Chaotic Dynamical Systems

by

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A thesis submitted to
The University of Birmingham
for the degree of
Doctor of Philosophy

School of Mathematics
The University of Birmingham
November 2011

Abstract

In this work, we look at the dynamics of four different spaces, the interval, the unit circle, subshifts of finite type and compact countable sets. We put our emphasis on chaotic dynamical system and exhibit sufficient conditions for the system on the interval, the unit circle and subshifts of finite type to be chaotic in three different types of chaos. On the interval, we reveal two weak conditions's role as a fast track to chaotic behavior. We also explain how a strong dense periodicity property influences chaotic behavior of dynamics on the interval, the unit circle and subshifts of finite type. Finally we show how dynamics property of compact countable sets effecting the structure of the sets.

Acknowledgements

It is a pleasure to thank those who made this PhD thesis possible. I wish to express my gratitude to my honorific supervisor, Dr Chris Good who was abundantly helpful and offered invaluable assistance, support and guidance. I am indebted to my graduate friends, especially group members; Andy, Amna and Kyriakoss for their assistance and encouragement. I would like to thank my best friend, Fina who always been there for me and being a shoulder to cry on. I am very grateful for my mother. She always lets me know that she is proud of me, which motivates me to work harder and do my best. The deepest gratitude is due to my devoted husband Rosli Muhammad and my children, Hanzalah and Zulaikha. Their love and support without any complaint or regret has enabled me to complete this Ph.D. project. Last but not least, I wish to express my love and gratitude to my beloved family members, for their understanding and endless love, through the duration of my studies. I owe my every achievement to all of them.

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November 2011

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Chapter 1

Introduction to Dynamical Systems

A dynamical system deals with the value of states in the system as they change over time. It describes how one state develops into another state over the course of time. Technically, a dynamical system is a smooth action of the reals or the integers on another object. When the reals are acting, the system is called a continuous dynamical system, and when the integers are acting, the system is called a discrete dynamical system.

If X is a set and f is a continuous function from X into itself, then the evolution of a variable $x \in X$ can be given by the formula $x_{n+1} = f(x_n)$ where $x_0 = x$ and $x_n = f^n(x)$ for every natural number n. Notice that f^n denotes the n-fold iterates of f. So for our purposes a discrete dynamical system is the $\mathbb{N}-$action of f on X and we study the behavior of $x, x_1, x_2 \cdots$ for every $x \in X$.

Before we go further in the study of dynamical systems, it is very important to introduce some preliminaries from calculus and some definitions from dynamics. After that, we will be ready to discuss some important tools in the study of dynamical systems, i.e. shift spaces, symbolic dynamics and topological conjugacy. These will be described in this chapter. In the second chapter, we will focus on one of the most important topics in the study of dynamical system i.e. chaos. There are many different definitions of chaos, but we put our emphasis on three of them which have been studied widely in the literature. By using tools discussed in this chapter, we study chaotic behavior in three spaces i.e. the interval, the unit circle and subshifts of finite type. We will give two new results; sufficient conditions for systems on the unit circle and subshifts of finite type to be chaotic. In chapter three we extend our study on chaotic behavior but restrict our attention on interval maps. We introduce two weak conditions which to our knowledge their importance for chaotic behavior are not generally known and show how these weak conditions imply chaotic behavior of interval maps. In chapter four, we introduce a strong dense periodicity property and discuss this property on the three spaces we discussed earlier. We will show how this strong property influences dynamical behavior and surprisingly the effects on the three different spaces are different. In chapter five, we study the dynamical behavior of another class of space, a compact countable space. We will show that the notion of chaotic behavior on this space is seems to be trivial and then explain some dynamical behavior on this space. In the last chapter, we give a conclusion of the whole finding in this work and give what is missing in our work and discuss possible future work.

1.1 Elementary Definitions and Principles

In this section we will give an introduction to some of the basic definitions, ideas and techniques from the calculus, topology and symbolic dynamics that will be used throughout this chapter and the sequel chapters. Before that we will fix some notation. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of complex numbers, real numbers, natural numbers and integers respectively. For any subset A of $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$ or $\mathbb{Z}$, $|A|$ denotes the cardinality of the set A.

For arbitrary functions f and g, the composition of f and g will be denoted as $f \circ g(x) = f(g(x))$. The $n-$fold composition of f with itself will be denoted as $f^n(x) = \underbrace{f \circ \cdots \circ f}_{n \ times}(x)$. For $n = 0$, f^0 is the identity function. It is important to note that f^n does not mean $f(x)$

raised to the n^{th} power, nor the n^{th} derivative of $f(x)$ which we denote by $f^{(n)}(x)$. If f is invertible (i.e. when f is bijective), then we write $f^{-n}(x) = f^{-1} \circ \cdots \circ f^{-1}(x)$.

We then will define some important classes of function.

Definition 1.1 *Let $f : X \to Y$ be a bijective continuous function and X and Y be any topological spaces, f will be called a homeomorphism whenever f^{-1} is continuous.*

There are three theories from elementary calculus that we will used in this work. The first feature is the Chain Rule.

Proposition 1.2 (Chain Rule) *If $f : Z \to Y$ and $g : X \to Z$ are differentiable functions, X, Y and Z are any subspaces of $\mathbb{R}$, and $x \in X$ then*

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

In particular, if $h(x) = f^n(x)$, for any $n \in \mathbb{Z}$ then

$$h'(x) = f'(f^{n-1}(x)) \cdot f'(f^{n-2}(x)) \cdot \ldots \cdot f'(x)$$

Other important notions from elementary calculus are the Mean Value Theorem and the Intermediate Value Theorem:

Theorem 1.3 (Mean Value Theorem) *Let $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 1.4 (Intermediate Value Theorem) *Let $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $f(a) = u$ and $f(b) = v$, then for any z between u and v, there exists c, $a \leqslant c \leqslant b$ such that $f(c) = z$.*

The following classes of set are also important to be defined.

Definition 1.5 *For a subset A of a topological space X, an element of the set $x \in A$ is a limit point (or accumulation point) of the set if every neighborhood of x also contains a point of A other than x itself. The set A is perfect if every element of A is a limit (or accumulation) point of A.*

Definition 1.6 *A subset A of $\mathbb{R}$ is totally disconnected if it contains no intervals. A is a Cantor set if it is closed, totally disconnected and perfect.*

Definition 1.7 *A subset A of a topological space X is a G_δ if there are open subsets U_n of X such that $A = \cap_{n \geqslant 0} U_n$.*

Definition 1.8 *Let $f : X \to X$ be a function and X be any space. A subset A of X is said to be (strongly) invariant under the function f if $f(A) = A$.*

Throughout this work we will always use invariant to mean strongly invariant.

Definition 1.9 *Let X be a metric space with metric d and A be a subset of X. The distance between A and any point $x \in X$ is denoted as $dist(x, A)$ and is defined as*

$$dist(x, A) = \inf\{d(x, a)|\ \textit{for all } a \in A\}$$

For any subset B of X ($A \neq B$), the distance between A and B is denoted as $dist(A, B)$ and is defined as

$$dist(A, B) = \inf\{d(a, b)|\ \textit{for all } a \in A, b \in B\}$$

The following definitions are common features from dynamical systems, about a function and its iterations of a point in its domain.

Definition 1.10 *For a function f and any point x in the domain of the function, the forward orbit of x (denoted by $O^+(x)$) is the set of points $\{x, f(x), f^2(x), \cdots\}$, the full orbit of x (denoted as $O(x)$) is the set of points $f^n(x)$ for $n \in \mathbb{Z}$ and the backward orbit of x (denoted as $O^-(x)$) is the set of points $\{x, f^{-1}(x), f^{-2}(x), \cdots\}$.*

For convenience, the forward orbit of x, $O^+(x)$ will be simply called the orbit of x.

Definition 1.11 *For a function f on a topological space X and a point $x \in X$, the ω-limit set of x, $\omega(x, f)$ is the set of all limit points of $O^+(x)$, that is*

$$\omega(x, f) = \bigcap_{n \in \mathbb{N}} \overline{\{f^k(x) | k \geqslant n\}}$$

Definition 1.12 *For a function f and any point x in the domain of the function, the point x is a fixed point for f if $f(x) = x$. The point x is a periodic point of period n if $f^n(x) = x$. The least positive n for which $f^n(x) = x$ is called the prime period of x.*

For a function f, $Fix(f)$ is the set of fixed points for f and $Per_l(f)$ is the set of periodic points of period l for f.

Finally, we will briefly define topological entropy since we will use this notion in the sequel. Originally, the definition that was introduced in 1965 by Adler, Konheim, and McAndrew [1] is topological entropy on a compact metric space by using open covers. Since then, different (but equivalent) definitions of topological entropy have been introduced. Here we will only give the original definition of topological entropy and the other definitions can be found in Section 7.2 in [45].

Consider a compact metric space X and a continuous map $f : X \to X$. A *finite cover* is a collection of sets $\mathcal{C} = \{C_1, \cdots, C_p\}$ such that $C_1 \cup \cdots \cup C_p = X$. It is an *open cover* if in addition the sets $C_1, \cdots, C_p$ are open. Topological entropy is usually defined for open covers only but some authors give the definition for any cover (for example, see [37]). If $\mathcal{C} = \{C_1, \cdots, C_p\}$ and $\mathcal{D} = \{D_1, \cdots, D_q\}$ are two covers, define the cover

$$\mathcal{C} \vee \mathcal{D} = \{C_i \cap D_j | 1 \leqslant i \leqslant p; 1 \leqslant j \leqslant q\}.$$

If $\mathcal{C}$ and $\mathcal{D}$ are open covers, then $\mathcal{C} \vee \mathcal{D}$ is an open cover too. The cover $\mathcal{D}$ is *finer* than $\mathcal{C}$ if every element of $\mathcal{D}$ is included in an element of $\mathcal{C}$ and we write it as $\mathcal{C} \prec \mathcal{D}$. If $\mathcal{C} = \{C_1, \cdots, C_p\}$ is a cover, define

$$N(\mathcal{C}) = \min\{n \mid \exists\ i_1, \cdots, i_n \in \{1, \cdots, p\}, X = C_{i_1} \cup \cdots \cup C_{i_n}\}$$

and

$$N_n(\mathcal{C}, f) = N\big(\mathcal{C} \vee f^{-1}(\mathcal{C}) \vee \cdots \vee f^{-(n-1)}(\mathcal{C})\big).$$

And the topological entropy of the cover $\mathcal{C}$ is

$$h_{top}(\mathcal{C}, f) = \lim_{n \geqslant 1} \frac{\log N_n(\mathcal{C}, f)}{n}.$$

Now we are ready to define the topological entropy of the compact metric space X.

Definition 1.13 *[45] Let X be a compact metric space and f a continuous map on X into itself. The (topological) entropy of the system (X, f) is defined as*

$$h_{top}(X, f) = \sup\{h_{top}(\mathcal{U}, f) \mid \mathcal{U} \textit{ finite open cover of } X\}$$

1.2 Full Shifts

Full Shifts and their subsets are topological spaces that play an important role in the study of dynamical systems. We will see the reason for this in the next section of this chapter, but before that let us introduce this topological space and its topological properties.

Let $\mathcal{A} = \{0, 1, \cdots, n-1\}$ be a set of n symbols, called an *alphabet*. Corresponding to $\mathcal{A}$ we have a product space $\mathcal{A}^{\mathbb{N}}$ or equivalently the space of the infinite sequences on the alphabet $\mathcal{A}$ of n symbols i.e. $\mathcal{A}^{\mathbb{N}} = \{\mathbf{s} = s_0 s_1 s_2 \cdots | s_j \in \mathcal{A} \text{ for all } j \in \mathbb{N}\}$. Every element of $\mathcal{A}^{\mathbb{N}}$ defined above is a one-sided sequence and therefore by *two-sided sequence*, we mean a sequence $\mathbf{s} = \cdots s_{-2} s_{-1} s_0 s_1 s_2 \cdots$ where, $s_j \in \mathcal{A}$ for all $j \in \mathbb{Z}$. A finite sequence on the alphabet $\mathcal{A}$ of n symbols with length l, $x_0 x_1 x_2 \cdots x_{l-1}$ will be called *l-block (l-word)* over $\mathcal{A}$, and the set $B_l(\mathcal{A}) = \mathcal{A}^l$ is the set of all l-blocks over $\mathcal{A}$. If $x \in \Sigma_n$ and w is a block over $\mathcal{A}$, we will say that w occurs in x if there are indices i and positive integer $m \geqslant 1$ such that $w = x_i x_{i+1} \cdots x_{i+m}$. For every $x_0 x_1 \cdots x_{l-1} \in B_l(\mathcal{A})$, there are n possibilities for x_i for each $i = 0, 1, \cdots, l-1$. Hence $|B_l(\mathcal{A})| = n^l$. Given $x \in B_l(\mathcal{A})$ and for every $i < j \leqslant l$ we denote by $x_{[i,j]}$ the block $x_i x_{i+1} \cdots x_j \in B_{j-i}(\mathcal{A})$.

We give a metric to the space to make it as a topological space, given as follows;

Lemma 1.14 *The set of sequences of $\mathcal{A}^{\mathbb{N}} = \{\mathbf{s} = \{s_i\}_{i \in \mathbb{N}} | s_i \in \mathcal{A}$ for every $i \in \mathbb{N}\}$ is a topological space equipped with a metric, d defined as*

$$d(\mathbf{s}, \mathbf{t}) := \begin{cases} 0 & \text{if } \mathbf{s} = \mathbf{t} \\ 2^{-j} & \text{if } \mathbf{s} \neq \mathbf{t} \end{cases}$$

where $j \in \mathbb{N}$ is the minimal number such that $s_j \neq t_j$.

Proof. To show that d is a metric on $\mathcal{A}^{\mathbb{N}}$, we will only prove the triangle inequality since the other conditions i.e. non-negative, identity of indiscernibles and symmetry are obvious from the definition of d. Let $d(\mathbf{r}, \mathbf{s}) = \frac{1}{2^m}$ and $d(\mathbf{s}, \mathbf{t}) = \frac{1}{2^n}$. Without loss of generality, assume $n \geqslant m$. So, for every $i = 0, 1, \cdots, m-1$, $r_i = s_i$ and $s_i = t_i$ but when $i = m$, $r_i \neq s_i$ and $s_i = t_m$. So, $d(\mathbf{r}, \mathbf{t}) = \frac{1}{2^m}$. Therefore $d(\mathbf{r}, \mathbf{s}) + d(\mathbf{s}, \mathbf{t}) \geqslant d(\mathbf{r}, \mathbf{t})$. $\square$

The metric d given above is not the only metric function used in literature. The other different metric used in [15] is as follows.

Lemma 1.15 *Let d_1 be a metric on $\mathcal{A}^{\mathbb{N}}$ that is defined as*

$$d_1(\mathbf{s}, \mathbf{t}) = \sum_{i=0}^{\infty} \frac{\alpha_i(\mathbf{s}, \mathbf{t})}{2^i}$$

(for $\mathbf{s}, \mathbf{t} \in \mathcal{A}^{\mathbb{N}}$) where $\alpha_i(\mathbf{s}, \mathbf{t}) = 0$ if $s_i = t_i$ and $\alpha_i(\mathbf{s}, \mathbf{t}) = 1$ if $s_i \neq t_i$. Then d_1 is equivalent to the metric d given in Lemma 1.14.

Proof. Firstly, we will show that d_1 is a metric. Clearly, $d_1(\mathbf{s}, \mathbf{t}) = 0$ iff $\alpha_i(\mathbf{s}, \mathbf{t}) = 0$ for every i iff $\mathbf{s} = \mathbf{t}$. Since $\alpha_i(\mathbf{s}, \mathbf{t}) = \alpha_i(\mathbf{t}, \mathbf{s})$ for every i, then $d_1(\mathbf{s}, \mathbf{t}) = d_1(\mathbf{t}, \mathbf{s})$. Also, for every i, $\alpha_i(\mathbf{r}, \mathbf{s}) = \alpha_i(\mathbf{s}, \mathbf{t})$ iff $\alpha_i(\mathbf{r}, \mathbf{t}) = 0$. This implies $\alpha_i(\mathbf{r}, \mathbf{s}) + \alpha_i(\mathbf{s}, \mathbf{t}) \geqslant \alpha_i(\mathbf{r}, \mathbf{t})$. Therefore, $d_1(\mathbf{r}, \mathbf{s}) + d_1(\mathbf{s}, \mathbf{t}) \geqslant d_1(\mathbf{r}, \mathbf{t})$. Now let, $d_1(\mathbf{s}, \mathbf{t}) = 0$. Since $\frac{1}{2^m} \neq 0$ for every m, so by the definition of d, $s_i = t_i$ for all i i.e. $\mathbf{s} = \mathbf{t}$. This completes the proof.

Now let $A \subset \mathcal{A}^{\mathbb{N}}$ be an open set with respect to d_1. Let $\mathbf{s} \in A$ and $n > 0$ such that whenever $\mathbf{r} \in \mathcal{A}^{\mathbb{N}}$ satisfy $d_1(\mathbf{s}, \mathbf{r}) < \frac{1}{2^n}$ then $\mathbf{r} \in A$. Now pick $k > n$ and $\mathbf{t} \in \mathcal{A}^{\mathbb{N}}$ such that $d(\mathbf{s}, \mathbf{t}) = \frac{1}{2^m} < \frac{1}{2^k}$. Since m is the minimal index such that $s_i \neq t_i$ and $m > k$, for every

$i = 0, 1, \cdots, k$, $s_i = t_i$, so

$$\begin{aligned} d_1(\mathbf{s}, \mathbf{t}) &= \sum_{i=0}^{k} \frac{\alpha_i(\mathbf{s}, \mathbf{t})}{2^i} + \sum_{i=k+1}^{\infty} \frac{\alpha_i(\mathbf{s}, \mathbf{t})}{2^i} \\ &= 0 + \sum_{i=k+1}^{\infty} \frac{\alpha_i(\mathbf{s}, \mathbf{t})}{2^i} \\ &\leqslant \sum_{i=k+1}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2^k} \\ &\leqslant \frac{1}{2^n} \end{aligned}$$

Therefore $\mathbf{t} \subset A$ and A is open with respect to d. Conversely, assume $A \subset \mathcal{A}^{\mathbb{N}}$ is open with respect to d. Suppose $\mathbf{s} \in A$ and $n > 0$ such that whenever $\mathbf{r}$ satisfy $d(\mathbf{s}, \mathbf{r}) < \frac{1}{2^n}$ then $\mathbf{r} \in A$. Pick $k > n$ and $\mathbf{t} \in \mathcal{A}^{\mathbb{N}}$ such that $d_1(\mathbf{s}, \mathbf{t}) < \frac{1}{2^k}$. Assume $s_j \neq t_j$ for some $j \leqslant k$, then $d_1(\mathbf{s}, \mathbf{t}) \geqslant \frac{1}{2^j} \geqslant \frac{1}{2^k}$ a contradiction. So, $s_j = t_j$ for all $j \leqslant k$. So $d(\mathbf{s}, \mathbf{t}) < \frac{1}{2^k} < \frac{1}{2^n}$. So $\mathbf{t} \in A$ and A is open with respect to d_1. □

For simplicity, we will always use the metric d as the metric on $\mathcal{A}^{\mathbb{N}}$ and therefore we are ready to define the full shifts as follow;

Definition 1.16 *The full-n-shift (denoted as Σ_n) is the topological space $(\mathcal{A}^{\mathbb{N}}, d)$.*

Since we will look at the sequences of the form $x, f(x), f^2(x), \cdots$ in X, compactness ensures this sequence has a limit.

Proposition 1.17 *Σ_n is a compact metric space.*

Proof. We want to show that every sequence inΣ $_n$ has a convergent subsequence. Let $\{\mathbf{s_i}\}_{i\in\mathbb{N}}$ be a sequence of points inΣ $_n$. For every i, the j-th element of the sequence $\mathbf{s_i}$ will be written as s_{ij}. If there exists $\mathbf{t} \in \Sigma_n$ such that $\mathbf{t}$ occurs infinitely often in

$\{\mathbf{s_i}\}_{i\in\mathbb{N}}$, then $\{\mathbf{s_i}\}_{i\in\mathbb{N}}$ has a convergent subsequence. So now let us assume the other case, i.e. there is no such $\mathbf{t}$. For every $i \in \mathbb{N}$, there are only n choices for s_{i0}, i.e. there exists $x_0 \in \{0, 1, \cdots, n-1\}$ such that $s_{i0} = x_0$ for infinitely many i's. Now let $N_0 = \{i \in \mathbb{N} | s_{i0} = x_0\} \subset \mathbb{N}$. So N_0 is an infinite set and $\{\mathbf{s_i}\}_{i\in N_0}$ is a subsequence of $\{\mathbf{s_i}\}_{i\in\mathbb{N}}$. Again, for every $i \in N_0$, there are only n choices for s_{i1}, i.e. there exists $x_1 \in \{0, 1, \cdots, n-1\}$ such that $s_{i1} = x_1$ for infinitely many i's in N_0. Then again define infinite subset of N_0 as $N_1 = \{i \in N_0 | s_{i1} = x_1\} = \{i \in \mathbb{N} | s_{i0} = x_0 \text{ and } s_{i1} = x_1\}$. By repeating this step we will get infinitely many sets $\cdots \subseteq N_i \subseteq N_{i-1} \subseteq \cdots N_1 \subseteq N_0 \subseteq \mathbb{N}$ with the property if $u, v \in N_i$ for any $i \in \mathbb{N}$ then for every $j \leqslant i$, $s_{uj} = s_{vj} = x_j$. Therefore by choosing one $u_i \in N_i$ for every $i \in \mathbb{N}$, then the subsequence $\{\mathbf{s_{u_i}}\}_{i\in\mathbb{N}}$ converges to $\mathbf{x} = x_0x_1 \cdots \in \Sigma_n$. So, Σ_n is compact. □

We now define a continuous map onΣ_n into itself as follows.

Definition 1.18 *The shift map $\sigma : \Sigma_n \to \Sigma_n$ is given by $\sigma(s_0s_1s_2\cdots) = s_1s_2s_3\cdots$.*

In other words, when the shift map is applied to a sequence $\mathbf{s}$ inΣ_n, the first entry of $\mathbf{s}$, s_0 is deleted to produce $\sigma(\mathbf{s})$. Clearly, σ is n-to-one map ofΣ_n, as s_0 may be either $0, 1, \cdots$ or $n-1$. Moreover, in the metric defined above, σ is a continuous map.

Proposition 1.19 *[15] $\sigma : \Sigma_n \to \Sigma_n$ is continuous.*

Proof. Let $\varepsilon > 0$ and $\mathbf{s} = s_0s_1s_2\cdots$ be an arbitrary element ofΣ_n. Pick k such that $\frac{1}{2^k} < \varepsilon$. Suppose $\delta = \frac{1}{2^k}$ and $\mathbf{t} = t_0t_1t_2\cdots$ satisfies $d(\mathbf{s}, \mathbf{t}) < \delta$. So, $s_i = t_i$ for $i = 0, 1, \cdots, k$. Then, $\sigma(\mathbf{s})_i = \sigma(\mathbf{t})_i$ for every $i = 0, 1, \cdots, k-1$ and therefore $d(\sigma(\mathbf{s}), \sigma(\mathbf{t})) \leqslant \frac{1}{2^k} < \varepsilon$. □

It is also easy to see that the periodic points of σ with period l are those with a repeating $l-$block,

$$\mathbf{s} = \overline{s_0s_1s_2\cdots s_{l-1}} = s_0s_1s_2\cdots s_{l-1}s_0s_1s_2\cdots s_{l-1}\cdots$$

Hence there are n^l periodic points of period l for σ inΣ_n, each generated by one of the n^l $l-$block of 0's, 1's, $\cdots$, $(l-1)'$s.

We will list some properties of σ inΣ_n that are important in the study of dynamical systems and the usage of these properties will be shown in the next chapter.

Proposition 1.20 *[15] Let σ be the shift map on a full-n-shift, Σ_n for any integer n. Then*

1. *The cardinality of $Per_l(\sigma)$ is n^l.*
2. *$Per(\sigma)$ is dense.*
3. *There exists a dense orbit for σ in Σ_n.*

Proof. Recall that, if $\mathbf{s}$ is a periodic point of period l for σ, then $\mathbf{s}$ is the repetition of a unique l-block in $B_l(\mathcal{A})$ and $|B_l(\mathcal{A})| = n^l$. So, $Per_l(\sigma) = |B_l(\mathcal{A})| = n^l$.
To show that $Per(\sigma)$ is dense, we need to show that for every $\mathbf{s} \in \Sigma_n$, there exists a sequence in $Per(\sigma)$ which converges to $\mathbf{s}$. Let us define a sequence $\{\tau_i\}_{i\in\mathbb{N}}$ where $\tau_i = \overline{s_0 s_1 \cdots s_i}$ is a periodic point of period i which agrees with $\mathbf{s}$ up to the i^{th} entry. So, $d(\tau_i, \mathbf{s}) \leqslant \frac{1}{2^{i+1}}$. Hence, as $i \to \infty$, $d(\tau_i, \mathbf{s}) \to 0$.
Any sequence $\mathbf{s}$ that contains all finite blocks in $B(\mathcal{A})$ will have dense orbit. □

1.3 Symbolic Dynamics

The properties of dynamical systems are sometimes easy to be seen in a symbol space. Therefore, symbolic dynamics was introduced as a method to analyze dynamical systems by using a discrete space. The discrete space is the shift space that we introduced previously. In fact, symbolic dynamics itself is a study of shift space. Recall that, in dynamical systems we focus on a point following its trajectory in a space. To apply symbolic dynamics to the system, we partition the space into finitely many pieces and label every partition element by a different symbol. A symbolic orbit is obtained by writing down the sequence of symbols corresponding to the successive partition elements visited by the point in its orbit. The resulting sequence is an element of the discrete space (shift space). We then use these sequences to model the original dynamical system.

To illustrate this, we write x for the point and f for the map in the original dynamical system and ϕ for the function taking the value in its orbit to the corresponding symbol. Then the resulting symbolic orbit is a sequence $\mathbf{s} = \phi(x)\phi(f(x))\phi(f^2(x))\cdots$ and will be called an *itinerary* of x. If we partition the space into n pieces, then denotes $\phi(f^i(x))$ as k for $k \in \{0, 1, \cdots, n-1\}$ if $f^i(x)$ lies in the $k-$th partition, then $\mathbf{s} \in \Sigma_n$. Now starting with $f(x)$ instead of x, the resulting symbolic orbit will be a sequence $\phi(f(x))\phi(f^2(x))\phi(f^3(x))\cdots$. Therefore, the dynamics on the discrete space of symbol sequences is shifting to the right, i.e. $s_0s_1s_2\cdots \mapsto s_1s_2s_3\cdots$. The map is the *shift map*. In other words, the coding of $f(x)$ is shifting the coding of x.

The point x will be represented symbolically by the itinerary sequence obtained, $\mathbf{s}$. Properties of the orbits of the original dynamical system are reflected in properties of the resulting itinerary space. For example, a point whose orbit is periodic gives a periodic itinerary since it is will be easy to see that the itinerary of any periodic point is a periodic sequence with respect to the shift map. Furthermore there are important cases where we can find those partitions and map ϕ such that there is a one-to-one correspondence between any point and the resulting sequences under the partition and map ϕ. Hence, the system is conjugate to the resulting shift space. Therefore we will later need a topological conjugacy to prove that symbolic dynamics and shift space are the best way to study dynamical systems. There are two running examples of dynamical systems which will be used to describe symbolic dynamics and topological conjugacy, in the next section.

1.3.1 Symbolic Dynamics for the Tent Map

The Tent Map will be used repeatedly throughout this chapter in order to illustrate some properties that are important in the study of dynamical systems. Now we will look at how symbolic dynamics work for the Tent Map.

Definition 1.21 (The Tent Map) *[16] The Tent Map, $T_\mu : I \to \mathbb{R}$ $(0 < \mu < \infty)$ is defined on the unit interval $I = [0, 1]$ by the expression*

$$T_\mu(x) = \mu\Big(\frac{1}{2} - \Big|\frac{1}{2} - x\Big|\Big).$$

There is a subset Λ of I such that Λ is invariant under $T_\mu|_\Lambda$. If $\mu > 2$, $T_\mu(x_0) > 1$ (i.e. not in I) whenever x_0 belongs to the open interval $A_0 = (a_0, b_0)$ of I whose endpoints are the solutions of the equation $T_\mu(x) = 1$. The set $I - A_0$ is the union of two closed intervals I_0 and I_1. I_0 and I_1 contain open subintervals $A_{1,1} \subset I_0$ and $A_{1,2} \subset I_1$ such that for every $x \in A_{1,1} \cup A_{1,2}$, $T_\mu^2(x) > 1$. Then let us denotes $A_1 = A_{1,1} \cup A_{1,2}$. In general, we are able to define an open set $A_k = A_{k,1} \cup A_{k,2} \cup \cdots \cup A_{k,2^k}$ in which $T_\mu^k(x) > 1$ for every $x \in A_k$ for any k and therefore $T_\mu^{k+1}(x) < 0$. Then, by discarding the open sets $A_k = A_{k,1} \cup A_{k,2} \cup \cdots \cup A_{k,2^k}$ for every k we arrive at uncountable set Λ,

$$\Lambda = I - \Big(\bigcup_{k=0}^{\infty} A_k\Big),$$

which is invariant under $T_\mu|_\Lambda$. We will show that the set Λ is a Cantor set.

The graphics of T_μ and T_μ^2 for some $\mu > 2$ (negative range not shown) are shown in the Figure 1.1. These give the picture of the first two steps of construction of the set Λ, as described above.

Proposition 1.22 *If $\mu > 2$, then Λ is a Cantor set.*

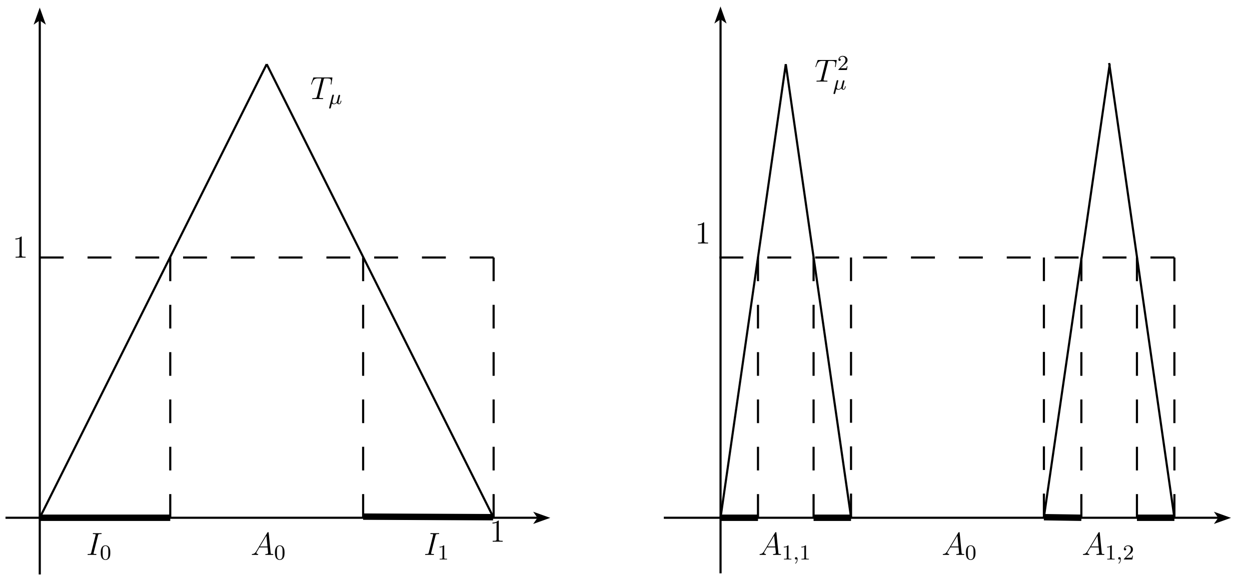

Figure 1.1: The graphics of T_μ and T_μ^2 for some $\mu > 2$

Proof. Firstly we will show that Λ is totally disconnected. By using contradiction, we may assume that there is an interval in Λ, say $[x, y] \subset \Lambda$. Since T_μ is piecewise linear function with gradient $\pm\mu$, T_μ^n is piecewise linear with gradient $\pm\mu^n$ except at critical points. Since every critical point of T_μ^n will be mapped by T_μ^n out from I, then every $a \in [x, y]$ is non-critical. Therefore $|(T_\mu^n)'(a)| = (\mu)^n$ for any n. We choose k such that $(\mu)^k|y - x| > 1$. Since $[x, y]$ will never leave I under any iteration of T_μ, $T_\mu^k : [x, y] \to I$ is continuous and by the Mean Value Theorem, there exists $a \in [x, y]$ such that

$$\begin{aligned} |T_\mu^k(y) - T_\mu^k(x)| &= |(T_\mu^k)'(a)||y - x| \\ &= (\mu)^k|y - x| > 1 \end{aligned}$$

This implies that at least one of $T_\mu^n(x)$ or $T_\mu^n(y)$ lies outside of I. This is a contradiction, and so Λ is totally disconnected.

Λ is closed since for every i, A_i is an open interval. So the complement of the infinite union of A_i is closed.

Lastly we need to show that Λ is a perfect set. By contradiction, let us assume that Λ has an isolated point, say $p \in \Lambda$. So, there exists a neighborhood $N(p)$ of p such that $N(p) \setminus \{p\} \cap \Lambda = \emptyset$. Therefore, for all $x \in N(P) \setminus \{p\}$, $x \in A_k$ for some k and $N(p) \setminus \{p\} \subseteq \bigcup_{k \geqslant 1} A_k$. Either there is a sequence $\{x_j\}$, x_j is an end point of one of the subintervals $A_{j,i}$ such that the sequence $\{x_j\}$ converges to p or else, there are $k, j \in \mathbb{N}$ such that $A_{k,s} = (a, p)$ and $A_{j,t} = (p, b)$ for some $s, t \in \mathbb{N}$. For the first case, since $x_j \in \Lambda$ for all j, therefore p is not isolated in Λ. From the way we construct $A_{j,i}$, the endpoints of $A_{j,i}$ will be mapped to 1 under T^j. So for the second case, whenever $k \neq j$ and p is an endpoint of $A_{k,s}$ and $A_{j,t}$, then $T^k(p) = T^j(p) = 1$. Without loss of generality, suppose $j > k$. Then $T^{j-k}(1) = T^{j-k}T^k(p) = T^j(p) = 1$ which is a contradiction since 1 is not a periodic point. Now let's assume $k = j$. So, by the way we construct A_k, $A_{j,t} = A_{j,s+1}$. Again, this is a contradiction since for every k, there exists $A_{l,i}$ such that $A_{l,i}$ lies between $A_{k,s}$ and $A_{k,s+1}$. □

All points in $\mathbb{R} \backslash \Lambda$ tend to $-\infty$ under iteration of T_μ. So, we restrict T_μ to Λ in order to understand its dynamics. So, by symbolic dynamics we will symbolically code every $x \in \Lambda$ with unique $\mathbf{s}$ inΣ_2. We code x by looking at the j-iteration of x under T_μ for every $j \in \mathbb{N}$. To do this we define a map ϕ_x (with respect to x) from $\mathbb{N}$ to $\{0, 1\}$ as follows

$$\phi_x(j) := \begin{cases} 0 & \text{if } T_\mu^j(x) \in I_0 \\ 1 & \text{if } T_\mu^j(x) \in I_1 \end{cases}$$

Then the symbolic code of x is called the *itinerary of* x, a sequence

$$\mathbf{s} = \phi_x(0)\phi_x(1)\phi_x(2)\cdots \in \Sigma_2.$$

When we iterate x under T_μ, the itinerary of $T_\mu(x)$ is a sequence

$$\mathbf{t} = \phi_x(1)\phi_x(2)\phi_x(3)\cdots = \sigma(\mathbf{s}).$$

So, the itinerary of $T_\mu(x)$ is the shift of the itinerary of x.

Although symbolic dynamics originated as a method to study general dynamical systems as described above, the techniques and ideas have found significant applications in data storage and transmission as well as linear algebra [32].

1.3.2 Symbolic Dynamics for the Doubling Map

Instead of partitioning the space into n partitions and coding every element in the space due to its trajectory, there are other significant coding methods depending on the map. The Doubling Map, F for example uses the binary expansion to code every element on the interval $[0, 1)$ with a sequence inΣ_2. As in the partition method, the coding method is significant in the dynamical study of the system because the coding of $i-$iteration of x under F is the i-shift of the coding of x.

Definition 1.23 (The Doubling Map) *The Doubling Map, $F : [0, 1) \to [0, 1)$ is defined on the interval $[0, 1)$ by the expression $x \mapsto 2x \pmod 1$.*

The graph of Doubling Map F as in the Figure 1.2 is not continuous at the point $\frac{1}{2}$. However F can be made continuous by identifying 0 and 1, as a circle map in Chapter 3 later. We code the orbits of the Doubling Map by using binary expansion, given by the theorem:

Theorem 1.24 *[14] Every $x \in [0, 1)$ has a binary expansion as follows*

$$x = \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}$$

where $x_n \in \{0, 1\}$ for all n.

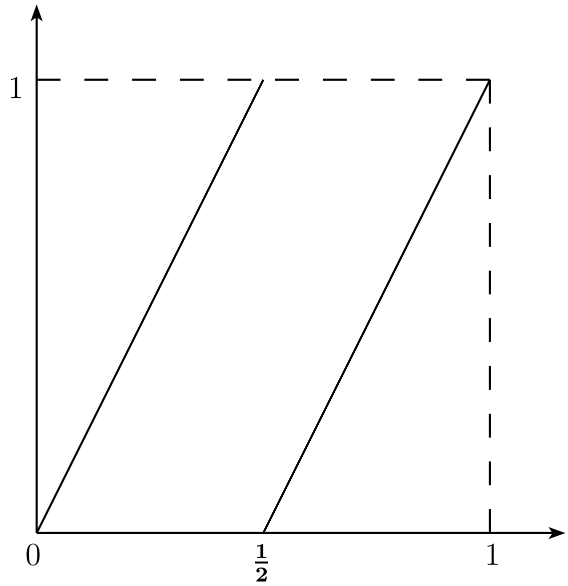

Figure 1.2: The graph of Doubing Map F

Therefore we can code x as the sequence $\mathbf{s} = x_0 x_1 \cdots \in \Sigma_2$. We now want to show that the coding of the 1-iteration of x under F is the shift of the coding of x. When we iterate x under F, we have $F(x) = x_0 + \sum_{n=1}^{\infty} \frac{x_n}{2^2} \pmod 1$. If $x_0 = 0$ then $F(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^2}$ and if $x_0 = 1$ then $F(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^2}$. So, the coding of $F(x)$ is $x_1 x_2 \cdots$ the shift of the code for x.

1.4 Topological Conjugacy

The goal of this section is to show that the coding made by symbolic dynamics is useful to represent the dynamics of the maps. Topological conjugacy guarantees that the original dynamical system and the resulting discrete space by symbolic dynamics are equivalent or conjugate. Consequently, if the dynamics of the discrete space (i.e the shift space) can be exposed, then the dynamics of the original system follow trivially. So, we can use symbolic dynamics to study dynamical systems.

Definition 1.25 *Let $f : X \to X$, $g : Y \to Y$ and $h : X \to Y$ satisfy $h \circ f = g \circ h$.*

If h is a continuous surjective function, then f and g are said to be topologically semi-conjugate and h is semi-conjugacy. If h is a homeomorphism, then f and g are said to be topologically conjugate and h is called a topological conjugacy.

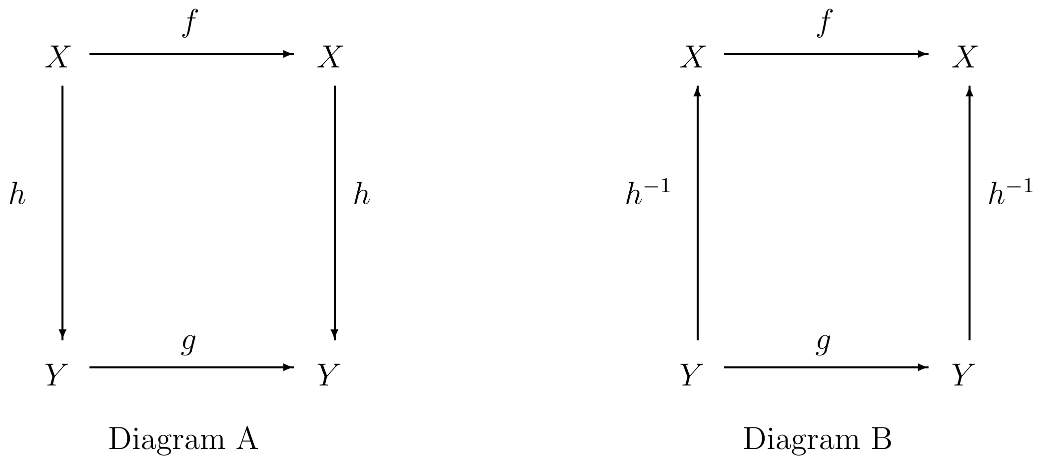

Figure 1.3: The illustration of semi-conjugacy and conjugacy between f and g

Semi-conjugacy and conjugacy are more commonly related to a commuted diagram, as given in Figure 1.3. If h is a semi-conjugacy from f to g, then Diagram A commutes. However, if h is a conjugacy for f and g, then h and h^{-1} is a semi-conjugacy from f to g and g to f respectively and the Diagram A and Diagram B commute representing semi-conjugation for h and h^{-1}.

In general, semi-conjugacy h from f to g carries some dynamical features of f across to identical features of g. So, g inherits some dynamical properties from f through h. On the other hand, h cannot guarantee that f also inherits some dynamical properties from g, unless h is a homeomorphism. This is because h^{-1} exists and becomes a semi-conjugacy from g to f. In this case, h is a conjugacy for f and g and therefore f and g share some dynamical properties. The following lemmas will help us to see how semi-conjugacy and topological conjugacy become an important tool in the study of iterated functions and more generally dynamical systems.

Lemma 1.26 *If h is a semi-conjugacy from f to g, then h is also a semi-conjugacy from f^n to g^n.*

Proof. As a semi-conjugacy from f to g, $hf = gh$. Therefore $hf^n = hff^{n-1} = ghff^{n-2} = \cdots = g^n h$, i.e. h is a semi-conjugacy from f^n to g^n. □

Lemma 1.27 *A semi-conjugacy from $f : X \to X$ to $g : Y \to Y$ sends orbits of f to orbits of g, and periodic points to periodic points.*

Proof. Let $x_0 \in X$ and $y_0 = h(x_0) \in Y$ where h is a semi-conjugacy from f to g. By definition

$$O^+(x_0) = \{f^n(x_0)|n > 0\}$$

Therefore

$$h(O^+(x_0)) = \{hf^n(x_0)|n > 0\} = \{g^n h(x_0)|n > 0\} = \{g^n(y_0)|n > 0\} = O^+(y_0)$$

So, h sends orbits of f to orbits of g.

Now let $x_0 \in X$ be a fixed point for f and $y_0 = h(x_0) \in Y$, then

$$g(y_0) = gh(x_0) = hf(x_0) = h(x_0) = y_0$$

Hence h is a continuous function from $Fix(f)$ to $Fix(g)$.

Finally, suppose x_0 is a point of prime period k for f and let $y_0 = h(x_0) \in Y$. Then

$$f^k(x_0) = x_0 \text{ and for } m < k,\ f^m(x_0) \neq x_0$$

So

$$g^k(y_0) = g^k h(x_0) = hf^k(x_0) = h(x_0) = y_0$$

and therefore y_0 is a periodic point of prime period at most k. □

Thus by these two lemmas we can see how a semi-conjugacy h from f to g plays its role to pass properties of f to g. In fact, if h is homeomorphism, a conjugacy for f and g, we can elaborate even more on this. The following lemma is an example, to compare with the previous lemma.

Corollary 1.28 *A conjugacy from $f : X \to X$ to $g : Y \to Y$ sends periodic points to periodic points of the same prime period.*

Proof. Let $x_0 \in X$ a periodic point of prime period k and $y_0 = h(x_0) \in Y$ where h is a semi-conjugacy from f to g. By Lemma 1.27, $g^k(y_0) = y_0$. For $m < k$, h as a bijection ensures that $f^m(x_0) \neq x_0$ implies $hf^m(x_0) \neq h(x_0)$ and then

$$g^m(y_0) = g^m h(x_0) = hf^m(x_0) \neq h(x_0).$$

Thus y_0 is a periodic point of prime period k for g. □

Therefore symbolics dynamics and topological conjugacy are a great combination of tools to study dynamical systems. For example, by knowing the dynamics of shift spaces, the dynamics of the Tent Map and the Doubling Map become apparent.

1.4.1 Conjugacy for the Tent Map and the Shift Map.

We will show that there exists a conjugacy for the Tent Map and the shift map, so that the shift map inΣ_n is in fact the same map as the Tent Map on Λ.

Definition 1.29 *The map from Λ to Σ_2 that assigns x to the itinerary $\mathbf{s}$ of x, is called the itinerary map, denoted as π.*

Proposition 1.30 *[15] Let T_μ be a Tent Map with $\mu > 2$, then the itinerary map $\pi : \Lambda \to \Sigma_2$ is a homeomorphism.*

Proof. We first will show that π is a one-to-one map. Let x, y be distinct points in Λ with the same itinerary, $\pi(x) = \pi(y)$. Then $T_\mu^n(x) \in I_0$ iff $T_\mu^n(y) \in I_0$ for every n. This implies that T_μ is monotonic on the interval between $T_\mu^n(x)$ and $T_\mu^n(y)$. Observe that, for every z in the interval between $T_\mu^n(x)$ and $T_\mu^n(y)$, $T_\mu^k(z) \in I_0$ iff $T_\mu^{n+k}(x) \in I_0$ for any $k \geqslant 0$. So, z remains in $I_0 \cup I_1$ under any iteration of T_μ, for any z in the closed interval. However in fact, there exists an element of any interval such that the element will leave $I_0 \cup I_1$ and tend to $-\infty$. So, this is a contradiction, due to totally disconnected.
To see that π is onto, we first need to note that if $J \subset I$ is a closed interval, then the pre-image of J under T_μ, i.e. $T_\mu^{-1}(J)$ consists of two intervals, one is a subinterval of I_0 and the other is a subinterval of I_1. For any $n > 0$ and any $(n+1)-$block over $\{0, 1\}$ i.e. $s_0 s_1 \cdots s_n$, we define a set

$$I_{s_0 s_1 \cdots s_n} = \{x \in I : x \in I_{s_0}, T_\mu(x) \in I_{s_1}, \cdots T_\mu^n(x) \in I_{s_n}\}.$$

Then we will have

$$I_{s_0 s_1 \cdots s_n} = I_{s_0} \cap T_\mu^{-1}(I_{s_1}) \cap \cdots \cap T_\mu^{-n}(I_{s_n}) \tag{1.31}$$

$$= I_{s_0 s_1 \cdots s_n} = I_{s_0} \cap T_\mu^{-1}(I_{s_1 s_2 \cdots s_n}) \tag{1.32}$$

$$= I_{s_0 \cdots s_{n-1}} \cap T_\mu^{-n}(I_{s_n}). \tag{1.33}$$

There are two arguments we will claim. The first argument is $I_{s_0 s_1 \cdots s_n}$ is a nonempty closed single subinterval of I. This can be proven by induction on n. It is true for $n = 0$ since I_{s_0}

is either I_0 or I_1. We then assume that $I_{s_1 \cdots s_n}$ is a nonempty closed single subinterval. So, $T_\mu^{-1}(I_{s_1 s_2 \cdots s_n})$ consists of two closed intervals, one is in I_0 and the other one is in I_1. Hence $I_{s_0} \cap T_\mu^{-1}(I_{s_1 s_2 \cdots s_n})$ is a closed single interval and so $I_{s_0 s_1 \cdots s_n}$ by equation (1.32). Secondly, we claim that the intervals $I_{s_0 \cdots s_n}$ are nested i.e. $I_{s_0 \cdots s_n} \subset I_{s_0 \cdots s_{n-1}} \subset I_{s_0 \cdots s_{n-2}} \subset \cdots \subset I_{s_0}$. This is true by equation (1.33). Therefore for an arbitrary element $\mathbf{s} = s_0 s_1 \cdots \in \Sigma_2$,

$$\bigcap_{n \geqslant 0} I_{s_0 s_1 \cdots s_n}$$

is nonempty and if $x \in \bigcap_{n \geqslant 0} I_{s_0 s_1 \cdots s_n}$ then $x \in I_{s_0}, T_\mu(x) \in I_{s_1} \cdots$. Hence $\pi(x) = \mathbf{s}$. So π is onto.

To show that π and π^{-1} are continuous, we first should note that for a fixed n, we have in total of 2^{n+1} intervals $I_{t_0 t_1 \cdots t_n}$ in the unit interval I. Every $I_{t_0 t_1 \cdots t_n}$ is a closed interval and they are all disjointed. Since the Tent Map has the same slope value (with either negative or positive)at every point, so all $I_{t_0 t_1 \cdots t_n}$ have the same length. Therefore the length of the interval is less or equal to $\frac{1}{2^{n+1}}$. To show the continuity of π, let $x \in \Lambda$ and the itinerary of x be $\pi(x) = s_0 s_1 s_2 \cdots$. Let $\varepsilon > 0$ and pick $n > 0$ such that $\frac{1}{2^n} < \varepsilon$. Choose $\delta = \frac{1}{2^{n+1}}$. So whenever $|x - y| < \delta$ and $y \in \Lambda$, then $y \in I_{s_0 s_1 \cdots s_n}$. Therefore, $\pi(y)$ agrees with $\pi(x)$ in the first n+1 terms. Hence

$$d[\pi(x), \pi(y)] < \frac{1}{2^n} < \varepsilon$$

and so π is continuous.

To show the continuity of the inverse of π, let $\mathbf{s} \in \Sigma_2$ and $\varepsilon > 0$. Let us pick $n > 0$ such that $\frac{1}{2^{n+1}} < \varepsilon$. Choose $\delta < \frac{1}{2^n}$, so whenever an arbitrary $\mathbf{t} \in \Sigma_2$ satisfies $d(\mathbf{s}, \mathbf{t}) < \delta$, $\pi^{-1}(\mathbf{t}), \pi^{-1}(\mathbf{s}) \in I_{s_0 s_1 \cdots s_n}$. Therefore $|\pi^{-1}(\mathbf{t}) - \pi^{-1}(\mathbf{s})| < \frac{1}{2^{n+1}} < \varepsilon$. So π^{-1} is continuous, and therefore π is a homeomorphism. □

Theorem 1.34 *The itinerary map, π is a topological conjugacy for the shift map, σ and the Tent Map, T_μ ($\mu > 2$) restricted to $\Lambda \subset I$.*

Proof. We want to show that $\pi \circ T_\mu = \sigma \circ \pi$. Note that since π is one-to-one, $\bigcap_{n\geqslant 0} I_{s_0 s_1 \cdots s_n}$ consists of a unique point. So, a point x in Λ may be defined uniquely by the nested sequence of intervals

$$\bigcap_{n\geqslant 0} I_{s_0 s_1 \cdots s_n}$$

determined by the itinerary of x. Then observe that

$$\begin{aligned}
T_\mu(I_{s_0 s_1 \cdots s_n}) &= T_\mu(I_{s_0} \cap T_\mu^{-1}(I_{s_1}) \cap \cdots \cap T_\mu^{-n}(I_{s_n})) \\
&= T_\mu(I_{s_0}) \cap T_\mu(T_\mu^{-1}(I_{s_1})) \cap \cdots \cap T_\mu(T_\mu^{-n}(I_{s_n})) \\
&= I \cap I_{s_1} \cap T_\mu^{-1}(I_{s_2}) \cap \cdots \cap T_\mu^{-n+1}(I_{s_n}) \\
&= I_{s_1} \cap T_\mu^{-1}(I_{s_2}) \cap \cdots \cap T_\mu^{-n+1}(I_{s_n}) \\
&= I_{s_1 s_2 \cdots s_n}
\end{aligned}$$

Hence,

$$\begin{aligned}
\pi T_\mu(x) &= \pi T_\mu\Big(\bigcap_{n=0}^{\infty} I_{s_0 s_1 \cdots s_n}\Big) \\
&= \pi\Big(\bigcap_{n=1}^{\infty} (I_{s_1 s_2 \cdots s_n})\Big) \\
&= s_1 s_2 \cdots s_n \\
&= \sigma\pi(x)
\end{aligned}$$

□

Now we can see how conjugacy plays an important role in the study of dynamics of the Tent Map. Next is the list of some properties of the Tent Map, as a consequence of being conjugate to the shift map.

Corollary 1.35 *Let* $T_\mu(x) = \mu\Big(\frac{1}{2} - \Big|\frac{1}{2} - x\Big|\Big)$ *with* $\mu > 2$. *Then*

1. *The cardinality of* $Per_n(T_\mu)$ *is* 2^n.

2. $Per(T_\mu)$ *is dense in* Λ.

3. T_μ *has a dense orbit in* Λ.

Proof. All of these statements will be proved by using the conjugacy, ϕ for the Tent Map, T_μ and the shift map, σ onΣ_2. Lemma 1.27 has shown that the conjugacy ϕ send orbits of T_μ and periodic points of T_μ to the orbits of σ and periodic points of σ with the same prime period, respectively. So, T_μ has the same properties as σ in terms of cardinality of set of periodic points, density of periodic points and existence of dense orbit. Therefore properties of σ onΣ_2 in Proposition 1.20 completes the proof. □

1.4.2 Semi-conjugacy for the Doubling Map and the Shift Map.

It is well known that there exists a semi-conjugacy for the Doubling Map and the shift map, so that the maps have some common dynamical properties.

Definition 1.36 *Define a projection map* $\pi : \Sigma_2 \to [0, 1)$ *by*

$$\pi(s_0 s_1 \cdots s_n) = \sum_{n=0}^{\infty} \frac{s_n}{2^{n+1}}$$

Theorem 1.37 π *is continuous surjection but not a homeomorphism.*

Proof. To show that π is continuous, let $\mathbf{s} = s_0 s_1 \cdots \in \Sigma_2$ and $\varepsilon > 0$. Pick n such that

$\frac{1}{2^{n+1}} < \varepsilon$. Let $\mathbf{t} \in \Sigma_2$ satisfy $d((\mathbf{s}, \mathbf{t})) < \frac{1}{2^n}$. Then $s_i = t_i$ for every $i \leqslant n$. Therefore

$$\begin{aligned}
|\pi(\mathbf{s}) - \pi(\mathbf{t})| &= \Big| \sum_{k=0}^{\infty} \frac{s_i - t_i}{2^{k+1}} \Big| \\
&\leqslant \sum_{k=0}^{\infty} \Big| \frac{s_i - t_i}{2^{k+1}} \Big| \\
&\leqslant \sum_{k=n+1}^{\infty} \Big| \frac{s_i - t_i}{2^{k+1}} \Big| \\
&= \frac{1}{2^n} \\
&\leqslant \varepsilon
\end{aligned}$$

π is a surjective by Theorem 1.24. π is not a homeomorphism since it is not injective. This is because $x \in [0, 1)$ with a base two expansion ending in all 0's has another expansion ending in all 1's. □

Theorem 1.38 *π is a semi-conjugacy for the shift map, σ and the Doubling Map, F*

Proof. We want to show that $\pi \circ \sigma = F \circ \pi$. Let $\mathbf{s} \in \Sigma_2$, and at the right hand side

$$\begin{aligned}
\pi(\sigma(s_0 s_1 \cdots)) &= \pi(s_1 s_2 \cdots) \\
&= \sum_{n=1}^{\infty} \frac{s_n}{2_n}
\end{aligned}$$

and at the left hand side,

$$\begin{aligned}
F(\pi(s_0 s_1 \cdots)) &= F\Big(\sum_{n=0}^{\infty} \frac{s_n}{2^{n+1}} \Big) \\
&= s_0 + \sum_{n=1}^{\infty} \frac{s_n}{2^n} \pmod 1 \\
&= \sum_{n=1}^{\infty} \frac{s_n}{2^n}
\end{aligned}$$

□

As in the dynamics of the Tent Map, the Doubling Map also has some similar dynamical properties with the shift map. However the semi-conjugacy only guarantee that the maps are sharing two of the dynamical properties.

Corollary 1.39 *Let F be a Doubling Map on $[0, 1)$. Then*

1. *$Per(F)$ is dense in $[0, 1)$.*

2. *F has a dense orbit in $[0, 1)$.*

Proof. Since F andΣ_2 are semi-conjugate by semi-conjugacy π Lemma 1.27 has shown that π send orbits of σ and periodic points of σ to the orbits of F and periodic points of F with possibly different prime periods, respectively. So, F has the same properties as σ in terms of density of periodic points and existence of dense orbit. Therefore the last two properties of σ onΣ_2 in Proposition 1.20 completes the proof. □

The properties in Corollary 1.35 and 1.39 will be used to show important dynamical behavior of the maps in the next chapter.

Chapter 2

Chaotic Dynamical System

Chaos is a hallmark of a complicated deterministic dynamical system with completely unpredictable behavior without perfect information. The theories of chaotic dynamical systems are applied to many fields such as sociology, economics, and biology [47]. One of the most active fields is brain science, see for example [27]. In addition, the presence of chaotic heartbeats have been implicated in human health, for example see [39] and [19]. In the field of economics, the use of chaos theory to explain the behavior of prices in financial, currency and a commodity market has been discussed in [36]. Therefore, over the past century chaotic behavior has become more apparent in a wide range of active topics. Furthermore, chaos theory has been so surprising because chaos can be found within almost trivial systems. The Tent Map is an example of a system with a simple equation but has very complex behavior. Conversely, a complex system can exhibit a non-chaotic system. For example [42] shows that biological systems are complex systems which has been described as "anti-chaos". Consequently, chaos is not easy to define and there is no universally agreed definition of chaos.

Much of chaos theory was developed almost entirely by mathematicians. Chaos has been defined mathematically in many different ways and they are not necessarily equivalent. The term chaos in mathematical terms was firstly introduced by Li and Yorke in

1975 [30] and the concept of chaos discussed in that paper is now referred to Li-Yorke chaos; it is one of the various definitions of chaos. The most widely utilized definition of chaos is chaos introduced by Robert L. Devaney in his book (see [15]), whose first edition was published in 1989. In spite of Devaney chaos and Li-Yorke chaos, there are a number of alternative ways to define chaos. An incomplete list of other popular methods includes distributional chaos [38], topological chaos [43], $\omega-$chaos [31], P-chaos [2], Block and Coppel chaos [4] and many more.

In this work we put our emphasizes on three sorts of chaos; Devaney chaos, Li-Yorke chaos and topological chaos. Hence in this chapter, we will describe in detail the three definitions of chaos in general and then restrict our attention on three different spaces, the interval, the unit circle and subshifts of finite type. In the literature, research on chaotic behavior of the circle map and the shift map are not yet done as much as research on the interval map. In this chapter, we will reveal that there are two sufficient conditions for chaos that apply to the interval, are also apply to circle maps and shift maps.

2.1 Li-Yorke chaos

In Li and Yorke's famous paper, *Period Three Implies Chaos* [30], they reported on a situation in which iterates of a point are very irregular. The main result in the paper was as follows;

Theorem 2.1 *[30] Let $f : I \to I$ be a continuous interval map. If there is a point $a \in I$ for which the points $b = f(a), c = f^2(a)$ and $d = f^3(a)$ satisfy $d \leqslant a < b < c$ or $d \geqslant a > b > c$ then I has a periodic point of all periods and there is an uncountable set $S \subset I$ (containing no periodic points), which satisfies the following conditions;*

1. $\liminf_{n\to\infty} \mid f^n(x) - f^n(y) \mid = 0$ *for all* $x, y \in S, x \neq y$

2. $\limsup_{n\to\infty} \mid f^n(x) - f^n(y) \mid > 0$ *for all* $x, y \in S, x \neq y$

3. $\limsup_{n\to\infty} \mid f^n(x) - f^n(p) \mid > 0$ *for all* $x \in S, p \in I$, *p periodic.*

They noted that this theorem establishes the irregularity of the behavior of iterates of points in an interval. So, there is a chaos definition introduced by them, although they did not formally define "chaos". The definition of chaos derived from here has been generalized to all metric spaces and has been called Li-Yorke chaos, defined as follows;

Definition 2.2 *[11] Let* $f : X \to X$ *be a map on a metric space* X *with metric* d*. The dynamical system* (X, f) *is said to be Li-Yorke chaotic if there exists a subset* $S \subset X$ *such that* S *is uncountable and satisfies;*

1. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ *for all* $x, y \in S, x \neq y$
2. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ *for all* $x, y \in S, x \neq y$

There are two conditions in the original idea of Li-Yorke chaos in Theorem 2.1 that have been dropped in Definition 2.2. Some authors use a definition with the additional conditionals, see [46] and [4] for examples of the different definitions. However Definition 2.2 has been accepted widely in the current literature.

There is another remarkable result called Sarkovskii's Theorem that relates closely to the Li and Yorke's theorem, Theorem 2.1. This theorem highlighted the importance of period three points in the real line, which gives a proof for the first part in the theorem of Li and Yorke. In particular, Sarkovskii's Theorem shows that existence of period three points implies the existence of periodic points of all periods. In fact, existence of period three points is equivalent to the hypothesis of the Li and Yorke's theorem.

Lemma 2.3 *Let* $f : I \to I$ *be a continuous interval map.* f *has a period three point iff there is a point* $a \in I$ *such that the points* $b = f(a), c = f^2(a)$ *and* $d = f^3(a)$ *satisfying* $d \leqslant a < b < c$ *or* $d \geqslant a > b > c$.

Proof. Let $a \in I$ such that the points $b = f(a), c = f^2(a)$ and $d = f^3(a)$ satisfy $d \leqslant a < b < c$ or $d \geqslant a > b > c$. If $I_1 = [a, b]$ and $I_2 = [b, c]$ then $I_2 \subset f(I_1), I_2 \subset f(I_2)$ and $I_1 \subset f(I_2)$. Therefore $I_1 \subset f^3(I_1)$. A simple consequence of the Intermediate Value Theorem gives a fixed point, x for f^3 in I_1. Since $I_1 \subset f(I_2)$, there exists $y \in I_2$ such that $x = f(y)$. Therefore x is not a fixed point for f. □

This equivalence proves the statement in the title of the famous paper by Li and York, *Period Three Implies Chaos.*

Sarkovskii's Theorem gives a complete accounting of which periods imply which other period, where this information is not provided by Li and Yorke's theorem. This information can be explained by an ordering of natural numbers, called *Sarkovskii ordering* [15];

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots$$
$$\rhd 2^3 \cdot 3 \rhd 2^3 \cdot 5 \rhd \cdots\cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$$

The ordering above starts with the odd numbers in increasing order, then 2 times the odds, 4 times the odds, 8 times the odds, etc., and at the end put the powers of two in decreasing order. Every natural number appears exactly once somewhere on this list. Sarkovskii's Theorem is then given as follows;

Theorem 2.4 (Sarkovskii's Theorem) *[15] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map and has a periodic point of prime period k. If $k \rhd l$ in the Sarkovskii ordering, then f also has periodic point of period l.*

Here is an example of system which is chaotic in the sense of Li and Yorke.

Example 2.5 *Full-2-shift, Σ_2 is Li-Yorke chaotic.*

Proof. Let us define an equivalence relation on$\Sigma\ _2$ as $\mathbf{r} \sim \mathbf{s}$ if and only if the cardinality of the set $\{i | r_i \neq s_i\}$ is finite. Since there are only countably many finite subset of $\mathbb{N}$, $[\mathbf{r}]$ has countably many points. SinceΣ_2 is uncountable, there are uncountably many equivalence classes $[\mathbf{r}]$ inΣ_2. For every $[\mathbf{r}]$ pick only one $\mathbf{s}_r \in [\mathbf{r}]$ and let A be the collection of all such $\mathbf{s}_r$. So A is an uncountable set with the property that for evert $\mathbf{u,v} \in A$, the set $\{i | u_i \neq v_i\}$ is infinite. Now define $B \subset \Sigma_2$ as $B = \{\mathbf{x}_s = s_0 s_1 0 s_2 00 s_3 000 s_4 \cdots |$ for every $\mathbf{s} \in A\}$. Since for every $\mathbf{x}_r$ and $\mathbf{x}_s \in B$ there are infinitely many i such that $r_i \neq s_i$ so it implies that $\limsup_{i\to\infty} d(\sigma^i(\mathbf{x}_r), \sigma^i(\mathbf{x}_s)) > 0$ and $\liminf_{i\to\infty} d(\sigma^i(\mathbf{x}_r), \sigma^i(\mathbf{x}_s)) = 0$. So σ onΣ_2 is Li-Yorke chaotic. □

2.2 Topological chaos.

A system with a strong property i.e. positive entropy is usually considered chaotic by mathematicians. Due to [10], without using the word chaos, Furstenberg in [17] chose to call "deterministic" all compact dynamical systems with zero topological entropy. Later Glasner and Weiss [18], in a discussion of Devaney's definition of chaos, proposed positive entropy as the essential criterion of chaos. Therefore positive entropy has been accepted as one kind of chaos definition, called *topological chaos*.

Definition 2.6 *[10] Let $f : X \to X$ be a map on a metric space X. The dynamical system (X, f) will be said to be topologically chaotic if the entropy is positive.*

On general spaces, topological chaos can be used to validate other notion of chaos, Li-Yorke chaos.

Theorem 2.7 *[11] Any topologically chaotic dynamical system is Li-Yorke chaotic.*

These are popular common results on the unit interval, I and the unit circle, S^1 about positive entropy or topological chaos.

Theorem 2.8 *[12] A dynamical system (X, f) where X is either the unit interval I or the unit circle S^1 has positive entropy whenever a periodic point of period $q2^p$ (q is odd number and p is any nonnegative integer) exists.*

2.3 Devaney chaos.

We also put our emphasize on other notion of chaos, Devaney chaos. There are three important conditions to be chaotic in the sense of Devaney. Two of them are notions called *sensitive dependence on initial conditions* and *topologically transitive*, defined as follows:

Definition 2.9 *[15] Let $f : X \to X$ be a function on a metric space X with metric d. f has sensitive dependence on initial conditions (SDIC) if there exists $\delta > 0$ such that, for any $x \in X$ and neighborhood N of x, there exists $y \in N$ and $n \geqslant 0$ such that $d(f^n(x), f^n(y)) > \delta$.*

Definition 2.10 *[15] Let $f : X \to X$ be a function on a metric space X. f is said to be topologically transitive (or simply transitive) if for any non-empty open subsets $U, V \subset X$, there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.*

When dealing with a transitive map, it is sometimes convenient to use the equivalent property, existence of dense orbit.

Theorem 2.11 *[37] Let $f : X \to X$ be a continuous map on a compact metric space X with no isolated point. f is transitive if and only if there exists a point of dense orbit. In this case the set of points of dense orbit is a dense G_δ-set. Moreover, if the orbit of x is dense, then $\omega(x, f) = X$ and the orbit of $f^n(x)$ is dense for all integers $n \geqslant 0$.*

Some authors prefer to use the existence of dense orbit instead of transitivity to describe Devaney chaos, see for example [41]. We then define Devaney chaos as in [15] as follows:

Definition 2.12 *[15] Let $f : X \to X$ be a map on a metric space X. The dynamical system (X, f) is said to be chaotic (in the sense of Devaney) if*

1. *f has sensitive dependence on initial conditions.*

2. *f is topologically transitive.*

3. *Periodic points are dense in X.*

However in 1992, Banks, Brooks, Cairns et al. [5] showed that the first condition on Devaney's definition of chaos is redundant by proving the following;

Theorem 2.13 *[5] Let $f : X \to X$ be a continuous function on an infinite metric space X. If f is transitive and has dense periodic points then f has sensitive dependence on initial conditions.*

Proof. Let X be a metric space with metric d and f be a transitive function with dense periodic points on X, and $x \in X$. We firstly claim that there exist $\delta_0 > 0$ such that for every $x \in X$ there exist a periodic point $q \in X$ where $dist(O^+(q), x) \geqslant \frac{\delta_0}{2}$. To prove this let q_1, q_2 be any two periodic points in X and $\delta_0 = dist(O^+(q_1), O^+(q_2))$. Let $q_1' \in O^+(q_1)$ and $q_2' \in O^+(q_2)$ such that $\delta_0 = d(q_1', q_2')$. For any $x \in X$, either $d(q_1', x) \geqslant d(q_2', x)$ or $d(q_1', x) \leqslant d(q_2', x)$. By the Triangle Inequality, $d(q_1', q_2') \leqslant d(q_1', x) + d(q_2', x)$. So, in the first case, one may have $2d(q_1', x) \geqslant d(q_1', q_2')$. Hence, $dist(O^+(q_1), x) \geqslant \frac{\delta_0}{2}$. In the other case, one may have, $dist(O^+(q_2), x) \geqslant \frac{\delta_0}{2}$.
Now is the time to show that f has sensitive dependence on initial conditions. As defined above, let $\delta = \frac{\delta_0}{8}$ and $N(x)$ be any open neighborhood of x. Let $B_\delta(x)$ be an open ball of x with radius δ, so $U = N(x) \cap B_\delta(x)$ is an open set. By density of periodic points, let p be a periodic point in U with prime period n. By our claim, there exists a periodic point, say q such that $dist(O^+(q), x) \geqslant \frac{\delta_0}{2} = 4\delta$. $q \neq p$ since $p \in B_\delta(x)$ and therefore $dist(O^+(p), x) < \delta$ .So let $V = \bigcap_{i=0}^{n} f^{-i}(B_\delta(f^i(q)))$. Since f is transitive, there exists $y \in U$ and $k > 0$ such that $f^k(y) \in V$. Now let $j \leqslant \frac{k}{n} + 1$ (i.e. $nj - k \leqslant n$). Hence, $f^{nj}(y) = f^{nj-k}(f^k(y)) \in f^{nj-k}(V)$. Let $a \in f^{nj-k}(V)$ and $b \in V$ such that

$a = f^{nj-k}(b)$. Since $b \in \bigcap_{i=0}^{n} f^{-i}(B_\delta(f^i(q)))$, then $a = f^{nj-k}(b) \in B_\delta(f^{nj-k}(q))$. Therefore, $f^{nj-k}(V) \subseteq B_\delta(f^{nj-k}(q))$. By the Triangle Inequality, $d(x, f^{nj-k}(q)) \leqslant d(x,p) + d(p, f^{nj}(y)) + d(f^{nj}(y), f^{nj-k}(q))$. Since, $p \in B_\delta(x)$, then $d(x,p) \leqslant \delta$. Also since, $f^{nj}(y) \in f^{nj-k}(V) \subseteq B_\delta(f^{nj-k}(q))$, so $d(f^{nj}(y), f^{nj-k}(q)) \leqslant \delta$. By our assumption, we have $d(x, f^{nj-k}(q)) \geqslant 4\delta$. All of these imply $d(p, f^{nj}(y)) \geqslant 2\delta$. Again, by Triangle Inequality, $d(p, f^{nj}(x)) + d(f^{nj}(x), f^{nj}(y)) \geqslant d(p, f^{nj}(y) \geqslant 2\delta$. If it is the case that $d(p, f^{nj}(x)) \leqslant d(f^{nj}(x), f^{nj}(y))$ then $d(p, f^{nj}(x)) \leqslant \delta$. Or otherwise, it should be $d(f^{nj}(x), f^{nj}(y)) > \delta$. So, there is a point $z \in N(x)$ (i.e. either y or p) such that $d(f^{nj}(x), f^{nj}(z)) > \delta$. □

Sensitivity is the only redundancy for a general function. D. Assaf and S. Gadbois [3] in 1992 proved that there is no other triviality in the Devaney definition of chaos in general. Surprisingly, two years after that it was shown that dense periodic points are also redundant in the definition of Devaney's chaos for maps on the interval. Therefore in the next section, we will see that for a continuous map from an interval (not necessarily finite) into itself, transitivity is equivalent to chaos in the sense of Devaney.

The surprising equivalence on the interval is because transitivity implies dense periodic points, but the converse is not necessarily true since M. Vellekoop and R. Berglund [44] found an example of a map which has sensitive dependence on initial conditions and whose periodic points are dense but the map is not transitive. We will discuss some examples to explain this in the next chapter. Note that this result cannot be generalized for higher dimensions or the unit circle because the proof of this result uses the ordering in $\mathbb{R}$ in an essential way. Furthermore, an irrational rotation on a circle is a transitive map but does not have any periodic points, which is a counterexample (a proof for this example can be found in the next section).

Devaney chaos can be considered as the strongest notion of chaos since it is stronger then Li-Yorke chaos in general spaces and in some specific spaces, it is stronger than topological chaos (will be explained in the next sections).

Theorem 2.14 *[26] Let $f : X \to X$ be continuous map on a compact metric space X. If the system is Devaney chaotic then the system is Li-Yorke chaotic.*

In particular Huang and Ye [26] showed that transitivity and the existence of a periodic point is sufficient for Li-Yorke chaos. However we will show in the next chapter that these two conditions can be weakened to implies Li-Yorke chaotic on the interval. Lastly, we will give some examples of a map which is chaotic in the sense of Devaney.

Example 2.15 *If $\mu > 2$, then the Tent Map, T_μ on Λ is Devaney chaotic.*

Proof. By Corollary 1.35 on Section 1.4.1, conjugacy has shown that the periodic points of the Tent Map are dense and the existence of a dense orbit. Theorem 2.11 shows that the Tent Map is a transitive map. So, the Tent Map is Devaney chaotic. □

Instead of conjugacy, semi-conjugacy is sufficient to show chaotic behaviors of some dynamical systems.

Example 2.16 *The Doubling Map, F on $[0, 1)$ is Devaney chaotic.*

Proof. By Corollary 1.39 on Section 1.4.2, semi-conjugacy has shown that the periodic points of the Tent Map are dense and the existence of a dense orbit. Theorem 2.11 shows that the Doubling Map is a transitive map. So the Doubling Map is Devaney chaotic. □

Example 2.17 *The shift map, σ on full-n-shift space, Σ_n is Devaney chaotic.*

Proof. By Proposition 1.20, σ has dense periodic points and is transitive onΣ_n. □

In the next subsections we will discuss Devaney chaos on three specific spaces i.e. the interval, unit circle and shift of finite type.

2.3.1 Devaney Chaos on the Interval

An interval is a connected subset of $\mathbb{R}$ of the form either $[a,b] = \{x \in \mathbb{R} | a \leqslant x \leqslant b\}$, or $(a,b) = \{x \in \mathbb{R} | a < x < b\}$ or $[a,b)$ or $(a,b]$ (defined similarly) for any $a,b \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $a < b$. The *unit interval* is the closed interval $[0,1]$. An interval is *non-degenerate* if its length is positive i.e. it is neither empty nor reduced to a single point.

Representative of the Interval Maps.

We say that a function $f : I \to I$ is an *interval map* if I is a non-degenerate interval and f is a continuous surjective map. The interval map f on I is a *piecewise linear* if the interval I can be divided into a finite number of subintervals on each of which f is linear and monotone (either non decreasing or non increasing). When dealing with an interval map one may assume that the interval is the unit interval. Indeed, for every interval map $f : [a,b] \to [a,b]$, there exists $g : [0,1] \to [0,1]$ such that f and g share the same dynamics. This can be shown by using topological conjugacy as follows;

Proposition 2.18 *For every interval map $f : [a,b] \to [a,b]$, there exists $g : [0,1] \to [0,1]$ such that f and g are conjugate.*

Proof. Let $g : [0,1] \to [0,1]$ defined by $g(x) = \frac{f(a+b-ax)-a}{b-a}$ and a linear $\varphi : [a,b] \to [0,1]$ defined by $\varphi(x) = a + (b-a)x$. g is well defined since for all $x \in [0,1], a + (b-a)x \in [a,b]$ and $\frac{f(a+b-ax)-a}{b-a} \in [0,1]$. φ is a conjugacy between f and g since $g = \varphi^{-1} \circ f \circ \varphi$. □

Hence when we talk about an interval map, we will consider the unit interval as the domain of the map, in general. Therefore, we will always denote the closed unit interval $[0,1]$ as I. When talking about topological notions in an interval, we always refer to the induced topology on the ambient interval. For example $[0, \frac{1}{2})$ is an open interval in [0,1]. The other thing to note about an interval map is that since the map is a continuous real-valued function on a compact set, then it will always take maximum and minimum values on any subinterval of I.

Over the past few decades, much research has been done on the three properties in Devaney chaos definition on the interval. Vellekoop and Berglund [44] proved that in spite of sensitive dependence on initial condition, dense periodic points is the other redundancy in Devaney chaos definition on the interval. However [7] showed the important role that dense periodic points can play on the interval to behave somehow chaotically. On the other hand, however weak sensitive dependence on initial conditions is rather intuitively a chaotic property. The idea of this property was originally discussed in [22] on the interval.

Devaney Chaotic Interval Maps

A special characteristic of chaotic behavior on the interval is that we do not need dense periodic points to show chaos in the sense of Devaney, because transitivity implies this condition. This implication does not work in general, even on the unit circle. To prove this, we need the following.

Lemma 2.19 *Let $g : I \to \mathbb{R}$ be a continuous map. If there exist $a, b \in I$ such that $a < b$ and $g(a) < 0 < g(b)$ then there exist $c \in (a, b)$ such that $g(c) = 0$.*

Proof. Let $h(x) = g(x) - x$ and then use the Intermediate Value Theorem. □

Lemma 2.20 *Let $g : I \to I$ be a continuous interval map, $J \subset I$ be an interval which contains no periodic points of g and $z \in J$ such that $g(z) \in J$. If $z < g(z)$ (or $z > g(z)$) then $g(z) < g^{k+1}(z)$ (or $g(z) > g^{k+1}(z)$) for all $k \geqslant 1$.*

Proof. Let k be the least such that $g^{k+1}(z) < g(z)$. Let $h(x) = g^k(x) - x$. Since $z < g^k(z)$ then $h(z) > 0$. Since $g^{k+1}(z) < g(z)$, $h(g(z)) > 0$. By Lemma 2.19 there exist $c \in [z, g(z)]$ such that $g^k(c) = c$, a contradiction. The other case can be shown by using the same argument. □

Lemma 2.21 *[44] Let $f : I \to I$ be a continuous interval map. If $J \subset I$ is an interval which contains no periodic points of f and z, $f^m(z)$ and $f^n(z) \in J$ with $0 < m < n$, then either $z < f^m(z) < f^n(z)$ or $z > f^m(z) > f^n(z)$.*

Proof. We will prove this by using contradiction. Let J be an interval map with no periodic points and $z \in J$ such that $f^m(z)$ and $f^n(z)$ are also in J for some $0 < m < n$. It is either $z < f^m(z)$ or $z > f^m(z)$. Let firstly consider $z < f^m(z)$ (called assumption (1)). By using contradiction, we assume that $f^n(z) < f^m(z)$ (called assumption (2)).
Let us define $g(x) = f^m(x)$. By assumption (1), $z < g(z)$. By Lemma 2.20 we will have $z < g(z) < g^{k+1}(z)$ for every $k \geqslant 1$. In particular $k = n - m > 0$ gives $z < f^{(n-m)m}(z)$ (called equation (1)).
Now let $g(x) = f^{n-m}(x)$. From assumption (2), $f^{n-m}f^m(z) < f^m(z)$ i.e. $g(f^m(z)) < f^m(z)$. By Lemma 2.20, we will have $g^{k+1}(f^m(z)) < gf^m(z) < f^m(z)$ i.e. $g^k(f^m(z)) < f^m(z)$ for every k. In particular, $k = n - m > 0$ giving $f^{(n-m)m}f^m(z) < f^m(z)$ (called equation (2)).
Now we are going to show our main aim. Let $h(x) = f^{(n-m)m}(x) - x$. So, from equation (1), $h(z) = f^{(n-m)m}(z) - z > 0$ and from equation (2), $h(f^m(z)) = f^{(n-m)m}f^m(z) - f^m(z) < 0$. Lastly, by Lemma 2.19, there exists c in $[z, f^m(z)]$ such that c is a $(n - m)m-$periodic point in J which is a contradiction.
The case for $z > f^m(z)$ can be shown analogously. □

Theorem 2.22 *[44] Let $f : I \to I$ be a continuous interval map. If f is transitive, then the periodic points of f are dense.*

Proof. By using contradiction, assume that f is a continuous interval map on I which is transitive but does not have dense periodic point. Then there exists an open interval, say J such that J does not have any periodic point. Take any $x \in J$ such that x is not an endpoint of J. Let $N(x) \subset J$ be a neighborhood of x and E be an open interval such that $E \subset J \setminus N(x)$. Since f is transitive, there is an $a \in f^m(N(x)) \cap E$ for some integer m. Let us write a as $a = f^m(y)$ for some $y \in N(x) \subset J$. Since J does not contain any periodic point, so $y \neq f^m(y)$. Therefore, by continuity we can find a neighborhood of y in J say $N(y)$, such that $f^m(N(y)) \cap N(y) = \emptyset$. Again, by transitivity there is $b \in f^k(f^m(N(y))) \cap N(y)$ for some integer k. Let us write b as $b = f^k f^m(z) \in N(y)$ for some $z \in N(y)$. Since $f^m(z) \in f^m(N(y))$ and $N(y)$ and $f^m(N(y))$ are disjoint, $f^m(z) \notin N(y)$. By Lemma 2.21 this is a contradiction, because $0 < m < k + m$ but $f^k f^m(z), z \in N(y)$ and $f^m(z) \notin N(y)$. □

The next examples will show that transitivity is the strongest condition in the definition of Devaney's chaos on the interval, since the other two conditions (dense periodic points and sensitive dependence on initial conditions) do not imply any other condition. The first example is a function, f_1 with dense periodic points but neither transitive nor sensitive dependence on initial conditions. The second example is a function, f_2 with sensitive dependence on initial conditions but again is neither transitive nor has a dense set of periodic points. The last example is a function f_3 with dense periodic points and sensitive dependence on initial conditions but not transitive.

Example 2.23 *Let f_1 be the identity map on the unit interval I, that is $f(x) = x$ for every $x \in I$. Therefore f_1 has dense periodic points but is neither transitive nor has sensitive dependence on initial conditions.*

Proof. For every x in I, x is a fixed point for f_1. So, f_1 has dense periodic points. f_1 is not transitive since every interval is invariant under f_1. Also, for every $\delta > 0$, there is a neighborhood of x, for any $x \in I$, such as an open ball of x with radius $\delta/3$, $B(x, \delta/3)$ such that for every y in the ball, $|f_1^n(x) - f_1^n(y)| = |x - y| < \delta/3$ for all n. So, f_1 does not has sensitive dependence on initial conditions. □

Example 2.24 *[44] Let* $f_2 : \left[0, \frac{3}{4}\right] \to \left[0, \frac{3}{4}\right]$ *be a piecewise linear map defined as follows;*

$$f_2(x) := \begin{cases} \frac{3}{2}x & \text{if } 0 \leqslant x < \frac{1}{2} \\ \frac{3}{2}(1-x) & \text{if } \frac{1}{2} \leqslant x < \frac{3}{4} \end{cases}$$

f_2 has sensitive dependence on initial conditions but is neither transitive nor have dense set of periodic points.

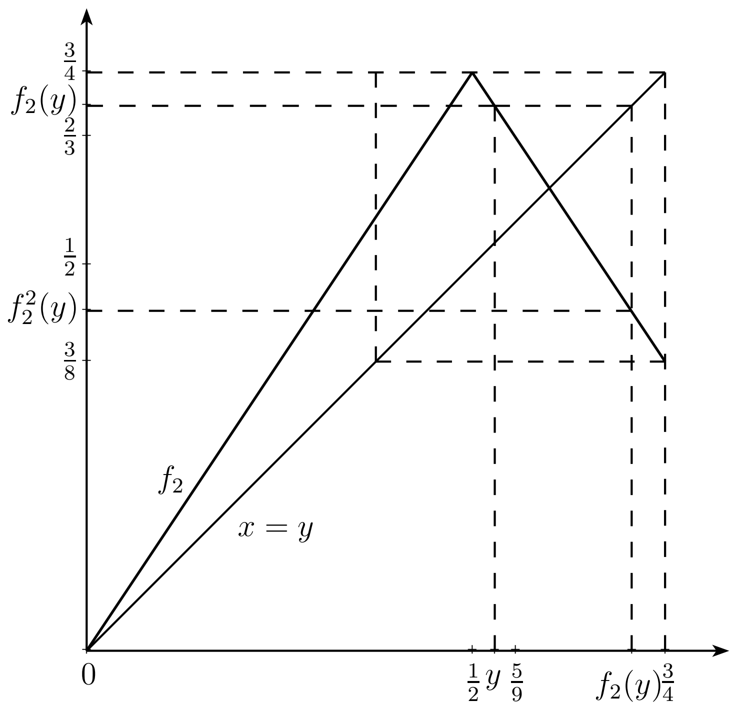

Figure 2.1: The graph of f_2 which has SDIC but neither transitive nor has dense periodic points

Proof. To show that f_2 has sensitive dependence on initial conditions, we claim that for any interval $A = (x, y)$ with length $|A| < \frac{1}{9}$, if $\frac{1}{2} \notin A$ then $|f_2(A)| > \frac{9}{8}|A|$, otherwise $|f_2^2(A)| > \frac{9}{8}|A|$. The first case is obvious since f_2 has slope $\frac{3}{2}$ everywhere except at $\frac{1}{2}$. To show the other case, let $A = (x, y)$ be an interval that contains $\frac{1}{2}$ where $x = \frac{1}{2} - a$, $y = \frac{1}{2} + b$ and $a \leqslant b \leqslant \frac{1}{18}$. Therefore $y \in \left(\frac{1}{2}, \frac{5}{9}\right)$ and $|A| < \frac{1}{9}$. From the graph of f_2 in

Figure 2.1, $f_2\left(\frac{1}{2}, y\right) = \left(f_2(y), \frac{3}{4}\right)$ and $f_2^2\left(\frac{1}{2}, y\right) = \left(\frac{3}{8}, f_2^2(y)\right)$ where $f_2(y) = \frac{3}{4} - \frac{3b}{2} \in \left(\frac{2}{3}, \frac{3}{4}\right)$ and $f_2^2(y) = \frac{3}{8} + \frac{9b}{4} \in \left(\frac{3}{8}, \frac{1}{2}\right)$. Therefore $|f_2^2(x, y)| = |f_2^2(\frac{1}{2}, y)| = \frac{9b}{4}$. Since $|A| = a + b < 2b$, then $|f_2^2(A)| > \frac{9}{8}|A|$. By the claim, for any $x \in [0, \frac{3}{4}]$ and any interval $A = (x - \varepsilon, x + \varepsilon)$ where $\varepsilon < \frac{1}{18}$, there exist m and n such that $|f_2^m(A)| > \left(\frac{9}{8}\right)^n |A| > \frac{1}{9}$. Therefore, with $\delta = \frac{1}{18}$, f_2 has sensitive dependence on initial conditions. f_2 is not transitive since the subinterval $[\frac{3}{8}, \frac{3}{4}]$ is invariant under f_2.

For every $z \in \left[0, \frac{3}{8}\right]$, there is k such that $f_2^i(z) > f_2^{i-1}(z)$ for any $i \leqslant k$ and $f_2^j(z) > \frac{3}{8}$ for any $j > k$. Therefore no point in $[0, \frac{3}{8}]$ will return to this interval again, hence f_2 does not have dense periodic points. □

Example 2.25 *Let f_3 be the Double Tent Map with $\mu = 2$, defined on $[-1, 1]$ as follows;*

$$f_3(x) := \begin{cases} -2x - 2 & \textit{if } -1 \leqslant x < \frac{-1}{2} \\ 2x & \textit{if } \frac{-1}{2} \leqslant x < \frac{1}{2} \\ 2 - 2x & \textit{if } \frac{1}{2} \leqslant x < 1 \end{cases}$$

f_3 is not transitive even though it has dense periodic points and has sensitive dependence on initial conditions.

Proof. f_3 is not transitive since the intervals $[-1, 0]$ and $[0, 1]$ are both invariant under f_3, as shown in Figure 2.2. To show that f_3 has dense periodic points and has sensitive dependence on initial conditions, we will look at $f_3|_{[-1,0]}$ and $f_3|_{[0,1]}$. Since $f_3|_{[-1,0]} = T_2$ (the Tent Map with $\mu = 2$), then $\Lambda = [0, 1]$ (Λ as defined in Section 1.3.1). Hence by using symbolic dynamics and conjugacy we are able to show that T_2 has dense periodic points and transitive on $[0, 1]$. By Theorem 2.13, T_2 has sensitive dependence on initial condition. The properties of $f_3|_{[-1,0]}$ are also follows from the properties of T_2 since for every $x \in [-1, 0]$, $f_3(x) = -T_2(-x)$. □

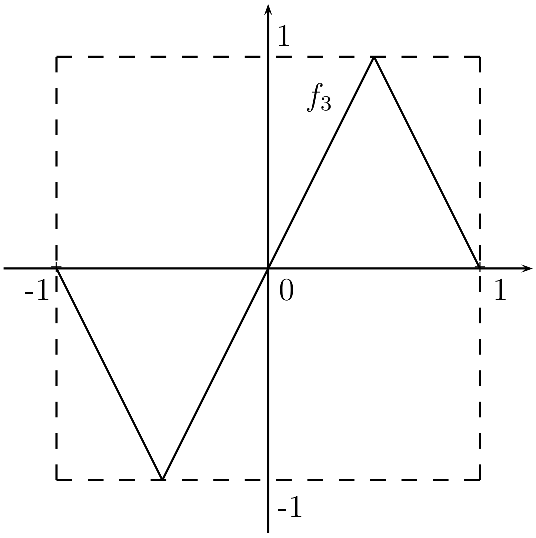

Figure 2.2: The graph of the Double Tent Map f_3 which has SDIC and a dense set of periodic points but not transitive

It is well known that Devaney chaos is stronger then Li-Yorke chaos in general space (Theorem 2.14). On the interval Devaney chaos is also stronger then topological chaos.

Theorem 2.26 *[31] Let f be a continuous interval map. The entropy of f is positive iff there exists a closed invariant subset $D \subset I$ such that $f|_D$ is Devaney chaotic.*

2.3.2 Devaney Chaos on the Circle

The unit circle can be represented in many different ways. For this work, we let S^1 denote the unit circle in the plane, i.e. $S^1 = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : x_1^2 + x_2^2 = 1\}$. By using basic trigonometry, for every $x \in S^1$, there exist a unique $\theta \in [0, 2\pi)$ such that $x = (x_1, x_2) = (\cos(\theta + 2\pi k), \sin(\theta + 2\pi k))$ for all natural numbers k. Therefore, every element of the unit circle x may sometimes be referred by its angle $\theta \in [0, 2\pi)$ measured in radians- in the standard manner. For every $x, y \in S^1$, the metric d of x and y will be the length of the minor arc between x and y. So, if $x = (\cos(\theta_1 + 2\pi k), \sin(\theta_2 + 2\pi k))$ and $y = (\cos(\theta_2 + 2\pi k), \sin(\theta_2 + 2\pi k))$ and $\theta_1 < \theta_2$ then $d(x, y) = \min\{\theta_2 - \theta_1, 2\pi - \theta_2 + \theta_1\}$ in radians. Therefore we denote $(x, y) = (y, x)$ as the collection of all elements in S^1 in the minor arc between x and y and call it subarc of S^1. The subarc (x, y) becomes a basic open set of S^1.

Representation of the Circle Maps.

Analogous to the interval map, we then call a function $f : S^1 \to S^1$ a *circle map.* When dealing with a circle map, we can think the unit circle S^1 as the interval $[0, 2\pi)$ with 0 and 2π identified. Therefore a continuous map f on S^1 will be associated with an interval map f' on $[0, 2\pi)$ which satisfies some properties.

Definition 2.27 *Let f be a circle map on S^1. Corresponding to f, we define an interval map f' on $[0, 2\pi) \subset \mathbb{R}$ by $f'(\theta) = f(\theta)$.*

We give an example of a circle map and its corresponding interval map.

Definition 2.28 (The Doubling Map) *The Doubling Map on the circle, $g : S^1 \to S^1$ is defined by the expression $g(\theta) = 2\theta$.*

Example 2.29 *Let $g : S^1 \to S^1$ be the Doubling Map. The interval map $g' : [0, 2\pi) \to [0, 2\pi)$ which can be associated to g is a piecewise linear defined by $g'(x) = 2x \pmod{2\pi}$. The graph of g' is given in the Figure 2.3.*

Referring to the graphs of g' for the Doubling Map g in Figure 2.3, g' is discontinuous at π and $g'(\pi) = g(\pi) = 0$. In particular, f' is discontinuous at $\theta \in [0, 2\pi)$ whenever $f'(\theta) = f(\theta) = 0$ and it satisfies one of the there cases;

1. $\lim_{x \to \theta^-} f'(x) = 0$ and $\lim_{x \to \theta^+} f'(x) = 2\pi$

2. $\lim_{x \to \theta^-} f'(x) = 2\pi$ and $\lim_{x \to \theta^+} f'(x) = 0$

3. $\lim_{x \to \theta^-} f'(x) = \lim_{x \to \theta^-} f'(x) = 2\pi$

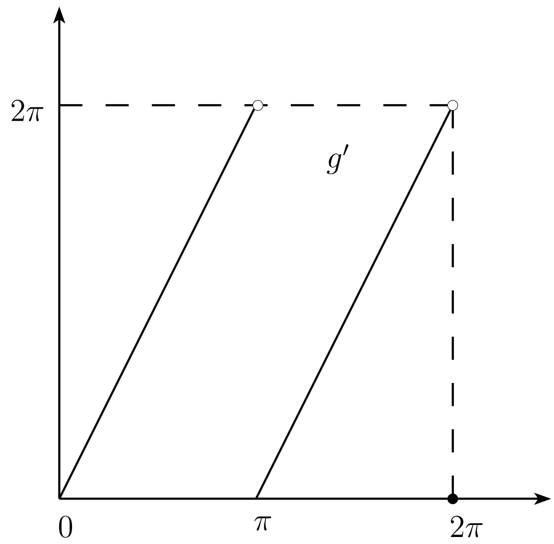

Figure 2.3: The graph of g' where g is the Doubling Map on the circle

Therefore either $\lim_{x\to\theta^-} f'(x) \neq \lim_{x\to\theta^+} f'(x)$ or $\lim_{x\to\theta} f'(x) \neq f(x)$ provides the discontinuity at x. So for any continuous circle map f, the corresponding interval map f' is continuous at θ if $f'(\theta) \neq 0$. Therefore there are subintervals of $[0, 2\pi)$ where f' is continuous in the real sense whenever we restricted to the subinterval.

Lemma 2.30 *Let f be a continuous circle map and f' be the interval map corresponding to f. If $(\theta_1, \theta_2) \subsetneq [0, 2\pi)$ is strongly invariant under f' and $0 \notin (\theta_1, \theta_2)$ then $f'|_{(\theta_1,\theta_2)}$ is continuous.*

Proof. Since $f'(\theta_1, \theta_2) = (\theta_1, \theta_2)$ and $0 \notin (\theta_1, \theta_2)$, then for every $\theta \in (\theta_1, \theta_2), f(\theta) \neq 0$. Therefore f' is continuous at θ. □

Finally we will give some remarks on the relationship between a circle map f and its corresponding interval map f'.

Remark 2.31 *The following are facts about f and f' for a continuous circle map f;*

1. *If f has dense periodic points then so does f'.*

2. *For any* $\theta \in [0, 2\pi)$, θ *is a periodic point of period* n *under* f *iff* θ *is a periodic point of period* n *under* f'.

3. *If* f' *is transitive, then so is* f.

4. *For any subset* $(\theta_1, \theta_2) \subset (0, 2\pi)$, (θ_1, θ_2) *is invariant under* f *iff* (θ_1, θ_2) *is invariant under* f'.

This representation of the interval maps are useful for our further discussion on the dynamics of the circle maps in the Chapter 4 later.

Devaney chaotic circle maps.

In this section, we discuss some of the basic ideas involved in studying the dynamics of continuous circle maps. We start by giving a comparison between dynamics of the interval maps as given in the previous section and dynamics of the circle maps, and then end up this section by giving a new result on a sufficient condition for the circle maps to be chaotic in the sense of Devaney.

Theorem 2.22 i.e transitivity implies dense periodic point on the interval cannot be totally carried over to the circle. The theorem relies heavily on the following feature of the unit interval: given any two points there is a unique minimal connected set containing them [41]. Therefore, the main obstruction on the circle is that there is more than one minimal connected set containing certain pair of points. We give a counterexample as follows:

Example 2.32 *[15] An irrational rotation* $L_\lambda : S^1 \to S^1$ *for an irrational* $\lambda \in \mathbb{R}$ *are translation of the circle defined by* $L_\lambda(\theta) = \theta + 2\pi\lambda$. L_λ *is transitive but the set of periodic points is not dense.*

Proof. We firstly claim that for any $\theta \in S^1$ and $m, n \in \mathbb{N}$ $(m \neq n)$, $L_\lambda^m(\theta) \neq L_\lambda^n(\theta)$. If $L_\lambda^m(\theta) = L_\lambda^n(\theta)$, then $(n-m)\lambda \in \mathbb{Z}$ and since λ is irrational, $n = m$. This claim proves that S^1 has no periodic points.

Let $\theta \in S^1$. By the previous proof, the point on the orbit of θ are distinct. By the compactness of S^1, any infinite set of points on the circle must have a limit point. Thus given any $\varepsilon > 0$, there exist integers n and m such that $d(L_\lambda^m(\theta) - L_\lambda^n(\theta)) < \varepsilon$. By letting $k = n - m$, $d(L_\lambda^k(\theta) - \theta) < \varepsilon$. Since L_λ preserves length in S^1, L_λ^k maps the arc connecting θ to $L_\lambda^k(\theta)$ to the arc connecting L_λ^k and $L_\lambda^{2k}(\theta)$ which has length less then ε. In particulars it follows that the points $\theta, L_\lambda^k, L_\lambda^{2k}, \cdots$ partition S^1 into arcs of lengths less than ε. Since ε is arbitrary, we can choose ε to be small enough so the orbit of θ is dense in S^1. □

However some remarkable results on the interval are still held in the circle, as follows;

Theorem 2.33 *[33] [12] If f be a continuous map on the unit circle, then the following statements are equivalent:*

1. *the entropy of f is positive,*
2. *there exists a closed invariant subinterval $D \subset S^1$ such that $f|_D$ is Devaney chaotic,*
3. *f has a periodic point of period $q2^p$ for an odd q and integer p.*

Even though transitivity implies Devaney chaos on the interval, but not on the circle. However Silverman [41] proved that transitivity almost implies Devaney chaos, as follows:

Theorem 2.34 *[41] Let $f : S^1 \to S^1$ is a transitive continuous circle map. If f is not one-to-one, then f is Devaney chaotic.*

Using this result, we will show that there is a stronger property than transitivity that implies transitive in the unit circle. The property is called *locally everywhere onto*. Some authors use *exact* term instead of locally everywhere onto, defined as follows;

Definition 2.35 *[21] Let $f : X \to X$ be a continuous map on a compact metric space X. The function f is said to be locally everywhere onto or simply* l.e.o *if for every open subset $U \subseteq X$ there exists a positive integer n such that $f^n(U) = X$.*

Locally everywhere onto is a sufficient condition for Devaney chaotic of both an interval map and a circle map.

Theorem 2.36 *Let $f : S^1 \to S^1$ be a continuous map on the circle. If f is l.e.o. then f is Devaney chaotic.*

Proof. We want to show that an l.e.o circle map is not a one-to-one map. Let U and V be two disjoint open sets. Since f is l.e.o, there exists k such that $f^k(U) = S^1$. Therefore for every $\theta_1 \in V$ such that $f^k(\theta_1) \in f^k(U)$ and therefore there exists $\theta_2 \in U$ such that $f^k(\theta_2) = f^k(\theta_1)$. Hence f is not one-to-one and by Theorem 2.34, f is Devaney chaotic.□

Example 2.37 *The Doubling Map g on the circle can be shown chaotic by using the Doubling Map F on the unit interval since $g' = F$, as explained in Example 2.29. Alternatively, without relates to any interval map, the simplest method is using Theorem 2.36 i.e. by showing that the map is l.e.o.. For any $\theta_1, \theta_2 \in S^1$ where $0 \leqslant \theta_1 < \theta_2 < 2\pi$, choose an integer k for which $kd(\theta_1, \theta_2) \geqslant 2\pi$, then $g^k(\theta_1, \theta_2) = S^1$.*

2.3.3 Devaney Chaos on Shifts of Finite Type

The most widely studied shift spaces are called *shift of finite type*. Shifts of finite type are useful in modeling dynamical systems and applications to data recording [32]. A shift of finite type is a subset of full$-k-$shift,Σ_k. So we will recall some definitions given in Section 1.2 about full shifts. For an alphabet $\mathcal{A} = \{0, 1, \cdots, k-1\}$ with k symbols, full$-k-$shiftΣ_k is the collection of all infinite sequences of the symbols in $\mathcal{A}$. An l-block is a finite sequence over $\mathcal{A}$ with length l. On the full-k-shift, we define a continuous map called the shift map $\sigma : \Sigma_k \to \Sigma_k$ that deletes the first entry of the element in

Σ_k to produce the image of the element under σ. There are two kinds of subset of the full$-k-$shift that we will define in this section. They are shift space and shift of finite type, where shift of finite type is a subset of shift space which will be defined first.

Definition 2.38 *Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$. A shift space is a subset X of full-k-shift, Σ_k if there exists a subset $\mathcal{F}$ of $\bigcup_{l\in\mathbb{N}} B_l(\mathcal{A})$ such that every block in $\mathcal{F}$ does not occur in any element of X. We called every block in $\mathcal{F}$ as a forbidden block for X.*

Shift spaces are invariant under the shift map σ and this common feature is called *shift invariance*. This amounts to the observation that the forbidden blocks cannot occur in any coordinates of $\mathbf{x} \in X$. Hence if $\mathbf{x} \in X$ does not contain any block in $\mathcal{F}$, then neither does $\sigma(\mathbf{x})$. For a shift space X, an $l-$block, w is allowed by X if there exists $\mathbf{x} \in X$ such that w occurs in $\mathbf{x}$. So then by $B_l(X)$ we denote the set of all $l-$blocks allowed by X. We also denote by $B(X)$ the set of all blocks allowed for X. The topology on $X \subset \Sigma_k$ is a subspace topology induced from topology on$\Sigma_{\ k}$ as described in Section 1.2. Hence for every $X_w = \{\mathbf{x} \in X : x_0x_1\cdots x_{l-1} = w\}$ is a basis open set generates by w.

By the definition of shift space, the set of forbidden blocks might prevent some symbols in $\mathcal{A}$ from occurring in all sequences in the shift space. For example, a shift space $X \subset X_5$ with forbidden block $\mathcal{F} = \{3\}$ is a collection of all sequences in full-5-shift without symbol 3. Hence, $\mathcal{F}$ prevent 3 to occur in any sequence in the shift space X. For convenience we will say that the shift space $X \subset \Sigma_4$ so that the alphabet $\{0, 1, 2, 3\}$ does not have any isolated symbol in the shift space X or in other words every symbol must exist at least in one sequence of shift space X. We will always use this assumption for our shift spaces in this work. This assumption generalizes that every block $w \in B_l(X)$ can be extended on both sides to another block $uwv \in B_m(X)$ such that $m > l + 2$. However, given two blocks u and v in $B(X)$, it may not be possible to find a block w so that $uwv \in B(X)$. Shift spaces for which two blocks can always be joined by a third block plays a special and important role.

Definition 2.39 *A shift space X is* irreducible *if for every ordered pair of blocks $u, v \in B(X)$ there is $w \in B(X)$ such that $uwv \in B(X)$.*

In shift space, irreducibility is an equivalent notion to topological transitivity, due to Oprocha and Wilczynski [35].

Theorem 2.40 *[35] A shift space X is transitive if and only if it is irreducible.*

Shift of finite type is a shift space with a condition, the forbidden blocks are finite and have been defined as follows,

Definition 2.41 *[32] A shift of finite type is a shift space that can be described by a finite set of forbidden blocks i.e. a shift space X having the form $X_{\mathcal{F}}$ for some finite set $\mathcal{F}$ of blocks.*

There is also a notion of "memory" for a shift of finite type.

Definition 2.42 *[32] A shift of finite type is M-step (or has memory M) if it can be described by a collection of forbidden blocks all of which have length $M + 1$.*

1-step shift of finite type is the simplest shift space since it can be easily characterized by finitely many blocks of length 2. For this shift space, if 2-blocks *ab* and *bc* are allowed, then 3-block *abc* is allowed. However this implication does not hold for general shift space. A shift space where 3-block *abc* is in the set of forbidden blocks, but not the blocks *ab* and *bc*, is the example.

Proposition 2.43 *[32] If X is a shift of finite type, then there is an $M \geqslant 0$ such that X is M-step shift of finite type.*

Proof. Take $M + 1$ as the length of the longest block in $\mathcal{F}$. Let $\mathcal{F}'$ be the collection of M-blocks over $\mathcal{A}$ such that at least one block in $\mathcal{F}$ occurs. Since $\mathcal{A}$ is finite, then $\mathcal{F}'$ is finite. X is described by $\mathcal{F}$ since any sequence over $\mathcal{A}$ is not in X if any element in $\mathcal{F}'$ occurs. Therefore, X can be described by $\mathcal{F}'$, a collection of finite M-blocks. $\square$

Representation of Shifts of Finite Type

By using machinery called the *Higher Block Presentation*, we will see that every shift of finite type shares some common features with a 1-step shift of finite type, which gives every shift of finite type a simple representation. Higher block presentation is a process to move a shift space to another shift space by replacing a single symbol in the original system to a block of consecutive symbols in the new system. The construction of the block is subject to the rule held in the original system and therefore this process provides an alternative description of the same shift space. For a shift of finite type $X \subset \Sigma_k$ and $N \geqslant 2$, the higher block presentation of X is a subset of $(B_N(X))^{\mathbb{N}}$, a full shift over symbols of allowed $N-$blocks of X. We define the *$N-$th higher block code,* $\beta_N : X \to (B_N(X))^{\mathbb{N}}$ by

$$(\beta_N(\mathbf{x}))_i = x_i x_{i+1} \cdots x_{i+N-1}.$$

Let us denote $(\beta_N(\mathbf{x}))_i^j$ as the $j-$th symbol of $(\beta_N(\mathbf{x}))_i$, then $(\beta_N(\mathbf{x}))_i^j = x_{i+j}$ for all integers $i \in \mathbb{N}$ and $j = 0, 1, \cdots, N-1$. Thus β_N replaces the $i-$th coordinate of x with the block of coordinates in x of length N starting at position i. This becomes clearer if we imagine the symbols in $B_N(X)$ as written vertically. Therefore we write

$$(\beta_N(\mathbf{x}))_i = \begin{bmatrix} x_{i+N-1} \\ \vdots \\ x_{i+1} \\ x_i \end{bmatrix}$$

So then the image of $\mathbf{x} = x_0x_1x_2x_3\cdots$ under β_N is

$$\beta_N(\mathbf{x}) = \begin{bmatrix} x_{N-1} \\ \vdots \\ x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix} \begin{bmatrix} x_N \\ \vdots \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} \begin{bmatrix} x_{N+1} \\ \vdots \\ x_5 \\ x_4 \\ x_3 \\ x_2 \end{bmatrix} \begin{bmatrix} x_{N+2} \\ \vdots \\ x_6 \\ x_5 \\ x_4 \\ x_3 \end{bmatrix} \cdots$$

Notice that every symbol of the image of $\mathbf{x}$ under β_N has a unique pattern depending on $\mathbf{x}$ i.e. consecutive symbols in the image of $\mathbf{x}$ under β_N overlap. Consequently, there are two important observations to note.

Observations:

1. $\{(\beta_N(\mathbf{x}))_i^j\}_{i\in\mathbb{N}} = \sigma^j(\mathbf{x})$ for all $j = 0, 1, \cdots, N-1$
2. For any two blocks $A, B \in B_N(X)$ where

$$A = \begin{bmatrix} a_{N-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}, B = \begin{bmatrix} b_{N-1} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}.$$

   AB is allowed in $\beta_N(X)$ iff $a_{i+1} = b_i$ for $i = 0, 1, \cdots, N-2$ and $a_0a_1\cdots a_{N-1}b_{N-1} \in B_N(X)$.

These two observations are very important in our discussion later.

Definition 2.44 *[32] Let X be a shift space. The* N-th higher block presentation of X *or the* $N-$th higher block shift $X^{[N]}$ *is the image of $\beta_N(X)$ in the full shift over $B_N(X)$.*

Therefore $\beta_M : X \to X^{[M]}$ is one-to-one and onto. By the second observation, we are able to explain how every shift of finite can be represented simply by another 1-step shift of finite type, i.e. its higher block presentation.

Proposition 2.45 *[32] If X is an $M-$step shift of finite type then $X^{[M]}$ is a 1-step shift of finite type.*

Proof. The $M+1$-block in X, $x_0 x_1 \cdots x_M$ is allowed iff the associated unique 2-block in $X^{[M]}$ (image under β_M),

$$\begin{bmatrix} x_{M-1} \\ \vdots \\ x_1 \\ x_0 \end{bmatrix} \begin{bmatrix} x_M \\ \vdots \\ x_2 \\ x_1 \end{bmatrix}$$

is allowed. Since X is an M-step shift of finite type, then $X^{[M]}$ is 1-step shit of finite type. □

X and $X^{[M]}$ share some common features that will help us to discover their dynamical properties. These two results will help us to show the similarity in dynamical properties.

Lemma 2.46

1. *If $\mathbf{y} \in X_w$ where $w = w_0 w_1 \cdots w_{k-1} \in B_k(X)$, then $\beta_M(\mathbf{y}) \in X_W^{[M]}$ where $W =$*

   $$W_0 W_1 \cdots W_{k-1} \in B_k(X^{[M]}) \text{ and } W_i = \begin{bmatrix} y_{i+M-1} \\ \vdots \\ w_{i+1} \\ w_i \end{bmatrix} \text{ for } i = 0, 1, \cdots, k-1.$$

2. *(The converse of (2)) If $\mathbf{Y} \in X_W^{[M]}$ where $W = W_0 W_1 \cdots W_{k-1}$, then $\beta_M^{-1}(\mathbf{Y}) = \mathbf{y} \in X_w$ where $w = w_0 w_1 \cdots \in B_k(X)$ and w_i is the first element of W_i for $i = 0, 1, \cdots, k-1$.*

3. (The implication of (2) and (3)) $\mathbf{y} \in X$ *is a periodic point of period* p *iff* $\beta_M(\mathbf{y}) \in X^{[M]}$ *is a periodic point of period* p.

Proof. (1) Let $\mathbf{y} = w_0 w_1 \cdots w_{k-1} y_k y_{k+1} \cdots \in X_w$ where $w = w_0 w_1 \cdots w_{k-1}$. Then the image of $\mathbf{y}$ under β_M is

$$\begin{aligned}\beta_M(\mathbf{y}) &= \begin{bmatrix} y_{M-1} \\ \vdots \\ w_1 \\ w_0 \end{bmatrix} \begin{bmatrix} y_M \\ \vdots \\ w_2 \\ w_1 \end{bmatrix} \cdots \begin{bmatrix} y_{M+k-2} \\ \vdots \\ y_k \\ w_{k-1} \end{bmatrix} \begin{bmatrix} y_{M+k-1} \\ \vdots \\ y_{k+1} \\ y_k \end{bmatrix} \cdots \\ &= W_0 W_1 \cdots W_{k-1} Y_k Y_{k+1} \cdots \\ &= W Y_k Y_{k+1} \cdots \end{aligned}$$

(2) Let $\mathbf{Y} = W_0 W_1 \cdots W_{k-1} Y_k Y_{k+1} \cdots \in X^{[M]}$ where $W = W_0 W_1 \cdots W_{k-1} \in B_k(X^{[M]})$. Since β_M is a bijection, there exists $\mathbf{y} \in X$ such that $\beta_M(\mathbf{y}) = Y$ and

$$\begin{aligned}\beta_M(\mathbf{y}) &= W_0 W_1 \cdots W_{k-1} Y_k Y_{k+1} \cdots \\ &= W Y_k Y_{k+1} \cdots \\ &= \begin{bmatrix} y_{M-1} \\ \vdots \\ w_1 \\ w_0 \end{bmatrix} \begin{bmatrix} y_M \\ \vdots \\ w_2 \\ w_1 \end{bmatrix} \cdots \begin{bmatrix} y_{M+k-2} \\ \vdots \\ y_k \\ w_{k-1} \end{bmatrix} \begin{bmatrix} y_{M+k-1} \\ \vdots \\ y_{k+1} \\ y_k \end{bmatrix} \cdots \end{aligned}$$

□

By our first observation, $\mathbf{y} = w_0 w_1 w_2 \cdots w_{k-1} y_k y_{k+1} \cdots \in X_w$ for $w = w_0 w_1 w_2 \cdots w_{k-1}$.
(3) The case when $\mathbf{y}$ and $\mathbf{Y}$ are both periodic can be shown by using the same arguments above since they are sequences of repeated blocks $w \in B_p(X)$ or $W \in B_p(X^{[M]})$ respectively.

Proposition 2.47 *Let X be an $M-$step shift of finite type. The following describe some common dynamical properties between X and X^M.*

1. *X is transitive iff $X^{[M]}$ is transitive.*

2. *X has dense periodic points iff $X^{[M]}$ has dense periodic point.*

Proof. Let X be transitive and A, B be two blocks in $X^{[M]}$ where

$$A = \begin{bmatrix} a_{M-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{M-1} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}$$

for $a_0, a_1, \cdots, a_{M-1}, b_0, b_1, \cdots, b_{M-1} \in \mathcal{A}$. Since X is transitive, there exists $w = w_0 w_1 \cdots w_{k-1} \in B_k(X)$ such that $u = a_0 a_1 \cdots a_{M-1} w b_0 b_1 \cdots b_{M-1} \in B_{2M+k}(X)$. For $\mathbf{y} = y_0 y_1 \cdots \in X_u$, (1) of Lemma 2.46 implies that $\beta_M(\mathbf{y}) \in X_U^{[M]}$ where $U = U_0 U_1 \cdots U_{2M+k-1} \in B_{2M+k-1}(X^{[M]})$, $U_0 = A$ and $U_{M+k} = B$. By taking $W = U_1 U_2 \cdots U_{M+k-1}$, $AWB \in B(X^{[M]})$ and $X^{[M]}$ is transitive.

In this proof, the property of transitivity will be replaced with irreducibility since they are equivalence by Theorem 2.40.

(1) Let $X^{[M]}$ be transitive and $a, b \in \mathcal{A}$ and $A, B \in B_M(X)$ be such that the first element of A and B are a and b respectively. Since $X^{[M]}$ is transitive, there exists a $k-$block, $W = W_1 W_2 \cdots W_{k-1}$ in $B_k(X^{[M]})$ such that AWB is allowed. Let us denote the first element of W_i (for every $i = 0, 2, \cdots, k-1$) as w_i. By (2) of Lemma 2.46, $aw_0 w_1 \cdots w_n b$ is allowed in X and therefore X is transitive.

(2) Let X has dense periodic points, $W = W_0 W_1 \cdots W_{k-1} \in B_k^{[M]}$ and $\mathbf{Y} \in X_W^{[M]}$. By (2) of Lemma 2.46, $\mathbf{y} = \beta_M^{-1} \mathbf{Y} \in X_w$ where $w = w_0 w_1 \cdots w_{k-1} \in B_k(X)$ and w_i is the first element of W_i for $i = 0, 1, \cdots, k-1$. Since X has dense periodic points, there exists

$\mathbf{z} \in X_w$ such that $\mathbf{z} = \overline{wv}$ is a periodic point for some $v \in B(X)$. By (1) and (3) of Lemma 2.46 and second observations that we made earlier, $\beta_M(\mathbf{z}) = \mathbf{Z} \in X_W^{[M]}$ is a periodic point. Let $X^{[M]}$ has dense periodic points, $w = w_0 w_1 \cdots w_{k-1} \in B_k(X)$ and $y \in X_w$. By Lemma 2.46 (1) $\beta_M(\mathbf{y}) \in X_W^{[M]}$ where $W = W_0 W_1 \cdots W_{k-1} \in B_k(X^{[M]})$ and the first element of W_i is u_i for all $i = 0, 1, \cdots, k-1$. Since $X^{[M]}$ has dense periodic points then there exists $\mathbf{Z} = \overline{WV} \in X_W^{[M]}$, a periodic point for some $V \in B(X^{[M]})$. By (1) and (2) of Lemma 2.46, if $\mathbf{z} = \beta_M^{-1}(\mathbf{Z})$ then $\mathbf{z} \in X_w$ is a periodic point. Therefore $\mathbf{z} \in X_u$ and X has dense periodic points. □

Devaney Chaotic Shift Map

This section discusses a role of transitivity on shift of finite type. Transitivity implies Devaney chaos on the interval, but not in the circle map. However l.e.o implies Devaney chaos for a continuous circle map. On shift of finite type, we answer the same question, the role of transitivity to Devaney chaos and as far as we known the following is new.

Theorem 2.48 *Let $X \subset \Sigma_k$ be any shift of finite type over k symbols. If the shift map, σ on X is transitive, then it is Devaney chaotic.*

Proof. Let X be an $M-$step shift of finite type (by Proposition 2.43) and $X^{[M]}$ is a 1-step shift of finite type (by Proposition 2.45). Let's assume that X is transitive. By Proposition 2.47 $X^{[M]}$ is also transitive. We claim that $X^{[M]}$ has dense periodic points. Let $W = W_0 W_1 \cdots W_{k-1} \in B_k(X^{[M]})$. By Theorem 2.40 there exists $A = A_0 A_1 \cdots A_{l-1} \in B_l(X^{[M]})$ such that $W_{k-1} A W_0$ is allowed in $X^{[M]}$. Since $X^{[M]}$ is a 1-step of finite type then the sequence $\mathbf{Y} = \overline{WA}$ is allowed and in $X_W^{[M]}$. Therefore $X^{[M]}$ has dense periodic points and so X by Proposition 2.47. □

Chapter 3

Chaos On The Interval

The purpose of this chapter is to introduce two weak conditions which to our knowledge, their importance is not generally known. We will describe how these properties play a variety of roles for being dynamically chaotic on the interval $I = [0, 1]$. One of the conditions is presented as having no (weakly) invariant proper (closed) interval (NIPS for short), defined as follows;

Definition 3.1 *Let $f : I \to I$ be a continuous interval map. f is said to has NIPS property whenever every proper closed subinterval of I is not weakly invariant under f.*

NIPS property is weaker than transitivity. By Theorem 2.26, transitivity implies positive entropy but not the converse. In particular, positive entropy implies the existence of an invariant closed transitive subsystem. So we realize that a system with NIPS property is essential to the occurrence of chaotic behavior. In fact, we can see later that a system on I with NIPS under f and with dense periodic points implies transitivity. This result therefore shows that periodicity of points affecting the course of chaotic behavior as well as NIPS.

In addition, a system with NIPS can require the existence of some fixed points sufficiently rather than dense periodic points to behave chaotically. Therefore the second weak property that we will emphasize in this chapter is having fixed points in the interval

I. Due to the Sarkovskii Theorem having a fixed point is weaker than dense periodic points. his will be shown in the section of this chapter.

3.1 NIPS and Dense Periodic Points

It was known earlier that dense periodic points is redundant for transitive system to be proven as a chaotic system in the sense of Devaney. Furthermore, the identity map is a counterexample to show that dense periodic point does not imply transitivity on the interval. From the observation on the graph of the identity map, we can see that the existence of invariant closed intervals everywhere appears to be transverse to transitivity. Therefore, NIPS is introduced and together with dense periodic points property, we will show that the system will be chaotic in the sense of Devaney and hence chaotic in the other senses. The following lemma is an obvious consequences of dense periodic points with an invariant proper subinterval and will be used frequently in the sequel.

Lemma 3.2 *Let f be a continuous interval map with dense periodic points. If f is not transitive, then there exists a closed subinterval $A \subsetneq I$ such that A is invariant.*

Proof. Let f be an interval map that has dense periodic points but that is not transitive. We firstly claim that there exists a basic open set $A \subset I$ (i.e. A is an open non-degenerate interval) such that the union of all iteration of A under f can be written as a finite union of non-degenerate disjoint intervals, as follows

$$\bigcup_{i\in\mathbb{N}} f^i(A) = \bigcup_{i=1}^{m} K_i \neq I$$

where $K_1, K_2, \cdots, K_m$ are single non-degenerate disjoint periodic intervals, for some integer m. Since f is not transitive, there exists $A \subset I$ such that $\overline{\bigcup_{i\in\mathbb{N}} f^i(A)} \neq I$. Let $a \in A$ such that $f^n(a) = a$ with the smallest periodic in A. Therefore for every $m \in \mathbb{N}$ and for every $i = 0, 1, 2, \cdots, n-1$, $f^{mn+i}(A)$ contain $f^i(a)$ and by dense periodic points,

$f^{mn+i}(A)$ is a non-degenerate interval. If not $f^{mn+i}(A) = \{a\}$ for some points a, means that A has at most one periodic point point while periodic points are everywhere in A by dense periodic points. Hence for every $i = 0, 1, 2, \cdots, n-1$, $K_i = \bigcup_{m \in \mathbb{N}} f^{mn+i}(A)$ is a non-degenerate single interval and $\bigcup_{i \in \mathbb{N}} f^i(A) = \bigcup_{i=0}^{n-1} K_i$. To complete the proof, we then are about to show the existence of an invariant proper closed subinterval of I. By the way we define K_i, $f^m(K_i) = K_i$ for every i and $f^m(\cup K_i) = \cup K_i$.

Let us assume $m > 2$. If f does not has any invariant proper closed subinterval, then there exist k such that $f(K_k) = K_l$ and either $f(K_{k+1}) = K_{l-n}$ or $f(K_{k-1}) = K_{l-n}$ for $1 < n < l$ will happen. If this is not happen, then $f(K_{i-1}) = K_{m-i}$ for every $1 \leqslant i \leqslant m$ and this implies the existence of an invariant proper closed subinterval, a contradiction. Therefore we assume that for some k, $f(K_k) = K_l$ and $f(K_{k+1}) = K_{l-n}$ for $1 < n < l$. Hence there exists an open neighborhood of x, N (x in between K_k and K_{k+1} but $x \notin \cup_i K_i$) such that $f(N) \subset K_{l-1}$. Since $\cup_i K_i$ is invariant, N does not has any periodic points.

The case when $m = 2$ can be seen by the fact f has dense periodic points and therefore no points in between K_0 and K_1 will be mapped to $K_0 \cup K_1$. Therefore the interval between K_0 and K_1 is invariant. The continuity of f implies that the existence of invariant interval is equivalent to the existence of closed invariant subinterval. □

We then get the main result in this section, shows the importance of NIPS.

Proposition 3.3 *Let $f : I \to I$ be a continuous interval map. If f has NIPS under f and has dense periodic points, then f is transitive on I.*

Proof. This is a direct application of Lemma 3.2. □

This gives a quick proof of a theorem of Barge and Martin in [7] about dense periodicity on the interval, will be explained in the next section. A system with no invariant subinterval does not implies Devaney chaotic without dense periodic points property, by the following example.

Example 3.4 *Let g be an interval map defined by the graph in the Figure 3.1. From the graph, g has NIPS and $\frac{1}{2}$ is the fixed point for g. $0, \frac{1}{2}$ and 1 are the only periodic points because every point will be repelled by $\frac{1}{2}$ to approach 0 and 1. Therefore g does not has dense periodic points and is not transitive.*

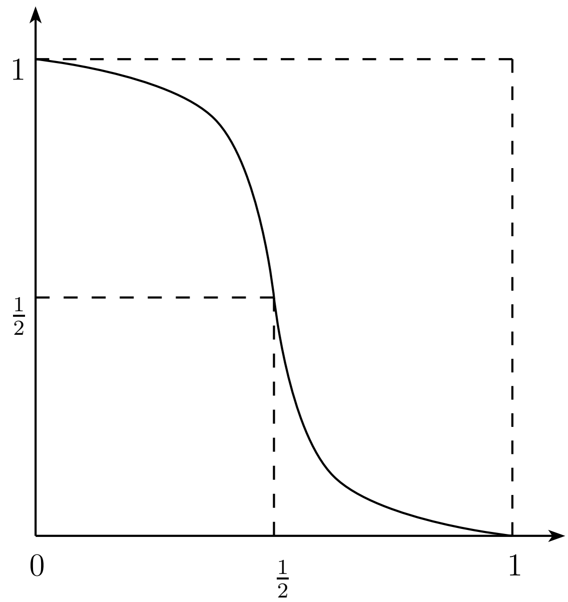

Figure 3.1: Graph of g which has NIPS but neither transitive nor has dense periodic points

3.2 Dense Periodicity on the Interval

The properties of interval maps with dense periodic points have been given in [7]. We will prove the same result by analyzing the graph and applying Proposition 3.3. We start by analyzing the graph of an interval map f with dense periodic points and with two additional conditions; I has invariant proper subintervals and the restriction of f^2 on any invariant proper subinterval is not the identity map. The first assumption is needed because the interval map f with NIPS is not our concern since Proposition 3.3 shows that such f is transitive. The second condition is needed because the dynamics of the identity map is trivial and fully understood. We start by getting the main form of the graph of f.

Lemma 3.5 *Let $f : I \to I$ be a continuous interval map. If f has dense periodic points and there exists an interval $A \subsetneq I$ such that $f(A) = A$, then it is either;*

- *Case 1: $I \setminus A = B \cup C$, $f(B) = C$ and $f(C) = B$*
- *Case 2: $I \setminus A = B \cup C$, $f(B) = B$ and $f(C) = C$*
- *Case 3: $I \setminus A = B$, $f(B) = B$*

for some subintervals B and C. For the Case 1, the endpoints of A are 2-cycle and the graph of f is as Graph A(Case 1) in Figure 3.2. For the Case 2, the endpoints of A are fixed and the graph of f is as Graph A(Case 2) in Figure 3.2. For the Case 3, the endpoints of A is fixed (except possibly for endpoints of I) and the graph of f is as Graph A(Case 3) in Figure 3.2.

Proof. Let $I \setminus A = B \cup C$ where B and C are disjoint. Let $x \in B$. By dense periodic points, $f(x)$ is either in B or C. If $x \in C$, then for every $y \in B$ $f(y) \in C$, otherwise continuity of f will send some elements of B into A, a contradiction. On the other hand, if $f(x) \in B$ then by the same argument $f(y) \in B$ for all $y \in B$. Therefore it is either $f(B) = C$ and $f(C) = B$ or else $f(B) = B$ and $f(C) = C$. If $I = A \cup B$ then B is also invariant because every element of B has no other place to be mapped into. □

For the Case 2 and 3, A, B and C are all invariant under f. If these intervals have NIPS under f then f is transitive in every interval, by Proposition 3.3. For the case 1, B and C are invariant under f^2 and again by the theorem, f^2 is transitive on every interval provided it has NIPS under f^2. However every interval has possibilities to have another invariant subinterval and in fact dense periodic points cannot guarantee the existence of an invariant subinterval with NIPS. On the other hand, we will show that we can rely on

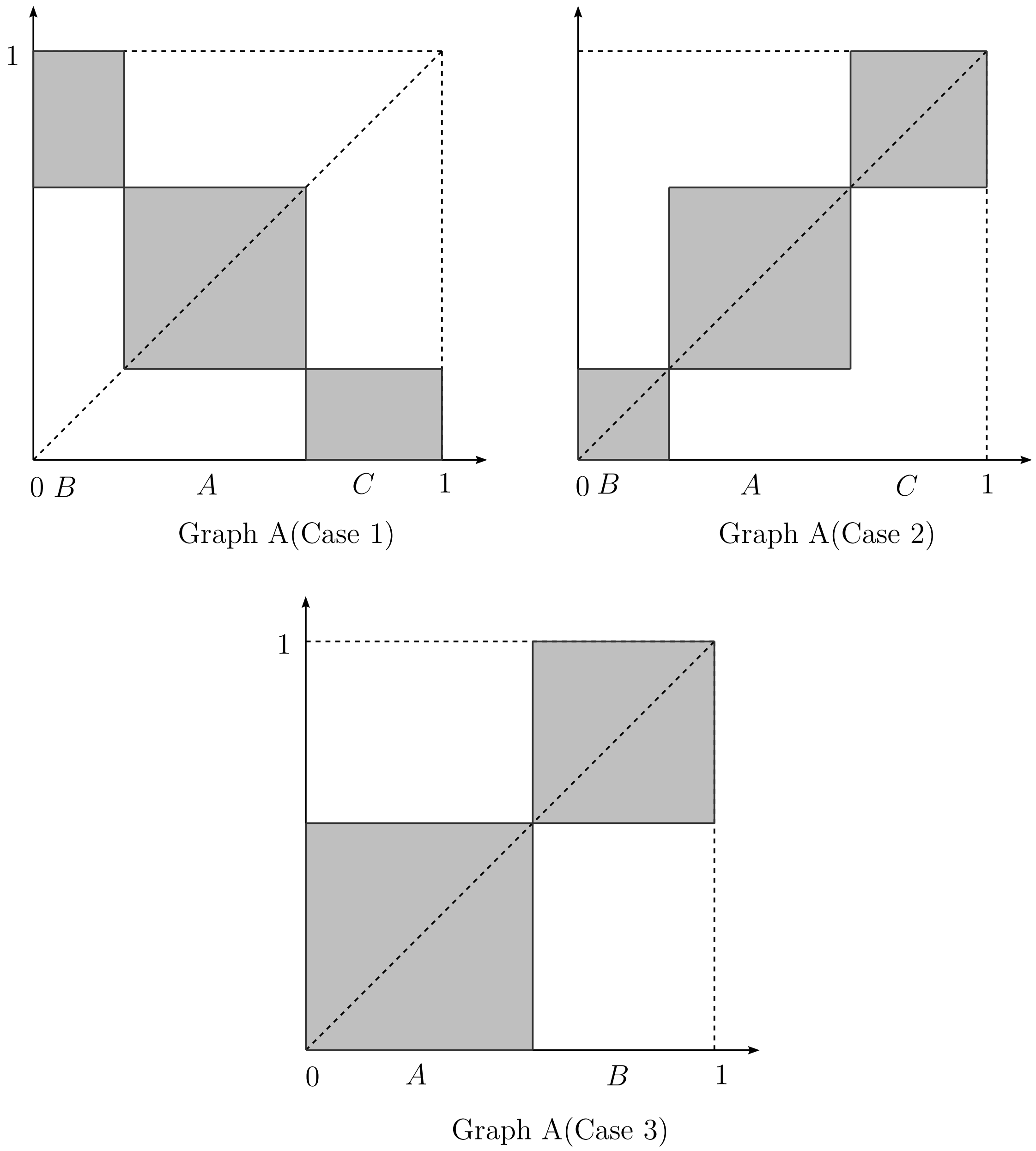

Figure 3.2: The possibilities of graph f with dense periodic points, graphs contained in the grey boxes.

the second additional conditional of f so that I has a proper invariant subinterval with NIPS.

Lemma 3.6 *Let $f : I \to I$ be a continuous interval map with dense periodic points. If f has an invariant proper subinterval and the restriction of f^2 on any subinterval is not the identity map, then there exists an invariant proper subinterval A of I such that f has NIPS on A.*

Proof. By using contradiction we assume that f has dense periodic point, any restriction of f^2 on any subinterval is not the identity map, I has proper invariant subintervals but every invariant subinterval always has invariant proper subinterval. By this assumption we firstly claim that for an arbitrary $x \in I$ and a sequence of invariant subintervals of I, $\{A_i\}_{i\in\mathbb{N}}$ satisfies $A_0 \supset A_1 \supset A_2 \supset \cdots$, if $x \in A_i$ for all i, then $f(x) = x$. To show this, let $A' = \bigcap_{i\in\mathbb{N}} A_i$. Therefore $A' \subset A_i$ for all i. If A' is a non-degenerate interval, then there exist another invariant subinterval B such that $B \subset A'$. This is a contradiction. Therefore $A' = \{x\}$. Since $f(A') \subseteq \bigcap_{i\in\mathbb{N}} f(A_i)$, then A' is invariant and hence $f(x) = x$. For the Case 2 and Case 3 in Lemma 3.5, every arbitrary $x \in I$ is in an invariant subinterval. Since every invariant subinterval always has invariant proper subinterval, we then are able to find a sequence as in our claim and therefore x is fixed. For the Case I, Lemma 3.5 we are widening our attention from f to f^2 and then use the same argument to show that every point is fixed under f^2, a contradiction. □

Proposition 3.7 *Let $f : I \to I$ be a continuous interval map with dense periodic points. If f has an invariant proper subinterval and the restriction of f^2 on any subinterval is not the identity map, then there exist closed subintervals $J_1, J_2, \cdots, J_k$ ($k \leqslant \infty$) such that $I = \bigcup_{i=1}^{k} J_i$ and f either satisfies Property A or Property B.*

Property A:

1. *J_i is invariant under f with NIPS for all i.*

2. *the endpoints of J_i for all i are fixed (except possibly for the endpoints of I),*

Property B:

1. *there exists unique integer j such that J_j is invariant under f and has NIPS,*

2. *J_i is invariant under f^2 with NIPS. In particular, for every $i = 1, 2, \cdots$ and $i < j$, $J_{j+i} = f(J_{j-i})$ and $f(J_{j+i}) = J_{j-i}$,*

3. *For every $i = 1, 2, \cdots$ the endpoints of J_i are 2-cycle (except possibly for the endpoints of I). In particular, if $J_i = (u_i, v_i)$, then $f(u_i) = v_{k-i+1}$ and $f(v_i) = u_{k-i+1}$.*

Proof. Let f be a continuous interval map with dense periodic points, has invariant proper subinterval and restriction of f^2 on any subinterval is not the identity map. By Lemma 3.6, there exist $A \subset I$ such that A is invariant and has NIPS. We then analyze the graph of f by considering its basic graph, Case 1,2 and 3 as stated in Lemma 3.5. We claim that Case 2 and 3 satisfy Property A and Case 1 satisfies Property B.
For the Case 2 and Case 3, the intervals B and C are invariant under f. By Lemma 3.6, there exist subintervals $B' \subseteq B$ and $C' \subseteq C$ such that B' and C' are respectively invariant under $f|_B$ and $f|_C$ and have NIPS. The compliment, $B \setminus B'$ and $C \setminus C'$ are also invariant under f and therefore by the same reason they will have another invariant subinterval which have NIPS. Repeating the same step until the compliment of the interval which has NIPS is empty and therefore the interval B and C can be decomposed into such intervals with NIPS property, prove the Property A(1). For the Property A(2), the endpoints of all of the intervals (except possibly for the endpoints of I) are fixed by the fact that the intervals are invariant.
For the Case 1, we analyze the graph of f^2 instead of f since the graph of f^2 is in the Case 2. Therefore the interval B and C can be decomposed into invariant subintervals i.e $B = B_1 \cup B_2 \cup \cdots$ and $C = C_1 \cup C_2 \cup \cdots$ with NIPS property with respect to f^2 and

$I = \left(\bigcup_i B_i\right) \cup A \cup \left(\bigcup_i C_i\right)$. Let us denote such intervals as $J_1, J_2, \cdots$ such that $I = \bigcup_i J_i$, $J_j = A$ for some integer j. j is unique since A is the only invariant subinterval and has NIPS. Since $J_j = A$, $J_i \subset B$ if $i < j$ and $J_j \subset C$ if $i > j$. Since $f(B) = C$ and $f(C) = B$, then for every i, there exist k such that $f(J_i) = J_k$ and $f(J_k) = J_i$. $f(J_{j-i}) = J_{j+i}$ for all $i < j$ can be shown inductively on i. By continuity of f, $f(J_{j-1} \cup J_j) = J_j \cup J_{j+l}$ must be a single non-degenerate interval, otherwise f is not continuous at the endpoints of J_j. Therefore $l = 1$. In general, whenever $f(J_{j-i}) = J_{j+i}$, then by continuity of f $f(J_{j-i-1} \cup J_{j-i}) = J_{k+i+m} \cup J_{k+i}$ must be a singe non-degenerate initerval. Therefore $m = 1$. Conversely, since $f^2(J_{j-i}) = J_{j-i}$, then $f(J_{j+i}) = J_{j-i}$, completes the proof for Property B(2). The Property B(3) is again obvious by using the same argument in the proof of Property A(2). □

Consequently, for the subintervals $J_1, J_2, \cdots J_k$ (k can be finite or infinite) of I as described in the proposition above, the graph of f with Property A is included in the grey boxes $J_i \times J_i$ in the Graph B(1) of Figure 3.3 and the graph of f with Property B is included in the grey boxes $J_{j-i} \times J_{j+i}$ in the Graph B(2) of Figure 3.3.

Following the above description, we then generalize the Property A and B (and Graph B(1) and B(2) of Figure 3.3) to became one kind of property and one graph.

Corollary 3.8 *Let $f : I \to I$ be a continuous interval map with dense periodic points. If f has an invariant proper subinterval and restriction of f^2 on any subinterval is not the identity map, then there exist closed subintervals $J_1, J_2, \cdots, J_k$ ($k \leqslant \infty$) such that :*

1. *J_i is invariant with NIPS with respect to f^2, for all $i = 1, 2, \cdots$,*
2. *The endpoints of J_i are 2-cycle (except possibly for the endpoints of I), for all $i = 1, 2, \cdots$.*

Now we will give the theorem of Martin and Barge [7] about an interval maps with dense periodic points, with our new proof. They studied the dynamics of interval maps f :

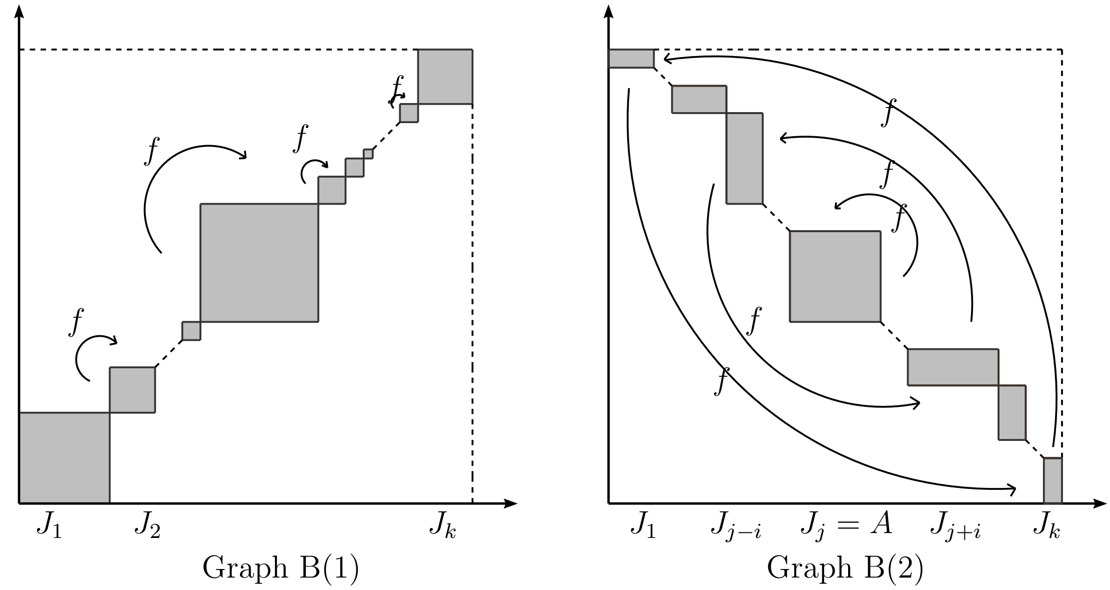

Figure 3.3: The graphics of function f with Property A or Property B

$I \to I$ by analyzing the *inverse limit space* of the maps, $(I, f) = \{(x_0, x_1, \cdots)|f(x_{i+1}) = x_i\}$, which is compact, connected metric space and is an example of what Bing [8] has called a snakelike continuum.

Theorem 3.9 *Suppose that $f : I \to I$ is continuous, and that the set of periodic points of f is dense in I. Then there are closed subintervals $J_1, J_2, \cdots, J_k$ ($k \leqslant \infty$) such that:*

1. $f^2(J_i) = J_i$,
2. *for each i, f is transitive on J_i,*
3. *if $x \in I \setminus \bigcup_{i=1}^{k} J_i$ then $f^2(x) = x$.*

Proof. Let us assume that the interval map f has dense periodic points. If I has NIPS, then by Proposition 3.3, I is transitive and therefore the collection of J_i is a single collection, $J_1 = I$, satisfies all of the properties.
On the other hand, let I have an invariant proper subinterval. The case of f^2 as the identity map means the collection of J_i is empty and every $x \in I$ satisfies the property 3.

So suppose f^2 is not the identity. There are two cases two considered. Either $f^2|A$ is not the identity for any interval A or there is an interval A such that f^2 is the identity map. For the first case, Corollary 3.8 implies that $I = \bigcup_i^k J_i$ (k can be finite or infinite) for closed subintervals J_i's, where $f^2(J_i) = J_i$ and J_i has NIPS under f^2. By Proposition 3.3, $f^2|_{J_i}$ is transitive for all i and so $f|_{J_i}$. Therefore it satisfies the three properties.
For the other case, i.e. whenever there exist K_j for $j = 1, 2, \cdots$ such that $f^2|_{K_j}$ is the identity map, we look at $I \setminus \bigcup_j K_j$. Since K_j is a non-degenerate interval, we have a collection of non-degenerate interval J_i for $i = 1, 2, \cdots$ such that $I \setminus \bigcup_j K_j = \bigcup_i J_i$. Then by the continuity and dense periodic points of f^2, J_i is invariant under f^2 for all i. Otherwise there exists an open subset of J_i that will be mapped to K_{i-1} or K_{i+1} makes it has no periodic points.

If J_i has NIPS, then Proposition 3.3 says that the interval is transitive. If not, then the graph of $f^2|J_i$ is in Graph A(2) of Figure 3.5 and by Proposition 3.7, J_i can be decomposed into subintervals $J_1', J_2', \cdots$ such that it is invariant and has NIPS under f^2 and by Proposition 3.3, $f^2|_{J_i'}$ is transitive, for every i. □

In particular, to show that every interval J_i in the above theorem is transitive, Martin and Barge [7] showed the existence of a point $x_i \in J_i$ such that $\{f^{4n}(x_i)|n > 0\}$ is dense in J_i.

3.3 NIPS and Existence of a Fixed Point.

In this section we will expand the possibilities of systems with the NIPS property to show that they are chaotic. Rather then dense periodic points, the system may require a weaker condition; having some fixed points in the interval I so that the system can be proven to behave topologically chaotic and Li-Yorke chaotic. However, Devaney chaos is not necessarily satisfied. Furthermore, the systems will satisfy some other properties that frequently occur on a chaotic system: existence of a periodic point of period $q2^p$ for any odd q and integer $p \geqslant 0$ and turbulence. Turbulence is another notion of chaos defined as follows;

Definition 3.10 *The map f will be said to be turbulent if there exist compact subintervals J, K with at most one common point such that $J \cup K \subseteq f(J) \cap f(K)$. It will be said strictly turbulent if the subinterval J, K can be chosen disjoint.*

Turbulence also implies the existence of cycles of all types for interval maps [12]. This makes turbulence as an important notion of chaos. We derive a tool to get an interval map which is turbulent and having 3-cycle which will be used frequently to gain the main result in this section.

Lemma 3.11 *If there exist $a < b < c$ (or $a > b > c$) in $(0, 1)$ such that a is fixed, $f(c) \leqslant a$ (or $f(c) \geqslant a$ respectively) and $c \leqslant f(b)$ (or $c \geqslant f(b)$ respectively), then I has a 3-cycle and f is turbulent.*

Proof. Let us assume $a < b < c$ be such that $f(a) = a$, $f(b) = p$ and $f(c) = q$ satisfy $q \leqslant a < b < c \leqslant p$). By the Intermediate Value Theorem, $f(b) = p, f(c) = q$ and $c \in (q, p]$ implies that there exists $x \in (b, c)$ such that $f(x) = c$. Again by the Intermediate Value Theorem, $f(a) = a, f(b) = p$ and $x \in (b, c) \subset (a, p)$ implies that there exists $y \in (a, b)$ such that $f(y) = x$. Therefore $q = f(c) < y < x = f(y) < c = f(x)$ which forces the system to have a 3-cycle, by Lemma 2.3.
To show that f is turbulent, let $J = (a, b)$ and $K = (b, c)$. Therefore $J \cup K = (a, c) \setminus \{b\}$ and $f(J) \cap f(K) = (a, p)$. Hence $J \cup K \subset f(J) \cap f(K)$ and f is turbulent. The proof for the other case is the same. $\square$

Proposition 3.12 *Let $f : I \to I$ have NIPS under f and $f(a) = a$ for some $a \in (0, 1)$. If one of the endpoints is fixed, then f has a 3-cycle and is turbulent in I.*

Proof. Let f be an interval map which has NIPS, $f(a) = a$ for some $a \in (0, 1)$ and 1 is a fixed point. We firstly assume that $f(0) \neq 1$ and then let $m = \sup\{f(x) : x \in [0, a]\}$. We have to consider either $m \neq 1$ or $m = 1$.

For $m \neq 1$, let $b = \inf\{x \in [a, m] : f(x) \geqslant m\}$. So, for all $x \in [a, b], f(x) \leqslant m$ and $f(b) = m$. b exists since $[0, m]$ is not invariant and m is the maximum value on $[0, a]$. Let $a' = \sup\{x \in [a, b] : f(x) = x\}$. So, $a' \in [a, b)$, $f(x) > x$ for all $x \in (a', b]$ and $f([a', b]) = [a', m]$. Let $n = \min\{f(x) : x \in [b, 1]\}$ and $c = \inf\{x \in [b, 1] : f(x) = n\}$. Therefore $n = f(c) < a'$. We have to consider either $c \neq 1$ or $c = 1$.

For $c \neq 1$ we claim that there exists $d \in (a', c)$ such that $f(d) > c$. To prove this, let us assume $c \neq 1$. If $c < m$, then $b = d$. If $c \geqslant m$, then by the way we choose m and b and invariant $(0, c)$, there exists $y \in (b, c)$ such that $f(y) > c$. So, choose $d = y$ to prove the claim. Hence, $a' < d < c$ with fixed a', $f(c) \leqslant f(a')$ and $c \leqslant f(d)$ forces the system to have 3-cycle and turbulence for f by Lemma 3.11.

Now, let $m = 1$ and $x \in [0, a]$ such that $f(x) = m = 1$. By our hypothesis $x \neq 0$. Let $n = \min\{f(x) : x \in [a, 1]\}$ with $f(c) = n$ for $c \in [a, 1]$. We have to consider either $n \neq 0$ or $n = 0$. For $n \neq 0$ and $n \geqslant x$, the invariant $[x, 1]$ implies the existence of $z \in (x, a)$ such that $f(z) < x$. Therefore $x < z < a$ with fixed a, $f(x) \geqslant f(a)$ and $x \geqslant f(z)$ forces the system to have 3-cycle and turbulence for f by Lemma 3.11.

For $n \neq 0$ and $n < x$, the invariant $[n, 1]$ implies the existence of $z \in (n, a)$ such that $f(z) < n$. z is either in (x, a) or (n, x). If $z \in (n, x)$, then $z < x < c$ with $f(z) < z$ and $f(x) > x$. By defining $g(\alpha) = f(\alpha) - \alpha$, Intermediate Value Theorem gives the existence of $a" \in (z, x)$ such that $g(a") = 0$ since $g(z) < 0$ and $g(x) > 0$. Therefore $a" \in (z, x)$ is fixed for f. So, $a'' < x < c$ with fixed a'', $f(c) \leqslant f(a'')$ and $c \leqslant f(x)$ forces the system to have 3-cycle and turbulence for f by Lemma 3.11.

If $z \in (x, a)$, $x < z < a$ with fixed a, $f(x) \geqslant f(a)$ and $x \geqslant f(z)$ forces the system to have 3-cycle and turbulence for f by Lemma 3.11.

For $n = 0$ i.e. $f(c) = 0$ for some $c \in (a, 1)$, by the Intermediate Value Theorem, there exist $z \in (a, c)$ and $w \in (c, 1)$ such that $f(z) = x$ and $f(w) = z$. Therefore by Lemma 2.3, there is a 3-cycle. Turbulence can be proved by considering $J = [x, c]$ and $K = [c, 1]$. For $f(0) = 1$ we assume $c \in (0, 1)$ such that $f(c) = 0$. If such point c is not exist then $p = \inf\{f(x)|x \in I\} > 0$ and therefore $[p, 0]$ is invariant under f. By the Intermediate Value Theorem, there exist $z \in (c, 1)$ such that $f(z) = c$. Therefore by Lemma 2.3, there is a 3-cycle. Turbulence can be proved by considering $J = [0, c]$ and $K = [c, w]$.
If 0 is the fixed endpoint, let us define an interval map g such that $g(x) = 1 - (f(1 - x))$. By a simple calculation, we can show that g has NIPS, a and 1 are fixed for g. By the previous proven argument, g has a 3-cycle and is turbulent and so is f. □

We then will show that any property of existence of fixed point in subinterval $(0, 1)$ and subset $\{0, 1\}$ cannot be excluded in the above proposition. Following examples are interval maps g_1 and g_2 with NIPS property but without a fixed point in $(0, 1)$ or $\{0, 1\}$ respectively and both interval maps have no 3-cycle. However the subsequent example shows that NIPS somehow can be left out for some interval maps with the two properties of fixed point existence to have the desired cycle somewhere in the interval I.

Example 3.13 *Let g_1 be an interval map on I defined by graph in Figure 3.4. g_1 has NIPS, and a fixed point in $(0, 1)$ but the endpoints are not fixed. There is no point $x \in I$ is a 3-cycle.*

Proof. From the graph of g_1 in Figure 3.4, $g_1[0, \frac{3}{4}] = [\frac{3}{4}, 1]$ and $g_1[\frac{3}{4}, 1] = [0, \frac{3}{4}]$. Therefore if $x \in [0, \frac{3}{4})$, then $g_1^3(x) \in (\frac{3}{4}, 1]$. Therefore x is not a 3-cycle. The other properties can be explained by the graph of g_1 in Figure 3.4. □

The property of the existence of a fixed point in $(0, 1)$ also cannot be excluded in Proposition 3.12, shown by the following example;

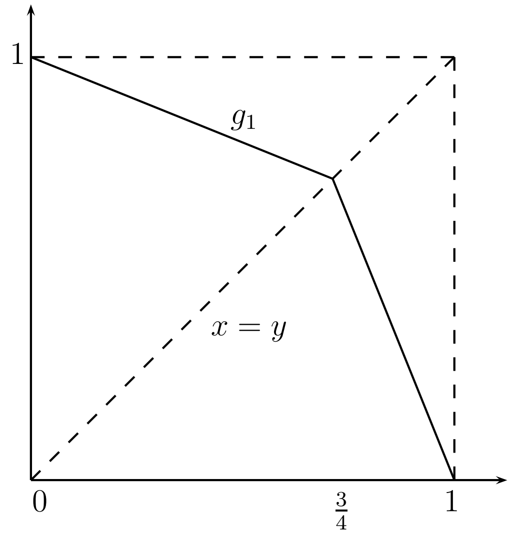

Figure 3.4: The graph of g_1 with NIPS and fixed point in $(0, 1)$

Example 3.14 *Let g_2 be a continuos interval map on I defined by $g_2(x) = x^3$. g_2 has NIPS, the endpoints are the only fixed points but no point in I is a 3-cycle.*

Proof. To show that g_2 has NIPS, let us assume $[x, y] \subsetneq I$ is invariant. $x \in [x, y]$ but $x^3 < x$, therefore $g_2(x) \notin [x, y]$ and $[x, y]$ is not weakly invariant. The other properties are also obvious since 0 and 1 are the only periodic points and they are fixed points. □

However, not all interval maps with fixed points in $(0, 1)$ and in $\{0, 1\}$ need NIPS to have 3-cycle. Here is the example;

Example 3.15 *Let g_3 be a continuous interval map on I defined by the graph in Figure 3.5. g_3 has fixed points in $(0, 1)$ and one of the endpoinds is fixed. Even though g_3 does not have NIPS property, but g_3 has a 3-cycle in I.*

Proof. The existence of fixed point in both set $(0, 1)$ and $\{0, 1\}$ are obvious from the graph in Figure 3.5, as well as the existence of invariant subintervals. 3-cycle is existed in the subinterval $\left[\frac{1}{4}, 1\right]$ since $g_3|_{\left[\frac{1}{4},1\right]}$ is conjugate to the Tent Map with $\mu = 2, T_2$. □

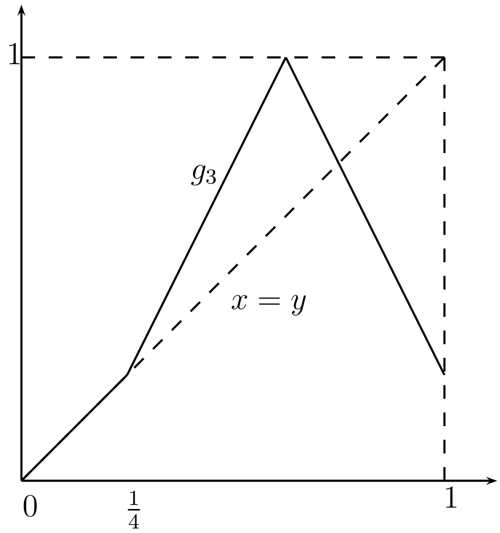

Figure 3.5: The graph of g_3 with fixed points in $(0, 1)$ and fixed endpoint but has weakly invariant proper closed interval

Collecting all results gives us the main theorem for this chapter.

Theorem 3.16 *Let $f : I \to I$. If f has NIPS under f, $f(a) = a$ for some $a \in (0, 1)$ and one of the endpoints is fixed, then f satisfies the following:*

1. *f has 3-cycle*
2. *f is turbulent on I*
3. *f is Li-Yorke chaotic*
4. *f is topologically chaotic*

Proof. The proof of Li-Yorke chaotic and topologically chaotic are due to Theorem 2.7 and 2.8. □

Chapter 4

A Strong Dense Periodicity Property

This chapter will focus on a stronger dense periodicity property as a fast track to chaos. In the definition of chaos in the sense of Devaney, dense periodic points does not play as important role as transitivity does in order to make the system behave chaotically. In general dense periodic points does not imply transitivity; the identity map which is not transitive, has dense periodic points, zero entropy and therefore is not chaotic in any sense.

Therefore we define a stronger property than dense periodic points and hope that this strong density property will imply Devaney chaotic or at least some other form of chaos. In this work we look at this problem on three different spaces i.e. on the interval, unit circle and shift of finite type. On the interval, a continuous interval map with no invariant proper subinterval (NIPS) with the strong dense periodicity property is Devaney chaotic and even has positive entropy. For a continuous interval map without NIPS property, the strong dense periodicity property implies that the whole system can be decomposed into chaotic Devaney subsystems and therefore the whole system has positive entropy. This conditions are not hold on a continuous circle map. Even with NIPS property, the whole system is not necessarily Devaney chaotic. However we will show that the system is topologically chaotic. Meanwhile on shifts of finite type, the property does not just imply positive entropy but is equivalent to Devaney chaotic.

4.1 P_n Dense for All n in General

Intuitively, the dynamics of a system with dense periodic points but only with the same small prime period (or even finitely many different small prime periods) does not necessarily behave chaotically, but having periodic points of large enough prime period everywhere is seems to be chaotic. Therefore we define a stronger dense periodic points as follows,

Definition 4.1 *For a space X, we say that it has P_n dense (for all $n \in \mathbb{N}$) if the set of periodic points P_n in X is dense, where the set is defined as*

$$P_n = \{x \in X : x \text{ is a periodic point of prime period } k \text{ for some } k \geqslant n\}$$

The property is well defined on an infinite space X by showing the following:

Theorem 4.2 *For a compact metric space X in which every basic open set is infinite, if $f : X \to X$ is Devaney chaotic, then P_n is dense for all n on X.*

Proof. Let f be a Devaney chaotic map but suppose P_n is not dense for some n. So, there exist an open subset $U \subset I$ and an integer $n > 0$ such that $U \cap P_n = \emptyset$. So, every periodic point in U has prime period less then n. Let k be the least common multiple of all prime period of U. So, every periodic point $p \in U$ will satisfies $f^k(p) = p$.

By transitivity and since U is infinite, there are $k+2$ disjoint open subsets of U, $W, V_1, V_2, \cdots, V_{k+1}$ and $k+1$ integers, $n_1 < n_2 < \cdots < n_{k+1}$ such that $f^{n_i}(W) \subset V_i$ for all $i \leqslant k+1$. Then there exist $W'_{k+i} \subseteq W'_k \subseteq W'_{k-1} \subseteq \cdots \subseteq W'_1 \subseteq W$ and $V'_i \subset V_i$ such that $f^{n_i}(W'_i) = V'_i$ for all $i \leqslant k+1$.

We first claim that $n_i \neq n_j$ for all i. If not, then there are $i < j$ such that $n_i = n_j$. Since

$W'_j \subseteq W'_i$, $f^{n_j}(W'_j) \subseteq f^{n_i}(W'_i)$ which implies $V'_j \subseteq V'_i$, a contradiction.

By this claim, $n_{k+1} \geqslant k+1$. If $n_{k+1} = k+1$ then $n_k = k$. By density of periodic point, there exists a periodic point $p \in W'_k$ such that $f^k(p) = p \in V'_k$ which is a contradiction as W and V_i are disjoint. If $n_{k+1} > k+1$ then there exists $i > j$ such that $n_i = n_j + mk$ for some $m > 0$. Again by density of periodic point, there exists a periodic point $q \in W'_{n_j}$ such that $f^{n_i}(q) \in V_{n_i}$. However $f^{n_j}(q) = f^{n_i+k}(q) = f^{n_i}(q)$ is a contradiction. □

Remark 4.3 *If $f : X \to X$ has dense P_n for all n, then f^k also has dense P_n for all n, for every positive integer k.*

When X is finite, Devaney chaos does not implies P_n dense for all n by the counterexample below;

Example 4.4 *Let $X = \{a, b\}$ with discrete topology and a function $f : X \to X$ defined by $f(a) = a$ and $f(b) = b$. The system is Devaney chaotic but P_3 is not dense since every point is 2-cycle.*

In fact, this property is not valid on a finite space X since periodic points of period $k > |X|$ is not exist. In general compact metric space, P_n dense for all n does not imply Devaney chaos since we have a following counterexample.

Example 4.5 *Let S^1 and I be the unit circle and the unit interval respectively and therefore $S^1 \times I$ will be a cylinder with diameter 1. We will write any element of S^1 as $0 < \theta \leqslant frm{-}epi$. Let us define a continuous function f on a the cylinder as below;*

$$\begin{aligned} f :& S^1 \times I \to S^1 \times I \\ & (\theta, a) \mapsto (\theta + 2\pi a, a) \end{aligned}$$

The dynamical system $(S^1 \times I, f)$ is not chaotic in the sense of Devaney even though it has P_n dense for all n.

Proof. The function f is obviously continuous since rotation on a circle and identity map are both continuous. The visualization of the function f is a rotation by an angle of $2\pi a$ on the unit circle at axis-a, $S^1 \times \{a\}$. Intuitively, the function f does not behave as a chaotic system since every point only move regularly within its own circle at the same axis. Indeed, we will show that the dynamical system is not transitive, but P_n is dense for all n and f has sensitive dependence on initial condition.

From the fact that f fixes $S^1 \times \{a\}$ for all $a \in I$, it is easy to show that there exist two disjoint open sets $U, V \subset S^1 \times I$ that will not hit each other after any iteration of f. We can choose U and V such that $\{a : (\theta, a) \in U\} \cap \{a : (\theta, a) \in V\} = \emptyset$. So, it will not be difficult to see that the system is not transitive and therefore is not chaotic in the sense of Devaney.

We then will show that for every n, P_n is dense i.e. for every open subset $A \times B \subset S^1 \times I$ and every n, there exists a periodic point of prime period greater or equal than n. We firstly claim that for every n, there exists an integer $q \geqslant n$ such that $\frac{p}{q} \in B$ for some integer p (p is neither q nor 1). To see this, let us choose large enough $q > 2$ such that when we cut I into q subintervals with same length, then there exists endpoint (except 0 and 1) of the subinterval which lies in B. The endpoint is in the form of $\frac{p}{q}$ where $p = 1, 2, \cdots, q-1$. Therefore, for every n, there exists $q \geqslant n$ and $q \neq 2$ such that $(\theta, \frac{p}{q}) \in A \times B$ for any $\theta \in A$ and $p = 1, 2, \cdots, q-1$. Then we will look at the value of these elements under $q-th$ iteration of f, as follows;

$$
\begin{aligned}
f^q\Big(\theta, \frac{p}{q}\Big) &= \Big(\theta + q\Big(2\pi\frac{p}{q}\Big), \frac{p}{q}\Big) \\
&= \Big(\theta + 2\pi p, \frac{p}{q}\Big) \\
&= \Big(\theta, \frac{p}{q}\Big)
\end{aligned}
$$

q is a prime period for $(\theta, \frac{p}{q})$. If not then there exists $k < q$ such that $\frac{2k\pi p}{q}$ is an integer, i.e. either $q = 2$, $q = k$ or $q = p$, contradict to the way we choose q. Therefore $(\theta, \frac{p}{q})$ is a periodic point of prime period q and therefore P_n is dense for every n. □

However we will show that in some spaces, P_n dense for all n implies positive entropy or even stronger than that, Devaney Chaotic.

4.2 P_n Dense for All n on the Interval

A dynamical system which has P_n dense for all n has a big tendency to behave chaotically, and even a stronger property than chaotic behaviour. On the interval, we will show that P_n dense for all n implies another special property i.e. having an invariant proper subinterval with no proper invariant subinterval (NIPS), which is the property discussed in the previous chapter. Such NIPS property is not implied by dense periodic points. For examples the maps $f(x) = x$ and $g^2(x) = x$ have dense periodic points but P_n $(n \geqslant 3)$ are not dense for f and g. For these examples, every invariant subinterval has invariant proper subinterval. By relying on this property, we get the main results on the interval as follows;

Theorem 4.6 *Let f be a continuous interval map. If f has dense P_n for all n, then there exist closed subintervals $J_1, J_2, \cdots, J_k$ of I (k can be finite or infinite) such that $I = \bigcup_{i=1}^{k} J_i$ and for every i,*

1. *there exists a periodic point of period 3 or 6 in J_i,*
2. *the entropy of f on J_i is positive,*
3. *either f or f^2 is turbulent on J_i,*
4. *f is transitive on J_i.*

Proof. Let f be a continuous interval map with P_n dense for all n. Either I has an invariant proper subinterval or not. If I does not have any invariant proper subinterval, then by Proposition 3.3, f is Devaney chaotic on I and the other properties are proven by Theorem 3.16.

For the case when I has an invariant proper subinterval, we firstly claim that for every invariant subinterval, $A \subset I$, the map $f^2|_A$ is not the identity map. If so, then P_n is not dense for all $n \geqslant 3$. Therefore by Corollary 3.8, there exist subinterval $J_1, J_2, \cdots, J_k$ (k can be finite or infinite) such that $I = \bigcup_{i=1}^{k} J_i$ and J_i is invariant with no proper invariant subinterval for f^2. By Proposition 3.3, f^2 is Devaney chaotic on J_i and so is f. The other properties of f on J_i are proven by Theorem 3.16. □

Corollary 4.7 *Let f be a continuous interval map. If f satisfies P_n dense for all n, then f is topologically chaotic and chaotic in the sense of Li and Yorke.*

Proof. This can be shown directly by the existence of periodic point of period $q2^p$ where q is odd and p is any nonnegative integer and application of Theorem 2.7 and Theoerm 2.8. □

In conclusion, the property of P_n dense for all n allows us to describe the role of dense periodicity points plays to have chaotic behavior. Dense periodic points alone is not sufficient for any sort of chaos. However P_n dense for all n does implies topological chaos and Li-Yorke chaos. Even though this stronger dense periodicity property cannot guarantee that the whole system is Devaney chaotic, nevertheless, P_n dense for all n can guarantee the existence of subsystem which is Devaney chaotic. So, an indecomposable system with P_n dense for all n is chaotic in the sense of Devaney.

4.3 P_n Dense for All n on the Unit Circle

In the study of dynamical systems of circle maps, one may consider a circle map as an interval map, as the arguments we used in the Chapter 2. However, in general circle maps and interval maps are not conjugate. Therefore the dynamics of circle maps are different from the dynamics of interval maps found in the previous section. In this section we will show that for circle maps, P_n dense for all n implies positive entropy, but this stronger density property cannot guarantee that the whole system is chaotic in the sense of Devaney even if it does not have any invariant proper subset. However we can guarantee the existence of a subsystem which is Devaney chaotic.

Firstly, let us recall some notations that we used in Section 2.3.2 to associate a circle map with an interval map. Have been shown earlier that every continuous circle map $f : S^1 \to S^1$ can be associate with an interval map (not necessarily continuous) $f' : [0, 2\pi) \to [0, 2\pi)$ defined by $f'(\theta) = \theta$ (for every $\theta \in [0, 2\pi)$). The relation between f and f' are as follows, as described in Remark 2.31.

Remark 4.8 *The following are facts about f and f' for a continuous circle map f;*

1. *If f has P_n dense for all n then so f'.*
2. *If f' is transitive, then so f.*
3. *For any connected interval $A \subset (0, 2\pi)$, A is invariant under f iff A is invariant under f'.*

Proposition 4.9 *Let $f : S^1 \to S^1$ be a continuous map on the unit circle S^1 with P_n dense for all n. If there exists a closed subarc $A \subset S^1$ which is strongly invariant under f and $0 \notin \overline{A}$, then there exists a closed subarc $B \subset S^1$ such that $f|_B$ is transitive.*

Proof. Let $A \subset S^1$ such that it is strongly invariant under f and $0 \notin \overline{A}$. By Remark 4.8, $\overline{A}$ is invariant under f'. Since $0 \notin \overline{A}$, Lemma 2.30 then gives that $f'|_{\overline{A}}$ is continuous. If $\overline{A}$ has NIPS, then $f'|_{\overline{A}}$ is transitive by Proposition 3.3 and so $f|_{\overline{A}}$. If $\overline{A}$ has an invariant proper closed subinterval, then by Lemma 3.5, there exists a closed subarc $B \subset A$ such that f' does not has any strongly invariant proper closed subinterval in B. By Proposition 3.3, $f'|_B$ is transitive and so $f|_B$. □

The next question is to ask either P_n dense for all n can assure the existence of the invariant open subarc of S^1 so that the system will have a chaotic subsytem (in sense of Devaney) and have positive entropy (topologically chaotic). We give the positive answer in the main theorem in this section as follows;

Theorem 4.10 *Let $f : S^1 \to S^1$ be a continuous map on the unit circle S^1. If f satisfies P_n dense for all n, then the entropy of f is positive.*

Proof. Suppose f has dense P_n for all n but is not transitive on S^1. We firstly claim that there exists a subarc $(\theta_1, \theta_2) \subset [0, 2\pi)$ such that $f^m(\theta_1, \theta_2) = (\theta_1, \theta_2)$ for some integer m. Since f is not transitive, there exists an open subarc A such that $\overline{\bigcup_{i\in\mathbb{N}} f^i(A)} \neq [0, 2\pi)$. Let $\theta \in A$ such that $f^n(\theta) = \theta$ with the smallest period in A. Therefore for every $m \in \mathbb{N}$ and for every $i = 0, 1, 2, \cdots, n-1$, $f^{mn+i}(A)$ contain $f^i(\theta)$. Since f has dense periodic points in $[0, 2\pi)$, for every i $f^{mn+i}(A)$ is a non-degenerate subarc i.e. it is neither empty nor reduced to a single point. Hence for every $i = 0, 1, 2, \cdots, n-1$, $K_i = \bigcup_{m\in\mathbb{N}} f^{mn+i}(A)$ is a non-degenerate subarc and $\bigcup_{i\in\mathbb{N}} f^i(A) = \bigcup_{i=0}^{n-1} K_i$. Therefore $\bigcup_{i=0}^{n-1} K_i$ is strongly invariant and $f^m(K_i) = K_i$ for every $i = 0, 1, 2, \cdots, n-1$ and for some $m \leqslant n-1$.
Suppose $(\theta_1, \theta_2) \subset [0, 2\pi)$ such that $f^m(\theta_1, \theta_2) = (\theta_1, \theta_2)$ for some integer m. Choose $\varphi \in [0, 2\pi)$ such that $0 \notin [\theta_1 + \varphi, \theta_2 + \varphi]$ and define a continuous circle map $g : S^1 \to S^1$ such that $g(\theta) = f^m(\theta - \varphi) + \varphi$. We then claim that $[\theta_1 + \varphi, \theta_2 + \varphi]$ is strongly invariant under g. By Remark 4.8, it is sufficient to show that the closed subinterval $[\theta_1 + \varphi, \theta_2 + \varphi]$ is strongly invariant under the interval map g'. Since $f^m(\theta_1, \theta_2) = (\theta_1, \theta_2)$ then $g(\theta_1 + \varphi, \theta_2 + \varphi) = (\theta_1 + \varphi, \theta_2 + \varphi)$ i.e. $g'(\theta_1 + \varphi, \theta_2 + \varphi) = (\theta_1 + \varphi, \theta_2 + \varphi)$. Since $0 \notin [\theta_1 + \varphi, \theta_2 + \varphi]$ then the interval map g' is continuous on the interval $[\theta_1 + \varphi, \theta_2 + \varphi]$. This gives $g'[\theta_1 + \varphi, \theta_2 + \varphi] = [\theta_1 + \varphi, \theta_2 + \varphi]$.
For $B = [\vartheta_1, \vartheta_2]$, we claim that $f|_{[\vartheta_1 - \varphi, \vartheta_2 - \varphi]}$ is transitive. So let (α_1, α_2) and (β_1, β_2) be any subarcs of $[\vartheta_1 - \varphi, \vartheta_2 - \varphi]$. Therefore $(\alpha_1 + \varphi, \alpha_2 + \varphi)$ and $(\beta_1 + \varphi, \beta_2 + \varphi)$ are subarcs of $B = (\vartheta_1, \vartheta_2)$. Since $g|_B$ is transitive, there exists an integer l such that

$g^l(x) \in (\beta_1 + \varphi, \beta_2 + \varphi)$ for some $x \in (\alpha_1 + \varphi, \alpha_2 + \varphi)$. Therefore $f^{ml}(x) \in (\beta_1, \beta_2)$ for some $x - \varphi \in (\alpha_1, \alpha_2)$. Hence $f|_{[\vartheta_1 - \varphi, \vartheta_2 - \varphi]}$ is transitive. By Theorem 2.33 f has positive entropy. $\square$

We will end this section by highlighting the different between consequences of P_n dense for alll n on the interval and on the unit circle. On the interval, P_n dense for all n implies that the whole system can be decomposed into subsystems where every system is Devaney chaotic. Therefore the system with NIPS property is Devaney chaotic whenever it satisfies this strong dense periodicity property. In fact, with or without NIPS property this strong property implies that the system has positive entropy. That also happen on the unit circle. P_n dense for all n can guarantee that the system on the unit circle has positive entropy i.e. is behaving topologically chaotic. This implication point out that the property of P_n dense for all n is more significance than dense periodic points since dense periodic points will not implies any sort of chaos.

Unlike what happen on the interval, this stronger density property cannot guarantee that the whole system is chaotic in the sense of Devaney even if it does not have any invariant proper subset. This is because the only invariant subinterval under f' is the whole interval $[0, 2\pi)$ which contains 0 and therefore discontinuity of f' at some points is possibly occurs. Hence its prevent us to use the same argument to show that the whole system is Devaney chaotic. However it is an interesting fact that this strong property can guarantee that a Devaney chaotic subsystem exists.

4.4 P_n Dense for All n on Shifts of Finite Type

Let us recall some notations and properties of subshifts of finite type as discussed in Section 2.3.3 and that will be used throughout this section. Let $\mathcal{A}$ be an alphabet of k symbols $0, 1, \cdots, k-1$ and Σ_k be the full-k-shift equipped with the shift map σ. A subshift $X \subset \Sigma_k$ is a shift of finite type if X can be described by finitely many forbidden blocks. The shift of finite type X is an m-step shift of finite type if X can be described by finitely many forbidden blocks of size m. For a shift space X, an $l-$block, w is allowed by X if there exists $\mathbf{x} \in X$ such that w occurs in $\mathbf{x}$. So then by $B_l(X)$ we denote the set of all $l-$blocks allowed by X. We also denote by $B(X)$ the set of all blocks allowed for X. The topology on $X \subset \Sigma_k$ is the subspace topology induced from topology onΣ_k as described in Section 1.2. Therefore, corresponding to $w \in B_l(X)$ for some integer l, we will denote the basic open set X_w of X as being generated by w (as described in Section 1.2). Hence $X_w = \{\mathbf{x} \in X | x_0 x_1 \cdots x_{l-1} = w\}$. Throughout this section, X denotes a subshift of finite type over k symbols in alphabet $\mathcal{A}$. The property of P_n dense for all n on shifts of finite type X has a slightly different implication on chaotic behavior compared to the spaces discussed in the two previous sections. Our strategy to solve this is to start looking at 1-step shift of finite type and then using Proposition 2.47 to generalize the result on 1-step shift of finite type to all kinds of shifts of finite type.

Lemma 4.11 *Let $X \subseteq \Sigma_k$ be a 1-step shift of finite type. X has P_n dense for all n is equivalent with the property; for all $a, b \in \mathcal{A}$ where $ab \in B_2(X)$ and for all positive integers n there exists $m \geqslant n$ and $w \in B_m(X)$ such that $bwa \in B(X)$ and w contains at least two different symbols in $\mathcal{A}$.*

Proof. Let X be a 1-step shift of finite type and P_n for all n be dense on X. Let $ab \in B_2(X)$ for some $a, b \in \mathcal{A}$. The existence of m-block $w \in B_m(X)$ where m is greater than an arbitrary integer n is from the periodic point of period m in the basic open set X_{ab}, directly from the definition of P_n dense for all n. Now we want to show that w contains at least two different symbols. Suppose the opposite i.e. w can only be in the form of $w = jjj \cdots jjj$ for some $j \in \mathcal{A}$. Therefore $jjj \cdots$ is the only point in the open set X_{jj} which contradicted to P_n dense for all n.

Conversely, let $a, b \in \mathcal{A}$ such that $ab \in B_2(X)$ and n be a positive integer. Let $u = u_0 u_1 \cdots u_l$ be an $l-$allowed block in X and X_u be the basic open set in X generated by the block u. By our assumption, there exist l many blocks $w_0, w_1, \cdots, w_{l-1}$ with size at least n such that $u_1w_0u_0$, $u_2w_1u_1$, $\cdots$, $u_lw_{l-1}u_{l-1}$ are allowed. Therefore we can choose $w = w_{l-1}u_{l-1}\cdots w_1u_1w_0u_0$, a block with size $m > n$ such that $\mathbf{x} = \overline{uw}$ is a periodic point in X_u with prime period $m > n$. So, X has P_n dense for all n. $\square$

Corollary 4.12 *Let $X \subset \Sigma_k$ be a 1-step shift of finite type with P_n dense for all n. For $a \in \mathcal{A}$, there exist $b, c \in \mathcal{A}$ such that $b \neq a$, $c \neq a$ (possibly $b = c$) and cab is allowed.*

Proof. Let $a \in \mathcal{A}$. If aa is allowed, Lemma 4.11 shows that there exist a block w such that w consist of at least two different symbols and awa is allowed. Without loss of generality, let us assume that w consists of only two different symbols, b and c. This implies that either $abca$ or $abcba$ is allowed. Therefore cab or bab is allowed, which completed the proof whenever aa is allowed.
If aa is not allowed, symbols b and c which satisfies $cab \in B_3(X)$ must exist, otherwise a is forbidden in X. $\square$

To show that P_n dense for all n implies transitivity on shift of finite type $X \subset \Sigma_k$, we firstly justify this argument in the case of $k = 2$ and $k = 3$.

Lemma 4.13 *Let $X \subset \Sigma_k$ be a 1-step shift of finite type. σ is transitive on X iff for every $a, b \in \mathcal{A}$, there exist a block $w \in B(X)$ such that $awb \in B(X)$ whenever $ab \notin B_2(X)$.*

Proof. Let σ be transitive and $a, b \in \mathcal{A}$ such that $ab \notin B_2(X)$. By transitivity there exist $\mathbf{x} \in X_a$ and $\mathbf{y} \in X_b$ such that $\sigma^k(\mathbf{x}) = \mathbf{y}$. Since ab is not allowed, $k \geqslant 2$ and therefore $\mathbf{x} = aw\mathbf{y} = awb\cdots$ where $w \in B_{k-1}(X)$.
Let X_u and X_v be any two disjoint basic open subsets of X and $u \in B_m(X)$ and $v \in B_n(X)$ with $u_{m-1} = a$ and $v_0 = b$. If ab is allowed then there exists $\mathbf{x} \in X_u$ such that $\mathbf{x} = uv\cdots$ and therefore $\sigma^m(\mathbf{x}) \in X_v$. If ab is not allowed, we let $w \in B_k(X)$ such that $awb \in B(X)$. Therefore there exist $\mathbf{x} \in X_u$ such that $\mathbf{x} = uwv\cdots$ and therefore $\sigma^{k+m-1}(\mathbf{x}) \in X_v$. Hence σ is transitive. $\square$

Proposition 4.14 *If $X \subseteq \Sigma_2$ is a 1-step shift of finite type with dense P_n for all n then X is transitive.*

Proof. Let X be a 1-step shift of finite type with P_n dense for all n. We will use property equivalent to transitivity in Lemma 4.13 to show that X is transitive. So let $a, b \in \{0, 1\}$. If $a \neq b$ then ab is allowed, otherwise X only has fixed points which contradicts to P_n dense for all n. Therefore ba also is allowed by the same reason. To complete the proof we assume aa is not allowed. Since ab and ba are allowed, we have aba is allowed to complete the proof. □

Proposition 4.15 *If $X \subset \Sigma_3$ is 1-step shift of finite type with P_n dense for all n, then X_A is transitive.*

Proof. We firstly claim that for any $a \neq b$ in $\{0, 1, 2\}$, if ab is not allowed then $c \in \{0, 1, 2\}$ where $c \neq b \neq a$, acb is allowed. To show this, let ac is not allowed. Since ab is not allowed then aa must be allowed, otherwise a is forbidden in X. However it contradicts to Corollary 4.12. So, ac must be allowed. By the same argument, we can easily show that cb is also allowed.

To show that X is transitive, by Lemma 4.13, we should show that for any $a, b \in \{0, 1, 2\}$, if ab is not allowed, then there exist an allowable block w such that awb is allowed. Therefore it is sufficient to show that either axb or $axyb$ is allowed for any $x, y \in \{0, 1, 2\}$. So let us assume that ab is not allowed. If $a \neq b$, then by our claim, acb is allowed and we are finished. If $a = b$, then aa is not allowed and by Corollary 4.12, either ab or ac is allowed and either ba or ca allowed where $a \neq b \neq c$. Let ab be not allowed and by our claim ac and cb are allowed. If ba is allowed, then $acba$ is allowed, and we are finished. If ca is allowed, then aca is allowed and we are finished. Now let ab and ac are both allowed. If ba is allowed, then aba is allowed, we are finished. Finally if ca is allowed, we have aca is allowed and we are completely finished. □

Proposition 4.16 *Let $X \subset \Sigma_k$ be a 1-step shift of finite type. If X has P_n dense for all n, then X is transitive.*

Proof. The case for $k = 2$ and $k = 3$ are shown in the previous propositions. Therefore we will be using the induction method to show that it is true for all k.

So, let us assume that if a 1-step shift of finite type $X \subset \Sigma_k$ has dense P_n for all n, then X is transitive. We aim to show that a 1-step shift of finite type $Y \subset \Sigma_{k+1}$ with P_n dense for all n is transitive.

For $j \in \{0, 1, \cdots, k\}$ let us define $Y_j \subset \Sigma_k$ as a shift space over the alphabets $\{0, 1, \cdots, k\} \backslash \{j\}$ such that for all $x, y \in \{0, 1, \cdots, k\} \backslash \{j\}$, xy is allowed in Y_j iff either xjy is allowed in Y or xy is also allowed in Y. We claim that Y_j is a subshift of finite type over k alphabets and has dense P_n for all n. Y_j is a subshift of finite type since every forbidden block in Y_j is also a forbidden block in Y and therefore Y_j has a finite number of forbidden blocks. To show that Y_j has dense P_n for all n, we assume that ab is allowed in Y_j and want to show that there exists an allowable $m-$block w where $m > n$ for an arbitrary n such that bwa is allowed in Y_j. We have to consider either ab is allowed or forbidden in Y. If ab is also allowed in Y, then by P_n dense for all n, there exists $m-$block w' in Y such that $abw'a$ is allowed in Y. By Lemma 4.11, there must exists i such that $w'_i \neq j$. After discarding all element j's in the block w', the new block w'' is allowed in Y_j such that $abw''a$ is allowed in Y_j. If the size of w'' is at least n, then we are done. So, let us assume the opposite. Without loss of generality, let us assume that there is only one i such that $w_i \neq j$. Let us say $w_i = c$. There are three possibilities of w' i.e. $jjj \cdots jjc$ or $jjj \cdots jcj \cdots jjj$ or $cjjj \cdots jjj$. Every case individually will implies that $abca$ is allowed in Y_j. Moreover, the cases imply that either jc is allowed, jcj is allowed or cj is allowed in Y. If jcj is allowed in Y, then cc is allowed in Y_j. If jc is allowed in Y then by P_n dense for all n there must exists an allowable block u such that $jcuj$ is allowed in Y and therefore cuc is allowed in Y_j. If cj is allowed in Y, then by the same argument there exists an allowable block v

such that cvc is allowed in Y_j. Therefore, the $m-$ block w is either $c\overline{uc}$ or $c\overline{cc}$ or $c\overline{vc}$.
If ab is not allowed in Y, then ajb is allowed in Y. So we can use the same argument to find an $m-$block such that $abwa$ is allowed in Y_j and $m \geqslant n$.
Now we are about to show that Y is transitive. Let ab is not allowed in Y. If ab is not allowed in Y_j, then we are done since Y_j is transitive. If ab is allowed in Y_j, then ajb is allowed in X and we are done. □

The main result on P_n dense for all n for shift spaces is as follows;

Theorem 4.17 *Let $X \subset \Sigma_k$ be a shift of finite type. If X has P_n dense for all n, then X is Devaney chaotic.*

Proof. By Proposition 2.43, X is an $M-$step shift of finite type, for some integer M. By Proposition 2.47 and Proposition 2.45 $X^{[M]}$ is a 1-step shift of finite type with P_n dense for all n. Therefore by Proposition 4.16, $X^{[M]}$ is transitive and Proposition 2.47 implies that X is transitive. □

Even though P_n dense for all n implies Devaney chaos on shift of finite type, but the converse is not true. However, Theorem 4.2 shows that the converse is true whenever the shift of finite type is infinite, which is the most common sort. On the other hand, on the interval and the unit circle, P_n dense for all n does not implies Devaney chaotic but the converse is true.

Chapter 5

Compact Countable Dynamical Syatems

Part of the study of dynamics is the study of invariant subsets. Understanding the dynamics on an invariant set such as $\omega-$limit sets helps us to understand the dynamics of the whole system. One reason for this is because a system on the interval which has positive entropy is equivalent to the existence of an infinite $\omega-$limit set containing a periodic orbit [12]. For any interval map, the $\omega-$limit set is a compact subset of the interval I. The subset may be finite union of intervals, the union of a Cantor set and a countable set or a countable set [12]. As far as we known there is only a small number of studies on the dynamics of the function restricted to these countable invariant sets. Therefore in this chapter we will discuss the dynamics of function on general compact countable sets. Understanding the dynamics of compact countable set helps us to show that the interval map with infinite countable $\omega-$limit set is topologically chaotic.

Chaos in dynamical system of countable sets has been discussed by Bobok, C. in [13]. He presented several results connected to so-called distributional chaos in the context of countable dynamical system. Distributional chaos is the other notion of chaos introduced in [38]. In [40] Shapovalov, S. A gave a new solution of a problem related to a set of ordinals, formulated by Birkhoff, G. D. in [9]. Recently, there are two sets of dynamicists described the dynamics of continuous map on compact countable sets. We will give two of their interesting results on the dynamics of compact countable spaces that motivated us to extend the study of dynamics on this set. The first is as follows;

Theorem 5.1 *[20] Let $f : X \to X$ be a surjective map. There is a compact countable Hausdorff topology on X to which f is continuous if and only if one of the following is satisfied;*

1. *f is not a homeomorphism*
2. *f is a homeomorphism and has both $\mathbb{Z}$-orbit and a cycle*
3. *f is a homeomorphism and has an n_i-cycle, for each $i \leqslant k$, with the property that whenever f has an n-cycle, then n is divisible by n_i, for some $i \leqslant k$.*

The second result is from [21] gives how the dynamics of the map influence the structure of the set.

Theorem 5.2 *[21] Let X be a compact metric space and let f be a finite-to-one self-map of X with repelling periodic point x. Then $lt_A(x)$ is not a limit ordinal for any closed, forward invariant, countable set A.*

5.1 Scattered Set

In this chapter, we are not merely focus on dynamics of the compact countable set but a set which is scattered. This is because, later in this section we will show that these two spaces are equivalent. Scattered sets are frequently studied in general topology (see [34] and [28] for examples) but are less well known in dynamics. Recently, there are two sets of dynamicists revealed some dynamics pattern of continuous maps on scattered sets in [21] and [20].

Definition 5.3 *[28] Let X be a topological space, S be a subset of X and x be any point in S. The point x is said to be an isolated point of S if there exists an open set $U \subset X$ such that $U \cap S = \{x\}$. A topological space X is said to be scattered if for every closed subset C of X, the set of isolated points of C is dense in C.*

In this section, we will briefly discuss some standard terminology about the Cantor Bendixson Derivative of a set which used in the interpretation of scattered sets.

Definition 5.4 *[21] Let X be a topological space and A be a subset of X. The derived set A' of a subset A of a space X is the set of limit points of the set A. A' will be called the Cantor-Bendixson derivative of A. The iterated Cantor-Bendixson derivatives of the space X are defined inductively by*

$$\begin{aligned} X^{(0)} &= X, \\ X^{(\alpha+1)} &= \left(X^{(\alpha)}\right)' \text{ for a successor } \alpha+1, \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)} \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Note that for every ordinal α, $X^{(\alpha)}$ is closed in X and if $\beta < \alpha$ is an ordinal, $X^{\alpha} \subset X^{\beta}$. Clearly $X^{(0)}, X^{(1)}, \cdots$ is a decreasing transfinite (beyond $\aleph_0$) sequence of subsets of X. Then there is either an ordinal δ so that $X^{(\delta)} = \emptyset$ or $X^{(\delta)} = X^{(\delta+1)}$.

Remark 5.5 *A topological space X is scattered if and only if there is an ordinal δ such that $X^{(\delta)} = \emptyset$.*

Therefore every point in the scattered set X has a well-defined rank, defined as follows;

Definition 5.6 *For a scattered set X, every point x of X has a rank, call limit type of $x, lt(x) = \alpha$ if $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. We call the least ordinal δ such that $X^{(\delta)} = \emptyset$ the limit type of X, $Lt(X)$. For an ordinal α, the collection of all point of X with limit type α is the set α-th level L_{α}.*

Clearly L_α is the set of isolated points of $X^{(\alpha)}$.

Lemma 5.7 *[21] If X is compact scattered set then $Lt(X)$ is a successor ordinal $\alpha+1$ and the level L_α is finite.*

Proof. If L_α is infinte, then it must have a limit, a contradiction since $X^{(\alpha)} = \emptyset$. □

By using the Cantor Bendixson Derivative we can consider a scattered set X as a union of disjoint sets, L_α $(\alpha = 0, 1, \cdots, Lt(X))$ and therefore any point in X has a scattered rank i.e. its limit type.

An example of scattered sets is an ordinal i.e. a totally ordered set of smaller ordinals or its own predecessors. Any ordinal α is a topological space by endowing α with the natural order topology. The set of limit points of an ordinal α is precisely the set of limit ordinals less then α. The successor ordinals (and zero) less then α are isolated in α since $\{\beta+1\} = (\beta,\beta+2)$ for all $\beta < \alpha$ and $\{0\} = [0, 1)$. Therefore α can be shown as scattered by taking the minimum element, β of a closed set $A \subset \alpha$ as the isolated point contained in A and open set $(0, \beta+1)$ in α. Moreover, if X is the ordinal space $\omega^\alpha + 1$ then $X^{(\alpha+1)} = \emptyset$ and therefore the limit type of X is $\alpha + 1$ [21].

It is also a standard topological fact that every countable compact Hausdorff space is not only scattered but homeomorphic to a countable successor (or compact) ordinal. In particular every compact subset of rationals is homeomorphic to a compact countable ordinal [21]. We will show that a countable compact Hausdorff space is not only scattered but is equivalent to a compact scattered metric space. The following is our main tool to approach the main results in this section;

Lemma 5.8 *Let X be a compact Hausdorff scattered set. For every $x \in X$ there exist an open set A containing x which satisfies:*

1. *$\overline{A} \cap L_\alpha = \{x\}$ where $lt(x) = \alpha$ for some ordinal α,*

2. $\overline{A} \setminus \{x\}$ *is not closed,*

3. $\overline{A} \subset \bigcup_{\beta \leqslant \alpha} L_\beta$.

Proof. Let $x \in L_\alpha$ for an ordinal α. Therefore x is not a limit point of $X^{(\alpha)}$ and there exists an open U such that $U \cap X^{(\alpha)} = \{\alpha\}$. Since $X^{(\alpha)} = \bigcup_{\beta \leqslant \alpha} L_\beta$ and $x \in L_\alpha$, then $U \cap L_\alpha = \{x\}$. By regularity, there exists an open A such that $x \in A \subset \overline{A} \subset U$. Hence $\overline{A} \cap L_\alpha = \{x\}$. Since x is a limit point of A, $\overline{A} \setminus \{x\}$ is not closed. $\overline{A} \subset \bigcup_{\beta \leqslant \alpha} L_\beta$ is obvious since $\overline{A} \subset U$ and $U \cap L_\alpha = \{x\}$ and $U \cap L_\beta = \emptyset$ for $\beta \geqslant \alpha$. Therefore $\overline{A} \subset U \subset \bigcup_{\beta \leqslant \alpha} L_\beta$. □

Proposition 5.9 *Let X be a compact Hausdorff metric space. If X is scattered, then X is countable.*

Proof. We claim that for every α, L_α is a countable set. For all α, $X^{(\alpha)}$ is a compact metric space, therefore is separable. Besides, L_α is a dense subset of $X^{(\alpha)}$. Since every point of L_α is isolated, no subset of L_α is dense in $X^{(\alpha)}$. Therefore L_α is countable since $X^{(\alpha)}$ is separable.

Therefore it is sufficient to show that $Lt(X)$ is a countable ordinal. To show that we let $Lt(X)$ be uncountable i.e. $Lt(X) \geqslant \omega_1$. Therefore the exists $x \in X$ such that $lt(x) = \omega_1$. By Lemma 5.8, let A be a closed neighborhood of x such that $A \cap L_{\omega_1} = \{x\}$ and $A \subset \bigcup_{\alpha \leqslant \omega_1} L_\alpha$. For every $\alpha \leqslant \omega_1$ choose one $x_\alpha \in L_\alpha \cap A$ and choose n_α such that $d(x_\alpha, x) \geqslant \frac{1}{n_\alpha}$. We have uncountably many α's but only countable many n_α's since $n_\alpha \in \mathbb{N}$ is countable. Therefore we have uncountably many same n_α's. Therefore there exist uncountable $W \leqslant \omega_1$ such that $n_\beta = n$ for all $\beta \in W$. Since $d(x_\alpha, x) \geqslant \frac{1}{n}$ for all $\alpha \in W$, x is not a limit point of $\{x_\alpha : \alpha \in W\} = B$. But A is closed so B has a limit point in A. This limit point must have limit type ω_1 so cannot be in A, a contradiction. □

We reach our important result in this section.

Theorem 5.10 *Let X be a compact Hausdorff metric space. X is scattered iff X is countable.*

5.2 Finite-to-one and infinite-to-one map

In this section we will discuss about the dynamics of a continuous map on a compact Hausdorff countable metric space (or equivalently a compact Hausdorff scattered space). In general, we will show how the dynamics of maps on this set influence the structure of the set. Moreover we will show that compactness of the set constitutes a difference that distinguishes between the dynamics of finite-to-one maps and infinite-to-one maps. In this section we will always assume that the map f on the space X is surjective, otherwise we can consider the core space, $Y = \bigcap_n f^n(X)$ which is also a compact Hausdorff scattered and take the restriction of f on Y [20].

For finite-to-one continuous map, we have the following result about the limit type of the image of any point under the map.

Proposition 5.11 *Let X be a compact countable metric Hausdorff space and f be a continuous surjective map on X. If f is finite-to-one then the limit type of $f(x)$ is at least as big as the limit type of x.*

Proof. Suppose not then there is a point x such that $f(x)$ has limit type β and x has limit type α where $\beta < \alpha$ and α is the least ordinal for which this happens.
By Lemma 5.8 there is an open V such that $x \in V$ and $\overline{V} \cap L_\alpha = \{x\}$. By Lemma 5.8 choose open W such that $f(x) \in W$ and $\overline{W} \cap L_\beta = \{f(x)\}$. Consider $A = f^{-1}(W) \cap V$. This is an open neighborhood of x. Since $f^{-1}(f(x))$ is finite there is $y \in A$ such that $y \notin f^{-1}(f(x))$ and the limit type of y is less than α but at least β. But $f(y) \in V$ and limit type of $f(y) \neq \beta$ so limit type $f(y) < \beta$ so less than limit type of y, which is a contradiction since α was the least ordinal for which this happens. □

In consequence of that, continuous finite-to-one maps cannot bring any point down to a lower limit type. On the other hand, the maps are allowed to bring a point up to a higher limit type, here is the example.

Example 5.12 *Let A be a scattered set defined as $A = \{a\} \cup \bigcup_{n \in \mathbb{N}} a_n$ where $\{a_n\}_{n \in \mathbb{N}}$ is a sequence converges to a. Define f as a continuous finite-to-one map on A as follows;*

$$f(x) := \begin{cases} a & \text{if } x = a_1 \text{ or } a \\ a_{n-1} & \text{if } x = a_n \text{ for all } n > 1 \end{cases}$$

then then there exists an isolated point a_1 such that $f(a_1) = a$ which is not isolated.

Proof. A is scattered with Cantor Bendixson derivatives sets as follows;
$A^{(1)} = \{a\}$, $A^{(2)} = \emptyset$,
$L_0 = \{a_n : \text{ for all } n \in \mathbb{N}\}$, $L_1 = \{a\}$.
f is a continuous map since every open set has an open pre-image under f.
f is a finite-to-one map since every point has either one or two pre-images and $f(L_0) = L_0 \cup L_1$. □

Proposition 5.11 will not hold for continuous infinite-to-one maps as shown by the following counter-example;

Example 5.13 *Let us choose infinitely many converging sequences from $\mathbb{R}$ as follows;*
$\{a_n\}_{n \in \mathbb{N}}$ where $\lim_{n \to \infty} a_n = 0$ and $0 < a_{n+1} < a_n$ for all n,
$\{a_{n_k}\}_{k \in \mathbb{N}}$ where $\lim_{k \to \infty} a_{n_k} = a_n$ and $a_n < a_{n_{k+1}} < a_{n_k} < a_{n-1}$ for all k.
Then let us define a set $A = \{0\} \cup \{\alpha_n : n \in \mathbb{N}\} \cup \{a_{n_k} : n \in \mathbb{N}, k \in \mathbb{N}\}$. We then define a continuos infinite-to-one map f on A as follows;

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \\ a_n & \text{if } x = a_{n+1} \\ a_{n_k} & \text{if } x = a_{n_{k+1}} \\ a_{1_1} & \text{if } x \in \{a_1\} \cup \{a_{1_k} : k \geqslant 1\} \end{cases}$$

f is a continuous infinite-to-one map on the scattered space A and does not satisfy Proposition 5.11.

Proof. A is scattered with Cantor Bendixson derivatives sets as follows;
$A^{(1)} = \{a_n : \text{ for all } n\}$, $A^2 = \{0\}$, $A^{(3)} = \emptyset$ and
$L_0 = \{a_{n_k} : \text{for all } n \text{ and } k\}$, $L_1 = \{a_n : \text{ for all } n\}$, $L_2 = \{0\}$.
f is obviously continuous since every closed set has closed pre-image under f. f is infinite-to-one since a_{1_1} has infinitely many pre-images under f. f does not satisfy Proposition 5.11 since $lt(a_1) = 2$ and $lt(f(a_1)) = lt(a_{1_1}) = 1$. □

Next will be our finding on limit type of pre-images of any point under more general continuous map.

Lemma 5.14 *[29] Every continuous map from a compact space to a Hausdorffs pace is closed.*

Theorem 5.15 *Let X be a compact countable metric Hausdorff space and f be a continuous surjective map on X. For every $x \in X$ there exists $y \in X$ such that $lt(x) \leqslant lt(y)$ and $f(y) = x$.*

Proof. Suppose not then there exists an ordinal $\alpha \neq 0$ such that α is the least ordinal such that there exists $x \in L_\alpha$ which $f^{-1}(x) \cap L_\gamma = \emptyset$ for every $\gamma \geqslant \alpha$. So there exists $\beta < \alpha$ such that every $y \in f^{-1}(x)$, $lt(y) \leqslant \beta$.
Now let us consider $g = f|_{X_\beta}$ a function on a closed compact subset of X, X_β To ensure surjectivity of g, we let $Y = \bigcap_{n\in\mathbb{N}} g^n(X_\beta)$ and consider a restriction of g on Y, $g|_Y$. We claim that $x \in Y$ and Y is a compact Hausdorff scattered space. To show this, let $y_0 \in L_\beta$ such that $f(y_0) = x$. Since $\beta < \alpha$ and by our first assumption on α, it implies that for every $z \in L_\beta$, there exists $z' \in X^{(\beta)}$ such that $g(z') = z$. Hence for every n, there exists $y_n \in X^{(\beta)}$ such that $y_0 = g^n(y_n)$ and therefore $y_0 \in g^n(X^{(\beta)})$ for every n. So $x \in Y$.

For the second claim, since X_β is closed, it is compact. By Lemma 5.14, $g : X_\beta \to X$ is a closed map. Therefore $g^n(X_\beta)$ closed for all n implies Y is closed and compact. As a subset of Hausdorff space, Y is therefore Hausdorff. Y is scattered as a subset of a scattered set.

By Lemma 5.8, there exists an open set A containing x such that $\overline{A} \cap L_\alpha = \{x\}$ and $A \setminus \{x\}$ is not closed. We then let $C = g^{-1}(\overline{A}) \setminus g^{-1}(x)$. Since every element of $g^{-1}(x)$ is isolated in Y, then $g^{-1}(x)$ is an open set. By continuity of g, $g^{-1}(\overline{A})$ is closed and therefore C is closed. Since g is a closed map, $g(C)$ must be closed. But $g(C) = \overline{A} \setminus \{x\}$ is not closed.

To complete the proof, let x be isolated in X. If there is no isolated point mapped to x, Lemma 5.11 implies that x has no pre-image, a contradiction. □

Corollary 5.16 *Let X be a compact Hausdorff countable set and f be a continuous map on X. If $Lt(X) = \gamma$ then there exists x in L_γ such that x is a periodic point.*

Proof. Since X is compact, L_γ is finite. Therefore Theorem 5.15 implies the existence of a periodic point in L_γ. □

This finding leads us to another interesting result on the dynamics on the interval. As mentioned earlier in this chapter, one of the case of a $\omega-$limit set of a point on the interval is a countable set. If the set is infinite then we will show that the interval map is somehow chaotic. Before that, let us gives an important result from [37] as follows;

Theorem 5.17 *[37] Let f be an interval map. The entropy of f is positive iff there exists an infinite $\omega-$limit set which contains a periodic orbit.*

Here is our result on the dynamics of the interval maps.

Theorem 5.18 *Let f be a interval map with an countable $\omega-$limit set. If the set is infinite, then f is topologically chaotic.*

Proof. The interval map has positive entropy since the infinite countable $\omega-$limit set contains a periodic orbit by Corollary 5.16. □

As a conclusion, finite-to-one continuous maps and infinite-to-one continuous maps have different dynamics. For finite-to-one continuous map, no point can go down to a lower limit type but for an infinite-to-one continuos map, any point has a possibility to go down upon iteration of f but not the pre-images under f. Therefore, any point in the top level stays at the same level and forms a cycle.

5.3 Devaney Chaos and its notions on Compact Countable Sets

In the previous chapters, we have discussed some of the most important terms and results about dynamical systems in general or pertaining to some specific spaces such as the interval. In this section we aim to apply those terms and results on compact countable Hausdorff sets. Some of them seem to be insignificant on this set. However some results which is not hold on general are still hold on this set.

Firstly we will look at Devaney chaos and its related concepts i.e. dense periodic points and transitivity. Dense periodic points has a unique meaning and becomes trivial on compact countable Hausdorff space, as follows;

Lemma 5.19 *For a compact countable metric Hausdorff space X, every isolated point $x \in X$ is periodic iff X has dense periodic points.*

Proof. Every open subset of X has an isolated point, therefore has a periodic point. For every isolated point $x \in X$, the singleton $\{x\}$ is open. By dense periodic point, x is periodic. □

Nevertheless, the property of having dense periodic points is redundant for Devaney chaotic not only on the interval but also on compact countable Hausdorff set.

Lemma 5.20 *For a compact countable Hausdorff set X, transitivity implies dense periodic points.*

Proof. Let $x, y \in X$ be any two different isolated points. Therefore the singletons $\{x\}$ and $\{y\}$ are open sets. By transitivity there exist positive integers m and n such that $f^m(x) = y$ and $f^n(y) = x$. Therefore x and y are periodic points of period $m + n$. □

However the notion of Devaney chaos lost its importance and meaning on compact countable Hausdorff set, explained as follows;

Proposition 5.21 *Let X be a compact countable Hausdorff set. If X has infinitely many isolated points then there is no function on X is Devaney chaotic.*

Proof. Let X be a scattered set with infinitely many isolated points and f on X is a continuous map. Let $x \in X$ such that x is isolated. If the system is Devaney chaotic, then there exists $n > 0$ such that $f^n(x) = x$ and the orbit $O^+(x)$ of x has cardinality n. Since X has infinitely many isolated points, there exists an isolated point $y \in X$ such that $f^k(x) \neq y$ for any integer k. But $\{x\}$ and $\{y\}$ are both open. Therefore the system is not transitive. □

The case for compact countable Hausdorff set X with finitely many isolated points is trivial because $X^{(1)} = \emptyset$.

Lastly, we will end up this section by showing that there is another significant result that hold on the interval but not on the compact countable Hausdorff set. As mentioned in Chapter 3, turbulence implies existence of all cycles for interval map. On the other hand, the example below shows that the implication is not hold on compact countable Hausdorff set.

Example 5.22 *Let A be a scattered set defined as $A = \bigcup_{n\in\mathbb{N}}(a_n \cup b_n) \cup \{a\}$ where $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are two converging sequences converge to a. Let f be a continuous map on*

A defined as follows;

$$f(x) := \begin{cases} a_n & \text{if } x = b_{2n-1}, \forall n \in \mathbb{N} \\ b_n & \text{if } x = b_{2n}, \forall n \in \mathbb{N} \\ b_n & \text{if } x = a_{2n-1}, \forall n \in \mathbb{N} \\ a_n & \text{if } x = a_{2n}, \forall n \in \mathbb{N} \\ a & \text{if } x = a \text{ or } b \end{cases}$$

f is turbulent on A but every periodic point is either a fixed point or periodic two point.

Proof. A is scattered with Cantor Bendixson derivatives sets as follows;

$A^{(1)} = \{a\}$, $A^{(2)} = \emptyset$ and

$L_0 = \{a_n : \text{ for all } n\} \cup \{b_n : \text{ for all } n\}$, $L_1 = \{a\}$.

A is continuous since every open set has an open pre-image under f.

Let $A_1 = \{a_n : \text{ for all } n\}$ and $A_2 = \{b_n : \text{ for all } n\}$. Therefore $f(A_1) = f(A_2) = A_1 \cup A_2$ and f is turbulent.

It is easy to see that if $x \in A$ and x is neither a, b, a_1 nor a_2 then there exist $k \geqslant 1$ such that $f^k(x) = a_1$. Since a_1 is a 2-cycle, then the point x is not a cycle. Therefore A only has fixed points and 2-cycles. □

In the previous section, we showed that an interval map with infinite countable compact $\omega-$limit set is topologically chaotic. However in this section we showed that the interval map restricted to the set is not chaotic in the sense of Devaney.

Chapter 6

Conclusion

This chapter is the last main division of this discourse, containing a summing up of the points reached in every chapter. We will also include some conjectures that have arisen from our results related to some results in the literature.

Our work generally was concerned about how the value of states in the system change over time. This study has been called a study of the dynamics of the system, as described in our first chapter. We restricted our attention to a discrete dynamical system of a space X with a continuous map f on it. Therefore the study of the dynamical system (X, f) is the study of the behavior of $x, f(x), f^2(x) \cdots$ for every $x \in X$. In general, we took X as a compact metric space and in particular we look at four different topological spaces, the interval, the unit circle, shifts of finite type and compact countable spaces.

One of the main topics of dynamical study is what has been called chaotic behavior. In Chapter 2, we presented, in detail, three definitions of chaos and managed to differentiate chaotic behavior between the interval, unit circle and shifts of finite type. The dynamics of interval map have been studied extensively. On the other hand, not much is known about dynamics on the unit circle and shifts of finite type. As a result, as far as we concern no one is aware that two well known conditions that individually imply Devaney chaos on the interval, also imply Devaney chaos on the unit circle and shifts of finite type.

These conditions are locally everywhere onto and transitive. In general, transitivity does not necessarily imply Devaney chaos, Example 2.32 is the counterexample on the circle map. However, it is well known that, transitivity implies Devaney chaos on the interval (Theorem 2.22). Even though this does not hold on the unit circle, we showed that it holds on shifts of finite type (Theorem 2.48). In addition, we manage to show that on the unit circle, locally everywhere onto is a necessary condition for Devaney chaos (Theorem 2.36). Since locally everywhere onto is stronger then transitivity, then locally everywhere onto implies Devaney chaos in these spaces. Moreover Devaney chaos is stronger than Li and Yorke chaos in general space (Theorem 2.14) and stronger then topological chaos on the interval and the circle (Theorem 2.26 and Theorem 2.33). Therefore we hope to extend our study to determine whether this implication does hold in a general compact metric space or not.

In Chapter 3, we furthered our study on chaotic dynamical system of the interval map. In this topic we studied the role of two conditions can play in determining chaotic behavior. The conditions are presented as "having no (weakly) invariant proper (closed) interval (NIPS for short)" and "existence of some fixed points". The conditions are closely related to transitivity and dense periodic points. In general, transitivity and the existence of dense periodic points imply Devaney chaos which is stronger then Li and Yorke chaos and topological chaos, as mentioned in the previous paragraph. In the first section of the chapter, we showed that an interval map with dense periodic points only require NIPS, rather than transitivity to be chaotic in the sense of Devaney (Proposition 3.3). By using this proposition we are able to easily describe the dynamics of an interval map with dense periodic points, discussed in the second section by Theorem 3.9.

Since we have weakened the property of transitivity to NIPS property, the next question is; can we weaken the property of dense periodic points? This question was discussed in the last section of Chapter 3. Since 1-cycle is the smallest cycle in Sarkovskii ordering, the existence of a fixed point in $(0, 1)$ is a weaker property than dense periodic points. However Example 3.4 show that NIPS and existence of a fixed point in $(0, 1)$ does not imply any chaotic behavior. However we managed to show that the interval maps with NIPS and existence of fixed points in $(0, 1)$ and $\{0, 1\}$ are somehow chaotic in the sense of Li and Yorke and topological chaos. In particular, we show that this implies the existence of 3-cycle and turbulence. This main result was explained in Theorem 3.16.

3-cycle is the biggest cycle in Sarkovskii ordering and its significance is explained by Sarkovskii Theorem and Li and Yorke's paper, Period Three Implies Chaos [30]. Turbulence as well as existence of 3-cycle is also a strong property of chaos [12]. All in all, in some unspecified way, showing that a system is having those properties is equal to showing that the system is chaotic. However we so far cannot verify its Devaney chaotic behavior. Nevertheless, having periodic points of all periods intuitively seems to relate closely to the property of having periodic points everywhere in the space. If such relation exists then taking account of Proposition 3.3 might help us to show that the system with NIPS and having some fixed points is chaotic in the sense of Devaney. If there is no such relation, then we can heavily rely on Theorem 2.26 to show that the system is Devaney chaotic. Recall that Theorem 2.26 said that the system with positive entropy implies the existence of a subset such that the restriction of the interval map on the subset is Devaney chaotic. If the subset is an interval, then it must be the whole interval, then it proves our desired result.

Realizing that weak conditions might not help us to find a shortcut to Devaney chaos, we on the other hand introduced a strong dense periodic points to replace two conditions in Devaney chaos. The strong dense periodic points was P_n dense for all n where $P_n = \{x : x \text{ is a periodic point of period greater or equal then } n\}$. This property and its implication on dynamical property were discussed in Chapter 4. In the first section, we investigate the relation between this property and Devaney chaos in general compact metric space. We found that if every basic open set is infinite then Devaney chaos implies P_n dense for all n (Theorem 4.2). However, the implication does not hold on finite sets, as shown by Example 4.4. The converse also fails to hold in general by Example 4.5. In spite of that we managed to show that the converse does hold on some spaces.

The second section of this chapter was devoted to the interval map. Since every basic open set of I is infinite, then Devaney chaos does imply P_n dense for all n. However, the converse does not hold whenever I has invariant proper subinterval. Nevertheless we shown that this system can be decomposed into subsystems such that the map is Devaney chaotic on every subsystem (Theorem 4.6). Moreover, the whole system is chaotic in the sense of Li-Yorke and topologically chaotic (Corollary 4.7). The third section was dedicated to the circle map. Here as well, Devaney chaos implies P_n dense for all n. However P_n dense for all n cannot guarantee that the unit circle S^1 can be decomposed into Devaney chaotic subsystems. Nevertheless, P_n dense for all n implies that the whole system is Li-Yorke chaotic and topologically chaotic (Theorem 4.10). In the last section, we focused on subshifts of finite type and we found that even though Devaney chaos does not implies P_n dense for all n, P_n dense for all n does implies Devaney chaotic (Theorem 4.17).

In Chapter 5, we discussed the dynamics of continuous maps on compact countable set. We gave some description how the dynamics of finite-to-one and infinite-to-one has a different influence on the structure of the set. For finite-to-one, no point can go down to a lower limit type (Theorem 5.11) but for an infinite-to-one continuous map, any point has a possibility to go down upon iteration of f (Example 5.13) but not the pre-images under f (Theorem 5.15). Therefore, any point in the top level stays at the same level and forms a cycle. Surprisingly, the notion of Devaney chaos lost its importance and meaning on this set (Theorem 5.21). However it does not necessarily mean that the system is non-chaotic in the other senses. Therefore determining the entropy of the system will be our next aim. Our results on the dynamics of compact countable set assisted us to expose one more dynamics property on the interval. The $\omega-$limit set of a point on the interval is either finite union of intervals, the union of Cantor set and a countable set or a countable set. If there exists an infinite countable $\omega-$limit set, then the interval map is topologically chaotic.

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