DIFFERENT PERSPECTIVES ON THE MIZOHATA-TAKEUCHI CONJECTURE

by

MICHELE FERRANTE

A thesis submitted to
The University of Birmingham

School of Mathematics
College of Engineering and Physical Sciences
The University of Birmingham
October 2024

Abstract

The Mizohata–Takeuchi conjecture states that one should be able to control the L^2 weighted norm of the Fourier extension operator associated with a measure μ supported on a smooth hypersurface in $\mathbb{R}^n$, with the L^∞ norm of the X-ray transform of the weight, i.e.

$$\int_{\mathbb{R}^n} |\widehat{gd\mu}|^2\, w \lesssim \|Xw\|_{L^\infty} \int |g|^2 d\mu,$$

where

$$\widehat{gd\mu}(x) = \int e^{ix\cdot\xi} g(\xi) d\mu(\xi)$$

and the X-ray transform is a function of lines defined as

$$Xw(\ell) = \int_\ell w.$$

We will discuss some possible generalization of the Mizohata–Takeuchi conjecture. In particular, we will see some results for measures not supported on smooth surfaces, we will discuss the possibility of introducing a general Mizohata–Takeuchi conjecture for measures satisfying only dimensionality hypotheses and we will face the Mizohata–Takeuchi problem in the setting of locally compact abelian group.

CONTENTS

INTRODUCTION

A classical problem in Harmonic Analysis is the study of the Fourier extension operator. Given a Radon measure μ, one defines the extension operator as

$$\widehat{gd\mu}(x) = \int e^{ix\cdot\xi} g(\xi) d\mu(\xi), \quad x \in \mathbb{R}^n.$$

A typical question concerns finding exponents $1 \leqslant p \leqslant \infty$ such that one has an L^p estimate for the operator of the form

$$\left\| \widehat{gd\mu} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{L^2(d\mu)} .$$

In particular, the restriction conjecture proposes necessary and sufficient conditions on the exponent p for the estimate to hold, when μ is the surface measure for the sphere. But another problem, formulated in the 80's, has risen recently a new wave of interest. That is the study of L^2 weighted estimates, instead of L^p ones, and, in particular, the so called Mizohata–Takeuchi conjecture. Like the restriction conjecture, this has been classically introduced for the Fourier extension operator for the sphere and it states that one can control the L^2 weighted norm of the operator with the L^∞ norm of the X-ray transform of the weight, i.e.

$$\int_{\mathbb{R}^n} |\widehat{gd\mu}|^2 \, w \lesssim \|Xw\|_{L^\infty} \int |g|^2 d\mu.$$

The X-ray transform of a function f is a function of lines in $\mathbb{R}^n$ defined as

$$Xf(\ell) = \int_\ell f,$$

where each line ℓ is equipped with induced Lebesgue measure. This is a classic object in Tomography, the study of a function through its integral along certain affine subspaces.

The weight w appearing in weighted estimates is usually assumed to be positive and at least locally integrable. Typically, we will need more regularity on w in order to have all our argument to make sense, for example in order to have $\|Xw\|_{L^\infty} < \infty$. We will not dwell much in this thesis on the technical hypotheses on the weight. Instead, we will assume w to have the required regularity for the results presented to be well posed. The reader can assume the w is positive, continuous and compactly supported, which will be more than enough for all our applications.

It still makes sense to formulate the Mizohata–Takeuchi conjecture if we consider any other compact smooth surface in place of the sphere. Our initial goal was to put ourselves in this same setting. What motivated our research and the content of this thesis was the discovery of a result, discussed in Chapter 4, that does not require the measure μ of the extension operator to be supported on a smooth surface, but just to respect some dimensional hypotheses, given in terms of boundedness of some energy integrals. We will typically ask that μ satisfies

$$\sup_{x \in \mathrm{spt}(\mu)} \int \frac{1}{|x-y|^{n-k-\epsilon}} d\mu(y) < \infty,$$

which in particular implies that μ must be supported on a set of Hausdorff dimension $n-k$. This led us to explore the possibility of expanding the boundaries of the state of the art for the Mizohata–Takeuchi conjecture and try to understand to what extent the conjecture can be generalized.

In Chapter 1 and Chapter 2 we will introduce some useful preliminary notions, respectively, some objects relevant in Tomography, like the X-ray transform, and some Geometric Measure Theory, like the definition of dimension and rectifiability.

In Chapter 3 will introduce the Fourier extension operator and the Mizohata–Takeuchi conjecture.

In Chapter 4 and Chapter 5 we will prove some positive results under only some dimensional hypotheses on the measure μ. In particular, we will use an idea presented in [12] to prove some variants of the Mizohata–Takeuchi estimate that involve some Sobolev norms and then we will study the case of radial weights.

In Chapter 6 we will show that a naive generalization of the Mizohata–Takeuchi conjecture, with only dimensionality hypotheses cannot hold, even if we consider some low regularity of the measure like rectifiability.

In Chapter 7 we turn back our attention to smooth surfaces, to prove a result for a particular class of weights, inspired by the ones that appear in the counterexamples constructed in Chapter 6.

In Chapter 8 we discuss the connection between two Mizohata–Takeuchi problems, the one on the sphere and the one on the paraboloid, which suggest that each measure needs its own tomographic transform in order to generalize the Mizohata–Takeuchi conjecture.

Finally, in Chapter 9 we introduce the Mizohata–Takeuchi problem in the setting of locally compact abelian groups and prove a result in the style of the ones in Chapter 4 for measures on the integer lattice $\mathbb{Z}^2$.

Notation

Given two non-negative quantities A and B, we will write $A \lesssim B$ if there is a constant $C > 0$ such that $A \leqslant CB$.

Similarly, we will write $A \gtrsim B$ if there is a constant $C > 0$ such that $A \geqslant CB$.

We will write $A \cong B$ if there is a constant $C > 0$ such that $A = CB$.

Moreover, we will write $A \sim B$ if both $A \lesssim B$ and $A \gtrsim B$ hold at the same time (observe that $A \cong B$ is not the same as $A \sim B$, since the two constant for which $A \lesssim B$ and $A \gtrsim B$ hold need not to be the same).

We denote by χ_A the characteristic function of a set A. In particular, $\chi_A(x) = 1$ if $x \in A$ or $\chi_A(x) = 0$ otherwise.

For a function f, we denote $\tilde{f}(x) = \overline{f(-x)}$.

CHAPTER 1

TOMOGRAPHY AND THE k-PLANE TRANSFORM

Tomography, from the Greek $\tau\acute{o}\mu o\varsigma$, meaning slice, is the art of recovering information of an object from its slices. Everyone is probably familiar with this term for its application in medical imaging. In mathematical terms, our object is a density function f, which we want to recover from its "slices", i.e. its integral along certain affine subspaces of the ambient space. In particular, in this chapter we will introduce the k-plane transform, which will play a major role in this thesis.

1.1 The k-plane transform on $\mathbb{R}^n$

For this section, we will mainly rely on Markoe's book "Analytic Tomography" ([35]).

Definition. Let $\mathcal{G}_{k,n}$ be the Grassmanian manifold of all k-dimensional subspaces in $\mathbb{R}^n$, and $M_{k,n}$ the manifold of all affine k-planes. Parametrize $M_{k,n}$ by (Ω, v), where $\Omega \in \mathcal{G}_{k,n}$ and $v \in \Omega^{\perp}$. For every $\Omega \in \mathcal{G}_{k,n}$, consider the induced Lebesgue measure on Ω, $d\lambda_\Omega$. Define the k-plane transform as

$$T_{k,n}f(\Omega, v) = \int_\Omega f(x+v)\,d\lambda_\Omega(x).$$

When there is no ambiguity, we will just write dx in place of $d\lambda_\Omega(x)$.

For $k = n-1$, the transform $T_{n-1,n}$ is usually referred to as Radon transform, which is the first object appearing in Tomography, introduced by by Johann Radon in 1917 ([45]). More in general, one refers to any operator that integrates a function on certain submanifolds of the ambient space as a *Radon-like transform*.

For $k = 1$, the transform $T_{1,n}$ is called the *X-ray transform*. For the X-ray transform we will use the notation

$$Xf(\omega, v) = \int_{\mathbb{R}} f(v + t\omega)dt,$$

where $\omega \in \mathbb{S}^{n-1}$ indicates the direction of the line we are integrating on and $v \in \langle\omega\rangle^{\perp}$. So we are essentially identifying $\mathcal{G}_{1,n}$ with $\mathbb{S}^{n-1}$. Observe that this is not a 1-to-1 correspondence, since for each $\omega \in \mathbb{S}^{n-1}$, both ω and $-\omega$ will describe the same line.

Similarly, if we consider an orthonormal basis of $\mathbb{R}^n$, $(\omega_1, ..., \omega_n)$, so that $(\omega_1, ..., \omega_k)$ spans $\Omega \in \mathcal{G}_{k,n}$, we can identify $\mathcal{G}_{k,n}$ with the set of k-tuples of unit vectors $(\omega_1, ..., \omega_k)$, where $\omega_1 \in \mathbb{S}^{n-1}$, $\omega_2 \in \mathbb{S}^{n-1} \cap \langle\omega_1\rangle^{\perp} \cong \mathbb{S}^{n-2}$ and so on. In particular we can think of $\mathcal{G}_{k,n}$ as the product of the spheres

$$\mathbb{S}^{n-1} \times \mathbb{S}^{n-2} \times ... \times \mathbb{S}^{n-k-1}.$$

Again, this is not a 1-to-1 correspondence, since we are counting each $\Omega \in \mathcal{G}_{k,n}$ multiple times.

This identification will be particularly useful when we will have to do explicit computations with the natural measure on the Grassmanian manifold.

On $\mathcal{G}_{k,n}$ we consider the unique, up to multiplicative constant, measure invariant under the action of $SO(n)$. Therefore, we can think of the measure on $\mathcal{G}_{k,n}$ as the product of measures on the spheres

$$d\sigma_{n-1}(\omega_1)d\sigma_{n-2}(\omega_2)\cdots d\sigma_{n-k-1}(\omega_k).$$

The fact that this identification is not a 1-to-1 correspondence will only account for a multiplicative constant depending only on n and k.
Alternatively, and equivalently, one can identify $\mathcal{G}_{k,n}$ with the group $O(n)/O(k) \times O(n-k)$ and define the measure as the Haar measure on the group.

Perhaps the most interesting properties of the k-plane transform are the behaviour of its Fourier transform and the fact that it can be used to establish an isometry on L^2.

Proposition 1.1.1. *Fix $\Omega \in \mathcal{G}_{k,n}$ and consider the Fourier transform of $T_{k,n}f(\Omega, v)$ in the variable v. We have that, for f integrable,*

$$\mathcal{F}_v(T_{k,n}f)(\Omega, \xi) = \widehat{f}(\xi), \quad \xi \in \Omega^{\perp}. \tag{1.1}$$

Proof. Since f is integrable, using Fubini's theorem, a simple computation shows that

$$\begin{aligned}\mathcal{F}_v(T_{k,n}f)(\Omega,\xi) &= \int_{\Omega^\perp} e^{iy\cdot\xi} T_{k,n}f(\Omega,y)d\lambda_{\Omega^\perp}(y) = \\ &= \int_{\Omega^\perp} e^{iy\cdot\xi} \int_{\Omega} f(x+y)d\lambda_\Omega(x)d\lambda_{\Omega^\perp}(y) = \\ &= \int_{\mathbb{R}^n} e^{iz\cdot\xi} f(z)dz = \widehat{f}(\xi),\end{aligned}$$

where $z = x + y$ and, since $\xi \in \Omega^\perp$ while $x \in \Omega$, $z \cdot \xi = y \cdot \xi$. □

Now that we know how the Fourier transform of $T_{k,n}f$ acts, we can define a class of pseudo-differential operators that can be applied to $T_{k,n}$.

Definition. Let $\alpha \in \mathbb{R}$ and fix $\Omega \in \mathcal{G}_{k,n}$. We define $(-\Delta_v)^{\frac{\alpha}{2}}$, the derivative operator of order α acting on functions of variable $v \in \Omega^\perp$, as the Fourier multiplier operator of symbol $|\xi|^\alpha$, i.e.

$$\mathcal{F}_v((-\Delta_v)^{\frac{\alpha}{2}} h)(\xi) = |\xi|^\alpha \widehat{h}(\xi), \quad \xi \in \Omega^\perp. \tag{1.2}$$

In particular, the composition of $(-\Delta_v)^{\frac{\alpha}{2}}$ and the k-plane transform will be defined as

$$\mathcal{F}_v((-\Delta_v)^{\frac{\alpha}{2}} T_{k,n}f)(\Omega,\xi) = |\xi|^\alpha \widehat{f}(\xi), \quad \xi \in \Omega^\perp. \tag{1.3}$$

These operators play a major role in the tomography theory because of the following result.

Proposition 1.1.2. *There is a constant $c_{k,n}$ such that $c_{k,n}(-\Delta_v)^{k/4}T_{k,n}$ is an isometry between $L^2(\mathbb{R}^n)$ and $L^2(\mathcal{G}_{k,n}, L^2(\Omega^\perp))$ or, equivalently,*

$$c_{k,n}^2 \langle (-\Delta_v)^{k/2} T_{k,n}f,\ T_{k,n}g\rangle = \langle f, g\rangle. \tag{1.4}$$

Proposition 1.1.2 follows directly from the following result, that we can refer to as *generalized polar coordinates*, in the sense that, for $k = 1$, it corresponds to the usual polar coordinates for integration on $\mathbb{R}^n$.

Theorem 1.1.3 (Generalized polar coordinate)**.** *For f non-negative measurable or integrable, we have*

$$\int_{\mathcal{G}_{k,n}} \int_{\Omega} |y|^{n-k} f(y) d\lambda_\Omega(y) d\Omega = |\mathcal{G}_{k-1,n-1}| \int_{\mathbb{R}^n} f(x)dx \tag{1.5}$$

and

$$\int_{\mathcal{G}_{k,n}} \int_{\Omega^\perp} |y|^k f(y) d\lambda_{\Omega^\perp}(y) d\Omega = |\mathcal{G}_{k-1,n-1}| \int_{\mathbb{R}^n} f(x) dx, \tag{1.6}$$

where $|\mathcal{G}_{k,n}|$ *is the total mass of the Grassmanian.*

Moreover,

$$\int_{\mathcal{G}_{k,n}} \int_{\Omega \cap \mathbb{S}^{n-1}} f(\omega) d\sigma(\omega) d\Omega = |\mathcal{G}_{k-1,n-1}| \int_{\mathbb{S}^{n-1}} f(\omega) d\sigma(\omega). \tag{1.7}$$

Proof. We start with proving (1.7).

Consider the linear functional defined on $C(\mathbb{S}^{n-1})$ by

$$\lambda(f) = \frac{1}{|\mathcal{G}_{k-1,n-1}|} \int_{\mathcal{G}_{k,n}} \int_{\Omega \cap \mathbb{S}^{n-1}} f(\omega) d\sigma(\omega) d\Omega.$$

Letting A be an orthogonal matrix and $f_A(\omega) = f(A\omega)$, if we consider the change of variable $\tau = A\omega$, we have

$$\begin{aligned}\lambda(f_A) &= \frac{1}{|\mathcal{G}_{k-1,n-1}|} \int_{\mathcal{G}_{k,n}} \int_{\Omega \cap \mathbb{S}^{n-1}} f(A\omega) d\sigma(\omega) d\Omega = \frac{1}{|\mathcal{G}_{k-1,n-1}|} \int_{\mathcal{G}_{k,n}} \int_{A\Omega \cap \mathbb{S}^{n-1}} f(\tau) d\sigma(\tau) d\Omega = \\ &= \frac{1}{|\mathcal{G}_{k-1,n-1}|} \int_{\mathcal{G}_{k,n}} \int_{\Omega \cap \mathbb{S}^{n-1}} f(\tau) d\sigma(\tau) d\Omega = \lambda(f),\end{aligned}$$

where we used the invariance of the measure on $\mathcal{G}_{k,n}$. It follows that the measure associated with the functional λ by the Riesz representation theorem is invariant under the group action on $\mathbb{S}^{n-1}$. Therefore, it must be the Haar measure on $\mathbb{S}^{n-1}$ and so there is $c > 0$ such that $\lambda(f) = c \int_{\mathbb{S}^{n-1}} f(\omega) d\sigma(\omega)$. By choosing $f \equiv 1$, we find that $c = 1$, proving (1.7).

Now, fix $\Omega \in \mathcal{G}_{k,n}$, and change to the k-dimensional polar coordinates on Ω in the following integral, by keeping in mind that $\mathbb{S}^{k-1} = \Omega \cap \mathbb{S}^{n-1}$:

$$\int_\Omega |y|^{n-k} f(y) dy = \int_{\mathbb{S}^{k-1}} \int_0^\infty r^{n-k} f(r\omega) r^{k-1} dr d\sigma(\omega) = \int_0^\infty r^{n-1} \left(\int_{\Omega \cap \mathbb{S}^{n-1}} f(r\omega) d\sigma(\omega) \right) dr,$$

where we used Fubini's theorem in the last identity. Integrating over $\mathcal{G}_{k,n}$, using Fubini's theorem and (1.7), we have

$$\begin{aligned}\int_{\mathcal{G}_{k,n}} \int_\Omega |y|^{n-k} f(y) dy d\Omega &= \int_0^\infty r^{n-1} \left(\int_{\mathcal{G}_{k,n}} \int_{\Omega \cap \mathbb{S}^{n-1}} f(r\omega) d\sigma(\omega) d\Omega \right) dr = \\ &= |\mathcal{G}_{k-1,n-1}| \int_0^\infty \int_{\mathbb{S}^{n-1}} f(r\omega) r^{n-1} d\sigma(\omega) dr = |\mathcal{G}_{k-1,n-1}| \int_{\mathbb{R}^n} f(x) dx.\end{aligned}$$

So we have proven (1.5). To prove (1.6), one just has to apply (1.5) with $n-k$ in place of k and use the fact that $\Omega \mapsto \Omega^{\perp}$ is a measure preserving homeomeorphism between $\mathcal{G}_{k,n}$ and $\mathcal{G}_{n-k,n}$ (see [35], Proposition 3.8). $\square$

Proof of Proposition 1.1.2. Since $f, g \in L^2$, then $\widehat{f}\widehat{\widetilde{g}} \in L^1$. Therefore, applying Plancherel's Theorem to (1.4), we obtain the proven identity (1.6), where the integrable function is $\widehat{f}\widehat{\widetilde{g}}$. $\square$

Remark. As Proposition (1.1.2) suggests, is is possible to deduce pointwise identities of the form

$$f = c_{k,n} T^*_{k,n} (-\Delta_v)^{k/2} T_{k,n} f, \tag{1.8}$$

under suitable hypotheses on f. Identities like (1.8) are commonly referred to as *inversion formulae.* The curious reader might look at [35] for some examples. In this thesis we will not use any inversion formulae, but just Proposition 1.1.2.

1.2 An analogue of the k-plane transform on groups

In this thesis we will work mainly on the Euclidean space $\mathbb{R}^n$. Nonetheless, a more general setting where Harmonic Analysis is developed is that of locally compact abelian (LCA) groups, of which $\mathbb{R}^n$ is an example. Therefore we can try to define an analogue of the k-plane Transform on a generic LCA group.

We recall some of the classical results of Fourier analysis on groups, which can be found in Rudin's book "Fourier Analysis on Groups" ([46]).

An LCA group is an abelian topological group whose topology is locally compact, i.e. every point has a compact neighbourhood, and is Hausdorff, i.e. for every pair of different points there are two disjoint neighbourhoods of each respective point. On every LCA group G one can define a unique, up to multiplicative constant, Radon measure μ_G that is translation invariant, i.e.

$$\mu_G(E) = \mu_G(gE), \quad \forall g \in G,$$

for all Borel set $E \subset G$, where $gE = \{ge |\ e \in E\}$. The measure μ_G is called the *Haar measure* of G.

A continuous homomorphism from G to the circle group $U(1)$ is called a *character* and set of characters is called the *Pontryagin dual* of G, referred to as $\widehat{G}$. If G is an LCA group, so is $\widehat{G}$ and the Pontryagin dual of $\widehat{G}$ is G. This duality works well with the direct product structure in the sense that, if G_1 and G_2 are two LCA groups, then the dual of their direct product $G_1 \times G_2$, which is still a LCA group when equipped with the product topology, is the product of the duals $\widehat{G_1} \times \widehat{G_2}$. In particular the characters in $\widehat{G_1} \times \widehat{G_2}$ are the given by the product in $U(1)$ of the characters in $\widehat{G_1}$ and $\widehat{G_2}$.

Given $f \in L^1(G)$, we can define the Fourier transform of f as

$$\widehat{f}(\xi) = \int \xi^{-1}(g) f(g) d\mu_G(g), \quad \xi \in \widehat{G}, \tag{1.9}$$

which is a complex valued function on $\widehat{G}$. Under suitable hypothesis on f, one can prove the inversion formula

$$f(g) = \int g(\xi)\widehat{f}(\xi) d\mu_{\widehat{G}}(\xi). \tag{1.10}$$

Plancherel's Theorem says that the Fourier transform can be extended to an isometry $\mathcal{F}_G : L^2(G) \to L^2(\widehat{G})$.

We can also define the convolution as

$$f_1 * f_2(g) = \int_G f_1(h) f_2(gh^{-1}) d\mu_G(h), \quad g \in G,$$

and we still have the property that the Fourier transform of the convolution is the product of the Fourier transforms.

If H is a closed subgroup of G, then both H and G/H are LCA groups (with an abuse of notation, we will write g in place of gH to indicate elements of G/H). One can normalize the Haar measures so that we have

$$\int f d\mu_G = \iint f(gh) d\mu_H(h) d\mu_{G/H}(g). \tag{1.11}$$

We can define the *annihilator* of H as

$$N = N(H) := \{\xi \in \widehat{G} | \ \xi(h) = 1 \ \forall h \in H\},$$

which is a closed subgroup of $\widehat{G}$.

Theorem 1.2.1. *N and $\widehat{G}/N$ are (isomorphically homeomorphic to) the dual groups of G/H and H, respectively.*

Now, we can see a k-plane $\Omega \in \mathcal{G}_{k,n}$ as a closed subgroup of $\mathbb{R}^n$ and taking the affine k-plane $\Omega + v$, with $v \in \Omega^{\perp}$, is like considering a coset of Ω.

Definition. If G is an LCA group, we can define, for each closed subgroup H of G, the following Radon-like transform of f:

$$Tf(H, g) = \int f(gh) d\mu_H(h), \quad g \in G/H, \tag{1.12}$$

Remark. One could arrive to this definition by following the framework introduced by Helgason. The broad idea is to work on space X, paired with a set Ξ, which represents the set of manifolds on which we want to integrate. On both X and Ξ there is an action of a locally compact group G, which is transitive. Both X and Ξ can be equipped with a measure that is invariant under the action of G. Fixing $x_0 \in X$, one can check that $X \cong G/K$, where K is the stabilizer of x_0, under the isomorphism $gK \mapsto g(x_0)$. Similarly, $\Xi \cong G/H$, where H is the stabilizer of a fixed $\xi_0 \in \Xi$. We say that $x \in X$ is *incident* to $\xi \in \Xi$ if they intersect as cosets in G. The set $\widehat{\xi} = \{x \text{ incident to } \xi\}$ naturally carries an invariant measure. Then one defines the Radon transform of a function f on X as

$$Rf(\xi) = \int_{\widehat{\xi}} f(x) dx.$$

In the case of the case of $T_{k,n}$, the group G is the group of affine isometries on $\mathbb{R}^n$, $X = \mathbb{R}^n$ and $\Xi = M_{k,n}$. In our case of LCA groups, with a slight abuse of notation, $X = G$, Ξ is the collection of cosets of subgroups of G and the acting group is $G \rtimes \mathrm{Aut}_{\mu_G}$, where Aut_{μ_G} is the group of automorphisms of G that preserve the Haar measure μ_G. The action on X and Ξ are given respectively by

$$(g,\phi)(x) = g\phi(x) \ \text{ and } \ (g,\phi)(xH) = g\phi(xH).$$

This construction, however more flexible and valid in much more general settings, is, in our opinion, more complex and less intuitive for the purpose of this thesis then the one we presented, so we will not use it. We refer the reader to Helgason's book [32] for further details.

With this definition, we can prove the analogue of Proposition 1.1.1.

Proposition 1.2.2.

$$\mathcal{F}_{G/H}(Tf(H,\cdot))(\xi) = \widehat{f}(\xi), \quad \xi \in N(H). \tag{1.13}$$

Proof. Since $\xi \in N(H)$, we have $\xi(gh) = \xi(g)$ for all $h \in H$. Therefore,

$$\begin{aligned}
\mathcal{F}_{G/H}(Tf)(H,\xi) &= \int \xi(g) \int f(gh) d\mu_H(h) d\mu_{G/H}(g) = \\
&= \iint \xi(gh) f(gh) d\mu_H(h) d\mu_{G/H}(g) = \\
&= \int \xi(a) f(a) d\mu_G(a) = \widehat{f}(\xi).
\end{aligned}$$

□

It is not clear if one can prove an analogue of Proposition 1.1.2 on this generality. In Chapter 9 will will put ourselves in a more specific setting where we are able to get close to an analogue of Proposition 1.1.2.

Remark. In the context of the Helgason–Radon transform, there is a wide literature on finding isometry properties and inversion formulae. The interested reader can look for example at [4], [5]. A suggestion we can recover from the Helgason construction is that, if we look for an isometry property, we might want to limit ourselves to look at cosets of subgroups that are all images of a base subgroup H_0 via automorphisms. Therefore they will all have similar structure. For example on $\mathbb{R}^n$ we only look at subgroups of a fixed dimension k.

CHAPTER 2

SOME NOTIONS OF GEOMETRIC MEASURE THEORY

In this chapter we will recall some notions of measure theory and geometric measure theory that will be helpful in this thesis. We will mainly refer to the books of Pertti Mattila ([39], [40], [41]).

2.1 Measures and dimensions

In this thesis we will always be working with Radon measures.

Definition. A *Borel measure* μ on $\mathbb{R}^n$ is a non-negative measure defined on the σ-algebra of Borel sets which is Borel regular, in the sense that, for any subset $A \subset \mathbb{R}^n$, there exist a Borel set $B \subset \mathbb{R}^n$ such that $A \subset B$ and $\mu(A) = \mu(B)$. A *Radon measure* is a Borel measure that is *locally finite* , i.e. every compact set has finite measure.

A Borel measure is uniquely determined by its values on Borel sets, which in particular implies that it is uniquely determined by the integrals of continuous functions with compact support.

Remark. The same definitions for Borel and Radon measures can be extended to any locally compact Hausdorff topological space. For example, the Haar measure of a LCA group G is a Radon measure.

Definition. The *support* of a measure μ is the smallest closed set F such that $\mu(\mathbb{R}^n \backslash F) = 0$ and it is denoted by $\text{spt}\mu$.

We will often be working with compactly supported measures. We refer to the set of measures with compact support contained in A as $\mathcal{M}(A)$.

We will now recall the notions of Hausdorff measure and dimension.

Definition. For a subset $A \in \mathbb{R}^n$, we define the *diameter of* A as

$$d(A) = \sup\{|x-y| \mid x, y \in A\}.$$

Definition. For $s \geqslant 0$, we define the *Hausdorff measure* $\mathcal{H}^s$ of a set A as

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where, for $0 < \delta \leqslant \infty$,

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_j \alpha(s) 2^{-s} d(E_j)^s | \ A \subset \bigcup_j E_j,\ d(E_j) < \delta\right\}.$$

Here, $\alpha(s)$ is a normalization constant that, for example, for $s = n$, $n \in \mathbb{N}$, is usually chosen to be the volume of the $n-$dimensional sphere. For simplicity, we will choose it so that $\alpha(s)2^{-s} = 1$, whenever s is not an integer.

Remark. One can prove that the Hausdorff measure is indeed a Borel measure (see [39], Chapter 4).

Definition. The Hausdorff dimension of $A \subset \mathbb{R}^n$ is

$$\dim A = \inf\{s | \ \mathcal{H}^s(A) = 0\} = \sup\{s | \ \mathcal{H}^s(A) = \infty\}.$$

One can prove that $\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s_\infty(A) = 0$ and, therefore,

$$\dim A = \inf\left\{s | \ \forall \epsilon > 0\ \exists \{E_j\}_j,\ E_j \subset \mathbb{R}^n,\ \text{s.t.}\ \sum_j d(E_j)^s < \epsilon,\ A \subset \bigcup_j E_j\right\}.$$

The following definition is connected to the one of Hausdorff measure.

Definition. The *upper s-density* of a set $A \subset \mathbb{R}^n$ at a point x is

$$\Theta^{*s}(A, x) = \limsup_{r \to 0} \alpha(s) r^{-s} \mathcal{H}^s(A \cap B_r(x)).$$

The *lower s-density* of a set $A \subset \mathbb{R}^n$ at a point x is

$$\Theta_*^s(A, x) = \liminf_{r \to 0} \alpha(s) r^{-s} \mathcal{H}^s(A \cap B_r(x)).$$

If the values of upper and lower density coincide, we call their common value the *s-density* $\Theta^s(A, x)$. Similarly, the *upper s-density* of a measure μ on $\mathbb{R}^n$ at a point x is

$$\Theta^{*s}(\mu, x) = \limsup_{r \to 0} \alpha(s) r^{-s} \mu(B_r(x)).$$

The *lower s-density* of a measure μ on $\mathbb{R}^n$ at a point x is

$$\Theta_*^s(\mu, x) = \liminf_{r \to 0} \alpha(s) r^{-s} \mu(B_r(x)).$$

If the values of upper and lower density coincide, we call their common value the *s-density* $\Theta^s(\mu, x)$.

Theorem 2.1.1. *Let μ be a Radon measure on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ and $0 < \lambda < \infty$.*

1) *If $\Theta^{*s}(\mu, x) \leqslant \lambda$ for $x \in A$, $\mu(A) \leqslant 2^s \lambda \mathcal{H}^s(A)$.*

2) *If $\Theta^{*s}(\mu, x) \geqslant \lambda$ for $x \in A$, $\mu(A) \geqslant \lambda \mathcal{H}^s(A)$.*

A proof can be found in [39] (Chapter 6).

We now introduce another notion of dimension.

Definition. For $\delta > 0$, the *$\delta-$neighbourhood of A* is the set

$$A(\delta) = \{x \in \mathbb{R}^n |\ d(x, A) < \delta\}.$$

Definition. The *lower Minkowski dimension* of a bounded set $A \subset \mathbb{R}^n$ is

$$\underline{\dim}_M A = \inf\{s > 0 |\ \liminf_{\delta \to 0} \delta^{s-n} \mathcal{L}^n A(\delta) = 0\}$$

and the *upper Minkowski dimension* of a bounded set $A \subset \mathbb{R}^n$ is

$$\overline{\dim}_M A = \inf\{s > 0 | \limsup_{\delta \to 0} \delta^{s-n} \mathcal{L}^n A(\delta) = 0\},$$

where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$.

In general we have that

$$\dim A \leqslant \underline{\dim}_M A \leqslant \overline{\dim}_M A. \tag{2.1}$$

If $\underline{\dim}_M A = \overline{\dim}_M A$ we say that A has Minkowski dimension $\dim_M A = \overline{\dim}_M A$.

In general, for the product of two sets, one can only show that

$$\dim(A \times B) \geqslant \dim A + \dim B. \tag{2.2}$$

However, we have the following result (see [39], Chapter 8):

Theorem 2.1.2. *If A and B are Borel sets in $\mathbb{R}^n$ and $\dim A = \overline{\dim}_M A$, then*

$$\dim(A \times B) = \dim A + \dim B. \tag{2.3}$$

2.2 Energy-integrals

A key item in our thesis will be the energy integral of a measure.

Definition. Given a Radon measure μ and $\alpha > 0$, we define the α-energy integral of μ as

$$I_\alpha(\mu) = \iint \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y). \tag{2.4}$$

We also introduce the following variant of the energy integral.

Definition. For a measure μ and $\alpha \in \mathbb{R}$, we define the quantity

$$S_\alpha(\mu) = \sup_{x \in \text{spt}(\mu)} \int \frac{1}{|x-y|^\alpha} d\mu(y)$$

The term energy integral comes from the world of Physics, where these types of objects arise naturally when discussing the potential and the energy of certain distributions like μ.

Remark. Observe that if $\mu(\mathbb{R}^n) < \infty$, for example if μ is compactly supported,

$$I_\alpha(\mu) \leqslant \mu(\mathbb{R}^n) \cdot S_\alpha(\mu).$$

Requiring that for a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ we have $S_\alpha(\mu) < \infty$ is related to the following condition:

$$\mu(B_r(x)) \lesssim r^\alpha, \quad 0 < r < \infty, \tag{2.5}$$

uniformly in $x \in \mathbb{R}^n$.

In fact, if (2.5) holds for some $\alpha' > \alpha$, we have

$$\begin{aligned}\int \frac{1}{|x-y|^\alpha} d\mu(y) &= \int_0^\infty \mu(\{y|\ |x-y|^{-\alpha} \geqslant \lambda\})d\lambda = \int_0^\infty \mu(B(x, \lambda^{-1/\alpha}))d\lambda = \\ &= \alpha \int_0^\infty \mu(B(x,r))r^{-\alpha-1}dr \lesssim \int_0^R r^{-1-\epsilon} < \infty,\end{aligned}$$

where $\epsilon = \alpha' - \alpha$. in particular $S_\alpha(\mu)$, and therefore $I_\alpha(\mu)$, are finite.

Vice versa, if $S_\alpha(\mu) < \infty$,

$$r^{-\alpha}\mu(B_r(x)) = \int_{B_r(x)} \frac{1}{|x-y|^\alpha} d\mu(y) \leqslant S_\alpha(\mu) < \infty.$$

We can use the energy integrals to introduce another notion of dimension.

Definition. The *capacitary dimension* of a set $A \subset \mathbb{R}^n$ is

$$\begin{aligned}\dim_c A &= \sup\{s|\ \exists \mu \in \mathcal{M}(A) \text{ s.t. } \mu(B_r(x)) \lesssim r^s,\ \forall x \in \mathbb{R}^n,\ r > 0\} = \\ &= \sup\{t|\ \exists \mu \in \mathcal{M}(A) \text{ s.t. } I_t(\mu) < \infty\}.\end{aligned}$$

The following result is useful to relate Hausdorff and capacitary dimensions.

Theorem 2.2.1 (Frostman's Lemma). *Let B be a Borel set. Then $\mathcal{H}^s(B) > 0$ if and only if there exists $\mu \in \mathcal{M}(B)$ such that $\mu(B_r(x)) \lesssim r^s,\ \forall x \in \mathbb{R}^n,\ r > 0$.*

In general, one only has that

$$\dim_c A \leqslant \dim A, \tag{2.6}$$

but, using Frostman's Lemma, one can prove that for Borel sets

$$\dim_c A = \dim A. \tag{2.7}$$

We end this section with the following definition.

Definition. We say that a measure μ is *α-Ahlfors-David regular*, or α-AD regular, if

$$\mu(B_r(x)) \sim r^\alpha,\ \forall x \in \mathbb{R}^n,\ r > 0.$$

2.3 The Fourier transforms of measures

We now want to introduce another notion of dimension connected to the Fourier transform.

First, for a finite Borel measure μ, one can define its Fourier transform as

$$\widehat{\mu}(\xi) = \int e^{ix\cdot\xi} d\mu(x).$$

One can prove the following characterization for energy integral (see [40], Section 3.5).

Proposition 2.3.1. *For $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $0 < s < n$,*

$$I_s(\mu) \cong \int |x|^{s-n} |\widehat{\mu}(x)|^2 dx. \tag{2.8}$$

It follows that, if $I_s(\mu) < \infty$ we must have that

$$|\widehat{\mu}(x)| \leqslant |x|^{-s/2}$$

for "most" $x \in \mathbb{R}^n$, in the sense that

$$\lim_{R\to\infty} \mathcal{L}^n(\{x \in B_R(0) |\ |\widehat{\mu}(x)| > |x|^{-s/2}\}) = 0.$$

Vice versa, if

$$|\widehat{\mu}(x)| \leqslant |x|^{-s/2}, \quad \forall x \in \mathbb{R}^n,$$

we have that $I_t(\mu) < \infty$, for all $t < s$.

This motivates the following definition.

Definition. The *Fourier dimension* of a set $A \subset \mathbb{R}^n$ is

$$\dim_f A = \sup\{s \leqslant n |\ \exists \mu \in \mathcal{M}(A) \text{ s.t. } |\widehat{\mu}(x)| \lesssim |x|^{-s/2},\ \forall x \in \mathbb{R}^n\}.$$

For a general Borel set we have

$$\dim_f A \leqslant \dim A. \tag{2.9}$$

If, for a Borel set A, we have

$$\dim_f A = \dim A, \tag{2.10}$$

we call A a *Salem set.*

We end this section by illustrating how the Gaussian curvature of a hypersurface determines its Fourier dimension. In particular, we will present a proof from Stein ([50]) that shows that a compact hypersurface with non-zero Gaussian curvature is a Salem set.

We will be using the following characterization (see Lee [33] for a proof).

Proposition 2.3.2. *Let $M \subset \mathbb{R}^{n+1}$ be an embedded hypersurface, $p \in M$ and $\kappa_1, ..., \kappa_n$ the principal curvatures at p. Then there is an isometry $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and a neighbourhood $U \subset \mathbb{R}^{n+1}$ of p such that $\varphi(U \cap M)$ is a graph of the form $x_{n+1} = f(x_1, ..., x_n)$, where*

$$f(x') = \frac{1}{2}\sum_{j=1}^{n} \kappa_j (x_j)^2 + O(|x'|^3).$$

Recall the two following classical results for an oscillatory integral

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)dx. \tag{2.11}$$

We refer to Stein ([50]) for the proofs.

Proposition 2.3.3. *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a smooth function with compact support and $\phi : \mathbb{R}^n \to \mathbb{R}$ a smooth function such that $\nabla\phi \neq 0$ in the support of ψ. Then*

$$I(\lambda) = O(\lambda^{-N}) \ \ \textit{as } \lambda \to \infty$$

for all $N \in \mathbb{N}$.

Proposition 2.3.4. *Let $x_0 \in \mathbb{R}^n$ such that $\phi(x_0) = 0$ and ϕ has a nondegenerate critical point in x_0. If ψ is supported in a small enough neighbourhood of x_0, then*

$$I(\lambda) = O(\lambda^{-n/2}).$$

Now we can prove the following:

Theorem 2.3.5. *Let S be a compact embedded hypersurface in $\mathbb{R}^{n+1}$ of nonzero Gaussian curvature and $d\mu$ a smooth measure on S. Then*

$$|\widehat{d\mu}(\xi)| \lesssim |\xi|^{-n/2}.$$

Proof. Since $d\mu$ has compact support, we just need to prove the result locally, around a point $p \in S$. By Proposition 2.3.2, we can assume S is the graph of a function f. Note that the transformation under which this assumption holds, is an isometry, so it does not change the integrals, since the module of the determinant of the Jacobian matrix for isometries is 1. In this coordinates, $d\sigma = \sqrt{1 + |\nabla f|^2} dx_1 ... dx_n$. Thus, we reduced ourselves to prove that

$$\left| \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,\eta)} \tilde{\psi}(x) dx \right| \lesssim \lambda^{-n/2}$$

where $\lambda = |\xi| > 0$, $\xi = \lambda\eta$, with $\eta \in \mathbb{S}^n$, and

$$\Phi(x, \eta) = \sum_{j=1}^{n} x_j \eta_j + f(x_1, ..., x_n)\eta_{n+1}.$$

We split the proof in three different cases:

1) η is in a small neighbourhood of $\eta_N = (0, ..., 0, 1)$;

2) η is in a small neighbourhood of $\eta_S = (0, ..., 0, -1)$;

3) η is far from both η_N and η_S.

In the case 1) we have $\nabla_x \Phi(x, \eta_N)\big|_{x=0} = 0$. By the Implicit Function Theorem, we have that, for each η in a neighbourhood of η_N, there is a unique $x = x(\eta)$, such that

$$\nabla_x \Phi(x, \eta)\big|_{x=x(\eta)} = 0.$$

In fact, by the hypothesis of nonzero Gaussian curvature,

$$\det\left[\frac{\partial^2 \Phi}{\partial x_j \partial x_k}\right](0, \eta_N) \neq 0.$$

Moreover, if the neighbourhood of η_N is small enough, we can assume, by continuity, that

$$\det\left[\frac{\partial^2 \Phi}{\partial x_j \partial x_k}\right](x(\eta), \eta) \neq 0.$$

Now we can invoke, for each fixed η, Proposition 2.3.4, with $x_0 = x(\eta)$.

The proof for the second case is the same.

Now we study case 3). By definition,

$$\nabla_x \Phi(x, \eta) = (\eta_1, ..., \eta_n) + \eta_{n+1} \nabla f(x).$$

But now, $\sqrt{\eta_1^2 + ... + \eta_n^2} \geqslant c > 0$. Together with the fact that $\nabla f(x) = O(|x|)$, as $x \to 0$, because $\partial_j^2 f \neq 0$, we can say that $|\nabla_x \Phi(x, \eta)| \geqslant \tilde{c} > 0$. Hence we can use Proposition 2.3.3. Thus the result is proved. $\square$

2.4 Rectifiability

We will now provide some background on another topic of Geometric Measure Theory, that is the notion of rectifiability.

Definition. A set $E \subset \mathbb{R}^n$ is called *m-rectifiable* if there are Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n,\ i \in \mathbb{N}$, such that

$$\mathcal{H}^m\left(E \backslash \left(\bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)\right)\right) = 0$$

A measure μ on $\mathbb{R}^n$ is called *m-rectifiable* if there are Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n,\ i \in \mathbb{N}$, such that

$$\mu\left(\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)\right)\right) = 0$$

Remark. Observe that a rectifiable measure is essentially supported on a rectifiable set. Moreover, working with an m-rectifiable set E, $0 < \mathcal{H}^m(E) < \infty$, is essentially equivalent to working with a rectifiable measure μ with almost everywhere positive and finite upper m-density, since, by Theorem 2.1.1, the measures μ and $\mathcal{H}^m\big|_E$ are mutually absolutely continuous.

For rectifiable sets we can give a notion of tangent space.

Definition. Define the cone

$$X(v, \Omega, s) = \{x \in \mathbb{R}^n |\ d(x, \Omega + v) < s|x - v|\}, \quad a \in \mathbb{R}^n,\ \Omega \in \mathcal{G}_{m,n},\ s > 0.$$

We say that $\Omega \in \mathcal{G}_{m,n}$ is an *approximate tangent plane* for a set $E \subset \mathbb{R}^n$ at a point x if $\Theta^{*m}(E, x) > 0$ and

$$\lim_{r \to 0} r^{-m} \mathcal{H}^m((E \cap B_r(x)) \setminus X(x, \Omega, s)) = 0, \quad \forall s > 0.$$

The notion of approximate tangent characterizes rectifiable sets. The following result is due to Mattila ([37]), who generalized an argument by Marstand ([36]) for two dimensional sets in $\mathbb{R}^3$.

Theorem 2.4.1. *If $E \subset \mathbb{R}^n$ is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(E) < \infty$, then E is m-rectifiable if and only if it has approximate tangent at $\mathcal{H}^m$ almost all of its points.*

Remark. There is an analogous notion of tangent measure to a measure that characterize rectifiable measures, but we will not present it, since we will not be using it.

To better understand the difference between approximate tangent and the classical notion of tangent, consider the following example, proposed by Mattila in [41].

Take an enumeration $\{q_j\}_{j \in \mathbb{N}}$ of the rational points in $\mathbb{R}^2$ and take

$$S_j = \{x \in \mathbb{R}^2 |\ |x - q_j| = 2^{-j}\}.$$

Then the set $E = \bigcup_j S_j$ is 1-rectifiable, since its the union of smooth curves, and

$$\mathcal{H}^1(E) \lesssim \sum_j 2^{-j} < \infty.$$

Now, each circle S_j has a tangent in the classical sense that is also an approximate tangent to E. But E does not have a tangent in the classical sense since it is dense in $\mathbb{R}^2$.

CHAPTER 3

THE EXTENSION OPERATOR AND THE MIZOHATA–TAKEUCHI CONJECTURE

3.1 The Fourier restriction and extension operators

If we consider a function $f \in L^1(\mathbb{R}^n)$, we know its Fourier transform $\widehat{f}$ is a continuous function over $\mathbb{R}^n$. So it makes sense to restrict $\widehat{f}$ to a a set of measure zero, like a submanifold of $\mathbb{R}^n$. In particular one has the estimate

$$\left\|\widehat{f}\right\|_{L^\infty(\mathbb{R}^n)} \leqslant \|f\|_{L^1(\mathbb{R}^n)}.$$

On the other hand, if f is just in $L^2(\mathbb{R}^n)$, then $\widehat{f}$ is again a generic L^2 function, so, in general, it does not make sense to restrict it. Therefore, one could ask if there exists a range of exponents $p,\ 1 < p < 2$, in which the restriction of the Fourier transform of a function in L^p to a set M of Lebesgue measure 0, is legitimate. In particular, given a measure μ supported in M, one can look for estimates of the form

$$\left\|\widehat{f}\Big|_M\right\|_{L^q(d\mu)} \leqslant C\,\|f\|_{L^p(\mathbb{R}^n)}.$$

This way, one can define a restriction in a canonical way on a dense subset of $L^1 \cap L^p$, and then extend it to a continuous operator on the whole of L^p.

If we call R_μ the restriction operator and consider its adjoint operator E_μ, called the extension operator, asking whether or not R_μ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(d\mu)$ is equivalent to asking if

E_μ is bounded from $L^{q'}(d\mu)$ to $L^{p'}(\mathbb{R}^n)$.

We can write the extension operator of a measure μ as

$$E_\mu g(x) = \widehat{gd\mu}(x) = \int e^{ix\cdot\xi} g(\xi) d\mu(\xi). \tag{3.1}$$

In fact,

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) g(\xi) d\mu(\xi) = \int_{\mathbb{R}^n} f(x) \widehat{gd\mu}(x) dx.$$

Let us focus for a moment on the case where μ is the surface measure of a hypersuface on $\mathbb{R}^n$, which is the most classical setting for this problem.

We want to look for necessary conditions for

$$\left\| \widehat{gd\mu} \right\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(d\mu)} \tag{3.2}$$

to hold.

The first observation is that we cannot have an extension estimate for an entirely flat surface, beyond the trivial $L^1 \to L^\infty$ bound. In fact, if we for example consider a function $g \in C_c^\infty(\mathbb{R}^n)$ and consider the extension operator for the plane $\{x_{n+1} = 0\}$ in $\mathbb{R}^{n+1}$, given by

$$Eg(x, x_{n+1}) = \int e^{i(x\cdot\xi + x_{n+1}\cdot 0)} g(\xi) d\xi = \int e^{ix\cdot\xi} g(\xi) d\xi,$$

we obtain a function constant in the variable x_{n+1}, and therefore with no hope of being in any L^p, for $p < +\infty$.

So, now consider a hypersurface M with non-zero gaussian curvature. Using Proposition 2.3.2, we can assume that, near a point of M, say the origin, M is approximately the graph of a quadratic function f and the normal vector to M at the origin is $N = (0, .., 0, 1)$. Consider a bump function φ supported in the ball $B_1(0)$. For a small $\delta > 0$, take $g(\xi) = \varphi(\xi/\delta)$ and consider the small region $D = M \cap B_\delta(0)$. Then, using the notation $x = (x_n, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$

and $\xi = (\xi_n, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$,

$$\widehat{gd\mu}(x) = \int_D e^{i[x_n f(\xi') + x' \cdot \xi']} \varphi(\xi/\delta) d\mu(\xi).$$

Now, consider, for a suitable small constant $c > 0$,

$$T_{c,\delta} = \{x \in \mathbb{R}^n | \; |x_n| \leqslant \left(\frac{c}{\delta}\right)^2, |x'| \leqslant \frac{c}{\delta}\},$$

which is a cylindrical tube which lies along the direction of the normal N, with base of radius c/δ and height $(c/\delta)^2$. If we take $x \in T_{c,\delta}$, then the phase $e^{ix\cdot\xi}$ has real part ~ 1. Hence, since $\mu(D) \sim \delta^{n-1}$,

$$\left|\widehat{gd\mu}(x)\right| \gtrsim \delta^{n-1}.$$

This is commonly referred to as Knapp's example and says that the extension operator applied to a small cap of a hypersurface with non-zero Gaussian curvature is essentially supported in a tube with central line parallel to the direction of the normal to the surface at the center of the cap.

Now, $T_{c,\delta}$ has measure $\sim \delta^{-(n+1)}$, so

$$\left\|\widehat{gd\mu}\right\|_{L^q(\mathbb{R}^n)} \gtrsim \delta^{n-1-(n+1)/q},$$

while

$$\|g\|_{L^p(d\mu)} \sim \delta^{(n-1)/p}.$$

In order to have, for $\delta \to 0$,

$$\delta^{n-1-(n+1)/q} \lesssim \delta^{(n-1)/p},$$

we need

$$q \leqslant \frac{n+1}{n-1} p'. \tag{3.3}$$

Recall that $|\widehat{\mu}(x)| = O(|x|^{-(n-1)/2})$. This is a sharp bound since, for example, for the sphere one can show

$$\widehat{\sigma}(x) \cong |x|^{-(n-1)/2} \cos(|x| - \pi/4) + O(|x|^{-(n+1)/2}).$$

Therefore we must have

$$q > \frac{2n}{n-1}, \tag{3.4}$$

for $\widehat{\mu}$ to be in L^q.

The restriction conjecture says that the conditions (3.3) and (3.4) are not only necessary, but also sufficient. This has been solved for curves in the plane, but it is still an open problem for higher dimensions. We refer the reader to [23] for more information on the restriction problem.

Nonetheless, we have the following result:

Theorem 3.1.1 (Tomas-Stein Theorem)**.** *Let M be a compact hypersurface in $\mathbb{R}^n$, whose Gaussian curvature is nowhere zero equipped with the standard surface measure μ. Then for all $q \geqslant 2(n+1)/(n-1)$*

$$\left\| \widehat{f d\mu} \right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^2(d\mu)} . \tag{3.5}$$

This result was published initially by Tomas for the sphere ([53]), excluding the endpoint case which was later proved by Stein, using a complex interpolation method (see [50]).

3.2 A generalization of the Tomas-Stein theorem

Mockenhaupt in [43] generalized the original Tomas argument and proved an analoguous result for a wider class of measures. The main idea is that the hypothesis on curvature, which is used to compute the Fourier dimension of the measure, is substituted by an a priori assumption on the Fourier dimension.

Theorem 3.2.1 (Mockenhaupt)**.** *Let μ be a compactly supported positive measure on $\mathbb{R}^n$ such that for some $\alpha, \beta \in (0, n)$ we have*

$$\mu(B(x,r)) \leqslant C_1 r^{\alpha} \quad \forall x \in \mathbb{R}^n \text{ and } r > 0, \tag{3.6}$$

$$|\widehat{\mu}(\xi)| \leqslant C_2 (1 + |\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^n. \tag{3.7}$$

Then for all $p > p_{n,\alpha,\beta} := 2(2n - 2\alpha + \beta)/\beta$, there is a $C = C(p) > 0$ such that

$$\left\| \widehat{g d\mu} \right\|_{L^p(\mathbb{R}^n)} \leqslant C \|g\|_{L^2(d\mu)} \tag{3.8}$$

for all $g \in L^2(d\mu)$.

Remark. The estimate at the endpoint $p = p_{n,\alpha,\beta}$ is also known to be true, thanks to a work of Bak and Seeger ([1]).

We present Mockenhaupt's proof of Theorem 3.2.1 (without covering the endpoint case) to show how the dimensional hypotheses on the measure are used.

Proof of Theorem 3.2.1. First, we restate the problem in term of the restriction operator. So, we want to prove that

$$\left\| \widehat{f} \right\|_{L^2(d\mu)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^n)} . \tag{3.9}$$

Using Hölder's inequality, we have

$$\left\| \widehat{f} \right\|_{L^2(d\mu)}^2 \leqslant \left\| f * \widehat{d\mu} \right\|_{L^p(\mathbb{R}^n)} \|f\|_{L^{p'}(\mathbb{R}^n)} .$$

Therefore the result follows if we prove that

$$\left\| f * \widehat{d\mu} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^n)} . \tag{3.10}$$

This is commonly referred to as the T^*T approach. In fact, if we want to prove that an operator T is bounded from L^p to L^2, it is enogh to prove that $T^*T : L^{p'} \to L^p$. In our case, T is the restriction operator and $T^*Tf = f * \widehat{d\mu}$.

Let $\varphi \in C^\infty$ be such that

$$\varphi(x) \begin{cases} = 1 \ \text{ if } |x| \geqslant 2 \\ \in (0,1) \ \text{ if } |x| \in (1,2) \\ = 0 \ \text{ if } |x| \leqslant 1 \end{cases} ,$$

and take $\phi(x) = \varphi(2x) - \varphi(x)$. Let $\phi_k(x) = \phi(2^{-k}x)$, so that each ϕ_k is supported in the annulus $\{2^{-(k-1)} \leqslant |x| \leqslant 2^{-k}\}$ for $k \in \mathbb{N}$. Let $\phi_0 = 1 - \sum_{k=1}^{\infty} \phi_k$, so we have

$$\sum_{k=0}^{\infty} \phi_k = 1,$$

with ϕ_0 supported in $B_1(0)$.

Now we decompose our operator according to this decomposition. So, let

$$f * \widehat{d\mu} = \sum_{k=0}^{\infty} T_k f,$$

where $T_k f = f * (\phi_k \widehat{d\mu})$.

Using hypothesis (3.7) we have that

$$\|T_k\|_{L^1 \to L^\infty} \leqslant \left\| \phi_k \widehat{d\mu} \right\|_\infty \leqslant C 2^{-k\frac{\beta}{2}}.$$

At the same time, by Plancherel's Theorem and using hypothesis (3.6) we have

$$\begin{aligned}
\|T_k\|_{L^2 \to L^2} &\leqslant C \left\| \widehat{\phi_k} * d\mu \right\|_\infty \leqslant C \sup_x \left| \int \widehat{\phi_k}(x-y) d\mu(y) \right| \leqslant \\
&\leqslant C 2^{kn} \sup_x \int \frac{1}{(1+2^k|x-y|)^N} d\mu(y) = \\
&= C' 2^{kn} \sup_x \int_0^\infty \mu(B_{2^{-k}r}(x))(1+r)^{-N-1} dr \leqslant \\
&\leqslant C'' 2^{kn} \sup_x \int_0^\infty r^\alpha (1+r)^{-N-1} dr \leqslant \tilde{C} 2^{k(n-\alpha)}.
\end{aligned}$$

Interpolating the two estimates we obtain that $\|T_k\|_{L^{p'} \to L^p} \leqslant C 2^{k\left(\frac{2n-2\alpha+\beta}{p} - \frac{\beta}{2}\right)}$.

Therefore $\|T\|_{L^{p'} \to L^p} \leqslant C$ if $p > 2(2n-2\alpha+\beta)/\beta$. $\square$

The sufficient condition $p \geqslant p_{n,\alpha,\beta}$ ($p < p_{n,\alpha,\beta}$ shown in Theorem 3.2.1 and $p = p_{n,\alpha,\beta}$ for the endpoint of Bak and Seeger) is also necessary in dimension $n = 1$.

Theorem 3.2.2. *Given $0 < \beta \leqslant \alpha < 1$, there exist a Borel probability measure ν on $\mathbb{R}$ supported on a compact set E and a sequence of characteristic functions (of finite union of intervals) $\{f_\ell\}_{\ell \geqslant 1}$ with $\|f_\ell\|_{L^2(d\nu)} > 0$ such that:*

$$|\widehat{\nu}(\xi)| \lesssim (1+|\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^n;$$

$$\sup_{\ell \geqslant 1} \frac{\left\| \widehat{f_\ell d\nu} \right\|_{L^q(\mathbb{R}^n)}}{\|f_\ell\|_{L^2(d\nu)}} = +\infty,$$

for all $2 \leqslant q < p_{1,\alpha,\beta}$;

Moreover,

$$|I|^{\alpha+\delta} \lesssim \nu(I) \lesssim |I|^{\alpha}$$

for all $\delta > 0$ and all intervals I centred in E with $|I| < 1/2$.

In particular, E is of Hausdorff and Minkowski dimension α.

This result was proved by Chen in [22], improving on a result of Hambrook and Łaba ([30]). The proof is rather complex, but the main idea is to construct the Salem set E via a randomized Cantor iteration in such a way that, despite the randomness, it contains smaller sets that come close to being arithmetically structured. In particular, E will contain subsets $E \cap F_\ell$, where F_ℓ is a finite iteration of a smaller Cantor set with endpoints in a generalized arithmetic progression. The functions f_ℓ will be characteristic functions of F_ℓ.

3.3 The Mizohata–Takeuchi conjecture

We now put aside L^p estimates for the extension operation and we look at weighted estimates instead.

Consider the extension operator for the surface measure of the sphere σ, which is the classical setting for this problem. We will look for some L^2-weighted estimates of the form

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w \lesssim Q(w) \int_{\mathbb{S}^{n-1}} |g|^2 d\sigma, \tag{3.11}$$

where $Q(w)$ will be a certain quantity depending on the weight w.

To understand what $Q(w)$ should be, consider again the Knapp example: take g to be the characteristic function of a small δ-cap on $\mathbb{S}^{n-1}$. We then know that $\left|\widehat{gd\sigma}(x)\right| \gtrsim \delta^{n-1}$ on a tube T with axis along the normal to $\mathbb{S}^{n-1}$ at the cap's centre. Therefore,

$$\delta^{2(n-1)} \int_T w \lesssim \int |\widehat{gd\sigma}|^2 w.$$

Since $\int |g|^2 d\mu \sim \delta^{n-1}$, the inequality (3.11) would imply that

$$\delta^{n-1} \int_T w \lesssim Q(w),$$

which constitutes our clue on how we should choose $Q(w)$.

Now, if that the axis of T is the line of direction ω, since T has a section of measure δ^{n-1}, then we can say that

$$\int_T w \leqslant \delta^{-(n-1)} \sup_{v \perp \omega} Xw(\omega, v)$$

and therefore, to ensure the inequality to hold, we can choose $Q(w)$ such that

$$\sup_{v \perp \omega} Xw(\omega, v) \lesssim Q(w).$$

Since we could argue with any tube along a normal direction to $\mathbb{S}^{n-1}$, a good candidate is to take

$$Q(w) = \sup_{\omega} \|Xw(\omega, \cdot)\|_{L^\infty} .$$

Moreover, call σ_R the measure on the sphere of radius R, we have

$$|\widehat{gd\sigma_R}|^2(x) = \int e^{ix\cdot\xi} g(\xi) d\sigma_R(\xi) = R^{n-1} \int e^{iRx\cdot\xi} g(R\xi) d\sigma(\xi) = R^{n-1} |\widehat{g_R d\sigma}|^2(Rx),$$

where $g_R(\xi) = g(R\xi)$. So,

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{gd\sigma_R}|^2(x) w(x) dx = R^{2(n-1)} \int_{\mathbb{R}^n} |\widehat{g_R d\sigma}|^2(Rx) w(x) dx = \\ = R^{n-2} \int_{\mathbb{R}^n} |\widehat{g_R d\sigma}|^2(y) w_{1/R}(y) dy \lesssim \\ \lesssim R^{n-2} Q(w_{1/R}) \int_{\mathbb{S}^{n-1}} |g_R|^2 d\sigma = R^{-1} Q(w_{1/R}) \int |g|^2 d\sigma_R.\end{aligned}$$

Therefore, if we want our estimate to be scale invariant we want $R^{-1}Q(w_{1/R}) = Q(w)$, which is satisfied by the choice $Q(w) = \|Xw\|_\infty$. In [2], the authors observe how $\|Xw\|_\infty$ is the smallest functional of w that satisfies this right scaling.

This bring us to the formulation of the Mizohata–Takeuchi conjecture for the sphere.

Conjecture. *For any weight w,*

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w \lesssim \sup_{\omega} \|Xw(\omega, \cdot)\|_{L^\infty} \int |g|^2 d\sigma, \tag{3.12}$$

where the supremum is taken over all normal directions to the support of g.

With analagous considerations, one can imagine to extend the conjecture to any a smooth compact hypersurface M, equipped with the surface measure μ, as

$$\int_{\mathbb{R}^n} |\widehat{gd\mu}|^2\, w \lesssim \sup_{\omega} \|Xw(\omega,\cdot)\|_{L^\infty} \int |g|^2 d\mu, \tag{3.13}$$

where the supremum is taken over all normal directions to M.

Remark. It is worth saying that is not clear what a general statement should be once we stray from the case of the sphere. For example we will see in Chapter 8 how, in the case of the paraboloid, one should consider a slightly different object from the X-ray transform. One of the goals of our thesis is try to understand if there might be a common statement for a wide class of extension operators and what this should be.

The conjecture can be further generalized to surfaces of codimension k in $\mathbb{R}^n$, by taking $T_{k,n}$ in place of X, by considering the inequality

$$\int_{\mathbb{R}^n} |\widehat{gd\mu}|^2\, w \lesssim \sup_{\Omega} \|T_{k,n}w(\Omega,\cdot)\|_{L^\infty} \int |g|^2 d\mu, \tag{3.14}$$

where the supremum is taken over all k-planes Ω normal to the surface.

Remark. Unlike in the L^p setting, we do not have the necessary condition of non-vanishing curvature. In fact, we are in the quite opposite situation, since the Mizohata–Takeuchi conjecture is true for k-planes. Consider, without loss of generality, the $n-k$-plane

$$M = \{x \in \mathbb{R}^n |\ x_1 = \cdots = x_k = 0\}.$$

The only normal k-plane to M is $\Omega = \{x_{k+1} = \cdots = x_n = 0\}$.

As similarly observed before,

$$Eg(x) = \int e^{i(x_{k+1}\xi_1 + \cdots + x_n \xi_{n-k})} g(\xi) d\xi = \widehat{g}(x_{k+1}, ..., x_n),$$

and therefore

$$\begin{aligned}\int_{\mathbb{R}^n} |Eg|^2 w &= \int_{\Omega}\int_M |\widehat{g}(x_{k+1}, ..., x_n)|^2 w(x) dx_1 ... dx_k dx_{k+1} ... dx_n \leqslant \\ &\leqslant \sup_{v \in M} T_{k,n} w(\Omega, v)\, \|g\|_2^2 .\end{aligned}$$

The origins of this conjecture can be traced back to the Japanese mathematician Yu Takeuchi, who in the late 70's was studying the well-posedness of the following Cauchy problem:

$$\begin{cases} -i\partial_t u(x,t) + \Delta_x u(x,t) + V(x) \cdot \nabla u(x,t) = F(x,t) \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}.$$

He claimed that the problem was L^2 well-posed if

$$\sup_{\omega,t,x} \left| \mathrm{Re} \int_0^t V(x+\omega s)ds \right| < +\infty, \tag{3.15}$$

where $\omega \in \mathbb{S}^{n-1},\ x \in \mathbb{R}^n,\ t \in \mathbb{R}$. Observe that the object taken into consideration is a kind of X-ray transform.

In his 1985 book "On the Cauchy Problem" ([42]), the Japanese mathematician Sigeru Mizohata contests Takeuchi's claim:

> "Takeuchi claimed that this condition is a sufficient condition condition for the L^2-wellposedness. However, it seems to the author his proof is not clear. [...] Even now, we do not know whether his claim is correct or not."

These types of problems are connected to estimates of the form of (3.13). For example in [3], Barceló, Ruiz and Vega prove the well-posedness of the problem

$$\begin{cases} -i\partial_t u(x,t) + \Delta_x u(x,t) + V(x,t) \cdot \nabla u(x,t) = F(x,t) \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}.$$

under some (3.15)-type condition and some additional hypothesis on the field V by using some (3.13)-type estimates (the difference with the problem discussed by Mizohata and Takeuchi is that the field V now depends on t). Incidentally, in the same paper, the authors prove the Mizohata–Takeuchi conjecture for the sphere with radial weights (the same has been proven independently in [19]).

Another perspective on why we are interested in a weighted inequality like (3.14) is that it gives us more information about the structure of the extension operator, compared to L^p estimates, which only describes the general size. For example, if f is a non-negative measurable function whose level sets have finite measure, one can define the *symmetric decreasing rearrangement* of f as

$$f^*(x) = \int_0^{+\infty} \chi_{\{y|\ f(y)>t\}}(x)dt.$$

The function f^* is symmetric and decreasing, and has level sets with the same size as those of f. The L^p norm cannot distinguish f from its rearrangement, in the sense that $\|f^*\|_p = \|f\|_p$. On the contrary, knowing how to control the weighted L^2 norm can give us more information about the level sets. In particular, if we take the weight w in the Mizohata–Takeuchi estimate to be the characteristic function of a level set of the operator, say $L_\lambda = \{x|\ |Eg| \sim \lambda\}$, then

$$\lambda^2 \cdot |L_\lambda| \sim \int_{L_\lambda} |Eg|^2 \lesssim \|T_{k,n}(\chi_{L_\lambda})\|_\infty \|g\|_2^2 .$$

This is somewhat telling us that the level sets roughly cluster on some k-planes.

A more ambitious conjecture, usually referred to as the Stein–Mizohata–Takeuchi conjecture, is to consider an inequality of the form

$$\int_{\mathbb{R}^n} |\widehat{gd\mu}(x)|^2 w(x)dx \lesssim \int \sup_v T_{k,n}w(\Omega_\xi, v)|g(\xi)|^2 d\mu(\xi), \tag{3.16}$$

where Ω_ξ is the k-plane normal to M at the point ξ.

Clearly this conjecture is stronger than the classical Mizohata–Takeuchi conjecture.

3.4 Known results on the Mizohata–Takeuchi conjecture

Although some partial results have been proven and some progress has been made, the Mizohata–Takeuchi conjecture still remains open today, even on $\mathbb{R}^2$, where the extension operators are better understood and the Restriction conjecture on L^p estimates has been proven (see for example [23]). We conclude this chapter by mentioning some of the know result on the Mizohata–Takeuchi problem that are present in the literature.

We already mentioned how in [3] and in [19], the authors prove the Mizohata–Takeuchi conjecture for the sphere with radial weights. What makes this case special is that, since we are dealing with the space $L^2(\mathbb{S}^{n-1})$, one can use spherical harmonic to decompose the operator. The coefficients appearing in this decomposition are given in terms of Bessel functions J_s and, since the weight is radial, the problem reduce to the study of estimates of the form

$$\int_0^{+\infty} |J_s(r)|^2 w(r) dr \lesssim \|Xw\|_\infty .$$

This is a simpler one dimensional problem that one can solve using known bounds for Bessel functions. Moreover, the X-ray of a radial function itself assume a simpler form, as we will see in Chapter 5.

In [17], the authors use a wave packet decomposition of the function g at scale R and a decoupling estimate ([29]) for a certain L^p norm of the extension operator of the packets to prove that, for a compact convex C^2 hypersuface, one has, for every $\epsilon > 0$

$$\int_{B_R} |\widehat{gd\mu}|^2 w \leqslant C_\epsilon R^\epsilon \sup_T \left(\int_T w^{\frac{n+1}{2}} \right)^{\frac{2}{n+1}} \int |g|^2 d\mu,$$

where T ranges over the family of all tubes of dimensions $R^{1/2} \times \cdots \times R^{1/2} \times R$. This in particular implies

$$\int_{B_R} |\widehat{gd\mu}|^2 w \leqslant C_\epsilon R^{\frac{n-1}{n+1}+\epsilon} \|Xw\|_{L^\infty} \int |g|^2 d\mu.$$

They also observe that the power $\frac{n-1}{n+1}$ is the best possible loss one can obtain with standard decoupling techniques.

In [48], the author studies some Mizohata–Takeuchi type estimates in the plane for Borel measures μ, under some hypothesis on the decay of $\widehat{\mu}$. For example, it is shown how, if the measure has a certain decay along a certain direction v, i.e.

$$|\widehat{\mu}(x)| \leqslant \frac{C}{|x \cdot v|},$$

one has

$$\int_{\mathbb{R}^2} |\widehat{gd\mu}|^2 w \lesssim \left(\sup_T \int_T w \right) \int |g|^2 d\mu,$$

where the supremum is taken over tubes with axis perpendicular to v. To do so, the author introduces a notion of "fractal dimension α" for a weight H and a related maximal function that are suited to exploit the decay hypothesis and allow to prove estimates of the form

$$\int |\widehat{gd\mu}|^2 H \lesssim A_\alpha(H) \int |g|^2 d\mu,$$

from which the Mizohata–Takeuchi estimate follows by choosing H appropriately, depending on w.

In [12] and [13], the authors introduce a tomographic approach to the problem that will be central in this thesis and we will discuss in the next chapters.

CHAPTER 4

SOBOLEV VARIANTS OF THE MIZOHATA–TAKEUCHI CONJECTURE

In this chapter we discuss some application of the inversion formula of the k-plane transform to the Mizohata–Takeuchi conjecture, starting from a result of Bennett and Nakamura ([12]) and then passing to some original results that were inspired by this.

4.1 L^2 Sobolev variant

In this section we introduce one of the main results of [12].

Bennett and Nakamura turned to study the following variant of the classical Mizohata–Takeuchi inequality for the sphere:

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 \, w \lesssim \left\| (-\Delta_v)^{\frac{n-1}{2q}} Xw \right\|_{L^\infty_\omega L^q_v} \int_{\mathbb{S}^{n-1}} |g|^2 d\sigma, \tag{4.1}$$

where $1 \leqslant q \leqslant \infty$ and the supremum of the L^∞ norm is taken over the direction normal to the support of g.

The key idea is here to substitute the L^∞ norm in the v variable with an alternative norm, which scales in the same way. To do so, the authors were inspired by the numerology of the standard Sobolev embeddings into L^∞.

Since we are working with the sphere, that has codimension 1, we will be using the X-ray transform $X = T_{1,n}$.

We can see the weighted norm as $\langle |\widehat{gd\sigma}|^2, w\rangle$, we can use property (1.4), and the self-adjonitness of $-\Delta_v$ to write

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w = c_n \left\langle (-\Delta_v)^{\frac{1}{2}} X(|\widehat{gd\sigma}|^2), Xw \right\rangle =$$
$$= c_n \left\langle (-\Delta_v)^{-\frac{n-1}{2q}} (-\Delta_v)^{\frac{1}{2}} X(|\widehat{gd\sigma}|^2), (-\Delta_v)^{\frac{n-1}{2q}} Xw \right\rangle =$$
$$= c_n \left\langle (-\Delta_v)^{\frac{1}{2}(1-\frac{n-1}{q})} X(|\widehat{gd\sigma}|^2), (-\Delta_v)^{\frac{n-1}{2q}} Xw \right\rangle .$$

Now an application of Hölder's inequality gives

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^2 w \lesssim \left\| (-\Delta_v)^{\frac{n-1}{2q}} Xw \right\|_{L^\infty_\omega L^q_v} \left\| (-\Delta_v)^{\frac{1}{2}(1-\frac{n-1}{q})} X(|\widehat{gd\sigma}|^2) \right\|_{L^1_\omega L^{q'}_v} .$$

So (4.1) would follow from the estimate

$$\left\| (-\Delta_v)^{\frac{1}{2}(1-\frac{n-1}{q})} X(|\widehat{gd\sigma}|^2) \right\|_{L^1_\omega L^{q'}_v} \lesssim \|g\|_{L^2(\mathbb{S}^{n-1})} . \tag{4.2}$$

In order to ensure finiteness in (4.1) and (4.2), we can consider the local variant

$$\int_{B_R} |\widehat{gd\sigma}|^2 \, w \lesssim R^\varepsilon \left\| (-\Delta_v)^{\frac{n-1}{2q}} Xw \right\|_{L^\infty_\omega L^q_v} \int_{\mathbb{S}^{n-1}} |g|^2 d\sigma. \tag{4.3}$$

Arguing as above, (4.3) would follow from the estimate

$$\left\| (-\Delta_v)^{\frac{1}{2}(1-\frac{n-1}{q})} X(\gamma_R |\widehat{gd\sigma}|^2) \right\|_{L^1_\omega L^{q'}_v} \lesssim R^\varepsilon \, \|g\|_{L^2(\mathbb{S}^{n-1})} , \tag{4.4}$$

where γ_R is a suitable smooth bump function adapted to B_R. It turns out this result is true in dimension 2 for $1 \leqslant q \leqslant 2$.

Theorem 4.1.1 (Bennett–Nakamura)**.** *Let $n = 2$. Then,*

$$\left\| (-\Delta_v)^{\frac{1}{4}} X(\gamma_R |\widehat{gd\sigma}|^2) \right\|_{L^1_\omega L^2_v} \lesssim \log(R) \, \|g\|^2_{L^2(\mathbb{S}^1)} , \tag{4.5}$$

and hence

$$\int_{B_R} |\widehat{gd\sigma}|^2 \, w \lesssim \log(R) \left\| (-\Delta_v)^{\frac{1}{4}} Xw \right\|_{L^\infty_\omega L^2_v} \int_{\mathbb{S}^1} |g|^2 d\sigma. \tag{4.6}$$

Remark. In a more recent work ([9]), the authors prove that, for $n > 2$, even if the analogue of (4.2) does not hold, an analogue inequality (4.1) still holds. The techniques used in this paper are different and the estimate obtained does not have the logarithmic loss $log(R)$ present in Theorem 4.1.1, but there is an ϵ-loss in the number of derivatives present in the Sobolev norm, in a sense analogue to the one that we will explore in the next sections.

Remark. The advantage of this tomographic approach is that we are able to introduce at the same time the X-ray transform and the derivatives, which help us at the same time to introduce these alternative Sobolev norms and to get better estimates on the objects like $X(|\widehat{gd\sigma}|^2)$. Moreover, the authors underline the importance of tomographic estimates for the extension operator by introducing a new conjecture that would slot in between the already mentioned restriction conjecture and the famous Kakeya maximal conjecture:

Conjecture 1. *For every $\epsilon > 0$ there is a constant $C_\epsilon < \infty$such that*

$$\left\| X(\gamma_R |\widehat{gd\sigma}|^{\frac{2}{n-1}}) \right\|_{L^n_\omega L^\infty_v} \leqslant C_\epsilon R^\epsilon \, \|g\|^{\frac{2}{n-1}}_{L^{\frac{2}{n-1}}(\mathbb{S}^{n-1})}, \tag{4.7}$$

for all $R > 0$.

In particular, one would have that

restriction conjecture $\implies$ Conjecture 1 $\implies$ Kakeya maximal conjecture.

4.2 Global L^1 Sobolev variant

We now want to argue similarly to Bennett and Nakamura, but using an L^1 Sobolev norm instead of the L^2 one. Observe that one can use the fundamental theorem of calculus to show the embedding $W^{d,1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. In fact, for $f \in C_c^\infty(\mathbb{R}^2)$, one can write

$$f(x) = \int \cdots \int \frac{\partial^d}{\partial x_1 \cdots \partial x_d} f(x_1, ..., x_d) dx_1 \cdots dx_d,$$

and, therefore,

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leqslant \|f\|_{W^{d,1}(\mathbb{R}^d)}.$$

Remark. It is worth to underline once more that we are not interested in the Sobolev embedding itself, but merely in knowing the right number of derivatives needed in order to introduce a new quantity with the same scaling as the L^∞ norm.

As we have already done earlier, we can use property (1.4), this time in the more general case of the k-plane transform $T_{k,n}$, and the self-adjonitness of $-\Delta_v$ to manipulate $\langle|\widehat{gd\sigma}|^2, w\rangle$ as follows:

$$\begin{aligned}\int |\widehat{gd\mu}|^2 w &\cong \langle(-\Delta_v)^{\frac{k-n}{2}} T_{k,n}|\widehat{gd\mu}|^2,\ T_{k,n}w\rangle \cong \\ &\cong \langle(-\Delta_v)^{\frac{2k-n}{2}} T_{k,n}|\widehat{gd\mu}|^2,\ (-\Delta_v)^{\frac{n-k}{2}} T_{k,n}w\rangle \lesssim \\ &\lesssim \left\|(-\Delta_v)^{\frac{2k-n}{2}} T_{k,n}(|\widehat{gd\mu}|^2)\right\|_{L^1_\Omega L^\infty_v} \left\|(-\Delta_v)^{\frac{n-k}{2}} T_{k,n}(w)\right\|_{L^\infty_\Omega L^1_v},\end{aligned}$$

where we used the Hölder's inequality to get the inequality.
By proving

$$\left\|(-\Delta_v)^{\frac{2k-n}{2}} T_{k,n}(|\widehat{gd\mu}|^2)\right\|_{L^1_\Omega L^\infty_v} \lesssim \|g\|^2_{L^2(d\mu)},$$

we can therefore obtain the estimate

$$\int |\widehat{gd\mu}|^2(x) w(x) dx \lesssim \left\|(-\Delta_v)^{\frac{n-k}{2}} T_{k,n}(w)\right\|_{L^\infty_\Omega L^1_v} \|g\|^2_{L^2(d\mu)}. \tag{4.8}$$

Remark. Observe that, for the previous argument to be rigorous, we need for $|\widehat{gd\mu}|^2$ to be in L^2 in order to use the isometry property of $T_{k,n}$. This is not always granted in general. For example, if we take μ to be the surface measure of the sphere $\mathbb{S}^{n-1}$, by looking at the numerology of Theorem 3.2.1, we can see that $|\widehat{gd\mu}|^2$ is in L^2 whenever $n \geqslant 3$, but not for $n = 2$. This also explains why the authors of [12] localized the problem discussed in the previous section. In this section, we will not worry about this technical hypothesis. In the next section we will address the problem by proving an analogous result in a localized version.

We start with some preliminary results.

Definition. Let $\Omega \in \mathcal{G}_{k,n}$ and consider an orthonormal basis of $\mathbb{R}^n$, $(\omega_1, ..., \omega_n)$, so that $(\omega_1, ..., \omega_k)$ span Ω. Define

$$\delta_\Omega(x) = \prod_{j=k+1}^{n} \delta(\omega_j \cdot x), \tag{4.9}$$

where δ is the usual 1-dimensional Dirac delta.

Proposition 4.2.1. *Let $\alpha \in \mathbb{R}$ and fix $\Omega \in \mathcal{G}_{k,n}$. Let μ be a Radon measure. Then, for any*

$\phi \in C_c^{\infty}(\Omega^{\perp})$, *we have*

$$\langle (-\Delta_v)^{\frac{\alpha}{2}} T_{k,n}(|\widehat{gd\mu}|^2), \phi \rangle_{\Omega^{\perp}} = \langle S(g)(\Omega, \cdot), \phi \rangle_{\Omega^{\perp}}, \tag{4.10}$$

where

$$S(g)(\Omega, v) = \iint g(x)\overline{g}(y)|x-y|^{\alpha} \delta_{\Omega^{\perp}}(x-y) e^{i(x-y)\cdot v} d\mu(x) d\mu(y). \tag{4.11}$$

Proof. We begin by applying Plancherel's theorem to get

$$\langle (-\Delta_v)^{\frac{\alpha}{2}} T_{k,n}(|\widehat{gd\mu}|^2), \phi \rangle_{\Omega^{\perp}} = \int_{\Omega^{\perp}} gd\mu * \widetilde{gd\mu}(\xi) |\xi|^{\alpha} \overline{\widehat{\phi}}(\xi) d\xi.$$

Recall that one may write the convolution using the Dirac delta as

$$f * h(\xi) = \iint f(x) h(y) \delta_0(\xi - x - y) dx dy.$$

So we have

$$\int_{\Omega^{\perp}} gd\mu * \widetilde{gd\mu}(\xi) |\xi|^{\alpha} \overline{\widehat{\phi}}(\xi) d\xi = \int_{\Omega^{\perp}} \iint g(x)\overline{g}(y) |\xi|^{\alpha} \overline{\widehat{\phi}}(\xi) \delta_0(\xi - x + y) d\mu(x) \mu(y) d\xi.$$

Now observe that one can write

$$\delta_0 = \delta_{\Omega^{\perp}} \cdot \delta_{\Omega}$$

and that, since $\xi \in \Omega^{\perp}$, $\delta_{\Omega^{\perp}}(\xi - x + y) = \delta_{\Omega^{\perp}}(x-y)$. Therefore, we can write

$$\begin{aligned}
&\int_{\Omega^{\perp}} \iint g(x)\overline{g}(y) |\xi|^{\alpha} \overline{\widehat{\phi}}(\xi) \delta_0(\xi - x + y) d\mu(x) \mu(y) d\xi = \\
&= \int_{\Omega^{\perp}} \iint g(x)\overline{g}(y) |\xi|^{\alpha} \overline{\widehat{\phi}}(\xi) \delta_{\Omega}(\xi - x + y) \delta_{\Omega^{\perp}}(x-y) d\mu(x) \mu(y) d\xi = \\
&= \iint g(x)\overline{g}(y) \delta_{\Omega^{\perp}}(x-y) \left(\int_{\Omega^{\perp}} |\xi|^{\alpha} \overline{\widehat{\phi}}(\xi) \delta_{\Omega}(\xi - x + y) d\xi \right) d\mu(x) \mu(y) = \\
&= \iint g(x)\overline{g}(y) \delta_{\Omega^{\perp}}(x-y) |x-y|^{\alpha} \overline{\widehat{\phi}}(\pi_{\Omega^{\perp}}(x-y)) d\mu(x) \mu(y),
\end{aligned}$$

where $\pi_{\Omega^{\perp}}$ is the orthogonal projection onto $\Omega^{\perp}$ and we used Fubini's theorem to integrate in the ξ variable first.

Now, writing explicitly the Fourier transform of ϕ and using once more Fubini's theorem, we get

$$\iint g(x)\overline{g}(y)\delta_{\Omega^\perp}(x-y)|x-y|^\alpha\overline{\widehat{\phi}}(\pi_{\Omega^\perp}(x-y))d\mu(x)\mu(y) =$$
$$= \int_{\Omega^\perp}\left(\iint g(x)\overline{g}(y)\delta_{\Omega^\perp}(x-y)|x-y|^\alpha e^{iv\cdot(x-y)}d\mu(x)\mu(y)\right)\overline{\phi}(v)dv,$$

which completes the proof. □

Corollary 4.2.2. *Assume that μ is compactly supported. Then*

$$(-\Delta_v)^{\frac{\alpha}{2}}T_{k,n}(|\widehat{gd\mu}|^2)(\Omega,v) = S(g)(\Omega,v) \tag{4.12}$$

holds pointwise almost everywhere.

Proof. Since μ is compactly supported, so is $gd\mu * \widetilde{gd\mu}$. Consider a smooth function ϕ such that $\widehat{\phi} \equiv 1$ on the support of the convolution, so that we have

$$gd\mu * \widetilde{gd\mu} = \widehat{\phi}\cdot gd\mu * \widetilde{gd\mu}$$

and therefore

$$|\widehat{gd\mu}|^2 = \phi * |\widehat{gd\mu}|^2.$$

Now we can write

$$(-\Delta_v)^{\frac{\alpha}{2}}T_{k,n}(\phi * |\widehat{gd\mu}|^2)(\Omega,v) = \int_{\Omega^\perp} e^{iv\cdot\xi}gd\mu * \widetilde{gd\mu}(\xi)|\xi|^\alpha\widehat{\phi}(\xi)d\xi.$$

Following the same steps as in the proof of Proposition 4.2.1, we get to the expression

$$\iint g(x)\overline{g}(y)\delta_{\Omega^\perp}(x-y)|x-y|^\alpha e^{iv\cdot(x-y)}\widehat{\phi}(x-y)d\mu(x)\mu(y).$$

But $\widehat{\phi}(x-y) = 1$ when $x, y \in \operatorname{spt}(\mu)$, so we have the thesis. □

Remark. In Proposition 4.2.1 and in Corollary 4.2.2 we have not worried about checking the finiteness of the quantity $S(g)$, which will in principle be depending on g, α and μ. However, when we will apply this results, we will give condition for which we obtain finiteness on the $L^1_\Omega L^\infty_v$ norm of $S(g)$ which, in particular, implies $S(g)$ is finite almost everywhere (see Theorem

4.2.4).

Remark. An alternative method of proof for the corollary could be to use the Fundamental Lemma of Calculus of Variations. In this case we would need to prove that both the right-hand side and left-hand side are continuous. In this case, one could also think about proving the continuity, without requiring for μ to be compactly supported.

We recall a classical result in analysis, the so called Schur's test.

Proposition 4.2.3 (Schur's test)**.** *Let X, Y be two measurable spaces and T be an integral operator with kernel $K(x,y)$ defined as*

$$Tf(x) = \int_Y K(x,y)f(y)dy.$$

If there is a constant C such that

$$\sup_{x\in X} \int_Y |K(x,y)|dy \leqslant C \quad \textit{and} \quad \sup_{y\in Y} \int_X |K(x,y)|dx \leqslant C,$$

then

$$\|T\|_{L^2\to L^2} \leqslant C$$

Proof. Using the Cauchy–Schwarz inequality we can write

$$\begin{aligned} |Tf(x)|^2 &= \left|\int_Y K(x,y)f(y)dy\right|^2 \leqslant \\ &\leqslant \left|\int_Y K(x,y)|f(y)|^2 dy\right| \cdot \left|\int_Y K(x,y)dy\right| \leqslant \\ &\leqslant C \int_Y |K(x,y)||f(y)|^2 dy. \end{aligned}$$

So we have

$$\|Tf\|_{L^2}^2 \leqslant C \int_X \int_Y |K(x,y)||f(y)|^2 dydx \leqslant C^2 \|f\|_{L^2}^2,$$

where we used Fubini's Theorem. □

We are now ready to prove the main result of this section. Although we are not able to prove exactly the inequality (4.8) with our methods, we can prove an alternative statement, in which we allow an ϵ-loss in terms of the number of derivatives in the Sobolev norm of the k-plane transform.

Recalling the we defined

$$S_\alpha(\mu) = \sup_{x \in \mathrm{spt}(\mu)} \int \frac{1}{|x-y|^\alpha} d\mu(y),$$

we have the following:

Theorem 4.2.4. *Let $\epsilon \in \mathbb{R}$. Assume that the measure μ is such that*

$$S_{n-k-\epsilon}(\mu) \lesssim 1. \tag{4.13}$$

Then

$$\left\| (-\Delta_v)^{\frac{2k-n+\epsilon}{2}} T_{k,n}(|\widehat{gd\mu}|^2) \right\|_{L^1_\Omega L^\infty_v} \lesssim \|g\|^2_{L^2(d\mu)}. \tag{4.14}$$

In particular, we have

$$\int |\widehat{gd\mu}|^2(x) w(x) dx \lesssim \left\| (-\Delta_v)^{\frac{n-k-\epsilon}{2}} T_{k,n}(w) \right\|_{L^\infty_\Omega L^1_v} \|g\|^2_{L^2(d\mu)}. \tag{4.15}$$

Proof. Estimate (4.15) follows from (4.14) after using (1.4) and Hölder's inequality.

We want to estimate $\sup_v (-\Delta_v)^{\frac{2k-n+\epsilon}{2}} T_{k,n}(|\widehat{gd\mu}|^2)$. To do so, we invoke Proposition 4.2.1.

$$\begin{aligned} &\sup_{v \in \Omega^\perp} (-\Delta_v)^{\frac{2k-n+\epsilon}{2}} T_{k,n}(|\widehat{gd\mu}|^2)(\Omega, v) \leqslant \\ &\leqslant \iint |g(x)||g(y)||x-y|^{2k-n+\epsilon} \delta_{\Omega^\perp}(x-y) d\mu(x) d\mu(y). \end{aligned}$$

Now we need to integrate in the variable Ω. Using the fact that $\delta_{\Omega^\perp}$ is homogeneous of degree $-k$, we just need to compute

$$\int_{\mathcal{G}_{k,n}} \delta_{\Omega^\perp}\left(\frac{x-y}{|x-y|}\right) d\Omega.$$

A purely geometrical argument shows that this is independent of x and y and it is in fact a finite constant as we will show in Lemma 4.2.5. Therefore we have

$$\left\| (-\Delta_v)^{\frac{2k-n+\epsilon}{2}} T_{k,n}(|\widehat{gd\mu}|^2) \right\|_{L^1_\Omega L^\infty_v} \lesssim \iint |g(x)||g(y)||x-y|^{k-n+\epsilon} d\mu(x) d\mu(y).$$

The right hand side can now be estimated using the Cauchy-Schwarz inequality,

$$\iint |g(x)||g(y)||x-y|^{k-n+\epsilon} d\mu(x) d\mu(y) \leqslant \|Fg\|_2 \, \|g\|_2 \,,$$

where

$$Fg(x) = \int \frac{g(y)}{|x-y|^{n-k-\epsilon}} d\mu(y).$$

Finally, observe that condition (4.13) corresponds with the hypothesis of Schur's test for the operator F. Therefore, $\|Fg\|_2 \lesssim \|g\|_2$, which concludes the proof of (4.14). $\square$

Remark. Condition (4.13) is easily satisfied, with $\epsilon > 0$, when μ is the $n-k$-dimensional Hausdorff measure of a compact $n-k$-dimensional manifold. In particular, is necessary to have ϵ strictly positive, since one can easily check that $S_{n-k}(\mu) = +\infty$. More in general, by Frostman's Lemma, one can always find such a measure on a Borel set of dimension $n-k$.

Remark. When stating Theorem 4.2.4, we took the liberty of letting the parameter ϵ to be any real number, since the proof would still hold in all cases, as long as (4.13) is satisfied. However, when considering the Mizohata–Takeuchi conjecture, one is usually interested in measure supported on sets of dimension $n-k$. In this sense, ϵ should be considered a small positive quantity, as usual.

Lemma 4.2.5. *For any $x \in \mathbb{S}^{n-1}$,*

$$\int_{\mathcal{G}_{k,n}} \delta_{\Omega^\perp}(x) d\Omega = C,$$

for some positive constant $C = C(k,n)$.

Proof. In this proof we use some delta calculus as described in [26]. If M is a smooth submanifold of $\mathbb{R}^n$, one can define a canonical measure σ_M, which is naturally induced by the Euclidean metric structure of $\mathbb{R}^n$. For a d-dimensional sphere $\mathbb{S}^d$, this correspond to the usual measure σ_d. Assume $M = \{x \in \mathbb{R}^n | \ F(x) = 0\}$, where $F : \mathbb{R}^n \to \mathbb{R}^{n-d}$ is a smooth function whose Jacobian matrix has max rank at any point. Then we can write the measure as

$$\int_M f(x) d\sigma_M(x) = \int_{\mathbb{R}^n} f(x)\delta(F(x)) J_F(x) dx,$$

where $J_F(x)$ is the the Jacobian determinant of the function F at x. In particular, $d\sigma_{n-1}(x) = \delta(|x|-1)dx$.

Call

$$I = \int_{\mathcal{G}_{k,n}} \delta_{\Omega^\perp}(x) d\Omega.$$

For the computation, we use the fact that $\Omega \mapsto \Omega^\perp$ is a measure preserving homeomeorphism between $\mathcal{G}_{k,n}$ and $\mathcal{G}_{n-k,n}$, and, therefore, we identify $\mathcal{G}_{k,n}$ with the product the spheres

$$\mathbb{S}^{n-1} \times \mathbb{S}^{n-2} \times ... \times \mathbb{S}^k.$$

So, we consider an orthonormal base of $\mathbb{R}^n$, $(\omega_1, ..., \omega_n)$, such that $(\omega_1, ..., \omega_k)$ spans Ω and where $\omega_{k+1} \in \mathbb{S}^{n-1}$, $\omega_{k+2} \in \mathbb{S}^{n-1} \bigcap \langle \omega_{k+1} \rangle^\perp \cong \mathbb{S}^{n-2}$ and so on.

For each $j = k+1, ..., n$, we can write

$$d\sigma_{n+k-j}(\omega_j) = \delta(|\omega_j| - 1) \left(\prod_{i=k+1}^{j-1} \delta(\omega_j \cdot \omega_i) \right) d\omega_j$$

Putting everything together, we have

$$\begin{aligned} I &\cong \int_{\mathbb{S}^{n-1}} ... \int_{\mathbb{S}^k} \left(\prod_{j=k+1}^{n} \delta(\omega_j \cdot x) \right) d\sigma_k(\omega_n) ... d\sigma_{n-1}(\omega_{k+1}) = \\ &= \int ... \int \prod_{j=k+1}^{n} \left[\delta(|\omega_j| - 1) \delta(\omega_j \cdot x) \left(\prod_{i=k+1}^{j-1} \delta(\omega_j \cdot \omega_i) \right) d\omega_j \right] = \\ &= \prod_{j=k+1}^{n} \left[\int_{\mathbb{S}^{n+k-j-1}} d\sigma_{n+k-j-1} \right] < +\infty. \end{aligned}$$

$\square$

4.3 Sharpness of the L^1 Sobolev variant

Let us try now to argue in a more naive way, under the same hypothesis that $S_{n-k-\epsilon}(\mu) \lesssim 1$. Consider the classical Laplacian on $\mathbb{R}^n$. We can write

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{gd\mu}|^2 w &= \int_{\mathbb{R}^n} (-\Delta)^{-\frac{n-k-\epsilon}{2}} |\widehat{gd\mu}|^2 (-\Delta)^{\frac{n-k-\epsilon}{2}} w \leqslant \\ &\leqslant \left\| (-\Delta)^{\frac{n-k-\epsilon}{2}} w \right\|_1 \left\| (-\Delta)^{-\frac{n-k-\epsilon}{2}} |\widehat{gd\mu}|^2 \right\|_\infty \lesssim \\ &\lesssim \left\| (-\Delta)^{\frac{n-k-\epsilon}{2}} w \right\|_1 \|g\|^2_{L^2(d\mu)} .\end{aligned}$$

Here the inequality

$$\left\| (-\Delta)^{-\frac{n-k-\epsilon}{2}} |\widehat{gd\mu}|^2 \right\|_\infty \lesssim \|g\|^2_{L^2(d\mu)}$$

follows by observing that we can write

$$(-\Delta)^{-\frac{n-k-\epsilon}{2}} |\widehat{gd\mu}|^2(z) = \iint \frac{e^{iz\cdot(x-y)}}{|x-y|^{n-k-\epsilon}} g(x)\overline{g(y)} d\mu(x) d\mu(y),$$

and applying the Schur's Test. Therefore we can wonder if the estimate we obtained in Theorem 4.2.4 via the use of the k-plane transform is really an improvement compared to this simpler estimate. The answer is yes since

$$\left\| (-\Delta_v)^{\frac{n-k-\epsilon}{2}} T_{k,n}(w) \right\|_{L^\infty_\Omega L^1_v} \leqslant \left\| (-\Delta)^{\frac{n-k-\epsilon}{2}} w \right\|_1 .$$

To show this, first observe that

$$(-\Delta_v)^{\frac{n-k-\epsilon}{2}} T_{k,n}(w) = T_{k,n}((-\Delta)^{\frac{n-k-\epsilon}{2}} w),$$

as one can for example by using the Fourier transform in the variable v.

Now

$$\begin{aligned}\left\| T_{k,n}((-\Delta)^{\frac{n-k-\epsilon}{2}} w) \right\|_{L^\infty_\Omega L^1_v} &= \sup_\Omega \int_{\Omega^\perp} \left| \int_\Omega (-\Delta)^{\frac{n-k-\epsilon}{2}} w(u+v) du \right| dv \leqslant \\ &\leqslant \sup_\Omega \int_{\Omega^\perp} \int_\Omega \left| (-\Delta)^{\frac{n-k-\epsilon}{2}} w(u+v) \right| du dv = \left\| (-\Delta)^{\frac{n-k-\epsilon}{2}} w \right\|_1 .\end{aligned}$$

So there is clearly an improvement, compared to a more naive estimate like the one obtained above, since the triangle inequality $\left|\int F\right| \leqslant \int |F|$ is strict for a general integrand F.

Another interesting question is whether or not inequality (4.14) is sharp, in the sense that we want to investigate if it is possible that inequality (4.14) becomes an equality for some measures.

The proof of (4.14) essentially boils down to proving

$$I_\alpha(g d\mu) \leqslant S_\alpha(\mu) \, \|g\|_2^2 \, ,$$

where $\alpha = n - k - \epsilon$. It would then seems natural to claim that, if the inequality is sharp, the constants are maximizers. In particular, if g is a constant, we have an identity if and only if

$$I_\alpha(\mu) = S_\alpha(\mu).$$

This is for example the case if μ is the usual surface measure on the sphere $\mathbb{S}^{n-k}$ (normalized to 1). In fact, in this case, thanks to the symmetry of the sphere, the integral

$$A_\alpha(x) = \int \frac{1}{|x-y|^\alpha} d\mu(y)$$

is a constant independent of x. It is curious to us how the constant are maximizers of this inequality for the sphere, just like constant are maximizers for the classical L^p estimates for the extension operator associated to the sphere, in many cases studied starting with [25], and many other works that follows the same ideas (for example [44], [20]).

We can say that constants are quasi-maximizers if $I_\alpha \sim S_\alpha$, or, equivalently, if $A_\alpha(x) \sim 1$.

A sufficient condition for this to happen is that μ is $(n-k)$-AD regular and supported on a compact set. In fact, under this hypothesis, we have

$$\begin{aligned} \int \frac{1}{|x-y|^\alpha} d\mu(y) &= \int_0^\infty \mu(\{y| \ |x-y|^{-\alpha} \geqslant \lambda\}) d\lambda = \int_0^\infty \mu(B(x, \lambda^{-1/\alpha})) d\lambda = \\ &= \alpha \int_0^\infty \mu(B(x,r)) r^{-\alpha-1} dr \sim \int_0^R r^{-1-\epsilon} \sim 1. \end{aligned}$$

More in general, it is enough that there exists a subset E of the support of μ such that $\mu(E) > 0$ and $\mu\big|_E$ is $(n-k)$-AD regular.

4.4 Local L^1 Sobolev variant

As in [12], we can adapt our argument to a local variant of (4.8). We will proceed following the same steps we had for the global estimate.

Let ψ be a smooth cut-off function on $\mathbb{R}^n$ and define, for $R > 0$, $\psi_R(x) := \psi(x/R)$.

Proposition 4.4.1. *Let $\alpha \in \mathbb{R}$ and fix $\Omega \in \mathcal{G}_{k,n}$. Let μ be a measure. Then, for any $\phi \in C_c^\infty(\Omega^\perp)$, we have*

$$\langle (-\Delta_v)^{\frac{\alpha}{2}} T_{k,n}(\psi_R|\widehat{gd\mu}|^2), \phi\rangle_{\Omega^\perp} = \langle S_R(g)(\Omega, \cdot), \phi\rangle_{\Omega^\perp}, \tag{4.16}$$

where

$$S_R(g)(\Omega, v) = \iint g(x)\overline{g}(y) K_{\Omega,v}(x,y) d\mu(x) d\mu(y) \tag{4.17}$$

and

$$K_{\Omega,v}(x,y) = \int_{\mathbb{R}^n} \widehat{\psi_R}(z)|x-y+z|^\alpha \delta_{\Omega^\perp}(x-y+z) e^{i(x-y+z)\cdot v} dz. \tag{4.18}$$

Remark. Observe how $S_R(g) \to S(g)$, as $R \to +\infty$, since $\widehat{\psi_R} \to \delta_0$. Therefore we recover Proposition 4.2.1.

Proof. We proceed as in the proof of Proposition 4.2.1. Applying Plancherel's theorem, we get

$$\begin{aligned}
\langle (-\Delta_v)^{\frac{\alpha}{2}} T_{k,n}(|\widehat{gd\mu}|^2), \phi\rangle_{\Omega^\perp} &= \int_{\Omega^\perp} \widehat{\psi_R} * gd\mu * \widetilde{gd\mu}(\xi)|\xi|^\alpha \overline{\widehat{\phi}}(\xi) d\xi = \\
= \int_{\Omega^\perp} \iiint g(x)\overline{g}(y)|\xi|^\alpha \overline{\widehat{\phi}}(\xi)\widehat{\psi_R}(z)\delta_0(\xi - z - x + y) d\mu(x)\mu(y) dz d\xi &= \\
= \int_{\Omega^\perp} \iiint g(x)\overline{g}(y)\widehat{\psi_R}(z)|\xi|^\alpha \overline{\widehat{\phi}}(\xi)\delta_\Omega(\xi - z - x + y)\delta_{\Omega^\perp}(x - y + z) d\mu(x)\mu(y) dz d\xi &= \\
= \iint g(x)\overline{g}(y)\delta_{\Omega^\perp}(x-y+z)\left(\int_{\Omega^\perp} |\xi|^\alpha \overline{\widehat{\phi}}(\xi)\delta_\Omega(\xi - z - x + y) d\xi\right) d\mu(x)\mu(y) dz &= \\
= \iiint g(x)\overline{g}(y)\delta_{\Omega^\perp}(x-y+z)|x-y+z|^\alpha \overline{\widehat{\phi}}(\pi_{\Omega^\perp}(x-y+z))\widehat{\psi_R}(z) d\mu(x)\mu(y) dz&,
\end{aligned}$$

where $\pi_{\Omega^\perp}$ is the orthogonal projection onto $\Omega^\perp$ and we used Fubini's theorem.

Writing explicitly the Fourier transform of ϕ and using Fubini's theorem, we obtain the thesis. $\square$

The analogue of Corollary 4.2.2 is the following:

Corollary 4.4.2. *Assume that μ is compactly supported. Then*

$$(-\Delta_v)^{\frac{\alpha}{2}} T_{k,n}(\psi_R|\widehat{gd\mu}|^2)(\Omega, v) = S_R(g)(\Omega, v) \tag{4.19}$$

holds pointwise almost everywhere.

Proof. Take ϕ as in the proof of Corollary 4.2.2 and write

$$(-\Delta_v)^{\frac{\alpha}{2}} T_{k,n}(\psi_R \cdot \phi * |\widehat{gd\mu}|^2)(\Omega, v) = \int_{\Omega^\perp} e^{iv\cdot\xi} \widehat{\psi_R} * \left(\widehat{\phi} gd\mu * \widetilde{gd\mu}\right)(\xi)|\xi|^\alpha(\xi)d\xi =$$
$$= \int_{\Omega^\perp} \iiint e^{iv\cdot\xi}|\xi|^\alpha(\xi)\widehat{\psi_R}(z)g(x)\overline{g}(y)\widehat{\phi}(\xi - z)\delta_0(\xi - z - x + y)d\mu(x)\mu(y)dzd\xi.$$

Following similar steps to the ones in the proof of Proposition , we arrive at the quantity

$$= \iiint g(x)\overline{g}(y)\delta_{\Omega^\perp}(x - y + z)|x - y + z|^\alpha \widehat{\psi_R}(z)\widehat{\phi}(x - y)d\mu(x)\mu(y)dz.$$

But $\widehat{\phi}(x - y) = 1$ when $x, y \in \mathrm{spt}(\mu)$, so we have the thesis. ☐

Theorem 4.4.3. *Let $\epsilon > 0$. Assume that the measure μ is such that*

$$S_{\frac{n-k}{1+\epsilon}}(\mu) \lesssim 1. \tag{4.20}$$

Then

$$\left\| (-\Delta_v)^{\frac{(2+\epsilon)k-n}{2(1+\epsilon)}} T_{k,n}(\psi_R|\widehat{gd\mu}|^2) \right\|_{L^1_\Omega L^\infty_v} \lesssim \|g\|^2_{L^2(d\mu)} . \tag{4.21}$$

In particular, we have

$$\int_{B_R} |\widehat{gd\mu}|^2(x)w(x)dx \lesssim R^{O(\epsilon)} \left\| (-\Delta_v)^{\frac{n-k}{2(1+\epsilon)}} T_{k,n}(w) \right\|_{L^\infty_\Omega L^{1+\epsilon}_v} \|g\|^2_{L^2(d\mu)} . \tag{4.22}$$

Proof. We first show how to obtain (4.22) from (4.21).

Observe that, by dominated convergence, it is enough to prove (4.22) with a smooth cut-off function ψ_R in place of χ_{B_R}, such that $\psi_R \geqslant \chi_{B_R}$. We start by using (1.4) and Hölder's inequality.

$$\int \psi_R(x)|\widehat{gd\mu}|^2(x)w(x)dx \lesssim$$
$$\lesssim \left\|(-\Delta_v)^{\frac{n-k}{2(1+\epsilon)}}T_{k,n}(w)\right\|_{L^\infty_\Omega L^{1+\epsilon}_v}\left\|(-\Delta_v)^{\frac{(2+\epsilon)k-n}{2(1+\epsilon)}}T_{k,n}(\psi_R|\widehat{gd\mu}|^2)\right\|_{L^1_\omega L_v^{\frac{1+\epsilon}{\epsilon}}}.$$

Observe that if f is supported on a ball or radius R, so is $T_{k,n}(f)$ with respect to the variable v. Therefore we can dominate the $L^{\frac{1+\epsilon}{\epsilon}}$ norm with the L^∞ norm times a contribution of the size of the support. In particular we can say that

$$\left\|(-\Delta_v)^{\frac{(2+\epsilon)k-n}{2(1+\epsilon)}}T_{k,n}(\psi_R|\widehat{gd\mu}|^2)\right\|_{L^1_\omega L_v^{\frac{1+\epsilon}{\epsilon}}} \leqslant$$
$$\leqslant R^{\frac{(n-k)\epsilon}{1+\epsilon}}\left\|(-\Delta_v)^{\frac{(2+\epsilon)k-n}{2(1+\epsilon)}}T_{k,n}(\psi_R|\widehat{gd\mu}|^2)\right\|_{L^1_\Omega L^\infty_v}.$$

Applying (4.21) we get (4.22).

To prove (4.21), we use (4.19).

$$\sup_v\left|(-\Delta_v)^{\frac{(2+\epsilon)k-n}{2(1+\epsilon)}}T_{k,n}(\psi_R|\widehat{gd\mu}|^2)\right| \lesssim$$
$$\lesssim \iiint |g(x)||g(y)||\widehat{\psi_R}(z)||x-y+z|^{\frac{-(n-k)}{(1+\epsilon)}}\delta_{\Omega^\perp}\left(\frac{x-y+z}{|x-y+z|}\right)dzd\mu(y)d\mu(x),$$

where we used the homogeneity of $\delta_{\Omega^\perp}$.

As in the proof of Theorem 4.2.4, after integrating in Ω, we are left to prove

$$\iiint |g(x)||g(y)||\widehat{\psi_R}(z)||x-y+z|^{\frac{-(n-k)}{(1+\epsilon)}}\delta_{\Omega^\perp}\left(\frac{x-y+z}{|x-y+z|}\right)dzd\mu(y) \lesssim \|g\|^2_{L^2(d\mu)}.$$

To do so we just need to show that the kernel

$$K(x,y)=\int |\widehat{\psi_R}(z)||x-y+z|^{\frac{-(n-k)}{(1+\epsilon)}}dz$$

satisfies the Schur test. This follows from (4.20), since

$$\sup_x\int K(x,y)d\mu(y) \lesssim S_{\frac{n-k}{1+\epsilon}}(\mu)\int|\widehat{\psi_R}(z)|dz \lesssim 1.$$

□

Remark. The quantities $S_{\frac{n-k}{1+\epsilon}}(\mu)$ and $S_{n-k-\epsilon}(\mu)$ are essentially the same. Therefore we can say that Theorem 4.2.4 and Theorem 4.4.3 share the same hypothesis.

Remark. In both Theorem 4.2.4 and Theorem 4.4.3 there is an ϵ-loss. In the global case it appears in the number of derivatives of the Sobolev norm. So we do not precisely have d derivatives like Sobolev embedding $W^{d,1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ would require. In the local case, the number of derivatives for the Sobolev embedding is precise for the chosen norm and the loss is instead given by the factor R^ϵ. This phenomenon can be contextualized within Tao's "epsilon removal lemmas". These are techniques introduced by Tao in [51] which allows one to go from local L^p estimates with R^ϵ loss to global estimates at the price of a loss in the exponent p.

CHAPTER 5

A MIZOHATA–TAKEUCHI ESTIMATE FOR RADIAL WEIGHTS

In this chapter we discuss the Mizohata–Takeuchi estimate, in its more classical form, and not in the Sobolev variants, for measures under similar hypotheses to the ones used in the previous chapter, in the particular case of radial weights. As already mentioned, this is a well studied case in classical setting of the Mizohata–Takeuchi conjecture for the spherical extension operator, for which it has been proven to hold ([3],[19]).

5.1 k-plane transform of radial weights

We consider in this chapter radial weights. To be precise, we assume that our weight w is such that there exist $w_0 : [0, +\infty) \to [0, +\infty)$ such that $w(x) = w_0(|x|^2)$. The choice of the modulus squared as our radial variable, instead of simply the modulus, is justified by the simplicity of some computations, as it will be clear later.

We want to compute the k-plane transform of w. Observe that, if $x \in \Omega$ and $v \in \Omega^\perp$, then $|x + v|^2 = |x|^2 + |v|^2$. Using polar coordinates on Ω and a change of variable, we can write

$$T_{k,n}(w)(\Omega, v) = \int_\Omega w(x + v) d\lambda_\Omega(x) = c_k \int_0^{+\infty} w_0(r^2 + |v|^2) r^{k-1} dr =$$
$$\cong \int_{|v|}^{+\infty} w_0(r^2)(r^2 - |v|^2)^{\frac{k-2}{2}} r dr,$$

where c_k is the surface area of $\mathbb{S}^{k-1}$.

Therefore we can see that, for a fixed radial weight w, $T_{k,n}(w)(\Omega, v)$ only depends on $|v|$. In particular we can write, if we call $d = |v|^2$,

$$T_{k,n}(w)(\Omega, v) \cong F(w_0)(d) = \int_0^{+\infty} w_0(r)[r-d]_+^{\frac{k-2}{2}}\, dr,$$

where $[.]_+$ is defined as

$$[a]_+ = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a \leqslant 0 \end{cases}.$$

The key observation is that, if $k \geqslant 2$, then we trivially have that $F(w_0)(d) \leqslant F(w_0)(0)$, for any $d > 0$. In particular $\|T_{k,n}(w)\|_\infty = T_{k,n}(w)(\Omega, 0)$, for any $\Omega \in \mathcal{G}_{k,n}$. So we have a linear behavior of the L^∞ norm in this case, which greatly simplifies the problem of proving a Mizohata–Takeuchi type estimate.

5.2 Mizohata–Takeuchi conjecture and spherical averages

We will now take advantage of the observation in the previous section to link the Mizohata–Takeuchi problem to another well known problem in harmonic analysis: that of spherical averages.

Let μ be a positive Radon measure with compact support in $\mathbb{R}^n$. Let $\mathbb{S}(r)$ be the $n-1$-dimensional sphere of radius r. Then we define the spherical averages of μ to be the quantities

$$\mathfrak{s}(\mu)(r) = \int_{\mathbb{S}(1)} |\widehat{\mu}|^2(r\omega)d\sigma(\omega), \tag{5.1}$$

for $r > 0$. One is interested in studying estimates of the form

$$\mathfrak{s}(\mu)(r) \lesssim r^{-\alpha},$$

for some $\alpha > 0$. This has different implications, but is mainly known for its connection with the Falconer's distance set problem (see for example Mattila [38], [40]).

Proposition 5.2.1. *Let $k \geqslant 2$ and $\mathbb{S}(r)$ be the $n-1$-dimensional sphere of radius r. Then the following are equivalent:*

- $$\int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(r\omega)d\sigma(\omega) \lesssim r^{-(n-k)} \|g\|^2_{L^2(d\mu)} ; \tag{5.2}$$

- $$\int_{\mathbb{R}^n} |\widehat{gd\mu}|^2(x)w(x)dx \lesssim \|T_{k,n}(w)\|_\infty \|g\|^2_{L^2(d\mu)} \tag{5.3}$$

for any radial weight w.

If $k = 1$, (5.2) implies (5.3), but the converse is false.

Proof. First we prove (5.2) implies (5.3).

Using polar coordinates, we can write

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{gd\mu}|^2(x)w(x)dx &= \int_0^\infty w_0(r^2)r^{n-1}\left(\int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(r\omega)d\sigma(\omega)\right)dr \lesssim \\ &\lesssim \left(\int_0^\infty w_0(r^2)r^{k-1}dr\right)\|g\|^2_{L^2(d\mu)} \cong \\ &\cong T_{k,n}(w)(\Omega,0)\|g\|^2_{L^2(d\mu)} \lesssim \|T_{k,n}(w)\|_\infty \|g\|^2_{L^2(d\mu)} .\end{aligned}$$

Now we prove (5.3) implies (5.2), for $k \geqslant 2$.

Therefore, assume (5.3) is true for any radial weight. In particular it holds for the characteristic function of the annulus $A = \{x \in \mathbb{R}^n |\ R < |x| < R + \epsilon\}$. For $k \geqslant 2$ we have

$$\|T_{k,n}(\chi_A)\|_\infty = T_{k,n}(\chi_A)(\Omega,0) = \int_R^{R+\epsilon} r^{k-1}dr \cong (R+\epsilon)^k - R^k \cong \epsilon R^{k-1} + o(\epsilon).$$

On the other hand, using the Mean Value Theorem, there exist $\tilde{R} \in (R, R+\epsilon)$ such that

$$\begin{aligned}\int_A |\widehat{gd\mu}|^2(x)dx &= \int_R^{R+\epsilon} r^{n-1}\left(\int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(r\omega)d\sigma(\omega)\right)dr = \\ &= \epsilon \int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(\tilde{R}\omega)d\sigma(\omega).\end{aligned}$$

So we have

$$\epsilon \int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(\tilde{R}\omega)d\sigma(\omega) \lesssim \left(\epsilon R^{k-1} + o(\epsilon)\right)\|g\|^2_{L^2(d\mu)} .$$

Dividing both sides by ϵ and letting $\epsilon \to 0$ we obtain (5.2). □

Remark. The proof we presented of (5.3) $\implies$ (5.2) fails in the case $k = 1$ because we do not have the linear behavior of $\|T_{k,n}(w)\|_\infty$ that we observed in the last section. In fact, we can never have (5.3) $\implies$ (5.2) for $k = 1$, since we already mentioned how, for the sphere, (5.3) is true, while one can easily check, using Knapp's example, that (5.2) fails.

An argument like the one in the proof of Proposition 5.2.1 enables us to use known results for spherical averages to obtain some Mizohata–Takeuchi type estimates, as we will do in the next section.

5.3 An estimate for general measures

A well studied problem concerning spherical averages, is to find the best possible decay β for the estimate $\mathfrak{s}(\mu)(r) \lesssim r^{-\beta}$. If a measure μ is α-dimensional, we generally have that $\beta < \alpha$ or at most $\beta = \alpha$ (see Lucà–Rogers [34]). Moreover, when trying to upgrade an estimate like $\mathfrak{s}(\mu)(r) \lesssim r^{-\alpha}$ to an estimate of the form

$$\int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(r\omega)d\sigma(\omega) \lesssim r^{-\alpha'} \|g\|^2_{L^2(d\mu)}$$

one usually has to pay an ϵ-loss, i.e. $\alpha' = \alpha - \epsilon$, if one can even prove that. Therefore it appears hard to use Proposition 5.2.1 to prove that Mizohata–Takeuchi conjecture is true for a general measure.

In this section we show that, if we localize the problem and allow the ϵ-loss, we are still able to use the idea of Proposition 5.2.1 to obtain a result, under a familiar hypothesis.

Theorem 5.3.1. *Let $\epsilon > 0$, $k + \epsilon \geqslant (n+1)/2$ and assume*

$$S_{n-k-\epsilon}(\mu) \lesssim 1. \tag{5.4}$$

Then

$$\int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(r\omega)d\sigma(\omega) \lesssim r^{-(n-k-\epsilon)} \|g\|^2_{L^2(d\mu)}. \tag{5.5}$$

In particular, for any radial weight w, we have

$$\int_{B_R} |\widehat{gd\mu}|^2(x)w(x)dx \lesssim R^\epsilon \|T_{k,n}(w)\|_\infty \|g\|^2_{L^2(d\mu)} \tag{5.6}$$

First we need a result for spherical averages. We use here the following theorem, due to P. Mattila ([38]).

Theorem 5.3.2 (Mattila)**.** *For $0 < \alpha \leqslant (n-1)/2$ and $r > 0$ we have $\mathfrak{s}(\mu)(r) \lesssim I_\alpha(\mu) r^{-\alpha}$.*

Proof of Theorem 5.3.1. Letting $\alpha = n - k - \epsilon$, with k in the given range,

$$\int_{\mathbb{S}(1)} |\widehat{\mu}|^2(r\omega) d\sigma(\omega) \lesssim I_\alpha(\mu) r^{-\alpha}.$$

So, as we already argued before, we have

$$\int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(r\omega) d\sigma(\omega) \lesssim r^{-\alpha} I_\alpha(gd\mu) \lesssim r^{-\alpha} S_\alpha(\mu) \|g\|^2_{L^2(d\mu)} .$$

If we now argue as in the proof of Proposition 5.2.1, we have

$$\begin{aligned}
\int_{B_R} |\widehat{gd\mu}|^2(x) w(x) dx &\cong \int_0^R w_0(r^2) r^{n-1} \left(\int_{\mathbb{S}(1)} |\widehat{gd\mu}|^2(r\omega) d\sigma(\omega) \right) dr \lesssim \\
&\lesssim \left(\int_0^R w_0(r^2) r^{k-1+\epsilon} dr \right) \|g\|^2_{L^2(d\mu)} \lesssim \\
&\lesssim R^\epsilon \left(\int_0^\infty w_0(r^2) r^{k-1} dr \right) \|g\|^2_{L^2(d\mu)} \lesssim \\
&\lesssim R^\epsilon T_{k,n}(w)(\Omega, 0) \|g\|^2_{L^2(d\mu)} \lesssim R^\epsilon \|T_{k,n}(w)\|_\infty \|g\|^2_{L^2(d\mu)} .
\end{aligned}$$

□

Remark. Theorem 5.3.1 does not recover the known result for the sphere with radial weights, not even up to the ϵ-loss. In fact, we cannot consider $k = 1$ while maintaining an arbitrary small value of ϵ. Indeed, to have arbitrary small ϵ, we need a large enough codimension k.

CHAPTER 6

THE MIZOHATA–TAKEUCHI CONJECTURE FOR GENERAL MEASURES

In the previous chapters we proved some results for a wide class of measures, only satisfying some dimensional hypotheses and not requiring smoothness. In this chapter we discuss the possibility of formulating a Mizohata–Takeuchi conjecture for such class of measures. In particular we will only require the support of the measure to have a certain Hausdorff dimension, without requiring any smoothness and, later, we will see what happens if we lower the smoothness hypothesis, by working with rectifiable sets. We will only consider integer dimensions, for which $T_{k,n}$ makes sense the way we defined it. In particular we will have $k \in \mathbb{N}$, $1 \leqslant k < n$, throughout, even if we will consider fractal measures at some point.

6.1 A tentative conjecture based on Hausdorff dimension

We observed a sufficient condition for (4.13) to hold with $\epsilon > 0$ is that μ is the surface measure of a smooth compact $n-k$-dimensional manifold. More in general, if we have a compact set M of Hausdorff dimension $n-k$, by Frostman's Lemma, there exist a measure μ supported inside M such that $\mu(B(x,r)) \leqslant r^{n-k}$. In particular, it follows that (4.13) is satisfied for any $\epsilon > 0$.

Conversely, if we assume that (4.13) holds for a measure μ compactly supported, for all $\epsilon > 0$, it follows from the characterization of Hausdorff dimension via the α-energy that μ must be supported on a set of dimension at least $n-k$. Of course the more interesting case is when the dimension is exactly $n-k$.

Moreover, using Proposition 2.3.1, which characterizes α-energies for μ, we can argue in the following way:

$$\begin{aligned}\int |\widehat{gd\mu}|^2(x)w(x)dx \leqslant \int |\widehat{gd\mu}|^2(x)\frac{1}{|x|^{k+\epsilon}}w(x)(1+|x|)^{k+\epsilon}dx \lesssim \\ \lesssim I_{n-k-\epsilon}(gd\mu)\left\|w(\cdot)(1+|\cdot|)^{k+\epsilon}\right\|_{L^\infty} \lesssim \\ \lesssim \left\|w(\cdot)(1+|\cdot|)^{k+\epsilon}\right\|_{L^\infty} S_{n-k-\epsilon}(\mu)\left\|g\right\|^2_{L^2(\mu)},\end{aligned}$$

where we used Hölder's inequality and the Schur test for the estimate

$$I_{n-k-\epsilon}(gd\mu) \lesssim S_{n-k-\epsilon}(\mu)\left\|g\right\|^2_{L^2(\mu)}.$$

The finiteness of the quantity $\left\|w(\cdot)(1+|\cdot|)^{k+\epsilon}\right\|_{L^\infty}$ is telling us something about the integrability of the weight on sets of dimension k. In particular, it is easy to see that

$$\left\|T_{k,n}w\right\|_{L^\infty} \lesssim \left\|w(\cdot)(1+|\cdot|)^{k+\epsilon}\right\|_{L^\infty}.$$

In fact

$$T_{k,n}w(\Omega,v) = \int_\Omega w(x+v)d\lambda_\Omega(x) \leqslant \left\|w(\cdot)(1+|\cdot|)^{k+\epsilon}\right\|_{L^\infty}\int_{\mathbb{R}^k}(1+|x|)^{-(k+\epsilon)}dx$$

and, $\int(1+|x|)^{-(k+\epsilon)}dx < +\infty$, for any $\epsilon > 0$.

Therefore it seems natural to ask ourselves if a generalization of the Mizohata–Takeuchi conjecture could be formulated for measures, perhaps compactly supported, under only dimensionality assumptions. In other words, let μ be a measure supported on a set of Hausdorff dimension $n-k$: could the following estimate be true for a constant C only depending on μ?

$$\int_{\mathbb{R}^n}|\widehat{gd\mu}|^2(x)w(x)dx \leqslant C\left\|T_{k,n}w\right\|_{L^\infty}\left\|g\right\|^2_{L^2(\mu)}. \tag{6.1}$$

6.2 A counter-example to the general conjecture

Unfortunately, we are immediately able to disprove (6.1).

We start by constructing a counter-example for $n=2,\ k=1$. Consider a measure ν, that is to be chosen later, supported on a set of Hausdorff and Minkowski dimension $1/2$ on $\mathbb{R}$, so that the measure μ defined on $\mathbb{R}^2$ as $\mu := \nu \otimes \nu$ has support of Hausdorff dimension 1. Similarly, consider $g = f \otimes f$ and $w = u \otimes u$. Then, (6.1) becomes

$$\int_{\mathbb{R}} |\widehat{fd\nu}|^2(x)u(x)dx \lesssim \sqrt{\|Xw\|_{L^\infty}}\, \|f\|_{L^2(\nu)}^2 . \tag{6.2}$$

If $\omega \in \mathbb{S}^1$,

$$Xw(\omega, v) = \int_{\mathbb{R}} u(\omega_1 t + v_1)u(\omega_2 t + v_2)dt.$$

Consider the case where u is the characteristic function of a set E. Since at least one between $|\omega_1|$ and $|\omega_2|$ is greater than $\sqrt{2}/2$, say $|\omega_1| \geqslant \sqrt{2}/2$, we have

$$|Xw(\omega, v)| \leqslant \int_{\mathbb{R}} \chi_E(\omega_1 t + v_1)dt \leqslant \sqrt{2}|E|.$$

Therefore we can bound

$$\sqrt{\|Xw\|_{L^\infty}} \lesssim \|u\|_{L^{2,1}} .$$

By duality then, (6.2) implies the weak estimate

$$\left\|\widehat{fd\nu}\right\|_{L^{4,\infty}} \lesssim \|f\|_{L^2(\nu)} .$$

We can choose the measure ν to be one as described by Theorem 3.2.2. In particular, ν satisfies the hypotheses of Theorem 3.2.1, and, therefore, we have that

$$\left\|\widehat{fd\nu}\right\|_{L^p} \lesssim \|f\|_{L^2(\nu)} ,$$

for $p \geqslant 6$. Now, we would have by interpolation that the bound

$$\left\|\widehat{fd\nu}\right\|_{L^p} \lesssim \|f\|_{L^2(\nu)} ,$$

would hold for all p, $4 < p < 6$. But this is a contradiction since Theorem 3.2.2 establish the bound cannot hold for any $p < 6$.

We can generalize this counterexample to any n and k. First, we need an higher dimensional analogue of the fact that if $\omega \in \mathbb{S}^1$, then either ω_1 or ω_2 is greater then $\sqrt{2}/2$. Recall the following classical result of linear algebra:

Lemma 6.2.1 (Cauchy-Binet). *Let* $A \in \mathrm{Mat}_{n\times k}(\mathbb{R})$ *and* $B \in \mathrm{Mat}_{k\times n}(\mathbb{R})$. *Let*

$$\mathfrak{S} = \{S \subset \{1, ..., n\} \mid |S| = k\}.$$

For $S \in \mathfrak{S}$, *define* $A_S \in \mathrm{Mat}_{k\times k}(\mathbb{R})$ *by taking the* k *rows of* A *indexed by* S; *similarly, define* B_S *by taking the* k *columns of* B *indexed by* S.

Then

$$\det(AB) = \sum_{S\in\mathfrak{S}} \det(A_S)\det(B_S).$$

Then we are able to prove the following corollary:

Lemma 6.2.2. *Let* $\omega^1, ..., \omega^k \in \mathbb{R}^n$, *with* $\langle \omega^i, \omega^j \rangle = \delta_{ij}$. *If* $A \in \mathrm{Mat}_{n\times k}(\mathbb{R})$ *is given by* $A_{ij} = \omega^i_j$, *then there exists* $S \in \mathfrak{S}$ *such that*

$$|\det(A_S)| \geqslant \binom{n}{k}^{-1/2}.$$

Proof. Applying the Cauchy-Binet Lemma with A and A^t, since $AA^t = I$, we have

$$1 = \sum_{S\in\mathfrak{S}} \det(A_S)^2.$$

So the result follows by observing that $|\mathfrak{S}| = \binom{n}{k}$. □

To generalize the counterexample, consider a measure ν, to be chosen later, supported on a set of Hausdorff and Minkowski dimension $(n-k)/n$ on $\mathbb{R}$, so that the measure μ defined on $\mathbb{R}^n$ as $\mu := \bigotimes_{m=1}^n \nu$ has support of Hausdorff dimension $n-k$. Similarly, consider $g = \bigotimes_{m=1}^n f$ and $w = \bigotimes_{m=1}^n u$. Then, (6.1) becomes

$$\int_{\mathbb{R}} |\widehat{fd\nu}|^2(x)u(x)dx \lesssim \left(\|T_{k,n}w\|_{L^\infty}\right)^{1/n} \|f\|^2_{L^2(\nu)}. \tag{6.3}$$

If $(\omega^1, ..., \omega^k)$ is an orthonormal basis of Ω,

$$T_{k,n}w(\Omega, v) = \int_{\mathbb{R}^k} \prod_{j=1}^n u\left(\sum_{h=1}^k \omega^h_j t_h + v_j\right) dt.$$

Consider the case where u is the characteristic function of a set E. By Lemma 6.2.2, without loss of generality, we can assume that the $k \times k$ matrix A, given by $A_{jh} = \omega_j^h$ has $\det(A) \geqslant c$. By using the change variables $At = x$, we can therefore estimate in the following way:

$$\begin{aligned} |T_{k,n}w(\Omega, v)| &\leqslant \int_{\mathbb{R}^k} \prod_{j=1}^k \chi_E \left(\sum_{h=1}^k \omega_j^h t_h + v_j \right) dt \lesssim \\ &\lesssim \prod_{j=1}^k \int_{\mathbb{R}} \chi_E \left(x_j + v_j \right) dx_j \lesssim |E|^k. \end{aligned}$$

Therefore we can bound

$$\|T_{k,n}w\|_{L^\infty}{}^{1/n} \lesssim \|u\|_{L^{n/k,1}} .$$

By duality then, (6.3) implies the weak estimate

$$\left\| \widehat{f d\nu} \right\|_{L^{\frac{2n}{n-k},\infty}} \lesssim \|f\|_{L^2(\nu)} .$$

Again, invoking Theorem 3.2.2, we can choose a measure ν which satisfies

$$\left\| \widehat{f d\nu} \right\|_{L^p} \lesssim \|f\|_{L^2(\nu)} ,$$

for $p \geqslant \frac{2n+2k}{n-k}$. By interpolating with the weak estimate found, we would have the $L^2 \to L^p$ bound for all $p > \frac{2n}{n-k}$, which cannot happen. In fact, Theorem 3.2.2 ensures the bound cannot hold for $p < \frac{2n+2k}{n-k}$, leading us to a contradiction.

6.3 L^p-type Mizohata–Takeuchi estimates

Since we showed that the estimate (6.1) fails for general measures, we can try and consider some weaker inequalities. We can consider the following family of estimates, that we call MTp:

$$\int |\widehat{g d\mu}|^2(x) w(x) dx \lesssim \|T_{k,n}(w)\|_{L^\infty_\Omega L^p_v} \|g\|^2_{L^2(d\mu)} , \tag{6.4}$$

for $1 \leqslant p \leqslant \infty$. The counterexample we constructed tells us that $MT\infty$ is definitely false for general measures.

On the contrary, $MT1$ is always true for compactly supported measures μ. In fact, it is straightforward to observe that $\|T_{k,n}(w)\|_{L^\infty_\Omega L^1_v} = \|w\|_{L^1}$. Therefore,

$$\int |\widehat{gd\mu}|^2(x)w(x)dx \leqslant \|w\|_{L^1} \left\|\widehat{gd\mu}\right\|_{L^\infty}^2 \lesssim \|T_{k,n}(w)\|_{L^\infty_\Omega L^1_v} \|g\|_{L^2(d\mu)}^2 .$$

So, we can expect there exists a certain $p_0 \in (1, +\infty)$ such that MTp always fails for $p > p_0$ for general measures. Slightly modifying our previous counterexample, we can say something about this range.

Again, consider a measure ν supported on a set of Hausdorff and Minkowski dimension $(n-k)/n$ on $\mathbb{R}$, take μ defined on $\mathbb{R}^n$ as $\mu := \bigotimes_{m=1}^n \nu$, $g = \bigotimes_{m=1}^n f$ and $w = \bigotimes_{m=1}^n u$. Writing $p = 1 + (p-1)$ and using Hölder's inequality, we have

$$\|T_{k,n}w(\Omega, .)\|_{L^p_v}^p = \int_{\Omega^\perp} \left| \int_{\mathbb{R}^k} \prod_{j=1}^n u\left(\sum_{h=1}^k \omega_j^h t_h + v_j\right) dt\right|^p d\lambda_{\Omega^\perp}(v) \leqslant I \cdot II,$$

where

$$I = \int_{\Omega^\perp} \int_{\mathbb{R}^k} \prod_{j=1}^n u\left(\sum_{h=1}^k \omega_j^h t_h + v_j\right) dt d\lambda_{\Omega^\perp}(v)$$

and

$$II = \sup_{v\in\Omega^\perp} \left| \int_{\mathbb{R}^k} \prod_{j=1}^n u\left(\sum_{h=1}^k \omega_j^h t_h + v_j\right) dt\right|^{p-1} .$$

A change of variable shows that $I = \|u\|_{L^1}^n$, while, using Hölder's inequality and Lemma 6.2.2, $II \lesssim (\|u\|_{L^1}^k \|u\|_{L^\infty}^{n-k})^{p-1}$.

By taking $u = \chi_E$, we have $(\|T_{k,n}(w)\|_{L^\infty_\Omega L^p_v})^{1/n} \lesssim |E|^{\frac{n-k+kp}{np}}$. So we get the weak estimate

$$\left\|\widehat{fd\nu}\right\|_{L^{\frac{2np}{(n-k)(p-1)},\infty}} \lesssim \|f\|_{L^2(\nu)} .$$

Again, we can invoke Theorem 3.2.2 and construct a measure satisfying our hypothesis and such that

$$\left\|\widehat{fd\nu}\right\|_{L^p} \lesssim \|f\|_{L^2(\nu)} ,$$

cannot hold for $p < \frac{2n+2k}{n-k}$, but is true for $p \geqslant \frac{2n+2k}{n-k}$. In order to not fall into a contradiction by interpolation, we must necessarily have $p \leqslant \frac{n+k}{k}$.

It is interesting to observe that $2 < \frac{n+k}{k}$, therefore the estimate $MT2$ seems like a good candidate if we are looking for positive results.

Estimates of this nature have some interest and have already appeared in literarure. For example, in a recent work of Dendrinos, Mustata and Vitturi ([24]), a $MT2$-type estimate is proved for a certain class of quadratic surfaces, like the paraboloid. The method used by the authors involve some application of the Cauchy-Schwarz inequality and the use of the Wigner distibution, on which we will come back later in Chapter 8.

If we put ourselves in a more general setting, we can try to exploit ourselves the Cauchy-Schwarz inequality and the isometry property of $T_{k,n}$ to argue as follows:

$$\langle |\widehat{gd\mu}|^2, w\rangle \cong \langle (-\Delta_v)^{k/2} T_{k,n}(|\widehat{gd\mu}|^2),\ T_{k,n}w\rangle \lesssim \left\|(-\Delta_v)^{k/2} T_{k,n}(|\widehat{gd\mu}|^2)\right\|_{L^2} \|T_{k,n}w\|_{L^2}.$$

Now, since the operator $(-\Delta)^{k/4}$ commutes with $T_{k,n}$, we can write, using Plancherel's Theorem,

$$\begin{aligned}\left\|(-\Delta_v)^{k/2} T_{k,n}(|\widehat{gd\mu}|^2)\right\|_{L^2} &= \left\|(-\Delta_v)^{k/4} T_{k,n}((-\Delta)^{k/4}|\widehat{gd\mu}|^2)\right\|_{L^2} \cong \\ &\cong \left\|(-\Delta)^{k/4}|\widehat{gd\mu}|^2\right\|_{L^2} = \left\||\cdot|^{k/2} gd\mu * \widetilde{gd\mu}\right\|_{L^2},\end{aligned}$$

where $\widetilde{f}(x) := \bar{f}(-x)$. Another application of Cauchy-Schwarz inequality shows that

$$|gd\mu * \widetilde{gd\mu}|^2 \leqslant |\mu * \widetilde{\mu}| \cdot |g|^2 d\mu * |g|^2 d\widetilde{\mu}.$$

Therefore, using Hölder's inequality,

$$\left\||\cdot|^{k/2} gd\mu * \widetilde{gd\mu}\right\|_{L^2}^2 \leqslant \left\||\cdot|^k \mu * \widetilde{\mu}\right\|_{L^\infty} \||g|^2 d\mu * |g|^2 d\widetilde{\mu}\|_{L^1} = \left\||\cdot|^k \mu * \widetilde{\mu}\right\|_{L^\infty} \|g\|_{L^2(\mu)}^4.$$

Combining everything together, we have that if

$$\left\||\cdot|^k \mu * \widetilde{\mu}\right\|_{L^\infty} < \infty, \tag{6.5}$$

then

$$\int |\widehat{gd\mu}|^2(x) w(x) dx \lesssim \|T_{k,n}(w)\|_{L^2} \|g\|_{L^2(d\mu)}^2. \tag{6.6}$$

Remark. Since $\mathcal{G}_{k,n}$ is compact, $\|T_{k,n}(w)\|_{L^2} \lesssim \|T_{k,n}(w)\|_{L^\infty_\Omega L^2_v}$. In particular, (6.6) implies $MT2$.

Objects like $\mu * \widetilde{\mu}$ are studied in Fourier restriction theory. For example, in a famous paper ([26]), Foschi takes advantage of the idea of accessing the L^p norm of the extension operator via convolutions of the measure when p is an even integer, to prove constants are extremizers for L^p estimates of the extension operator for $\mathbb{S}^2$.

An example of a measure that satisfies (6.5) is the surface measure σ_{n-1} of the sphere $\mathbb{S}^{n-1}$, for $n \geqslant 3$, since

$$\sigma_{n-1} * \sigma_{n-1}(\xi) = \frac{V_{n-2}}{|\xi|} \left[1 - \frac{|\xi|}{4}\right]_+^{\frac{n-3}{2}},$$

where V_{n-2} denotes the surface measure of $\mathbb{S}^{n-2}$.

Further Problem. Condition (6.5) is essentially a geometrical hypothesis on the measure μ. Even if we have not been able to produce an example at the time of the writing of this thesis, it is not unthinkable that one could be able to construct a measure satisfying (6.5) while not being supported on a regular manifold.

6.4 Mizohata–Takeuchi conjecture for rectifiable measures

We saw how the sole condition on the Hausdorff dimension in not enough to obtain a Mizohata–Takeuchi estimate. Since the idea behind the formulation of the Mizohata–Takeuchi conjecture is based on the existence of normal and tangent spaces of the support of the measure, which of course are not defined for fractal measures, we can try to see what happens if we consider rectifiable measures. This represent a wider class compared to the one canonical measures of smooth manifolds, but we still have a notion of tangent space in this setting.

So, let μ be the $n-k$-Hausdorff measure restricted to a, perhaps compact, $n-k$-rectifiable set E. Then, is the estimate

$$\int_{R^n} |\widehat{gd\mu}|^2(x) w(x) dx \lesssim \|T_{k,n} w\|_{L^\infty} \|g\|^2_{L^2(\mu)} \tag{6.7}$$

true?

Let us first introduce a slightly different problem. Consider the following bilinear extension estimate on the plane:

$$\left\| \widehat{g_1 d\sigma_1} \widehat{g_2 d\sigma_2} \right\|_{L^2(\mathbb{R}^2)} \lesssim \|g_1\|_{L^2(d\sigma_1)} \|g_2\|_{L^2(d\sigma_2)} . \tag{6.8}$$

We say that two curves S_1 and S_2 on the plane are transverse if any pair of unitary vectors n_1, n_2, respectively normal to S_1 and S_2, are linearly independent and α-transverse if $|n_1 \wedge n_2| > \alpha$, for some $\alpha > 0$.

Proposition 6.4.1. *Let, for $i = 1, 2$, $S_i = \{x \in A_i \subset \mathbb{R}^d \mid F_i(x) = 0\}$ be two smooth α-transverse compact hypersurfaces. Then* (6.8) *holds.*

The content of Proposition 6.4.1 is well know and goes back to some arguments of Fefferman and Sjölin (we refer the reader to [8] for a more complete overview on the topic of multilinear restriction). Here, we will give a simple proof using the following result ([13]), which will also be usefull to us in the Chapter 7.

Theorem 6.4.2 (Bennett-Nakamura-Shiraki)**.** *Suppose that M is a k-dimensional C^1 submanifold of $\mathbb{R}^n$ and Ω is a k-dimensional subspace of $\mathbb{R}^n$ for which*

$$T_\xi M \cap \Omega^\perp = \{0\} \quad \textit{for all} \quad \xi \in M, \tag{T}$$

and

$$\langle \xi - \eta \rangle \cap \Omega^\perp = \{0\} \quad \textit{for all} \quad \xi, \eta \in M. \tag{GT}$$

Then

$$T_{k,n}(|\widehat{gd\sigma}|^2)(\Omega, v) = \int_M \frac{|g(\xi)|^2}{|(T_\xi M)^\perp \wedge \Omega|} d\sigma(\xi) \tag{6.9}$$

Proof of Proposition 6.4.1. Using Plancherel's Theorem and the Cauchy-Schwarz inequality, we write

$$\left\| \widehat{g_1 d\sigma_1} \widehat{g_2 d\sigma_2} \right\|_{L^2} = \|g_1 d\sigma_1 * g_2 d\sigma_2\|_{L^2} \leqslant \|d\sigma_1 * d\sigma_2\|_{L^\infty} \|g_1\|_{L^2(d\sigma_1)} \|g_2\|_{L^2(d\sigma_2)} .$$

To study the convolution, first observe that the transversality hypothesis ensures that $S_i - \{x\} \cap S_2$ is a well defined $d-2$-dimensional manifold, for any $x \in S_1 + S_2$ and compactness ensures that, calling μ_x its natural surface measure, $\mu_x(\mathbb{R}^d) \leqslant K < +\infty$, uniformly in x. We use again delta calculus as in [26].

$$g_1 d\sigma_1 * g_2 d\sigma_2(x) = \int \delta \begin{pmatrix} F_1(x-y) \\ F_2(y) \end{pmatrix} g_1(x-y) g_2(y) |\nabla F_1(x-y)| |\nabla F_2(y)| dy =$$
$$= \int g_1(x-y) g_2(y) \frac{|\nabla F_2(y)| |\nabla F_1(x-y)|}{|\nabla_y F_1(x-y) \wedge \nabla F_2(y)|} d\mu_x(y).$$

Now, using $|\nabla_y F_1(x-y) \wedge \nabla F_2(y)| > \alpha$, $|\nabla F_1(x-y)|$ and $|\nabla F_2(y)| \leqslant C$ and $\mu_x(\mathbb{R}^d) \leqslant K$, we get the estimate

$$\|g_1 d\sigma_1 * g_2 d\sigma_2\|_{L^\infty} \lesssim \|g_1\|_{L^\infty} \|g_2\|_{L^\infty},$$

which in particular implies $\|d\sigma_1 * d\sigma_2\|_{L^\infty} \lesssim 1$. □

Now consider two curves S_1 and S_2 on the plane that are α-transverse, and assume that for each of them the Mizohata–Takeuchi conjecture holds. Then

$$\left\| \widehat{g_1 d\sigma_1} \widehat{g_2 d\sigma_2} \right\|^2_{L^2(\mathbb{R}^2)} \lesssim \left\| X(|\widehat{g_2 d\sigma_2}|^2) \right\|_{L^\infty} \|g_1\|_{L^2(d\sigma_1)},$$

where the L^∞ norm of the X-ray transform is taken over directions normal to S_1. Therefore, thanks to the transversality hypothesis, we can use the Theorem 6.4.2 to conclude that

$$\left\| X(|\widehat{g_2 d\sigma_2}|^2) \right\|_{\infty} \lesssim \|g_2\|^2_{L^2(d\sigma_2)}.$$

In particular we have that the Mizohata–Takeuchi conjecture implies the bilinear estimate (6.8).

In [16], the authors (Carbery, Hänninen and Valdimarsson) prove in more general settings how inequalities like the Mizohata–Takeuchi one imply multilinear estimates like (6.8).

We include a proof of Theorem 6.4.2 for completeness and to show how the transversality hypothesis comes into play.

Proof of Theorem 6.4.2. The condition (GT) guarantees that M intersects any translate of $\Omega^\perp$ in at most one point. Therefore M can be viewed as a graph of a function φ over Ω.

The condition (T) tells us that all tangent spaces to M meet $\Omega^\perp$ transversely, and ensures that the function φ is of class C^1.

In particular, calling $U \subset \Omega$ the orthogonal projection of M onto Ω, and $u \in U$, then $\varphi(u)$ is the unique element of $M \cap (\{u\} + \Omega^{\perp}) - \{u\}$. By construction, we can represent M as

$$M = \{u + \varphi(u) | \ u \in U\},$$

where $\varphi : U \to \Omega^{\perp}$.

Now, we can write the extension operator in the following way:

$$\widehat{gd\sigma}(x) = \int_U e^{ix \cdot \Sigma(u)} g(\Sigma(u)) J(u) d\lambda_{\Omega}(u),$$

where $\Sigma(u) = u + \varphi(u)$ and

$$J(u) = \left| \frac{\partial \Sigma}{\partial u_1} \wedge \dots \wedge \frac{\partial \Sigma}{\partial u_k} \right|.$$

Then, using Plancherel's theorem on Ω, we have

$$\begin{aligned} T_{k,n}(|\widehat{gd\sigma}|^2)(\Omega, v) &= \int_{\Omega} |\widehat{gd\sigma}|^2 (x + v) d\lambda_{\Omega}(x) = \\ &= \int_{\Omega} \left| \int_U e^{i[x \cdot u + v \cdot \varphi(u)]} g(u + \varphi(u)) J(u) d\lambda_{\Omega}(u) \right|^2 d\lambda_{\Omega}(x) = \\ &= \int_{\Omega} \left| e^{iv \cdot \varphi(u)} g(u + \varphi(u)) J(u) \right|^2 d\lambda_{\Omega}(u) = \\ &= \int_{\Omega} \left| g(u + \varphi(u)) J(u)^{\frac{1}{2}} \right|^2 J(u) d\lambda_{\Omega}(u) = \int_M |g(\xi)|^2 J(u(\xi)) d\sigma(\xi), \end{aligned}$$

where $u(\xi)$ is the orthogonal projection of $\xi \in M$ onto Ω.

Therefore we only have to show that

$$J(u) = \frac{1}{|T_{\Sigma(u)} M \wedge \Omega^{\perp}|},$$

since $|(T_{\xi} M)^{\perp} \wedge \Omega| = |T_{\Sigma(u)} M \wedge \Omega^{\perp}|$.

Calling $e_1, ..., e_n$ the standard basis of $\mathbb{R}^n$, we can assume, without loss of generality, that Ω =span$(e_1, ..., e_k)$. First, observe that, since φ has image in $\Omega^{\perp}$,

$$\left| \frac{\partial \Sigma}{\partial u_1} \wedge \dots \wedge \frac{\partial \Sigma}{\partial u_k} \wedge e_{k+1} \wedge \dots \wedge e_n \right| = |e_1 \wedge \dots \wedge e_n| = 1.$$

Next, we construct an orthogonal basis $v_1, ..., v_k$ of $T_{\Sigma(u)} M$ via the Gram–Schmidt algorithm, using as starting vectors $\frac{\partial \Sigma}{\partial u_1}, ..., \frac{\partial \Sigma}{\partial u_k}$. So we have that

$$\frac{\partial \Sigma}{\partial u_1} \wedge ... \wedge \frac{\partial \Sigma}{\partial u_k} = v_1 \wedge ... \wedge v_k$$

and, in particular,

$$J(u) = \left| \frac{\partial \Sigma}{\partial u_1} \wedge ... \wedge \frac{\partial \Sigma}{\partial u_k} \right| = |v_1 \wedge ... \wedge v_k| = |v_1| \cdot ... \cdot |v_k|,$$

since the vectors $v_1, ..., v_k$ form an orthogonal basis. Therefore we have

$$\begin{aligned} |T_{\Sigma(u)} M \wedge \Omega^{\perp}| &= \left| \frac{v_1}{|v_1|} \wedge ... \wedge \frac{v_k}{|v_k|} \wedge e_{k+1} \wedge ... \wedge e_n \right| = \\ &= \frac{|v_1 \wedge ... \wedge v_k \wedge e_{k+1} \wedge ... \wedge e_n|}{|v_1| \cdot ... \cdot |v_k|} = \frac{1}{J(u)}, \end{aligned} \tag{6.10}$$

which concludes the proof. □

It is then tempting to believe that if the Mizohata–Takeuchi conjecture is true for rectifiable sets, then so is (6.8), under a suitable transversality hypothesis. Again the answer is negative.

Theorem 6.4.3. *Estimate* (6.8) *fails for a general pair of transverse rectifiable sets. Moreover, the Mizohata–Takeuchi conjecture is false for a general rectifiable set.*

Proof. Consider

$$M = \bigcup_{k \in \mathbb{N}} \{2^{-k}\} \times [0, 2^{-k}] \subset \mathbb{R}^2$$

and

$$N = [-2, 2] \times \{0\} \subset \mathbb{R}^2,$$

and equip them with Hausdorff measure, called respectively μ and ν.

If (6.8) holds, then we must have, by using Plancherel, for any sequence of functions $\{g_N\}$,

$$\left\| \widehat{g_N d\mu} \widehat{d\nu} \right\|_{L^2(\mathbb{R}^2)} = \|g_N d\mu * d\nu\|_{L^2(\mathbb{R}^2)} \lesssim \|g_N\|_{L^2(d\mu)}.$$

First, we want to compute $g_N d\mu * d\nu$. The geometry of the problem suggest that this only depends on the vertical coordinate. In particular, if $h \sim 2^{-k}$,

$$g_N d\mu * d\nu(0,h) = \sum_{j=0}^{k} g_N(2^{-j},h)$$

and, therefore,

$$\|g_N d\mu * d\nu\|_{L^2(\mathbb{R}^2)}^2 \cong \sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \left| \sum_{j=0}^{k} g_N(2^{-j},h) \right|^2 dh.$$

On the other side,

$$\|g_N\|_{L^2(d\mu)}^2 = \sum_{k=0}^{\infty} \int_0^{2^{-k}} \left| g_N(2^{-k},h) \right|^2 dh.$$

If we choose

$$g_N = \sum_{j=0}^{N} a_j \cdot \chi_{\{2^{-j}\}\times[0,2^{-N}]},$$

where $\{a_j\}$ is a sequence of positive numbers, then the estimate would imply

$$\left| \sum_{j=0}^{N} a_j \right|^2 \lesssim \sum_{j=0}^{N} |a_j|^2,$$

which cannot hold uniformly in N, as the optimal constant for the inequality between ℓ^1 and ℓ^2 norms in $\mathbb{R}^N$ is $\sqrt{N}$.

In particular this constitutes a counterexample to the "rectifiable" Mizohata–Takeuchi inequality itself, if we take the weight to be $w(x,y) = |\widehat{d\nu}|^2(x)$. Observe that, with this choice, $w(x,y) \sim \sin^2(x)/x^2$, which is a positive weight with finite X-ray transform in the normal direction to the set M. $\square$

CHAPTER 7

A MIZOHATA–TAKEUCHI ESTIMATE FOR TENSOR WEIGHTS

In this chapter we prove the Mizohata–Takeuchi estimate for smooth manifolds with weights of the tensor form we already started to study in the previous chapter when constructing the counterexamples presented.

7.1 The k-plane transform for tensor weights

In this chapter we will refer as *tensor weights* to weights on $\mathbb{R}^n$ of the form

$$w(x_1, ..., x_n) = w_1(x_1) \cdots w_n(x_n).$$

These behave particularly well when taking the L^∞ norm of the k-plane transform, as the following proposition shows. Recall the notation $\mathfrak{S} = \{S \subset \{1, ..., n\} \mid |S| = k\}$.

Proposition 7.1.1. *Consider a tensor weight on $\mathbb{R}^n$ of the form*

$$w(x_1, ..., x_n) = w_1(x_1) \cdots w_n(x_n).$$

Then

$$\|T_{k,n} w\|_{L^\infty} \sim \max_{S \in \mathfrak{S}} \left\{ \prod_{i \in S} \|w_i\|_1 \cdot \prod_{j \in S^c} \|w_j\|_\infty \right\}.$$

Proof. Call

$$Q = \max_{S \in \mathfrak{S}} \left\{ \prod_{i \in S} \|w_i\|_1 \cdot \prod_{j \in S^c} \|w_j\|_\infty \right\}.$$

For S in $\mathfrak{S}$, consider the associated k-plane Ω_S =span$\{e_i \mid i \in S\}$. Then

$$T_{k,n}w(\Omega_S, 0) = \prod_{i \in S} \|w_i\|_1 \cdot \prod_{j \in S^c} \|w_j\|_\infty .$$

Therefore $Q \lesssim \|T_{k,n}w\|_\infty$.

To prove the reverse inequality, consider $\Omega \in \mathcal{G}_{k,n}$ and let $(\omega^1, ..., \omega^k)$ be an orthonormal basis of Ω. Applying Lemma 6.2.2 to $\omega^1, ..., \omega^k$, we can assume, without loss of generality, that $\det(A_S) \gtrsim 1$ for $S = \{1, ..., k\}$ (recall that A_S is the $k \times k$ minor of the matrix $A = (\omega^1 \cdots \omega^k)$ defined in Lemma 6.2.2). Then,

$$T_{k,n}w(\Omega, v) = \int_{\mathbb{R}^k} \prod_{j=1}^n w_j \left(\sum_{h=1}^k \omega_j^h t_h + v_j \right) dt.$$

By using the change variables $A_S t = x$, we have

$$\begin{aligned}
|T_{k,n}w(\Omega, v)| &\leqslant \prod_{j=k+1}^n \|w_j\|_\infty \int_{\mathbb{R}^k} \prod_{j=1}^k w_j \left(\sum_{h=1}^k \omega_j^h t_h + v_j \right) dt \lesssim \\
&\lesssim \prod_{j=k+1}^n \|w_j\|_\infty \cdot \prod_{j=1}^k \int_{\mathbb{R}} w_j (x_j + v_j)\, dx_j \lesssim \\
&\lesssim \prod_{j=k+1}^n \|w_j\|_\infty \cdot \prod_{i=1}^k \|w_i\|_1 .
\end{aligned}$$

Therefore $Q \gtrsim \|T_{k,n}w\|_\infty$ and the constant is given by Lemma 6.2.2. □

Remark. Observe that taking the maximum of the quantities $\prod_{i \in S} \|w_i\|_1 \cdot \prod_{j \in S^c} \|w_j\|_\infty$ is necessary if we want to consider the L^∞ norm over all possible orientation of k-planes. When dealing with a Mizohata–Takeuchi estimate however, depending on the normal bundle, we can get away with just considering less planes, or even just one.

We first look at the case when all normal k-planes to a manifold M are close to a main k-plane Ω.

Theorem 7.1.2. *Let M be a k-dimensional C^1 manifold and w be a weight of the form of Proposition 7.1.1. Let Ω is a k-plane of the form $\Omega_S = \operatorname{span}\{e_j | j \in S\}$, for some $S \in \mathfrak{S}$. Assume that M and Ω satisfy the hypothesis of Theorem 6.4.2 and, moreover, that $|T_\xi M \wedge \Omega_S| \geqslant c > 0$, for all $\xi \in M$. Then the Mizohata–Takeuchi conjecture is true for M.*

Proof. Without loss of generality, assume that $\Omega = \text{span}\{e_1, ..., e_k\}$.

First observe that $w \leqslant \tilde{w}$ pointwise a.e., where

$$\tilde{w}(x_1, ..., x_n) = w_1(x_1)...w_k(x_k) \prod_{j=k+1}^{n} \|w_j\|_\infty .$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 w(x)dx \leqslant \int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 \tilde{w}(x)dx \leqslant \\ \leqslant \prod_{j=k+1}^{n} \|w_j\|_\infty \cdot \prod_{i=1}^{k} \|w_i\|_1 \, T_{k,n}(|\widehat{gd\sigma}|^2)(\Omega, 0).\end{aligned}$$

Then by Theorem 6.4.2 and Proposition 7.1.1 we have the result. □

7.2 Decomposition of a manifold

We now want to prove that, given a manifold, we can decompose it into pieces on which we can apply Theorem 7.1.2.

Theorem 7.2.1. *Let M be a compact C^1-manifold of codimension k and w be a weight of the form of Proposition 7.1.1. Then the Mizohata-Takeuchi conjecture holds for M equipped with the canonical surface measure σ and weight w.*

Proof. For $S \in \mathfrak{S}$, call $\Omega_S = \text{span}\{e_i | \ i \in S\}$. First observe that for all $\xi \in M$ there is $S \in \mathfrak{S}$ such that

$$|T_\xi M \wedge \Omega_S| > 0$$

and, in particular, $T_\xi M \cap \Omega_S = \{0\}$. In fact, if $\omega_1, ..., \omega_{n-k}$ is an orthonormal base of $T_\xi M$, then there must exist a minor $n-k \times n-k$ of the matrix given by the vectors $\omega_1, ..., \omega_{n-k}$ as columns with non-zero determinant. In particular, there is an $S \in \mathfrak{S}$ such that the matrix given by the vectors $\omega_1, ..., \omega_{n-k}$ and $\{e_i | \ i \in S\}$ as columns has non-zero determinant equal to $|T_\xi M \wedge \Omega_S| = c$. Observe that we can choose S optimally if we select the one that gives the greatest determinant.

Since M is C^1- regular we can find a whole neighbourhood $U = U(\xi)$ of ξ such that $|T_\eta M \wedge \Omega_S| \geqslant c/10 > 0, \ \forall \eta \in U$.

We now show that if $U(\xi)$ is chosen small enough, then $\langle x - y\rangle \cap \Omega_S = \{0\}$ for all $x, y \in U$. If this was false, then we would be able to find two sequences $\{x_m\}$ and $\{y_m\}$ converging to ξ such that $\langle x_m - y_m\rangle \subset \Omega_S$.

Up to passing to a subsequence, we have that

$$\lim_{m\to\infty} \frac{x_m - y_m}{|x_m - y_m|} = v \in \Omega_S,$$

since $(x_m - y_m)/|x_m - y_m| \in \Omega_S$ and Ω_S is closed. If we prove that v belongs to $T_\xi M$, we get a contradiction because we would have $|T_\xi M \wedge \Omega_S| = 0$.

If $U(\xi)$ is chosen small enough, we can assume there is a C^1- homeomorphism $\varphi : B \to U$, where B is a ball in $\mathbb{R}^{n-k}$ and $\varphi(0) = \xi$. We can then write $x_m = \varphi(s_m)$ and $y_m = \varphi(t_m)$, where the sequences $\{s_m\}$ and $\{t_m\}$ converge to 0. $T_\xi M$ can be characterized as the image of the jacobian matrix $J\varphi(0)$ of φ at 0. Since B is convex, we can use the mean value theorem to write

$$x_m - y_m = \varphi(s_m) - \varphi(t_m) = J\varphi(z_m)(s_m - t_m),$$

for a sequence $\{z_m\} \subset B$, where each z_m lies in the line segment between s_m and t_m. Dividing both sides by $|x_m - y_m|$, and and since $|x_m - y_m| = |\varphi(s_m) - \varphi(t_m)|$, we obtain

$$\frac{x_m - y_m}{|x_m - y_m|} = J\varphi(z_m)\frac{s_m - t_m}{|\varphi(s_m) - \varphi(t_m)|}, \tag{7.1}$$

Clearly, $z_m \to 0$. In fact,

$$|z_m| \leqslant |s_m| + |z_m - s_m| \leqslant |s_m| + |t_m - s_m| \to 0 \ \text{ as } m \to \infty.$$

Passing to the limit in (7.1), the left-hand side converges to v and $J\varphi(z_m)$ converges to $J\varphi(0)$. So, we only need to show that $w_m = (s_m - t_m)/|\varphi(s_m) - \varphi(t_m)|$ converges to some vector w. Now,

$$\frac{\varphi(s_m) - \varphi(t_m)}{|s_m - t_m|} = J\varphi(z_m)\frac{s_m - t_m}{|s_m - t_m|}$$

is bounded and therefore, up to passing to a subsequence, it converges to a non-zero vector, because φ is a non-degenerate map. In particular, it follows that $|w_m|$ is definitely bounded, and so, up to passing to a subsequence, convergent.

Using compactness, we can now choose a finite number of such sets U that cover the whole manifold. We can modify the sets so that, if they intersect, we consider the intersection only once. On each U we can apply Theorem 7.1.2 with $\Omega = \Omega_S^{\perp}$ and, summing all the pieces we get the wanted inequality for the whole M. $\square$

Remark. It is worth mentioning that one might consider an optimal decomposition in the following way: given the manifold M, consider, for each $S \in \mathfrak{S}$,

$$M_S = \left\{ \xi \in M | \; |T_\xi M \wedge \Omega_S| \geqslant |T_\xi M \wedge \Omega_{S'}| \; \forall S' \in \mathfrak{S} \right\}.$$

In fact, this decomposition both minimizes the number of components and the constant appearing in the estimate which is $\sup_\xi |T_\xi M \wedge \Omega_S|^{-1}$. However, there are some problems. First, each M_S is not necessarily connected. For example, if M is the sphere $\mathbb{S}^{n-1}$, each M_S has 2 connected components, at opposite poles. This is easily fixable by considering each connected component separately. The main issue is that, in the proof of Theorem 7.2.1 we deduce condition (GT) from condition (T) using the fact that we consider small enough pieces of M. We do not have a way to prove (GT) just from (T) with this decomposition for a general M.

This is still an optimal decomposition for any specific manifold for which we can verify (GT) directly, like one can do, for example, in the case of $\mathbb{S}^{n-1}$.

Remark. Observe that Proposition 7.1.1 still holds for weight of the form

$$w(x) = \prod_{j=1}^{n} w_j(v_j \cdot x),$$

where $\{v_j\}_{j=1}^n$ are linearly independent vectors in $\mathbb{R}^n$. In fact the two cases only differ by a linear change of coordinates and we will only get a constant that depends on the geometry of the vectors $\{v_j\}$. Similarly, we can apply a linear change of coordinates to extend the results of Theorem 7.1.2 and Theorem 7.2.1 to this type of weights by recovering to the proven case.

Further Problem. A result like Proposition 7.1.1, can be seen as type of Brascamp-Lieb inequality in the sense that we are proving an estimate of the form

$$\int_{\Omega+v} \prod w_j \lesssim \prod_{j=k+1}^{n} \|w_j\|_\infty \cdot \prod_{i=1}^{k} \|w_i\|_1 .$$

One can therefore wonder if it is possible to argue similarly to what done in this chapter, to prove a Mizohata–Takeuchi estimate for more general “Bracamp-Lieb” weights of the form

$$w(x) = \prod_{j+1}^{m} w_j(L_j x),$$

for some linear maps $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$, via comparing

$$\|T_{k,n} w\|_\infty \sim \prod_{j+1}^{m} \|w_j\|_{p_j}.$$

CHAPTER 8

A MIZOHATA–TAKEUCHI CONJECTURE FOR THE PARABOLOID

In this chapter we will introduce a variant of the Mizohata–Takeuchi conjecture for the paraboloid in $\mathbb{R}^n$ and give a proof of the fact that the Mizohata–Takeuchi inequality for the sphere implies the one for the paraboloid, by using an approximation argument via ellipsoids.

8.1 Wigner distribution and Schrödinger equation

The following observation is due to a work of Bennett, Gutiérrez, Nakamura and Oliveira ([9]).

Let $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ be a solution to the Schrödinger equation

$$i\frac{\partial u}{\partial t} = \Delta u, \tag{8.1}$$

with initial data $u_0 \in L^2(\mathbb{R}^d)$.

One can see the solution u as the Fourier extension operator associated with the paraboloid in $\mathbb{R}^{d+1}$

$$\mathbb{P} = \{(\xi, \xi_{d+1}) \in \mathbb{R}^d \times \mathbb{R} |\ \xi_{d+1} = |\xi|^2\},$$

defined as

$$E_{\mathbb{P}}g(x,t) = \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)} g(\xi)d\xi.$$

In fact we have $u(x,t) = E_{\mathbb{P}}\widehat{u_0}(x,t)$.

Now, let us introduce the Wigner transform, defined as

$$W(g_1, g_2)(x, v) = \int_{\mathbb{R}^d} g_1(x-y)\overline{g_2(x+y)}e^{iv\cdot y}dy, \tag{8.2}$$

for $g_1, g_2 \in L^2(\mathbb{R}^d)$.

We refer the reader to [21] for more information about the Wigner transform and its role in Quantum Mechanics. In particular, we will use some of its properties.

One can show that

$$f(x, v, t) := W(u(\cdot, t), u(\cdot, t))(x, v)$$

satisfies the kinetic transport equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0.$$

Therefore we have

$$f(x, v, t) = f_0(x - tv, v),$$

where $f_0 = W(u_0, u_0)$ is the Wigner distribution of the initial data u_0.

The Wigner distribution has the following property:

$$\int_{\mathbb{R}^d} W(g, g)(x, v)dv = |g(x)|^2.$$

Therefore we can write

$$|u(x, t)|^2 = \int_{\mathbb{R}^d} f(x, v, t)dv = \int_{\mathbb{R}^d} f_0(x - tv, v)dv =: \rho(f_0)(x, t). \tag{8.3}$$

The operator ρ, which is referred to as a *velocity averaging operator* in kinetic theory, resembles an adjoint space-time X-ray transform. In fact, we have that its adjoint is

$$\rho^*(g)(x, v) = \int_{\mathbb{R}} g(x - tv, t)dt,$$

which is of course an integral of the space-time function g along the line through the point $(x, 0)$ with direction $(-v, 1)$.

If we now look at the weighted L^2 norm of u, we can use (8.3) to write

$$\int_{\mathbb{R}^d\times\mathbb{R}} |u(x,t)|^2 w(x,t)dxdt = \int_{\mathbb{R}^d\times\mathbb{R}^d} W(u_0,u_0)(x,v)\rho^* w(x,v)dxdv. \tag{8.4}$$

Now, if we choose the initial data u_0 to be a real gaussian, then $W(u_0,u_0)$ is also a real gaussian. We can then use the non-negativity of the terms, together with Hölder's inequality, to get

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}} |u(x,t)|^2 w(x,t)dxdt &\leqslant \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} W(u_0,u_0)(x,v)dx\right)\sup_x \rho^* w(x,v)dv \\ &= \int_{\mathbb{R}^d} |\widehat{u}_0(v)|^2 \sup_x \rho^* w(x,v)dv,\end{aligned}$$

which in particular implies that

$$\int_{\mathbb{R}^d\times\mathbb{R}} |u(x,t)|^2 w(x,t)dxdt \leqslant \|\rho^* w\|_\infty \|u_0\|_2^2,$$

It is seems therefore natural to ask if we can have

$$\int_{\mathbb{R}^d\times\mathbb{R}} |E_{\mathbb{P}}g(x,t)|^2 w(x,t)dxdt \lesssim \int_{\mathbb{R}^d} |g(v)|^2 \sup_x \rho^* w(x,v)dv, \tag{8.5}$$

or the weaker

$$\int_{\mathbb{R}^d\times\mathbb{R}} |E_{\mathbb{P}}g(x,t)|^2 w(x,t)dxdt \lesssim \sup_{\substack{x\in\mathbb{R}^d \\ v\in \mathrm{spt}(g)}} \rho^* w(x,v)\ \|g\|_2^2 \tag{8.6}$$

for general $g \in L^2$. The inequalities (8.6) and (8.5) represent respectively and equivalent for the paraboloid of the Mizohata–Takeuchi conjecture and the Stein–Mizohata–Takeuchi conjecture.

Observe that it is enough to prove the conjecture for a compact piece of the paraboloid, for it to hold for the full paraboloid.

Proposition 8.1.1. *Let*

$$\mathbb{P}_R = \{(\xi,\xi_{d+1}) \in \mathbb{R}^d \times \mathbb{R} |\ \xi_{d+1} = |\xi|^2,\ \xi \in B_R(0)\}$$

be the truncated paraboloid and let $E_{\mathbb{P}_R}$ be the associated Fourier extension operator.

Then, if (8.5)*, or* (8.6)*, holds for $E_{\mathbb{P}_R}$ with constant C, so it does for $E_{\mathbb{P}}$, with the same constant.*

Proof. We prove the statement for (8.5). The other is analogous.

Let $g \in C_c^\infty(\mathbb{R}^n)$. Then, there exist $\lambda > 0$ such that $g_\lambda(\xi) := g(\lambda\xi)$ is supported in $B_R(0)$.

We can write

$$E_{\mathbb{P}}(g)(x,t) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} g(\xi) d\xi = \lambda^n \int_{\mathbb{R}^n} e^{i(\lambda x\cdot\eta + \lambda^2 t|\eta|^2)} g(\lambda\eta) d\eta = \lambda^n E_{\mathbb{P}_R} g_\lambda(\lambda x, \lambda^2 t).$$

In particular we have

$$\begin{aligned} \int |E_{\mathbb{P}} g(x,t)|^2 w(x,t) dxdt &= \lambda^{2n} \int |E_{\mathbb{P}_R} g_\lambda(\lambda x, \lambda^2 t)|^2 w(x,t) dxdt = \\ &= \lambda^{n-2} \int |E_{\mathbb{P}_R} g_\lambda(x,t)|^2 w(x/\lambda, t/\lambda^2) dxdt \leqslant \\ &\leqslant C\lambda^{n-2} \int |g_\lambda(\xi)|^2 \sup_x \rho^* w_\lambda(x,\xi) d\xi, \end{aligned}$$

where $w_\lambda(x,t) := w(x/\lambda, t/\lambda^2)$.

Now,

$$\begin{aligned} \rho^* w_\lambda(x,\xi) &= \int w((x+t\xi)/\lambda, t/\lambda^2) dt = \int w((x+t\xi)/\lambda, t/\lambda^2) dt = \\ &= \lambda^2 \int w(\lambda x + t\lambda\xi, t) = \lambda^2 \rho^* w(\lambda x, \lambda\xi) dt. \end{aligned}$$

Since taking a supremum over x or λx are equivalent, we obtain, by changing variable once again,

$$\int |E_{\mathbb{P}} g(x,t)|^2 w(x,t) dxdt \leqslant C\lambda^n \int |g_\lambda(\xi)|^2 \sup_x \rho^* w(x, \lambda\xi) d\xi = C \int |g(\xi)|^2 \sup_x \rho^* w(x,\xi) d\xi.$$

The thesis now follows via a standard approximation argument, using that C_c^∞ is dense in L^2. $\square$

8.2 Approximating the parabola with ellipses

We want to embed the parabola into a family of quadratic surfaces which contains both the paraboloid and the sphere, in order to study the connection between the respective Mizohata–Takeuchi inequalities.

We start by studying the planar case, so consider the family of conic sections $\mathcal{E}_e$ on the plane, with focus on the origin and parametrized by their eccentricity $e \in [0,1]$, given by the polar equation

$$r = r(\theta) = \frac{1}{1 + e\cos(\theta)}. \tag{8.7}$$

In particular $\mathcal{E}_e$ can be parametrized by the function

$$\varphi_e(\theta) = \left(\frac{\cos(\theta)}{1 + e\cos(\theta)}, \frac{\sin(\theta)}{1 + e\cos(\theta)} \right), \quad \theta \in (-\pi, \pi). \tag{8.8}$$

Of course here $\mathcal{E}_0$ corresponds to the circle $\mathbb{S}^1$, $\mathcal{E}_1$ to the parabola and $\mathcal{E}_e$, for $0 < e < 1$, are ellipses.

It is worth observing that, when defining the extension operator, we are not equipping the paraboloid with the usual surface measure, but the so called *affine surface measure*. In the 2-dimensional setting we talk about the *affine arc-length measure*, which is defined for a C^2-curve as

$$d\mu = k^{1/3} d\sigma, \tag{8.9}$$

where k is the curvature of the curve and $d\sigma$ is the usual arc-length measure of a rectifiable curve. Alternatively, and equivalently, given a parametrization $\phi(t)$ of the curve, we can use the formula

$$d\mu(t) = \left(\det \begin{pmatrix} \phi'(t) & \phi''(t) \end{pmatrix} \right)^{1/3} dt, \tag{8.10}$$

since $d\sigma(t) = \|\phi'(t)\|\, dt$ and

$$k(t) = \det \begin{pmatrix} \phi'(t) & \phi''(t) \end{pmatrix} \cdot \left\| \phi'(t) \right\|^{-3}. \tag{8.11}$$

We refer the reader to Guggenheimer ([28]) to understand the idea behind the definition of affine arc-length for curves in $\mathbb{R}^n$ and to the survey paper [47] for its generalization, that we will discuss in the next section.

We equip each $\mathcal{E}_e$ with its affine arc-length measure μ_e which we compute using our polar coordinate parametrization. We have

$$\varphi_e'(\theta) = \begin{pmatrix} \frac{-\sin(\theta)}{(1+e\cos(\theta))^2} \\ \frac{e+\cos(\theta)}{(1+e\cos(\theta))^2} \end{pmatrix}, \quad \varphi_e''(\theta) = \begin{pmatrix} \frac{-\cos(\theta)(1+e\cos(\theta))-2e\sin^2(\theta)}{(1+e\cos(\theta))^3} \\ \frac{-\sin(\theta)(1+e\cos(\theta))+2e(e+\cos(\theta))\sin(\theta)}{(1+e\cos(\theta))^3} \end{pmatrix},$$

and so we obtain

$$d\mu_e(\theta) = \frac{d\theta}{1+e\cos(\theta)}. \tag{8.12}$$

In particular, we have that the curvature of the curve $\mathcal{E}_e$ is

$$k_e(t) = \frac{1}{(\sqrt{1+2\cos(\theta)+e^2})^3}.$$

Now we need to define a suitable "X-ray transform" for each $\mathcal{E}_e$, which has to coincide with the known cases for the circle and the parabola. From the computation of the tangent vector φ_e', we observe that a normal vector to each $\mathcal{E}_e$ at the point of parameter θ is

$$\mathbf{n}_e(\theta) = \left(\frac{e+\cos(\theta)}{1+e\cos(\theta)}, \frac{\sin(\theta)}{1+e\cos(\theta)} \right). \tag{8.13}$$

This corresponds to the unit normal vectors to the circle for $e = 0$ and to vectors of the form $(1, y)$ for a parabola of coordinates (x, y), like the ones we saw in the previous section in the definition of ρ^*.

Therefore, define the following variant of the X-ray transform, adapted to each conic:

$$X_e f(\mathbf{n}_e, v) = \int_{\mathbb{R}} f(\mathbf{n}_e t + v)dt, \quad v \in \mathbb{R}^2. \tag{8.14}$$

This can be seen as an extension of the domain of the classical X-ray transform, since we are not only considering unitary vectors for the direction of the lines we integrate on.

Remark. X_1 correspond to ρ^* up to some constants, depending on the fact that the paraboloid $\mathcal{E}_1$ is not the paraboloid of equation $x_n = \sum x_j^2$, which is the one considered on previous section. The two are of course equivalent up to an affine transformation.

We can now conjecture the two following inequalities for the whole family of conics:

$$\int_{\mathbb{R}^2} |\widehat{gd\mu_e}|^2(x)w(x)dx \lesssim \int_{-\pi}^{\pi} |g(\theta)|^2 \sup_v X_e w(\mathbf{n}_e(\theta), v)d\mu_e(\theta), \tag{8.15}$$

$$\int_{\mathbb{R}^2} |\widehat{gd\mu_e}|^2(x)w(x)dx \lesssim \|X_e w\|_\infty \int_{-\pi}^{\pi} |g(\theta)|^2 d\mu_e(\theta). \tag{8.16}$$

Proposition 8.2.1. *If* (8.15) *is true for a value of* $e \in [0,1)$*, then its true for every* $0 \leqslant e < 1$ *with the same constant. Similarly, for* (8.16).

Proof. From (8.7), we can deduce that the equation of $\mathcal{E}_e$ in Cartesian coordinates, for $0 \leqslant e < 1$, is

$$(1-e^2)^2\left(x_1 + \frac{e}{1-e^2}\right)^2 + (1-e^2)x_2^2 = 1. \tag{8.17}$$

So we can define the affine map $T_e : \mathbb{S}^1 \to \mathcal{E}_e$ as

$$T_e(\omega) = A_e^{-1}\omega - b_e,$$

where

$$A_e = \begin{pmatrix} (1-e^2) & 0 \\ 0 & \sqrt{1-e^2} \end{pmatrix}, \quad b_e = \begin{pmatrix} \frac{e}{1-e^2} \\ 0 \end{pmatrix},$$

that brings the circle to the ellipse bijectively. Observe that T_1 is not well defined. In fact, there is no affine function, nor continuous function, that maps the sphere to the parabola, since the sphere is compact. We prove that if (8.15) is true for $\mathbb{S}^1$, then its true for $\mathcal{E}_e$. The converse implication follows from the same argument, but using the map T_e^{-1}. The proposition then follows by transitivity.

From the definition (8.10), we can easily check that affine arc-length measure is preserved by affine change of coordinates, up to the cubic root of the determinant of the matrix of the affine map. So we can write

$$\begin{aligned}\widehat{gd\mu_e}(x) &= \int e^{ix\cdot\xi}g(\xi)d\mu_e(\xi) = (1-e^2)^{-1/2}\int e^{ix\cdot T_e(\eta)}g(T_e(\eta))d\sigma(\eta) = \\ &= (1-e^2)^{-1/2}e^{-ix\cdot b_e}\int e^{ix\cdot A_e^{-1}(\eta)}g_e(\eta)d\sigma(\eta) = (1-e^2)^{-1/2}e^{-ix\cdot b_e}\widehat{g_e d\sigma}(A_e^{-1}x),\end{aligned}$$

where $g_e(\eta) = g(T_e(\eta))$.

Therefore we have

$$\int |\widehat{gd\mu_e}|^2(x)w(x)dx = (1-e^2)^{-1}\int |\widehat{g_e d\sigma}|^2(A_e^{-1}x)w(x)dx =$$
$$= (1-e^2)^{1/2}\int |\widehat{g_e d\sigma}|^2(y)w(A_e y)dy = (1-e^2)^{1/2}\int |\widehat{g_e d\sigma}|^2(y)w_e(y)dy,$$

where $w_e(y) = w(A_e y)$.

By hypothesis, we then have

$$\int |\widehat{gd\mu_e}|^2(x)w(x)dx \lesssim (1-e^2)^{1/2}\int |g_e(\omega)|^2 \sup_v Xw_e(\omega, v)d\sigma(\omega).$$

Now, we change variable on the right-hand side, by writing $\omega = T_e^{-1}(\xi) = A_e(\xi + b_e)$, $\xi \in \mathcal{E}_e$. so we get

$$\int |\widehat{gd\mu_e}|^2(x)w(x)dx \lesssim (1-e^2)\int |g(\xi)|^2 \sup_v Xw_e(T_e^{-1}(\xi), v)d\mu_e(\xi).$$

So we only need to prove that

$$(1-e^2)Xw_e(T_e^{-1}(\xi), v) = X_e w(\mathbf{n}_e\xi, \tilde{v}),$$

for some translation variable $v, \tilde{v}$.

We have

$$Xw_e(T_e^{-1}(\xi), v) = \int w(A_e^2(\xi + b_e)t + A_e v)dt$$

and

$$A_e^2(\xi + b_e) = A_e^2 T_e(\omega) + A_e^2 b_e = A_e\omega.$$

First we want to show that $A_e\omega$ is normal to $\mathcal{E}_e$ at the point $T_e(\omega)$.

Call

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is the matrix that correspond to a rotation of angle $\pi/2$.

If $\varphi(t)$ is a parametrization of $\mathbb{S}^1$, then $T_e \circ \varphi(t)$ is a parametrization of $\mathcal{E}_e$. Let $\omega = \varphi(t)$. The tangent vector to $\mathbb{S}^1$ at ω can be written as $\varphi'(t) = R\omega$. The normal of $\mathcal{E}_e$ at the point $T_e(\omega)$ is given by a rotation of angle $\pi/2$ of the tangent vector, which can be written, using the parametrization, as

$$\mathbf{n}_e T_e(\omega) = R\frac{dT_e \circ \varphi}{dt}(t) = RA_e^{-1}\varphi'(t) = RA_e^{-1}R\omega.$$

So we want that $RA_e^{-1}R\omega = \lambda A_e\omega$, for some $\lambda \in \mathbb{R}$. But this is true since

$$RA_e^{-1}RA_e^{-1} = \det(A_e)^{-1}\mathrm{Id}$$

as one can check by simply observing that multiplying a diagonal matrix left and right by the matrix R exchanges the values of the diagonal terms. Therefore we know that,

$$Xw_e(T_e^{-1}(\xi), v) = Xw(c\mathbf{n}_e\xi, \tilde{v}) = c^{-1}X_e w(\mathbf{n}_e\xi, \tilde{v}),$$

where

$$c = \frac{\|A_e\omega\|}{\|\mathbf{n}_e T_e(\omega)\|}.$$

Using the polar coordinate representation of $\mathcal{E}_e$,

$$\|\mathbf{n}_e(\theta)\|^2 = \frac{(e+\cos(\theta))^2 + \sin^2(\theta)}{(1+e\cos(\theta))^2}$$

and

$$\|A_e\omega\|^2 = \frac{(1+2e\cos(\theta)+e^2)(1-e^2)^2}{(1+e\cos(\theta))^2},$$

where $T_e(\omega)$ and θ represent the same point on $\mathcal{E}_e$. In particular, $c = 1 - e^2$ and therefore the thesis is proved.

The proof of the statement about (8.16) is analogous. □

Proposition 8.2.2. *If* (8.16) *is true for* $e = 0$ *(circle), then it is true for* $e = 1$ *(parabola) for positive weights.*

Proof. We prove the statement for a compact piece of the parabola, since we saw that this implies the result for the full parabola.

In this proof, instead of working with the conics themselves, we will use the polar representation to just work on the interval $[-\pi/2, \pi/2]$, equipped with the different measures μ_e. In particular, we start with a smooth function g defined on $[-\pi/2, \pi/2]$.

For $0 \leqslant e \leqslant 1$, consider the family of maps from $(-\pi, \pi)$ to the normal bundle $\mathbf{N}\mathcal{E}_e$ of $\mathcal{E}_e$

$$\begin{aligned} F_e \colon (-\pi, \pi) &\to \mathbf{N}\mathcal{E}_e \\ \theta &\mapsto \mathbf{n}_e(\theta). \end{aligned}$$

Observe that on $[-\pi/2, \pi/2]$ the maps F_e are bijective.

Define $g_e(\theta) = g(F_1^{-1} \circ F_e(\theta))$. We have to think of g_e as the function g, initially defined on the parabola, glued on the ellipse $\mathcal{E}_e$ in a way so that the set of normal directions to the parabola on the support of g is preserved.

By hypothesis, (8.16) is true for the circle and therefore for every ellipse. So we can consider the family of inequalities

$$\int |\widehat{g_e d\mu_e}|^2(x)w(x)dx \lesssim \|X_e w\|_\infty \int_{-\pi/2}^{\pi/2} |g_e(\theta)|^2 d\mu_e(\theta),$$

for $0 < e < 1$. We want to obtain the thesis by taking the limit for $e \to 1$. For the left-hand side, observe that $\widehat{g_e d\mu_e} \to \widehat{g d\mu_1}$ converge pointwise by dominated convergence. Therefore we can use Fatou's Lemma to get

$$\int |\widehat{g d\mu_1}|^2(x)w(x)dx \lesssim \lim_{e \to 1} \int |\widehat{g_e d\mu_e}|^2(x)w(x)dx.$$

On the right-hand side, we have to deal with two terms. The L^2 norm simply converges by dominated convergence, since the measures are uniformly bounded on $[-\pi/2, \pi/2]$. To take the limit on the X-ray transform we have to be more careful. First observe that, since $\|\mathbf{n}_e\| \geqslant 1$, with a change of variable we have

$$X_e f(\mathbf{n}_e, v) \leqslant X f\left(\frac{\mathbf{n}_e}{\|\mathbf{n}_e\|}, v\right).$$

Now, thanks to the way we defined the g_e, the family of vectors $\mathbf{n}_e / \|\mathbf{n}_e\|$ does not depend on e. So we have

$$\int |\widehat{g d\mu_1}|^2(x)w(x)dx \lesssim \|Xw\|_\infty \|g\|_{L^2(\mu_1)}^2.$$

But now, since we are working on a compact subset of the parabola, the normal vectors have a maximum norm and therefore we can find a constant such that $\|Xw\|_\infty \lesssim \|X_1 w\|_\infty$. □

Proposition 8.2.3. *If* (8.15) *is true for* $e = 0$ *(circle), then it is true for* $e = 1$ *(parabola) for weights of class* C_c *(continuous and compactly supported).*

Proof. The proof is the same as the one for the previous Proposition and it only differs when taking the limit on the right-hand side of our inequality

$$\int_{-\pi/2}^{\pi/2} |g_e(\theta)|^2 \sup_v X_e w(\mathbf{n}_e(\theta), v) d\mu_e(\theta).$$

Observe that, since $w \in C_c$, $\|Xw\|_\infty < \infty$, so we know that the integrands are uniformly bounded and we can use dominated convergence. We only need to show that

$$\sup_v X_e w(\mathbf{n}_e(\theta), v) \longrightarrow \sup_v X_1 w(\mathbf{n}_1(\theta), v)$$

pointwise, as $e \to 1$. Now,

$$X_e w(\mathbf{n}_e(\theta), v) = \int_{\mathbb{R}} w(\mathbf{n}_e(\theta)t + v)dt.$$

Clearly $\mathbf{n}_e \to \mathbf{n}_1$ and so, by continuity of w, $X_e w(\mathbf{n}_e(\theta), v) \to X_1 w(\mathbf{n}_1(\theta), v)$.

Since w is compactly supported and continuous, the same holds for $W(e, v) := X_e w(\mathbf{n}_e(\theta), v)$, which is then, in particular, uniformly continuous. Therefore the convergence is uniform over the parameter v, and so we have our thesis. □

8.3 Approximating the paraboloid with ellipsoids

We now want to extend the results of previous section to higher dimensions.

We define our family of quadratic surfaces as rotational hypersurfaces, starting from the 2-dimensional setting we already studied. In other words, consider the spherical coordinates $(r, \theta_1, ..., \theta_{n-1})$ in $\mathbb{R}^n$ and define the family $\mathcal{E}_e$ of quadratic hypersurfaces parametrized by their eccentricity $e \in [0, 1]$, via the equation

$$r = r(\theta_1) = \frac{1}{1 + e\cos(\theta_1)}. \tag{8.18}$$

In particular $\mathcal{E}_e$ can be parametrized by the function

$$\begin{aligned}\varphi_e(\theta_1, ..., \theta_{n-1}) = \Bigg(&\frac{\cos(\theta_1)}{1+e\cos(\theta_1)}, \frac{\sin(\theta_1)\cos(\theta_2)}{1+e\cos(\theta_1)}, ... \\ &..., \frac{1}{1+e\cos(\theta_1)}\left(\prod_{\ell=1}^{j-1}\sin(\theta_\ell)\right)\cos(\theta_j), ... \\ &..., \frac{\sin(\theta_1)...\sin(\theta_{n-2})\cos(\theta_{n-1})}{1+e\cos(\theta_1)}, \frac{\sin(\theta_1)...\sin(\theta_{n-1})}{1+e\cos(\theta_1)}\Bigg),\end{aligned} \tag{8.19}$$

$\theta_1 \in (-\pi, \pi)$, $\theta_2, ..., \theta_{n-1} \in (-\pi/2, \pi/2)$.

$\mathcal{E}_0$ corresponds to the sphere $\mathbb{S}^1$, $\mathcal{E}_1$ to the paraboloid and $\mathcal{E}_e$, for $0 < e < 1$, are ellipsoids.

The affine surface measure of an hypersurface is defined as

$$d\mu = k^{1/(n+1)} d\sigma, \tag{8.20}$$

where k is the Gaussian curvature, which is given by the product of the principal curvatures, and $d\sigma$ is the standard surface measure.

Given a parametrization $\phi(t_1, ..., t_{n-1})$, the affine surface measure can be written as

$$d\mu(t) = \left|\det\left(\det\left(\frac{\partial \phi}{\partial t_1}, ..., \frac{\partial \phi}{\partial t_{n-1}}, \frac{\partial^2 \phi}{\partial t_i \partial_j}\right)\right)_{i,j=1}^{n-1}\right|^{\frac{1}{n+1}} dt_1...dt_{n-1}. \tag{8.21}$$

In our case, this measure is easy to compute since we are working with a rotational hypersurface. The surface area is given by the the arc-length measure of the profile curve and the surface area of the $n-2$-dimensional sphere of rotation of radius given by the distance of the curve from the rotation axis. In spherical coordinates this is

$$[\sin(\theta_1) r(\theta_1)]^{n-2} [r(\theta_1)\sqrt{1+2\cos(\theta_1)+e^2}]^2 \sin^{n-2}(\theta_2)...\sin(\theta_{n-2}) d\theta_1...d\theta_{n-1}.$$

The principal curvatures are given by the curvature of the profile curve $(\sqrt{1+2\cos(\theta_1)+e^2})^{-3}$ and and the curvature along the directions of rotation. To compute these we can use Meusnier's formula:

Proposition 8.3.1 (Meusnier's formula)**.** *Let C be a curve on a manifold M passing through $p \in M$ and with unit tangent vector V. Then the curvature of M in direction V at p is $k\cos(\alpha)$,* *where k is the curvature of C at p and α is the angle between the normal of α and the normal of M at p.*

See [49] for a proof.

Given the symmetry of the the problem, the other $n-2$ principal curvatures are all equal. We compute the curvature along the circle of coordinates $(\theta_1, \theta_2, 0, ..., 0)$. This circle has curvature $[\sin(\theta_1)r(\theta_1)]^{-1}$ and normal $N = (0, 1, 0, ..., 0)$, while

$$\cos(\alpha) = \frac{\langle N, \partial_{\theta_1}\varphi_e\rangle}{\|\partial_{\theta_1}\varphi_e\|} = \frac{\sin(\theta_1)r(\theta_1)^3}{r(\theta_1)^2(\sqrt{1+2\cos(\theta_1)+e^2})}.$$

Therefore, the principal curvatures are $[\sqrt{1+2\cos(\theta_1)+e^2}]^{-1}$ and the affine surface measure is

$$d\mu_e(\theta) = r(\theta_1)^n \sin^{n-1}(\theta_1)\sin^{n-2}(\theta_2)...\sin(\theta_{n-2})d\theta_1...d\theta_{n-1}.$$

Remark. Observe that $d\mu_e(\theta)$ has the same form of the Lebesgue measure in $\mathbb{R}^n$ in polar coordinates, with the difference that the radius variable depends on the angular one.

We define the variant of the X-ray transform as in the planar case:

$$X_e f(\mathbf{n}_e, v) = \int_{\mathbb{R}} f(\mathbf{n}_e t + v)dt, \quad v \in \mathbb{R}^n, \tag{8.22}$$

where the normal vectors $\mathbf{n}_e$ are just the vectors of the planar case rotated around the rotational axis.

We can now conjecture the two following inequalities for the whole family of quadratic hypersurfaces:

$$\int_{\mathbb{R}^n} |\widehat{gd\mu_e}|^2(x)w(x)dx \lesssim \int |g(\theta)|^2 \sup_v X_e w(\mathbf{n}_e(\theta), v)d\mu_e(\theta), \tag{8.23}$$

$$\int_{\mathbb{R}^n} |\widehat{gd\mu_e}|^2(x)w(x)dx \lesssim \|X_e w\|_\infty \int |g(\theta)|^2 d\mu_e(\theta). \tag{8.24}$$

Proposition 8.3.2. *If* (8.23) *is true for a value of* $e \in [0, 1)$, *then its true for every* $0 \leqslant e < 1$ *with the same constant. Similarly, for* (8.24).

Proof. From (8.18), we can deduce that the equation of $\mathcal{E}_e$ in Cartesian coordinates, for $0 \leqslant e <$

1, is

$$(1-e^2)^2\left(x_1+\frac{e}{1-e^2}\right)^2+(1-e^2)\sum_{j=2}^{n}x_j^2=1. \tag{8.25}$$

So we can define the affine map $T_e:\mathbb{S}^{n-1}\to\mathcal{E}_e$ as

$$T_e(\omega)=A_e^{-1}\omega-b_e,$$

where

$$A_e=\begin{pmatrix}(1-e^2) & 0 & 0 & \dots & 0\\ 0 & \sqrt{1-e^2} & 0 & \dots & 0\\ 0 & 0 & \sqrt{1-e^2} & \dots & 0\\ \vdots & & & & \vdots\\ 0 & 0 & \dots & 0 & \sqrt{1-e^2}\end{pmatrix},\quad b_e=\begin{pmatrix}\frac{e}{1-e^2}\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

T_e brings the sphere to the ellipsoids bijectively. Again, observe that T_1 is not well defined. We proceed as in the planar case and prove that if (8.23) is true for $\mathbb{S}^n$, then its true for $\mathcal{E}_e$. The converse implication follows from the same argument, but using the map T_e^{-1}. The proposition then follows by transitivity.

From the definition (8.21), we can easily check that affine surface measure is preserved by affine change of coordinates, up to the $\frac{n-1}{n+1}$ power of the determinant of the matrix of the affine map. So we can write

$$\begin{aligned}\widehat{gd\mu_e}(x)&=\int e^{ix\cdot\xi}g(\xi)d\mu_e(\xi)=|(1-e^2)^{\frac{n+1}{2}}|^{-\frac{n-1}{n+1}}\int e^{ix\cdot T_e(\eta)}g(T_e(\eta))d\sigma(\eta)=\\ &=(1-e^2)^{-\frac{n-1}{2}}e^{-ix\cdot b_e}\int e^{ix\cdot A_e^{-1}(\eta)}g_e(\eta)d\sigma(\eta)=(1-e^2)^{-\frac{n-1}{2}}e^{-ix\cdot b_e}\widehat{g_e d\sigma}(A_e^{-1}x),\end{aligned}$$

where $g_e(\eta)=g(T_e(\eta))$.

Therefore, we have

$$\begin{aligned}\int|\widehat{gd\mu_e}|^2(x)w(x)dx&=(1-e^2)^{1-n}\int|\widehat{g_e d\sigma}|^2(A_e^{-1}x)w(x)dx=\\ &=(1-e^2)^{\frac{3-n}{2}}\int|\widehat{g_e d\sigma}|^2(y)w(A_e y)dy=(1-e^2)^{\frac{3-n}{2}}\int|\widehat{g_e d\sigma}|^2(y)w_e(y)dy,\end{aligned}$$

where $w_e(y)=w(A_e y)$.

By hypothesis, we then have

$$\int |\widehat{gd\mu_e}|^2(x)w(x)dx \lesssim (1-e^2)^{\frac{3-n}{2}} \int |g_e(\omega)|^2 \sup_v Xw_e(\omega, v)d\sigma(\omega).$$

Now, we change variable on the right-hand side, by writing $\omega = T_e^{-1}(\xi) = A_e(\xi + b_e)$, $\xi \in \mathcal{E}_e$. So we get

$$\int |\widehat{gd\mu_e}|^2(x)w(x)dx \lesssim (1-e^2) \int |g(\xi)|^2 \sup_v Xw_e(T_e^{-1}(\xi), v)d\mu_e(\xi).$$

So we only need to prove that

$$(1-e^2)Xw_e(T_e^{-1}(\xi), v) = X_e w(\mathbf{n}_e\xi, \tilde{v}),$$

for some translation variable $v, \tilde{v}$.

We have

$$Xw_e(T_e^{-1}(\xi), v) = \int w(A_e^2(\xi + b_e)t + A_e v)dt$$

and

$$A_e^2(\xi + b_e) = A_e^2 T_e(\omega) + A_e^2 b_e = A_e\omega.$$

First we want to show that $A_e\omega$ is normal to $\mathcal{E}_e$ at the point $T_e(\omega)$. But this follows from the 2-dimensional case, since both vectors are rotations of the vectors appearing in the 2-dimensional case. Therefore we know that,

$$Xw_e(T_e^{-1}(\xi), v) = Xw(c\mathbf{n}_e\xi, \tilde{v}) = c^{-1}X_e w(\mathbf{n}_e\xi, \tilde{v}),$$

where

$$c = \frac{\|A_e\omega\|}{\|\mathbf{n}_e T_e(\omega)\|}.$$

Again, the norms are the same of the 2-dimensional vectors, so $c = 1 - e^2$ and therefore the thesis is proved.

The proof of the statement about (8.24) is analogous. □

Proposition 8.3.3. *If* (8.24) *is true for* $e = 0$ *(sphere), then it is true for* $e = 1$ *(paraboloid) for positive weights.*

Proof. The proof is analogous to the planar case. We only need to be carefulwe defining the functions g_e. In fact we only need to modulate the function g on the variable θ_1. So, given the maps F_e as in the previous section,

$$g_e(\theta_1, \theta_2, ..., \theta_{n-1}) = g(F_1^{-1} \circ F_e(\theta_1), \theta_2, ..., \theta_{n-1}).$$

$\square$

Proposition 8.3.4. *If* (8.23) *is true for $e = 0$ (sphere), then it is true for $e = 1$ (paraboloid) for weights of class C_c.*

Proof. Again, the proof follows the planar case. In particular, we use that $w \in C_c$ to prove that $\|Xw\|_\infty < \infty$ and

$$\sup_v X_e w(\mathbf{n}_e(\theta), v) \longrightarrow \sup_v X_1 w(\mathbf{n}_1(\theta), v)$$

pointwise in θ, so that the thesis follows from dominated convergence. $\square$

Further Problem. In this chapter we saw how it was necessary to introduce for each ellipsoid, its own X-ray transform. This might suggest us that, in order to formulate a very general Mizohata–Takeuchi conjecture for a measure μ, it might be necessary to introduce a particular X-ray tranform depending on the measure μ. This approach is somewhat investigated in [9], where the authors introduce a certain Wigner transform on each surface, which in turn induces a certain X-ray type transform, like in the case of the paraboloid.

CHAPTER 9

THE MIZOHATA–TAKEUCHI PROBLEM ON GROUPS

We have already introduced an analogue of the k-plane transform in the setting of LCA groups. We now want to use it to try formulate an analogue of the Mizohata–Takeuchi conjecture in this more abstract setting.

9.1 Setting up the problem

This work was partially inspired by a work of Bennett and Jeong ([11]) on Brascamp-Lieb inequalities.

Let $H_1, ..., H_m$ be euclidean spaces and consider a subspace H of $H_1 \times \cdots \times H_m$. Given an m-tuple of exponents $\mathbf{p} = (p_1, ..., p_m) \in [1, +\infty]^m$, the Brascamp-Lieb problem consists in establishing the best constant $BL(H, \mathbf{p})$ for the Brascamp-Lieb inequality

$$\left| \int_H f_1 \otimes \cdots \otimes f_m \right| \leqslant BL(H, \mathbf{p}) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(H_j)} \tag{9.1}$$

to hold. In [7] it was established that

$$BL(H, \mathbf{p}) \cong BL(H^\perp, \mathbf{p}'), \tag{9.2}$$

where $\mathbf{p}'$ is the Hölder conjugate of $\mathbf{p}$ and $H^\perp$ is the orthogonal complement of H.

In [11] this result is generalized to the setting of LCA groups. In particular the concept of "orthogonal complement" gets replaced by a notion of Fourier duality: if H is a subgroup of G, then the role of $H^\perp$ is played by the annihilator of H in $\widehat{G}$, $N(H)$.

Our aim is to study an analogue of the Mizohata–Takeuchi conjecture in the setting of LCA groups, in which we try to replace the notion of normal to the support of a measure using the Fourier duality. In particular, given an LCA group G and a measure σ supported on a proper subset of G, we can consider the associated extension operator

$$E_\sigma g(\xi) = \widehat{gd\sigma}(\xi) = \int_G \xi(x)g(x)d\sigma(x), \quad \xi \in \widehat{G}, \tag{9.3}$$

which will be a function defined on $\widehat{G}$, assuming that $g \in L^1(d\sigma)$.

We would then be looking at estimates of the form

$$\int_{\widehat{G}} |\widehat{gd\sigma}(\xi)|^2 w(\xi) d\mu_{\widehat{G}}(\xi) \lesssim \|Tw\|_\infty \int |g|^2 d\sigma, \tag{9.4}$$

where the supremum for the transform T is taken over a suitable set of subgroups of $\widehat{G}$, possibly chosen by having some duality relation with the support of σ, living in G.

For a very general measure σ on G it is not clear which subgroups we should consider, so we will be happy with just taking the supremum over all possible subgroups, like we did when discussing general measures in the euclidean case.

9.2 Some simple results

We list here some simple results we already proved in the euclidean case, that are still true for LCA groups.

One simple case where it is easy to define what is the “normal” to the support of a measure is when we consider a subgroup H of G equipped with its Haar measure μ_H. In fact we would only be dealing with the annihilator $N(H)$. This correspond to the euclidean case of the extension operator for a k-plane in $\mathbb{R}^n$, for which we saw that the Mizohata–Takeuchi inequality holds.

This is still true for a generic LCA group.

Proposition 9.2.1. *Let G be an LCA group and H a closed subgroup of G. Then*

$$\int_{\widehat{G}} |\widehat{gd\mu_H}(\xi)|^2 w(\xi) d\mu_{\widehat{G}}(\xi) \leqslant \|Tw(N(H), \cdot))\|_\infty \int_H |g|^2 d\mu_H. \tag{9.5}$$

Proof. The proof is analogue to the one in the euclidean case.

If $\eta \in N(H)$, then $\widehat{gd\mu_H}(\eta\xi) = \widehat{gd\mu_H}(\xi)$, for any $\xi \in \widehat{G}$. So we have

$$\begin{aligned}\int_{\widehat{G}} |\widehat{gd\mu_H}|^2 w d\mu_{\widehat{G}} &= \int_{\widehat{G}/N} \int_N |\widehat{gd\mu_H}(\eta\xi)|^2 w(\eta\xi) d\mu_N(\eta) d\mu_{\widehat{G}/N}(\xi) = \\ &= \int_{\widehat{G}/N} |\widehat{gd\mu_H}(\xi)|^2 \int_N w(\eta\xi) d\mu_N(\eta) d\mu_{\widehat{G}/N}(\xi) \leqslant \\ &\leqslant \|Tw(N(H),\cdot))\|_\infty \int_{\widehat{G}/N} |\widehat{gd\mu_H}(\xi)|^2 d\mu_{\widehat{G}/N}(\xi).\end{aligned}$$

Now the result follows by using Plancherel's theorem after observing that $\widehat{G}/N$ is isomorphic to $\widehat{H}$ and that $\widehat{gd\mu_H}$ is just the fourier transform of g in H. So,

$$\int_{\widehat{G}/N} |\widehat{gd\mu_H}|^2 d\mu_{\widehat{G}/N} = \int_{\widehat{H}} |\widehat{g}|^2 d\mu_{\widehat{H}} = \int_H |g|^2 d\mu_H.$$

□

When discussing what we called MTp estimates in the euclidean case, we observed how the $MT1$ estimate is always true for any compactly supported measure. This is true for LCA groups as well. In fact we still have that $\|Tw\|_{L^\infty_H L^1_v} = \|w\|_{L^1}$ and therefore

$$\int |\widehat{gd\sigma}|^2 w \leqslant \|w\|_{L^1} \left\|\widehat{gd\sigma}\right\|^2_{L^\infty} \leqslant \|Tw\|_{L^\infty_H L^1_v} \|g\|^2_{L^1(d\sigma)}$$

and $\|g\|_{L^1(d\sigma)} \lesssim \|g\|_{L^2(d\sigma)}$, if σ is a finite measure.

In order to mimic some other results that we explored in this thesis, we will specialize the problem to a particular group.

9.3 The 2-dimensional torus and $\mathbb{Z}^2$

Our goal for the next two sections is to study the Mizohata–Takeuchi problem for measures on the group $\mathbb{Z}^2$. To do so, we need to focus on the study of our Radon-like transform on its dual.

If we look at the circle group $\mathbb{S}^1$ as the quotient group $\mathbb{R}/\mathbb{Z}$, we can easily identify its characters acting on $\theta \in \mathbb{S}^1$ as the exponentials

$$e^{2\pi i n\theta}, \quad n \in \mathbb{Z}.$$

So we can identify the dual of $\mathbb{S}^1$ with $\mathbb{Z}$, and, therefore, we have the duality pairing of $\mathbb{Z}^2$ and $\mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$.

The Haar measure on $\mathbb{Z}^2$ is the counting measure. Since $\mathbb{T}^2 \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, the Haar measure on the torus can be seen as the Lebesgue measure on the square $[0,1]^2 \subset \mathbb{R}^2$.

On $\mathbb{Z}^2$, we are interested in the class of maximal 1-generated subgroups, and we call $\widehat{\mathcal{H}}$ their collection.

If H is such a subgroup, then there is a pair $(p,q) \in \mathbb{Z}^2$ such that $H = (p,q)\cdot\mathbb{Z}$, which is unique up to fixing for example the $q > 0$. Since H is maximal, it follows that $\mathrm{MCD}(p,q) = 1$. In fact, if $\mathrm{MCD}(p,q) = d > 1$, we would have $H \subset (p/d, q/d)\cdot\mathbb{Z}$. This means that every pair (p,q) correspond to the element $p/q \in \mathbb{Q}$, with the exception of the pair $(1,0)$, which we can make correspond to the point ∞. So, if we call $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, we have a bijection between $\widehat{\mathcal{H}}$ and $\overline{\mathbb{Q}}$. Using this correspondence, for $\mathfrak{q} \in \overline{\mathbb{Q}}$, we will refer to the corresponding subgroup in $\widehat{\mathcal{H}}$ as $N_{\mathfrak{q}}$. The subgroups in $\widehat{\mathcal{H}}$ are essentially discrete lines in the plane with rational angular coefficient passing through the origin, with N_∞ being the vertical line.

On $\mathbb{T}^2$, we consider the collection of subsets

$$\mathcal{H} = \{H \subset \mathbb{T}^2 |\ \exists \mathfrak{q} \in \overline{\mathbb{Q}} \text{ s.t. } N(H) = N_{\mathfrak{q}}\}.$$

To understand what these subgroups are, consider the quotient map

$$\mathrm{P} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2.$$

If $H \in \mathcal{H}$, since $\widehat{H} \cong \mathbb{Z}$, we must have $H \cong \mathbb{S}^1$ and $\mathrm{P}^{-1}(H)$ will be a collection of parallel lines in the plane, of which only one will be passing through the origin. Call this line ℓ. Say that

$N(H) = N_{\mathfrak{q}}$, where $\mathfrak{q} = (p, q)$. The relation between H and $N(H)$ tells us that

$$1 = e^{2\pi i n(p\theta_1 + q\theta_2)}, \quad \forall n \in \mathbb{Z},\ \theta \in \mathbb{T}^2,$$

which implies that

$$\ell = \{x \in \mathbb{R}^2 |\ px_1 + qx_2 = 0\} = (-q, p) \cdot \mathbb{R},$$

which is again a line with rational angular coefficient.

In particular, if we identify H with the line ℓ, $N(H)$ corresponds to the orthogonal complement of H in a classical sense, since they are related to a pair of perpendicular lines in the plane.

If we consider the map

$$\begin{aligned} \overline{\mathbb{Q}} &\longrightarrow \overline{\mathbb{Q}} \\ \mathfrak{q} = (p, q) &\longmapsto -\frac{1}{\mathfrak{q}} = (-q, p), \end{aligned}$$

and call $\ell_{\mathfrak{q}}$ the line with direction $\mathfrak{q}$ and $H_{\mathfrak{q}} = \mathrm{P}(\ell_{\mathfrak{q}})$, we have the relation quite elegant relation

$$N(H_{\mathfrak{q}}) = N_{-1/\mathfrak{q}}. \tag{9.6}$$

We are now ready to work with our transform on $\mathbb{T}^2$.

It is useful, as we we will see, to normalize the Haar measure on each $H \in \mathcal{H}$ to 1. In this way we can write, for a function f on $\mathbb{T}^2$,

$$Tf(H_{\mathfrak{q}}, v) = \int_0^1 f(\mathfrak{q}t + v)dt, \quad v \in \ell_{-1/\mathfrak{q}}. \tag{9.7}$$

We will refer to this operator as the X-ray transform on $\mathbb{T}^2$, by the clear analogy with the classical X-ray transform, since we are essentially integrating on lines.

Remark. It would be more correct to talk about Radon transform in this setting. In fact we started our argumentation by defining the orthogonal complement of our subgroups as 1-generated. Therefore it would be more correct to say that we are integrating on sets of codimension 1, as in the case of the Radon transform. Since we are working on total dimension 2, all these theoretical differences do not exist.

Remark. We could run a similar construction in higher dimensions, namely on $\mathbb{T}^d$. In this case we could be starting considering on $\mathbb{Z}^d$ k-generated subgroups, with $k \geqslant 1$, just like on $\mathbb{R}^n$ we have $T_{k,n}$, with $k \geqslant 1$. Most of what we discussed and will be discussing still works in higher dimension if considering 1-generated subgroups on $\mathbb{Z}^d$, but we have not dealt with the case of $k > 1$.

We are able to prove a result which is almost an analogue to the isometry property of the k-plane transform.

Proposition 9.3.1. *Let $f, g \in L^2(\mathbb{T}^2)$ and assume $\widehat{f}(0) = 0$. Then*

$$\sum_{\mathfrak{q}\in\overline{\mathbb{Q}}} \langle Tf(H_{\mathfrak{q}}, \cdot), Tg(H_{\mathfrak{q}}, \cdot)\rangle = \langle f, g\rangle. \tag{9.8}$$

Proof. We begin by observing that any $(n, m) \in \mathbb{Z}^2 \backslash \{0\}$ belongs to one and only one subgroup $N_{\mathfrak{q}}$. In particular

$$\mathbb{Z}^2 \backslash \{0\} = \bigcup_{\mathfrak{q}\in\overline{\mathbb{Q}}} (N_{\mathfrak{q}} \backslash \{0\}).$$

Using Plancherel's theorem on each subgroup we have

$$\sum_{\mathfrak{q}\in\overline{\mathbb{Q}}} \langle Tf(H_{\mathfrak{q}}, \cdot), Tg(H_{\mathfrak{q}}, \cdot)\rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(\mathfrak{q}n)\widehat{g}(\mathfrak{q}n) = \sum_{\xi\in\mathbb{Z}^2} \widehat{f}(\xi)\widehat{g}(\xi) = \langle f, g\rangle.$$

$\square$

Remark. Proposition 9.3.1 has two main differences with the euclidean case. First of all, we need the technical hypothesis $\widehat{f}(0) = 0$. This is due to the fact that all subgroups contain the origin, which is therefore counted infinitely many times if we were to simply base the proof on the fact that

$$\mathbb{Z}^2 = \bigcup_{\mathfrak{q}\in\overline{\mathbb{Q}}} N_{\mathfrak{q}}.$$

This also happens on $\mathbb{R}^n$, but in that case the origin has measure zero and, moreover, the pseudo-differential operator appearing helps removing singularities at the origin. The second difference is in fact that we do not have an analogue of the pseudo-differential operator appearing in the euclidean case. We do not exclude that such an operator might exist and that it might be needed to get rid of the problem at the origin, but we were not able to identify one. It would seem strange to us that this operator, that appears so naturally in the euclidean case, would be something very obscure in this setting.

9.4 L^1-Sobolev–Mizohata–Takeuchi for $\mathbb{Z}^2$

We now will discuss the Mizohata–Takeuchi problem for measures on $\mathbb{Z}^2$.

A measure on $\mathbb{Z}^2$ for us will typically be the characteristic function of a subset $\Lambda \subset \mathbb{Z}^2$, that we will keep calling μ. The corresponding extension operator will be defined as

$$Eg(\theta) = \widehat{g\mu}(\theta) = \sum_{x\in\Lambda} e^{2\pi i x\cdot\theta} g(x), \quad \theta \in \mathbb{T}^2. \tag{9.9}$$

As we did in Chapter 4, we want to use a Sobolev norm in place of the $\|Tw(H_{\mathfrak{q}},\cdot)\|_\infty$. Our "translation variable" lives on the space $\mathbb{T}^2/H_{\mathfrak{q}}$, that, as we discussed, can be associated to the line $\ell_{-1/\mathfrak{q}}$. So, since we want to take the L^1 norm on a line in place of the L^∞ norm, we need one derivative.

Define the differential operator of direction $\mathfrak{q}$, $(-\Delta_{\mathfrak{q}})^{\frac{\alpha}{2}}$, acting on functions on $\mathbb{T}^2/H_{-1/\mathfrak{q}}$ via the Fourier transform as

$$\mathcal{F}\left((-\Delta_{\mathfrak{q}})^{\frac{\alpha}{2}} F\right)(x) = |x|^\alpha \widehat{F}(x), \quad x \in \mathbb{N}_{\mathfrak{q}}. \tag{9.10}$$

In particular,

$$\mathcal{F}\left((-\Delta_{\mathfrak{q}})^{\frac{\alpha}{2}} Tf(H_{-1/\mathfrak{q}},\cdot)\right)(x) = |x|^\alpha \widehat{f}(x), \quad x \in \mathbb{N}_{\mathfrak{q}}. \tag{9.11}$$

Remark. Recall that, if $x \in \mathbb{N}_{\mathfrak{q}}$, $x = \mathfrak{q}n$ for some $n \in \mathbb{Z}$.

Our goal is then to prove an estimate of the form

$$\int_{\mathbb{T}^2} |\widehat{g\mu}|^2 w \lesssim \sup_{\mathfrak{q}\in\overline{\mathbb{Q}}} \left\|(-\Delta_{\mathfrak{q}})^{\frac{1}{2}} Tw(H_{-1/\mathfrak{q}},\cdot)\right\|_{L^1} \sum_{\Lambda} |g|^2. \tag{9.12}$$

Since we do not have a true isometry property for our X-ray transform, we either have to assume that $\widehat{w}(0) = \int_{\mathbb{T}^2} w = 0$, which is an uninteresting case, because we want positive weights, or, like we are going to do, we have to discuss the origin term separately.

Unsurprisingly, we will still incur some ϵ-losses.

Theorem 9.4.1. *Assume that, for $\epsilon > 0$,*

$$\sup_{x\in\Lambda} \sum_{y\in\Lambda\setminus\{x\}} \frac{1}{|x-y|^{1+\epsilon}} < \infty. \tag{9.13}$$

Then

$$\int_{\mathbb{T}^2} |\widehat{g\mu}|^2 w \lesssim \left(\|Tw\|_{L^\infty} + \sup_{\mathfrak{q}\in\overline{\mathbb{Q}}} \left\|(-\Delta_{\mathfrak{q}})^{\frac{1+\epsilon}{2}} Tw(H_{-1/\mathfrak{q}},\cdot)\right\|_{L^1} \right) \sum_{\Lambda} |g|^2. \tag{9.14}$$

Proof. Applying Plancherel's theorem, we have

$$\int_{\mathbb{T}^2} |\widehat{g\mu}|^2 w = \widehat{w}(0)\mathcal{F}(|\widehat{g\mu}|^2)(0) + \sum_{\mathbb{Z}^2\setminus\{0\}} \widehat{w}\mathcal{F}(|\widehat{g\mu}|^2).$$

First, we discuss the term at the origin. For the weight, we have

$$\widehat{w}(0) = \int_{\mathbb{T}^2} w \leqslant \int_{\mathbb{T}^2/H_{\mathfrak{q}}} \left(\int_{H_{\mathfrak{q}}} w \right) \leqslant \|Tw\|_{L^\infty}.$$

On the other side, by Plancherel's theorem

$$\mathcal{F}(|\widehat{g\mu}|^2)(0) = \int_{\mathbb{T}^2} |\widehat{g\mu}|^2 = \sum_{\Lambda} |g|^2.$$

To deal with the rest of the terms, the proof will be analogous to what we have done in the euclidean case. After multiplying and dividing by $|\cdot|^{1+\epsilon}$, we can use Proposition 9.3.1 and Hölder's inequality to get.

$$\sum_{z\in\mathbb{Z}^2\setminus\{0\}} |z|^{1+\epsilon}\widehat{w}(z)\mathcal{F}(|\widehat{g\mu}|^2)(z)\frac{1}{|z|^{1+\epsilon}} = \sum_{\mathfrak{q}\in\overline{\mathbb{Q}}} \sum_{n\in\mathbb{Z}\setminus\{0\}} |\mathfrak{q}n|^{1+\epsilon}\widehat{w}(\mathfrak{q}n)\mathcal{F}(|\widehat{g\mu}|^2)(\mathfrak{q}n)\frac{1}{|\mathfrak{q}n|^{1+\epsilon}} \leqslant$$
$$\leqslant \sup_{\mathfrak{q}\in\overline{\mathbb{Q}}} \left\|(-\Delta_{\mathfrak{q}})^{\frac{1+\epsilon}{2}} Tw(H_{-1/\mathfrak{q}},\cdot)\right\|_{L^1} \cdot \sum_{\mathfrak{q}\in\overline{\mathbb{Q}}} \left\|K(-\Delta_{\mathfrak{q}})^{-\frac{1+\epsilon}{2}} T(|\widehat{g\mu}|^2)(H_{-1/\mathfrak{q}},\cdot)\right\|_{L^\infty},$$

where $Kf = \mathcal{F}^{-1}(\widehat{f} - \widehat{f}(0))$.

So we are left with proving that

$$\sum_{\mathfrak{q}\in\overline{\mathbb{Q}}} \left\|K(-\Delta_{\mathfrak{q}})^{-\frac{1+\epsilon}{2}} T(|\widehat{g\mu}|^2)(H_{-1/\mathfrak{q}},\cdot)\right\|_{L^\infty} \lesssim \|g\|_2^2$$

We have that

$$K(-\Delta_{\mathfrak{q}})^{-\frac{1+\epsilon}{2}}T(|\widehat{g\mu}|^2)(H_{-1/\mathfrak{q}},v) = \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{2\pi i v\cdot \mathfrak{q}n}\frac{1}{|\mathfrak{q}n|^{1+\epsilon}} g\mu * \widetilde{g\mu}(\mathfrak{q}n) =$$
$$= \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{2\pi i v\cdot \mathfrak{q}n}\frac{1}{|\mathfrak{q}n|^{1+\epsilon}} \sum_{x\in\Lambda}\sum_{y\in\Lambda} g(x)\overline{g}(y)\delta(\mathfrak{q}n - x + y),$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore

$$\sum_{\mathfrak{q}\in\overline{\mathbb{Q}}} \left\| K(-\Delta_{\mathfrak{q}})^{-\frac{1+\epsilon}{2}}T(|\widehat{g\mu}|^2)(H_{-1/\mathfrak{q}},\cdot)\right\|_{L^\infty} \leqslant \sum_{\mathfrak{q}\in\overline{\mathbb{Q}}}\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{|\mathfrak{q}n|^{1+\epsilon}} \sum_{x\in\Lambda}\sum_{y\in\Lambda} |g(x)||g(y)|\delta(\mathfrak{q}n - x + y) =$$
$$= \sum_{z\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{|z|^{1+\epsilon}} \sum_{x\in\Lambda}\sum_{y\in\Lambda} |g(x)||g(y)|\delta(z - x + y) =$$
$$= \sum_{x\in\Lambda}\sum_{y\in\Lambda\setminus\{x\}} \frac{1}{|x-y|^{1+\epsilon}}|g(x)||g(y)|.$$

So the result follows by using hypothesis (9.13) and the Schur's test. □

Remark. If μ is not the characteristic function of Λ, but a more general function supported on Λ, we lose the property that $\int_{\mathbb{T}^2}|\widehat{g\mu}|^2 = \|g\|_2^2$. However we can still use Cauchy-Schwarz inequality to show that

$$|g\mu * \widetilde{g\mu}|^2 \leqslant |\mu * \widetilde{\mu}| \cdot |g|^2\mu * |g|^2\widetilde{\mu}.$$

So, using Plancherel's theorem and Hölder's inequality, we have

$$\int_{\mathbb{T}^2} |\widehat{g\mu}|^2 \leqslant \|\mu * \widetilde{\mu}\|_\infty \sum_{\mathbb{Z}^2} |g|^2\mu * |g|^2\widetilde{\mu} \leqslant \|\mu * \widetilde{\mu}\|_\infty \|g\|_{L^2(\mu)}^2 .$$

In particular, our result would still hold under the additional hypothesis that $\|\mu * \widetilde{\mu}\|_\infty < \infty$ and substituting (9.13) with the more general energy hypothesis

$$\sup_{x\in\Lambda}\sum_{y\in\Lambda\setminus\{x\}} \frac{\mu(y)}{|x-y|^{1+\epsilon}} < \infty. \tag{9.15}$$

Remark. Hypothesis (9.13) is analogue to hypothesis (4.13) and is essentially an energy condition on the distribution of points Λ. Object like these are studied in potential theory. In particular, people are interested in discrete N-points configurations on certain sets that minimize some energy functionals of the form, for example,

$$\sum_{i \neq j} \frac{k(x_i, x_j)}{|x_i - x_j|^{\alpha}},$$

and its asymptotic for $N \to \infty$ (see for example [14], [15], [31]). In these settings, the points are usually taken on some regular surfaces and, for $N \to \infty$, the discrete energy will tend to the energy integral for the Hausdorff measure of the surface. We are more interested in the related problem of configuration with finite energy. We can ourselves consider points on a manifold. For example, for a function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(\mathbb{Z}) \subset \mathbb{Z}$, we can consider $\Lambda_\phi = \{(n, \phi(n)) | \ n \in \mathbb{Z}\}$, which satisfies (9.13), since

$$\sup_{x \in \Lambda_\phi} \sum_{y \in \Lambda_\phi \setminus \{x\}} \frac{1}{|x - y|^{1+\epsilon}} \leqslant \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(1 + |n|)^{1+\epsilon}}.$$

In fact, as in the euclidean case, we can think of (9.13) as a condition on the "dimension" of the set Λ. So, in our case, we are happy to work with analogue of 1-dimensional objects.

Remark. In a work of Bennett, Gutiérrez, Nakamura and Oliveira, still unpublished, the authors prove the equivalence between the Mizohata–Takeuchi conjecture for the paraboloid and an Mizohata–Takeuchi conjecture for discrete points of the paraboloid, in the same sense that we discussed in this chapter. Moreover, they prove an L^2-Sobolev–Mizohata–Takeuchi estimate for the discrete paraboloid, which is connected to the periodic Schrödinger equation.

LIST OF REFERENCES

[1] J.G. Bak, A. Seeger, *Extensions of the Stein-Tomas theorem*, Mathematical Research Letter. (4)18, pp. 767–781, 2011.

[2] J.A. Barceló, J. Bennett, A. Carbery, *A Note on Localised Weighted Inequalities for the Extension Operator*, Journal of the Australian Mathematical Society, 84(3), pp. 289-299, 2008.

[3] J.A. Barceló, A. Ruiz, L. Vega, *Weighted Estimates for the Helmholtz Equation and Some Applications*, Journal of Functional Analysis, Volume 150, Issue 2, pp. 356-382, 1997.

[4] F. Bartolucci, F. De Mari, M. Monti, *Unitarization of the Horocyclic Radon Transform on Symmetric Spaces*, In: De Mari, F., De Vito, E. (eds) Harmonic and Applied Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser, Cham., 2021

[5] F. Bartolucci, F. De Mari, M. Monti, *Unitarization of the Horocyclic Radon Transform on Homogeneous Trees*, J Fourier Anal Appl 27, 84, 2021.

[6] D. Beltran, L. Vega, *Bilinear identities involving the k-plane transform and Fourier extension operators*, Proc. of the Royal Society of Edinburgh, Vol. 150, Issue 6, pp. 3349-3377, 2020.

[7] J. Bennett, N. Bez, S. Buschenhenke, M. G. Cowling and T. C. Flock, *On the nonlinear Brascamp–Lieb inequality*, Duke Math. J. 169, no. 17, pp. 3291–3338, 2020.

[8] J. Bennett, A. Carbery, T. Tao, *On the multilinear restriction and Kakeya conjectures*, Acta Math., 196(2):261–302, 2006.

[9] J. Bennett, S. Gutiérrez, S. Nakamura, I. Oliveira, *A phase-space approach to weighted Fourier extension inequalities*, preprint at arxiv.org/abs/2406.14886, 2024.

[10] J. Bennett, M. Iliopoulou, *A multilinear Fourier extension identity on* $\mathbb{R}^n$, Mathematical Research Letters. 25, 4, pp. 1089-1108, 2018.

[11] J. Bennett, E. Jeong, *Fourier duality in the Brascamp–Lieb inequality*, Mathematical Proceedings of the Cambridge Philosophical Society, 173(2), pp. 387-409, 2022.

[12] J. Bennett, S. Nakamura, *Tomography bounds for the Fourier extension operator and applications*, Math. Ann. 380, pp. 119–159, 2021.

[13] J. Bennett, S. Nakamura, S. Shiraki, *Tomographic Fourier Extension Identities for Submanifolds of* $\mathbb{R}^n$, preprint at arxiv.org/abs/2212.12348, 2022

[14] S.V. Borodachov, D.P. Hardin, E. B. Saff, *Asymptotics for discrete weighted minimal Riesz energy problems on rectifiable sets*, Trans. Amer. Math. Soc., 360(3), pp. 1559–1580, 2008.

[15] S.V. Borodachov, D.P. Hardin, A. Reznikov, E. B. Saff, *Optimal Discrete Measures for Riesz Potentials*, Transactions of the American Mathematical Society 370, no. 10, pp. 6973–93, 2018.

[16] A. Carbery, T. Hänninen, S.I. Valdimarsson, *Disentanglement, multilinear duality and factorisation for nonpositive operators*, Analysis & PDE, 16(2), pp. 511-543, 2023.

[17] A. Carbery, M. Iliopoulou, H. Wang, *Some sharp inequalities of Mizohata–Takeuchi-type*, Rev. Mat. Iberoam., 2024.

[18] A. Carbery, E. Romera, F. Soria, *Radial weights and mixed norm inequalities for the disc multiplier*, Journal of Functional Analysis, Vol. 109, pp. 52-75, 1992.

[19] A. Carbery, F. Soria, *Pointwise Fourier inversion and localisation in* $\mathbb{R}^n$, Journal of Fourier Analysis and Applications 3, pp. 847–858, 1997.

[20] E. Carneiro, G. Negro, D. Oliveira e Silva, *Stability of sharp Fourier restriction to spheres*, preprint at arxiv.org/abs/2108.03412, 2021.

[21] W. B. Case, *Wigner functions and Weyl transforms for pedestrians*, Am. J. Phys., 76 (10), pp. 937–946, 2008.

[22] X. Chen, *Sets of Salem Type and Sharpness of the* L^2*-Fourier Restriction Theorem*, Transactions of the Americ. Math. Soc., Volume 368, Number 3, pp. 1959–1977, 2016.

[23] C. Demeter, *Fourier Restriction, Decoupling, and Applications*, Cambridge University Press, 2020.

[24] S. Dendrinos, A. Mustata, M. Vitturi, *A restricted 2-plane transform related to Fourier Restriction for surfaces of codimension 2*, preprint at arxiv.org/abs/2209.15530, 2022.

[25] D. Foschi, *Global maximizers for the sphere adjoint Fourier restriction inequality*, J. Funct. Anal., 268(3), pp. 690–702, 2015.

[26] D. Foschi, D. Oliveira e Silva, *Some recent progress on sharp Fourier restriction theory*, Anal Math 43, pp. 241–265, 2017.

[27] L. Grafakos, *Classical Fourier Analysis*, 3rd ed., Springer, New York, 2014.

[28] H.W. Guggenheimer, *Differential Geometry*, Dover, New York, 1977.

[29] L. Guth, A. Iosevich, Y. Ou, H. Wang, *On Falconer's distance set problem in the plane*, Invent. Math. 219, no. 3, pp. 779-830, 2020.

[30] K. Hambrook,I. Łaba, *On the Sharpness of Mockenhaupt's Restriction Theorem*, Geom. Funct. Anal. 23, pp. 1262–1277, 2013.

[31] D.P. Hardin, E.B. Saff, *Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds*, Advances in Mathematics, Volume 193, Issue 1, pp. 174-204, 2005

[32] S. Helgason, *The Radon Transform*, 2nd edition, Springer, 2013.

[33] J.M. Lee, *Introduction to Riemannian Manifolds*, 2nd ed., Springer, New York, 2018.

[34] R. Lucà, K.M. Rogers, *Average decay of the Fourier transform of measures with applications*, J. Eur. Math. Soc. 21, no. 2, pp. 465–506, 2019.

[35] A. Markoe, *Analytic Tomography*, Cambridge University Press, 2006.

[36] J.M. Marstrand, *Hausdorff two-dimensional measure in 3-space*, Proc. London. Math. Soc. (3) 11, pp. 91–108, 1961.

[37] P. Mattila, *Hausdorff m-regular and rectifiable sets in n-space*, Trans. Amer. Math. Soc. 205, pp. 263–274, 1975.

[38] P. Mattila, *Spherical averages of Fourier transforms of measures with finite energy; dimensions of intersections and distance sets*, Mathematika, 34, pp. 207-228, 1987.

[39] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability* (Cambridge Studies in Advanced Mathematics), Cambridge, Cambridge University Press, 1995.

[40] P. Mattila, *Fourier Analysis and Hausdorff Dimension* (Cambridge Studies in Advanced Mathematics), Cambridge, Cambridge University Press, 2015.

[41] P. Mattila, *Rectifiability: A Survey*, Cambridge University Press, 2023.

[42] S. Mizohata, *On the Cauchy Problem*, Notes and Reports in Mathematics in Science and Engineering, Vol. 3, Academic Press, Inc., 1985.

[43] G. Mockenhaupt, *Salem sets and restriction properties of Fourier transforms*, Geometric and Functional Analysis, 10, pp. 1579–1587, 2000.

[44] D. Oliveira e Silva, R. Quilodrán, *Global maximizers for adjoint Fourier restriction inequalities on low dimensional spheres*, Journal of Functional Analysis, Volume 280, Issue 7, 2021.

[45] J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, Berichte über die Verhandlungen der Königlich-Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse [Reports on the Proceedings of the Royal Saxonian Academy of Sciences at Leipzig, Mathematical and Physical Section] (69), Leipzig: Teubner, pp. 262–277, 1917.

[46] W. Rudin, *Fourier Analysis on Groups*, John Wiley & Sons, 1990.

[47] C. Schütt, E.M. Werner, *Affine surface area*, preprint at arxiv.org/2204.01926, 2022.

[48] B. Shayya, *Mizohata–Takeuchi estimates in the plane*, Bull. London Math. Soc., 55, pp. 2176-2194, 2023.

[49] T. Shifrin, *Differential Geometry: A First Course in Curves and Surfaces*, University of Georgia, 2023.

[50] E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[51] T. Tao, *The Bochner-Riesz conjecture implies the restriction conjecture*, Duke Math. J. 96 (2), pp. 363 - 375, 1999.

[52] T. Tao, Lecture Notes from the course 247B, UCLA Department of Mathematics, 2020, available at terrytao.wordpress.com/2020/03/29/247b-notes-1-restriction-theory/.

[53] P. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. 81, pp. 477-478, 1975.