

FIBRED CONTEXTUAL QUANTUM PHYSICS

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SUMMARY

Inspired by the recast of the quantum mechanics in a toposical framework, we develop a contextual quantum mechanics using only the geometric mathematics to propose a quantum contextuality adaptable in every topos. The contextuality adopted here corresponds to the belief that the quantum world must only be seen from the classical viewpoints à la Bohr and consequently putting forth the notion of a context, while retaining a realist understanding. Mathematically, the cardinal object is a spectral Stone bundle $\Sigma \longrightarrow \mathcal{B}$ (between stably-compact locales) permitting a treatment of the kinematics, fibre by fibre and fully point-free. In leading naturally to a new notion of point, the geometricity permits to understand those of the base space \mathcal{B} as the contexts C — the commutative C^* -algebras of a noncommutative C^* -algebra — and those of the spectral locale Σ as the couples (C, ψ) , with ψ a state of the system from the perspective of such a C . The contexts are furnished with a natural order, the aggregation order which is installed as the specialization on \mathcal{B} and Σ thanks to (one part of) the Priestley's duality adapted geometrically as well as to the effectuality of the lax descent of the Stone bundles along the perfect maps.

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I — INTRODUCTION

I.1 — CONTEXTUALITY

The quantum mechanics is famous for its statistical predictions in spite of a determinist partial differential equation (of the first order) — the Schrödinger’s equation — concerning the temporal evolution of the quantum state. Equally known is the (il)locality of the theory [BelAsp04] while a characteristic perhaps less noticed regards the negative result advocated by the Bell–Kochen–Specker theorem [Bel66; KocSpe68] on the necessity to rebut at least one of the assumptions on the quantum observables — the selfadjoint operators acting on the (complex) Hilbert space of pure states of the system — expected naïvely to be inherited from their analogues in classical mechanics, here grossly exposed [Hel13],

1° that all the observables possess a value at every time, via a *valuation function* (attached to a state)

2° that the value which an observable outputs upon a measurement be independent of the potentiality to simultaneously measure another observable

The standpoint that the value of an observable indeed depend upon the (parts of the) commensurable observables whereto the said observable belongs is designated as « the *contextuality* of the quantum physics ». ^① In keeping true the first property, the *definiteness*, this stance is described as *(neo)realist*. The appendix A at page 117 explains less succinctly the ins and outs of the foregoing affair whereas we expose subsequently what a predicative mathematics for a finite contextual quantum mechanics may be.

I.2 — TOPOSICAL CONTEXTUAL PHYSICS

The cardinal notion that we must employ for a contextual kinematics is the one of a (localic) *bundle*, let us say f . This is constituted of a *base space* \mathcal{B} of all the possible *contexts* C — or of all the classical viewpoints or of all the possible collections of some physical commensurable variables — and a quantum state space taking the form of a *spectral space* Σ . Diagrammatically,

$$\begin{array}{c} \Sigma \\ \downarrow f \\ \mathcal{B} \end{array}$$

and over each context C exists a *spectral fibre* Σ_C expected to be the classical state space for the classical point of view that C is. A point (C, ψ) over C of the bundle f is in this manner a complete (classical) description ψ — identical to a classical state — of the system from the vista C . The object Σ itself — in company of the arrow f — must somehow be created from these classical state spaces. At first glance, it could be the coproduct of those fibres, but this position may be inadequate topologically. Fortunately, to help us along in reaching for a good

^① Let us underline that ordinarily, the contextuality is approached by the negative. That is to say, by the analysis of the incontextual quantum theories also called « noncontextual hidden variable theories » as typically found in the Eistein–Podosky–Rosen paradox. Let us also mention that, a priori, the toposical framework does not purport the idea of incorporating these.

topology, we can draw inspiration from a natural notion of refinement of context as well as of a specialization order (as on a topological space). The major task is thus to collate the fibres in a desired manner.

In details, even though remaining on a mathematically idealized position, a point C of the contextual base \mathcal{B} is realized as a commutative (unital) sub C^* -algebra of the noncommutative C^* -algebra \mathcal{A} of all the possible bounded operators acting on a finite Hilbert space \mathcal{H} (of dimension n) of a system — in one word, the typical noncommutative C^* -algebra of the traditional quantum theory. In finite dimension, the prime illustration of a C^* -algebra which is noncommutative is the algebra of matrices of size n by n ; and the typical expression of a context C , of size ℓ , of the C^* -algebra $\mathcal{A} = \text{Mat}_n(\mathbb{C})$ stems from the duality by Gelfand through the direct sum $C \simeq \bigoplus_{j \leq \ell \leq n} \mathbb{C}p_j \subseteq \mathcal{A}$ of ℓ projectors p (as matrices of size n by n) subject to two properties,

1° their completeness,

$$\text{Id}_{\mathcal{H}} = \sum_{j \leq \ell} p_j$$

2° their orthogonality,

$$\forall i, j \leq \ell, i \neq j \Rightarrow p_i p_j = 0 = p_j p_i$$

In practice though, we do not manipulate directly the contexts as algebras but, for instance, in as ideals^① of a poset playing a rôle quite important in the toposical contextuality. Indeed, a capital feature enjoyed by the contextual space is the one of *refinement* (or *aggregation*), which is nothing more nor less than the one of the subset inclusion of the underlying sets of the C^* -algebras. Incidentally, we can form a *contextual poset* $\mathcal{C}(\mathcal{A})$ of all the contexts and their inclusions as commutative C^* -algebras; and its topos $\text{Sh}(\text{Idl}(\mathcal{C}(\mathcal{A})))$ of the sheaves on a topology of the ideals $\text{Idl}(\mathcal{C}(\mathcal{A}))$ of the poset. Since the knowledge of a few projectors suffices to generate a commutative C^* -algebra — hence a context ideally — we prefer to establish, even over the ideals, the projector sequences satisfying the two aforementioned properties as the basic entities that we handle from the outset.

As evoked previously, the state space Σ of the quantum states has the possibility to bear a topological flavour connected to the contextual refinement lifted to its level. However, being naturally led to place ourselves in a topos differing from **Set** while still desiring to work topologically, we must use a different kind of topological tools from the traditional ones focusing on the points (of the topological spaces). To wit, without elaborating here, the theory of the locales done in a predicative manner — or *geometrically*^② or point-free — for the geometric framework allows (most of) the traditional topological theorems to hold in every topos once translated accordingly. Let us underline that is it not enough to be constructive-impredicative, even though it is authorized in every topos.

^① The directed subsets closed downwardly for our order.

^② This adverb is disconnected from the field of the geometry.

The main shift is about the notion of space itself as the topological ones in **Set** become the locales (in the ambient topos) — the locales in **Set** and their continuous maps become the category **Loc**. Yet the notion of point is altered too; in the sense that the (categorical) global points of the topological spaces become the generalized points of the locales. Furthermore, a subtle consideration raised by the geometricity regards a finiteness, unrelated to a topological consideration — on first thought only. Informally, a finite set is a set wherefrom we can list the elements, in a finite manner but with the possibility of a redundancy — precisely, it is finite in the *sense of Kuratowski*. A definition having the oddity, in appearance, to entail that a subset of a finite set may not be finite by a lack of an equality being decidable. We choose to consider only the finite contexts — obtained from the finite underlying sets of the matrix algebra — for they remain the sole ones to study classically.

Returning to the quantum contextuality, there exist two inequivalent methods to obtain such an object Σ ; and indeed, for both, this spectrum is a locale. The first version is also the simplest for it denies outwardly every topological idea. The spectrum Σ is the *externalization* — the étale bundle — of a *spectral sheaf* in the topos $\text{Sh}(\text{Idl}((\mathcal{C}(\mathcal{A}))^{\text{op}}))$ of the sheaves on the (ideal completion of the) dual of the contextual poset $\mathcal{C}(\mathcal{A})$. Concretely, to each context $C \simeq \bigoplus_{j \leq \ell} \mathbb{C}p_j$ is associated, through the spectral sheaf, the set — in localic terms, the discrete locale — $\ell \doteq \dim(C)$ with a decidable equality and consisting of ℓ elements isomorphic to the set of points of its Gelfand spectrum; that is to say, isomorphic to the set of its *characters*, those functionals $C \rightarrow \mathbb{C}$ multiplicative, unital and linear. The spectral bundle f takes in effect the form of a (localic) coproduct over \mathcal{B} . The behaviour of (the fibres of) the state locale Σ under the contextual refinement when we refine a small context into a bigger one is *covariant* in the sense that the direction remains, on the fibres, identical to the one between their contexts. This kind of variance is strongly linked to the discreteness of the spectral space and categorically, the bundle is an opfibration. This framework is advocated in [DoeIsh08].

Of more direct interest to us, the second version found in [HeuLanSpi09a] produces equally well a bundle $f: \Sigma \rightarrow \mathcal{B}$ as the externalization of a locale $\underline{\Sigma}$, internal to $\text{Sh}(\text{Idl}(\mathcal{C}(\mathcal{A})))$, itself dual to a sheaf bearing (internally) the property of being also a C^* -algebra unital and commutative. The duality is once more the one from Gelfand (adapted geometrically in order to work in every topos) which classically remains on the level of the category of the topological spaces and the category of the commutative and unital C^* -algebras,

$$\text{uAbCStarAlg} \simeq (\text{KHausSp})^{\text{op}}$$

sending such a C^* -algebra to its compact Hausdorff topological space of characters; sending a Hausdorff topological space to its algebra of functions. The localic analogues of those spaces being the compact regular^① locales,

$$\text{uAbCStarAlg} \simeq (\text{KRegLoc})^{\text{op}} \simeq \text{KRegFrm}$$

and in finite dimension, the internal locale $\underline{\Sigma}$ is also discrete. When we move downwards in the refinement of contexts, we move upwards in its analogue on the states, on the fibres. The refinements are *contravariant*. The conclusion is that the external bundle f is now compact and regular. Categorically, the bundle is a fibration.

^① Read Hausdorff when it comes to the topological spaces.

We must retain that in practice, the contextual state space Σ is not a (localic) coproduct, but an entity slightly more elaborated, even though it remains true to claim that its points are pairs (C, ψ) where C is a context and ψ a classical state as an element of the spectral fibre Σ_C — upon the condition to broaden the notion of point. The arrow f constituting the bundle sends a doublet (C, ψ) to its context C . Regardless of the variance, the Bell–Kochen–Specker result asserts that the bundle f is devoid of cross-sections, id est of global points.

I.3 — TOPOLOGY

The two approaches suffer from a few disadvantages. Regardless of the dimension, the covariant one systematically sees its spectral bundle always as a discrete locale since, after all, the spectral presheaf takes its values in **Set**. Moreover, the discreteness goes against the idea that the fibres ought to be the spectra from the Gelfand’s duality. As for the contravariant manner, it has for base the completion by its ideals of the poset $\mathcal{C}(\mathcal{A})$ which takes the form, in finite dimension, of a disjoint union [Cas+09],

$$\mathcal{C}(\mathcal{A}) \simeq \coprod_{\vec{t}} \mathcal{C}_{\vec{t}} / \sim_{\vec{t}}$$

of the locales $\mathcal{C}_{\vec{t}}$ whereof the points are sequences \vec{p} of projectors p being both mutually orthogonal and complete in the uncommutative C^* -algebra \mathcal{A} . The diverse traces are put into the sequences \vec{t} called *types* which can see their components ordered numerically, from the smallest to the biggest let us say. Each sequence \vec{p} generates a commutative C^* -algebra and therefore is the datum of a context. However, two sequences of projectors lead to the same context precisely when one is the permutation of the other — the permutation must leave the type, the trace sequence intact though. Hence the quotient by a relation of equivalence at fixed type. The final expression of a point of the poset is in terms of the flag manifolds; those sequences of linear subspaces, increasing in dimension, embedded in the following one in the sequence.

This being said, when we manipulate the topos of sheaves, we concretely use the completion by the ideals of the poset $\mathcal{C}(\mathcal{A})$, not the poset itself. The completion becomes a locale when endowed with its Scott topology or, equivalently, with the opens defined as the subsets (of the poset) upper saturated — this latter notion of openness pertains to the Alexandrov’s topology — whereas we evidently wish to manipulate directly the contexts as much as possible; and in a predicative setting if possible. Thus, even after the quotients, we have too many points constructively in the completion for an ideal is not necessarily principal. The situation worsens in the case where we force ourselves to consider the locale (or topological space) of the points of the completion,

$$\text{pt}(\text{Idl}(\mathcal{C}(\mathcal{A}))) \simeq \coprod_{\vec{t}} \mathcal{C}_{\vec{t}} / \sim_{\vec{t}}$$

since it leads to a dissatisfaction on the level of the constructive mathematics. Indeed, the isomorphism is valid in classical logic for every locale involved is spatial — that is to say, isomorphic to the topological space that its collection of points is. However, when we desire to remain fully constructive, we cannot assume that the locale is well represented by its points and must deal with the locale itself — in one word, the completion.

The second defect of the completion as the base space concerns the topology itself — the Alexandrov's topology — and is best illustrated by the qbit [Cas+09], which we analyse explicitly to clarify all the constructions exposed thus far. We give ourselves the uncommutative C^* -algebra $\mathcal{A} = \text{Mat}_{n=2}(\mathbb{C})$ and its poset of commutative $\text{sub}C^*$ -algebras of \mathbb{C}^2 as the Hilbert space of the system. Evidently, when we do not focus much on the symmetries coming from the unitaries, there is the complex numbers \mathbb{C} , as the least context generated by the biggest projector that is the identity and the contexts of length two given by the pairs,

$$(\mathbf{p}, 1 - \mathbf{p})$$

where \mathbf{p} is any projector of \mathcal{A} of unitary trace — since we discard the nil projector and the identity goes into the smallest context. The projectors are becomingly parametrized by three real numbers living on the sphere \mathbb{S}^2 . Indeed, we have the following characterization,

1° of the projectivity,

$$\mathbf{p}^2 = \mathbf{p} \iff (2\mathbf{p} - \text{Id})^2 = \text{Id}$$

2° of the hermicity,

$$\mathbf{p}^\dagger = \mathbf{p} \iff (2\mathbf{p} - \text{Id})^\dagger = 2\mathbf{p} - \text{Id}$$

3° of the unitary trace,

$$\text{Tr}(\mathbf{p}) = 1 \iff \text{Tr}(2\mathbf{p} - \text{Id}) = 0$$

whose consequence is that a general projector \mathbf{p} is given by a unitary \mathbf{u} ,

$$\forall (\mathbf{a}, \mathbf{b}, e) \in \mathbb{S}^2 \subset \mathbb{R}^3, \mathbf{u} \doteq 2\mathbf{p} - \text{Id} = \begin{pmatrix} e & \mathbf{a} + i\mathbf{b} \\ \mathbf{a} - i\mathbf{b} & -e \end{pmatrix}$$

It remains to identify two sequences of projectors when they generate the same context, the same commutative C^* -algebra. Typically, they are the sequences $(\mathbf{p}, \text{Id} - \mathbf{p})$ and $(\text{Id} - \mathbf{p}, \mathbf{p})$ which leads to, once translated on the sphere, the identification of the points diametrically opposed. We find that the (discrete) poset is,

$$\{\mathbb{C} \leq \mathbb{R}\mathbb{P}^2\}$$

while we put the Alexandrov topology on it — the sole elaborated open is thus the projective plane. Instead, we wish that the poset (and especially $\mathbb{R}\mathbb{P}^2$) have its subspace topology as an embedding into (some part of) the complex numbers. Naturally, we desire this in every dimension.

I.4—FOR A FIBRED CONTEXTUALITY

It rests that we must be predicative mathematically as we expected to have to settle on a topos differing from **Set**. Whence the inadequacy of the aforementioned picture. In recalling that we are in finite dimension, the work exposed here has a few advantages for,

1° it is entirely geometric

2° it treats the two frameworks in a sole one, up to a certain point in the development

3° it provides the contextual space \mathcal{B} with a « manifold topology »

Nevertheless, the last two goals are rigorously reached in classical mathematics at the very end, in removing the possibility of some infinite contexts (permitted in constructive logic).

In order to rectify the situation, we believe that we must focus on and apply (an extension of) the patch topology to a (compact regular) locale X of projector sequences \vec{p} , after having constructed this latter as a coproduct of sublocales of (some power of) the real numbers \mathbb{R} . The patch in fact applies to the locale X supplemented with the action of a category encoding the aggregations. In essence, the abstract reasoning is in the continuation of the duality between discrete posets and compact Hausdorff ones [Tow08]; duality itself reminiscent of the one between open and proper locale maps [Tow06]. Let us illustrate the case on the discrete posets; a set with a partial order associated to its discrete topology outputs the locale of ideals of the poset; the opens are the upper sets for the partial order. To retrieve the initial poset, we define the topology to be the upper sets intersected with the lower sets and the overall effect is to render the partial order as the specialization order of a new space. However, classically, the points of the two locales remain identical.

When we deal in effect with a closed partial order, the construction of the patch becomes a categorical equivalence between the category $\mathbf{KRegPos}$ of compact regular locales supplemented with a closed partial order (and (necessarily proper) monotone localic maps) and the category \mathbf{SbKLoc} of stably compact ones with the perfect arrows of \mathbf{Loc} ,

$$\mathcal{C} : \mathbf{KRegPos} \simeq \mathbf{StbKLoc}_{\text{perfect}}$$

The pullback of the order of specialization of a stably compact locale along the counit of the adjunction gives back the initial closed order. In our case, when we generalize to the compact regular presets, we loose the duality in retaining only one direction,

$$\mathcal{C} : \mathbf{KRegPreset} \longrightarrow \mathbf{StbKLoc}_{\text{perfect}}$$

Cursively, we apply the functor on a compact regular locale X — hence discreteness of its order of specialization — equipped with the closed preorder of refinement on it and we make it the specialization of a new locale X' . Fortunately, by the principle of geometricity, we are not required to look at the opens and verify the continuity of any construction we do on the projector sequences since we stick to the geometric elaborations. The continuity is automatic. Besides, to work fully externally and geometrically — as opposed to work internally in manipulating the sheaves over a site — provides a more concise and more explicit exposition of the physical concepts as we manipulate the points and not the ideals and their opens. We equally look at the theory of the lax descent in \mathbf{Loc} to analyse what happens to the bundles over the compact regular presets. This is carried out in establishing that the fibrewise Stone bundles over the compact regular locales descent down the perfect surjective locale maps.

I.5 — OUTLINE

The section II at page 9 exposes the geometric logic concisely from the literature, especially the propositional part, and states the few results we must rely upon in the remaining work. We explicitly develop the algebraic structure needed on the locale of the real numbers in II.3.1 at page 28.

The section III.3.2 at page 70 regards our personal mathematical results required on the extension of the patch topology to the compact regular locales supplemented with a closed preorder. Furthermore, it presents in III.4.2 at page 80 and thereafter our personal theorem upon the lax descent in **Loc** of the fibrewise Stone bundles.

The section IV at page 88 concerns the construction of our finite (geometric) spectral bundle as we desire it. In IV.3 at page 97, we define the closed preorder which is required for encoding the embeddings of the finite commutative subC^* -algebras — and not solely a poset of the inclusions for diverging from the dichotomy of the variances contra/cova. In IV.4 at page 103, the results from III.3.2 and III.4.2 are applied to the bundle with the purpose to generate an arrow $\Sigma \longrightarrow \mathcal{B}$ of a contextual locale \mathcal{B} , with its preorder of specialization given by the aforementioned closed preorder, equipped with its spectral bundle Σ compact and regular inside the topos $\text{Sh}(\mathcal{B})$.

Before each (sub)part, an overview is given with the relevant bibliography, sometimes restated for the important theorems.

II—GEOMETRIC LOGIC

II.01 — OVERVIEW

Several paths are permissible to establish the concept of a locale ; we can begin from a poset to gradually arrive at a frame, then the category of the frames and their morphisms and eventually define its dual to be the category of the locales — as in III.1 at page 46. The disappointment from the frames is their lack of geometricity, which means that a frame internal to a category is not preserved by the inverse image functor (parts of the geometric morphisms). A better manner for the locales to exist is the presentation of their frames, presentations which are geometric when they are developed from those (structured) sets which are presented themselves geometrically. Each presentation carries a logical theory — structural logical rules plus some axioms — and this theory possesses a Lindenbaum algebra consisting of the set of sentences quotiented by a syntactical equivalence. For the propositional geometric theories, these algebras are the frame and the logical theories themselves are the locales — for the predicate ones, these algebras are the classifying toposes.^① Naturally, we can start from scratch to build a logical theory with the right logical tools — but we can also use the notion of site (or of coverage), build a theory (of flat functors) from it and whose topos of sheaves is the classifying one.

In remaining on this intuitive level, we clarify the crucial notion of points since this latter is not prime, but derived from the framework which goes against the tide in taking for granted the opens. A point of a locale is always a (generalized) model of its theory ; an element of its Lindenbaum algebra is systematically an open of the locale and corresponds to a logical proposition of the theory. Abstractly, an open is a property, finitely observable by a program. On the level of the toposes, we can view the topos (of sheaves over a site) itself as the predicate theory, just as a locale is for a propositional one ; the points are its generalized models, the objects of the topos are the sheaves which are generalisations of the opens, when we view the traditional opens as simply subsheaves of the terminal sheaf.

In forgetting the distinction between predicate and proposition, we thus claim that a point of the locale is in a supposed open when the model satisfies the proposition, when the proposition symbolized by the open is true in the model. Indeed, when we desire to fix a formula of a logical theory, the open^② is all the possible variables making the formula true in the model — here, we think traditionally of the model as a carrier set whose elements are the interpretations of the (propositional) variables of the theory. Consequently, we can pair an open with a model to define a satisfaction relation \models wherefrom we derive the specialization (pre)order \sqsubseteq on the points of the locale ; for a point x to be smaller than another one y in \sqsubseteq means that it satisfies fewer properties than y . In other words, every proposition that the model x makes true, y does it so ; however, y can make other opens true as well, hence our desire to view it bigger, of a greater importance. The upshot is that the points of a locale are its models, the opens are the formulas, the specialization order is the model homomorphisms ; the category **Loc** is a 2-category.

^① Sometimes, the predicate theories are called the Grothendieck toposes while the propositional theories are the locales.

^② In truth, the open is a class of equivalence of formulas, where two formulas are equivalent if and only if they are deductible syntactically from each other.

In the following, we prefer to concentrate on the locales and their logics of propositions rather than on the toposes and their sheaves and predicates. We shall use the concrete situation of the locale of the reals to expose a predicate theory, even though it is equivalent to a propositional one; whereby the crafted topos is localic. Many mathematical concepts are already described geometrically; we can list all the algebraic theories involving some finitary axioms such the monoid, the group, the ring, the poset, the finite lattice; there exists also the patch topology (for locales) [Tow96c; Tow97; Esc99; Esc01; Coq03a; Vig04; Tow08]; the powerlocale (of a locale) [Vic04a; Vic04b]; the integral and the analogue of the measures [Vic08a; CoqSpi09b; Vic11]; the Banach's algebra [CoqSpi10] but also, canonically in every topos, the (Set-indexed) colimits of the locales and their finite limits; \mathbb{N} , \mathbb{Z} , \mathbb{Q} each equipped with their universal property; and naturally, there is a locale of the reals \mathbb{R} , the complex numbers \mathbb{C} — however, they are only geometric when viewed as locales, logical theories or frame presentations, but never as frames nor as their collections of points, that is to say as topological spaces. Once recast in the appropriate language, various ordinary theorems hold geometrically amongst the theorems by Alaoglu [MulPel82; Coq03a], by Heine and Borel [FouGra82; Vic97], by Tychonov [Ban88; Coq92; Coq03b; Vic05], by Hahn and Banach [MulPel91; Coq06], by Stone and Weierstrass [Ban97; Coq01], by Gelfand [Mul79; BanMul00a; BanMul00b; BanMul06], [Coq05b; CoqSpi05; CoqSpi08; CoqSpi09a; CoqSpi10] et cetera. These theorems permit us to do topology in a predicative fashion — in the sense that the powerset (of a set) is never used (as a type), nor the axiom of choice nor the excluded middle. We present succinctly the locales from their frames in III.1 at page 46. The typical categorical account is exposed in [Joh77; Mac98; Bor08]. ◇

II.1 — LOGIC OF PROPOSITIONS

II.02 — OVERVIEW

The clearest approach to the locales is through the logical aspect. As it is customary in this field, the first chapter regards the syntactic rules of the logic — grossly, the grammar; it essentially dictates what strings of formal symbols are permissible and which must be thought of the sentences of the logic. The second part deals with the semantics — the vocabulary; it dictates how we can determine the truth of the syntactical elements with the notion of model. The third aspect is the calculus, that is to say the derivation of some new sentences in our language. For the geometric logic, we use the sequent instead of the famous hilbertian system and its logical implication. In logic, the first study is the one of the propositions, then the one of the predicates. To illustrate quickly, the propositions are those rigid sentences of the kind « today it rains » and whose truth is constant for a given model, while the predicates — in varying the variables — are of the kind « today it x » where x is a variable of the type « verb for whatever weather we can conceive » and its truth depends upon the assignment of the various variables through the model.

Notwithstanding the above, our explanation is equally well achieved in restricting ourselves to the propositional theories. The abridged expositions are in [Vic93 ; Vic95b ; Vic09b ; Vic10 ; Vic13]. For a more complete picture, we refer to [MakRey77 ; FouMulSco79] and to [Vic07a] at a level quite intermediate. This sections attempts to define as many localic concepts as possible on a logical ground, without referring to the frames. Since generally, these are used for the ordinary definitions, ours are not standard and require a few concepts exposed later on — let us say for a Stone locale — in III.1 at page 46 in a traditional manner. \diamond

II.1.1 — ONTOLOGY

The ontology is the bridge between the formal strings of symbols and what the logic is suitable for, in the real world let us say; although going beyond the semantics since the ontology includes the rules of inference. The description of the one of the geometric logic is short and subtle since not so many logical connectors are used, though at the same time differ slightly from the classical ones through the syntax and in consequence, through the semantics. Overall, the geometric logic is more suitable to answer « is this observation ascertained or not ? » rather than « is this proposition true or false ? » as in classical logic. More is revealed in [Vic10 ; Vic13]. A justification to use the toposes — the characterization by Giraud of what is a topos of sheaves — for the logic of observation is found in [Vic91].

The custom is to present this logic as one of observation for several reasons; the first is the positivity by lack of negation^①; the second is the gain of a knowledge retrospective to an observation (by the sequent \vdash); the third is the finite work required to acquire a knowledge. This third characteristic stems from the finite logical connectors employed. The commutative finite meet \wedge of (two) formulas signifies that we observe both of them simultaneously — or rather that the choice of their order is irrelevant; the finite join \vee of formulas means that we observe at least one; but if this is the desired interpretation, nothing prevents us to extend the finite joins to the infinite case \bigvee . A collection of formulas can be infinite, as long as we are able to observe one of its member, we are in position to observe its infinite join. Moreover, we obtain the existence of the finite universal quantifier since in a formula $\forall x, \phi(x)$ we must verify, one by one, that each x makes the formula ϕ true; this verification is possible only for the finite sets^②; however we allow the existential quantifier \exists to quantify over the infinite sets as this latter keeps the traditional meaning of the existence of an element; the size of the set wherein we are looking for a variable to make a formula true is irrelevant.

The axioms of a theory differ equally from the classical logic. Indeed, in our case, the axioms are sequents $\alpha \vdash \beta$ holding for expression the possibility to conclude the knowledge β from the observation of (the formula) α . We intend the sequents as a source of derivations of new formulas, differently from the usual logical arrows \longrightarrow by Hilbert, even in the constructive setting. In the predicate setting, the sequent $\alpha \vdash \beta$ is even more than that for it is a sequent $\alpha \vdash_{\vec{x}} \beta$ in context^③ \vec{x} ; that is to say, it carries the supplementary knowledge that the variables

^① In the sense that there is a difficulty to discriminate. We still can implement a symbol of difference in giving it a geometric theory; typically, an element different from itself implies false and true implies that either two variables are equal or different.

^② Also, as another intuition, the inverse image function in set theory violates the preservation of the infinite intersections.

^③ Naturally, the notion of context here is disconnected from the physical one.

in the formulas α and β not subject to the influence of a quantifier — the free variables — must be present in the finite list \vec{x} of typed variables x_i of type X_i . The ontology of the contextual sequents affirms that this sequent $\alpha \vdash_{\vec{x}} \beta$ becomes true under some logical interpretation ω as soon as we can certify that the datum of a finite list \vec{a} of abstract elements a_i in the diverse abstract sets $\omega(X_i)$ making the formula α true turns into the certain existence — existence and not a stronger notion of derivation ! — of a finite list \vec{b} of elements b_i in the various abstract sets $\omega(X_i)$ such that the formula β be also true.

That this assurance of existence be the ontology of the sequents instead of an algorithm for determining the object \vec{b} as in the intuitionist ontology illustrates well what is a function in the geometric logic. In order to be a graph of a function $f: A \rightarrow B$, the predicate $\Gamma \subseteq A \times B$ of type $A \times B$ must be defined everywhere and singled valued. Typically, it is always true that for a variable a of type A , there must exist a variable b of type B such that $\Gamma(a, b)$ do hold and equally that, if we have $\Gamma(a, b)$ and $\Gamma(a, b')$ at the same instant, then we can ascertain the equality of b with b' . However, we have no means for the derivation of a b (of type B) given a a of type A and such that $\Gamma(a, b)$. In the geometric logic, a (graph of a) function ascertains solely the belongings of a pair (a, b) of type $A \times B$ to the graph; though it does not provide the manners of finding the b 's knowing only the a 's. In the intuitionist logic, it is more than the existence of b that we are able to establish, it is an algorithm to find it. Some additional subtleties are present about the cut rule and the substitution procedure [Vic07a].

II.1.2 — SYNTAX

II.03 — OVERVIEW

The syntax of the geometric logic is weak compared to the classical one for we must not use the negation, nor the implication, nor the universal quantifier (of variables) over arbitrary infinite sets; briefly said, all these sweets classical logic allows and we gorge ourselves upon. The sequent \vdash is used to conceptualize the derivation of the conclusion lying on its right from the premises lying on its left — instead of the implication \rightarrow . Because we are propositional, the contexts are inexistent and therefore, naturally, we could replace a sequent by an arrow if it were not for the interdiction for the sequent to take as argument another sequent; contrary to the system à la Hilbert where an implication is permitted to be an argument of another implication. ◇

II.04 — DEFINITION — GEOMETRIC LOGIC OF PROPOSITIONS, SIGNATURE, FORMULA

All the geometric theories (for propositions) share a common structural fragment of the geometric logic (for propositions) which exhaustively contains,

- 1° a propositional **signature** Σ consisting in a set — possibly infinite — of various (propositional) variables which are illogical
- 2° various logical symbols,
 - a) a symbol \top (for true)
 - b) a symbol \perp (for false)
 - c) a finite conjunction \wedge
 - d) a finite — little — disjunction \vee

e) a infinite — big — disjunction \bigvee

f) a parenthesis^① left (and right)

The collection $\text{Sen}(\Sigma)$ of (well-formed) formulas of Σ is the set of all the possible strings subject to the inductive principle that,

1° all the propositional symbols be sentences

2° $\bigwedge_j \alpha_j$ be a sentence, when α_j is sentence for the indices j in a finite set

3° $\bigvee_j \alpha_j$ be a sentence, when α_j is sentence for the indices j in a finite set

4° $\bigvee_j \alpha_j$ be a sentence, when α_j is sentence for the indices j in every set ◇

II.05—NOTE

The signature changes from a theory to another one whereas the logical operators remain. The illogical symbols are devoid of sense at this stage ; only the semantics sheds light on what they are. The little conjunction is interpreted as a logical « and » and the big disjunction as a logical « or ». From the foregoing rule of induction and the frame distributivity, a general sentence is a disjunct of finite conjuncts, $\bigvee_j \bigwedge_{i_j} \alpha_{i_j}$. ◇

II.06—DEFINITION — SEQUENT, THEORY, LOCALE, SUBLOCALE, CONSTRUCTIVE NEGATION

A **sequent** for a given signature is a string,

$$\alpha \vdash \gamma$$

where α, γ are formulas.

A **propositional geometric theory** or a **locale** is the datum of a signature and a collection of sequents. In addition, a « **sublocale** of a locale » is the same theory having additional sequents.

The « **geometric negation** $\neg a$ » of a formula α is^②,

$$\alpha \vdash \perp \quad \diamond$$

II.07—NOTE

A sequent is really read as an entailment between the premises and the conclusion. The premises will be, in the frame, less than the conclusion and a sequent will turn out to be an (pre)order on the formulas. An open U of the locale is a sublocale which is open precisely when we add the axiom $\top \vdash U$, closed when we add $U \vdash \perp$. When it comes to the sublocales, by the presence of more sequents to satisfy, there are less models — the points — for the sublocale than for the bigger one. ◇

^① In all evidence, nobody writes the parentheses in the courses of manipulations.

^② Loosely, a formula is false when it implies falsum.

II.1.3 — CALCULUS

II.08 — NOTE — SOUNDNESS, COMPLETENESS

The immediate step after the construction of our sentences is to derive new ones in using a few structural rules. In short, they are a system of proofs stipulating what laws to infer — as the modus ponens. Whereas in classical logic both the sequents and the traditional system by Hilbert are possible, in geometric logic the (logical) implication behave too poorly to be usable — for it would be concreted as the Heyting implication which is not preserved by the geometric morphisms — and whereat the sequents are favoured. The sequent \vdash relates more to the syntax, yet will be sound with the satisfaction relation \models which pertains to the semantic side of the logic. Briefly said, the *soundness* means that what is derivable formally from the syntax will be true semantically, while the converse — the *completeness* of the logic — means that each (true) theorem has a proof — consequently true syntactically. The equivalence between the soundness and the completeness exists for the classical logic and the geometric one involving only the finite joins — coherent logic [Coq05a]; the full geometric variant is not complete even though the soundness of its structural rules is guaranteed, which is the least that we require from every calculus.

After all, by completeness, there may be classical theorems stating the existence of an object while failing to give an algorithm to obtain and apprehend it; case inadmissible constructively, whereupon geometrically, for to prove the existence of a thing signifies the ability to construct it by hand, not merely saying it must exist because its inexistence is impossible or disastrous. The incompleteness of the logic translates on the level of the topology, as the lack of global points of many locales, and incidentally is a good advocate for the geometricity — this principle calling for a more natural definition of point permitting to retrieve the traditional theorems in topology. \diamond

II.09 — DEFINITION — LAW OF INFERENCE

A *law of inference* in geometric logic is a schema,

$$\frac{\alpha_1, \dots, \alpha_n}{\gamma}$$

where each α_j is a sequent part of the premises of the rule and γ is a concluding sequent. \diamond

II.10 — NOTE

Such a schema is read as « given the sundry hypotheses α_j we can derive the theorem γ ». \diamond

II.11 — DEFINITION — LAW OF INFERENCE FOR PROPOSITIONAL GEOMETRIC LOGIC

The rules of inference are,

1° the *identity*; we can derive every formula for free,

$$\frac{}{\alpha \vdash \alpha}$$

2° the *cut*; if a formula is both a conclusion and a premise of another one, we can discard it,

$$\frac{\alpha \vdash \beta \quad \beta \vdash \gamma}{\alpha \vdash \gamma}$$

3° the *conjunction*,

$$\frac{\alpha \wedge \beta}{\alpha} \quad \text{and} \quad \frac{\alpha \wedge \beta}{\beta}$$

$$\frac{\alpha \vdash \beta \quad \alpha \vdash \gamma}{\alpha \vdash \beta \wedge \gamma}$$

4° the *disjunction* ; in observing a formula belonging to a set, we are able to observe the corresponding join ; when S is a subset of $\text{Sen}(\Sigma)$ and α a member of S ,

$$\overline{\alpha \vdash \bigvee S}$$

and

$$\frac{\alpha \vdash \beta}{\bigvee S \vdash \beta}$$

5° the *distributivity of frames* ; when S is a subset of $\text{Sen}(\Sigma)$,

$$\overline{\alpha \wedge \bigvee S \vdash \bigvee \{\alpha \wedge s \mid s \in S\}}$$

We prefer to refer to [Vic07a] for what others are possible. \diamond

II.12 — NOTE

As in every calculus, there exist some rules of introduction and elimination of a connective — here it is for the conjunction. When there are not any premises, it means that it is always true ; in other words, it comes for free in a proof. \diamond

II.13 — DEFINITION — LINDENBAUM ALGEBRA

The « *Lindenbaum algebra* $\Omega(\mathcal{T})$ of a geometric theory \mathcal{T} » is its set of sentences quotiented by the syntactical relation of equivalence stating that two formulas are equivalent when they syntactically imply one another under the axioms of \mathcal{T} ,

$$\forall \alpha, \beta \in \text{Sen}(\Sigma), \alpha \simeq_{\mathcal{T}} \beta \iff \frac{\mathcal{T}}{\alpha \vdash \neg \beta} \quad \diamond$$

II.14 — PROPOSITION

A Lindenbaum algebra of a geometric theory is a frame. \diamond

PROOF

Indeed, the sequent of entailment is a partial order on the quotient and the logical rule of the frame distributivity assures that the joins distribute over the meets in the quotient. \bullet

II.15 — NOTE

For the case of the classical logic, the algebra is boolean. \diamond

II.16 — PROPOSITION — BARR'S THEOREM^①

Classically, when a fully geometric statement is deductible from a geometric theory and in using the axiom of choice and the classical logic then there exists a constructive proof of the statement. \diamond

II.17 — NOTE

Being only classical, this theorem does not state that it suffices to carry out a classical reasoning geometrically to be geometric — but it is the first step towards a geometric work. \diamond

^① [MacMoe06].

II.1.4 — SEMANTICS

II.18 — OVERVIEW

In the semantic field, we attribute a meaning, in the sense of a truth value, to our propositions. As the frames becoming toposes, the predicate and the propositional logics begin also to differ on this level, with, as usual, the predicate version generalizing the one for propositions. We delay the predicate case as the concept of interpretation is far simpler to expose in the case of the frames. An interpretation is a function of sets, from the set of variables to a frame ; evidently, an interpretation must be compatible with the joins and meets and carries over the Lindenbaum algebra of a theory to become a model precisely when it is compatible with the present sequents. It is from the semantics that we are able to manipulate the points of a locale. We define first two decisive locales, namely the terminal locale $\mathbf{1}$ and the Sierpinski's one \mathbb{S} . Intuitively, the interpretations of a theory \mathcal{T} in the Lindenbaum algebra of $\mathbf{1}$ is all the global points of the locale \mathcal{T} . The interpretations of \mathcal{T} into the Lindenbaum algebra of \mathbb{S} are all the opens of \mathcal{T} — this is not the ordinary manner to introduce the opens. These two Lindenbaum algebras generalize in the form of the terminal topos **Set** and the classifying topos of the object classifier \mathbb{O} on the predicate level. The global points of a predicate locale are its models in **Set**. The sheaves over a locale are thus the interpretations of the theory in the classifying topos of \mathbb{O} . \diamond

II.19 — DEFINITION — INITIAL & TERMINAL & SIERPINSKI LOCALE, POINTS, OPENS

The *initial locale* $\mathbf{0}$ is the geometric theory lacking of propositional variables, but coming with the sole axiom of inconsistency,

$$1^\circ \top \vdash \perp$$

The *terminal locale* $\mathbf{1}$ is the geometric theory without propositional variables and devoid of axiomatic.

The *Sierpinski locale* \mathbb{S} consists of the signature of one propositional letter — the signature is the singleton — subject to no axioms. \diamond

II.20 — NOTE

What are the frames ? The frame of the initial locale is the singleton ; the frame of the terminal locale has two elements classically, it is the powerset of the singleton ; the locale of Sierpinski has three opens and two points constructively.

The frame of Sierpinski is also the free frame on one generator ; a point of \mathbb{S} in a ambient topos \mathcal{E} is a frame morphism from the frame presentation of \mathbb{S} to $\Omega(1)_{\mathcal{E}}$; that is to say, a function from the singleton to $\Omega(1)_{\mathcal{E}}$, that is to say a global point of $\Omega(1)_{\mathcal{E}}$. \diamond

II.21 — PROPOSITION — LOCALIC DISTRIBUTIVE LATTICE ON \mathbb{S}

In jumping ahead of II.51 at page 25, the locale \mathbb{S} possesses a structure of a distributive lattice for it is the localic ideal completion of the poset $\mathbf{2}$ of two elements. \diamond

II.22 — DEFINITION — INTERPRETATION, GENERALIZED & STANDARD & GENERIC MODEL, SATISFACTION, VALIDITY

An « *interpretation* ω of a geometric signature Σ into a frame F » is a set function,

$$\begin{aligned}\omega &: \Sigma \longrightarrow F \\ p &\longmapsto \omega(p)\end{aligned}$$

It can be lifted to a set function,

$$\begin{aligned}\vec{\omega} &: \text{Sen}(\Sigma) \longrightarrow F \\ p &\longmapsto \vec{\omega}(p)\end{aligned}$$

converting entailment on $\text{Sen}(\Sigma)$ into the partial order on F , finite conjunctions into finite meets, big disjunctions into arbitrary joins. It is a « *generalized model* of a propositional geometric theory \mathcal{T} in F » if and only if it makes the axioms true, in the sense that « a sequent $\alpha \vdash \beta$ of \mathcal{T} is *true* under $\vec{\omega}$ » when,

$$\vec{\omega}(\alpha) \leq \vec{\omega}(\beta)$$

A « *standard model* of \mathcal{T} » is a generalized model into the frame $\Omega(\mathbf{1})$; the « *generic model* G of \mathcal{T} » is the generalized model into the Lindenbaum algebra $\Omega(\mathcal{T}) \doteq \text{Sen}(\Sigma)_{/\simeq_{\mathcal{T}}}$ such that it be the lift of the interpretation,

$$\begin{aligned}\Sigma &\longrightarrow \text{Sen}(\Sigma)_{/\simeq_{\mathcal{T}}} \\ p &\longmapsto [p]\end{aligned}$$

An « interpretation lifted to a model $\vec{\omega}$ in a frame F *satisfies* a formula p , $\vec{\omega} \models p$ » when it assigns the top element \top_F of the frame to it. A formula is *valid* if and only if it is satisfied under every interpretation. Idem for the sequents. \diamond

II.23 — PROPOSITION — FUNDAMENTAL THEOREM OF THE GEOMETRIC LOGIC

There exists a bijective correspondence between the (generalized) models $\Sigma_{\mathcal{T}} \longrightarrow F$ of a geometric theory \mathcal{T} into a frame F with the frame morphisms $\Omega(\mathcal{T}) \longrightarrow F$ from the Lindenbaum algebra $\Omega(\mathcal{T})$ of \mathcal{T} into F . In other words, there exists a bijective correspondence between geometric transformations of models of two theories — to turn the axioms of one into theorems of the other — and morphisms of their frames. \diamond

PROOF

Given a model ω of a geometric \mathcal{T} in a frame F , we define,

$$\begin{aligned}\phi &: \Omega(\mathcal{T}) \longrightarrow F \\ [\alpha] &\longmapsto \omega(\alpha)\end{aligned}$$

and verifies that the output does not depend on the representative of the equivalent class by antisymmetry of the order on the subjacent poset of the frame F .

Now, for a morphism of frames $f: \Omega(\mathcal{T}) \longrightarrow F$, the model is immediately created by composition,

$$\omega = f \circ G$$

with the generic model of the theory. \bullet

II.24 — NOTE — GEOMETRICITY FOR CONTINUITY, LOCALIC ARROW

The fundamental result justifies and enables us to define what a localic arrow between two theories^① is; it is the frame morphism going in the opposite direction.

This is also the result justifying that « the geometricity be continuity » for a locale arrow gives a frame morphism between the topologies — in sending an open to an open — and their generalizations as geometric morphisms between the (localic) toposes of sheaves.

The theorem also underlines the crucial rôle of the generic model G in order to define the points of a locale. ◇

II.25 — PROPOSITION

Every frame is the Lindenbaum algebra of a geometric theory with,

1° the signature composed of its elements

2° the axioms, the order of its subjacent poset ◇

II.26 — DEFINITION — GENERALIZED & GENERIC POINT OF A LOCALE

The « **(generalized) points** of a locale X » are all the localic arrows with codomain X ; consequently, a (generalized) point of X is a generalized model of (the theory of) X .

The generic point of a locale X is the localic arrow Id_X . ◇

II.27 — PROPOSITION — ORDER OF SPECIALIZATION

For any locales X, Y , the set $\mathbf{Loc}(X, Y)$ of « points of Y at the level X » is a dcpo once specified the **specialization order** given as the pointwise order,

$$\begin{aligned} \forall f, g \in \mathbf{Loc}(X, Y), f \leq g &\iff \forall U \in \Omega(Y), f \models U \Rightarrow g \models U \\ &\iff \forall U \in \Omega(Y), \top \leq f^*(U) \Rightarrow \top \leq g^*(U) \end{aligned}$$

The directed joins are the directed unions and the finite meets are the finite intersections. ◇

II.28 — NOTE — GLOBAL POINT

The set of all the « **global points** of a locale X » is the dcpo $\mathbf{Loc}(1, X)$.

The generic point of a locale X leads to a simplification of the specialization order. Indeed, on first thoughts, we must define the order for all the points x of X ; thanks to the generic model, we know that it suffices to look at the behaviour of the opens U under f and g , as opposed to the behaviour of U under $f \circ x$ and $g \circ x$ for every x . The details are in III.2.3 at page 60.

The point f is a « **specialization** of g » or g a « **generification** of f ». ◇

II.29 — DEFINITION — FRAME, TOPOS OF SHEAVES, AXIOMS OF SEPARATION

The « **frame** $\Omega(X)$ of a (propositional) locale X » is its Lindenbaum algebra or the dcpo $\mathbf{Loc}(X, \mathbb{S})$ of opens of X (once acquired the algebra of the locale \mathbb{S} of truth values; to wit, the finite meets, infinite joins and their distributivity).

The « **topos of sheaves** $\text{Sh}(X)$ of a locale X » is the topos of sheaves over the site that is its frame.

A locale is T_0 when the preorder of specialization is **antisymmetric**^②; a locale is R_0 when it is **symmetric**. ◇

^① And the theories are the logical definitions of the locales; whence these arrows are the new continuous functions.

^② A property which always holds for a locale.

II.30—NOTE

Each global point of X has a parallel as a morphism of frames which is itself equivalent to a complete prime filter (of the frame), namely its true kernel. Classically, the prime elements of the frame and the false kernels of frame morphisms are also points of the locale.

We systematically consider the T_0 locales; in other words, not a single locale will see its order of specialization missing its antisymmetry. A few classical characterizations of the separations for a topological space; it is T_0 when the points are distinguishable topologically; R_0 when every open set is the union of closed sets; T_1 when T_0 and R_0 simultaneously, equivalently when the singletons are the closed sets; T_2 when Hausdorff, when the points are separable, when the diagonal is proper et cetera. \diamond

II.31—PROPOSITION

Incidentally, **Loc** is enriched by a poset and is thus a 2-category. The arrows of **Loc** are necessarily Scott continuous. \diamond

II.32—DEFINITION — STONE LOCALE

A locale X is of the kind of Stone when its frame is isomorphic, as a dcpo, to $\text{Idl}(\mathbf{Loc}(X, \mathbf{2}))$. \diamond

II.33—NOTE

This definition is sound from the following fact, III.2.1 at page 55. The frame $\Omega(\mathbf{2})$ of the two-element poset $\mathbf{2}$ is isomorphic to the localic ideal completion $\text{Idl}(\mathbf{4})$ of the four-element poset. In incidence, by the universal property of the completion, a frame morphism from $\Omega(\mathbf{2})$ to $\Omega(X)$ is equivalent to a morphism of posets from $\mathbf{4}$ to $\Omega(X)$. Such a morphism of posets is the knowledge of an open and its boolean complement. \diamond

II.2—SWIFT ACCOUNT ON THE LOGIC OF PREDICATES

II.34—OVERVIEW — TERM, BOUNDED & FREE VARIABLE

The first order world is more creative than the one bounded to the propositions. The elementary claims of our fresh universe of discourse will be the equalities of two terms — or more generally, that a collection of terms are in relation, for a given relation. Concretely, for the notion of term, we can think of a monoid and the various expressions of its monoidal axioms as well as the theorems valid for a monoid. These basic claims take the form of an equality between two members of the set, members which may clearly come beforehand from an application of the internal law. Therefore, abstractly, we must have in possession, a set whose elements will be the terms, a function on the set for the internal monoidal law, an equality between terms and a constant symbol for the neutral element. Such is the notion of term.

The use of the quantifiers brings a subtlety in the concept of free and bound variables; a « variable x is **bounded** in the formula $\exists x, \alpha$ » as soon as there is an occurrence of the variable x in the formula α . A variable is **free** when it is not bounded. And whereas the classical first order logic is quite unconstrained in some sense on the (free) variables it uses for its calculus, some logics whereto the geometric one belongs favour the concept of context for a sequent. A theory remains given by a collection of sequents (in contexts). As always, once given syntax and calculus, we immediately quotient the set of formulas by syntactical equivalence under the axioms of a theory.

For the predicate geometric logic, a generalized model is given by a flock of objects in a Grothendieck topos while the Lindenbaum algebra — seen as what furnishes the new opens, the sheaves — is effectively the topos classifying the theory. Furthermore, there exists now a **category of models** of a theory in a given topos and this one is cardinal because it permits to generalize the fundamental theorem of the geometric logic for the propositions — the bijectivity between generalized models and frame morphisms. Explicitly, the models of a theory \mathcal{T} in a topos \mathcal{E} are exactly the geometric morphisms emanating from \mathcal{E} and taking values in the classifying topos of \mathcal{T} . This classifying topos deserves to be seen as the algebra of Lindenbaum of \mathcal{T} .

In order to help the comprehension, we remark informally that a formula with (free) variables in a model is equivalent to the set of all the elements of the carrier sets making the formula true — a formula is true as a set when it is the whole of its types, when it is the maximal subobject of the types involved. When a formula is devoid of free variables, it is mandatory a subset of the carrier set to the power of zero; that is to say to the singleton set. A proposition is true when it is the whole singleton; false when it is the empty set. Since all happens in a topos, a formula without free variables is no longer a subset of true — the whole space symbolized by \top in the frame — but a subsheaf of the terminal object $\mathbf{1}$ (of the topos wherein we interpret the theory). \diamond

II.35 — DEFINITION — SYNTAX

The signature of the first order geometric logic incorporates all the logical connectors of the propositional geometric logic plus,

1° some illogical symbols comprising,

- a) some **types** or sorts (which will be (generalized) carrier sets (possibly empty) once interpreted in a topos (possibly other than **Set**))
- b) a few **predicates** or relations (which will be subsets of products — or arities — of the types wherefrom they take their input)
- c) a few **function symbols** which are predicates whose arities give also the sorts of their resulting actions (id est the codomain, besides the arity of the domain)^①

2° some logical symbols,

- a) a quantifier \exists described as **existential**
- b) a quantifier \forall described as **universal** but bounded over a finite set^②
- c) an **equality** = whose inputs must possess identical sorts \diamond

^① The constant functions have no arguments, only a sort for their outputs. Typically, a function symbol is used when we describe the neutral element 0, 1 which are constant in a ring, let us say.

^② Not expressly contained in the logic, included only after we understand that the finite powerset $\mathcal{P}(X)$ of a locale X is a geometric type, and for a variable S of type $\mathcal{P}X$, $(\forall x \in S, \alpha(x))$ is sensible as a formula in context $\{x\} \subseteq S$.

II.36—CONSPECTUS — TERMS, FORMULAS

The *terms* are various elements of the sorts, once taken the values of their variables dictated by a model. A term is compelled to be seen as a function from the collection of the types of its inputs to another sort. The novelty of the geometric logic (compared to the classical one) is the use of contextual formulas; that is to say, those formulas comprising their free variables amongst the elements — not necessarily all — of the finite sets \vec{X} of typed propositional variables x, y, z, \dots . A formula always possesses a context. The *formulas well formed* and in context are constituted by all the possible strings of formal symbols subject to the inductive rule that,

- 1° the variables and constant functions be terms and formulas
- 2° $f(\vec{t})$ be a term and a formula, if f is a function symbol of arity n and t_1, \dots, t_n are terms typed for f
- 3° the string $t = s$ be a formula, if t, s are terms of identical types
- 4° the string $r(\vec{t})$ be a formula, if r is a predicate symbol of arity n and t_1, \dots, t_n are terms typed for r
- 5° the string $\exists y, \alpha$ be a formula in context $\vec{x} \cup \{y\}$, if α is a formula in context \vec{x} , y a variable not in \vec{x} ◇

II.37—DEFINITION — SEQUENT

A *sequent of predicates* is of the form,

$$\alpha \vdash_{\vec{x}} \beta$$

where α, β are formulas comprising all their free variables from the context \vec{x} , that is to say, from a finite set of (typed) variables — typed from the product \vec{X} , possibly infinite. ◇

II.38—NOTE

This new sequent is felt secretly as,

$$\forall \vec{x} \in \vec{X}, (\alpha \vdash \beta)$$

or more classically,

$$\forall \vec{x} \in \vec{X}, (\alpha \longrightarrow \beta)$$

and corresponds to the universal closure of a formula which is open in classical logic. Further extended rules of inferences for the calculus we do not wish to mention are explicit in [Vic07a]. ◇

II.39—CONSPECTUS — INTERPRETATION, CATEGORY OF MODELS

The semantics in a topos is prescribed by an *interpretation* when,

- 1° every sort becomes an object
- 2° every predicate becomes a subobject of the product of its sorts
- 3° the propositions become subobjects of the subobject classifier
- 4° every term becomes a morphism between its sorts
- 5° the existential quantifier becomes the image of an image factorization

- 6° every little conjunct becomes a pullback
- 7° every big disjunct becomes a coproduct
- 8° the equality becomes an equalizer
- 9° every formula is a subobject of the sorts of its free variables
- 10° the sequent is the order between subobjects

There exists a category $\text{Mod}(\mathcal{T}, \mathcal{E})$ of all the *models of the theory \mathcal{T} in the topos \mathcal{E}* . Typically, a model is an interpretation respecting the sequents. In full, the models are such that when a formula is a premise of a theorem, then the interpreted premise arrow — subobject of the types of the hypothesis — factorizes throughout the theorem arrow. The morphisms in this category are the flocks of the arrows of \mathcal{E} between the old types and the new ones, respecting the relations and predicates et cetera. \diamond

II.40 — DEFINITION — GEOMETRIC MORPHISM

The functors between the categories of models are the *geometric morphisms*,

$$f: \mathcal{E} \longrightarrow \mathcal{F} \iff f^*: \mathcal{F} \longrightarrow \mathcal{E} \dashv f_*: \mathcal{E} \longrightarrow \mathcal{F}$$

with f^* exact on the left — preserving the finite limits, is flat.

There exists a category **Topos** of toposes and geometric morphisms. \diamond

II.41 — NOTE

We favour the geometric morphisms over the logical functors $F: \mathcal{E} \longrightarrow \mathcal{F}$ between the toposes [Mar09] because the latter turn each interpretation — not each model ! — of \mathcal{T} in \mathcal{E} into an interpretation of \mathcal{T} in \mathcal{F} . However, the logical functor F does not restrict to a functor between the models of \mathcal{T} in the two toposes. Moreover, when the toposes are localic, the geometric morphisms are in equivalence with the localic morphisms. \diamond

II.42 — CONSPECTUS — CLASSIFYING TOPOS, GENERIC MODEL

Instead of explicitly giving the construction of the topos $\mathcal{C}(\mathcal{T})$ which « *classifies* the predicate theory \mathcal{T} », we give the universal property characterizing it; for every topos \mathcal{E} , there exists an equivalence of categories,

$$\mathbf{Topos}(\mathcal{E}, \mathcal{C}(\mathcal{T})) \simeq \text{Mod}(\mathcal{T}, \mathcal{E})$$

taking a geometric morphism f to the model $f^*(G)$ where G is the *generic model* of \mathcal{T} corresponding to the identity as a geometric morphism on $\mathcal{C}(\mathcal{T})$. To write it in full; the *classifying topos* of \mathcal{T} is the topos whose points are all the possible models of \mathcal{T} in all the possible toposes. \diamond

II.43 — DEFINITION — EMPTY THEORY, GLOBAL POINTS, Set, OBJECT CLASSIFIER, SHEAVES

The *empty predicate theory* has none axiom, none relation et cetera.

The theory \mathbb{O} of the *object classifier* is the predicate theory having one sort and none other datum. \diamond

II.44 — NOTE

The empty theory has for classifying topos the topos **Set**. The global points of the classifying topos of a geometric theory are all its models in **Set**.

A model of \mathbb{O} in a topos $\text{Sh}(X)$ is nothing but the datum of an object of $\text{Sh}(X)$, a sheaf. We conclude that the points of \mathbb{O} — as the models of \mathbb{O} in $\text{Sh}(\mathbf{1}) \simeq \mathbf{Set}$ — are the sets of the geometric logic or the sets allowed by this logic. This set theory is not classical because we cannot employ the classical connectors. \diamond

II.45 — DEFINITION — POINT & SHEAF & FRAME & TOPOS OF A THEORY, STALK

The « *points* of a predicate theory Y » are the generalized models of Y , all the geometric morphisms with codomain Y .

The « *topos of sheaves* (or the Lindenbaum algebra) $\text{Sh}(X)$ of a (predicate) theory X » is the classifying topos of the theory presenting X .

The « *stalk* $F(x)$ of a sheaf F at the point x of a predicate theory X » is the set $F(x)$ as point of \mathbb{O} .

The « *frame* of a (predicate) theory X » is the Heyting algebra consisting in the subsheaves of the terminal sheaf $\mathbf{1}$ in $\text{Sh}(X)$. \diamond

II.46 — NOTE

As we have seen, the topos of sheaves $\text{Sh}(X)$ is also the category $\mathbf{Topos}(X, \mathbb{O})$ of all the interpretations of the theory \mathbb{O} into the classifying topos of X ; it is supplemented with all the algebraic tools on \mathbb{O} respected by the pullbacks of the geometric morphisms. Explicitly, this is the finite limits, the finite colimits and the arbitrary (**Set**-)colimits.

For any locales X, Y , the set $\mathbf{Topos}(X, Y)$ is a filtered category and hereby provides a 2-categorical structure on **Topos**. \diamond

II.47 — DEFINITION — THEORY ESSENTIALLY PROPOSITIONAL

A predicate theory may not need (to implement) other types than those offered by the geometric logic, those predicate constructions inside a topos; in this case it is *essentially propositional*. \diamond

II.3 — ILLUSTRATIONS OF THE GEOMETRIC REASONING

II.48 — OVERVIEW

The geometric logic is more stringent than the impredicative-constructive one for its theorems must be constructive, but also respected by the pullbacks in **Loc** — see III.2.3 page 60. A good illustration is the notion of a complete lattice since it is constructive while the arbitrary joins and meets are altered by the pullbacks, whence our privilege of the frame presentations when it comes to the frames.

The presentations are expressions of constructions of structures out of another one. For instance; there exist the presentation of the free dcpo from a poset [Tow05b]; a presentation of the frame out of generators and relations — the most general — [Vic04b; Vic11]; a presentation of a frame out of a distributive lattice [JunMosVic08]; a presentation of a frame out of a given preframe [VicJoh91; Tow96a]; a presentation of a compact locale from a

distributive lattice [CedCoq98]; a presentation of a compact regular locale out of a normal distributive lattice [SpiVicWol13].

A presentation amounts to a set (equipped with a structure), a lot of axioms and a universal object which is the smallest object satisfying the axioms since any other one miming them must factually stem from it by a universal property. Nevertheless, to us, not all presentations are treated on an equal footing for solely those being geometric are relevant. These are stable under pullbacks of the geometric morphisms between the sheaf toposes. For instance, a presentation for the powerlocale of a locale — the points are some particular sublocales of a the said entity — is easily defined using its frame whence it fails to be geometric since these are not so. To make it so, we must use a presentation involving only geometric tools [Vic04b].

Typically a presentation (of a structure) is manifestly geometric when the generators are a kind of lattice, as long as there does not exist arbitrary meets somewhere. Another flavour of the presentations is present in the theory of the formal topology because it sets forth the purpose to use the notion of coverage and begins from a poset to do the topology fully predicatively [Coq96; Sam03; Vic06; Vic07b]. We give a few presentations whereon we shall rely in the construction of our spectral bundle. \diamond

II.49 — DEFINITION — PRESENTATION

A presentation $E \doteq A \langle G \text{ qua } S \mid R \rangle$ consists of a set E with structure A made out of generators G such that the structure S , on G , is preserved in E wherein the various relations R on G also hold. Such a presentation E does present the set with structure A when there is a morphism^① of structure S from G to E such that it satisfies the relations R and is universal amongst these arrows.

In other terms, when there is a morphism f of structure S from G to a set F with structure A , there exists a unique morphism $!_f$ of structure A from E to F such that the diagram commutes,

$$\begin{array}{ccc} G & \xrightarrow{\quad} & E \\ \downarrow \forall f & \searrow !_f & \\ F & & \end{array}$$

This unique morphism is an interpretation of the theory R (presented by the presentation) in F . \diamond

II.50 — NOTE

Concretely, the universality simplifies the morphism f into a morphism f' from the generators to the structured set F wherein the formal relation hold as equalities. When a presentation is supposed to present a frame, the axioms of the geometric theory are the relations R . By universal property, a point of a locale presented by some generators and relations is a morphism from the generators to the set of two elements — and which must respect the relations. \diamond

^① Not necessarily injective when G is deprived of a structure for instance.

II.51 — CONSPECTUS — IDEAL COMPLETION OF A POSET^①

Let there be a poset (P, \leq) . The presentation of the frame $\Omega(\text{Idl}(P))$ of the Alexandrov opens of the posets P is,

$$\begin{aligned} \Omega(\text{Idl}(P)) = \mathbf{Frm}\langle \{ \uparrow p \mid p \in P \} \mid \text{if } (p \leq_p q) \text{ then } \uparrow q \vdash \uparrow p \\ \top_p \vdash \bigvee^{\text{dir}} \{ \uparrow p \mid p \in P \} \\ \uparrow p \wedge \uparrow q \vdash \bigvee^{\text{dir}} \{ \uparrow r \mid p, q \leq_p r \} \rangle \end{aligned}$$

The frame presentation displays the behaviour of the subbasic opens of our locale. The elements p of P are manipulated as the (formal) generators $\uparrow p$ — rigorously, $\uparrow\{p\}$ — for the future frame. The relations of the presentation of $\Omega(\text{Idl}(P))$ are the axioms of the geometric theory, essentially propositional, of an ideal when we replace each $\uparrow p$ with a predicate $J(p)$ on the poset, read as « the element p lies in the ideal J ». \diamond

II.52 — PROPOSITION

The presentation of a frame $\Omega(\text{Idl}(P))$ leads to a locale whose points are the ideals of the poset — the directed subsets closed downwardly, see III.09 at page 48. \diamond

PROOF

A global point ω of the locale is a frame arrow $\omega: \Omega(\text{Idl}(P)) \longrightarrow \Omega(\mathbf{1})$ and before being a model, it is an interpretation,

$$\begin{aligned} \omega: \{ \uparrow p \mid p \in P \} &\longrightarrow \Omega(\mathbf{1}) \\ \uparrow p &\longmapsto \omega(\uparrow p) \leq \top \end{aligned}$$

with the respect of all the relations. We regard all the propositional symbols sent to true via the set,

$$\omega = \{ \uparrow p \mid p \in P \text{ and } \omega(\uparrow p) = \top \}$$

Since the model must not violate the first axiom, this set must be lower closed. Since it must verify the second axiom, the set must be inhabited. Since it verifies the third one, it must be directed from,

$$\omega(\uparrow p \wedge \uparrow q) = \top \Rightarrow \exists r \in P, p, q \leq r \text{ and } \uparrow r \in \omega$$

Our conclusion is that a model is nothing else than an ideal of the poset. The satisfaction relation is,

$$\begin{aligned} \omega \models \uparrow p &\iff \omega(\uparrow p) = \top \\ &\iff \uparrow p \in \omega \\ &\iff \exists \downarrow q \in \omega, \uparrow p = \downarrow q \\ &\iff \exists q \in \omega, q = p \end{aligned}$$

since an ideal is always the directed join of the principal ideals of its elements. \bullet

II.53 — NOTE

Almost by definition, the geometric ideal completion of a poset is a locale which is Alexandrov — see III.80 at page 59 — having for consequence that the locale is spatial, hereby justifying to focus on the global points. \diamond

II.54 — DEFINITION — KURATOWSKI'S FINITENESS^①

A set X is finite in the *sense of Kuratowski* when, as an element of the powerset $\mathcal{P}X$, it is in the \cup -subsemilattice $\mathcal{F}X$ generated by the singleton. \diamond

II.55 — NOTE — FINITE POWERSET

A finite subset of a set must be understood as a finite list of elements, with possible redundancy. The union is a mere collage of lists and in consequence can always be performed. Nonetheless, the intersection of two finite subsets is cursed when the equality remains undecidable for we must tell whether an element of one finite subset lies in the other ones or not. Moreover, a subset of a finite set may not be finite. Indeed, typically a subset is given by a (geometric) rule to determine if an element belongs to it; however, to list all those, we must also determine the ones which do not verify the rule, explicitly those which verify its impredicative negation. \diamond

II.56 — DEFINITION — LOCALE OF THE SURJECTIVE SET FUNCTIONS

We desire to define geometrically a locale,

$$\vec{\mathcal{A}} \doteq \{g : \ell \longrightarrow h \mid dg, cg \in \mathbb{N}\} = \coprod_{\ell} \coprod_{h \leq \ell} \vec{\mathcal{A}}_{h \leq \ell}$$

of all the surjective set functions on the finite subsets of \mathbb{N} . Two theories achieve this goal.

The locale $\vec{\mathcal{A}}_{h \leq \ell}$ of the surjective set functions f from the discrete locale ℓ to the discrete locale h is given by the predicate geometric theory with the signature,

$$(\ell, h, \text{graph}(f) \subseteq \ell \times h)$$

under the yoke of the axiomatic,

- 1° assuring that the types ℓ and h be interpreted as the finite subsets of \mathbb{N} with respectively ℓ and h elements
- 2° $\forall x \in \ell, \forall y, z \in h, ((x, y) \in \text{graph}(f) \wedge (x, z) \in \text{graph}(f)) \vdash y = z$
- 3° $\forall x \in \ell, \top \vdash \exists y \in h, (x, y) \in \text{graph}(f)$
- 4° $\forall y \in h, \top \vdash \exists x \in \ell, (x, y) \in \text{graph}(f)$

Or given natural numbers ℓ and h less than ℓ , the locale $\vec{\mathcal{A}}_{h \leq \ell}$ of the surjective set functions from the discrete locale ℓ to the discrete locale h is given by the propositional geometric theory with the signature,

$$\{P_{x,y} \doteq (x, y) \mid x \in \ell, y \in h\}$$

subject to the axiomatic,

- 1° $(P_{x,y} \wedge P_{x,z}) \vdash \bigvee \{\top \mid y = z\}$
- 2° $\forall x \in \ell, \top \vdash \bigvee \{P_{x,y} \mid y \in h\}$
- 3° $\forall y \in h, \top \vdash \bigvee \{P_{x,y} \mid x \in \ell\}$

We can sum these little locales into a bigger one $\vec{\mathcal{A}}$ for ease of manipulation. \diamond

^① [Vic99; Vic04a].

^① [VicDaw94; Vic02].

II.57—NOTE

The difference between the two theories takes more after our will to be in a general sheaf topos than in **Set**. In **Set**, we know that we can define geometrically the locale \mathbb{N} and its finite subsets such that we can privilege the propositions $P_{x,y}$ indexed by their elements. And when we use them, we express the existences via the (infinite) joins. In a general topos, we use the internal types constructed geometrically and favour a predicate « graph of a function ». We must use the existential quantifier for the existence. \diamond

II.58—DEFINITION — VALUATION^①

A « **probablistic valuation** on a locale X » is a function into an interval of the lower reals,

$$\begin{aligned} \mu : \Omega(X) &\longrightarrow [0, 1] \\ \mathbf{u} &\longmapsto \mu(\mathbf{u}) \\ \perp &\longmapsto 0 \\ \top &\longmapsto 1 \end{aligned}$$

under the constraint of the modular law,

$$\forall \mathbf{u}, \mathbf{v} \in \Omega(X), \mu(\mathbf{u}) + \mu(\mathbf{v}) = \mu(\mathbf{u} \wedge \mathbf{v}) + \mu(\mathbf{u} \vee \mathbf{v})$$

and of the continuity with respect to the Scott topology; explicitly, the commutation with the directed joins,

$$\forall S \subseteq_{\text{dir}} \Omega(X), \mu\left(\bigvee^{\text{dir}} S\right) = \bigvee^{\text{dir}} \mu(S)$$

The order of specialization is the pointwise comparison of the valuations.

The construction,

$$\begin{aligned} \mathcal{V} : \mathbf{Loc} &\longrightarrow \mathbf{Loc} \\ X &\longmapsto \mathcal{V}(X) \\ f &\longmapsto \mathcal{V}(f) : \mathcal{V}(df) \longrightarrow \mathcal{V}(cf) \\ &\mu \longmapsto \mu \circ \Omega(f) \end{aligned}$$

is functorial on **Loc**. \diamond

II.59—NOTE

The geometric construction \mathcal{V} can be applied to every locale when we wish to have some valuations, probabilistic or not. The value of an open under such a valuation is a lower real. The Riesz' theorem for the valuations in the lower reals [Vic08a; Vic11] integrates the functions from the initial locale to the lower reals strictly positive. Notwithstanding, there exists a second concept of valuation [CoqSpi09b] restricted to an application on the compact regular locales whose valuation locale remains equally compact and regular. The valuations take their values in the lower reals anew, but the Riesz' theorem is modified in the sense that it integrates some functions from the said compact regular locale to the entirety of the Dedekind reals, positive and negative. When they are probabilistic, the two notions of valuation coincide. \diamond

^① [Vic08a; CoqSpi09b; Vic11].

II.3.1 — THE LOCALIC REAL NUMBERS

II.60 — OVERVIEW

A lower real is nothing more nor less than a subset, rounded above for the (ordinary) numerical order, of the rational numbers. It is not negative when it does include all the negative rationals. Simply put, a lower real is an ideal of the discrete locale \mathbb{Q} with its strict numerical order — the general mechanism [Vic93 ; Gie+03] is a generalization of the ideal completion of a poset. Together they form the locale \mathbb{R} of the lower reals and their (natural, canonical) topology is the lower semicontinuity ; to wit that we can approximate by a rational number a lower real as close as we wish from below. Their specialization order is the inclusion of subsets of \mathbb{Q} . The lower reals are half of what constitutes the Dedekind reals. Constructively, a real number is a lower real and an upper real — the upper reals are dual to the lower ones and possess the upper semicontinuity — carrying the traditional topology of the open balls. The locale is regular and therefore the specialization order is the identity. Since we rely on them, to convince ourselves that the geometric logic is not completely vapid, we expose the details of their construction and especially their algebraic laws — full construction seeming to be lacking in the literature likely due to its simplicity. The reference used for the theories of the reals is [Vic07a], though [Joh82] surveys them equally.

How can we be certain that our definitions are continuous arrows — let us say, the addition — between locales ? Naturally, we could verify the continuity by the traditional manner of taking an open of the codomain, pull it back on the codomain with the inverse image (of the addition) and see for ourselves that it is an open. This is too tedious to be worthy. As a substitute, we summon the principle of geometricity after having been sure that our definitions are geometric, that is to say, we have used only geometric constructions. The quickest and infallible rule to verify that an arrow be illocalic is to focus on the orders of the specialization. If the orders are not respected, then the arrow does not exist. The subtraction of the two semireal numbers (of identical nature) is forbidden for instance for it turns a lower real into an upper real ; the specializations of their two locales are the inclusions of the subsets of \mathbb{Q} , direct for the lower reals, but reversed for the upper reals. As for the Dedekind cuts, the specialization order is the identity whereupon all is well ; we possess all the ordinary tools that the traditional sequences of Cauchy have even though, we only need them up to the square root. \diamond

II.3.1.1 — LOWER REAL NUMBER

II.61 — DEFINITION — THEORY OF A (NON-NEGATIVE) LOWER REAL

A « *lower real* L » is a (logical) model of the (predicate) geometric theory with signature,

$$\{\mathcal{Q}, L(q) \doteq (q < x) \subseteq \mathcal{Q}\}$$

alongside the axiomatic,

1° assuring that the carrier of the sort \mathcal{Q} is the rational object of the ambient topos

2° of the rounding of the lower cut,

$$\forall q \in \mathcal{Q}, L(q) \vdash \exists p \in \mathcal{Q}, L(p) \wedge q < p$$

A « *non-negative lower real* L » is a (logical) model of the (predicate) geometric theory for a lower real with the additional axiomatic,

$$1^\circ \forall q \in \mathcal{Q}, q < 0 \vdash L(q)$$

Equivalently, the predicate theory is essentially propositional for a lower real is a model of the propositional geometric theory with signature^①,

$$\{L(\mathbf{q}) \doteq (\mathbf{q}, +\infty) \mid \mathbf{q} \in \mathbb{Q}\}$$

alongside the axiomatic,

$$1^\circ L(\mathbf{q}) \vdash \bigvee \{L(\ell) \mid \mathbf{q} < \ell\} \quad \diamond$$

II.62—PROPOSITION

The models of the predicate and propositional theories of a lower real are equivalent in every topos of Grothendieck, as explained in [Vic07a, 4.7].

And in **Set**, it appears that a lower real L corresponds to a set L such that,

$$1^\circ L \subseteq \mathbb{Q}$$

$$2^\circ L = \downarrow L$$

and can equally be expressed as a directed supremum of its principal ideals,

$$L = \bigcup_{\ell \in L}^{\text{dir}} \downarrow \ell = \bigcup_{\ell \in L}^{\text{dir}} \{\mathbf{q} \in \mathbb{Q} \mid \mathbf{q} < \ell\} \quad \diamond$$

PROOF

Given an interpretation ω of our propositional theory in the frame F of classical truth values, we define a set L as,

$$L \doteq \{\mathbf{q} \in \mathbb{Q} \mid \omega(L(\mathbf{q})) = \top_F\} = \{\mathbf{q} \in \mathbb{Q} \mid \bar{\omega} \models L(\mathbf{q})\}$$

which verifies the properties desired.

And given a subset L of \mathbb{Q} arbitrary and downwardly closed, we can derive a model in a frame F as,

$$\begin{aligned} \bar{\omega} : \text{Sen}(\Sigma) &\longrightarrow F \\ L(\mathbf{q}) &\longmapsto \top_F \iff \mathbf{q} \in L \end{aligned}$$

because, for each choice of rational \mathbf{q} , the symbol $L(\mathbf{q})$ is a proposition whereto must be assigned a truth value by every model. •

II.63—NOTE

A lower real can be the empty set, corresponding then to $-\infty$; but could equally be \mathbb{Q} itself thus behaving more as $+\infty$. ◊

II.64—PROPOSITION — LOCALE OF THE LOWER REALS

The locale \mathbb{R} whose points are all the lower reals is such that its (sub)basic opens be of the form,

$$\left(\mathbf{q}, +\infty \right]$$

where \mathbf{q} is a rational. The specialization order of this locale is the inclusion of subset of \mathbb{Q} . The topology coming out of these basic opens is the Scott's topology. ◊

^① Since the rational numbers are supplied by the geometric logic, we are not even in the need to bring them as types; it is all implicit — when we assume there exists a natural number object in the ambient topos.

PROOF

Because the locale of the lower reals is the completion of \mathbb{Q} by its rounded ideals for the strict preorder $<$, we know that a (sub)basic open is a principal rounded filter (for $<$) of rational numbers [Gie+03, p. 250], [Vic93]; however, we can trade each rational number — especially those in the filters — for its principal rounded ideal.

Explicitly, to obtain a subbasic open, we must take a term \mathbf{p} consisting of a rational p and must determine all the lower reals (as subsets of \mathbb{Q}) where the propositional formula $L(\mathbf{p})$ is factually true; in other words, in those lower reals wherein the rational p lies.

We immediately know that \mathbf{p} is not in its embedding \underline{p} in $\underline{\mathbb{R}}$. Equally, it is immediate that if the rational p is in a lower real K , then it is also in every lower real numerically greater than K ; the converse need not be true. Naturally, the rational p is in the lower real $\mathbb{Q} = +\infty$.

Let us explicit the specialization order; we take two ordered points of $\underline{\mathbb{R}}$, namely some lower reals $K \sqsubseteq L$ and we note that,

$$\forall k \in K, K \in (\underline{k}, +\infty] \Rightarrow k \in L$$

but if now the points are numerically ordered as $K < L$, when we take a general open \mathcal{U} of the form $\bigvee_{\mathcal{J}} \bigwedge_{\mathcal{S}_j} (\underline{q}_{i_j}, +\infty]$,

$$\begin{aligned} K \in \mathcal{U} &\Rightarrow \exists j \in \mathcal{J}, \forall i_j \in \mathcal{S}_j, q_{i_j} \in K \Rightarrow \exists j \in \mathcal{J}, \forall i_j \in \mathcal{S}_j, q_{i_j} \in L \\ &\Rightarrow L \in \mathcal{U} \end{aligned}$$

To prove that the final topology coming from these basic opens is indeed Scott, we only need to state that an arbitrary open in a locale is automatically a Scott open.

To prove the converse, we take an open for Scott \mathcal{U} of lower reals, a lower real L in \mathcal{U} and we note that,

$$\begin{aligned} \bigcup_{\ell \in L}^{\text{dir}} \downarrow \ell = L \in \mathcal{U} &\Rightarrow \exists \ell \in L, \downarrow \ell \in \mathcal{U} \\ &\Rightarrow \mathcal{U} \subseteq \bigcup_{q \in \mathbb{Q}}^{\text{dir}} \{(\underline{q}, +\infty] \mid \downarrow q \in \mathcal{U}\} \end{aligned}$$

Now, the reverse inclusion derives from,

$$\begin{aligned} \forall q \in \mathbb{Q}, \downarrow q \in \mathcal{U} &\Rightarrow \forall L \in (\underline{q}, +\infty], \downarrow q \sqsubseteq L \\ &\Rightarrow \forall L \in (\underline{q}, +\infty], L \in \mathcal{U} \end{aligned}$$

for \mathcal{U} is closed upwardly in the specialization order. •

II.65 — NOTE — INCLUSION OF THE INFINITIES

The delimiter after a $+\infty$ must always be a bracket — not a parenthesis — for the filtered colimit of \mathbb{Q} is \mathbb{Q} itself. Moreover, the infinities $\pm\infty$ are always preincluded in $\underline{\mathbb{R}}$ for the directed joins of every subset of points must be included. ♦

II.66—PROPOSITION — MONOID OF THE LOWER REALS

The locale $\underline{\mathbb{R}}$ of all the lower reals forms a commutative monoid with neutral element,

$$\underline{0} \doteq \{q \in \mathbb{Q} \mid q < 0\}$$

and the infinities behave in effect as infinities. \diamond

PROOF

We can define the addition in the following way; for every K, L in $\underline{\mathbb{R}}$,

$$\begin{aligned} K + L &\doteq \{k + \ell \mid k \in K \text{ and } \ell \in L\} = \{q \in \mathbb{Q} \mid \exists k \in K, \exists \ell \in L, q < k + \ell\} \\ &= \bigcup_{k \in K, \ell \in L} \{q \in \mathbb{Q} \mid q < k + \ell\} \end{aligned}$$

which is a lower real for since K, L are closed downwardly, so is their sum,

$$\begin{aligned} \forall p \in \mathbb{Q}, \forall k \in K, \forall \ell \in L, p < k + \ell &\Rightarrow p - k \in L \\ &\Rightarrow p - k + k = p \in K + L \end{aligned}$$

and $K + L$ is rounded above as, for a k in K and a ℓ in L ,

$$((\exists q_k \in K, k < q_k) \text{ and } (\exists q_\ell \in L, \ell < q_\ell)) \Rightarrow k + \ell < q_k + q_\ell \in K + L$$

The addition on the lower reals is associative and commutative because the one on \mathbb{Q} is so. The algebra of the infinities is immediate. \bullet

II.67—NOTE

We could have equally described and established the addition and its properties logically from the axioms, in using the sequents. \diamond

II.68—PROPOSITION — SEMIRING OF THE NON-NEGATIVE LOWER REALS

The locale $\underline{\mathbb{R}}^+$ of the non-negative lower reals is a semiring. \diamond

PROOF

Restricted to non-negative elements, the addition of the lower reals defines an addition on $\underline{\mathbb{R}}^+$.

We define the multiplication of two non-negative lower reals K, L as,

$$\begin{aligned} KL &\doteq \{k\ell \mid 0 < k \in K \text{ and } 0 < \ell \in L\} \cup \{0 \mid 0 < k \in K \text{ and } 0 < \ell \in L\} \cup \{q \in \mathbb{Q} \mid q < 0\} \\ &= \underline{0} \cup \bigcup_{\substack{0 < k \in K \\ 0 < \ell \in L}} \{q \in \mathbb{Q} \mid q < k\ell\} \end{aligned}$$

and show that it is downwardly closed via,

$$\begin{aligned} 1^\circ \forall k \in K^+, \forall \ell \in L^+, \forall p, q \in \mathbb{Q}, p < q < k\ell &\Rightarrow p < k\ell \\ 2^\circ \forall q, p \in \mathbb{Q}, p < q < 0 &\Rightarrow p < 0 \end{aligned}$$

The roundedness is split into two cases,

$$1^\circ \text{ for } k \text{ in } K^+, \ell \text{ in } L^+,$$

$$\exists q_k \in K, \exists q_\ell \in L, (0 < k < q_k \text{ and } 0 < \ell < q_\ell) \Rightarrow (k\ell < q_k q_\ell \text{ and } q_k q_\ell \in KL)$$

$$2^\circ \forall q \in \mathbb{Q}, q < 0 \Rightarrow \exists \frac{q}{2} \in \mathbb{Q}, q < \frac{q}{2} < 0$$

To prove that $\underline{0} \doteq \{q \in \mathbb{Q} \mid q < 0\}$ annihilates all the multiplications is immediate by our definition. Let us directly prove that $\underline{1} \doteq \{q \in \mathbb{Q} \mid q < 1\}$ is the multiplicative neutral element,

1° $L \times \underline{1} \subseteq L$ because,

$$\forall q \in \mathbb{Q}, \forall 0 < \ell \in L, 0 < q < 1 \Rightarrow q\ell < \ell \Rightarrow q\ell \in L$$

2° $L \subseteq L \times \underline{1}$ in noting that,

$$\forall k \in L, \exists \ell \in L, k < \ell$$

in such a manner that,

a) if k is strictly positive, then ℓ is so too, hereby making the product $\frac{k}{\ell}\ell$ as an element of $\underline{1} \times L$

b) if k is strictly negative, then it lies already in the product

c) if k is nil, then for any rational p between zero and one, k lies in $\downarrow p\ell$

Let us check the distributivity of the multiplication over the addition in supposing H, K, L to be some non-negative lower reals in noticing immediately that,

$$L \times (K + H) \subseteq LH + LK$$

comes from the following cases,

1° the negative rationals are in $LH + LK$

2° for some rational q strictly less than $\ell(h + k)$ with ℓ strictly positive lying in L and $h + k$ strictly positive in $H + K$,

a) if h and k are positive, everything is fine

b) by symmetry between H and K , we focus on a negative k and we notice that h is positive, $q - \ell h$ is negative and write q as $q - \ell h + \ell h$

In order to prove the reverse inclusion, we split the case,

1° if p, q are not positive as rationals, then their sum is anew negative and lies in the product $L \times (K + H)$

2° if q is strictly negative in LH , for a rational p strictly positive but strictly less than $k\ell$ for some non-negatives k in K and ℓ in L ,

$$0 < q + p \Rightarrow q + p \leq q + k\ell = \ell\left(\frac{q}{\ell} + k\right) \in L \times (K + H)$$

3° if we have a rational q strictly less than ℓh and p strictly less than $m k$ for for some non-negatives ℓ, m in L, h in H and k in K ,

a) if m is not greater than ℓ then the proof ends for $q + p$ remains strictly less than $\ell(h + \frac{m}{\ell}k)$ and is thus in the required set because K is a down set and ℓ, m, h, k are positive

b) if ℓ is strictly less than m , we factorise, this time, m and conclude in the same manner

We check that this multiplication is commutative and associative essentially because these laws on \mathbb{Q} are so and because a multiplication of two rationals is positive if and only if the two multiplied are of the same sign. •

II.69 — NOTE

In the definition of KL, we do not need both k and ℓ to be positive, only one suffices but would yield a uncommutative multiplication. We equally see that the multiplication of the general lower reals cannot exist for, otherwise, it would not be continuous in the Scott topology for it would reverse the specialization order when dealing with the negative lower reals.

We put explicitly the lower real $\underline{0}$ in the definition of a multiplication to get a set closed downwardly. \diamond

II.3.1.2 — UPPER REAL NUMBER

II.70 — DEFINITION — GEOMETRIC THEORY OF A (NON-NEGATIVE) UPPER REAL

An « **upper real** » is a (logical) model of the (predicate) geometric theory with signature,

$$\Sigma = \{\mathcal{Q}, R(q) \subseteq \mathcal{Q}\}$$

alongside the axiomatic,

1° assuring that the type \mathcal{Q} is the rational object of the ambient topos

2° $\forall q \in \mathcal{Q}, R(q) \vdash \neg \exists r \in \mathcal{Q}, R(r) \wedge (r < q)$

A « **non-negative upper real** » is a (logical) model of the (predicate) geometric theory for a upper real with the additional axiom,

1° $\forall q \in \mathcal{Q}, 0 < q \vdash R(q)$ \diamond

II.71 — PROPOSITION — LOCALE OF THE (NON-NEGATIVE) UPPER REALS

An upper real R is a subset of \mathbb{Q} , rounded below and upwardly closed in the numerical order of \mathbb{Q} .

Besides, the locale \overline{R} whose points are the upper real numbers has for (sub)basic opens,

$$[-\infty, \overline{q})$$

where q is a rational. The specialization order is the opposite of the inclusion of subset of \mathbb{Q} . Its topology is the coAlexandrov one. \diamond

II.72 — PROPOSITION — ALGEBRA OF THE NON-NEGATIVE UPPER REALS

The locale \overline{R} is a commutative monoid and the locale \overline{R}^+ of non-negative upper reals is a semiring. The multiplication of two elements K, R is,

$$K.R \doteq \bigcup_{\substack{0 < k \in K \\ 0 < \ell \in L}} \{0 < q \in \mathbb{Q} \mid 0 < q < k r\} \quad \diamond$$

II.3.1.3 — ALGEBRA OF THE DEDEKIND CUTS

II.73 — DEFINITION — GEOMETRIC THEORY OF A DEDEKIND CUT

A « *Dedekind cut* » is a (logical) model of the (predicate) geometric theory with signature,

$$(\mathcal{Q}, L(q) \subseteq \mathcal{Q}, R(q) \subseteq \mathcal{Q})$$

subject to the axioms of,

1° the assurance that the type \mathcal{Q} is the rational object of the ambient topos

2° the inhabitation of the cuts,

$$\top \vdash \exists \ell, r \in \mathcal{Q}, L(\ell) \wedge R(r)$$

3° the rounding of the lower cut,

$$\forall q \in \mathcal{Q}, L(q) \vdash \exists p \in \mathcal{Q}, L(p) \wedge q < p$$

4° the rounding of the upper cut,

$$\forall q \in \mathcal{Q}, R(q) \vdash \exists p \in \mathcal{Q}, R(p) \wedge p < q$$

5° the separation of the two cuts,

$$\forall q \in \mathcal{Q}, L(q) \wedge R(q) \vdash \perp$$

6° the locatedness of the cuts,

$$\forall p, q \in \mathcal{Q}, p < q \vdash L(p) \vee R(q) \quad \diamond$$

II.74 — NOTE

The locatedness from [Vic07a] is equivalent to the more usual axiom [Joh77],

$$\forall \epsilon \in \mathcal{Q}, 0 < \epsilon \vdash \exists \ell, r \in \mathbb{Q}, L(\ell) \wedge R(r) \wedge (r - \ell) < \epsilon \quad \diamond$$

II.75 — PROPOSITION

A Dedekind cut is an ordered pair (L, R) of a lower real L and an upper real R , both inhabited and verifying,

$$1^\circ \forall q \in \mathbb{Q}, (q \in L \text{ and } q \in R) \Rightarrow \perp$$

$$2^\circ \forall p, q \in \mathbb{Q}, p < q \Rightarrow (p \in L \text{ or } q \in R) \quad \diamond$$

II.76 — PROPOSITION — LOCALE OF THE DEDEKIND REALS

The sublocale $\mathbb{R} \mapsto \underline{\mathbb{R}} \times \overline{\mathbb{R}}$ of all the Dedekind cuts x has for (sub)basis the opens of the form,

$$(p, q) \doteq \{x \in \mathbb{R} \mid p < x < q\}$$

for some rational p, q . Furthermore, the specialization order is the pair $(\sqsubseteq_{\underline{\mathbb{R}}}, \sqsubseteq_{\overline{\mathbb{R}}})$. \diamond

PROOF

Let us prove that the basic opens of \mathbb{R} are of the kind open interval. We know that a subbasic open — and in effect a basic open — in the localic product $\underline{\mathbb{R}} \times \overline{\mathbb{R}}$ is of the form,

$$(\underline{p}, +\infty] \otimes [-\infty, \overline{q})$$

for some rationals p, q . Taking our three extra axioms into account, we notice that,

$$\begin{aligned} \forall p, q \in \mathbb{Q}, \forall L \otimes R \in (\underline{p}, +\infty] \otimes [-\infty, \overline{q}), (p \in L \text{ and } q \in R) &\Rightarrow (q < p \Rightarrow q \in L) \\ &\Rightarrow \perp \\ &\Rightarrow p \leq q \end{aligned}$$

Conversely,

$$\forall p, q \in \mathbb{Q}, \forall (L, R) \in (p, q), p \leq q \Rightarrow L \otimes R \in (\underline{p}, +\infty] \otimes [-\infty, \overline{q}) \quad \bullet$$

II.77—NOTE

The locale \mathbb{R} is not the whole of the locale $\underline{\mathbb{R}} \times \overline{\mathbb{R}}$ because there are three additional axioms that a pair of lower and upper reals must satisfy to be Dedekind.

We effectively fall back on the euclidean topology ! ◇

II.78—PROPOSITION — RING OF THE DEDEKIND REALS

The locale \mathbb{R} of the real numbers forms a ring. ◇

PROOF

With the lower and upper reals together, we can manage to get a commutative group in using the minus (sign) function,

$$\begin{aligned} -: \underline{\mathbb{R}} &\longrightarrow \overline{\mathbb{R}} \\ L &\longmapsto -L \end{aligned}$$

which is well defined, continuous (for Scott) because it preserves the directed joins.^①

Let us prove then that the real $0 \doteq (0, \overline{0})$ is the sum of any finite real and its additive opposite; for any real $x \doteq (L, R)$, we check first the lower part,

1° for a negative rational q in using the alternative axiom from [Joh77],

$$\begin{aligned} 0 < -q &\Rightarrow \exists \ell, r \in \mathbb{Q}, L(\ell) \wedge R(r) \wedge (r - \ell) < -q \\ &\Rightarrow q \in L - R \end{aligned}$$

2° let us suppose thus a rational ℓ in L , a rational r in R and note that,

$$\begin{aligned} r - \ell \leq 0 &\Rightarrow r \in L \\ &\Rightarrow \perp \end{aligned}$$

in such a manner that we proved that $\neg(r \leq \ell)$, namely, the proposition $\ell < r$

^① Naturally, there is the same one from upper to lower reals.

3° the equality $R - L = \bar{0}$ of subsets of \mathbb{Q} is done in the same fashion

In order to define the multiplication for arbitrary reals, we first define the function,

$$\begin{aligned} \max: \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (\mathbf{x} \doteq (L_x, R_x), \mathbf{y} \doteq (L_y, R_y)) &\longmapsto (L_x \cup L_y, R_x \cap R_y) \end{aligned}$$

and prove that it exists via the following notes,

1° $(L_x \cup L_y, R_x \cap R_y)$ is indeed in the product $\underline{\mathbb{R}} \times \bar{\mathbb{R}}$

2° for every rational \mathbf{q} in both $L_x \cup L_y$ and $R_x \cap R_y$, we get falsum

3° for every pair of rationals \mathbf{p} strictly less than \mathbf{q} , we derive,

$$\begin{aligned} (\mathbf{p} < \mathbf{x} \vee \mathbf{x} < \mathbf{q}) \wedge (\mathbf{p} < \mathbf{y} \vee \mathbf{y} < \mathbf{q}) &= (\mathbf{p} < \mathbf{x} \wedge \mathbf{p} < \mathbf{y}) \vee (\mathbf{p} < \mathbf{x} \wedge \mathbf{y} < \mathbf{q}) \\ &\quad \vee (\mathbf{x} < \mathbf{q} \wedge \mathbf{p} < \mathbf{y}) \vee (\mathbf{x} < \mathbf{q} \wedge \mathbf{y} < \mathbf{q}) \\ &\Rightarrow (\mathbf{p} < \mathbf{x} \wedge \mathbf{p} < \mathbf{y}) \vee \mathbf{y} < \mathbf{q} \\ &\quad \vee \mathbf{x} < \mathbf{q} \vee (\mathbf{x} < \mathbf{q} \wedge \mathbf{y} < \mathbf{q}) \\ &\Rightarrow (\mathbf{p} < \mathbf{x} \vee \mathbf{p} < \mathbf{y}) \vee (\mathbf{p} < \mathbf{x} \wedge \mathbf{p} < \mathbf{y}) \end{aligned}$$

We conclude that the function,

$$\begin{aligned} \max(0, -): \mathbb{R} &\longrightarrow \mathbb{R}^+ \\ (\mathbf{x} \doteq (L_x, R_x)) &\longmapsto (L_x \cup \underline{0}, R_x \cap \bar{0}) \end{aligned}$$

is not negative. Consequently, for every real $\mathbf{x} \doteq (L, R)$ in \mathbb{R} , we pose $\mathbf{x}^\pm \doteq \max(0, \pm \mathbf{x})$ and show that \mathbf{x} equals $\mathbf{x}^+ - \mathbf{x}^-$ in obtaining the followings lines,

1° let \mathbf{r} be in \mathbb{R} ,

a) if \mathbf{r} is not negative, then it lies in $\mathbb{R}(\mathbf{x}^+ - \mathbf{x}^-) = \{\mathbf{r} + \mathbf{p} \mid 0 < \mathbf{r} \in \mathbb{R}, (0 \leq \mathbf{p} \text{ or } \mathbf{p} \in \mathbb{R})\}$

b) if \mathbf{r} is negative, there exists a \mathbf{r}' in \mathbb{R} ,

$$\mathbf{r}' < \mathbf{r} < 0 \Rightarrow \mathbf{r} = (\mathbf{r} - \mathbf{r}') + \mathbf{r}'$$

2° let the sum $\mathbf{r} + \mathbf{p}$ be in $\mathbb{R}(\mathbf{x}^+ - \mathbf{x}^-)$,

a) if $0 \leq \mathbf{p}$, then $\mathbf{r} \leq \mathbf{p} + \mathbf{r}$ such that $\mathbf{r} + \mathbf{p}$ is in \mathbb{R}

b) if \mathbf{p} is in \mathbb{R} , $\mathbf{p} < \mathbf{r} + \mathbf{p}$ and we derive the same conclusion

3° L is a subset of $\underline{0} \cup L + -(\bar{0} \cap (-L)) = \underline{0} \cup L + L^-$ since,

$$\forall \ell \in L, \exists \mathbf{k} \in L, \ell < \mathbf{k}$$

implies that,

a) if \mathbf{k} is negative or nil, we only need to write $\ell = (\ell - \mathbf{k}) + \mathbf{k}$

b) if \mathbf{k} is strictly positive, 0 is in L and we write $\ell \leq \mathbf{k} + 0$

4° if $\ell + \mathbf{k}$ is in the sum $L(\mathbf{x}^+ - \mathbf{x}^-)$,

$$\begin{aligned} \mathbf{k} \in L \text{ and } \mathbf{k} \leq 0 &\Rightarrow \mathbf{k} + \ell \leq \ell \\ &\Rightarrow \mathbf{k} + \ell \in L \end{aligned}$$

Finally, we posit the multiplication via the function,

$$\begin{aligned} \times : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x^+y^+ + x^-y^- - x^+y^- - x^-y^+ \end{aligned}$$

where we use the multiplications of the non-negative upper and lower reals to express it. In knowing that $\underline{0}$ and $\bar{0}$ are neutral elements for the additions of our semireals and in remarking that 0^\pm is nil, the real 0 annihilates any real number multiplied by it. To prove that $1 \doteq (\underline{1}, \bar{1})$ is the neutral element of \times , we understand that $1^+ = 1$ and $1^- = 0$ in such a manner that,

$$\forall x \in \mathbb{R}, x \times 1 = x^+1 - x^-1$$

and again it comes to the neutrality of the numbers $\underline{1}$ and $\bar{1}$ to conclude. Whereas the commutativity of \times is immediate, we focus swiftly first on associativity in noticing that,

$$1^\circ \forall x \in \mathbb{R}, x^- = (-x)^+$$

$$2^\circ \forall x \in \mathbb{R}, x^+ = (-x)^- = -\min(0, -x)$$

$$3^\circ \forall x, y \in \mathbb{R}^+, (x - y)^+ = x - \min(x, y)$$

4° for every real x, y, z , it is true that $\max(x, y) + z$ equals $\max(x + z, y + z)$ because, in concentrating on one inclusion for the upper part as the lower one is immediate, if there is a rational q written as a sum of some rational r_x in \mathbb{R}_x with a rational r_z in \mathbb{R}_z as well as a sum of an element r_y in \mathbb{R}_y and t_z in \mathbb{R}_z ,

$$q \leq \max(r_x, r_y) + \max(r_z, t_z)$$

such that q is in $(\mathbb{R}_x \cap \mathbb{R}_y) + \mathbb{R}_z$

5° for every non-negative real x, y , for every real z , the distributivity,

$$(x - y)z = xz - yz$$

holds since, given our multiplication in \mathbb{R} , it shrewdly suffices to restrict to non-negative z 's,

$$(x - y)z = (x - y)^+z - (x - y)^-z = (x - \min(x, y))z - (y - \min(y, x))z = xz - yz$$

wherefrom follow the equalities, for every reals x, y, z ,

$$\begin{aligned} (xy)z &= (xy)^+z^+ + (xy)^-z^- - (xy)^+z^- - (xy)^-z^+ \\ &= (x^+y^+ + x^-y^- - (x^+y^- + x^-y^+))^+z^+ + (-xy)^+z^- - (xy)^+z^- - (-xy)^+z^+ \\ &= (x^+y^+ + x^-y^- - \gamma)z^+ + ((x^+y^- + x^-y^+) - \gamma)z^- - (x^+y^+ + x^-y^- - \gamma)z^- \\ &\quad - ((x^+y^- + x^-y^+) - \gamma)z^+ \\ &= (x^+y^+ + x^-y^-)z - (x^+y^- + x^-y^+)z \\ &= x(yz) \end{aligned}$$

since all the numbers $\gamma \doteq \min(x^+y^+ + x^-y^-, x^+y^- + x^-y^+)$ appearing do cancel off.

The distributivity is carried out in an analogous way; for every reals x, y, z ,

$$\begin{aligned} (x+y)z &= (x+y)^+z^+ + (x+y)^-z^- - (x+y)^+z^- - (x+y)^-z^+ \\ &= (x^+ + y^+ - (x^- + y^-))^+z^+ + (-x - y)^+z^- - (x+y)^+z^- - (-x - y)^-z^+ \\ &= (x^+ + y^+)z - (x^- + y^-)z \\ &= xz + yz \end{aligned}$$

and we conclude the proof. •

II.79—PROPOSITION — FIELD OF THE DEDEKIND REALS

The localic ring \mathbb{R} can be turned into a field. More precisely, the points of \mathbb{R} in the open complement of $\{0\}$ are invertible; these reals x are such that,

$$\exists p \in \mathbb{Q}, (x < p < 0 \text{ or } 0 < p < x) \quad \diamond$$

PROOF

For a non-nil real $x \doteq (L, R)$ in the localic product $\underline{\mathbb{R}}^+ \times \overline{\mathbb{R}}^+$, that is to say a x in the open complement of $\{0\}$, we define the inverse y of x via,

$$y \doteq \left(\bigcup_{0 < r \in \mathbb{R}} \{q \in \mathbb{Q} \mid rq < 1\}, \bigcup_{0 < \ell \in L} \{q \in \mathbb{Q} \mid 1 < q\ell\} \right)$$

and we prove that it constitutes a pair of lower and upper reals for, in focusing on the lower part L_y , L_y is inhabited, closed downwardly and, if there exists a positive r in R such that there exists a rational q with rq strictly less than 1,

1° if q is not positive, then immediately by positivity of r ,

$$qr \leq 0 < 1$$

2° otherwise if q is negative,

$$\exists s \in \mathbb{R}, 0 < s < r \Rightarrow \exists q \frac{r}{s} \doteq p \in \mathbb{Q}, ps = qr < 1$$

At present, we show that R_y is a upper real in the same fashion and move on to prove that they are practically of Dedekind. Indeed, if we take a rational q in both L_y and R_y , then,

$$\exists 0 < \ell \in L, 1 < \ell q \Rightarrow \left(q \neq 0 \text{ and } \frac{1}{q} \in L \right)$$

yet simultaneously,

$$\exists 0 < r \in R, rq < 1 \Rightarrow \frac{1}{q} \in R$$

which implies falsum and thereby the separation.

If now we possess a pair of rationals p strictly less than q ,

1° if p is positive, we have that,

$$\begin{aligned} \frac{1}{q} < \frac{1}{p} &\Rightarrow \left(\frac{1}{q} \in L \text{ or } \frac{1}{p} \in R \right) \\ &\Rightarrow (p \in L_y \text{ or } q \in R_y) \end{aligned}$$

2° if p is not positive, immediately,

$$\begin{aligned} \underline{0} \not\subseteq L &\Rightarrow \exists \ell \in L, 0 < \ell \\ &\Rightarrow \exists r \in R, 0 < \ell < r \\ &\Rightarrow pr < 1 \\ &\Rightarrow p \in L_y \\ &\Rightarrow (p \in L_y \text{ or } q \in R_y) \end{aligned}$$

since ℓ is in L and R is inhabited and ℓ is always strictly lower than anything in R

Let us demonstrate now that the (lower part of) the product of y by x gives the (lower part of) the multiplicative neutral $1 \doteq (\underline{1}, \bar{1})$,

1° we already have $\underline{0} \not\subseteq \underline{1}$

2° if for some rational q , there exist some positive rationals ℓ , k with ℓ in L and k in L_y such that q is strictly less than their product ℓk ,

a) if the said product is strictly less than one, the proof ends

b) otherwise,

$$\frac{1}{k} < \ell \Rightarrow \frac{1}{k} \in L$$

in such a manner that,

$$\begin{aligned} k \in L_y &\Rightarrow \exists 0 < r \in R, 0 < r < \frac{1}{k} \\ &\Rightarrow \frac{1}{k} \in R \\ &\Rightarrow \perp \end{aligned}$$

3° if there is a rational q strictly less than one, then with the knowledge of the existence of a positive ℓ in L ,

$$\begin{aligned} \forall r \in R, 0 < \ell < r &\Rightarrow \forall r \in R, 0 < \ell(1-q) < r(1-q) \\ &\Rightarrow q < 1 - \frac{\epsilon}{r} \end{aligned}$$

where we define a positive rational $\epsilon \doteq \ell(1-q)$; choosing such a positive r in R ,

$$\forall k \in \mathbb{Q}, 0 < kr < 1 \Rightarrow q < 1 - \frac{\epsilon}{r} < 1 - \epsilon k = 1 - (r - \ell)k < k\ell$$

but since such a positive rational k always exists, the proposition is proved

We finish with the uniqueness of the multiplicative inverse; because the real 1 is positive, given a positive real $x \doteq (L, R)$, it is sufficient to suppose a positive real $y \doteq (L_y, R_y)$ such that xy is 1. In targeting the lower part of this multiplication, the proof splits into,

1° if a rational q is in $L(\frac{1}{x})$,

$$\exists 0 < r \in R, rq < 1$$

hence in applying our hypothesis,

a) if rq is not positive, the proof ends

b) if $r\mathbf{q}$ is strictly less than $k\ell$ for some positive rational k in L_y and some positive rational ℓ in L ,

$$\begin{aligned} \ell < r &\Rightarrow \mathbf{q} < \mathbf{q} \frac{r}{\ell} < k \\ &\Rightarrow \mathbf{q} \in L_y \end{aligned}$$

2° if \mathbf{q} is in L_y ,

a) if \mathbf{q} is not positive, since there exists some positive rational in R , their multiplication is less than 1

b) when \mathbf{q} is not negative,

$$\exists l_y \in L_y, 0 < \frac{\mathbf{q}}{l_y} \leq 1$$

implying,

$$\exists 0 < \alpha \in L, \exists 0 < \gamma \in L\left(\frac{1}{x}\right), \frac{\mathbf{q}}{l_y \alpha} < \gamma \Rightarrow \frac{\mathbf{q}}{l_y \alpha} \in L\left(\frac{1}{x}\right)$$

however by roundedness of L_y and L , we know that \mathbf{q} is in $L\left(\frac{1}{x}\right)$. •

II.80 — PROPOSITION — SQUARE ROOT OF REAL NUMBERS

The following localic function on \mathbb{R}^+ ,

$$\begin{aligned} \sqrt{(-)}: \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto \left(\sqrt{L} \doteq \underline{0} \cup \bigcup_{0 < \ell \in L} \{\mathbf{q} \in \mathbb{Q} \mid \mathbf{q}^2 < \ell\}, \sqrt{R} \doteq \bar{0} \cup \bigcup_{0 < r \in R} \{\mathbf{q} \in \mathbb{Q} \mid r < \mathbf{q}^2\} \right) \end{aligned}$$

is perfectly well defined and constitutes the *square root* function. ◇

PROOF

For a non-negative real $x \doteq (L, R)$, its square root is Dedekind since,

1° the lower real $\underline{0}$ assures us \sqrt{L} is closed downward

2° if a rational p lies in \sqrt{L} ,

a) if p is not negative, because p^2 is in L ,

$$\begin{aligned} \exists k \in L, 0 < p^2 < k &\Rightarrow p < \sqrt{k} \\ &\Rightarrow \exists \ell \in L, \left(0 < k < \ell \text{ and } \sqrt{k}^2 = k < \ell \right) \end{aligned}$$

b) if p is negative, so is its half which is strictly greater than p

3° idem for the upper part of our square root

4° if a rational q is an element of both \sqrt{L} and \sqrt{R} , its square q^2 is in both L and R whereby leading to falsum

5° if p is strictly less than q as rationals,

a) if p remains positive or nil,

$$\begin{aligned} 0 \leq p < q &\Rightarrow p^2 < q^2 \\ &\Rightarrow p^2 \in L \text{ or } q^2 \in R \end{aligned}$$

where we can use the roundedness of L or a logical weakening to get our conclusion

b) if p is negative, it is in \sqrt{L}

We verify that our square root gives indeed a root with the following two inclusions,

1° all the negative rationals are in L as well as in the product $\sqrt{L}\sqrt{L}$

2° for a rational q less than some product uv with u and v both positive and lying in \sqrt{L} ,

$$\exists 0 < k, \ell \in L, (uu < k \text{ and } vv < \ell)$$

hence,

$$k < \ell \Rightarrow q < uv < \sqrt{k}\sqrt{\ell} < \ell \Rightarrow q \in L$$

while the same conclusion holds in the case where ℓ is less then or equal to k

3° when a non-negative ℓ is in L ,

$$\exists 0 \leq \ell < k \in L, k = \sqrt{k}\sqrt{k} \Rightarrow 0 < \sqrt{k} \in \sqrt{L}$$

in using the roundedness of L for k

Let us switch to the uniqueness of our square root ; let us suppose a non-negative real $x \doteq (L, R)$; it is enough to use a non-negative real $y \doteq (L_y, R_y)$ whose square is x . Now,

1° if a rational q is in \sqrt{L} ,

a) if q is negative, it is in L_y

b) if q is not negative such that its square is strictly less than some positive element in L , q^2 is in L ; by our hypothesis, q^2 could be negative hence bringing falsum and so our conclusion ; otherwise,

$$\exists 0 < k, \ell \in L_y, q^2 < k\ell$$

and supposing $k < \ell$ let us say,

$$\begin{aligned} 0 \leq q^2 < \ell^2 &\Rightarrow q < \ell \\ &\Rightarrow q \in L_y \end{aligned}$$

2° for the reverse inclusion, if a positive rational q is in L_y , by roundedness of L_y followed by our hypothesis, q is in L •

II.81 — DEFINITION — LOCALE OF THE COMPLEX NUMBERS

The locale \mathbb{C} of all the complex numbers is the locale $\mathbb{R}^2 \doteq \mathbb{R} \times \mathbb{R}$ whose basic opens are thus necessarily of the form,

$$(p, q) \otimes (r, s)$$

with (p, q) and (r, s) some (sub)basic opens of \mathbb{R} . The order of specialization on \mathbb{C} is $(\sqsubseteq_{\mathbb{R}}, \sqsubseteq_{\mathbb{R}})$. ◊

II.82 — PROPOSITION — ALGEBRA OF THE COMPLEX NUMBERS

The locale of the complex numbers can be turned into a localic algebra. \diamond

PROOF

We transpose all the internal laws on \mathbb{R} defining it as a field to the product $\mathbb{C} \doteq \mathbb{R} \times \mathbb{R}$; in particular, we posit the multiplication on \mathbb{C} to read,

$$\begin{aligned} \times : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{C} \\ ((x, y), (u, v)) &\longmapsto (xu - yv, xv + yu) \end{aligned}$$

We set a localic function on \mathbb{C} via,

$$\begin{aligned} \overline{(-)} : \mathbb{C} &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto (x, -y) \end{aligned}$$

which is indeed an involution because first, it is sesquilinear, it is an antiautomorphism and is isometric for the norm defined as,

$$\begin{aligned} \text{mod}(-) : \mathbb{C} &\longrightarrow \mathbb{R}^+ \\ z &\longmapsto \sqrt{z\bar{z}} \end{aligned}$$

which is itself sound since the product $z\bar{z}$ in \mathbb{C} for a complex z is non-negative. \bullet

II.3.1.4 — SHORT ANALYSIS

II.83 — NOTE

For the missing definitions see III.1 at page 46. \diamond

II.84 — PROPOSITION — REGULARITY OF THE DEDEKIND REALS

The locale \mathbb{R} of the real numbers is regular. \diamond

PROOF

For a general open $U \doteq \bigcup_{j \in \mathcal{J}} (p_j, q_j)$ of \mathbb{R} and an open V well inside it, there exists some open G such that,

$$\bigcup_{j \in \mathcal{J}} G \cup (p_j, q_j) = \mathbb{R}$$

and taking a real x in \mathbb{R} means that there exists some j in \mathcal{J} such that x be an element of the open $G \cup (p_j, q_j)$. In other terms,

$$\begin{aligned} \bigvee \{V \in \Omega(\mathbb{R}) \mid V \prec U\} &= \bigvee \{V \in \Omega(\mathbb{R}) \mid \exists j \in \mathcal{J}, V \prec (p_j, q_j)\} \\ &= \bigvee_{j \in \mathcal{J}} (p_j, q_j) \\ &= U \end{aligned}$$

Let us prove the penultimate equality; specifically,

$$(p, q) = \bigvee \{V \in \Omega(\mathbb{R}) \mid V \prec (p, q)\}$$

Indeed, for an open U of a locale X , an open V is well inside U implies that,

$$\begin{aligned} \exists G \in \Omega(X), (V \wedge G = 0 \text{ and } U \vee G = 1) &\Rightarrow V = V \wedge 1 = V \wedge (U \vee G) = V \wedge U \\ &\Rightarrow V \leq U \end{aligned}$$

Let us pose $g \doteq (-\infty, q) \cup (r, +\infty)$; it remains to derive for a basic open $U \doteq (p, t) =$

$\bigcup_{p < q < r < t} (q, r)$ that,

$$\forall p < q < r < t \in \mathbb{Q}, (g \wedge (q, r) = 0 \text{ and } g \vee (p, t) = \mathbb{R}) \Rightarrow (q, r) < (p, t)$$

granting our wish. •

II.85—PROPOSITION

The following holds,

1° for every real numbers x and r ,

$$0 \leq x^2 \leq r \Rightarrow (x \leq \sqrt{r} \text{ and } -x \leq \sqrt{r})$$

2° if x is a real number,

$$x^2 \leq x \Rightarrow 0 \leq x \leq 1$$

3° if a real number is its square, then it is either zero or one

4° if x is a real number,

$$x^2 \leq x \Rightarrow 0 \leq x \leq 1 \quad \diamond$$

PROOF

1° in order to prove that x is less than or equal to \sqrt{r} , let us prove \perp in supposing that,

$$\exists q \in \mathbb{Q}, \sqrt{r} < q < x$$

which is equivalent to,

$$\exists q \in \mathbb{Q}, (r < q^2 \text{ and } q < x)$$

whence the deduction,

$$\exists q \in \mathbb{Q}, r < q^2 < x^2$$

and falsum follows

2° indeed, we notice that,

$$0 \leq \left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4} \leq \frac{1}{4}$$

hence,

$$x - \frac{1}{2} \leq \frac{1}{2} \Rightarrow x \leq 1$$

and,

$$\frac{1}{2} - x \leq \frac{1}{2} \Rightarrow 0 \leq x$$

3° same proof as for the inequality

4° we note that for a real x ,

$$0 \leq x \leq 1 \Rightarrow 0 \leq 1 - x \Rightarrow 0 \leq x(1 - x) \quad \bullet$$

II.86 — PROPOSITION — LOCALE OF THE COMPLEX DISK

The complex disk $\mathcal{U}(r)$ of radius r is the compact regular sublocale of all complex numbers with modulus less than r formed by the pullback,

$$\begin{array}{ccc}
 \mathcal{U}(r) & \xrightarrow{\quad} & \mathbb{C} \\
 \downarrow & \lrcorner & \downarrow \text{abs} \\
 [-r, r] & \xrightarrow{\quad} & \mathbb{R}
 \end{array}$$

◇

PROOF

This locale $\mathcal{U}(r)$ is compact in noting that it is a sublocale of the square $[-r, r] \times [-r, r]$ as a sublocale of $\mathbb{R} \times \mathbb{R}$ by the existence of the localic function,

$$\begin{aligned}
 \mathcal{U}(r) &\hookrightarrow [-r, r] \times [-r, r] \\
 \mathbf{z} \doteq x + iy &\mapsto (x, y)
 \end{aligned}$$

as this is immediate after,

$$\forall \mathbf{z} \doteq x + iy \in \mathcal{U}(r), 0 \leq x^2 \leq x^2 + y^2 \leq r^2 \Rightarrow -r \leq x \leq r$$

•

II.87 — PROPOSITION — KRONECHER'S DELTA

The Kronecher's delta,

$$\begin{aligned}
 \delta: \mathbb{Q} \times \mathbb{Q} &\longrightarrow \{0, 1\} \\
 (p, q) &\mapsto \begin{cases} 0 & \text{when } p \text{ differs from } q \\ 1 & \text{otherwise} \end{cases}
 \end{aligned}$$

is continuous by discreteness of the topology on \mathbb{Q} as well as by decidability of the equality on \mathbb{Q} .

◇

PROOF

Indeed, the discreteness guarantees that the frame of the localic rational numbers is their powerset. In incidence, the diagonal of \mathbb{Q} must be open. By decidability, the set complement is equally open and is the boolean complement.

•

III — LAX COEQUALIZER OF \mathbf{Loc}

III.01 — OVERVIEW

We must acknowledge the usefulness of the frames when they are manipulated constructively, even though these are not geometric. We use them more precisely in III.3 at page 66 for the Hausdorff systems [Tow96b; Tow96c], those compact regular locales X equipped with a closed relation R on them such that R equal its square $R \circ R$ — in full, that R be a closed relation both transitive and interpolative. The convenience of these systems lies in their ability to generalize the Priestley’s duality in several ways; in the sense for instance, that it now applies between compact regular locales (and necessarily proper) monotone maps (for R) and stably compact locales equipped of perfect localic maps — there exists equally a geometric duality involving some appropriate morphisms of Hausdorff systems and preframe morphisms, formally reversed. Indeed, the traditional localic Priestley duality for the locales is an equivalence of categories from the ordered stone locales with the coherent locales (and perfect maps).

Briefly, from a coherent locale we take the patch topology consisting in the original compact opens intersected with the compacts closed subsets. The order on the ordered stone locale is the original specialization on the coherent locale. On the other hand, from a stone locale with a given closed order, we manufacture a coherent one by taking only the opens closed upwardly (for this order). The equivalence takes the form of a functor $\text{Patch}(-)$ which in effect takes a coherent locale in order to send it to the ideal completion of the free boolean algebra of the distributive lattice of its compact opens. Let us note that a priori, the construction $\text{Patch}(-)$ chooses a Hausdorff system amongst many others isomorphic to it after the application of the functor to a stably compact locale. We privilege the extended localic duality concerning the case of these aforementioned closed relations, but we restrict ourselves to closed preorders. In section III.3.2 on 70, we notice that the study simplifies for the better since with the closed preorders, we still have the perfect maps as the outputs of the functor $\text{Patch}(-)$, in lieu of the general preframe arrows.

This personal analysis is prolonged once cast in the descent theory, in the lax form, of sheaves down the surjective perfections of \mathbf{Loc} in the section III.4.2 at page 80 and the section thereafter. Such a study is the capital result to make the connection between our personal construction of the contextual locale (with the suitable topology of the reals) with the one of [Cas+09] (with the Alexandrov’s topology on the contextual poset). Beforehand, though, we must look at the category \mathbf{Frm} of the frames; in other words, we must focus on the lattices as exposed in the literature. \diamond

III.1 — FRAMES VIA LATTICES

III.02 — OVERVIEW

We set forth various definitions of sets (plus extra structures) whose elements must be conceptualized as data. The elementary object of study is the structure of a poset whose elements are pieces of knowledge and the order is an abstract entailment — a member is less than another one when it entails the bigger. The perspective taken brings the crucial notion of termination of an observation, id est the termination of the derivation of a new information from a bunch of primitive ones. In one word, an observation is a program taking a knowledge as input and outputting a new knowledge when it halts, or in fact never halts at all. The most basic programs provided are the conjunction \wedge and disjunction \vee of pieces of information. In order to observe the meet of two members, we must compute both pieces; in order to observe the join of elements, we must observe at least one of them.

Moreover, an order on the elements is suggested by the informational view; to wit, the relation of approximation. An element does approximate a second one when to observe the latter necessarily involves the observation of the former knowledge by a finite process — the approximation is also dubbed the way-below relation. The immediate notion stemming from this and the join operation is the one of a directed join \bigvee^{dir} rendering more apparent the topological flavours at this point. Briefly said, a directed join is a filtered colimit of pieces of information^① whereas a subset possesses a direction when it is inhabited and for any two of its members, there exists a bigger one already in the subset — that is to say, such that the said two elements do approximate the biggest. Informally, when we have two pieces of knowledge from a directed subset, we know how to observe, in a finite time, another one carrying more information than the first two. The join of a directed set — a directed join — symbolizes the best knowledge derivable from the (sub)set.

All the relevant arrows between the sundry structures of information must preserve the entailment; a thing for posets transcribed as continuity of the arrows with respect to the famous Scott's topology. Naturally, we are lead to gather the members into parts to create new structures, richer than a poset. When we ask for all the directed joins to indeed exist, we mathematically move from the poset to the algebraic dcpo that is its ideal completion. When we generalize this notion of dcpo, we obtain a domain wherein every element is truly the (directed) join of the elements, in the poset, approximating it (in the way-below relation). The relevant (monotone) functions must preserve the directions of the existing directed subsets. They effectively do this when they are continuous with respect to the Scott's topology.

The connection with the locales becomes clear through the notion of a frame; a frame is a poset with the requirement of the existence of the finite meets, but more importantly of the arbitrary joins. In other terms, the joins of elements indexed by a set which can be infinite — even though, these infinite joins are only the directed ones of other finite joins. Furthermore, the finite meets must distribute over the (arbitrary) joins, just as the multiplication of numbers distributes over their addition. We eventually define a morphism of frames as the morphisms of the subjacent sets respecting the order relation and all the joins. When we group them, they form a category **Frm** and a locale is nothing else than an object in the dual category **Loc**.

^① The term is manifestly unbecoming since a direction truly pertains to a subset of the poset, not at all to a join.

The explicit link with the point-set topology appears after we notice that a topology (of a topological space) does constitute a frame, since the infinite unions are permissible and after all, they do distribute over finite intersections. Equally important is the remark that the sets $\mathbf{Loc}(X, Y)$ of localic arrows have a structure of a dcpo coming from the specialization, just as $\mathbf{Top}(X, Y)$ has one where the order is given traditionally by defining $x \sqsubseteq y$ if and only if the point x lies in the closure of the singleton $\{y\}$. The localic topology modifies the definition in generalizing it in a natural manner.

Because we wish to be as clear as possible and equally to minimize the rôle played by the frames, we remain concise and favour the definitions against the proofs. The compendium or handbook [SamGabMai99; Gie+03] are useful for the various notions exposed here; the locales from their frames are studied in [Vic89] and naturally [Joh82] which contains all the relevant proofs regarding them — the classical theory can be found in [PedTho04; PicPul11]. \diamond

III.03 — DEFINITION — PRESET, PREORDER SET, TOP, BOTTOM, MONOTONY

A « **preset** (P, \leq) » is a set P equipped with a relation $\leq \hookrightarrow P \times P$ which is a preorder, that is to say which is reflexive and transitive.

When needed, there exists also a **bottom element** bot , which is lower than any element and there is also a **top** \top , the biggest.

A morphism between two presets is a monotone functions of the underlying sets. \diamond

III.04 — PROPOSITION

There is a (co)complete category of presets and morphisms of presets. ^① \diamond

III.05 — NOTE

In the subsequent exposition, the extreme elements top \top and bottom \perp exist.

We can view each preset as a category; the elements of the preset are the objects and an arrow between two elements exists if and only if the domain is smaller than the codomain; the reflexivity of the order provides the identities while transitivity assures the compositions and their unicities. \diamond

III.06 — DEFINITION — EMBEDDING, INFLATIONARY & CLOSURE FUNCTION

An « **embedding** of the preset X into a preset Y » is a preset morphism $f: X \longrightarrow Y$ under the constraint,

$$\forall x, z \in X, f(x) \leq f(z) \Rightarrow x \leq z$$

When a monotone function f on a preset X is greater than the identity in the pointwise order, it is **inflationary**. A **closure** on a preset is a function monotone, inflationary and which equals its square. \diamond

III.07 — DEFINITION — ADJUNCTION

Two monotone arrows f, g between presets are « **in adjunction** $f \dashv g$ » when they go in opposite directions and,

$$\forall x \in df, \forall y \in cf, f(x) \leq y \iff x \leq g(y)$$

and f is then the « **left adjoint** of its **right adjoint** g ». \diamond

III.08 — NOTE

Both arrows must be monotone initially. ◇

III.09 — DEFINITION — JOIN, MEET, DIRECTION, DIRECTED SUBSET, IDEAL, FILTER

On a preset X , for one of its subset S ,

1° « a bound from below » is any element of X smaller than every element of S

2° the « *infimum* $\bigwedge S$ » is the biggest of its lower bounds, should the infimum exist at all

3° a « bound from above » is an element of X greater than each one of its elements

4° the « *supremum* $\bigvee S$ » is the least of its upper bounds, should the supremum exist at all

5° S is « *directed* » when every finite subset of S has an upper bound already in S ^①

6° S is « *filtered* » dually when each one of its finite subsets possesses a lower bound already on S

7° S is an *ideal* when closed downwardly and directed

8° S is a *filter* when closed upwardly and filtered^②

9° S is a *principal ideal* when it is the down set of an element

When the meets are available (as in a lattice), S is a *prime ideal* when it is proper and,

$$\forall x, y \in X, x \wedge y \in S \Rightarrow (x \in S \text{ or } y \in S) \quad \diamond$$

III.10 — NOTE

Classically, the complement of a prime filter is a prime ideal. The prime filters are crucial for they are the points of the spectrum of a lattice. ◇

III.11 — DEFINITION — POSET, (PARTIALLY) ORDERED SET

A « *poset* (P, \leq) » is a set P equipped with a relation $\leq \hookrightarrow P \times P$ reflexive, transitive and antisymmetric. A poset is *bounded* when it possesses a top and bottom. A morphism of posets is the morphism between the underlying presets. ◇

III.12 — PROPOSITION

For a monotone arrow f of posets,

1° its left or right adjoint is unique, should it exist

2° the arrow g is its adjoint on the right if and only if,

$$f \circ g \leq \text{Id} \iff \text{Id} \leq g \circ f$$

3° if it is adjoint on the left, of some monotone arrow g , it preserves suprema; and when the suprema exist, the general formula for g is,

$$\forall y \in cf, g(y) = \bigvee^{\text{dir}} \{x \in df \mid f(x) \leq y\}$$

4° if it is adjoint on the right (of some monotone arrow), it preserves infima ◇

^① Let us note that an isomorphism in this category is more than a bijective function of sets by monotonicity; an embedding differs from a monic, an injection.

^① Which means that some upper bound for bottom is in S whence S inhabited and every finite join (of elements) of S has an upper bound in S .

^② Also codirected.

III.13 — DEFINITION — EQUIVALENCE RELATION

An « **equivalence relation** on a set P » is a relation on P which is reflexive, transitive and symmetric. \diamond

III.14 — PROPOSITION

There is a (co)complete category of posets and morphisms of posets which is full subcategory of the category of presets. This functor is a reflection. \diamond

PROOF

Indeed, the forgetful functor from the category of posets to the one of presets is right adjoint to the functor turning a preset (P, \leq) canonically into a poset $(Q, <)$, imposing the antisymmetry by the following construction. The set Q is the quotient $P_{/\simeq}$ where \simeq is a relation of equivalence on P defined by,

$$\forall m, p \in P, m \simeq p \iff (m \leq p \text{ and } p \leq m)$$

and the partial order on Q is,

$$\forall [m], [p] \in Q, [m] < [p] \iff m \leq p$$

Every monotone function between a preset and a poset extends in a unique manner to a monotonous function between the posets. \bullet

III.15 — DEFINITION — SEMILATTICE

The notion of a lattice and its generalizations turn an order into an algebra.

A « **bounded semilattice** L » is a set L equipped with a binary idempotent operation, associative, commutative and possessing a neutral element. An arrow of semilattices is a set function between the carrier sets which must commute with the binary operations and must respect the neutral elements. \diamond

III.16 — PROPOSITION

There exists a category of semilattices and their morphisms. \diamond

III.17 — PROPOSITION — CORRESPONDENCE BETWEEN (BOUND) POSETS AND (BOUND) SEMILATTICES

We can turn a poset into a semilattice (on the same subjacent set) when it has all the finite meets — the nullary and binary meets by recurrence.

For a semilattice with juxtaposition as internal law, the order on the underlying set is given by,

$$x \leq y \iff y = xy \quad \diamond$$

III.18 — DEFINITION — LATTICE, DISTRIBUTIVE & COMPLETE & SUP LATTICE

A **lattice** is a poset possessing all the finite joins and the finite meets. A lattice is **distributive** when the joins distribute over the meets. A lattice is **complete** when either the arbitrary joins or the arbitrary meets exist. A lattice morphism is a morphism of posets commuting with all the finite meets and joins; the complete version of the morphisms must be coherent with the arbitrary joins and meets.

A poset is a **suplattice** when it is a complete lattice. The morphisms of suplattices are the morphisms of posets which preserve only all the joins. \diamond

III.19—PROPOSITION

The lattices and their morphisms constitute a category; idem for the complete lattices and their morphisms. \diamond

III.20—NOTE

A subset of a lattice is a priori not a sublattice, even when it is a lattice on its own; the meets and joins on it could differ from the original one — the best example is the bounded operators on a hilbertian space, their (bound) sum is not the same as their sum as mere operators.

The morphisms of complete lattices have a right adjoint as monotone function between posets. \diamond

III.21—PROPOSITION — ALGEBRAIC CHARACTERIZATION OF LATTICES

A lattice is a set L equipped with two binary laws \wedge, \vee which turn it into a meet-semilattice and a join-semilattice and such that the interconnection of the laws are verified,

$$\forall a, b \in L, a \vee (a \wedge b) = a = a \wedge (a \vee b) \quad \diamond$$

III.22—PROPOSITION — DECOMPOSITION OF THE JOIN

A join which is infinite in a lattice is decomposable into a directed join of finite ones. \diamond

PROOF

Indeed, we break a join $\bigvee L$ in a complete lattice down to a directed join $\bigvee^{\text{dir}} H$ of collections H of little joins $\bigvee K$ of elements K of the finite powerset of the lattice, but imposing that all the K 's remain subsets of L . \bullet

III.23—NOTE

The directed joins must be perceived as the joins which are not finite; a join which is arbitrary in a complete lattice is a directed join of finite ones. \diamond

III.24—DEFINITION — PREFRAME, FRAME, LOCALE

A **preframe** is a poset whose directed joins do exist and furthermore distribute over the finite meets. A morphism of preframes must preserve finite meets and directed joins; they form a category.

A **frame** is a lattice whose arbitrary joins do exist and further, distribute over the finite meets^①. A morphism of frames must be compatible with the joins and finite meets; they form a category \mathbf{Frm} ^②.

A **locale** is an object of the category \mathbf{Loc} , dual to \mathbf{Frm} . \diamond

III.25—NOTE

The frames are not selfdual in general, not because of the existence of joins and meets, rather by their distributivity. \diamond

^① There exists also a characterization of a frame in terms of a complete Heyting algebra, to wit a complete lattice for which there exists an arrow of implication. However, the implication is not preserved under the geometric morphisms, idem for the frame themselves.

^② The frame morphisms do not commute with the arbitrary meets nor the implications.

III.26 — DEFINITION — HEYTING ARROW, PSEUDOCOMPLEMENT

The « *Heyting arrow* $a \longrightarrow b$ of two elements a, b of a poset P » is the join (when it exists),

$$a \longrightarrow b \doteq \bigvee \{d \in P \mid a \wedge d \leq b\}$$

In other terms, for every element a , it exists when the monotone function ($a \longrightarrow -$) is right adjoint to the monotone function ($a \wedge -$).

A « *pseudocomplement* $\neg a$ of an element a of a frame F » is the directed join,

$$a \longrightarrow \perp \doteq \bigvee^{\text{dir}} \{b \in F \mid b \wedge a = \perp\}$$

whose existence is assured by distributivity.

A « *complement* of an element a in a frame F » is every element b of F with the property,

$$b \wedge a = \perp \text{ and } b \vee a = \top \quad \diamond$$

III.27 — DEFINITION — HEYTING ALGEBRA

A *Heyting algebra* is a lattice such that there exists a Heyting arrow for each pair of its elements. ◇

III.28 — DEFINITION — BOOLEAN ALGEBRA

A boolean algebra is a distributive lattice such that each one of its elements has a complement, necessarily unique and corresponding to the pseudocomplement.

A morphism of boolean algebras is a morphism of distributive lattices preserving the complementation. ◇

III.29 — NOTE

In a distributive lattice, the complements are the pseudocomplements, yet may not exist; hence the study of the boolean algebras.

The unicity of the complements follows from the distributivity. ◇

III.30 — DEFINITION — WELL-INSIDE RELATION, RELATIVE CLOSEDNESS, T_3

In a lattice P , an element (x, y) is in the *well-inside relation* $\ll \hookrightarrow P \times P$ when,

$$x \ll y \iff \exists z \in P, (z \wedge x = \perp \text{ and } z \vee y = \top) \quad \diamond$$

III.31 — DEFINITION — REGULAR PRIME FILTER

A prime filter F on lattice L is *regular* when,

$$\forall u \in L, (u \in F \Rightarrow \exists v \in L, u \ll v \text{ and } v \in F)$$

that is to say, it is rounded for the relative closedness. ◇

III.32 — NOTE

The regular prime filters are the points of the regular spectrum of a normal distributive lattice; this is why they are fundamental. ◇

III.33 — PROPOSITION

When we have a Heyting algebra X ,

$$\forall x, y \in X, x \ll y \iff \neg x \vee y = \top$$

and when we have a boolean algebra X ,

$$\forall x, y \in X, x \ll y \iff \neg x \vee y = \top \iff x \leq y \quad \diamond$$

III.34—NOTE

The boolean case is utterly handy when we analyse the regular ideals of a boolean algebra, they are only the ideals. \diamond

III.35—DEFINITION — NORMALITY, T_4

A lattice L is *normal* when,

$$\forall a, b \in L, a \vee b = \top \Rightarrow \exists x, y \in L, (x \vee a = \top = y \vee b \text{ and } x \wedge y = \perp) \quad \diamond$$

III.36—NOTE

Typically, the normality means that we are able to plane down a by the amount corresponding to its overlap with b . \diamond

III.37—PROPOSITION

A lattice L is normal if and only if,

$$\forall a, b \in L, a \vee b = \top \Rightarrow \exists e \in L, (e \leq a \text{ and } e \vee b = \top) \quad \diamond$$

III.38—PROPOSITION

Every boolean algebra is normal. \diamond

PROOF

If the meet of a and b is top, then we choose $x = \neg a$ and $y = \neg b$ to get,

$$x \vee a = \top = y \vee b \text{ and } x \wedge y = \perp$$

since we have access to the boolean complements. \bullet

III.39—DEFINITION — SCOTT'S TOPOLOGY, INACCESSIBILITY BY DIRECTED JOINS

The « *Scott's topology* on a poset (P, \leq) » has for opens the subsets U of P which are,

1° closed upwardly for \leq ,

$$U = \uparrow U \doteq \{p \in P \mid \exists u \in U, u \leq p\}$$

2° *inaccessible by directed joins*,

$$\forall S \subseteq_{\text{dir}} P, \bigvee^{\text{dir}} S \in U \Rightarrow \exists s \in S, s \in U$$

The concision is for the Scott closedsets; a closed is Scott when it is lower closed and the joins of its directed subsets are amongst its elements. \diamond

III.40—PROPOSITION — SCOTT'S TOPOLOGY AND MONOTONICITY

A monotone function between posets is continuous for the Scott's topology if and only if it preserves all the possible directed joins. \diamond

III.41—DEFINITION — DIRECTED COMPLETE PARTIAL ORDER SET, DCPO MORPHISM, IDEAL COMPLETION

When a poset has all the directed joins for all its possible directed subsets, it is a *directed complete partial order set* — a dcpo. In short, all the ideals have their joins. A morphism between dcpos is a morphism between the underlying posets which preserves the directed joins.

When gathered up, the ideals of a poset under the order of inclusion form its *ideal completion*. \diamond

III.42 — NOTE

The ideals of a poset form necessarily the free dcpo on the poset. ◇

III.43 — PROPOSITION

There exists the (co)complete category of dcpos and morphisms of dcpos which is a subcategory of the category of posets. ◇

III.44 — NOTE

The cocompleteness is not a trivial result. ◇

III.45 — DEFINITION — WAY-BELOW RELATION, RELATIVE COMPACTNESS, COMPACT ELEMENT

In a dcpo^① P , an element (x, y) is in the *way-below relation* $\ll \hookrightarrow P \times P$ when, x is compact relative to y ; when a cover of y has a finite cover of x ; when for every directed subset S of P having a directed join, as soon as y is smaller than the (directed) join of S , there does exist an element s of S above x ,

$$x \ll y \iff \forall S \subseteq_{\text{dir}} P, y \leq \bigvee^{\text{dir}} S \Rightarrow \exists s \in S, x \leq s$$

An element y of a dcpo P is *compact* when it is way below itself,

$$\forall y \in P, y \ll y$$

On a preframe, the way-below relation is *stable under the finite conjunction* when,

$$\forall a, b, d \in P, a \ll b \text{ and } a \ll d \Rightarrow a \ll b \wedge d \quad \diamond$$

III.46 — DEFINITION — ALGEBRAIC & CONTINUOUS DCPO, DOMAIN, CONTINUOUS POSET

A dcpo P is *algebraic* when,

$$\forall y \in P, y = \bigvee^{\text{dir}} \{x \in P \mid x \ll x \text{ and } x \leq y\}$$

A dcpo is *continuous*^② — is a *domain* — when all the following joins exist, are directed and verify,

$$\forall y \in P, y = \bigvee^{\text{dir}} \downarrow y \doteq \bigvee^{\text{dir}} \{x \in P \mid x \ll y\} \quad \diamond$$

III.47 — PROPOSITION

A dcpo which is algebraic is always isomorphic to its poset of compact elements. ◇

III.48 — DEFINITION — COMPACTNESS

A dcpo is *compact* when,

$$T \ll T$$

explicitly, the top element is compact relatively to itself. ◇

III.49 — NOTE

Let us illustrate the compactness; since T is the whole open space, the traditional compactness is immediate. For the converse, when we possess a cover of a topological space, we know that we can break its join into a directed join and a finite one; but the finite joins are members of the set whereof we take the directed join. ◇

^① This works for posets as well.

^② When the poset is the one of a frame, we can say *locally compact*.

III.50 — PROPOSITION

For a domain P ,

$$\downarrow \dashv \bigvee^{\text{dir}} \dashv \downarrow$$

with \bigvee^{dir} the monotone function,

$$\begin{aligned} \bigvee^{\text{dir}}: \text{Idl}(P) &\longrightarrow P \\ S &\longmapsto \bigvee^{\text{dir}} S \end{aligned}$$

◇

III.51 — NOTE

The adjunction $\downarrow \dashv \bigvee^{\text{dir}}$ characterizes the domains.

◇

III.52 — DEFINITION — ALEXANDROV'S TOPOLOGY

The opens of the **Alexandrov topology** for a preset are its subsets closed upwardly, its upper sets. A base is given by the principal upper sets.

◇

III.53 — PROPOSITION

It emanates that the Alexandrov topology of a preset is the finest topology so that the preorder become the specialization order.

◇

III.54 — NOTE

This topology is useful when the compactness of elements is present ; indeed, we know that an object of the poset is compact if and only if its upward closure (for the primitive order) is Scott open.

◇

III.55 — PROPOSITION

The Scott's topology on the ideal completion of a poset is isomorphic, as a frame, to the Alexandrov's topology on the poset. Consequently, the order of specialization on the algebraic dcpo is the original partial order.

◇

III.2 — LOCALE THEORY IN **Frm** AND **Loc**

III.56 — OVERVIEW

The theory of the locales seems dichotomous in appearance for, on one side, we emphasis the frame and thereby the opens as their elements ; on the other side, we focus on the locales themselves privileging the points ; not the global ones as we would in classical mathematics, but the generalized points. Besides, all the concepts of the lattices were formulated in set theory, therefore making sense in the topos **Set** since, after all, the lattices are structured sets. Naturally, we wish to be inside a topos of sheaves, completely different from **Set**. The passage from one view to the other accompany well this idea as it relies on the landmark result consisting of the externalization from [JoyTie84] which allows us to extract an internal frame $\Omega(X)$ in a topos of sheaves $\text{Sh}(Y)$ in order to view it rather as a localic bundle $X \longrightarrow Y$ coming from a geometric morphism $\text{Sh}(X) \longrightarrow \text{Sh}(Y)$. This elegance is spoilt by the notion of topological space, as [Joh02, C1.6] argues.

Externally, we have the ability to describe what a sublocale is as well as the notions of compactness, openness, properness, perfection, discreteness. And some of these properties enjoy a stability under the pullbacks of **Loc**. We refer to [Tow03 ; Tow05a ; Tow09] for the undertaking to categorically characterize **Loc**.

◇

III.2.1 — SPECTRAL ADJUNCTION

III.57 — DEFINITION — INITIAL FRAME, SUBOBJECT CLASSIFIER

The *initial frame* — or *subobject classifier* — $\Omega(\mathbf{1})$ in the category **Frm** consists of the discrete topology given by the powerset $\mathcal{P}\mathbf{1}$ of the singleton $\mathbf{1} \doteq \{\text{pt}\}$ ^①. \diamond

III.58 — PROPOSITION

The frame of the singleton $\mathbf{1}$ is isomorphic to,

$$\Omega(\mathbf{1}) \simeq \text{Idl}(\mathbf{2})$$

the ideal completion of the two-element poset $\mathbf{2}$. Conclusion : an element p of $\Omega(\mathbf{1})$ is,

1° a join $\bigvee^{\text{dir}} \{\{\emptyset\} \cup \{\text{pt}\} \mid \text{pt} \in p\}$ ^②

2° a subset p of $\{\text{pt}\}$

3° a proposition p in the ambient topos \diamond

PROOF

Indeed, an ideal of $\mathbf{2} \doteq \{0 \leq 1\}$ is a directed subset being also closed downwardly. In consequence, an ideal of $\mathbf{2}$ is the knowledge of the inclusion of the number 1 in the given ideal. However, such a knowledge is precisely the one of a subset of $\mathbf{1}$. \bullet

III.59 — PROPOSITION

The frame of the two-element poset $\mathbf{2}$ is,

$$\Omega(\mathbf{2}) \simeq \text{Idl}(\mathbf{4})$$

the ideal completion of the four-element poset $\mathbf{4}$. \diamond

III.60 — DEFINITION — SIERPINSKI LOCALE

The Scott's topology on the ideal completion of the poset $\mathbf{2}$ with two elements $\{0 \leq 1\}$ constitutes the frame of the *Sierpinski locale* \mathbb{S} . \diamond

III.61 — PROPOSITION

We have,

$$\Omega(\mathbb{S}) \simeq \text{Alex}[\mathbf{2}] \simeq \text{Flt}(\mathbf{3}) \simeq \text{Idl}((\mathbf{3})^{\text{op}}) \quad \diamond$$

PROOF

The ideal containing only the number 2 corresponds to the empty set, the ideal of two and one together is the empty set united to top and the ideal constituted by the whole poset $\mathbf{3}$ is the whole topology on $\mathbf{2}$. \bullet

III.62 — NOTE

This is in continuity with the localic definition of \mathbb{S} as the localic ideal completion of $\mathbf{2}$ in II.21 at page 16.

The frame is also the free frame on one generator. \diamond

^① Also denoted \top or $\{*\}$.

^② The writing of a logical proposition — defined constructively as a subset of the singleton — as a join is welcomed for it is directed. And we know that the good maps preserve those.

III.63 — CONSPECTUS — SPATIAL LOCALE, SOBER TOPOLOGICAL SPACE^①

To every topological space X , we are in capacity to associate it to its topology $\Omega(X)$, via a covariant functor,

$$\begin{aligned}\Omega(-): \text{Top} &\longrightarrow \text{Loc} \\ X &\longmapsto \Omega(X) \\ f &\longmapsto (\Omega(f) \doteq f^* \doteq f^{-1})^{\text{op}}\end{aligned}$$

On the other hand, we can also send a locale X to its *spectrum*, to wit the topological space $\text{pt}X$,

$$(\text{pt}X \doteq \text{Loc}(\mathbf{1}, X), \{\chi_u \doteq \{f^* \mid f^*(u) = \mathbf{1}\} \mid u \in \Omega(X)\})$$

of its global points.

Or we can also replace the localic morphisms by the completely prime filters on $\Omega(X)$ since the frame morphisms f^* into the initial frame are in bijective correspondence with these,

$$f^*: \Omega(X) \longrightarrow \Omega(\mathbf{1}) \simeq \{\mathbf{u} \in \Omega(X) \mid \text{pt} \in f^*(\mathbf{u})\}$$

It appears that the construction $\text{pt}(-)$ is functorial and in effect, constructively in adjunction with $\Omega(-)$,

$$\Omega(-) \dashv \text{pt}(-)$$

Eventually, a locale is *spatial* when its frame is isomorphic to the topology on its spectrum via the unit of the adjunction. In a dual manner, a topological space is *sober*^② when it is isomorphic, as a set of points, to the spectrum of its topology via the counit of the adjunction. \diamond

III.64 — NOTE — SOBRIETY LOGICALLY

Logically and more simply, a topological space X is sober when its propositional theory is characterized in the following manner. We already know that a point x of X is a model because we can discriminate the opens U wherein x lies and can send them to true. However, a priori, the models are not in bijection with the points. When they are so, the space is sober. \diamond

III.65 — PROPOSITION — EQUIVALENCE OF SPATIAL LOCALES AND SOBER TOPOLOGICAL SPACES

The category of spatial locales is equivalent to the category of sober topological spaces. \diamond

III.66 — NOTE

Since the theory of the frames and locales emphasises the opens, we cannot expect a locale, completely arbitrary, to be in bijective correspondence with a topological space. When two points are in the same opens, they are indistinguishable topologically. Incidentally, it makes sense to consider the sobriety of the spaces; the final task is to state it in a constructive manner as we exposed. \diamond

III.67 — DEFINITION — LOCAL HOMEOMORPHISM OF TOPOLOGICAL SPACES

Let $f: Y \longrightarrow X$ be a continuous set function between some topological spaces. The arrow f is a local homeomorphism when every element y of the domain Y possesses a neighbourhood (open) U such that $f|_U$ is a homeomorphism onto a neighbourhood (open) of $f(y)$. \diamond

^① [PedTho04].

^② Classically, sober means that the space is T_0 and the irreducible closed subsets are closures of a singleton. Classically, the Hausdorff spaces are sober.

III.68 — PROPOSITION — EQUIVALENCE OF LOCAL HOMEOMORPHISMS AND SHEAVES^①

There exists an equivalence between the local homeomorphisms of topological spaces, with codomain X , and the sheaves over X . \diamond

III.2.2 — SUBLOCALE

III.69 — DEFINITION — SUBLOCALE, OPEN & CLOSED SUBLOCALE

An *embedding* in \mathbf{Loc} or a *sublocale* is a regular monic^② in \mathbf{Loc} or a regular epic arrow in \mathbf{Frm} ^③. For a locale X , every open \mathbf{a} defines two sublocales,

1° one *open*,

$$\begin{aligned} X_{\mathbf{a}} : \Omega(X) &\longrightarrow \downarrow \mathbf{a} \\ u &\longmapsto \mathbf{a} \wedge u \end{aligned}$$

2° one *closed*,

$$\begin{aligned} X - \mathbf{a} \doteq X_{-\mathbf{a}} : \Omega(X) &\longrightarrow \uparrow \mathbf{a} \\ u &\longmapsto \mathbf{a} \vee u \end{aligned}$$

\diamond

III.70 — NOTE

Naturally the logical formulation is clearer. The acquisition of a sublocale \mathbf{a} which is open is done by the additional axiom $\top \vdash \mathbf{a}$ which will become $\top \leq \mathbf{a} \leq \top$ in the new space ; concretely the open \mathbf{a} becomes the totality of a new space, the open sublocale \mathbf{a} . Now for the closed, we desire the complement, which is nothing else than the geometric negation.

There are sundry equivalent definitions of a sublocale when we focus on the frame ; we can cite the congruences, and the nuclei. \diamond

III.71 — PROPOSITION — LATTICE OF SUBLOCALES^④

The sublocales of a locale form a complete lattice with the pullback of sublocales playing the rôle of the meet.

A closed sublocale is complemented by its open counterpart, that is to say when they are generated by the same element of the frame. Another manner to state this is that the closed complements of the open sublocales are their boolean complements in the lattice of the sublocales. \diamond

III.72 — NOTE — DISSIMILITUDE OF PSEUDO AND BOOLEAN COMPLEMENTS OF SUBLOCALES

For a discrete locale X , there appears to be a crucial, subtle difference between the closed complement of a sublocale U which is open and living in the lattice of the sublocales and the set complement $X \setminus U$ precisely because of the decidability of the equality on the points of X . Indeed, the closed complement in the lattice of the sublocales is a boolean complement — for the open and closed sublocales see their meets as bottom and their joins as top. However, the set complement is only a pseudo complement of U because we are not able to ascertain $X \subseteq U \cup X \setminus U$ until the decidability holds. \diamond

^① [Vic10].

^② The notation for a regular monic is \hookrightarrow . Let us recall that a monic in \mathbf{Top} is a function which is injective and becomes only an embedding when it is equally regular.

^③ Therefore is a surjective.

^④ [Vic89].

III.73—DEFINITION — DECIDABLE EQUALITY

A discrete locale X has a « **decidable equality** for its points » when the diagonal Δ_X possesses a boolean complement as a subset of $X \times X$; in other words, as an element of the powerset $\mathcal{P}(X \times X)$. \diamond

III.74—DEFINITION — OPEN & PERFECT & PROPER LOCALIC MORPHISM

An arrow f of locales is,

1° **open** [JoyTie84] when,

- a) f^* has a left adjoint $f_!$, necessarily of suplattices
- b) $\forall a \in df, \forall b \in cf, f_!(a \wedge f^*(b)) \geq f_!(a) \wedge b$

2° semi-proper or **perfect** or a perfection [KorLab07] when its right adjoint f_* is a preframe morphism

3° **proper** [Ver94] when,

- a) it is perfect
- b) $\forall a \in df, \forall b \in cf, f_*(a \vee f^*(b)) \leq f_*(a) \vee b$

III.75—NOTE

Intuitively, an open map is a map whereof the direct image preserves the openness; a proper is a map whereof the direct image preserves the closedness; a perfect map is map whereof the downward closure of the direct image of a closed subspace remains closed.

The perfection and properness are useful concepts for a locale map is proper if and only if its frame counterpart is Scott continuous for the Scott topologies on the frames [Vic89]. A locale map between stably compact locales is perfect if and only if its frame counterpart preserves the relative compactness [Vig04]. \diamond

III.76—PROPOSITION — STABILITY UNDER PULLBACKS OF OPENNESS & PROPERNESS, BECK–CHEVALLEY^①

The open sublocales are open localic arrows; the closed sublocales are proper localic arrows.

The properness and the openness are concepts stable under the pullbacks in **Loc**. Furthermore, the **condition of Beck–Chevalley** does hold for them; if the square,

$$\begin{array}{ccc}
 \dots & \xrightarrow{g^*(f)} & Y \\
 \downarrow f^*(g) & \lrcorner & \downarrow g \\
 Z & \xrightarrow{f} & X
 \end{array}$$

is a pullback of,

1° g open then,

$$(f^*(g))_! \circ \Omega(g^*(f)) \simeq f^* \circ g_!$$

2° g proper then, or proper along f then,

$$(f^*(g))_* \circ \Omega(g^*(f)) \simeq f^* \circ g_*$$

III.77—NOTE

The perfection is not stable under the pullbacks. \diamond

^① [Tow96c].

III.78 — PROPOSITION

A perfect surjective f in **Loc** verifies,

$$f_* \circ f^* = \text{Id}_{\Omega(cf)} \quad \diamond$$

III.79 — NOTE

This is handy in order to prove the perfection of some map. ◇

III.80 — DEFINITION — ALEXANDROV & SPECTRAL & COHERENT & DISCRETE LOCALE

A locale is,

- 1° of the kind *Alexandrov* when it is the localic ideal completion of a discrete poset ;
when its frame is the upper sets of some discrete poset
- 2° *discrete* when its frame is isomorphic to the powerset of some set
- 3° *spectral* or *coherent*^① when its frame is the ideal completion of a distributive lattice ◇

III.81 — NOTE

The discreteness of a locale is preserved by the finite limits and the general colimits. The fibrewise discreteness is also equivalent to the openness of the diagonal map. ◇

III.82 — PROPOSITION — SPATIAL ALEXANDROV LOCALES

Every Alexandrov locale is spatial (constructively). Its points are the ideals of the poset. ◇

III.83 — PROPOSITION

When X is a discrete locale finite and decidable, its power set is a normal distributive lattice. ◇

III.84 — NOTE

We recall that the adjective finite is in the sense of Kuratowski, II.54 at page 26. ◇

III.85 — DEFINITION — STABLY (LOCALLY) COMPACT & REGULAR LOCALE

A locale X is,

- 1° *locally compact* when the underlying poset of its frame is continuous
- 2° *stably locally compact* when it is locally compact and the way-below relation \ll is stable under the finite meet
- 3° *compact* when the top open in its frame is so
- 4° *stably compact* when compact and stably locally compact
- 5° *regular* when its frame satisfies,

$$\forall y \in \Omega(X), y = \bigvee^{\text{dir}} \{x \in \Omega(X) \mid x \ll y\} \quad \diamond$$

III.86 — PROPOSITION — EXPONENTIABLE LOCALE^②

A locale is exponentiable in **Loc** if and only if locally compact. ◇

^① That is to say the logic of the locale does not involve the infinite joins, only the finite ones.

^② [Hyl81].

III.87—PROPOSITION

The category **KRegLoc** of compact regular locales is complete and cocomplete ; its limits and colimits are those of **Loc**. This category is full subcategory of the one **StbKLoc** of the stably compact locales.

Furthermore, in a compact regular locale X ,

$$\forall x, y \in \Omega(X), x \leq y \iff x \ll y$$

the way-below relation coincide with the relative compactness. \diamond

III.88—PROPOSITION — EXPONENTIABILITY, DISCRETENESS, COMPACT-REGULARITY

For some locales X, Y , when Y is discrete and X is compact regular,

1° the localic exponential Y^X exists and is discrete [[Hyl81](#) ; [Vic04b](#)]

2° the localic exponential X^Y exists and is compact regular [[Vic04b](#)] \diamond

III.89—DEFINITION — POINT-FREE SPECTRUM, STONE LOCALE^①

The *spectrum of a distributive lattice* is the locale whose points are the prime filters and whose frame is the ideal completion of this lattice.

The *(regular) spectrum of a normal distributive lattice* is the compact regular locale whose points are the regular prime filters and whose frame is the completion by its regular ideals.

The *spectrum of a boolean algebra* is its *Stone locale* that is to say, the locale whose points are the prime filters and whose frame is the ideal completion of this lattice.

The *spectrum of a frame* is the locale whose points are the complete prime filters. \diamond

III.2.3—ABUNDANCE OF POINTS

III.90—PROPOSITION — COMPLETENESS, COCOMPLETENESS

The category **Loc** is complete, cocomplete. The product in **Loc** is the coproduct **Frm**, itself the tensor product in the category of the suplattices or in a manner completely equivalent, the tensor product in the category of the preframes. \diamond

III.91—PROPOSITION — EQUIVALENCE BETWEEN LOCALIC AND GEOMETRIC MORPHISMS

There exists an equivalence of posets between localic arrows and geometric morphisms,

$$\mathbf{Loc}(X, Y) \simeq \mathbf{Topos}(\mathbf{Sh}(X), \mathbf{Sh}(Y))$$

\diamond

III.92—NOTE

This is the justification to study these functors.

We recall, [II.29](#) at page 18, that the posets are formed via the specialization orders thanks to its cogent geometric definition on the generalized points. In fact, these two homsets are dcpo. \diamond

^① [[Joh82](#) ; [Coq05b](#)].

III.93 — PROPOSITION — EXTERNALISATION OF LOCALES^①

In general, a locale Y internal to a topos of sheaves $\text{Sh}(X)$ is the equivalent datum of an object in $\mathbf{Loc}/_X$, to wit an arrow $Z \longrightarrow X$ in \mathbf{Loc} , in other words, a localic arrow $Z \longrightarrow X$ in \mathbf{Set} ,

$$\mathbf{Loc}(\text{Sh}(X)) \doteq (\mathbf{Frm}(\text{Sh}(X)))^{\text{op}} \simeq (\mathbf{Loc}/_X)_{\mathbf{Set}}$$

as categories. The external locale Z is typically the internal frame Y applied, as a sheaf over $\Omega(X)$, to the top open, X . The internal frame $\Omega(Y)$ corresponding to $Z \longrightarrow X$ is the direct image $f_*(\Omega_Z)$ of the subobject classifier Ω_Z of $\text{Sh}(Z)$. \diamond

III.94 — PROPOSITION — ÉTALE BUNDLE

Let X be a locale and F an object of its topos $\text{Sh}(X)$ of sheaves. Then F leads to a new sheaf $\mathcal{P}F$ which can be interpreted as the frame of a discrete locale. These discrete locales are in bijective correspondence with the étale bundles over the locale X ; that is to say that the discrete locales internal to the sheaf topos give the local homeomorphisms once externalized. \diamond

III.95 — PROPOSITION — GLOBAL POINT, CROSS SECTION

A « **global point** of a locale Y internal to the topos $\text{Sh}(X)$ » is a **cross section** $s : X \longrightarrow Z$ of the external bundle $Z \longrightarrow X$ corresponding to Y internally; the diagram commutes,

$$\begin{array}{ccc} & & Z \\ & \nearrow s & \downarrow f \\ X & \xrightarrow{\text{Id}} & X \end{array}$$

III.96 — DEFINITION — ABUNDANCE OF GLOBAL POINTS

A locale X **possesses all its global points** or has enough (global) points when,

$$\forall U, V \in \Omega(X), (\forall x \in \mathbf{Loc}(1, X), x \in U \Rightarrow x \in V) \Rightarrow U \subseteq V$$

equivalently, we have that the frame morphism $\Omega(X) \longrightarrow \mathcal{P}(\text{pt}X)$ is injective or that the locale X is spatial. \diamond

III.97 — NOTE — LOGICAL INCOMPLETENESS AS PAUCITY OF POINTS

We understand better the incompleteness of the logic as the lack of (global) points of a theory in order to distinguish the opens,

$$\exists U, V \in \Omega(X), (\forall x \in \mathbf{Loc}(1, X), x \models U \Rightarrow x \models V) \not\Rightarrow U \vdash V$$

Rephrased even more logically, indeed if we have a theorem, a true formula V in a model \mathbf{x} — symbolically $\mathbf{x} \models V$ — whose truth is deductible from the truth of the hypothesis U equally holding in \mathbf{x} , then we are not able to systematically demonstrate syntactically V from U . In one word, a locale misses its points when the theory is not complete. \diamond

^① [JoyTie84].

III.98 — CONSPECTUS — BUNDLE OF LOCALES, FIBRES, PRINCIPLE OF GEOMETRICITY

A **bundle** f between two locales is a locale arrow f , however perceived as a coproduct of a collection of (localic) **fibres** $f^*(x)$, suspended over a point x of cf , in such a manner that a generalized point of the domain df is seized as a pair,

$$(x, y) \in cf \times f^*(x)$$

where each fibre $f^*(x)$ is a locale constructed geometrically from the knowledge of a generalized point x of cf . Or categorically, the domain df is perceived as the union $\coprod_{x \in cf} f^*(dx)$ of the pullbacks,

$$\begin{array}{ccc} x^*(df) \doteq f^*(dx) & \longrightarrow & df \\ \downarrow & \lrcorner & \downarrow f \\ X & \xrightarrow{x} & cf \end{array}$$

A « **morphism of bundles** over a locale X » is a morphism in $\mathbf{Loc}/_X$ or the (usual) commutative triangle in \mathbf{Loc} .

A « **geometric construction** \mathcal{F} on \mathbf{Loc} » is a transformation of locales which commutes with the pullbacks of the category. More concretely, the transformation \mathcal{F} can be done fibrewise through its sliced variant \mathcal{F}_{cf} in $\mathbf{Loc}/_{cf}$ on every bundle f and carries over the fibres of the pullback of the bundle f along any morphism g of locales (with $cg = cf$). The **principle of geometricity** takes the form of an isomorphism,

$$\mathcal{F}_{dg}(g^*(df)) \simeq g^*(\mathcal{F}_{cg}(df))$$

as the expression of the aforementioned commutativity. ◇

III.99 — NOTE — FRAMES AND POWERSSETS ARE NOT GEOMETRIC

A frame inside a topos of sheaves is no longer so after it is pulled back by a geometric morphism by lack of the infinite joins indexed by the domain of the geometric morphism. The geometric theory it presents is geometric though. ◇

III.100 — DEFINITION — FIBREWISE

A locale $Y \longrightarrow X$ has a **fibrewise** geometric property when the internal locale associated to Y in the sheaf topos $\mathbf{Sh}(X)$ has this property.

Externally, fibrewise means that each fibre of the bundle possesses the geometric property. ◇

III.101 — PROPOSITION — ABUNDANCE OF GLOBAL POINTS UNDER GEOMETRIC CONSTRUCTIONS

When the hypothesis of the definition of abundance of points,

$$(\forall x \in \mathbf{Loc}(1, X), x \models U \Rightarrow x \models V) \Rightarrow U \vdash V$$

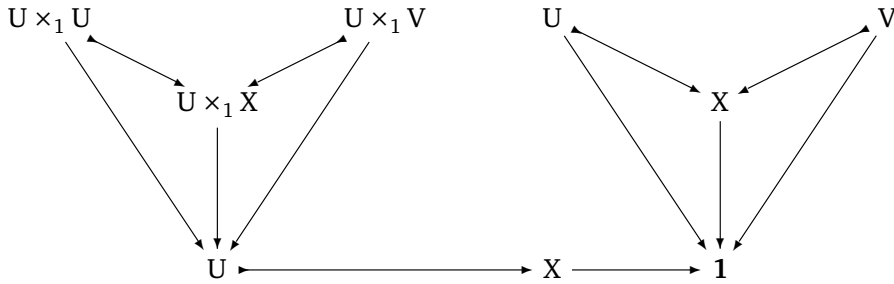
is verified geometrically when U, V are open sublocales of a locale X , then the locale X has all its generalized points.

Indeed, if \mathcal{F} is a geometric transformation of global points of the open U of a locale X into some global points of an open V of X — and thus correspond to the logical \Rightarrow in the premiss — then it corresponds to a locale arrow $f : U \longrightarrow V$ and acts on the generalized points of U by composition,

$$\forall W \in \mathbf{Loc}, \forall \omega \in \mathbf{Loc}(W, U), \mathcal{F}_U(\omega) = f \circ \omega$$
◇

PROOF

We expose the proof from [Vic08b]. Let us take two opens $V, U \multimap X$ of X and let us contrive a factorization $U \multimap V$ of U through V (as (open) sublocales of X) after a transformation \mathcal{F} of the global points of U into global points of V . We view this diagram as fibred over the terminal locale $\mathbf{1}$, whereof we pull back everything along U ,



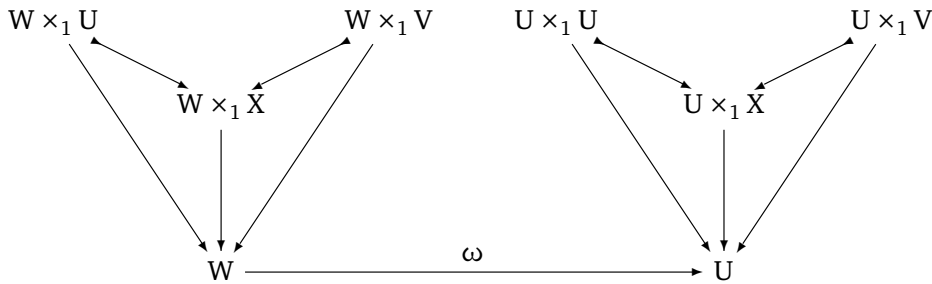
As \mathcal{F} is geometric, it can be carried out in every sheaf topos and so in particular, in the one of sheaves over U . From the locale U , U itself becomes $U \times U \longrightarrow U$ and equally for V , its counterpart over U is the pullback $U \times_1 V \longrightarrow U$. Nonetheless, at present, we can manipulate a very special global point of U , to wit, the diagonal $\Delta_U : U \multimap U \times U$ and can apply \mathcal{F} henceforth to it. The construction acts to output precisely a global point of the counterpart of V in $\text{Sh}(U)$, concretely an arrow,

$$U \longrightarrow U \times_1 V$$

This arrow is the datum,

$$\langle \text{Id}_U, f \rangle$$

for some f in $\text{Loc}(U, V)$. We must check that the factorization f outputs a generalized point of V when it takes a generalized point $\omega : W \longrightarrow U$ of U ; we anew take the pullback,



and apply \mathcal{F} on the global point $\langle \text{Id}_W, \omega \rangle : W \longrightarrow W \times U \xrightarrow{\omega} U$ seen from W — to output an arrow,

$$W \xrightarrow{\langle \text{Id}_W, f \circ \omega \rangle} W \times V$$

which is a point of V and corresponds to the pullback of the points $\mathcal{F}_U(\Delta_U)$ along ω . The geometricity of \mathcal{F} is equivalent to the isomorphy,

$$\omega^*(\mathcal{F}_U(\Delta_U)) = \langle \text{Id}_W, f \circ \omega \rangle \simeq \mathcal{F}_W(\omega^*(\Delta_U)) = \mathcal{F}_W(\langle \text{Id}_W, \omega \rangle) \quad \bullet$$

III.102—PROPOSITION — SPECIALIZATION ORDER, SEPARATION OF POINTS, T_0 , R_0

For any locales X, Y , the set $\mathbf{Loc}(X, Y)$ is a dcpo whereon the order is the (pre)order of specialization \sqsubseteq given by the pointwise order^①,

$$\forall f, g \in \mathbf{Loc}(X, Y), f \sqsubseteq g \iff f \leq g \iff \forall U \in \Omega(Y), f^*(U) \leq g^*(U)$$

The directed joins and the meets are computed pointwise. Incidentally, \mathbf{Loc} is enriched by a poset and is thus a 2-category. \diamond

III.103—PROPOSITION

Every localic arrow respects the specialization order by composition of frame morphisms. \diamond

III.104—PROPOSITION — DISCRETENESS OF SPECIALIZATION ORDER^②

The regular locales have their specialization orders as the identities on the points. Idem for the discrete locales. \diamond

III.105—NOTE

The finite powerset of a discrete locale is discrete and the full powerset being its localic ideal completion, the latter has the subset inclusion as the order of specialization. \diamond

III.106—PROPOSITION — CHARACTERIZATION OF DISCRETENESS & COMPACTNESS & COMPACT-REGULARITY^③

A localic arrow $f : Y \longrightarrow X$ is,

- 1° a local homeomorphism or a sheaf or étale if and only if it is fibrewise discrete if and only if its internal locale in $\mathbf{Sh}(cf)$ is discrete if and only if,

$$X \xleftarrow{f} Y \xrightarrow{\Delta} Y \times_X Y$$

are both open

- 2° is fibrewise compact if and only if its internal locale in $\mathbf{Sh}(cf)$ is compact if and only if,

$$X \xleftarrow{f} Y$$

is proper

- 3° is fibrewise compact and regular if and only if its internal locale in $\mathbf{Sh}(cf)$ is compact and regular if and only if,

$$X \xleftarrow{f} Y \xrightarrow{\Delta} Y \times_X Y$$

are both proper \diamond

^① The traditional definition for the topological spaces concerns only the global points $1 \longrightarrow Y$.

^② [Joh82].

^③ [JoyTie84; Tow96c; Vic10].

III.107 — PROPOSITION — GENERAL PROPERTIES IN \mathbf{Loc} ^①

The following holds,

- 1° an embedding in \mathbf{Loc} is closed if and only if the localic arrow is proper
- 2° a sublocale in \mathbf{Loc} is open if and only if it is open as a localic arrow.
- 3° a proper surjective is always the coequalizer of its kernel pair; idem for a open surjection
- 4° the surjections which are either proper or open are stable under pullbacks; idem for the embeddings
- 5° a surjective perfection is always the lax coequalizer of its lax kernel pair [KorLab07]
- 6° a sublocale of a (completely) regular locale is itself (completely) regular
- 7° an equalizer of a regular locale is necessarily proper
- 8° a closed sublocale of a compact regular locale is itself compact
- 9° a compact sublocale of a closed regular locale is itself closed
- 10° every localic arrow between compact regular locales is proper
- 11° with the axiom of dependent choice, each normal regular locale is completely regular
- 12° every monic in $\mathbf{KRegLoc}$ is regular monic in \mathbf{Loc} and also proper
- 13° if a set is finite decidable, then it is compact regular as a discrete locale \diamond

III.108 — DEFINITION — POWERLOCALES^{②③}

There are four principal manners to form the sublocales of a locale X ,

- 1° the *upper powerlocale* $\mathbb{P}_u X$ is the locale of all the sublocales of X being compact and fitted
- 2° the *lower powerlocale* $\mathbb{P}_\ell X$ is the locale of all the sublocales of X being weakly closed and having an open domain
- 3° the *double power locale* $\mathbb{P}X \simeq \mathbb{P}_u \mathbb{P}_\ell X \simeq \mathbb{P}_\ell \mathbb{P}_u X$ which is homeomorphic to $\mathbb{S}^{\mathbb{S}^X}$ when X is locally compact
- 4° the *Vietoris' powerlocale* $\mathbb{P}_v X$ is the locale of all the sublocales of X compact, overt and weakly semifitted \diamond

III.109 — PROPOSITION — CRITERION OF OVERTNESS AND COMPACTNESS VIA THE POWERLOCALES^④

A locale X is compact if and only if $\mathbb{P}_u X \longrightarrow \mathbf{1}$ has a left adjoint. A locale is overt if and only if $\mathbb{P}_\ell X \longrightarrow \mathbf{1}$ has a right adjoint. \diamond

① [Joh82].

② For general knowledge to us.

③ [Vic04b; Vic09a].

④ [Vic95a].

III.110—DEFINITION — FIBRATION, OPFIBRATION, FIBRE MAP

A bundle f is a **fibration** when we are certain of the existence of a (contravariant) **fibre map** $f: \top^*(df) \longrightarrow \perp^*(df)$, for $\perp \sqsubseteq \top$ in cf , factorizing via $df^{\mathbb{S}}$ thanks to an adjoint which has for its counit the identity.

A bundle f is an **opfibration** when we are certain of the existence of a (contravariant) **fibre map** $f: \perp^*(df) \longrightarrow \top^*(df)$, for $\perp \sqsubseteq \top$ in cf , factorizing via $df^{\mathbb{S}}$ thanks to an adjoint which has for its unit the identity. \diamond

III.111—PROPOSITION — FIBREWISE COMPACT REGULARITY AND FIBRATION^①

The sheaves or local homeomorphisms over a locale are opfibrations.

The fibrewise compact regular bundles over a locale are fibrations. \diamond

III.112—NOTE — TOPOSICAL CONTEXTULITY AS A (OP)FIBRATION

The theorem asserts that the variances of the aggregations on the spectral bundles of the covariant and contravariant approaches is truly determined by the kind of object that their bundle is. For the contravariant one, the spectral bundle is a fibration whereby establishing the variance of the aggregations on the fibres — hence the name. \diamond

III.3—CLOSED PREORDER AS HAUSDORFF SYSTEM

III.113—OVERVIEW — PATCH, COCOMPACT TOPOLOGY, PRIESTLEY’S DUALITY FOR PARTIAL ORDERS

We put the theory of the Hausdorff systems (X, R) [Tow96b; Tow96c] to good use since they give us a generalization of a coequalizer of a closed equivalence relation on a compact regular locale. Indeed, traditionally, we quotient the equivalence relation stemming from a preorder on a set to obtain the poset of the diverse equivalent classes. Yet categorically, the lax coequalizer is equally useful when its codomain is a locale for it has the effect to turn a closed preorder into the specialization \sqsubseteq of the codomain; and since all our locales are systematically T_0 , the identification of two elements $x \leq y \leq x$ in the original preorder is carried out by the antisymmetry of \sqsubseteq for we have $x \sqsubseteq y \sqsubseteq x$ only if $x = y$ in the codomain. Mathematically, we can generalize the situation in considering not merely closed preorders on some locales but in focusing on a compact regular locale X furnished with a closed relation R — required to be idempotent, $R \circ R = R$ — on them. These are the Hausdorff systems.

The categorical equivalence,

$$\mathcal{C}: \text{HausSyst} \simeq \text{StbKLoc}_{\text{prefrm hm}}$$

to be explained involves the interesting stably compact locales however equipped with preframe morphisms. This setting results from the extension of the (generalized) duality from Priestley, in its localic form,

$$\mathcal{C}: \text{KRegPos} \simeq \text{StbKLoc}_{\text{perfect}}$$

to the compact regular posets (X, \leq) and anew the stably compact locales $Z \doteq \mathcal{C}(X, \leq)$, but this time with perfect localic arrows. Under this duality, the compact regular poset is the famous **patch** $\text{Patch}(Z)$ of Z . Equivalently, the locale X is the locale whose frame is the one of

^① [FauVic11].

the perfect nuclei on the frame of Z [Esc99; Esc01; Coq03a] — or stated more geometrically, the perfect sublocales of Z . In classical mathematics [JunKegMos01], the patch of a topological space X is anew a topological space on the same set of points, while the topology given is the join of the original one with the *cocompact topology*. This new hybrid topology has for typical opens the intersections of an original open with the complement of a compact saturated (for the specialization). The patch becomes an ordered topological space once equipped with the partial order of specialization of the original X .

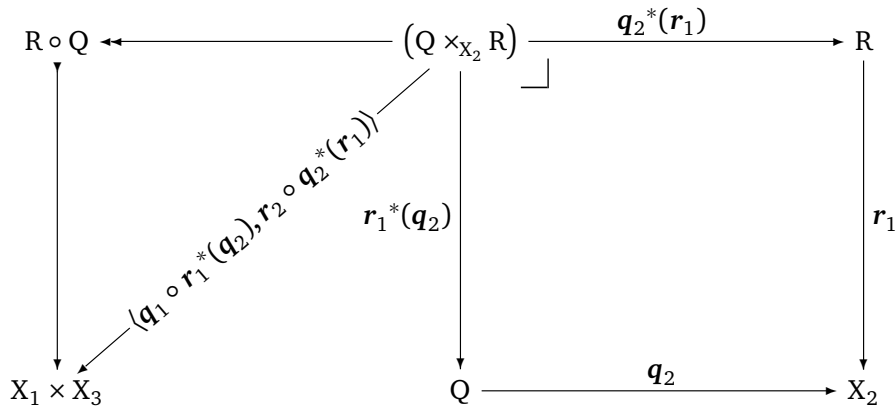
The point-free work of [Kli11] furthers a generalization in keeping the two topologies separated working with bitopological spaces and of D-frames. The original duality from Priestley is adapted locally in [Tow97; Vig04] but as said, it is generalized firstly, by the compact regular posets [Tow08], and secondly, by the aforementioned Hausdorff systems [Tow96b; Tow96c].

When we concentrate on the closed preorders, as in our personal development in section III.3.2 on 70, this time, we loose the patch direction of the equivalence — the passage of a stably compact locale to a compact regular poset — but hopefully, it is not the one we wish to keep. We only desire to establish the crucial properties of the locale map (as the former counit of the adjunction-equivalence) $\psi_R: X \rightarrow \mathcal{C}(X, R)$ (for R a closed preorder on X compact and regular) that it is the lax coequalizer of R and that R is its lax kernel pair. In staying between the compact regular posets and the general Hausdorff systems, we manage to keep the localic arrows, on the stably compact side, perfect. Naturally, ψ remains the counit of the more general equivalence with the Hausdorff systems. We must begin with a few general facts on the composition of locales and the Hausdorff systems. \diamond

III.3.1 — HAUSDORFF SYSTEMS

III.114 — DEFINITION — RELATIONAL COMPOSITION IN Loc

Given two sublocales $Q \multimap X_1 \times X_2$ and $R \multimap X_2 \times X_3$ interpreted as relations between some locales, their composition $Q \circ R$ is defined after the epi-regular-mono factorization in the following diagram,



More formally, the locale $R \circ Q$ is the sublocale given by the geometric type theory,

$$\forall (r, q) \in X_3 \times X_1, \top \vdash \exists s \in X_2, ((r, s) \in R \wedge (s, q) \in Q) \quad \diamond$$

III.115 — CONSPECTUS — RELATIONS IN **KRegLoc**^①

With the definition of the relational composition in **Loc**, we wish to be certain that it always exists. The categorical mechanism asserts that it is enough for a category to be regular in order to create its category of relations. Fortunately, **KRegLoc** is so : **KRegLoc** is finitely complete ; the kernel pair of any of its morphisms admits a coequalizer ; the pullback of a regular epimorphism along every morphism is anew regular and epimorphic. Its category of relations is precisely the one having for objects the monomorphisms in **KRegLoc** — becoming regular monomorphisms in **Loc** — with the aforementioned composition.

Finally, the closedness of the relations — or the properness of the regular monics — in **KRegLoc** assures their alternative characterization as preframe morphisms and, furthermore, that this correspondence be an isomorphism of posets but reversing the orders. \diamond

III.116 — PROPOSITION — EQUIVALENCE BETWEEN CLOSED RELATIONS AND PREFRAME MORPHISMS

In the case of the compact regular locales, there exists a bijection,

$$\begin{aligned} \mathbf{PreFrm}(\Omega(Y), \Omega(X)) &\longrightarrow \Omega(X) \otimes \Omega(Y) (\doteq \Omega(X \times Y) \simeq (\neg(\Omega(X \times Y)))^{\text{op}}) \\ \psi_* &\longmapsto (\psi_* \otimes \text{Id}_X)(\mathbf{a}_{\Delta_X}) \end{aligned}$$

between the poset $\mathbf{PreFrm}(\Omega(Y), \Omega(X))$ of preframe morphisms between the (underlying preframes of the) frames $\Omega(Y)$ and $\Omega(X)$ and opens of the product $X \times Y$ of the locales. It sends a preframe morphism $\psi_* : \Omega(Y) \longrightarrow \Omega(X)$ to the open resulting of the application of $(\psi_* \otimes \text{Id}_X)$ on the open \mathbf{a}_{Δ_X} corresponding to the closed diagonal Δ_X . This bijection is in effect an isomorphism reversing the orders on the posets.

A closed relation $R \subseteq X \times Y$ translates as the unique preframe morphism,

$$\begin{aligned} (\Downarrow_R)^{\text{op}} \doteq \psi_{R*} \doteq \psi_* : \Omega(X) &\longrightarrow \Omega(Y) \\ \mathbf{a} &\longmapsto (\Downarrow_R)^{\text{op}}(\mathbf{a}) \end{aligned}$$

such that,

$$\forall \mathbf{a} \in \Omega(X), \neg(\Downarrow_R)^{\text{op}}(\mathbf{a}) = \Downarrow_R(\neg \mathbf{a})$$

$$\forall \mathbf{a} \in \Omega(X), \neg((\Downarrow_R)^{\text{op}}(\mathbf{a})) = \Downarrow_R(\neg(\mathbf{a}))$$

where \Downarrow_R is the downward closure of the relation R , on the closedsets of X . \diamond

III.117 — NOTE — LOWER CLOSURE

We call very loosely the arrow ψ_* (or solely ψ) the « **lower closure** of R », yet we must remember that the real one is \Downarrow_R .

This is the compact regular version of the duality, for a discrete poset X , between the fixed points of the upward closure $\uparrow(-)$, as an endofunction on $\mathcal{P}(X)$, of the order on X and the subsets of X closed upwardly for the order. More generally, the opens of $X \times Y$ are the morphisms of suplattices from $\Omega(X)$ to $\Omega(Y)$ [Tow08]. \diamond

^① [Tow96c].

III.118 — DEFINITION — HAUSDORFF SYSTEM, APPROXIMABLE MAPPING^①

Every Hausdorff system is the datum of a compact regular locale X equipped with a closed relation R subject to its idempotency,

$$R \circ R = R$$

A « **upper approximable semi-mapping** $T: (X, R) \longrightarrow (Y, Q)$ of Hausdorff systems » is an arrow $T \longmapsto X \times Y$ being closed and verifying,

$$T = Q \circ T \circ R$$

A « **approximable mapping** $T: (X, R) \longrightarrow (Y, Q)$ of Hausdorff systems » is an arrow T being upper approximable and verifying,

$$1^\circ (\Downarrow_R)^{\text{op}}(X) \leq_{\text{SubLoc}(X)} Y \circ T$$

2° for all closed sublocales W and Z of Y ,

$$(\Downarrow_R)^{\text{op}}(W \circ T \wedge Z \circ T) \leq_{\text{SubLoc}(X)} ((\Downarrow_Q)^{\text{op}}(W) \wedge (\Downarrow_Q)^{\text{op}}(Z)) \circ T \quad \diamond$$

III.119 — NOTE — HAUSDORFF SYSTEM AND INFORMATIONAL SYSTEM

Historically, the first analysis in [Vic93] concerned a discrete locale X supplemented with a relation R which is idempotent. Thanks to being regular, the category of the discrete locales has a category of relations and the relations become a morphism of suplattice on its frame, namely its powerset; the fixed points of this morphism constitute a remarkable completely distributive lattice useful in domain theory. The development of the Hausdorff systems is motivated from its belonging to a broader analogy between open and proper maps as argued in [Tow06; Tow08] for instance. ◇

III.120 — PROPOSITION

The Hausdorff systems and their morphisms constitute a category.

The explicit statement of a general isomorphism $T: (X, R) \longleftrightarrow (Y, Q): S$ in the category of Hausdorff systems is the identities,

$$T \circ S = R \text{ and } S \circ T = Q$$

which translates on the level of the preframe morphisms as,

$$\psi_S \circ \psi_T = \Downarrow_R \text{ and } \psi_T \circ \psi_S = \Downarrow_Q$$

and expectedly the respects of the relations under these morphisms — monotonicity. ◇

III.121 — CONSPECTUS — EXTENDED PRIESTLEY'S DUALITY VIA HAUSDORFF SYSTEMS^②

The category of the Hausdorff systems and the upper approximable semi-mappings is isomorphic to the category of the stably compact locales with preframe morphisms, formally reversed,

$$\mathcal{C}: \text{HausSyst} \simeq \text{StbKloc}_{\text{prefrm hm}}$$

The category of the Hausdorff systems with the approximable mappings is isomorphic to the category of the stably compact locales with frame morphisms, formally reversed,

$$\mathcal{C}: \text{HausSyst} \simeq \text{StbKloc}_{\text{frm hm}}$$

^① [Tow96b].

^② [Tow96c].

Concretely, to every Hausdorff system $(\text{Patch}(X'), R)$ corresponds a stably compact locale $X' \doteq \mathcal{C}(\text{Patch}(X'), R)$ whose frame is the subpreframe — not subframe in general ! — of the one of $\text{Patch}(X')$ corresponding to the collection of the fixed points of the lower closure of R ,

$$\Omega(X') \doteq \{a \in \Omega(\text{Patch}(X')) \mid \psi_*(a) = a\}$$

Nonetheless, there exists also a simple preframe injection,

$$\begin{aligned} \psi^* : \Omega(X') &\longrightarrow \Omega(\text{Patch}(X')) \\ a &\longmapsto a = \psi_*(a) \end{aligned}$$

The theory tells us that the finite joins \vee' in $\Omega(X')$ are computed as,

$$\forall a, b \in \Omega(X'), a \vee' b \doteq \psi_*(a \vee b)$$

In the reverse direction, a stably compact locale X' is sent to the Hausdorff system $\mathcal{B}(X') \doteq (\text{Patch}(X'), R)$ with $\Omega(\text{Patch}(X'))$ being the injection of $\Omega(X')$ in the ideal completion of its free boolean algebra qua distributive lattice. The frame of the locale $\text{Patch}(X')$ is the frame of the perfect nuclei [Esc99; Esc01]. The closed relation equipping $\text{Patch}(X')$ is the (pullback along the localic surjection from $\text{Patch}(X')$ into X' of the) specialization order on X' . \diamond

III.3.2—PREORDERS ON COMPACT REGULAR LOCALES

III.122—PROPOSITION — PREORDER AS HAUSDORFF SYSTEM

A closed preorder R on a compact regular locale $\text{Patch}(X')$ is a Hausdorff system.

Besides, the frame $\Omega(X')$ of the fixed points of its lower closure ψ_* — concretely, it is the stably compact equivalent of the Hausdorff system $(\text{Patch}(X'), R)$ — is not only a subpreframe of $\Omega(\text{Patch}(X'))$, but also a subframe.

We are thus compelled to see $\psi : \text{Patch}(X') \longrightarrow X'$ as a surjective perfection going from a compact regular locale $\text{Patch}(X')$ onto a stably compact one X' . \diamond

PROOF

We immediately have,

$$\Delta \leq R \Rightarrow R \leq R \circ R$$

because Δ is the identity arrow for the relational composition.

Regarding the subframe, indeed, denoting $\Omega(X')$ the frame of opens of the locale $\text{Patch}(X')$ under the equivalence between Hausdorff systems and stably compact locales, when the relation R is reflexive, its translation into operator ψ_* on the frame of opens is deflationary and,

$$\forall a, e \in \Omega(X'), (e, a \leq e \vee' a) \Rightarrow e \vee a = \psi_*(e) \vee \psi_*(a) \leq e \vee' a \leq e \vee a$$

bearing the consequence that $\Omega(X')$ is factually a subframe of $\Omega(\text{Patch}(X'))$. But now, this injection is the left adjoint to ψ_* , for ψ_* is transitive and reflexive,

$$\forall a \in \Omega(X'), \psi^*(a) = a \Rightarrow \psi_*(\psi^*(a)) = \psi_*(a) = a$$

likewise,

$$\forall a \in \Omega(\text{Patch}(X')), \psi_*(a) \leq a \Rightarrow \psi^*(\psi_*(a)) \leq \psi^*(a) = a$$

and we can conclude that ψ_* is surjective and perfect as a locale map. \bullet

III.123 — PROPOSITION

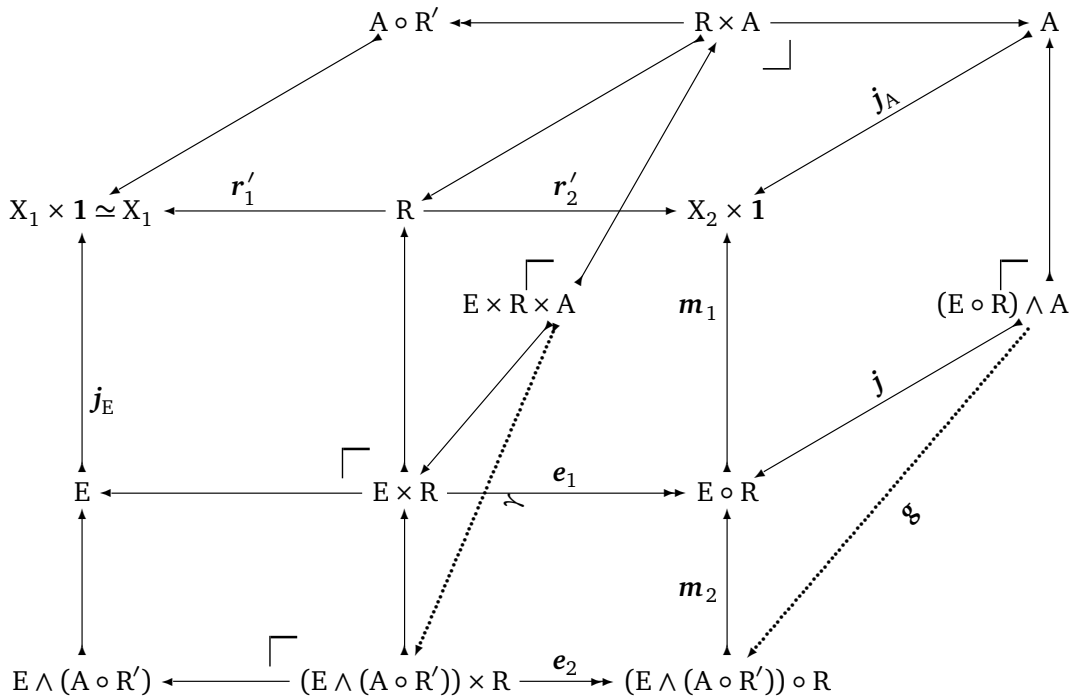
For a closed relations R on a locale X and closed sublocales E and A of X , each one compact and regular,

$$(E \circ R) \wedge A \leq (E \wedge (A \circ R')) \circ R$$

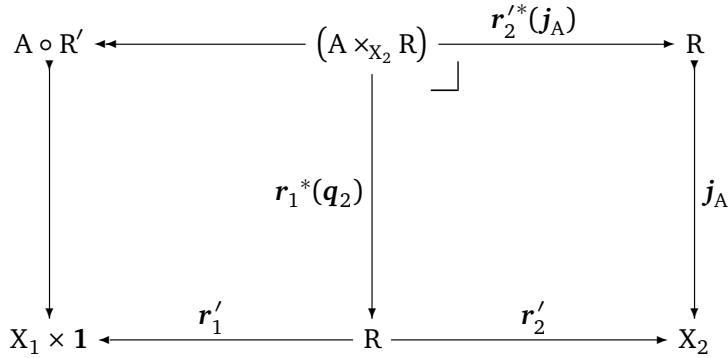
with R' the opposite of R and \leq is the order of the inclusion of the sublocales. ^① ◇

PROOF

We choose to note the inverse $R' \doteq \tau R$ of the relation $\langle r_1, r_2 \rangle: R \rightarrow X_1 \times X_2$ by the maps $\langle r'_1, r'_2 \rangle: R' \doteq R \rightarrow X_1 \times X_2$ with τ the twist operator and locales $X_2 = X_1 \doteq X_2 = X_1 \doteq X$ with its sublocales $A \rightarrow X_2 \times 1$ and $E \rightarrow 1 \times X_1$. We build up the following diagram, bit by bit,

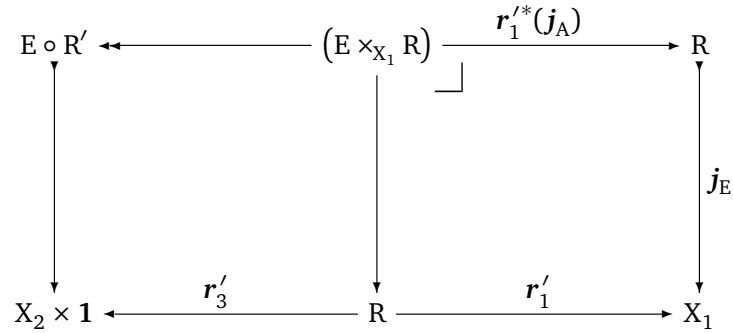


To begin, we form the pullback $R \times_{X_2} A$ then immediately take its image $A \circ R \rightarrow X_1$,

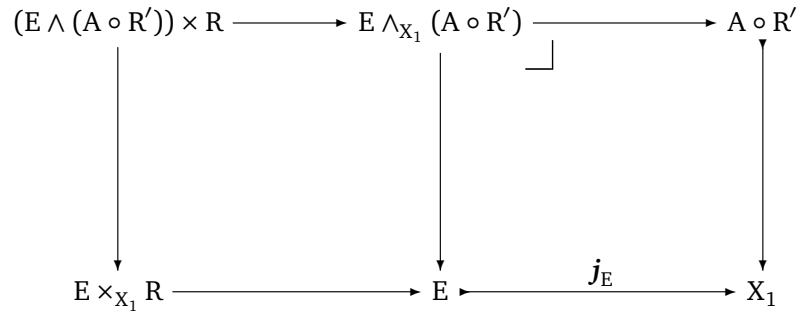


^① The composite $E \circ R$ is really the composition of the relation $E \times 1 \rightarrow X \times 1$ with R and where we project the output of the procedure to keep a closed relation on X .

We transpose it to the undisplayed case of the sublocale E in order to form the sublocale $E \circ R \rightarrow 1 \times X_2$,

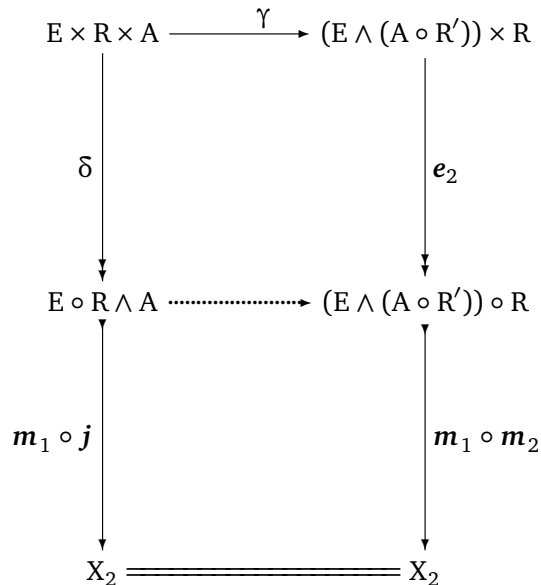


The second part consists in taking the pullback $E \wedge_{X_1} (A \circ R')$ of E and $A \circ R'$, then in taking its pullback with $E \times_{X_1} R$,



There exists a unique arrow γ from $E \times R \times A$ to $(E \wedge (A \circ R')) \times R$ by two universalities of pullbacks. In the same vein, there exists a unique (undisplayed) arrow δ from $E \times R \times A$ to $(E \circ R) \wedge_{X_2} A$. By Beck–Chevalley in **KRegLoc** [Ver94, 4.3], it must be epic because all our arrows are necessarily proper since all our locales are compact regular by hypothesis.

The following diagram is thus constituted,



and indeed commutes ; we exhume a unique arrow g from $(E \circ R) \wedge A$ to $(E \wedge (A \circ R')) \circ R$ such that,

$$j = m_2 \circ g \quad \bullet$$

III.124—NOTE

This proposition is useful for the theorem about the closed equivalence relation. \diamond

III.125—PROPOSITION — HAUSDORFF SYSTEM AND EQUIVALENCE RELATION

Considering a Hausdorff system $(\text{Patch}(X'), R)$ where R is a relation of equivalence, the corresponding stably compact locale $\mathcal{C}(\text{Patch}(X'), R)$ is regular. \diamond

PROOF

We must prove that the sub(pre)frame $\Omega(X') \doteq \Omega(\mathcal{C}(\text{Patch}(X'), R))$ of $\Omega(\text{Patch}(X'))$ is regular. We know that $\text{Patch}(X')$ being regular, the demonstration is the equality,

$$\forall a \in \Omega(X'), \bigvee^{\text{dir}} \{b \in \Omega(\text{Patch}(X')) \mid b \ll a\} = \bigvee^{\text{dir}'} \{b \in \Omega(X') \mid b \ll' a\}$$

For the proof, we merely need to go from the closure $\Omega(X')$ to the frame $\Omega(\text{Patch}(X'))$; let us then suppose, when a, b are in $\Omega(X')$, that,

$$\exists e \in \Omega(\text{Patch}(X')), e \wedge b = \perp \text{ and } e \vee a = \top$$

and we engage ourselves into proving $(\Downarrow)^{\text{op}}(e)$ is the required element of $\Omega(X')$ to conclude the implication holds ; that is to say,

$$(\Downarrow)^{\text{op}}(e) \wedge' b = \perp' \text{ and } (\Downarrow)^{\text{op}}(e) \vee' a = \top'$$

where, the meet, join, top and bottom belong to the frame $\Omega(\text{Patch}(X'))$ as $\Omega(X')$ is one of its subframe.

We trade relations on opens against relations on closed sublocales in such a manner that we focus on proving,

$$\neg e \circ R \wedge \neg a \leq \perp = \neg(e \vee a)$$

as closed sublocales. From III.123, we know the general fact that,

$$\neg e \circ R \wedge \neg a \leq (\neg e \wedge \neg a \circ R^{-1}) \circ R$$

for the relations on the compact regular locale ; implying that,

$$\perp \leq \neg e \circ R \wedge \neg a \leq \neg(e \vee a) \circ R = \perp \circ R = \perp$$

since R is its opposite. \bullet

III.126—PROPOSITION — LAX COEQUALIZER OF PREORDER

For a closed preorder \leq on a locale $\text{Patch}(X')$ compact and regular, its downward closure ψ is its lax coequalizer. Moreover, the lax kernel pair of ψ is \leq . \diamond

PROOF

When we take the lax kernel pair of a localic arrow ψ with a domain $\text{Patch}(X')$ compact regular, we retrieve a preorder on $\text{Patch}(X')$ [Vig04, 2.]. In the case where the codomain is locally compact and the arrow is perfect, the lax kernel pair is closed and its open complement is [Vig04, 2.],

$$s = \bigvee \{ \psi^*(a) \otimes \neg \psi^*(a) \mid a \in \Omega(X') \} = \bigvee \{ a \otimes \neg a \mid a \in \Omega(X') \} \\ = (\psi_* \otimes \text{Id}_{\text{Patch}(X')})(a_{\Delta_{\text{Patch}(X')}})$$

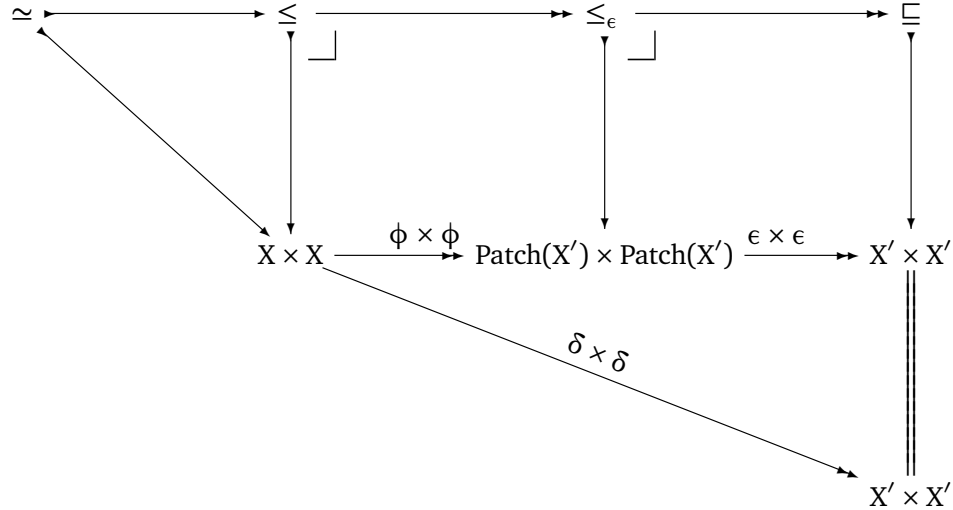
The lax coequalizer of this closed lax kernel pair is necessarily perfect [KorLab07]; and the complement of this closed sublocale keeps the same expression as s , this time replacing ψ_* by the lax coequalizer. By the isomorphy between preframe morphisms and closed relations, ψ_* and the lax coequalizer must correspond to the complement of s ; namely the lax kernel pair of ψ . •

III.127 — PROPOSITION — LAX COEQUALIZER OF A LAX KERNEL PAIR

The lax coequalizer of a closed preorder \leq on a compact regular locale is the composite of the strict coequalizer of its equivalence relation $\simeq \doteq \leq \wedge (\leq)^{\text{op}}$ — giving rise to a closed partial order on the compact regular quotient — with the counit ϵ of the adjunction between the compact regular locales and the stably compact ones — ϵ is the counit of the functor $\text{Patch}(-)$. ◊

PROOF

Let us establish the notations and the intermediate claims that we must obtain in order to conclude the proposition. Diagrammatically, we construct step by step,



in giving ourselves a perfect surjection δ and its lax kernel \leq on a compact regular locale $d\delta$, whereof we can also construct the kernel pair \simeq which is a closed relation of equivalence on the compact regular locale $X \doteq d\delta$ and which itself has a proper coequalizer ϕ onto a compact regular locale $\text{Patch}(X')$ itself giving the existence of a unique perfect surjection ϵ onto the stably compact locale X' that $c\delta$ is — and in such a manner that δ is the composite $\epsilon \circ \phi$.

It is naturally desired to create the lax kernel \leq_ϵ of ϵ on X' whose pullback along the proper surjection $\phi \times \phi$ is in effect the lax kernel of δ . It emanates as well that the lax kernel \leq_ϵ has

the property of the antisymmetry whereof the usefulness appears after we notice that the Hausdorff system it does constitute on $\text{Patch}(X')$ is the true patch of X' bearing the partial order gotten in pulling back the (partial) order of specialization \sqsubseteq on X' .

In the end, we merely need to consider δ instead of the chain ϕ, ϵ . Let us detail further. We are assured that a proper surjective ϕ is always the coequalizer of its kernel pair [Ver94, 5.3]. And by the theory of the Hausdorff systems applied to the closed equivalence relations, we do know that its codomain is compact regular. By universal property of every coequalizer,

$$\exists! \epsilon \in \mathbf{Loc}(\text{Patch}(X'), X'), \delta = \epsilon \circ \phi$$

given by,

$$\epsilon^* \doteq \phi_* \circ \delta^*$$

But the uniqueness of the adjuncts permits to claim that,

$$\delta_* = \epsilon_* \circ \phi_* \Rightarrow \delta_* \circ \phi^* = \epsilon_*$$

imposing the perfection of ϵ .

Now, we take anew a lax kernel pair \leq_ϵ and show that its pullback along $\phi \times \phi$ is the original preorder. The lax kernel pair of δ laxly coequalizes the arrow ϵ ; whence the existence of a unique morphism between the two preorders such that the diagram commutes,

$$\begin{array}{ccc} \leq & \xrightarrow{\dots\dots\dots} & \leq_\epsilon \\ \downarrow & & \downarrow \\ d\delta \times d\delta & \xrightarrow{\phi \times \phi} & \text{Patch}(X') \times \text{Patch}(X') \end{array}$$

We reverse the unique arrow from the primitive preorder to the pullback in using the universal property of the preorder on $d\delta$; the two legs of \leq_ϵ lead immediately to the pair of legs of their pullbacks along ϕ which possesses additionally the property to laxly coequalize δ since, after all, the pullback of the proper surjection ϕ is again surjective proper.

It remains to say that the preorder \leq_ϵ is in effect a partial order. For this purpose we only need to construct the pullback,

$$\begin{array}{ccccc} \dots & \xrightarrow{\quad} & \leq_\epsilon \wedge \geq_\epsilon & & \\ \downarrow & \lrcorner & \downarrow & & \\ d\delta \times d\delta & \xrightarrow{\phi \times \phi} & X' \times X' & \xrightarrow{\epsilon} & X' \times X' \end{array}$$

and prove that it is a sublocale of \simeq . There exists indeed the desired embedding for when we compose by ϵ the two legs of the relation $\leq_\epsilon \wedge \geq_\epsilon \longrightarrow X'$, the equality is forced by antisymmetry of the specialization on X' . •

III.128 — PROPOSITION

Let there be g an arrow of locales, with domain and codomain compact regular and each one equipped with a closed preorder on them; when g is monotone with respect to the preorders, then there exists a perfect arrow g' between the codomains of the lax coequalizers of the preorders. \diamond

PROOF

All is explained in [Vig04, 2.]. The arrow g' does exist by the universality of a lax coequalizer. It is perfect essentially by uniqueness of the adjunction defining a locale map. \diamond

III.129 — NOTE

The true difference in working with the closed preorders instead of closed partial orders is that we loose the converse of the proposition. If we have a perfect locale map between some stably compact locales, we know how to manufacture a continuous monotone proper map, between the patches of the locales. All would be well if the patches were the original compact regular locales corresponding to the images of the stably compact codomains of the lax coequalizers under the patch duality; but in general the patches are more special than a general locale compact and regular. The posets are more special than the presets. \diamond

III.4—LAX DESCENT IN **Loc**

III.4.1—DESCENT STRICT AND LAX

III.130 — OVERVIEW — DESCENT

We have studied, more or less explicitly, the perfect lax coequalizers f of **Loc** and have seen that they are tied to the quotientage of the closed preorders \leq on the compact regular locales df . The quotient cf itself is in effect a stably compact locale. The immediate question is the becoming of the bundles g over the locale df once we look at them from the codomain cf . The (lax) descent carries out this task.

The notion of (*lax*) *descent* (in **Loc**) regards the passage of some properties of the bundles g over the domain df , of a locale arrow f , to the bundles over the codomain cf . In fact, we equally impose to the bundles over the domain df to be the pullbacks of the bundles over the codomain. What are their essence? They are the bundles $g : dg \rightarrow df$ over df which are equipped with a supplementary datum, a *descent datum*, for descending down the arrow f . Whereas f itself must be given (at least) a preorder $\leq \xrightarrow{\pi_1 \leq \pi_2} df \times df$.

For what kinds of localic arrows f is the descent possible? For the strict descent, we see f as a proper surjective [Ver94] bearing the consequence that it is the coequalizer of its kernel pair, kernel pair being in effect a closed equivalence relation (on the domain df) — the open arrows and their kernel pairs are also adequate for descending strictly [JoyTie84]. For the lax descent, we see f as a perfect surjective bearing the consequence that it is the lax coequalizer of its lax kernel pair, lax kernel pair being effectively a closed preorder when the domain df is compact and regular and the codomain is locally compact. We know that the localic surjective perfections are of effective lax descent from [VerMoe97] for the sheaves, because these localic arrows are the relatively tidy geometric morphisms between the sheaf toposes [Vig04].

We offer to show that the Stone locales are equally good candidates to laxly descend down the surjective perfections in **Loc**. First, we prove that the products of the lax data is what we

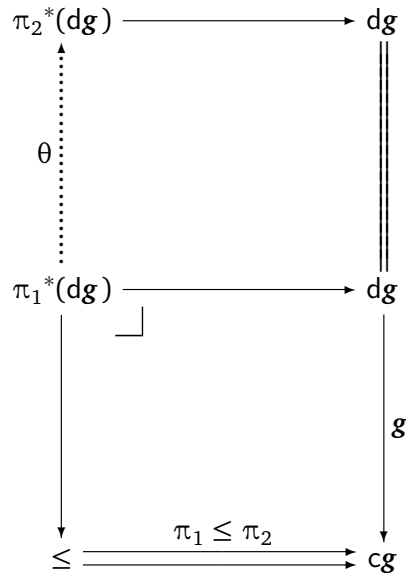
expect; to wit that they are the product the components. Secondly, we demonstrate that the boolean algebras in the category of the data are the boolean algebras for the bundle plus a compatibility concerning the maps. It only remains then to verify that the specialization order in the codomain of our lax coequalizers (permitting to laxly descend) gives a lax datum which must be preserved over the domain, once pulled back. \diamond

III.131 — CONSPECTUS — ACTION OF PREORDER, COUNIT & COCYCLE CONDITION

The descent datum that is required by the theory of descent is a specialized notion of the more general concept of (the datum for) an action by a preorder \leq upon a bundle g . The **datum** — or an **action** — is concretized as an arrow of locales,

$$\theta: \pi_1^*(dg) \longrightarrow \pi_2^*(dg)$$

between the pullbacks,



of (the legs $\pi_{1,2}$ of) the preorder \leq on cg along the bundle g . Every datum is also subject to,

- 1° a **unit condition** expressing the compatibility of the datum with the reflexivity of the preorder as the pullback of θ along the diagonal Δ_{cg} of cg matching the identity on the domain of the bundle g ,

$$\Delta_{cg}^*(\theta) \simeq Id_{dg}$$

- 2° a **cocycle condition** expressing the compatibility of the datum with respect to the transitivity of the preorder as the irrelevance of the existence of x_1 in the knowledge $x_0 \leq x_1 \leq x_2$ in order to conclude $x_0 \leq x_2$,

$$\pi_{12}^*(\theta) \circ \pi_{01}^*(\theta) \simeq \pi_{02}^*(\theta)$$

where some isomorphisms are hidden and the π 's are all the possible projections to the locale \leq ,

$$(\leq \times_{cg} \leq) \xrightarrow{\pi_{01}, \pi_{02}, \pi_{12}} \leq$$

from the pullback,

$$\begin{array}{ccc}
 (\leq \times_{\mathbf{Cg}} \leq) & \xrightarrow{\pi_2^*(\pi_1)} & \leq \\
 \pi_1^*(\pi_2) \downarrow & \lrcorner & \downarrow \pi_1 \\
 \leq & \xrightarrow{\pi_2} & \mathbf{Cg}
 \end{array}$$

◇

III.132 — DEFINITION — CATEGORY OF ACTIONS OF PREORDER, LAX DATUM

For a preorder \leq on a locale X , the category of its *actions* — or *lax data* — has for objects the various pairs,

$$(g, \theta)$$

where g is a bundle over X and θ is an action of \leq for g . The arrows are the arrows of \mathbf{Loc}_X which are compatible with the actions; for every bundle arrow $h: f \rightarrow g$ over X , the diagram,

$$\begin{array}{ccc}
 \pi_2^*(df) & \xrightarrow{\pi_2^*(h)} & \pi_2^*(dg) \\
 \theta_f \uparrow & & \uparrow \theta_g \\
 \pi_1^*(df) & \xrightarrow{\pi_1^*(h)} & \pi_1^*(dg)
 \end{array}$$

must commute.

The composition and the equality of arrows are the one of \mathbf{Loc}_X plus the relevant diagrams over \leq .

◇

III.133 — PROPOSITION

The product of the actions (F, θ) and (G, ψ) over a locale X and its preorder \leq is the lax datum,

$$(F \times_X G, \theta \times_{\leq} \psi)$$

◇

PROOF

Firstly, we do have a geometric result on the product,

$$j = 1, 2 \Rightarrow \pi_j^*(F \times_X G) \simeq \pi_j^*(F) \times_{\leq} \pi_j^*(G)$$

The counit condition behaves nicely,

$$\Delta_X^*((\theta \times_{\leq} \psi)) \simeq (\Delta_X^*(\theta) \times_X \Delta_X^*(\psi)) \simeq (\text{Id}_F \times_X \text{Id}_G) \simeq \text{Id}_{(F \times_X G)}$$

And the one of the cocycle is similar,

$$\begin{aligned}
 \pi_{12}^*((\theta \times_{\leq} \psi)) \circ \pi_{01}^*((\theta \times_{\leq} \psi)) &\simeq (\pi_{12}^*(\theta) \times_{(\leq \times_X \leq)} \pi_{12}^*(\psi)) \circ (\pi_{01}^*(\theta) \times_{(\leq \times_X \leq)} \pi_{01}^*(\psi)) \\
 &\simeq (\pi_{12}^*(\theta) \circ \pi_{01}^*(\theta) \times_{(\leq \times_X \leq)} \pi_{12}^*(\psi) \circ \pi_{01}^*(\psi)) \\
 &\simeq \pi_{02}^*((\theta \times_{\leq} \psi))
 \end{aligned}$$

If we have a lax datum (Q, ϕ) and some morphism $\alpha: (Q, \phi) \longrightarrow (F, \theta)$ then we know that it is the knowledge of two morphisms $\alpha_{1,2} \doteq \pi_{1,2}^*(\alpha)$ making the square of the lax data commutative. Idem with a morphism $\beta: (Q, \phi) \longrightarrow (G, \psi)$. It results that we can form the product $(\alpha \times \beta)_{1,2} \doteq \pi_{1,2}^*(\alpha \times \beta)$ in order to obtain a morphism of lax data from (Q, ϕ) to $(F \times_X G, \theta \times_{\leq} \psi)$. •

III.134 — PROPOSITION — COALGEBRA^{①②}

The codomain \mathcal{E} of every surjection $f: \mathcal{F} \longrightarrow \mathcal{E}$ between toposes is equivalent to the category of coalgebras for the comonad f^*f_* on \mathcal{F} .

When Beck–Chevalley holds, as it does for the pullbacks and their lax variants, id est the isomorphism,

$$f^* \circ f_* \simeq \pi_{1*} \circ \pi_2^*$$

is present, the colagbras are turned into the data for a (lax) descent via adjunction $f^* \dashv f_*$. ◇

III.135 — DEFINITION — STRICT & LAX (EFFECTIVE) DESCENT (OF SHEAVES)

Let there be f a locale arrow and its geometric morphism $f: \mathcal{F} \longrightarrow \mathcal{E}$. The arrow f permits (the sheaves^③) to **(laxly) descend** (down itself) when the pullback functor is fully faithful,

$$\begin{aligned} f^*: \mathcal{E} &\longrightarrow \text{Sh(L)Des}(f) \\ E &\longmapsto (f^*(E), \phi_E: (f \circ \pi_1)^*(E) \longrightarrow (f \circ \pi_2)^*(E)) \end{aligned}$$

where $\text{Sh(L)Des}(f)$ is the category of the actions (g, θ) for the (lax) kernel pair of f with the special requirement that g be a local homeomorphism over \mathcal{F} . The arrow ϕ_E is,

1° for the strict descent, an isomorphism coming from the kernel pair of f which is a relation of equivalence

2° for the lax descent, a 2-cell $(f \circ \pi_1)^*(-) \Rightarrow (f \circ \pi_2)^*(-)$ given by the specialization \sqsubseteq of cf

The (lax) descent (down f) is **effective** when the functor is an equivalence of categories. ◇

III.136 — DEFINITION — RELATIVE TIDINESS^④

A geometric morphism f between some toposes over a base topos \mathcal{T} is **relatively tidy** when the functor f_* of the direct image respects the filtered colimits externally indexed by \mathcal{T} —these are the same as in the internal logic of \mathcal{T} .

Incidentally, over the base topos **Set**, a geometric morphism f between some toposes is **relatively tidy** when the functor f_* of the direct image respects the filtered colimits. ◇

III.137 — PROPOSITION — PERFECTION AND RELATIVE TIDINESS^⑤

The perfect arrows in **Loc** restricted between stably compact locales have their geometric morphisms (between the sheaf toposes) relatively tidy. ◇

^① [Joh02].

^② Disconnected from the motivation of the descent.

^③ The traditional focus in descent is on the sheaves; that is why we put it into parenthesis.

^④ [VerMoe97].

^⑤ [Vig04].

III.138 — PROPOSITION — EFFECTUALITY IN **Loc**

The open surjectives in **Loc** are effective morphisms of descent (for sheaves) [JoyTie84].

The proper surjective localic arrows are of effective descent (for sheaves) [Ver94].

The perfect surjective localic arrows are of effective lax descent (for sheaves) [VerMoe97; Vig04]. ◇

III.4.2—LAX DESCENT AND STONE BUNDLE

III.139 — PROPOSITION

The finitary algebraic structures of the category of actions of a preorder \leq on a locale X are the objects (g, θ) where,

- 1° g must be of a nature algebraically identical in $\mathbf{Loc}/_X$
- 2° the action θ commutes with the (algebraic) operations

The morphisms $h: (g, \theta_g) \longrightarrow (k, \theta_k)$ of algebras are the morphisms h of the bundles in $\mathbf{Loc}/_X$ with a supplementary knowledge of the commutativity of the diagram,

$$\begin{array}{ccc}
 \pi_2^*(dg) & \xrightarrow{\pi_2^*(h)} & \pi_2^*(dk) \\
 \uparrow \theta_g & & \uparrow \theta_k \\
 \pi_1^*(dg) & \xrightarrow{\pi_1^*(h)} & \pi_1^*(dk)
 \end{array}$$

◇

PROOF

The proof is mostly the analysis of the existence of a algebraic structure on dg for some bundle g over X . If $(F \doteq dg, \theta)$ is an action which is algebraic at the same time, we have a few commutativities of diagrams to express the booleaness. These diagrams are more precisely diagrams over X involving a few arrows for the operations over X plus their pullbacks, along the legs of \leq , to get them over \leq . If we forget the diagrams from the takings of the pullbacks, we can conclude that F must be a algebraic bundle over X . When g is an algebra in $\mathbf{Loc}/_X$, and is supplemented with a lax datum commuting with the algebraic connectors, we can conclude that the operations are indeed the morphisms of action.

At present the morphisms. A morphism h of algebras in the category of actions is a morphism of the same algebras in $\mathbf{Loc}/_X$ and several commutative diagrams over \leq , one whereof is the diagram involving h and the actions of its domain and codomain. If, on the other hand, we begin with a algebraic morphism $h: g \longrightarrow k$ in $\mathbf{Loc}/_X$ and the diagram of the compatibility, we possess indeed a algebraic morphism h in the category of actions by composition of the arrows of the actions. ●

III.140—PROPOSITION

The boolean algebras from the category of the actions of a preorder \leq on a locale X are the objects (\mathbf{g}, θ) , subject to the constraints that,

1° \mathbf{g} be a boolean algebra in \mathbf{Loc}/X

2° the action θ commute with the (finite) meets and (finite) joins and also with the complementation ; for example,

$$\pi_2^*(\wedge) \circ (\theta \times_{\leq} \theta) = \theta \circ \pi_1^*(\wedge) \quad \diamond$$

III.141—DEFINITION — OPLAX (EFFECTIVE) DESCENT OF FIBREWISE STONE BUNDLES

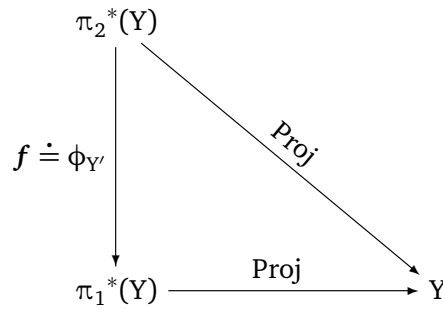
Let there be δ a map and its geometric morphism $\delta: \mathcal{F} \longrightarrow \mathcal{E}$. The arrow δ permits the fibrewise Stone bundles over \mathcal{F} to *oplaxly descend* (down itself) for the opposite $(\leq)^{op}$ of its lax kernel pair \leq when the arrow δ permits the boolean sheaves to laxly descend down itself for its kernel pair \leq .

In functorial terms, the pullback functor is fully faithful,

$$\begin{aligned} \delta^*: \text{StoneLoc}(\mathcal{E}) &\longrightarrow \text{StoneOpLDes}(\delta) \\ E &\longmapsto (\delta^*(E), \phi_E: (\delta \circ \pi_2)^*(E) \longrightarrow (\delta \circ \pi_1)^*(E)) \end{aligned}$$

where $\text{StoneOpLDes}(\delta)$ is the category of the actions (\mathbf{g}, θ) for the opposite $(\leq)^{op}$ of the lax kernel pair \leq of δ with the special requirement that \mathbf{g} be a fibrewise Stone bundle over \mathcal{F} .

The functor sends a Stone locale Y' over $c\delta$ to a Stone bundle $\delta^*(Y')$ over $d\delta$ plus a contravariant *fibre map*, to wit a contravariant lax datum $\phi_{Y'}$,



obtained by pullback, along $\delta \times \delta$, of the canonical action, of the opposite of the specialization of \mathcal{E} on Y' taking the form of a 2-cell $(f \circ \pi_2)^*(-) \Rightarrow (f \circ \pi_1)^*(-)$.

The oplax descent (down δ) is *effective* when the functor is an equivalence of categories. ◇

III.142—PROPOSITION

The functorial definition of the oplax (effective) descent of fibrewise Stone bundles is sound ; it is indeed a functor between the given categories. ◇

PROOF

Let us take a locale arrow δ with domain $\text{Sh}(X)$ and codomain $\text{Sh}(X')$. We must recall that the Stone's duality is geometric [Joh82; Tow96b]; equally that a Stone locale Y' over $c\delta$ is dual to a boolean algebra B' over $c\delta$.

This latter sheaf B' can be turned by the pullback functor δ^* into a local homeomorphism over $d\delta$ which is also a boolean algebra. Indeed, the pullback functor preserves the local homeomorphisms and the lattice structure — we pull it back in act.

The boolean algebra B' over X' possesses a covariant fibre map $\Theta_{B'}$ (for \sqsubseteq') whereof the pullback along $\delta \times \delta$ is a covariant lax datum θ_B , for \leq , associated to B . This arrow $\Theta_{B'}$ is a covariant fibre map for the lax descent down the identity $\text{Id}_{X'}$, that is to say, for the lax descent down the generic point of X' . The geometric theory tells us that every other lax descent is the pullback of the generic lax descent along some (suitable) localic arrow. The arrow $\Theta_{B'}$ is compatible with every localic arrow \mathbf{h} in $\mathbf{Loc}/_{X'}$ whose domain \mathbf{g} and codomain \mathbf{k} are sheaves in the sense that, diagrammatically, the following square is commutative,

$$\begin{array}{ccc} \pi_2^*(d\mathbf{g}) & \xrightarrow{\pi_2^*(\mathbf{h})} & \pi_2^*(d\mathbf{k}) \\ \Theta_{\mathbf{g}} \uparrow & & \uparrow \Theta_{\mathbf{k}} \\ \pi_1^*(d\mathbf{g}) & \xrightarrow{\pi_1^*(\mathbf{h})} & \pi_1^*(d\mathbf{k}) \end{array}$$

Concretely we first prove that the arrow over $\sqsubseteq_{X'}$,

$$\pi_1^*(d\mathbf{g}) \xrightarrow{\Theta_{\mathbf{g}}} \pi_2^*(d\mathbf{g}) \xrightarrow{\pi_2^*(\mathbf{h})} \pi_2^*(d\mathbf{k})$$

is a map between the two pullbacks that we desire; indeed, in denoting Π 's the pullback projections of (the domain of) the bundles over X' ,

$$\begin{aligned} \mathbf{k} \circ \Pi_{2,\mathbf{k}} \circ \pi_2^*(\mathbf{h}) \circ \Theta_{\mathbf{g}} &= \mathbf{k} \circ \mathbf{h} \circ \Pi_{2,\mathbf{g}} \circ \Theta_{\mathbf{g}} = \mathbf{g} \circ \Pi_{2,\mathbf{g}} \circ \Theta_{\mathbf{g}} \\ &= \pi_2 \circ \pi_2^*(\mathbf{g}) \circ \Theta_{\mathbf{g}} = \pi_2 \circ \pi_1^*(\mathbf{g}) \\ &= \pi_2 \circ \pi_1^*(\mathbf{k} \circ \mathbf{h}) = \pi_2 \circ \pi_2^*(\mathbf{k}) \circ \Theta_{\mathbf{k}} \circ \pi_1^*(\mathbf{h}) \\ &= \mathbf{k} \circ \Pi_{2,\mathbf{k}} \circ \Theta_{\mathbf{k}} \circ \pi_1^*(\mathbf{h}) \end{aligned}$$

And,

$$\begin{aligned} \pi_2 \circ \pi_2^*(\mathbf{k}) \circ \pi_2^*(\mathbf{h}) \circ \Theta_{\mathbf{g}} &= \pi_2 \circ \pi_2^*(\mathbf{g}) \circ \Theta_{\mathbf{g}} = \mathbf{g} \circ \Pi_{2,\mathbf{g}} \circ \Theta_{\mathbf{g}} \\ &= \pi_2 \circ \pi_1^*(\mathbf{k} \circ \mathbf{h}) = \pi_2 \circ \pi_1^*(\mathbf{k}) \circ \pi_1^*(\mathbf{h}) \\ &= \pi_2 \circ \pi_2^*(\mathbf{k}) \circ \Theta_{\mathbf{k}} \circ \pi_1^*(\mathbf{h}) \end{aligned}$$

We conclude that $\pi_2^*(\mathbf{h}) \circ \Theta_{\mathbf{g}}$ and $\Theta_{\mathbf{k}} \circ \pi_1^*(\mathbf{h})$ are identical. When we take \mathbf{h} to be the boolean connectors — which are geometric concepts — we conclude that the pullback along δ of the fibre map over X' is again compatible with the boolean structure, pulled back along δ , on B .

The arrow $\theta_B \doteq \delta^*(\Theta_{B'})$ constitutes a lax datum because $\Theta_{B'}$ verifies the cocycle and counit condition for $\sqsubseteq_{X'} \simeq X'^{\mathbb{S}}$ [VerMoe97, 6.]. For instance, in the case of the counit, in denoting j_i the isomorphism $\pi_{ik}^*(\pi_1) \simeq \pi_{\ell i}^*(\pi_2)$ with i, k, ℓ between 0, 1, 2,

$$\begin{aligned} \pi_2 \circ \pi_2^*(g) \circ \Pi_2 \circ j_2 &= g \circ g^*(\pi_2) \circ \Pi_2 \circ j_2 = g \circ g^*(\pi_1) \circ \Pi_1 \circ j_1 \\ &= g \circ g^*(\pi_2) \circ \Theta_{B'} \circ \Pi_1 \circ j_1 = g \circ g^*(\pi_2) \circ \Pi_2 \circ j^*(\Theta_{B'}) \circ j_1 \\ &= \pi_2 \circ \pi_2^*(g) \circ \Pi_2 \circ j^*(\Theta_{B'}) \circ j_1 \end{aligned}$$

We conclude that the square of the counit condition commutes with the monicity of the arrow π_2 .

By the Stone's duality, a contravariant lax datum between the Stone spectra $\pi_2^*(Y)$ and $\pi_1^*(Y)$ is given. The conditions of the counit and the cocycle stems from the functoriality of the duality.

By geometricity of the Stone's duality, the locale Y must be the pullback of the locale Y' whose frame is the ideal completion of the boolean algebra B' over X' which is the result of the descent of B .

The functor is fully faithful by the effectual lax descent on the sheaves and the isomorphism between boolean morphisms and spectral maps. •

III.143 — PROPOSITION — EFFECTIVE LAX DESCENT OF STONE LOCALE, FIBRE MAP

If an arrow δ has effective lax descent for the sheaves, it has effective oplax descent for the Stone bundles. In effect the square of Stone bundles commutes,

$$\begin{array}{ccccc} \pi_2^*(Y) & \xrightarrow{\quad} & Y & \xrightarrow{\quad \epsilon \quad} & Y' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ & & p & & \\ \leq & \xrightarrow{(\pi_1 \leq) \pi_2} & X & \xrightarrow{\quad \delta \quad} & X' \end{array} \quad \diamond$$

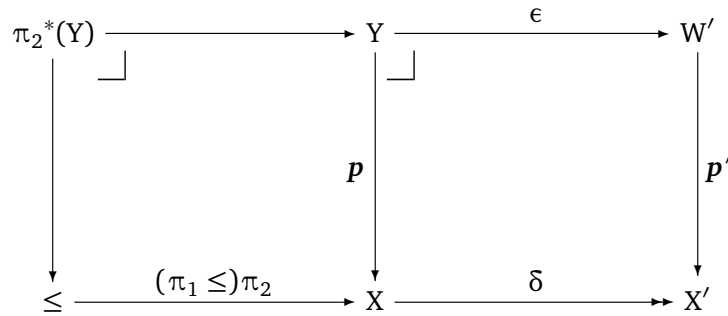
PROOF

We suppose a perfect surjection δ and use the effectuality of the descent for sheaves through the (geometric) Stone's duality. Categorically, we have a few equivalences; typically the category $\text{BoolAlg}(\text{ShLDes})$ of the boolean actions for δ and its lax kernel pair is equivalent to the category ActBoolAlg of the actions (g, θ) on the already boolean bundles g in Loc/X . By the duality, $\text{BoolAlg}(\text{Sh}(X'))$ is dual to the category $\text{StoneLoc}(X')$ of fibrewise Stone locales over X' — the Stone locales in $\text{Sh}(X')$ — and the category ActBoolAlg is dual to the category StoneOpLDes of Stone oplax descent data. By effective descent, the category $\text{Sh}(X')$ and ShLDes are equivalent. In consequence, we must conclude that StoneOpLDes is equivalent to the category $\text{StoneLoc}(X')$. •

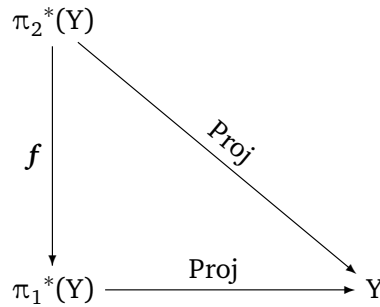
III.4.3 — PULLBACK OF LAX COEQUALIZER

III.144 — PROPOSITION — FIBRE MAP AND LAX COEQUALIZER

Let us suppose a compact regular locale X provided with a closed preorder \leq and its lax coequalizer δ . Let us also suppose a boolean algebra B supplied with a lax datum θ (for \leq). In this case, there exists a locale W' obtained from the Stone dual Y to B by a lax descent and W' is Stone as well,



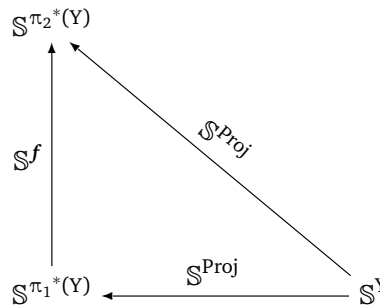
The localic arrow $\epsilon: Y \rightarrow W'$ is the lax coequalizer of the diagram,



where the *fibre map* f is the Stone dual arrow of the lax datum θ on the boolean algebra B . \diamond

PROOF

The locale Y is Stone and thereby compact regular and locally compact. It is thus exponentiable; and because the exponentiation is geometric, there exists a Stone locale W' — with bundle p' — whereof Y is the pullback by lax descent; moreover, W' is necessarily exponentiable for p' being fibrewise stone, p' is exponentiable over X' and secondly, X' is locally compact so exponentiable too. And we must demonstrate the existence of its exponential $\mathbb{S}^{W'}$ as the lax equalizer of the exponentiated arrows,



over the terminal locale $\mathbf{1}$; that is to say, as locales in **Loc** — not some locales in the toposes of sheaves over X nor X' , let us say.

We must demonstrate that the points of the lax equalizer \mathcal{E} ,

$$\mathcal{E} \doteq \text{IEq} \longleftarrow \mathbb{S}^Y \xrightarrow{\mathbb{S}^{\text{Proj} \circ f}, \mathbb{S}^{\text{Proj}}} \mathbb{S}^{\pi_2^*(Y)}$$

are isomorphic to the points of the exponential of W' ,

$$\mathcal{E} \simeq \mathbb{S}^{W'}$$

that is to say, to the opens of W' . From this purpose, it becomes clear that we must analyse what is the nature of the points of the locales of the kind $\mathbb{S}^{(-)}$. A global point e (in **Loc**) of \mathcal{E} is an arrow,

$$\mathbf{1} \xrightarrow{e} \mathbb{S}^Y$$

verifying in $\mathbb{S}^{\pi_2^*(Y)}$,

$$\mathbb{S}^{\text{Proj} \circ f}(e) \sqsubseteq \mathbb{S}^{\text{Proj}}(e)$$

This formula is a shorthand for the expression of the belonging, once we think of e as only an open $U \longleftarrow Y$ of Y , of the points of the open $\mathbb{S}^{\text{Proj} \circ f}(e)$ (of $\mathbb{S}^{\pi_2^*(Y)}$) to the open $\mathbb{S}^{\text{Proj}}(e)$. Explicitly, a point $(\mathbf{x}_1 \leq \mathbf{x}_2, \mathbf{y}_2)$ of $\pi_2^*(U)$ — which verifies the condition that $f(\mathbf{y}_2)$ is in $Y_{\mathbf{x}_1}$ — must belong, from the onset, to the open $\mathbb{S}^{\text{Proj}}(e)$; that is to say, that it must verify \mathbf{y}_2 in $Y_{\mathbf{x}_2}$.

In order to conclude, we must introduce a few locales from the point of view of X and X' — thus far, the aforementioned locales are in **Loc**. We must change the viewpoint in turning the point e into the equivalent arrow [SpiVicWol13],

$$\begin{aligned} E: X &\longrightarrow \mathbb{S}_X^Y \\ \mathbf{x} &\longmapsto (\mathbf{x}, J_{\mathbf{x}} \in \text{Idl}(B_{\mathbf{x}})) \end{aligned}$$

where \mathbb{S}_X^Y is the externalisation (over X) of the internal exponential \mathbb{S}^Y of the internal locale Y in $\text{Sh}(X)$ by Sierpinski. The geometricity compels us to see the points of \mathbb{S}_X^Y as the choice of a point \mathbf{x} of X together with an open in the frame $\Omega(Y_{\mathbf{x}})$ of the Stone fibre $Y_{\mathbf{x}}$ (of the bundle $Y \longrightarrow X$) over \mathbf{x} . Such an open is, by the duality from Stone, an ideal $J_{\mathbf{x}}$ of a boolean algebra $B_{\mathbf{x}}$. We must in consequence turn a cross section E of $\mathbb{S}_X^Y \longrightarrow X$ into a cross section of $\mathbb{S}_{X'}^{W'} \longrightarrow X'$ belonging to \mathcal{E} .

The fibre map becomes under this perspective,

$$\begin{aligned} \mathbb{S}^{\text{Proj} \circ f}: \mathbb{S}^Y &\longrightarrow \mathbb{S}^{\pi_2^*(Y)} \\ E: X \longrightarrow \mathbb{S}_X^Y &\longmapsto \mathcal{V}(f): \leq \longrightarrow \mathbb{S}^{\pi_2^*(Y)}_{\leq} \\ \mathbf{x} \longmapsto (\mathbf{x}, J \in \text{Idl}(B_{\mathbf{x}})) &\longmapsto (\mathbf{x}_1, \mathbf{x}_2) \longmapsto (\mathbf{x}_1 \leq \mathbf{x}_2, (\text{Proj} \circ f)^*(J_{\mathbf{x}_1}) \in \text{Idl}(\pi_2^*(B))) \end{aligned}$$

with $(f \circ \text{Proj})^*(J_{\mathbf{x}_1})$ the open of the pullback, along $f \circ \text{Proj}$, of the open pertaining to the pair $E(\mathbf{x}_1)$. And the condition on E to belong to the lax equalizer \mathcal{E} becomes the claim that when $(\mathbf{x}_1, \mathbf{x}_2)$ is in \leq ,

$$(\text{Proj} \circ f)^*(J_{\mathbf{x}_1}) \sqsubseteq J_{\mathbf{x}_2}$$

The remaining task is to compose each point E of \mathcal{E} with the function,

$$\begin{aligned} \phi: \mathbb{S}_X^Y &\longrightarrow \mathbb{S}_{X'}^{W'} \\ (\mathbf{x}, J \in \text{Idl}(B_{\mathbf{x}})) &\longmapsto (\delta(\mathbf{x}), J \in \text{Idl}(B_{\mathbf{x}})) \end{aligned}$$

since the fibres of the pullback behave geometrically as,

$$W'_{\delta(x)} \simeq \delta^*(W')_x \doteq Y_x$$

idem for the boolean algebras in place,

$$B'_{\delta(x)} \simeq \delta^*(B')_x \doteq B_x$$

The composite $\phi \circ E$ is used in order to decompose it into two arrows involving the stably compact locale X' , one whereof is a point of $\mathbb{S}^{W'}$,

$$\begin{array}{ccc} X & \xrightarrow{\delta} & X' \\ & \searrow \phi \circ E & \vdots E' \\ & & \mathbb{S}^{W'}_{X'} \end{array}$$

by the universal property of a lax coequalizer. For this, we must prove that the arrow $\phi \circ E$ turns the specialization on X into the specialization of its codomain, $\mathbb{S}^{W'}_{X'}$. Let us analyse what this specialization is and in passing let us recall that this bundle is an exponentiation of our (fibrewise) Stone bundle $W' \longrightarrow X'$ with its contravariant fibre map θ' , Stone dual to the boolean morphism Θ' on B' by the premiss. The article [SpiVicWol13] informs us on how to obtain the action on our exponential; it is essentially given by Θ' ,

$$\begin{aligned} \mathbb{S}^{\theta'} : \pi'_{\perp}{}^*(\mathbb{S}^{W'}_{X'}) &\longrightarrow \pi'_{\top}{}^*(\mathbb{S}^{W'}_{X'}) \simeq \mathbb{S}^{\pi'_{\top}{}^*(W')}_{\square'} \simeq \pi'_{\top}{}^*(\text{Idl}(B')) \\ (\mathbf{x}'_{\perp} \sqsubseteq' \mathbf{x}'_{\top}, J'_{\perp} \in \text{Idl}(B'_{\perp})) &\longmapsto (\mathbf{x}'_{\perp} \sqsubseteq' \mathbf{x}'_{\top}, \downarrow \Theta'(J'_{\perp}) \sqsubseteq J'_{\top} \in \text{Idl}(B'_{\top})) \end{aligned}$$

However, we manipulate ideals; consequently the lower set $\downarrow \Theta'(J'_{\perp})$ is already $\Theta'(J'_{\perp})$ and the condition to belong to \mathcal{E} (with the fibre map f) for a point E becomes, over X' and after the duality, the inverse image whereof f is the pullback, to wit, Θ' . Conclusion : we derive a localic arrow $X' \longrightarrow \mathbb{S}^{W'}_{X'}$ as a global point $\mathbf{1} \longrightarrow \mathbb{S}^{W'}$ of $\mathbb{S}^{W'}$. The derivation gives an arrow $\Phi : \mathcal{E} \longrightarrow \mathbb{S}^{W'}$.

The reverse direction follows from the universal property of the lax equalizer \mathcal{E} . All we need to demonstrate is that the arrow \mathbb{S}^{ϵ} factorizes throughout it. This is true thanks to the functoriality of the exponentiation $\mathbb{S}^{(-)}$ for, when we take a point of $\mathbb{S}^{W'}$ such a map,

$$\begin{aligned} E' : X' &\longrightarrow \mathbb{S}^{W'}_{X'} \\ \mathbf{x}' &\longmapsto (\mathbf{x}', J'_{\mathbf{x}'} \in \text{Idl}(B'_{\mathbf{x}'})) \end{aligned}$$

the application of \mathbb{S}^{ϵ} to it is justly the inverse image ϵ^* to the ideals over X' . Concretely, we must prove that,

$$\forall E' \in \mathbb{S}^{W'}, \mathbb{S}^{\epsilon \circ \text{Proj} \circ f}(E') \sqsubseteq_{\mathbb{S}^{\pi_2^*(Y)}} \mathbb{S}^{\epsilon \circ \text{Proj}}(E')$$

An arrow such as $\mathbb{S}^{\epsilon \circ \text{Proj} \circ f}$ takes a point E' to output a point E of $\mathbb{S}^{\pi_2^*(Y)}$,

$$\begin{aligned} E : \leq &\longrightarrow \pi_2^*(\mathbb{S}^Y) \\ (\mathbf{x}_1 \leq \mathbf{x}_2) &\longmapsto (\mathbf{x}_1 \leq \mathbf{x}_2, (\epsilon \circ \text{Proj} \circ f)^*(J'_{E'(\delta(\mathbf{x}_1))}) \in \text{Idl}((\delta \circ \pi_2)^*(B'))) \end{aligned}$$

We must notice that the doublet $(x_1 \leq x_2)$ expresses that,

$$\delta(x_1) \sqsubseteq_{X'} \delta(x_2)$$

and incidentally, we have the certitude that the aforementioned map E is well defined — the codomain is indeed $\pi_2^*(\mathbb{S}^Y)$ — and that,

$$\Theta'(J'_{E'(\delta(x_1))}) \sqsubseteq J'_{E'(\delta(x_2))} \in \text{Idl}(\pi_2^*(B'))$$

which gives, once pulled back over X and \leq , the condition of the belonging to \mathcal{E} . By the universal property of the lax equalizer \mathcal{E} , we obtain an arrow $\psi: \mathbb{S}^{W'} \longrightarrow \mathcal{E}$.

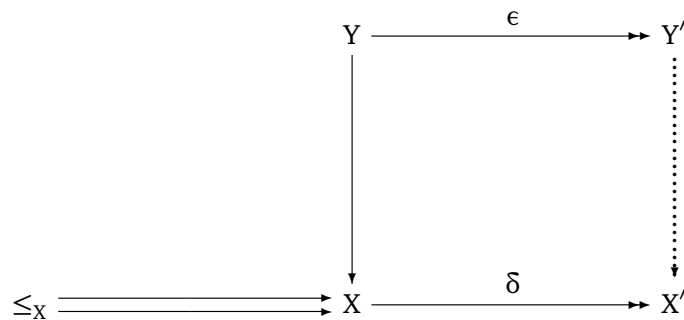
The two maps ψ and Φ are inverse of each other for, on the first hand, the preservation of the specialization of \mathbb{S}^Y by every localic arrow; and on the other hand, for $\phi \circ \mathbb{S}^e$ preserves the points E' . •

III.145 — NOTE

The proof uses the Sierpinski exponentials for these are geometric, even though conceptually isomorphic to the frames of opens; however by now, we do know that these later are not geometric. ◊

III.4.3.1 — FORTHCOMING APPLICATION

The study of the closed preorders from the Hausdorff systems will be used in IV4 at page 103 in order to turn a fibrewise Stone bundle $Y \longrightarrow X$ into a square,



and more precisely, a perfect map $Y' \longrightarrow X'$ between stably compact locales.

The effectuality of the lax descent for the Stone bundles is employed instantaneously thereafter in order to prove that $Y' \longrightarrow X'$ remains fibrewise Stone and that the square is a pullback.

IV — SPECTRAL BUNDLE

IV.01 — OVERVIEW

After having introduced the traditional algebraic definitions, we construct our geometric version of the base space X comprising the precontexts as sequences of projectors, complete and mutually orthogonal. The locale X has at least two bundles over it; first the bundle $Y \longrightarrow X$ for the pure states and the bundle $\mathcal{V}_X(Y) \longrightarrow X$ for the impure ones — in fact, the context locale X is itself fibred over the locale T of the types. We develop an action of a preorder on these three compact-regular locales which becomingly identifies, once quotiented, the projector sequences leading to identical contexts — the action naturally lifts to the level of the states Y . Since we want a topology from the real numbers, and not the one of Alexandrov on the poset, we begin with the localic Hilbert space \mathbb{K}^n with $\mathbb{K} = \mathbb{C}$ and construct all we can from it; in particular, we refer to II.3.1 at page 28 for their construction. Finally, the books [Weg93; Bla06] present the theory of operators.

The little proofs about the projectors (and the effects) are a consequence (quite immediate) from the Gelfand's duality adapted geometrically. Notwithstanding, we wish to delay its insertion in the exposition in order to suggest explicitly that the proofs can be totally carried out in a geometric manner. \diamond

IV.1 — PROJECTOR OF C^* -ALGEBRA

IV.02 — CONSPECTUS — EFFECTS

The effects are those positive selfadjoints (of a C^* -algebra) less than the identity — equivalently, they are all the squares of selfadjoints and the squares are less than the identity; the projectors in a C^* -algebra are the effects equating their squares. The effects are to the positive-operator-valued measures what the projectors are to the projection-valued measures, [Tro03; dMuy06]. More physically, the effects are handy to model a noisy measurement or a measurement where the outcome is not perfectly determinable.

Instead of beginning directly with the projectors, we mention the effects only to present them locally as they seem promising from their version of the Gleason's theorem — as it is extended to dispersion-free valuations on the effects on the Hilbert spaces of dimension strictly greater than one (as opposed to two for the projectors) by an argument about the continuity and the linearity of the valuations. In short, there does not exist a set function $\nu: \text{Eff}(n) \longrightarrow \{0, 1\}$ satisfying the additivity condition [BusSin98; Bre03; Bus03; Cav+04]. We can see this as a Bell–Kochen–Specker theorem proper to the effects. However, the question to assign some classical valuations to a set of effects seems to remain open; that is why our construction focuses on (the spectrum from) the projectors. All the subsequent treatment concerning the effects can be restricted directly to the projectors. \diamond

IV03 — DEFINITION — NORMAL & SELFADJOINT & POSITIVE & UNITARY OPERATOR, PROJECTOR, EFFECTS

In a (incommutative) C*-algebra, an operator \mathbf{a} ,

1° is *normal*^① when it commutes with its adjoint,

$$\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a}$$

2° is *selfadjoint* when it equals its adjoint,

$$\mathbf{a} = \mathbf{a}^\dagger$$

3° is *positive* when there exists an operator \mathbf{b} selfadjoint such that,

$$\mathbf{a} = \mathbf{b}^\dagger\mathbf{b}$$

4° is *unitary* when the adjoint is the inverse,

$$\mathbf{a}^{-1} = \mathbf{a}^\dagger$$

5° is a *projection* when it equals its square and its adjoint,

$$\mathbf{a}^2 = \mathbf{a} = \mathbf{a}^\dagger$$

6° is an *effect* when it is positive as well as its subtraction from the identity,

$$0 \leq \mathbf{a} \text{ and } 0 \leq \text{Id} - \mathbf{a} \quad \diamond$$

IV04 — PROPOSITION — SPECTRUM OF AN OPERATOR

The spectrum of a selfadjoint element is part of the real line ; of a positive one is part of the positive real line ; of a unitary is part of the complex numbers with modulus one ; of an effect is part of the real unit interval ; of a projector is part of the set $\{0, 1\}$.

When the operator is normal, these logical implications are logical equivalences. ◇

IV05 — PROPOSITION — CRITERION OF POSITIVITY

An operator \mathbf{a} in a C*-algebra is positive,

1° classically, if and only if there exists an operator \mathbf{b} such that,

$$\mathbf{a} = \mathbf{b}^\dagger\mathbf{b}$$

2° classically and constructively, if and only if it is selfadjoint and the elements $\langle \mathbf{v}, \mathbf{A}\mathbf{v} \rangle$ for every vector \mathbf{v} of a Hilbert space — whereon the operator \mathbf{a} acts as a matrix \mathbf{A} — are not negative ◇

PROOF

1° indeed, classically, we have access to the unique square root (abstract, in the C*-algebra) of every positive element [Bla06] ; so when we know that the operator \mathbf{a} has some \mathbf{b} such that it equals $\mathbf{b}^\dagger\mathbf{b}$, we know that the square root of $\mathbf{b}^\dagger\mathbf{b}$ does exist and is moreover selfadjoint such that its square is \mathbf{a} anew ; the converse is immediate by definition

^① The normality conveys the assurance and necessity to be unitarily diagonalizable. All our operators are normal.

2° the proof uses the spectral theorem to decompose (the matrix representation A of the abstract operator) \mathbf{a} in its diagonal form of real eigenvalues classically [Lan87] and constructively [Spi03] and when in addition these are positive or nil, we can take their square roots and \mathbf{a} is indeed a square of a selfadjoint and so is positive in the C*-algebra; for the converse, we also work with its matrix representative A of $\mathbf{a} = \mathbf{b}^\dagger \mathbf{b}$ and we note that the positivity in the C*-algebra implies that,

$$\forall \mathbf{v} \in \mathcal{H}, 0 \leq \langle \mathbf{v}, A\mathbf{v} \rangle = \|B\mathbf{v}\|^2$$

where B is the representative of the selfadjoint \mathbf{b} •

IV.06—PROPOSITION — CRITERION FOR EFFECTS

It emanates that an operator \mathbf{e} on a finite Hilbert space is constructively an effect if and only if it is hermitian and bigger than its square (in the order of positivity). ◊

PROOF

We know that the positivity of an operator takes the form of its hermicity and the positivity of all the elements of the kind $\langle \mathbf{v}, \mathbf{e}\mathbf{v} \rangle$ with \mathbf{v} a vector. For a selfadjoint \mathbf{e} bigger than its square \mathbf{e}^2 , let us note that by hermicity, all the components on the diagonal are real; we must conclude that,

$$\forall i, j \leq n, i = j \Rightarrow (e^2)_{ij} \doteq \sum_{k \leq n} e_{ik} e_{kj} = \sum_{k \leq n} e_{ik} e_{jk}^\dagger \leq e_{ij}$$

and,

$$\forall j, k \leq n, 0 \leq \text{abs}(e_{jk}^2) = \text{abs}(e_{jk})^2 \leq \sum_{k \leq n} \text{abs}(e_{jk})^2 \leq e_{jj}$$

From the analysis in II.85 at page 43, the (real) diagonal elements verifies,

$$\forall j \leq n, 0 \leq e_{jj} \leq 1$$

This result implies that, for the elements outside of the diagonal,

$$\forall k, j \leq n, \text{abs}(e_{kj}) = \text{abs}(e_{jk}) \leq \sqrt{e_{jj}} \leq 1$$

as real numbers. An effect sees its components reals and bounded in every basis. Since we must verify the positivity for every vector \mathbf{v} , not solely those of the initial basis chosen arbitrarily, we must choose the basis of its constructive spectral decomposition [Spi03] in order to conclude.

For the converse, we choose anew the constructive spectral decomposition of a selfadjoint operator and notice that the spectral decomposition of the square of a positive operator is only the one given by the squares of the eigenvalues. Also, if a real number is less than one, it is bigger than its square. •

IV.07—DEFINITION — LOCALE OF SELFADJOINTS

The locale $\mathbb{K}_{\text{sa}}^{n^2}$ of all the selfadjoints on a Hilbert space \mathbb{K}^n is the equalizer corresponding to the geometric type theory,

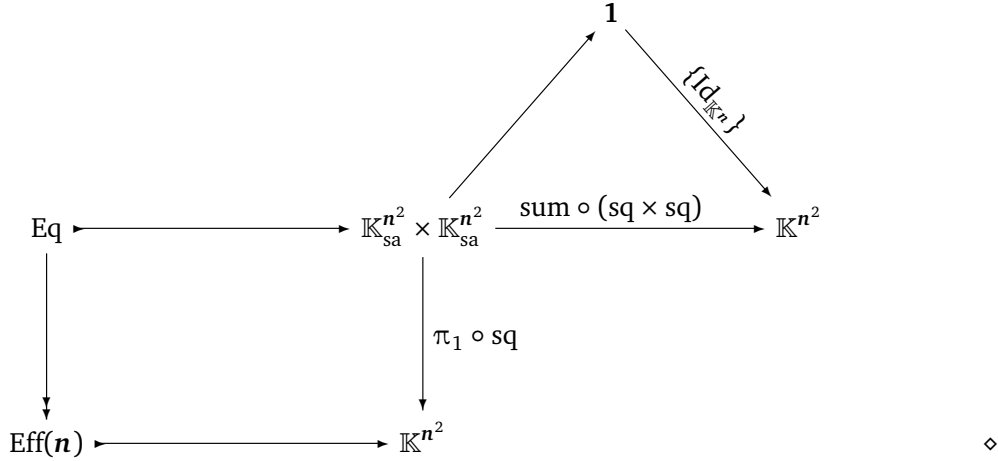
$$1^\circ \forall \mathbf{a} \in \mathbb{K}^{n^2}, \top \vdash \mathbf{a}^\dagger = \mathbf{a} \quad \diamond$$

IV.08 — DEFINITION — LOCALE OF EFFECTS

Naturally, we desire a locale $\text{Eff}(\mathbf{n})$ whose points are all the effects on the C*-algebra \mathbb{K}^n ; it is given by the geometric type theory,

$$1^\circ \forall e \in \mathbb{K}_{\text{sa}}^{n^2}, \top \vdash \exists a \in \mathbb{K}_{\text{sa}}^{n^2}, e^2 + a^2 = \text{Id}_{\mathbb{K}^n}$$

diagrammatically constructed as the image $\text{Eff}(\mathbf{n})$,



IV.09 — PROPOSITION

The locale $\text{Eff}(\mathbf{n})$ of all the effects of a C*-algebra is compact, closed and regular; thereby it is necessarily normal.

PROOF

To prove that $\text{Eff}(\mathbf{n})$ is compact as a sublocale of \mathbb{K}^{n^2} , we wish to define a localic arrow,

$$\text{Eff}(\mathbf{n}) \dashrightarrow \mathcal{U}(r)^{n^2}$$

into the complex disk of radius r ; it is necessary and sufficient to consider n^2 localic arrows,

$$\begin{aligned} \text{abs} \circ \pi_{ij} : \text{Eff}(\mathbf{n}) &\longrightarrow \mathbb{R} \subseteq \mathbb{K} \\ e &\longmapsto \text{abs}(e_{ij}) \end{aligned}$$

and by boundedness (constructively) (by the real number 1) of each component of an effect, this proves that the arrow $\text{abs} \circ \pi_{ij}$ factorizes through the sublocale $[-r, r]$ with r equal to 1 for all indices i, j . Compactness and closedness follow by application of constructive Heine–Borel [FouGra82; Vic97]. Furthermore, the operator a sees its components bounded,

$$\forall j, k \leq n, \text{abs}(a_{jk}) \leq e_{jj} \leq 1$$

We conclude that the equalizer is a sublocale compact, regular and proper. Its image is regular and compact and the surjection $\text{Eq} \dashrightarrow \text{Eff}(\mathbf{n})$ is proper. •

IV.10 — DEFINITION — LOCALE OF PROJECTORS

The locale Proj of the projectors on the Hilbert space \mathbb{K}^n is the geometric type theory,

$$1^\circ \forall p \in \text{Eff}(\mathbf{n}), \top \vdash p^2 = p$$

◊

IV.11 — PROPOSITION

The locale remains closed compact and regular. \diamond

IV.12 — PROPOSITION — TRACE OF A PROJECTOR

The trace of a projector is a natural number. \diamond

PROOF

Given a projector, we know that it is an effect and necessarily has constructively its diagonal elements as real numbers between zero and one. However, being a projector, the diagonal elements equals their squares; in consequence, they are the real number zero or one from II.85 at page 43. Naturally, we can see them as natural numbers. \bullet

IV.13 — DEFINITION — COMMENSURABLE OBSERVABLES

Two *commensurable observables* are two selfadjoints jointly diagonalizable. \diamond

IV.14 — PROPOSITION

Two selfadjoints do commute if and only if their spectral decompositions commute equally. \diamond

PROOF

Indeed, if the selfadjoints $a = \sum_i a_i q_i$ and $b = \sum_k b_k p_k$ commute, with the numbers a_j and b_i being real,

$$\begin{aligned} ab = ba &\Rightarrow \sum_{i \leq \#\vec{q}} a_i q_i b = b \sum_{i \leq \#\vec{q}} q_i a_i \\ &\Rightarrow \forall i, j \leq \#\vec{q}, q_i b q_j (a_i - a_j) = 0 \end{aligned}$$

Now let us split the case on the indices,

1° when i is not j ,

$$q_i b q_j = 0$$

2° when i is j , we notice that,

$$q_i b q_i = \left(\text{Id} - \sum_{h \neq i} q_h \right) b q_i = b q_i = q_i b$$

after noticing that if more than one projector are associated to the nil eigenvalue, we can add them to make only one projector for this one

By symmetry, the operator a commutes with all the p_j . By the same reasoning, we replace the operator a with a p_j and conclude the first implication. The reverse one is immediate. \bullet

IV.15 — PROPOSITION

When $\vec{p}\vec{q}$ is a common spectral decomposition of two commuting selfadjoints (with spectral decomposition \vec{p} and \vec{q}), there exist at most n non-nil projectors of the form $p_j q_i$. \diamond

PROOF

Indeed, we know that we have the spectral resolution,

$$\text{Id} = \sum_{\substack{j \leq \# \vec{p} \\ i \leq \# \vec{q}}} p_j q_i$$

and by the linearity of the trace,

$$n = \sum_{\substack{j \leq \# \vec{p} \\ i \leq \# \vec{q}}} \text{Tr}(p_j q_i)$$

we manifestly possess a partition of n into numbers less than it. When the natural number k of non-nil projectors is strictly bigger than n , then n will be strictly bigger than itself and we have an absurdity. •

IV.16 — PROPOSITION — CRITERION OF MUTUAL ORTHOGONALITY

A sum of projectors equal to the identity is equivalent to the mutual orthogonality of the projectors. ◊

PROOF

For every pair of projectors p_i, p_j , their triproduct is positive,

$$p_i p_j p_i = (p_j p_i)^\dagger p_j p_i \Rightarrow 0 \leq p_i p_j p_i$$

but we also know that when a collection of projectors sums to one,

$$\begin{aligned} \text{Id} = \sum_k p_k &\Rightarrow \forall i \leq \ell, p_i = p_i + \sum_{k \neq i} p_k p_i \\ &\Rightarrow \forall i, j \leq \ell, \sum_{k \neq i, j} p_i p_k p_i + p_i p_j p_i = 0 \end{aligned}$$

We are bound to see $-p_i p_j p_i$ as positive for a sum of positive operators remains so, even constructively via the duality from Gelfand.^① Since its opposite is also positive, it must be nil constructively. Now, because the derivation is symmetric in the indices i, j ,

$$p_i p_j p_i = p_j p_i p_j$$

whence the conclusion follows after we notice that the formula is equivalent to,

$$p_i p_j = p_j p_i$$

for,

$$(p_i p_j - p_j p_i)^\dagger (p_i p_j - p_j p_i) = -(p_i p_j - p_j p_i)^2 \quad \bullet$$

IV.17 — NOTE

This theorem avoids the explicit reference to the finiteness of the framework when we are led to implement the orthogonality through a plethora of equalizers. It is also quicker.

The result holds when a sum of effects is less than the identity. ◊

IV.18 — PROPOSITION

There cannot be twice or more the same non-nil projector in every complete sequence of orthogonal projectors. ◊

^① It is immediate in $\mathcal{B}(\mathcal{H})$.

IV.19 — DEFINITION — SEQUENCE OF PROJECTORS MUTUALLY ORTHOGONAL

We define the sublocale $\Sigma\text{Proj}(\mathbf{n}, \ell)$ ^① constituted of the sequences of length ℓ of projectors summing to the identity. Its geometric type theory is,

$$1^\circ \forall \vec{p} \in \text{Proj}(\mathbf{n})^\ell, \top \vdash \sum_{i \leq \ell} p_i = \text{Id}_{\mathbb{K}^n} \quad \diamond$$

IV.20 — NOTE

From the analysis of the commensurable observables, we avoid the explicit creation of the locale having all the biproducts of projector sequences; they are all in X already. \diamond

IV.21 — PROPOSITION

The locale $\Sigma\text{Proj}(\mathbf{n}, \ell)$ remains closed, compact and regular. \diamond

IV.2 — PRECONTEXT AND PRESTATE

IV.22 — OVERVIEW

We search for a construction $Y \longrightarrow X$ in order that Y be the state locale and that X be the locale of contexts. The enterprise is to create X as the locale of the (pre)contexts, to wit a sequence of projectors plus its trace sequence; and Y as the locale whereof the fibres over a precontext is the Gelfand's (geometric) spectrum of the commutative C^* -algebra generated by the said precontext — formally the contexts will be gotten after having analysed the permutations of the sequences; informally, we do not distinguish between the two presently. In consequence, a point of Y ought to be a context together with a choice of a state in its given spectrum. In short, here, we put the emphasis on the objects of study; the morphisms will follow. \diamond

IV.23 — DEFINITION — TYPE

For a non-nil natural number ℓ , a « **type \vec{t} of length ℓ** » is a sequence of ℓ natural numbers whose sum has value \mathbf{n} , the dimension of the Hilbert space.

A type is thus a point of the locale T_ℓ compact, closed and regular from the geometric type theory,

$$1^\circ \forall \vec{t} \in \llbracket 1, \mathbf{n} \rrbracket^\ell, \top \vdash \sum_{i \leq \ell} t_i = \mathbf{n}$$

IV.24 — PROPOSITION

The collection T of types is a discrete locale. \diamond

IV.25 — DEFINITION — PRECONTEXT

We pair the types of length ℓ and the sequences of projectors into a locale X_ℓ of « **precontexts of length ℓ** »; for this purpose, we take the pullbacks,

$$\begin{array}{ccccc} X_{\vec{t}} & \xrightarrow{\quad} & X_\ell & \xrightarrow{\quad} & \Sigma\text{Proj}(\mathbf{n}, \ell) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \overrightarrow{\text{Tr}(-)} \\ \mathbf{1} & \xrightarrow{\vec{t}} & T_\ell & \xrightarrow{\quad} & \llbracket 1, \mathbf{n} \rrbracket^\ell \end{array}$$

and manufactures equally a locale $X_{\vec{t}}$ of all the projectors having for type the type \vec{t} . \diamond

^① Σ reminds us that the components of a sequent must sum to the identity.

IV.26 — PROPOSITION

From the definition of a precontext, we conclude firstly that,

$$\coprod_{\substack{\vec{\tau} \\ \#\vec{\tau}=\ell}} X_{\vec{\tau}} \simeq X_{\ell}$$

and secondly, that X_{ℓ} is compact regular, as well as any of the $X_{\vec{\tau}}$, just as the locale $X = \coprod_{\ell} X_{\ell}$ which is the locale of all the precontexts. They form a bundle $X \longrightarrow T$. \diamond

PROOF

The locale X remains compact by finiteness of the coproducts. \diamond

IV.27 — DEFINITION — GEOMETRIC GELFAND'S DUALITY^①

A sequence \vec{p} of length ℓ of projectors generates canonically a commutative C^* -algebra $C \doteq \bigoplus_{j \leq \ell} \mathbb{C}p_j$ whose *spectrum* Spec_{ℓ} is a locale compact, completely regular and whose,

1° points are the non-nil unital multiplicative linear morphisms $\phi : C \longrightarrow \mathbb{C}$

2° (sub)basic opens are,

$$D(\mathbf{a}) \doteq \{\phi \in \text{Spec}_{\ell} \mid 0 < \hat{a}(\phi) \doteq \phi(\mathbf{a})\}$$

(the usual (sub)basic opens) indexed by the formal symbols \mathbf{a} (interpreted as the selfadjoints) of type C_{sa} as an object of **Set** which generate the weak star topology altogether \diamond

IV.28 — PROPOSITION — STONE SPECTRUM

The localic spectrum of a finite commutative C^* -algebra $C \doteq \bigoplus_{j \leq \ell} \mathbb{C}p_j$ is the discrete locale $(\ell \simeq) \llbracket 1, \ell \rrbracket$. Besides, its (sub)basic opens are of the form $D(\mathbf{a}) = \{i \leq \ell \mid 0 < \mathbf{a}_i\}$ with \mathbf{a} a selfadjoint in $C \simeq \mathbb{C}^{\ell}$; in short, a basic open is the (collection of the) positive parts of a selfadjoint.

In consequence, in finite dimension, the localic spectrum of a finite commutative C^* -algebra is discrete, finite and more importantly decidable in the sense that we can decide of the equality of each couple of members. This entails equally that the spectrum is compact and regular; also of kind of Stone. \diamond

PROOF

For every index $j \leq \ell$, we can assign the j -th projection from the vector space C to \mathbb{C} .

For every ϕ in the spectrum, by the multiplicative property, each $\phi(e_j)$ is zero or one — where we use the typical basis of C whereof a member e_j consists in the tuple of length ℓ with zeros everywhere but a one in the j -th position; explicitly, e_j is the projector p_j . From its properties to be unitary and positive, we can conclude that,

$$\exists j \leq \ell, \phi(e_j) = 1$$

And this existence is unique; for when e_i and e_j are subject to the property^②,

$$\phi(e_i e_j) = 1 = \delta(i, j) \phi(e_i) = \delta(i, j) 1 \iff i = j$$

^① [Coq05b; CoqSpi05; CoqSpi08; CoqSpi09a; CoqSpi10].

^② Let us recall that we have the skill to judge of the equality of two natural numbers.

The regularity of the spectrum and its quality of being Stone follows from the decidability. Indeed, the frame of a discrete locale is always its powerset. When the finiteness is taken into account, constructively, the frame remains dissimilar to the finite powerset, even though it is expressible as its ideal completion. At present, the decidability entails that the meet and the complement of the finite subsets are possible and becoming in defining the boolean structure on the finite powerset. By the geometric Stone's duality, we understand that the original locale must be Stone since its spectrum is a boolean algebra. •

IV.29 — NOTE — SPECTRUM IN A TOPOS^①

Since two sequences of projectors of identical length generate two isomorphic C^* -algebras, their spectra are isomorphic as well. When we consider the transformations of the sequences, we shall be attentive to transform accordingly the states by an isomorphism, in general differing from the identity.

The description works regardless of the concept of dimension of the C^* -algebra but this latter must be over the complexified rational numbers. On the side of the frames, the construction amounts to taking the positive cone, to noticing that there exists a structure of a lattice and, moreover, to remarking that it carries over its quotient by relation of equivalence. The final product is a distributive lattice enjoying the property of normality whereof it generates the frame of a compact regular locale, the spectrum.

To obtain the spectrum in a topos \mathcal{E} , it is sufficient to pullback the theory or its frame presentation via the unique morphism $!_{\mathcal{E}}: \mathcal{E} \rightarrow \mathbf{Set}$ since \mathbf{Set} is the terminal object of the category of toposes. Concretely, there must be a type \mathcal{Q} as the object of the rationals in \mathcal{E} — but assured automatically by geometricity; the type for the selfadjoints become $!_{\mathcal{E}}^*(\mathcal{A}_{sa}) = \coprod_{a \in \mathcal{A}_{sa}} \mathbf{1}_{\mathcal{E}}$, that is to say, the coproduct of the terminal object of \mathcal{E} , as many times as there are elements of \mathcal{A}_{sa} . Becomingly, the theory in the form of its axioms remains exactly the same. The result is necessarily a normal distributive lattice internal to \mathcal{E} generating a compact regular locale, still internal; by externalization, we end up with a compact locale over \mathcal{E} ; it is this latter that we use. ◇

IV.30 — DEFINITION — SPECTRAL BUNDLE, PURE QUANTUM STATE

We pair a sequence of projectors in the locale $X_{\vec{t}}$ with a state in the compact regular locale $\llbracket 1, \# \vec{t} \rrbracket$ via the theory,

$$1^\circ \forall (\vec{t}, \vec{p}) \in X_{\vec{t}}, \forall \ell \leq n, \forall \phi \leq \ell, \top \vdash (\vec{t}, \vec{p}, \phi) \wedge (\# \vec{t} = \ell)$$

to form the locale $Y \doteq \coprod_{\vec{t}} Y_{\vec{t}}$ of the *pure quantum states*. ◇

IV.31 — PROPOSITION

The state locale Y is fibrewise compact regular and even fibrewise Stone. It constitutes itself into a (surjective^②) bundle $f: Y \rightarrow X$ which possesses the additional property to be a local homeomorphism, that is to say, a sheaf over X . ◇

^① [Cas+09; SpiVicWol13].

^② Surjective because every context has its fibre inhabited.

PROOF

Our indices belong to a finite set and a finite (co)product of compact regular locales remains so. Equally regarding the quality of being of the kind Stone, for this duality is geometric [Tay00; CirSam08].

The bundle f is a sheaf because $\llbracket 1, \# \vec{t} \rrbracket$ is discrete which means that the arrow $\llbracket 1, \# \vec{t} \rrbracket \longrightarrow \mathbf{1}$ is open as well as the diagonal [JoyTie84]. The pullback of a open localic arrow is anew open which leads to, once bundled up, an open bundle. Idem for the diagonal of the bundle ; in conclusion the $Y \longrightarrow X$ is fibrewise discrete. •

IV.32 — NOTE

The frame of the spectrum $\llbracket 1, \# \vec{t} \rrbracket$ is its powerset. Nevertheless, the negative information is available, in the sense that we can take, as its pleases us, the complements of finite subsets because the equality of elements is decidable. ◊

IV.33 — DEFINITION — IMPURE STATES

The geometric construction of the localic valuations II.58 at page 27 is applied on the (compact regular) locale $Y \longrightarrow X$ materialized in the form of the valuation monad \mathcal{V}_X (on $\mathbf{Loc}/_X$) applied to each fibre of the quantum state space Y to form the locale $\mathcal{V}_X(Y)$ whose points are the « *impure states* on Y » to wit,

$$\mathcal{V}_X(Y) \simeq \coprod_{\vec{t}} \mathcal{V}_{X_{\vec{t}}}(Y_{\vec{t}}) \simeq \coprod_{\vec{t}} X_{\vec{t}} \times \mathcal{V}(\llbracket 1, \# \vec{t} \rrbracket) \quad \diamond$$

IV.34 — PROPOSITION

The locale of the impure states forms a compact regular bundle $\mathcal{V}_X(Y) \longrightarrow X \longrightarrow T$. The unit of the monad is the Dirac's measure. In our finite case,

$$\mathcal{V}(\llbracket 1, \# \vec{t} \rrbracket) \simeq \Delta_{\# \vec{t}}$$

the collections of the (probabilistic) valuations is isomorphic to the diverse simplices. ◊

IV.3 — PREORDER OF AGGREGATION

IV.35 — OVERVIEW

We henceforth focus on the morphisms between the precontexts for it is time to understand how to aggregate the projectors and to quotient the surplus of contexts generated thus far ; indeed, typically, we desire to identify (p, q) with (q, p) since, after all, the order of a sequence is irrelevant physically. The components of our types are natural numbers wherefrom we have the ability to order them by, let us say, increasing value. Should it be the path chosen, it would remain to postulate that two sequences of projectors are equivalent when they differ by a permutation of two (or more) projectors of equal rank ; the types would be unaltered. It is nevertheless judicious to leave our types unordered, and equally our projectors, and to define an action on a sequence as an aggregation of some of its constituents — we go from a long sequence to a shorter one in summing projectors. In this perspective, the permutations are aggregations in disguise for a general action is a sum and a permutation. ◊

IV.36—DEFINITION — LOCALIC CATEGORY

The locale \mathcal{A} is a category having,

1° for objects, the natural numbers less than n , collected in the locale $\text{Ob}.\mathcal{A}$

2° for arrows, the surjective set functions collected in the locale $\vec{\mathcal{A}}$ defined in II.56 at page 26 ◇

IV.37—PROPOSITION — AGGREGATION

We can define a (sound) categorical action on our spectral bundle $Y \longrightarrow X \longrightarrow T$ with the arrow,

$$\begin{aligned} \lambda_T : T &\longrightarrow \text{Ob}.\mathcal{A} \\ \vec{t} &\longmapsto \# \vec{t} \end{aligned}$$

which can be composed with the bundle — as its morphisms are nothing more nor less than the projections, we forget the state and the projector sequence in going to T . The second part of the action is given directly by the action on the states,

$$\begin{aligned} \alpha : (\vec{\mathcal{A}} \times_{\text{Ob}.\mathcal{A}} Y) &\longrightarrow Y \\ (g, \vec{t}, \vec{p}, \phi, dg = \# \vec{t}) &\longmapsto \left(\left(s_j \doteq \sum_{i \in g^{-1}(\{j\})} t_i = \sum_{i \leq \ell} \delta(g(i), j) t_i \right)_{j \leq cg \leq dg}, \right. \\ &\left(q_j \doteq \sum_{i \leq \ell} \delta(g(i), j) p_i \right)_{j \leq cg \leq dg}, \\ &\left. \text{Spec}_g(\phi) \right) \end{aligned}$$

where, for a surjective $g : \ell \longrightarrow h$, we employ the surjective set function,

$$\begin{aligned} \text{Spec}_g : [[1, \ell]] &\longrightarrow [[1, h]] \\ \phi : \mathbb{C}^\ell &\longrightarrow \mathbb{C} \longmapsto \phi \circ \mathbb{C}^g : \mathbb{C}^h \longrightarrow \mathbb{C} \end{aligned}$$

via the contravariant injective transformation $\mathbb{C}^g : \mathbb{C}^h \longrightarrow \mathbb{C}^\ell$ of C^* -algebras,

$$\begin{aligned} \mathbb{C}^g : \mathbb{C}^h &\longrightarrow \mathbb{C}^\ell \\ \vec{z} &\longmapsto \{w_i \doteq z_{g(i)}\}_{i \leq \ell} \end{aligned}$$

The action α reduces to an action on the precontexts X and the types T in forgetting the state component. It also furnishes the action $\mathcal{V}(\alpha)$ on the impure states simply by application of the functor \mathcal{V} on the arrow. ◇

PROOF

We must prove firstly that α is coherent when composed with λ_Y ; it is so indeed, for the length of the type outputted by $\lambda_Y \circ \alpha$ is given by the arrow,

$$c \circ \pi_1$$

The arrow α is functorial for it concords equally with the composition.

We demonstrate its transitivity on the types; if we have two composable surjections g, f and their composite $g \circ f$, we can be sure that the action is transitive; indeed,

$$\begin{aligned} \forall \vec{t} \in T_{df}, \forall k \leq cg, g(f(\vec{t}))_k &= \sum_{j \leq cf} \delta(g(j), k) \sum_{u \leq df} \delta(f(u), j) t_u \\ &= \sum_{u \leq df} \sum_{j \leq cf} \delta(g(j), k) \delta(f(u), j) t_u \\ &= \sum_{u \leq df} \delta(g(f(u)), k) t_u \\ &= ((g \circ f) \vec{t})_k \end{aligned}$$

Some identical equalities hold for the projector sequences. The functoriality on the states via $\text{Spec}_{(-)}$ is also valid thanks to the functoriality of the exponential $\mathbb{C}^{(-)}$.

In order to obtain the action on the prestate locale X , we compose α with the spectral bundle $Y \longrightarrow X$. In effect, it factorizes as,

$$\begin{array}{ccc} & & (\vec{\mathcal{A}} \times_{\text{Ob.}\mathcal{A}} X) \\ & \nearrow & \downarrow \\ (\vec{\mathcal{A}} \times_{\text{Ob.}\mathcal{A}} Y) & \longrightarrow & X \end{array}$$

producing an arrow,

$$(\vec{\mathcal{A}} \times_{\text{Ob.}\mathcal{A}} X) \longrightarrow X$$

which remains the one of an action. Idem for the types.

In the case of the valuation monad, the action becomes,

$$\begin{aligned} \mathcal{V}(\alpha): (\vec{\mathcal{A}} \times_{\text{Ob.}\mathcal{A}} \mathcal{V}_X(Y)) &\longrightarrow \mathcal{V}_X(Y) \\ (g, \vec{t}, \vec{p}, \mu, dg = \# \vec{t}) &\longmapsto \left(\left(s_j \doteq \sum_{i \in g^{-1}(\{j\})} t_i = \sum_{i \leq l} \delta(g(i), j) t_i \right)_{j \leq cg \leq dg}, \right. \\ &\left. \left(q_j \doteq \sum_{i \leq l} \delta(g(i), j) p_i \right)_{j \leq cg \leq dg}, \right. \\ &\left. \mathcal{V}(\text{Spec}_g)(\mu) \right) \end{aligned}$$

thanks to its functoriality and geometricity. •

IV.38—PROPOSITION

Not much can be derived when two aggregations f, g get two projector sequences \vec{p}, \vec{q} in agreement, only that,

$$\forall j \leq dg, \forall i \leq df, g(j) \neq f(i) \Rightarrow q_j p_i = 0 = p_i q_j \quad \diamond$$

PROOF

Indeed, let us suppose two aggregations f, g rendering two projector sequences \vec{p} and \vec{q} equal,

$$\forall h \leq cg, \forall j \leq dg, \delta(g(j), h) q_j = \sum_{k \leq df} \delta(f(k), h) p_k q_j$$

and in particular,

$$\forall j \leq dg, q_j = \sum_{k \leq df} \delta(f(k), g(j)) p_k q_j = \sum_{k \leq df} \delta(f(k), g(j)) q_j p_k$$

and immediately, by orthogonality of the p 's,

$$\forall i, j \leq dg, g(j) \neq f(i) \Rightarrow q_j p_i = 0 = p_i q_j \quad \bullet$$

IV.39—PROPOSITION — PREORDER OF AGGREGATION

The categorical action of \mathcal{A} on X, Y and $\mathcal{Y}_X(Y)$ reduces to an action of a closed preorder. The bundle maps $Y \rightarrow X$ and $\mathcal{Y}_X(Y) \rightarrow X$ are monotone. \diamond

PROOF

The proof follows from the presence of the projectors as the components of the points of these locales. Since the transitivity is assured by the functoriality, we must essentially show the reflexivity and the uniqueness of the arrow. We write W for either the compact regular locale Y or $\mathcal{Y}_X(Y)$ but let us first analyse the case of X . We define the relation \leq_X on the precontextual locale X as the image,

$$\begin{array}{ccc} & & \leq_X \\ & \nearrow & \downarrow \\ (\vec{\mathcal{A}} \times_{\text{Ob.}\mathcal{A}} X) & \xrightarrow{\langle \alpha, \text{Proj} \rangle} & X \times X \end{array}$$

Because it is a fibred product, the pullback $(\vec{\mathcal{A}} \times_{\text{Ob.}\mathcal{A}} X)$ corresponds to,

$$(\vec{\mathcal{A}} \times_{\text{Ob.}\mathcal{A}} X) \simeq \prod_{\ell} \prod_{h \leq \ell} (\mathcal{A}_{h, \ell} \times X_{\ell})$$

which demonstrates that the pullback is compact and regular. And we now prove that it is also isomorphic to \leq_X .

Firstly, it is proper for the image of a proper arrow remains so [Ver94]. We understand secondly that a surjective $g : \mathbf{h} \rightarrow \mathbf{h}$ of sets is mandatory bijective, when \mathbf{h} is a finite natural. By the excluded third in \mathbb{N} , if two numbers x, y in \mathbf{h} are mapped to the same image under g , the image could not longer be fully \mathbf{h} precisely for the domain is \mathbf{h} itself. There would not be enough elements in the domain for that the image be \mathbf{h} .

Next, we notice that the locale $\mathcal{A}_{\ell\ell}$ has all the surjections on ℓ — whereupon the bijections — and we note Id_ℓ the identity function; in considering the little arrows $\langle \{\text{Id}_\ell\} \circ!_{X_\ell}, \text{Id}_{X_\ell} \rangle$, the next diagram commutes,

$$\begin{array}{ccccc}
 \prod_{\ell} X_{\ell} & \xrightarrow{\prod_{\ell} \langle \{\text{Id}_{\ell}\} \circ!_{X_{\ell}}, \text{Id}_{X_{\ell}} \rangle} & \prod_{\ell} \mathcal{A}_{\ell\ell} \times X_{\ell} & \twoheadrightarrow & \prod_{\ell, h \leq \ell} \mathcal{A}_{h, \ell} \times X_{\ell} \\
 \Delta \downarrow & & & & \downarrow \\
 X \times X & \xrightarrow{\hspace{10em}} & & & \leq_X \\
 \parallel & & & & \downarrow \\
 X \times X & \xlongequal{\hspace{10em}} & & & X \times X
 \end{array}$$

by definition of the image \leq_X , and shows that the relation is reflexive.

Now, for the uniqueness of the composition law, let us suppose two projector sequences \vec{p} and \vec{q} , of length ℓ and h respectively, such that the first is longer than the second; perhaps, these two are incompatible, yet when there exist some surjectives $f, g : \ell \twoheadrightarrow h$ such that,

$$\forall j \leq h, q_j = \sum_{i \leq \ell} \delta(g(i), j) p_i = \sum_{i \leq \ell} \delta(f(i), j) p_i$$

we see that this amounts to,

$$\forall j \leq h, \forall i \leq \ell, q_j p_i = \delta(g(i), j) p_i = \delta(f(i), j) p_i$$

if and only if,

$$\forall j \leq h, \forall i \leq \ell, \delta(g(i), j) = \delta(f(i), j)$$

by lack of nullity of the projectors; however this equality only happens when f is g by definition of the Dirac's deltas after we derive,

$$\forall j \leq h, \forall i \leq \ell, \delta(g(i), j) j = \delta(f(i), j) j$$

and we sum over the collection of the j 's sharing this property — which is the set h .

The arrow $\alpha \times \text{Proj}$ is necessarily a monic of compact regular locales because the arrow is injective as we have just seen. Which bears the consequence of being more than a monic; it is an embedding in **Loc**.

The same construction applies on $W \twoheadrightarrow X$. Eventually, the universal characterization of images furnishes the commutativities of the preorder squares,

$$\begin{array}{ccccc}
 \leq_W & \xrightarrow{\hspace{2em}} & \leq_X & \xrightarrow{\hspace{2em}} & \leq_T \\
 \downarrow & & \downarrow & & \downarrow \\
 W \times W & \xrightarrow{\hspace{2em}} & X \times X & \xrightarrow{\hspace{2em}} & T \times T
 \end{array}$$

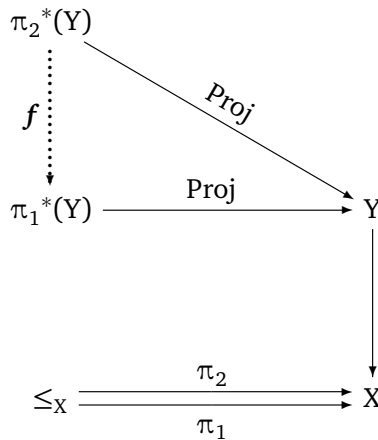
in other words, the monotonicity of the diverse bundle maps. •

IV.40 — NOTE

The monotonicity guarantees the actions are coherent on the various levels ; indeed, for the degenerate case of the qbit, suppose we have two isomorphic sequences \vec{p} and $\vec{q} \doteq g(\vec{p}) \doteq (p_2, p_1)$ of projectors, tied together by a permutation $g : 2 \longleftrightarrow 1$. Consequently, the three possible spectra are (\vec{p}, f_1, f_2) , $g(\vec{p}, f_1, f_2) \doteq (\vec{q}, e_1 \doteq f_2, e_2 \doteq f_1)$ and (\vec{q}, f_1, f_2) and our goal is to tie, under g , the element (\vec{p}, f_1) with (\vec{q}, e_2) and the other element (\vec{p}, f_2) with (\vec{q}, e_1) — which is an appropriate transformation for spectra — but not (\vec{p}, f_1) with (\vec{q}, e_1) nor (\vec{p}, f_2) with (\vec{q}, e_2) . \diamond

IV.41 — PROPOSITION — FIBRE MAP

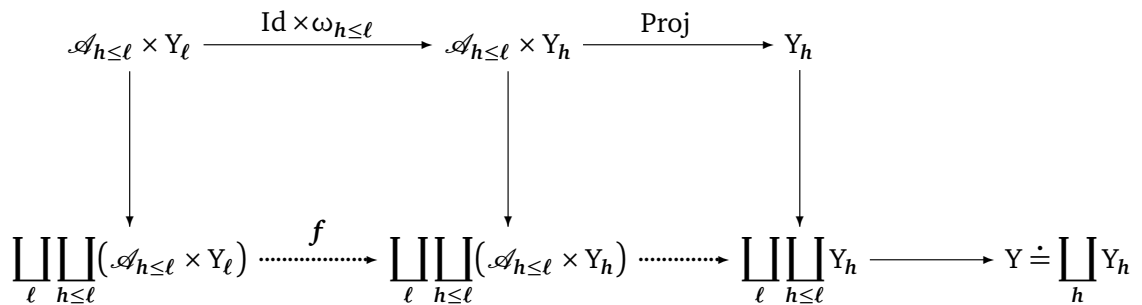
There exists a **fibre map** f between the pullbacks of the bundle $Y \longrightarrow X$ along the legs $(\pi_1 \leq \pi_2)$ of \leq_X ,



This fibre map encodes the action of \mathcal{A} . \diamond

PROOF

The fibre map f corresponds to the global action on Y when we take into account (the pullback along the legs of) the action on X . Indeed, when we work locally, in other words, when we focus on a pair of natural numbers ℓ and h , there exists a local action, say $\omega_{h \leq \ell}$. These little arrows give back the global action when we sum them and thanks to the fibrewise property of the pullbacks, we notice that there exists a factorization f ,



throughout $\coprod_{\ell} \coprod_{h \leq \ell} (\mathcal{A}_{h \leq \ell} \times Y_h)$ instead of going directly to Y as α does. It results that the fibre map f for a bundle $Y \longrightarrow X$ over the precontextual locale X assures the commutativity of the

square,

$$\begin{array}{ccc}
 \pi_2^*(Y) & \longrightarrow & Y \\
 \downarrow & & \downarrow \\
 \leq_X & \xrightarrow{\pi_1} & X
 \end{array}$$

since we have the isomorphism,

$$\pi_2^*(Y) \simeq \coprod_{\ell} \coprod_{h \leq \ell} (\mathcal{A}_{h,\ell} \times Y_{\ell})$$

as well as,

$$\pi_1^*(Y) \simeq \coprod_{\ell} \coprod_{h \leq \ell} (\mathcal{A}_{h,\ell} \times Y_h)$$

as the preoder \leq_X is also $\coprod_{\ell} \coprod_{h \leq \ell} (\mathcal{A}_{h,\ell} \times X_{\ell})$. •

IV.42 — NOTE

In the same manner, the fibre map $\mathcal{V}_{\leq}(f)$ exists naturally for the bundle $\mathcal{V}_X(Y) \longrightarrow X$. ◇

IV.4 — CONTEXT AND STATE

IV.43 — OVERVIEW — PULLBACK AND COEQUALIZER

Thus far, we did not distinguish heavily between a precontext and a context. Eventually, yet, we must quotient our preorders on the various compact regular locales. For illustration, we can quotient the aggregations on the precontexts X and the prestates Y in following the traditional approach consisting in forming the bundle $Y_{/\simeq} \longrightarrow X_{/\simeq}$ of posets whereby leaving us with a partial order on the quotients since the relation of equivalence \simeq of the preorder \leq is taken out. From the preservation of the relation of equivalence on Y once its points projected on X , we are given a commutative square of coequalizers,

$$\begin{array}{ccc}
 Y & \longrightarrow & Y_{/\simeq} \\
 \downarrow \lrcorner & & \downarrow \\
 X & \longrightarrow & X_{/\simeq}
 \end{array}$$

which is in effect a pullback. A rigorous proof is by strict descent. Classically or when we deal instead with the topological spaces, we can invoke the following reasoning. The points of the pullback constitute the set,

$$\{(\vec{p}, [\vec{q}, \psi]) \mid [\vec{p}] = [\vec{q}]\}$$

which implies, together with the uniqueness of the aggregations on the contexts, that we can pick a unique bijection and apply it to ψ ; indeed, let us suppose two representatives (\vec{q}, ψ) and (\vec{r}, ψ') of the equivalence class $[(\vec{q}, \psi)]$; we know that \vec{p} is a permutation of \vec{q} and of \vec{r} , but the latter is itself a permutation of \vec{q} . The uniqueness of the composition of the

aggregations between two contexts shows that there is only one transformation to consider, let us say the one between \vec{q} and \vec{p} , and we can apply it to ψ to obtain a state over \vec{p} . Overall, we have an isomorphism between the pullback and Y .

This is exactly what we desire to avoid for the aggregations on the quotients seem tedious, at best. Instead, for the bundle $T \rightarrow \mathbb{N}$ of the types, the proof is again about discrete preordered locales so all goes well. For the trickier spectral bundle $Y \rightarrow X$, we appeal to the theory of the descent in **Loc**; but in its lax form in order to deal with a preorder and its lax coequalizer, as exposed in III.130 at page 76. The overall process is not to use Y directly for its fibre map is contravariant as a lax datum. The result holds in a unique theorem, but we wish to explicit the steps for we believe that the mechanism is clearer when we go to the other side of the Stone's duality; that is to say, when we look at the finite powerset which is the boolean algebra to consider. \diamond

IV.44—PROPOSITION

Let us suppose that we order the types^① T and let us consider their preorder \leq_T of aggregation; the square,

$$\begin{array}{ccc}
 \leq_T \rightrightarrows & T & \xrightarrow{\epsilon} \text{Idl}(T) \\
 & \downarrow f \doteq \# & \downarrow g \\
 \leq \rightrightarrows & \mathbb{N} & \xrightarrow{\delta} \text{Idl}(\mathbb{N})
 \end{array}$$

PROOF

It is appropriate to use the points because our locales are discrete; in other words, they are sets. In the discrete case, the locale $\text{Idl}(\mathbb{N})$ has for points the ideals of the preset (\mathbb{N}, \leq) whose opens are its Alexandrov ones. The lax coequalizer take a point of \mathbb{N} to its principal ideal [Vic93]. When we reason on the points, one element of the pullback is a pair,

$$(\ell, J) \in \mathbb{N} \times \text{Idl}(T)$$

subject to the constraint,

$$\delta(\ell) = \downarrow \ell = g(J) \doteq \bigvee^{\text{dir}} \{\downarrow f(t) \mid t \in J\}$$

whose equivalent formula is,

$$\forall h \in \mathbb{N}, h \leq \ell \iff \exists t \in J, h \leq f(t)$$

but the existential quantifier does not give a unique element a priori; when we take h to be ℓ , there exists a type t in the ideal J verifying the equivalence for ℓ ; by direction of J , there exists a u in J that is bigger than t . Manifestly,

$$\ell \leq f(t) \leq f(u) \leq \ell$$

and we conclude that we are able to send bijectively a point of T to the pullback via,

$$\begin{array}{l}
 T \longrightarrow \text{Pullback} \\
 t \longmapsto (f(t), \downarrow t)
 \end{array}$$

^① Always possible as done in [Cas+09].

and via,

Pullback \longrightarrow T

$(\ell, J) \longmapsto$ type of length ℓ with components ordered numerically

because the components of the types are ordered natural numbers. •

IV.4.1 — LAX DESCENT OF THE SPECTRAL BUNDLE

IV.45 — PROPOSITION — THE FIBRE MAPS AS A CONTRAVARIANT LAX DATUM

The fibre map $f: \pi_2^*(Y) \longrightarrow \pi_1^*(Y)$ associated to the spectral bundle $Y \longrightarrow X$ is contravariant, but does satisfy the counit and the cocycle condition of the closed preorder \leq_X . ◇

PROOF

It is immediate after the definition of the fibre map since this latter is the factorization of the action (of the category \mathcal{A}) on the prestates. The counit condition stipulates that on the same fibres, the action must be an isomorphism. We have seen that this is true, since the surjective set functions on a set ℓ must be bijective. The cocycle condition is the transitivity of the action. •

IV.46 — NOTE

This is the reason why we look at the finite powerset of our locale Y . When we dualize in going over to the boolean algebras, we turn the contravariant fibre map into a covariant lax datum and all goes well. ◇

IV.47 — DEFINITION — FINITE POWERSET

The bundle $\mathcal{F}_X(Y) \longrightarrow X$ of the finite powerset of the prestate bundle $Y \longrightarrow X$ is the locale taking the form of the finite powerset of the spectra,

$$\mathcal{F}_X(Y) \doteq \coprod_{\vec{t}} X_{\vec{t}} \times \mathcal{F}(\llbracket 1, \# \vec{t} \rrbracket)$$

using its geometricity. ◇

IV.48 — NOTE

We recall that Y being fibrewise stone, its finite powerset is its boolean algebra of clopens; the frame of Y being the full powerset or simpler, the ideal completion of the finite powerset. ◇

IV.49 — PROPOSITION

The bundle $\mathcal{F}_X(Y) \longrightarrow X$ remains a sheaf. ◇

PROOF

By geometricity. •

IV.50 — PROPOSITION — NORMALITY, LAX DATUM FOR THE SPECTRAL FINITE POWERSET

There exists a datum of lax descent for the bundle $\mathcal{F}_X(Y) \longrightarrow X$ thanks to the action,

$$\begin{aligned} \theta: \pi_1^*(\mathcal{F}_X(Y)) &\longrightarrow \pi_2^*(\mathcal{F}_X(Y)) \\ (\mathbf{g}, \mathbf{g}(\vec{p}) \leq_X \vec{p}, A) &\longmapsto (\mathbf{g}, \mathbf{g}(\vec{p}) \leq_X \vec{p}, \mathbf{g}^{-1}(A)) \end{aligned}$$

In effect, there is more to it for the arrow θ respects the structure of the distributive lattice and even better, the booleanity of the finite powerset. ◇

PROOF

The arrow is (well) defined locally because any $\mathbf{g}^{-1}(A)$ is indeed finite as we manipulate an equality decidable.

Our action does respect the unit condition because the pullback of $\mathcal{F}_X(Y) \longrightarrow X$ along the composite arrow $X \xrightarrow{\Delta} \leq_X \xrightarrow{\pi_{1,2}} X$ imposes the constraint that all the projectors sequences (and all the possible aggregations) equate their aggregates; thus the aggregations must be the identity on the sequences in the first place — as opposed to a general permutation thereof. The arrow $\Delta^*(\theta)$ is indeed the identity. Additionally, the cocycle condition is equally verified by the composition of the inverse functions.

We note immediately that the bundle $\mathcal{F}_X(Y) \longrightarrow X$ is a normal distributive lattice; typically the finite meets and finite joins are calculated fibrewise^①. The pair $(\mathcal{F}_X(Y) \times_X \mathcal{F}_X(Y), \theta \times_X \theta)$ is a product of lax descent data by the preservation of the products under the pullbacks,

$$j = 1, 2 \Rightarrow \pi_j^*(\mathcal{F}_X(Y) \times_X \mathcal{F}_X(Y)) \simeq \pi_j^*(\mathcal{F}_X(Y)) \times_{\leq_X} \pi_j^*(\mathcal{F}_X(Y))$$

Equally, meets and joins are morphisms of lax descent data because the functions of the inverse images commute with the meets and joins of the finite subsets^②. This being said, we can go further in noticing that the lax datum that is θ evidently agrees well with the complementation present on the bundle $\mathcal{F}_X(Y)$ since every inverse function in set theory does. •

IV.51 — PROPOSITION — REGULARITY

The prestate locale $Y \longrightarrow X$ over the contextual locale X is a pullback of a Stone locale Y' , over the codomain X' of the perfect lax coequalizer δ of the closed preorder \leq_X of the aggregations on X ,

$$\begin{array}{ccccc} \leq_Y & \xrightarrow{\quad\quad\quad} & Y & \xrightarrow{\quad \epsilon \quad} & Y' \\ & & \downarrow \lrcorner & & \vdots \\ & & j & & \mathbf{g} \\ \leq_X & \xrightarrow{\quad\quad\quad} & X & \xrightarrow{\quad \delta \quad} & X' \end{array}$$

Besides, the pullback square is equally the square of the lax coequalizers of Y and X . Explicitly, ϵ is the lax coequalizer of the order \leq_Y corresponding to the action of the aggregations on the prestates.

A partial order \sqsubseteq of specialization is naturally present on Y' . Two points (\mathbf{x}', ψ') , (\mathbf{z}', ϕ') in Y' are in this order when, in posing $(\mathbf{x}', \psi') = (\delta(\mathbf{x}), \psi') = \epsilon(\mathbf{x}, \psi)$, $(\mathbf{z}', \phi') = (\delta(\mathbf{z}), \phi') = \epsilon(\mathbf{z}, \phi)$,

$$\begin{aligned} (\mathbf{x}', \psi') \sqsubseteq (\mathbf{z}', \phi') &\iff \mathbf{x}' \sqsubseteq_{X'} \mathbf{z}' \text{ and } \psi = f(\phi) \\ &\iff \mathbf{x} \leq_X \mathbf{z} \text{ and } \psi = f(\phi) \end{aligned}$$

where f is the fibre map implementing the aggregations on the prestates. ◊

^① Because a meet or join is an arrow $\mathcal{F}_X(Y) \times_X \mathcal{F}_X(Y) \longrightarrow \mathcal{F}_X(Y)$, not from the product over $\mathbf{1}$, but over X .

^② The inverse set functions violates the infinite intersection.

PROOF

The bundle $Y \longrightarrow X$ is fibrewise Stone and in consequence is the pullback of some fibrewise Stone locale Y' over X' . Nonetheless, we know that the projection ϵ is geometrically proven to be the lax coequalizer of the fibre map resulting from the duality by Stone applied to the lax datum in relation with the finite powerset whereof Y is the Stone spectrum. This is precisely the fibre map defining our aggregations on the prestates. •

IV.4.2 — THE MANIFOLD TOPOLOGY

IV.52 — CONSPECTUS

Our initial motivation is to furnish the contextual base space of [Cas+09] with a kind of a manifold topology while keeping, at least classically (in order to remove the predicate oddity of the potential infinite subsets of a finite set), the same points of their spectral space. We show here that our construction grants our wish. ◊

IV.53 — PROPOSITION — MANIFOLD TOPOLOGY

Classically, the spectral bundle $Y \longrightarrow X \longrightarrow T$ — and its descended version — is in agreement with the bohrified quantum phase space of [Cas+09] while embedding itself into some sublocale of the complex numbers. ◊

PROOF

We must begin by recalling cursively the formation of the spectral internal locale presented in the article [Cas+09]. It proves that the internal frame of the internal Gelfand spectrum of the tautological sheaf (being simultaneously a commutative C^* -algebra) of a matrix algebra $\text{Mat}_n(\mathbb{C})$ is identified as a subfunctor of boolean sheaf sending a context to its boolean algebra of projectors. The conclusion is that the internal spectrum is a Stone locale internal to the topos of sheaves over the ideal completion of the contextual poset. The frame is isomorphic to the ideal completion of this internal boolean algebra and its prime filters biject with the points of its Gelfand spectrum. This result is in agreement with our Y .

This being said, it is also noted that the poset $\mathcal{C}(\mathcal{A})$ to consider is the one of the projector sequences $\mathcal{C}_{\vec{t}}$ (for each type \vec{t}) quotiented by a relation $\sim_{\vec{t}}$ (for each type \vec{t}), encoding the permutation of their components,

$$\mathcal{C}(\mathcal{A}) \simeq \coprod_{\vec{t}} \mathcal{C}_{\vec{t}} / \sim_{\vec{t}}$$

The types from [Cas+09] are always ordered. Overall, this calculation motivates to work over $X' \longrightarrow T'$ instead of $X \longrightarrow T$; in other terms, it motivates to finely quotient the aggregations. Our process from III.127 at page 74 distinguishes the locale T from the patch $\text{Patch}(T')$ precisely in the sense that the points of the patch are the classes of the types, under the relation of equivalence generated from the injective surjections. We conclude that the points of $\text{Patch}(T')$ are the ordered types of the article. Classically, the points of T' biject with the ones of its patch.

Whereupon, in taking into account the precontexts, the bundle of [Cas+09] coincide with the one of the patches since it does not embed the aggregations into a unique locale; they remain a partial order outside of it. Anew, classically, the bundle of the patches is the bundle of the stably compact locales with the aggregations as the specializations.

Regarding the manifold topology, we must prove the existence of an embedding from X into some locale Z derived from the complex numbers. And that this embedding carries over the side of the stably compact locales, classically their equivalents.

The bundle $Y' \longrightarrow X'$ remaining fibrewise stone, the locale Y remains a Stone locale internally in $\text{Sh}(X')$ whose fibres are isomorphic to the ones over X by geometricity in such a manner that the points can be considered identical. By construction, our locale X is and embeds in,

$$X \doteq \coprod_{\ell \leq n} X_\ell \xrightarrow{j} Z \doteq \coprod_{\ell \leq n} \prod_{i=1}^{\ell} \mathcal{U}(1)^{n^2}$$

for each component X_ℓ embeds in some power of the complex disk $\mathcal{U}(1)$ of radius 1,

$$\forall \ell \leq n, \exists j_\ell, X_\ell \xrightarrow{j_\ell} Z_\ell \doteq \prod_{i=1}^{\ell} \mathcal{U}(1)^{n^2}$$

as we proved that the complex projectors see their matrix components bounded by one (in absolute value).

The category \mathcal{A} of the surjective set functions equally acts on the arbitrary matrices in Z . This time, the action remains categorical for its property of being a preorder pertains to the manipulation of projectors only. Naturally, the embedding is monotone for the action of the category \mathcal{A} since if a sequence of projectors is gotten by a summation from another one, then it remains so as a sequence of arbitrary matrices. The (image of the) action remains transitive and interpolative.

On the other hand, Z remaining a compact regular locale by finiteness, we know that the Hausdorff system it constitutes together with the action of \mathcal{A} outputs a stably compact locale Z' . Furthermore, j being monotone by construction, the mechanism of the patch outputs a unique arrow j' ,

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X' \\ \downarrow j & & \downarrow j' \\ Z & \xrightarrow{\quad} & Z' \end{array}$$

between the stably compact locales.

The arrow $\Omega(j')$ is understood as a frame morphism as opposed to a mere preframe morphism for it is a mapping which is also approximable,

1° by the reflexivity,

$$j^*(\perp) \leq (\Downarrow_X)^{\text{op}}(\perp)$$

2° by the property of j to be a frame morphism and monotone, for every \mathbf{a}, \mathbf{b} in $\Omega(Z)$,

$$\begin{aligned} j^*((\Downarrow_{Z'})^{\text{op}}(\mathbf{a}) \vee (\Downarrow_{Z'})^{\text{op}}(\mathbf{b})) &\leq (j^* \circ (\Downarrow_{Z'})^{\text{op}})(\mathbf{a} \vee \mathbf{b}) \\ &\leq ((\Downarrow_X)^{\text{op}} \circ j^*)(\mathbf{a} \vee \mathbf{b}) \end{aligned}$$

The continuous map j' remains regular monic for its frame counterpart is surjective. Indeed, the frame $\Omega(Z')$ is all the fixed opens of the relation created from the action of \mathcal{A} on Z . Idem

for the frame $\Omega(X')$ which in effect uses the same relation restricted on $\Omega(X)$. The frame morphism $\Omega(j)$ being surjective, we conclude that $\Omega(j')$ is equally so. •

IV.54 — NOTE — LAX DESCENT FOR THE OPPOSITE OF THE ORDER OF AGGREGATION

When we use the opposite order from the aggregations, we can directly apply the theory of the lax descent to our bundle $Y \longrightarrow X$ because the action of aggregation is covariant, from IV.45 at page 105. It results a sheaf $Y' \longrightarrow X'$ closer in spirit, this time, to the spectral sheaf of [DoeIsh08]. ◊

IV.5 — MILD EXTENSIONS

IV.55 — CONSPECTUS

Finally, we study two mild generalizations in the sense that we now attach the index n of the dimension to our locale X of contexts and analyse how it interacts with another $X(m)$. A finite direct sum of matrix algebras is the general form that a C^* -algebra being finite as a vector space takes [Dav96]. Our formalism extends to this case essentially for the tools we employ are all geometric. We begin with the verification that the morphisms of matrix algebras are monotone thanks to their linearity. ◊

IV.5.1 — MORPHISMS OF MATRIX ALGEBRAS

IV.56 — DEFINITION — LOCALE OF INJECTIVE UNITAL STAR MORPHISMS

Let us assume two natural numbers $n = n_1 = n_2$ and $m = m_1 = m_2$ with the first less than the second. Given the two Hilbert spaces \mathbb{K}^{n^2} and \mathbb{K}^{m^2} , the (necessarily injective^①) unital star morphisms from \mathbb{K}^{n^2} to \mathbb{K}^{m^2} are the points of the locale $\text{sMorphi}(n, m)$, sublocale of $\mathbb{K}^{n_1 \times n_2 \times m_1 \times m_2}$ itself having for points the arrows,

$$\begin{aligned} M: n_1 \times n_2 \times m_1 \times m_2 &\longrightarrow \mathbb{C} \\ (i, i', d, d') &\longmapsto (M_{ii'})_{dd'} \end{aligned}$$

(concretely sending a pair (i, i') of $n_1 \times n_2$ to a matrix of size $m \times m$).

The unital star morphisms are subject to the axioms stating,

1° the respect of the involution,

$$\forall M \in \mathbb{K}^{n_1 \times n_2 \times m_1 \times m_2}, \top \vdash (M_{ii'})^\dagger = M_{i'i}$$

2° the orthogonality,

$$\forall M \in \mathbb{K}^{n_1 \times n_2 \times m_1 \times m_2}, \forall i, i', j, j' \leq n, \top \vdash M_{ii'} M_{jj'} = \delta(i', j) M_{ij'}$$

3° the completeness,

$$\forall M \in \mathbb{K}^{n_1 \times n_2 \times m_1 \times m_2}, \top \vdash \sum_{i \leq n} M_{ii} = \text{Id}_{\mathbb{K}^{m^2}}$$

of the maps M . ◊

^① The injectivity follows instantaneously from the orthogonality and the linearity.

IV.57 — PROPOSITION — ACTION & MONOTONICITY OF THE INJECTIVE MORPHISMS OF MATRIX ALGEBRAS

For two natural numbers n less than m , we can define an action of the locale $\text{sMorphi}(n, m)$ on the precontexts $X(n)$ as,

$$\begin{aligned} \text{act} : \text{sMorphi}(n, m) \times X(n) &\longrightarrow X(m) \\ (j, \text{Tr}(\vec{p}), \vec{p}) &\longmapsto (\text{Tr}(j_{/\langle \vec{p} \rangle}(\vec{p})), j_{/\langle \vec{p} \rangle}(\vec{p})) \end{aligned}$$

where $\langle \vec{p} \rangle$ stands for the commutative C^* -algebra generated by the projector sequence \vec{p} .

And this action is monotone with respect to the action of aggregation from $\mathcal{A}(n)$. Explicitly, for each morphism j in $\text{sMorphi}(n, m)$, the following square commutes,

$$\begin{array}{ccccc} \leq_n & \xRightarrow{\quad} & X(n) & \xrightarrow{\delta_n} & X'(n) \\ & & \downarrow j & & \downarrow j' \\ \leq_m & \xRightarrow{\quad} & X(m) & \xrightarrow{\delta_m} & X'(m) \end{array}$$

where j' is the unique morphism of locales rendering the commutativity possible. ◇

IV.58 — NOTE

An identical result holds for the precontexts since the linearity applies equally well. ◇

IV.5.2 — FINITE C^* -ALGEBRAS

IV.59 — DEFINITION — DIRECT SUM OF MATRIX ALGEBRAS

We establish the direct sum $\mathbb{K}^{n^2} \oplus \mathbb{K}^{m^2}$ of two matrix algebras \mathbb{K}^{n^2} and \mathbb{K}^{m^2} as the locale $\mathbb{K}^{n^2} \times \mathbb{K}^{m^2}$ supplemented with the geometric operations — for the elements are finite — of,

1° the multiplication by the scalars,

$$\forall \alpha \in \mathbb{K}, \forall N \in \mathbb{K}^{n^2}, \forall M \in \mathbb{K}^{m^2}, \top \vdash \alpha(N, M) = (\alpha N, \alpha M)$$

2° the addition,

$$\forall N_1, N_2 \in \mathbb{K}^{n^2}, \forall M_1, M_2 \in \mathbb{K}^{m^2}, \top \vdash (N_1, M_1) + (N_2, M_2) = (N_1 + N_2, M_1 + M_2)$$

3° the multiplication,

$$\forall N_1, N_2 \in \mathbb{K}^{n^2}, \forall M_1, M_2 \in \mathbb{K}^{m^2}, \top \vdash (N_1, M_1) \times (N_2, M_2) = (N_1 \times N_2, M_1 \times M_2) \quad \diamond$$

IV.60 — PROPOSITION — LAX DESCENT FOR THE FINITE C^* -ALGEBRAS

Given two spectral bundles $Y_i \longrightarrow X_i \longrightarrow T_i$, for $i = 1, 2$, their localic product is well defined, remains fibrewise Stone and laxly descent down the counit of the patch adjunction. It remains a fibrewise Stone product over the stably compact locales $X'_1 \times X'_2$. Idem for the localic direct sum. ◇

PROOF

The product of the bundles gives indeed a bundle $Y_1 \times Y_2 \longrightarrow X_1 \times X_2$. The locale $X_1 \times X_2$ remains compact and regular by the Tychonoff's theorem and $Y_1 \times Y_2$ is equally a Stone locale as this property is stable under the pullbacks of **Loc**. The frame of $Y_1 \times Y_2$ is the tensor product $\text{Idl}(\mathcal{F}Y_1) \otimes \text{Idl}(\mathcal{F}Y_2)$ of the two frames; but thanks to the powerset being a right adjoint in the Stone's duality, we know that we can multiply the boolean algebras,

$$\text{Idl}(\mathcal{F}Y_1) \otimes \text{Idl}(\mathcal{F}Y_2) \simeq \text{Idl}(\mathcal{F}Y_1 \times \mathcal{F}Y_2)$$

Moreover, the product $\leq_1 \times \leq_2$ of the closed preorders on the X 's is anew a closed preorder as the two locales X_1 and X_2 are not mixed under the product, the two categories $\mathcal{A}(\mathbf{n})$ and $\mathcal{A}(\mathbf{m})$ acting on separated components of the product. The fibre map on the fibres of the product $Y_1 \times Y_2$ becomes the product of the fibres map for the product commutes with the pullbacks. The last thing to verify is the lax descent which indeed is effective for the products of the fibrewise Stone locales as this latter remains fibrewise Stone. The sheaf as the product of the boolean algebras over $X_1 \times X_2$ descends down to produce the products of the two individual descents. For the X 's, we only need to argue that the functor $\text{Patch}(-)$ is right adjoint in the adjunction and in incidence, that it copes well with the products. For the Y 's, we must say that the counit of the adjunction being the product of the individual counits, the geometric morphism concreting the lax descent commutes also with the product of the lax descent data — the category of the lax data for the product being the product of the two initial categories. In consequence, we obtain the product $Y'_1 \times Y'_2$ of the fibrewise Stone locales Y'_1 and Y'_2 over the stably compact locale $X'_1 \times X'_2$. The bijectivity of the pullback functor assures that the direct sums remain so once descended laxly. •

V — PROSPECTING

V.01 — OVERVIEW

We re-exhibit the main results swiftly to focus essentially on the analysis of the lax descent for the compact regular locales. \diamond

V.1 — CONCLUSION

In being so tightly tied to the toposical mathematics, the contextuality argues itself for a full geometric treatment, especially when we desire to study the base space \mathcal{B} of contexts and its various bundles inside of a topos we may find physically appropriate but simultaneously lacking the excluded third. The geometricity of this development corresponds to a fibration of the contextuality because it irremediably manufactures a contextual locale X compact and regular wherever live our two privileged bundles : the Stone locale Y of the pure states and the compact regular locale $\mathcal{V}_X(Y)$ of the impure ones — in the form of the statistical valuations on the state locale Y .

Besides, it deals with, a priori, a more pleasant topology for the contextual locale X is defined as,

$$X \doteq \coprod_{\vec{t}} X_{\vec{t}}$$

a coproduct of compact regular locales $X_{\vec{t}}$ indexed by the types ; in other words, indexed by all the possible decompositions of the dimension n (of the Hilbert space \mathcal{H}) in sums of natural numbers. Each $X_{\vec{t}}$ is the locale of the projector sequences \vec{p} , of length ℓ , of type \vec{t} and thus is a sublocale of $(\mathbb{C}^{n^2})^\ell$. It is indeed what we desire for, in finite dimension, the typical expression of a context is through the direct sum $C \doteq \bigoplus_{j \leq \ell} \mathbb{C}p_j$ of ℓ projectors p (as matrices of size n by n) subject to two properties,

1° their completeness,

$$\text{Id}_{\mathcal{H}} = \sum_{j \leq \ell} p_j$$

2° their orthogonality,

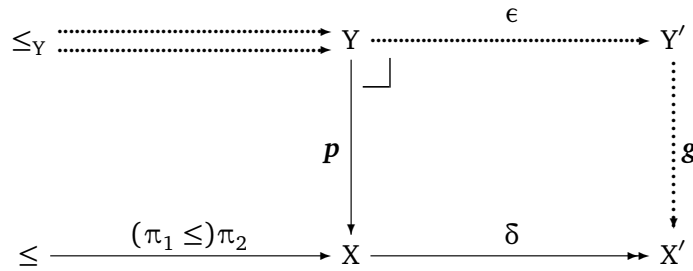
$$\forall i, j \leq \ell, i \neq j \Rightarrow p_i p_j = 0 = p_j p_i$$

In a detailed manner, the locale X is more precontextual than contextual for it incorporates an abundance of precontexts — the projector sequences — in regard to the number of actual contexts — the commutative subC^* -algebras. This multiplicity is the result of the possibility to permute the sequences and incidentally must be reduced in quotienting laxly the compact regular locale X by a closed preorder \leq of permutation — the aggregations — on it to obtain a stably compact locale X' of contexts — because (X, \leq) constitutes a Hausdorff system. However, we must refine a bit the duality — and loose it — in act,

$$\mathcal{C} : \text{KRegPreset} \longrightarrow \text{StbKLoc}_{\text{perf}}$$

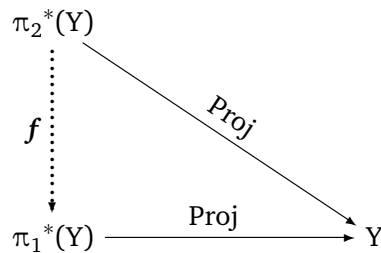
for we wish to keep the transformation of the proper maps between the compact regular presets into the perfect maps between the stably compact locales — instead of mere preframe

morphisms, formally reversed. In effect, we are left with a diagram of the kind,



where we must find some closed preorder \leq_Y on Y and a locale Y' stably compact (with p monotone) in order to have a pullback square of lax coequalizers. This technique has also the advantage to settle the closed preorder \leq on X as the specialization order on X' and analogously, to settle the aggregations on the states as their specialization.

It is then the moment to use the results in the field of the descent in **Loc**. We establish that the fibrewise Stone bundle that constitutes the spectral bundle $Y \rightarrow X$ does indeed descent laxly down the perfect surjection δ . In a concrete manner, the locale Y' does exist, is fibrewise Stone and ϵ is the lax quotient of the fibre map f ,



implementing the aggregations on the states.

V.2—LAX DESCENT AND COMPACT REGULARITY

V.02—OVERVIEW

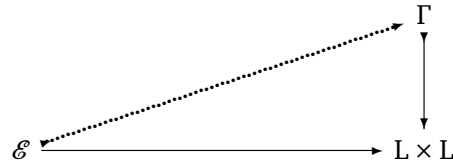
The sole explicit disadvantage of the geometric spectral bundle pertains to the necessity, for every (geometric) construction over the contextual locale X , to be in concord with the lax descent down the counit of the patch. After the analysis of the pure states, we must focus equally on the impure ones in questioning if the compact regular locales descent down the relatively tidy geometric morphisms. The strategy follows a path quite identical to the boolean case. We must understand the normal distributive lattices in the category of actions ; we must check that the pullback functor is fully faithful. In this case, we must also verify that the pullback square from the descent is equally a pullback square of lax coequalizers and, in application to the quantum mechanics, we must prove that it is the square obtained after the application of the valuation monad on the pullback square from the lax descent of the Stone locale of the pure prestates. We already know that the valuation monad preserves the fibrations [FauVic14]. ◇

V.2.1 — NORMALITY

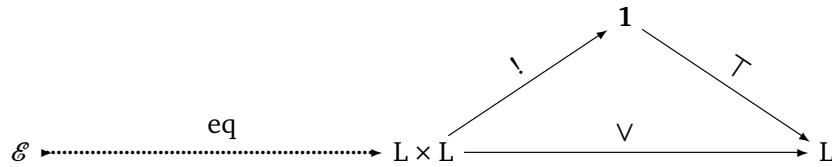
Before any conjecture, let us analyse what the normality becomes in the category of actions. We begin by recalling the normality for a lattice, in the traditional way. A lattice L is normal when,

$$\forall a, b \in L, a \vee b = \top \Rightarrow \exists x, y \in L, (x \vee a = \top = y \vee b \text{ and } x \wedge y = \perp)$$

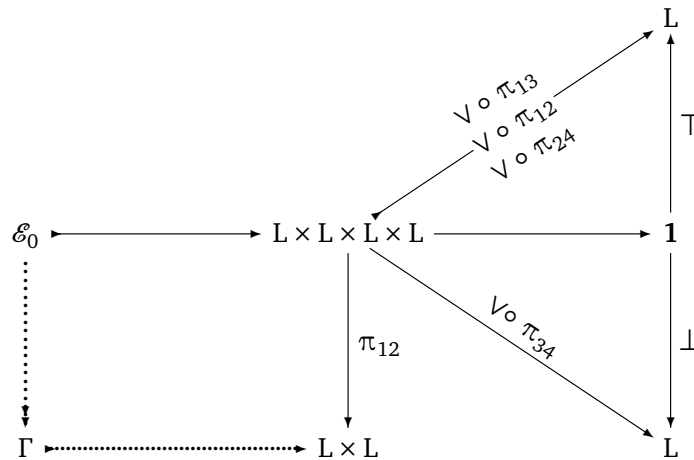
Categorically, the normality is the factorization,



of the equalizer \mathcal{E} ,



through the image Γ of the equalizer \mathcal{E}_0 ,



in order to code the existential quantifier in the definition of the normality.

We reasonably conjecture that a normal distributive lattice (F, θ) in the category of the actions for some preorder \leq (with legs $\pi_{1,2}$) on a locale X is,

- 1° a normal distributive lattice F in \mathbf{Loc}_X
- 2° a compatibility of θ with respect to the diagrams of normality

Concretely, we conjecture that we can assign some lax data to the domains and codomains of the above diagrams and that the morphisms appearing in them are compatible with the data thanks to the geometricity.

V.2.2 — EFFECTUALITY

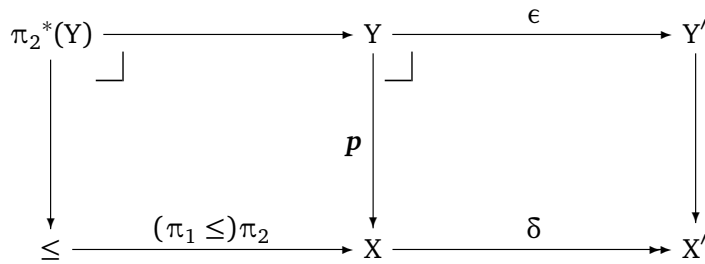
Once we are able to identify the normal distributive lattices, we must analyse the effectuality of the lax descent of the compact regular locales. The difficulty comes from the loss of the bijection between boolean arrows between some boolean algebras and their continuous counterparts between the Stone spectra. In the boolean case, a lattice morphism from a boolean algebra to the completion by its ideals of another boolean algebra is equivalent to a boolean morphisms (between the two boolean structures). It is no longer the case for the normal distributive lattices. Instead, we expect to search for a bijection between some suitable relations on normal distributive lattices and the continuous spectral maps.

V.2.3 — PULLBACK OF LAX COEQUALIZERS

When the effectuality does hold, meaning that we have the fully faithfulness and essential surjectivity of the pullback functor,

$$\begin{aligned} \delta^* : \text{KRegLoc}_{/c\delta} &\longrightarrow \text{KRegOpLDes}(\delta) \\ Y' &\longmapsto (Y \doteq \delta^*(Y'), \phi_Y : (\delta \circ \pi_2)^*(Y') \longrightarrow (\delta \circ \pi_1)^*(Y')) \end{aligned}$$

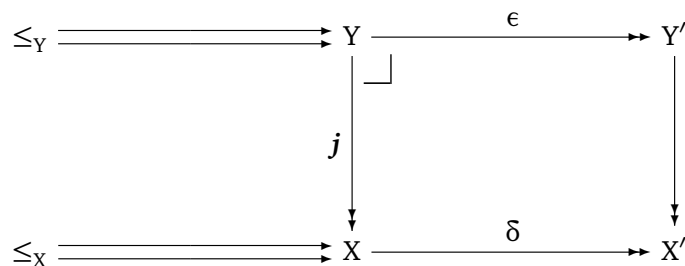
the spirit of the proof that the diagram,



is indeed a pullback of lax coequalizers carries over in a manner remaining identical to the one in III.144 at page 84.

V.2.4 — LAX COEQUALIZER FOR VALUATIONS

This is also a novelty compared to the boolean case for the pure states. The locale $\mathcal{V}_X(Y) \longrightarrow X$ over the contextual locale X is compact and regular for it is the probabilistic valuation monad that we use. By its geometricity, we know that we can apply this construction to the whole diagram,



of the pure states. The output is another pullback by geometricity of the functor \mathcal{V} ,

$$\begin{array}{ccc}
 \mathcal{V}_{\leq}(\leq_Y) \doteq \leq_{\mathcal{V}_X(Y)} & \xrightarrow{\quad} & \mathcal{V}_X(Y) \xrightarrow{\mathcal{V}_{\delta}(\epsilon)} \mathcal{V}_{X'}(Y') \\
 & & \downarrow i \quad \lrcorner \quad \downarrow g \\
 \leq_X & \xrightarrow{\quad} & X \xrightarrow{\delta} X'
 \end{array}$$

If the lax descent does happen for the compact regular locales, we anticipate that this pullback be also a pullback from the lax descent of $\mathcal{V}_X(Y)$ with the fibre map corresponding to the action of the aggregations on the prestates,

$$\begin{array}{ccc}
 \mathcal{V}_{\leq}(\pi_2^*(Y)) & & \\
 \downarrow \mathcal{V}_{\leq}(f) & \searrow \text{Proj} & \\
 \pi_1^*(\mathcal{V}_X(Y)) & \xrightarrow{\text{Proj}} & \mathcal{V}_X(Y)
 \end{array}$$

More precisely, we conjecture that the arrow $\mathcal{V}_{\delta}(\epsilon)$ universally laxly coequalizes the action on $\mathcal{V}_X(Y)$. In effect, the conservatism of the pullback functor is incident to the effectuality of the lax descent. What must be shown is that the identity on $\mathcal{V}_X(Y)$ is morphism of the two lax data that we can define from the two preorders; naturally, all is done so that it be the case.

A — PROLEGOMENON ON THE CONTEXTUALITY

A.01 — OVERVIEW

We attempt an exposition, at once more motivated and more elaborated, to the notion of the contextuality in contrasting the situations from the mechanics in its classical and quantum forms. A little bibliography is provided for the two toposical contextualities. The articles refer either directly to the subject or to a possible broadening. \diamond

A.1 — THE CLASSICAL AND QUANTUM ASSUMPTIONS

Even though the will to reformulate the rules and mechanisms of the quantum physics is the will of the many, the discussion remains generally on the level of the interpretation whereas the mathematical framework keeps the traditional Hilbert spaces. In the subsequent exposition on the toposical contextuality, both the physics and the mathematics are (deeply) altered. We illustrate the matter gradually in beginning with the case of the classical mechanics. By tradition, the first step consists in giving ourselves the data of a space of phases, at each moment in time, whereof the points are the data of (generalized) coordinates and of velocities in order to determine fully the solutions of the equations of second order of the motion. A lagrangian — representing the kinematic energy less the one of the potential — is also given as a function (of time) from the tangent bundle (over a manifold) that the phase space is to the real numbers. A path of the system is a cross-section of the tangent bundle. We also note that the sums of lagrangians transcribe the sums of independent systems and that further, the multiplication by the real numbers of the lagrangian does not modify the equations of motion. Naturally, the pointwise multiplication of arbitrary observables — the abstract functionals from the state space to the reals — is permissible as well as their differentiations. We must conclude that it is cogent to give to the set of observables an algebra. After a Legendre's transformation on the coordinates of the system, we acquire a state consisting of a (spatial) coordinate and a momentum and manage to turn the previous partial differential equations of the second order into a multitude of the first order. The lagrangian is turned into the hamiltonian representing the sum of the kinematic and potential energies — mathematically a function from the cotangent bundle to the reals. Eventually, in statistical physics, we can introduce some measures on the set of the states to weaken our knowledge of the system, but for the pure ones, we are certain that a couple of observables can be measured accurately, without restriction — interpretation of their vanishing variances.

When we favour the hamiltonian, the abstract situation is therefore the one of a manifold, of a state space, of (a construction of) the real numbers along with their topology and of a commutative C^* -algebra of observables which are functions between these two last topological spaces. Beside to obtain the results of the observables paired with the states, there equally well exist some propositions (and their truths) on the system which we must in consequence include into the general scheme. These consist of the questions « what are the permissible states wherein the system must be, such that the (real) value outputted upon an observation through the action of a given observable lie in a given part of the real line^① ? » and mathematically encoded in the characteristic functions from the parts of the reals to the parts of the state space. The collection of these basic propositions forms a boolean algebra. The negation

^① In truth, in a Borel's set of the reals.

of a proposition is the result of taking the complement of the set of the states for which the proposition does hold true^①; the straight consequence being that every state renders a proposition true or false^②. And the disjunct, conjunct, logical implication of propositions have their set-theoretic translations — the disjunction are the unions of set, the conjunctions are the intersections, the implications are derived from these last two by the de Morgan's laws. Such an interpretation is qualified of realist precisely for all the observables possess, at every moment of time, a (real) value. In a word, every proposition possesses a truth value. Besides this property of *definite values* stands the *incontextuality* in the form of an independence of the output of an observable from (a part of) the other possible measurements of the system. For a concrete illustration, the observable consisting of the square of the energy does exist at every time — hence a *realism* — and outputs the square of the value of the energy — *functoriality* — at the time of the measurement — *faithfulness*. Moreover, the value of the energy is independent of the potentiality to simultaneously measure the size of the system, let us say; giving in incidence a physics manifestly incontextual. The incontextuality takes equipollently the form of a bijection between the observables and the operators modelling them on the state space.

Briefly said, the classical logic governs the logic of the classical mechanics. And even though the one of the relativities remains confined^③, the quantum logic has seen a wealth of interests in its sundry axiomatizations [Pav92; GreHenWei09], surely because the original one was, along with its disappointments, at the heart of its modern formulation through the Hilbert spaces and the famous orthocomplemented lattices [BirVon36]. In this latter, there does not exist a logical implication for instance^④ and we know that we must be careful when we wish to express the disjunctions of some propositions and all the more for the conjunctions since for these latter, the conjunction is only defined in the case of the commuting selfadjoints^⑤. The only propositions leading to a sharp truth^⑥ are modelled by the projectors on the Hilbert space; projectors themselves equivalent to closed subspaces of the space — incidentally, we move from the characteristic functions and their subsets in classical mechanics to the projectors and their closed subspaces in the quantum one. The Gleason's theorem [Gle57; CooKeaMor85] and its consequence by Bell, Kochen and Specker [Bel66; KocSpe68] tell us that there cannot be a realist interpretation of quantum mechanics, as long as we wish to stick to the incontextuality, as long as we wish to believe the outputs of an observable remains independent of our ability to perform other experiments, commensurable with it or not. When a theory desires to be incontextual and about hidden variables as well as to

^① A proposition is thus a part of the states wherefore it is unconditionally true.

^② We recognize here the principle of excluded middle. In a boolean lattice, for every element p , whose interpretation is a proposition of the system, the proposition $p \vee \neg p = \top$ does hold; to wit that, either p or its negation does hold; this allows the reductio ad absurdum and also that every Hilbert space has a basis. Such a logical reasoning is lost when we depart from the boolean topos **Set** to an arbitrary one.

^③ Which appears equally classical by the logical connectors and methods employed [Cas02], [Mar06; Kei09; MarPan11].

^④ A thing that cannot be deemed worthy of a decent logic according to [PirJau70].

^⑤ These connectors are so unsatisfactory, they lead to a no less silly excluded middle. A proposition is true when its probability is one for the considered state, false when the probability is zero; the consequence is that not to be true differs greatly than to be false.

^⑥ Thereby we mean true or false, nothing between.

model the results of the quantum mechanics, it is impossible to assign a real value at once to every set of selfadjoint operators in a consistent and somewhat natural, classical manner. All together, there is none space of classical states for the quantum physics. Logically, the theorem asserts that the conjunction (in the metalogic of the physics, supposed classical) of the measurements as prescribed in the quantum mechanics, their realism, their faithfulness and their incontextuality lead to an incoherence. In order to avoid it, we must rebut at least one of the assumptions ; we choose the incontextuality.

Notwithstanding, a collection of (complete) commuting projectors possesses everything a classical logic enjoys, for the lattice that is generated is boolean. We are thus compelled to see that there does exist a minute bit of a classical logic in quantum mechanics. Naturally, we will not go far if we take a sole (complete) set of commuting projectors — something raising question of the choice of the best one [HalCli01] while this can quickly leads to a shortcoming [OkoSud13]; the novelty resides in grasping them all together, collectively. When we emphasize on this idea, we develop a contextual (neo)realist view of the quantum physics ; mathematically amounting to departing from the topos **Set** for the classical mechanics to a (undetermined) topos of (pre)sheaves for the quantum one ; logically moving from the classical logic to the constructive one. The change is as much physical (logically) as it is mathematical (logically) for the permissible mathematics itself changes — the rules are only weakened to become the constructive-impredicative ones instead of classical ones. This being said, there is no inquietude to have since some crucial theorems in mathematical physics are already established constructively ; typically the cornerstones that the Gleason's theorem [Bil97 ; RicBri99] and the spectral theorem [Spi03 ; Spi05 ; BriVit06 ; Fen11] are furnish already the best illustration of the sheer contingency of classical mathematical logic in physical science. An now the toposical framework trades the excluded middle of the (traditional) quantum logic to regain the distributivity law (of finite meets over arbitrary joins) wherewith we remain closer physically to a classical physics even though done mathematically in a constructive manner. Some advocate the mathematical logic of any physics — contextual or not — must in effect be intuitionist [Bri99 ; BriSvo00] ; others go even further and conclude that « the physics is logic » for any decent one must be preserved in every topos [Heu+08] and actually gets pinned down to the logic which must be the geometric one — it is the predicate part of the intuitionist one ; to wit that, we must not use the negation, nor the infinite universal quantifier — this fragment of the intuitionism is precisely the one preserved by the geometric morphisms between the toposes.^① We prefer the motto in the form « be wise, be fibrewise ».

Let us also underline the point that, a priori, the framework does not purport the idea incorporating some hidden variables. The prime motivation [DoeIsh08] is to disintertwine the various occurrences of the real numbers, to provide a better logic for the (quantum) physics and to forget the instrumentalism and its problems in the hope to apply it to the quantum cosmology [ButIsh00 ; Ish06 ; Doe13]. This perspective comes also close to the complementarity by Bohr advocating that to know completely a system is to know what information we possess in all the complementary experiments on the system, those experiments which cannot fully display simultaneously the phenomena associated to the concepts involved — in the manner that the

^① Grothendieck names these functors the continuous maps between the toposes. The adjective geometric is historical and completely disconnected from the geometry, in appearance.

Young’s slits cannot simultaneously display fully the corpuscular and ondulatory character of the matter, or that we cannot determine the spin along two different axes in only one experiment. In one word, as we loose the simultaneity^① in going from the classical mechanics to the relativities, we loose the commensurability in settling down on the quantum level. And it is a good thing.

In analogy to the logicism and the axiomatization of the physics, the trend to depart from the manifolds and Hilbert spaces to favour the algebraic outlooks has been present for the relativities [Ger72; Hes03] but perhaps even more dominant in quantum physics [HalMue06; Dav12]. The beginning of this perspective is the bijection between the points of a space and their evaluation functions on the set of all the morphisms (of a given structure) from the space to the complex numbers. In the commutative quantum case, the custom of the (abstract) algebraic formulation involves a C^* -algebra, whose part of the selfadjoints represents the observables, and a topological space of states — seen as continuous star-morphisms from the C^* -algebra to the complex numbers; they are also called the characters — with their weak star topology. By the famous Riesz–Markov theorem, every state of the C^* -algebra corresponds to a measure on the state space in such a manner that the pairing of such a state with an observable gives its expectation value mathematically expressed as the integral of the observable against the said measure. The interpretation is consequently in tune with the classical statistical mechanics. The Hilbert spaces and the density matrices are recovered via the representations of the C^* -algebras and the construction from Gelfand–Neumark–Seagal.

Nevertheless, with a view to the contextuality, we prefer to categorify/toposify the poset whose carrier set is the collect of all the commutative sub C^* -algebras and whose order is the simple inclusion of set. The right place to go further into the construction is the topos of (pre)sheaves on this poset — id est, each element of the poset is termed a context whereto is prescribed a set by each presheaf. Amongst all the possible presheaves, the fundamental forgetful one associates to each commutative sub C^* -algebra its carrier set and bears becomingly the structure of a commutative C^* -algebra (as an object of the topos). We can attribute to this commutative C^* -algebra a spectrum — in the sense of Gelfand — which itself remains internal. In effect, there is more to it because it carries a structure of a topology; in other words, it is not only the set of pure states — one state set for each context but altogether unified — that we recover, but also their weak star topologies. A way is contrived to associate a spectrum to a uncommutative C^* -algebra, even though there can be others [vBerHeu12].

A.2 — CONTEXTUALITY VIA PRESHEAVES

In few more details, there is a first manner [DoeIsh08]^②, historically speaking [ButIsh98; ButIsh99; ButIsh00; HamButIsh00; ButIsh02], to implement the idea of a context. It relies on the basic tool that is the set of all the subalgebras, commutative and unital, of the traditional von Neumann algebra of all the bounded operators on a fixed Hilbert space \mathcal{H} ; the prime illustration being the finite matrix algebras [Dor10]. This collection is a poset $\mathcal{V}(\mathcal{H})$ once stuffed with the arrows corresponding to the inclusions of subset. The physical input lies in the result that each element of the poset is a realist, classical viewpoint of the system

^① Let us say that the simultaneity is a temporal contextuality where a context is then a referential system.

^② The articles [DorIsh08a; DorIsh08b; DorIsh08c; DorIsh08d] are the scattered initial versions.

precisely because all the observables of a context are commensurable. A minuscule context has a minuscule perspective of the system — to the tune of being merely capable of acting on it via the identity, if it were not excluded as a context in this first approach — whereas a context rather enormous has plenty of émincé projectors, of low rank, to refine the propositions we can ask. The physical assumption is that the poset and what can be derived from it contains an essential part of the (quantum) physics.

In consequence, to each context C , we are able to give its set of characters Σ_C by Gelfand–Neumark; and where there is an inclusion $C \subseteq D$ of contexts, there is a restriction,

$$r_{DC}: \Sigma_D \longrightarrow \Sigma_C$$

on them. Indeed, it suffices to take a state $\psi_D: D \longrightarrow \mathbb{C}$ in Σ_D and to restrict it to C thus,

$$\psi_{D/C}: C \longrightarrow \mathbb{C}$$

The reversion of the direction of the restrictions (with respect to the inclusions) translates the contravariance of the spectral functor $\Sigma: \mathcal{V}(\mathcal{H}) \longrightarrow \mathbf{Set}$ on $\mathcal{V}(\mathcal{H})$; whence its covariance on the dual poset $(\mathcal{V}(\mathcal{H}))^{\text{op}}$ ^①. When we favour these covariant functors, we work in the category $[(\mathcal{V}(\mathcal{H}))^{\text{op}}, \mathbf{Set}]$ of « *presheaves* on $\mathcal{V}(\mathcal{H})$ ». Informally, a presheaf^② F on a category \mathcal{C} is a datum of sets indexed by the objects of the category,

$$\begin{aligned} F: \mathcal{C} &\longrightarrow \mathbf{Set} \\ C &\longmapsto F(C) \\ f: C &\longrightarrow D \longmapsto F(f): F(C) \longleftarrow F(D) \end{aligned}$$

The immediate mathematical question is « how much of the primitive uncommutative von Neumann algebra belongs to its spectral presheaf ? ». Without elaborating, it is the Jordan algebra which can be recovered integrally [HarDoe10; Doe12a] — since the usual purpose of this kind of algebras is to break the product into a symmetric one [JorvNeuWig34], the antisymmetric part, the rest of the algebra remains in the one of Lie.

The wealth of the approach takes equally a logical side as the framework continues naturally to extend itself with the refinement of physical propositions. The general idea matching the classical mechanics is that the fundamental spectral presheaf serves as the state space and indeed we are encouraged to manipulate it as a classical one. Each projector p of the Hilbert space \mathcal{H} leads to a (clopen) subobject of the spectral presheaf in approximating p , in every context, by a projector from above as well as from below. Because the category $[(\mathcal{V}(\mathcal{H}))^{\text{op}}, \mathbf{Set}]$ is a topos, we do know that the collection of subpresheaves of each presheaf constitute an Heyting algebra — in the present case, a biHeyting algebra [Doe12c]. These algebras are models of the intuitionist-impredicative logic; this logic which refutes the excluded middle and the axiom of choice but keep the rest of the classical one, thereby pushing forward the logical implication over the excluded middle. Inside a topos, the mathematical logic must be intuitionist as well — but it is not respected by the good morphisms between the topos, only the constructive-predicative one is so.

^① The dual poset possesses the exact same objects, but we formally reverse the arrows, the order.

^② See [Lei10] for a quick introduction. A preheaf is as much an amalgamation of sets as a vector is one of numbers.

The purpose of the refinement is to approximate an observable of the initial Hilbert space \mathcal{H} in each one of its contexts; the simplest observables are as usual the projectors p — typically coming from the spectral decomposition of an observable a in asking if the output of a (once paired with a given state) lies in a real interval Δ , [dGro04; dGro06; dGro08a; dGro08b; dGro13] — and the intuitive formulas in order to go as near as possible from p in a context C are,

$$p_o \doteq \inf\{q \in \text{Proj}(C) \mid p \leq q\}$$

from above and from below,

$$p_i \doteq \sup\{q \in \text{Proj}(C) \mid p \geq q\}$$

where the partial order \leq on the collection of projectors of \mathcal{H} is the traditional one; to wit that,

$$p \leq q \iff pq = p$$

In other words, it is the inclusions of the closed subspaces of \mathcal{H} — when p and q are commuting, the operator pq is a projector abstracting the closed subspace of \mathcal{H} rendered by the intersection of the ones of the two projectors. Because the closed subspaces of \mathcal{H} form a (orthomodular) complete lattice under the inclusion of subset [Red98], the lattice of the projections is in the same manner complete, whereby guaranteeing the existences of the two approximations. A classical theorem in mathematical physics asserts that this lattice is in fact the boolean algebra of the clopens of the Stone spectrum of the von Neumann algebra. Even better, the boolean algebra is complete and leads to more than a compact Hausdorff topological space, the Stone spectrum is in fact hyperstonean.

On the other hand, we can use shrewdly the Gelfand transform of an operator to turn the approximation p_o into an arrow $\Sigma(C) \longrightarrow \mathbb{C}$, continuous in **Set**. Its inverse image applied on the singleton $\{1\}$ furnishes a clopen subset of Σ_C . The upshot is necessarily that each projector of the quantum mechanics gives a clopen subpresheaf of the spectral one, in a contextual manner — and this global equivalence is what distinguishes the toposical approach to the modal one. The mechanism lifts to the observables [Dor05b; DoeDew12a]. This being said, the entirety of the Hilbert space is recoverable, via its lattice of propositions, from its boolean subparts [ConDoe13].

The definitions match the will to coarse-grain the propositions in the following sense; if we know an inclusion $C \subseteq D$ of contexts then the approximation (of a projector) from above will be bigger (or worse) in the coarse context C and finer in the finest one D . Idem for the lower approximation. The Gelfand transformation turning a projector whereon the states act equally works for the observables after we employ the spectral theorem to decompose them in some projectors (indexed by the real numbers). In consequence, in generalizing the order on the projectors we extend their (two) refinements to all the possible observables on \mathcal{H} — equivalently, the two approximations are the two sums of the ones of the projectors of the spectral decomposition. Without digressing, we note that the object in the topos $[(\mathcal{V}(\mathcal{H}))^{\text{op}}, \mathbf{Set}]$ characterizing the real numbers — the presheaf sending every context to the set of the real numbers — cannot be used as the object wherein our observables take their value. It is precisely the consequence (somehow desired) of the approximations. Finally, the

density matrices from the primitive Hilbert space are managed as internal valuations — the analogue of the measures — on the collection of the clopen subobjects of the spectral presheaf. Whereupon, [DoeDew12b] demonstrates that the spectral presheaf can be a sample space for all the observables and pushes further the concept of a toposical impure state from [Dor08; DoeIsh11].

Let us return now to the pure toposical logic. A fundamental input of the theory of the toposes is the Heyting algebra constituted by the collection of all the subobjects of a given presheaf equipped thus with all the logical connectors wished. The logic is no longer classical, but constructive^①, both mathematically and physically. Concretely, the conjunctions, implications, disjunctions and constructive negations are tolerated and conceptualized as categorical devices. Even better, the distributivity law of the conjunction over the disjunction is present, contrary to the (original) quantum logic — existence whereof their fathers hold for characteristic of the classical logic [BirVon36]. The truth of a proposition must be, as always, a sieve for each context for the truth in a topos is given by its subobject classifier Ω . Explicitly, this presheaf on a category \mathcal{C} is,

$$\begin{aligned} \Omega: \mathcal{C} &\longrightarrow \mathbf{Set} \\ C &\longmapsto \Omega(C) \doteq \{F \longmapsto \mathcal{C}(-, C)\} \end{aligned}$$

which gives, to each object C of \mathcal{C} , a set $\Omega(C)$ which must be the collection of subpresheaves F of the hom functor $\mathcal{C}(-, C)$. Such a subpresheaf is termed a sieve and takes the form, for our poset, of a part of the collection of all its subcontexts; but this part must be closed downwards for the inclusion of context; a sieve is a principal ideal^②. In other terms, we pair a state with a proposition \mathbf{p} whereof the truth in a context C is all the subcontexts of C where \mathbf{p} is turned into the probability one by the said state. If this collection of subcontexts is the one of all the subcontexts of C , then the proposition is true; otherwise in a nuance of false. In this topos the truth necessarily is contextual since it changes from contexts to contexts, but equally multivalued for a sieve can be something else than true — all the subcontexts of the one we are looking at — and false — the empty sieve. In the topos \mathbf{Set} , the subobject classifier is only the boolean algebra $\{\perp, \top\}$ whereby we understand that \mathbf{Set} is the topos for the classical physics.

^① Constructive here means constructive-impredicative — the use of the powerset is permissible — but without axiom of choice nor excluded middle.

^② An ideal is a lower set (for the order) which is also directed; in one word, it is a filtered colimit. An ideal is principal when it is the down set of a single element.

The theory of categories tells us that this topos of presheaves is equivalent to the topos of sheaves^① over the site constituted by the ideal completion of $(\mathcal{V}(\mathcal{H}))^{\text{op}}$ with its Scott topology^② whose opens bijectively correspond to the subsets of $\mathcal{V}(\mathcal{H})$ closed downwardly for its partial order — the inclusion. Explicitly, a presheaf F becomes on a topological space an attribution of a set to every element of the topology,

$$\begin{aligned} F: \text{Alex}[(\mathcal{V}(\mathcal{H}))^{\text{op}}] &\longrightarrow \mathbf{Set} \\ \downarrow C &\longmapsto FC \\ \downarrow C \subseteq \downarrow D &\longmapsto FC \subseteq FD \end{aligned}$$

A sheaf F is a presheaf F with a condition of coherence for the elements lying in an intersection of the sets given by the presheaf. When an open U of a topological space is factually the union $\bigcup_j U_j$ of opens $e_j: U_j \hookrightarrow U$, we must have,

$$\forall s, t \in F(U), \forall j \in \mathcal{J}, F(e_j)(s) = F(e_j)(t) \Rightarrow s = t$$

and also, for the embeddings $e_{ij}: U_i \cap U_j \hookrightarrow U_i$,

$$\forall i, j \in \mathcal{J}, \forall s \in F(U_j \cap U_i), F(e_{ij})(s) = F(e_{ji})(s) \Rightarrow \exists t \in F(U), \forall j \in \mathcal{J}, F(e_j)(t) = s_j$$

The negative result that the poset $\mathcal{V}(\mathcal{H})$ is unfortunately not continuous for the von Neumann algebras in infinite dimension is proven in [DoeBar11]. Becomingly, in finite dimension, with the faculty to externalize every sheaf as an object in a topos to put it in \mathbf{Set} , the mechanism simplifies and the pictorial summary becomes a spectral set Σ over a base space \mathcal{B} of contexts,

$$\begin{array}{c} \Sigma \\ \downarrow f \\ \mathcal{B} \end{array}$$

with the particularities such that,

- 1° the locale^③ \mathcal{B} be the ideal completion of the dual of the poset $\mathcal{V}(\mathcal{H})$ together with the coAlexandrov topology of $\mathcal{V}(\mathcal{H})$
- 2° the fibre of a context C be the set Σ_C of all its states with, as said, the discrete topology
- 3° the locale arrow f be a local homeomorphism, the locale Σ be the union of all the fibres Σ_C , be discrete; its frame be isomorphic to the powerset of some set
- 4° the restriction (on the fibres) be covariant with respect to the refinement on the dual of $\mathcal{V}(\mathcal{H})$
- 5° the Bell–Kochen–Specker theorem be equivalent to the lack of continuous cross sections of f [Dor05a]

A prolonged analysis is present in [Doe12b] which studies the two temporal evolutions in quantum physics and in [Flo13] focusing on the generalization of this presehaf to the normal operators [BreFlo12] and the introduction of the KMS states [GelFlo12].

^① A set is an object of the topos \mathbf{Set} , a generalized set is an object of a topos of sheaves.

^② It is equivalently the filter completion of $\mathcal{V}(\mathcal{H})$ with the coAlexandrov opens of $\mathcal{V}(\mathcal{H})$.

^③ We only mention that a topology on a topological space is a frame which represents a (spatial) locale.

A.3 — CONTEXTUALITY VIA COPRESHEAVES

Only sketched heretofore, we concentrate henceforth on the explication of the second manner to contextualize a physics. All of the content from the covariant one was classical on the face of it but we can see this one as a first attempt to use only the mathematical logic in a topos [Cas+09; HeuLanSpi09a; HeuLanSpi09b], with a swift account in [Heu+11]. First, we use a C^* -algebra \mathcal{A} unital, noncommutative whose purpose is to abstract the bounded operators of a Hilbert space \mathcal{H} ; and, anew, we look at the poset $\mathcal{C}(\mathcal{A})$ of its sub C^* -algebras unital, commutative whose order remains the set inclusions. This time we keep the trivial context consisting of the identity operator — the corresponding commutative C^* -algebra is thus \mathbb{C} . Whereas the previous approach send a context to its spectrum, we now send it to its carrier set in favouring the topos $[\mathcal{C}(\mathcal{A}), \mathbf{Set}]$ of copresheaves on the poset $\mathcal{C}(\mathcal{A})$. As a copresheaf, the forgetful functor — *bohri-fied* uncommutative C^* -algebra [HeuLanSpi09c] — is a commutative C^* -algebra; in other words, as in \mathbf{Set} expressed diagrammatically, there exist various commutative diagrams (of objects of the topos) for the existence of an addition, a multiplication, a number zero et cetera on this (pre)sheaf. From the traditional work by Gelfand adapted constructively [Mul79; BanMul00a; BanMul00b; BanMul06], [Coq05b; CoqSpi05; CoqSpi08; CoqSpi09a; CoqSpi10], we know to give to every internal commutative unital C^* -algebra in any topos a spectral internal frame $\underline{\Omega\Sigma}$. Classically, we remain on the level of the topological spaces and the famous duality by Gelfand between the category of commutative and unital C^* -algebras on the one hand, and the compact Hausdorff spaces on the other,

$$\mathbf{uAbCStarAlg} \simeq (\mathbf{KHausSp})^{\text{op}}$$

sending a suitable C^* -algebra to its topological space of characters, sending a Hausdorff topological space to its algebra of functions. As a side note, the link between the spectral presheaf and the forgetful functor appears nicely; the spectral presheaf of Imperial is the spectrum of the internal commutative C^* -algebra of Nijmegen [Doe12a].

However, predicatively^①, the appropriate notion of topology (in a topos) is the topology via the locales rather than the one via the topological spaces. In first instance, a locale is to its associated frame what is a topological space to its topology. Actually, the locales and their frames are exactly the topological spaces when these former are spatial — the characteristic of these topological spaces being the sobriety. But many locales are not so — typically the real numbers — and when they are done via the geometric logic, many classical theorems on the topological spaces are recovered whereas whenever we try to use the classical logic with them, the results fail. Naturally, often, the usual concepts do not transcribe exactly in the constructive-predicate world and in effect must be adapted constructively which may be disconcerting at first glance. In any case, the constructive-impredicative and geometric logical reasonings differ on the notion of points. The traditional notion of global points as arrows $1 \longrightarrow X$ (in a topos) of an object X does not suffice in the sense that the locales do not have the necessary amount of points for distinguishing between two opens — the element of the

^① Predicative is synonymous of constructive to us; it means that we must not use any powerset — so the axiom of powerset in every topos suggests that a topos is already far too much accommodating to us [MaiVic10; vBer12] — and the logical separation scheme claiming that the collection of the elements making a (logical) formula true is always a set.

frame of the locale. When we take the topological space coming with every locale we find, far too often to be a good notion, the empty set. Fortunately, geometrically, all the locales have all their points, on the sole condition to extend the definition. A point of a locale X becomes thereby nothing more nor less than a general localic arrow $W \longrightarrow X$. Concretely, this is tantamount to the restriction to the geometric constructions on the points; so typically we must not use the power object of a topos. Even more concretely, the mathematics are conducted without the axiom of choice, without the excluded third, without the classical negation, without the universal quantifier over infinite set. All that remains is the existential quantifier and the arbitrary disjuncts of formulas. Manifestly, these two suffice. Moreover, as said previously, the geometric logic is becomingly the one to be preserved under the interesting arrows between the toposes. In consequence, the suitable duality to use becomes,

$$\mathbf{uAbCStarAlg} \simeq (\mathbf{KRegLoc})^{\text{op}} \simeq \mathbf{KRegFrm}$$

The internal state space $\underline{\Sigma}$ is precisely the Gelfand spectrum of the internal commutative C^* -algebra as the forgetful sheaf. Explicitly, we choose to move on from \mathbf{Set} as in classical mechanics, to the topos $\mathbf{Sh}(\mathcal{C}(\mathcal{A}), \mathbf{Alex}[\mathcal{C}(\mathcal{A})])$ of sheaves over the site that is the ideal completion of the poset with its Alexandrov topology on the poset $\mathcal{C}(\mathcal{A})$ of contexts. The subbasic opens are all the subsets closed upwardly for the inclusion and we can define the sheaf $\underline{\mathcal{A}}$ on the (sub)basic opens by,

$$\begin{aligned} \underline{\mathcal{A}} : \mathbf{Alex}[\mathcal{C}(\mathcal{A})] &\longrightarrow \mathbf{Set} \\ \uparrow C &\longmapsto C \\ \uparrow C \subseteq \uparrow D &\longmapsto D \subseteq C \end{aligned}$$

which turns out to possess an internal structure in this topos of a commutative C^* -algebra. Whereafter, to the internal commutative C^* -algebra $\underline{\mathcal{A}}$ is associated a frame $\underline{\Omega\Sigma}$ compact and regular, as the sheaf defined on the basic opens,

$$\begin{aligned} \underline{\Omega\Sigma} : \mathbf{Alex}[\mathcal{C}(\mathcal{A})] &\longrightarrow \mathbf{Set} \\ \uparrow C &\longmapsto \bigotimes_{D \in \uparrow C} \Omega(\Sigma_D) \\ \uparrow C \subseteq \uparrow D &\longmapsto \bigotimes_{E \in \uparrow D} \Omega(\Sigma_E) \longrightarrow \bigotimes_{F \in \uparrow C} \Omega(\Sigma_F) \\ &U \longmapsto U \cap \bigsqcup_{F \in \uparrow C} \Sigma_F \end{aligned}$$

where for a commutative sub C^* -algebra C of \mathcal{A} , the frame $\bigotimes_{D \in \uparrow C} \Omega(\Sigma_D)$ is the subspace topology of the disjoint union of the diverse Gelfand spectra Σ_D of the various commutative sub C^* -algebras D containing C which is induced by the topology on $\bigsqcup_{D \in \mathcal{C}(\mathcal{A})} \Sigma_D$ whose opens U are its subsets U such that,

- 1° $\forall C \in \mathcal{C}(\mathcal{A}), U \cap \Sigma_C \in \Omega(\Sigma_C)$
- 2° $\forall C \subseteq D \in \overline{\mathcal{C}(\mathcal{A})}, \forall s \in \Sigma_D, s|_C \in U \cap \Sigma_C \Rightarrow s \in U \cap \Sigma_D$

where the set Σ_C of pure algebraic states of the commutative sub C^* -algebra C has the weak star-topology. It is then proved [HeuLanSpi09b] that the quasi-states^① on the uncommutative C^* -algebra are in bijective correspondence with the analogue of the measures on the internal state space that is $\underline{\Omega}\Sigma$.

Once more, we externalize the construction to work over a locale. The mechanism consists in transforming a frame $\Omega(Z)$ in a topos $\text{Sh}(X)$ of sheaves over X , into a locale Z over the locale X in **Set**, via a continuous arrow $Z \longrightarrow X$. Becomingly, our locales here are spatial, whence their treatments as topologies and all that is required is a base frame $\Omega(\mathcal{B}) \doteq \Omega(\text{Idl}(\mathcal{C}(\mathcal{A})))$ for the contexts, a spectral frame $\Omega(\Sigma) \doteq \bigotimes_{D \in \uparrow C} \Omega(\Sigma_D) = \Omega\left(\prod_{D \in \mathcal{B}} \Sigma_D\right)$ and last but not least a frame morphism $f^* \doteq f^{-1}: \Omega(\mathcal{B}) \longrightarrow \Omega(\Sigma)$ from the frame of the base space to the frame of the spectrum. Without being simplistic, we privilege the concept of a frame morphism such as f^* which must truly mean a locale arrow $f: \Sigma \longrightarrow \mathcal{B}$. The external view matching the first one is now the localic bundle in **Set**,

$$\begin{array}{ccc} & & \Sigma \\ & \nearrow \sigma & \downarrow f \\ \mathcal{B} & \xrightarrow{\text{Id}} & \mathcal{B} \end{array}$$

such that a (necessarily continuous) cross-section σ associates, to each context C , an element of its spectrum Σ_C . Precisely, in analysing the literature [Cas+09],

- 1° the locale \mathcal{B} is the ideal completion of the poset $\mathcal{C}(\mathcal{A})$ (together with the Alexandrov topology of $\mathcal{C}(\mathcal{A})$)
- 2° the fibre of a context C is the set Σ_C of all its states with the weak-star topology; the frame is the distributive lattice of the projections of C
- 3° the locale arrow f is closed, the locale Σ is compact^②
- 4° the internal spectrum $\underline{\Omega}\Sigma$ is a frame compact, (completely) regular
- 5° the restrictions (on the fibres) are contravariant with respect to the refinements on the poset $\mathcal{C}(\mathcal{A})$
- 6° the Bell–Kochen–Specker theorem is equivalent to the lack of continuous cross sections of f

^① A quasi-state is only a map from \mathcal{A} to the complex numbers \mathbb{C} with the duty to be a state on the sub C^* -algebras and that it agree well on the decomposition of the (arbitrary) operators in their sum of selfadjoints.

^② The first approach then assigns to each context the locale having for points the Gelfand spectrum, but not directly the topology.

The important nuances from the spectral presheaf are analysed in [Wol10; Vák12; Wol13a; Wol13b]; typically on what can be done with the newly quantum logic, whether it has good logical connectors or not. The work in [Wol13a] explores also the temporal evolution of the fibration.

The technique in [Spi11] eases the definition of a site inside a topos; something desired since a becoming topos to consider is the one of sheaves over the spacetime regions where to is assigned, traditionally, a uncommutative C^* -algebra, [FraNabTso06, p. 198]. The axiom of microcausality corresponds to a condition on a descent by the local geometric surjections [Nui11; Nui12; Wol13a; WolHal13].

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