presented groups. This appears to be more effective than using **FPGroup**, with permutation groups for example.

```
function AP(g,h,b,iso);

// input is g,h, b, b a subgroup of g, iso a map b-> h
// output is Y, the free amalgamated product of g and h
// with amalgamation over b and the maps g->Y,h->Y

x,phi1:=FPGroupStrong(g);
phi1i:=Inverse(phi1);
y,mu1:=FPGroupStrong(h);
```

```
mu1i:=Inverse(mu1);

X:=FreeProduct(x,y);
gensX:=Generators(X);

gensxinX:=[ X.i : i in [1..#Generators(x)] ];
gensyinX:=[X.i : i in [#Generators(x)+1..#Generators(X)] ];

phi2:=hom<x->X | {x.i->gensxinX[i] :
        i in [1..#Generators(x)] } >;
mu2:=hom<y->X | {y.i->gensyinX[i] :
        i in [1..#Generators(y)] } >;

R:= {  (d@phi1i@phi2)*((((d@iso)@mu1i)@mu2)^-1)
        : d in Generators(b)};

Y,quo:=quo<X|R>;

return Y,phi1i*phi2*quo,mu1i*mu2*quo;

end function;
```

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