CONNECTIVITY OF HURWITZ SPACES

by

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Abstract

Let G be a finite group and $\boldsymbol{C} = (C_1, \ldots, C_r)$ a collection of conjugacy classes of G. The *Hurwitz space* $\mathcal{H}^{\text{in}}(G, \boldsymbol{C})$ is the space of Galois covers of $\mathbb{P}^1\mathbb{C}$ with monodromy group G, and ramification type $\boldsymbol{C}$. Points of the Hurwitz space can be parameterised combinatorially by Nielsen tuples: tuples in G^r with product one. There is a correspondence between connected components of $\mathcal{H}(G, \boldsymbol{C})$ and orbits of the braid group on the set of Nielsen tuples.

In this thesis we consider the problem of determining the number of components of the Hurwitz space for A_5 and A_6. For both groups we give a complete classification of the braid orbits for all types $\boldsymbol{C}$. We show that when there exists more than one orbit then Fried's lifting invariant distinguishes these orbits.

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CHAPTER 1

INTRODUCTION

Let X be a compact Riemann surface of genus g. Let $\phi : X \to \mathbb{P}^1\mathbb{C}$ be a degree n normal cover. For all but finitely many points $x \in \mathbb{P}^1\mathbb{C}$, the fibre $\phi^{-1}(x)$ has cardinality n. We call those points, whose fibre has cardinality strictly less than n, *branch points* of ϕ, and we let $B = B_\phi = \{b_1, \ldots, b_r\}$ be the set of branch points of ϕ.

For any $x \in \mathbb{P}^1\mathbb{C} \setminus B$, the fundamental group $\pi_1(P^1\mathbb{C} \setminus B, x)$ acts transitively on the n elements of the fibre. This action is known as the *monodromy action*, and it induces a homomorphism

$$\phi_*\colon \pi_1(\mathbb{P}^1\mathbb{C} \setminus B, x) \to S_n. \tag{1.1}$$

The image of ϕ_* is called the *monodromy group* of ϕ. The function ϕ and the basepoint x determine the monodromy group. Changing the basepoint results in a conjugate monodromy group. For every $1 \leq i \leq r$, let γ_i be the closed curve winding once around the point b_i. Then $\pi_1(\mathbb{P}^1\mathbb{C} \setminus B, x)$ is generated by the homotopy classes of the γ_i (which we also denote by γ_i), and the γ_i satisfy the single relation

$$\gamma_1 \cdots \gamma_r = 1.$$

The function ϕ_* takes the generators $\gamma_1, \ldots, \gamma_r$ to non-identity elements $g_1, \ldots, g_r$, which generate G and which satisfy the product-one condition:

$$g_1 \cdots g_r = 1.$$

If C_i denotes the conjugacy class of g_i then the tuple $\boldsymbol{C} = (C_1, \ldots, C_r)$ is called the *ramification type* (or simply *type*) of f.

Consider the set of all possible monodromy homomorphisms of the form (1.1). Each homomorphism is determined by the images of the standard generators. Hence the set of monodromy homomorphisms for a fixed group G is given by

$$\mathcal{E}_r(G) = \{(g_1, \dots, g_r) \mid g_i \neq 1, 1 \leq i \leq r, \langle g_1, \dots, g_r \rangle = G \text{ and } g_1 \cdots g_r = 1\}.$$

A tuple $(g_1, \dots, g_r) \in \mathcal{E}_r(G)$ is called a *Nielsen tuple* (or *Hurwitz tuple*). If $\boldsymbol{g} = (g_1, \dots, g_r)$ and $\boldsymbol{g'} = (g'_1, \dots, g'_r)$ are two Nielsen tuples such that there exists $h \in G$ satisfying $g_i^h = g'_i$ for all i, then we write $\boldsymbol{g}^h = \boldsymbol{g'}$. Conjugate tuples correspond to equivalent monodromy homomorphisms.

Let $\mathcal{B}_r = \langle Q_1, \dots, Q_{r-1} \rangle$ be the Artin braid group on r strands (see [2]). The braid group acts on the set of Nielsen tuples and these orbits, called *braid orbits*, are known to correspond to connected components of the *Hurwitz space*. The Hurwitz space $\mathcal{H}^{\text{in}}(G, \boldsymbol{C})$ is a topological space parameterizing covers of $\mathbb{P}^1\mathbb{C}$ with monodromy group G and ramification type $\boldsymbol{C}$. The study of braid orbits and Hurwitz spaces goes back as far as Clebsch and Hurwitz.

There is renewed interest in determining the properties of braid orbits driven by connections to the regular inverse Galois problem [29], the theory of modular towers [1], and l-adic representations of Shimura varieties [13]. It has become clear that an understanding of the braid orbits on Nielsen tuples may provide an understanding of these arithmetic problems.

In this thesis we consider the problem of determining the number of braid orbits on Nielsen tuples for a fixed group G and an arbitrary tuple of conjugacy classes of G. The following is an outline of the contents of this thesis.

Our exposition begins in Chapter 2 with a brief overview of Riemann surfaces and their coverings. Much of this material is likely to be familiar to the reader, but is included for completeness. The highlight of this chapter, at least for the purposes of the rest of the thesis, is Riemann's existence theorem which gives a correspondence between meromorphic functions on a Riemann surface and equivalence classes of monodromy homomorphisms. This result allows us to give a combinatorial description of Hurwitz spaces. The Hurwitz spaces are the subject of Chapter 3.

Traditionally the Hurwitz space $\mathcal{H}_{g,G}$ is defined as the space of surface-function pairs (X, φ) such that

$$\varphi : X \to \mathbb{P}^1\mathbb{C}$$

is a meromorphic function with associated monodromy group G. This space has a higher genus counterpart, the space of Riemann surfaces of genus g with a group of automorphisms isomorphic to G. When $g = 0$ these two notions coincide. Of course there are infinitely many such covers and so often refinements of the Hurwitz spaces where the covers in question are ramified over a set of points with prescribed ramification types are considered. Following Fried and Volklein, [36] and [37], we explain how points of this space correspond to Nielsen tuples, and how braid orbits on Nielsen tuples determine topological properties of the Hurwitz space. Finally the chapter ends with a discussion of a selection of comparable results in the field. The result of greatest influence on this thesis is the classification of braid orbits for 3-cycle types due to Fried [11]. Fried defined an invariant of braid orbits. Using this invariant, which is usually called the *lifting invariant* or *lift invariant*, he showed that the braid orbits for alternating groups, with type consisting of 3-cycles, are distinguished by said lifting invariant.

The principle theoretical achievements of this thesis are the classifications of braid orbits for the alternating groups A_5, Theorem 5.3.1, and A_6, Theorem 4.3.1, found in Chapters 4 and 5 respectively. Such classifications represent the first complete descriptions of braid orbits for all types for a nonsoluble group. The classifications themselves are satisfyingly simple: Fried's lifting invariant is extended to all types, and we show that, outside of a number of small length exceptions, two Nielsen tuples with the same lifting invariant lie in the same braid orbit. Perhaps of greater importance, is the pattern which these two classifications suggest: that, almost always, the lifting invariant separates the braid orbits for alternating groups. Unfortunately this thesis does not contain a more general result for an arbitrary alternating group A_n. However evidence is presented, in Chapter 7 particularly, which lends itself to this conclusion.

A necessary byproduct of the work undertaken during the production of this thesis is the creation of computational tools and techniques designed to calculate and analyse braid orbits. Chapter 6 presents an overview of these tools, which is distributed with the computational algebra system GAP. One hopes that improving the ease of use and performance of these tools will encourage others to complete classifications for further groups, and to aid the development of more general theory.

CHAPTER 2

COVERINGS AND SYMMETRIES OF RIEMANN SURFACES

This chapter is an introduction to Riemann surfaces, their covering spaces and their symmetries. The aim is to provide an overview of the basic results in the area. None of the material contained in this chapter is needed to understand the proofs or techniques used in the later chapters to prove the main results of this thesis, however it does provide motivation for the problem considered. The material in this chapter can be found in most introductory texts on Riemann surfaces, such as [31, 24, 7].

2.1 Definitions and Examples

To begin we define Riemann surfaces and consider some of their properties.

Definition 2.1.1. A *complex chart* on a topological space X is a homeomorphism $\varphi : U \to V$, from an open set $U \subset X$ to an open set $V \subset \mathbb{C}$. The set U is called the *domain* of the chart, and if $\varphi(p) = 0$ for some $p \in U$ we say that the chart is *centred* at p.

Thus, a complex chart is merely a set of local complex coordinates for a space. Two such sets of coordinates are compatible if the change of coordinates map is holomorphic.

Definition 2.1.2. Two charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ on a topological space X are said to be *compatible* if the map

$$\varphi_2 \circ \varphi^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is holomorphic. The function $\varphi_2 \circ \varphi^{-1}$ is called the *transition map.*

Local charts can be patched together to cover X.

Definition 2.1.3. An *atlas of complex charts for* X is a set $\mathcal{A}$ of pairwise compatible charts that cover X.

Two atlases, $\mathcal{A}$ and $\mathcal{B}$, are equivalent if every chart of $\mathcal{A}$ is compatible with every chart of $\mathcal{B}$. An equivalence class of atlases is called a *complex structure*.

Definition 2.1.4. A *Riemann surface*, X, is a second-countable Hausdorff space with an accompanying complex structure.

2.2 Examples

This section will focus on some examples of Riemann surfaces. Note that examples are given in terms of charts rather than their equivalence classes.

The first nontrivial example of a Riemann surface is often the *Riemann sphere*, also know as the *projective line*, $\mathbb{P}^1\mathbb{C}$. There are two ways of viewing the Riemann sphere: algebraically and geometrically. This dichotomy is common within the study of Riemann surfaces.

The geometric approach to defining the Riemann Sphere is as follows: Define $\widehat{\mathbb{C}}$ to be the *extended complex plane* $\mathbb{C} \cup \{\infty\}$. The topology of points in $\mathbb{C}$ is the standard topology. The basic neighbourhoods of ∞ are given by

$$D_R(\infty) = \{z \in \mathbb{C} \mid |z| > \frac{1}{R}\} \cup \{\infty\}.$$

Define two charts on $\widehat{\mathbb{C}}$ by

$$U_0 = \mathbb{C}, \qquad \phi_0(z) = z$$

$$U_\infty = \widehat{\mathbb{C}} \setminus \{0\}, \qquad \phi_\infty(z) = \begin{cases} 1/z & z \neq \infty \\ 0 & z = \infty \end{cases}$$

It is straightforward to see that this pair of charts forms an atlas.

Alternatively one may consider an algebraic approach. Recall that, for a given field $\mathbb{K}$, the projective line over $\mathbb{K}$ is the set of all homogeneous coordinates:

$$\mathbb{P}^1(\mathbb{K}) = \{[x_1 : x_2] \mid x_1, x_2 \in \mathbb{K},\ x_1 \neq 0,\ \text{or } x_2 \neq 0\},$$

where $[x_1 : x_2]$ denotes the one-dimensional subspace of $\mathbb{K}^2$ containing x_1 and x_2. The point $[1 : 0]$ corresponds to the additional point ∞. Note that for any $\lambda \in \mathbb{K}$

$$[x_1 : x_2] = [\lambda x_1, \lambda x_2].$$

Define a sequence of charts for $\mathbb{P}^1\mathbb{C} = \mathbb{P}^1(\mathbb{C})$ by

$$U_0 = \{[x_1 : x_2] \in \mathbb{P}^1\mathbb{C} \mid x_1 \neq 0\}, \qquad \varphi_0([x_1 : x_2]) = \frac{x_2}{x_1}$$
$$U_1 = \{[x_1 : x_2] \in \mathbb{P}^1\mathbb{C} \mid x_2 \neq 0\}, \qquad \varphi_0([x_1 : x_2]) = \frac{x_1}{x_2}.$$

Again, one may easily check that this forms an atlas. It is not immediately obvious that these two structures are equivalent. Using the stereographic projection one can verify that $\mathbb{P}^1\mathbb{C}$ may be viewed as a sphere. The algebraic nature of this definition allows for easily calculation.

Another important class of examples of Riemann surfaces are *complex tori.* Such examples illustrate how one may attach a complex structure to a Riemann surface acted on by a group. Let Λ be a discrete subgroup of $\mathbb{C}$. In particular

$$\Lambda = \mathbb{Z} \oplus \lambda\mathbb{Z},$$

for some $\lambda \in \mathbb{C}$. Consider the quotient projection

$$\pi : \mathbb{C} \to \mathbb{C}/\Lambda.$$

This quotient can be given a complex structure, producing a Riemann surface. Observe that $\mathbb{C}/\Lambda$ is equipped with the natural quotient topology: $U \subset \mathbb{C}/\Lambda$ is open if and only if its preimage, $\pi^{-1}(U)$, is open in $\mathbb{C}$.

There exists some ε such that for any $z \in \mathbb{C}$ the open ball $D_\varepsilon(z)$ and its G-translates are

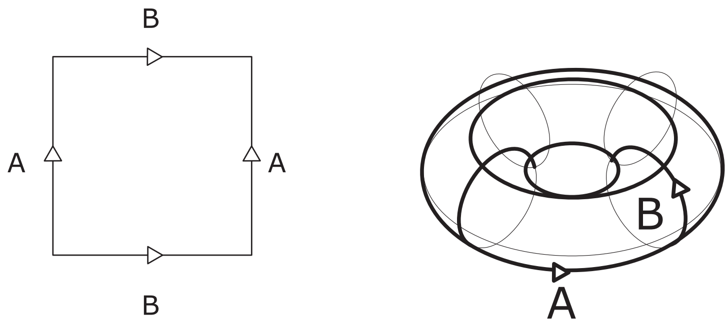


Figure 2.1: A torus realised by the identification of the edges of a square.

disjoint, i.e.,

$$D_\varepsilon(z) \cap g \cdot D_\varepsilon(z) = \emptyset \text{ for all } g \in G.$$

A complex atlas on $\mathbb{C}/\Lambda$ is defined as follows: For each $z \in \mathbb{C}$ we let $D_\varepsilon(z)$ be the domain of its chart. The chart map itself is the inverse function $\pi \mid_{D_\varepsilon(z)}^{-1}$. One can then verify that the transition maps are holomorphic [31]. In fact, this process of using the charts for the domain of a continuous function to provide charts for the codomain is the same as the one used for a group acting on a given Riemann surface. As seen in Figure 2.1, a complex torus is a torus in the topological sense.

2.3 Topology of Riemann surfaces

As the name suggests, Riemann surfaces, are topological surfaces. Indeed, it is clear that if X is a Riemann surface then an atlas attached to X ensures that X is a one dimensional complex manifold, or equivalently, a two dimensional real manifold. Moreover the insistence that the transition functions are holomorphic ensures that the 2-manifold is orientable [24]. Therefore the classification of compact orientable surfaces applies.

Theorem 2.3.1 ([19])**.** *Every orientable compact Riemann surface X is homeomorphic to a sphere with $g \geq 0$ handles for some integer g. The integer g is called the* genus*, and is denoted $g(X)$.*

A presentation for the fundamental group of surfaces is also well known. By the previous result this presentation applies to Riemann surfaces.

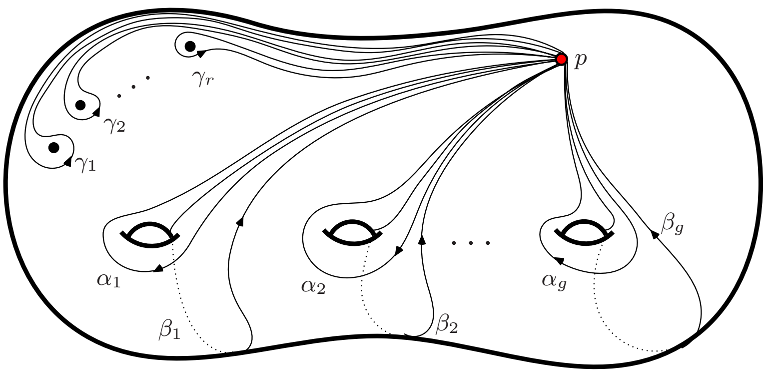


Figure 2.2: Generators for the fundamental group.

Theorem 2.3.2. *The compact orientable surface of genus g with n punctures, has fundamental group with presentation:*

$$\pi_1(S) := \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots \gamma_n \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_n = 1 \rangle.$$

The geometric interpretation of these generators is given in Figure 2.2. In particular, if the surface is an n-punctured sphere, then its fundamental group has presentation:

$$\langle \gamma_1, \ldots, \gamma_n \mid \gamma_1 \cdots \gamma_n = 1 \rangle.$$

2.4 Maps between Riemann Surfaces

Definition 2.4.1. A map $F : X \to Y$ is said to be *holomorphic* at a point $p \in X$ if there exist charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ for X and Y respectively, such that $p \in U_1$ and $F(p) \in U_2$, and such that the composition $\varphi_2 \circ F \circ \varphi_1^{-1}$ is holomorphic at $\varphi_1(p)$ in the usual sense of complex functions $\mathbb{C} \to \mathbb{C}$.

A function is said to be holomorphic on some domain if it is holomorphic at every point on the given domain, and F is called *holomorphic* if it is holomorphic on the whole of X.

A bijective holomorphism is called an *isomorphism* and self-isomorphisms are, as expected, called *automorphisms*. The set of automorphisms of a Riemann surface X forms a group under composition. This group, called the *automorphism group of* X, is denoted $Aut(X)$.

If Y is the Riemann sphere, $\mathbb{P}^1\mathbb{C}$, then a holomorphic map $F : X \to Y$ is often called a *meromorphic function*.

Proposition 2.4.2. *Let $F : X \to Y$ and $G : Y \to Z$ be holomorphic maps. Then*

- *if F is holomorphic, then F is continuous;*
- *the composition $G \circ F : X \to Z$ is holomorphic.*

Recall that if h is holomorphic on the annulus

$$A = \{z \in \mathbb{C} \mid 0 < |z - c| < R\}$$

then h can be expressed as a Laurent series, i.e.,

$$h(z) = \sum_{-\infty}^{\infty} a_n (z - c)^n$$

for some constants a_n, which may be determined by evaluating suitable line integrals. Suppose that p is a point in X, and that there exists a holomorphic map $f : X \to Y$ in a punctured neighbourhood of p. Choose a chart $\varphi : U \to V$ centered at p. Then $f \circ \varphi^{-1}$ is holomorphic in a neighbourhood of $0 = \varphi(p)$, therefore, letting $z = \phi(x)$,

$$f \circ \varphi(z) = \sum_{n=N} a_n z^n.$$

The integer N is called the order of f at p, and is denoted $\mathrm{ord}_p(f)$. Note that the order does not depend on the chart: if φ' is a different chart then, since transition maps are biholomorphic, the composition $\varphi \circ \varphi'^{-1}(z)$ has a Laurent series whose leading term has degree 1. So

$$\varphi \circ \varphi'^{-1}(z) = \sum_{n=1} b_n z^n$$

and therefore, as

$$\begin{aligned} f \circ \varphi'^{-1} &= (f \circ \varphi^{-1}) \circ (\varphi \circ \varphi'^{-1}) \\ &= a_N (\sum_{n=1} b_n z^n)^N + a_{N+1} (\sum_{n=1} b_n z^n)^{N+1} + \cdots , \end{aligned}$$

the degree of the leading term of $f \circ \varphi'^{-1}$ is N.

Definition 2.4.3. Let X and Y be Riemann surfaces, and let $p \in X$. Suppose that $f : X \to Y$ is a holomorphism and that $f(p) = q$. Define the *multiplicity* of f at p, written $\text{mult}_f(p)$, to be $\text{ord}(\varphi \circ f)$ for some chart φ centered at q. By the above, this does not depend on the choice of the chart φ.

Most points will have multiplicity 1, those points which do not are sites of unusual behaviour for the holomorphic map.

Definition 2.4.4. Let $f : X \to Y$ be a non-constant holomorphic map. A point $p \in X$ is called a *ramification point* if $\text{mult}_f(p) \geq 2$. A point $y \in Y$ is called a *branch point* for f if it is the image of a ramification point. Points in X which are not ramification points are called *regular points.* A holomorphic map with ramification points is said to be *ramified* or *branched.*

Note that the set of branch points is discrete because these are points where the derivative of $\varphi \circ \varphi'^{-1}$ vanishes [15].

We can give a precise description about the local behaviour of holomorphic maps between Riemann surfaces.

Theorem 2.4.5 (Local Normal Form [31])**.** *If $f : X \to Y$ is a nonconstant holomorphic map and $p \in X$, then there is a unique nonnegative integer n such that f looks like $z \mapsto z^n$. To be more precise, for every chart $\varphi_2 : U_2 \to V_2$ on Y centered at $f(p)$, there exists some chart $\varphi_1 : U_1 \to V_1$, and an integer $n \geq 0$, such that φ_1 is centred at p, and $\varphi_2(f(\varphi_1^{-1}(z))) = z^n$ for every $z \in \varphi_1^{-1}(V_1)$.*

The integer n is unique, as can be seen by observing that this n is given by the topological properties of the map. If local coordinates are chosen so that f is viewed as $z \mapsto z^n$, then there are exactly n preimages of points in a suitably chosen neighbourhood of $f(p)$, and this number is independent of the coordinates chosen. In fact, the integer n coincides with the multiplicity of f at p [31].

Definition 2.4.6. Let X and Y be Riemann surfaces and let $f : X \to Y$ be a nonconstant holomorphic map. Then define $d_y(f)$ to be

$$d_y(f) = \sum_{p \in f^{-1}(y)} \text{mult}_p(f).$$

It can be shown that $d_f(y)$ does not depend on the point y and depends only on f [31]. The constant $\deg(f) = d(f) = d_y(f)$ is called the *degree of* f.

Let $f : X \to Y$ be a holomorphic map. An important consequence of the local structure for holomorphic maps is that, should f be unbranched, then $\chi(X) = \chi(Y)\deg(f)$, where $\chi(X)$ denotes the Euler characteristic of X. If however, f is ramified, then we must subtract a term which takes the additional multiplicity of the branch points into account. These two facts are captured by the celebrated Riemann-Hurwitz formula:

Theorem 2.4.7. *(Riemann-Hurwitz Formula) Let $f : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Then*

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum_{p \in X} (\mathrm{mult}_p(f) - 1).$$

Observe that the sum $\sum_{p\in X}(\mathrm{mult}_p(f) - 1)$ is finite because the set of ramification points is discrete.

2.5 Automorphisms of Riemann Surfaces

Definition 2.5.1. Let G be a finite group acting on a Riemann surface X. We say that G acts *effectively (or faithfully)* on X if for any distinct $g, h \in X$ there exists some $x \in X$ such that $g \cdot x \neq h \cdot x$. An action is said to be *holomorphic* if for every $g \in G$ the map sending $x \in X$ to its image under g is a holomorphic map from X to itself.

If G acts on a Riemann surface X then the quotient space is also a Riemann surface.

Theorem 2.5.2 ([24], [31])**.** *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. Then we can construct a complex structure for X/G which makes X/G a Riemann surface. Moreover, the quotient map $\pi : X \to X/G$ is a holomorphic map of degree $|G|$ and* $\mathrm{mult}_p(\pi) = |G_p|$ *for any point $p \in X$.*

If a group acts on a Riemann surface then we have an alternative version of the local normal form for this action, which says that, if g stabilises a point p in X, then the local action is linear.

Theorem 2.5.3 ([31])**.** *Let G be a finite group acting holomorphically and effectively on a Riemann surface, X. Fix a point, $p \in X$, with nontrivial stabilizer of order m. Choose a* *generator, g, for G_p. Then there is a local coordinate z centred at p such that the action of g is given by $g(z) = \lambda z$, for some primitive m^{th} root of unity λ.*

The element g is called the *distinguished generator* for G_p. The distinguished generator is unique up to conjugation in G. We will call the conjugacy class of g the *ramification type of* p. For regular points the stabilizer is trivial so the above theorem does not apply.

The following is a restatement of the Riemann-Hurwitz formula for groups acting on Riemann surfaces which follows from the two previous results.

Corollary 2.5.4. *Let G be a finite group acting holomorphically and effectively on a Riemann surface X. Let $\pi : X \to X/G =: Y$ be the quotient map. Then for each branch point $y \in Y$ there exists some integer $r \geq 2$ such that $\pi^{-1}(y)$ consists of exactly $\frac{|G|}{r}$ points, and π has multiplicity r at each of these points. Moreover, if there are exactly k branch points, $y_1, \ldots, y_k$, with π having multiplicity r_i at the points above y_i then*

$$\begin{aligned} 2g(X) - 2 &= |G|(2g(X/G) - 2) + \sum_{i=1}^{k} \frac{|G|}{r_i}(r_i - 1) \\ &= |G|[2g(X/G) - 2 + \sum_{i=1}^{k}(1 - \frac{1}{r_i})]. \end{aligned} \tag{†}$$

A corollary to this version of the Riemann-Hurwitz theorem is a bound on the size of a finite group acting on a Riemann surface in terms of only the genus.

Theorem 2.5.5. *(Hurwitz' Theorem) Let G be a finite group acting holomorphically and effectively on a Riemann surface, X, of genus $g \geq 2$. Then*

$$|G| \leq 84(g - 1).$$

In fact, as can be seen in [9] or [24] for example, the automorphism group $\mathrm{Aut}(X)$ of any Riemann surface of genus at least 2 is a finite group and so also satisfies this bound, as the automorphism group acts holomorphically and effectively on X.

2.6 Ramified Covers, Monodromy, and Riemann's Existence Theorem

The results of the previous section demonstrate that, under certain reasonable assumptions, holomorphic maps between Riemann surfaces are well understood. In particular it is clear that if $f : X \to Y$ is a holomorphic map between compact Riemann surfaces, then the restriction of f to regular points is a local homeomorphism. By restricting the domain, the map f is guaranteed to be a covering. Such covering maps have a well-understood structure derived from the local normal form.

This section begins with a discussion of some results from covering space theory. Proofs and more general statements can be found in most texts on algebraic topology, such as those by Hatcher [19] or Massey [30]. Let X and Y be topological spaces and recall the following key definitions.

Definition 2.6.1.

- A map $f : X \to Y$ is said to be a *covering map* if around every point $y \in Y$ there is some open neighbourhood U such that $f^{-1}(U)$ is a disjoint union of open sets, each of which are mapped homeomorphically onto U by f. The pair (X, f) is called a *covering space*. Often the map is often omitted from the pair.

- Two covering maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are said to be *equivalent* or *isomorphic* if there is some homeomorphism $\gamma : X_1 \to X_2$, such that $\gamma \circ f_1 = f_2$. The group of automorphisms of a cover $f : X \to Y$ is usually called the *covering group of f* and is denoted $\mathrm{Aut}(X, f)$ or $\mathrm{cov}(X, f)$.

- The covering space (X, f) is said to be *universal* if the fundamental group of X is trivial. Such a space, if it exists, is unique up to equivalence, and has the following universal property: If (X', f') is also a covering space of X then there exists a covering $g : X \to X'$ such that $f = f' \circ g$.

- A map $f : X \to Y$ is said to be *proper* if the inverse image of any compact subset of Y is compact. Any proper local homeomorphism is a covering.

Recall that if $f : X \to Y$ is a covering and γ is a path in Y, then $\tilde{\gamma}$ is said to be a *lift* of γ if

$\gamma = f \circ \tilde{\gamma}$. An important property of covering spaces is that paths can always be lifted, this is known as the *path-lifting property.*

Proposition 2.6.2 (Path-lifting property)**.** *Let $\gamma : I \to Y$ be a path in the Riemann surface Y and $f : X \to Y$ is a covering. Suppose that $\gamma(0) = p_0 \in Y$ and that $\tilde{p_0} \in f^{-1}(p_0)$. Then there is a unique, up to homotopy, path $\tilde{\gamma}$ which lifts γ and has initial point $\tilde{\gamma}(0) = \tilde{p_0}$.*

The path-lifting property ensures that, if Y is path connected, then the fibres have the same cardinality. Indeed, let p_0 and p_1 be distinct points in Y. Let γ be a path in Y which joins p_0 to p_1. Then consider the map which takes $a \in f^{-1}(p_0)$ to the endpoint of the lift of γ to a path in X with initial point a. Clearly the end point is an element of the fibre $f^{-1}(p_1)$. By the path lifting property this map is a bijection.

If Y is a Riemann surface then X must also be a Riemann surface, inheriting a complex structure so that the covering map is holomorphic [15]. Therefore all results of the previous section concerning holomorphic functions are applicable.

Let G be a finite group, and let X be a topological surface such that G acts freely on X, i.e., without fixed points; and *properly discontinuously*, by which we shall mean that, for every $x \in X$, there exists an open neighbourhood U of x such that the set

$$\{g \in G \mid g(U) \cap U \neq \emptyset\}$$

is finite. The map $X \to Y := X/G$ is a covering map with covering group G [15].

The set of all covering spaces can be shown to correspond to the set of all conjugacy classes of subgroups of $\pi_1(Y)$. This correspondence is known as the *classification of covering spaces.*

Theorem 2.6.3 ([15])**.** *Let Y be a connected topological surface space. Then Y has a unique universal cover $\pi : \tilde{Y} \to Y$, and:*

- *The covering group $Aut(\tilde{Y}, y)$ is isomorphic to the fundamental group of the base space $\pi_1(Y)$.*

- *The covering group acts freely and properly discontinuously on $\tilde{Y}$. This action permutes elements of each fibre transitively. Such covers, which act transitively on the elements of each fibre, are said to be* normal *or* Galois.

- *The covering group action gives rise to a homeomorphism*

$$\tilde{Y}/Aut(\tilde{Y}, \pi) \to Y.$$

In addition any covering of Y is isomorphic to the covering induced by some subgroup G of $Aut(\tilde{Y}, \pi)$:

$$\tilde{Y}/G \to \tilde{Y}/Aut(\tilde{Y}, \pi) \cong Y.$$

- *These coverings are holomorphic maps and any other holomorphic map is isomorphic to one of these coverings.*

Theorem 2.6.3 provides us with a complete description of covers of Riemann surfaces, and unramified holomorphic maps between Riemann surfaces. If however we have an nonempty branch set, then it is not the case that the holomorphic map is a cover. The following result says that, if the branch points and ramification points are removed, then holomorphic maps are covering spaces.

Proposition 2.6.4. *Let $f : X \to Y$ be a nonconstant holomorphic map between connected compact Riemann surfaces. Let $B \subset Y$ be the set of branch points for f, and let $R \subset X$ be the set of ramified points for f. Define new surfaces $X' = X \setminus R$ and $Y' = Y \setminus B$. Then the map*

$$f' : X' \to Y'$$

is a covering.

Consider covers of the punctured disc. These determine the behaviour of holomorphic maps local to the ramified points. The result is unsurprising given the local normal form of holomorphic maps.

Proposition 2.6.5. *Let $\mathbb{D}^*$ denote the punctured unit disc, then the universal cover of $\mathbb{D}^*$ is*

$$\mathbb{H} \to \mathbb{D}^*$$

$$z \mapsto e^{2\pi i z}$$

Moreover, $\mathbb{D}^*$ has exactly one cover of each degree. The cover of degree n is given by

$$\mathbb{D}^* \mapsto \mathbb{D}^*$$
$$z \mapsto z^n.$$

This is the map corresponding to the action of the subgroup $\mathbb{Z}/n\mathbb{Z} \leq \mathbb{Z} \cong \pi_1(\mathbb{D}^*)$ in Theorem 2.6.3.

Example 2.6.6. *The map $f : \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C}$, defined by $z \mapsto z^d$ is a proper holomorphic map with branch point set $\{0, \infty\}$ and ramification point set $\{0, \infty\}$. The restriction $\mathbb{P}^1\mathbb{C} \setminus \{0, \infty\} \to \mathbb{P}^1\mathbb{C} \setminus \{0, \infty\}$ is a degree d covering.*

Theorem 2.6.3 and Proposition 2.6.4 show that, if f is a holomorphic map between Riemann surfaces then it is associated to a conjugacy class of subgroups in the fundamental group, and in fact the degree of f equals the index of its corresponding subgroup in $\pi_1(Y)$. However, this correspondence is unhelpful because, without returning to the proof of the covering space correspondence, determining which cover corresponds to which class of subgroups is not possible. The *monodromy representation* explicitly realises this association.

Let $f' : X' \to Y'$ be a degree d covering as outlined above. Pick a basepoint, y_0 in Y, for the fundamental group $\pi_1(Y', y_0)$. We outline a procedure which defines a representation $\rho : \pi_1(Y, y_0) \to S_d$ encoding the desired information about our covering space.

- Since f is a degree d map, there are exactly d points $\{x_1, \ldots, x_d\}$ which lie above y_0.
- Let γ be a loop in Y based at y_0.
- For each x_i, lift the path γ to a path $\tilde{\gamma_i}$ starting at x_i.
- Each $\tilde{\gamma_i}$ has an endpoint x_j lying above y_0.
- Thus to each γ there is an associated permutation σ_γ which takes i to j where x_j is the endpoint of $\tilde{\gamma_i}$.

The above gives a representation $\rho : \pi_1(Y, y_0) \to S_d$. This map is well defined and independent of the choice of representative γ [30]. The choice of basepoint will, on the other hand, give rise to a conjugate representation; as will a relabelling of the points of the fibre. This representation is called the *monodromy representation* and the image of such a map is called the *monodromy group* of the cover. A converse to this construction exists: from a monodromy representation and a Riemann surface, a covering space may be obtained. Variants of this result are often referred to as *Riemann's existence theorem.*

Theorem 2.6.7 (Riemann's existence theorem)**.** *Let Y be a connected Riemann surface and Δ a discrete subset of Y. Let ρ be a transitive permutation representation $\pi_1(Y \setminus \Delta) \to S_d$ for some $d \geq 1$. There is a unique connected Riemann surface X, and a unique proper holomorphic map $f : X \to Y$ such that ρ is the associated monodromy representation.*

Corollary 2.6.8. *Let Y be a compact Riemann surface, and let Δ be a discrete set of points in Y. The following are in one-to-one correspondence:*

- *Transitive permutation representations $\rho : \pi_1(Y \setminus \Delta) \to S_d$.*
- *Pairs (X, f) of a Riemann surface X, and a holomorphic map $f : X \to Y$ of degree d such that the branch points lie in Δ.*

The correspondence takes the representation ρ to a map whose corresponding monodromy representation is equivalent to ρ.

This correspondence plays an important role in the rest of this thesis and can be used to give a combinatorial description of the space of covers of the Riemann surface with a fixed genus and type. This is the subject of the next chapter.

CHAPTER 3

HURWITZ SPACES

Riemann's existence theorem, and more specifically Corollary 2.6.8, show that branched coverings of a compact Riemann surface may be parameterized as permutation representations. Thus, a geometric problem is transformed into an algebraic problem. In this chapter we consider the two related questions:

- *What can be said about the space of branched Galois covers of Riemann surfaces of genus g with monodromy group G?*
- *What can be said about the space of Riemann surfaces of genus g whose group of automorphisms contains a subgroup isomorphic to G?*

The spaces parameterising these collections are known as *Hurwitz spaces* and are the subject of this chapter. Immediately one must ask whether such collections are indeed equivalent; this is shown in Section 3.2. Riemann's existence theorem can be used to give a combinatorial property which may be used to determine the connectivity of the Hurwitz spaces.

The focus of this chapter is an exposition of the genus zero Hurwitz space following the book by Völklein [37] and the survey by Wewers and Romagny [32]. In the process, important subtopics, such as the mapping class groups, braid groups, and the braiding action are defined and discussed. The final section of the chapter is a less complete discussion of the Hurwitz spaces for higher genus. The higher genus Hurwitz spaces have fewer applications, and the key results of the thesis concern only the genus zero Hurwitz space. Still, much of the theory remains true and the computational techniques presented in Chapter 6 were designed to cope with calculations in these spaces.

3.1 The Hurwitz Space

Definition 3.1.1. Let $\mathcal{H}^{\text{in}}_{r,G}$ denote the set of all inner equivalence classes of Galois covers of $\mathbb{P}^1\mathbb{C}$ branched over r points, with monodromy group isomorphic to G. This space is called the *(inner) Hurwitz space.*

Generally when people talk of Hurwitz spaces they are usually referring to the space of Galois covers of the Riemann sphere, as above, and do not consider, as we shall later, the space of covers of an arbitrary genus Riemann surface.

By Corollary 2.6.8 elements of $\mathcal{H}^{\text{in}}_{r,G}$ correspond to equivalence classes of pairs (Δ, ρ) where Δ is a set of r points in $\mathbb{P}^1\mathbb{C}$ and ρ is a monodromy monomorphism into the finite group G:

$$\rho : \pi_1(\mathbb{P}^1\mathbb{C} \setminus \Delta) \to G.$$

Pairs, (Δ, ρ) and (Δ', ρ') are equivalent if $\Delta = \Delta'$ and $\rho' = \theta \circ \rho$ for some inner-automorphism θ. We use $[\Delta, \rho]$ to denote the equivalence class containing the pair (Δ, ρ). This equivalence is a sensible one because two pairs are equivalent if the monodromy maps differ by a change of basepoint or a change of the labeling of the branch points.

Since a monodromy homomorphism is required to be surjective then it is completely determined by its action on the standard generators of the fundamental group, $\pi_1(\mathbb{P}^1\mathbb{C} \setminus \Delta)$. This group, we recall, has presentation

$$\langle \gamma_1, \ldots, \gamma_r \mid \gamma_1 \ldots \gamma_r \rangle,$$

and thus a monodromy homomorphism ρ is equivalent to a tuple $\boldsymbol{g} = (g_1, \ldots, g_r) \in G^r$ such that:

$$g_1 \ldots g_r = 1; \text{ and} \tag{3.1}$$

$$\langle g_1, \ldots, g_r \rangle = G. \tag{3.2}$$

Moreover, since each of the r points of Δ should be a proper branch point, we insist that $g_i \neq 1$. Such tuples are called *Nielsen tuples.* If $\boldsymbol{g}$ and $\boldsymbol{g}' = (g'_1, \ldots, g'_r)$ correspond to equivalent pairs then there exists some $h \in G$ such that $g_i^h = g'_i$ for $1 \leq i \leq r$.

Definition 3.1.2. Denote by $\varepsilon_r(G)$ the set

$$\{(g_1, \dots, g_r) \mid g_i \neq 1,\ \langle g_1, \dots, g_r \rangle = G\}$$

of generating r-tuples with product one. The *(inner) Nielsen class* is the set of equivalence classes of all such Nielsen tuples:

$$\mathrm{Ni}^{\mathrm{in}}(r, G) = \varepsilon_r(G)/\operatorname{Inn}(G).$$

There is an obvious correspondence between elements of the Nielsen class, $\mathrm{Ni}^{\mathrm{in}}(r, G)$, and elements of the Hurwitz space.

Fix a set of r branch points $\Delta = \{\delta_1, \dots, \delta_r\} \subset \mathbb{P}^1\mathbb{C} \setminus \infty$. The point ∞ will play the role of the basepoint of $\mathbb{P}^1\mathbb{C}$. Let C_i be the ramification type of the branch point δ_i. Recall that this means that C_i is the conjugacy class of the distinguished generator for the stabiliser at this point.

Definition 3.1.3. If $\varphi : X \to \mathbb{P}^1\mathbb{C}$ is a cover branched over $\Delta = \{\delta_1, \dots, \delta_r\} \subset \mathbb{P}^1\mathbb{C}$ and C_i is the ramification type of δ_i then the tuple

$$\boldsymbol{C} = (C_1, \dots, C_r)$$

is called the *ramification type* (or simply *type*) of φ.

Note that some authors define the ramification type to be the tuple of orders of elements of the conjugacy class and not the conjugacy classes themselves. In this thesis the set of orders is called the *signature*.

Instead of considering all branched covers of the Riemann sphere we may want to restrict ourselves to branched covers of a given type.

Definition 3.1.4. For a finite group G, and type $\boldsymbol{C} = (C_1, \dots, C_r)$ we define $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ to be the set of all branched covers of $\mathbb{P}^1\mathbb{C}$ with ramification type $\boldsymbol{C}$.

For a fixed ramification type $\boldsymbol{C}$ and group G, the genus of a covering surface is enforced by the Riemann-Hurwitz formula. Thus all covering surfaces in $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ have the same genus. The Nielsen class for tuples of a given type is defined analogously.

Definition 3.1.5. Let $\boldsymbol{C} = (C_1, \ldots, C_r)$ be a ramification type. Define the *Nielsen class of type* $\boldsymbol{C}$ to be the set

$$\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C}) = \{(g_1, \ldots, g_r) \in \mathrm{Ni}^{\mathrm{in}}(r, G) \mid \text{ for some } \sigma \in S_r, \ \forall \ g_i \in C_{\sigma(i)}\}$$

This definition allows Nielsen tuples in which the ith component does not lie in the *ith* conjugacy class. However, it is sometimes advantageous to restrict ourselves to such tuples. The set of all such tuples:

$$\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C}) = \{(g_1, \ldots, g_r) \in \mathrm{Ni}^{\mathrm{in}}(r, G) \mid g_i \in C_i\},$$

is called the *pure Nielsen class of type* $\boldsymbol{C}$, and is denoted by $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$.

Following [37] we argue that topological properties of $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ are determined by computable properties of $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$. At this moment it makes no sense to talk of "topological properties" for the set $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$. Rectifying this we define a basis for a topology on $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$.

Choose a point $[\Delta = \{\delta_1, \ldots, \delta_r\}, \rho] \in \mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$. Around each branch point δ_i choose an open neighbourhood $U_i \subset \mathbb{P}^1\mathbb{C}$ of δ_i such that U_i lies within the interior of the standard generator $\gamma_i \in \pi_1(\mathbb{P}^1\mathbb{C} \setminus \Delta, \infty)$. Let $\mathcal{U}$ be the product of the open neighbourhoods $U_1 \times \cdots \times U_r$. For any $\Delta' = (\delta'_1, \ldots, \delta'_r) \in \mathcal{U}$ the loop γ_i is homotopic to a small loop around δ'_i. Thus, for each $\mathcal{U}$, define $\mathcal{N}_\rho(\mathcal{U})$ to be the set of pairs $[\Delta', \rho']$ where $\Delta' \in \mathcal{U}$ and ρ' is equal to the composition of ρ with the isomorphism

$$\pi_1(\mathbb{P}^1\mathbb{C} \setminus \Delta', \infty) \to \pi_1(\mathbb{P}^1\mathbb{C} \setminus \Delta, \infty).$$

Equip $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ with the topology with basis consisting of the sets $\mathcal{N}_\rho(\mathcal{U})$ as $\mathcal{U}$ and ρ range over all possibilities. This topology is well defined [12, 37]. Let $\mathcal{O}_r$ be the set of all r-tuples of distinct elements in $\mathbb{P}^1\mathbb{C}$, equipped with the product topology. Let $\Psi_r : \mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C}) \to \mathcal{O}_r$ be the projection:

$$[\Delta, \rho] \mapsto \Delta.$$

This map is in fact a cover.

Proposition 3.1.6 ([37])**.** *The projection $\Psi_r : \mathcal{H}^{in}(G, \boldsymbol{C}) \to \mathcal{O}_r$ is a topological covering map.*

It is natural to ask what the degree of the covering Ψ_r is. This is, by definition, the number of equivalence classes of monodromy representation ρ. This set is finite and equal to the size of the Nielsen class $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$.

Proposition 3.1.7. *The degree of* Ψ_r *is equal to the size of the Nielsen class* $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$.

Since $\mathcal{O}_r$ is a complex manifold of dimension r and the Hurwitz space $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ is a covering space of $\mathcal{O}_r$ by Proposition 3.1.6, $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ is also a complex manifold of dimension r.

Lemma 3.1.8. *Let* $\Delta \in \mathcal{O}_r$ *and let* $\gamma_1, \ldots, \gamma_r$ *be the standard generators for the fundamental group* $\pi_1(\mathbb{P}^1\mathbb{C} \setminus \Delta)$. *Then the map* $\Psi_r^{-1}(\Delta) \to \mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ *given by*

$$[\Delta, \rho] \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_r))$$

is a bijection.

Proposition 3.1.9. *Let* X *and* Y *be topological spaces,* y_0 *a point in* Y *and* $f : X \to Y$ *a covering. Then the components of* X *are in one-to-one correspondence with the orbits of the monodromy action of the fundamental group* $\pi_1(Y, y_0)$ *on the preimage* $f^{-1}(y_0)$.

Thus we can ask whether or not $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ is connected by computing the orbits of the fundamental group on fibres. This is the question we try to resolve for small alternating groups in Chapter 4 and Chapter 5. The fundamental group of $\mathcal{O}_r$ is well understood and is often called the braid group on r strands. In the next subsection the braid group is discussed in more detail.

3.1.1 The Braid Group

Definition 3.1.10. The *configuration space of* r *points in* $\mathbb{C}$, denoted $\mathcal{C}(\mathbb{C}, r)$, is defined to be the set of cardinality r subsets of $\mathbb{C}$, i.e.,

$$\mathcal{C}(\mathbb{C}, r) = \mathcal{F}(\mathbb{C}, r)/S_r,$$

where

$$\mathcal{F}(\mathbb{C}, r) = \{(c_1, \ldots, c_r) \in \mathbb{C}^r \mid c_i \neq c_j \text{ for } i \neq j\}$$

and S_r is the symmetric group acting by permuting the entries of the tuple.

Definition 3.1.11. The *braid group on* r *strands* denoted $\mathcal{B}_r$ is the fundamental group $\pi_1(\mathcal{C}(\mathbb{C}, r), \boldsymbol{c}_0)$, where $\boldsymbol{c}_0 \in \mathcal{C}(\mathbb{C}, r)$.

Choose $\boldsymbol{c}_0 = (\infty, \ldots, \infty)$ as our basepoint and note the bijection between $\mathcal{O}_r$ and $\mathcal{C}(\mathbb{C}, r)$. Then the following theorem holds.

Proposition 3.1.12.

$$\pi_1(\mathcal{O}_r) \cong \pi_1(\mathcal{C}(\mathbb{C}, r), \boldsymbol{c}_0) = \mathcal{B}_r$$

A presentation for the braid group is well known.

Proposition 3.1.13 ([2])**.** *The braid group on r strands has a presentation with generators $Q_1, \ldots, Q_r$ and relations*

$$Q_iQ_j = Q_jQ_i \quad \textit{where } |i-j| > 1$$
$$Q_iQ_{i+1}Q_i = Q_{i+1}Q_iQ_i \quad \textit{for } i \neq j$$

Whilst elements of the braid groups are paths in $\mathcal{C}(\mathbb{C}, r)$ these paths are viewed as r distinct strands joining two tuples of r elements in $\mathbb{C}$. Such a set of strands can be represented by diagrams as seen in Figure 3.5. Note that the generator Q_i corresponds to the crossing of strand i over strand $i+1$. With this in mind the generation of $\mathcal{B}_r$ by the Q_i is obvious.

If G is a finite group then the braid group acts on tuples $\boldsymbol{g} \in G^r$. The action of the generator Q_i is given by

$$(g_1, \ldots, g_r) \mapsto (g_1, \ldots, g_ig_{i+1}g_i^{-1}, g_i, g_{i+2}, \ldots, g_r). \tag{3.3}$$

And in fact this action also restricts to an action on the set of Nielsen tuples. Indeed, if $g_1 \cdots g_r = 1$ then

$$g_1 \cdots g_ig_{i+1}g_i^{-1}g_ig_{i+2} \cdots g_r = 1;$$

and if $\langle g_1, \ldots, g_r \rangle = G$ then

$$\langle g_1, \ldots, g_ig_{i+1}g_i^{-1}, g_i, g_{i+2}, \ldots, g_r \rangle = G.$$

One important subgroup of the braid group is the group of braids which do not permute the components of the endpoints. Label the r coordinates of the basepoint $\boldsymbol{c}_0$ with the integers

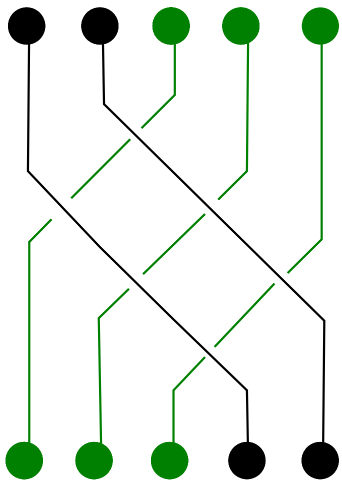


Figure 3.1: The shown braid Q has associated permutation $\sigma(Q) = (1, 4, 2, 5, 3)$.

$1, \dots, r$. There is a natural homomorphism

$$\sigma : \mathcal{B}_r \to S_r,$$

where $\sigma(Q)$ is the permutation of the coordinates of the basepoint induced by Q. For example consider the braid in Figure 3.1, then the associated permutation σ is $(1, 2)(2, 5)(3, 1)(4, 2)(5, 3) = (1, 4, 2, 5, 3)$.

Definition 3.1.14. The kernel of the homomorphism $\sigma : \mathcal{B}_r \to S_r$ is known as the *pure braid group on r strands.*

As with the braid group the pure braid group is generated by

$$P_{i,j} = Q_i \cdots Q_{j-2} Q_{j-1}^2 Q_{j-2} \cdots Q_i, \quad 0 < i < j \leq r.$$

and the generator $P_{i,j}$ acts on tuples by

$$P_{i,j}(\boldsymbol{g}) = (g_1, \dots, g_{i-1}, g_i^{(g_i \cdots g_j)^{-1}}, g_{i+1}^{g_i^{-1}}, \dots, g_j^{g_i^{-1}}, g_{j+1}, \dots, g_r).$$

Observe that pure braids act by conjugation. This property of pure braids is exploited in later chapters.

There is a geometric interpretation of the braid group as the *mapping class group of the punctured disc.*

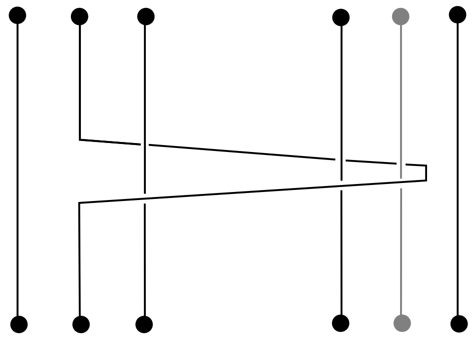


Figure 3.2: The pure braid generator $P_{2,j}$ passing the second strand under the grey colored jth strand and all strands in between.

3.1.2 Mapping Class Groups

Definition 3.1.15. Suppose that S is a surface with (possibly empty) boundary ∂S. Denote by $\mathrm{Homeo}^+(S, \partial S)$ the group of orientation-preserving homeomorphisms of the surface S which restrict to the identity on ∂S. Let $\mathrm{Homeo}_0(S, \partial S)$ denote the subgroup of $\mathrm{Homeo}^+(S, \partial S)$ which consists of those elements isotopic to the identity.

Proposition 3.1.16. $\mathrm{Homeo}_0(S, \partial S)$ *is a normal subgroup of* $\mathrm{Homeo}^+(S, \partial S)$.

Definition 3.1.17. The *mapping class group* of a surface, S, written $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation preserving homeomorphisms of S that are the identity on the boundary ∂S, that is

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S).$$

If S is a punctured surface then we view the punctures as marked points. Then homeomorphisms, up to isotopy, may either permute these marked points, as the above definition allows, or fix them pointwise. The group of isotopy classes of orientation preserving homeomorphisms of S which fix the puncture set pointwise is known as the *pure mapping class group of* S and is denoted $\mathrm{PMod}(S)$.

The mapping class groups of a surface depends entirely on the topological type of the surface. If S is an r punctured, genus g compact Riemann surface then we write $\mathrm{Mod}_{g,r} = \mathrm{Mod}(S)$. The pure mapping class groups is denoted $\mathrm{PMod}_{g,r}$.

The braid group is a mapping class group. In particular, the braid group on r strands is isomorphic to the mapping class group of the r punctured disc. In order to show this we first must attempt to understand the homeomorphisms that are possible on the punctured disc. In fact it can be shown that all such homeomorphisms correspond to the permuting of punctures [9].

Definition 3.1.18. Let D be a subsurface of a surface S that is homeomorphic to an open disc containing exactly two punctures. Let a be a simple arc joining the two punctures. Consider the twice-punctured plane:

$$X = \{(r, \theta)\colon 0 \leq 2, 0 \leq \theta \leq 2\pi\} \setminus \{(1, 0), (1, \pi)\}$$

and the homeomorphism $f : X \to X$ given by

$$f(r, \theta) = \begin{cases} (r, \theta) & \text{if } r < 3/4 \text{ or } r > 5/4 \\ (r, \theta + 4\pi(r - 3/4)) & \text{if } 3/4 \leq r \leq 5/4 \end{cases}$$

Let $\psi : D \to X$ be an orientation preserving homeomorphism taking punctures to punctures. The *half twist* about a is the homeomorphism H_a of D defined by

$$H_a(x) = \begin{cases} x & \text{if } x \in S \setminus D \\ \psi^{-1} \circ f \circ \psi(x) & \text{if } x \in D \end{cases}$$

Sometimes half twists are called *braid twists* due to their relationship with the braid group. The half twists generate the mapping class group of the punctured disc. This result is part of a larger description of the mapping class groups for compact connected surfaces.

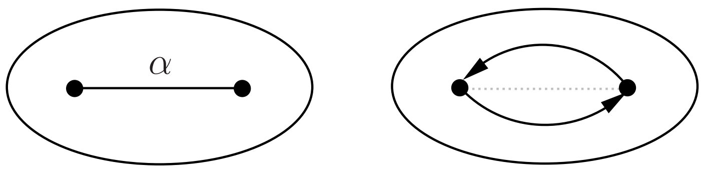


Figure 3.3: A half twist.

The half twist is primarily described by its action on a curve which lies between the two punctures, as in Figure 3.4.

Proposition 3.1.19. *Let D_r denote the r-punctured disc. Label the punctures in order with $p_1, \dots p_r$. For $1 \leq i \leq r-1$ let h_i denote the half twist permuting the two adjacent punctures p_i*

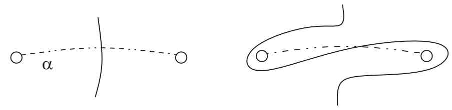


Figure 3.4: The effect of the half twist H_α on an intersecting curve.

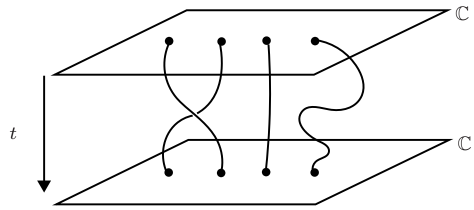


Figure 3.5: A braid as a path.

and p_{i+1}. *Then the* q_i *generate* $Mod(D_r, \partial D_n)$.

Note that it appears as though a consequence of the above proposition is that the unpunctured disc has trivial mapping class group. This result, known as *Alexander's Theorem* or *Alexander's Trick* is actually the basis for the proof of the above (see [9]).

Consider a disc D which has been punctured r times. The punctures may be viewed as marked points and can be labelled $p_1, \ldots, p_n$. Identify this disc with an open subset of $\mathbb{C}$. Let ϕ be an arbitrary homeomorphism of D. We sketch the correspondence between ϕ and an element of the braid group $\mathcal{B}_r$.

Imagine filling the punctures of the disc. By Alexander's theorem, this unpunctured disc is now homeomorphic to the open disc and hence the homeomorphism ϕ is isotopic to the trivial map. The restriction of this isotopy to the set of marked points $p_1, \ldots, p_n$ corresponds to a path in the configuration space. This is the path traced by the marked points as the isotopy is performed. By Proposition 3.1.12 this path can be viewed as a braid. This association is in fact an isomorphism [9]. The half-twist h_i corresponds to the braid generator Q_i.

Theorem 3.1.20 ([2]). *Let* D_r *be the closed disc with* r *punctures. Then* $\mathrm{Mod}(D_r, \partial D_r)$ *is isomorphic to the braid group on* r *strands.*

We now return to our discussion of the Hurwitz spaces. Our next step is to notice that the braid action on Nielsen tuples actually corresponds to the monodromy action of the fundamental group.

Proposition 3.1.21 ([37]). *The action of the braid group on Nielsen tuples shown in* (3.3) *corresponds to the monodromy action of the fundamental group* $\pi_1(\mathbb{P}^1\mathbb{C} \setminus \Delta, p)$ *on the preimage* $\Psi^{-1}(p)$ *via the bijection in Proposition 3.1.9.*

Recall Proposition 3.1.9 which states that if $F : R \to S$ is a covering, and $s \in S$ then there is a one-to-one correspondence between connected components of S and the orbits of $f^{-1}(s)$ under the monodromy action. Therefore the correspondence from Proposition 3.1.9 yields the following theorem.

Theorem 3.1.22. *There is a one-to-one correspondence between components of the Hurwitz space* $\mathcal{H}^{in}(G, \boldsymbol{C})$ *and orbits of* $\mathcal{B}_r$ *on the Nielsen class* $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$.

Orbits of the braid group $\mathcal{B}_r$ on the Nielsen class $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ are called *braid orbits*. The orbits of the *pure* braid group on the Nielsen class $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ are called *pure braid orbits*. The above result is the basis of the rest of the work presented in this thesis. Using this result we determine the number of components of the Hurwitz spaces $\mathcal{H}^{\mathrm{in}}(A_5, \boldsymbol{C})$ and $\mathcal{H}^{\mathrm{in}}(A_6, \boldsymbol{C})$ for an arbitrary type $\boldsymbol{C}$.

3.2 Hurwitz Spaces for Positive Genus

It is fortunate that up to isomorphism there is a single compact Riemann surface of genus 0. The approach taken in the previous section was simplified because of this fact. Unfortunately, for any given positive integer g there are many Riemann surfaces of this genus. Therefore, should we wish to study branched covers of Riemann surfaces of genus g then we must consider the space of all such surfaces. This parameter space is called the *moduli space of Riemann surfaces of genus* g.

Definition 3.2.1. For g a non-negative integer, write $\mathcal{M}_g$ to denote the space of isomorphism classes of Riemann surfaces of genus $g \geq 0$. We call this set the *moduli space of Riemann surfaces of genus* g.

We may also consider the analogous space of Riemann surfaces of genus g and with r punctures. This space is denoted by $\mathcal{M}_{g,r}$.

The moduli space of genus zero Riemann surfaces, $\mathcal{M}_0$, consists of just a single point: the Riemann sphere.

We now consider pairs of Riemann surfaces, and subgroups of their automorphism groups.

Definition 3.2.2. Let X_1 and X_2 be compact, connected Riemann surfaces of genus at least 2, and let G be a finite group. Suppose further that there exist subgroups H_1 and H_2 of $\mathrm{Aut}(X_1)$ and $\mathrm{Aut}(X_2)$ respectively, and isomorphisms $\theta_i : G \to H_i$ for $i = 1, 2$. The pairs (X_1, H_1) and (X_2, H_2) are equivalent if there is a holomorphic homeomorphism $\varphi : X_1 \to X_2$ such that for every $g \in G$ the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ {\scriptstyle \theta_1(g)}\downarrow & & \downarrow{\scriptstyle \theta_2(g)} \\ X_1 & \xrightarrow[\varphi]{} & X_2 \end{array}$$

Write $[X, G]$ for the equivalence class containing the pair (X, G) and call the set,

$$\mathcal{H}^{\text{in}}_{g,G} = \{[X, H] \mid X \text{ has genus } g \text{ and } H \text{ is isomorphic to } G\},$$

of equivalence classes of genus g pairs, the *Hurwitz space* for g and G.

If two pairs, (X_1, H_1) and (X_2, H_2), are equivalent if and only if there exists a biholomorphism

$$\varphi' : X_1/H_1 \to X_2/H_2$$

between the two quotient spaces (see [16]), which we recall are Riemann surfaces themselves. Thus a viable alternative to considering surface-group pairs is to consider the space of quotient surfaces. If the genus g is fixed and the ramification data is fixed, then the genus g_0 of the quotient surface is also fixed. As before, observe that Corollary 2.6.8 provides a parameterisation of the set of all such Riemann surfaces. Recall the following result from the theory of covering spaces.

Proposition 3.2.3 ([19])**.** *Let X be a topological space and G a group acting on X such that for each $x \in X$ there is an open neighbourhood, U, such that all the G-translates of U are disjoint. Then:*

1. *The quotient map $p : X \to X/G = Y$ is a Galois covering space.*

2. *G is the covering group of deck transformations provided that X is path-connected. (Note that the monodromy group and the covering group coincide when the covering map is*

Galois.)

3. *If X is path-connected and locally path-connected then G is isomorphic to $\pi_1(X/G)/p_*(\pi_1(X))$, where p_* is the map $\pi_1(X) \to \pi_1(X/G)$ induced by the cover.*

The second of the above items implies that $\mathrm{Aut}(X)$ has a subgroup which is isomorphic to G. Recall from Chapter 2 that, if X is a Riemann surface and $G \leq \mathrm{Aut}(X)$, then the map

$$\pi : X \to X/G$$

is a holomorphic map of degree $|G|$ and for $p \in X$ then $\mathrm{mult}_p(\pi) = |G_p|$. In particular if π has ramified points then the action of G is not free and so the hypotheses of Proposition 3.2.3 are not satisfied. As in Chapter 2 the branch points and the ramification points may be removed, in which case the hypotheses of Proposition 3.2.3 are satisfied. Therefore if Δ is the set of branch points for π and R is the set of ramification points for π, and we let $X' = X - R$ and $Y' = X/G - \Delta$, then the map

$$\hat{\pi} : X' \to Y'$$

is a normal covering of degree $|G|$ and with monodromy group and deck transformation group isomorphic to G.

Definition 3.2.4. The pair $[X, G]$ is said to be of *(ramification) type* $(g_0; C_1, \cdots, C_r)$ if

- The quotient space $Y = X/G$ has genus g_0.
- The quotient cover $X' \to Y'$ induces a monodromy epimorphism $\rho : \pi_1(Y') \to G$.
- The branch points can be labelled $\{p_1, \cdots, p_r\}$ such that p_i has ramification type C_i, and $C_i \neq 1$. Under these conditions the monodromy epimorphism is said to be *admissible*.

That the branch point p_i has ramification type C_i is equivalent to saying that if γ_{p_i} is the standard generator winding around p_i then C_i is the conjugacy class in G of the image of γ_{p_i} under the monodromy map. The condition that the maps be admissible ensures that there is a proper ramification over the points $p_1, \ldots, p_r$. The Riemann-Hurwitz formula says that surface-group pairs, $[X, G]$, of ramification type $(g_0; C_1, \cdots, C_r)$, and with X of genus g, must satisfy

$$2g - 2 = |G|[2g_0 - 2 + \sum_i (1 - 1/c_i)] \tag{†}$$

where c_i is the common order of the elements in C_i. In particular this means that if we fix g_0, G, and $C_1, \ldots, C_r$ then the orbit genus g is also determined. The redundancy of the parameter g given the list of conjugacy classes allows us to omit the genus g from the ramification type.

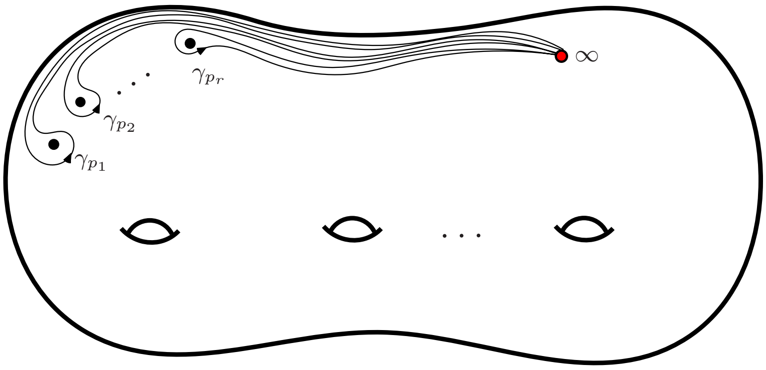


Figure 3.6: The loops γ_{p_i} around the points $p_1, \ldots, p_r \in P$

With the above definition of ramification type we define the Hurwitz space of surfaces with a given type.

Definition 3.2.5. Let $r \geq 1, g_0 \geq 0$ be integers, and G a finite group. Let $\boldsymbol{C} = (C_1, \ldots, C_r)$ be a collection of conjugacy classes of G. Let $\mathcal{H}^{\text{in}}(g, g_0, G, \boldsymbol{C}) = \mathcal{H}^{\text{in}}(g_0, G, \boldsymbol{C})$ denote the set,

$$\{[X, G] \in \mathcal{H}^{\text{in}}_{g,G} \mid [X, G] \text{ is of ramification type } (g_0; \boldsymbol{C})\},$$

of surface-group pairs of ramification type, $(g_0; \boldsymbol{C})$.

The space $\mathcal{H}^{\text{in}}(g, g_0, G, \boldsymbol{C})$ embeds into the Hurwitz space $\mathcal{H}^{\text{in}}_{g,G}$. By Riemann's existence theorem, the set $\mathcal{H}^{\text{in}}(g, g_0, \boldsymbol{C})$ is in bijective correspondence with the set of equivalence classes of triples of the form

$$[\Delta, \rho, Y]$$

where Δ is a set of points in Y (over which our quotient projection is ramified), ρ is the monodromy map, and Y is an isomorphism class of Riemann surfaces of genus g_0. The monodromy map $\rho : \pi_1(Y \setminus \Delta) \to G$ is a surjection and so is determined by the images of the standard generators for $\pi_1(Y \setminus \Delta)$. Recall that the fundamental group of an R-punctured, genus g_0 Riemann surface is generated by loops $\alpha_1, \beta_1, \ldots, \alpha_{g-0}, \beta_{g_0}, \gamma_1, \ldots, \gamma_r$ subject to the relation

$$\prod_{i=1}^{g_0} [\alpha_i, \beta_i] \prod_{i=1}^{r} \gamma_i = 1, \tag{3.4}$$

where $[\alpha_i, \beta_i] = \alpha_i^{-1} \beta_i^{-1} \alpha_i \beta_i$ is the commutator of α_i and β_i. Therefore, the set of all monodromy homomorphisms is in bijective correspondence with the set of all length $2g_0 + r$ generating G-tuples satisfying (3.4).

Definition 3.2.6. Let G be a finite group, $\boldsymbol{C}$ a ramification type, and g_0, r nonnegative integers. Let $\mathcal{E}(g_0, G, \boldsymbol{C})$ denote the set of tuples

$$\boldsymbol{g} = (a_1, b_1, \ldots, a_{g_0}, b_{g_0}, c_1, \ldots, c_r) \in G^{2g_0 + r}$$

such that $c_i \in C_{\sigma(i)}$for some permutation $\sigma \in S_r$ and for all i; $\boldsymbol{g}$ satisfies (3.4); and the elements of $\boldsymbol{g}$ generate G.

Let $\mathrm{Ni}^{\mathrm{in}}(g_0, G, \boldsymbol{C})$ denote the quotient

$$\mathcal{E}(g_0, G, \boldsymbol{C}) / \operatorname{Inn}(G).$$

The set $\mathrm{Ni}^{\mathrm{in}}(g_0, G, \boldsymbol{C})$ is called the *Nielsen class of type $\boldsymbol{C}$ and genus g_0*, its elements are called *Nielsen tuples.* Note that $\mathrm{Ni}^{\mathrm{in}}(0, G, \boldsymbol{C}) = \mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ from earlier in the chapter.

With the above definitions, the two sets $\mathcal{H}^{\mathrm{in}}(g_0, G, \boldsymbol{C})$ and $\mathrm{Ni}^{\mathrm{in}}(g_0, G, \boldsymbol{C})$ are in one-to-one correspondence via the map

$$[\Delta, \rho, Y] \mapsto (\rho(\alpha_1), \rho(\beta_1), \ldots, \rho(\alpha_{g_0}), \rho(\beta_{g_0}), \rho(\gamma_1), \ldots, \rho(\gamma_r)) \tag{3.5}$$

where $\alpha_i, \beta_i, \gamma_i$ are the standard generators for the fundamental group of $\pi_1(Y \setminus \Delta)$.

In developing the correspondence between components of $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ and orbits of the braid group on tuples the key observation was that the map

$$[\Delta, \rho] \mapsto \Delta$$

is a covering space. This implied that connected components corresponded to orbits of fibres under the action of the fundamental group of base space $\mathcal{O}_r$. An identical argument will not work in the general genus case. Consider the projection

$$\Psi : [\Delta, \rho, Y] \mapsto Y \setminus \Delta.$$

This map Ψ is a cover, but the base space $\mathcal{M}_{g_0,r}$ is simply connected and so has trivial fundamental group [9]. However, the *orbifold fundamental group* of $\mathcal{M}_{g_0,r}$ is isomorphic to $\mathrm{Mod}(Y \setminus \Delta)$.

Fix an r-punctured Riemann surface Y with puncture set Δ. Then a representative homomorphism φ from a mapping class $[\varphi] \in \mathrm{Mod}(Y \setminus \Delta)$ acts on points $[\Delta, \rho, Y]$ of the fibre by

$$[\Delta, \rho, Y] \mapsto [\Delta, \rho \circ \varphi_*, Y],$$

where φ_* is the map $\pi_1(Y \setminus \Delta) \to \pi_1(Y \setminus \Delta)$ induced by φ. Triples which lie in the same orbit under this action differ only up to the choice of generators for the fundamental groups. Thus the monodromy maps are equivalent. By using the orbifold fundamental group, one is able to show that this action classifies the components of $\mathcal{H}^{\mathrm{in}}(g_0, G, \boldsymbol{C})$ [4]. In particular we have the following correspondence.

Theorem 3.2.7 ([4]). *The following sets are in one-to-one correspondence:*

- *Connected components of the Hurwitz space* $\mathcal{H}^{in}(g_0, G, \boldsymbol{C}, r)$.
- *Elements of* $(\mathrm{Ni}^{\mathrm{in}}(g_0, G, \boldsymbol{C})/Mod_{g_0,r})/\,\mathrm{Inn}(G)$

The orbits of $\mathrm{Ni}^{\mathrm{in}}(g_0, G, \boldsymbol{C}, r)$ *under the the action of* $Mod_{g_0,r}$ *are called* mapping class orbits. *Note that the mapping class group action and the action of conjugation commute.*

Theorem 3.2.7 says that the components of the Hurwitz space are determined by the action of the mapping class group on generators for the fundamental group. In the next section we look at a generating set for the mapping class group and determine how these generators act on the standard generators for the fundamental group.

3.2.1 Generating the Mapping Class Group

To begin, consider a set of simple homeomorphisms of our surface.

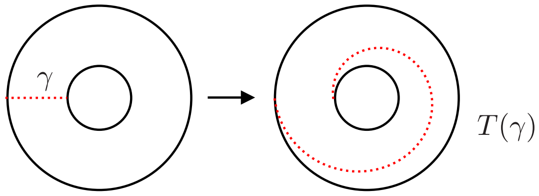


Figure 3.7: A twisted annulus as in the definition of a Dehn twist.

Definition 3.2.8. Let $A = \{(r,\theta) \in \mathbb{R}^2 : 1 \leq r \leq 2\}$ be an annulus in the plane and let $T : A \to A$, as in Figure 3.7, be given by

$$T(r,\theta) = (r, \theta + 2\pi r).$$

Let S be a surface and let α be a simple closed curve in S. Pick a regular neighbourhood N of α and let $\psi : A \to S$ be an orientation preserving map whose image is N. Then the *Dehn twist about* α, which will be denoted by T_α, is the homeomorphism $T_\alpha : S \to S$ given by

$$T_\alpha(x) = \begin{cases} x & \text{if } x \in S \backslash N \\ \psi \circ T \circ \psi^{-1}(x) & \text{if } x \in N \end{cases}$$

As indicated in Figure 3.7, a Dehn twist can be seen as the process of cutting along a curve and then twisting one component by 2π and 'gluing' the two ends back together. It is clear that a Dehn twist is a self-homeomorphism of the surface, and the isotopy class of T_α does not depend on the choice of N nor on the homeomorphism ψ. Therefore, if a is the isotopy class of α, then T_a is a well defined element of $\text{Mod}(S)$.

One way to investigate T_a is to consider its behaviour with respect to curves. Clearly if b is an isotopy class of curves and $i(a,b) = 0$, then $T_a(b) = b$. If however, $i(a,b) \neq 0$, then T_a twists b. This twisting of curves is illustrated in Figure 3.8.

Proposition 3.2.9 ([9])**.** *If a is the isotopy class of an essential curve in S then T_a is a nontrivial mapping class.*

Consider the set of $3g - 1$ curves in Figure 3.9. Dehn twists about these curves are known as the *Lickorish generators* or *Lickorish twists.* These twists generate $\text{Mod}(S)$.

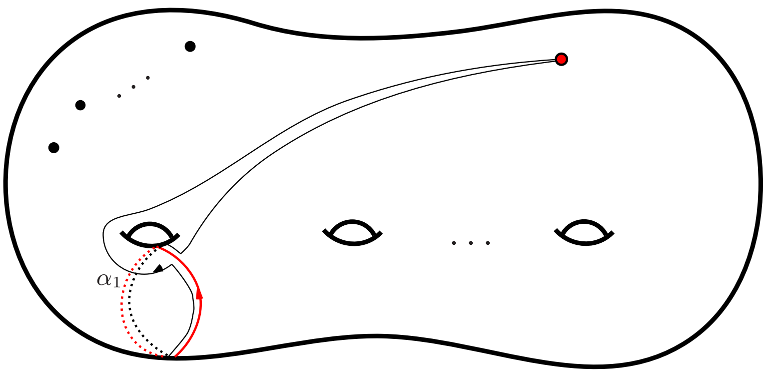


Figure 3.8: A Dehn twist takes a curve, it turns right, and goes all the way around before turning continuing along its previous path.

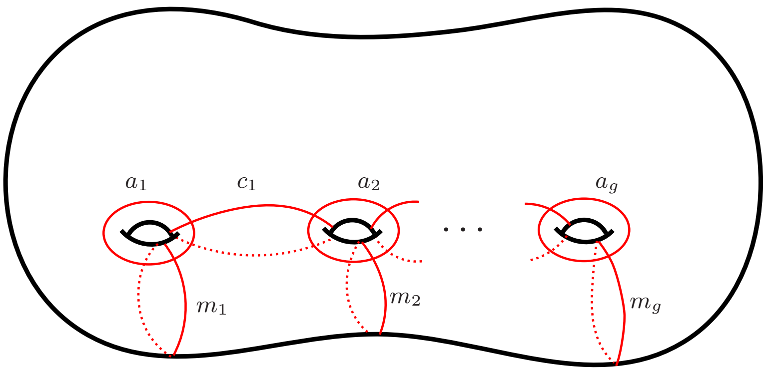


Figure 3.9: The Lickorish twists.

Theorem 3.2.10 ([9]). *Let $S = S_g$ be the closed surface of genus $g \geq 1$. The Dehn twists about the isotopy classes*

$$a_1, \ldots, a_g, m_1 \ldots, m_g, c_1, \ldots, c_{g-1}$$

shown in Figure 3.9, generate Mod(S).

Surprisingly it can be shown that we only need twists about two of the curves around handles to generate the mapping class group. These twists are known as the *Humphries generators* and are shown in Figure 3.10. It can be shown, [9, Proposition 7.4], that we need at least $2g + 1$ twists to generate Mod(S) and so, as a subset of the Lickorish twists, the Humphries generators can be considered best possible.

Theorem 3.2.11 ([21]). *Suppose $S = S_g$, the surface with genus $g \geq 0$. Then* Mod(S) *is*

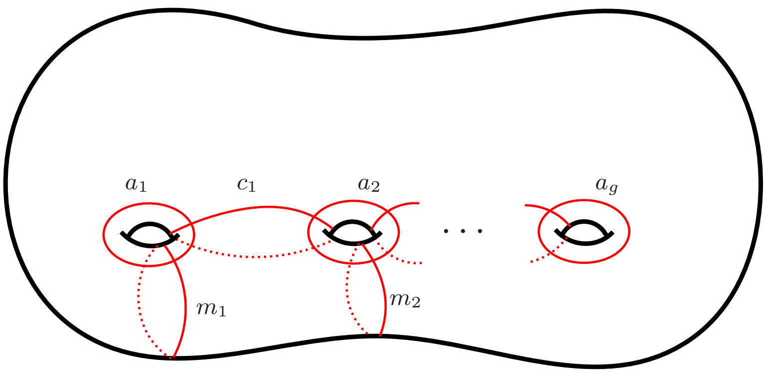


Figure 3.10: The Humphries generators.

generated by Dehn twists about the $2g + 1$ *non-separating curves*

$$a_1 \dots, a_g, c_1, \dots, c_{g-1}, m_1, m_2$$

shown in Figure 3.10.

If our surface is punctured then we must add in half-twists to permute the punctures. Thus we have the following set of generators for the mapping class group of the punctured surface.

Theorem 3.2.12. *Suppose* $S = S_{g,n}$ *is the surface with* n *punctures and genus* $g \geq 2$*. Then* $\mathrm{Mod}(S)$ *is generated by Dehn twists about the* $2g + n + 1$ *non-separating curves*

$$a_1 \dots, a_g, c_1, \dots, c_{g-1}, f_1, \dots, f_n, m_1, m_2$$

and half twists about the $n - 1$ *arcs*

$$h_1, \dots, h_{n-1}$$

shown in Figure 3.11.

We now consider the action of the generators for the mapping class group on the standard set of generators for the fundamental group. Take, for example, the homeomorphism $T_{f_0} : S \to S$, which is the Dehn twist about a curve isotopic to f_0, as shown in Figure 3.12. Let T_* denote the isomorphism $\pi_1(S \setminus P, p) \to \pi_1(S \setminus P, p)$ induced by T_{f_0}. The homomorphism T_{f_0} clearly preserves the puncture set. Suppose that there is a function $\phi : \pi_1(S \setminus P, p) \to G$, and its corresponding tuple is

$$(a_1, b_1, \dots, a_{g_0}, b_{g_0}, c_1, \dots, c_r)$$

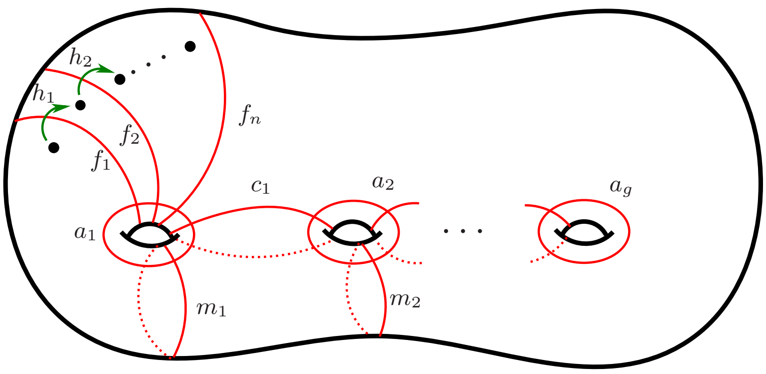


Figure 3.11: Generators for the mapping class group.

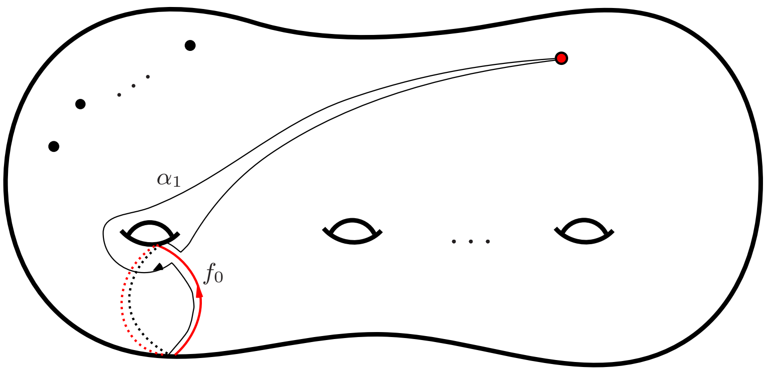


Figure 3.12: Computing the action on α_1.

where $a_1 = \phi(\alpha_i)$, $b_1 = \phi(\beta_i)$ and $c_i = \phi(\gamma_i)$. Assume that ϕ' is another map such that $\phi' = \phi \circ T_*$. Then T_* acts on the standard generators for $\pi_1(S \setminus P, p)$ by:

$$\alpha_1 \mapsto \beta_1^{-1}\alpha_1,$$

with all other curves being left fixed (see Figure 3.12). The tuple corresponding to $\phi' = \phi \circ T_{a_1}$ is the tuple

$$(a_1', b_1', \ldots, a_{g_0}', b_{g_0}', c_1', \ldots, c_r'),$$

where, for $j \neq 1$, we have, $a_j' = a_j$, $b_j' = b_j$ and $c_j' = c_j$, and

$$a_1' = \phi(T_*(\alpha_1)) = \phi(\beta_1^{-1}\alpha_1) = b_1^{-1}a_1.$$

One can easily verify that this new tuple satisfies the single relation for the fundamental group.

Indeed

$$[a_1', b_1'] = [b_1^{-1}a_1, b_1] = a_1^{-1}b_1b_1^{-1}b_1^{-1}a_1b_1 = [a_1, b_2].$$

Using the above technique and the standard set of generators for the mapping class group, as in Figure 3.13, we can determine the mapping class orbits of a tuple. The following theorem explicitly describes this action on tuples.

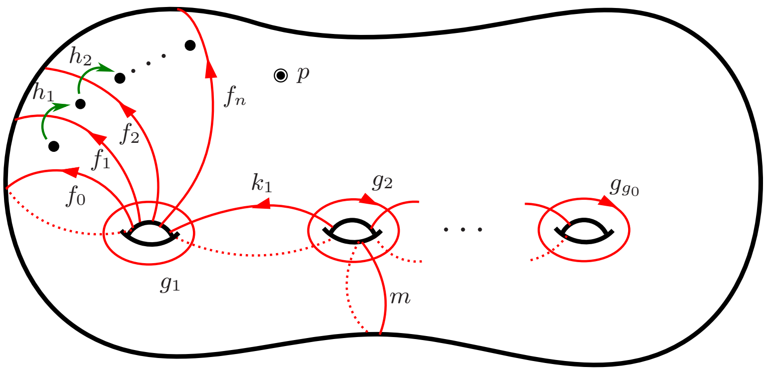


Figure 3.13: Generators for the mapping class group.

Theorem 3.2.13. *Using the notation defined in the preceding section, the action of the standard generators for the mapping class group induces the following action on tuples:*

- f_i *action:*
  - $a_1' = a_1c_r^{-1}\dots c_{i+1}^{-1}c_1\dots c_r a_1^{1-}b_1^{-1}a_1$
  - $a_j' = a_j$ for $j > 1$
  - $b_j' = b_j$
  - $c_j' = c_1\dots c_r a_1^{-1}b_1^{-1}a_1c_r^{-1}\dots c_{i+1}{}^{-1}c_jc_{i+1}\dots c_ra_1^{-1}b_1a_1c_r^{-1}\dots c_1^{-1}$ for $i < j$
  - $c_j' = c_j$ if $j < i$
- g_i *action:*
  - $a_i' = b_ia_i$
  - $a_j' = a_j$ for $j \neq i$
  - $b_l' = b_j$

- $c'_j = c_j$

- h_i *action:*
  - $a'_j = a_j$
  - $b'_j = b_j$
  - $c'_i = c_{i+1}$
  - $c'_{i+1} = c_{i+1}^{-1} c_i c_{i+1}$
  - $c'_j = c_j$ for $j \neq i, i+1$

- k_i *action:*
  - $a'_i = b_i a_{i+1}^{-1} b_{i+1}^{-1} a_{i+1} a_i$
  - $a'_{i+1} = b_{i+1} a_{i+1} b_i^{-1}$
  - $a'_j = a_j$ for $j \neq i, i+1$
  - $b'_i = b_i a_{i+1}^{-1} b_{i+1}^{-1} a_{i+1} b_i a_{i+1}^{-1} b_{i+1} a_{i+1} b_i^{-1}$
  - $b'_j = b_j$ for $j \neq i$
  - $c'_j = c_j$

- m
  - $a'_2 = b_1^{-1} a_2$
  - $a'_j = a_j$ for $j \neq i$
  - $b'_j = b_j$
  - $c'_j = c_j$

Thus we now have a correspondence between components of the Hurwitz space and a combinatorial action on Nielsen tuples.

Theorem 3.2.14. *There is a one-to-one correspondence between*

- *Components of the Hurwitz space* $\mathcal{H}^{in}(G, \boldsymbol{C})$.
- *Orbits of* $\mathrm{Ni}^{\mathrm{in}}(g_0, G, \boldsymbol{C})$ *under the action described in Theorem 3.2.13.*

3.3 Variants of Hurwitz space

The name *Hurwitz space* is attached to many different geometric and algebraic collections. In order to place the results in context we briefly consider some common variants and discuss their connections to the objects which in this thesis have been called Hurwitz spaces.

Let $\mathrm{Ni}^{\mathrm{tot}}(G, \boldsymbol{C})$ denote the *total Nielsen class* of tuples of type C:

$$\mathrm{Ni}^{\mathrm{tot}}(G, \boldsymbol{C}) = \{(g_1, \ldots, g_r) \in \mathcal{E}_r(G) \mid \text{for some } \sigma \in S_r, g_i \in C_{\sigma(i)}\}.$$

Therefore,

$$\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C}) = \mathrm{Ni}^{\mathrm{tot}}(G, \boldsymbol{C}) / \operatorname{Inn}(G).$$

It is therefore clear why the space $\mathrm{Ni}^{\mathrm{in}}(g, \boldsymbol{C})$ is called the *inner* Hurwitz space, and is denoted $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$. If however we consider tuples up to an alternative notion of equivalence then we have a different space classifying a different set of objects. One natural action is componentwise application of an outer automorphism.

Definition 3.3.1. Let G be a finite group and $\boldsymbol{C}$ a type. Let $\operatorname{Abs}(G, \boldsymbol{C})$ denote the group of automorphisms preserving the type, i.e.,

$$\operatorname{Abs}(G, \boldsymbol{C}) = \{\iota \in \operatorname{Aut}(G) \mid \iota(C_i) = C_{\pi(i)} \text{ for some } \pi \in S_r \text{ and all } 1 \leq i \leq r\}.$$

The action of the absolute group, $\operatorname{Abs}(G, \boldsymbol{C})$, acting componentwise on the tuples of $\mathrm{Ni}^{\mathrm{tot}}(G, \boldsymbol{C})$, commutes with the braid action. The set of equivalence classes of tuples up to absolute equivalence is denoted by

$$\mathrm{Ni}^{\mathrm{abs}}(G, \boldsymbol{C}) = \mathrm{Ni}^{\mathrm{tot}}(G, \boldsymbol{C}) / \operatorname{Abs}(G, \boldsymbol{C}).$$

Whilst points in $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ correspond to Galois covers of $\mathbb{P}^1\mathbb{C}$, points of $\mathrm{Ni}^{\mathrm{abs}}(G, \boldsymbol{C})$ correspond to covers (which are not necessarily normal) of $\mathbb{P}^1\mathbb{C}$ with ramified points of type $\boldsymbol{C}$, which are points of the absolute Hurwitz space $\mathcal{H}^{\mathrm{abs}}(G, \boldsymbol{C})$. Points in the inner Hurwitz space are the normal closures of the points of the absolute Hurwitz spaces [12].

There is a natural embedding of the inner Nielsen class with the absolute Nielsen class

$$\Psi : \mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C}) \to \mathrm{Ni}^{\mathrm{abs}}(G, \boldsymbol{C})$$

given by the action of the $\mathrm{Abs}(G, \boldsymbol{C})/\operatorname{Inn}(G)$. Moreover this gives rise to a covering map between the Hurwitz spaces

$$\Phi : \mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C}) \to \mathcal{H}^{\mathrm{abs}}(G, \boldsymbol{C}).$$

The degree of the map Φ is $\mid \mathrm{Abs}(G, \boldsymbol{C}) : \mathrm{Inn}(G) \mid$.

3.4 Known Results

The first result concerning the connectivity of Hurwitz spaces was due to Clebsch who showed the connectivity of the space of simple covers. A cover $f : Y \to \mathbb{P}^1\mathbb{C}$ of degree n is said to be simple if the number of preimages of any point in $\mathbb{P}^1\mathbb{C}$ is either n or $n-1$. Under these circumstances, the type of the cover must consist solely of transpositions, and the monodromy group is S_n. So $\boldsymbol{C} = (C_1, \ldots, C_r)$ where $C_i = C$ is the transposition class. Clebsch showed that $\mathrm{Ni}^{\mathrm{in}}(S_n, \boldsymbol{C})$ is non-empty if and only if r is even and $r \geq 2(n-1)$. This follows from the Riemann-Hurwitz formula. Furthermore if $\mathrm{Ni}^{\mathrm{in}}(S_n, \boldsymbol{C})$ is non-empty then the braid group $\mathcal{B}_r$ acts transitively. In order to prove this result Clebsch established a normal form for tuples, showing that every tuple is braid equivalent to a tuple of the form

$$(g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_k, g_k^{-1}).$$

Establishing a normal form for tuples is a commonly used approach, and one we use in Chapter 4 and Chapter 5. Tuples of the above form are sometimes said to be in *Harbater-Mumford* form. Hurwitz used this connectivity result to establish the connectivity of the moduli space of curves.

In their 1991 paper Fried and Völklein gave a construction (very similar to the one in this chapter) of the *moduli space of covers of the Riemann sphere*. An appendix in this paper gave a detailed proof of an asymptotic result concerning the connectivity of Hurwitz space. This result was a previously unpublished result due to Conway and Parker, and is often called the Conway-Parker theorem or the Conway-Parker-Fried-Völklein theorem. The statement of this theorem is the following.

Theorem 3.4.1. *Let G be a finite group and let $r \geq 3$ be an integer. Suppose that the Schur-Multiplier of G is generated by commutators. Then there exists a positive integer N such that if each nontrivial conjugacy class C of G appears in the type $\boldsymbol{C}$ with multiplicity at least N then the Hurwitz space $\mathcal{H}^{in}(G, \boldsymbol{C})$ is connected.*

This result is very strong, however, no nontrivial bounds on N exist and so its practical uses are limited. The results of later chapters also suggest that the condition that every conjugacy class appears often is too strong and that only particular classes must appear in the type. Dunfield and Thurston has proven a similar asymptotic result for $g > 0$ [8].

To this point all results have concluded with the Hurwitz space being connected. Fried conjectured that there are infinitely many examples of groups and types for which the braid group fails to act transitively on the Nielsen class. He proved his own conjecture true with the following result:

Theorem 3.4.2 ([11])**.** *Let C be the conjugacy class of 3-cycles in A_n. Let $\boldsymbol{C} = (C_1, \ldots, C_r)$ be a type where, for each i, we have $C_i = C$. Then*

- *If $r = n - 1$ then $\mathcal{B}_r$ acts transitively on the Nielsen class $\mathrm{Ni}^{\mathrm{in}}(A_n, \boldsymbol{C})$.*
- *If $r \geq n$ then there are two braid orbits.*

The case $r = n - 1$ corresponds to covers of $\mathbb{P}^1\mathbb{C}$ of genus 0. To establish that there are two orbits, Fried defined an invariant of the braid orbits for alternating groups for braid orbits. The key observation is that the alternating group A_n has a double cover $2 \cdot A_n$. A 3-cycle in A_n can be lifted to a unique element $\hat{g}$ of order 3. The product of this lifted tuple lies in the centre of $2 \cdot A_n$ which is the group of order 2. The value of this product is an invariant of the braid orbit called the *lifting invariant*. Fried demonstrated that for $r \geq n$ there are tuples of length r for both possible values of the lifting invariant, and moreover those tuples with equal lift invariant are braid equivalent.

For dihedral groups, and semidihedral groups, Sia has provided a classification for braid orbits of all types [35]. Independently Catanese, Lönne and Perroni have also completed a classification for mapping class orbits of dihedral groups [5]. They showed that the number of orbits depends only on the orders of the elements of the type. In a later paper, the same authors provide the same classification using a generalized lifting invariant, which extends Fried's lifting invariant to an arbitrary group [4]. Like Fried's invariant this invariant is closely related to the Schur multiplier and in fact when G is A_n for $n \neq 6, 7$ the two coincide. They show that this invariant is a fine invariant for dihedral groups, and conjecture that this invariant distinguishes all braid orbits for tuples of sufficient length.

The evidence, namely the results contained in this thesis, the work of Fried, the Conway-Parker theorem, the Dunfield-Thurston theorem, and the contributions from Catanese, Lönne, and Perroni, suggests that the Schur multiplier is key to determining the number of braid orbits. The results combine to say that the Hurwitz space is as connected as possible.

An alternative approach to the questions asked in this thesis is to fix the genus of the covering space rather than fixing the orbit genus. Magaard, Shpectorov and Völklein considered the problem of determining the locus of curves of genus g whose automorphism group contains a subgroup isomorphic to a given finite group G [26]. Thus they wanted to compute within the Hurwitz space $\mathcal{H}^{\text{in}}_{g,G}$. They calculated the number of components of this locus for $g \leq 10$. This is possible because, by Hurwitz's theorem, the order of G must be less than $84(g-1)$. The Riemann-Hurwitz theorem heavily restricts the list of possible types. Such a list of types was computed by Breuer [3]. The orbits were computed using BRAID, a precursor to the MapClass package.

Similarly, Liu and Osserman consider the problem of determining the connectivity of the space $\mathcal{H}_{d,r}$ of genus 0, degree d covers of $\mathbb{P}^1\mathbb{C}$, branched over r points up to absolute equivalence. In particular they consider the problem when the type of such covers is *pure cycle*, i.e., consisting purely of elements whose cycle shape is just a single cycle. Suppose that $\boldsymbol{C}$ consists of cycles of lengths $e_1, \ldots, e_r$, then the Riemann-Hurwitz formula may be rewritten as

$$2d - 2 = \sum_{i=1}^{r} (e_i - 1).$$

This formula is sometimes called the *planarity condition*. Liu and Osserman show that, given the above conditions, the Hurwitz space is connected. The proof of this result relies on a reduction to the case when $r = 4$. Fried has since considered this case for $G = A_n$ using inner equivalence rather than absolute equivalence [11].

CHAPTER 4

A_5 BRAID ORBITS

In this chapter we provide a complete classification of the braid orbits of A_5 for all types. The main result of this chapter, Theorem 4.3.1, gives a simple condition on the type $\boldsymbol{C}$ for determining the number of components the Hurwitz space $\mathcal{H}^{\text{in}}(A_5, \boldsymbol{C})$. There are very few configurations of group and type for which such a condition is known. The contents of this chapter may also be found in [22]

4.1 Notation

Throughout this chapter we use the following shorthand for the conjugacy classes of A_5:

- $2A = (1,2)(3,4)^{A_5}$
- $3A = (1,2,3)^{A_5}$
- $5A = (1,2,3,4,5)^{A_5}$
- $5B = (1,2,3,5,4)^{A_5}$

Let $\boldsymbol{C}$ be a type and C is conjugacy class of A_5 then $n_C(\boldsymbol{C})$ denotes the number of occurrences of C within the type $\boldsymbol{C}$.

The group A_5 has a non-trivial outer automorphism which permutes the two classes of 5-cycles but fixes all other conjugacy classes. This automorphism corresponds to conjugation by $(4,5) \in S_5$.

4.2 Covers and Lifting Invariants

For the purposes of the classification a method of determining whether two tuples lie in different braid orbit is required. Fried introduced an invariant of braid orbits which he used to give a classification of braid orbits for the alternating groups where the type consists of 3-cycles. This invariant, called the *lifting invariant*, uses the existence of a double cover for the alternating groups.

Definition 4.2.1. Let G be a finite group. A group $\widehat{G}$ is called a *covering group* of G if $Z(\widehat{G}) \leq \widehat{G}'$ and $\widehat{G}/Z(\widehat{G}) \cong G$. It is not unusual to find both the covering group $\widehat{G}$ and the covering homomorphism $\theta : \widehat{G} \to G$ referred to as the *cover of* G. The index $[\widehat{G} : Z(\widehat{G})]$ is called the *degree* of the cover.

When G is a perfect group, i.e., when $G' = G$, there exists a unique maximal cover which is universal, in the sense that all other covering groups are quotients of the maximal covering group. The centre of the maximal covering group is called the *Schur multiplier of* G.

Schur showed that the alternating group A_n has a unique degree 2 cover, also known as a *double cover*. For $n \neq 6, 7$ the double cover is maximal. This double cover is denoted $2 \cdot A_n$. A property of the double cover that we wish to exploit is that every odd order element g in A_n has a unique odd order lift to the double cover denoted $\hat{g}$.

Lemma 4.2.2. *Let G be a finite group and let $\widehat{G}$ be a covering group G, and let $\theta : \widehat{G} \to G$ be the corresponding covering homomorphism. Let K denote the centre $Z(\widehat{G})$. Suppose that $g \in G$ and $(o(g), |K|) = 1$. Then there exists a unique $h \in \widehat{G}$ such that $\theta(h) = g$ and $o(h) = o(g)$.*

Proof. Let h be a preimage of g. Let n denote the order of g and k denote the size of K. Suppose that $h^n = x \in K$ such that x is nontrivial. Since n and k are coprime, there exists a unique $y \in K$ such that $y^n = x$. Therefore

$$(hy^{-1})^n = h^n y^{-n} = xx^{-1} = 1.$$

Thus hy^{-1} is a preimage of g with order n.

To show uniqueness let $l = hy^{-1}$ and note that the preimage of g is the set lK. Since $(p, k) = 1$, then $o(kl) = o(k)o(l)$ for any $k \in K$. □

The previous result allows us to define the lifting invariant.

Definition 4.2.3. Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ be a Nielsen tuple in A_n consisting of odd order elements. Let $\hat{\boldsymbol{g}} = (\hat{g_1}, \ldots, \hat{g_r})$ be the lifted tuple in $2 \cdot A_n$. The *lifting invariant of* $\boldsymbol{g}$, written $\mathrm{LI}(\boldsymbol{g})$, is given by

$$\mathrm{LI}(t) = \hat{g_1} \cdots \hat{g_r}.$$

The lifting invariant is a lift of 1 and is hence central in $\widehat{G}$. For convenience we identify the centre with the multiplicative group $\{1, -1\}$. The lifting invariant takes values in this set. The lifting invariant is an invariant of the braid orbit [11], and is not defined for types involving classes of elements of even order.

4.3 Discussion of Main Results

In this remainder of this chapter a classification of the braid orbits for $G = A_5$ and all types is proven. We show that the double transposition class plays a key role in ensuring connectivity of the Hurwitz space. The results in the chapter can also be found in [22]. The main result of this chapter is the following.

Theorem 4.3.1. *For $G = A_5$ and a type $\boldsymbol{C} = (C_1, \ldots, C_r)$, $r \geq 3$, the Nielsen class $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ is non-empty if and only if $\boldsymbol{C}$ is not listed in Table 4.1. Furthermore, given that $\boldsymbol{C}$ is not in Table 4.1,*

- *if $n_{2A}(\boldsymbol{C}) > 0$ the $\mathcal{B}_r$ acts transitively on $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$;*
- *if $n_{2A}(\boldsymbol{C}) = 0$ then there are two braid orbits on $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ if and only if $\boldsymbol{C}$ is not listed in Table 4.2; moreover, the two orbits are distinguished by the lifting invariant.*

Note that the tables shown are in condensed form. The actual list of exceptional types are obtained from Table 4.1 and Table 4.2 by permutations and the outer automorphism.

The following is a translation of Theorem 4.3.1 into the language of Hurwitz spaces.

Theorem 4.3.2. *For $G = A_5$ and a type $\boldsymbol{C}$ of length at least 3, the Hurwitz space $\mathcal{H}^{in}(A_5, \boldsymbol{C})$ is nonempty if and only if $\boldsymbol{C}$ is not in Table 4.1. Furthermore, if nonempty, $\mathcal{H}^{in}(A_5, \boldsymbol{C})$ is connected if and only if $\boldsymbol{C}$ is in Table 4.2 or $n_{2A}(\boldsymbol{C}) > 0$. In all other cases, $\mathcal{H}^{in}(A_5, \boldsymbol{C})$ has exactly two components, which are distinguished by the lifting invariant.*

Type	Genus
$(2A, 2A, 2A)$	-1
$(2A, 2A, 3A)$	-1
$(2A, 2A, 5A)$	0
$(2A, 3A, 3A)$	-1
$(2A, 5A, 5A)$	1
$(3A, 3A, 3A)$	-1
$(5A, 5A, 5B)$	2
$(2A, 2A, 2A, 2A)$	0

Table 4.1: Types $\boldsymbol{C}$, up to permutation and outer automorphism, for which the Nielsen class $\mathrm{Ni}^{\mathrm{in}}(A_5, \boldsymbol{C})$ is empty.

Type	LI
$(3A, 3A, 5A)$	-1
$(3A, 5A, 5A)$	-1
$(3A, 5A, 5B)$	1
$(5A, 5A, 5A)$	-1
$(3A, 3A, 3A, 3A)$	1
$(3A, 5A, 5A, 5A)$	1
$(5A, 5A, 5A, 5A)$	1
$(5A, 5A, 5A, 5B)$	-1
$(5A, 5A, 5A, 5A, 5A)$	-1

Table 4.2: Odd types, also up to permutation and outer automorphism, for which only one braid orbit on $\mathrm{Ni}^{\mathrm{in}}(A_5, \boldsymbol{C})$ exists; the right column shows the lifting invariant of the orbit.

4.4 Braids and Partitions

The proof of Theorem 4.3.1 requires us to consider subgroups of the braid group which fix a partition in a particular way. This section discusses such subgroups. The material in this section is a special case of the theory of *mixed braids*, in which, in the language of [17], all interior braids are trivial.

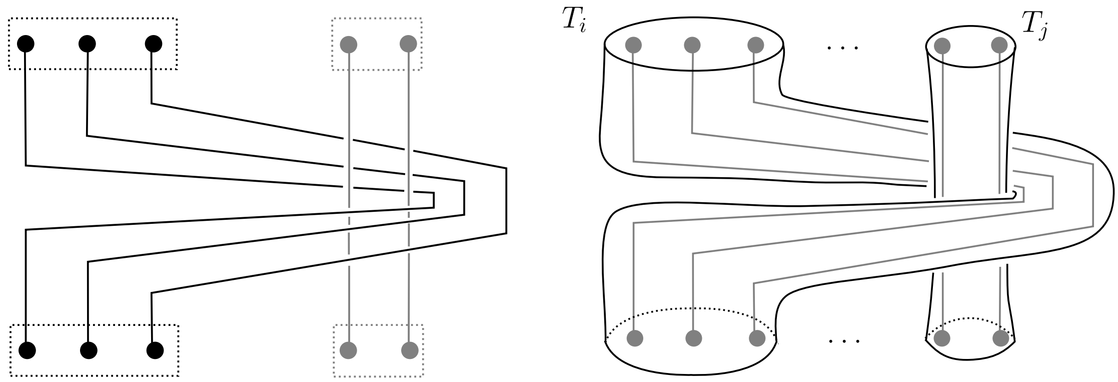
Definition 4.4.1. A partition of $\{1, \ldots, r\}$ will be called *admissible* if each block consists of a consecutive subsequence of the integers $1, \ldots, r$. Thus an admissible partition will have the form

$$\{\{1, \ldots, n_1\}, \{n_1 + 1, \ldots, n_2\}, \ldots, \{n_{k-1} + 1, \ldots, r\}\},$$

for some $1 \leq n_1 < n_2 < \cdots < n_{k-1} < r$. The notation $[a_1, a_2, \ldots, a_l]$ will be used to denote an admissible partition of $\{1, \ldots, r\}$ as above, where block i has size a_i. In particular, the above partition can be written as

$$[n_1, n_2 - n_1, \ldots, r - n_{k-1}].$$

Figure 4.1: Ribbon braids.



(a) The ribbon generator $R_{i,j}$. The elements of block i are braided under block j and all intermediate blocks.

(b) Ribbon braids can be seen as braidings of blocks of braids. Strands in the same block all lie in embedded cylinders.

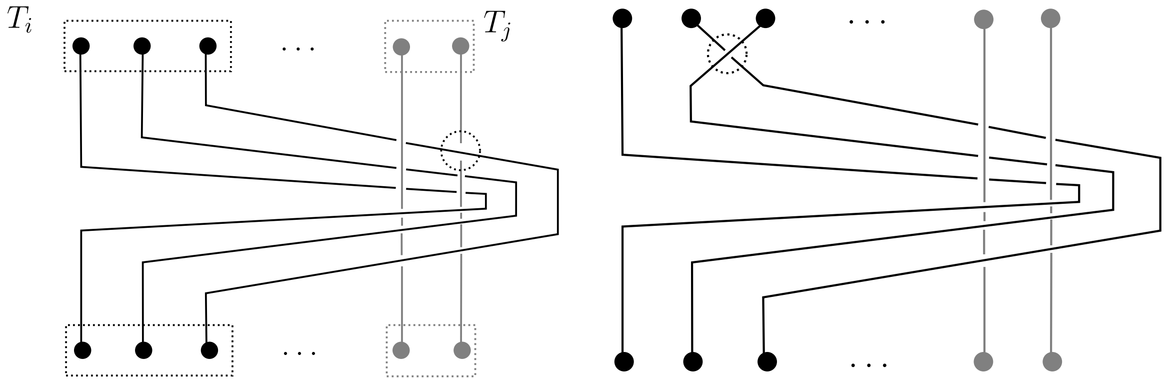
Fix an admissible partition P of $\{1\ldots,r\}$, with k blocks $T_1,\ldots,T_k$. We say that a pure braid $Q\in\mathcal{PB}_r$ is *tubular with respect to* P if for each $1\le i\le k$ we can embed a cylinder $D_i\cong D\times[0,1]$ into $\mathbb{C}\times[0,1]$ such that any two cylinders are disjoint and every every strand corresponding to a point in T_i lies in the interior of D_i (see Figure 4.1b; a more thorough description can be found in [17]). To each cylinder D_i we can associate a braid in $\mathcal{B}_{|T_i|}$ by restricting to the interior of D_i. Such a braid is called an *interior braid.* If a tubular braid with respect to P is such that all interior braids are trivial then we call this braid a *ribbon braid.* Define $\mathcal{PR}_{r,P}$ to be the subgroup of $\mathcal{PB}_r$ consisting of all ribbon braids, for the partition P. We call $\mathcal{PR}_{r,P}$ the *ribbon braid group with respect to* P.

Remark 4.4.2.

- Ribbon braids preserve the partition P. In fact they do more than that. The order of the strands within a block is preserved, since the braids are pure, and the order is preserved throughout time (see Figure 4.2b). The name ribbon braid is sometimes used to include the possibility that we rotate the strands in a cylinder by some multiple of 2π [10]. Our definition excludes such braids.

- We think of ribbon braids as those braids obtained by combining all the strands in a partition block into a single strand. For $1\le i<j\le r$, define $R^P_{i,j}$ to be the ribbon braid which braids block i under block j, in analogy with the pure braid generator $P_{i,j}$ (see Figure 4.1a).

Figure 4.2: Examples of partition preserving braids which are not ribbon braids.



(a) All braids from a single block must "stay together" throughout time. This is *not* an example of a ribbon braid.

(b) Strands in the same block must also maintain their ordering throughout time. If we restrict the braids to our partition P_j then the sub-braid must be trivial. The pictured braid is *not* a ribbon.

The ribbon braid group $\mathcal{PR}_{r,P}$ depends only on the number of blocks in the associated partition. The next proposition makes this relationship clear.

Proposition 4.4.3. *Suppose that P is an admissible partition of $\{1, \ldots, r\}$, and that P has k blocks. Then there is an isomorphism $\phi_P : \mathcal{PR}_{r,P} \to \mathcal{PB}_k$.*

Proof. To show the claim we appeal to a result of Gonzàlez-Meneses and Wiest from [17]. Let $\mathcal{MB}_P$ denote the group of all braids fixing the partition P. In particular $\mathcal{MP}_B$ is defined as the ribbon braid group except that we drop the condition that interior braids must be trivial. Consider the homomorphism $\mathcal{MB}_P \to \mathcal{PB}_k$ taking the mixed braid to its corresponding tubular braid. Then Gonzàlez-Meneses and Wiest show that the kernel of this map is set of all mixed braids where the interior braids are non-trivial. Therefore restricting this map gives

$$\mathcal{PR}_{r,P} \to \mathcal{PB}_k$$

is a bijection. $\square$

The presentation for mixed braid groups in [17] encapsulates the above proposition.

For the remainder of this section we want to examine ribbon braids and their action on tuples of length r. This action will be used later to reduce the size of tuples to a more manageable length.

Definition 4.4.4. Suppose that $P = [n_1, \dots, n_k]$ is an admissible partition of $\{1. \dots, r\}$. Given a tuple $\boldsymbol{g} = (g_1, \dots, g_r)$ of elements from a finite group G we denote by $\boldsymbol{g}^P$ the tuple:

$$\boldsymbol{g}^P = (g_1 \cdots g_{n_1}, g_{n_1+1} \cdots g_{n_1+n_2}, \dots, g_{n_1+\dots+n_{k-1}+1} \cdots g_r).$$

We call such a tuple a *coalesced tuple.*

For example, if we have a tuple, $(g_1, g_2, g_3, g_4, g_5)$ with five elements, and an admissible partition $P = [2, 1, 2]$. Then

$$\boldsymbol{g}^P = (g_1 g_2, g_3, g_4 g_5)$$

and $\boldsymbol{g}^P$ is a tuple with three elements. Coalescing reduces the size of the tuple whilst maintaining the product one condition. Note that we do not allow the identity element in Nielsen tuples, so implicit in our definition is that none of the products in the coalesced tuple are allowed to be the identity.

The following observation does not require proof.

Lemma 4.4.5. *Suppose that P is an admissible partition of $\{1 \dots, r\}$, with k blocks. Let $R \in \mathcal{PR}_{r,P}$, $Q = \phi_P(R) \in \mathcal{PB}_k$, and $\boldsymbol{g} = (g_1, \dots, g_r)$ be a Nielsen tuple of length r. Then*

$$(R(\boldsymbol{g}))^P = Q(\boldsymbol{g}^P).$$

□

It is clear that ribbon braids, as pure braids, act on the elements of the tuple by conjugation. Furthermore, the action on elements of the same partition block is identical.

Lemma 4.4.6. *Let $P = [n_1, \dots, n_k]$ be an admissible partition of $\{1, \dots, r\}$. If $\boldsymbol{g}$ is a Nielsen tuple and $R \in \mathcal{PR}_{r,P}$ then*

$$R(\boldsymbol{g}) = (g_1^{s_1}, \dots, g_{m_1}^{s_1}, g_{m_1+1}^{s_2}, \dots, g_{m_2}^{s_2}, \dots, g_{m_{k-1}+1}^{s_k}, \dots, g_r^{s_k})$$

for some $s_i \in G$ and where $m_i = n_1 + \dots + n_i$.

Proof. It suffices to consider the case where $R = R^P_{i,j}$ is one of the generators of $\mathcal{PR}_{r,P}$, and then the claim is clear. □

Under certain circumstances we may undo the coalescing in a unique way as the following proposition suggests.

Proposition 4.4.7. *Let G be a finite group, m a natural number, and $g \in G$ such that $(m, o(g)) = 1$. Suppose that $P = [\underbrace{1, \ldots, 1}_{i}, m, 1, \ldots, 1]$ is an admissible partition of $\{1, \ldots, r\}$ into $k = r-m+1$ blocks and that*

$$\boldsymbol{g} = (g_1, \ldots, g_i, \underbrace{g, \ldots, g}_{m}, g_{i+m+1}, \ldots, g_r)$$

$$\boldsymbol{g'} = (g'_1, \ldots, g'_i, \underbrace{g', \ldots, g'}_{m}, g'_{i+m+1}, \ldots, g'_r)$$

are two Nielsen tuples of the same type with $Q(\boldsymbol{g}^P) = \boldsymbol{g'}^P$ for some $Q \in \mathcal{PB}_k$. Then $R(\boldsymbol{g}) = \boldsymbol{g'}$, where $R = \phi_P^{-1}(Q) \in \mathcal{PR}_{r,P}$.

Proof. By Lemma 4.4.6, $R(\boldsymbol{g}) = (h_1, \ldots, h_i, h, \ldots, h, h_{i+m+1}, \ldots, h_r)$ for suitable elements $h_i, h \in G$. Since $R(\boldsymbol{g})^P = Q(\boldsymbol{g}^P) = \boldsymbol{g'}^P$ by Lemma 4.4.5, it follows that $h_j = g'_j$ for $1 \leq j \leq i$ and $i + m + 1 \leq j \leq r$, and hence also that $h^m = (g')^m$. Since $o(g') = o(g) = o(h)$ and $(m, o(g)) = 1$, we have that $h = \boldsymbol{g'}$ and so $R(\boldsymbol{g}) = \boldsymbol{g'}$. □

4.5 A_5 Braid Orbits

In this section we aim to provide a complete description of braid orbits for A_5. For the rest of this article we will call an r-tuple which contains a class of double transpositions an *even tuple*. Tuples containing no class of double transpositions will be called *odd tuples*. By *parity* of a tuple we refer to whether the tuple is even or odd. The same terminology also applies to types.

The proof will consider the action of $\mathcal{PB}_r$ instead of $\mathcal{B}_r$. Every braid $Q \in \mathcal{B}_r$ acting on a tuple of type $\boldsymbol{C}$ produces a tuple of type $\boldsymbol{C}^{\sigma_Q}$, where $\sigma_Q \in S_r$ is the permutation associated with Q. Consequently, we have:

Lemma 4.5.1. *The action of $Q \in \mathcal{B}_r$ on $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ is a bijection between this class and the class $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C}^{\sigma_Q})$. This bijection takes pure braid orbits to pure braid orbits and, for odd types, it preserves the lifting invariant.* □

Clearly, this observation implies the following reduction.

Proposition 4.5.2. *The number of braid orbits on* $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ *is at most the number of pure braid orbits on* $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ *(and this is true whether or not the conjugation is included in the actions). If the pure braid orbits are distinguished by the lifting invariant then the same is true for the braid orbits and the number of braid orbits and pure braid orbits is the same.* □

In turn, this proposition allows us to deduce our Theorem 4.3.1 from the following "pure braid" equivalent.

Theorem 4.5.3. *For* $G = A_5$ *and a type* $\boldsymbol{C} = (C_1, \ldots, C_r)$*,* $r \geq 3$*, the pure Nielsen class* $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ *is non-empty if and only if* $\boldsymbol{C}$ *is not a permutation of a type from Table 4.1. Furthermore, if* $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ *is non-empty then:*

- *for even types* $\boldsymbol{C}$*,* $\mathcal{PB}_r$ *acts transitively on* $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$*;*
- *for odd types* $\boldsymbol{C}$*, there are two pure braid orbits on* $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ *if and only if* $\boldsymbol{C}$*, up to permutation, is not listed in Table 4.2; moreover, the two orbits are distinguished by the lifting invariant.*

We now commence proving Theorem 4.5.3.

4.5.1 Basis for Induction.

The proof is by induction on r, the length of $\boldsymbol{C}$. The following lemma anchors the induction.

Lemma 4.5.4. *If* $\boldsymbol{C}$ *is a type of length* $3 \leq r \leq 7$ *then the conclusion of Theorem* 4.5.3 *holds.* □

The lemma is established by explicit calculation of all orbits for all types of length $3 \leq r \leq 7$. Note that in view of Lemma 4.5.1, we only need to consider lexicographically ordered types. For these types, the computation was done using our MapClass package [23] for the GAP computational algebra system [14]. Data for this computation can be found in Appendix A. More details about how such a computation is performed can be found in Chapter 6 or in [27]. In particular, all exceptional cases were found in this computation.

4.5.2 A Normal Form for Tuples

Definition 4.5.5. We say that a tuple

$$(g_1, \ldots, g_{i-1}, g, g, g_{i+2}, \ldots, g_r)$$

is in *odd repetitive form* if g is an odd-order element. If the position of the repeated element is important then we shall say that the tuple is in odd repetitive form at position i. A tuple

$$(g_1, \ldots, g_{i-1}, g, g, g, g_{i+3}, \ldots, g_r),$$

where g is a double transposition, is said to be in *even repetitive form* at position i.

For a tuple $\boldsymbol{g}$ in repetitive form at position i, define $P_{\boldsymbol{g}}$ to be the partition $[\underbrace{1, \ldots, 1}_{i-1}, 2, 1, \ldots, 1]$ or $[\underbrace{1, \ldots, 1}_{i-1}, 3, 1, \ldots, 1]$, depending on the parity of the form.

Similarly, we will talk about types in odd or even repetitive form at position i. In the first case, this means that $C_i = C_{i+1}$ is an odd-order class; in the second, $C_i = C_{i+1} = C_{i+2}$ is the double-transposition class.

The tuples in repetitive form are useful for our induction because coalescing such tuples with respect to $P = P_{\boldsymbol{g}}$ preserves all the salient properties. The following lemma makes this precise. Note, first of all, that the coalesced element g^2 or g^3 in the respective cases is not identity and so coalescing makes sense.

Recall that, for every odd order element $g \in A_5$, the unique odd order lift of g to $2 \cdot A_5$ is denoted by $\hat{g}$.

Lemma 4.5.6. *Suppose $\boldsymbol{g}$ is a tuple in a repetitive form, $P = P_{\boldsymbol{g}}$, and $\boldsymbol{h} = \boldsymbol{g}^P$. Then*

- *$\boldsymbol{g}$ and $\boldsymbol{h}$ generate the same subgroup of G;*
- *$\boldsymbol{g}$ and $\boldsymbol{h}$ have the same parity; and*
- *if they are odd then $LI(\boldsymbol{g}) = LI(\boldsymbol{h})$.*

Proof. The second statement is obvious. For the first claim, if $\boldsymbol{g}$ is in odd repetitive form at position i then, since $g_i = g_{i+1}$ is of odd order, $\langle g_i, g_{i+1}\rangle = \langle g_i g_{i+1}\rangle$, and this yields the claim. If $\boldsymbol{g}$ is in even repetitive form then $\boldsymbol{g}$ and $\boldsymbol{h}$ contain the same elements and so again the claim follows.

Now suppose $\boldsymbol{g}$ is odd and in particular it is in odd repetitive form at position i. Let

$g = g_i = g_{i+1}$. Note that $\widehat{g^2} = (\hat{g})^2$. Therefore,

$$\begin{aligned} LI(\boldsymbol{g}) &= \hat{g}_1 \cdots \hat{g}_{i-1}(\hat{g}\hat{g})\hat{g}_{i+2} \cdots \hat{g}_r \\ &= \hat{g}_1 \cdots \hat{g}_{i-1}\widehat{(g^2)}\hat{g}_{i+2} \cdots \hat{g}_r \\ &= LI(\boldsymbol{h}), \end{aligned}$$

completing the proof. □

In the remainder of this subsection we prove the following.

Proposition 4.5.7. *Let $\boldsymbol{C} = (C_1, \ldots, C_r)$, $r \geq 6$, be a type in a repetitive form at position i. Then*

- *either every pure braid orbit on $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ contains a tuple in repetitive form at position i,*
- *or $r = 6$ and either*
  - $\boldsymbol{C} = (2A, 2A, 2A, 2A, 2A, 2A)$, *or*
  - $\boldsymbol{C} = (5A, 5A, 5A, 5A, 5B, 5B)$ *or* $(5A, 5A, 5B, 5B, 5B, 5B)$ *up to permutation and, furthermore, $C_i = C_{i+1} = 5B$ or $5A$, respectively.*

The proof of this proposition will be given in three lemmas. Throughout the proof we assume the hypotheses of Proposition 4.5.7 and each lemma establishes the conclusion of the proposition for different values of r and i.

Lemma 4.5.8. *The claim holds if $r \leq 7$.*

Proof. Consider a pure braid orbit Ω on $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$.

Suppose first that $C = C_i = C_{i+1}$ is an odd order class. Define the type

$$\boldsymbol{D} = (C_1, \ldots, C_{i-1}, C^2, C_{i+2}, \ldots, C_r),$$

where C^2 is the class containing the squares of elements from C. Hence, if $C = 3A$ then also $C^2 = 3A$. If $C = 5A$ (respectively, $5B$) then $C^2 = 5B$ (respectively, $5A$).

Note that the types $\boldsymbol{C}$ and $\boldsymbol{D}$ have the same parity. If they are even then there is only one orbit on $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ by Lemma 4.5.4. Thus, $\Omega = \mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$. Since $\boldsymbol{D}$ has length $r - 1 \geq 5$, the pure Nielsen class $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{D})$ is not empty. Select any tuple $\boldsymbol{g} = (g_1, \ldots, g_{i-1}, g, g_{i+2}, \ldots, g_r)$ from this class. Since g has odd order, there is a unique $h \in \langle g \rangle$ such that $g = h^2$. Note that this h is contained in C and so the tuple $\boldsymbol{h} = (g_1, \ldots, g_{i-1}, h, h, g_{i+2}, \ldots, g_r)$ is of type $\boldsymbol{C}$. By Lemma 4.5.6, since $\boldsymbol{g}$ is generating, $\boldsymbol{h}$ must also be generating and so it is a Nielsen tuple. Hence $\boldsymbol{h}$ is in Ω and it is evidently in odd repetitive form at position i.

Now suppose that $\boldsymbol{C}$ and $\boldsymbol{D}$ are odd. By Lemma 4.5.4, Ω is one of two pure braid orbits on $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$, and let ϵ be the lifting invariant of Ω. Note that $\boldsymbol{D}$ is in Table 4.2 only if it is $(5A, 5A, 5A, 5A, 5A)$ (or its S_5-conjugate $(5B, 5B, 5B, 5B, 5B)$). This leads to the second exceptional case above. Otherwise, the same Lemma 4.5.4 implies that there is a pure braid orbit on $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{D})$ with lifting invariant ϵ. Let $\boldsymbol{g} = (g_1, \ldots, g_{i-1}, g, g_{i+2}, \ldots, g_r)$ be a tuple from that orbit. As above, select $h \in \langle g \rangle$ such that $g = h^2$ and set $\boldsymbol{h} = (g_1, \ldots, g_{i-1}, h, h, g_{i+2}, \ldots, g_r)$. Again, we note that $\boldsymbol{h}$ is a Nielsen tuple of type $\boldsymbol{C}$. Furthermore, by Lemma 4.5.6, it has lifting invariant ϵ. Therefore, $\boldsymbol{h}$ is in Ω and it is in the required repetitive form.

Finally, suppose C is the double-transposition class. Let

$$\boldsymbol{D} = (C_1, \ldots, C_{i-1}, C, C_{i+3}, \ldots, C_r).$$

Then $\boldsymbol{D}$ is an even type of length at least four. If it is in Table 4.1 , it must be $(2A, 2A, 2A, 2A)$, leading to the first exceptional case above. Otherwise, $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{D})$ is nonempty. Select $\boldsymbol{g} = (g_1, \ldots, g_{i-1}, g, g_{i+3}, \ldots, g_r) \in \mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{D})$ and set $\boldsymbol{h} = (g_1, \ldots, g_{i-1}, g, g, g, g_{i+3}, \ldots, g_r)$. Clearly, $\boldsymbol{h}$ is a Nielsen tuple of type $\boldsymbol{C}$. Since $\boldsymbol{C}$ is even, $\Omega = \mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ by Lemma 4.5.4, and so $\boldsymbol{g}$ is in Ω. □

We note that all exceptions in this lemma are bona fide, that is, for each of these types there exists a pure braid orbit containing no tuple in repetitive form at position i.

Next, let us consider the case where the repeated classes are at the end of the type.

Lemma 4.5.9. *Let $r \geq 8$ and suppose that $i = r - 1$ in the odd repetitive form case and $i = r - 2$ in the even repetitive form case. Then the claim holds.*

Proof. We are proving this by induction on r, with $r = 6$ and 7 serving as base cases. Let Ω be a pure braid orbit of type $\boldsymbol{C}$. Select a tuple $\boldsymbol{g} = (g_1, \ldots, g_r) \in \Omega$. We first claim that $\boldsymbol{g}$ can be chosen so that g_{r-1} and g_r do not commute. Indeed, if this is not the case then

$g_{r-1} \in C = C_G(g_r)$. Since $\boldsymbol{g}$ is generating, there is g_j that does not normalize C. Applying the pure braid $P_{j,r-1}$, we obtain a new tuple from Ω, where in position $r-1$ we find $g_{r-1}^{g_j^{-1}}$, which is not in C. Here we use the property of $G = A_5$ that C^g is either equal to C or intersects C trivially.

From now on we assume that g_{r-1} and g_r do not commute. In particular, $\langle g_{r-1}, g_r \rangle$ is either a maximal subgroup of G or it is all of G. (This is again a property of $G = A_5$.) We will call the repeated positions, $[i, \ldots, r]$, at the end the *special* positions and all others *general.*

We will first try to coalesce $\boldsymbol{g}$ using two adjacent general positions, j and $j+1$. Hence $j < i-1$ and the partition used is $P = [\underbrace{1, \ldots, 1}_{j-1}, 2, 1, \ldots, 1]$. Suppose that the resulting tuple $\boldsymbol{h} = (g_1, \ldots, g_{j-1}, g_j g_{j+1}, g_{j+2}, \ldots, g_r)$ is a Nielsen tuple. Since $r-1 \geq 7$, by induction, there exists a pure braid $Q \in \mathcal{PB}_{r-1}$ such that $Q(\boldsymbol{h})$ is in repetitive form at position $i-1$. Let $R \in \mathcal{PR}_{r,P}$ be the ribbon braid such that $\phi_P(R) = Q$. By Lemma 4.4.5, $(R(\boldsymbol{g}))^P = Q(\boldsymbol{h})$. This means that $R(\boldsymbol{g})$ is in repetitive form at position i, and so the claim holds.

We will now see that if this trick does not work for $\boldsymbol{g}$ for any j then this tuple has a very restricted shape.

In view of our choice, since $\boldsymbol{g}$ is generating, there is a general position k such that g_k and the elements in the special positions generate G. By the above, we can assume now that $g_j g_{j+1} = 1$ for any two adjacent general positions disjoint from k. Indeed, if $g_j g_{j+1} \neq 1$ then the coalesced tuple $\boldsymbol{h}$ is a Nielsen tuple, as it clearly contains generating elements for G. Hence, $g_{j+1} = g_j^{-1}$ for all j as above. Since $r \geq 8$, there are at least five general positions. This means that either in front of g_k or after g_k there are at least two general positions. The proof is symmetric for these two possibilities, so let us assume the former. Hence $k \geq 3$.

We know that $g_{j+1} = g_j^{-1}$ for all $j < k-1$. We claim that the same holds for $j = k-1$. If not, the tuple $\boldsymbol{h}$ does not contain the identity element and it is generating, since $\langle g_{k-2}, g_{k-1}, g_k \rangle = \langle x, x^{-1}, g_k \rangle = \langle x, x^{-1} g_k \rangle = \langle g_{k-2}, g_{k-1} g_k \rangle$. Thus, $g_k = g_{k-1}^{-1}$. Therefore, in $\boldsymbol{g}$ we have the sequence $x, x^{-1}, x, x^{-1}, \ldots$ in positions 1 through k. In particular, $g_1 = g_k$ or g_k^{-1}. This means that we could have chosen $k = 1$. Now applying the symmetric argument (for the general positions after g_1), we conclude the all elements in the general positions obey the same pattern $x, x^{-1}, x, x^{-1}, \ldots$.

Now we try another trick, namely, we coalesce with respect to $P = [3, 1, \ldots, 1]$. The gives $\boldsymbol{h} = (x, g_4, \ldots, g_r)$, which is clearly a Nielsen tuple. Note that this time $\boldsymbol{h}$ has length $r-2 \geq 6$.

The above inductive argument with the pure braid Q and the corresponding ribbon braid R works whenever $\boldsymbol{h}$ does not fall into one of the exceptional cases from Proposition 4.5.7. In all these cases $r-2=6$ and so $r=8$.

We deal with the two exceptional cases in turn. Suppose that $\boldsymbol{h}$ is of type $(2A, 2A, 2A, 2A, 2A, 2A)$ and so $\boldsymbol{g}$ is of type $(2A, 2A, 2A, 2A, 2A, 2A, 2A, 2A)$. By the above, $\boldsymbol{g} = (x, x, x, x, x, g_6, g_7, g_8)$. This implies that the tuple (x, g_6, g_7, g_8) is a generating tuple, with product 1 and of type $(2A, 2A, 2A, 2A)$. However, this is impossible, since this type is on Table 4.1 and so the corresponding pure Nielsen class is empty.

In the second exceptional case $\boldsymbol{h}$ is odd and so $\boldsymbol{g}$ is odd, too. By the above, $\boldsymbol{g} = (x, x^{-1}, x, x^{-1}, x, x^{-1}, g_7, g_8)$. Now the product-one condition implies that $g_7 g_8 = 1$, which is a contradiction, since g_7 and g_8 do not commute. This completes the proof. □

Finally, we can do the general case.

Lemma 4.5.10. *If $r \geq 8$ then the claim holds.*

Proof. Consider $Q = Q_1 \cdots Q_{r-1} \in \mathcal{B}_r$. This braid rotates each tuple, sending $(g_1, \ldots, g_r)$ to $(g_r, g_1, \ldots, g_{r-1})$.

Now suppose that Ω is a pure braid orbit of type $\boldsymbol{C}$ and let $\boldsymbol{g} = (g_1, \ldots, g_r) \in \Omega$. Let j be the final position of our repeated classes in $\boldsymbol{C}$. So $j = i+1$ in the odd repetitive form case and $j = i+2$ in the even repetitive form case. Note that $Q^{r-j}(\boldsymbol{g})$ is of type $\boldsymbol{C}^{\sigma^{r-j}}$, where $\sigma = \sigma_Q$ is the cycle $(1, 2, \ldots, r) \in S_r$. Hence the repeated classes are now at the end of the tuple and so Lemma 4.5.9 applies. Therefore, there exists a pure braid R such that $RQ^{r-j}(\boldsymbol{g})$ has the repeated elements in the last two or three positions depending on the form parity. Finally, $Q^{-(r-j)}RQ^{r-j}(\boldsymbol{g})$ is again of type $\boldsymbol{C}$ and it is in the required repetitive form at position i.

It remains to notice that $Q^{-(r-j)}RQ^{r-j} = R^{Q^{r-j}}$ is a conjugate of R and hence it is a pure braid. □

This completes the proof of Proposition 4.5.7.

4.5.3 Proof of Theorem 4.5.3

Let $r \geq 8$. In view of Lemma 4.5.1 we can assume that $\boldsymbol{C} = (C_1, \ldots, C_r)$ is lexicographically ordered.

To begin, we consider the question of existence.

Lemma 4.5.11. *There exists a Nielsen tuple of type $\boldsymbol{C}$. Moreover, if the type is odd then there are Nielsen tuples for both possible values of the lifting invariant.*

Proof. We use induction on r. Since $r \geq 8$, $\boldsymbol{C}$ must contain some conjugacy class twice. Since $\boldsymbol{C}$ is lexicographically ordered, we can assume that $C_i = C_{i+1}$. The type $\boldsymbol{D} = (C_1, \ldots, C_{i-1}, C_{i+2}, \ldots, C_r)$ is of length at least $r-2 \geq 6$ and lexicographically ordered. Hence by induction (and since Table 4.1 contains no type of such length) there exists a Nielsen tuple $(g_1, \ldots, g_{i-1}, g_{i+2}, \ldots, g_r)$ of type $\boldsymbol{D}$. Pick $g \in C_i$ and note that $g^{-1} \in C_i = C_{i+1}$. Therefore the tuple $(g_1, \ldots, g_{i-1}, g, g^{-1}, g_{i+2}, \ldots, g_r)$ is a Nielsen tuple of type $\boldsymbol{C}$.

Suppose now that $\boldsymbol{C}$ is odd and let $\epsilon \in \{1, -1\}$. We proceed in exactly the same way as above. By induction (and since Table 4.2 contains no type of length $r - 2 \geq 6$) we can select $(g_1, \ldots, g_{i-1}, g_{i+2}, \ldots, g_r)$ with lifting invariant ϵ, in which case the extended tuple $(g_1, \ldots, g_{i-1}, g, g^{-1}, g_{i+2}, \ldots, g_r)$ also has lifting invariant ϵ, since $\widehat{g^{-1}} = (\hat{g})^{-1}$. □

We can now complete the proof of Theorem 4.5.3.

Proof of Theorem 4.5.3. Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ and $\boldsymbol{g'} = (g'_1, \ldots, g'_r)$ be two Nielsen tuples of type $\boldsymbol{C}$. If the type is odd let us assume that they have the same lifting invariant. We need to show that there exists a pure braid $R \in \mathcal{PB}_r$ such that $R(\boldsymbol{g}) = \boldsymbol{g'}$.

If $\boldsymbol{C}$ has the same odd class twice then, since $\boldsymbol{C}$ is lexicographically ordered, it is in odd repetitive form at some position i. Otherwise, $\boldsymbol{C}$ contains no more than three odd classes and hence it contains the double-transposition class at least five times. In particular, $\boldsymbol{C}$ in this case is in even repetitive form at position $i = 1$. In either case, by Proposition 4.5.7, each of $\boldsymbol{g}$ and $\boldsymbol{g'}$ are conjugate by pure braids to some Nielsen tuples in repetitive form at position i. Hence, without loss of generality we can assume that $\boldsymbol{g}$ and $\boldsymbol{g'}$ are themselves in this form.

Let $P = P_{\boldsymbol{g}} = [\underbrace{1, \ldots, 1}_{i-1}, 2, 1, \ldots, 1]$ or $[3, 1, \ldots, 1]$ depending on the case. Let $\boldsymbol{h} = \boldsymbol{g}^P$ and $\boldsymbol{h'} = (\boldsymbol{g'})^P$. By Lemma 4.5.6, $\boldsymbol{h}$ and $\boldsymbol{h'}$ are Nielsen tuples and if they are odd then they have the same lifting invariant. Clearly, $\boldsymbol{h}$ and $\boldsymbol{h'}$ are of the same type. Note that in the odd form case the new type may not be lexicographically ordered! This does not matter since the statement of Theorem 4.5.3 does not require this. We also note that by a more careful choice of i we could in fact ensure that the coalesced type be lexicographically ordered.

By induction and since $r-2 \geq 6$, $\boldsymbol{h}$ and $\boldsymbol{h}'$ lie in the same pure braid orbit and hence there exists a pure braid $Q \in \mathcal{PB}_{r-1}$ or $\mathcal{PB}_{r-2}$, respectively, such that $Q(\boldsymbol{h}) = \boldsymbol{h}'$. Let $R \in \mathcal{PR}_{r,P}$ such that $\phi_P(R) = Q$. By Proposition 4.4.7, $R(\boldsymbol{g}) = \boldsymbol{g}'$. $\square$

It remains to discuss how Theorem 4.5.3 implies the Theorem 4.3.1. For each type $\boldsymbol{C}$, the Nielsen class $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ (when non-empty) is the union of the pure Nielsen classes $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C}^{\sigma})$ for all $\sigma \in S_r$. If $\boldsymbol{C}$ is even then $\mathcal{PB}_r$ acts transitively on each of these pure Nielsen classes, while $\mathcal{B}_r$ also fuses them together into a single orbit (see Lemma 4.5.1), since $\mathcal{B}_r$ maps to S_r surjectively.

Similarly, if $\boldsymbol{C}$ is odd and $\epsilon \in \{1, -1\}$ then $\mathcal{B}_r$ fuses the pure braid orbits with lifting invariant ϵ (when such orbits exist) into a single braid orbit with lifting invariant ϵ. Again, this follows from Lemma 4.5.1.

This completes the proof of Theorem 4.3.1.

CHAPTER 5

A_6 BRAID ORBITS

In this chapter we extend our classification of braid orbits to A_6. This case is of particular interest because the structure of the covers of A_n is exceptional for $n = 6$ or 7. In these exceptional cases then A_n has a six-fold cover, and this will be reflected in the classification of braid orbits. It is hoped that by extending the classification to A_6 the nature of those exceptional types which do not fit within our classification will become clear. The classification of A_6 braid orbits is more complicated than that of A_5 but the general pattern is maintained.

Throughout this chapter we will label the six nontrivial conjugacy classes of A_6, as in [6], by:

- $2A = (1,2)(3,4)^{A_6}$;
- $3A = (1,2,3)^{A_6}$;
- $3B = (1,2,3)(4,5,6)^{A_6}$;
- $4A = (1,2,3,4)(5,6)^{A_6}$;
- $5A = (1,2,3,4,5)^{A_6}$;
- $5B = (1,2,3,5,4)^{A_6}$.

If C is a conjugacy class of A_6 then we let $n_C(\boldsymbol{C})$ denote the number of occurences of C within the type $\boldsymbol{C}$. Additionally, let $n_3(\boldsymbol{C})$ denote the sum $n_{3A}(\boldsymbol{C}) + n_{3B}(\boldsymbol{C})$; and $n_5(\boldsymbol{C})$ denote the sum $n_{5A}(\boldsymbol{C}) + n_{5B}(\boldsymbol{C})$.

5.1 Covers of A_6 and the Lift Invariant

The structure of the covering groups for A_n is well known [38]. For $n \neq 6, 7$, A_n has a single isomorphism class of covers, and these covers have degree 2 [33][34]. For $n = 6$ and $n = 7$ there are further exceptional covers of higher degrees. In particular, A_6 has a universal central extension $6 \cdot A_6$:

$$1 \to C_6 \to 6 \cdot A_6 \stackrel{\theta_6}{\to} A_6 \to 1.$$

of degree 6. The action of the subgroups $\mathbb{Z}_2, \mathbb{Z}_3 \leq \mathbb{Z}_6 \cong Z(6 \cdot A_6)$ gives rise to the covers of degree 3 and 2 respectively [34]. Denote the isomorphism classes of the covering homomorphisms corresponding to the covering groups $2 \cdot A_6$, $3 \cdot A_6$ and $6 \cdot A_6$ by

$$\begin{aligned} &\theta_2 : 2 \cdot A_6 \to A_6, \\ &\theta_3 : 3 \cdot A_6 \to A_6, \text{ and} \\ &\theta_6 : 6 \cdot A_6 \to A_6, \end{aligned}$$

respectively. To simplify statements θ_1 is used to denote the trivial cover of A_6 by itself.

Recall that lifting invariant, introduced in Chapter 4, was defined only for odd order elements. This was due to the fact that an element of order 2 cannot be uniquely lifted to the double cover $2 \cdot A_5$. An examination of the orders of the preimages of an element under the respective covers shows that this problem also occurs for A_6, and is in fact more severe. However, by Lemma 4.2.2, an element of order 2 can be lifted uniquely to the cover $3 \cdot A_6$ and so the definition of the lifting invariant must be extended to admit the possibility. In particular, the lifts of an element of order 4 must be treated carefully. By Lemma 4.2.2 an element of order 4 has a unique lift of order 4 to the covering group $3 \cdot A_6$. However when lifting to the double cover then this is not so.

Proposition 5.1.1. *Let $g \in 4A$, then $\theta_2^{-1}(g)$ contains two nonconjugate elements $\hat{g}_1, \hat{g}_2$ of order 8. Furthermore, if $h \in \theta_2^{-1}(g)$ then either h is conjugate to $\hat{g}_1$ or h is conjugate to $\hat{g}_2$.*

This follows from a more general result classifying the splitting of conjugacy classes upon lifting.

Theorem 5.1.2 ([20] Theorem 3.9)**.** *Let C be a conjugacy class of A_n. Then C splits in $2 \cdot A_n$ if:*

- *elements of C can be decomposed into disjoint cycles such that all cycles are of odd length; or*
- *elements of C can be decomposed into disjoint cycles such that no two cycles have the same length (including 1), and there is at least one cycle of even length.*

The lifting invariant can now be defined. By appealing to Proposition 5.1.1 and Lemma 4.2.2 we can ensure that the lift invariant is well-defined.

Definition 5.1.3. Let $\boldsymbol{C}$ be a type. Define

$$\theta(\boldsymbol{C}) = \begin{cases} \theta_1, & n_{2A}(\boldsymbol{C}) > 0 \text{ and } n_3(\boldsymbol{C}) > 0 \\ \theta_2, & n_{2A}(\boldsymbol{C}) = 0 \text{ and } n_3(\boldsymbol{C}) > 0 \\ \theta_3, & n_{2A}(\boldsymbol{C}) > 0 \text{ and } n_3(\boldsymbol{C}) = 0 \\ \theta_6, & n_{2A}(\boldsymbol{C}) = 0 \text{ and } n_3(\boldsymbol{C}) = 0. \end{cases}$$

We call $\theta(\boldsymbol{C})$ the *covering for type* $\boldsymbol{C}$. The covering of a given type is defined in this way, so as to be the maximal cover for which a unique lifting can be defined.

Definition 5.1.4. Let $g \in A_6$and let $\theta : H \to A_6$ be a covering homomorphism of A_6 with kernel K.

- If $(|K|, o(g)) = 1$ then there exists a unique $h \in H$ such that $o(h) = o(g)$ and $\theta(h) = g$. Let $\widehat{g}$ denote this unique element h.

- If $o(g) = 4$ and $n = (4, |K|) \neq 1$ then by Proposition 5.1.1 the preimage $\theta^{-1}(4A)$ splits into two conjugacy classes denoted C_1 and C_2. Define $\widehat{g}$ to be the element $h \in \theta^{-1}(g)$ such that $h \in C_1$ and $o(h) = 4n$.
- If neither of the above occur then $\widehat{g}$ is undefined for this cover.

The above definition relies on the chosen cover. In particular, we lift an element of type $g \in 4A$ using the first rule if the cover in question is of degree 3; however, if the cover in question is of degree 2 or 6 then we lift using the second rule. The elements of order 4 are particularly troublesome in this respect.

Conjugacy Class	θ_2	θ_3	θ_6
$2A$	$\{4,4\}$	$\{2,6,6\}$	$\{4,4,12,12,12,12\}$
$3A$	$\{3,6\}$	$\{3,3,3\}$	$\{3,3,3,6,6,6\}$
$3B$	$\{3,6\}$	$\{3,3,3\}$	$\{3,3,3,6,6,6\}$
$4A$	$\{8,8\}$	$\{4,12,12\}$	$\{8,8,24,24,24,24\}$
$5A$	$\{5,10\}$	$\{5,15,15\}$	$\{5,10,15,15,30,30\}$
$5B$	$\{5,10\}$	$\{5,15,15\}$	$\{5,10,15,15,30,30\}$

Table 5.1: For each cover θ of A_6 and each conjugacy class C we list the orders of the elements in the preimage of a member of said class.

Let $\boldsymbol{g} = (g_1, \ldots, g_n)$ be a tuple of type $\boldsymbol{C}$. Let $\theta = \theta(\boldsymbol{C})$ be the cover for type $\boldsymbol{C}$. The *lifting invariant of* $\boldsymbol{g}$ is defined by

$$\mathrm{LI}(\boldsymbol{g}) = \mathrm{LI}_\theta(\boldsymbol{g}) = \widehat{g}_1\widehat{g}_2\cdots\widehat{g}_r.$$

When $\boldsymbol{g}$ is a Nielsen tuple, such that the lifting invariant is defined, then the lifting invariant is an element of the kernel of the corresponding cover.

Lemma 5.1.5. *Let θ be a covering homomorphism of A_6, let K denote the kernel of θ, and let $g \in A_6$be such that $\widehat{g}$ is defined for θ. Assume that $(o(g), |K|) = 1$. Then*

$$\widehat{g}^n = \widehat{g^n}.$$

Proof. Let $h = \widehat{g}^n$, let $m = o(g)$ and let $k = |K|$. Clearly $h = a\widehat{g^n}$ for some $a \in K$. Since $(o(g), |K|) = 1$,

$$1 \neq h^k = (a\widehat{g^k}) = a^k\widehat{g^n}^k = \widehat{g^n}^k,$$

and therefore $\widehat{g}^n = h = \widehat{g^n}$. □

The above statement is not true when lifting an element of order $4A$ to the double cover. In fact $((1,2,3,4)(5,6))^2 = (1,3)(2,4)$ and therefore $(1,2,3,4)(5,6)$ cannot be lifted. Inverses, on the other hand, do behave nicely.

Lemma 5.1.6. *Let θ be a covering homomorphism of A_6, let K denote the kernel of θ, and let $g \in A_6$be such that $\widehat{g}$ is defined for θ. Then*

$$\widehat{g}^{-1} = \widehat{g^{-1}}.$$

Type	Genus
$(2A, 2A, 2A, 2A)$	-1
$(2A, 2A, 2A, 3A)$	-1
$(2A, 2A, 3A, 3A)$	-1
$(2A, 3A, 3A, 3A)$	-1
$(3A, 3A, 3A, 3A)$	-1

Table 5.2: Types $\boldsymbol{C}$, up to permutation and outer automorphism, for which the Nielsen class $\mathrm{Ni}^{\mathrm{in}}(A_6, \boldsymbol{C})$ is empty. All such cases then the Riemann-Hurwitz formula predicts that the genus of the corresponding cover is negative. Of course, this cannot occur. This is in contrast to the A_5 exceptions in Table 4.1.

Type	Number of orbits	Expected	Genus
$(2A, 2A, 2A, 5A)$	2	3	0
$(3A, 3A, 3A, 3B)$	1	2	0
$(3A, 3A, 3A, 5A)$	1	2	0
$(3A, 3A, 4A, 4A)$	3	2	1
$(3A, 3A, 5A, 5B)$	3	2	1
$(4A, 4A, 5A, 5A)$	9	6	3
$(4A, 4A, 5A, 5B)$	8	6	3
$(5A, 5A, 5A, 5A)$	12	6	3
$(5A, 5A, 5A, 5B)$	5	6	3
$(5A, 5A, 5B, 5B)$	9	6	3
$(2A, 2A, 2A, 2A, 2A)$	2	3	0
$(3A, 3A, 3A, 3A, 3A)$	1	2	0

Table 5.3: Types for which the lifting invariant does not entirely determine the pure braid orbits on $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$. The third column indicates how many orbits we might expect there to be.

5.2 Discussion of Main Result

The main theorem of this chapter is the following:

Theorem 5.2.1. *For $G = A_6$ and a type $\boldsymbol{C} = (C_1, \ldots, C_r)$, $r \geq 4$, the Nielsen class $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ is non-empty if and only if $\boldsymbol{C}$ is not listed in Table 5.2. Furthermore, given that $\boldsymbol{C}$ is not in Table 5.3,*

- *for the appropriate cover $\theta = \theta(\boldsymbol{C})$ and lifting invariant ϵ, there exists a Nielsen tuple of type $\boldsymbol{C}$ and lifting invariant ϵ.*
- *Any two tuples in $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ with the same lifting invariant are pure braid equivalent.*

Note that the lists shown are in condensed form. The complete lists of exceptional types are obtained from Table 5.2 and Table 5.3 by permutations and outer automorphisms. The orbits of length 3 have also been calculated and may be found in Appendix B. We also remark that the

Type	Number of orbits	Expected	Genus
$(2A, 2A, 2A, 5A)$	2	3	0
$(3A, 3A, 3A, 3B)$	1	2	0
$(3A, 3A, 3A, 5A)$	1	2	0
$(3A, 3A, 4A, 4A)$	3	2	1
$(3A, 3A, 5A, 5B)$	3	2	1
$(4A, 4A, 5A, 5A)$	9	6	3
$(4A, 4A, 5A, 5B)$	8	6	3
$(5A, 5A, 5A, 5B)$	5	6	3
$(5A, 5A, 5B, 5B)$	7	6	3
$(2A, 2A, 2A, 2A, 2A)$	2	3	0
$(3A, 3A, 3A, 3A, 3A)$	1	2	0

Table 5.4: Types for which the lifting invariant does not entirely determine the braid orbits on $Ni(A_6, \boldsymbol{C})$. The 'Expected' column indicates how many orbits we might expect there to be.

above are exceptions for the pure braid orbits and *not* necessarily the braid orbits. For example, the type $(5A, 5A, 5A, 5A)$ has twelve orbits under the pure braid action and just six under the regular braid action, and these six orbits are distinguished by the lifting invariant. Therefore this type should no longer be considered exceptional with respect to the braid action.

In many cases where there are more orbits than expected then any two orbits with the same lifting invariant differ by an outer automorphism. Therefore, up to the action of the full automorphism group these types maynot be considered exceptional. Recall that Fried's 3-cycle classification showed that there are two braid orbits on $\mathrm{Ni}^{\mathrm{in}}(A_n, \boldsymbol{C})$ except when $\boldsymbol{C}$ has length $n = r - 1$ in which case braid group acts transitively. These cases correspond to genus 0 coverings. Looking at the types in Table 5.3 we observe that the types with fewer than the expected number of orbits all have genus 0 except for the type $(5A, 5A, 5A, 5B)$. Thus Fried's condition is not sufficient for determining when there are fewer orbits than expected.

The pure braid result is much stronger and so this is proven. However for applications the regular braid action is usually preferred, and so statements are given in terms of the regular braid group action. Table 5.4 lists those types which are exceptional for the regular braid action. We see that only the type $(5A, 5A, 5A, 5A)$ appears in Table 5.3 but not in Table 5.4. We also note that the type $(5A, 5A, 5B, 5B)$ has 9 orbits under the pure braid action, but only 7 orbits under the regular braid action. It is clearly true that $\mathrm{Ni}^{\mathrm{in}}(A_6, \boldsymbol{C})$ is empty if and only if $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ is empty so there is no need for us to have an alternative version of Table 5.2 for regular braid orbits.

As with A_5 we can translate the main result into a statement concerning the connectivity of the corresponding Hurwitz spaces.

Theorem 5.2.2. *For $G = A_6$ and a type $\boldsymbol{C}$ of length $r \geq 3$, the Hurwitz space $\mathcal{H}(G, \boldsymbol{C})$ is nonempty if and only if $\boldsymbol{C}$ is not in Table 5.2. Furthermore if $\boldsymbol{C}$ is not in Table 5.4 then:*

- *If $n_{2A}(\boldsymbol{C}) > 0$ and $n_3(\boldsymbol{C}) > 0$ then $\mathcal{H}(G, \boldsymbol{C})$ is connected.*
- *If $n_{2A}(\boldsymbol{C}) > 0$ and $n_3 = 0$ then $\mathcal{H}(G, \boldsymbol{C})$ has three components.*
- *If $n_{2A}(\boldsymbol{C}) = 0$ and $n_3 > 0$ then $\mathcal{H}(G, \boldsymbol{C})$ has two components.*
- *If $n_{2A}(\boldsymbol{C}) = 0$ and $n_3 = 0$ then $\mathcal{H}(G, \boldsymbol{C})$ has six components.*

If there is more than one components then separated components are distinguished by the lifting invariant for the appropriate cover.

5.3 A_6 Braid Orbits

Instead of proving Theorem 5.2.1 we prove a pure braid analogue. This simplifies the argument and gives a stronger result.

Theorem 5.3.1. *For $G = A_6$ and a type $\boldsymbol{C} = (C_1, \ldots, C_r)$, $r \geq 3$, the pure Nielsen class $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ is non-empty if and only if $\boldsymbol{C}$ is not a permutation of a type from Table 5.2. Furthermore, if $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ is non-empty and $\boldsymbol{C}$ does not appear in Table 5.3 then the pure braid orbits are distinguished by the lifting invariant appropriate for the given type. Moreover, for every value the lifting invariant might possibly take there exists a tuple of type $\boldsymbol{C}$ realizing this value.*

The above theorem can be broken down into the following classification: Suppose that $\boldsymbol{C}$ is a type not listed in Table 5.2 or Table 5.3. Then

- if $n_{2A}(\boldsymbol{C}) > 0$ and $n_3(\boldsymbol{C}) > 0$ then there is exactly one pure braid orbit;
- if $n_{2A}(\boldsymbol{C}) = 0$ and $n_3(\boldsymbol{C}) > 0$ then there are two pure braid orbits;
- if $n_{2A}(\boldsymbol{C}) > 0$ and $n_3(\boldsymbol{C}) = 0$ then there are three pure braid orbits;
- if $n_{2A}(\boldsymbol{C}) = 0$ and $n_3(\boldsymbol{C}) = 0$ then there are six pure braid orbits.

Type	Orbits
$(2A, 2A, 3A, 4A, 4A, 5A, 5B)$	1
$(2A, 2A, 3A, 3B, 4A, 4A, 5A)$	1
$(2A, 2A, 2A, 2A, X, Y, Z)$	
$(3A, 3A, 3A, 3A, 3A, Y, Z)$	
$(2A, 2A, 2A, 2A, 2A, 2A, 2A, 2A)$	3

Table 5.5: A list of longer types computed with MapClass. For the generic types $(2A, 2A, 2A, 2A, X, Y, Z)$ and $(3A, 3A, 3A, 3A, 3A, X, Y)$ we allow X, Y and Z to range over all conjugacy classes.

In all of the above cases the lifting invariant distinguishes orbits. In the exceptional cases either there is no tuple of the length, in which case the tuple appears in Table 5.2, or there are either too many or too few pure braid orbits; these exceptional types appear in Table 5.3. Notice that we never obtain the correct number orbits but find two of these orbits share the same lift invariant.

5.3.1 Basis for induction.

The proof is by induction on r, the length of $\boldsymbol{C}$. The following lemma anchors the induction.

Lemma 5.3.2. *If $\boldsymbol{C}$ is a type of length $3 \leq r \leq 6$, or $\boldsymbol{C}$ is in Table 5.5 then the conclusion of Theorem 5.3.1 holds.* □

The lemma is established by explicit calculation of all orbits for all types of length $3 \leq r \leq 6$. Note that in view of Lemma 4.5.1, we only need to consider lexicographically ordered types. However we do frequently reorder types and so it is not a condition of any of the arguments that the type shall be ordered. For these lexiographically ordered types, the computation was completed using the MapClass package [23] for the GAP computational algebra system [14]. Data for this computation is included in Appendix B. It was unfeasible for us to calculate braid orbits for all types of length 7 but our induction requires some longer types. These were computed using the splitting method discussed in Chapter 6.

5.3.2 A Normal Form for Tuples

The normal form introduced in Chapter 4 is extended to tuples in A_6.

Definition 5.3.3. Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ be a Nielsen tuple. The tuple $\boldsymbol{g}$ is said to be:

- In *odd normal form* at position i if $g_i = g_{i+1} = g$ for some element g of odd order.
- In *even normal form* at position i if $g_i = g$, $g_{i+1} = g^{-1}$, and $g_{i+2} = g$ for some element g of even order. Note that if g is an even order element in A_6 then g^{-1} and g lie in the same conjugacy class.
- In $(2, 2, 4)$-*form* at position i if $g_i = g_{i+1} = g^2$ and $g_{i+2} = g$ for some element g of order 4.

The odd normal form and even normal form are collectively called the *repetitive normal forms.*

For each of the normal forms, a *normal partition*, denoted $P_{\boldsymbol{g}}$ is defined. Coalescing with respect to the normal partition preserves generation and the lifting invariant. The normal partitions for tuples in odd, even and $(2, 2, 4)$-form are given by

$$[\underbrace{1, \ldots, 1}_{i-1}, 2, 1, \ldots, 1],$$

$$[\underbrace{1, \ldots, 1}_{i-1}, 3, 1, \ldots, 1] \text{ and,}$$

$$[\underbrace{1, \ldots, 1}_{i-1}, 2, 1, \ldots, 1],$$

respectively. The elements in position $i, i+1$ and $i+2$ play an important role and so are called the *distinguished elements* of the tuple.

The three definitions of normal form can also be applied to types. A type $\boldsymbol{C} = (C_1, \ldots, C_r)$ is said to be:

- In even normal form at position i, if $C_i = C_{i+1} = C_{i+2} = C$ for conjugacy class C whose elements have even order;
- In odd normal form at position i, if $C_i = C_{i+1} = C$ for conjugacy class C whose elements have odd order;
- In $(2, 2, 4)$-form at position i, if $C_i = C_{i+1} = 2A$ and $C_{i+2} = 4A$.

Often we are not concerned about the order of the conjugacy classes in the type only that each conjugacy class appears often enough. If a type $\boldsymbol{C}$ is in normal form up to permuting the components of the type, then it is said to be in *unordered normal form.* Note that a type $\boldsymbol{C}$ of length $r \geq 7$ must be in one of the above normal forms, possibly unordered, or

$$\boldsymbol{C} = (2A, 4A, 4A, 3A, 3B, 5A, 5B).$$

Our main result, Theorem 5.2.2, says that the number of components of the Hurwitz space, or alternatively, the number of braid orbits, is controlled by the pair of integers $(n_{2A}(\boldsymbol{C}), n_3(\boldsymbol{C}))$. We say that

- $\boldsymbol{C}$ has $(2,3)$-*shape* if $n_{2A}(\boldsymbol{C}) > 0$ and $n_3(\boldsymbol{C}) > 0$.
- $\boldsymbol{C}$ has 2-*shape* if $n_{2A}(\boldsymbol{C}) > 0$ and $n_3(\boldsymbol{C}) = 0$.
- $\boldsymbol{C}$ has 3-*shape* if $n_{2A}(\boldsymbol{C}) = 0$ and $n_3(\boldsymbol{C}) > 0$.

The tuples in one of the above normal forms are useful for our induction because coalescing such tuples with respect to $P = P_{\boldsymbol{g}}$ gives shorter tuples whilst preserving all the salient properties. The following lemma makes this precise. In all cases the coalescing produces a tuple which does not contain trivial elements.

Lemma 5.3.4. *Suppose $\boldsymbol{g}$ is a tuple in a normal form, $P = P_{\boldsymbol{g}}$, and $\boldsymbol{h} = \boldsymbol{g}^P$. Then*

- *$\boldsymbol{g}$ and $\boldsymbol{h}$ generate the same subgroup of G;*
- *If $\boldsymbol{h}$ and $\boldsymbol{g}$ have the same shape then $LI_\theta(\boldsymbol{g}) = LI(\boldsymbol{h})$ for the appropriate cover θ.*

When discussing braid orbits for A_5 the analogue of the above lemma also claimed that coalescing with respect to the normal partition gives tuples of the same shape. This is not necessarily so here; coalesced tuples in $(2,2,4)$ form may no longer contain an element of order 2. However, if such a type is not in repetitive form (possibly unordered) then it has a very limited structure: $n_C(\boldsymbol{C}) \leq 1$ for every odd conjugacy class, and $n_C(\boldsymbol{C}) \leq 2$ for every even order conjugacy class. These types are small enough that they may be dealt with separately.

Proof. For the first claim we consider each of the possible forms:

- $\boldsymbol{g}$ is in odd repetitive form, then $\langle g^2 \rangle = \langle g, g \rangle$ for odd g, therefore $\langle \boldsymbol{g}^P \rangle = \langle \boldsymbol{g} \rangle$.
- $\boldsymbol{g}$ is in even repetitive form, then $\langle g \rangle = \langle g, g^{-1}, g \rangle$ for even g, therefore $\langle \boldsymbol{g}^P \rangle = \langle \boldsymbol{g} \rangle$.
- $\boldsymbol{g}$ is in $(2,2,4)$-form, then $\langle g^2, g^2, g \rangle = \langle g \rangle$ for g of order 4, therefore $\langle \boldsymbol{g}^P \rangle = \langle \boldsymbol{g} \rangle$.

For the above forms it is obvious that $LI(\boldsymbol{g}) = LI(\boldsymbol{g}^{P_g})$. This follows from Lemma 5.1.5. We demonstrate for elements in $(2,2,4)$-form. The other cases are similar.

$$\begin{aligned} LI(\boldsymbol{g}) &= \hat{g}_1 \cdots \hat{g}_{i-1}(\widehat{g^2}\widehat{g^2}\hat{g})\hat{g}_{i+3}\cdots \hat{g}_r \\ &= \hat{g}_1 \cdots \hat{g}_{i-1}\hat{g}\hat{g}_{i+3}\cdots \hat{g}_r \\ &= LI(\boldsymbol{g}^{P_g}), \end{aligned}$$

completing the proof. □

We wish to establish that, outside of a small list of exceptional types, a tuple of type $\boldsymbol{C}$ may be placed in one of the normal forms listed previously. We shall be considering subtypes of $\boldsymbol{C}$ and thus introduce the following definition.

Definition 5.3.5. Let $\boldsymbol{C} = (C_1, \ldots, C_r)$ be a type of length r, and let $1 \leq k \leq r$ be an integer. Then a subtype of the form $\boldsymbol{D} = (C_k, \ldots, C_r)$ is called a *suffix-subtype* of $\boldsymbol{C}$. A subtype of the form $\boldsymbol{D} = (C_1, \ldots, C_k)$ is called a *prefix-subtype* of $\boldsymbol{C}$.

The argument used is one which will be repeated throughout this chapter, and is encapsulated by the following lemma.

Lemma 5.3.6. *Let $\boldsymbol{C} = (C_1, \ldots, C_r)$ be a type, and suppose that the pure braid orbits of $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ are distinguished entirely by the lifting invariant for the appropriate cover θ. Let $\boldsymbol{D} = (C_k, \ldots, C_r)$ be a suffix-subtype of $\boldsymbol{C}$; suppose that $\boldsymbol{C}$ and $\boldsymbol{D}$ are of the same shape and that there exists a Nielsen tuple $\boldsymbol{h} = (h_k, \ldots, h_r)$ of type $\boldsymbol{D}$ with lifting invariant ϵ. If there exists $g_1, \ldots, g_{k-1}$ such that $LI((g_1, \ldots, g_{k-1})) = 1$ and $g_1 \cdots g_{k-1} = 1$ then every tuple of type $\boldsymbol{C}$ and lifting invariant ϵ is pure braid equivalent to*

$$(g_1, \ldots, g_{k-1}, h_k, \ldots, h_r).$$

Proof. The tuple $\boldsymbol{g} = (g_1, \ldots, g_{k-1}, h_k, \ldots, h_r)$ is clearly a Nielsen tuple. The lifting invariant of $\boldsymbol{g}$ is equal to $LI((g_1, \ldots, g_{k-1}))LI(\boldsymbol{h}) = \epsilon$. Therefore, as the pure braid orbits are distinguished by the lifting invariant, we are done. □

This process of building a larger type from a suffix-subtype is repeated frequently. The following lemma shows Lemma 5.3.6 in action. It shows that types in repetitive form almost always contain a tuple in repetitive form.

Lemma 5.3.7. *Let $\boldsymbol{C} = (C_1, \dots, C_r)$ be type of length $r = 6$ and assume that $\boldsymbol{C}$ is in repetitive form at position i and assume that $\boldsymbol{C}$ is not, up to permutation and automorphism, one of the following exceptional types:*

- $(3A, 3A, 3A, 3A, 3A, 3A)$*;*
- $(2A, 2A, 2A, 2A, 2A, 3A)$*;*
- $(2A, 2A, 2A, 2A, 2A, 5A)$*; or*
- $(2A, 2A, 2A, 2A, 2A, 2A)$*.*

Then every pure braid orbit of $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ *contains a tuple in repetitive normal form at position* i*.*

Proof. Let $\boldsymbol{C} = (C_1, \dots, C_r)$ and assume that $\boldsymbol{C}$ is in odd repetitive form at position i, therefore $C_i = C_{i+1} = C$ is of odd type. Let C^2 denote the conjugacy class of the squares of elements in class C. In particular, if $C = 5A$ (respectively $5B$) then $C^2 = 5B$ (respectively $5A$) and if $C = 3A$ (respectively $3B$) then $C^2 = 3A$ (respectively $3B$). Let $\boldsymbol{D}$ denote the coalesced type

$$(C_1, \dots, C_{i-1}, C^2, C_{i+2}, \dots, C_r)$$

which we know is of length 5 and hence not in Table 5.3 unless $\boldsymbol{C}$ is the type $(3A, 3A, 3A, 3A, 3A, 3A)$ (or $\boldsymbol{C} = (3B, 3B, 3B, 3B, 3B, 3B)$). Leaving these exceptions aside, for each possible value ϵ that the lifting invariant may take there exists a single pure braid orbit Ω with the property that a tuple lies in Ω if and only if it has lifting invariant ϵ. Choose a Nielsen tuple $\boldsymbol{h} = (h_1, \dots, h_{i-1}, h, h_{i+1}, \dots, h_{r-1})$ lying in Ω. So in particular $\boldsymbol{h}$ has lifting invariant ϵ. Since h is of odd order then there exists a unique g such that $g^2 = h$. Let $\boldsymbol{g} = (h_1, \dots, h_{i-1}, g, g, h_{i+1}, \dots, h_{r-1})$. Then $\boldsymbol{g}$ is a Nielsen tuple of lifting invariant ϵ and the claim holds by Lemma 5.3.6.

Now assume that $\boldsymbol{C}$ contains no repeated odd conjugacy class and that $\boldsymbol{C}$, is of even repetitive type at position i. Therefore $C_i = C_{i+1} = C_{i+2} = C$ and $C = 2A$ or $C = 4A$. As in the previous argument we consider the smaller type $\boldsymbol{D} = (C_1, \dots, C_{i-1}, C, C_{i+3}, \dots, C_r)$ of length 4. Note that it could well be the case that $\boldsymbol{C}$ and $\boldsymbol{D}$ are in Table 5.2 or Table 5.3. However when this occurs there is almost always a repeated class of odd order elements, contradicting our assumption that there is no repeated odd conjugacy class. The only true exceptions are when $\boldsymbol{D} = (2A, 2A, 2A, 2A)$, $(2A, 2A, 2A, 3A)$, or $(2A, 2A, 2A, 5A)$ (up to permutation and automorphism), i.e., $\boldsymbol{C} = (2A, 2A, 2A, 2A, 2A, 2A)$ or $\boldsymbol{C} = (2A, 2A, 2A, 2A, 2A, 3A)$, or $\boldsymbol{C} = (2A, 2A, 2A, 2A, 2A, 5A)$. These are remaining exceptional cases from the statement of the lemma, so let us assume that $\boldsymbol{D}$ is not in Table 5.2 or Table 5.3.

Again we observe that for all possible values the lifting invariant may take there is a single non-empty pure braid orbit Ω of type $\boldsymbol{D}$. Pick a tuple

$$\boldsymbol{h} = (h_1, \dots, h_{i-1}, h, h_{i+1}, \dots, h_{r-2})$$

in Ω, where $h \in C$. Consider the length r tuple

$$\boldsymbol{g} = (h_1, \dots, h_{i-1}, h, h^{-1}, h, h_{i+1}, \dots, h_{r-2}).$$

Note that this tuple is of type $\boldsymbol{C}$, has lifting invariant ϵ, is generating, and has product one. Hence, by Lemma 5.3.6 the pure braid orbit of $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ with lifting ϵ contains $\boldsymbol{g}$: a tuple in even repetitive form at position i. □

The following lemma is a similar result to Lemma 5.3.7 but for types in $(2, 2, 4)$-form.

Lemma 5.3.8. *Let $\boldsymbol{C}$ be a type of length $r = 6$ in $(2, 2, 4)$-form at position i. Furthermore, suppose that $\boldsymbol{C}$ is not in unordered repetitive form. Then either*

- $\boldsymbol{C} = (2A, 2A, 4A, 4A, 5A, 5B)$*; or*
- *every pure braid orbit Ω of $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ contains a tuple in $(2, 2, 4)$-form at position i.*

Proof. First assume that $n_3(\boldsymbol{C}) = 0$. Then either $\boldsymbol{C}$ contains a repeated odd class, $\boldsymbol{C}$ contains a triple of repeated even classes – both of which would mean the type is in repetitive normal form – or $\boldsymbol{C} = (4A, 4A, 2A, 2A, 5A, 5B)$, the exceptional case.

Now suppose that $\boldsymbol{C}$ is in $(2, 2, 4)$-form at the first position, that $\boldsymbol{C} = (2A, 2A, 4A, C_4, C_5, C_6)$, and $n_3(\boldsymbol{C}) > 0$. In particular $\mathcal{PB}_r$ acts transitively on $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ by Lemma 5.3.2. Let $\boldsymbol{D} = (4A, C_4, C_5, C_6)$. There are no types in Table 5.2 such that $n_{4A} > 0$, therefore $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{D})$ is nonempty. Moreover, $n_3(\boldsymbol{D}) > 0$ and so does $\boldsymbol{D}$ does not appear in Table 5.3. Let $\boldsymbol{h} = (h =$

$h_1, h_2, h_3, h_4)$ be a tuple in $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{D})$. Then apply Lemma 5.3.6 to the tuple $(h^2, h^2, h, h_2, h_3, h_4)$ which is of type $\boldsymbol{C}$. Therefore every pure braid orbit of $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ must contain a tuple in $(2,2,4)$-form at position 1, and hence by Lemma 5.3.22, position i. □

At this point we summarise our progress: if $\boldsymbol{C}$ is a type of length 6 in normal form then every pure braid orbit of $\mathrm{PNi}^{\mathrm{in}}(G, \boldsymbol{C})$ contains a tuple in normal form except for the following types:

- $(2A, 2A, 2A, 2A, 2A, 2A)$
- $(2A, 2A, 2A, 2A, 2A, 3A)$
- $(2A, 2A, 2A, 2A, 2A, 5A)$
- $(2A, 2A, 4A, 4A, 5A, 5B)$
- $(3A, 3A, 3A, 3A, 3A, 3A)$

These types are genuine exceptions. Consider, for example, the type

$$\boldsymbol{C} = (2A, 2A, 2A, 2A, 2A, 5A).$$

Suppose that for each of its three orbits, Ω_1, Ω_2 and Ω_3, corresponding to the three values the lifting invariant takes, there exists a tuple in even repetitive form. Label the three tuples. Coalescing with respect to the normal partitions gives three tuples of type $(2A, 2A, 2A, 5A)$, and each with a different lifting invariant. However we know that there are just two pure braid orbits of $\mathrm{PNi}^{\mathrm{in}}(G, (2A, 2A, 2A, 5A))$.

5.3.3 Generating Subtuples

In order to extend the results of the previous chapter to types of greater length we want to coalesce with respect to a partition whilst still maintaining key properties such as the shape and generation. This subsection deals with these concerns. To begin, we comment generally on how we can generate A_6. The approach taken is to consider chains of maximal subgroups of G, showing that by appropriately braiding we can always ensure the inclusions in such chains are proper and that the length of such chains is small. This approach does not rely on properties of A_6, other than for the calculation of lengths of chains, and so can be applied more generally. No claims are made concerning the optimality of these results.

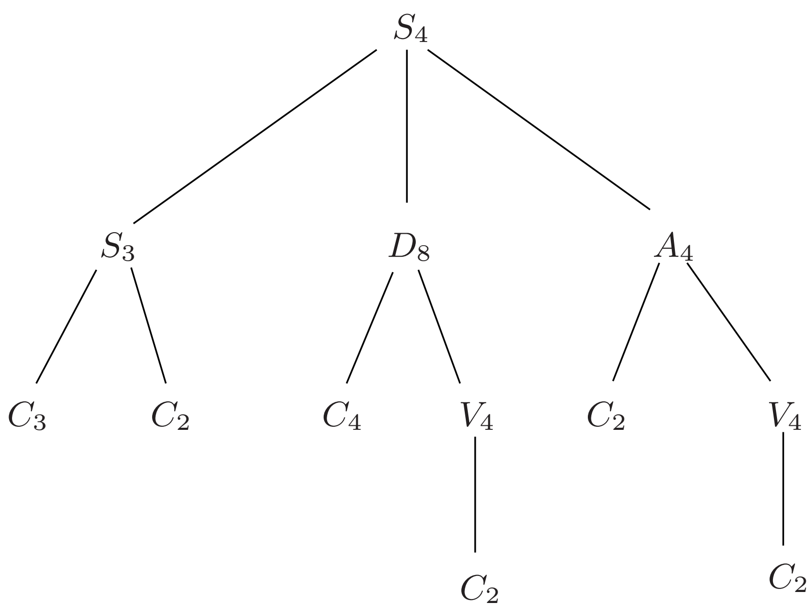


Figure 5.1: Chains of isomorphism classes of subgroups for the maximal subgroup S_4

Lemma 5.3.9. *Suppose that $H_1, \ldots, H_k$ are proper subgroups of $G = A_6$ and that*

$$H_1 \subsetneqq H_2 \subsetneqq \cdots \subsetneqq H_k.$$

Then $k \leq 4$. Moreover, if H_1 is cyclic and of order 4 then $k \leq 3$.

Proof. This follows by considering the longest possible chains of maximal subgroups in G. The maximal subgroups of A_6 can be found in [6]. Figure 5.1, Figure 5.3 and 5.2 show the maximal subgroup chains for the three isomorphism classes of maximal subgroups of A_6: A_5, S_4 and $(C_3 \times C_3) \rtimes C_4$. □

Using the previous lemma we argue that it is always possible, via a sequence of braid moves, to transform our tuple into one in which the initial elements generate.

Definition 5.3.10. A tuple $\boldsymbol{g} = (g_1, \ldots, g_r)$ is said to have a *generating n-head* if the initial n elements $g_1, \ldots, g_n$ generate G. Of course a Nielsen tuple of length r always has a generating $(r-1)$-head.

To avoid overusing chains of complex braid moves we describe two useful sequences which are used repeatedly. We recall from Chapter 4 that the pure braids act by conjugation. Recall Lemma 4.4.6:

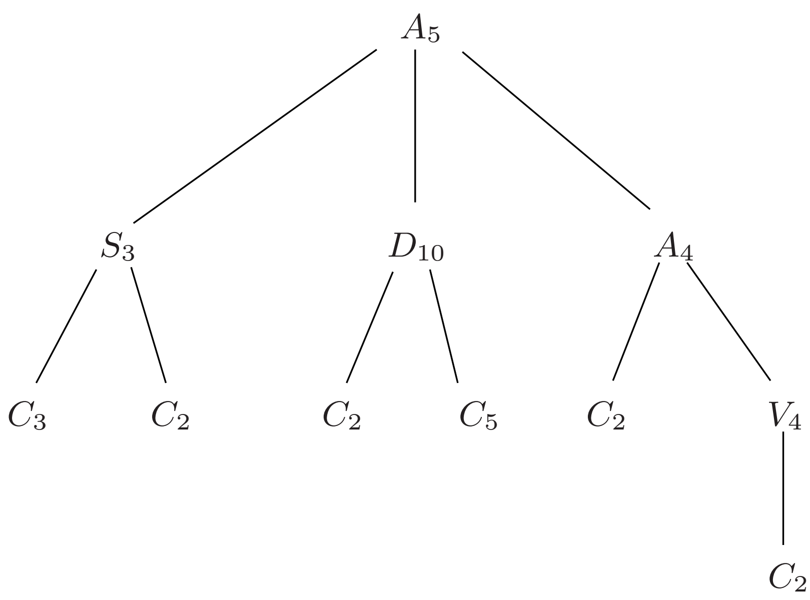


Figure 5.2: Chain of isomorphism classes of subgroups for the maximal subgroup A_5

Lemma 5.3.11 (Lemma 4.4.6)**.** *Let $P = [n_1, \ldots, n_k]$ be an admissible partition of $\{1, \ldots, r\}$. If $\boldsymbol{g}$ is a Nielsen tuple and $R \in PR_{r,P}$ then*

$$R(\boldsymbol{g}) = (g_1^{s_1}, \ldots, g_{m_1}^{s_1}, g_{m_1+1}^{s_2}, \ldots, g_{m_2}^{s_2}, \ldots, g_{m_{k-1}+1}^{s_k}, \ldots, g_r^{s_k})$$

for some $s_i \in G$ and where $m_i = n_1 + \cdots + n_i$.

If a pure braid is used to conjugate a component of the tuple as described by the lemma, then we say that we have *conjugated via pure braids.* In addition to being able to act via conjugation we can also choose our braid moves so that we can shift a component of our tuple.

Proposition 5.3.12. *Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ be a Nielsen tuple. Then there exists $Q \in \mathcal{B}_r$ such that the j-th component of $Q(\boldsymbol{g})$ is g_i. Moreover $Q(g_k) = g_k$ for all g_k not lying between g_i and g_j.*

Proof. Suppose that $i < j$, and let

$$Q = Q_i Q_{i+1} \cdots Q_{j-1}.$$

This braid behaves as required. □

The invariance of those elements that lie outside of the range of permutation is important. It

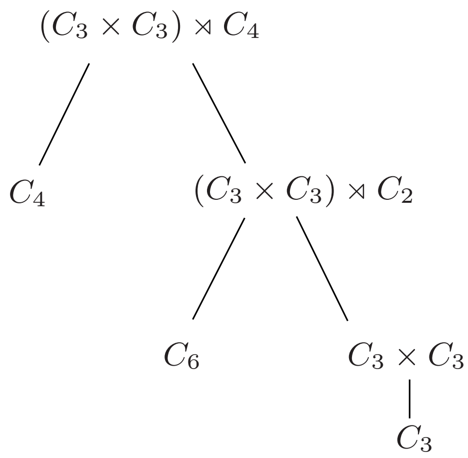


Figure 5.3: Chains of isomorphism classes of subgroups for the maximal subgroup $(C_3 \times C_3) \rtimes C_4$

is easy to see that in fact we may move blocks of adjacent components in a similar manner by using the ribbon braid equivalents of the braid in the proof of Proposition 5.3.12.

Definition 5.3.13. Let $Q \in \mathcal{B}_r$, let $A \subset \{1, \ldots, r\}$, and let $\sigma \in S_r$ be the permutation associated to Q. The braid Q is said to *act purely on* A if $\sigma(a) = a$ for all $a \in A$. Moreover, if $A_{\boldsymbol{g}} = \{g_a \in \boldsymbol{G} \mid a \in A\}$ then Q is said to act purely on $A_{\boldsymbol{g}}$.

Lemma 5.3.14. *Let $\boldsymbol{g} = (g_1, \ldots, g_r)$ be a Nielsen tuple and let $g_1, g_2, g_3 \in 2A$. There exists a pure braid $P \in \mathcal{PB}_r$ such that $P(\boldsymbol{g}) = (g_1', \ldots, g_r')$, $K = \langle g_1, g_2', g_3' \rangle$, and either $K = G$ or there exist proper subgroups K_1, K_2 of K such that*

$$1 \lneqq K_1 \lneqq K_2 \lneqq K.$$

Proof. The following claim is shown first.

Claim. *There exists a pure braid $W \in \mathcal{PB}_r$ such that, if $W(\boldsymbol{g}) = (w_1, \ldots, w_r)$, then $\langle w_1 \rangle \neq \langle w_2 \rangle$, $\langle w_1 \rangle \neq \langle w_3 \rangle$, $\langle w_2 \rangle \neq \langle w_3 \rangle$.*

To prove the claim observe that we may assume $g_2 \notin \langle g_1 \rangle$. Otherwise ${g_2}^{g_s} \notin \langle g_1 \rangle$ for some $s \geq 3$. Therefore, by Lemma 5.3.11, there exists a pure braid which conjugates g_2 out of $\langle g_1 \rangle$. Similarly we may also assume that $g_3 \notin \langle g_1 \rangle$. Therefore $g_1 \neq g_2$ and $g_1 \neq g_3$. If $g_3 \neq g_2$ then the claim holds, so assume that $g_3 = g_2$.

Let H denote the initial subgroup $\langle g_1, g_2 \rangle$. Since $g_1 \cdots g_r = 1$ then any $r-1$ cardinality subset of $\{g_1, \ldots, g_r\}$ generates G. In particular $S = \{g_2, \ldots, g_r\}$ also generates G. Since $\langle g_2 \rangle$ is not normal in G, and S generates then there exists $s \geq 4$ such that ${g_3}^{g_s} \notin \langle g_2 \rangle$. Without loss of generality assume that $s = 4$. If ${g_3}^{g_4} \notin H$ then the conclusion of the lemma holds, as applying the braid ${Q_2}^2$ gives a new tuple whose initial three elements are $(g_1, g_2, {g_3}^{g_4})$ and

$$\langle g_1 \rangle \gneqq \langle g_1, g_2 \rangle \gneqq \langle g_1, g_2, {g_3}^{g_4} \rangle.$$

If, however, ${g_3}^{g_4} \in H$ then the claim holds unless $\langle g_3^{g_4} \rangle = \langle g_1 \rangle$. Therefore assume $g_1 = {g_3}^{g_4}$.

The set $S' = \{g_1, g_2, g_3, g_5, \ldots, g_r\}$ generates G and $g_3^{g_s} = g_2^{g_s} \in H$ for all $g_s \in S'$. However, $H = \langle g_1, g_2 \rangle$ is not normal in G and so there exists some $t \geq 5$ such that $g_1^{g_t} \notin H$. Therefore

$${g_3}^{g_4 g_t} = {g_1}^{g_t} \notin H.$$

The pure braid $Q_3^{-1} Q_4 \cdots Q_{t-1} Q_t^{-1} \cdots Q_4^{-1} Q_3$ transforms the initial three elements

$$(g_1, g_2, g_3) \mapsto (g_1, g_2, g_3^{g_4 g_t}) = (g_1, g_2, g_1^{g_t}).$$

and therefore

$$\langle g_1 \rangle \gneqq \langle g_1, g_2 \rangle \gneqq \langle g_1, g_2, g_1^{g_t} \rangle.$$

This concludes the proof of the claim. Therefore assume that $g_1 \neq g_2$, $g_1 \neq g_3$ and $g_2 \neq g_3$. Then the conclusion of the lemma holds unless $\langle g_1, g_2 \rangle = \langle g_1, g_2, g_3 \rangle$ and similarly

$$\langle g_1, g_3 \rangle = \langle g_1, g_2, g_3 \rangle, \tag{5.1}$$

$$\langle g_2, g_3 \rangle = \langle g_1, g_2, g_3 \rangle. \tag{5.2}$$

Let $K = \langle g_1, g_2, g_3 \rangle$ and suppose that $g_3^{g_s} \in K$ for all $s \geq 5$. If not then there exists some $s \geq 5$ such that $g_3^{g_s} \notin K$ and so by Lemma 5.3.11 there exists a pure braid transforming the initial three elements of the tuple:

$$(g_1, g_2, g_3) \mapsto (g_1, g_2, g_3^{g_s}),$$

giving

$$\langle g_1 \rangle \gneqq \langle g_1, g_2 \rangle = K \gneqq \langle g_1, g_2, g_3^{g_s} \rangle,$$

and the conclusion of the lemma holds.

However $K = \langle g_2, g_3 \rangle = \langle g_3, g_2^{g_3} \rangle$. So as g_3 and $g_2^{g_3}$ generate K, and K is not normal in G, then there must exist some $s \geq 5$ such that $(g_2^{g_3})^{g_S} \notin K$. Applying the braid $Q = Q_2^{-1} Q_3 \cdots Q_{s-1} Q_s^2 Q_{s_1}^{-1} \cdots Q_3^{-1}$ transforms the initial three elements of the tuple in the following way:

$$(g_1, g_2, g_3) \mapsto (g_1, g_3, g_2^{g_3 g_s}).$$

Therefore we have

$$\langle g_1 \rangle \subsetneqq \langle g_1, g_3 \rangle = K \subsetneqq \langle g_1, g_3, g_2^{g_3 g_s} \rangle.$$

Finally applying the braid Q_2 makes the braiding pure and does so without altering the group that the initial three elements generate. □

Lemma 5.3.15. *Let $\boldsymbol{C} = (C_1, \ldots, C_r)$ be a type of length $r \geq 7$ in normal form in the first position, and let $\boldsymbol{g}$ be a Nielsen tuple of type $\boldsymbol{C}$. Then there exists $Q \in \mathcal{B}_r$ such that $Q(\boldsymbol{g}) = (g'_1, \ldots, g'_r)$ has a generating head of length 5. Furthermore, Q may be chosen so that it acts purely on the initial distinguished elements of $\boldsymbol{g}$ (two if $\boldsymbol{g}$ is in odd normal form and three otherwise).*

Proof. By Lemma 5.3.9 it suffices to show that there exists a braid $P \in \mathcal{PB}_r$ such that, if $P(\boldsymbol{g}) = (g'_1, \ldots, g'_5)$, and $K = \langle g'_1, \ldots, g'_5 \rangle$, then one of the following holds

- There exists a chain of proper subgroups

$$1 \neq H_1 \subsetneqq \cdots \subsetneqq H_5 = K$$

- There exists a chain of proper subgroups

$$1 \neq H_1 \subsetneqq \cdots \subsetneqq H_4 = K$$

  and H_1 is cyclic of order 4.

First suppose that $\boldsymbol{C} = (C_1, \ldots, C_r)$ is in even normal form and $C_1 = C_2 = C_3 = 2A$. Then by Lemma 5.3.14 we may assume that $K = \langle g_1, g_2, g_3 \rangle$ is equal to the whole group K or there exists proper subgroups $K_1, K_2 \subsetneqq K$ such that

$$1 \neq K_1 \subsetneqq K_2 \subsetneqq K.$$

By Lemma 5.3.9 it suffices to show that this chain can be extended to a chain of length 5 using braids acting purely on the three initial elements. If $K \neq G$ then there exists some g_j such that $g_j \notin K$. By Proposition 5.3.12 there exists braid Q' such that g_j is in the fourth position of the tuple, Q' acts purely on the three initial elements and

$$K \gneqq \langle g_1, g_2, g_3, g_j \rangle.$$

Repeat this process to find a braid $W \in \mathcal{B}_r$, such that $W(\boldsymbol{g}) = (g_1, g_2, g_3, g_j, h_5, \ldots, h_r)$, $h_5 \notin \langle g_1, \ldots, g_j \rangle$ and therefore

$$K \gneqq \langle g_1, g_2, g_3, g_j \rangle \gneqq \langle g_1, g_2, g_3, g_j, h_5 \rangle$$

as required.

The second case considered is when $\boldsymbol{C} = (C_1, \ldots, C_r)$ is in even normal form and $C_1 = C_2 = C_3 = 4A$. Then we may assume that $\langle g_1 \rangle \gneqq \langle g_1, g_2 \rangle$ by conjugating g_2 out of $\langle g_1 \rangle$. Therefore $C_4 \cong \langle g_1 \rangle \gneqq \langle g_1, g_2 \rangle = K$. Assume that $g_3 \in K$, if not then we have a chain of proper subgroups of length 3:

$$\langle g_1 \rangle \gneqq \langle g_1, g_2 \rangle \gneqq g_1, g_2, g_3 \rangle,$$

and so argue as in the previous paragraph. Thus $\langle g_1, g_2, g_3 \rangle = K$. As before, note that if $K \neq G$ then there exist $g_j \notin H$. By Proposition 5.3.12 there exists a braid $Q_1 \in \mathcal{B}_r$, acting purely on the initial elements of the tuple, such that g_j is moved into the fourth position of the tuple. Repeat if necessary producing a new tuple

$$(g_1', \ldots, g_r')$$

such that there is the following chain of subgroups

$$\langle g_1' \rangle \gneqq \langle g_1', g_2' \rangle \gneqq \langle g_1', g_2', g_4' \rangle \gneqq \langle g_1', g_2', g_4', g_5' \rangle$$

However, $\langle g_1 \rangle$ is cyclic of order 4 and so we are done.

The third case considered is when $\boldsymbol{C}$ is in $(2, 2, 4)$-form. Apply the braid $Q_2 Q_1$ transforming the tuple:

$$(g_1, g_2, g_3, \ldots, g_r) \mapsto \langle g_3, g_1^{g_3}, g_2^{g_3}, g_4, \ldots, g_r) = \boldsymbol{g}'.$$

This is a tuple whose first element generates a cyclic subgroup of order 4. Argue as in the previous paragraph to find a braid $W \in \mathcal{B}_r$, acting purely on the first three elements of the tuple, such that, if $W(\boldsymbol{g}') = (w_1, \ldots, w_r)$ and $K = \langle w_1, \ldots, w_5 \rangle = G$. Therefore, the conjugate of W by $(Q_2 Q_1)^{-1}$ acts purely on the initial elements of $\boldsymbol{g}'$ and the first 5 elements of $W^{Q_1^{-1} Q_2^{-1}}(\boldsymbol{g})$ generate G.

Finally suppose that $\boldsymbol{C}$ is in odd normal form. In this case then it is necessary to act purely only on the two initial elements of $\boldsymbol{g}$. As before, assume that $\langle g_1 \rangle \neq \langle g_2 \rangle$. Let $H = \langle g_1, g_2 \rangle$ and note that because $\boldsymbol{g}$ generates G then there exists some g_j such that $g_j \notin H$. Therefore, there is some braid, acting purely on the initial two elements, which moves g_j into the third position, giving

$$\langle g_1 \rangle \lneqq \langle g_1, g_2 \rangle \lneqq \langle g_1, g_2, g_3 \rangle.$$

Repeat this process if necessary to find a pure braid Q whose action on $\boldsymbol{g}$ gives a generating head of length 5. $\square$

Using the above lemma we may always argue that a tuple has a generating head of length 5. If one were to coalesce with respect to the partition $[1, \ldots, 1, r-5]$ then the coalesced tuple may not be a Nielsen tuple because the coalesced tail has product one (recall that we require all elements of a Nielsen tuple be nontrivial). The following lemma demonstrates that such occurrences can always be avoided.

Lemma 5.3.16. *Suppose that $\boldsymbol{g} = (g_1, \ldots, g_r)$ is a Nielsen tuple and that $g_1, \ldots, g_k$ generate G. If $g_{k+1} g_{k+2} \cdots g_r = 1$ then there exists $P \in \mathcal{PB}_r$ such that, if $P(\boldsymbol{g}) = (g'_1 \ldots, g'_k)$, then $g'_1, \ldots, g'_k, g'_{k+1} g'_{k+2} \cdots g'_r$ generate G and $g'_{k+1} \cdots g'_r \neq 1$. In particular coalescing $P(\boldsymbol{g})$ with respect to the partition $[1, \ldots, 1, r-k]$ yields a Nielsen tuple.*

Proof. Since $g_1, \ldots, g_k$ generate G and G is simple, there must be some $1 \leq l < m \leq r$ such that $g_l^{g_m} \neq g_l$. Without loss of generality assume that $l = k$ and $m = k+1$. Applying the square Q_k^{-2} transforms $\boldsymbol{g}$:

$$(g_1, \ldots, g_r) \mapsto (g_1, \ldots, g_{k-1}, g_k^{g_{k+1}}, g_{k+1}^{g_k g_{k+1}}, g_{k+2}, \ldots, g_r)$$

Since $g_k^{g_{k+1}} \neq g_k$ then

$$g_{k+1}^{g_k g_{k+1}} g_{k+2} \cdots g_r \neq 1,$$

and

$$g_k^{g_{k+1}} g_{k+1}^{g_k g_{k+1}} g_{k+2} \cdots g_r = g_k.$$

Therefore, since $\{g_1, \ldots, g_k\}$ is a generating set, the set

$$\{g_1, \ldots, g_k^{g_{k+1}}, g_{k+1}^{g_k g_{k+1}} \cdots g_r\}$$

also generates G. Thus coalescing with respect to $[1, \ldots, 1, r-k]$ gives a Nielsen tuple. □

5.3.4 Longer Tuples

Lemma 5.3.7 and Lemma 5.3.8 form the basis for our proof that tuples of a given length may always be placed in normal form. In this section we prove this result. We are required to consider longer types in order to ensure that all types have tuples in normal form, because, as we have already seen there are some types of length six for which we can guarantee that there is not a tuple of normal form.

Lemma 5.3.17. *Let $\boldsymbol{g}$ be a tuple of length 7 whose type, $\boldsymbol{C}$, is in normal form at position i. Further suppose that $\boldsymbol{C} \neq (2A)^7$; then every pure braid orbit of $\boldsymbol{g}$ contains a tuple in normal form at position i.*

Proof. Without loss of generality assume that the type $\boldsymbol{C}$ is in normal form in the first position. It follows from Proposition 5.3.12 that we may argue in this fashion. By Lemma 5.3.15, there exists some braid $Q \in \mathcal{B}_r$ such that the initial 5 elements of $Q(\boldsymbol{g})$ generate $G = A_6$. Currently there is no guarantee that Q is a pure braid; however Lemma 5.3.15 says that the action on distinguished positions has been pure, i.e., subject only to pure braid moves. Coalesce with respect to the partition $P = [1, \ldots, 1, 2]$ obtaining a new tuple $\boldsymbol{g'} = (g'_1, \ldots, g'_6)$ of length 6. Note that $\boldsymbol{g'}$ generates G. Suppose that upon coalescing $g'_6 = 1$, in which case $\boldsymbol{g'}$ is not a Nielsen tuple. Then by Lemma 5.3.16, there exists $S \in \mathcal{PB}_r$ such that $(S(\boldsymbol{g}))^P$ contains no trivial elements. Therefore, assume that g'_6 is nontrivial and $\boldsymbol{g'}$ is a Nielsen tuple. Let $\boldsymbol{D}$ denote the type of the coalesced tuple. Since care was taken to maintain the distinguished classes at the front of the tuple, $\boldsymbol{D}$ is in normal form in the first position. Consider a case distinction.

First suppose that $\boldsymbol{C}$ is in odd normal form in the first position. Also suppose that $\boldsymbol{D} \neq (3A, 3A, 3A, 3A, 3A)$, then by Lemma 5.3.7, there exists some pure braid S' such that $S'(\boldsymbol{g'})$ is in repetitive form. Therefore, by Proposition 4.4.7, there is some ribbon braid $R \in \mathcal{PR}_{7,P}$ such that $R(g')$ is in repetitive form in the first position.

If, on the other hand, $\boldsymbol{D} = (3A)^6$, then

$$\boldsymbol{C} = (3A, 3A, 3A, 3A, 3A, X, Y)$$

for conjugacy classes X and Y. Lemma 5.3.20 says that in such cases every braid orbit of $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ contains a tuple in odd normal form as required.

Next, consider the case when $\boldsymbol{C}$ is in even repetitve form in the first position. And suppose that $\boldsymbol{D}$ is not one of the following types:

$$(2A, 2A, 2A, 2A, 2A, 2A)$$
$$(2A, 2A, 2A, 2A, 2A, 3A)$$
$$(2A, 2A, 2A, 2A, 2A, 5A).$$

These cases are treated separately by Lemma 5.3.18. Then there are two possibilites:

1. The coalesced type $\boldsymbol{D}$ is not in (unordered) odd repetitive form. Therefore $n_C(\boldsymbol{D}) \leq 1$ for every odd conjugacy class C.

2. The coalesced type $\boldsymbol{D}$ is in odd repetitive form. Thus $\boldsymbol{D}$ contains an odd conjugate pair.

Suppose that the first of these possibilities is true. By Lemma 5.3.7 there exists some pure braid $Q \in \mathcal{PB}_6$ such that $Q(\boldsymbol{g}')$ is in even repetitive form in the first position and hence, by Proposition 4.4.7, there is some ribbon braid $R \in \mathcal{PR}_{7,P}$ such that $R(\boldsymbol{g})$ is in repetitive even form in the first posiiton as required.

If the second possibility occurs then arguing as in the previous paragraph fails because $\boldsymbol{D}$ is in odd repetitive form but Lemma 5.3.7 only asserts that there is a Nielsen tuple in odd repetitive form in every orbit; it says nothing about the existence of tuples in even normal form if the type is in odd repetitive form. Consider the length 4 suffix-subtype $\boldsymbol{E}$ of $\boldsymbol{D}$, which consists of the final four conjugacy classes of $\boldsymbol{C}$, then $\boldsymbol{E}$ may be appear in Table 5.2 or Table 5.3. This occurs when $\boldsymbol{E}$, which we recall must have at least one even order class andexactly one repeated odd classes, is one of the following possible types (up to permutation/automorphism):

$$(2A, 2A, 3A, 3A),$$
$$(4A, 4A, 3A, 3A),$$
$$(4A, 4A, 5A, 5A).$$

Therefore $\boldsymbol{C}$ must be in one of the following forms:

$$(2A, 2A, 2A, 2A, 3A, X, Y),$$
$$(4A, 4A, 4A, 4A, 4A, X, Y),$$
$$(4A, 4A, 4A, 4A, 5A, X, Y),$$

where X and Y are types such that $\boldsymbol{C}$ is not in odd repetitive form. Notice that these length 7 types are amongst those for which we initially computed the length in Lemma 5.3.2. Lemma 5.3.18 and Lemma 5.3.19 show that the conclusion of the current lemma holds in these cases.

Finally suppose that $\boldsymbol{C}$ is in $(2, 2, 4)$-form at the first position and that $\boldsymbol{C}$ is not in either of the repetitive form. Then in fact there is just the one possibility for $\boldsymbol{C}$:

$$\boldsymbol{C} = (2A, 2A, 4A, 3A, 3B, 5A, 5B).$$

The conclusion of Theorem 5.3.1 holds for this type as stated in Lemma 5.3.2. Let $\boldsymbol{D}$ be the suffix-subtype $(4A, 3A, 3B, 5A, 5B)$. Choose any tuple $\boldsymbol{k} = (g_1, \ldots, g_5)$ in $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{D})$. Let $\boldsymbol{h}$ be the tuple $(g_1^2, g_1^2, g_1, g_2, g_3, g_4, g_5) \in \mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$. This tuple is in $(2, 2, 4)$-form and since the braid group acts transitively on $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$, there exists some braid Q such that $Q(\boldsymbol{g}) = \boldsymbol{h}$ as required. □

The three following lemmas complete the above proof. We begin by considering length 7 types which, upon coalescing, may become a length 6 tuple of the form $(2A)^6$ or may find themselves in odd normal form.

Lemma 5.3.18. *Suppose* $\boldsymbol{C}$ *is a type of the form:*

$$(2A, 2A, 2A, 2A, X, Y, Z)$$

for conjugacy classes X, Y *and* Z*, such that* $\boldsymbol{C} \neq (2A)^7$*. Then every braid orbit of* $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ *contains a tuple in repetitive normal form in the first position.*

Furthermore, every pure braid orbit of type $\mathrm{PNi}^{\mathrm{in}}(A_6, (2A)^8)$ *contains a tuple in repetitive normal form in the first position.*

Proof. We repeat the process used previously. Consider the suffix-subtype

$$\boldsymbol{D} = (2A, 2A, X, Y, Z).$$

This is a type of length 5, and one of X, Y and Z is not $2A$. Therefore, $\boldsymbol{D}$ does not appear in Table 5.2 or Table 5.3. Thus for every value of lifting invariant ϵ there is a tuple $\boldsymbol{h} = (h_1, \ldots, h_5)$ of length 5 with lifting invariant ϵ. The tuple

$$\boldsymbol{g} = (h_1, h_1, h_1, h_2, \ldots, h_5)$$

is a Nielsen tuple of type $\boldsymbol{C}$, has lifting invariant ϵ, and is in even repetitive form in the first position. Thus by Lemma 5.3.2 each pure braid orbit contains a tuple in repetitive normal form.

A similar argument applies for the type

$$\boldsymbol{C} = (2A, 2A, 2A, 2A, 2A, 2A, 2A, 2A).$$

The suffix-subtype type now has length 6 and so is not exceptional. It is necessary for us to consider length 8 tuples because the claim of the lemma does not apply for the type

$$(2A, 2A, 2A, 2A, 2A, 2A, 2A).$$

□

The same method of proof also works for types of the form $(4A, 4A, 4A, 4A, 4A, X, Y)$

Lemma 5.3.19. *Suppose $\boldsymbol{C}$ is a type of the form:*

$$(4A, 4A, 4A, 4A, 4A, X, Y)$$

for conjugacy classes X, Y. Then every braid orbit of $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ *contains a tuple in repetitive normal form in the first position.*

Next, consider the case where there are many classes of three cycles.

Lemma 5.3.20. *Suppose $\boldsymbol{C}$ is a type of the form:*

$$(3A, 3A, 3A, 3A, 3A, X, Y)$$

Then every braid orbit of $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ *contains a tuple in repetitive normal form in the first position.*

Proof. Consider the suffix-subtype $\boldsymbol{D} = (3A, 3A, 3A, 3A, X, Y)$. This is a type of length 6, and hence does not appear in Table 5.3. Thus for every possible value ϵ the lifting invariant may take there is a tuple $\boldsymbol{h} = (h_1, \ldots, h_6)$ of length 6 with lifting invariant ϵ. Choose the unique $g \in 3A$ such that $g^2 = h_1$. Then

$$\boldsymbol{g} = (g, g, h_2, \ldots, h_6)$$

is a Nielsen tuple of type $\boldsymbol{C}$, has lifting invariant ϵ and so by Lemma 5.3.6 every pure braid orbit of $\mathrm{PNi}^{\mathrm{in}}(A_6, \boldsymbol{C})$ contains a tuple in repetitive normal form in the first position. □

In the statement of Lemma 5.3.17 there is no requirement that our braid be pure. In fact it can be shown that we may insist the braid is pure as the next result demonstrates.

Lemma 5.3.21. *Let $\boldsymbol{g}$ be a Nielsen tuple whose type $\boldsymbol{C} = (C_1, \ldots, C_r)$ is in normal form at the first position, and there exists $Q \in \mathcal{B}_r$ such that $Q(\boldsymbol{g})$ is in normal form in the first position. Furthermore suppose that Q acts purely on the initial distinguished positions. Then there exists a pure braid $S \in \mathcal{PB}_r$ such that $S(\boldsymbol{g})$ is in normal form in the first position.*

Proof. Let $T \in \mathcal{B}_r$ be a braid which reorders the tuple $Q(\boldsymbol{g})$ into its original order, so that $S = T \circ Q \in \mathcal{PB}_r$. There are many such braids. Since Q acts purely on the initial distinguished positions then S can be chosen so that it acts trivially on the inital positions of $Q(\boldsymbol{g})$. Thus $T \circ Q$ is a pure braid and $S(\boldsymbol{g}) = T(Q(\boldsymbol{g}))$ is in normal form in the first position. □

The following makes it clear that our decision to restrict types to being in normal form in the first position was inconsequential.

Lemma 5.3.22. *Let $\boldsymbol{g}$ be a tuple in normal form in the first position. There exists $Q \in \mathcal{B}_r$ such that $Q(\boldsymbol{g})$ is in normal form in the ith position.*

Proof. This follows from Proposition 5.3.12. □

Any sufficiently large tuple whose type is in normal form can be placed into normal form.

Lemma 5.3.23. *Let $\boldsymbol{C}$ be a type, of length at least 8, and in normal form at position i. Let $\boldsymbol{g}$ be a tuple of type $\boldsymbol{C}$. Then the pure braid orbit of $\boldsymbol{g}$ contains a tuple in normal from in position i.*

Proof. Without loss of generality assume that $i = 1$. Proceed by induction using Lemma 5.3.17 as the basis for the induction. By Lemma 5.3.15 there exists $Q \in \mathcal{B}_r$ such that $Q(\boldsymbol{g}) = \boldsymbol{g}'$ has a generating head of length 5. Coalesce with respect to the partition, $P = [1, \ldots, 1, 2]$ to obtain a tuple $\boldsymbol{h}$ of length $r - 1 \geq 7$ and type $\boldsymbol{D}$. Since the initial 5 elements of $\boldsymbol{g}'$ generate G then $\boldsymbol{h}$ is a generating tuple; and by Lemma 5.3.16, we can assume that coalescing produces no trivial elements. Therefore $\boldsymbol{h}$ is a Nielsen tuple. Suppose that $\boldsymbol{D} \neq (2A)^7$, then by inductive hypothesis there exists some pure braid $P \in \mathcal{B}_r$ such that $P(\boldsymbol{h})$ is in normal form in the first position. Therefore, by Proposition 4.4.7 there exists a ribbon braid $R \in \mathcal{RB}_r$ such that $R(\boldsymbol{g}')$ is in normal form in the first position.

If, on the other hand, $\boldsymbol{D} = (2A)^7$, then

$$\boldsymbol{C} = (2A, 2A, 2A, 2A, 2A, 2A, X, Y).$$

Therefore, either $X = Y$, or, $X \neq Y$ and one of the two conjugacy classes is not equal to $2A$. In the first of these possibilities and $\boldsymbol{C} = (2A)^8$, in which case Lemma 5.3.18 says that every pure braid orbit of $\mathrm{Ni}^{\mathrm{in}}(G, \boldsymbol{C})$ contains a tuple in even repetitve normal form in the first position. So suppose that there $X \neq Y$ and without loss of generality assume that $Y \neq 2A$. Instead of coalescing $\boldsymbol{g}'$ with respect to the partition $[1, \ldots, 1, 2]$ we coalesce with respect to the partition $P' = [1, \ldots, 1, 2, 1]$. The resulting tuple $\boldsymbol{h}'$ again generates G as the initial 5 elements of $\boldsymbol{g}'$, which are also the initial 5 elements of $\boldsymbol{h}'$, generate G. Moreover, $\boldsymbol{h}'$ is not of type $(2A)^7$ and therefore, by Lemma 5.3.17, there is a pure braid $S \in \mathcal{PB}_7$ such that $S(\boldsymbol{h}')$ is in repetitive normal form in the first position. Therefore there exists a ribbon braid $R \in \mathcal{PR}_{8,P'}$ such that $R(\boldsymbol{g}')$ is in repetitive normal form.

The proof so far only demonstrates that there exists a braid $B \in \mathcal{B}_r$, acting purely on the initial positions, such that $B(\boldsymbol{g})$ is in normal form. However by Lemma 5.3.21 we see that in fact a pure braid suffices. □

For any type of large enough length the classes can always be permuted to give a tuple in normal form in some position. Together with Lemma 5.3.23 this observation says that any tuple of large enough type is braid equivalent to a tuple in normal form. This argument forms the basis of our proof of Theorem 5.3.1.

5.3.5 Proof of Theorem 5.3.1

First consider the question of existence. We remark that a result due to Guralnick and Tiep actually establishes the existence of tuples with all possible lifting invariant values in great generality [18]. However, it is not applicable in all of the cases needed here so existence is established independently of their result.

Lemma 5.3.24. *For all types $\boldsymbol{C}$ of length $r \geq 7$ there exists a Nielsen tuple of type $\boldsymbol{C}$. Moreover, for the appropriate cover θ there are Nielsen tuples for all possible values the lifting invariant LI_θ may take.*

Proof. The proof is by induction on the length of the type r using Lemma 5.3.2 as the basis for the argument.

Consider the shape of the type. Suppose that $\boldsymbol{C}$ has $(2,3)$-shape, i.e., $n_{2A}(\boldsymbol{C}) > 0$ and $n_3(\boldsymbol{C}) > 0$. Then Theorem 5.3.1 claims that there is a single braid orbit. Since $r \geq 7$ there is a repeated conjugacy class C. Since the order of the conjugacy classes within the type can be permuted with braids assume that $C_1 = C_2 = C$. Then let $\boldsymbol{D}$ be the type

$$\boldsymbol{D} = (C_3, \ldots, C_r).$$

This has length $r - 2 \geq 5$. Table 5.3 contains no types of length 5; therefore by our inductive hypothesis there exists a tuple $\boldsymbol{h} = (h_1, \ldots, h_{r-2})$ of type $\boldsymbol{D}$. Pick $g \in C$ and note that $g^{-1} \in C$, therefore

$$\boldsymbol{g} = (g, g^{-1}, h_1, \dots, h_{r-2})$$

is a Nielsen tuple of length r and type C.

In the remaining cases then there is an additional obstacle the existence of tuples of type $\boldsymbol{C}$ and with lifting invariant ϵ for every possible value that the lifting invariant for the appropriate cover may take. If the repeated conjugacy class, C, of $\boldsymbol{C}$, contains elements of order 5 or 4 then proceed as above. Let $\boldsymbol{D}$ be the suffix-subtype of $\boldsymbol{C}$ of length $r-2 \geq 5$. Note that $\boldsymbol{C}$ and $\boldsymbol{D}$ have the same shape. Assume that $\boldsymbol{D}$ is not in Table 5.3. Therefore there is a tuple

$$\boldsymbol{h} = (h_1, \dots, h_{r-2})$$

of length $r-2 \geq 5$ with lifting invariant ϵ. As before, let

$$\boldsymbol{h} = (g, g^{-1}, h_1, \dots, h_{r-2})$$

and observe that since $\hat{g}^{-1} = \widehat{g^{-1}}$ then $LI(\boldsymbol{g}) = LI(\boldsymbol{h}) = \epsilon$.

On the other hand suppose that $n_{5A}(\boldsymbol{C}) < 2$; $n_{5B}(\boldsymbol{C}) < 2$; and $n_{4A}(\boldsymbol{C}) < 2$ and either

$$n_{2A}(\boldsymbol{C}) = 0; \text{ or}$$
$$n_{3A}(\boldsymbol{C}) = 0.$$

Then $n_{2A}(\boldsymbol{C}) \geq 3$ or $n_{3A}(\boldsymbol{C}) \geq 3$ or $n_{3B}(\boldsymbol{C}) \geq 3$. In any of these eventualities choose $C = C_1 = C_2 = C_3$ to be thrice repeated class, and as before let $\boldsymbol{D}$ be the suffix-subtype type $(C_3, \dots, C_r)$. The types $\boldsymbol{C}$ and $\boldsymbol{D}$ have the same shape. Therefore proceed as in the previous paragraph.

Finally let us suppose that $\boldsymbol{D}$ does appear in Table 5.3. If $\boldsymbol{D} = (2A)^5$ then $\boldsymbol{C}$ must be a type of the form:

$$(X, X, 2A, 2A, 2A, 2A, 2A)$$

where X is conjugacy class whose elements have order not equal to 3. If $X \neq 2A$ we may instead let $\boldsymbol{D} = (X, X, 2A, 2A, 2A, 2A)$ and argue as before. The final case $\boldsymbol{C} = (2A)^7$ was previously calculated in Lemma 5.3.18. We may argue similarly for the type $\boldsymbol{D} = (3A)^5$. This time noting that the case $\boldsymbol{C} = (3A)^7$ does not require extra computation because of the result of Fried which says that if $r > n$ and $C = (3A)^r$ then there are two braid orbits which are distnguished by the lifting invariant. □

The proof of Theorem 5.3.1 can now be completed.

Proof of Theorem 5.3.1. Let $\boldsymbol{g} = (g_1, \dots g_r)$ and $\boldsymbol{g'} = (g'_1, \dots, g'_r)$ be two Nielsen tuples of type $\boldsymbol{C}$, length $r \geq 7$ and assume that they have the same lifting invariant. Furthermore suppose that $\boldsymbol{C}$ is not one of the following types for which the result has already been established:

- $(2A, 2A, 2A, 2A, 2A, 2A, 2A)$
- $(2A, 2A, 2A, 2A, 2A, 2A, 3A)$
- $(2A, 2A, 2A, 2A, 2A, 2A, 5A)$
- $(2A, 2A, 3A, 3B, 5A, 5B, 4A)$

Then $\boldsymbol{C}$ has must be in (possibly unordered) normal form. Since we may reorder the tuple by braiding then we can assume that $\boldsymbol{C}$ is in normal form in the first position. By Lemma 5.3.23 there exists pure braids $Q, Q' \in \mathcal{PB}_r$ such that $\boldsymbol{h} = Q(\boldsymbol{g})$ and $\boldsymbol{h}' = Q(g')$ are in normal form at the first position. If the type is in more than one unordered normal form then we choose which normal form to use based on the following preference: odd repetitive form, even repetitive form, $(2, 2, 4)$-form.

Note that we may conjugate our tuples so that the first elements of $\boldsymbol{h}$ and $\boldsymbol{h'}$ are equal. Coalesce $\boldsymbol{h}$ and $\boldsymbol{h}'$ with respect to the normal partitions $P_{\boldsymbol{h}}$ and $P_{\boldsymbol{h}'}$. Let $\boldsymbol{k}$ and $\boldsymbol{k'}$ denote the coalesced tuples, and let their common type be denoted $\boldsymbol{D}$. The tuples $\boldsymbol{k}$ and $\boldsymbol{k'}$ are Nielsen tuples and by Lemma 5.3.4 they have the same lifting invariant. If $\boldsymbol{D}$ is not in Table 5.3 then by our inductive hypothesis the tuples $\boldsymbol{h}$ and $\boldsymbol{h'}$ are pure braid equivalent. Thus there is a ribbon braid R such that $R(\boldsymbol{h}) = \boldsymbol{h'}$. Therefore there exists a pure braid $S \in \mathcal{PB}_r$ such that $S(\boldsymbol{g}) = \boldsymbol{g'}$ as required.

Alternatively $\boldsymbol{D}$ may be in Table 5.3. Then $\boldsymbol{D}$ has length 5 and $\boldsymbol{C}$ must have length $r = 7$. Furthermore $\boldsymbol{D} \neq (3A)^5$ since otherwise, by our earlier choice of precedence of the normal forms, we would have placed our tuples in odd repetitive form. Therefore we must have coalesced a triple of even order elements. Thus

$$\boldsymbol{C} = (2A, 2A, 2A, 2A, 2A, 2A, 2A)$$

or

$$\boldsymbol{C} = (4A, 4A, 4A, 2A, 2A, 2A, 2A).$$

Both of these possibilities are resolved by Lemma 5.3.18. □

Thus we have competed our proof of Theorem 5.3.1 and hence Theorem 5.2.2.

CHAPTER 6

THE MAPCLASS PACKAGE FOR GAP

The results of the previous chapters rely on the calculation of braid orbits for all types. This calculation used a package for GAP written for this purpose. This package, called MAPCLASS, is now distributed with GAP, for versions ≥ 4.5.

For the remainder of the chapter fix the following data:

- A group G.
- An integer g_0, which corresponds to the orbit genus.
- A tuple $\boldsymbol{C} = (C_1, \cdots, C_r)$ of conjugacy classes in G.

Using this information the MAPCLASS package computes the corresponding mapping class orbit. This chapter describes the key functionality and implementation details.

The package is derived from the package BRAID [27]. BRAID was limited to computing braid orbits, and so the new package has more functionality. Additionally, MAPCLASS also differs in a number of ways in its implementation, and its efficiency. In particular the problem of determining whether two tuples are conjugate is dealt with more effectively. The computations required for the calculation of the results contained in this thesis were made feasible by the performance increases gained.

6.1 Overview of Main Functions

The MAPCLASS package has two main functions:

- `AllMCOrbits(group, genus, tuple)`

- `GeneratingMCOrbits(group, genus, tuple)`

where `tuple` is in fact a tuple of conjugacy class representatives.

Both functions compute mapping class orbits, but in the case of `AllMCOrbits` we drop the condition that the tuples must generate G. The following sample session demonstates how one can use the package.

```
gap>group:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> tuple:=[ (1,2)(3,4), (1,2)(3,4), (1,2)(3,4) ]
[ (1,2)(3,4), (1,2)(3,4), (1,2)(3,4) ]
gap> orbits:=AllMCOrbits(group,1,tuple);;

Total Number of Tuples: 189120



Collecting 20 random tuples... done

Cleaning done; 20 random tuples remaining



Orbit 1:

Length=3072
Generating Tuple  =[ (1,2,4,5,3), (1,4,5,2,3), (1,2)(4,5),
(1,4)(2,3), (2,5)(3,4) ]
Generated subgroup size=60
Centralizer size=1
4800 tuples remaining
Cleaning current orbit...
Cleaning a list of 20 tuples
Random Tuples Remaining: 0
Cleaning done; 0 random tuples remaining



Collecting 20 random tuples... done
```

```
Cleaning orbit 1
Cleaning a list of 20 tuples
Random Tuples Remaining: 0

Cleaning done; 0 random tuples remaining



Collecting 40 random tuples... done

Cleaning orbit 1
Cleaning a list of 40 tuples
Random Tuples Remaining: 3

Cleaning done; 3 random tuples remaining



Orbit 2:

Length=32
Generating Tuple =[ (1,4)(2,3), (1,2)(3,4), (1,4)(2,3), (1,2)(3,4),
(1,3)(2,4) ]
Generated subgroup size=4
Centralizer size=4
4320 tuples remaining
Cleaning current orbit...
Cleaning a list of 3 tuples
Random Tuples Remaining: 2
Cleaning done; 2 random tuples remaining



Orbit 3:

Length=72
Generating Tuple =[ (1,5,2), (1,3,2), (1,2)(3,5), (1,3)(2,5),
(1,3)(2,5) ]
Generated subgroup size=12
Centralizer size=1
```

```
0 tuples remaining
Cleaning current orbit...
Cleaning a list of 2 tuples
Random Tuples Remaining: 0
Cleaning done; 0 random tuples remaining
```

A sample session

We refer the reader to the documentation provided on the package website for more details and documentation of other functions [23].

6.2 Overview of Routine

In this section the behaviour of `GeneratingMCOrbits` when called on a group G, with r conjugacy classes $C_1, \cdots, C_r$ and genus g_0 is described.

The first step the program takes is to compute the total number of tuples it has to account for. It must calculate this number beforehand otherwise the routine will not count the number of orbits correctly or it will enter an infinite loop. To calculate this we use two formulae due to Frobenius, which calculate the number of ways in which an element of G can be written as a product of r elements $g_1, \cdots, g_r$ with $g_i \in C_i$, and calculate the number of ways in which an element of G can be written as a product of g_0 commutators. We discuss these formulae in Section 6.4.

After computing the number of tuples and observing that it is positive, we select a number of random tuples of length $2g_0 + r$ where the elements at index $2g_0 + 1, \cdots, 2g_r + r$ lie in the conjugacy classes $C_1, \cdots, C_r$. Also, at this point all tuples chosen generate G, because we are using the generating version of the algorithm.

Take the first random tuple, and begin applying the generators for the mapping class action, recording new tuples in a table. When no new tuples can be found for this orbit we stop and record the orbit. Taking the next random tuple we repeat this process (first checking the tuple is not in a preexisting orbit) until all orbits are accounted for. Note that this is done up to conjugacy in G and in particular this makes the routine for calculating whether two tuples are conjugate the most frequently accessed routine of the program.

A tuple minimisation routine is used to speed up the process of determining whether two tuples are conjugate in G. This routine takes a tuple and calculates the minimal conjugate tuple. This minimal tuple is unique in a given orbit. This technique supercedes the previous fingerprinting technique used in the BRAID package, largely because of its superior performace on p-groups and Frobenius groups. The fingerprinting technique can outperform the new technique on certain classes of groups and so is available for use within the package.

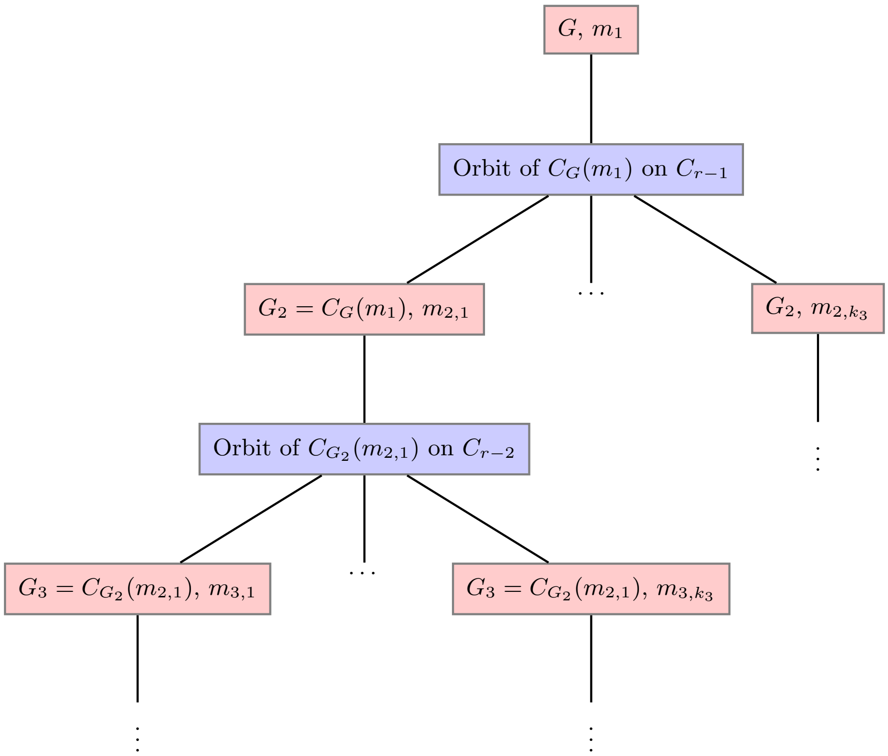
6.3 Tuple Minimization

In the previous section we noted that the routine uses tuple minimisation to detect duplicate tuples appearing withing the orbit. In this section we describe this process in more detail. The tuple minimisation routines aim to solve the following problem: given two tuples $\tau = (t_1, \ldots, t_n)$ and $\tau' = (t'_1, \ldots, t'_n)$ does there exist a g such that $t_i^g = t'_i$ for all $i = 1, \ldots, n$.

The process that we outline has two parts: a preprocessing function dependent on G, and function called on each tuple we want to minimise.

6.3.1 Preprocessing

For this section we fix a group G and a sequence of conjugacy classes $C_1, \ldots, C_r$. A tree of groups and minimums is constructed recursively. The root of the tree is a pair (G, x), where x is the minimal element of our final conjugacy class C_r. Suppose we have a node (H, y) in the tree at level $k-1$. The children of (H, y) are pairs $(C_H(y), m_i)$ where the m_i are the minima of the orbits of $C_H(x)$ on the conjugacy class C_k. Then for each element of the conjugacy class we keep track of the minimal element which lies in the same orbit. We continue until all groups are trivial or until we run out of conjugacy classes. The tree looks as follows:



6.3.2 Minimisation Process

Given the minimisation tree as described in the previous section, and a tuple t we minimise as follows:

- For t_i we select the corresponding minimal element, m_i using the preprocessed tree.
- Conjugate the whole tuple by the h_i taking t_i to m_i.
- Continue the minimisation on the new tuple. Note that all further conjugation will fix the preceding subtuple because we are conjugating by an element of the intersection of the centralizers.

This process is equivalent to finding a path through the minimisation tree.

Consider the case when $g = 0$, $r = 3$ and $G = A_5$. The tuple we aim to minimize is

$$[(1,2)(3,4),(1,4)(2,3),(1,4,5)].$$

The routine works as follows:

- The minimal element of the third conjugacy class C_3 is $(3,4,5)$ (where by minimal we order by where the points on which our group acts are moved – GAP's default ordering of permutations).
- The element $(1,4,5)$ is taken to $(3,4,5)$ by $(1,3,2)$. Let G_2 be the centralizer $C_G((3,4,5))$. We conjugate the whole tuple by $(1,3,2)$:

$$[(1,2)(3,4),(1,4)(2,3),(1,4,5)] \mapsto [(1,3)(2,4),(1,2)(3,4),(3,4,5)]$$

- Then we continue by calculating the orbits of G_2 on the conjugacy class C_2 containing $(1,4)(2,3)$. We take the minimal element in the orbit containing $(1,2)(3,4)$. This is $(1,2)(4,5)$. We now conjugate the tuple by the element $(3,4,5)$ taking $(1,2)(3,4)$ to $(1,2)(4,5)$.

$$[(1,3)(2,4),(1,2)(3,4),(3,4,5)] \mapsto [(1,4)(2,5),(1,2)(4,5),(3,4,5)].$$

  Finally we have to consider the centralizer of $(1,2)(4,5)$ in G_2, which is just trivial, and so in fact this tuple is our minimum.

Note that the program actually does the minimisation and the comparison term-by-term which significantly reduces the amount of time taken.

6.4 How Many Tuples are There?

The routine needs to know exactly how many tuples there are in total in order for it to determine when to stop looking for new orbits. The question then is: For a finite group G and conjugacy classes $C_1, \dots, C_r$, in how many different ways can we write 1 as a product

$$[a_1,b_1]\dots[a_{g_0},b_{g_0}]c_1\dots c_r$$

of elements of G, where $c_i \in C_i$? We shall denote this number by $\Lambda(G;g_0;C_1,\dots,C_r)$, and we note that we are actually counting homomorphisms from a Fuchsian group (and particularly surface groups) to G. This question was answered by Frobenius, although the statement we give is due to Liebeck and Shalev [25]

Proposition 6.4.1.

$$\Lambda(G; g_0; C_1, \ldots, C_r) = |G|^{2g_0-1}|C_1| \ldots |C_r| \sum_{\chi \in Irr(G)} \frac{\chi(C_1) \ldots \chi(C_r)}{\chi(1)^{r-2+2g_0}}$$

This formula can be used for calculating the total number of tuples; if we are only concerned with the generating case then we have to use inclusion-exclusion on subgroups of G which can be generated by some tuple.

At this point it would be remiss not to draw attention to the $|G|^{2g_0-1}$ term in the above theorem. This term dominates the growth of Λ. This means that for large g_0, and without a drastic change in our method of calculation, the number of total tuples to account for becomes prohibitively large.

6.5 The Splitting Routine

The previous section showed that the growth of orbits is exponential in the length of the tuple. Typically one would approach the problem of an algorithm having exponential search space with a "divide-and-conquer" style algorithm which divides the process into smaller subprocess, which upon iteration reduces the algorithmic complexity of the problem. In this section we explain a process introduced by Magaard, Shpectorov and Wang which does just this [28]. Finally we discuss how this approach can be used alongside an invariant of braid orbits and give an application.

We now outline this splitting process. For a given type $\boldsymbol{C} = (C_1, \ldots, C_r)$ define $\boldsymbol{C}_l$ and $\boldsymbol{C}_r$ to be the subtypes $(C_1, \ldots, C_k)$ and $(C_{k+1}, \ldots, C_r)$ for some $1 \leq k \leq r$ chosen to be roughly the middle. For each conjugacy class C of G we pick a representative elements x_C, and we compute all braid orbits (including non-generating orbits) for the types $(\boldsymbol{C}_l, C)$ and $(C^{-1}, \boldsymbol{C}_l)$. Where C^{-1} means the conjugacy class containing x_C^{-1}. We call these the *left* and *right* orbits respectively. Note that every element of the left orbit of type $(\boldsymbol{C}_l, C)$ can be written, possibly after conjugation, in the form

$$(g_1, \ldots, g_r, x_C),$$

and every element of the left orbit of type $(C^{-1}, \boldsymbol{C}_r)$ can be written in the form

$$(x_C^{-1}, g_{k+1}, \ldots, g_r).$$

Thus these two tuples have product one and the tuple $(g_1, \ldots, g_r)$ is a Nielsen tuple provided it generates G. The aim therefore is to match the smaller left and right orbits to get orbits of type $\boldsymbol{C}$.

Let us suppose that $\mathcal{O}_l$ is an orbit of type $(\boldsymbol{C}_l, C)$ and $\mathcal{O}_r$ is a orbit of type $(C^{-1}, \boldsymbol{C}_r)$. Choose representative tuples $\boldsymbol{g}_l = (g_1, \ldots, g_k, x_C)$ and $\boldsymbol{g}_r = (x_C^{-1}, g_{k+1}, \ldots, g_r)$ which are are in the forms specified in the previous paragraph. We say that this pair of tuples, is *matched by h* if $h \in C_G(x_C)$ and the tuple

$$(g_1^h, \ldots, g_k^h, g_{k+1}, \ldots, g_r)$$

generates G. The pair $(\mathcal{O}_l, \mathcal{O}_r)$ along with the element h are stored for later use. Such pairs are called *matching pairs* in the literature.

We form a graph $\mathcal{G}$ whose vertices are the matching pairs. Two vertices are connected if their corresponding matching pairs lie within the same large braid orbit. Thus, components of the graph correspond to braid orbits. To determine the connectivity of the graph the edges of $\mathcal{G}$ must be computed. This edge finding process is implemented incrementally. For each of the components of our graph we collect a set representative tuples. Missing generators of the braid group, i.e., those which braid across the partition, are applied to $\boldsymbol{g}$, generating a set N of neighbours of our representative tuples. It is then checked whether these neighbours lie within another component. If so we merge the components of the graph. If the graph is connected the process terminates. Therefore this routine can only determine if the braid group acts transitively. Additionally, if a large number of the missing braidings must be applied in order to establish connectivity then this process is less efficient than direct calculation of the orbits. At first glance it appears as though this routine is not of great use when we have more than one orbit. However, using an invariant of our braid orbits, such as the lifting invariant, we call tell the routine to stop trying to connect the graph once our invariant separates the orbits.

The splitting routine played an important role in our classification of A_6. The braid orbits for $G = A_6$ types of length at least 7 become so large that they can be problematic to calculate. For example the Nielsen class for the type $\boldsymbol{C} = (2A)^8$ contains 46116604800 generating tuples in total and 128101680 tuples up to conjugation. In this case we do not want to calculate the whole orbit as doing so may take many weeks. For the calculations found in Table 5.5 we used the splitting routine described above. In many of these cases then there is more than one orbit. In our routine we allow the user to suggest how many braid orbits there might be and to provide a function to distinguish between these orbits. In particular we provide a function that calculates the lifting invariant. Thus for type $\boldsymbol{C} = (2A)^7$ then we predict that there will be exactly three orbits. We allow the routine to run until it has ascertained that there are at most three orbits. We then calculate the lifting invariant for a representative tuple from each orbit. If the lift invariants are distinct then we have guaranteed that there are exactly three orbits.

CHAPTER 7

EXPERIMENTS AND EXTENSIONS

In the chapter we look back over the work in the previous chapters, assess the results contained, consider extensions of this work, and present some experimental data.

7.1 Experiments

Given the results of the previous chapter the immediate question to ask is: Can the results of Chapters 4 and 5 be extended to larger alternating groups? We state a conjecture and then consider the evidence for and against this conjecture. It is unlikely that said conjecture has not been made before; Catanese, Lönne and Perroni have suggested a generalization of this conjecture for their own generalized lifting invariant [4].

Conjecture 7.1.1. *For $G = A_n$, $n \geq 5$ then there exists $k \in \mathbb{Z}$ such that for all types $\boldsymbol{C}$ of length $r \geq k$ the Hurwitz space $\mathcal{H}^{in}(G, \boldsymbol{C})$ is non-empty and the components are distinguished by lifting invariants.*

First observe that the Conway-Parker-Fried-Völklein Theorem says that there exists some N such that if the Schur multiplier is generated by commutators, and every conjugacy class appears at least N times within $\boldsymbol{C}$, then the braid group acts transitively on tuples [12]. Therefore, should each conjugacy class appear often enough then the conclusion of the conjecture holds. This seems to match our intuition; it is expected that two very long tuples be braid equivalent simply because the extra length gives us a greater degree of freedom when making braid moves. On the other hand, Fried 3-cycle resultsays that it is not simply enough for the tuples to be long. The conjecture takes this into account.

We now consider further experimental evidence for the truth of Conjecture 7.1.1. The first step towards a complete classification would of course be to consider the $G = A_7$ case. Since the Schur multiplier is also exceptional in this case then weexpect a result similar to that of Theorem 5.2.1. When $G = A_7$ has 8 nontrival conjugacy classes:

- $2A = (1,2)(3,4)^{A_7}$
- $3A = (1,2,3)^{A_7}$
- $3B = (1,2,3)(4,5,6)^{A_7}$
- $4A = (1,2,3,4)(5,6)^{A_7}$
- $5A = (1,2,3,4,5)^{A_7}$
- $6A = (1,2,3)(4,5)(6,7)^{A_7}$
- $7A = (1,2,3,4,5,6,7)^{A_7}$
- $7B = (1,2,3,4,5,7,6)^{A_7}$

If $\boldsymbol{C}$ is a type and C is a conjugacy class of G then we define, as in Chapter 5, $n_C(\boldsymbol{C})$ to be the number of occurences of C within $\boldsymbol{C}$. Let $n_3(\boldsymbol{C})$ be the sum $n_{3A}(\boldsymbol{C}) + n_{3B}(\boldsymbol{C})$. Additionally define

$$n_{2,6}(\boldsymbol{C}) = n_{2A}(\boldsymbol{C}) + n_{6A}(\boldsymbol{C}).$$

Then we make the following conjecture.

Conjecture 7.1.2. *There exists an integer k such that for all types $\boldsymbol{C}$ of length $r \geq k$ the Hurwitz space $\mathcal{H}^{in}(G, \boldsymbol{C})$ is nonempty. Furthermore*

- *If $n_{2,6}(\boldsymbol{C}) > 0$ and $n_3(\boldsymbol{C}) > 0$ then $\mathcal{H}^{in}(G, \boldsymbol{C})$ is connected.*
- *If $n_{2,6}(\boldsymbol{C}) > 0$ and $n_3(\boldsymbol{C}) = 0$ then $\mathcal{H}^{in}(G, \boldsymbol{C})$ has three components.*
- *If $n_{2,6}(\boldsymbol{C}) = 0$ and $n_3(\boldsymbol{C}) > 0$ then $\mathcal{H}^{in}(G, \boldsymbol{C})$ has two components.*
- *If $n_{2,6}(\boldsymbol{C}) = 0$ and $n_3(\boldsymbol{C}) = 0$ then $\mathcal{H}^{in}(G, \boldsymbol{C})$ has six components.*

If there is more than one component then these components are distinguished by lift invariants.

Type	Genus
$(2A, 2A, 2A, 2A, 2A)$	0
$(2A, 2A, 2A, 2A, 3A)$	0
$(2A, 2A, 2A, 3A, 3A)$	0
$(2A, 2A, 3A, 3A, 3A)$	0
$(2A, 3A, 3A, 3A, 3A)$	0
$(3A, 3A, 3A, 3A, 3A)$	0

Table 7.1: Types of length 5 for which $\mathrm{Ni}^{\mathrm{in}}(A_7, \boldsymbol{C})$ is empty.

Initial calculations show that $k \geq 6$. Indeed, the orbits for all types of lengths $r = 4$ and $r = 5$ were computed using MapClass. Table 7.1 and Table 7.2 show the types of length $r = 5$ which types which do not adhere to the classification suggested by Conjecture 7.1.2. Note that Fried's 3-cycle result says that $\mathcal{H}^{\mathrm{in}}(A_7, (3A)^n)$ is connected for $n \leq 6$ and has two components for $n \geq 7$. Therefore the k in the conjecture must be at least 7.

It is evident that in trying to prove this result we may argue as we have in Chapter 5, the only obstacle currently is a computational one. The largest length tuples it is feasible to consider are those of length at most 6. However A_7, with its 8 nontrivial conjugacy classes, would require us to compute all types of a larger length to proceed as before.

The group A_7 is also of further interest because, unlike A_7, both A_5 and A_6 are both isomorphic to $PSL_2(q)$ for some q. Perhaps then the conjecture should not concern alternating groups but groups of the form $PSL_2(q)$?

7.2 Extensions

We can now consider other questions which have arisen but are not directly related to Conjecture 7.1.1. The first question arises from our treatment of mapping class groups and the higher genus Hurwitz spaces.

Question 7.2.1. *Can we classify the components of $\mathcal{H}^{in}(g_0, A_n, \boldsymbol{C})$ for nonzero g_0? Perhaps even for just for $G = A_5$ or $G = A_6$?*

Type	Number of Orbits	Expected	Genus
$(3A, 3A, 3A, 3A, 3B)$	1	2	-
$(3A, 3A, 3A, 3A, 5A)$	1	2	-

Table 7.2: Types $\boldsymbol{C}$ of length 5 for which the number of components of $\mathcal{H}^{\mathrm{in}}(G, \boldsymbol{C})$ is not as predicted by Conjecture 7.1.2.

As we mentioned in Chapter 3 a result of Dunfield and Thurston shows that if G is a simple group and $g_0 >> 0$ then the space of unramified covers is connected. However using MapClass a short calculation shows that for $g_0 = 1$, $G = A_5$ and for odd types of length 4, there are two orbits as in our classification.

7.3 Conclusions

Our classifications for A_5 and A_6 relied on us being able to establish a base case for our induction. Our approach relied on the explicit computation of braid orbits for short types. This of course is a limitation of the approach taken. For $n \geq 9$ it is unlikely that it will be possible to establish a similar base case with the current computational techniques. Still there is value in the analysis of small cases. In particular:

- We hope to be able to establish a pattern for exceptional cases. Fried's 3-cycle theorem gives a very simple pattern for those exceptionalcases: they appear when the genus of the covering space is 0. Unfortunately the data we have suggests that this pattern does not continue to hold nor is there an immediately obvious common property of those types found in Table 4.2 and Table 5.3. The covering genus can not solely be responsible for exceptional types. It may well be too much to hope that there is such a nice reason for types to be exceptional.

- The A_6 case is a particularly interesting example. We have already noted that this is one of the two special cases for which the Schur cover is of degree 6 not degree 2. Thus providing evidence for the general role that the covering group plays. We also note that this establishes an infinte set of examples for which we have more than 2 orbits. This is the first nontrivial set of examples known to the author.

- Our explicit computations provide actual data. In particular during the production of this thesis every orbit computed has been saved and is made available. Thus for those small exceptional types we can actually inspect these orbits to try and explain these hypotheses. In fact in trying to establish that all tuples can be placed in a normal form it was first established computationally by an anlysis of the stored orbits.

APPENDIX A

A_5 RESULTS

In this appendix we collect the data for our calculations of pure braid orbits for A_5. The results also happen to coincide with the braid orbits.

Table A.1: Pure braid orbits for types of length 4, their respective lengths and lifting invariants.

Tuple	Orbits	Lengths
$(2A, 2A, 2A)$	0	
$(2A, 2A, 3A)$	0	
$(2A, 2A, 5A)$	0	
$(2A, 2A, 5B)$	0	
$(2A, 3A, 3A)$	0	
$(2A, 3A, 5A)$	1	1
$(2A, 3A, 5B)$	1	1
$(2A, 5A, 5A)$	0	
$(2A, 5A, 5B)$	1	1
$(2A, 5B, 5B)$	0	
$(3A, 3A, 3A)$	0	
$(3A, 3A, 5A)$	1	$1\ (-1)$
$(3A, 3A, 5B)$	1	$1\ (-1)$
$(3A, 5A, 5A)$	1	$1\ (-1)$
$(3A, 5A, 5B)$	1	$1\ (1)$
$(3A, 5B, 5B)$	1	$1\ (-1)$
$(5A, 5A, 5A)$	1	$1\ (-1)$

(5A, 5A, 5B)	0	
(5A, 5B, 5B)	0	
(5B, 5B, 5B)	1	1 (−1)

Table A.2: Pure braid orbits for types of length 4, their respective lengths and lifting invariants.

Tuple	Orbits	Lengths
(2A, 2A, 2A, 2A)	0	
(2A, 2A, 2A, 3A)	1	18
(2A, 2A, 2A, 5A)	1	10
(2A, 2A, 2A, 5B)	1	10
(2A, 2A, 3A, 3A)	1	18
(2A, 2A, 3A, 5A)	1	15
(2A, 2A, 3A, 5B)	1	15
(2A, 2A, 5A, 5A)	1	10
(2A, 2A, 5A, 5B)	1	5
(2A, 2A, 5B, 5B)	1	10
(2A, 3A, 3A, 3A)	1	24
(2A, 3A, 3A, 5A)	1	20
(2A, 3A, 3A, 5B)	1	20
(2A, 3A, 5A, 5A)	1	12
(2A, 3A, 5A, 5B)	1	12
(2A, 3A, 5B, 5B)	1	12
(2A, 5A, 5A, 5A)	1	4
(2A, 5A, 5A, 5B)	1	8
(2A, 5A, 5B, 5B)	1	8
(2A, 5B, 5B, 5B)	1	4
(3A, 3A, 3A, 3A)	1	18 (1)
(3A, 3A, 3A, 5A)	2	15 (1) 10 (−1)
(3A, 3A, 3A, 5B)	2	15 (1) 10 (−1)
(3A, 3A, 5A, 5A)	2	15 (1) 2 (−1)

$(3A, 3A, 5A, 5B)$	2	5 (1) 12 (−1)
$(3A, 3A, 5B, 5B)$	2	15 (1) 2 (−1)
$(3A, 5A, 5A, 5A)$	1	9 (1)
$(3A, 5A, 5A, 5B)$	2	6 (−1) 3 (1)
$(3A, 5A, 5B, 5B)$	2	3 (1) 6 (−1)
$(3A, 5B, 5B, 5B)$	1	9 (1)
$(5A, 5A, 5A, 5A)$	1	10 (1)
$(5A, 5A, 5A, 5B)$	1	4 (−1)
$(5A, 5A, 5B, 5B)$	2	5 (1) 2 (−1)
$(5A, 5B, 5B, 5B)$	1	4 (−1)
$(5B, 5B, 5B, 5B)$	1	10 (1)

Table A.3: Pure braid orbits for types of length 4, their respective lengths and lifting invariants.

Tuple	Orbits	Lengths
$(2A, 2A, 2A, 2A, 2A)$	1	192
$(2A, 2A, 2A, 2A, 3A)$	1	270
$(2A, 2A, 2A, 2A, 5A)$	1	150
$(2A, 2A, 2A, 2A, 5B)$	1	150
$(2A, 2A, 2A, 3A, 3A)$	1	360
$(2A, 2A, 2A, 3A, 5A)$	1	225
$(2A, 2A, 2A, 3A, 5B)$	1	225
$(2A, 2A, 2A, 5A, 5A)$	1	120
$(2A, 2A, 2A, 5A, 5B)$	1	145
$(2A, 2A, 2A, 5B, 5B)$	1	120
$(2A, 2A, 3A, 3A, 3A)$	1	468
$(2A, 2A, 3A, 3A, 5A)$	1	300
$(2A, 2A, 3A, 3A, 5B)$	1	300
$(2A, 2A, 3A, 5A, 5A)$	1	180
$(2A, 2A, 3A, 5A, 5B)$	1	180
$(2A, 2A, 3A, 5B, 5B)$	1	180

$(2A, 2A, 5A, 5A, 5A)$	1	120
$(2A, 2A, 5A, 5A, 5B)$	1	100
$(2A, 2A, 5A, 5B, 5B)$	1	100
$(2A, 2A, 5B, 5B, 5B)$	1	120
$(2A, 3A, 3A, 3A, 3A)$	1	576
$(2A, 3A, 3A, 3A, 5A)$	1	400
$(2A, 3A, 3A, 3A, 5B)$	1	400
$(2A, 3A, 3A, 5A, 5A)$	1	240
$(2A, 3A, 3A, 5A, 5B)$	1	240
$(2A, 3A, 3A, 5B, 5B)$	1	240
$(2A, 3A, 5A, 5A, 5A)$	1	144
$(2A, 3A, 5A, 5A, 5B)$	1	144
$(2A, 3A, 5A, 5B, 5B)$	1	144
$(2A, 3A, 5B, 5B, 5B)$	1	144
$(2A, 5A, 5A, 5A, 5A)$	1	64
$(2A, 5A, 5A, 5A, 5B)$	1	96
$(2A, 5A, 5A, 5B, 5B)$	1	80
$(2A, 5A, 5B, 5B, 5B)$	1	96
$(2A, 5B, 5B, 5B, 5B)$	1	64
$(3A, 3A, 3A, 3A, 3A)$	2	432 (-1) 252 (1)
$(3A, 3A, 3A, 3A, 5A)$	2	300 (-1) 225 (1)
$(3A, 3A, 3A, 3A, 5B)$	2	225 (1) 300 (-1)
$(3A, 3A, 3A, 5A, 5A)$	2	220 (-1) 105 (1)
$(3A, 3A, 3A, 5A, 5B)$	2	205 (1) 120 (-1)
$(3A, 3A, 3A, 5B, 5B)$	2	220 (-1) 105 (1)
$(3A, 3A, 5A, 5A, 5A)$	2	144 (-1) 45 (1)
$(3A, 3A, 5A, 5A, 5B)$	2	105 (1) 84 (-1)
$(3A, 3A, 5A, 5B, 5B)$	2	84 (-1) 105 (1)
$(3A, 3A, 5B, 5B, 5B)$	2	144 (-1) 45 (1)
$(3A, 5A, 5A, 5A, 5A)$	2	108 (-1) 9 (1)

$(3A, 5A, 5A, 5A, 5B)$	2	81 (1) 36 (−1)
$(3A, 5A, 5A, 5B, 5B)$	2	72 (−1) 45 (1)
$(3A, 5A, 5B, 5B, 5B)$	2	81 (1) 36 (−1)
$(3A, 5B, 5B, 5B, 5B)$	2	108 (−1) 9 (1)
$(5A, 5A, 5A, 5A, 5A)$	1	96 (−1)
$(5A, 5A, 5A, 5A, 5B)$	2	45 (1) 12 (−1)
$(5A, 5A, 5A, 5B, 5B)$	2	40 (−1) 30 (1)
$(5A, 5A, 5B, 5B, 5B)$	2	30 (1) 40 (−1)
$(5A, 5B, 5B, 5B, 5B)$	2	45 (1) 12 (−1)
$(5B, 5B, 5B, 5B, 5B)$	1	96 (−1)

Table A.4: Pure braid orbits for types of length 4, their respective lengths and lifting invariants.

Tuple	Orbits	Lengths
$(2A, 2A, 2A, 2A, 2A, 2A)$	1	2880
$(2A, 2A, 2A, 2A, 2A, 3A)$	1	4212
$(2A, 2A, 2A, 2A, 2A, 5A)$	1	2500
$(2A, 2A, 2A, 2A, 2A, 5B)$	1	2500
$(2A, 2A, 2A, 2A, 3A, 3A)$	1	5562
$(2A, 2A, 2A, 2A, 3A, 5A)$	1	3375
$(2A, 2A, 2A, 2A, 3A, 5B)$	1	3375
$(2A, 2A, 2A, 2A, 5A, 5A)$	1	2050
$(2A, 2A, 2A, 2A, 5A, 5B)$	1	1925
$(2A, 2A, 2A, 2A, 5B, 5B)$	1	2050
$(2A, 2A, 2A, 3A, 3A, 3A)$	1	7416
$(2A, 2A, 2A, 3A, 3A, 5A)$	1	4500
$(2A, 2A, 2A, 3A, 3A, 5B)$	1	4500
$(2A, 2A, 2A, 3A, 5A, 5A)$	1	2700
$(2A, 2A, 2A, 3A, 5A, 5B)$	1	2700
$(2A, 2A, 2A, 3A, 5B, 5B)$	1	2700
$(2A, 2A, 2A, 5A, 5A, 5A)$	1	1540

$(2A, 2A, 2A, 5A, 5A, 5B)$	1	1640
$(2A, 2A, 2A, 5A, 5B, 5B)$	1	1640
$(2A, 2A, 2A, 5B, 5B, 5B)$	1	1540
$(2A, 2A, 3A, 3A, 3A, 3A)$	1	9720
$(2A, 2A, 3A, 3A, 3A, 5A)$	1	6000
$(2A, 2A, 3A, 3A, 3A, 5B)$	1	6000
$(2A, 2A, 3A, 3A, 5A, 5A)$	1	3600
$(2A, 2A, 3A, 3A, 5A, 5B)$	1	3600
$(2A, 2A, 3A, 3A, 5B, 5B)$	1	3600
$(2A, 2A, 3A, 5A, 5A, 5A)$	1	2160
$(2A, 2A, 3A, 5A, 5A, 5B)$	1	2160
$(2A, 2A, 3A, 5A, 5B, 5B)$	1	2160
$(2A, 2A, 3A, 5B, 5B, 5B)$	1	2160
$(2A, 2A, 5A, 5A, 5A, 5A)$	1	1400
$(2A, 2A, 5A, 5A, 5A, 5B)$	1	1240
$(2A, 2A, 5A, 5A, 5B, 5B)$	1	1320
$(2A, 2A, 5A, 5B, 5B, 5B)$	1	1240
$(2A, 2A, 5B, 5B, 5B, 5B)$	1	1400
$(2A, 3A, 3A, 3A, 3A, 3A)$	1	12672
$(2A, 3A, 3A, 3A, 3A, 5A)$	1	8000
$(2A, 3A, 3A, 3A, 3A, 5B)$	1	8000
$(2A, 3A, 3A, 3A, 5A, 5A)$	1	4800
$(2A, 3A, 3A, 3A, 5A, 5B)$	1	4800
$(2A, 3A, 3A, 3A, 5B, 5B)$	1	4800
$(2A, 3A, 3A, 5A, 5A, 5A)$	1	2880
$(2A, 3A, 3A, 5A, 5A, 5B)$	1	2880
$(2A, 3A, 3A, 5A, 5B, 5B)$	1	2880
$(2A, 3A, 3A, 5B, 5B, 5B)$	1	2880
$(2A, 3A, 5A, 5A, 5A, 5A)$	1	1728
$(2A, 3A, 5A, 5A, 5A, 5B)$	1	1728

$(2A, 3A, 5A, 5A, 5B, 5B)$	1	1728
$(2A, 3A, 5A, 5B, 5B, 5B)$	1	1728
$(2A, 3A, 5B, 5B, 5B, 5B)$	1	1728
$(2A, 5A, 5A, 5A, 5A, 5A)$	1	896
$(2A, 5A, 5A, 5A, 5A, 5B)$	1	1088
$(2A, 5A, 5A, 5A, 5B, 5B)$	1	1024
$(2A, 5A, 5A, 5B, 5B, 5B)$	1	1024
$(2A, 5A, 5B, 5B, 5B, 5B)$	1	1088
$(2A, 5B, 5B, 5B, 5B, 5B)$	1	896
$(3A, 3A, 3A, 3A, 3A, 3A)$	2	6912 (-1) 9090 (1)
$(3A, 3A, 3A, 3A, 3A, 5A)$	2	5625 (1) 5000 (-1)
$(3A, 3A, 3A, 3A, 3A, 5B)$	2	5000 (-1) 5625 (1)
$(3A, 3A, 3A, 3A, 5A, 5A)$	2	3825 (1) 2600 (-1)
$(3A, 3A, 3A, 3A, 5A, 5B)$	2	3600 (-1) 2825 (1)
$(3A, 3A, 3A, 3A, 5B, 5B)$	2	3825 (1) 2600 (-1)
$(3A, 3A, 3A, 5A, 5A, 5A)$	2	2385 (1) 1440 (-1)
$(3A, 3A, 3A, 5A, 5A, 5B)$	2	2040 (-1) 1785 (1)
$(3A, 3A, 3A, 5A, 5B, 5B)$	2	2040 (-1) 1785 (1)
$(3A, 3A, 3A, 5B, 5B, 5B)$	2	2385 (1) 1440 (-1)
$(3A, 3A, 5A, 5A, 5A, 5A)$	2	1665 (1) 648 (-1)
$(3A, 3A, 5A, 5A, 5A, 5B)$	2	1368 (-1) 945 (1)
$(3A, 3A, 5A, 5A, 5B, 5B)$	2	1008 (-1) 1305 (1)
$(3A, 3A, 5A, 5B, 5B, 5B)$	2	945 (1) 1368 (-1)
$(3A, 3A, 5B, 5B, 5B, 5B)$	2	1665 (1) 648 (-1)
$(3A, 5A, 5A, 5A, 5A, 5A)$	2	1161 (1) 216 (-1)
$(3A, 5A, 5A, 5A, 5A, 5B)$	2	864 (-1) 513 (1)
$(3A, 5A, 5A, 5A, 5B, 5B)$	2	729 (1) 648 (-1)
$(3A, 5A, 5A, 5B, 5B, 5B)$	2	648 (-1) 729 (1)
$(3A, 5A, 5B, 5B, 5B, 5B)$	2	864 (-1) 513 (1)
$(3A, 5B, 5B, 5B, 5B, 5B)$	2	1161 (1) 216 (-1)

$(5A, 5A, 5A, 5A, 5A, 5A)$	2	975 (1) 40 (−1)
$(5A, 5A, 5A, 5A, 5A, 5B)$	2	560 (−1) 200 (1)
$(5A, 5A, 5A, 5A, 5B, 5B)$	2	352 (−1) 510 (1)
$(5A, 5A, 5A, 5B, 5B, 5B)$	2	456 (−1) 355 (1)
$(5A, 5A, 5B, 5B, 5B, 5B)$	2	352 (−1) 510 (1)
$(5A, 5B, 5B, 5B, 5B, 5B)$	2	200 (1) 560 (−1)
$(5B, 5B, 5B, 5B, 5B, 5B)$	2	975 (1) 40 (−1)

Table A.5: Pure braid orbits for types of length 4, their respective lengths and lifting invariants.

Tuple	Orbits	Lengths
$(2A, 2A, 2A, 2A, 2A, 2A, 2A)$	1	47040
$(2A, 2A, 2A, 2A, 2A, 2A, 3A)$	1	63180
$(2A, 2A, 2A, 2A, 2A, 2A, 5A)$	1	37500
$(2A, 2A, 2A, 2A, 2A, 2A, 5B)$	1	37500
$(2A, 2A, 2A, 2A, 2A, 3A, 3A)$	1	84240
$(2A, 2A, 2A, 2A, 2A, 3A, 5A)$	1	50625
$(2A, 2A, 2A, 2A, 2A, 3A, 5B)$	1	50625
$(2A, 2A, 2A, 2A, 2A, 5A, 5A)$	1	30000
$(2A, 2A, 2A, 2A, 2A, 5A, 5B)$	1	30625
$(2A, 2A, 2A, 2A, 2A, 5B, 5B)$	1	30000
$(2A, 2A, 2A, 2A, 3A, 3A, 3A)$	1	112212
$(2A, 2A, 2A, 2A, 3A, 3A, 5A)$	1	67500
$(2A, 2A, 2A, 2A, 3A, 3A, 5B)$	1	67500
$(2A, 2A, 2A, 2A, 3A, 5A, 5A)$	1	40500
$(2A, 2A, 2A, 2A, 3A, 5A, 5B)$	1	40500
$(2A, 2A, 2A, 2A, 3A, 5B, 5B)$	1	40500
$(2A, 2A, 2A, 2A, 5A, 5A, 5A)$	1	24600
$(2A, 2A, 2A, 2A, 5A, 5A, 5B)$	1	24100
$(2A, 2A, 2A, 2A, 5A, 5B, 5B)$	1	24100
$(2A, 2A, 2A, 2A, 5B, 5B, 5B)$	1	24600

$(2A, 2A, 2A, 3A, 3A, 3A, 3A)$	1	149184
$(2A, 2A, 2A, 3A, 3A, 3A, 5A)$	1	90000
$(2A, 2A, 2A, 3A, 3A, 3A, 5B)$	1	90000
$(2A, 2A, 2A, 3A, 3A, 5A, 5A)$	1	54000
$(2A, 2A, 2A, 3A, 3A, 5A, 5B)$	1	54000
$(2A, 2A, 2A, 3A, 3A, 5B, 5B)$	1	54000
$(2A, 2A, 2A, 3A, 5A, 5A, 5A)$	1	32400
$(2A, 2A, 2A, 3A, 5A, 5A, 5B)$	1	32400
$(2A, 2A, 2A, 3A, 5A, 5B, 5B)$	1	32400
$(2A, 2A, 2A, 3A, 5B, 5B, 5B)$	1	32400
$(2A, 2A, 2A, 5A, 5A, 5A, 5A)$	1	18880
$(2A, 2A, 2A, 5A, 5A, 5A, 5B)$	1	19680
$(2A, 2A, 2A, 5A, 5A, 5B, 5B)$	1	19280
$(2A, 2A, 2A, 5A, 5B, 5B, 5B)$	1	19680
$(2A, 2A, 2A, 5B, 5B, 5B, 5B)$	1	18880
$(2A, 2A, 3A, 3A, 3A, 3A, 3A)$	1	198000
$(2A, 2A, 3A, 3A, 3A, 3A, 5A)$	1	120000
$(2A, 2A, 3A, 3A, 3A, 3A, 5B)$	1	120000
$(2A, 2A, 3A, 3A, 3A, 5A, 5A)$	1	72000
$(2A, 2A, 3A, 3A, 3A, 5A, 5B)$	1	72000
$(2A, 2A, 3A, 3A, 3A, 5B, 5B)$	1	72000
$(2A, 2A, 3A, 3A, 5A, 5A, 5A)$	1	43200
$(2A, 2A, 3A, 3A, 5A, 5A, 5B)$	1	43200
$(2A, 2A, 3A, 3A, 5A, 5B, 5B)$	1	43200
$(2A, 2A, 3A, 3A, 5B, 5B, 5B)$	1	43200
$(2A, 2A, 3A, 5A, 5A, 5A, 5A)$	1	25920
$(2A, 2A, 3A, 5A, 5A, 5A, 5B)$	1	25920
$(2A, 2A, 3A, 5A, 5A, 5B, 5B)$	1	25920
$(2A, 2A, 3A, 5A, 5B, 5B, 5B)$	1	25920
$(2A, 2A, 3A, 5B, 5B, 5B, 5B)$	1	25920

$(2A, 2A, 5A, 5A, 5A, 5A, 5A)$	1	16240
$(2A, 2A, 5A, 5A, 5A, 5A, 5B)$	1	15280
$(2A, 2A, 5A, 5A, 5A, 5B, 5B)$	1	15600
$(2A, 2A, 5A, 5A, 5B, 5B, 5B)$	1	15600
$(2A, 2A, 5A, 5B, 5B, 5B, 5B)$	1	15280
$(2A, 2A, 5B, 5B, 5B, 5B, 5B)$	1	16240
$(2A, 3A, 3A, 3A, 3A, 3A, 3A)$	1	261120
$(2A, 3A, 3A, 3A, 3A, 3A, 5A)$	1	160000
$(2A, 3A, 3A, 3A, 3A, 3A, 5B)$	1	160000
$(2A, 3A, 3A, 3A, 3A, 5A, 5A)$	1	96000
$(2A, 3A, 3A, 3A, 3A, 5A, 5B)$	1	96000
$(2A, 3A, 3A, 3A, 3A, 5B, 5B)$	1	96000
$(2A, 3A, 3A, 3A, 5A, 5A, 5A)$	1	57600
$(2A, 3A, 3A, 3A, 5A, 5A, 5B)$	1	57600
$(2A, 3A, 3A, 3A, 5A, 5B, 5B)$	1	57600
$(2A, 3A, 3A, 3A, 5B, 5B, 5B)$	1	57600
$(2A, 3A, 3A, 5A, 5A, 5A, 5A)$	1	34560
$(2A, 3A, 3A, 5A, 5A, 5A, 5B)$	1	34560
$(2A, 3A, 3A, 5A, 5A, 5B, 5B)$	1	34560
$(2A, 3A, 3A, 5A, 5B, 5B, 5B)$	1	34560
$(2A, 3A, 3A, 5B, 5B, 5B, 5B)$	1	34560
$(2A, 3A, 5A, 5A, 5A, 5A, 5A)$	1	20736
$(2A, 3A, 5A, 5A, 5A, 5A, 5B)$	1	20736
$(2A, 3A, 5A, 5A, 5A, 5B, 5B)$	1	20736
$(2A, 3A, 5A, 5A, 5B, 5B, 5B)$	1	20736
$(2A, 3A, 5A, 5B, 5B, 5B, 5B)$	1	20736
$(2A, 3A, 5B, 5B, 5B, 5B, 5B)$	1	20736
$(2A, 5A, 5A, 5A, 5A, 5A, 5A)$	1	11520
$(2A, 5A, 5A, 5A, 5A, 5A, 5B)$	1	12800
$(2A, 5A, 5A, 5A, 5A, 5B, 5B)$	1	12288

$(2A, 5A, 5A, 5A, 5B, 5B, 5B)$	1	12544
$(2A, 5A, 5A, 5B, 5B, 5B, 5B)$	1	12288
$(2A, 5A, 5B, 5B, 5B, 5B, 5B)$	1	12800
$(2A, 5B, 5B, 5B, 5B, 5B, 5B)$	1	11520
$(3A, 3A, 3A, 3A, 3A, 3A, 3A)$	2	181440 (-1) 160020 (1)
$(3A, 3A, 3A, 3A, 3A, 3A, 5A)$	2	103125 (1) 110000 (-1)
$(3A, 3A, 3A, 3A, 3A, 3A, 5B)$	2	110000 (-1) 103125 (1)
$(3A, 3A, 3A, 3A, 3A, 5A, 5A)$	2	58125 (1) 70000 (-1)
$(3A, 3A, 3A, 3A, 3A, 5A, 5B)$	2	68125 (1) 60000 (-1)
$(3A, 3A, 3A, 3A, 3A, 5B, 5B)$	2	58125 (1) 70000 (-1)
$(3A, 3A, 3A, 3A, 5A, 5A, 5A)$	2	33525 (1) 43200 (-1)
$(3A, 3A, 3A, 3A, 5A, 5A, 5B)$	2	37200 (-1) 39525 (1)
$(3A, 3A, 3A, 3A, 5A, 5B, 5B)$	2	37200 (-1) 39525 (1)
$(3A, 3A, 3A, 3A, 5B, 5B, 5B)$	2	43200 (-1) 33525 (1)
$(3A, 3A, 3A, 5A, 5A, 5A, 5A)$	2	28080 (-1) 18045 (1)
$(3A, 3A, 3A, 5A, 5A, 5A, 5B)$	2	20880 (-1) 25245 (1)
$(3A, 3A, 3A, 5A, 5A, 5B, 5B)$	2	24480 (-1) 21645 (1)
$(3A, 3A, 3A, 5A, 5B, 5B, 5B)$	2	20880 (-1) 25245 (1)
$(3A, 3A, 3A, 5B, 5B, 5B, 5B)$	2	28080 (-1) 18045 (1)
$(3A, 3A, 5A, 5A, 5A, 5A, 5A)$	2	9045 (1) 18576 (-1)
$(3A, 3A, 5A, 5A, 5A, 5A, 5B)$	2	15525 (1) 12096 (-1)
$(3A, 3A, 5A, 5A, 5A, 5B, 5B)$	2	13365 (1) 14256 (-1)
$(3A, 3A, 5A, 5A, 5B, 5B, 5B)$	2	13365 (1) 14256 (-1)
$(3A, 3A, 5A, 5B, 5B, 5B, 5B)$	2	15525 (1) 12096 (-1)
$(3A, 3A, 5B, 5B, 5B, 5B, 5B)$	2	18576 (-1) 9045 (1)
$(3A, 5A, 5A, 5A, 5A, 5A, 5A)$	2	12960 (-1) 3645 (1)
$(3A, 5A, 5A, 5A, 5A, 5A, 5B)$	2	10125 (1) 6480 (-1)
$(3A, 5A, 5A, 5A, 5A, 5B, 5B)$	2	9072 (-1) 7533 (1)
$(3A, 5A, 5A, 5A, 5B, 5B, 5B)$	2	8829 (1) 7776 (-1)
$(3A, 5A, 5A, 5B, 5B, 5B, 5B)$	2	7533 (1) 9072 (-1)

$(3A, 5A, 5B, 5B, 5B, 5B, 5B)$	2	6480 (−1) 10125 (1)
$(3A, 5B, 5B, 5B, 5B, 5B, 5B)$	2	12960 (−1) 3645 (1)
$(5A, 5A, 5A, 5A, 5A, 5A, 5A)$	2	10080 (−1) 1050 (1)
$(5A, 5A, 5A, 5A, 5A, 5A, 5B)$	2	3040 (−1) 6450 (1)
$(5A, 5A, 5A, 5A, 5A, 5B, 5B)$	2	4425 (1) 5680 (−1)
$(5A, 5A, 5A, 5A, 5B, 5B, 5B)$	2	5100 (1) 4800 (−1)
$(5A, 5A, 5A, 5B, 5B, 5B, 5B)$	2	5100 (1) 4800 (−1)
$(5A, 5A, 5B, 5B, 5B, 5B, 5B)$	2	5680 (−1) 4425 (1)
$(5A, 5B, 5B, 5B, 5B, 5B, 5B)$	2	6450 (1) 3040 (−1)
$(5B, 5B, 5B, 5B, 5B, 5B, 5B)$	2	10080 (−1) 1050 (1)

APPENDIX B

A_6 RESULTS

In this appendix we collect the data for our calculations on pure braid orbits for A_6. In particular it lists orbits of length 3. These orbits were not included in our classification. For the types of length $3 \leq r \leq 5$ we also list the lifting invariant as a member of $\langle(1, 2, 3, 4, 5, 6)\rangle$.

Table B.1: Pure braid orbits for types of length 3, their respective lengths, and lifting invariants

Tuple	Orbits	Lengths
$(2A, 2A, 2A)$	0	
$(2A, 2A, 3A)$	0	
$(2A, 2A, 3B)$	0	
$(2A, 2A, 4A)$	0	
$(2A, 2A, 5A)$	0	
$(2A, 2A, 5B)$	0	
$(2A, 3A, 3A)$	0	
$(2A, 3A, 3B)$	0	
$(2A, 3A, 4A)$	0	
$(2A, 3A, 5A)$	0	
$(2A, 3A, 5B)$	0	
$(2A, 3B, 3B)$	0	
$(2A, 3B, 4A)$	0	
$(2A, 3B, 5A)$	0	
$(2A, 3B, 5B)$	0	
$(2A, 4A, 4A)$	0	

$(2A,\,4A,\,5A)$	2	1 (()) 1 ((1, 3, 2))
$(2A,\,4A,\,5B)$	2	1 ((1, 3, 2)) 1 (())
$(2A,\,5A,\,5A)$	2	1 ((1, 3, 2)) 1 ((1, 2, 3))
$(2A,\,5A,\,5B)$	0	
$(2A,\,5B,\,5B)$	2	1 ((1, 2, 3)) 1 ((1, 3, 2))
$(3A,\,3A,\,3A)$	0	
$(3A,\,3A,\,3B)$	0	
$(3A,\,3A,\,4A)$	0	
$(3A,\,3A,\,5A)$	0	
$(3A,\,3A,\,5B)$	0	
$(3A,\,3B,\,3B)$	0	
$(3A,\,3B,\,4A)$	2	1 (()) 1 ((1, 2))
$(3A,\,3B,\,5A)$	1	1 (())
$(3A,\,3B,\,5B)$	1	1 (())
$(3A,\,4A,\,4A)$	0	
$(3A,\,4A,\,5A)$	2	1 ((1, 2)) 1 (())
$(3A,\,4A,\,5B)$	2	1 ((1, 2)) 1 (())
$(3A,\,5A,\,5A)$	0	
$(3A,\,5A,\,5B)$	1	1 ((1, 2))
$(3A,\,5B,\,5B)$	0	
$(3B,\,3B,\,3B)$	0	
$(3B,\,3B,\,4A)$	0	
$(3B,\,3B,\,5A)$	0	
$(3B,\,3B,\,5B)$	0	
$(3B,\,4A,\,4A)$	0	
$(3B,\,4A,\,5A)$	2	1 (()) 1 ((1, 2))
$(3B,\,4A,\,5B)$	2	1 (()) 1 ((1, 2))
$(3B,\,5A,\,5A)$	0	
$(3B,\,5A,\,5B)$	1	1 ((1, 2))
$(3B,\,5B,\,5B)$	0	

$(4A, 4A, 4A)$	4	$1\ (())\ 1\ ((1,4)(2,5)(3,6))\ 1\ (())\ 1\ ((1,4)(2,5)(3,6))$
$(4A, 4A, 5A)$	4	$1\ (())\ 1\ (())\ 1\ ((1,6,5,4,3,2))\ 1\ ((1,2,3,4,5,6))$
$(4A, 4A, 5B)$	4	$1\ ((1,2,3,4,5,6))\ 1\ ((1,6,5,4,3,2))\ 1\ (())\ 1\ (())$
$(4A, 5A, 5A)$	4	$1\ ((1,4)(2,5)(3,6))\ 1\ (())\ 1\ (())\ 1\ ((1,4)(2,5)(3,6))$
$(4A, 5A, 5B)$	4	$1\ ((1,6,5,4,3,2))\ 1\ ((1,3,5)(2,4,6))\ 1\ ((1,5,3)(2,6,4))\ 1\ ((1,2,3,4,5,6))$
$(4A, 5B, 5B)$	4	$1\ (())\ 1\ ((1,4)(2,5)(3,6))\ 1\ ((1,4)(2,5)(3,6))\ 1\ (())$
$(5A, 5A, 5A)$	2	$1\ (())\ 1\ (())$
$(5A, 5A, 5B)$	2	$1\ ((1,2,3,4,5,6))\ 1\ ((1,6,5,4,3,2))$
$(5A, 5B, 5B)$	2	$1\ ((1,6,5,4,3,2))\ 1\ ((1,2,3,4,5,6))$
$(5B, 5B, 5B)$	2	$1\ (())\ 1\ (())$

Table B.2: Length 4 braid orbits.

Tuple	Orbits	Lengths
$(2A, 2A, 2A, 2A)$	0	
$(2A, 2A, 2A, 3A)$	0	
$(2A, 2A, 2A, 3B)$	0	
$(2A, 2A, 2A, 4A)$	2	$24\ ((1,3,2))\ 24\ (())$
$(2A, 2A, 2A, 5A)$	2	$15\ ((1,2,3))\ 15\ ((1,3,2))$
$(2A, 2A, 2A, 5B)$	2	$15\ ((1,3,2))\ 15\ ((1,2,3))$
$(2A, 2A, 3A, 3A)$	0	
$(2A, 2A, 3A, 3B)$	1	$18\ (0)$
$(2A, 2A, 3A, 4A)$	1	$48\ (0)$
$(2A, 2A, 3A, 5A)$	1	$30\ (0)$
$(2A, 2A, 3A, 5B)$	1	$30\ (0)$
$(2A, 2A, 3B, 3B)$	0	
$(2A, 2A, 3B, 4A)$	1	$48\ (0)$
$(2A, 2A, 3B, 5A)$	1	$30\ (0)$
$(2A, 2A, 3B, 5B)$	1	$30\ (0)$
$(2A, 2A, 4A, 4A)$	3	$24\ (())\ 40\ ((1,3,2))\ 24\ ((1,2,3))$
$(2A, 2A, 4A, 5A)$	3	$30\ ((1,3,2))\ 30\ (())\ 40\ ((1,2,3))$

$(2A, 2A, 4A, 5B)$	3	30 $((1,3,2))$ 40 $((1,2,3))$ 30 $(())$
$(2A, 2A, 5A, 5A)$	3	15 $((1,3,2))$ 30 $(())$ 15 $((1,2,3))$
$(2A, 2A, 5A, 5B)$	3	30 $((1,3,2))$ 10 $(())$ 30 $((1,2,3))$
$(2A, 2A, 5B, 5B)$	3	30 $(())$ 15 $((1,2,3))$ 15 $((1,3,2))$
$(2A, 3A, 3A, 3A)$	0	
$(2A, 3A, 3A, 3B)$	1	20 (0)
$(2A, 3A, 3A, 4A)$	1	32 (0)
$(2A, 3A, 3A, 5A)$	1	20 (0)
$(2A, 3A, 3A, 5B)$	1	20 (0)
$(2A, 3A, 3B, 3B)$	1	20 (0)
$(2A, 3A, 3B, 4A)$	1	58 (0)
$(2A, 3A, 3B, 5A)$	1	40 (0)
$(2A, 3A, 3B, 5B)$	1	40 (0)
$(2A, 3A, 4A, 4A)$	1	96 (0)
$(2A, 3A, 4A, 5A)$	1	90 (0)
$(2A, 3A, 4A, 5B)$	1	90 (0)
$(2A, 3A, 5A, 5A)$	1	60 (0)
$(2A, 3A, 5A, 5B)$	1	60 (0)
$(2A, 3A, 5B, 5B)$	1	60 (0)
$(2A, 3B, 3B, 3B)$	0	
$(2A, 3B, 3B, 4A)$	1	32 (0)
$(2A, 3B, 3B, 5A)$	1	20 (0)
$(2A, 3B, 3B, 5B)$	1	20 (0)
$(2A, 3B, 4A, 4A)$	1	96 (0)
$(2A, 3B, 4A, 5A)$	1	90 (0)
$(2A, 3B, 4A, 5B)$	1	90 (0)
$(2A, 3B, 5A, 5A)$	1	60 (0)
$(2A, 3B, 5A, 5B)$	1	60 (0)
$(2A, 3B, 5B, 5B)$	1	60 (0)
$(2A, 4A, 4A, 4A)$	3	36 $(())$ 96 $((1,3,2))$ 96 $((1,2,3))$

$(2A,\, 4A,\, 4A,\, 5A)$	3	75 $(())$ 75 $((1,2,3))$ 50 $((1,3,2))$
$(2A,\, 4A,\, 4A,\, 5B)$	3	50 $((1,3,2))$ 75 $(())$ 75 $((1,2,3))$
$(2A,\, 4A,\, 5A,\, 5A)$	3	20 $((1,2,3))$ 72 $(())$ 72 $((1,3,2))$
$(2A,\, 4A,\, 5A,\, 5B)$	3	80 $((1,2,3))$ 42 $((1,3,2))$ 42 $(())$
$(2A,\, 4A,\, 5B,\, 5B)$	3	72 $(())$ 72 $((1,3,2))$ 20 $((1,2,3))$
$(2A,\, 5A,\, 5A,\, 5A)$	2	60 $((1,3,2))$ 60 $((1,2,3))$
$(2A,\, 5A,\, 5A,\, 5B)$	3	36 $((1,3,2))$ 40 $(())$ 36 $((1,2,3))$
$(2A,\, 5A,\, 5B,\, 5B)$	3	36 $((1,2,3))$ 40 $(())$ 36 $((1,3,2))$
$(2A,\, 5B,\, 5B,\, 5B)$	2	60 $((1,3,2))$ 60 $((1,2,3))$
$(3A,\, 3A,\, 3A,\, 3A)$	0	
$(3A,\, 3A,\, 3A,\, 3B)$	1	12 $((1,2))$
$(3A,\, 3A,\, 3A,\, 4A)$	2	16 $((1,2))$ 16 $(())$
$(3A,\, 3A,\, 3A,\, 5A)$	1	10 $(())$
$(3A,\, 3A,\, 3A,\, 5B)$	1	10 $(())$
$(3A,\, 3A,\, 3B,\, 3B)$	2	18 $(())$ 8 $((1,2))$
$(3A,\, 3A,\, 3B,\, 4A)$	2	24 $(())$ 24 $((1,2))$
$(3A,\, 3A,\, 3B,\, 5A)$	2	15 $(())$ 20 $((1,2))$
$(3A,\, 3A,\, 3B,\, 5B)$	2	20 $((1,2))$ 15 $(())$
$(3A,\, 3A,\, 4A,\, 4A)$	3	48 $((1,2))$ 48 $(())$ 8 $((1,2))$
$(3A,\, 3A,\, 4A,\, 5A)$	2	40 $(())$ 40 $((1,2))$
$(3A,\, 3A,\, 4A,\, 5B)$	2	40 $(())$ 40 $((1,2))$
$(3A,\, 3A,\, 5A,\, 5A)$	2	30 $(())$ 20 $((1,2))$
$(3A,\, 3A,\, 5A,\, 5B)$	3	10 $((1,2))$ 30 $(())$ 5 $(())$
$(3A,\, 3A,\, 5B,\, 5B)$	2	20 $((1,2))$ 30 $(())$
$(3A,\, 3B,\, 3B,\, 3B)$	1	12 $((1,2))$
$(3A,\, 3B,\, 3B,\, 4A)$	2	24 $(())$ 24 $((1,2))$
$(3A,\, 3B,\, 3B,\, 5A)$	2	20 $((1,2))$ 15 $(())$
$(3A,\, 3B,\, 3B,\, 5B)$	2	20 $((1,2))$ 15 $(())$
$(3A,\, 3B,\, 4A,\, 4A)$	2	36 $(())$ 40 $((1,2))$
$(3A,\, 3B,\, 4A,\, 5A)$	2	40 $(())$ 40 $((1,2))$

$(3A, 3B, 4A, 5B)$	2	40 $(())$ 40 $((1,2))$
$(3A, 3B, 5A, 5A)$	2	40 $((1,2))$ 27 $(())$
$(3A, 3B, 5A, 5B)$	2	22 $(())$ 40 $((1,2))$
$(3A, 3B, 5B, 5B)$	2	27 $(())$ 40 $((1,2))$
$(3A, 4A, 4A, 4A)$	2	108 $((1,2))$ 108 $(())$
$(3A, 4A, 4A, 5A)$	2	90 $(())$ 90 $((1,2))$
$(3A, 4A, 4A, 5B)$	2	90 $(())$ 90 $((1,2))$
$(3A, 4A, 5A, 5A)$	2	72 $((1,2))$ 72 $(())$
$(3A, 4A, 5A, 5B)$	2	72 $((1,2))$ 72 $(())$
$(3A, 4A, 5B, 5B)$	2	72 $((1,2))$ 72 $(())$
$(3A, 5A, 5A, 5A)$	2	54 $((1,2))$ 45 $(())$
$(3A, 5A, 5A, 5B)$	2	60 $(())$ 48 $((1,2))$
$(3A, 5A, 5B, 5B)$	2	60 $(())$ 48 $((1,2))$
$(3A, 5B, 5B, 5B)$	2	45 $(())$ 54 $((1,2))$
$(3B, 3B, 3B, 3B)$	0	
$(3B, 3B, 3B, 4A)$	2	16 $(())$ 16 $((1,2))$
$(3B, 3B, 3B, 5A)$	1	10 $(())$
$(3B, 3B, 3B, 5B)$	1	10 $(())$
$(3B, 3B, 4A, 4A)$	3	48 $(())$ 48 $((1,2))$ 8 $((1,2))$
$(3B, 3B, 4A, 5A)$	2	40 $(())$ 40 $((1,2))$
$(3B, 3B, 4A, 5B)$	2	40 $(())$ 40 $((1,2))$
$(3B, 3B, 5A, 5A)$	2	30 $(())$ 20 $((1,2))$
$(3B, 3B, 5A, 5B)$	3	30 $(())$ 10 $((1,2))$ 5 $(())$
$(3B, 3B, 5B, 5B)$	2	30 $(())$ 20 $((1,2))$
$(3B, 4A, 4A, 4A)$	2	108 $((1,2))$ 108 $(())$
$(3B, 4A, 4A, 5A)$	2	90 $((1,2))$ 90 $(())$
$(3B, 4A, 4A, 5B)$	2	90 $(())$ 90 $((1,2))$
$(3B, 4A, 5A, 5A)$	2	72 $(())$ 72 $((1,2))$
$(3B, 4A, 5A, 5B)$	2	72 $((1,2))$ 72 $(())$
$(3B, 4A, 5B, 5B)$	2	72 $((1,2))$ 72 $(())$

$(3B, 5A, 5A, 5A)$	2	45 (()) 54 ((1, 2))
$(3B, 5A, 5A, 5B)$	2	48 ((1, 2)) 60 (())
$(3B, 5A, 5B, 5B)$	2	48 ((1, 2)) 60 (())
$(3B, 5B, 5B, 5B)$	2	54 ((1, 2)) 45 (())
$(4A, 4A, 4A, 4A)$	10	24 ((1, 4)(2, 5)(3, 6)) 72 ((1, 6, 5, 4, 3, 2)) 72 ((1, 2, 3, 4, 5, 6)) 24 ((1, 3, 5)(2, 4, 6)) 24 ((1, 5, 3)(2, 6, 4)) 40 (()) 24 ((1, 4)(2, 5)(3, 6)) 40 (()) 40 (()) 24 ((1, 4)(2, 5)(3, 6))
$(4A, 4A, 4A, 5A)$	6	80 ((1, 4)(2, 5)(3, 6)) 60 ((1, 6, 5, 4, 3, 2)) 80 (()) 60 ((1, 2, 3, 4, 5, 6)) 60 ((1, 5, 3)(2, 6, 4)) 60 ((1, 3, 5)(2, 4, 6))
$(4A, 4A, 4A, 5B)$	6	80 (()) 80 ((1, 4)(2, 5)(3, 6)) 60 ((1, 6, 5, 4, 3, 2)) 60 ((1, 2, 3, 4, 5, 6)) 60 ((1, 5, 3)(2, 6, 4)) 60 ((1, 3, 5)(2, 4, 6))
$(4A, 4A, 5A, 5A)$	9	64 (()) 40 (()) 30 ((1, 3, 5)(2, 4, 6)) 30 ((1, 5, 3)(2, 6, 4)) 40 ((1, 6, 5, 4, 3, 2)) 40 ((1, 2, 3, 4, 5, 6)) 80 ((1, 4)(2, 5)(3, 6)) 2 ((1, 6, 5, 4, 3, 2)) 2 ((1, 2, 3, 4, 5, 6))
$(4A, 4A, 5A, 5B)$	8	72 ((1, 6, 5, 4, 3, 2)) 60 ((1, 3, 5)(2, 4, 6)) 72 ((1, 2, 3, 4, 5, 6)) 24 (()) 60 ((1, 5, 3)(2, 6, 4)) 20 (()) 10 ((1, 4)(2, 5)(3, 6)) 10 ((1, 4)(2, 5)(3, 6))
$(4A, 4A, 5B, 5B)$	9	40 ((1, 2, 3, 4, 5, 6)) 64 (()) 80 ((1, 4)(2, 5)(3, 6)) 30 ((1, 5, 3)(2, 6, 4)) 30 ((1, 3, 5)(2, 4, 6)) 40 (()) 40 ((1, 6, 5, 4, 3, 2)) 2 ((1, 6, 5, 4, 3, 2)) 2 ((1, 2, 3, 4, 5, 6))
$(4A, 5A, 5A, 5A)$	6	80 (()) 24 ((1, 3, 5)(2, 4, 6)) 24 ((1, 6, 5, 4, 3, 2)) 80 ((1, 4)(2, 5)(3, 6)) 24 ((1, 2, 3, 4, 5, 6)) 24 ((1, 5, 3)(2, 6, 4))
$(4A, 5A, 5A, 5B)$	6	48 ((1, 6, 5, 4, 3, 2)) 48 ((1, 2, 3, 4, 5, 6)) 48 ((1, 5, 3)(2, 6, 4)) 48 ((1, 3, 5)(2, 4, 6)) 32 (()) 32 ((1, 4)(2, 5)(3, 6))
$(4A, 5A, 5B, 5B)$	6	48 ((1, 6, 5, 4, 3, 2)) 32 (()) 32 ((1, 4)(2, 5)(3, 6)) 48 ((1, 3, 5)(2, 4, 6)) 48 ((1, 5, 3)(2, 6, 4)) 48 ((1, 2, 3, 4, 5, 6))
$(4A, 5B, 5B, 5B)$	6	24 ((1, 3, 5)(2, 4, 6)) 80 (()) 24 ((1, 6, 5, 4, 3, 2)) 80 ((1, 4)(2, 5)(3, 6)) 24 ((1, 2, 3, 4, 5, 6)) 24 ((1, 5, 3)(2, 6, 4))

$(5A, 5A, 5A, 5A)$	12	80 ((1,4)(2,5)(3,6)) 30 (()) 30 (()) 2 ((1,2,3,4,5,6)) 2 ((1,2,3,4,5,6)) 30 (()) 15 ((1,3,5)(2,4,6)) 15 ((1,5,3)(2,6,4)) 2 ((1,2,3,4,5,6)) 2 ((1,6,5,4,3,2)) 2 ((1,6,5,4,3,2)) 2 ((1,6,5,4,3,2))
$(5A, 5A, 5A, 5B)$	5	42 ((1,2,3,4,5,6)) 45 ((1,3,5)(2,4,6)) 18 (()) 42 ((1,6,5,4,3,2)) 45 ((1,5,3)(2,6,4))
$(5A, 5A, 5B, 5B)$	9	24 ((1,6,5,4,3,2)) 24 ((1,2,3,4,5,6)) 20 ((1,4)(2,5)(3,6)) 44 (()) 30 ((1,5,3)(2,6,4)) 20 ((1,4)(2,5)(3,6)) 30 ((1,3,5)(2,4,6)) 5 (()) 5 (())
$(5A, 5B, 5B, 5B)$	5	42 ((1,6,5,4,3,2)) 45 ((1,3,5)(2,4,6)) 42 ((1,2,3,4,5,6)) 45 ((1,5,3)(2,6,4)) 18 (())
$(5B, 5B, 5B, 5B,)$	12	30 (()) 2 ((1,2,3,4,5,6)) 80 ((1,4)(2,5)(3,6)) 15 ((1,5,3)(2,6,4)) 30 (()) 30 (()) 15 ((1,3,5)(2,4,6)) 2 ((1,2,3,4,5,6)) 2 ((1,6,5,4,3,2)) 2 ((1,6,5,4,3,2)) 2 ((1,2,3,4,5,6)) 2 ((1,6,5,4,3,2))

Table B.3: Pure braid orbits for types of length 5, their respective lengths, and lifting invariants

Tuple	Orbits	Lengths
$(2A, 2A, 2A, 2A, 2A)$	2	432 ((1,3,2)) 432 ((1,2,3))
$(2A, 2A, 2A, 2A, 3A)$	1	864 (0)
$(2A, 2A, 2A, 2A, 3B)$	1	864 (0)
$(2A, 2A, 2A, 2A, 4A)$	3	960 ((1,2,3)) 864 (()) 864 ((1,3,2))
$(2A, 2A, 2A, 2A, 5A)$	3	675 ((1,3,2)) 600 (()) 675 ((1,2,3))
$(2A, 2A, 2A, 2A, 5B)$	3	675 ((1,2,3)) 600 (()) 675 ((1,3,2))
$(2A, 2A, 2A, 3A, 3A)$	1	720 (0)
$(2A, 2A, 2A, 3A, 3B)$	1	1080 (0)
$(2A, 2A, 2A, 3A, 4A)$	1	2448 (0)
$(2A, 2A, 2A, 3A, 5A)$	1	1800 (0)
$(2A, 2A, 2A, 3A, 5B)$	1	1800 (0)
$(2A, 2A, 2A, 3B, 3B)$	1	720 (0)

$(2A,\ 2A,\ 2A,\ 3B,\ 4A)$	1	2448 (0)
$(2A,\ 2A,\ 2A,\ 3B,\ 5A)$	1	1800 (0)
$(2A,\ 2A,\ 2A,\ 3B,\ 5B)$	1	1800 (0)
$(2A,\ 2A,\ 2A,\ 4A,\ 4A)$	3	1944 (()) 1944 ($(1,2,3)$) 1464 ($(1,3,2)$)
$(2A,\ 2A,\ 2A,\ 4A,\ 5A)$	3	1575 (()) 1575 ($(1,3,2)$) 1400 ($(1,2,3)$)
$(2A,\ 2A,\ 2A,\ 4A,\ 5B)$	3	1575 (()) 1575 ($(1,3,2)$) 1400 ($(1,2,3)$)
$(2A,\ 2A,\ 2A,\ 5A,\ 5A)$	3	800 (()) 1305 ($(1,2,3)$) 1305 ($(1,3,2)$)
$(2A,\ 2A,\ 2A,\ 5A,\ 5B)$	3	1080 ($(1,2,3)$) 1200 (()) 1080 ($(1,3,2)$)
$(2A,\ 2A,\ 2A,\ 5B,\ 5B)$	3	1305 ($(1,2,3)$) 800 (()) 1305 ($(1,3,2)$)
$(2A,\ 2A,\ 3A,\ 3A,\ 3A)$	1	504 (0)
$(2A,\ 2A,\ 3A,\ 3A,\ 3B)$	1	972 (0)
$(2A,\ 2A,\ 3A,\ 3A,\ 4A)$	1	2112 (0)
$(2A,\ 2A,\ 3A,\ 3A,\ 5A)$	1	1500 (0)
$(2A,\ 2A,\ 3A,\ 3A,\ 5B)$	1	1500 (0)
$(2A,\ 2A,\ 3A,\ 3B,\ 3B)$	1	972 (0)
$(2A,\ 2A,\ 3A,\ 3B,\ 4A)$	1	2322 (0)
$(2A,\ 2A,\ 3A,\ 3B,\ 5A)$	1	1800 (0)
$(2A,\ 2A,\ 3A,\ 3B,\ 5B)$	1	1800 (0)
$(2A,\ 2A,\ 3A,\ 4A,\ 4A)$	1	4896 (0)
$(2A,\ 2A,\ 3A,\ 4A,\ 5A)$	1	4050 (0)
$(2A,\ 2A,\ 3A,\ 4A,\ 5B)$	1	4050 (0)
$(2A,\ 2A,\ 3A,\ 5A,\ 5A)$	1	3060 (0)
$(2A,\ 2A,\ 3A,\ 5A,\ 5B)$	1	3060 (0)
$(2A,\ 2A,\ 3A,\ 5B,\ 5B)$	1	3060 (0)
$(2A,\ 2A,\ 3B,\ 3B,\ 3B)$	1	504 (0)
$(2A,\ 2A,\ 3B,\ 3B,\ 4A)$	1	2112 (0)
$(2A,\ 2A,\ 3B,\ 3B,\ 5A)$	1	1500 (0)
$(2A,\ 2A,\ 3B,\ 3B,\ 5B)$	1	1500 (0)
$(2A,\ 2A,\ 3B,\ 4A,\ 4A)$	1	4896 (0)
$(2A,\ 2A,\ 3B,\ 4A,\ 5A)$	1	4050 (0)

$(2A, 2A, 3B, 4A, 5B)$	1	4050 (0)
$(2A, 2A, 3B, 5A, 5A)$	1	3060 (0)
$(2A, 2A, 3B, 5A, 5B)$	1	3060 (0)
$(2A, 2A, 3B, 5B, 5B)$	1	3060 (0)
$(2A, 2A, 4A, 4A, 4A)$	3	3456 ((1, 2, 3)) 3456 ((1, 3, 2)) 4260 (())
$(2A, 2A, 4A, 4A, 5A)$	3	2925 ((1, 2, 3)) 2925 (()) 3250 ((1, 3, 2))
$(2A, 2A, 4A, 4A, 5B)$	3	3250 ((1, 3, 2)) 2925 ((1, 2, 3)) 2925 (())
$(2A, 2A, 4A, 5A, 5A)$	3	2980 ((1, 2, 3)) 2160 (()) 2160 ((1, 3, 2))
$(2A, 2A, 4A, 5A, 5B)$	3	2610 (()) 2610 ((1, 3, 2)) 2080 ((1, 2, 3))
$(2A, 2A, 4A, 5B, 5B)$	3	2160 ((1, 3, 2)) 2980 ((1, 2, 3)) 2160 (())
$(2A, 2A, 5A, 5A, 5A)$	3	1620 ((1, 2, 3)) 1620 ((1, 3, 2)) 2340 (())
$(2A, 2A, 5A, 5A, 5B)$	3	1660 (()) 1980 ((1, 2, 3)) 1980 ((1, 3, 2))
$(2A, 2A, 5A, 5B, 5B)$	3	1980 ((1, 3, 2)) 1980 ((1, 2, 3)) 1660 (())
$(2A, 2A, 5B, 5B, 5B)$	3	2340 (()) 1620 ((1, 2, 3)) 1620 ((1, 3, 2))
$(2A, 3A, 3A, 3A, 3A)$	1	384 (0)
$(2A, 3A, 3A, 3A, 3B)$	1	816 (0)
$(2A, 3A, 3A, 3A, 4A)$	1	1824 (0)
$(2A, 3A, 3A, 3A, 5A)$	1	1200 (0)
$(2A, 3A, 3A, 3A, 5B)$	1	1200 (0)
$(2A, 3A, 3A, 3B, 3B)$	1	944 (0)
$(2A, 3A, 3A, 3B, 4A)$	1	2032 (0)
$(2A, 3A, 3A, 3B, 5A)$	1	1600 (0)
$(2A, 3A, 3A, 3B, 5B)$	1	1600 (0)
$(2A, 3A, 3A, 4A, 4A)$	1	4584 (0)
$(2A, 3A, 3A, 4A, 5A)$	1	3600 (0)
$(2A, 3A, 3A, 4A, 5B)$	1	3600 (0)
$(2A, 3A, 3A, 5A, 5A)$	1	2640 (0)
$(2A, 3A, 3A, 5A, 5B)$	1	2640 (0)
$(2A, 3A, 3A, 5B, 5B)$	1	2640 (0)
$(2A, 3A, 3B, 3B, 3B)$	1	816 (0)

$(2A, 3A, 3B, 3B, 4A)$	1	2032 (0)
$(2A, 3A, 3B, 3B, 5A)$	1	1600 (0)
$(2A, 3A, 3B, 3B, 5B)$	1	1600 (0)
$(2A, 3A, 3B, 4A, 4A)$	1	4284 (0)
$(2A, 3A, 3B, 4A, 5A)$	1	3600 (0)
$(2A, 3A, 3B, 4A, 5B)$	1	3600 (0)
$(2A, 3A, 3B, 5A, 5A)$	1	2880 (0)
$(2A, 3A, 3B, 5A, 5B)$	1	2880 (0)
$(2A, 3A, 3B, 5B, 5B)$	1	2880 (0)
$(2A, 3A, 4A, 4A, 4A)$	1	10008 (0)
$(2A, 3A, 4A, 4A, 5A)$	1	8100 (0)
$(2A, 3A, 4A, 4A, 5B)$	1	8100 (0)
$(2A, 3A, 4A, 5A, 5A)$	1	6480 (0)
$(2A, 3A, 4A, 5A, 5B)$	1	6480 (0)
$(2A, 3A, 4A, 5B, 5B)$	1	6480 (0)
$(2A, 3A, 5A, 5A, 5A)$	1	5040 (0)
$(2A, 3A, 5A, 5A, 5B)$	1	5040 (0)
$(2A, 3A, 5A, 5B, 5B)$	1	5040 (0)
$(2A, 3A, 5B, 5B, 5B)$	1	5040 (0)
$(2A, 3B, 3B, 3B, 3B)$	1	384 (0)
$(2A, 3B, 3B, 3B, 4A)$	1	1824 (0)
$(2A, 3B, 3B, 3B, 5A)$	1	1200 (0)
$(2A, 3B, 3B, 3B, 5B)$	1	1200 (0)
$(2A, 3B, 3B, 4A, 4A)$	1	4584 (0)
$(2A, 3B, 3B, 4A, 5A)$	1	3600 (0)
$(2A, 3B, 3B, 4A, 5B)$	1	3600 (0)
$(2A, 3B, 3B, 5A, 5A)$	1	2640 (0)
$(2A, 3B, 3B, 5A, 5B)$	1	2640 (0)
$(2A, 3B, 3B, 5B, 5B)$	1	2640 (0)
$(2A, 3B, 4A, 4A, 4A)$	1	10008 (0)

$(2A, 3B, 4A, 4A, 5A)$	1	8100 (0)
$(2A, 3B, 4A, 4A, 5B)$	1	8100 (0)
$(2A, 3B, 4A, 5A, 5A)$	1	6480 (0)
$(2A, 3B, 4A, 5A, 5B)$	1	6480 (0)
$(2A, 3B, 4A, 5B, 5B)$	1	6480 (0)
$(2A, 3B, 5A, 5A, 5A)$	1	5040 (0)
$(2A, 3B, 5A, 5A, 5B)$	1	5040 (0)
$(2A, 3B, 5A, 5B, 5B)$	1	5040 (0)
$(2A, 3B, 5B, 5B, 5B)$	1	5040 (0)
$(2A, 4A, 4A, 4A, 4A)$	3	6144 $((1,2,3))$ 7776 (()) 7776 $((1,3,2))$
$(2A, 4A, 4A, 4A, 5A)$	3	5600 (()) 6300 $((1,3,2))$ 6300 $((1,2,3))$
$(2A, 4A, 4A, 4A, 5B)$	3	6300 $((1,3,2))$ 6300 $((1,2,3))$ 5600 (())
$(2A, 4A, 4A, 5A, 5A)$	3	5400 (()) 3800 $((1,3,2))$ 5400 $((1,2,3))$
$(2A, 4A, 4A, 5A, 5B)$	3	5600 $((1,3,2))$ 4500 $((1,2,3))$ 4500 (())
$(2A, 4A, 4A, 5B, 5B)$	3	5400 $((1,2,3))$ 3800 $((1,3,2))$ 5400 (())
$(2A, 4A, 5A, 5A, 5A)$	3	4464 (()) 2720 $((1,2,3))$ 4464 $((1,3,2))$
$(2A, 4A, 5A, 5A, 5B)$	3	3744 $((1,3,2))$ 3744 (()) 4160 $((1,2,3))$
$(2A, 4A, 5A, 5B, 5B)$	3	3744 $((1,3,2))$ 3744 (()) 4160 $((1,2,3))$
$(2A, 4A, 5B, 5B, 5B)$	3	4464 (()) 4464 $((1,3,2))$ 2720 $((1,2,3))$
$(2A, 5A, 5A, 5A, 5A)$	3	1440 (()) 3888 $((1,3,2))$ 3888 $((1,2,3))$
$(2A, 5A, 5A, 5A, 5B)$	3	2736 $((1,3,2))$ 3680 (()) 2736 $((1,2,3))$
$(2A, 5A, 5A, 5B, 5B)$	3	3312 $((1,3,2))$ 2560 (()) 3312 $((1,2,3))$
$(2A, 5A, 5B, 5B, 5B)$	3	2736 $((1,2,3))$ 2736 $((1,3,2))$ 3680 (())
$(2A, 5B, 5B, 5B, 5B)$	3	3888 $((1,2,3))$ 3888 $((1,3,2))$ 1440 (())
$(3A, 3A, 3A, 3A, 3A)$	1	192 $((1,2))$
$(3A, 3A, 3A, 3A, 3B)$	2	432 (()) 264 $((1,2))$
$(3A, 3A, 3A, 3A, 4A)$	2	768 (()) 768 $((1,2))$
$(3A, 3A, 3A, 3A, 5A)$	2	600 $((1,2))$ 300 (())
$(3A, 3A, 3A, 3A, 5B)$	2	600 $((1,2))$ 300 (())
$(3A, 3A, 3A, 3B, 3B)$	2	432 $((1,2))$ 378 (())

$(3A, 3A, 3A, 3B, 4A)$	2	960 (()) 960 ((1, 2))
$(3A, 3A, 3A, 3B, 5A)$	2	825 (()) 600 ((1, 2))
$(3A, 3A, 3A, 3B, 5B)$	2	825 (()) 600 ((1, 2))
$(3A, 3A, 3A, 4A, 4A)$	2	2064 (()) 2064 ((1, 2))
$(3A, 3A, 3A, 4A, 5A)$	2	1600 ((1, 2)) 1600 (())
$(3A, 3A, 3A, 4A, 5B)$	2	1600 ((1, 2)) 1600 (())
$(3A, 3A, 3A, 5A, 5A)$	2	1020 (()) 1200 ((1, 2))
$(3A, 3A, 3A, 5A, 5B)$	2	945 (()) 1300 ((1, 2))
$(3A, 3A, 3A, 5B, 5B)$	2	1200 ((1, 2)) 1020 (())
$(3A, 3A, 3B, 3B, 3B)$	2	378 (()) 432 ((1, 2))
$(3A, 3A, 3B, 3B, 4A)$	2	832 ((1, 2)) 832 (())
$(3A, 3A, 3B, 3B, 5A)$	2	625 (()) 800 ((1, 2))
$(3A, 3A, 3B, 3B, 5B)$	2	625 (()) 800 ((1, 2))
$(3A, 3A, 3B, 4A, 4A)$	2	1944 (()) 1960 ((1, 2))
$(3A, 3A, 3B, 4A, 5A)$	2	1600 (()) 1600 ((1, 2))
$(3A, 3A, 3B, 4A, 5B)$	2	1600 (()) 1600 ((1, 2))
$(3A, 3A, 3B, 5A, 5A)$	2	1305 (()) 1240 ((1, 2))
$(3A, 3A, 3B, 5A, 5B)$	2	1240 ((1, 2)) 1330 (())
$(3A, 3A, 3B, 5B, 5B)$	2	1240 ((1, 2)) 1305 (())
$(3A, 3A, 4A, 4A, 4A)$	2	4320 ((1, 2)) 4320 (())
$(3A, 3A, 4A, 4A, 5A)$	2	3600 ((1, 2)) 3600 (())
$(3A, 3A, 4A, 4A, 5B)$	2	3600 (()) 3600 ((1, 2))
$(3A, 3A, 4A, 5A, 5A)$	2	2880 ((1, 2)) 2880 (())
$(3A, 3A, 4A, 5A, 5B)$	2	2880 ((1, 2)) 2880 (())
$(3A, 3A, 4A, 5B, 5B)$	2	2880 ((1, 2)) 2880 (())
$(3A, 3A, 5A, 5A, 5A)$	2	2115 (()) 2340 ((1, 2))
$(3A, 3A, 5A, 5A, 5B)$	2	2010 (()) 2400 ((1, 2))
$(3A, 3A, 5A, 5B, 5B)$	2	2400 ((1, 2)) 2010 (())
$(3A, 3A, 5B, 5B, 5B)$	2	2340 ((1, 2)) 2115 (())
$(3A, 3B, 3B, 3B, 3B)$	2	432 (()) 264 ((1, 2))

$(3A,\ 3B,\ 3B,\ 3B,\ 4A)$	2	960 $((1,2))$ 960 $(())$
$(3A,\ 3B,\ 3B,\ 3B,\ 5A)$	2	600 $((1,2))$ 825 $(())$
$(3A,\ 3B,\ 3B,\ 3B,\ 5B)$	2	825 $(())$ 600 $((1,2))$
$(3A,\ 3B,\ 3B,\ 4A,\ 4A)$	2	1944 $(())$ 1960 $((1,2))$
$(3A,\ 3B,\ 3B,\ 4A,\ 5A)$	2	1600 $(())$ 1600 $((1,2))$
$(3A,\ 3B,\ 3B,\ 4A,\ 5B)$	2	1600 $((1,2))$ 1600 $(())$
$(3A,\ 3B,\ 3B,\ 5A,\ 5A)$	2	1305 $(())$ 1240 $((1,2))$
$(3A,\ 3B,\ 3B,\ 5A,\ 5B)$	2	1330 $(())$ 1240 $((1,2))$
$(3A,\ 3B,\ 3B,\ 5B,\ 5B)$	2	1240 $((1,2))$ 1305 $(())$
$(3A,\ 3B,\ 4A,\ 4A,\ 4A)$	2	4644 $(())$ 4644 $((1,2))$
$(3A,\ 3B,\ 4A,\ 4A,\ 5A)$	2	3600 $((1,2))$ 3600 $(())$
$(3A,\ 3B,\ 4A,\ 4A,\ 5B)$	2	3600 $(())$ 3600 $((1,2))$
$(3A,\ 3B,\ 4A,\ 5A,\ 5A)$	2	2880 $(())$ 2880 $((1,2))$
$(3A,\ 3B,\ 4A,\ 5A,\ 5B)$	2	2880 $(())$ 2880 $((1,2))$
$(3A,\ 3B,\ 4A,\ 5B,\ 5B)$	2	2880 $(())$ 2880 $((1,2))$
$(3A,\ 3B,\ 5A,\ 5A,\ 5A)$	2	2160 $((1,2))$ 2484 $(())$
$(3A,\ 3B,\ 5A,\ 5A,\ 5B)$	2	2160 $((1,2))$ 2439 $(())$
$(3A,\ 3B,\ 5A,\ 5B,\ 5B)$	2	2160 $((1,2))$ 2439 $(())$
$(3A,\ 3B,\ 5B,\ 5B,\ 5B)$	2	2484 $(())$ 2160 $((1,2))$
$(3A,\ 4A,\ 4A,\ 4A,\ 4A)$	2	9936 $((1,2))$ 9792 $(())$
$(3A,\ 4A,\ 4A,\ 4A,\ 5A)$	2	8100 $((1,2))$ 8100 $(())$
$(3A,\ 4A,\ 4A,\ 4A,\ 5B)$	2	8100 $((1,2))$ 8100 $(())$
$(3A,\ 4A,\ 4A,\ 5A,\ 5A)$	2	6480 $((1,2))$ 6480 $(())$
$(3A,\ 4A,\ 4A,\ 5A,\ 5B)$	2	6480 $((1,2))$ 6480 $(())$
$(3A,\ 4A,\ 4A,\ 5B,\ 5B)$	2	6480 $((1,2))$ 6480 $(())$
$(3A,\ 4A,\ 5A,\ 5A,\ 5A)$	2	5184 $((1,2))$ 5184 $(())$
$(3A,\ 4A,\ 5A,\ 5A,\ 5B)$	2	5184 $(())$ 5184 $((1,2))$
$(3A,\ 4A,\ 5A,\ 5B,\ 5B)$	2	5184 $(())$ 5184 $((1,2))$
$(3A,\ 4A,\ 5B,\ 5B,\ 5B)$	2	5184 $(())$ 5184 $((1,2))$
$(3A,\ 5A,\ 5A,\ 5A,\ 5A)$	2	3960 $(())$ 4104 $((1,2))$

$(3A,\, 5A,\, 5A,\, 5A,\, 5B)$	2	4176 $((1,2))$ 4050 $(())$
$(3A,\, 5A,\, 5A,\, 5B,\, 5B)$	2	4005 $(())$ 4140 $((1,2))$
$(3A,\, 5A,\, 5B,\, 5B,\, 5B)$	2	4050 $(())$ 4176 $((1,2))$
$(3A,\, 5B,\, 5B,\, 5B,\, 5B)$	2	3960 $(())$ 4104 $((1,2))$
$(3B,\, 3B,\, 3B,\, 3B,\, 3B)$	1	192 $((1,2))$
$(3B,\, 3B,\, 3B,\, 3B,\, 4A)$	2	768 $(())$ 768 $((1,2))$
$(3B,\, 3B,\, 3B,\, 3B,\, 5A)$	2	600 $((1,2))$ 300 $(())$
$(3B,\, 3B,\, 3B,\, 3B,\, 5B)$	2	600 $((1,2))$ 300 $(())$
$(3B,\, 3B,\, 3B,\, 4A,\, 4A)$	2	2064 $(())$ 2064 $((1,2))$
$(3B,\, 3B,\, 3B,\, 4A,\, 5A)$	2	1600 $(())$ 1600 $((1,2))$
$(3B,\, 3B,\, 3B,\, 4A,\, 5B)$	2	1600 $((1,2))$ 1600 $(())$
$(3B,\, 3B,\, 3B,\, 5A,\, 5A)$	2	1200 $((1,2))$ 1020 $(())$
$(3B,\, 3B,\, 3B,\, 5A,\, 5B)$	2	1300 $((1,2))$ 945 $(())$
$(3B,\, 3B,\, 3B,\, 5B,\, 5B)$	2	1200 $((1,2))$ 1020 $(())$
$(3B,\, 3B,\, 4A,\, 4A,\, 4A)$	2	4320 $(())$ 4320 $((1,2))$
$(3B,\, 3B,\, 4A,\, 4A,\, 5A)$	2	3600 $(())$ 3600 $((1,2))$
$(3B,\, 3B,\, 4A,\, 4A,\, 5B)$	2	3600 $(())$ 3600 $((1,2))$
$(3B,\, 3B,\, 4A,\, 5A,\, 5A)$	2	2880 $(())$ 2880 $((1,2))$
$(3B,\, 3B,\, 4A,\, 5A,\, 5B)$	2	2880 $((1,2))$ 2880 $(())$
$(3B,\, 3B,\, 4A,\, 5B,\, 5B)$	2	2880 $((1,2))$ 2880 $(())$
$(3B,\, 3B,\, 5A,\, 5A,\, 5A)$	2	2115 $(())$ 2340 $((1,2))$
$(3B,\, 3B,\, 5A,\, 5A,\, 5B)$	2	2400 $((1,2))$ 2010 $(())$
$(3B,\, 3B,\, 5A,\, 5B,\, 5B)$	2	2010 $(())$ 2400 $((1,2))$
$(3B,\, 3B,\, 5B,\, 5B,\, 5B)$	2	2340 $((1,2))$ 2115 $(())$
$(3B,\, 4A,\, 4A,\, 4A,\, 4A)$	2	9792 $(())$ 9936 $((1,2))$
$(3B,\, 4A,\, 4A,\, 4A,\, 5A)$	2	8100 $((1,2))$ 8100 $(())$
$(3B,\, 4A,\, 4A,\, 4A,\, 5B)$	2	8100 $((1,2))$ 8100 $(())$
$(3B,\, 4A,\, 4A,\, 5A,\, 5A)$	2	6480 $(())$ 6480 $((1,2))$
$(3B,\, 4A,\, 4A,\, 5A,\, 5B)$	2	6480 $((1,2))$ 6480 $(())$
$(3B,\, 4A,\, 4A,\, 5B,\, 5B)$	2	6480 $((1,2))$ 6480 $(())$

$(3B, 4A, 5A, 5A, 5A)$	2	5184 $(())$ 5184 $((1,2))$
$(3B, 4A, 5A, 5A, 5B)$	2	5184 $(())$ 5184 $((1,2))$
$(3B, 4A, 5A, 5B, 5B)$	2	5184 $(())$ 5184 $((1,2))$
$(3B, 4A, 5B, 5B, 5B)$	2	5184 $(())$ 5184 $((1,2))$
$(3B, 5A, 5A, 5A, 5A)$	2	4104 $((1,2))$ 3960 $(())$
$(3B, 5A, 5A, 5A, 5B)$	2	4176 $((1,2))$ 4050 $(())$
$(3B, 5A, 5A, 5B, 5B)$	2	4005 $(())$ 4140 $((1,2))$
$(3B, 5A, 5B, 5B, 5B)$	2	4050 $(())$ 4176 $((1,2))$
$(3B, 5B, 5B, 5B, 5B)$	2	4104 $((1,2))$ 3960 $(())$
$(4A, 4A, 4A, 4A, 4A)$	6	8624 $(())$ 6912 $((1,3,5)(2,4,6))$ 6912 $((1,5,3)(2,6,4))$ 8624 $((1,4)(2,5)(3,6))$ 6912 $((1,2,3,4,5,6))$ 6912 $((1,6,5,4,3,2))$
$(4A, 4A, 4A, 4A, 5A)$	6	6950 $(())$ 6050 $((1,4)(2,5)(3,6))$ 5625 $((1,3,5)(2,4,6))$ 6075 $((1,6,5,4,3,2))$ 6075 $((1,2,3,4,5,6))$ 5625 $((1,5,3)(2,6,4))$
$(4A, 4A, 4A, 4A, 5B)$	6	6075 $((1,2,3,4,5,6))$ 5625 $((1,3,5)(2,4,6))$ 6950 $(())$ 6075 $((1,6,5,4,3,2))$ 6050 $((1,4)(2,5)(3,6))$ 5625 $((1,5,3)(2,6,4))$
$(4A, 4A, 4A, 5A, 5A)$	6	4320 $((1,6,5,4,3,2))$ 4320 $((1,5,3)(2,6,4))$ 4320 $((1,3,5)(2,4,6))$ 5960 $((1,4)(2,5)(3,6))$ 5960 $(())$ 4320 $((1,2,3,4,5,6))$
$(4A, 4A, 4A, 5A, 5B)$	6	4160 $((1,4)(2,5)(3,6))$ 5220 $((1,5,3)(2,6,4))$ 5220 $((1,3,5)(2,4,6))$ 4160 $(())$ 5220 $((1,6,5,4,3,2))$ 5220 $((1,2,3,4,5,6))$
$(4A, 4A, 4A, 5B, 5B)$	6	5960 $(())$ 5960 $((1,4)(2,5)(3,6))$ 4320 $((1,3,5)(2,4,6))$ 4320 $((1,5,3)(2,6,4))$ 4320 $((1,6,5,4,3,2))$ 4320 $((1,2,3,4,5,6))$
$(4A, 4A, 5A, 5A, 5A)$	6	4880 $((1,4)(2,5)(3,6))$ 5168 $(())$ 3240 $((1,5,3)(2,6,4))$ 3384 $((1,6,5,4,3,2))$ 3240 $((1,3,5)(2,4,6))$ 3384 $((1,2,3,4,5,6))$

$(4A, 4A, 5A, 5A, 5B)$	6	3960 ((1,3,5)(2,4,6)) 4104 ((1,2,3,4,5,6)) 4104 ((1,6,5,4,3,2)) 3960 ((1,5,3)(2,6,4)) 3728 (()) 3440 ((1,4)(2,5)(3,6))
$(4A, 4A, 5A, 5B, 5B)$	6	3960 ((1,5,3)(2,6,4)) 3440 ((1,4)(2,5)(3,6)) 4104 ((1,6,5,4,3,2)) 4104 ((1,2,3,4,5,6)) 3728 (()) 3960 ((1,3,5)(2,4,6))
$(4A, 4A, 5B, 5B, 5B)$	6	3384 ((1,2,3,4,5,6)) 3240 ((1,5,3)(2,6,4)) 4880 ((1,4)(2,5)(3,6)) 5168 (()) 3384 ((1,6,5,4,3,2)) 3240 ((1,3,5)(2,4,6))
$(4A, 5A, 5A, 5A, 5A)$	6	2304 ((1,3,5)(2,4,6)) 4736 (()) 2304 ((1,6,5,4,3,2)) 2304 ((1,2,3,4,5,6)) 2304 ((1,5,3)(2,6,4)) 4736 ((1,4)(2,5)(3,6))
$(4A, 5A, 5A, 5A, 5B)$	6	2432 (()) 3456 ((1,2,3,4,5,6)) 3456 ((1,6,5,4,3,2)) 2432 ((1,4)(2,5)(3,6)) 3456 ((1,3,5)(2,4,6)) 3456 ((1,5,3)(2,6,4))
$(4A, 5A, 5A, 5B, 5B)$	6	2880 ((1,6,5,4,3,2)) 3584 (()) 2880 ((1,5,3)(2,6,4)) 3584 ((1,4)(2,5)(3,6)) 2880 ((1,2,3,4,5,6)) 2880 ((1,3,5)(2,4,6))
$(4A, 5A, 5B, 5B, 5B)$	6	3456 ((1,5,3)(2,6,4)) 3456 ((1,3,5)(2,4,6)) 3456 ((1,6,5,4,3,2)) 2432 ((1,4)(2,5)(3,6)) 2432 (()) 3456 ((1,2,3,4,5,6))
$(4A, 5B, 5B, 5B, 5B)$	6	2304 ((1,6,5,4,3,2)) 2304 ((1,3,5)(2,4,6)) 4736 (()) 2304 ((1,5,3)(2,6,4)) 2304 ((1,2,3,4,5,6)) 4736 ((1,4)(2,5)(3,6))
$(5A, 5A, 5A, 5A, 5A)$	6	4572 (()) 4400 ((1,4)(2,5)(3,6)) 1548 ((1,2,3,4,5,6)) 1548 ((1,6,5,4,3,2)) 1485 ((1,3,5)(2,4,6)) 1485 ((1,5,3)(2,6,4))
$(5A, 5A, 5A, 5A, 5B)$	6	2700 ((1,3,5)(2,4,6)) 2700 ((1,5,3)(2,6,4)) 2952 ((1,6,5,4,3,2)) 1760 ((1,4)(2,5)(3,6)) 1614 (()) 2952 ((1,2,3,4,5,6))

$(5A, 5A, 5A, 5B, 5B)$	6	2600 $(())$ 2295 $((1,5,3)(2,6,4))$ 2640 $((1,4)(2,5)(3,6))$ 2295 $((1,3,5)(2,4,6))$ 2484 $((1,6,5,4,3,2))$ 2484 $((1,2,3,4,5,6))$
$(5A, 5A, 5B, 5B, 5B)$	6	2600 $(())$ 2484 $((1,2,3,4,5,6))$ 2295 $((1,3,5)(2,4,6))$ 2295 $((1,5,3)(2,6,4))$ 2484 $((1,6,5,4,3,2))$ 2640 $((1,4)(2,5)(3,6))$
$(5A, 5B, 5B, 5B, 5B)$	6	2700 $((1,5,3)(2,6,4))$ 1760 $((1,4)(2,5)(3,6))$ 2952 $((1,2,3,4,5,6))$ 2952 $((1,6,5,4,3,2))$ 2700 $((1,3,5)(2,4,6))$ 1614 $(())$
$(5B, 5B, 5B, 5B, 5B)$	6	4400 $((1,4)(2,5)(3,6))$ 4572 $(())$ 1485 $((1,3,5)(2,4,6))$ 1485 $((1,5,3)(2,6,4))$ 1548 $((1,2,3,4,5,6))$ 1548 $((1,6,5,4,3,2))$

Table B.4: For each type of length 6 we state the number of braid orbits and their respective lengths.

Tuple	Orbits	Lengths
$(2A, 2A, 2A, 2A, 2A, 2A)$	3	19440 19440 16560
$(2A, 2A, 2A, 2A, 2A, 3A)$	1	51840
$(2A, 2A, 2A, 2A, 2A, 3B)$	1	51840
$(2A, 2A, 2A, 2A, 2A, 4A)$	3	43632 43632 39360
$(2A, 2A, 2A, 2A, 2A, 5A)$	3	30000 33750 33750
$(2A, 2A, 2A, 2A, 2A, 5B)$	3	30000 33750 33750
$(2A, 2A, 2A, 2A, 3A, 3A)$	1	44496
$(2A, 2A, 2A, 2A, 3A, 3B)$	1	50058
$(2A, 2A, 2A, 2A, 3A, 4A)$	1	113184
$(2A, 2A, 2A, 2A, 3A, 5A)$	1	87750
$(2A, 2A, 2A, 2A, 3A, 5B)$	1	87750
$(2A, 2A, 2A, 2A, 3B, 3B)$	1	44496
$(2A, 2A, 2A, 2A, 3B, 4A)$	1	113184
$(2A, 2A, 2A, 2A, 3B, 5A)$	1	87750
$(2A, 2A, 2A, 2A, 3B, 5B)$	1	87750

$(2A, 2A, 2A, 2A, 4A, 4A)$	3	82512 88048 82512
$(2A, 2A, 2A, 2A, 4A, 5A)$	3	67500 67500 70000
$(2A, 2A, 2A, 2A, 4A, 5B)$	3	70000 67500 67500
$(2A, 2A, 2A, 2A, 5A, 5A)$	3	51975 55950 51975
$(2A, 2A, 2A, 2A, 5A, 5B)$	3	55350 55350 49450
$(2A, 2A, 2A, 2A, 5B, 5B)$	3	51975 55950 51975
$(2A, 2A, 2A, 3A, 3A, 3A)$	1	37440
$(2A, 2A, 2A, 3A, 3A, 3B)$	1	44820
$(2A, 2A, 2A, 3A, 3A, 4A)$	1	99936
$(2A, 2A, 2A, 3A, 3A, 5A)$	1	76500
$(2A, 2A, 2A, 3A, 3A, 5B)$	1	76500
$(2A, 2A, 2A, 3A, 3B, 3B)$	1	44820
$(2A, 2A, 2A, 3A, 3B, 4A)$	1	101898
$(2A, 2A, 2A, 3A, 3B, 5A)$	1	81000
$(2A, 2A, 2A, 3A, 3B, 5B)$	1	81000
$(2A, 2A, 2A, 3A, 4A, 4A)$	1	226368
$(2A, 2A, 2A, 3A, 4A, 5A)$	1	182250
$(2A, 2A, 2A, 3A, 4A, 5B)$	1	182250
$(2A, 2A, 2A, 3A, 5A, 5A)$	1	143100
$(2A, 2A, 2A, 3A, 5A, 5B)$	1	143100
$(2A, 2A, 2A, 3A, 5B, 5B)$	1	143100
$(2A, 2A, 2A, 3B, 3B, 3B)$	1	37440
$(2A, 2A, 2A, 3B, 3B, 4A)$	1	99936
$(2A, 2A, 2A, 3B, 3B, 5A)$	1	76500
$(2A, 2A, 2A, 3B, 3B, 5B)$	1	76500
$(2A, 2A, 2A, 3B, 4A, 4A)$	1	226368
$(2A, 2A, 2A, 3B, 4A, 5A)$	1	182250
$(2A, 2A, 2A, 3B, 4A, 5B)$	1	182250
$(2A, 2A, 2A, 3B, 5A, 5A)$	1	143100
$(2A, 2A, 2A, 3B, 5A, 5B)$	1	143100

$(2A, 2A, 2A, 3B, 5B, 5B)$	1	143100
$(2A, 2A, 2A, 4A, 4A, 4A)$	3	161412 174528 174528
$(2A, 2A, 2A, 4A, 4A, 5A)$	3	138375 133250 138375
$(2A, 2A, 2A, 4A, 4A, 5B)$	3	138375 138375 133250
$(2A, 2A, 2A, 4A, 5A, 5A)$	3	113400 113400 101300
$(2A, 2A, 2A, 4A, 5A, 5B)$	3	114800 106650 106650
$(2A, 2A, 2A, 4A, 5B, 5B)$	3	113400 101300 113400
$(2A, 2A, 2A, 5A, 5A, 5A)$	3	76800 91260 91260
$(2A, 2A, 2A, 5A, 5A, 5B)$	3	87400 85860 85860
$(2A, 2A, 2A, 5A, 5B, 5B)$	3	85860 87400 85860
$(2A, 2A, 2A, 5B, 5B, 5B)$	3	76800 91260 91260
$(2A, 2A, 3A, 3A, 3A, 3A)$	1	30672
$(2A, 2A, 3A, 3A, 3A, 3B)$	1	39312
$(2A, 2A, 3A, 3A, 3A, 4A)$	1	88608
$(2A, 2A, 3A, 3A, 3A, 5A)$	1	66000
$(2A, 2A, 3A, 3A, 3A, 5B)$	1	66000
$(2A, 2A, 3A, 3A, 3B, 3B)$	1	40464
$(2A, 2A, 3A, 3A, 3B, 4A)$	1	90288
$(2A, 2A, 3A, 3A, 3B, 5A)$	1	72000
$(2A, 2A, 3A, 3A, 3B, 5B)$	1	72000
$(2A, 2A, 3A, 3A, 4A, 4A)$	1	203112
$(2A, 2A, 3A, 3A, 4A, 5A)$	1	162000
$(2A, 2A, 3A, 3A, 4A, 5B)$	1	162000
$(2A, 2A, 3A, 3A, 5A, 5A)$	1	126000
$(2A, 2A, 3A, 3A, 5A, 5B)$	1	126000
$(2A, 2A, 3A, 3A, 5B, 5B)$	1	126000
$(2A, 2A, 3A, 3B, 3B, 3B)$	1	39312
$(2A, 2A, 3A, 3B, 3B, 4A)$	1	90288
$(2A, 2A, 3A, 3B, 3B, 5A)$	1	72000
$(2A, 2A, 3A, 3B, 3B, 5B)$	1	72000

$(2A, 2A, 3A, 3B, 4A, 4A)$	1	200556
$(2A, 2A, 3A, 3B, 4A, 5A)$	1	162000
$(2A, 2A, 3A, 3B, 4A, 5B)$	1	162000
$(2A, 2A, 3A, 3B, 5A, 5A)$	1	129600
$(2A, 2A, 3A, 3B, 5A, 5B)$	1	129600
$(2A, 2A, 3A, 3B, 5B, 5B)$	1	129600
$(2A, 2A, 3A, 4A, 4A, 4A)$	1	454680
$(2A, 2A, 3A, 4A, 4A, 5A)$	1	364500
$(2A, 2A, 3A, 4A, 4A, 5B)$	1	364500
$(2A, 2A, 3A, 4A, 5A, 5A)$	1	291600
$(2A, 2A, 3A, 4A, 5A, 5B)$	1	291600
$(2A, 2A, 3A, 4A, 5B, 5B)$	1	291600
$(2A, 2A, 3A, 5A, 5A, 5A)$	1	231120
$(2A, 2A, 3A, 5A, 5A, 5B)$	1	231120
$(2A, 2A, 3A, 5A, 5B, 5B)$	1	231120
$(2A, 2A, 3A, 5B, 5B, 5B)$	1	231120
$(2A, 2A, 3B, 3B, 3B, 3B)$	1	30672
$(2A, 2A, 3B, 3B, 3B, 4A)$	1	88608
$(2A, 2A, 3B, 3B, 3B, 5A)$	1	66000
$(2A, 2A, 3B, 3B, 3B, 5B)$	1	66000
$(2A, 2A, 3B, 3B, 4A, 4A)$	1	203112
$(2A, 2A, 3B, 3B, 4A, 5A)$	1	162000
$(2A, 2A, 3B, 3B, 4A, 5B)$	1	162000
$(2A, 2A, 3B, 3B, 5A, 5A)$	1	126000
$(2A, 2A, 3B, 3B, 5A, 5B)$	1	126000
$(2A, 2A, 3B, 3B, 5B, 5B)$	1	126000
$(2A, 2A, 3B, 4A, 4A, 4A)$	1	454680
$(2A, 2A, 3B, 4A, 4A, 5A)$	1	364500
$(2A, 2A, 3B, 4A, 4A, 5B)$	1	364500
$(2A, 2A, 3B, 4A, 5A, 5A)$	1	291600

$(2A, 2A, 3B, 4A, 5A, 5B)$	1	291600
$(2A, 2A, 3B, 4A, 5B, 5B)$	1	291600
$(2A, 2A, 3B, 5A, 5A, 5A)$	1	231120
$(2A, 2A, 3B, 5A, 5A, 5B)$	1	231120
$(2A, 2A, 3B, 5A, 5B, 5B)$	1	231120
$(2A, 2A, 3B, 5B, 5B, 5B)$	1	231120
$(2A, 2A, 4A, 4A, 4A, 4A)$	3	330048 354976 330048
$(2A, 2A, 4A, 4A, 4A, 5A)$	3	280000 270000 270000
$(2A, 2A, 4A, 4A, 4A, 5B)$	3	270000 270000 280000
$(2A, 2A, 4A, 4A, 5A, 5A)$	3	210600 235000 210600
$(2A, 2A, 4A, 4A, 5A, 5B)$	3	224100 224100 208000
$(2A, 2A, 4A, 4A, 5B, 5B)$	3	210600 210600 235000
$(2A, 2A, 4A, 5A, 5A, 5A)$	3	166320 192160 166320
$(2A, 2A, 4A, 5A, 5A, 5B)$	3	170560 177120 177120
$(2A, 2A, 4A, 5A, 5B, 5B)$	3	177120 170560 177120
$(2A, 2A, 4A, 5B, 5B, 5B)$	3	166320 192160 166320
$(2A, 2A, 5A, 5A, 5A, 5A)$	3	127440 162280 127440
$(2A, 2A, 5A, 5A, 5A, 5B)$	3	144720 128040 144720
$(2A, 2A, 5A, 5A, 5B, 5B)$	3	145160 136080 136080
$(2A, 2A, 5A, 5B, 5B, 5B)$	3	128040 144720 144720
$(2A, 2A, 5B, 5B, 5B, 5B)$	3	127440 127440 162280
$(2A, 3A, 3A, 3A, 3A, 3A)$	1	24320
$(2A, 3A, 3A, 3A, 3A, 3B)$	1	34752
$(2A, 3A, 3A, 3A, 3A, 4A)$	1	77312
$(2A, 3A, 3A, 3A, 3A, 5A)$	1	56000
$(2A, 3A, 3A, 3A, 3A, 5B)$	1	56000
$(2A, 3A, 3A, 3A, 3B, 3B)$	1	35776
$(2A, 3A, 3A, 3A, 3B, 4A)$	1	81280
$(2A, 3A, 3A, 3A, 3B, 5A)$	1	64000
$(2A, 3A, 3A, 3A, 3B, 5B)$	1	64000

$(2A, 3A, 3A, 3A, 4A, 4A)$	1	181536
$(2A, 3A, 3A, 3A, 4A, 5A)$	1	144000
$(2A, 3A, 3A, 3A, 4A, 5B)$	1	144000
$(2A, 3A, 3A, 3A, 5A, 5A)$	1	110400
$(2A, 3A, 3A, 3A, 5A, 5B)$	1	110400
$(2A, 3A, 3A, 3A, 5B, 5B)$	1	110400
$(2A, 3A, 3A, 3B, 3B, 3B)$	1	35776
$(2A, 3A, 3A, 3B, 3B, 4A)$	1	78976
$(2A, 3A, 3A, 3B, 3B, 5A)$	1	64000
$(2A, 3A, 3A, 3B, 3B, 5B)$	1	64000
$(2A, 3A, 3A, 3B, 4A, 4A)$	1	179136
$(2A, 3A, 3A, 3B, 4A, 5A)$	1	144000
$(2A, 3A, 3A, 3B, 4A, 5B)$	1	144000
$(2A, 3A, 3A, 3B, 5A, 5A)$	1	115200
$(2A, 3A, 3A, 3B, 5A, 5B)$	1	115200
$(2A, 3A, 3A, 3B, 5B, 5B)$	1	115200
$(2A, 3A, 3A, 4A, 4A, 4A)$	1	401472
$(2A, 3A, 3A, 4A, 4A, 5A)$	1	324000
$(2A, 3A, 3A, 4A, 4A, 5B)$	1	324000
$(2A, 3A, 3A, 4A, 5A, 5A)$	1	259200
$(2A, 3A, 3A, 4A, 5A, 5B)$	1	259200
$(2A, 3A, 3A, 4A, 5B, 5B)$	1	259200
$(2A, 3A, 3A, 5A, 5A, 5A)$	1	204480
$(2A, 3A, 3A, 5A, 5A, 5B)$	1	204480
$(2A, 3A, 3A, 5A, 5B, 5B)$	1	204480
$(2A, 3A, 3A, 5B, 5B, 5B)$	1	204480
$(2A, 3A, 3B, 3B, 3B, 3B)$	1	34752
$(2A, 3A, 3B, 3B, 3B, 4A)$	1	81280
$(2A, 3A, 3B, 3B, 3B, 5A)$	1	64000
$(2A, 3A, 3B, 3B, 3B, 5B)$	1	64000

$(2A, 3A, 3B, 3B, 4A, 4A)$	1	179136
$(2A, 3A, 3B, 3B, 4A, 5A)$	1	144000
$(2A, 3A, 3B, 3B, 4A, 5B)$	1	144000
$(2A, 3A, 3B, 3B, 5A, 5A)$	1	115200
$(2A, 3A, 3B, 3B, 5A, 5B)$	1	115200
$(2A, 3A, 3B, 3B, 5B, 5B)$	1	115200
$(2A, 3A, 3B, 4A, 4A, 4A)$	1	407592
$(2A, 3A, 3B, 4A, 4A, 5A)$	1	324000
$(2A, 3A, 3B, 4A, 4A, 5B)$	1	324000
$(2A, 3A, 3B, 4A, 5A, 5A)$	1	259200
$(2A, 3A, 3B, 4A, 5A, 5B)$	1	259200
$(2A, 3A, 3B, 4A, 5B, 5B)$	1	259200
$(2A, 3A, 3B, 5A, 5A, 5A)$	1	207360
$(2A, 3A, 3B, 5A, 5A, 5B)$	1	207360
$(2A, 3A, 3B, 5A, 5B, 5B)$	1	207360
$(2A, 3A, 3B, 5B, 5B, 5B)$	1	207360
$(2A, 3A, 4A, 4A, 4A, 4A)$	1	906768
$(2A, 3A, 4A, 4A, 4A, 5A)$	1	729000
$(2A, 3A, 4A, 4A, 4A, 5B)$	1	729000
$(2A, 3A, 4A, 4A, 5A, 5A)$	1	583200
$(2A, 3A, 4A, 4A, 5A, 5B)$	1	583200
$(2A, 3A, 4A, 4A, 5B, 5B)$	1	583200
$(2A, 3A, 4A, 5A, 5A, 5A)$	1	466560
$(2A, 3A, 4A, 5A, 5A, 5B)$	1	466560
$(2A, 3A, 4A, 5A, 5B, 5B)$	1	466560
$(2A, 3A, 4A, 5B, 5B, 5B)$	1	466560
$(2A, 3A, 5A, 5A, 5A, 5A)$	1	371520
$(2A, 3A, 5A, 5A, 5A, 5B)$	1	371520
$(2A, 3A, 5A, 5A, 5B, 5B)$	1	371520
$(2A, 3A, 5A, 5B, 5B, 5B)$	1	371520

$(2A, 3A, 5B, 5B, 5B, 5B)$	1	371520
$(2A, 3B, 3B, 3B, 3B, 3B)$	1	24320
$(2A, 3B, 3B, 3B, 3B, 4A)$	1	77312
$(2A, 3B, 3B, 3B, 3B, 5A)$	1	56000
$(2A, 3B, 3B, 3B, 3B, 5B)$	1	56000
$(2A, 3B, 3B, 3B, 4A, 4A)$	1	181536
$(2A, 3B, 3B, 3B, 4A, 5A)$	1	144000
$(2A, 3B, 3B, 3B, 4A, 5B)$	1	144000
$(2A, 3B, 3B, 3B, 5A, 5A)$	1	110400
$(2A, 3B, 3B, 3B, 5A, 5B)$	1	110400
$(2A, 3B, 3B, 3B, 5B, 5B)$	1	110400
$(2A, 3B, 3B, 4A, 4A, 4A)$	1	401472
$(2A, 3B, 3B, 4A, 4A, 5A)$	1	324000
$(2A, 3B, 3B, 4A, 4A, 5B)$	1	324000
$(2A, 3B, 3B, 4A, 5A, 5A)$	1	259200
$(2A, 3B, 3B, 4A, 5A, 5B)$	1	259200
$(2A, 3B, 3B, 4A, 5B, 5B)$	1	259200
$(2A, 3B, 3B, 5A, 5A, 5A)$	1	204480
$(2A, 3B, 3B, 5A, 5A, 5B)$	1	204480
$(2A, 3B, 3B, 5A, 5B, 5B)$	1	204480
$(2A, 3B, 3B, 5B, 5B, 5B)$	1	204480
$(2A, 3B, 4A, 4A, 4A, 4A)$	1	906768
$(2A, 3B, 4A, 4A, 4A, 5A)$	1	729000
$(2A, 3B, 4A, 4A, 4A, 5B)$	1	729000
$(2A, 3B, 4A, 4A, 5A, 5A)$	1	583200
$(2A, 3B, 4A, 4A, 5A, 5B)$	1	583200
$(2A, 3B, 4A, 4A, 5B, 5B)$	1	583200
$(2A, 3B, 4A, 5A, 5A, 5A)$	1	466560
$(2A, 3B, 4A, 5A, 5A, 5B)$	1	466560
$(2A, 3B, 4A, 5A, 5B, 5B)$	1	466560

$(2A, 3B, 4A, 5B, 5B, 5B)$	1	466560
$(2A, 3B, 5A, 5A, 5A, 5A)$	1	371520
$(2A, 3B, 5A, 5A, 5A, 5B)$	1	371520
$(2A, 3B, 5A, 5A, 5B, 5B)$	1	371520
$(2A, 3B, 5A, 5B, 5B, 5B)$	1	371520
$(2A, 3B, 5B, 5B, 5B, 5B)$	1	371520
$(2A, 4A, 4A, 4A, 4A, 4A)$	3	647520 698112 698112
$(2A, 4A, 4A, 4A, 4A, 5A)$	3	533000 553500 553500
$(2A, 4A, 4A, 4A, 4A, 5B)$	3	533000 553500 553500
$(2A, 4A, 4A, 4A, 5A, 5A)$	3	453600 453600 405200
$(2A, 4A, 4A, 4A, 5A, 5B)$	3	426600 459200 426600
$(2A, 4A, 4A, 4A, 5B, 5B)$	3	453600 405200 453600
$(2A, 4A, 4A, 5A, 5A, 5A)$	3	367200 367200 315200
$(2A, 4A, 4A, 5A, 5A, 5B)$	3	345600 345600 358400
$(2A, 4A, 4A, 5A, 5B, 5B)$	3	345600 358400 345600
$(2A, 4A, 4A, 5B, 5B, 5B)$	3	367200 367200 315200
$(2A, 4A, 5A, 5A, 5A, 5A)$	3	231680 304128 304128
$(2A, 4A, 5A, 5A, 5A, 5B)$	3	269568 300800 269568
$(2A, 4A, 5A, 5A, 5B, 5B)$	3	286848 286848 266240
$(2A, 4A, 5A, 5B, 5B, 5B)$	3	269568 300800 269568
$(2A, 4A, 5B, 5B, 5B, 5B)$	3	231680 304128 304128
$(2A, 5A, 5A, 5A, 5A, 5A)$	3	161920 254016 254016
$(2A, 5A, 5A, 5A, 5A, 5B)$	3	212544 212544 244480
$(2A, 5A, 5A, 5A, 5B, 5B)$	3	216960 226368 226368
$(2A, 5A, 5A, 5B, 5B, 5B)$	3	226368 216960 226368
$(2A, 5A, 5B, 5B, 5B, 5B)$	3	212544 244480 212544
$(2A, 5B, 5B, 5B, 5B, 5B)$	3	254016 161920 254016
$(3A, 3A, 3A, 3A, 3A, 3A)$	2	11880 6080
$(3A, 3A, 3A, 3A, 3A, 3B)$	2	12852 17040
$(3A, 3A, 3A, 3A, 3A, 4A)$	2	33792 33792

$(3A, 3A, 3A, 3A, 3A, 5A)$	2	26250 20000
$(3A, 3A, 3A, 3A, 3A, 5B)$	2	26250 20000
$(3A, 3A, 3A, 3A, 3B, 3B)$	2	17550 15072
$(3A, 3A, 3A, 3A, 3B, 4A)$	2	36352 36352
$(3A, 3A, 3A, 3A, 3B, 5A)$	2	26875 30000
$(3A, 3A, 3A, 3A, 3B, 5B)$	2	26875 30000
$(3A, 3A, 3A, 3A, 4A, 4A)$	2	81792 81920
$(3A, 3A, 3A, 3A, 4A, 5A)$	2	64000 64000
$(3A, 3A, 3A, 3A, 4A, 5B)$	2	64000 64000
$(3A, 3A, 3A, 3A, 5A, 5A)$	2	45200 50850
$(3A, 3A, 3A, 3A, 5A, 5B)$	2	44200 51725
$(3A, 3A, 3A, 3A, 5B, 5B)$	2	45200 50850
$(3A, 3A, 3A, 3B, 3B, 3B)$	2	14418 16384
$(3A, 3A, 3A, 3B, 3B, 4A)$	2	35328 35328
$(3A, 3A, 3A, 3B, 3B, 5A)$	2	28000 28875
$(3A, 3A, 3A, 3B, 3B, 5B)$	2	28000 28875
$(3A, 3A, 3A, 3B, 4A, 4A)$	2	78624 78688
$(3A, 3A, 3A, 3B, 4A, 5A)$	2	64000 64000
$(3A, 3A, 3A, 3B, 4A, 5B)$	2	64000 64000
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$(3A, 3A, 3A, 4A, 4A, 5A)$	2	144000 144000
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$(3A, 3A, 3A, 4A, 5A, 5A)$	2	115200 115200
$(3A, 3A, 3A, 4A, 5A, 5B)$	2	115200 115200
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$(3A, 3A, 3A, 5A, 5A, 5A)$	2	88200 92115
$(3A, 3A, 3A, 5A, 5A, 5B)$	2	92940 87600

$(3A, 3A, 3A, 5A, 5B, 5B)$	2	87600 92940
$(3A, 3A, 3A, 5B, 5B, 5B)$	2	92115 88200
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$(3A, 3A, 3B, 3B, 3B, 4A)$	2	35328 35328
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$(3A, 3A, 3B, 3B, 3B, 5B)$	2	28875 28000
$(3A, 3A, 3B, 3B, 4A, 4A)$	2	80992 80928
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$(3A, 3A, 3B, 3B, 5A, 5A)$	2	52875 49600
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$(3A, 3A, 3B, 4A, 4A, 4A)$	2	180576 180576
$(3A, 3A, 3B, 4A, 4A, 5A)$	2	144000 144000
$(3A, 3A, 3B, 4A, 4A, 5B)$	2	144000 144000
$(3A, 3A, 3B, 4A, 5A, 5A)$	2	115200 115200
$(3A, 3A, 3B, 4A, 5A, 5B)$	2	115200 115200
$(3A, 3A, 3B, 4A, 5B, 5B)$	2	115200 115200
$(3A, 3A, 3B, 5A, 5A, 5A)$	2	91260 92880
$(3A, 3A, 3B, 5A, 5A, 5B)$	2	91485 92880
$(3A, 3A, 3B, 5A, 5B, 5B)$	2	91485 92880
$(3A, 3A, 3B, 5B, 5B, 5B)$	2	91260 92880
$(3A, 3A, 4A, 4A, 4A, 4A)$	2	407160 406440
$(3A, 3A, 4A, 4A, 4A, 5A)$	2	324000 324000
$(3A, 3A, 4A, 4A, 4A, 5B)$	2	324000 324000
$(3A, 3A, 4A, 4A, 5A, 5A)$	2	259200 259200
$(3A, 3A, 4A, 4A, 5A, 5B)$	2	259200 259200
$(3A, 3A, 4A, 4A, 5B, 5B)$	2	259200 259200
$(3A, 3A, 4A, 5A, 5A, 5A)$	2	207360 207360
$(3A, 3A, 4A, 5A, 5A, 5B)$	2	207360 207360

$(3A, 3A, 4A, 5A, 5B, 5B)$	2	207360 207360
$(3A, 3A, 4A, 5B, 5B, 5B)$	2	207360 207360
$(3A, 3A, 5A, 5A, 5A, 5A)$	2	162000 168030
$(3A, 3A, 5A, 5A, 5A, 5B)$	2	161280 167940
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$(3A, 3A, 5A, 5B, 5B, 5B)$	2	167940 161280
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$(3A, 3B, 3B, 3B, 3B, 4A)$	2	36352 36352
$(3A, 3B, 3B, 3B, 3B, 5A)$	2	26875 30000
$(3A, 3B, 3B, 3B, 3B, 5B)$	2	26875 30000
$(3A, 3B, 3B, 3B, 4A, 4A)$	2	78624 78688
$(3A, 3B, 3B, 3B, 4A, 5A)$	2	64000 64000
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$(3A, 3B, 3B, 3B, 5A, 5A)$	2	49275 53200
$(3A, 3B, 3B, 3B, 5A, 5B)$	2	49150 53200
$(3A, 3B, 3B, 3B, 5B, 5B)$	2	49275 53200
$(3A, 3B, 3B, 4A, 4A, 4A)$	2	180576 180576
$(3A, 3B, 3B, 4A, 4A, 5A)$	2	144000 144000
$(3A, 3B, 3B, 4A, 4A, 5B)$	2	144000 144000
$(3A, 3B, 3B, 4A, 5A, 5A)$	2	115200 115200
$(3A, 3B, 3B, 4A, 5A, 5B)$	2	115200 115200
$(3A, 3B, 3B, 4A, 5B, 5B)$	2	115200 115200
$(3A, 3B, 3B, 5A, 5A, 5A)$	2	91260 92880
$(3A, 3B, 3B, 5A, 5A, 5B)$	2	91485 92880
$(3A, 3B, 3B, 5A, 5B, 5B)$	2	92880 91485
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$(3A, 3B, 4A, 4A, 5A, 5A)$	2	259200	259200
$(3A, 3B, 4A, 4A, 5A, 5B)$	2	259200	259200
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$(3A, 3B, 4A, 5A, 5B, 5B)$	2	207360	207360
$(3A, 3B, 4A, 5B, 5B, 5B)$	2	207360	207360
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$(3A, 3B, 5A, 5A, 5A, 5B)$	2	163053	168480
$(3A, 3B, 5A, 5A, 5B, 5B)$	2	168480	163458
$(3A, 3B, 5A, 5B, 5B, 5B)$	2	168480	163053
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$(3A, 4A, 4A, 4A, 4A, 4A)$	2	909792	909792
$(3A, 4A, 4A, 4A, 4A, 5A)$	2	729000	729000
$(3A, 4A, 4A, 4A, 4A, 5B)$	2	729000	729000
$(3A, 4A, 4A, 4A, 5A, 5A)$	2	583200	583200
$(3A, 4A, 4A, 4A, 5A, 5B)$	2	583200	583200
$(3A, 4A, 4A, 4A, 5B, 5B)$	2	583200	583200
$(3A, 4A, 4A, 5A, 5A, 5A)$	2	466560	466560
$(3A, 4A, 4A, 5A, 5A, 5B)$	2	466560	466560
$(3A, 4A, 4A, 5A, 5B, 5B)$	2	466560	466560
$(3A, 4A, 4A, 5B, 5B, 5B)$	2	466560	466560
$(3A, 4A, 5A, 5A, 5A, 5A)$	2	373248	373248
$(3A, 4A, 5A, 5A, 5A, 5B)$	2	373248	373248
$(3A, 4A, 5A, 5A, 5B, 5B)$	2	373248	373248
$(3A, 4A, 5A, 5B, 5B, 5B)$	2	373248	373248
$(3A, 4A, 5B, 5B, 5B, 5B)$	2	373248	373248
$(3A, 5A, 5A, 5A, 5A, 5A)$	2	297000	297216
$(3A, 5A, 5A, 5A, 5A, 5B)$	2	296568	299835
$(3A, 5A, 5A, 5A, 5B, 5B)$	2	298890	296784

$(3A, 5A, 5A, 5B, 5B, 5B)$	2	296784 298890
$(3A, 5A, 5B, 5B, 5B, 5B)$	2	299835 296568
$(3A, 5B, 5B, 5B, 5B, 5B)$	2	297000 297216
$(3B, 3B, 3B, 3B, 3B, 3B)$	2	11880 6080
$(3B, 3B, 3B, 3B, 3B, 4A)$	2	33792 33792
$(3B, 3B, 3B, 3B, 3B, 5A)$	2	26250 20000
$(3B, 3B, 3B, 3B, 3B, 5B)$	2	26250 20000
$(3B, 3B, 3B, 3B, 4A, 4A)$	2	81792 81920
$(3B, 3B, 3B, 3B, 4A, 5A)$	2	64000 64000
$(3B, 3B, 3B, 3B, 4A, 5B)$	2	64000 64000
$(3B, 3B, 3B, 3B, 5A, 5A)$	2	45200 50850
$(3B, 3B, 3B, 3B, 5A, 5B)$	2	51725 44200
$(3B, 3B, 3B, 3B, 5B, 5B)$	2	50850 45200
$(3B, 3B, 3B, 4A, 4A, 4A)$	2	177984 177984
$(3B, 3B, 3B, 4A, 4A, 5A)$	2	144000 144000
$(3B, 3B, 3B, 4A, 4A, 5B)$	2	144000 144000
$(3B, 3B, 3B, 4A, 5A, 5A)$	2	115200 115200
$(3B, 3B, 3B, 4A, 5A, 5B)$	2	115200 115200
$(3B, 3B, 3B, 4A, 5B, 5B)$	2	115200 115200
$(3B, 3B, 3B, 5A, 5A, 5A)$	2	92115 88200
$(3B, 3B, 3B, 5A, 5A, 5B)$	2	92940 87600
$(3B, 3B, 3B, 5A, 5B, 5B)$	2	87600 92940
$(3B, 3B, 3B, 5B, 5B, 5B)$	2	92115 88200
$(3B, 3B, 4A, 4A, 4A, 4A)$	2	406440 407160
$(3B, 3B, 4A, 4A, 4A, 5A)$	2	324000 324000
$(3B, 3B, 4A, 4A, 4A, 5B)$	2	324000 324000
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$(3B, 3B, 5A, 5A, 5A, 5A)$	2	162000 168030
$(3B, 3B, 5A, 5A, 5A, 5B)$	2	167940 161280
$(3B, 3B, 5A, 5A, 5B, 5B)$	2	161640 167985
$(3B, 3B, 5A, 5B, 5B, 5B)$	2	161280 167940
$(3B, 3B, 5B, 5B, 5B, 5B)$	2	162000 168030
$(3B, 4A, 4A, 4A, 4A, 4A)$	2	909792 909792
$(3B, 4A, 4A, 4A, 4A, 5A)$	2	729000 729000
$(3B, 4A, 4A, 4A, 4A, 5B)$	2	729000 729000
$(3B, 4A, 4A, 4A, 5A, 5A)$	2	583200 583200
$(3B, 4A, 4A, 4A, 5A, 5B)$	2	583200 583200
$(3B, 4A, 4A, 4A, 5B, 5B)$	2	583200 583200
$(3B, 4A, 4A, 5A, 5A, 5A)$	2	466560 466560
$(3B, 4A, 4A, 5A, 5A, 5B)$	2	466560 466560
$(3B, 4A, 4A, 5A, 5B, 5B)$	2	466560 466560
$(3B, 4A, 4A, 5B, 5B, 5B)$	2	466560 466560
$(3B, 4A, 5A, 5A, 5A, 5A)$	2	373248 373248
$(3B, 4A, 5A, 5A, 5A, 5B)$	2	373248 373248
$(3B, 4A, 5A, 5A, 5B, 5B)$	2	373248 373248
$(3B, 4A, 5A, 5B, 5B, 5B)$	2	373248 373248
$(3B, 4A, 5B, 5B, 5B, 5B)$	2	373248 373248
$(3B, 5A, 5A, 5A, 5A, 5A)$	2	297000 297216
$(3B, 5A, 5A, 5A, 5A, 5B)$	2	296568 299835
$(3B, 5A, 5A, 5A, 5B, 5B)$	2	298890 296784
$(3B, 5A, 5A, 5B, 5B, 5B)$	2	298890 296784
$(3B, 5A, 5B, 5B, 5B, 5B)$	2	296568 299835
$(3B, 5B, 5B, 5B, 5B, 5B)$	2	297000 297216
$(4A, 4A, 4A, 4A, 4A, 4A)$	6	650592 726240 650592 669600 694944 669600

(4A, 4A, 4A, 4A, 4A, 5A)	6	560000 540000 560000 540000 540000 540000
(4A, 4A, 4A, 4A, 4A, 5B)	6	540000 560000 540000 540000 540000 560000
(4A, 4A, 4A, 4A, 5A, 5A)	6	475400 418500 418500 464600 423900 423900
(4A, 4A, 4A, 4A, 5A, 5B)	6	445500 450900 445500 421400 450900 410600
(4A, 4A, 4A, 4A, 5B, 5B)	6	423900 418500 464600 423900 418500 475400
(4A, 4A, 4A, 5A, 5A, 5A)	6	332640 332640 384320 332640 332640 384320
(4A, 4A, 4A, 5A, 5A, 5B)	6	354240 354240 341120 354240 354240 341120
(4A, 4A, 4A, 5A, 5B, 5B)	6	341120 354240 341120 354240 354240 354240
(4A, 4A, 4A, 5B, 5B, 5B)	6	332640 332640 332640 384320 332640 384320
(4A, 4A, 5A, 5A, 5A, 5A)	6	254880 326720 330176 256608 256608 254880
(4A, 4A, 5A, 5A, 5A, 5B)	6	289440 291168 261056 257600 291168 289440
(4A, 4A, 5A, 5A, 5B, 5B)	6	273888 273888 272160 295616 272160 292160
(4A, 4A, 5A, 5B, 5B, 5B)	6	289440 261056 291168 291168 257600 289440
(4A, 4A, 5B, 5B, 5B, 5B)	6	256608 254880 330176 326720 256608 254880
(4A, 5A, 5A, 5A, 5A, 5A)	6	193536 284672 193536 284672 193536 193536
(4A, 5A, 5A, 5A, 5A, 5B)	6	235008 235008 201728 235008 235008 201728
(4A, 5A, 5A, 5A, 5B, 5B)	6	221184 221184 221184 221184 229376 229376
(4A, 5A, 5A, 5B, 5B, 5B)	6	221184 221184 221184 229376 229376 221184
(4A, 5A, 5B, 5B, 5B, 5B)	6	235008 235008 201728 235008 235008 201728
(4A, 5B, 5B, 5B, 5B, 5B)	6	284672 193536 284672 193536 193536 193536
(5A, 5A, 5A, 5A, 5A, 5A)	6	138240 261030 138240 141750 256800 141750
(5A, 5A, 5A, 5A, 5A, 5B)	6	149370 193320 193320 195075 145600 195075
(5A, 5A, 5A, 5A, 5B, 5B)	6	190080 173745 173745 171288 194034 171288
(5A, 5A, 5A, 5B, 5B, 5B)	6	171702 167840 182304 182304 184410 184410
(5A, 5A, 5B, 5B, 5B, 5B)	6	173745 190080 171288 173745 194034 171288
(5A, 5B, 5B, 5B, 5B, 5B)	6	193320 145600 195075 195075 193320 149370
(5B, 5B, 5B, 5B, 5B, 5B)	6	138240 261030 141750 256800 141750 138240

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