Extremal problems in graphs

By

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ABSTRACT

In the first part of this thesis we will consider degree sequence results for graphs. An important result of Komlós [39] yields the asymptotically exact minimum degree threshold that ensures a graph G contains an H-tiling covering an x-proportion of the vertices of G (for any fixed $x \in (0, 1)$ and graph H). In Chapter 2, we give a degree sequence strengthening of this result. A fundamental result of Kühn and Osthus [46] determines up to an additive constant the minimum degree threshold that forces a graph to contain a perfect H-tiling. In Chapter 3, we prove a degree sequence version of this result.

We close this thesis in the study of asymmetric Ramsey properties in $G_{n,p}$. Specifically, for fixed graphs $H_1, \ldots, H_r$, we study the asymptotic threshold function for the property $G_{n,p} \to (H_1, \ldots, H_r)$. Rödl and Ruciński [61, 62, 63] determined the threshold function for the general symmetric case; that is, when $H_1 = \cdots = H_r$. Kohayakawa and Kreuter [33] conjectured the threshold function for the asymmetric case. Building on work of Marciniszyn, Skokan, Spöhel and Steger [51], in Chapter 4, we reduce the 0-statement of Kohayakawa and Kreuter's conjecture to a more approachable, deterministic conjecture. To demonstrate the potential of this approach, we show our conjecture holds for almost all pairs of regular graphs.

COAUTHORS

Chapter 2 is joint work with Hong Liu and Andrew Treglown, and is based on [29].

Chapter 3 is joint work with Andrew Treglown, and is based on [30].

Chapter 4 is solely my own work and is based on [28].

DEDICATION

To my bucket: Hugo, Clyde, Jumbo, Philippe, Louie, Doughnut, Howie, Mel, Small blob, Hugo Jr. and Huugoo!

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All of the noise takes me to the outside where there's all

Creations joining in celebrating happiness and joy

All around the world on land and in the sea

Cardiacs, "Big Ship"

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Chapter One

Introduction

1.1 Matchings and tilings

A fundamental problem in extremal graph theory is to ascertain whether a given graph contains some desired substructure. One may ask which conditions on a given graph G necessitate it containing another graph H. For instance, for $r \in \mathbb{N}$, Turán [70] determined the number of edges in G required to guarantee K_r as a subgraph, with the equivalent result for general graphs H being supplied later by Erdős and Stone [16]. Often though, one is interested in finding a spanning substructure of a graph since this gives us information about the global structure of the graph. One such structure is a *perfect matching*: a set of vertex-disjoint edges that cover every vertex of the graph. (More generally, a *matching* is a set of vertex-disjoint edges in a graph, not necessarily covering every vertex.)

Edmonds [12] showed that the decision problem of whether a graph contains a perfect matching is solvable in polynomial time and, according with this, Tutte [71] characterised those graphs that contain a perfect matching. Given a general graph H, an *H-tiling* in a graph G is a collection of vertex-disjoint copies of H in G. We can generalise the problem of finding a perfect matching in a graph G to finding a *perfect H-tiling* in G; namely, an

H-tiling in G covering every vertex of G.[1]

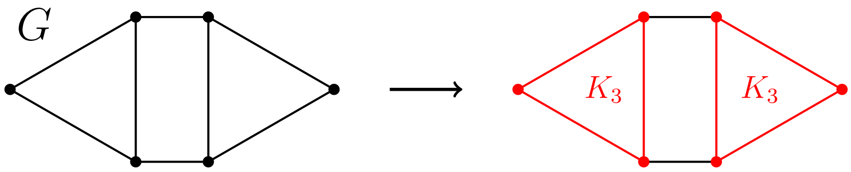

Figure 1.1: A perfect K_3-tiling.

In contrast to Edmonds' [12] result, Hell and Kirkpatrick [26] showed that the decision problem of whether a graph contains a perfect H-tiling is NP-complete precisely when H has a component on at least 3 vertices, that is, when H is not a matching or the union of a matching with isolated vertices. Hence, for such H it is useful to find sufficient conditions that guarantee a graph G contains a perfect H-tiling. One could ask for results in the mode of Erdős and Stone, where we desire to know the number of edges that guarantee a perfect H-tiling. However, this immediately becomes not so interesting if we consider the following example: Let G be the graph on n vertices made up of a clique on $n-1$ vertices and an isolated vertex (so G is very dense) and let H be a graph such that $|H|$ divides n and H has no isolated vertices. Then G does not contain a perfect H-tiling. A more illuminating approach is to consider minimum degree conditions that guarantee perfect H-tilings.

1.1.1 Corrádi and Hajnal's theorem: Cycle-tilings

Since any perfect H-tiling of G requires that $|H|$ divides $|G|$, we assume in every perfect tiling result from now on that $|H|$ divides $|G|$.[2]

[1]Perfect H-tilings are also referred to as perfect H-packings, H-factors and perfect H-matchings.
[2]In later sections in this chapter we assume a similar criteria.

When $H = K_3$, Corrádi and Hajnal [8] proved the following.

Theorem 1.1.1 (Corrádi and Hajnal [8]). *If G is a graph on $n \geq 3$ vertices and $\delta(G) \geq 2n/3$ then G has a perfect K_3-tiling.*

In fact, they proved a more general result that for all $1 \leq k \leq n/3$, if $\delta(G) \geq 2k$ then G contains k vertex-disjoint cycles. This result was generalised by Enomoto [13] and Wang [72]:

Theorem 1.1.2 (Enomoto [13], Wang [72]). *Let G be a graph and*

$$\sigma_2(G) := \min\{d_G(x) + d_G(y) \mid x, y \in V(G), xy \notin E(G)\}.$$

If $|G| = n \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k vertex-disjoint cycles.

Theorem 1.1.2 is called an *Ore-type*[3] version of Theorem 1.1.1. Other versions of Theorem 1.1.1 have been proven, including a density version due to Justesen [32, Theorem 4] for $k \geq 2$ and a graph G on $n \geq 3k$ vertices, and a more specific asymptotic density version due to Allen, Böttcher, Hladký and Piguet [1] who consider the number of edges in G that guarantee $k + 1$ disjoint triangles. By 'density version' here, we are referring to how the central condition in these theorems relates to $e(G)$.

1.1.2 Hajnal and Szemerédi's theorem: Perfect clique-tilings

The $k = n/3$ case of Theorem 1.1.1 is a specific case of the celebrated theorem of Hajnal and Szemerédi [24]:

[3]Here we are referring to Ore's theorem [57] concerning Hamilton cycles.

Theorem 1.1.3 (Hajnal and Szemerédi [24]). *Let $r \in \mathbb{N}$. If G is a graph on n vertices with*

$$\delta(G) \geq \left(1 - \frac{1}{r}\right) n,$$

then G contains a perfect K_r-tiling.

Theorem 1.1.3 is best possible as there exist graphs with $\delta(G) \geq (1 - 1/r)n - 1$ that do not contain a perfect K_r-tiling (for example, the complete r-partite graph with one vertex class of size $n/r - 1$, one vertex class of size $n/r + 1$ and $(r - 2)$ vertex classes of size n/r). Let G be a graph and $r \in \mathbb{N}$. We say a graph G has an *equitable r-colouring* if there exists a (proper) r-colouring of $V(G)$ with colour classes $V_1, V_2, \ldots, V_r$ such that $||V_i| - |V_j|| \leq 1$ for all $1 \leq i \neq j \leq r$. Theorem 1.1.3 was originally stated in terms of equitable colourings and answered a conjecture of Erdős [14]:

Theorem 1.1.4 (Hajnal and Szemerédi [24]). *Let G be a graph. If $\Delta(G) \leq r - 1$ then G has an equitable r-colouring.*

Let G be a graph and $\bar{G}$ be the complement graph of G ($V(\bar{G}) := V(G)$ and $E(\bar{G}) := \{xy : xy \notin E(G), x, y \in V(G)\}$). Let r divide n. One can see that Theorem 1.1.3 applied to G with r and Theorem 1.1.4 applied to $\bar{G}$ with n/r are equivalent: observe that $\delta(G) \geq n - n/r$ if and only if $\Delta(\bar{G}) \leq n/r - 1$. If $\Delta(\bar{G}) \leq n/r - 1$ then by Theorem 1.1.4 we have an equitable n/r-colouring in $\bar{G}$. Since r divides n we must have that the colour classes of this equitable n/r-colouring in $\bar{G}$ are equal in size, which immediately gives us a perfect K_r-tiling in G (each colour class yields a copy of K_r in G). On the other hand, if $\delta(G) \geq n - n/r$ then by Theorem 1.1.3 we have a perfect K_r-tiling in G which immediately gives us an equitable n/r-colouring in $\bar{G}$.

As with Theorem 1.1.1, certain versions and generalisations of the Theorem 1.1.3 have been proven. Kierstead and Kostochka [34] proved an Ore-type version and also provided a shorter proof of Theorem 1.1.3 (see [33]). In a slightly different vein, given $n, r, D \in \mathbb{N}$, Balogh, Kostochka and Treglown [4] were able to determine the minimum edge density that ensures a graph G on n vertices with $\delta(G) \geq D$ contains a perfect K_r-tiling. Further, a discrepancy version of Theorem 1.1.3 has recently been proven by Balogh, Csaba, Pluhár and Treglown [3], as well as a deficiency version by Freschi, Treglown and the author [19] which generalises earlier work of Nenadov, Sudakov and Wagner [55].

1.1.3 Alon and Yuster's theorem and Komlós' theorem: H-tilings

After considering cycle and clique tilings, a natural problem to consider is which minimum degree conditions guarantee H-tilings for *any* graph H. An important milestone in this study was the following theorem of Alon and Yuster [2].

Theorem 1.1.5 (Alon and Yuster [2]). *Let $\eta > 0$ and H be a graph. Then there exists $n_0 := n_0(\eta, H) \in \mathbb{N}$ such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1 - 1/\chi(H) + \eta)n$ contains a perfect H-tiling.*

Komlós, Sárközy and Szemerédi [41] were able to improve on Theorem 1.1.5 by replacing ηn with a constant dependent only on the graph H.

Observe that $\chi(K_r) = r$, so this result and Theorem 1.1.3 may suggest that the parameter governing whether a graph G contains a perfect H-tiling is $\chi(H)$. However, Komlós [39] showed that another parameter $\chi_{cr}(H)$ determines whether a graph G contains an H-tiling covering all but at most an arbitrarily small linear number of vertices. Such an H-tiling is informally known as an *almost perfect H-tiling.* We define

$$\chi_{cr}(H) := (\chi(H) - 1)\frac{|H|}{|H| - \sigma(H)},$$

where $\sigma(H)$ denotes the size of the smallest possible colour class in any $\chi(H)$-colouring of H. Observe that $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$ and $\chi(H) = \chi_{cr}(H)$ if and only if in every $\chi(H)$-colouring of H every colour class has the same size (See Figure 1.2 for examples of $\chi_{cr}(H)$ in comparison to $\chi(H)$ for several graphs).

Theorem 1.1.6 (Komlós [39]). *Let $\eta > 0$ and H be a graph. Then there exists an $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that every graph G on $n \geq n_0$ vertices with*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n$$

contains an H-tiling covering all but at most ηn vertices.

Note that the minimum degree condition in Theorem 1.1.6 is best possible in the sense that one cannot replace the $(1 - 1/\chi_{cr}(H))$ term with any smaller fixed constant and still guarantee an almost perfect H-tiling. However, Shoukoufandeh and Zhao [65] were able to strengthen Theorem 1.1.6 by improving the number of uncovered vertices to a constant dependent only on H. For all $x \in (0, 1)$, define

$$g_H(x) := x\left(1 - \frac{1}{\chi_{cr}(H)}\right) + (1 - x)\left(1 - \frac{1}{\chi(H) - 1}\right).$$

For $x \in (0, 1)$, Komlós [39] gave the optimal minimum degree condition that ensures an H-tiling covering at least an x-proportion of the vertices of G. The condition depends on both $\chi(H)$ and $\chi_{cr}(H)$:

Theorem 1.1.7 (Komlós [39]). *Let H be a graph, $\eta > 0$ and $x \in (0,1)$. Then there exists an $n_0 = n_0(\eta, x, H) \in \mathbb{N}$ such that every graph G on $n \geq n_0$ vertices with*

$$\delta(G) \geq g_H(x)n$$

contains an H-tiling covering at least $(x - \eta)n$ vertices.

Theorem 1.1.7 implies Theorem 1.1.6. A consequence of the theorem of Erdős and Stone mentioned earlier is that every n-vertex graph G with $\delta(G) \geq \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) n$ contains a copy of H. So a way to interpret Theorem 1.1.7 is that, for very small $x > 0$, the minimum degree threshold is governed essentially by the value of $\chi(H) - 1$; however, as one increases x, the value of $\chi_{cr}(H)$ plays an increasing role in the value of the threshold. Hladký, Hu and Piguet [27] were able to prove both graphon and stability versions of Theorem 1.1.7. Further, Ore-type generalisations of Theorems 1.1.5, 1.1.6 and 1.1.7 are given in [68].

1.1.4 Kühn and Osthus's theorem: Perfect H-tilings

In the previous section we noted how $\chi(K_r)$ is the parameter in the minimum degree condition governing whether G has a perfect K_r-tiling (Theorem 1.1.3), whereas $\chi_{cr}(H)$ is the parameter in the minimum degree condition governing whether G has an almost perfect H-tiling (Theorem 1.1.6). Hence, for a graph G on sufficiently many vertices and a graph H, one may ask the question of whether $\chi(H)$ or $\chi_{cr}(H)$ (or something else) is the parameter governing whether G has a perfect H-tiling. This question was fully settled by Kühn and Osthus [44, 46]. To state Kühn and Osthus' result we require several definitions. Let H be a graph. We say that a proper colouring c of H is *optimal* if c uses precisely $\chi(H)$ colours. Let C_H be the set of all optimal colourings of H. Given an optimal colouring $c \in C_H$, let

$x_{c,1} \le x_{c,2} \le \cdots \le x_{c,\chi(H)}$ denote the sizes of the colour classes of c. Define

$$\mathcal{D}(c) := \{x_{c,j+1} - x_{c,j} \mid j = 1, \dots, \chi(H) - 1\}$$

and take

$$\mathcal{D}(H) := \bigcup_{c \in C_H} \mathcal{D}(c).$$

Let $\mathrm{hcf}_\chi(H)$ be the highest common factor of all integers in $\mathcal{D}(H)$. In the case $\mathcal{D}(H) = \{0\}$, we set $\mathrm{hcf}_\chi(H) := \infty$. Note that $\mathcal{D}(H) = \{0\}$ if and only if $\chi(H) = \chi_{cr}(H)$. Let $\mathrm{hcf}_c(H)$ be the highest common factor of all the orders of the components of H. If $\chi(H) \neq 2$ we say that $\mathrm{hcf}(H) = 1$ if $\mathrm{hcf}_\chi(H) = 1$. If $\chi(H) = 2$ then we say that $\mathrm{hcf}(H) = 1$ if both $\mathrm{hcf}_c(H) = 1$ and $\mathrm{hcf}_\chi(H) \le 2$. We provide a few examples in the following table.

H	$\chi(H)$	$\chi_{cr}(H)$	$\mathrm{hcf}_\chi(H)$	$\mathrm{hcf}_c(H)$	$\mathrm{hcf}(H)$
C_{2k+1}	3	$2 + 1/k$	1	$2k + 1$	1
K_k	k	k	∞	k	$\neq 1$
$K_{1,2} \cup C_6$	2	$9/5$	1	3	$\neq 1$
$K_{1,4} \cup C_6$	2	$11/7$	3	1	$\neq 1$
$K_{1,2} \cup K_{1,4}$	2	$4/3$	2	1	1

Figure 1.2: Examples of $\chi(H)$, $\chi_{cr}(H)$, $\mathrm{hcf}_\chi(H)$, $\mathrm{hcf}_c(H)$ and $\mathrm{hcf}_(H)$ for various graphs

When $\mathrm{hcf}(H) = 1$, Kühn and Osthus showed that $\chi_{cr}(H)$ is the parameter governing the minimum degree condition that ensures a perfect H-tiling. When $\mathrm{hcf}(H) \neq 1$, $\chi(H)$ is the parameter. This is summarised in the following theorem.

Theorem 1.1.8 (Kühn and Osthus [44, 46])**.** *Let H be a graph. Let*

$$\chi^*(H) := \begin{cases} \chi_{cr}(H) & \text{if } \mathrm{hcf}(H) = 1; \\ \chi(H) & \text{otherwise.} \end{cases}$$

Further, let $\delta(H, n)$ denote the smallest k such that every graph G on n vertices with $|H|$ dividing n and $\delta(G) \ge k$ contains a perfect H-tiling. Then there exists a constant $C = C(H)$

such that

$$\left(1-\frac{1}{\chi^*(H)}\right)n-1\leq\delta(H,n)\leq\left(1-\frac{1}{\chi^*(H)}\right)n+C.$$

If $\mathrm{hcf}(H)=1$ or $\chi(H)\geq 3$ then the -1 on the left hand side can be removed. Furthermore, there exist graphs H for which the constant C is necessary to ensure a perfect H-tiling. We remark that just as Ore-type versions of the Theorems 1.1.3 and 1.1.7 have been proven, an Ore-type version of the Theorem 1.1.8 was proven by Kühn, Osthus and Treglown [48].

Summarising, the parameter governing whether a graph contains a perfect H-tiling is $\chi^*(H)$ and the parameter governing the minimum degree condition that ensures an almost perfect H-tiling is $\chi_{cr}(H)$.

1.1.5 Degree sequence conditions

Another way we can generalise minimum degree results is by instead considering *degree sequence* conditions. We say a graph G on n vertices has degree sequence $d_1\leq d_2\leq\cdots\leq d_n$ if there exists an ordering of the vertices $v_1,v_2,\dots,v_n$ of G such that $d(v_i)=d_i$ for all $1\leq i\leq n$. Pósa's [59] and Chvátal's [7] theorems are based on such degree sequence conditions. Both generalise Dirac's [11] theorem, which states that, for a graph G on $n\geq 3$ vertices, if $\delta(G)\geq n/2$ then G contains a Hamilton cycle.

Erdős [15] was able to characterise those degree sequences which ensure the existence of a copy of K_r. For perfect K_r-tilings, Balogh, Kostochka and Treglown [4] conjectured the following degree sequence version of Theorem 1.1.3.

Conjecture 1.1.9 (Balogh, Kostochka and Treglown [4]). *Let n, $r \in \mathbb{N}$ such that r divides n. Suppose that G is a graph on n vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

$$(\alpha) \quad d_i \geq (r-2)n/r + i \quad \textit{for all } i < n/r;$$

$$(\beta) \quad d_{n/r+1} \geq (r-1)n/r.$$

Then G contains a perfect K_r-tiling.

Conjecture 1.1.9 is best possible in the sense that there are examples (see [4, Section 4]) showing that one cannot replace (α) with $d_i \geq (r-2)n/r + i - 1$ for a *single* i or (β) with $d_{n/r+1} \geq (r-1)n/r - 1$. Treglown [69] proved the following asymptotic version of Conjecture 1.1.9.

Theorem 1.1.10 (Treglown [69]). *Let $\eta > 0$ and $r \in \mathbb{N}$. Then there exists an integer $n_0 = n_0(\eta, r)$ such that the following holds: Suppose that G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

$$d_i \geq (r-2)n/r + i + \eta n \quad \textit{for all } i < n/r.$$

Then G contains a perfect K_r-tiling.

Note that Treglown's degree sequence allows for almost n/r vertices to have degree below the Hajnal–Szemerédi threshold of $(1-1/r)n$.

Moreover, Treglown [69] proved the following strong generalisation of Theorem 1.1.5.

Theorem 1.1.11 (Treglown [69]). *Let $\eta > 0$ and H be a graph with $\chi(H) = r \geq 2$. Then there exists an integer $n_0 = n_0(\eta, H)$ such that the following holds: Suppose that G is a graph* *on $n \geq n_0$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that:*

$$d_i \geq (r-2)n/r + i + \eta n \quad \textit{for all } i \leq n/r.$$

Then G contains a perfect H-tiling.

One may ask whether the ηn term in the degree sequence in Theorem 1.1.11 can be replaced by a constant depending only on H, just as in Komlós, Sárközy and Szemerédi's [41] improvement of Theorem 1.1.5. For many graphs however, if we replace the ηn term with a $o(\sqrt{n})$ term *for every* $i \in \{1, \ldots, n\}$ then we cannot guarantee a perfect H-tiling. Note that this does not rule out the possibility that one could replace the ηn term with a constant $C = C(\eta, H)$ for each $i \geq n/r - \eta n + C$ and still ensure a perfect H-tiling. Such an improvement of Theorem 1.1.11 would accord with Komlós, Sárközy and Szemerédi's [41] improvement of Theorem 1.1.5. We also note that Knox and Treglown [36] proved a degree sequence result for embedding spanning bipartite graphs of small bandwidth[4] and Staden and Treglown [66] proved a degree sequence version of Pósa's conjecture.

1.1.6 Our degree sequence results

Let $x \in (0, 1)$ and H be a graph. In Chapter 2, we prove degree sequence versions of Theorems 1.1.6 and 1.1.7 for almost perfect H-tilings and H-tilings covering at least an x-proportion of the vertices in G. These results are similar in form to Theorem 1.1.3, particularly in how our degree sequences allow for a significant proportion of the vertices in G to have degree less than the corresponding Komlós thresholds. Also, in Chapter 3, we use Theorem 1.1.6 to prove a degree sequence version of the following theorem of Kühn and Osthus.

[4] In fact, they prove a much stronger result where the degree sequence condition is relaxed to a certain robust expansion property.

Theorem 1.1.12 (Kühn and Osthus [44]). *Let $\eta > 0$ and H be a graph with $\mathrm{hcf}_\chi(H) = 1$ and $\chi(H) \geq 3$. Then there exists an integer $n_0 = n_0(\eta, H)$ such that the following holds: Let G be a graph on $n \geq n_0$ vertices such that $|H|$ divides n and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)} + \eta\right) n.$$

Then G contains a perfect H-tiling.

(One can see that Theorem 1.1.12 is a (weaker) error-term version of the $\mathrm{hcf}_\chi(H) = 1$ case of Theorem 1.1.8.) In fact, we will prove a degree sequence version of Theorem 1.1.12 for all graphs H with $\mathrm{hcf}(H) = 1$ and $\chi(H) \geq 2$.

1.2 Ramsey theory

In this thesis we will also consider a particular problem in Ramsey theory, one of the most studied areas in modern combinatorics. Let $r \in \mathbb{N}$ and let $G, H_1, \ldots, H_r$ be graphs. We write $G \to (H_1, \ldots, H_r)$ to denote the property that whenever we colour the edges[5] of G with colours from the set $[r] := \{1, \ldots, r\}$ there exists $i \in [r]$ and a copy of H_i in G monochromatic in colour i. In this notation, the classical result of Ramsey [60] is as follows.

Theorem 1.2.1 (Ramsey [60]). *Let $H_1, \ldots, H_r$ be graphs and n be sufficiently large. Then*

$$K_n \to (H_1, \ldots, H_r).$$

One may posit that Theorem 1.2.1 is only true because K_n is very dense, but, building on work of Folkman [17], Nešetřil and Rödl [56] proved that for any graph H there is a graph

[5]Here we will only be interested in colouring edges, not vertices, but there has been significant interest in vertex Ramsey results (see e.g. [10, 43, 50]).

G such that $\omega(H) = \omega(G)$ and $G \to (H_1, \ldots, H_r)$ when $H_1 = \cdots = H_r = H$.

1.2.1 Symmetric Ramsey properties: Rödl and Ruciński's theorem

If we transfer our study of the Ramsey property to the random setting, we discover that such graphs G are in fact very common. Let $G_{n,p}$ be the binomial random graph with n vertices and edge probability p. In this thesis, we will say that $p = f(n)$ is a *threshold for the property* $G_{n,p} \to (H_1, \ldots, H_r)$ if there exist positive constants $b, B > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \to (H_1, \ldots, H_r)] = \begin{cases} 0 & \text{if } p \leq bf(n), \\ 1 & \text{if } p \geq Bf(n). \end{cases}$$

The statement for $p \leq bf(n)$ is known as a *0-statement* and the statement for $p \geq Bf(n)$ is known as an *1-statement*. For ease of reading, we will write out this definition in full in later theorems. Improving on earlier work of Frankl and Rödl [18], Łuczak, Ruciński and Voigt [50] proved that $p = n^{-1/2}$ is a threshold for the property $G_{n,p} \to (K_3, K_3)$. Following this, Rödl and Ruciński [61, 62, 63] determined a threshold for the general symmetric case. For a graph H, we define

$$d_2(H) := \begin{cases} (e_H - 1)/(v_H - 2) & \text{if } H \text{ is non-empty with } v(H) \geq 3, \\ 1/2 & \text{if } H \cong K_2, \\ 0 & \text{otherwise} \end{cases}$$

and the *2-density of* H to be

$$m_2(H) := \max\{d_2(J) : J \subseteq H\}.$$

We say that a graph H is *2-balanced* if $d_2(H) = m_2(H)$, and *strictly 2-balanced* if for all proper subgraphs $J \subset H$, we have $d_2(J) < m_2(H)$.

Theorem 1.2.2 (Rödl and Ruciński [63]). *Let $r \geq 2$ and let H be a non-empty graph such that at least one component of H is not a star. If $r = 2$, then in addition restrict H to having no component which is a path on 3 edges. Then there exist positive constants $b, B > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \to \underbrace{(H, \ldots, H)}_{r \text{ times}}] = \begin{cases} 0 & \text{if } p \leq bn^{-1/m_2(H)}, \\ 1 & \text{if } p \geq Bn^{-1/m_2(H)}. \end{cases}$$

The assumption on the structure of H in Theorem 1.2.2 is necessary. If every component of H is a star then $G_{n,p} \to (H, \ldots, H)$ as soon as sufficiently many vertices of degree $r(\Delta(H)-1)+1$ appear in $G_{n,p}$. A threshold for this property in $G_{n,p}$ is $p = n^{-1-1/(r(\Delta(H)-1)+1)}$, but $m_2(H) = 1$. For the case when $r = 2$ and at least one component of H is a path on 3 edges while the others are stars, the 0-statement of Theorem 1.2.2 becomes false. Indeed, one can show that, if $p = cn^{-1/m_2(P_3)} = cn^{-1}$ for some $c > 0$, then the probability that $G_{n,p}$ contains a cycle of length 5 with an edge pending at every vertex is bounded from below by a positive constant $d = d(c)$. One can check that every colouring of the edges of this augmented 5-cycle with 2 colours yields a monochromatic path of length 3. This special case was missed in [63], and was eventually observed by Friedgut and Krivelevich [20], who corrected the 0-statement to have the assumption $p = o(n^{-1/m_2(H)})$ instead. Note that Nenadov and Steger [53] produced a short proof of Theorem 1.2.2 using the hypergraph container method.

The intuition behind the threshold in Theorem 1.2.2 is as follows: Firstly, assume H is 2-balanced. The expected number of copies of a graph H in $G_{n,p}$ is $\Theta(n^{v(H)}p^{e(H)})$ and the expected number of edges is $\Theta(n^2p)$. For $p = n^{-1/m_2(H)}$ (the threshold in Theorem 1.2.2), these two expectations are of the same order since H is 2-balanced. That is to say, if the expected number of copies of H at a fixed edge is smaller than some small constant c, then we can hope to colour without creating a monochromatic copy of H: very roughly speaking, each copy will likely contain an edge not belonging to any other copy of H, so by colouring these edges with one colour and all other edges with a different colour we avoid creating monochromatic copies of H. If the expected number of copies of H at a fixed edge is larger than some large constant C then a monochromatic copy of H may appear in any r-colouring since the copies of H most likely overlap heavily.

1.2.2 Asymmetric Ramsey properties: The Kohayakawa-Kreuter Conjecture

Here we are interested in asymmetric Ramsey properties of $G_{n,p}$, that is, finding a threshold for the property $G_{n,p} \to (H_1 \ldots, H_r)$ when $H_1, \ldots, H_r$ are not all the same graph. In classical Ramsey theory, the study of asymmetric Ramsey properties sparked off many interesting routes of research (see, e.g. [6]), including the seminal work of Kim [35] on establishing an asymptotically sharp lower bound on the Ramsey number $R(3,t)$. In $G_{n,p}$, asymmetric Ramsey properties were first considered by Kohayakawa and Kreuter [33]. For graphs H_1 and H_2 with $m_2(H_1) \geq m_2(H_2)$, we define

$$d_2(H_1, H_2) := \begin{cases} \frac{e(H_1)}{v(H_1)-2+\frac{1}{m_2(H_2)}} & \text{if } H_2 \text{ is non-empty and } e(H_1) \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

and the *asymmetric 2-density of the pair* (H_1, H_2) to be

$$m_2(H_1, H_2) := \max\left\{d_2(J, H_2) : J \subseteq H_1\right\}.$$

We say that H_1 is *balanced with respect to (w.r.t.)* $d_2(\cdot, H_2)$ if we have $d_2(H_1, H_2) = m_2(H_1, H_2)$ and *strictly balanced w.r.t.* $d_2(\cdot, H_2)$ if for all proper subgraphs $J \subset H_1$ we have $d_2(J, H_2) < m_2(H_1, H_2)$. Note that $m_2(H_1) \geq m_2(H_1, H_2) \geq m_2(H_2)$ (see Proposition 4.3.1).

Kohayakawa and Kreuter [33] conjectured the following generalisation of Theorem 1.2.2. (We give here a slight rephrasing of the conjecture: we consider r colours (instead of 2) and add the assumption of Kohayakawa, Schacht and Spöhel [38] that H_1 and H_2 are not forests.[6])

Conjecture 1.2.3 (Kohayakawa and Kreuter [33]). *Let $r \geq 2$ and suppose that $H_1, \ldots, H_r$ are non-empty graphs such that $m_2(H_1) \geq \cdots \geq m_2(H_r)$ and $m_2(H_2) > 1$. Then there exist constants $b, B > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \to (H_1, \ldots, H_r)] = \begin{cases} 0 & \text{if } p \leq bn^{-1/m_2(H_1,H_2)}, \\ 1 & \text{if } p \geq Bn^{-1/m_2(H_1,H_2)}. \end{cases}$$

Observe that we would always need $m_2(H_2) \geq 1$ as an assumption, otherwise $m_2(H_2) = 1/2$ (that is, H_2 is the union of a matching and some isolated vertices) and we would have that $m_2(H_1, H_2) = e_J/v_J$ for some non-empty subgraph $J \subseteq H_1$. For any constant $B > 0$, the probability that $G_{n,p}$ with $p = Bn^{-1/m_2(H_1,H_2)}$ contains no copy of H_1 exceeds a positive constant $C = C(B)$; see, e.g. [31]. We include the assumption of Kohayakawa, Schacht and Spöhel [38] in this thesis, that $m_2(H_2) > 1$, to avoid possible complications arising from H_2 (and/or H_1) being certain forests, such as those excluded in the statement of Theorem 1.2.2.

The intuition behind the threshold in Conjecture 1.2.3 is most readily explained in the case of $r = 3$, $H_2 = H_3$ and when $m_2(H_1) > m_2(H_1, H_2)$. (The following explanation is adapted from [23].) Firstly, observe that we can assign colour 1 to every edge that does not

[6]This version of the conjecture is the same as that given in [52].

lie in a copy of H_1. Since $m_2(H_1) > m_2(H_1, H_2)$, we expect that the copies of H_1 in $G_{n,p}$ with $p = \Theta(n^{-1/m_2(H_1,H_2)})$ do not overlap much (by similar reasoning as in the intuition for the threshold in Theorem 1.2.2). Hence the number of edges left to be coloured is of the same order as the number of copies of H_1, which is $\Theta(n^{v(H_1)}p^{e(H_1)})$. If we further assume that these edges are randomly distributed (which is not correct, but gives good intuition) then we get a random graph G^* with edge probability $p^* = \Theta(n^{v(H_1)-2}p^{e(H_1)})$. Now we colour G^* with colours 2 and 3, and apply the intuition from the symmetric case (as $H_2 = H_3$): if the copies of H_2 are heavily overlapping then we cannot hope to colour without getting a monochromatic copy of H_2, but if not then we should be able to colour. As observed before, a threshold for this property is $p^* = n^{-1/m_2(H_2)}$. Solving $n^{v(H_1)-2}p^{e(H_1)} = n^{-1/m_2(H_2)}$ for p then yields $p = n^{-1/m_2(H_1,H_2)}$, the conjectured threshold.

Actually, it turns out that if $p < bn^{-1/m_2(H_1,H_2)}$ then we do not even need colour 3. That is, we can colour G^* with colours 1 and 2 and avoid monochromatic copies of H_1 and H_2 in their respective colours. This is why the conjectured threshold only relies on the two graphs with the largest 2-density.

After earlier work (see e.g. [23, 25, 33, 38, 51]), the 1-statement of Conjecture 1.2.3 was proven by Mousset, Nenadov and Samotij [52].

We are interested in the 0-statement of Conjecture 1.2.3, which has so far only been proven when H_1 and H_2 are both cycles [33], both cliques [51] and, recently, when H_1 is a clique and H_2 is a cycle [49]. (We also note that the authors of [23] prove, under certain balancedness conditions, the 0-statement of a generalised version of Conjecture 1.2.3 which allows $H_1, \ldots, H_r$ to be uniform hypergraphs.) In Chapter 4, we prove a reduction of the 0-statement of Conjecture 1.2.3 to a particular deterministic subproblem and then solve this subproblem for almost all pairs of regular graphs, thus resolving the 0-statement for these graphs.

1.3 Notation and Definitions

Let G be a graph. We define $V(G)$ to be the vertex set of G and $E(G)$ to be the edge set of G. We will also denote (particularly in Chapter 4) the number of vertices of G by $v(G) = v_G := |V(G)|$ and the number of edges of G by $e(G) = e_G := |E(G)|$. Moreover, for graphs H_1 and H_2 we let $v_1 := |V(H_1)|$, $e_1 := |E(H_1)|$, $v_2 := |V(H_2)|$ and $e_2 := |E(H_2)|$.

Let $X \subseteq V(G)$. Then $G[X]$ is the *graph induced by X on G* and has vertex set X and edge set $E(G[X]) := \{xy \in E(G) : x, y \in X\}$. We also define $G \setminus X$ to be the graph with vertex set $V(G) \setminus X$ and edge set $E(G \setminus X) := \{xy \in E(G) : x, y \in V(G) \setminus X\}$. For each $x \in V(G)$, we define the *neighbourhood of x in G* to be $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ and define $d_G(x) := |N_G(x)|$. We drop the subscript G if it is clear from context which graph we are considering. We write $d_G(x, X)$ for the number of edges in G that x sends to vertices in X. Given a subgraph $G' \subseteq G$, we will write $d_G(x, G') := d_G(x, V(G'))$. Let $A, B \subseteq V(G)$ be disjoint. Then we define $e_G(A, B) := |\{xy \in E(G) : x \in A, y \in B\}|$.

Let $t \in \mathbb{N}$. We define the *blow-up $G(t)$* to be the graph constructed by first replacing each vertex $x \in V(G)$ by a set V_x of t vertices and then replacing each edge $xy \in E(G)$ with the edges of the complete bipartite graph with vertex sets V_x and V_y.

We write $0 < a \ll b \ll c < 1$ to mean that we can choose the constants a, b, c from right to left. More precisely, there exist non-decreasing functions $f : (0, 1] \to (0, 1]$ and $g : (0, 1] \to (0, 1]$ such that for all $a \leq f(b)$ and $b \leq g(c)$ our calculations and arguments in our proofs are correct. Larger hierarchies are defined similarly. Note that $a \ll b$ implies that we may assume e.g. $a < b$ or $a < b^2$.

Chapter Two

A degree sequence Komlós theorem

This chapter is joint work with Hong Liu and Andrew Treglown, and is based on [29]. Our main work here is proving the following degree sequence strengthening of Theorem 1.1.6. Recall that, for a graph H, $\sigma(H)$ denotes the size of the smallest possible colour class in any $\chi(H)$-colouring of H.

Theorem 2.0.1. *Let $\eta > 0$ and H be a graph with $\chi(H) = r$. Let $\sigma := \sigma(H)$, $h := |H|$ and $\omega := (h - \sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{h}\right) n + \frac{\sigma}{\omega} i \quad \text{for all } 1 \leq i \leq \frac{\omega n}{h}.$$

Then G contains an H-tiling covering all but at most ηn vertices.

Note that if one considers an r-partition of H with smallest vertex class of size $\sigma = \sigma(H)$ and set $i = \omega n/h$ then we obtain that $(1 - (\omega + \sigma)/h)\, n + \sigma i/\omega = 1 - 1/\chi_{cr}(H)$. Thus, Theorem 2.0.1 is a significant strengthening of Theorem 1.1.6. Indeed, Theorem 2.0.1 allows for up to $\omega n/h$ vertices to have degree below that in Theorem 1.1.6. In particular, when H is bipartite, the degree sequence condition in Theorem 2.0.1 starts at $d_1 \geq 1$ and allows for at least half of the vertices of H to have degree less than that required by Theorem 1.1.6. Figure 2.1 gives a visualisation of the degree sequence in Theorem 2.0.1. Figure 2.2 presents some key properties of the degree sequence in Theorem 2.0.1 for several graphs. Here, 'angle of slope' refers to the value σ/ω.

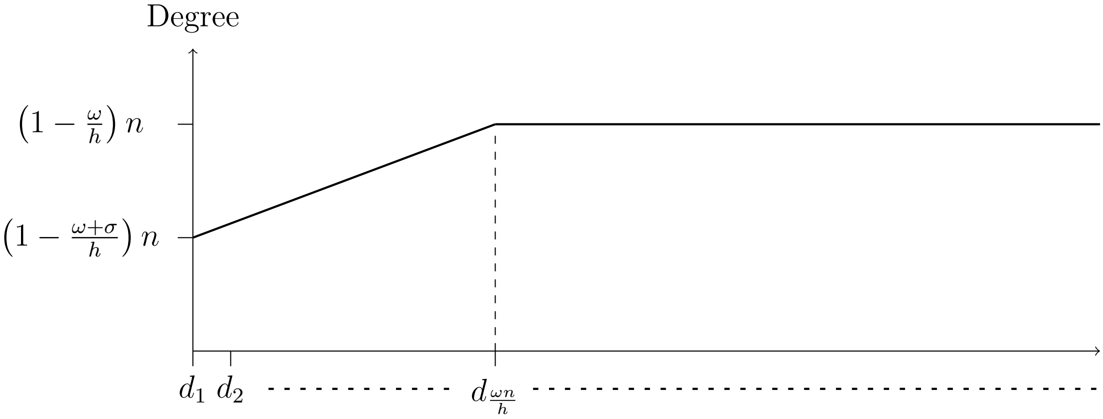

Figure 2.1: The degree sequence in Theorem 2.0.1.

Graph	Bound on d_1	Bound on $d_{\frac{\omega n}{h}}$	Angle of slope
C_5	$2n/5$	$3n/5$	$1/2$
$K_{1,t}$	1	$n/(t+1)$	$1/t$
K_t	$(t-2)n/t$	$(t-1)n/t$	1
$K_{2,4,6}$	$5n/12$	$7n/12$	$2/5$

Figure 2.2: Values of the start points, end points and angles of the slope in Theorem 2.0.1 for certain graphs.

The degree sequence in Theorem 2.0.1 is best possible in more than one sense for many graphs H. For all graphs H, one cannot allow significantly more than $\omega n/h$ vertices to have degree below the 'Komlós threshold', so in this sense the bound on the number of 'small degree' vertices in Theorem 2.0.1 is tight. Further, for many graphs H, we show that the degree sequence cannot start at a lower value and the angle of the 'slope' in Figure 2.1 is best possible. This is discussed in more depth in Section 2.1.

Theorem 2.0.1 deals with almost perfect tilings. A natural question now is whether such a degree sequence strengthening also exists for tilings covering an x-proportion of vertices, as in Theorem 1.1.7. Indeed, the following result is a straightforward consequence of Theorem 2.0.1. Recall for a graph H and $x \in (0,1)$ that

$$g_H(x) := x\left(1-\frac{1}{\chi_{cr}(H)}\right)+(1-x)\left(1-\frac{1}{\chi(H)-1}\right).$$

Theorem 2.0.2. *Let $x \in (0,1)$ and H be a graph with $\chi(H)=r$. Set $\eta > 0$. Let $\sigma := \sigma(H)$, $h := |H|$ and $\omega := (h-\sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, x, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(g_H(x) - \frac{x\sigma}{h}\right)n + \frac{(r-1)x\sigma}{h-x\sigma}i \quad \text{for all } 1 \leq i \leq \left(\frac{h-x\sigma}{(r-1)h}\right)n.$$

Then G contains an H-tiling covering at least $(x-\eta)n$ vertices.

Theorem 2.0.2 is an improvement on Theorem 1.1.7. Indeed, Theorem 2.0.2 allows for almost $(h-x\sigma)n/(r-1)h$ vertices to have degree below $g_H(x)n$. Observe that as x approaches 0, the degree sequence condition in Theorem 2.0.2 tends towards the condition $\delta(G) \geq (1-1/(r-1))n$, and thus accords with the Erdős–Stone theorem.

Piguet and Saumell [58, Theorem 1.3] recently proved another generalisation of Theorem 1.1.7. In their result they only require a certain fraction of the vertices to satisfy the degree condition of Theorem 1.1.7, and all other vertices have *no* restriction on their degree (so some could even be isolated vertices). Note though that our result allows for more vertices to have small degree (i.e. smaller than the bound in Theorem 1.1.7), at a price of having some restriction of the degrees of these vertices. In the case of almost perfect H-tilings, Theorem 2.0.1 allows a large proportion of the vertices to have small degree, whilst in this

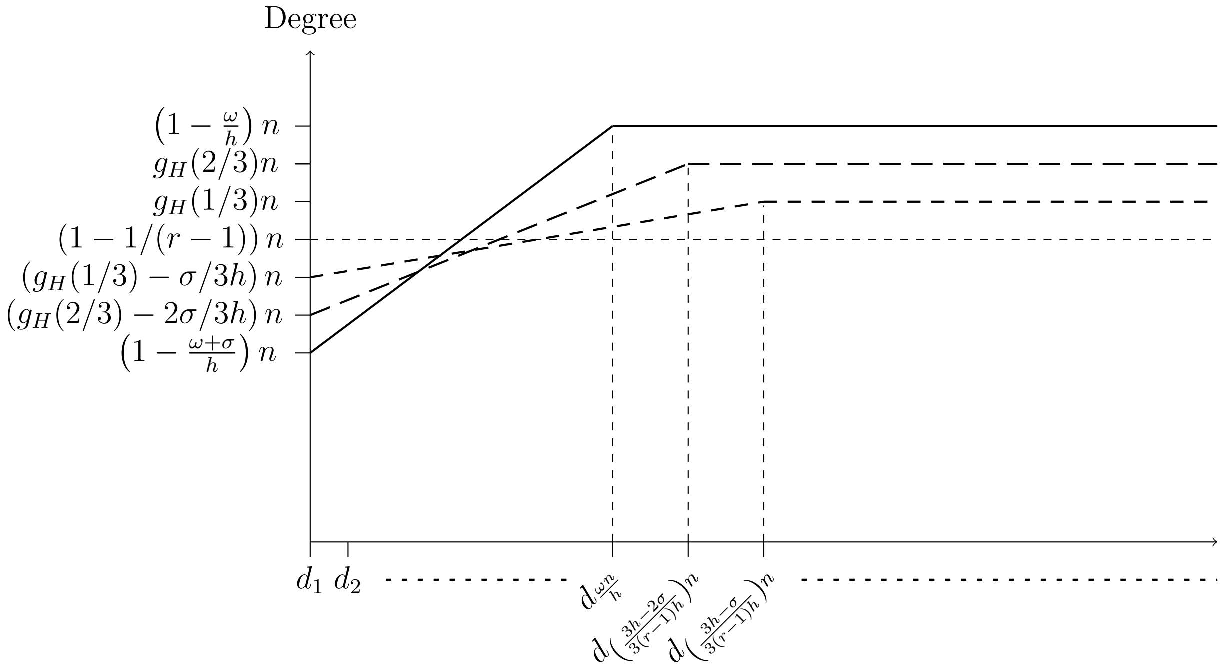

Figure 2.3: The degree sequence in Theorem 2.0.2 for $x = 2/3$ (long dashed), $x = 1/3$ (medium dashed).

case [58, Theorem 1.3] corresponds precisely to Theorem 1.1.6.

As well as considering minimum degree and degree sequence conditions, it is also natural to seek conditions on the density of a graph G that forces an H-tiling covering a given fraction of the vertices of G. We remark though that only limited progress has been made on this question (though Allen, Böttcher, Hladký and Piguet [1] did resolve this problem in the case of K_3-tilings).

Organisation The rest of Chapter 2 is organised as follows. In Section 2.1 we give extremal examples for both Theorems 2.0.1 and 2.0.2. We then introduce an ‘error term’ version (Theorem 2.2.1) of Theorem 2.0.1 in Section 2.2 and show that it implies Theorem 2.0.1. Szemerédi’s Regularity lemma and several auxiliary results are presented in Section 2.3. Then in Section 2.4 we provide the tools that we will need to prove Theorem 2.2.1. We prove a result that iteratively constructs an almost perfect H-tiling and then use this result to prove Theorem 2.2.1 in Section 2.5. To conclude Section 2.5, we show that Theorem 2.0.1 implies Theorem 2.0.2.

2.1 Extremal Examples

In this section we present three extremal examples. We require the following definition. Set $r, \sigma, \omega \in \mathbb{N}$ and $\sigma < \omega$. We define the *r-partite bottle graph B with neck σ and width ω* to be the complete r-partite graph with one vertex class of size σ and $(r-1)$ vertex classes of size ω. The first demonstrates that the 'slope' of the degree sequence in Theorem 2.0.1 is best possible for bottle graphs. The second shows that for many graphs H, the degree sequence in Theorem 2.0.1 'starts' at the correct place. The third shows that, for any graph H, to ensure an H-tiling covering at least $(x - \eta)n$ vertices we cannot have significantly more than $(h - x\sigma)n/(r-1)h$ vertices with degree below the 'Komlós threshold' of $g_H(x)n$.

Extremal Example 1. Set $\eta \in \mathbb{R}$. Let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$. The following extremal example G on n vertices demonstrates that Theorem 2.0.1 is best possible for such graphs B, in the sense that G satisfies the degree sequence of Theorem 2.0.1 except for a small linear part that only just fails the degree sequence, but does not contain a B-tiling covering all but at most ηn vertices.

Proposition 2.1.1. *Set $\eta \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $0 < 1/n \ll \eta \ll 1$. Let B be an (r-partite) bottle graph with neck σ and width ω, where $b := |B|$. Additionally assume that b divides n. Then for any $1 \leq k < \omega n/b - 2\eta n$, there exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{b}\right) n + \frac{\sigma}{\omega} i \quad \text{for all } i \in \{1, \ldots, k-1, k + 2\eta n + 1, \ldots, \omega n/b\},$$

$$d_i = \left(1 - \frac{\omega + \sigma}{b}\right) n + \left\lceil \frac{\sigma}{\omega} k \right\rceil \quad \textit{for all } k \leq i \leq k + 2\eta n,$$

but such that G does not contain a B-tiling covering all but at most ηn vertices.

Proof. Let G be the graph on n vertices with r vertex classes $V_1, \dots, V_r$ where $|V_1| = \sigma n/b$ and $|V_2| = |V_3| = \cdots = |V_r| = \omega n/b$. Label the vertices of V_1 as $a_1, a_2, \dots, a_{\sigma n/b}$. Similarly, label the vertices of V_2 as $c_1, c_2, \dots, c_{\omega n/b}$. The edge set of G is constructed as follows.

Firstly, let G have the following edges:

- All edges with an endpoint in V_1 and the other endpoint in $V(G) \setminus V_2$, in particular $G[V_1]$ is complete;
- All edges with an endpoint in V_i and the other endpoint in $V(G) \setminus (V_1 \cup V_i)$ for $2 \leq i \leq r$;
- Given any $1 \leq i \leq \omega n/b$ and $j \leq \lceil \sigma i/\omega \rceil$ include all edges $c_i a_j$.

So at the moment G does satisfy the degree sequence in Theorem 2.0.1; we therefore modify G slightly. For all $k + 1 \leq i \leq k + 2\eta n$ and $\lceil \sigma k/\omega \rceil + 1 \leq j \leq \lceil \sigma(k + 2\eta n)/\omega \rceil$ delete each edge between c_i and a_j. One can easily check that G satisfies the degree sequence in the statement of the proposition. In particular, the vertices of degree $\left(1 - \frac{\omega+\sigma}{b}\right) n + \lceil \frac{\sigma}{\omega} k \rceil$ are $c_k, \dots, c_{k+2\eta n}$.

Define $A := \{a_1, \dots, a_{\lceil \sigma k/\omega \rceil}\}$ and $C := \{c_1, \dots, c_{k+2\eta n}\}$. Note that there are no edges between C and $V_1 \setminus A$ in G.

Claim 2.1.2. *Let T be a B-tiling of G. Then T does not cover at least $3\eta n/2$ vertices in C.*

Consider any copy B' of B in G that contains an element of C. As C is an independent set in G, B' contains at most ω elements from C. Since there are no edges between C and $V_1 \setminus A$ in G, B' contains at least σ vertices in A. This implies that at most $\lceil \sigma k/\omega \rceil (\omega/\sigma) <$

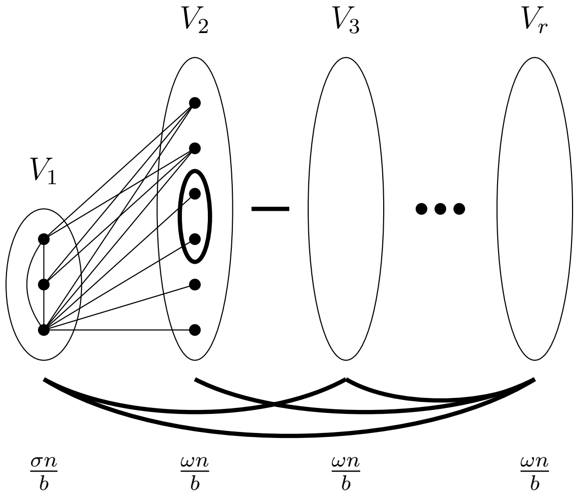

Figure 2.4: An example of a graph G in Proposition 2.1.1 where $\sigma = 1, \omega = 2$.

$k + \eta n/2$ vertices in C can be covered by T. Since $|C| = k + 2\eta n$, we have that T does not cover at least $3\eta n/2$ vertices in C. Therefore, Claim 2.1.2 holds. Hence G does not have a B-tiling covering all but at most ηn vertices. □

Proposition 2.1.1 implies that for bottle graphs B, the degree sequence in Theorem 2.0.1 cannot be lowered significantly in a small part of the degree sequence and still ensure an almost perfect B-tiling; so the 'slope' of the degree sequence in Theorem 2.0.1 cannot be improved upon. It would be interesting to find other classes of graphs H for which the slope in Theorem 2.0.1 is also best possible; we suspect though that there are graphs H where the slope is not best possible.

Extremal Example 2. The next example shows that for many graphs H, Theorem 2.0.1 is best possible in the sense that we cannot start the degree sequence at a significantly lower value.

Proposition 2.1.3. *Let H be an r-chromatic graph so that, for every $x \in V(H)$, $H[N(x)]$ is $(r-1)$-chromatic. Let $h := |H|$, $\sigma := \sigma(H)$ and set $\omega := (h-\sigma)/(r-1)$. Additionally suppose $\sigma < \omega$. Let $0 < 1/n \ll \eta \ll (\omega - \sigma)/h$ where h divides n. Then there is an n-vertex graph G with*

(i) $\lfloor \eta n \rfloor + 1$ vertices of degree $(1 - \frac{\omega+\sigma}{h})n$,

(ii) all other vertices have degree at least $(1 - 1/\chi_{cr}(H))n = (1 - \omega/h)n$,

and G does not have an H-tiling covering all but at most ηn vertices.

Proof. Let G be the complete r-partite graph on n vertices with vertex classes $V_1, \dots, V_r$ where $|V_1| = \sigma n/h + \lfloor \eta n \rfloor + 1$, $|V_2| = \omega n/h - \lfloor \eta n \rfloor - 1$ and $|V_3| = \cdots = |V_r| = \omega n/h$. Let $V' \subseteq V_1$ be of size $\lfloor \eta n \rfloor + 1$. Delete from G all edges with one endpoint in V' and the other in V_2. By construction G satisfies (i) and (ii). Note that since the neighbourhood of any $x \in V'$ induces an $(r-2)$-partite subgraph of G, no vertex in V' lies in a copy of H in G. So G does not have an H-tiling covering all but at most ηn vertices. $\square$

Extremal Example 3. Set $\eta \in \mathbb{R}$ and $x \in (0, 1]$. Let H be a graph with $\chi(H) = r$. Let $h := |H|$, $\sigma := \sigma(H)$ and set $\omega := (h - \sigma)/(r-1)$. Define $g_H(1) := 1 - \omega/h$. We give an extremal example G on n vertices which satisfies the degree sequence of Theorem 2.0.2 except that $(h - x\sigma)n/(r-1)h + \eta n$ vertices have degree at most $(g_H(x) - \eta)n$, but does not contain an H-tiling covering at least $(x - \eta)n$ vertices.

Proposition 2.1.4. *Set $\eta \in \mathbb{R}$ and $x \in (0, 1]$. Let H be a graph with $\chi(H) = r$. Let $h := |H|$, $\sigma := \sigma(H)$ and set $\omega := (h - \sigma)/(r-1)$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i = (g_H(x) - \eta)\, n \quad \text{for all} \quad i \leq \frac{h - x\sigma}{(r-1)h} n + \eta n,$$

$$d_i \geq g_H(x)n \quad \textit{for all} \;\; i > \frac{h - x\sigma}{(r-1)h}n + \eta n,$$

but such that G does not contain an H-tiling covering at least $(x-\eta)n$ vertices.

Proof. Let G be the complete r-partite graph on n vertices with vertex classes $V_1, \dots, V_r$ such that

- $|V_1| = \frac{x\sigma n}{h} - \eta n$,
- $|V_2| = \frac{(h-x\sigma)n}{(r-1)h} + \eta n$,
- $|V_3| = \cdots = |V_r| = \frac{(h-x\sigma)n}{(r-1)h}$.

Consider any H-tiling T of G. Observe that T can contain at most $xn/h - \eta n/\sigma$ copies of H. Indeed, to attain this bound one requires that all colour classes of size σ in copies of H are placed into V_1. Hence at most $x(r-1)\omega n/h - (r-1)\omega\eta n/\sigma$ vertices are covered by T in $V_2 \cup \cdots \cup V_r$. Thus at most $(x-\eta)n - (r-1)\omega\eta n/\sigma$ vertices are covered by T. Hence G does not contain an H-tiling covering at least $(x-\eta)n$ vertices □

2.2 Deriving Theorem 2.0.1 from a weaker result

To prove Theorem 2.0.1 we will first prove the following 'error term' version.

Theorem 2.2.1. *Let $\eta > 0$ and H be a graph with $\chi(H) = r$. Let $h := |H|$, $\sigma := \sigma(H)$ and set $\omega := (h-\sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{h}\right)n + \frac{\sigma}{\omega}i + \eta n \;\; \textit{for all} \; 1 \leq i \leq \frac{\omega n}{h}.$$

Then G has an H-tiling covering all but at most ηn vertices.

Theorem 2.2.1 implies Theorem 2.0.1. Indeed, a simple argument (as in [39]) allows us to remove the error terms.

Proof of Theorem 2.0.1. Set $0 < \tau \ll \eta, 1/h$ and let $n \geq n_0$. Suppose G is an n-vertex graph as in the statement of Theorem 2.0.1. Let A be a set of τn vertices and define G^* to be the graph with vertex set $V(G) \cup A$ and edge set $E(G^*) := E(G) \cup \{xy : x \in V(G) \cup A, y \in A, x \neq y\}$. Then G^* has degree sequence $d_{G^*,1} \leq d_{G^*,2} \leq \cdots \leq d_{G^*,\,(1+\tau)n}$ where

$$\begin{aligned} d_{G^*,i} \geq d_{G,i} + \tau n \geq \left(1 - \frac{\omega+\sigma}{h}\right) n + \frac{\sigma}{\omega} i + \tau n &= \left(1 - \frac{\omega+\sigma}{h}\right)(1+\tau)n + \frac{\sigma}{\omega} i + \frac{\sigma\tau + \omega\tau}{h} n \\ &\geq \left(1 - \frac{\omega+\sigma}{h}\right)(1+\tau)n + \frac{\sigma}{\omega} i + \frac{\omega\tau}{2h}(1+\tau)n \end{aligned}$$

for all $1 \leq i \leq \frac{\omega n}{h}$ and

$$d_{G^*,i} \geq d_{G,\,\omega n/h} + \tau n \geq \left(1 - \frac{\omega}{h}\right) n + \tau n \geq \left(1 - \frac{\omega+\sigma}{h}\right)(1+\tau)n + \frac{\sigma}{\omega} i + \frac{\omega\tau}{2h}(1+\tau)n$$

for all $\frac{\omega n}{h} \leq i \leq \frac{\omega(1+\tau)n}{h}$. Indeed, for $i = \frac{\omega(1+\tau)n}{h}$, we have

$$\begin{aligned} \left(1 - \frac{\omega+\sigma}{h}\right)(1+\tau)n + \frac{\sigma}{\omega} i + \frac{\omega\tau}{2h}(1+\tau)n &= \left(1 - \frac{\omega}{h}\right) n + \tau n + \frac{\omega\tau}{2h}(1+\tau)n - \frac{\omega\tau}{h} n \\ &\leq \left(1 - \frac{\omega}{h}\right) n + \tau n. \end{aligned}$$

Applying Theorem 2.2.1 (with $\frac{\omega\tau}{2h}$), we have that G^* has an H-tiling T covering all but at most $\frac{\omega\tau}{2h}(1+\tau)n$ vertices.

Now, remove every copy of H from T that contains a vertex in A. Then we have removed at most $(h-1)\tau n$ vertices from $V(G) \subset V(G^*)$. Moreover, this implies that there exists an H-tiling in G covering all but at most $(h-1)\tau n + \frac{\omega\tau}{2h}(1+\tau)n$ vertices. Since $(h-1)\tau n + \frac{\omega\tau}{2h}(1+\tau)n < \eta n$, Theorem 2.0.1 holds. $\square$

Outline of the proof of Theorem 2.2.1. The aim of the rest of this chapter is to prove Theorem 2.2.1; we now outline the proof of this result.

We first show that it suffices to prove Theorem 2.2.1 in the case when $H = B$, a bottle graph with neck σ and width ω (where $\sigma < \omega$). In particular, Theorem 2.2.1 is already known in the case when H is a balanced r-partite graph [69].

We then employ a variant of an idea of Komlós [39]. Roughly speaking the idea is as follows: Let B^* be a suitably large blown-up copy of B. We apply the Regularity lemma (Lemma 2.3.2) to obtain a reduced graph R of G. If R contains an almost perfect B^*-tiling then one can rather straightforwardly conclude that G contains an almost perfect B-tiling, as required (for this we apply Lemma 2.4.1). Otherwise, suppose that the largest B^*-tiling in R covers precisely $d \leq (1 - o(1))|R|$ vertices. We then show that, for some $t \in \mathbb{N}$, there is a B^*-tiling in the blow-up $R(t)$ of R covering substantially more than dt vertices. Thus, crucially, the largest B^*-tiling in $R(t)$ covers a higher proportion of vertices than the largest B^*-tiling in R. By repeating this argument, we obtain a blow-up R' of R that contains an almost perfect B^*-tiling. We then show that this implies G contains an almost perfect B-tiling, as desired.

Other applications of this general method have been used in the past [9, 22, 69]. Note however, our approach has different challenges. Indeed, the process of moving from a B^*-tiling $\mathcal{B}$ in R to a proportionally larger B^*-tiling in $R(t)$ is rather subtle. In particular, what we would like to do is conclude that one can find a tiling $\mathcal{B}_0$ (not necessarily of copies of B^*) in R that covers a larger proportion of the vertices in R *and* when one takes a suitable blow-up $R(t)$ of R, then $\mathcal{B}_0$ corresponds to a B^*-tiling in $R(t)$. However, the vertices in R that are uncovered by $\mathcal{B}$ could perhaps all be 'small degree' vertices (i.e. they do not have degree as large as that in Theorem 1.1.6). This is a barrier to finding such a special tiling $\mathcal{B}_0$. (Intuitively, one can imagine that if one has large degree vertices outside of $\mathcal{B}$ then one can glue such vertices onto $\mathcal{B}$ in such a way to obtain our desired tiling $\mathcal{B}_0$.) In this case, one has to (through perhaps many steps) modify $\mathcal{B}$ and then blow-up R to obtain an intermediate blow-up $R(t')$ of R such that (i) there is a B^*-tiling $\mathcal{B}'$ in $R(t')$ that covers the same proportion of vertices compared to the tiling $\mathcal{B}$ in R and (ii) many of the vertices in $R(t')$ uncovered by $\mathcal{B}'$ are now such that they can be 'glued' onto $\mathcal{B}'$ to obtain our desired larger tiling $\mathcal{B}_0$.

Despite these technicalities the proof of Theorem 2.2.1 is perhaps surprisingly short. The main work of the proof is encoded in Lemma 2.5.1, which ensures one can modify the tiling $\mathcal{B}$ as above.

2.3 Szemerédi's Regularity lemma and auxiliary results

A key tool in the proof of Theorem 2.2.1 is Szemerédi's Regularity lemma [67]. To state this lemma we will need the following notion of ε-regularity.

Definition 2.3.1. Let G be a bipartite graph with vertex classes A and B. We define the *density* of G to be

$$d_G(A,B) := \frac{e_G(A,B)}{|A||B|}.$$

Set $\varepsilon > 0$. We say that G is *ε-regular* if for all $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$ we have that $|d_G(X,Y) - d_G(A,B)| < \varepsilon$.

Lemma 2.3.2 (Degree form of Szemerédi's Regularity lemma [67])**.** *Let $\varepsilon \in (0,1)$ and $M' \in \mathbb{N}$. Then there exist natural numbers M and n_0 such that, for any graph G on $n \geq n_0$ vertices and any $d \in (0,1)$, there is a partition of the vertices of G into subsets $V_0, V_1, \dots, V_k$ and a spanning subgraph G' of G such that the following hold:*

- $M' \leq k \leq M$;
- $|V_0| \leq \varepsilon n$;
- $|V_1| = \cdots = |V_k| =: q$;
- $d_{G'}(x) > d_G(x) - (d+\varepsilon)n$ *for all* $x \in V(G)$;
- $e(G'[V_i]) = 0$ *for all* $i \geq 1$;
- *For all* $1 \leq i, j \leq k$ *with* $i \neq j$, *the pair* $(V_i, V_j)_{G'}$ *is* ε*-regular and has density either* 0 *or at least* d.

We call $V_1, \dots, V_k$ the *clusters* of our partition, V_0 the *exceptional set* and G' the *pure graph*. We define the *reduced graph* R of G with parameters ε, d and M' to be the graph whose vertex set is $V_1, \dots, V_k$ and in which V_iV_j is an edge if and only if $(V_i, V_j)_{G'}$ is ε-regular with density at least d. Note also that $|R| = k$.

The proof of the next result is analogous to that of [69, Lemma 5.2]. It states that the degree sequence of G in Theorem 2.2.1 is 'inherited' by its reduced graph R.

Lemma 2.3.3. *Set $M', n_0 \in \mathbb{N}$ and $\varepsilon, d, \eta, b, \omega, \sigma$ to be positive constants such that $1/n_0 \ll 1/M' \ll \varepsilon \ll d \ll \eta, 1/b$ and where $\omega + \sigma \leq b$. Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \frac{b-\omega-\sigma}{b}n + \frac{\sigma}{\omega}i + \eta n \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{b}. \tag{2.1}$$

Let R be the reduced graph of G with parameters ε, d and M' and set $k := |R|$. Then R has degree sequence $d_{R,1} \leq d_{R,2} \leq \cdots \leq d_{R,k}$ such that

$$d_{R,i} \geq \frac{b-\omega-\sigma}{b}k + \frac{\sigma}{\omega}i + \frac{\eta k}{2} \quad \textit{for all } 1 \leq i \leq \frac{\omega k}{b}. \tag{2.2}$$

Proof. Let $V_1, \dots, V_k$ be the clusters of G and V_0 the exceptional set, and let G' be the pure graph of G. Set $q := |V_1| = \cdots = |V_k|$. Clearly we may assume $d_R(V_1) \leq d_R(V_2) \leq \cdots \leq d_R(V_k)$. Now consider any $i \leq \frac{\omega k}{b}$. Set $S := \cup_{1 \leq j \leq i} V_j$. Then $|S| = qi \leq \frac{\omega q k}{b} \leq \frac{\omega n}{b}$. Thus by (2.1) there exists a vertex $x \in S$ such that $d_G(x) \geq d_{qi} \geq \frac{b-\omega-\sigma}{b}n + \left(\frac{\sigma}{\omega}\right)qi + \eta n$. Suppose that $x \in V_j$ where $1 \leq j \leq i$. Since we have that $kq \leq n$, Lemma 2.3.2 implies that

$$\begin{aligned} d_R(V_j) \geq \frac{d_{G'}(x) - |V_0|}{q} &\geq \frac{1}{q}\left(\frac{b-\omega-\sigma}{b}n + \left(\frac{\sigma}{\omega}\right)qi + \eta n - (d+2\varepsilon)n\right) \\ &\geq \frac{b-\omega-\sigma}{b}k + \frac{\sigma}{\omega}i + \frac{\eta k}{2}. \end{aligned}$$

Since $d_{R,i} = d_R(V_i) \geq d_R(V_j)$ we have that (2.2) holds. $\square$

We will also apply the following well-known fact.

Fact 2.3.4. *Let $0 < \varepsilon < \alpha$ and $\varepsilon' := \max\{\varepsilon/\alpha, 2\varepsilon\}$. Let (A, B) be an ε-regular pair of density d. Suppose $A' \subseteq A$ and $B' \subseteq B$ where $|A'| \geq \alpha|A|$ and $|B'| \geq \alpha|B|$. Then (A', B') is an ε'-regular pair with density d' where $|d' - d| < \varepsilon$.*

We will also need the following lemma, which we will use in a couple of proofs.

Lemma 2.3.5 (Key Lemma [42])**.** *Suppose that $0 < \varepsilon < d$, that $q, t \in \mathbb{N}$ and that R is a graph where $V(R) = \{v_1, \dots, v_k\}$. We construct a graph G as follows: Replace every vertex $v_i \in V(R)$ by a set V_i of q vertices and replace each edge of R by an ε-regular pair of density at least d. For each $v_i \in V(R)$, let U_i denote the set of t vertices in $R(t)$ corresponding to v_i. Let H be a subgraph of $R(t)$ with maximum degree Δ, and set $h := |H|$. Set $\delta := d - \varepsilon$*

and $\varepsilon_0 := \delta^\Delta/(2+\Delta)$*. If* $\varepsilon \leq \varepsilon_0$ *and* $t-1 \leq \varepsilon_0 q$ *then there are at least* $(\varepsilon_0 q)^h$ *labelled copies of* H *in* G *so that if* $x \in V(H)$ *lies in* U_i*, then* x *is embedded into* V_i *in* G*.*

2.4 Tools for proving Theorem 2.2.1

In this section we provide further tools that we will need to prove Theorem 2.2.1. We will need the following notation.

The following lemma is a special case of [39, Lemma 11] (which in turn is easily implied by the Key lemma above).

Lemma 2.4.1. *Set* $0 < \beta < 1/2$ *and let* B *be the bottle graph with neck* σ *and width* ω*. Set* $d \in (0,1)$*. Then there exists an* $\varepsilon' > 0$ *such that for all* $\varepsilon \leq \varepsilon'$ *the following holds for all* $q \in \mathbb{N}$*: Let* G *be a graph constructed from* B *by replacing every vertex of* B *by* q *vertices and replacing the edges of* B *with* ε*-regular pairs of density at least* d*. Then* G *has a* B*-tiling covering all but at most a* β*-proportion of the vertices in* G*.*

Given a bottle graph B, the next lemma ensures various blown-up copies of graphs contain perfect B-tilings. Let $v \in \mathbb{N}$. For brevity we will sometimes refer to a vertex class of size v of G as a *v-class* of G.

Lemma 2.4.2. *Set* $m \in \mathbb{N}$*. Let* B *be an* r*-partite bottle graph with neck* σ *and width* ω*, where* $b := |B|$ *and* $\sigma < \omega$*. Define* B' *to be the* r*-partite bottle graph with neck* σ *and width* $\omega - 1$ *and let* $B^* := B(m)$*. Define* $t := (\omega - \sigma)b$*. Then* $B(mt)$*,* $B^*(mt)$*,* $B'(mt)$ *and* $K_r(mt)$ *all have perfect* B^**-tilings.*

Proof. Clearly $B(mt)$ and $B^*(mt)$ both have perfect B^*-tilings. It remains to show that $B'(mt)$ and $K_r(mt)$ have perfect B^*-tilings.

For $K_r(mt)$, tile $(\omega - \sigma)r$ copies of B^* into $K_r(mt)$ such that their (σm)-classes are distributed evenly amongst the r vertex classes of $K_r(mt)$. Indeed, we can view this as tiling $(\omega - \sigma)$ collections of r copies of B^* into $K_r(mt)$ such that, for each collection C, each vertex class of $K_r(mt)$ contains the (σm)-class of precisely one copy of B^* in C.

For $B'(mt)$, firstly tile $(\omega - 1 - \sigma)b$ vertex-disjoint copies of B^* into $B'(mt)$ such that each (σm)-class is placed into the (σmt)-class in $B'(mt)$. So our current B^*-tiling covers all but $\sigma mt - \sigma m(\omega - 1 - \sigma)b = \sigma mb$ vertices in the (σmt)-class in $B'(mt)$ and all but $(\omega - 1)mt - \omega m(\omega - 1 - \sigma)b = \sigma mb$ vertices in each (ωmt)-class in $B'(mt)$. Then the remaining vertices to be covered in $B'(mt)$ form a $K_r(\sigma mb)$ which can be tiled with σr copies of B^*. □

The next result states that the degree sequence of G in Theorem 2.2.1 is inherited by any blown-up copy of G.

Proposition 2.4.3. *Let $n, s \in \mathbb{N}$ and $b, \omega, \sigma > 0$ such that $\omega n > b$ and $\omega + \sigma \leq b$. Set $\eta > 0$. Suppose G is a graph on n vertices with degree sequence $d_{G,1} \leq d_{G,2} \leq \cdots \leq d_{G,n}$ such that*

$$d_{G,i} \geq \frac{b - \omega - \sigma}{b} n + \frac{\sigma}{\omega} i + \eta n \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{b}.$$

Then $\bar{G} := G(s)$ has degree sequence $d_{\bar{G},1} \leq d_{\bar{G},2} \leq \cdots \leq d_{\bar{G},ns}$ such that

$$d_{\bar{G},i} \geq \frac{b - \omega - \sigma}{b} ns + \frac{\sigma}{\omega} i + \left(\eta n - \frac{\sigma}{\omega}\right) s \quad \textit{for all } 1 \leq i \leq \frac{\omega ns}{b}.$$

Proof. For any $1 \leq j \leq ns$ we see that

$$d_{\bar{G},j} = s \cdot d_{G,\lceil j/s \rceil}.$$

Suppose that $j \leq \frac{\omega ns}{b} - s$. Then $\lceil j/s \rceil \leq \frac{\omega n}{b}$ and we have

$$d_{\bar{G},j} \geq \frac{b-\omega-\sigma}{b}ns + \frac{\sigma}{\omega}\lceil j/s \rceil s + \eta ns \geq \frac{b-\omega-\sigma}{b}ns + \frac{\sigma}{\omega}j + \eta ns.$$

In particular, if we take any $i \leq \frac{\omega ns}{b}$ we have

$$d_{\bar{G},i} \geq \frac{b-\omega-\sigma}{b}ns + \frac{\sigma}{\omega}(i-s) + \eta ns = \frac{b-\omega-\sigma}{b}ns + \frac{\sigma}{\omega}i + \left(\eta n - \frac{\sigma}{\omega}\right)s.$$

$\square$

The following result acts as a springboard from which to begin the proof of Lemma 2.5.1.

Proposition 2.4.4. *Set $\eta > 0$ and $m \in \mathbb{N}$, and let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$. Define $B^* := B(m)$. Then there exists $n_0 \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ where*

$$d_i \geq \frac{b-\omega-\sigma}{b}n + \frac{\sigma}{\omega}i + \eta n \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{b}.$$

Then there exists a copy of B^* in G.

Proof. Set $\Delta := \Delta(B^*)$. Let n be sufficiently large and define constants $\varepsilon, d > 0$ and $M' \in \mathbb{N}$ such that $0 < 1/n \ll 1/M' \ll \varepsilon \ll d \ll 1/b, \eta, 1/\Delta$. Let G be an n-vertex graph as in the statement of the proposition. Applying Lemma 2.3.2 with parameters ε, d and M' to G, we obtain clusters $V_1, \dots, V_k$, an exceptional set V_0 and a pure graph G'. Set $q := |V_1| = \cdots = |V_k|$. Let R be the reduced graph of G with parameters ε, d and M', where $k := |R|$. By Lemma 2.3.3 we have that R has degree sequence $d_{R,1} \leq d_{R,2} \leq \cdots \leq d_{R,k}$

where

$$d_{R,i} \geq \frac{b-\omega-\sigma}{b}k + \frac{\sigma}{\omega}i + \frac{\eta k}{2} \quad \text{for all} \quad 1 \leq i \leq \frac{\omega k}{b}. \tag{2.3}$$

By doing the following steps, we find a set $\{x_1, \ldots, x_r\} \subseteq V(R)$ such that $\{x_1, \ldots, x_r\}$ induces a copy of K_r in R:

Step 1: Choose a vertex $x_1 \in V(R)$ such that

$$d_R(x_1) \geq k - \frac{\omega}{b}k + \frac{\eta k}{3}.$$

Such a vertex exists by (2.3).

Step i for each $i \in \{2, \ldots, r-1\}$: We have that $\{x_1, x_2, \ldots, x_{i-1}\}$ induces a copy of K_{i-1} in R and

$$d_R(x_1), d_R(x_2), \ldots, d_R(x_{i-1}) \geq k - \frac{\omega}{b}k + \frac{\eta k}{3}.$$

Let $N_R(x_1, x_2, \ldots, x_{i-1}) := N_R(x_1) \cap N_R(x_2) \cap \cdots \cap N_R(x_{i-1})$. Then

$$\begin{aligned}|N_R(x_1, x_2, \ldots, x_{i-1})| &\geq k - \frac{(i-1)\omega}{b}k + \frac{(i-1)\eta k}{3} \\ &\geq \frac{b-(r-2)\omega}{b}k + \frac{(i-1)\eta k}{3} = \frac{\omega+\sigma}{b}k + \frac{(i-1)\eta k}{3}.\end{aligned}$$

Here the last equality follows as $b = \sigma + (r-1)\omega$. Hence by (2.3) there exists $y \in N_R(x_1, x_2, \ldots, x_{i-1})$ such that $d_R(y) \geq k - \frac{\omega}{b}k + \frac{\eta k}{3}$. Let $x_i := y$.

Step r: We have that $\{x_1, x_2, \ldots, x_{r-1}\}$ induces a copy of K_{r-1} in R. Moreover,

$$|N_R(x_1, x_2, \ldots, x_{r-1})| \geq \frac{\sigma}{b}k + \frac{(r-1)\eta k}{3}.$$

Choose x_r to be any vertex in $N_R(x_1, x_2, \dots, x_{r-1})$.

Therefore there exists a copy of K_r in R, which implies that there exists a copy of B^* in $R(\omega m)$. By Lemma 2.3.5 we have that there exists a copy of B^* in G. □

A crucial tool in the proof of Theorem 2.2.1 is Lemma 2.5.1 below. Before stating this lemma, we need two more definitions.

Definition 2.4.5. Set $\ell \in \mathbb{N}$. Let G be a graph on n vertices and B be a bottle graph with neck σ and width ω. Suppose that there exists a B-tiling T of G and let $\{z_1, \dots, z_\ell\} \subseteq V(G) \setminus V(T)$. We say that $\{z_1, \dots, z_\ell\}$ is an *expanding set of size ℓ for T in G* if the following is true: there exists an injection $f : \{z_1, \dots, z_\ell\} \to T$ such that z_i has a neighbour in every ω-vertex class of $f(z_i)$ for each $1 \leq i \leq \ell$.

Definition 2.4.6. Set $k, \ell, m \in \mathbb{N}$. Let G be a graph on n vertices and let $(v_1, v_2, \dots, v_n)$ be an ordering of the vertices of G. Let B be a bottle graph with neck σm and width ωm. Suppose that there exists a B-tiling T of G and let $\{z_1, \dots, z_\ell\} \subseteq V(G) \setminus V(T)$. Denote by Ω_T the set of all vertices in $V(G)$ that belong to ωm-classes of copies of B in T. Let $z \in \{z_1, \dots, z_\ell\}$ and $y \in \Omega_T$, and denote by B_y the copy of B in T that contains y. Then there exist $1 \leq i, j \leq n$ such that $z := v_i$, $y := v_j$ and $i \neq j$. We say that (z, y) is a *k-swapping pair with respect to $(v_1, \dots, v_n)$* if the following is true: z is adjacent to at least σ vertices in the σm-class of B_y; z is adjacent to at least ω vertices in each ωm-class of B_y that does not contain y; and $j \geq i + k$. We say that $\{z_1, \dots, z_\ell\}$ is a *k-swapping set of size ℓ for T in G with respect to $(v_1, \dots, v_n)$* if there exists a set of ℓ vertices $\{y_1, \dots, y_\ell\} \subseteq \Omega_T$ such that (z_i, y_i) is a k-swapping pair with respect to $(v_1, \dots, v_n)$ for each $1 \leq i \leq \ell$ and $B_{y_p} \neq B_{y_q}$ for all $p \neq q$.

Suppose $\mathcal{B}$ is a B-tiling in a reduced graph R. Very roughly speaking the purpose of expanding sets is to extend $\mathcal{B}$ to a larger tiling whilst swapping sets allow us to 'rotate' which vertices are uncovered by our tiling (which helps for future expansion of $\mathcal{B}$ to a larger tiling).

2.5 Almost perfect H-tilings in graphs

Lemma 2.5.1. *Let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$. Set $\eta, \gamma > 0$ and $n, m \in \mathbb{N}$ such that $0 < 1/n \ll \gamma \ll 1/m \ll \eta \ll 1/b$. Set $B^* := B(m)$. Let G be a graph on n vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ where*

$$d_i \geq \frac{b-\omega-\sigma}{b}n + \frac{\sigma}{\omega}i + \eta n \quad \text{for all } 1 \leq i \leq \frac{\omega n}{b}.$$

Let $V(G) = \{v_1, \dots, v_n\}$ be such that $d_G(v_i) = d_i$ for all $1 \leq i \leq n$. Suppose the largest B^*-tiling in G covers precisely $n' \leq (1-\eta)n$ vertices. Then for any B^*-tiling T covering n' vertices in G there exists an expanding set of size γn for T in G or an $\frac{\omega\gamma n}{\sigma}$-swapping set of size γn for T in G with respect to $(v_1, \dots, v_n)$.

Proof. By repeatedly applying Proposition 2.4.4, we see that $n' \geq \eta n/2$. Define a bijection $I : V(G) \to [n]$ where $I(x) = i$ implies that $d_G(x) = d_i$. Let $V(G) := \{v_1, \dots, v_n\}$ such that $I(v_i) = i$. Set $n'' := n - n'$ and let $G'' := G \setminus V(T)$. Let $V(G'') = \{x_1, \dots, x_{n''}\}$ where $I(x_1) < I(x_2) < \cdots < I(x_{n''})$. For each $1 \leq i \leq n''$, set $s_i := I(x_i)$. Then $d_G(x_i) = d_{s_i}$. Choose j to be the largest integer such that

$$d_G(x_j) \leq \frac{b-\omega}{b}n + (\eta - 2\gamma)n.$$

Notice that $s_j \leq \omega n/b$. We will refer to $x_1, \dots, x_j$ as *small* vertices and $x_{j+1}, \dots, x_{n''}$ as *big* vertices.

Case 1: Suppose we have γn big vertices $z_1, \dots, z_{\gamma n} \in V(G'')$ such that

$$d_G(z_i, G'') \leq \frac{b-\omega}{b} n'' + \frac{\eta n}{4} \quad \text{for all} \quad 1 \leq i \leq \gamma n. \tag{2.4}$$

Then

$$d_G(z_i, T) \geq \frac{b-\omega}{b} n' + \frac{\eta n}{4} \quad \text{for all} \quad 1 \leq i \leq \gamma n.$$

Set $\omega^* := \omega m$. For each $1 \leq i \leq \gamma n$, we see that z_i can be adjacent to at most a $\frac{b-\omega}{b}$-proportion of the vertices in T without having a neighbour in each ω^*-class of some copy of B^* in T. Since $\gamma \ll 1/m \ll \eta \ll 1/b$, for each $1 \leq i \leq \gamma n$ there are at least

$$\frac{\left(\frac{b-\omega}{b} n' + \frac{\eta n}{4}\right) - \left(\frac{b-\omega}{b} n'\right)}{\omega^*} = \frac{\eta n}{4\omega^*} \geq \gamma n$$

copies of B^* in T that have at least one neighbour of z_i in each of their ω^*-classes. Thus we can define an injection $f : \{z_1, \dots, z_{\gamma n}\} \to T$ such that z_i has a neighbour in each ω^*-class of $f(z_i)$ for each $1 \leq i \leq \gamma n$. Hence $\{z_1, \dots, z_{\gamma n}\}$ is an expanding set of size γn for T in G.

Case 2: We may assume there does not exist an expanding set of size γn for T in G.

In particular, there are at most $\gamma n - 1$ vertices in $V(G'')$ that have a neighbour in every ω^*-class of γn copies of B^* in T. (Note that these could be small or big vertices.) Remove such vertices from $V(G'')$ and call the remaining graph G''_1. In particular, no big vertex in G''_1 satisfies (2.4). Set $n''_1 := |G''_1|$.

Subcase A: Suppose we have γn small vertices $x_{i_1}, \dots, x_{i_{\gamma n}} \in V(G''_1)$ such that

$$d_G(x_{i_\ell}, G''_1) \leq \frac{b-\omega-\sigma}{b} n''_1 + \frac{\sigma}{\omega} i_\ell + 2\gamma n \quad \text{for all} \quad 1 \leq \ell \leq \gamma n. \tag{2.5}$$

Then by (2.5) and the degree sequence condition of the lemma, we have

$$d_G(x_{i_\ell}, T) \geq \frac{b-\omega-\sigma}{b}n' + \left(\frac{\sigma}{\omega}s_{i_\ell} - \frac{\sigma}{\omega}i_\ell\right) + \frac{\eta n}{2} \quad \text{for all} \quad 1 \leq \ell \leq \gamma n. \tag{2.6}$$

Let $k \in \{1, \dots \gamma n\}$. Denote by Ω_T^* the set of all vertices in G that belong to ω^*-classes of copies of B^* in T. Set $\sigma^* := \sigma m$. We aim to count the number of vertices $y \in \Omega_T^*$ such that (x_{i_k}, y) is an $\frac{\omega\gamma n}{\sigma}$-swapping pair (with respect to $(v_1, \dots, v_n)$). Let T_1 denote the subcollection of copies B_1 of B^* in T such that x_{i_k} is adjacent to a vertex in every ω^*-class of B_1. Then since we removed earlier all vertices that have a neighbour in every ω^*-class of γn copies of B^* in T, we have

$$d_G(x_{i_k}, T_1) \leq (\gamma n - 1)bm.$$

Suppose $y \in \Omega_T^*$ and let B_y^* be the copy of B^* in T containing y. We say y is *swappable* with x_{i_k} if x_{i_k} is adjacent to at least σ vertices in the σ^*-class of B_y^* and at least ω vertices in each ω^*-class of B_y^* that does not contain y. Denote the set of vertices that are swappable with x_{i_k} by $S(x_{i_k})$. Let T_2 denote the subcollection of copies B_2 of B^* in $T \setminus T_1$ such that B_2 does not contain any vertex in $S(x_{i_k})$. Then

$$d_G(x_{i_k}, T_2) \leq (bm - \omega m - \sigma m + \sigma - 1)|T_2|.$$

Note that the $-\omega m$ term is present since x_{i_k} cannot be adjacent to a vertex in every ω^*-class of any copy of B^* in $T \setminus T_1$. Let $T_3 := T \setminus (T_1 \cup T_2)$. Then

$$d_G(x_{i_k}, T_3) \leq (bm - \omega m)|T_3|.$$

Observe that $|T_1|+|T_2|+|T_3|=n'/bm$. Then

$$\begin{align} 2d_G(x_{i_k},T) &= d_G(x_{i_k},T_1)+d_G(x_{i_k},T_2\cup T_3) \tag{2.7}\\ &\leq (\gamma n-1)bm+\left(\frac{b-\omega-\sigma}{b}n'+\frac{(\sigma-1)}{bm}n'\right)+\sigma m|T_3|. \tag{2.8}\end{align}$$

Using (2.6) and (2.7) we see that

$$|T_3|\geq \frac{s_{i_k}-i_k}{\omega m}+\frac{\eta n}{2\sigma m}-\frac{(\gamma n-1)b}{\sigma}-\frac{(\sigma-1)n'}{b\sigma m^2}\geq \frac{s_{i_k}-i_k}{\omega m}+\frac{\eta n}{8\sigma m},$$

where the the last inequality follows as $\gamma\ll 1/m\ll\eta\ll 1/b$.

Note that as $T_3\cap T_2=\emptyset$, every copy B_3 of B^* in T_3 must contain a vertex from $S(x_{i_k})$. By definition of swappable, this in fact implies that every copy B_3 of B^* in T_3 must contain ω^* vertices from $S(x_{i_k})$. Hence there are at least $s_{i_k}-i_k+\frac{\omega\eta n}{8\sigma}$ vertices in $S(x_{i_k})$. Not all vertices in $S(x_{i_k})$ may form an $\omega\gamma n/\sigma$-swapping pair with x_{i_k} (with respect to $(v_1,\dots,v_n)$). Indeed, there are at most $s_{i_k}-i_k+\frac{\omega\gamma n}{\sigma}$ vertices $y\in S(x_{i_k})$ with $I(y)<s_{i_k}+\frac{\omega\gamma n}{\sigma}$ (and so do not form an $\omega\gamma n/\sigma$-swapping pair with x_{i_k}). Hence, since $\gamma\ll 1/m,\eta,1/b$, there are at least

$$\frac{\omega\eta n}{16\sigma}\geq bm\gamma n$$

vertices $y\in\Omega_T^*$ such that (x_{i_k},y) is an $\frac{\omega\gamma n}{\sigma}$-swapping pair. Therefore, since $k\in\{1,\dots,\gamma n\}$ was arbitrary, for each $\ell\in\{1,\dots,\gamma n\}$ there exist at least $bm\gamma n$ vertices $y\in\Omega_T^*$ such that (x_{i_ℓ},y) is an $\frac{\omega\gamma n}{\sigma}$-swapping pair. Hence there exists a set of vertices $\{y_1,\dots,y_{\gamma n}\}\subseteq\Omega_T^*$ such that (x_{i_ℓ},y_ℓ) is an $\frac{\omega\gamma n}{\sigma}$-swapping pair for each $1\leq\ell\leq\gamma n$ and $B^*_{y_i}\neq B^*_{y_j}$ for all $i\neq j$. Thus $\{x_{i_1},\dots,x_{i_{\gamma n}}\}$ is an $\frac{\omega\gamma n}{\sigma}$-swapping set of size γn for T in G.

Subcase B: Assume there are at most $\gamma n - 1$ small vertices $x \in V(G''_1)$ that satisfy (2.5).

Remove such vertices from $V(G''_1)$, call the remaining graph G''_2 and set $n''_2 := |G''_2|$. Then for every small vertex $x_i \in V(G''_2)$ we have

$$d_{G''_2}(x_i) \geq \frac{b-\omega-\sigma}{b} n''_2 + \frac{\sigma}{\omega} i + \gamma n''_2.$$

For every big vertex $y \in V(G''_2)$, recall that y does not satisfy (2.4). So since $|G'' \setminus G''_2| \leq 2\gamma n$, we have

$$d_{G''_2}(y) \geq \frac{b-\omega}{b} n''_2 + \gamma n''_2.$$

Thus, G''_2 has degree sequence $d_{G''_2,1} \leq d_{G''_2,2} \leq \ldots \leq d_{G''_2,n''_2}$ such that

$$d_{G''_2,i} \geq \frac{b-\omega-\sigma}{b} n''_2 + \frac{\sigma}{\omega} i + \gamma n''_2 \quad \text{for all} \quad 1 \leq i \leq \frac{\omega n''_2}{b}.$$

Hence, by Proposition 2.4.4 there exists a copy of B^* in G''_2, contradicting that the largest B^*-tiling in G covers n' vertices.

Thus Lemma 2.5.1 holds. $\square$

With Lemma 2.5.1 at hand we now can prove Theorem 2.2.1.

Proof of Theorem 2.2.1. If $\sigma = \omega$, then Theorem 2.2.1 is equivalent to the (non-directed) graph version of [69, Theorem 4.2].

So we may assume that $\sigma < \omega$. Set $\sigma' := (r-1)\sigma$ and $\omega' := (r-1)\omega$. Let B be the r-partite bottle graph with neck σ' and width ω', set $b := |B|$ and observe that B has a perfect H-tiling. Let $t := (\omega' - \sigma')b$. Note that it suffices to prove the theorem under the additional assumption that $\eta \ll 1/b$. Define additional constants $\varepsilon, d, \gamma \in \mathbb{R}$ and $M', m \in \mathbb{N}$

such that

$$0 < 1/n \ll 1/M' \ll \varepsilon \ll d \ll \gamma \ll 1/m \ll \eta \ll 1/b.$$

Let $B^* := B(m)$ and set

$$S := \frac{2\sigma}{\omega\gamma^2},\ Q := \lceil 1/\gamma \rceil \ \text{ and } \ z := Q(S+1).$$

Note that B^* has a perfect H-tiling.

Suppose G is an n-vertex graph as in the statement of the theorem. Apply Lemma 2.3.2 with parameters ε, d and M' to G. This gives us clusters $V_1, \dots, V_k$, an exceptional set V_0 and a pure graph G', where $|V_0| \leq \varepsilon n$ and $|V_1| = \cdots = |V_k| =: q$. Let R be the reduced graph of G with parameters ε, d and M'; so $k = |R|$. By Lemma 2.3.3, R has degree sequence $d_{R,1} \leq d_{R,2} \leq \cdots \leq d_{R,k}$ such that

$$d_{R,i} \geq \left(1 - \frac{\omega+\sigma}{h}\right)k + \frac{\sigma}{\omega}i + \frac{\eta k}{2} = \left(1 - \frac{\omega'+\sigma'}{b}\right)k + \frac{\sigma'}{\omega'}i + \frac{\eta k}{2} \ \text{ for all } 1 \leq i \leq \frac{\omega k}{h} = \frac{\omega' k}{b}.$$

In what follows when we consider an s-swapping set in some blow-up $R(w)$ of R, we always implicitly mean an s-swapping set in $R(w)$ with respect to $(v_1, \dots, v_{kw})$ where $V(R(w)) = \{v_1, \dots, v_{kw}\}$ and $d_{R(w)}(v_1) \leq d_{R(w)}(v_2) \leq \cdots \leq d_{R(w)}(v_{kw})$. That is, each blow-up $R(w)$ of R comes equipped with an ordering of its vertices based on the degrees; these orderings are defined by the functions I_j below.

Claim 2.5.2. *$R' := R((mt)^z)$ contains a B^*-tiling $\mathcal{T}$ covering at least $(1-\eta/2)k(mt)^z = (1-\eta/2)|R'|$ vertices.*

Proof of Claim 2.5.2. If R contains a B^*-tiling covering at least $(1-\eta/2)k$ vertices then

Lemma 2.4.2 implies that Claim 2.5.2 holds. Suppose then that the largest B^*-tiling T in R covers exactly c vertices where $c < (1-\eta/2)k$. Then by Lemma 2.5.1, there exists an expanding set of size γk for T in R or an $\frac{\omega\gamma k}{\sigma}$-swapping set of size γk for T in R. Define B' to be the r-partite bottle graph with neck σ' and width $\omega'-1$. Set $\omega^* := \omega' m$.

Step 1: Find a B^*-tiling covering at least $(c+\gamma k)(mt)^{S+1}$ vertices in $R((mt)^{S+1})$.

Case 1: There exists an expanding set $\{z_1, \dots, z_{\gamma k}\}$ for T, and hence also an associated injection $f : \{z_1, \dots, z_{\gamma k}\} \to T$.

We need a few definitions. Let $i \in \mathbb{N}$ and $\mathcal{H}$ be a (finite) family of graphs. We define an *$\mathcal{H}$-tiling in G* to be a collection of vertex-disjoint copies of graphs from $\mathcal{H}$ in G.

In this case we do the following: For each $1 \leq i \leq \gamma k$, separate $R[z_i \cup f(z_i)]$ into a copy of K_r (containing z_i and one vertex from each ω^*-class of $f(z_i)$), a copy of B' and a copy of $B(m-1)$. Then we have a $(B^*, B(m-1), B', K_r)$-tiling in R covering at least $c+\gamma k$ vertices. By Lemma 2.4.2, $R(mt)$ contains a B^*-tiling covering at least $(c+\gamma k)mt$ vertices. Further applying Lemma 2.4.2 we obtain a B^*-tiling covering at least $(c+\gamma k)(mt)^{S+1}$ vertices in $R((mt)^{S+1})$, as desired.

Case 2: There does not exist an expanding set of size γk for T in R.

For each $1 \leq j \leq S$, Proposition 2.4.3 implies that $R((mt)^j)$ has degree sequence $d_{R((mt)^j),1} \leq d_{R((mt)^j),2} \leq \cdots \leq d_{R((mt)^j),k(mt)^j}$ such that

$$d_{R((mt)^j),i} \geq \left(1 - \frac{\omega+\sigma}{h}\right) k(mt)^j + \frac{\sigma}{\omega} i + \left(\frac{\eta k}{2} - \frac{\sigma}{\omega}\right)(mt)^j \quad \text{for all } 1 \leq i \leq \frac{\omega k (mt)^j}{h}.$$

Define for $0 \leq j \leq S$ bijections $I_j : V(R((mt)^j)) \to [k(mt)^j]$ where $I_j(x) := i$ implies that $d_{R((mt)^j)}(x) = d_{R((mt)^j),i}$. In particular, suppose that $x \in V(R)$ and let $x_1, \dots, x_{(mt)^j}$ denote the $(mt)^j$ vertices in $R((mt)^j)$ that correspond to x. Suppose that $I_0(x) = i$. Then we may assume that

$$I_j(x_s) = (i-1)(mt)^j + s > (I_0(x)-1)(mt)^j \text{ for each } 1 \leq s \leq (mt)^j. \tag{2.9}$$

To put all this another way, one can view I_0 as an ordering of the vertices in R in terms of the vertex degrees; I_j is the ordering of $R((mt)^j)$ 'inherited' from the ordering I_0.

Note that for all $0 \leq j \leq S$,

$$\left(\sum_{x \in V(R((mt)^j))} I_j(x) \right) \leq k^2 (mt)^{2j}. \tag{2.10}$$

Denote by Ω_T^* the set of all vertices in $V(R)$ that belong to ω^*-classes of copies of B^* in T. As there does not exist an expanding set of size γk for T in R, then there exists an $\frac{\omega\gamma k}{\sigma}$-swapping set $\{z_1, \dots, z_{\gamma k}\}$ for T in R. Hence there also exists a set $\{y_1, \dots, y_{\gamma k}\} \subseteq \Omega_T^*$ such that (z_i, y_i) is an $\frac{\omega\gamma k}{\sigma}$-swapping pair for each $1 \leq i \leq \gamma k$, such that $B^*_{y_i} \neq B^*_{y_j}$[1] for all $i \neq j$, and such that $I_0(y_i) \geq I_0(z_i) + \frac{\omega\gamma k}{\sigma}$ for all $1 \leq i \leq \gamma k$.

For each $1 \leq i \leq \gamma k$, note that $R[(z_i \cup V(B^*_{y_i})) \setminus \{y_i\}]$ can be separated into a copy of B containing z_i and a copy of $B(m-1)$. Then we have a $(B^*, B(m-1), B)$-tiling T_1 covering c vertices in R. Further, since each (z_i, y_i) is an $\frac{\omega\gamma k}{\sigma}$-swapping pair, we have that

$$\left(\sum_{x \in V(R) \setminus V(T_1)} I_0(x) \right) \geq \left(\sum_{x \in V(R) \setminus V(T)} I_0(x) \right) + \frac{\omega\gamma^2 k^2}{\sigma}. \tag{2.11}$$

By Lemma 2.4.2, $T_1(mt)$ contains a perfect B^*-tiling T', i.e. T' is a B^*-tiling covering $c(mt)$ vertices in $R(mt)$. Observe that T' in $R(mt)$ covers proportionally the same amount

[1] As in Definition 2.4.6.

of vertices as T in R. Further, (2.9) and (2.11) imply that

$$\begin{aligned}\sum_{x\in V(R(mt))\setminus V(T')} I_1(x) &\geq \left(\sum_{x\in V(R)\setminus V(T_1)} (I_0(x)-1)\right)(mt)^2\\ &\geq \left(\left(\sum_{x\in V(R)\setminus V(T)} I_0(x)\right)+\frac{\omega(\gamma k)^2}{2\sigma}\right)(mt)^2. \qquad (2.12)\end{aligned}$$

Denote by $\Omega^*_{T'}$ the set of all vertices in $R(mt)$ that belong to ω^*-classes of copies of B^* in T'. Suppose that there does not exist an expanding set of size γkmt for T' in $R(mt)$. Then by Lemma 2.5.1 there must exist an $\frac{\omega\gamma kmt}{\sigma}$-swapping set of size γkmt for T' in $R(mt)$. As before we can produce a $(B^*, B(m-1), B)$-tiling T'_1 covering $c(mt)$ vertices in $R(mt)$. Then by Lemma 2.4.2, $T'_1(mt)$ contains a perfect B^*-tiling T'', i.e. T'' is a B^*-tiling covering $c(mt)^2$ vertices in $R((mt)^2)$. Observe, similarly as before, that T'' in $R((mt)^2)$ covers proportionally the same amount of vertices as T in R and

$$\begin{aligned}\sum_{x\in V(R((mt)^2))\setminus V(T'')} I_2(x) &\geq \left(\left(\sum_{x\in V(R(mt))\setminus V(T')} I_1(x)\right)+\frac{\omega(\gamma kmt)^2}{2\sigma}\right)(mt)^2\\ &\overset{(2.12)}{\geq} \left(\left(\sum_{x\in V(R)\setminus V(T)} I_0(x)\right)+\frac{\omega(\gamma k)^2}{\sigma}\right)(mt)^4.\end{aligned}$$

Note that (2.10) implies that one can repeat this argument at most S times; that is, for some $j \leq S$ we must obtain an expanding set of size $\gamma k(mt)^j$ in $R((mt)^j)$. More precisely, we obtain a B^*-tiling $T^{(j)}$ in $R((mt)^j)$ covering $c(mt)^j$ vertices, such that there exists an expanding set of size $\gamma k(mt)^j$ for $T^{(j)}$ in $R((mt)^j)$. Then as before, one can use this expanding set and Lemma 2.4.2 to obtain a B^*-tiling covering at least $(c+\gamma k)(mt)^{S+1}$ vertices in $R((mt)^{S+1})$, as desired.

General Step:

Repeating the whole argument from Step 1 at most Q times we see that $R((mt)^{Q(S+1)}) = R((mt)^z) = R'$ has a B^*-tiling $\mathcal{T}$ covering at least $(1-\eta/2)|R'|$ vertices. Thus Claim 2.5.2 holds.

$\square$

Now for each $1 \le i \le k$, partition V_i into classes $V_i^*, V_{i,1}, \dots, V_{i,(mt)^z}$ where $q' := |V_{i,j}| = \lfloor q/(mt)^z \rfloor \ge q/(2(mt)^z)$ for all $1 \le j \le (mt)^z$. Lemma 2.3.2 implies that $qk \ge (1-\varepsilon)n$, therefore

$$q'|R'| = \lfloor q/(mt)^z \rfloor k(mt)^z \ge qk - k(mt)^z \ge (1-2\varepsilon)n. \tag{2.13}$$

Fact 2.3.4 tells us that for each ε-regular pair $(V_{i_1}, V_{i_2})_{G'}$ with density at least d we have that $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon(mt)^z$-regular with density at least $d-\varepsilon \ge d/2$ (for all $1 \le j_1, j_2 \le (mt)^z$). Note that $2\varepsilon(mt)^z \le \varepsilon^{1/2}$. So we can label the vertex set of R' so that $V(R') = \{V_{i,j} : 1 \le i \le k, 1 \le j \le (mt)^z\}$ and see that if $V_{i_1,j_1}V_{i_2,j_2} \in E(R')$ then $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $\varepsilon^{1/2}$-regular with density at least $d/2$.

We know by Claim 2.5.2 that R' has a B^*-tiling $\mathcal{T}$ that covers at least $(1-\eta/2)|R'|$ vertices. Let $\hat{B}^*$ be a copy of B^* in $\mathcal{T}$ and label the vertices of $\hat{B}^*$ so that $V(\hat{B}^*) = \{V_{i_1,j_1}, V_{i_2,j_2}, \dots, V_{i_{bm},j_{bm}}\}$. Set $V' := V_{i_1,j_1} \cup V_{i_2,j_2} \cup \dots \cup V_{i_{bm},j_{bm}}$. Applying Lemma 2.4.1 with $\eta^2, q', d/2, \varepsilon^{1/2}$ playing the roles of β, q, d, ε, we have that $G'[V']$ has a B^*-tiling covering at least $(1-\eta^2)q'bm$ vertices. Applying Lemma 2.4.1 in this way to each copy of B^* in $\mathcal{T}$ we see that $G' \subseteq G$ has a B^*-tiling covering at least

$$\left(\left(1-\eta^2\right)q'bm\right) \times \left(\left(1-\eta/2\right)|R'|\right)/bm \overset{(2.13)}{\ge} \left(1-\eta^2\right)\left(1-\eta/2\right)\left(1-2\varepsilon\right)n \ge (1-\eta)n$$

vertices. Since each copy of B^* has a perfect H-tiling, G contains an H-tiling covering all but at most ηn vertices. $\square$

Theorem 2.0.1 easily implies Theorem 2.0.2.

Proof of Theorem 2.0.2. Let H, $x \in (0,1)$ and $\eta > 0$ be as in the statement of the theorem. Suppose n is sufficiently large and let G be an n-vertex graph as in the statement of the theorem.

Note that it suffices to prove the result in the case when $x \in (0,1) \cap \mathbb{Q}$. Thus, there exist $a, b \in \mathbb{N}$ such that $x = a/b$. Define $\sigma_1 := a(r-1)\sigma$ and $\omega_1 := a(r-1)\omega + (b-a)h = bh - a\sigma$. Let H_1 be the r-partite bottle graph with neck σ_1 and width ω_1, and observe that $\sigma_1 < \omega_1$ and $|H_1| = b(r-1)h$.

Claim. *H_1 contains an H-tiling covering $x|H_1|$ vertices.*

The claim follows since one can tile H_1 with $a(r-1)$ copies of H where each σ-class lies in the σ_1-class of H_1. Thus, we have an H-tiling covering $a(r-1)h = x|H_1|$ vertices in H_1, as desired.

Note that

$$d_i \geq \left(g_H(x) - \frac{x\sigma}{h}\right)n + \frac{(r-1)x\sigma}{h - x\sigma}i = \left(1 - \frac{\omega_1 + \sigma_1}{b(r-1)h}\right)n + \frac{\sigma_1}{\omega_1}i$$

for all $i \leq \left(\frac{h - x\sigma}{(r-1)h}\right)n = \frac{\omega_1 n}{b(r-1)h}$. Thus, applying Theorem 2.0.1 with H_1 playing the role of H, we produce an H_1-tiling in G covering all but at most ηn vertices. Then the claim implies that we have an H-tiling in G covering at least $x(1-\eta)n > (x-\eta)n$ vertices. $\square$

2.6 Concluding remarks

In this chapter we have given a particular degree sequence condition that forces a graph to contain an almost perfect H-tiling (Theorem 2.0.1). In fact, in general for a fixed graph H, Theorem 2.0.1 yields a whole class of degree sequences that force an almost perfect H-tiling. Indeed, we have the following consequence of Theorem 2.0.1.

Theorem 2.6.1. *Let $\eta > 0$ and H be a graph with $\chi(H) = r$ and $h := |H|$. Set $\sigma \in \mathbb{R}$ such that $\sigma(H) \leq \sigma \leq h/r$ and $\omega := (h - \sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, \sigma, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{h}\right) n + \frac{\sigma}{\omega} i \quad \text{for all } 1 \leq i \leq \frac{\omega n}{h}.$$

Then G contains an H-tiling covering all but at most ηn vertices.

Proof. Note that it suffices to prove the theorem under the assumption that $\sigma \in \mathbb{Q}$. To prove Theorem 2.6.1, we define a certain bottle graph H^* and then apply Theorem 2.0.1 with input H^* to conclude our result.

Since $\sigma \in \mathbb{Q}$, there exist $a, b \in \mathbb{N}$ such that $\sigma = a/b$. Let $\omega(H) := (h - \sigma(H))/(r-1)$ and $t := b(r-1)(\omega(H) - \sigma(H))$. We define H^* to be the r-partite bottle graph with neck σt and width ωt (note $\sigma t, \omega t \in \mathbb{N}$). Also, notice that $|H^*| = ht$.

Claim. *H^* contains a perfect H-tiling.*

We tile t copies of H into H^*. Firstly, tile $b(r-1)(\omega(H) - \sigma)$ copies of H into H^* such that the $\sigma(H)$-classes are all placed in the σt-class of H^* and the rest of the vertex classes of H are equally distributed amongst the vertex classes of H^*. This leaves

$$\sigma b(r-1)(\omega(H)-\sigma(H))-\sigma(H)b(r-1)(\omega(H)-\sigma)=\omega(H)b(r-1)(\sigma-\sigma(H))$$

vertices in the σt-class of H^* to be covered and

$$\begin{aligned}
&\omega b(r-1)(\omega(H)-\sigma(H))-b\omega(H)(r-1)(\omega(H)-\sigma)\\
&=b((h-\sigma)(\omega(H)-\sigma(H))-(h-\sigma(H))(\omega(H)-\sigma))\\
&=b(h-\omega(H))(\sigma-\sigma(H))
\end{aligned}$$

vertices in each ωt-class of H^* to be covered. Let $\hat{H}$ be the r-partite complete graph with one vertex class of size $(r-1)\omega(H)$ and $(r-1)$ vertex classes of size $(r-2)\omega(H)+\sigma(H)$. Observe that $\hat{H}$ has a perfect H-tiling (using $r-1$ copies of H). To cover the remaining vertices of H^*, tile $b(\sigma-\sigma(H))$ copies of $\hat{H}$ into H^* such that every vertex class of size $(r-1)\omega(H)$ is placed in the σt class of H^*. Observe that

$$((r-2)\omega(H)+\sigma(H))b(\sigma-\sigma(H))=b(h-\omega(H))(\sigma-\sigma(H)).$$

Hence H^* contains a perfect H-tiling and the claim holds.

Suppose G is as in the statement of Theorem 2.6.1. Applying Theorem 2.0.1 with input G and H^*, we obtain that G contains an H^*-tiling covering all but at most ηn vertices. (Note the degree sequence in Theorem 2.6.1 is precisely the degree sequence of Theorem 2.0.1 with input H^*.) Since each copy of H^* has a perfect H-tiling, we conclude that G contains an H-tiling covering all but at most ηn vertices. $\square$

In a similar way, Theorem 2.0.2 yields a class of degree sequences forcing an almost x-proportional H-tiling in G.

Theorem 2.6.2. *Let $x \in (0,1)$ and H be a graph with $\chi(H) = r$ and $h := |H|$. Set $\eta > 0$. Let $\sigma \in \mathbb{R}$ such that $\sigma(H) \leq \sigma \leq h/r$ and $\omega := (h-\sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, x, \sigma, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(g_H(x) - \frac{x\sigma}{h}\right) n + \frac{(r-1)x\sigma}{h - x\sigma} i \quad \textit{for all } 1 \leq i \leq \left(\frac{h - x\sigma}{(r-1)h}\right) n.$$

Then G contains an H-tiling covering at least $(x-\eta)n$ vertices.

Proof. Define H^* as in the proof of Theorem 2.6.1. Applying Theorem 2.0.2 with input H^*, we obtain that G contains an H^*-tiling covering all but at most $(x-\eta)n$ vertices. Since each copy of H^* has a perfect H-tiling, we conclude that G contains an H-tiling covering all but at most $(x-\eta)n$ vertices. $\square$

Chapter Three

A degree sequence version of the Kühn–Osthus tiling theorem

This chapter is joint work with Andrew Treglown and is based on [30]. Recall the following definitions pertaining to Theorem 1.1.8. Let H be a graph and $\chi(H) =: r$. We say that a proper colouring c of H is *optimal* if c uses precisely r colours. Let C_H be the set of all optimal colourings of H. Given an optimal colouring $c \in C_H$, let $x_{c,1} \leq x_{c,2} \leq \cdots \leq x_{c,r}$ denote the sizes of the colour classes of c. Define

$$\mathcal{D}(c) := \{x_{c,j+1} - x_{c,j} \mid j = 1, \ldots, r-1\}$$

and take

$$\mathcal{D}(H) := \bigcup_{c \in C_H} \mathcal{D}(c).$$

Let $\mathrm{hcf}_\chi(H)$ be the highest common factor of all integers in $\mathcal{D}(H)$. In the case $\mathcal{D}(H) = \{0\}$, we set $\mathrm{hcf}_\chi(H) := \infty$. Note that $\mathcal{D}(H) = \{0\}$ if and only if $\chi(H) = \chi_{cr}(H)$. Let $\mathrm{hcf}_c(H)$ be the highest common factor of all the orders of the components of H. If $\chi(H) \neq 2$ we say that $\mathrm{hcf}(H) = 1$ if $\mathrm{hcf}_\chi(H) = 1$. If $\chi(H) = 2$ then we say that $\mathrm{hcf}(H) = 1$ if both $\mathrm{hcf}_c(H) = 1$

and $\mathrm{hcf}_\chi(H) \leq 2$. Note also that there are graphs H which have $\mathrm{hcf}_\chi(H) = 1$, but for every optimal colouring c of H we have that the highest common factor of the integers in $\mathcal{D}(c)$ is greater than one.[1] That is to say that no particular colouring of H necessarily 'certifies' that $\mathrm{hcf}_\chi(H) = 1$. Let $\sigma(H)$ denote the size of the smallest possible colour class in any r-colouring of H.

Recall that in [44], Kühn and Osthus proved the following theorem:

Theorem 1.1.12 ([44])**.** *Let $\eta > 0$ and H be a graph with $\mathrm{hcf}_\chi(H) = 1$ and $\chi(H) =: r \geq 3$. Then there exists an integer $n_0 = n_0(\eta, H)$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices such that $|H|$ divides n and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)} + \eta\right) n.$$

Then G contains a perfect H-tiling.

When H is such that $\mathrm{hcf}_\chi(H) = 1$ and $r \geq 3$, Theorem 1.1.12 is a weaker version of Theorem 1.1.8 due to the additional error term required in the minimum degree condition. In this chapter we prove the following degree sequence strengthening of Theorem 1.1.12, including a case when $r = 2$ and $\mathrm{hcf}(H) = 1$.

Theorem 3.0.1. *Let $\eta > 0$ and H be a graph with $\mathrm{hcf}(H) = 1$ and $\chi(H) =: r \geq 2$. Let $\sigma := \sigma(H)$, $h := |H|$ and $\omega := (h - \sigma)/(r-1)$. Then there exists an integer $n_0 = n_0(\eta, H)$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices such that h divides n and G has degree sequence $d_1 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{h}\right) n + \frac{\sigma}{\omega} i + \eta n \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{h}.$$

Then G contains a perfect H-tiling.

[1] An example is $K_{1,4,6}$ with an extra vertex attached to every vertex in the vertex class of size 4.

Observe that $\omega/h = 1/\chi_{cr}(H)$. Hence, Theorem 3.0.1 is a strengthening of Theorem 1.1.12. Note that Theorem 3.0.1 applies to all graphs H with $\text{hcf}(H) = 1$, not just graphs H with $\chi(H) \geq 3$ and $\text{hcf}_\chi(H) = 1$ (as in Theorem 1.1.12). One can also see that the statement of Theorem 3.0.1 is essentially the same as Theorem 2.0.1 except we have an additional error term throughout the degree sequence and get a perfect H-tiling instead of one covering all but at most ηn vertices. See Extremal Example 1 and Section 3.3 regarding whether this additional error term can be removed or not.

Theorem 3.0.1 (and Theorem 1.1.11) is best-possible for many graphs H in the sense that we cannot replace the ηn-term with a $o(\sqrt{n})$-term (see Proposition 3.2.1). Theorem 3.0.1 is also best possible for all graphs H in the sense that there are n-vertex graphs G with only slightly more than $\omega n/h$ vertices with degree (slightly) below $(1 - \omega/h + \eta)n$ that do not contain a perfect H-tiling (see Proposition 3.2.4). Thus, it is not possible to allow significantly more 'small' degree vertices in Theorem 3.0.1. Extremal examples are discussed in more detail in Section 3.2.

Recall Theorem 1.1.11, the degree sequence Alon–Yuster theorem proved by Treglown [69]:

Theorem 1.1.11 (Treglown [69]). *Let $\eta > 0$ and H be a graph with $\chi(H) =: r$. Then there exists an integer $n_0 = n_0(\eta, H)$ such that the following holds: Suppose that G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq \dots \leq d_n$ such that*

$$d_i \geq (r-2)n/r + i + \eta n \quad \text{for all } i < n/r$$

and $|H|$ divides n. Then G contains a perfect H-tiling.

Theorem 3.0.1 complements Theorem 1.1.11. Indeed, combining Theorems 3.0.1 and 1.1.11 yields the following degree sequence version of Theorem 1.1.8.

Theorem 3.0.2. *Let $\eta > 0$ and H be a graph with $\chi(H) =: r \geq 2$. Let $\sigma := \sigma(H)$, $h := |H|$ and $\omega := (h - \sigma)/(r-1)$. Then there exists an integer $n_1 = n_1(\eta, H)$ such that if G is a graph on $n \geq n_1$ vertices, h divides n and either* (i) *or* (ii) *(below) holds, then G contains a perfect H-tiling.*

(i) $\mathrm{hcf}(H) = 1$ *and G has degree sequence $d_1 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{h}\right) n + \frac{\sigma}{\omega} i + \eta n \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{h}.$$

(ii) $\mathrm{hcf}(H) \neq 1$ *and G has degree sequence $d_1 \leq \cdots \leq d_n$ such that*

$$d_i \geq (r-2)n/r + i + \eta n \quad \textit{for all } 1 \leq i \leq n/r.$$

One can in fact obtain the following generalisation of Theorem 3.0.1.

Theorem 3.0.3. *Let $\eta > 0$ and H be a graph with* $\mathrm{hcf}(H) = 1$ *and $\chi(H) =: r \geq 2$. Let $h := |H|$. Set $\sigma \in \mathbb{R}$ such that $\sigma(H) \leq \sigma < h/r$ and $\omega := (h - \sigma)/(r-1)$. Then there exists an integer $n_1 = n_1(\eta, H)$ such that the following holds: Let G be a graph on $n \geq n_1$ vertices such that h divides n and G has degree sequence $d_1 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{h}\right) n + \frac{\sigma}{\omega} i + \eta n \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{h}.$$

Then G contains a perfect H-tiling.

Observe that for graphs H with $\mathrm{hcf}(H) = 1$, Theorem 3.0.3 interpolates between Theorem 1.1.11 and Theorem 3.0.1. In Section 3.7, we will prove Theorem 3.0.3 directly. The proof of Theorem 3.0.3 follows that of Theorem 1.1.12 in [44] closely. The main novelty of our proof is how we avoid divisibility barriers. For this we make use of an elementary number theoretic result for graphs with $\mathrm{hcf}(H) = 1$ (see Theorem 3.5.2). We also make use of Theorem 2.6.1 from the preceding chapter.

Since the choice of $\sigma \in [\sigma(H), h/r)$ is arbitrary, note that Theorem 3.0.3 provides an infinite collection of degree sequences that force a perfect H-tiling. Having a higher value of σ lowers the starting point of the degree sequence condition, but at the price of a steeper 'slope' and higher value of $d_{\omega n/h}$ (see Figure 3.1). As with Theorem 3.0.1, for many graphs H, each of these degree sequences is best-possible in the sense that we cannot replace the ηn-term with a $o(\sqrt{n})$-term (see Section 3.2). Note too that we cannot extend Theorem 3.0.3 to the case when $\sigma < \sigma(H)$. Indeed, in this case, if we set $\eta \ll 1$ then the degree sequence condition in Theorem 3.0.3 would allow all vertices in G to have degree below $(1 - 1/\chi_{cr}(H))n - 1$; however, we know from Theorem 1.1.8 that there are graphs G that satisfy this condition and that do not contain perfect H-tilings.

3.1 Organisation

The rest of Chapter 3 is organised as follows. In the next section we give several extremal examples for Theorems 3.0.1 and 3.0.3. We also ask whether one can improve Theorem 3.0.1 by suitably 'capping' the bounds on the degrees of the vertices (see Question 3.3.1). In Section 3.4 we give a number of auxiliary results and definitions, including a generalisation of [46, Lemma 12] (Lemma 3.4.5). We then prove a new elementary number theoretic result (Theorem 3.5.2) in Section 3.5 which will be a crucial tool in overcoming divisibility barriers during the proof of Theorem 3.0.3. In Section 3.6 we give an overview of the proof of Theorem 3.0.3. In Section 3.7 we prove Theorem 3.0.3.

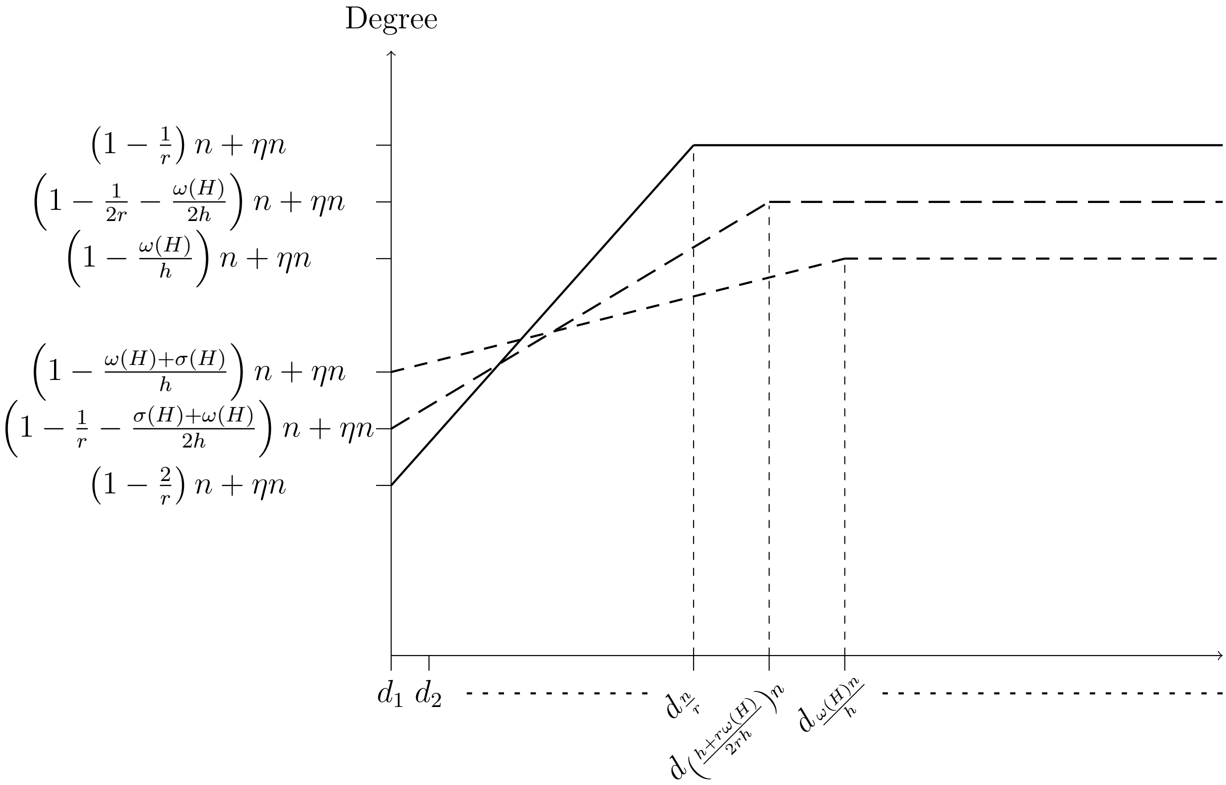

Figure 3.1: The degree sequence in Theorem 3.0.3 for a fixed graph H given $\sigma = \sigma(H)$ (medium dashed); $\sigma = \frac{h+r\sigma(H)}{2r}$ (long dashed); $\sigma = \frac{h}{r}$ (full).

3.2 Extremal examples for Theorems 3.0.1 and 3.0.3

The following 3 extremal examples demonstrate ways in which Theorems 3.0.1 and 3.0.3 are best possible. The first shows that we cannot significantly lower *every* term in the degree sequence conditions of Theorems 3.0.1 and 3.0.3 and still ensure a perfect H-tiling for complete r-partite graphs. The second shows that that the 'slope' of the degree sequence in Theorem 3.0.1 is best possible for bottle graphs. The third demonstrates that for any graph H, to ensure a perfect H-tiling (or even an 'almost' perfect H-tiling) in a graph G on n vertices we cannot have significantly more than $\omega n/h$ vertices that have degree below $\left(1-\frac{1}{\chi_{cr}(H)}+\eta\right)n$.

Extremal Example 1. The following construction (a simple adaption of [69, Proposition 3.1]) demonstrates that for most complete r-partite graphs H, one cannot replace the ηn-term in Theorems 3.0.1 and 3.0.3 with a $o(\sqrt{n})$-term.

Proposition 3.2.1. *Let $r \geq 3$ and $H := K_{t_1,\dots,t_r}$ with $t_i \geq 2$ (for all $1 \leq i \leq r$). Let $h := |H|$. Set $\sigma \in \mathbb{R}$ such that $\sigma(H) \leq \sigma < h/r$ and $\omega := (h-\sigma)/(r-1)$. Let $n \in \mathbb{N}$ be sufficiently large so that $\sqrt{n}$ is an integer that is divisible by $6h^2$. Set $C := \sqrt{n}/3h^2$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \dots \leq d_n$ satisfies*

$$d_i \geq \left(1 - \frac{\omega+\sigma}{h}\right)n + \frac{\sigma}{\omega}i + C \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{h}$$

but such that G does not contain a perfect H-tiling.

Proof. Let G denote the graph on n vertices consisting of r vertex classes $V_1, \dots, V_r$ with $|V_1| = 1$, $|V_2| = \omega n/h + 1 + Cr$, $|V_3| = (\sigma+\omega)n/h - 2 - 3C$ and $|V_i| = \omega n/h - C$ if $4 \leq i \leq r$ and which contains the following edges:

- All possible edges with an endpoint in V_3 and the other endpoint in $V(G) \setminus V_1$ (in particular, $G[V_3]$ is complete);
- All edges with an endpoint in V_i and the other endpoint in $V(G) \setminus V_i$ for $i = 2$ and $4 \leq i \leq r$;
- There are $\sqrt{n}/2$ vertex-disjoint stars in V_2, each of size $\lfloor 2|V_2|/\sqrt{n} \rfloor$, $\lceil 2|V_2|/\sqrt{n} \rceil$, which cover all of V_2.

In particular, note that the vertex $v \in V_1$ sends all possible edges to $V(G) \setminus V_3$ but no edges to V_3.

Let $d_1 \leq \dots \leq d_n$ denote the degree sequence of G. Notice that every vertex in V_i for $3 \leq i \leq r$ has degree at least $(1-\omega/h)n + C$. Note that $\lfloor 2|V_2|/\sqrt{n} \rfloor \geq 2\sqrt{n}/h = 6Ch \geq 6Cr$.

Thus, there are $\sqrt{n}/2$ vertices (namely those at the centers of the stars) in V_2 of degree at least

$$(1-\omega/h)n - 1 - Cr + (6Cr-1) \geq (1-\omega/h)n + C.$$

The remaining $\omega n/h + 1 + Cr - \sqrt{n}/2 \leq \omega n/h - \sqrt{n}/3 - 1$ vertices in V_2 have degree at least

$$(1-\omega/h)n - Cr \geq (1-\omega/h)n - \sigma\sqrt{n}/3\omega + C.$$

Since $d_G(v) \geq \left(1 - \frac{\omega+\sigma}{h}\right)n + C + \sigma/\omega$ for the vertex $v \in V_1$ we have that

$$d_i \geq \left(1 - \frac{\omega+\sigma}{h}\right)n + \frac{\sigma}{\omega}i + C \quad \text{for all } 1 \leq i \leq \frac{\omega n}{h}.$$

Suppose that $v \in V_1$ lies in a copy H' of H in G. Then by construction of G, two of the vertex classes U_1, U_2 of H' must lie entirely in V_2. By definition of H, $H'[U_1 \cup U_2]$ contains a path of length 3. However, $G[V_2]$ does not contain a path of length 3, a contradiction. Thus, v does not lie in a copy of H and so G does not contain a perfect H-tiling. □

Extremal Example 2. Recall the following definitions. Let $v \in \mathbb{N}$. We will refer to a vertex class of size v of G as a *v-class* of G. Set $r, \sigma, \omega \in \mathbb{N}$ and $\sigma < \omega$. We define the *r-partite bottle graph B with neck σ and width ω* to be the complete r-partite graph with one σ-class and $(r-1)$ ω-classes.

Let $\eta > 0$ be fixed. Let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$. The following extremal example (adapted from Proposition 2.1.1) G on n vertices demonstrates that Theorem 3.0.1 is best possible for such graphs B, in the sense that G satisfies the degree sequence of Theorem 3.0.1 except for a small linear part that only just fails the degree sequence, but does not contain a perfect B-tiling. In fact, G does not contain a B-tiling that covers all but at most ηn vertices.

Proposition 3.2.2. *Let $\eta > 0$ be fixed and $n \in \mathbb{N}$ such that $0 < 1/n \ll \eta \ll 1$. Let $r \geq 3$ be an integer. Let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$. Additionally assume that b divides n. Then for any $1 \leq k < \omega n/b - (rb+1)\eta n$, there exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i \geq \left(1 - \frac{\omega+\sigma}{b}\right) n + \frac{\sigma}{\omega} i + \eta n \quad \textit{for all } i \in \{1, \dots, k-1, k + rb\eta n + 1, \dots, \omega n/b\},$$

$$d_i = \left(1 - \frac{\omega+\sigma}{b}\right) n + \left\lceil \frac{\sigma}{\omega} k \right\rceil + \eta n \quad \textit{for all } k \leq i \leq k + rb\eta n,$$

but such that G does not contain a B-tiling covering all but at most ηn vertices.[2]

Proof. Let G be the graph on n vertices with $r+1$ vertex classes $V_1, \dots, V_{r+1}$ where

- $|V_1| = \sigma n/b$;
- $|V_2| = \omega n/b - \eta n$;
- $|V_3| = \cdots = |V_r| = \omega n/b - (\eta n + 1)$;
- $|V_{r+1}| = (r-1)(\eta n + 1) - 1$.

Label the vertices of V_1 as $a_1, a_2, \dots, a_{\sigma n/b}$. Similarly, let us label the vertices of V_2 as $c_1, c_2, \dots, c_{\omega n/b - \eta n}$. The edge set of G is constructed through the following process.

Initially, let G have the following edges:

- All edges with an endpoint in V_1 and the other endpoint in $V(G) \setminus V_2$, in particular $G[V_1]$ is complete;
- All edges with an endpoint in V_i and the other endpoint in $V(G) \setminus (V_1 \cup V_i)$ for $2 \leq i \leq r+1$;

[2]Proposition 3.2.2 essentially implies Proposition 2.1.1, but with worse constants.

- All edges with both endpoints in V_{r+1}, in particular $G[V_{r+1}]$ is complete;
- Given any $1 \leq i \leq \omega n/b - \eta n$ and $j \leq \lceil \sigma i/\omega \rceil$ include all edges $c_i a_j$.

So at the moment G indeed satisfies the degree sequence in Theorem 3.0.1; we therefore modify G slightly. For all $k \leq i \leq k + rb\eta n$ and $\lceil \sigma k/\omega \rceil + 1 \leq j \leq \lceil \sigma(k + rb\eta n)/\omega \rceil$ delete each edge between c_i and a_j. One can easily check that G satisfies the degree sequence in the statement of the proposition. In particular, the vertices of degree $\left(1 - \frac{\omega+\sigma}{b}\right) n + \lceil \frac{\sigma}{\omega} k \rceil + \eta n$ are $c_k, \dots, c_{k+rb\eta n}$.

Define $A := \{a_1, \dots, a_{\lceil \sigma k/\omega \rceil}\}$ and $C := \{c_1, \dots, c_{k+rb\eta n}\}$. Note that there are no edges between C and $V_1 \setminus A$ in G.

Claim 3.2.3. Let T be a B-tiling of G. Then T does not cover at least $3\eta n/2$ vertices in C.

Firstly, consider any copy B' of B in T that contains at least one vertex in V_{r+1}. Since C is an independent set in G, observe that B' contains at most ω vertices from C. Thus there are at most $\omega|V_{r+1}| = \omega(r-1)\eta n + \omega(r-2)$ vertices in C covered by copies of B in T that each contain at least one vertex in V_{r+1}.

Secondly, consider any copy B' of B in T that contains at least one vertex from C and no vertices from V_{r+1}. As before, since C is an independent set in G, we have that B' contains at most ω vertices from C. Since there are no edges between C and $V_1 \setminus A$ in G, B' contains at least σ vertices in A.

These two observations, alongside the fact that $b = \sigma + \omega(r-1) \geq \omega(r-1) \geq \omega(r-2)$, imply that at most $\omega(r-1)\eta n + \omega(r-2) + \lceil \sigma k/\omega \rceil (\omega/\sigma) < k + 1 + b(\eta n + 1)$ vertices in C can be covered by T. Since $|C| = k + rb\eta n$, we have that T does not cover at least $3\eta n/2$ vertices in C. Therefore, Claim 3.2.3 holds. Hence G does not have a B-tiling covering all but at most ηn vertices. $\square$

Extremal Example 3. Let H be an h-vertex graph, $\chi(H) =: r$, $\sigma := \sigma(H)$ and $\omega := (h-\sigma)/(r-1)$. The following extremal example demonstrates that there are n-vertex graphs G for which all but $(\omega/h + o(1))n$ vertices have degree above $(1 - 1/\chi_{cr}(H) + o(1))n$, with the remaining $(\omega/h + o(1))n$ vertices having degree $(1 - 1/\chi_{cr}(H) - o(1))n$, and yet G does not contain a perfect H-tiling. Thus, this shows that one cannot have significantly more than $\omega n/h$ 'small' degree vertices in Theorem 3.0.1.

Proposition 3.2.4. *Let $0 < \eta \ll 1$ be fixed. Let H be a graph with $\chi(H) =: r$. Let $h := |H|$, $\sigma := \sigma(H)$ and set $\omega := (h-\sigma)/(r-1)$. Then there exists a graph G on n vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies*

$$d_i = (1 - \omega/h - (r-1)\eta)\, n = (1 - 1/\chi_{cr}(H) - (r-1)\eta)\, n \quad \textit{for all} \quad i \leq (\omega/h + (r-1)\eta)n,$$

$$d_i \geq (1 - \omega/h + \eta)\, n = (1 - 1/\chi_{cr}(H) + \eta)\, n \quad \textit{for all} \quad i > (\omega/h + (r-1)\eta)n,$$

but such that G does not contain an H-tiling covering all but at most ηn vertices.[3]

Proof. Let G be the complete r-partite graph on n vertices with vertex classes $V_1, \dots, V_r$ such that

- $|V_1| = \frac{\sigma n}{h} - \eta n$,
- $|V_2| = \frac{\omega n}{h} + (r-1)\eta n$,
- $|V_3| = \cdots = |V_r| = \frac{\omega n}{h} - \eta n$.

Then G satisfies the degree sequence condition in the proposition. The choice in size of V_1 ensures that any H-tiling in G covers at most $|V_1|h/\sigma < n - \eta n$ vertices, as desired. $\square$

[3]For $x = 1$, Proposition 3.2.4 implies Proposition 2.1.4, but with a worse constant on some error terms.

3.3 A possible strengthening of Theorem 3.0.1

Whilst Proposition 3.2.1 demonstrates that we cannot lower every term in the degree sequence condition in Theorem 3.0.1 by much, perhaps one can cap the degrees as follows.

Question 3.3.1. *Can the degree sequence condition in Theorem 3.0.1 be replaced by*

$$d_i \geq \min\left\{\left(1-\frac{\omega+\sigma}{h}\right)n+\frac{\sigma}{\omega}i+\eta n, \left(1-\frac{1}{\chi_{cr}(H)}\right)n+C\right\} \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{h}$$

where C is a constant dependent only on H?

Note that Theorem 3.0.2 does not quite imply Theorem 1.1.8 due to the ηn-terms. On the other hand, an affirmative answer to Question 3.3.1, together with an analogous 'capped' version of Theorem 3.0.2(ii), would fully imply the upper bound in Theorem 1.1.8.

3.4 Auxiliary results

The results in this section will be employed in our proof of Theorem 3.0.3. First we need the following definition.

Definition 3.4.1. Given $\varepsilon > 0$, $d \in [0, 1]$ and $G = (A, B)$ a bipartite graph, we say that G is (ε, d)-*superregular* if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy that $d(X, Y) > d$, that $d_G(a) > d|B|$ for all $a \in A$ and that $d_G(b) > d|A|$ for all $b \in B$.

Recall the the degree form of Szemerédi's Regularity lemma [67] we used in Chapter 2.

Lemma 2.3.2 (Degree form of Szemerédi's Regularity lemma [67])**.** *Let $\varepsilon \in (0, 1)$ and $M' \in \mathbb{N}$. Then there exist natural numbers M and n_0 such that for any graph G on $n \geq n_0$*

vertices and any $d \in (0,1)$ there is a partition of the vertices of G into subsets $V_0, V_1, \dots, V_k$ and a spanning subgraph G' of G such that the following hold:

- *$M' \leq k \leq M$;*
- *$|V_0| \leq \varepsilon n$;*
- *$|V_1| = \cdots = |V_k| =: q$;*
- *$d_{G'}(x) > d_G(x) - (d+\varepsilon)n$ for all $x \in V(G)$;*
- *$e(G'[V_i]) = 0$ for all $i \geq 1$;*
- *For all $1 \leq i, j \leq k$ with $i \neq j$ the pair $(V_i, V_j)_{G'}$ is ε-regular and has density either 0 or at least d.*

Recall also that we call $V_1, \dots, V_k$ the *clusters* of our partition, V_0 the exceptional set and G' the *pure graph*. We define the *reduced graph* R of G with parameters ε, d and M' to be the graph whose vertex set is $V_1, \dots, V_k$ and in which V_iV_j is an edge if and only if $(V_i, V_j)_{G'}$ is ε-regular with density at least d. Note also that $|R| = k$.

Recall Lemmas 2.3.4 and 2.3.5.

Fact 2.3.4. *Let $0 < \varepsilon < \alpha$ and $\varepsilon' := \max\{\varepsilon/\alpha, 2\varepsilon\}$. Let (A, B) be an ε-regular pair of density d. Suppose $A' \subseteq A$ and $B' \subseteq B$ where $|A'| \geq \alpha|A|$ and $|B'| \geq \alpha|B|$. Then (A', B') is an ε'-regular pair with density d' where $|d' - d| < \varepsilon$.*

Fact 2.3.5 (Key lemma [42])**.** *Suppose that $0 < \varepsilon < d$, that $q, t \in \mathbb{N}$ and that R is a graph with $V(R) = \{v_1, \dots, v_k\}$. We construct a graph G as follows: Replace every vertex $v_i \in V(R)$ with a set V_i of q vertices and replace each edge of R with an ε-regular pair of density at least d. For each $v_i \in V(R)$, let U_i denote the set of t vertices in $R(t)$ corresponding* *to v_i. Let H be a subgraph of $R(t)$ with maximum degree Δ and set $h := |H|$. Set $\delta := d - \varepsilon$ and $\varepsilon_0 := \delta^\Delta/(2+\Delta)$. If $\varepsilon \leq \varepsilon_0$ and $t - 1 \leq \varepsilon_0 q$ then there are at least*

$$(\varepsilon_0 q)^h \text{ labelled copies of } H \text{ in } G$$

so that if $x \in V(H)$ lies in U_i in $R(t)$, then x is embedded into V_i in G.

Let G and H be graphs and R be a reduced graph of G. Let $\mathcal{H}$ be a perfect H-tiling in R. The following result ensures that after removing only a few vertices from each cluster in R every edge in each copy of $H \in \mathcal{H}$ corresponds to a superregular pair. This will be essential to apply Lemma 3.4.3 in Section 3.7.6.

Proposition 3.4.2 ([47]). *Let G be a graph, $\varepsilon, d \in (0,1)$ and $M', \Delta \in \mathbb{N}$. Apply Lemma 2.3.2 to G with parameters ε, M' and d to obtain a reduced graph R with clusters of size q. Let H be a subgraph of the reduced graph R with $\Delta(H) \leq \Delta$ and label the vertices of H as $V_1, \dots, V_{|H|}$. Then each vertex V_i of H contains a subset V_i' of size $(1 - \varepsilon\Delta)q$ such that for every edge V_iV_j of H the graph $(V_i', V_j')_{G'}$ is $(\varepsilon/(1-\varepsilon\Delta), d - (1+\Delta)\varepsilon)$-super-regular.*

The following fundamental result of Komlós, Sárközy and Szemerédi, known as the *Blow-up lemma*, essentially says that (ε, d)-superregular pairs behave like complete bipartite graphs with respect to containing bounded degree subgraphs.

Lemma 3.4.3 (Blow-up lemma [40]). *Given a graph F on vertices $\{1, \dots, f\}$ and $d, \Delta > 0$, there exists an $\varepsilon_0 = \varepsilon_0(d, \Delta, f) > 0$ such that the following holds: Given $L_1, \dots, L_f \in \mathbb{N}$ and $\varepsilon \leq \varepsilon_0$, let F^* be the graph obtained from F by replacing each vertex $i \in F$ with an independent set V_i of L_i new vertices and joining all vertices in V_i to all vertices in V_j whenever ij is an edge of F. Let G be a spanning subgraph of F^* such that for every edge $ij \in F$ the pair $(V_i, V_j)_G$ is (ε, d)-superregular. Then G contains a copy of every subgraph H of F^* with $\Delta(H) \leq \Delta$.*

In [44], the following result of Kühn and Osthus is essential to their proof of Theorem 1.1.12.

Lemma 3.4.4 (Lemma 12 in [46]). *Let H be a graph with $\chi(H) =: r \geq 2$ and $\mathrm{hcf}(H) = 1$. Let $h := |H|$ and $\omega(H) := (h - \sigma(H))/(r-1)$. Let β_1, λ_1 be positive constants such that $0 < \beta_1 \ll \lambda_1 \ll \sigma(H)/\omega(H)$, $1 - \sigma(H)/\omega(H)$, $1/h$. Suppose that F is a complete r-partite graph with vertex classes $U_1, \dots, U_r$ such that: $0 < 1/|F| \ll \beta_1$; $|F|$ is divisible by h; $(1 - \lambda_1^{1/10})|U_r| \leq \sigma(H)|U_i|/\omega(H) \leq (1 - \lambda_1)|U_r|$ for all $i < r$; $||U_i| - |U_j|| \leq \beta_1|F|$ whenever $1 \leq i < j < r$. Then F contains a perfect H-tiling.*

We will use the Blow-up lemma in tandem with the following generalisation of Lemma 3.4.4 to conclude that a particular tiling that we construct in a reduced graph R guarantees a perfect H-tiling in our original graph G.

Lemma 3.4.5. *Let H be a graph with $\chi(H) =: r \geq 2$ and $\mathrm{hcf}(H) = 1$. Let $h := |H|$. Set $\sigma \in \mathbb{R}$ such that $\sigma(H) \leq \sigma < h/r$ and $\omega := (h - \sigma)/(r-1)$. Let $0 < \beta_2 \ll \lambda_2 \ll \sigma/\omega$, $1 - \sigma/\omega$, $1/h$ be positive constants. Suppose that F is a complete r-partite graph with vertex classes $U_1, \dots, U_r$ such that: $0 < 1/|F| \ll \beta_2$; $|F|$ is divisible by h; $(1 - \lambda_2^{1/10})|U_r| \leq \sigma|U_i|/\omega \leq (1 - \lambda_2)|U_r|$ for all $i < r$; $||U_i| - |U_j|| \leq \beta_2|F|$ whenever $1 \leq i < j < r$. Then F contains a perfect H-tiling.*

Proof. Note we may assume that $\sigma > \sigma(H)$ as otherwise the result follows immediately from Lemma 3.4.4. We choose $\beta_2 \ll \beta_1 \ll \lambda_2 \ll \lambda_1$ where β_1 and λ_1 are as in Lemma 3.4.4. Additionally we may assume $\beta_2, \lambda_2 \ll (\sigma/\omega - \sigma(H)/\omega(H))$.

Let F be as in the statement of the lemma. Set H^* to be the complete balanced r-partite graph on rh vertices (that is, each vertex class of H^* has size h). Observe that H^* has a perfect H-tiling using precisely r copies of H.

Repeatedly delete disjoint copies of H^* from F (and therefore update the classes $U_1, \ldots, U_r$) until the first point for which there is some $i < r$ such that $(1 - \lambda_1^{1/10}/2)|U_r| \leq \sigma(H)|U_i|/\omega(H) \leq (1 - 2\lambda_1)|U_r|$. Call the resulting graph F'. Note that $\sigma/\omega > \sigma(H)/\omega(H)$, so we can indeed obtain F'. Further note that our (sufficiently small) choice of β_2 ensures each class U_j still contains at least a $\beta_2^{1/2}$-proportion of the vertices it started with. So now $||U_i| - |U_j|| \leq \beta_2|F| \leq \beta_2^{1/2}|F'| \leq \beta_1|F'|$ whenever $1 \leq i < j < r$. Moreover, this implies $(1 - \lambda_1^{1/10})|U_r| \leq \sigma(H)|U_j|/\omega(H) \leq (1 - \lambda_1)|U_r|$ for all $j < r$. Thus, by Lemma 3.4.4, F' contains a perfect H-tiling and therefore, so too does F, as desired. □

In [44], Kühn and Osthus begin their proof of Theorem 1.1.12 by applying Theorem 1.1.6. In our proof of Theorem 3.0.3 we will use Theorem 2.6.1. We state it again here for reference.

Theorem 2.6.1 ([29])**.** *Let $\eta > 0$ and H be a graph with $\chi(H) = r$ and $h := |H|$. Set $\sigma \in \mathbb{R}$ such that $\sigma(H) \leq \sigma \leq h/r$ and $\omega := (h - \sigma)/(r - 1)$. Then there exists an integer $n_0 = n_0(\eta, \sigma, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{h}\right) n + \frac{\sigma}{\omega} i \quad \textit{for all } 1 \leq i \leq \frac{\omega n}{h}.$$

Then G contains an H-tiling covering all but at most ηn vertices.

To justify a certain claim (Claim 3.7.1) we will apply the following Chernoff bound.

Lemma 3.4.6 (See e.g. [31])**.** *If $X \in \mathrm{Bi}(n, p)$ and $\varepsilon \in (0, 3/2]$, then we have that*

$$\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 2e^{\left(-\frac{\varepsilon^2}{3}\mathbb{E}X\right)}.$$

To prove Theorem 3.5.2 we will need the following elementary arithmetic result.

Lemma 3.4.7 (Bézout's Lemma). *Let $a_1, a_2, \dots, a_t \in \mathbb{Z}$. Then there must exist integers $y_1, y_2, \dots, y_t \in \mathbb{Z}$ such that*

$$\sum_{i=1}^{t} y_i a_i = \text{hcf}(a_1, a_2, \dots, a_t)$$

where $\text{hcf}(a_1, a_2, \dots, a_t)$ *is the highest common factor of* $a_1, a_2, \dots, a_t$.

3.5 A tool for the proof of Theorem 3.0.3

In this section, we prove a theorem (Theorem 3.5.2) that will be used in Sections 3.7.4 and 3.7.5 of the proof of Theorem 3.0.3. At the beginning of Section 3.7.3, we will have a certain $\hat{B}$-tiling $\hat{\mathcal{B}}$ of a reduced graph R (the graph $\hat{B}$ will be defined later). Denote the copies of $\hat{B}$ in $\hat{\mathcal{B}}$ by $\hat{B}_1, \hat{B}_2, \dots, \hat{B}_{\hat{k}}$. For applications of Lemma 3.4.5 required at the end of our proof of Theorem 3.0.3, we will need $|V_G(\hat{B}_i)|$ to be divisible by h for each $1 \leq i \leq \hat{k}$. The following theorem is the crucial tool for ensuring we can remove copies of H from G to achieve this.

For a graph H with $\chi(H) = r$, recall that C_H is the set of all optimal colourings of H and that given an optimal colouring $c \in C_H$ we let $x_{c,1} \leq x_{c,2} \leq \dots \leq x_{c,r}$ denote the sizes of the colour classes of c. We require the following definitions.

Definition 3.5.1. Let H be a graph with $\chi(H) =: r$. Fix $1 \leq p \leq r-1$. For each $c \in C_H$, define D_c to be the multiset $[x_{c,1}, x_{c,2}, \dots, x_{c,r}]$. We say that A is a *p-subset contained in D_c* if A is a multiset, $|A| = p$ and $A = [x_{c,j_1}, x_{c,j_2}, \dots, x_{c,j_p}]$ where $j_1, j_2, \dots, j_p \in \{1, \dots, r\}$ are distinct. Let z_p be the number of p-subsets contained in D_c and observe that $z_p \leq \binom{r}{p}$. For each colouring $c \in C_H$, label the p-subsets contained in D_c by $A_{p,c,1}, A_{p,c,2}, \dots, A_{p,c,z_p}$. Let $S_{p,c,J} = \sum_{x \in A_{p,c,J}} x$ for each $c \in C_H$, $1 \leq J \leq z_p$.

Theorem 3.5.2. *Let H be an r-partite graph and let $h := |H|$. Fix $1 \leq p \leq r-1$. Let b be the number of components of H and $t_1, \dots, t_b$ be the sizes of the components of H. Then*

- *if $r = 2$ and $\mathrm{hcf}_c(H) = 1$, there exists a collection of non-negative integers $\{a_i : 1 \leq i \leq b\}$ such that*
$$\sum_{i=1}^{b} a_i t_i \equiv 1 \mod h.$$
- *if $r \geq 3$ and $\mathrm{hcf}_\chi(H) = 1$, there exists a collection of non-negative integers $\{a_{p,c,i} : c \in C_H, 1 \leq i \leq z_p\}$ such that*
$$\sum_{c \in C_H} \sum_{i=1}^{z_p} a_{p,c,i} S_{p,c,i} \equiv 1 \mod h.$$

For each $1 \leq p \leq r-1$, $c \in C_H$ and $j \in \{1, \dots, r\}$, let $Z_{p,c,j}$ be the multiset defined by the following table:

Colour class size	$x_{c,1}$	$\cdots$	$x_{c,j-1}$	$x_{c,j}$	$x_{c,j+1}$	$\cdots$	$x_{c,r}$
Multiplicity in $Z_{p,c,j}$	p	$\cdots$	p	$p+1$	p	$\cdots$	p

The following fact will be useful in our proof of Theorem 3.5.2.

Fact 3.5.3. *For any $1 \leq J, L \leq r$ and $1 \leq p \leq r-1$, we can partition $Z_{p,c,J}$ into $\{x_{c,L}\}$ and r p-subsets contained in D_c.*

Proof of Fact 3.5.3. We assume wlog that $x_{c,1} \neq x_{c,2} \neq \ldots \neq x_{c,r}$.[4] If $L = J$ then clearly there is such a partition. So assume $L \neq J$. Partition $Z_{p,c,J}$ into $\{x_{c,J}\}$ and r many p-subsets

[4] If there exist $i_1 \neq i_2$ such that $x_{c,i_1} = x_{c,i_2}$, then we label $x_{c,i}$ with $x'_{c,i}$ for each $1 \leq i \leq r$, define D'_c to be the *set* $\{x'_{c,1}, x'_{c,2}, \dots, x'_{c,r}\}$ and define a *p-subset contained in* D'_c to be a subset of size p in D'_c. Further, let $Z'_{p,c,J}$ be the multiset defined in the same way as $Z_{p,c,J}$, but with $x'_{c,i}$ replacing $x_{c,i}$ for each $1 \leq i \leq r$. The proof of Fact 3.5.3 is then exactly the same, replacing $Z_{p,c,J}$ with $Z'_{p,c,J}$, D_c with D'_c and $x_{c,i}$ with $x'_{c,i}$ for $i = 1, \dots, r$. The resulting partition of $Z'_{p,c,J}$ into $x'_{c,L}$ and r many p-subsets contained in D'_c clearly yields a partition of $Z_{p,c,J}$ into $x_{c,L}$ and r many p-subsets contained in D_c.

contained in D_c. Denote this partition by P. Assume there exists a p-subset contained in D_c in P which contains $x_{c,L}$ and does not contain $x_{c,J}$ and denote it by A. Then remove the element $x_{c,L}$ from A to form a $(p-1)$-subset contained in D_c, denoted by A'. Concatenate A' and $\{x_{c,J}\}$ to form a p-subset contained in D_c. Then Lemma 3.5.3 holds.

So assume no such A exists. Observe that since $p \leq r-1$ there exists a p-subset contained in D_c in P that does not contain $x_{c,J}$. Denote this p-subset contained in D_c by B_1. By assumption, B_1 does not contain $x_{c,L}$. Let B_2 be a p-subset contained in D_c in P that does contain $x_{c,L}$. Then by assumption, B_2 must also contain $x_{c,J}$. Observe that there must exist an element $x_{c,K}$ such that $K \neq L, J$ and we have that B_1 contains $x_{c,K}$ and B_2 does not contain $x_{c,K}$. Remove the element $x_{c,K}$ from B_1 to form a $(p-1)$-subset contained in D_c; denote it by B_1'. Further, remove the element $x_{c,L}$ from B_2 to form a $(p-1)$-subset contained in D_c; denote it by B_2'. Concatenate B_1' and $\{x_{c,J}\}$ and concatenate B_2' and $\{x_{c,k}\}$ to form p-subsets contained in D_c. Hence Fact 3.5.3 holds.

Proof of Theorem 3.5.2. Firstly, we will consider the case when $r = 2$ and $\mathrm{hcf}(H) = 1$. In this case, as $\mathrm{hcf}(H) = 1$ we must have that $\mathrm{hcf}_c(H) = 1$. Hence H must have multiple components. The sizes of these components of H are $t_1, t_2, \dots, t_b$. Since $\mathrm{hcf}_c(H) = 1$, by Bezout's Lemma (Lemma 3.4.7) there exist integers $a_1', \dots, a_b'$ such that

$$\sum_{i=1}^{b} a_i' t_i = \mathrm{hcf}(t_1, \dots, t_b) = 1.$$

Since $\sum_{i=1}^{b} t_i = h$, there exists $\hat{a} \in \mathbb{N} \cup \{0\}$ such that

$$\sum_{i=1}^{b} (a_i' + \hat{a}) t_i \equiv 1 \mod h$$

and

$$a_i' + \hat{a} \geq 0 \ \text{ for all } \ 1 \leq i \leq b.$$

For each $1 \leq i \leq b$, take $a_i := a'_i + \hat{a}$.

Next consider when $r \geq 3$. Instead of explicitly calculating $a_{p,c,i}$ for each $c \in C_H$, $1 \leq i \leq z_p$, we will construct for each $c \in C_H$ a multiset X_c of bounded size which can be partitioned into p-subsets contained in D_c. Further, we will construct our multisets X_c such that

$$\sum_{c \in C_H} \sum_{x \in X_c} x \equiv 1 \mod h.$$

Observe that constructing such multisets X_c immediately yields a collection of non-negative integers $\{a_{p,c,i} : c \in C_H, 1 \leq i \leq z_p\}$ that satisfy the conditions in Theorem 3.5.2. Indeed, for each $c \in C_H$ and $1 \leq i \leq z_p$, we take $a_{p,c,i}$ to be precisely the number of times $A_{p,c,i}$ occurs in the partition of X_c into p-subsets.

In order to start constructing our multisets X_c, we define the following multiset:

$$\mathcal{D}^*(H) := \bigcup_{c \in C_H} [x_{c,j+1} - x_{c,j} \mid j = 1, \dots, r-1].$$

Since $\text{hcf}(H) = 1$ we know that $\text{hcf}_\chi(H) = 1$. Hence we can apply Lemma 3.4.7 to the multiset $\mathcal{D}^*(H)$ to get for each $c \in C_H$, $1 \leq j \leq r-1$ integers $b_{c,j}$ such that the following holds:

$$\sum_{c \in C_H} \sum_{j=1}^{r-1} b_{c,j}(x_{c,j+1} - x_{c,j}) \equiv 1 \mod h. \tag{3.1}$$

We now construct our multisets X_c. Fix $c \in C_H$. Choose $t_c \in \mathbb{N}$ to be the smallest natural number such that

$$pt_c \geq \max\{|b_{c,1}|, |b_{c,1} - b_{c,2}|, |b_{c,2} - b_{c,3}|, \dots, |b_{c,r-2} - b_{c,r-1}|, |b_{c,r-1}|\}.$$

Then $pt_c - b_{c,1}, pt_c + b_{c,1} - b_{c,2}, pt_c + b_{c,2} - b_{c,3}, \dots, pt_c + b_{c,r-2} - b_{c,r-1}, pt_c + b_{c,r-1}$ are non-negative integers. Let Y_c be the multiset defined by the following table:

Colour class size	$x_{c,1}$	$x_{c,2}$	$x_{c,3}$	$\cdots$	$x_{c,r-1}$	$x_{c,r}$
Multiplicity in Y_c	$pt_c - b_{c,1}$	$pt_c + b_{c,1} - b_{c,2}$	$pt_c + b_{c,2} - b_{c,3}$	$\cdots$	$pt_c + b_{c,r-2} - b_{c,r-1}$	$pt_c + b_{c,r-1}$

Then $|Y_c| = rpt_c$. If we can partition Y_c into p-subsets contained in D_c then we take $X_c := Y_c$. Assume we cannot. Then the multiplicities of $x_{c,1}, \dots, x_{c,r}$ in Y_c must be sufficiently different from one another. We employ the following algorithm which transforms Y_c into a multiset which can be partitioned into p-subsets contained in D_c using Fact 3.5.3. To state the algorithm we require the following definitions.

Definition 3.5.4. For each $c \in C_H$, $1 \le i \le r$, let $m_{c,i}$ be the multiplicity of $x_{c,i}$ in Y_c. Let $J, L \in \{1, \dots, r\}$ be such that

- $m_{c,J} \ge \frac{\sum_{i=1}^{r} m_{c,i}}{r}$;
- $m_{c,L} \le \frac{\sum_{i=1}^{r} m_{c,i}}{r}$;
- $m_{c,L} + 1 \ne m_{c,J}$;
- $m_{c,L} \ne m_{c,J}$.

Let $Y'_c := Y_c - \{x_{c,J}\} + \{x_{c,L}\}$.[5] Then we say that Y'_c is *more balanced than* Y_c.

Algorithm.

1) Let $Q := \emptyset$.

2) If $|m_{c,i} - m_{c,j}| = 0$ *for all* $1 \le i, j \le r$, *output* Y_c *and* Q. *Otherwise, choose* $J, L \in \{1, \dots, r\}$ *such that* $Y'_c := Y_c - \{x_{c,J}\} + \{x_{c,L}\}$ *is more balanced than* Y_c.[6]

[5] That is, Y'_c is the multiset Y_c except with $x_{c,J}$ having multiplicity $m_{c,J} - 1$ and $x_{c,L}$ having multiplicity $m_{c,L} + 1$.

[6] Observe that we cannot ever have $|m_{c,i} - m_{c,j}| \le 1$ for all i and j, with equality for some i and j, by definition of Y_c.

3) Add the colour classes sizes of p copies of H with colouring c to Y_c (that is, $x_{c,i}$ now has multiplicity $m_{c,i}+p$ in Y_c for each $1 \leq i \leq r$).

4) Take $Z_{p,c,J}$ to be the union of $\{x_{c,J}\}$ and the colour class sizes of the p copies of H we just added. By Fact 3.5.3 there exists a partition of $Z_{p,c,J}$ into $\{x_{c,L}\}$ and r p-subsets contained in D_c.

5) Place into Q these r p-subsets contained in D_c.

6) Take $Y_c := Y'_c$ and update the value of each $m_{c,i}$ (that is, $m_{c,J}$ has decreased by 1 and $m_{c,L}$ has increased by 1). Go to Step 2.

Therefore, at the end of the algorithm $|Y_c| = rpt_c$ and $|m_{c,i} - m_{c,j}| = 0$ for all $1 \leq i, j \leq r$. In particular, it is easy to see that Y_c now has a partition Q_{Y_c} into p-subsets contained in D_c. Let t'_c be the number of collections of colour class sizes of p copies of H added during the algorithm and define $\hat{t}_c := t_c + t'_c$. Then the multiset $\hat{Y}_c$, defined by the table below, can be partitioned into p-subsets contained in D_c using the partition $Q \cup Q_{Y_c}$:

Colour class size	$x_{c,1}$	$x_{c,2}$	$x_{c,3}$	$\cdots$	$x_{c,r-1}$	$x_{c,r}$
Multiplicity in $\hat{Y}_c$	$p\hat{t}_c - b_{c,1}$	$p\hat{t}_c + b_{c,1} - b_{c,2}$	$p\hat{t}_c + b_{c,2} - b_{c,3}$	$\cdots$	$p\hat{t}_c + b_{c,r-2} - b_{c,r-1}$	$p\hat{t}_c + b_{c,r-1}$

Take $X_c := \hat{Y}_c$. We now confirm that our multisets X_c satisfy

$$\sum_{c \in C_H} \sum_{x \in X_c} x \equiv 1 \mod h.$$

By (3.1) and the definition of X_c for each $c \in C_H$ we have

$$\begin{aligned}
&\sum_{c\in C_H}\sum_{x\in X_c} x\\
&= \sum_{c\in C_H}\left(\sum_{j=1}^{r-1} b_{c,j}(x_{c,j+1}-x_{c,j}) + p\hat{t}_c\left(\sum_{j=1}^{r} x_{c,j}\right)\right)\\
&= \sum_{c\in C_H}\sum_{j=1}^{r-1} b_{c,j}(x_{c,j+1}-x_{c,j}) + \left(p\sum_{c\in C_H}\hat{t}_c\right)h\\
&\overset{(3.1)}{\equiv} 1 \mod h.
\end{aligned}$$

Therefore, recalling the discussion earlier in this proof, there must exist the desired collection of non-negative integers $\{a_{p,c,i} : c \in C_H, 1 \le i \le z_p\}$, and we take a to be the maximum element in this collection. $\square$

3.6 Proof Overview

The rest of this chapter will be devoted to proving Theorem 3.0.3 and here we outline the proof. As noted previously, our proof follows closely Kühn and Osthus' proof of Theorem 1.1.12 in [44].

Let H, G, η and σ be as in the statement of the theorem. In particular, $h := |H|$ and $\omega := (h-\sigma)/(r-1)$. Note that it suffices to prove the result in the case when $\sigma \in \mathbb{Q}$. First we define a bottle graph B that contains a perfect H-tiling.

Definition 3.6.1. *Let $a, b \in \mathbb{N}$ such that $\sigma = a/b$. Let $\omega(H) := (h-\sigma(H))/(r-1)$ and $\hat{c} := b(r-1)(\omega(H)-\sigma(H))$. Define B to be the r-partite bottle graph with neck $\sigma\hat{c}$ and width $\omega\hat{c}$ (note that $\sigma\hat{c}, \omega\hat{c} \in \mathbb{N}$). Observe that $|B| = h\hat{c}$; $\sigma(B) = \sigma\hat{c}$; $\omega(B) = \omega\hat{c}$. Further,*

$$\chi_{cr}(B) = r - 1 + \sigma/\omega = h/\omega.$$

Since $|B| = h\hat{c}$; $\sigma(B) = \sigma\hat{c}$; $\omega(B) = \omega\hat{c}$, we have that G satisfies the hypothesis of Theorem 2.6.1 with B, $\sigma(B)$ and $\omega(B)$ playing the roles of H, σ and ω respectively. That is, G contains an almost perfect B-tiling. In fact, as the reduced graph R of G almost inherits the degree sequence of G, Theorem 2.6.1 ensures that R contains an almost perfect B-tiling $\mathcal{B}$. Further note that the choice of $\hat{c}$ implies that B has a perfect H-tiling consisting of $\hat{c}$ copies of H. Indeed, this follows as B has a perfect tiling of $a - \sigma(H)b$ copies of $K_{h,\dots,h}$ and $hb - ar$ copies of H, where $K_{h,\dots,h}$ is the complete r-partite graph with each vertex class having size h.

Ideally one would like to use $\mathcal{B}$ as a framework to build the perfect H-tiling in G. However, as explained shortly, we need more flexibility in our tiling in R. Therefore, we introduce the following 'modified' version of B.

Definition 3.6.2. *Let $s \in \mathbb{N}$ be sufficiently large and $\lambda \in \mathbb{R}^+$ be sufficiently small where $\sigma(1+\lambda)s/w \in \mathbb{N}$. Recall that $\sigma < \omega$. Let $\hat{B}$ be the r-partite bottle graph with neck $\sigma(1+\lambda)s/\omega$ and width s.*[7] *Moreover, we choose λ and s such that $\hat{B}$ contains a perfect B-tiling. Hence $\hat{B}$ contains a perfect H-tiling. Note that*

$$\chi_{cr}(\hat{B}) = r - 1 + \sigma(1 + \lambda)/\omega.$$

Since λ is chosen to be small (and so $\chi_{cr}(\hat{B})$ is very close to $\chi_{cr}(B)$), one can still apply Theorem 2.6.1 on inputs $\hat{B}$ and R. That is, R contains an almost perfect $\hat{B}$-tiling $\hat{\mathcal{B}}$. Denote the copies of $\hat{B}$ in $\hat{\mathcal{B}}$ by $\hat{B}_1, \hat{B}_2, \dots, \hat{B}_{\hat{k}}$. By removing a small number of vertices from each cluster in R, we can ensure the edges of each $\hat{B}_i$ correspond to superregular pairs. Let V_0 denote the set of all vertices in G not contained in the clusters lying in the tiling $\hat{\mathcal{B}}$.

For each $1 \le i \le \hat{k}$, let $\hat{G}_i$ be the r-partite subgraph of G whose jth vertex class is

[7] We have that $\sigma(1 + \lambda)/\omega < 1$ by our choice of λ and that $\sigma < \omega$.

the union of all those clusters contained in the jth vertex class of $\hat{B}_i$, for each $1 \leq j \leq r$. Let G_i^* be the complete r-partite graph on the same vertex set as $\hat{G}_i$. We introduce the graph $\hat{B}$ (rather than just working with B) since $\hat{B}$ has the following crucial property: For each $1 \leq i \leq \hat{k}$ we can arbitrarily delete a small number of vertices from G_i^* (and correspondingly $\hat{G}_i$) and, provided $|V(G_i^*)|$ is now divisible by h, the resulting graph satisfies the hypothesis of Lemma 3.4.5. That is, this graph contains a perfect H-tiling. Then the Blow-up lemma (Lemma 3.4.3) implies that each $\hat{G}_i$ contains a perfect H-tiling.

We make use of this property of $\hat{B}$ as follows: In Section 3.7.2 we remove copies of H from G that cover all vertices in V_0, as well as a small (possibly zero) number of vertices from each $\hat{G}_i$; call this H-tiling (formed from these copies of H) $\mathcal{H}_1$. Deleting these covered vertices from each $\hat{G}_i$, if $|V(\hat{G}_i)|$ $(= |V(G_i^*)|)$ is still divisible by h for each $1 \leq i \leq \hat{k}$ then each $\hat{G}_i$ now contains a perfect H-tiling (by our argument above). However, for some i, we may have that $|V(\hat{G}_i)|$ is not divisible by h. So in Section 3.7.3 we remove a further bounded number of copies of H, forming an H-tiling $\mathcal{H}_2$, to ensure $|V(\hat{G}_i)|$ $(= |V(G_i^*)|)$ is divisible by h for each $1 \leq i \leq \hat{k}$. Thus, we now have that each $\hat{G}_i$ contains a perfect H-tiling $\hat{\mathcal{H}}_i$. Combining the tilings $\mathcal{H}_1, \mathcal{H}_2, \hat{\mathcal{H}}_1, \dots, \hat{\mathcal{H}}_{\hat{k}}$ yields a perfect H-tiling in G, as desired.

Our argument in Section 3.7.3 will split into two cases, the first being when $\chi(H) \geq 3$ and the latter when H is bipartite. This is where our proof differs the most from that in [44] as we must make use of Theorem 3.5.2 to find suitable copies of H to ensure each $|V(\hat{G}_i)|$ is divisible by h.

3.7 Proof of Theorem 3.0.3

3.7.1 Applying the regularity lemma

Note that it suffices to prove the theorem in the case when $\sigma \in \mathbb{Q}$. Let n be sufficiently large and fix constants that satisfy the following hierarchy

$$0 < 1/n \ll 1/M' \ll \varepsilon \ll d \ll \eta_1 \ll \beta \ll \alpha \ll \lambda \ll \eta, \sigma/\omega, 1 - \sigma/\omega, 1/h. \tag{3.2}$$

As discussed in Section 3.6, we choose $s \in \mathbb{N}$ sufficiently large and define $\hat{B}$ to be the r-partite bottle graph with neck $\sigma(1+\lambda)s/\omega$ and width s. As before, we choose λ and s such that $\hat{B}$ contains a perfect B-tiling, which implies that $\hat{B}$ contains a perfect H-tiling. Note again that

$$\chi_{cr}(\hat{B}) = r - 1 + \sigma(1+\lambda)/\omega.$$

Moreover, choose η_1 and M' such that

$$\eta_1 \ll 1/|\hat{B}| \text{ and } M' \geq n_0(\eta_1, \sigma(\hat{B}), \hat{B}),$$

where n_0 is defined as in Theorem 2.6.1. Let G be an n-vertex graph as in the statement of Theorem 3.0.3. Apply Lemma 2.3.2 with parameters ε, d and M' to G to obtain clusters $V_1, \dots, V_k$, an exceptional set V_0 and a pure graph G', where $q := |V_1| = \cdots = |V_k|$ and $k \geq M'$. Let R be the corresponding reduced graph. Using (3.2), we may apply Lemma 2.3.3 with parameters $M', n, \varepsilon, d, \eta, h, \omega, \sigma$ to conclude that R has degree sequence $d_{R,1} \leq d_{R,2} \leq \cdots \leq d_{R,k}$ where

$$d_{R,i} \geq \left(1 - \frac{\omega+\sigma}{h}\right)k + \frac{\sigma}{\omega}i + \frac{\eta k}{2} \text{ for all } 1 \leq i \leq \frac{\omega k}{h}. \tag{3.3}$$

For a graph F, recall that $\sigma(F)$ denotes the size of the smallest possible colour class in any $\chi(F)$-colouring of F and $\omega(F) := (|F| - \sigma(F))/(\chi(F) - 1)$.

We aim to apply Theorem 2.6.1 to conclude that there is a $\hat{B}$-tiling covering all but at most $\eta_1 k$ vertices in R. To do so we need to conclude that R has degree sequence $d_{R,1} \leq \cdots \leq d_{R,k}$ such that

$$d_{R,i} \geq \left(1 - \frac{\omega(\hat{B}) + \sigma(\hat{B})}{|\hat{B}|}\right)k + \frac{\sigma(\hat{B})}{\omega(\hat{B})}i \ \text{ for all } 1 \leq i \leq \frac{\omega(\hat{B})k}{|\hat{B}|}. \tag{3.4}$$

Recall that $\chi_{cr}(\hat{B}) = r - 1 + \sigma(1+\lambda)/\omega$. Since $\lambda \ll \eta$ and $|\hat{B}| = s\chi_{cr}(\hat{B})$ we have that

$$-\frac{\omega k}{h} + \frac{\eta k}{4} \geq -\frac{sk}{|\hat{B}|}.$$

Hence by (3.3),

$$d_{R,i} \geq \left(1 - \frac{s}{|\hat{B}|} - \frac{\sigma}{h}\right)k + \frac{\sigma}{\omega}i + \frac{\eta k}{4} \ \text{ for all } 1 \leq i \leq \frac{\omega k}{h}.$$

Further, observe that

$$-\frac{\sigma k}{h} > -\frac{\sigma(1+\lambda)sk}{\omega|\hat{B}|} \ \text{ and } \ \frac{\omega k}{h} \geq \frac{sk}{|\hat{B}|}.$$

Thus

$$d_{R,i} \geq \left(1 - \frac{s + \frac{\sigma(1+\lambda)s}{\omega}}{|\hat{B}|}\right)k + \frac{\sigma}{\omega}i + \frac{\eta k}{4} \ \text{ for all } 1 \leq i \leq \frac{sk}{|\hat{B}|}.$$

Using again the fact that $\lambda \ll \eta$ we have that

$$\frac{\sigma}{\omega}i + \frac{\eta k}{4} \geq \frac{\sigma(1+\lambda)}{\omega}i \ \text{ for all } 1 \leq i \leq \frac{sk}{|\hat{B}|}.$$

Then, $\sigma(\hat{B}) = \sigma(1+\lambda)s/\omega$ and $\omega(\hat{B}) = s$ we have that

$$d_{R,i} \geq \left(1 - \frac{\omega(\hat{B}) + \sigma(\hat{B})}{|\hat{B}|}\right) k + \frac{\sigma(\hat{B})}{\omega(\hat{B})} i \quad \text{for all } 1 \leq i \leq \frac{\omega(\hat{B})k}{|\hat{B}|},$$

as desired.

Since $|V(R)| = k \geq M' \geq n_0(\eta_1, \sigma(\hat{B}), \hat{B})$ and (3.4) holds, we apply Theorem 2.6.1 to find a $\hat{B}$-tiling $\hat{\mathcal{B}}$ covering all but at most $\eta_1 k$ vertices in R. Denote the copies of $\hat{B}$ in $\hat{\mathcal{B}}$ by $\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_{\hat{k}}$. Now delete all clusters not contained in some $\hat{B}_i$ from R and add the vertices in these clusters to V_0. Therefore now

$$|V_0| \leq \varepsilon n + \eta_1 n \leq 2\eta_1 n.$$

Let R' be the reduced graph induced by all the remaining clusters and let $k' := |V(R')|$. Since $\eta_1 \ll \eta$, (3.3) implies that R' has degree sequence $d_{R',1} \leq d_{R',2} \leq \cdots \leq d_{R',k'}$ where

$$d_{R',i} \geq \left(1 - \frac{\omega + \sigma}{h}\right) k' + \frac{\sigma}{\omega} i + \frac{\eta k'}{4} \quad \text{for all } 1 \leq i \leq \frac{\omega k'}{h}. \tag{3.5}$$

For each $\hat{B}_i$ in $\hat{\mathcal{B}}$, let $\mathcal{B}_i$ be a perfect B-tiling in $\hat{B}_i$ (recall that earlier we chose s and λ such that $\hat{B}$ contains a perfect B-tiling). Let $\mathcal{B} := \bigcup \mathcal{B}_i$ and observe that $\mathcal{B}$ is a perfect B-tiling in R'. To aid with calculations we will sometimes work with $\mathcal{B}$ instead of $\hat{\mathcal{B}}$.

Let $q' := (1 - \varepsilon|\hat{B}|)q$. By Proposition 3.4.2, for all $1 \leq i \leq \hat{k}$ we can remove $\varepsilon|\hat{B}|q$ vertices from each cluster V_a belonging to $\hat{B}_i$ so that each edge V_aV_b in $\hat{B}_i$ now corresponds to a $(2\varepsilon, d/2)$-superregular pair $(V_a, V_b)_{G'}$. Further, all clusters now have size q' and for each edge V_aV_b in $\hat{B}_i$ the pair $(V_a, V_b)_{G'}$ is a 2ε-regular pair with density at least $d/2$ (by Fact 2.3.4).

Add all the vertices we removed from the clusters to V_0 and observe that now, since $\varepsilon \ll \eta_1, 1/|\hat{B}|$,

$$|V_0| \leq 3\eta_1 n. \tag{3.6}$$

From now on, we will refer to the subclusters of size q' as the clusters of R'.

By considering a random partition of each cluster V_a, and applying a Chernoff bound, one can obtain the following partition of each cluster.

Claim 3.7.1. *Let V_a be a cluster. Then there exists a partition of V_a into a red part V_a^{red} and a blue part V_a^{blue} such that*

$$\left| \left|V_a^{red}\right| - \left|V_a^{blue}\right| \right| \leq \varepsilon q'$$

and

$$\left| \left|N_G(x) \cap V_a^{red}\right| - \left|N_G(x) \cap V_a^{blue}\right| \right| < \varepsilon q' \quad \textit{for all } x \in V(G).$$

Proof of Claim 3.7.1. Let (V_a^{red}, V_a^{blue}) be a partition of V_a. Independently of all other vertices in V_a, place each vertex $v \in V_a$ into V_a^{red} with probability $1/2$ and into V_a^{blue} with probability $1/2$. Let $X := |V_a^{red}|$ and observe that $\mathbb{E}X = q'/2$ and $|V_a^{blue}| = q' - X$. Then applying Lemma 3.4.6 we have that

$$\begin{aligned}
\mathbb{P}\left(\left| \left|V_a^{red}\right| - \left|V_a^{blue}\right| \right| < \varepsilon q'\right) = \mathbb{P}(|2X - q'| < \varepsilon q') &= \mathbb{P}(|X - q'/2| < \varepsilon q'/2) \\
&= 1 - \mathbb{P}(|X - q'/2| \geq \varepsilon q'/2) \\
&\geq 1 - 2e^{\left(\frac{-\varepsilon^2 q'}{6}\right)}.
\end{aligned}$$

Observe that given any $x \in V(G)$, $\left|N_G(x) \cap V_a^{red}\right| = \left|N_{V_a}(x) \cap V_a^{red}\right|$ and $\left|N_G(x) \cap V_a^{blue}\right| = \left|N_{V_a}(x) \cap V_a^{blue}\right|$. Let $Y := \left|N_{V_a}(x) \cap V_a^{red}\right|$ and observe that $\mathbb{E}Y = |N_{V_a}(x)|/2$ and $\left|N_{V_a}(x) \cap V_a^{blue}\right| = |N_{V_a}(x)| - Y$. Clearly $q' \geq |N_{V_a}(x)|$.

Let $x \in V(G)$. If $|N_{V_a}(x)| < \varepsilon q'$, then

$$\mathbb{P}\left(\left|\,\left|N_{V_a}(x) \cap V_a^{red}\right| - \left|N_{V_a}(x) \cap V_a^{blue}\right|\,\right| \leq \varepsilon q'\right) = 1.$$

If $|N_{V_a}(x)| \geq \varepsilon q'$, then applying Fact 3.4.6 we have that

$$\begin{aligned}
\mathbb{P}\left(\left|\,\left|N_{V_a}(x) \cap V_a^{red}\right| - \left|N_{V_a}(x) \cap V_a^{blue}\right|\,\right| < \varepsilon q'\right) &= \mathbb{P}\left(\left|2Y - |N_{V_a}(x)|\,\right| < \varepsilon q'\right) \\
&\geq \mathbb{P}\left(\left|2Y - |N_{V_a}(x)|\,\right| < \varepsilon\,|N_{V_a}(x)|\right) \\
&= 1 - \mathbb{P}\left(\left|Y - |N_{V_a}(x)|\,/2\right| \geq \varepsilon\,|N_{V_a}(x)|\,/2\right) \\
&\geq 1 - 2e^{\left(\frac{-\varepsilon^2\left|N_{V_a}(x)\right|}{6}\right)}.
\end{aligned}$$

From Lemma 2.3.2, we know that $q' \geq (1 - 3\eta_1)n/M$. Hence

$$\begin{aligned}
\mathbb{P}\left(\forall x \in V(G),\ \left|\,\left|N_G(x) \cap V_a^{red}\right| - \left|N_G(x) \cap V_a^{blue}\right|\,\right| < \varepsilon q'\right) &\geq 1 - 2ne^{\frac{-\varepsilon^2\left|N_{V_a}(x)\right|}{6}} \\
&\geq 1 - 2ne^{-\frac{\varepsilon^3(1-3\eta_1)n}{6M}}.
\end{aligned}$$

Hence we can choose the partition required for Claim 3.7.1. □

Apply Claim 3.7.1 to every cluster to yield a partition of $V(G) - V_0$ into red and blue vertices. In the next section, we will remove vertices of particular copies of H in G from their respective clusters and do so in such a way that we avoid all the red vertices of G. After removing these vertices, we will be able to conclude that that each (modified) pair $(V_a, V_b)_{G'}$ is $(5\varepsilon, d/5)$-superregular[8] since V_a^{red} and V_b^{red} will have had no vertices removed from them. After the next section, we will only remove a bounded number of vertices from the clusters, which will not affect the superregularity of pairs of clusters in any significant way.

[8]Where V_aV_b is any edge in any $\hat{B}_i$ in $\hat{\mathcal{B}}$.

3.7.2 Covering the exceptional vertices

As in [44], given $x \in V_0$, we call a copy of $B \in \mathcal{B}$ *useful for* x if there exist $r-1$ clusters in B, each belonging to a different vertex class of B, such that x has at least $\alpha q'$ neighbours in each cluster. Denote by k_x the number of copies of B in $\mathcal{B}$ that are useful for x. The following calculation demonstrates that

$$k_x \beta q' \geq |V_0|.$$

By (3.2) and (3.6), we have that

$$\begin{aligned}
&k_x|B|q' + (|\mathcal{B}| - k_x)(|B|q' - (1-\alpha)q'\hat{c}(\omega+\sigma)) \\
&\geq d_G(x) - |V_0| \\
&\geq \left(1 - \frac{\omega+\sigma}{h} + \frac{\eta}{2}\right) q'|B||\mathcal{B}|,
\end{aligned}$$

which implies

$$(|\mathcal{B}| - k_x)(-(1-\alpha)q'\hat{c}(\omega+\sigma)) \geq \left(-\frac{\omega+\sigma}{h} + \frac{\eta}{2}\right) q'h\hat{c}|\mathcal{B}|.$$

Rearranging, we get

$$k_x \geq \frac{|\mathcal{B}|\left(\frac{h\eta}{2} - \alpha(\omega+\sigma)\right)}{(\omega+\sigma)(1-\alpha)}.$$

Since $\alpha \ll \eta$, we have that

$$k_x \geq \frac{\eta|\mathcal{B}|}{4}.$$

Now as $|\mathcal{B}|q' \geq \frac{n}{2|B|}$ and $\eta_1 \ll \beta, \eta, 1/h$ we have that

$$k_x\beta q' \geq \eta|\mathcal{B}|\beta q'/4 > 3\eta_1 n \geq |V_0|.$$

Hence we can assign greedily each vertex $x \in V_0$ to a copy B_x that is useful for x and do so in such a way that at most $\beta q'$ vertices in V_0 are assigned to the same copy $B \in \mathcal{B}$. Then for each copy $B_x \in \mathcal{B}$ that is useful for some $x \in V_0$ we can apply Lemma 2.3.5 to find a copy of H containing x which contains no red vertices. We do this as follows:

For each x, since $\varepsilon \ll \alpha$ and x has at least $\alpha q'$ neighbours in $r-1$ clusters belonging to different vertex classes of B_x, Claim 3.7.1 implies that x has at least $\alpha q'/4$ blue neighbours in each of these $r-1$ clusters. Further, we can find $\alpha q'/4$ blue vertices in a cluster belonging to the vertex class of B_x that does not necessarily contain any neighbours of x. Then it is easy to see that we can find subclusters $S_1, \dots, S_r$ of r clusters in B_x such that: all vertices in $S_1 \cup \dots \cup S_r$ are blue vertices; $|S_i| = \alpha q'/4$ for each i; every vertex in $S_1 \cup \dots \cup S_{r-1}$ is a neighbour of x in G. By Fact 2.3.4, each pair (S_i, S_j), $1 \leq i < j \leq r$, corresponds to an $(8\varepsilon/\alpha)$-regular pair in G' with density at least $d/3$. Using Lemma 2.3.5 with parameters $8\varepsilon/\alpha, d/3, \alpha q'/4, h-1$, we find a copy of H containing x. Since each $B \in \mathcal{B}$ has been assigned to at most $\beta q'$ vertices in V_0 and $\beta \ll \alpha$ (from (3.2)), we may repeat the above argument to find copies of H that contain each exceptional vertex in such a way that the copies are disjoint and contain no red vertices. Denote the H-tiling induced by these copies of H by $\mathcal{H}_1$. Remove all the vertices lying in these copies of H from their respective clusters. Observe that currently, for each i,

$$(1 - \beta h)q' \leq |V_i| \leq q'.$$

3.7.3 Making the blow-up of each $B \in \mathcal{B}$ divisible by h

For a subgraph $S \subseteq R'$, let $V_G(S)$ denote the union of the clusters in S. We aim to apply Lemma 3.4.3 to each $\hat{B}_i$ in $\hat{\mathcal{B}}$ to find an H-tiling that covers every vertex of $V_G(\hat{B}_i)$. Combining these H-tilings with $\mathcal{H}_1$ will result in a perfect H-tiling in G as desired. Recall that, for each $1 \leq i \leq \hat{k}$, $\hat{G}_i$ is the r-partite subgraph of G' whose jth vertex class is the union of all those clusters contained in the jth vertex class of $\hat{B}_i$, for each $1 \leq j \leq r$. Further, recall that G_i^* is the complete r-partite graph on the same vertex set as $\hat{G}_i$. To apply Lemma 3.4.3 to each $\hat{B}_i$ in $\hat{\mathcal{B}}$ we require that each G_i^* contains a perfect H-tiling. To guarantee the existence of these perfect H-tilings we will apply Lemma 3.4.5. To use Lemma 3.4.5 on G_i^* we require that $|V(\hat{G}_i^*)|$ is divisible by h. When we first chose our $\hat{B}$-tiling this was the case. Indeed, as each $\hat{B}_i$ contained a perfect H-tiling and every cluster V_i was the same size, $|V(G_i^*)|$ was divisible by h. However, in the last section we took out vertices from G in a greedy way, changing the sizes of the clusters in R'. Hence we cannot guarantee that $|V(G_i^*)|$ is still divisible by h for each i. Now we will take out a further bounded number of copies of H in G to ensure $|V(G_i^*)|$ is divisible by h for each $1 \leq i \leq \hat{k}$. In fact, we will ensure $|V_G(B)|$ is divisible by h for each $B \in \mathcal{B}$.

We now split into two cases: when $r \geq 3$ and when $r = 2$. When $r \geq 3$ we have that $\text{hcf}_\chi(H) = 1$ and this property will be central to our argument. For $r = 2$, we have that $\text{hcf}_c(H) = 1$ and $\text{hcf}_\chi(H) \leq 2$. The former property will provide us an easy way of removing copies of H from $V(G)$ to ensure $|V_G(B)|$ is divisible by h for each $B \in \mathcal{B}$. Further, we will not need to use the property that $\text{hcf}_\chi(H) \leq 2$ in our argument. The only time we (implicitly) use the property that $\text{hcf}_\chi(H) \leq 2$ will be when we apply Lemma 3.4.5.

3.7.4 Case 1: $r \geq 3$

For a subgraph S of R', let $V_{R'}(S)$ denote the vertex set of S. To assist in our argument, we define an auxiliary graph F whose vertices are the copies of B in $\mathcal{B}$ and for $B_1, B_2 \in V(F)$, we let B_1B_2 be an edge in F if and only if there exists a vertex x in $V_{R'}(B_1)$ and $r-1$ vertices in $V_{R'}(B_2)$, all in different vertex classes of B_2, (or vice versa) such that these r vertices induce a K_r in R'. Assume F is connected and let B_1B_2 be an edge in F. Then we may apply Lemma 2.3.5 to find $h-1$ disjoint copies of H which each have one vertex in $V_G(B_1)$ and all other vertices in $V_G(B_2)$ (or vice versa). This means that we can remove at most $h-1$ copies of H to ensure $V_G(B_1)$ is divisble by h. Continuing in this way we can 'shift the remainders mod h' along a spanning tree of F to ensure $|V_G(B)|$ is divisible by h for each $B \in \mathcal{B}$. (Indeed, since n is divisible by h we have that $\sum_{B \in \mathcal{B}} |V_G(B)|$ is divisible by h.)

So assume F is not connected. Let $\mathcal{C}$ be the set of all components of F. For $C \in \mathcal{C}$ we will write $V_{R'}(C)$ for the set of vertices in R' belonging to copies of B in C and $V_G(C)$ for the union of the clusters corresponding to the vertices in $V_{R'}(C)$. In what follows our aim is to remove a bounded number of copies of H to ensure that for each component $C \in \mathcal{C}$ we have that $|V_G(C)|$ is divisible by h. Then we can apply our previous argument to spanning trees of each component to achieve that $|V_G(B)|$ is divisible by h for each $B \in \mathcal{B}$.

Call vertices in R' of degree at least

$$(1 - \omega/h + \eta/4)k' \tag{3.7}$$

big. If a vertex is not big, call it *small*. Note by (3.5) that all but at most $\omega k'/h - 1$ vertices in R' are big.

Claim 3.7.2. *Let $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and let $a \in V_{R'}(C_2)$. Then*

$$|N_{R'}(a) \cap V_{R'}(C_1)| < \left(1 - \frac{\omega + \sigma}{h} + \frac{\eta}{4}\right) |V_{R'}(C_1)|.$$

Proof. Recall that B has width $\omega \hat{c}$. Suppose Claim 3.7.2 is false. Then there exists some $B_0 \in \mathcal{B}$ such that $B_0 \in C_1$ and

$$|N_{R'}(a) \cap B_0| \geq \left(1 - \frac{\omega + \sigma}{h} + \frac{\eta}{4}\right) |B_0| = (r-2)\omega\hat{c} + \frac{\eta h \hat{c}}{4}.$$

Thus a must have neighbours in at least $r-1$ vertex classes of B_0. We can therefore construct a copy of K_r in R' which consists of a together with $r-1$ of its neighbours in B_0. But by definition of the auxiliary graph F, we must have that B_0 is adjacent in F to the copy of B in $\mathcal{B}$ that contains a. This contradicts that C_1 and C_2 were different components of F. Thus Claim 3.7.2 holds. $\square$

Claim 3.7.3. *There exist components $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, a big vertex $x_1 \in V(R')$ and another (not necessarily big) vertex $x_2 \in V(R')$ such that $x_1 \in V(C_1)$, $x_2 \in V(C_2)$ and $x_1x_2 \in E(R')$.*

Proof. Take some big vertex $x \in R'$. Then x is in $V_{R'}(C_x)$ for some component C_x of F. We may assume $|C_x| \geq (1-\omega/h+\eta/4)k'$, as otherwise x has a neighbour in R' outside of C_x and we are done. Since $r \geq 3$,

$$|R' \setminus V_{R'}(C_x)| \leq (\omega/h - \eta/4)k' < (1-\omega/h+\eta/4)k'.$$

If $R' \setminus V_{R'}(C_x)$ contains any big vertex y, then y has a neighbour in $V_{R'}(C_x)$ since $|R' \setminus V_{R'}(C_x)| < (1-\omega/h+\eta/4)k'$ and we are done. Hence assume all big vertices are in $V_{R'}(C_x)$. Then all vertices in $R' \setminus V_{R'}(C_x)$ are small vertices. Let z be a small vertex in $R' \setminus V_{R'}(C_x)$. Since $r \geq 3$,

$$d_{R'}(z) \geq (1-(\omega+\sigma)/h+\eta/4)k' \geq (\omega/h+\eta/4)k'.$$

Since there are at most $\omega k'/h - 1$ small vertices in R', we have that z has a neighbour w which is a big vertex. But then $w \in V_{R'}(C_x)$. Thus Claim 3.7.3 holds. $\square$

Claim 3.7.4. *There exists a copy K' of K_r in R' which has vertices in at least two components of F.*

Proof. By Claim 3.7.3, there exist components $C_1, C_2 \in \mathcal{C}$, a big vertex $x_1 \in V(R')$ and another vertex $x_2 \in V(R')$ such that $x_1 \in V_{R'}(C_1)$, $x_2 \in V_{R'}(C_2)$ and $x_1x_2 \in E(R')$. By (3.5) and (3.7), x_1 and x_2 have a common neighbourhood of size at least

$$((r-3)\omega/h + \eta/2)k'.$$

If $r = 3$, then we choose x_3 in the common neighbourhood of x_1 and x_2, and we are done. So assume $r \geq 4$. Since there are at most $\omega k'/h$ small vertices, we can choose a big vertex x_3 in the common neighbourhood of x_1 and x_2. Then x_1, x_2 and x_3 have a common neighbourhood of size at least

$$((r-4)\omega/h + 3\eta/4)k'.$$

If $r = 4$, then we choose x_4 in the common neighbourhood of x_1, x_2 and x_3 and we are done. Otherwise $r \geq 5$ and we continue as before. Thus Claim 3.7.4 holds. $\square$

For such a copy K' of K_r in R', we now show that we can take out a bounded number of copies of H from the clusters corresponding to the vertices of K' in such a way that leaves one of the components $C \in \mathcal{C}$ with $|V_G(C)|$ divisible by h. We use Theorem 3.5.2 and Lemma 2.3.5 to achieve this. We will then repeat this process to ensure $|V_G(B)|$ is divisible by h for each $B \in \mathcal{B}$.

Claim 3.7.5. *There exists $t \in \mathbb{N}$ such that by removing at most $t + (|\mathcal{B}| - |\mathcal{C}|)(h-1)$ copies of H from G we can ensure $|V_G(B)|$ is divisible by h for each $B \in \mathcal{B}$.*

Proof. Firstly, for each component $C \in \mathcal{C}$ we will remove copies of H to ensure $|V_G(C)|$ is divisible by h. Apply Claim 3.7.4 to find a copy K' of K_r in R which has vertices in at least two components of F. Let C^* be a component of F which contains at least one vertex of K'. Let p be the number of vertices of K' contained in C^* and observe that $1 \leq p \leq r-1$. Let $0 \leq g \leq h-1$ such that $|V_G(C^*)| \equiv g \mod h$. If $g = 0$ then $|V_G(C^*)|$ is divisible by h and we consider the graphs $F_1 := F - V(C^*)$ and $R_1 := R - V_R(C^*)$. So assume $1 \leq g \leq h-1$.

Observe that we can apply Lemma 2.3.5 to find any bounded number of disjoint copies of H in G in the clusters of K' (see the end of Section 3.7.2). For any copy H' of H in G in the clusters of K' there are precisely p colour classes of some colouring c of H' contained in the clusters of K' in $V_G(C^*)$. Moreover, given any colouring c of H and p-subset P contained in D_c (recall Definition 3.5.1) we can find any bounded number of disjoint copies H' of H in G with colouring c in the clusters of K' so that the colour classes of H' in $V_G(C^*)$ correspond to the p-subset P. So there exists $j \in \{1, \dots, z_p\}$ such that $P = A_{p,c,j}$ (recall this notation from Definition 3.5.1). Thus, removing such a copy H' of H from G would result in removing precisely $S_{p,c,j}$ vertices from $V_G(C^*)$.

By Theorem 3.5.2, there exist a collection of non-negative integers $\{a_{p,c,i} : c \in C_H, 1 \leq i \leq z_p\}$ such that

$$g \cdot \sum_{c \in C_H} \sum_{i=1}^{z_p} a_{p,c,i} S_{p,c,i} \equiv g \mod h.$$

Let $\bar{a} = \max a_{p,c,i} : c \in C_H, 1 \leq i \leq z_p$. Hence we can remove

$$g \cdot \sum_{c \in C_H} \sum_{i=1}^{z_p} a_{p,c,i} \leq (h-1)\bar{a}|C_H|z_p$$

suitable disjoint copies of H in G in the clusters of K' to make $|V_G(C^*)|$ divisible by h.

Next we consider graphs $F_1 := F - V(C^*)$ and $R_1 := R' - V_R(C^*)$. Let $k_1 := |R_1|$. Claim 3.7.2 and (3.5) together give us that R_1 has degree sequence $d_{R_1,1} \leq \cdots \leq d_{R_1,k_1}$ where

$$d_{R_1,i} \geq \left(1 - \frac{\omega + \sigma}{h}\right) k_1 + \frac{\sigma}{\omega} i + \frac{\eta k_1}{4} \quad \text{for all } 1 \leq i \leq \frac{\omega k_1}{h}.$$

Suppose $|\mathcal{C}| \geq 2$. Arguing as in Claims 3.7.3 and 3.7.4 we can find a copy K_1' of K_r in R_1 which has vertices in at least two components of F_1.

Let C^{**} be a component of F which contains at least one vertex of K_1'. As before by removing at most $(h-1)\bar{a}|C_H|z_p$ copies of H from the clusters of K_1' we can make $|V_G(C^{**})|$ divisible by h. Since $|G|$ is divisible by h, we can continue in this way to make $|V_G(C)|$ divisible by h for each component $C \in \mathcal{C}$. We then apply the 'shifting the remainders mod h' argument mentioned earlier during the 'F connected' case to guarantee that $|B|$ is divisible by h for each $B \in \mathcal{B}$. In this process we removed at most $(|\mathcal{C}|-1)(h-1)\bar{a}|C_H|z_p$ disjoint copies of H from G. Each time we use the 'shifting the remainders mod h' argument on a connected component $C \in \mathcal{C}$ we remove at most $(|C|-1)(h-1)$ disjoint copies of H in G. Hence overall we remove at most $(|\mathcal{C}|-1)(h-1)\bar{a}|C_H|z_p + (|\mathcal{B}|-|\mathcal{C}|)(h-1)$ disjoint copies of H in G. Denote this H-tiling (formed from these copies of H) by $\mathcal{H}_2$. $\square$

Observe that now

$$(1-2h\beta)q' \leq |V_i| \leq q'$$

for each i since we only removed a bounded number of vertices from G.

3.7.5 Case 2: $r = 2$

As in the statement of Theorem 3.5.2, let b be the number of components of H and $t_1, \dots, t_b$ be the sizes of the components of H. By Theorem 3.5.2, there exists a collection of non-negative integers $\{a_i : 1 \leq i \leq b\}$ such that

$$\sum_{i=1}^{b} a_i t_i \equiv 1 \mod h.$$

Let $B_1, B_2 \in \mathcal{B}$. If $|V_G(B_1)| \equiv 0 \mod h$, define $\mathcal{B}_1 := \mathcal{B} \setminus B_1$. If not, let $p \in \{1, \dots, h-1\}$ such that $|V_G(B_1)| \equiv p \mod h$. Remove $p\sum_{i=1}^{b} a_i$ copies of H from $V_G(B_1) \cup V_G(B_2)$ in the following way: For each $1 \leq i \leq b$, remove pa_i copies of H from $V_G(B_1) \cup V_G(B_2)$ such that the component of order t_i is in $V_G(B_1)$ and all other components are in $V_G(B_2)$.[9] Since

[9]We use Lemma 2.3.5 to do this.

$p\sum_{i=1}^{b} a_i t_i \equiv p \mod h$, by removing these $p\sum_{i=1}^{b} a_i$ copies of H from $V_G(B_1) \cup V_G(B_2)$ we now have that $|V_G(B_1)|$ is divisible by h. Define $\mathcal{B}_1 := \mathcal{B} \setminus B_1$.

Let $B_1', B_2' \in \mathcal{B}_1$. If $|V_G(B_1')| \equiv 0 \mod h$, define $\mathcal{B}_2 := \mathcal{B}_1 \setminus B_1'$. If not, let $p' \in \{1, \dots, h-1\}$ such that $|V_G(B_1')| \equiv p' \mod h$. Remove $p'\sum_{i=1}^{b} a_i$ copies of H from $V_G(B_1') \cup V_G(B_2')$ in the same way as before. Define $\mathcal{B}_2 := \mathcal{B}_1 \setminus B_1'$ and let $\bar{a} = \max\{a_i : 1 \leq i \leq b\}$. Continuing in the same way, we see that by removing at most

$$(|\mathcal{B}| - 1)(h - 1)b\bar{a} \tag{3.8}$$

copies of H we can ensure that $|B|$ is divisible by h for each $B \in \mathcal{B}$.[10] Denote this H-tiling (formed from these copies of H) by $\mathcal{H}_2$.

Observe that now

$$(1 - 2h\beta)q' \leq |V_i| \leq q'$$

for each i since we only removed a bounded number of vertices.

3.7.6 Completing the perfect tiling

As we related at the beginning of Section 3.7.3, we aim to apply Lemma 3.4.3 to each $\hat{B}_i \subseteq R'$ $(1 \leq i \leq \hat{k})$ where the vertices of R' are the now modified clusters – modified by the removing of copies of H in previous sections. Recall that, for each $1 \leq i \leq \hat{k}$, $\hat{G}_i$ is the r-partite subgraph of G' whose jth vertex class is the union of all those clusters contained in the jth vertex class of $\hat{B}_i$, for each $1 \leq j \leq r$. Observe that in Section 3.7.3 we made

[10] Since $\sum_{B \in \mathcal{B}} |V_G(B)|$ is divisible by h, after we remove copies of H from $\mathcal{B}_{|\mathcal{B}|-2} = \{B_1^{(|\mathcal{B}|-2)}, B_2^{(|\mathcal{B}|-2)}\}$ (if necessary), both $|V_G(B_1^{(|\mathcal{B}|-2)})|$ and $|V_G(B_2^{(|\mathcal{B}|-2)})|$ will be divisible by h. This explains the presence of the $(|\mathcal{B}| - 1)$ term in (3.8).

$|\hat{G}_i| = |V_G(\hat{B}_i)|$ divisible by h for each i. Further,

$$(1 - 2h\beta)q' \leq |V_i| \leq q'$$

for each i. Recall that G_i^* is the complete r-partite graph on the same vertex set as $\hat{G}_i$. Since $0 < 2h\beta \ll \sigma/\omega, 1 - \sigma/\omega, 1/h$ by (3.2), we can apply Lemma 3.4.5 to conclude that each G_i^* contains a perfect H-tiling.

Furthermore, pairs of clusters that correspond to edges of $\hat{B}_i$ are still $(6\varepsilon, d/6)$-superregular. Indeed, in Section 3.7.2 we removed copies of H which avoided red vertices, resulting in each pair of clusters (in a copy of H) being $(5\varepsilon, d/5)$-superregular. Then, in Section 3.7.4, or Section 3.7.5 if $r = 2$, we removed only a constant number of vertices from each cluster. Hence each pair of clusters (in a copy of H) is $(6\varepsilon, d/6)$-superregular.

We now have all we need to apply Lemma 3.4.3 to find a perfect H-tiling $\hat{\mathcal{H}}_i$ in $\hat{G}_i$ for each $1 \leq i \leq \hat{k}$. Then

$$\mathcal{H}_1 \cup \mathcal{H}_2 \cup \hat{\mathcal{H}}_1 \cup \cdots \cup \hat{\mathcal{H}}_{\hat{k}}$$

is a perfect H-tiling in G. Hence we have proved Theorem 3.0.3.

Chapter Four

Towards the 0-statement of the Kohayakawa-Kreuter Conjecture

This chapter is based on [28]. Recall from Chapter 1 that in this chapter we will prove a reduction of Conjecture 1.2.3 to a certain deterministic subproblem (Conjecture 4.0.4). To state Conjecture 4.0.4 we will require a significant preamble. We begin by recalling several definitions from Chapter 1.

We write $G \to (H_1, \ldots, H_r)$ to denote the property that whenever we colour the edges of G with colours from the set $[r] := \{1, \ldots, r\}$ there exists $i \in [r]$ and a copy of H_i in G monochromatic in colour i.

For a graph H, we define

$$d_2(H) := \begin{cases} (e_H - 1)/(v_H - 2) & \text{if } H \text{ is non-empty with } v(H) \geq 3, \\ 1/2 & \text{if } H \cong K_2, \\ 0 & \text{otherwise} \end{cases}$$

and the *2-density of H* to be

$$m_2(H) := \max\{d_2(J) : J \subseteq H\}.$$

We say that a graph H is *2-balanced* if $d_2(H) = m_2(H)$, and *strictly 2-balanced* if for all proper subgraphs $J \subset H$, we have $d_2(J) < m_2(H)$.

For graphs H_1 and H_2 with $m_2(H_1) \geq m_2(H_2)$, we define

$$d_2(H_1, H_2) := \begin{cases} \frac{e(H_1)}{v(H_1)-2+\frac{1}{m_2(H_2)}} & \text{if } H_2 \text{ is non-empty and } e(H_1) \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

and the *asymmetric 2-density of the pair* (H_1, H_2) to be

$$m_2(H_1, H_2) := \max\left\{d_2(J, H_2) : J \subseteq H_1\right\}.$$

We say that H_1 is *balanced with respect to (w.r.t.)* $d_2(\cdot, H_2)$ if we have $d_2(H_1, H_2) = m_2(H_1, H_2)$ and *strictly balanced w.r.t.* $d_2(\cdot, H_2)$ if for all proper subgraphs $J \subset H_1$ we have $d_2(J, H_2) < m_2(H_1, H_2)$. Note that $m_2(H_1) \geq m_2(H_1, H_2) \geq m_2(H_2)$ (see Proposition 4.3.1).

We define the following for brevity.

Definition 4.0.1. Let H_1 and H_2 be non-empty graphs. We say that a graph G has a *valid colouring for H_1 and H_2* if there exists a red/blue colouring of the edges of G that does not produce a red copy of H_1 or a blue copy of H_2.

To prove the 0-statement of Conjecture 1.2.3, one only needs to show that $G = G_{n,p}$ has a valid colouring for H_1 and H_2 asymptotically almost surely (a.a.s.) (that is, in accordance with the intuition for the threshold in Conjecture 1.2.3, we can ignore $H_3, \ldots, H_r$

and colours $3, \ldots, r$). Further, when $m_2(H_1) > m_2(H_2)$ we can assume when proving the 0-statement of Conjecture 1.2.3 that H_2 is strictly 2-balanced and H_1 is strictly balanced w.r.t. $d_2(\cdot, H_2)$. Indeed, if either of these assumptions do not hold then one can replace H_1 and H_2 with subgraphs $H_1' \subseteq H_1$ and $H_2' \subseteq H_2$ such that H_2' is strictly 2-balanced and H_1' is strictly balanced w.r.t. $d_2(\cdot, H_2')$. Then we would aim to show that G has a valid colouring for H_1' and H_2' a.a.s.[1] Similarly, when $m_2(H_1) = m_2(H_2)$, we can assume when proving the 0-statement of Conjecture 1.2.3 that both H_1 and H_2 are strictly 2-balanced.

In past work on attacking 0-statements of Ramsey problems (e.g. Conjecture 1.2.3 and Theorem 1.2.2), researchers have applied variants of a standard and natural approach (see e.g. [33, 49, 51, 61]). In Chapter 4, we prove that every step of this approach, except one, holds w.r.t. Conjecture 1.2.3. That is, we reduce Conjecture 1.2.3 to a single subproblem. To state this subproblem we require a number of definitions adapted from [51].

Definition 4.0.2. For any graph G we define the families

$$\mathcal{R}_G := \{R \subseteq G : R \cong H_1\} \text{ and } \mathcal{L}_G := \{L \subseteq G : L \cong H_2\}$$

of all copies of H_1 and H_2 in G, respectively. Furthermore, we define

$$\mathcal{L}_G^* := \{L \in \mathcal{L}_G : \forall e \in E(L)\ \exists R \in \mathcal{R}_G \text{ s.t. } E(L) \cap E(R) = \{e\}\} \subseteq \mathcal{L}_G,$$

$$\mathcal{C} = \mathcal{C}(H_1, H_2) := \{G = (V, E) : \forall e \in E\ \exists (L, R) \in \mathcal{L}_G \times \mathcal{R}_G \text{ s.t. } E(L) \cap E(R) = \{e\}\}$$

and

$$\mathcal{C}^* = \mathcal{C}^*(H_1, H_2) := \{G = (V, E) : \forall e \in E\ \exists L \in \mathcal{L}_G^* \text{ s.t. } e \in E(L)\}.$$

[1]Which would immediately imply that G has a valid colouring for H_1 and H_2.

Note that $\mathcal{C}^*(H_1, H_2) \subseteq \mathcal{C}(H_1, H_2)$. These sets will be very important when analysing the algorithms in this chapter.

Definition 4.0.3. Let H_1 and H_2 be non-empty graphs such that $m_2(H_1) \geq m_2(H_2) > 1$. Let $\varepsilon := \varepsilon(H_1, H_2) > 0$ be a constant. Define $\hat{\mathcal{A}} = \hat{\mathcal{A}}(H_1, H_2, \varepsilon)$ to be

$$\hat{\mathcal{A}} := \begin{cases} \{A \in \mathcal{C}^*(H_1, H_2) : m(A) \leq m_2(H_1, H_2) + \varepsilon \wedge A \text{ is 2-connected}\} \text{ if } m_2(H_1) > m_2(H_2), \\ \{A \in \mathcal{C}(H_1, H_2) : m(A) \leq m_2(H_1, H_2) + \varepsilon \wedge A \text{ is 2-connected}\} \text{ if } m_2(H_1) = m_2(H_2). \end{cases}$$

We now state our subproblem as the following conjecture.

Conjecture 4.0.4. *Let H_1 and H_2 be non-empty graphs such that $H_1 \neq H_2$ and $m_2(H_1) \geq m_2(H_2)$. Assume H_2 is strictly 2-balanced. Moreover, assume H_1 is strictly balanced w.r.t. $d_2(\cdot, H_2)$ if $m_2(H_1) > m_2(H_2)$ and strictly 2-balanced if $m_2(H_1) = m_2(H_2)$. Then there exists a constant $\varepsilon := \varepsilon(H_1, H_2) > 0$ such that the set $\hat{\mathcal{A}}$ is finite and every graph in $\hat{\mathcal{A}}$ has a valid colouring for H_1 and H_2.*

Notice that we can assume $H_1 \neq H_2$ as the $H_1 = H_2$ case of Conjecture 1.2.3 is handled by Rödl and Ruciński's theorem (Theorem 1.2.2).

Theorem 1.2.2 (Rödl and Ruciński [63])**.** *Let $r \geq 2$ and let H be a non-empty graph such that at least one component of H is not a star. If $r = 2$, then in addition restrict H to having no component which is a path on 3 edges. Then there exist positive constants $b, B > 0$ such that*

$$\lim_{n \to \infty} \mathbb{P}[G_{n,p} \to (\underbrace{H, \ldots, H}_{r \text{ times}})] = \begin{cases} 0 & \text{if } p \leq bn^{-1/m_2(H)}, \\ 1 & \text{if } p \geq Bn^{-1/m_2(H)}. \end{cases}$$

To be clear, the main purpose of the work in this chapter is to show that if Conjecture 4.0.4 holds then the rest of a variant of a standard approach for attacking the 0-statement of Conjecture 1.2.3 falls into place (see Section 4.1). That is, Conjecture 4.0.4 is a natural subproblem of Conjecture 1.2.3. Thus we prove the following theorem.

Theorem 4.0.5. *If Conjecture 4.0.4 is true then the 0-statement of Conjecture 1.2.3 is true.*

We prove Conjecture 4.0.4 for almost every pair of regular graphs, which, by Theorem 4.0.5, significantly extends the class of graphs for which the 0-statement of Conjecture 1.2.3 is resolved.

Theorem 4.0.6. *Let H_1 and H_2 meet the criteria in Conjecture 4.0.4. In addition, let H_1 and H_2 be regular graphs, excluding the cases when (i) H_1 and H_2 are a clique and a cycle, (ii) H_2 is a cycle and $|V(H_1)| \geq |V(H_2)|$ or (iii) $(H_1, H_2) = (K_3, K_{3,3})$. Then Conjecture 4.0.4 is true for H_1 and H_2.*

Such pairs of graphs (H_1, H_2) that satisfy the criteria in Theorem 4.0.6 include $(K_k, K_{k,k})$ for all $k \geq 4$. We note that the 0-statement of Conjecture 1.2.3 is proved in [49] for pairs of graphs in case (i). We also think it probable that $\hat{\mathcal{A}}(K_3 K_{3,3}) \neq \emptyset$. As a natural subproblem of the 0-statement of Conjecture 1.2.3, we believe that Conjecture 4.0.4 is a considerably more approachable problem than the 0-statement of Conjecture 1.2.3. Indeed, the techniques used in the proof of Theorem 4.0.6 are elementary and uncomplicated. Thus, we hope that a full resolution of Conjecture 1.2.3 can be achieved via Theorem 4.0.5.

4.1 Overview of the proof of Theorem 4.0.5

As mentioned earlier, to prove Theorem 4.0.5 we will employ a variant of a standard approach for attacking 0-statements of Ramsey problems. For attacking the 0-statement of Conjecture 4.0.4, this standard approach is as follows:

- For $G = G_{n,p}$, assume $G \to (H_1, H_2)$;
- Use structural properties of G (resulting from this assumption) to show that G contains at least one of a sufficiently small collection of non-isomorphic graphs $\mathcal{F}$;
- Show that there exists a constant $b > 0$ such that for $p \leq bn^{-1/m_2(H_1,H_2)}$ we have that G contains no graph in $\mathcal{F}$ a.a.s.;
- Conclude, by contradiction, that $G \not\to (H_1, H_2)$ a.a.s.

The variant of this approach we will use is due to Marciniszyn, Skokan, Spöhel and Steger [51], who proved Conjecture 1.2.3 for cliques. In [51], for $r > \ell \geq 3$, they employ an algorithm Asym-Edge-Col which either produces a valid colouring for K_r and K_ℓ of G (showing that $G \not\to (K_r, K_\ell)$) or encounters an error. Instead of assuming $G \to (K_r, K_\ell)$, they assume Asym-Edge-Col encounters an error, and proceed with the standard approach from there. One of the advantages of this approach is that it provides an algorithm for constructing a valid colouring for K_r and K_ℓ, rather than just proving the existence of such a colouring.

4.1.1 On Conjecture 4.0.4

As mentioned in Chapter 1, we provide all but one step, Conjecture 4.0.4, of this approach. Let us consider how Conjecture 4.0.4 relates to previous work on the 0-statement of Conjecture 1.2.3. Firstly, Conjecture 4.0.4 was implicitly proven for pairs of cliques in [51] and pairs of a clique and a cycle in [49]. More specifically, when H_1 and H_2 are both cliques (except when $H_1 = H_2 = K_3$)[2], the authors of [51] prove a slightly more general version of

[2]The case $H_1 = H_2$ of Conjecture 1.2.3 is, of course, covered by Theorem 1.2.2.

Conjecture 4.0.4 ([51, Lemma 8]) where $\hat{\mathcal{A}}(H_1, H_2, \varepsilon)$ is replaced with the set

$$\mathcal{A}(H_1, H_2) := \{A \in \mathcal{C}(H_1, H_2) : m(A) \leq m_2(H_1, H_2) + 0.01 \wedge A \text{ is 2-connected}\}.$$

Note that the proof of [51, Lemma 8] shows that $\mathcal{A}(H_1, H_2) \neq \emptyset$ for certain pairs of cliques H_1 and H_2. When H_1 is a clique, H_2 is a cycle and $H_1 \neq H_2$ (that is, excluding again the case when $H_1 = H_2 = K_3$), the proof of [49, Lemma 3.3] implies that there exists a constant $\varepsilon > 0$ such that $\hat{\mathcal{A}}(H_1, H_2, \varepsilon) = \emptyset$.

For reference, we note here the places in our proof of Theorem 4.0.5 where we specifically need Conjecture 4.0.4 to hold:

- the proof of Lemma 4.4.1;
- the proofs of Claims 4.5.6 and 4.6.6;
- the definition of $\gamma = \gamma(H_1, H_2)$ in Section 4.5.

4.1.2 Proof sketch of Theorem 4.0.5

Let us now proceed with describing the proof of Theorem 4.0.5 in detail. In what follows, we write *(Result A; Result B)* to mean that 'Result B in [51] fulfils the same role (in [51]) as Result A does in our proof of Theorem 4.0.5'. This is to illustrate how we indeed provide every step bar one (Conjecture 4.0.4) of a proof of the 0-statement of Conjecture 1.2.3.

Firstly, as in [51], we give an algorithm Asym-Edge-Col that, assuming Conjecture 4.0.4 holds, produces a valid colouring for H_1 and H_2 of $G = G_{n,p}$ provided it does not encounter an error (Lemma 4.4.2; Lemma 11). Our aim then is to prove that Asym-Edge-Col does not encounter an error a.a.s. (Lemma 4.4.3; Lemma 12), that is, $G \not\rightarrow (H_1, H_2)$

a.a.s. We split our proof of Lemma 4.4.3 into two cases: when $m_2(H_1) > m_2(H_2)$ and when $m_2(H_1) = m_2(H_2)$.

Suppose for a contradiction that Asym-Edge-Col encounters an error. Let $G' \subseteq G$ be the graph that Asym-Edge-Col got stuck on when it encountered this error. In the $m_2(H_1) > m_2(H_2)$ case, we input G' into an auxiliary algorithm Grow which always outputs a subgraph $F \subseteq G'$ (Claim 4.5.1; Claim 13) belonging to a sufficiently small collection of non-isomorphic graphs $\mathcal{F}$. The definition of $\mathcal{F}$ will be such that w.h.p. no copy of any $F \in \mathcal{F}$ will be present in $G_{n,p}$ provided that $|\mathcal{F}|$ is sufficiently small.

In order to show $|\mathcal{F}|$ is sufficiently small, we carefully analyse the possible outputs of Grow. Assuming Conjecture 4.0.4 holds, we show that only a constant number of graphs can be produced by Grow if one of two special cases occurs. If neither of these special cases occur, then, starting from a copy of H_1, in each step of Grow our subgraph F is constructed iteratively by either (i) appending a copy of H_1 to F or (ii) appending a 'flower-like' structure to F, consisting of a central copy of H_2 with 'petals' that are appended copies of H_1. We say an iteration is *degenerate* if it is of type (i) or, loosely speaking, of type (ii) where 'the flower is folded in on itself or into F'. Otherwise an iteration is called *non-degenerate*. Denote by $\lambda(F)$ the order of magnitude of the expected number of copies of F in $G_{n,p}$ with $p = bn^{-1/m_2(H_1,H_2)}$. Key to showing $|\mathcal{F}|$ is sufficiently small is proving that $\lambda(F)$ stays the same after a non-degenerate iteration (Claim 4.5.2; Claim 14) and decreases by a *constant* amount after a degenerate iteration (Claim 4.5.3; Claim 15). Indeed, one of the termination conditions for Grow is that $\lambda(F) < -\gamma$ (where $\gamma = \gamma(H_1, H_2, \varepsilon) > 0$ is defined later in Section 4.5, given $\varepsilon = \varepsilon(H_1, H_2) > 0$, the constant acquired from assuming Conjecture 4.0.4 holds), that is, only a constant number of such *degenerate* steps occur before Grow terminates (Claim 4.5.4; Claim 16). Proving Claim 4.5.3 is the main work of this chapter. An important step in proving it is showing that if an iteration of type (ii) occurs where, loosely speaking, 'the flower is folded in on itself', we get a helpful inequality comparing this iteration with a non-degenerate iteration (Lemma 4.5.8; Lemma 21). Indeed, the most novel work in this chapter is the proof of Lemma 4.5.8.

The proof of Lemma 4.4.3 in the $m_2(H_1) = m_2(H_2)$ case is both similar and significantly simpler. Notably, we use a different algorithm, Grow-Alt, to grow our subgraph $F \subseteq G'$. Our analysis of Grow-Alt is much quicker than that of Grow, allowing us to easily prove a result analogous to Claim 4.5.3.

4.2 Organisation

The rest of this chapter is organised as follows. In Section 4.3, we collect together notation, density measures and several useful results we will need. In Section 4.4, we give our algorithm Asym-Edge-Col for producing a valid colouring for H_1 and H_2 of $G = G_{n,p}$ provided it does not encounter an error (and Conjecture 4.0.4 holds for H_1 and H_2). In Sections 4.5-4.5.3, we prove that Asym-Edge-Col does not encounter an error a.a.s. (Lemma 4.4.3) in the case when $m_2(H_1) > m_2(H_2)$. In Section 4.6, we prove Lemma 4.4.3 in the case when $m_2(H_1) = m_2(H_2)$. In Section 4.7, we prove Theorem 4.0.6, before providing some concluding remarks in Section 4.8.

4.3 Notation, density measures and useful results

As far as possible we keep to the notation used in [51]. Also, we repeat several definitions introduced in Chapter 1 for ease of reference.

Let H be a graph. The most well-known density measure is

$$d(H) := \begin{cases} e_H/v_H & \text{if } v(H) \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Taking the maximum value of d over all subgraphs $J \subseteq H$, we have the following measure:

$$m(H) := \max\{d(J) : J \subseteq H\}.$$

(We say that a graph H is *balanced w.r.t* d, or just *balanced*, if we have $d(H) = m(H)$. Moreover, we say H is *strictly balanced* if for every proper subgraph $J \subset H$, we have $d(J) < m(H)$.)

In [61], Rödl and Ruciński introduced the following so-called 2-*density measure*:

$$d_2(H) := \begin{cases} (e_H - 1)/(v_H - 2) & \text{if } H \text{ is non-empty with } v(H) \geq 3, \\ 1/2 & \text{if } H \cong K_2, \\ 0 & \text{otherwise.} \end{cases}$$

As with d, we have an associated measure based on maximising d_2 over subgraphs of H:

$$m_2(H) := \max\left\{d_2(J) : J \subseteq H\right\}.$$

Analogously to the notion of balancedness, we say that a graph H is 2-*balanced* if $d_2(H) =$

$m_2(H)$, and *strictly 2-balanced* if for all proper subgraphs $J \subset H$, we have $d_2(J) < m_2(H)$.

Regarding asymmetric Ramsey properties, in [37], Kohayakawa and Kreuter introduced the following generalisation of d_2. Let H_1 and H_2 be any graphs, and define

$$d_2(H_1, H_2) := \begin{cases} \frac{e_1}{v_1 - 2 + \frac{1}{m_2(H_2)}} & \text{if } H_2 \text{ is non-empty and } e_1 \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly to before, we have the following measure based on maximising d_2 over all subgraphs $J \subseteq H_1$:

$$m_2(H_1, H_2) := \max \{ d_2(J, H_2) : J \subseteq H_1 \}.$$

We say that H_1 is *balanced w.r.t.* $d_2(\cdot, H_2)$ if we have $d_2(H_1, H_2) = m_2(H_1, H_2)$ and *strictly balanced w.r.t.* $d_2(\cdot, H_2)$ if for all proper subgraphs $J \subset H_1$ we have $d_2(J, H_2) < m_2(H_1, H_2)$.

Observe that $m_2(\cdot, \cdot)$ is not symmetric in both arguments. The following result will be very useful for us, and illuminates the relationship between the one and two argument m_2 measures. It can be readily proven using elementary arguments.

Proposition 4.3.1. *Suppose that H_1 and H_2 are non-empty graphs with $m_2(H_1) \geq m_2(H_2)$. Then we have*

$$m_2(H_1) \geq m_2(H_1, H_2) \geq m_2(H_2).$$

Moreover,

$$m_2(H_1) > m_2(H_1, H_2) > m_2(H_2) \text{ whenever } m_2(H_1) > m_2(H_2).$$

Proof.

We will only prove here the case when for $i \in \{1, 2\}$ we have

$$m_2(F_i) = \max \left\{ \frac{e_J - 1}{v_J - 2} : J \subseteq F_i \wedge e(J) \geq 1, v(J) \geq 3 \right\},$$

that is, we assume F_1 and F_2 are neither edges nor have zero edges. Assume for a contradiction that $m_2(H_1) < m_2(H_1, H_2)$. Then there exists $F' \subseteq H_1$ with $e(F') \geq 1$ and $v(F') \geq 3$ such that

$$m_2(H_1) < \frac{e(F')}{v(F') - 2 + \frac{1}{m_2(H_2)}}.$$

Since $m_2(H_1) \geq m_2(H_2)$ we have

$$m_2(H_1) < \frac{e(F')}{v(F') - 2 + \frac{1}{m_2(H_1)}}$$

and so

$$m_2(H_1) < \frac{e(F') - 1}{v(F') - 2},$$

contradicting the definition of $m_2(H_1)$.

Now assume for a contradiction that $m_2(H_1, H_2) < m_2(H_2)$. For *every* graph $J \subseteq H_1$ with $e(J) \geq 1$ and $v(J) \geq 3$ we have that

$$\frac{e(J)}{v(J) - 2 + \frac{1}{m_2(H_2)}} < m_2(H_2).$$

Since $m_2(H_1) \geq m_2(H_2)$, for *every* graph $J \subseteq H_1$ with $e(J) \geq 1$ and $v(J) \geq 3$ we have that

$$\frac{e(J) - 1}{v(J) - 2} < m_2(H_1),$$

contradicting the definition of $m_2(H_1)$. Hence $m_2(H_1) \geq m_2(H_1, H_2) \geq m_2(H_2)$, and so $m_2(H_1) = m_2(H_1, H_2) = m_2(H_2)$ whenever $m_2(H_1) = m_2(H_2)$.

It remains to show that $m_2(H_1) > m_2(H_1, H_2) > m_2(H_2)$ whenever $m_2(H_1) > m_2(H_2)$. To do this we will show that if $m_2(H_1) = m_2(H_1, H_2)$ then $m_2(H_1) = m_2(H_2)$, and if $m_2(H_1, H_2) = m_2(H_2)$ then, also, $m_2(H_1) = m_2(H_2)$. So let us firstly assume $m_2(H_1) = m_2(H_1, H_2)$. Then there exists $F' \subseteq H_1$ with $e(F') \geq 1$ and $v(F') \geq 3$ such that

$$m_2(H_1) = \frac{e(F')}{v(F') - 2 + \frac{1}{m_2(H_2)}}.$$

Assume for a contradiction that $m_2(H_1, H_2) > m_2(H_2)$. Then

$$\frac{e(F') - 1}{v(F') - 2} > m_2(H_2).$$

Thus

$$m_2(H_1) < \frac{e(F')}{v(F') - 2 + \frac{v(F')-2}{e(F')-1}} = \frac{e(F') - 1}{v(F') - 2},$$

contradicting the definition of $m_2(H_1)$. Hence $m_2(H_1, H_2) = m_2(H_2)$ and so $m_2(H_1) = m_2(H_2)$. Now assume $m_2(H_1, H_2) = m_2(H_2)$. For every graph $J \subseteq H_1$ with $e(J) \geq 1$ and $v(J) \geq 3$ we have that

$$m_2(H_1, H_2) \geq \frac{e(J)}{v(J) - 2 + \frac{1}{m_2(H_2)}}.$$

Thus for every graph $J \subseteq H_1$ with $e(J) \geq 1$ and $v(J) \geq 3$ we have that

$$m_2(H_2) \geq \frac{e(J) - 1}{v(J) - 2}.$$

Moreover, there exists a graph $F' \subseteq H_1$ with $e(F') \geq 1$ and $v(F') \geq 3$ such that

$$m_2(H_2) = m_2(H_1, H_2) = \frac{e(F')}{v(F') - 2 + \frac{1}{m_2(H_2)}}$$

which implies

$$m_2(H_2) = \frac{e(F') - 1}{v(F') - 2}.$$

Thus

$$m_2(H_2) = \max\left\{\frac{e(J) - 1}{v(J) - 2} : J \subseteq H_1 \wedge e(J) \geq 1, v(J) \geq 3\right\} = m_2(H_1).$$

$\square$

Note that if $m_2(H_1) = m_2(H_2)$ and H_1 and H_2 are non-empty graphs, then H_1 cannot be strictly balanced w.r.t. $d_2(\cdot, H_2)$ unless $H_1 \cong K_2$. Indeed, otherwise, by Proposition 4.3.1 we would then have that

$$m_2(H_2) = m_2(H_1, H_2) > d_2(K_2, H_2) = m_2(H_2).$$

The following fact will be useful in the proofs of Lemmas 4.3.3 and 4.5.8.

Fact 4.3.2. *For $a, c, C \in \mathbb{R}$ and $b, d > 0$, we have*

$$(i)\ \ \frac{a}{b} \leq C \ \wedge\ \frac{c}{d} \leq C \implies \frac{a+c}{b+d} \leq C \ \ \textit{and}\ \ (ii)\ \ \frac{a}{b} \geq C \ \wedge\ \frac{c}{d} \geq C \implies \frac{a+c}{b+d} \geq C$$

and similarly, if also $b > d$,

$$(iii)\ \ \frac{a}{b} \leq C \ \wedge\ \frac{c}{d} \geq C \implies \frac{a-c}{b-d} \leq C \ \ \textit{and}\ \ (iv)\ \ \frac{a}{b} \geq C \ \wedge\ \frac{c}{d} \leq C \implies \frac{a-c}{b-d} \geq C.$$

The following result will be very useful for us, creating an important connection between types of balancedness and 2-connectivity.

Lemma 4.3.3. *Let H_1 and H_2 be graphs such that $m_2(H_1) > m_2(H_2) > 1$, H_2 is strictly 2-balanced and H_1 is strictly balanced w.r.t. $d_2(\cdot, H_2)$. Then H_1 and H_2 are both 2-connected.*

Proof. By [54, Lemma 3.3], H_2 is 2-connected. We now use a very similar method to that of the proof of Lemma 3.3 to show that H_1 is 2-connected. Since H_1 is strictly balanced w.r.t. $d_2(\cdot, H_2)$, we have that H_1 is connected. Indeed, assume not. Let H_1 have $k \geq 2$ components and denote the number of vertices and edges in each component by $u_1, \ldots, u_k$ and $d_1, \ldots, d_k$, respectively. Then since H_1 is strictly balanced w.r.t. $m_2(\cdot, H_2)$, we must have that

$$\frac{\sum_{i=1}^k d_i}{\sum_{i=1}^k u_i - 2 + \frac{1}{m_2(H_2)}} > \frac{d_1}{u_1 - 2 + \frac{1}{m_2(H_2)}}$$

and

$$\frac{\sum_{i=1}^k d_i}{\sum_{i=1}^k u_i - 2 + \frac{1}{m_2(H_2)}} > \frac{\sum_{i=2}^k d_i}{\sum_{i=2}^k u_i - 2 + \frac{1}{m_2(H_2)}}.$$

Since $m_2(H_2) > 1$, by Fact 4.3.2(i) we get that

$$\frac{\sum_{i=1}^k d_i}{\sum_{i=1}^k u_i - 2 + \frac{1}{m_2(H_2)}} \geq \frac{\sum_{i=1}^k d_i}{\sum_{i=1}^k u_i - 4 + \frac{2}{m_2(H_2)}} > \frac{\sum_{i=1}^k d_i}{\sum_{i=1}^k u_i - 2 + \frac{1}{m_2(H_2)}},$$

a contradiction.

Assume H_1 is not 2-connected. Then there exists a cut[3] vertex $v \in V(H_1)$. Further, using Proposition 4.3.1 alongside that H_1 is strictly balanced w.r.t. $d_2(\cdot, H_2)$ and $m_2(H_1) > m_2(H_2) > 1$, we can show that H_1 does not contain any vertex of degree 1. Indeed, otherwise $\frac{e_1-1}{v_1-3+\frac{1}{m_2(H_2)}} > \frac{e_1}{v_1-2+\frac{1}{m_2(H_2)}} = m_2(H_1, H_2)$, contradicting that H_1 is strictly balanced w.r.t. $d_2(\cdot, H_2)$. Thus there exist subgraphs J_1 and J_2 of H_1 such that $|E(J_1)|, |E(J_2)| \geq 1$, $J_1 \cup J_2 = H_1$ and $V(J_1) \cap V(J_2) = \{v\}$. Using Fact 4.3.2(i) and that H_1 is strictly balanced w.r.t. $d_2(\cdot, H_2)$, we have that

$$\begin{aligned} e_1 = e_{J_1} + e_{J_2} &< m_2(H_1, H_2)\left(v_{J_1} - 2 + \frac{1}{m_2(H_2)} + v_{J_2} - 2 + \frac{1}{m_2(H_2)}\right) \\ &= m_2(H_1, H_2)\left(v_1 - 3 + \frac{2}{m_2(H_2)}\right). \end{aligned}$$

However, since $m_2(H_2) > 1$ we also have that

$$\frac{e_1}{v_1 - 3 + \frac{2}{m_2(H_2)}} > \frac{e_1}{v_1 - 2 + \frac{1}{m_2(H_2)}} = m_2(H_1, H_2),$$

contradicting the inequality above. Hence H_1 is 2-connected. $\square$

[3]That is, removing v and its incident edges from H_1 produces a disconnected graph.

4.4 Algorithm for computing valid colourings: ASYM-EDGE-COL

To prove Theorem 4.0.5, we can clearly assume H_1 and H_2 are non-empty graphs satisfying the criteria of Conjecture 4.0.4 and that Conjecture 4.0.4 itself holds. Suppose $G = G_{n,p}$ and $p \leq bn^{-1/m_2(H_1,H_2)}$ where b will be a small constant defined later. As noted earlier, to prove Conjecture 1.2.3 we can show that a.a.s. G has a valid colouring for H_1 and H_2. We construct our valid colouring using an algorithm ASYM-EDGE-COL (see Figure 4.1). In order to state the algorithm succinctly, we need to define a considerable amount of notation, almost all of which we keep very similar to that in [51].

Recall Definitions 4.0.2 and 4.0.3. That is, for any graph G we have the families

$$\mathcal{R}_G := \{R \subseteq G : R \cong H_1\} \text{ and } \mathcal{L}_G := \{L \subseteq G : L \cong H_2\}$$

of all copies of H_1 and H_2 in G, respectively. Also,

$$\mathcal{L}^*_G := \{L \in \mathcal{L}_G : \forall e \in E(L)\ \exists R \in \mathcal{R}_G \text{ s.t. } E(L) \cap E(R) = \{e\}\}.$$

We highlight here that if $E(L) \cap E(R) = \{e\}$ for some $L \in \mathcal{L}_G$ and $R \in \mathcal{R}_G$ then it is still possible that $|V(L) \cap V(R)| > 2$.

Intuitively, the graphs in $\hat{\mathcal{A}}$ are the building blocks of the graphs $\hat{G}$ which may remain after the edge deletion process in ASYM-EDGE-COL (described later).

For any graph G, define

$$\mathcal{S}_G := \{S \subseteq G : S \cong A \in \hat{\mathcal{A}} \wedge \nexists S' \supset S \text{ with } S' \subseteq G,\ S' \cong A' \in \hat{\mathcal{A}}\},$$

that is, the family $\mathcal{S}_G$ contains all maximal subgraphs of G isomorphic to a member of $\hat{\mathcal{A}}$. Hence, there are no two members $S_1, S_2 \in \mathcal{S}_G$ such that $S_1 \subset S_2$. For any edge $e \in E(G)$, let

$$\mathcal{S}_G(e) := \{S \in \mathcal{S}_G : e \in E(S)\}.$$

We call G an $\hat{\mathcal{A}}$-*graph* if, for all $e \in E(G)$, we have

$$|\mathcal{S}_G(e)| = 1.$$

In particular, an $\hat{\mathcal{A}}$-graph is an edge-disjoint union of graphs from $\hat{\mathcal{A}}$. In an $\hat{\mathcal{A}}$-graph G, a copy of H_1 or H_2 can be a subgraph of G in two particular ways: either it is a subgraph of an $S \in \mathcal{S}_G$ or it is a subgraph with edges in at least two different graphs from $\mathcal{S}_G$. The former we call *trivial* copies of H_1 and H_2, and we define

$$\mathcal{T}_G := \left\{ T \subseteq G : (T \cong H_1 \vee T \cong H_2) \wedge \left| \bigcup_{e \in E(T)} \mathcal{S}_G(e) \right| \geq 2 \right\}$$

to be the family of all *non-trivial* copies of H_1 and H_2 in G. We say that an $\hat{\mathcal{A}}$-graph G is (H_1, H_2)-*sparse* if $\mathcal{T}_G = \emptyset$. Our next lemma asserts that (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graphs are easily colourable, provided Conjecture 4.0.4 holds.

Lemma 4.4.1. *There exists a procedure* A-COLOUR *that returns for any* (H_1, H_2)*-sparse* $\hat{\mathcal{A}}$*-graph* G *a valid colouring for* H_1 *and* H_2*.*

Proof. By Conjecture 4.0.4, there exists a valid colouring for H_1 and H_2 of every $A \in \hat{\mathcal{A}}$. Using this we define a procedure A-COLOUR(G) as follows: Assign a valid colouring for H_1 and H_2 to every subgraph $S \in \mathcal{S}_G$ locally, that is, regardless of the structure of G. Since G is an (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph, we assign a colour to each edge of G without producing a red copy of H_1 or a blue copy of H_2, and the resulting colouring is a valid colouring for H_1 and

H_2 of G. □

Note that we did not use that $\hat{\mathcal{A}}$ is finite, as given by Conjecture 4.0.4, in our proof of Lemma 4.4.1, only that 'every graph in $\hat{\mathcal{A}}$ has a valid colouring for H_1 and H_2'. The finiteness of $\hat{\mathcal{A}}$ will be essential later for the proofs of Claims 4.5.6 and 4.6.6.

Now let us describe the algorithm Asym-Edge-Col which if successful outputs a valid colouring of G. In Asym-Edge-Col, edges are removed from and then inserted back into a working copy $G' = (V, E')$ of G. Each edge is removed in the first while-loop only when it is not the unique intersection of the edge sets of some copy of H_1 and some copy of H_2 in G' (line 6). It is then 'pushed[4]' onto a stack s such that when we reinsert edges (in reverse order) in the second while-loop we can colour them to construct a valid colouring for H_1 and H_2 of G; if at any point G' is an (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph, then we combine the colouring of these edges with a valid colouring for H_1 and H_2 of G' provided by A-Colour. We also keep track of the copies of H_2 in G and push abstract representations of some of them (or all of them if G' is never an (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph during Asym-Edge-Col) onto s (lines 8 and 15) to be used later in the colour swapping stage of the second while-loop (lines 30-32).

Let us consider algorithm Asym-Edge-Col in detail. In line 5, we check whether G' is an (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph or not. If not, then we enter the first while-loop. In line 6, we choose an edge e which is not the unique intersection of the edge sets of some copy of H_1 and some copy of H_2 in G' (if such an edge e exists). Then in lines 7-12 we push each copy of H_2 in G' that contains e onto s before pushing e onto s as well. Now, if every edge $e \in E'$ is the unique intersection of the edge sets of some copy of H_1 and some copy of H_2 in G', then we push onto s a copy L of H_2 in G' which contains an edge that is not the unique intersection of the edge set of L and the edge set of some copy of H_1 in G'. If no such

[4]For clarity, by 'push' we mean that the object is placed on the top of the stack s.

```
procedure ASYM-EDGE-COL($G = (V, E)$)
    $s \leftarrow$ EMPTY-STACK()
    $E' \leftarrow E$
    $\mathcal{L} \leftarrow \mathcal{L}_G$
    while $G' = (V, E')$ is no $(H_1, H_2)$-sparse $\hat{\mathcal{A}}$-graph do
        if $\exists e \in E'$ s.t. $\nexists (L, R) \in \mathcal{L} \times \mathcal{R}_{G'} : E(L) \cap E(R) = \{e\}$ then
            for all $L \in \mathcal{L} : e \in E(L)$ do
                $s$.PUSH($L$)
                $\mathcal{L}$.REMOVE($L$)
            end for
            $s$.PUSH($e$)
            $E'$.REMOVE($e$)
        else
            if $\exists L \in \mathcal{L} \setminus \mathcal{L}^*_{G'}$ then
                $s$.PUSH($L$)
                $\mathcal{L}$.REMOVE($L$)
            else
                error "stuck"
            end if
        end if
    end while
    A-COLOUR($G' = (V, E')$)
    while $s \neq \emptyset$ do
        if $s$.TOP() is an edge then
            $e \leftarrow s$.POP()
            $E'$.ADD($e$)
            $e$.SET-COLOUR(blue)
        else
            $L \leftarrow s$.POP()
            if $L$ is entirely blue then
                $f \leftarrow$ any $e \in E(L)$ s.t. $\nexists R \in \mathcal{R}_{G'} : E(L) \cap E(R) = \{e\}$
                $f$.SET-COLOUR(red)
            end if
        end if
    end while
end procedure
```

Figure 4.1: The implementation of algorithm ASYM-EDGE-COL.

copies L of H_2 exist, then the algorithm has an error in line 18. If Asym-Edge-Col does not run into an error, then we enter the second while-loop with input G'. Observe that G' is either the empty graph on vertex set V or some (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph. By Lemma 4.4.1, G' has a valid colouring for H_1 and H_2. The second while-loop successively removes edges (line 25) and copies of L (line 29) from s in the reverse order in which they were added onto s, with the edges added back into E'. Each time an edge is added back it is coloured blue, and if a monochromatic blue copy L of H_2 is constructed, we make one of the edges of L red (lines 30-32). This colouring process is then repeated until we have a valid colouring for H_1 and H_2 of G.

The following lemma confirms that our colouring process in the second while-loop produces a valid colouring for H_1 and H_2 of G.

Lemma 4.4.2. *Algorithm* Asym-Edge-Col *either terminates with an error in line 18 or finds a valid colouring for H_1 and H_2 of G.*

Proof. Our proof is almost identical to the proof of [51, Lemma 11]. We include it here for completeness.

Let G^* denote the argument in the call to A-Colour in line 22. By Lemma 4.4.1, there is a valid colouring for H_1 and H_2 of G^*. It remains to show that no forbidden monochromatic copies of H_1 or H_2 are created when this colouring is extended to a colouring of G in lines 23-35.

Firstly, we argue that the algorithm never creates a blue copy of H_2. Observe that *every* copy of H_2 that does not lie entirely in G^* is pushed on the stack in the first while-loop (lines 5-21). Therefore, in the execution of the second loop, the algorithm checks the colouring of every such copy. By the order of the elements on the stack, each such test is performed only after all edges of the corresponding copy of H_2 were inserted and coloured.

For every blue copy of H_2, one particular edge f (see line 31) is recoloured to red. Since red edges are never flipped back to blue, no blue copy of H_2 can occur.

We need to show that the edge f in line 31 always exists. Since the second loop inserts edges into G' in the reverse order in which they were deleted during the first loop, when we select f in line 31, G' has the same structure as at the time when L was pushed on the stack. This happened either in line 8 when there exists no copy of H_1 in G' whose edge set intersects with L on some particular edge $e \in E(L)$, or in line 15 when L is not in $\mathcal{L}^*_{G'}$ due to the if-clause in line 14. In both cases we have $L \notin \mathcal{L}^*_{G'}$, and hence there exists an edge $e \in E(L)$ such that the edge sets of all copies of H_1 in G' do not intersect with L exactly in e.

It remains to prove that changing the colour of some edges from blue to red by the algorithm never creates an entirely red copy of H_1. By the condition on f in line 31 of the algorithm, at the moment f is recoloured there exists no copy of H_1 in G' whose edge set intersects L exactly in f. So there is either no copy of H_1 containing f at all, or every such copy contains also another edge from L. In the latter case, those copies cannot become entirely red since L is entirely blue. $\square$

To prove Theorem 4.0.5, it now suffices to prove the following lemma.

Lemma 4.4.3. *There exists a constant $b = b(H_1, H_2) > 0$ such that for $p \leq bn^{-1/m_2(H_1,H_2)}$ algorithm* Asym-Edge-Col *terminates on $G_{n,p}$ without error a.a.s.*

We split our proof of Lemma 4.4.3 into two cases: (1) when $m_2(H_1) > m_2(H_2)$ and (2) when $m_2(H_1) = m_2(H_2)$. Notice that this accords with our definition of $\hat{\mathcal{A}}$.

4.5 Case 1: $m_2(H_1) > m_2(H_2)$.

We will prove Case 1 of Lemma 4.4.3 using an auxiliary algorithm Grow (see Figure 4.2). If Asym-Edge-Col has an error, then Grow computes a subgraph $F \subseteq G$ which is either too large in size or too dense to appear in $G_{n,p}$ a.a.s. (with p as in Lemma 4.4.3). Indeed, letting $\mathcal{F}$ be the class of all graphs that can possibly be returned by Grow, we will show that the expected number of copies of graphs from $\mathcal{F}$ contained in $G_{n,p}$ is $o(1)$, which with Markov's inequality implies that $G_{n,p}$ a.a.s. contains no graph from $\mathcal{F}$. This in turn implies Lemma 4.4.3 by contradiction. Note that algorithm Grow is only used for proving Lemma 4.4.3 and hence does not add anything on to the run-time of Asym-Edge-Col.

To state Grow we require the following definitions. Let

$$\gamma = \gamma(H_1, H_2) := \frac{1}{m_2(H_1, H_2)} - \frac{1}{m_2(H_1, H_2) + \varepsilon(H_1, H_2)} > 0,$$

where $\varepsilon(H_1, H_2)$ is the constant in Conjecture 4.0.4. For any graph F, let

$$\lambda(F) := v(F) - \frac{e(F)}{m_2(H_1, H_2)}.$$

The definition of $\lambda(F)$ is motivated by the fact that the expected number of copies of F in $G_{n,p}$ with $p = bn^{-1/m_2(H_1,H_2)}$ has order of magnitude

$$n^{v(F)} p^{e(F)} = b^{e(F)} n^{\lambda(F)}.$$

For any graph F and edge $e \in E(F)$, we say that e is *eligible for extension in* Grow if it satisfies

$$\nexists L \in \mathcal{L}_F^* \text{ s.t. } e \in E(L),$$

and observe that F is in $\mathcal{C}^*$ (see Definition 4.0.2) if and only if it contains no edge that is eligible for extension in Grow.

Algorithm Grow has as input the graph $G' \subseteq G$ that Asym-Edge-Col got stuck on. Let us consider the properties of G' when Asym-Edge-Col got stuck. Because the condition in line 6 of Asym-Edge-Col fails, G' is in the family $\mathcal{C}$, where we recall

$$\mathcal{C} = \mathcal{C}(H_1, H_2) := \{G = (V, E) : \forall e \in E\ \exists (L, R) \in \mathcal{L}_G \times \mathcal{R}_G \text{ s.t. } E(L) \cap E(R) = \{e\}\}.$$

In particular, every edge of G' is contained in a copy $L \in \mathcal{L}_{G'}$ of H_2, and, because the condition in line 14 fails, we can assume in addition that L belongs to $\mathcal{L}^*_{G'}$. Hence, G' is actually in the family $\mathcal{C}^* = \mathcal{C}^*(H_1, H_2)$ where we recall

$$\mathcal{C}^* = \mathcal{C}^*(H_1, H_2) := \{G = (V, E) : \forall e \in E\ \exists L \in \mathcal{L}^*_G \text{ s.t. } e \in E(L)\}.$$

Lastly, G' is not an (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph because Asym-Edge-Col ended with an error.

We now outline algorithm Grow. Firstly, Grow checks whether either of two special cases occur (lines 2-9). If neither occurs, it chooses a suitable graph $R \in \mathcal{R}_{G'}$ (line 11) and makes it the seed F_0 for a growing procedure. In each iteration i of the while-loop, the growing procedure extends F_i to F_{i+1} in one of two ways. The first (lines 14-15) is by attaching a copy of H_1 in G' that intersects F_i in at least two vertices but is not contained in F_i. The second is more involved and begins with calling a function Eligible-Edge which maps F_i to an edge $e \in E(F_i)$ which is eligible for extension in Grow (we will show that such an edge always exists). Importantly, Eligible-Edge selects this edge e to be *unique up to isomorphism of* F_i, that is, for any two isomorphic graphs F and F', there exists an isomorphism ϕ with $\phi(F) = F'$ such that

$$\phi(\text{Eligible-Edge}(F)) = \text{Eligible-Edge}(F').$$

In particular, our choice of e depends only on F_i and not on the surrounding graph G' or any previous graph F_j with $j < i$ (indeed, there may be many ways that Grow could construct a graph isomorphic to F_i). One could implement Eligible-Edge by having an enormous table of representatives for all isomorphism classes of graphs with up to n vertices. What is important is that Eligible-Edge does not itself increase the number of graphs F that Grow can output.

Once we have our edge $e \in E(F_i)$ eligible for extension in Grow, we apply a procedure Extend-L which attaches a graph $L \in \mathcal{L}^*_{G'}$ that contains e to F_i (line 18). We then attach to each new edge $e' \in E(L) \setminus E(F_i)$ a graph $R_{e'} \in \mathcal{R}_{G'}$ such that $E(L) \cap E(R_{e'}) = \{e'\}$ (lines 4-6 of Extend-L). (We will show later that such a graph L and graphs $R_{e'}$ exist and that $E(L) \setminus E(F_i)$ is non-empty.) The algorithm comes to an end when either $i \geq \log(n)$ or $\lambda(\tilde{F}) \leq -\gamma$ for some subgraph $\tilde{F} \subseteq F_i$. In the former, the algorithm returns F_i (line 23); in the latter, the algorithm returns a subgraph $\tilde{F} \subseteq F_i$ that minimises $\lambda(\tilde{F})$ (line 25). For each graph F, the function Minimising-Subgraph(F) returns such a minimising subgraph that is *unique up to isomorphism*. Once again, this is to ensure that Minimising-Subgraph(F) does not itself artificially increase the number of graphs that Grow can output. As with function Eligible-Edge, one could implement Minimising-Subgraph using an enormous look-up table.

We will now argue that Grow terminates without error, that is, Eligible-Edge always finds an edge eligible for extension in Grow and all 'any'-assignments in Grow and Extend-L are always successful. In order to argue such, recall the properties of G' when Asym-Edge-Col got stuck: $G' \in \mathcal{C}^*(H_1, H_2) = \{G = (V, E) : \forall e \in E\ \exists L \in \mathcal{L}^*_G \text{ s.t. } e \in E(L)\}$ and G' is not an (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph.

Claim 4.5.1. *Algorithm* Grow *terminates without error on any input graph $G' \in \mathcal{C}^*$ that is no (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph. Moreover, for every iteration i of the while-loop, we have $e(F_{i+1}) > e(F_i)$.*

procedure GROW($G' = (V, E)$)
  if $\forall e \in E : |\mathcal{S}_{G'}(e)| = 1$ **then**
    $T \leftarrow$ any member of $\mathcal{T}_{G'}$
    return $\bigcup_{e \in E(T)} \mathcal{S}_{G'}(e)$
  end if
  if $\exists e \in E : |\mathcal{S}_{G'}(e)| \geq 2$ **then**
    $S_1, S_2 \leftarrow$ any two distinct members of $\mathcal{S}_{G'}(e)$
    return $S_1 \cup S_2$
  end if
  $e \leftarrow$ any $e \in E : |\mathcal{S}_{G'}(e)| = 0$
  $F_0 \leftarrow$ any $R \in \mathcal{R}_{G'} : e \in E(R)$
  $i \leftarrow 0$
  while $(i < \ln(n)) \wedge (\forall \tilde{F} \subseteq F_i : \lambda(\tilde{F}) > -\gamma)$ **do**
    if $\exists R \in \mathcal{R}_{G'} \setminus \mathcal{R}_{F_i} : |V(R) \cap V(F_i)| \geq 2$ **then**
      $F_{i+1} \leftarrow F_i \cup R$
    else
      $e \leftarrow$ ELIGIBLE-EDGE(F_i)
      $F_{i+1} \leftarrow$ EXTEND-L(F_i, e, G')
    end if
    $i \leftarrow i + 1$
  end while
  if $i \geq \ln(n)$ **then**
    return F_i
  else
    return MINIMISING-SUBGRAPH(F_i)
  end if
end procedure

procedure EXTEND-L(F, e, G')
  $L \leftarrow$ any $L \in \mathcal{L}^*_{G'}$: $e \in E(L)$
  $F' \leftarrow F \cup L$
  for all $e' \in E(L) \setminus E(F)$ **do**
    $R_{e'} \leftarrow$ any $R \in \mathcal{R}_{G'} : E(L) \cap E(R) = \{e'\}$
    $F' \leftarrow F' \cup R_{e'}$
  end for
  return F'
end procedure

Figure 4.2: The implementation of algorithm GROW.

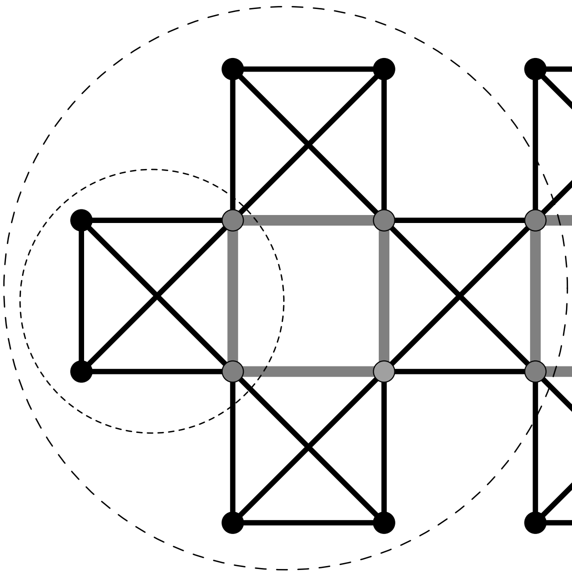

Figure 4.3: A graph F_2 resulting from two non-degenerate iterations for $H_1 = K_4$ and $H_2 = C_4$. The two central copies of H_2 are shaded.

Proof. Our proof is very similar to the proof of [51, Claim 13].

We first show that the special cases in lines 2-9 always function as desired. The first case occurs if and only if G' is an $\hat{\mathcal{A}}$-graph. By assumption, G' is not (H_1, H_2)-sparse, hence the family $\mathcal{T}_{G'}$ is not empty. Hence the assignment in line 3 is successful. Clearly, the assignment in line 7 is always successful due to the if-condition in line 6.

One can also easily see that the assignments in lines 10 and 11 are successful. Indeed, neither of the two special cases occur so we must have an edge $e \in E$ that is not contained in any $S \in \mathcal{S}_{G'}$. Also, there must exist a member of $\mathcal{R}_{G'}$ that contains e because G' is a member of $\mathcal{C}^* \subseteq \mathcal{C}$.

Next, we show that the call to Eligible-Edge in line 17 is always successful. Indeed, suppose for a contradiction that no edge in F_i is eligible for extension in Grow for some $i \geq 0$. Then every edge $e \in E(F_i)$ is in some $L \in \mathcal{L}^*_{F_i}$, by definition. Hence $F \in \mathcal{C}^*$. Recall that H_1 and H_2 satisfy the criteria of Conjecture 4.0.4. Hence H_2 is strictly 2-balanced, H_1

is strictly balanced w.r.t. $d_2(\cdot, H_2)$) and $m_2(H_1) \geq m_2(H_2) > 1$. Then, by Lemma 4.3.3, H_1 and H_2 are 2-connected, hence F_i is 2-connected by construction. However, our choice of F_0 in line 11 guarantees that F_i is not in $\hat{\mathcal{A}}$. Indeed, the edge e selected in line 10 satisfying $|\mathcal{S}_{G'}(e)| = 0$ is an edge of F_0 and $F_0 \subseteq F_i \subseteq G'$. Thus, by the definition of $\hat{\mathcal{A}}$ and that $m_2(H_1) > m_2(H_2)$, we have that $m(F_i) > m_2(H_1, H_2) + \varepsilon$. Thus, there exists a non-empty graph $\tilde{F} \subseteq F_i$ with $d(\tilde{F}) = m(F_i)$ such that

$$\begin{aligned}
\lambda(\tilde{F}) &= v(\tilde{F}) - \frac{e(\tilde{F})}{m_2(H_1, H_2)} \\
&= e(\tilde{F}) \left(\frac{1}{m(F_i)} - \frac{1}{m_2(H_1, H_2)} \right) \\
&< e(\tilde{F}) \left(\frac{1}{m_2(H_1, H_2) + \varepsilon} - \frac{1}{m_2(H_1, H_2)} \right) \\
&= -\gamma e(\tilde{F}) \leq -\gamma.
\end{aligned}$$

Thus Grow terminates in line 13 without calling Eligible-Edge, and so every call to Eligible-Edge is successful and returns an edge e. Since $G' \in \mathcal{C}^*$, the call to Extend-L(F_i, e, G') is also successful and thus there exist suitable graphs $L \in \mathcal{L}^*_{G'}$ with $e \in E(L)$ and $R_{e'}$ for each $e' \in E(L) \setminus E(F_i)$.

It remains to show that for every iteration i of the while-loop, we have $e(F_{i+1}) > e(F_i)$. Since a copy R of H_1 found in line 14 is a copy of H_1 in G' but not in F_i (and H_1 is connected), we must have that $F_{i+1} = F_i \cup R$ contains at least one more edge than F_i.

So assume lines 17 and 18 are called in iteration i and let e be the edge chosen in line 17 and L the subgraph selected in line 2 of Extend-L(F_i, e, G'). By the definition of $\mathcal{L}^*_{G'}$, for each $e' \in E(L)$ there exists $R_{e'} \in \mathcal{R}_{G'}$ such that $E(L) \cap E(R_{e'}) = e'$. If $|E(L) \setminus E(F_i)| > 0$, then $e(F_{i+1}) \geq e(F_i \cup L) > e(F_i)$. Otherwise, $L \subseteq F_i$. But since e is eligible for extension in Grow, we must have $L \notin \mathcal{L}^*_{F_i}$. Thus there exists $e' \in L$ such that $R_{e'} \in \mathcal{R}_{G'} \setminus \mathcal{R}_{F_i}$ and $|V(R_{e'}) \cap V(F_i)| \geq 2$, contradicting that lines 17 and 18 are called in iteration i. □

4.5.1 Proof of Lemma 4.4.3

We consider the evolution of F_i now in more detail. We call iteration i of the while-loop in algorithm Grow *non-degenerate* if all of the following hold:

- The condition in line 14 evaluates to false (and Extend-L is called);
- In line 3 of Extend-L, we have $V(F) \cap V(L) = e$;
- In every execution of line 6 of Extend-L, we have $V(F') \cap V(R_{e'}) = e'$.

Otherwise, we call iteration i *degenerate*. Note that, in non-degenerate iterations, there are only a *constant* number of graphs F_{i+1} that can result from any given F_i since Eligible-Edge determines the exact position where to attach the copy L of H_2, $V(F_i) \cap V(L) = e$ and for every execution of line 6 of Extend-L we have $V(F') \cap V(R_{e'}) = e'$ (recall that the edge e found by Eligible-Edge(F_i) is *unique up to isomorphism of F_i*).

Claim 4.5.2. *If iteration i of the while-loop in procedure* Grow *is non-degenerate, we have*

$$\lambda(F_{i+1}) = \lambda(F_i).$$

Proof. In a non-degenerate iteration we add $v_2 - 2$ vertices and $e_2 - 1$ edges for the copy of H_2 and then $(e_2 - 1)(v_1 - 2)$ new vertices and $(e_2 - 1)(e_1 - 1)$ new edges to complete the copies of H_1. This gives

$$\begin{aligned}\lambda(F_{i+1}) - \lambda(F_i) &= v_2 - 2 + (e_2 - 1)(v_1 - 2) - \frac{(e_2 - 1)e_1}{m_2(H_1, H_2)} \\ &= v_2 - 2 + (e_2 - 1)(v_1 - 2) - (e_2 - 1)\left(v_1 - 2 + \frac{1}{m_2(H_2)}\right) \\ &= 0,\end{aligned}$$

where we have used in the penultimate equality that H_1 is (strictly) balanced w.r.t. $d_2(\cdot, H_2)$ and in the final inequality that H_2 is (strictly) 2-balanced. □

When we have a degenerate iteration i, the structure of F_{i+1} may vary considerably and also depend on the structure of G'. Indeed, if F_i is extended by a copy R of H_1 in line 15, then R could intersect F_i in a multitude of ways. Moreover, there may be several copies of H_1 that satisfy the condition in line 14. The same is true for graphs added in lines 3 and 6 of Extend-L. Thus, degenerate iterations cause us difficulties since they enlarge the family of graphs algorithm Grow can return. However, we will show that at most a constant number of degenerate iterations can happen before algorithm Grow terminates, allowing us to bound from above sufficiently well the number of non-isomorphic graphs Grow can return. Pivotal in proving this is the following claim.

Claim 4.5.3. *There exists a constant $\kappa = \kappa(H_1, H_2) > 0$ such that if iteration i of the while-loop in procedure* Grow *is degenerate then we have*

$$\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa.$$

We prove Claim 4.5.3 in Section 4.5.2. Together, Claims 4.5.2 and 4.5.3 yield the following claim.

Claim 4.5.4. *There exists a constant $q_1 = q_1(H_1, H_2)$ such that algorithm* Grow *performs at most q_1 degenerate iterations before it terminates, regardless of the input instance G'.*

Proof. By Claim 4.5.2, the value of the function λ remains the same in every non-degenerate iteration of the while-loop of algorithm Grow. However, Claim 4.5.3 yields a constant κ, which depends solely on H_1 and H_2, such that

$$\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa$$

for every degenerate iteration i.

Hence, after at most

$$q_1 := \frac{\lambda(F_0) + \gamma}{\kappa}$$

degenerate iterations, we have $\lambda(F_i) \leq -\gamma$, and algorithm Grow terminates. $\square$

For $0 \leq d \leq t < \lceil \ln(n) \rceil$, let $\mathcal{F}(t, d)$ denote a family of representatives for the isomorphism classes of all graphs F_t that algorithm Grow can possibly generate after exactly t iterations of the while-loop with exactly d of those t iterations being degenerate. Let $f(t, d) := |\mathcal{F}(t, d)|$.

Claim 4.5.5. *There exist constants $C_0 = C_0(H_1, H_2)$ and $A = A(H_1, H_2)$ such that*

$$f(t, d) \ \leq \lceil \ln(n) \rceil^{(C_0+1)d} \cdot \ A^{t-d}$$

for n sufficiently large.

Proof. By Claim 4.5.1, in every iteration i of the while-loop of Grow, we add new edges onto F_i. These new edges span a graph on at most

$$K := v_2 + (e_2 - 1)(v_1 - 2)$$

vertices. Thus $v(F_t) \leq v_1 + Kt$. Let $\mathcal{G}_K$ denote the set of all graphs on at most K vertices. In iteration i of the while-loop, F_{i+1} is uniquely defined if one specifies the graph $G \in \mathcal{G}_K$ with edges $E(F_{i+1}) \setminus E(F_i)$, the number y of vertices in which G intersects F_i, and two ordered lists of vertices from G and F_i respectively of length y, which specify the mapping of the intersection vertices from G onto F_i. Thus, the number of ways that F_i can be extended to

F_{i+1} is bounded from above by

$$\sum_{G \in \mathcal{G}_K} \sum_{y=2}^{v(G)} v(G)^y v(F_i)^y \leq |\mathcal{G}_K| \cdot K \cdot K^K (v_1 + Kt)^K \leq \lceil \ln(n) \rceil^{C_0}$$

where C_0 depends only on v_1, v_2 and e_2, and n is sufficiently large. The last inequality follows from the fact that $t < \ln(n)$ as otherwise the while-loop would have already ended.

Recall that, since Eligible-Edge determines the exact position where to attach the copy of H_2, in non-degenerate iterations i there are at most

$$2e_2(2e_1)^{e_2-1} =: A$$

ways to extend F_i to F_{i+1}, where the coefficients of 2 correspond with the orientations of the edge of the copy of H_2 we attach to F_i and the edges of the copies of H_1 we attach to said copy of H_2. Hence, for $0 \leq d \leq t < \lceil \ln(n) \rceil$,

$$f(t,d) \leq \binom{t}{d} (\lceil \ln(n) \rceil^{C_0})^d \cdot A^{t-d} \leq \lceil \ln(n) \rceil^{(C_0+1)d} \cdot A^{t-d},$$

where the binomial coefficient corresponds to the choice of when in the t iterations the d degenerate iterations happen. □

A reader of [51] may observe that Claim 4.5.5 is not analogous to [51, Claim 17]. Indeed, since we have a constant number of non-degenerate iterations, instead of a unique non-degenerate iteration as in [51], we truncated the proof of Claim 17 in order to have the appropriate bound to prove the following claim. Let $\mathcal{F} = \mathcal{F}(H_1, H_2, n)$ be a family of representatives for the isomorphism classes of *all* graphs that can be outputted by Grow (whether Grow enters the while-loop or not). Note that the proof of the following claim requires Conjecture 4.0.4 to be true; in particular, we need that $\hat{\mathcal{A}}(H_1, H_2, \varepsilon)$ is finite when $m_2(H_1) > m_2(H_2)$.

Claim 4.5.6. *There exists a constant $b = b(H_1, H_2) > 0$ such that for all $p \leq bn^{-1/m_2(H_1,H_2)}$, $G_{n,p}$ does not contain any graph from $\mathcal{F}(H_1, H_2, n)$ a.a.s.*

Proof. We first consider the two special cases in lines 2-9 of Grow. Let $\mathcal{F}_0 = \mathcal{F}_0(H_1, H_2) \subseteq \mathcal{F}$ denote the class of graphs that can be outputted by Grow if one of these two cases happens. We can see that any $F \in \mathcal{F}_0$ is either of the form

$$F = \bigcup_{e \in E(T)} \mathcal{S}_{G'}(e)$$

for some graph $T \in \mathcal{T}_{G'}$, or of the form

$$F = S_1 \cup S_2$$

for some edge-intersecting $S_1, S_2 \in \mathcal{S}_{G'}$. Whichever of these forms F has, since every element of $\mathcal{S}_{G'}$ is 2-connected and in $\mathcal{C}^*$, and T is 2-connected[5], we have that F is 2-connected and in $\mathcal{C}^*$. On the other hand, $F \subseteq G'$ is not in $\mathcal{S}_{G'}$ and thus not isomorphic to a graph in $\hat{\mathcal{A}}$. Indeed, otherwise the graphs S forming F would not be in $\mathcal{S}_{G'}$ due to the maximality condition in the definition of $\mathcal{S}_{G'}$. It follows that $m(F) > m_2(H_1, H_2) + \varepsilon(H_1, H_2)$. Since we assumed Conjecture 4.0.4 holds, the family $\mathcal{F}_0$ is finite. Hence Markov's inequality yields that $G_{n,p}$ contains no graph from $\mathcal{F}_0$ a.a.s.

Let $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(H_1, H_2, n)$ denote a family of representatives for the isomorphism classes of all graphs that can be the output of Grow with parameters n and $\gamma(H_1, H_2)$ on any input instance G' for which it enters the while-loop. Observe that $\mathcal{F} = \mathcal{F}_0 \cup \tilde{\mathcal{F}}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the classes of graphs that algorithm Grow can output in lines 23 and 25, respectively.

[5]Since $T \cong H_1$ or $T \cong H_2$ and Lemma 4.3.3 holds.

For each $F \in \mathcal{F}_1$, we have that $e(F) \geq \ln(n)$, as F was generated in $\lceil \ln(n) \rceil$ iterations, each of which introduces at least one new edge by Claim 4.5.1. Moreover, Claims 4.5.2 and 4.5.3 imply that $\lambda(F_i)$ is non-increasing. Thus, we have that $\lambda(F) \leq \lambda(F_0)$ for all $F \in \mathcal{F}_1$. For all $F \in \mathcal{F}_2$, we have that $\lambda(F) \leq -\gamma$ due to the condition in line 13 of Grow. Let $A := A(H_1, H_2)$ be the constant found in the proof of Claim 4.5.5. Since we have chosen $F_0 \cong H_1$ as the seed of the growing procedure, it follows that for

$$b := (Ae)^{-\lambda(F_0)-\gamma} \leq 1,$$

the expected number of copies of graphs from $\tilde{\mathcal{F}}$ in $G_{n,p}$ with $p \leq bn^{-1/m_2(H_1,H_2)}$ is bounded by

$$\begin{aligned}
\sum_{F \in \tilde{\mathcal{F}}} n^{v(F)} p^{e(F)} &\leq \sum_{F \in \tilde{\mathcal{F}}} b^{e(F)} n^{\lambda(F)} && (4.1)\\
&\leq \sum_{F \in \mathcal{F}_1} (eA)^{(-\lambda(F_0)-\gamma)\ln(n)} n^{\lambda(F_0)} + \sum_{F \in \mathcal{F}_2} b^{e(F)} n^{-\gamma}\\
&= \sum_{F \in \mathcal{F}_1} A^{(-\lambda(F_0)-\gamma)\ln(n)} n^{-\gamma} + \sum_{F \in \mathcal{F}_2} b^{e(F)} n^{-\gamma}.
\end{aligned}$$

Observe that, since $m_2(F_2) \geq 1$, we have that

$$\lambda(F_0) = v_1 - \frac{e_1}{m_2(F_1, F_2)} = 2 - \frac{1}{m_2(F_2)} \geq 1 \tag{4.2}$$

By Claims 4.5.1, 4.5.4 and 4.5.5, and (4.2), we have that

$$\begin{aligned}\sum_{F\in\mathcal{F}_1} A^{(-\lambda(F_0)-\gamma)\ln(n)}n^{-\gamma} &\le \sum_{d=0}^{\min\{t,q_1\}} f(\lceil\ln(n)\rceil,d)A^{(-\lambda(F_0)-\gamma)\ln(n)}n^{-\gamma} \\ &\le (q_1+1)\lceil\ln(n)\rceil^{(C_0+1)q_1}\cdot A^{\lceil\ln(n)\rceil}A^{(-\lambda(F_0)-\gamma)\ln(n)}n^{-\gamma} \\ &\le (\ln(n))^{2(C_0+1)q_1}n^{-\gamma}.\end{aligned} \tag{4.3}$$

Observe that, by Claim 4.5.1, if some graph $F\in\mathcal{F}_2$ is the output of Grow after precisely t iterations of the while-loop then $e(F)\ge t$. Since $b<1$, this implies

$$b^{e(F)} \le b^t \tag{4.4}$$

for such a graph F. Using (4.4) and Claims 4.5.1, 4.5.4 and 4.5.5, we have that

$$\begin{aligned}\sum_{F\in\mathcal{F}_2} b^{e(F)}n^{-\gamma} &\le \sum_{t=0}^{\lceil\ln(n)\rceil}\sum_{d=0}^{\min\{t,q_1\}} f(t,d)b^t n^{-\gamma} \\ &\le \sum_{t=0}^{\lceil\ln(n)\rceil}\sum_{d=0}^{\min\{t,q_1\}} \lceil\ln(n)\rceil^{(C_0+1)d}\cdot A^{t-d}(Ae)^{(-\lambda(F_0)-\gamma)t}n^{-\gamma} \\ &\le (\lceil\ln(n)\rceil+1)(q_1+1)\lceil\ln(n)\rceil^{(C_0+1)q_1}n^{-\gamma} \\ &\le (\ln(n))^{2(C_0+1)q_1}n^{-\gamma}.\end{aligned} \tag{4.5}$$

Thus, by (4.1), (4.3) and (4.5), we have that $\sum_{F\in\tilde{\mathcal{F}}} n^{v(F)}p^{e(F)} = o(1)$. Consequently, Markov's inequality implies that $G_{n,p}$ a.a.s. contains no graph from $\tilde{\mathcal{F}}$.

Combined with the earlier observation that $G_{n,p}$ a.a.s. contains no graph from $\mathcal{F}_0$, we have that $G_{n,p}$ a.a.s. contains no graph from $\mathcal{F}=\mathcal{F}_0\cup\tilde{\mathcal{F}}$. □

Proof of Lemma 4.4.3 Case 1. Suppose that the call to Asym-Edge-Col(G) gets stuck for some graph G, and consider $G' \subseteq G$ at this moment. Then Grow(G') returns a copy of a graph $F \in \mathcal{F}(H_1, H_2, n)$ that is contained in $G' \subseteq G$. Provided Claim 4.5.3 holds, by Claim 4.5.6 this event a.a.s. does not occur in $G = G_{n,p}$ with p as claimed. Thus Asym-Edge-Col does not get stuck a.a.s and, by Lemma 4.4.2, finds a valid colouring for H_1 and H_2 of $G_{n,p}$ with $p \leq bn^{-1/m_2(H_1,H_2)}$ a.a.s.

4.5.2 Proof of Claim 4.5.3

Our strategy for proving Claim 4.5.3 revolves around comparing our degenerate iteration i of the while-loop of algorithm Grow with any non-degenerate iteration which could have occurred instead. In accordance with this strategy, we have the following technical lemma which will be crucial in proving Claim 4.5.3.[6] The lemma will play the same role as [51, Lemma 21], but is considerably different. In order to state our technical lemma, we define the following families of graphs.

Definition 4.5.7. Let F, H_1 and H_2 be graphs and $\hat{e} \in E(F)$. We define $\mathcal{H}(F, \hat{e}, H_1, H_2)$ to be the family of graphs constructed from F in the following way: Attach a copy $H_{\hat{e}}$ of H_2 to F such that $E(H_{\hat{e}}) \cap E(F) = \{\hat{e}\}$ and $V(H_{\hat{e}}) \cap V(F) = \hat{e}$. Then, for each edge $f \in E(H_{\hat{e}}) \setminus \{\hat{e}\}$, attach a copy H_f of H_1 to $F \cup H_{\hat{e}}$ such that $E(F \cup H_{\hat{e}}) \cap E(H_f) = \{f\}$ and $(V(F) \setminus \hat{e}) \cap V(H_f) = \emptyset$.

Notice that, during construction of a graph $J \in \mathcal{H}(F, \hat{e}, H_1, H_2)$, the edge of $H_{\hat{e}}$ intersecting at $\hat{e}$ and the edge of each copy H_f of H_1 intersecting at an edge $f \in E(H_{\hat{e}}) \setminus \{\hat{e}\}$ are not stipulated. That is, we may end up with different graphs after the construction process if we choose different edges of $H_{\hat{e}}$ to intersect F at $\hat{e}$ and different edges of the copies H_f of H_1 to intersect the edges in $E(H_{\hat{e}}) \setminus \{\hat{e}\}$. Observe that although $E(F \cup H_{\hat{e}}) \cap E(H_f) =$

[6] More specifically, in proving Claim 4.5.10, stated later.

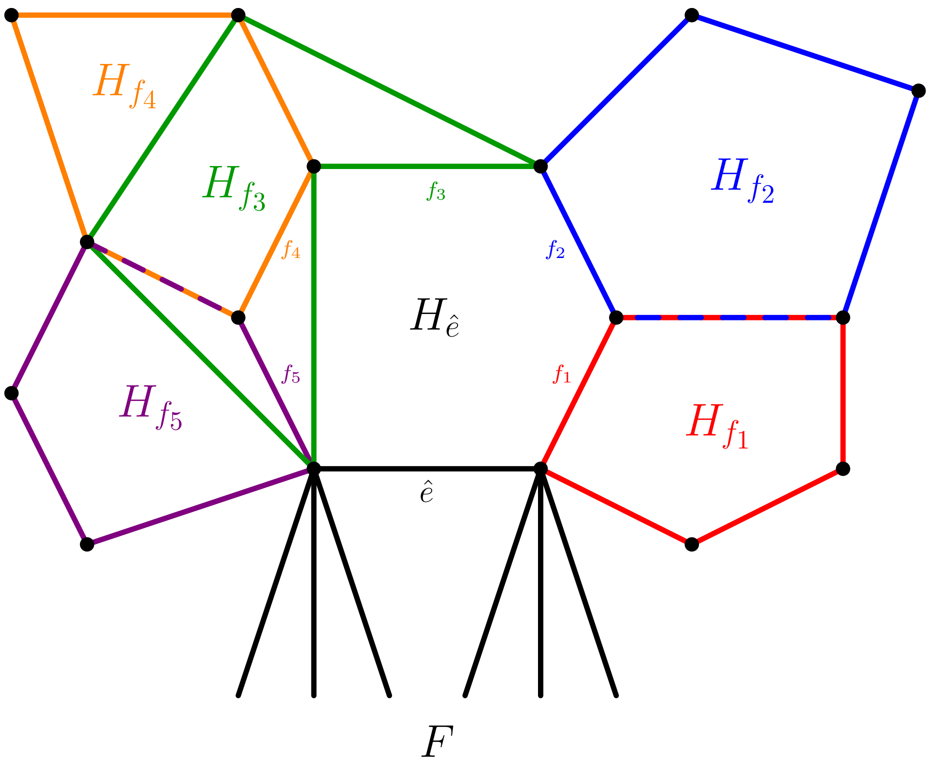

Figure 4.4: A graph $J \in \mathcal{H}(F, \hat{e}, C_5, C_6) \setminus \mathcal{H}^*(F, \hat{e}, C_5, C_6)$.

$\{f\}$ and $(V(F) \setminus \hat{e}) \cap V(H_f) = \emptyset$ for each $f \in E(H_{\hat{e}}) - \{\hat{e}\}$, the construction may result in one or more graphs H_f intersecting $H_{\hat{e}}$ in more than two vertices, including possibly in vertices of $\hat{e}$ (e.g. H_{f_3} in Figure 4.4). Also, the graphs H_f may intersect with each other in vertices and/or edges (e.g. H_{f_1} and H_{f_2} in Figure 4.4).

Borrowing notation and language from [51], for any $J \in \mathcal{H}(F, \hat{e}, H_1, H_2)$ we call the vertices in $V_J := V(H_{\hat{e}}) \setminus \hat{e}$ the *inner vertices of* J and $E_J := E(H_{\hat{e}}) \setminus \{\hat{e}\}$ the *inner edges of* J. Let $H_{\hat{e}}^J$ be the *inner graph* on vertex set $V_J \,\dot{\cup}\, \hat{e}$ and edge set E_J and observe that $H_{\hat{e}}^J$ is isomorphic to a copy of H_2 minus some edge. Further, for each copy H_f of H_1, we define $U_J(f) := V(H_f) \setminus f$ and $D_J(f) := E(H_f) \setminus \{f\}$ and call

$$U_J := \bigcup_{f \in E_J} U_J(f)$$

the set of *outer vertices* of J and

$$D_J := \bigcup_{f \in E_J} D_J(f)$$

the set of *outer edges* of J. Observe that the sets $U_J(f)$ may overlap with each other and, as noted earlier, with $V(H_{\hat{e}}^J)$. However, the sets $D_J(f)$ may overlap only with each other. Further, define $\mathcal{H}^*(F, \hat{e}, H_1, H_2) \subseteq \mathcal{H}(F, \hat{e}, H_1, H_2)$ such that for any $J^* \in \mathcal{H}^*(F, \hat{e}, H_1, H_2)$ we have $U_{J^*}(f_1) \cap U_{J^*}(f_2) = \emptyset$ and $D_{J^*}(f_1) \cap D_{J^*}(f_2) = \emptyset$ for all $f_1, f_2 \in E_{J^*}$, $f_1 \neq f_2$, and $U_{J^*}(f) \cap V(H_{\hat{e}}^{J^*}) = \emptyset$ for all $f \in E_{J^*}$; that is, the copies of H_1 are, in some sense, pairwise disjoint. Note that each $J^* \in \mathcal{H}^*(F, \hat{e}, H_1, H_2)$ corresponds with a non-degenerate iteration i of the while loop of algorithm Grow when $F = F_i$, $J^* = F_{i+1}$ and $\hat{e}$ is the edge chosen by Eligible-Edge(F_i). This observation will be very helpful several times later. For any $J \in \mathcal{H}(F, \hat{e}, H_1, H_2)$, define

$$v^+(J) := |V(J) \setminus V(F)| = v(J) - v(F)$$

and

$$e^+(J) := |E(J) \setminus E(F)| = e(J) - e(F),$$

and call $\frac{e^+(J)}{v^+(J)}$ the *F-external density of J*. The following lemma relates the F-external density of any $J^* \in \mathcal{H}^*(F, \hat{e}, H_1, H_2)$ to that of any $J \in \mathcal{H}(F, \hat{e}, H_1, H_2) \setminus \mathcal{H}^*(F, \hat{e}, H_1, H_2)$.

Lemma 4.5.8. *Let F be a graph and $\hat{e} \in E(F)$. Then for any $J \in \mathcal{H}(F, \hat{e}, H_1, H_2) \setminus \mathcal{H}^*(F, \hat{e}, H_1, H_2)$ and any $J^* \in \mathcal{H}^*(F, \hat{e}, H_1, H_2)$, we have*

$$\frac{e^+(J)}{v^+(J)} > \frac{e^+(J^*)}{v^+(J^*)}.$$

We prove Lemma 4.5.8 in Section 4.5.3.

Claim 4.5.3 will follow from the next two claims. We say that algorithm Grow encounters a *degeneracy of type 1* in iteration i of the while-loop if line 14 returns true, that is, $\exists R \in \mathcal{R}_{G'} \setminus \mathcal{R}_{F_i} : |V(R) \cap V(F_i)| \geq 2$. Note that the following claim requires that $m_2(H_1) > m_2(H_2)$.

Claim 4.5.9. *There exists a constant $\kappa_1 = \kappa_1(H_1, H_2) > 0$ such that if procedure* Grow *encounters a degeneracy of type 1 in iteration i of the while-loop, we have*

$$\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa_1.$$

Proof. Let $F := F_i$ be the graph before the operation in line 15 is carried out (that is, before $F_{i+1} \leftarrow F_i \cup R$), let R be the copy of H_1 merged with F in line 15 and let $F' := F_{i+1}$ be the output from line 15. We aim to show there exists a constant $\kappa_1 = \kappa_1(H_1, H_2) > 0$ such that

$$\lambda(F) - \lambda(F') = v(F) - v(F') - \frac{e(F) - e(F')}{m_2(H_1, H_2)} \geq \kappa_1.$$

Choose any edge $\hat{e} \in E(F)$ (the edge $\hat{e}$ need not be in the intersection of R and F). Let $F^* \in \mathcal{H}^*(F, \hat{e}, H_1, H_2)$. Our strategy is to compare our degenerate outcome F' with F^*. As noted earlier, F^* corresponds to a non-degenerate iteration of the while loop of algorithm Grow (if $\hat{e}$ was the edge chosen by Eligible-Edge). Then Claim 4.5.2 gives us that $\lambda(F) = \lambda(F^*)$. Then

$$\lambda(F) - \lambda(F') \quad = \lambda(F^*) - \lambda(F') = v(F^*) - v(F') - \tfrac{e(F^*) - e(F')}{m_2(H_1, H_2)}.$$

Hence we aim to show that there exists $\kappa_1 = \kappa_1(H_1, H_2) > 0$ such that

$$v(F^*) - v(F') - \frac{e(F^*) - e(F')}{m_2(H_1, H_2)} \geq \kappa_1. \tag{4.6}$$

Define R' to be the graph with vertex set $V' := V(R) \cap V(F)$ and edge set $E' := E(R) \cap E(F)$, and let $v' := |V'|$ and $e' := |E'|$. Observe that $R' \subset R$. Since F^* corresponds with a non-degenerate iteration of the while-loop of algorithm Grow, H_2 is (strictly) 2-balanced and H_1 is (strictly) balanced w.r.t. $d_2(\cdot, H_2)$, we have

$$\begin{aligned}
v(F^*) - v(F') - \frac{e(F^*) - e(F')}{m_2(H_1, H_2)} &= (e_2 - 1)(v_1 - 2) + (v_2 - 2) - (v_1 - v') \\
&\quad - \frac{(e_2 - 1)e_1 - (e_1 - e')}{m_2(H_1, H_2)} \\
&= (e_2 - 1)(v_1 - 2) + (v_2 - 2) \\
&\quad - (e_2 - 1)\left(v_1 - 2 + \frac{1}{m_2(H_2)}\right) \\
&\quad + \frac{e_1 - e'}{m_2(H_1, H_2)} - (v_1 - v') \\
&= \frac{e_1 - e'}{m_2(H_1, H_2)} - (v_1 - v') \\
&= v' - 2 + \frac{1}{m_2(H_2)} - \frac{e'}{m_2(H_1, H_2)}.
\end{aligned} \tag{4.7}$$

Also, since Grow encountered a degeneracy of type 1, we must have $v' \geq 2$. Hence, if $e' = 0$ and $v' \geq 2$, then

$$v' - 2 + \frac{1}{m_2(H_2)} - \frac{e'}{m_2(H_1, H_2)} \geq \frac{1}{m_2(H_2)} > 0.$$

If $e' \geq 1$, then since R is a copy of H_1, H_1 is *strictly* balanced w.r.t. $d_2(\cdot, H_2)$ and $R' \subset R$ with $|E(R')| = e' \geq 1$, we have that $0 < d_2(R', H_2) < m_2(H_1, H_2)$, and so

$$-\frac{1}{m_2(H_1, H_2)} > -\frac{1}{d_2(R', H_2)}. \tag{4.8}$$

Then by (4.7) and (4.8), we have that

$$\begin{aligned} v(F^*) - v(F') - \frac{e(F^*) - e(F')}{m_2(H_1, H_2)} &= v' - 2 + \frac{1}{m_2(H_2)} - \frac{e'}{m_2(H_1, H_2)} \\ &> v' - 2 + \frac{1}{m_2(H_2)} - \frac{e'}{d_2(R', H_2)} \\ &= 0, \end{aligned}$$

using the definition of $d_2(R', H_2)$. Thus (4.6) holds for

$$\kappa_1 = \min_{R' \subset R} \left\{ \frac{1}{m_2(H_2)},\ v' - 2 + \frac{1}{m_2(H_2)} - \frac{e'}{m_2(H_1, H_2)} \right\}.$$

□

We say that algorithm Grow encounters a *degeneracy of type 2* in iteration i of the while-loop if, when we call Extend-L(F_i, e, G'), the graph L found in line 2 overlaps with F_i in more than 2 vertices, or if there exists some edge $e' \in E(L) \setminus E(F_i)$ such that the graph $R_{e'}$ found in line 5 overlaps in more than 2 vertices with F'. The following result corresponds to [51, Claim 22]. As in the proof of [51, Claim 22], we transform F' into the output of a non-degenerate iteration F^* in three steps. However, we swap the order of the latter two steps in our proof. More precisely, we transform F' into a graph $F^2 \in \mathcal{H}(F_i, e, H_1, H_2)$ in the first two steps, then transform F^2 into a graph $F^3 := F^* \in \mathcal{H}^*(F_i, e, H_1, H_2)$. In this last step we require Lemma 4.5.8.

Claim 4.5.10. *There exists a constant $\kappa_2 = \kappa_2(H_1, H_2) > 0$ such that if procedure* Grow *encounters a degeneracy of type 2 in iteration i of the while-loop, we have*

$$\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa_2.$$

Proof. Let $F := F_i$ be the graph passed to EXTEND-L and let $F' := F_{i+1}$ be its output. We aim to show that there exists a constant $\kappa_2 = \kappa_2(H_1, H_2) > 0$ such that

$$\lambda(F) - \lambda(F') = v(F) - v(F') - \frac{e(F) - e(F')}{m_2(H_1, H_2)} \geq \kappa_2. \tag{4.9}$$

Recall that F' would be one of a constant number of graphs if iteration i was non-degenerate. Our strategy is to transform F' into the output of such a non-degenerate iteration F^* in three steps

$$F' =: F^0 \overset{(i)}{\to} F^1 \overset{(ii)}{\to} F^2 \overset{(iii)}{\to} F^3 := F^*,$$

with each step carefully resolving a different facet of a degeneracy of type 2. By Claim 4.5.2, we have $\lambda(F) = \lambda(F^*)$, hence we have that

$$\begin{aligned} \lambda(F) - \lambda(F') &= \lambda(F^*) - \lambda(F') = \textstyle\sum_{j=1}^{3} \left(\lambda(F^j) - \lambda(F^{j-1})\right) \\ &= \textstyle\sum_{j=1}^{3} \left(v(F^j) - v(F^{j-1}) - \frac{e(F^j) - e(F^{j-1})}{m_2(H_1, H_2)}\right). \end{aligned}$$

We shall show that there exists $\kappa_2 = \kappa_2(H_1, H_2) > 0$ such that

$$\left(v(F^j) - v(F^{j-1}) - \frac{e(F^j) - e(F^{j-1})}{m_2(H_1, H_2)}\right) \geq \kappa_2 \tag{4.10}$$

for each $j \in \{1, 2, 3\}$, whenever F^j and F^{j-1} are not isomorphic. In each step we will look at a different structural property of F' that may result from a degeneracy of type 2. We do not know the exact structure of F', and so, for each j, step j may not necessarily modify F^{j-1}. However, since F' is not isomorphic to F^*, as F' resulted from a degeneracy of type 2, we know that for at least one j that F^j is not isomorphic to F^{j-1}. This will allow us to conclude (4.9) from (4.10).

We will now analyse the graph that Extend-L attaches to F when a degeneracy of type 2 occurs. First of all, Extend-L attaches a graph $L \cong H_2$ to F such that $L \in \mathcal{L}^*_{G'}$. Let x be the number of new vertices that are added onto F when L is attached, that is, $x = |V(L) \backslash (V(F) \cap V(L))|$. Since L overlaps with the edge e determined by Eligible-Edge in line 17 of Grow, we must have that $x \leq v_2 - 2$. Further, as $L \in \mathcal{L}^*_{G'}$, every edge of L is covered by a copy of H_1. Thus, since the condition in line 14 of Grow came out as false in iteration i, we must have that

$$\text{for all } u, v \in V(F) \cap V(L), \text{ if } uv \in E(L) \text{ then } uv \in E(F). \tag{4.11}$$

(By Claim 4.5.1, (4.11) implies that $x \geq 1$ since F must be extended by at least one edge.)

Let $L' \subseteq L$ denote the subgraph of L obtained by removing every edge in $E(F) \cap E(L)$. Observe that $|V(L')| = |V(L)| = v_2$ and $|E(L')| \geq 1$ (see the remark above). Extend-L attaches to each edge $e' \in E(L')$ a copy $R_{e'}$ of H_1 in line 6 such that $E(L') \cap E(R_{e'}) = \{e'\}$. As the condition in line 14 of Grow came out as false, each graph $R_{e'}$ intersects F in at most one vertex and, hence, zero edges. Let

$$L'_R := L' \cup \bigcup_{e' \in E(L')} R_{e'}.$$

Then F' is the same as $F \cup L'_R$, and since every graph $R_{e'}$ contains at most one vertex of F, we have that $E(F') = E(F) \,\dot{\cup}\, E(L'_R)$. Therefore,

$$e(F') - e(F) = e(L'_R).$$

Observe that $|V(F) \cap V(L')| = v_2 - x$ and so

$$
\begin{aligned}
v(F') - v(F) &= v(L'_R) - |V(F) \cap V(L'_R)| \\
&= v(L'_R) - (v_2 - x) - |V(F) \cap (V(L'_R) \setminus V(L'))|.
\end{aligned}
$$

Transformation (i): $F^0 \to F^1$. If $|V(F) \cap (V(L'_R) \setminus V(L'))| \geq 1$, then we apply transformation (i), mapping F^0 to F^1: For each vertex $v \in V(F) \cap (V(L'_R) \setminus V(L'))$, transformation (i) introduces a new vertex v'. Every edge incident to v in $E(F)$ remains connected to v and all those edges incident to v in $E(L'_R)$ are redirected to v'. In L'_R we replace the vertices in $V(F) \cap (V(L'_R) \setminus V(L'))$ with the new vertices. So now we have $|V(F) \cap (V(L'_R) \setminus V(L'))| = 0$. Since $E(F) \cap E(L'_R) = \emptyset$, the output of this transformation is uniquely defined. Moreover, the structure of L'_R is completely unchanged. Hence, since $|V(F) \cap (V(L'_R) \setminus V(L'))| \geq 1$, and $|E(F')| = |E(F) \,\dot\cup\, E(L'_R)|$ remained the same after transformation (i), we have that

$$
v(F^1) - v(F^0) - \frac{e(F^1) - e(F^0)}{m_2(H_1, H_2)} = |V(F) \cap (V(L'_R) \setminus V(L'))| \geq 1.
$$

Transformation (ii): $F^1 \to F^2$. Recall the definition of $\mathcal{H}(F, e, H_1, H_2)$. If $x \leq v_2 - 3$, then we apply transformation (ii), mapping F^1 to F^2 by replacing L'_R with a graph L''_R such that $F \cup L''_R \in \mathcal{H}(F, e, H_1, H_2)$.

If $x = v_2 - 2$, observe that already $F \cup L'_R \in \mathcal{H}(F, e, H_1, H_2)$ and we continue to transformation (iii). So assume $x \leq v_2 - 3$. Consider the proper subgraph $L_F := L[V(F) \cap V(L)] \subset L$ obtained by removing all x vertices in $V(L) \setminus V(F)$ and their incident edges from L. Observe that $v(L_F) = v_2 - x \geq 3$ and also that $L_F \subseteq F$ by (4.11). Assign labels to $V(L_F)$ so that $V(L_F) = \{y, z, w_1, \ldots, w_{v_2-(x+2)}\}$ where $e = \{y, z\}$ and $w_1, \ldots, w_{v_2-(x+2)}$ are arbitrarily assigned. At the start of transformation (ii), we create $v_2 - (x+2)$ new vertices $w'_1, \dots, w'_{v_2-(x+2)}$ and also new edges such that $\{y, z, w'_1, \dots, w'_{v_2-(x+2)}\}$ induces a copy $\hat{L}_F$ of L_F, and for all $i, j \in \{1, \dots, v_2 - (x+2)\}$, $i \neq j$,

$$\begin{aligned}
&\text{if } w_i w_j \in E(L_F) \text{ then } w'_i w'_j \in E(\hat{L}_F);\\
&\text{if } w_i y \in E(L_F) \text{ then } w'_i y \in E(\hat{L}_F);\\
&\text{if } w_i z \in E(L_F) \text{ then } w'_i z \in E(\hat{L}_F);\\
&\text{and } e = yz \in E(\hat{L}_F).
\end{aligned}$$

We also transform L'_R. For each edge in $E(L'_R)$ incident to a vertex w_i in L_F, redirect the edge to w'_i, and remove $w_1, \dots, w_{v_2-(x+2)}$ from $V(L'_R)$. Hence the structure of L'_R remains the same except for the vertices $w_1, \dots, w_{v_2-(x+2)}$ that we removed. Define $L'' := L'_R \cup \hat{L}_F$ and observe that $V(L'') \cap V(F) = e$.

Continuing transformation (ii), for each $e' \in E(\hat{L}_F) \setminus \{e\}$, attach a copy $R_{e'}$ of H_1 to L'' such that $E(R_{e'}) \cap E(L'' \cup F) = \{e'\}$ and $V(R_{e'}) \cap V(L'' \cup F) = e'$. That is, all these new copies $R_{e'}$ of H_1 are, in some sense, pairwise disjoint. Observe that $E(L'') \cap E(F) = \{e\}$ and define

$$L''_R := L'' \cup \bigcup_{e' \in E(\hat{L}_F) \setminus \{e\}} R_{e'}.$$

Then $F \cup L''_R \in \mathcal{H}(F, e, H_1, H_2)$. (See Figure 4.5 for an example of transformation (ii).)

Let $F^2 := F \cup L''_R$. Then,

$$
\begin{aligned}
& v(F^2) - v(F^1) - \frac{e(F^2) - e(F^1)}{m_2(H_1, H_2)} \\
&= (e(\hat{L}_F) - 1)(v_1 - 2) + v(\hat{L}_F) - 2 - \frac{(e(\hat{L}_F) - 1)e_1}{m_2(H_1, H_2)} \\
&= v(\hat{L}_F) - 2 - \frac{(e(\hat{L}_F) - 1)}{m_2(H_2)} \\
&= \frac{(v(\hat{L}_F) - 2)\left(m_2(H_2) - \frac{e(\hat{L}_F)-1}{v(\hat{L}_F)-2}\right)}{m_2(H_2)} \\
&\geq \delta_1
\end{aligned}
$$

for some $\delta_1 = \delta_1(H_1, H_2) > 0$, where the second equality follows from H_1 being (strictly) balanced w.r.t. $d_2(\cdot, H_2)$, the third equality follows from $v(\hat{L}_F) = v(L_F) \geq 3$ and the last inequality follows from $\hat{L}_F$ being a copy of $L_F \subset L \cong H_2$ and H_2 being *strictly* 2-balanced.

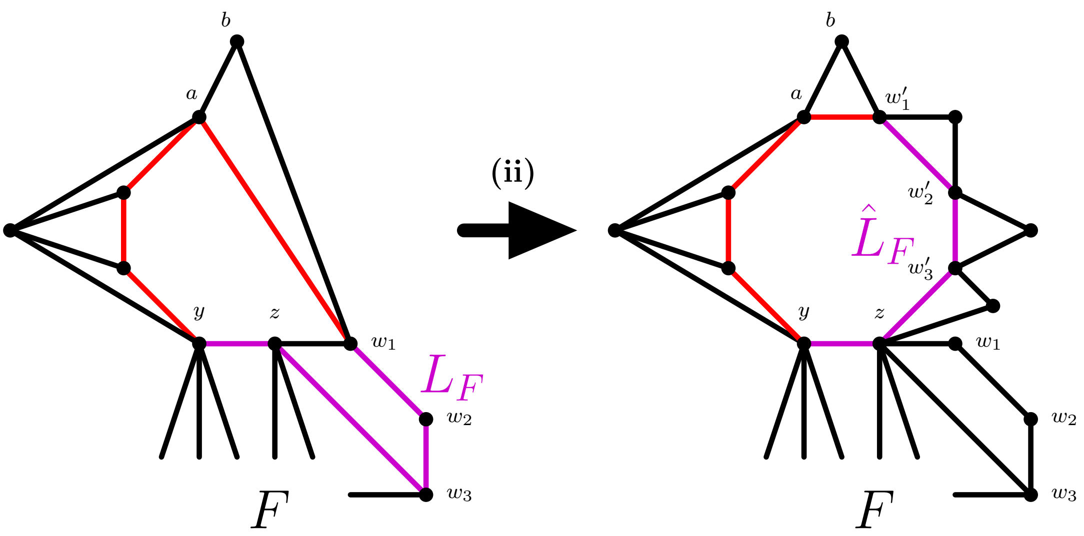

Figure 4.5: An example of transformation (ii) where $H_1 = K_3$ and $H_2 = C_8$. Observe that edges aw_1 and bw_1 are replaced by edges aw_1' and bw_1'.

Transformation (iii): $F^2 \to F^3$. Recall that for any $J \in \mathcal{H}(F, \hat{e}, H_1, H_2)$, we define $v^+(J) := |V(J) \setminus V(F)| = v(J) - v(F)$ and $e^+(J) := |E(J) \setminus E(F)| = e(J) - e(F)$. Remove the edge e from $E(L'')$ (and $E(L''_R)$) to give $E(L'') \cap E(F) = \emptyset$. Then

$$e^+(F \cup L''_R) = e(L''_R)$$

and

$$v^+(F \cup L''_R) = v(L''_R) - 2.$$

If $F^2 = F \cup L''_R \in \mathcal{H}^*(F, e, H_1, H_2)$, then transformation (iii) sets $F^3 := F^2$. Otherwise $F \cup L''_R \in \mathcal{H}(F, e, H_1, H_2) \setminus \mathcal{H}^*(F, e, H_1, H_2)$. Let $F^3 := J^*$ where J^* is any member of $\mathcal{H}^*(F, e, H_1, H_2)$ and recall that, indeed, J^* is a possible output of a non-degenerate iteration of the while-loop of Grow.

Then, in transformation (iii), we replace $F \cup L''_R$ with the graph J^*. Since H_2 is (strictly) 2-balanced and H_1 is (strictly) balanced w.r.t. $d_2(\cdot, H_2)$, we have that

$$m_2(H_1, H_2) = \frac{e_1}{v_1 - 2 + \frac{1}{m_2(H_2)}} = \frac{e_1(e_2 - 1)}{(v_1 - 2)(e_2 - 1) + v_2 - 2} = \frac{e^+(J^*)}{v^+(J^*)}. \tag{4.12}$$

Using (4.12) and Lemma 4.5.8, and that H_2 is (strictly) 2-balanced and H_1 is (strictly) balanced w.r.t. $d_2(\cdot, H_2)$, we have that

$$
\begin{aligned}
& v(F^3) - v(F^2) - \frac{e(F^3) - e(F^2)}{m_2(H_1, H_2)} \\
&= v(J^*) - v(F \cup L''_R) - \frac{e(J^*) - e(F \cup L''_R)}{m_2(H_1, H_2)} \\
&= v^+(J^*) - v^+(F \cup L''_R) - \frac{e^+(J^*) - e^+(F \cup L''_R)}{m_2(H_1, H_2)} \\
&\overset{L.4.5.8}{>} v^+(J^*) - v^+(F \cup L''_R) - \frac{e^+(J^*) - e^+(J^*)\left(\frac{v^+(F \cup L''_R)}{v^+(J^*)}\right)}{m_2(H_1, H_2)} \\
&= \left(v^+(J^*) - v^+(F \cup L''_R)\right)\left(1 - \frac{e^+(J^*)}{v^+(J^*) m_2(H_1, H_2)}\right) \\
&= 0.
\end{aligned}
$$

Since $v^+(J^*)$, $v^+(F \cup L''_R)$, $e^+(J^*)$, $e^+(F \cup L''_R)$ and $m_2(H_1, H_2)$ only rely on H_1 and H_2, there exists $\delta_2 = \delta_2(H_1, H_2) > 0$ such that

$$
v(F^3) - v(F^2) - \frac{e(F^3) - e(F^2)}{m_2(H_1, H_2)} \geq \delta_2.
$$

Taking

$$
\kappa_2 := \min\{1, \delta_1, \delta_2\}
$$

we see that (4.10) holds. $\square$

As stated earlier, Claim 4.5.3 follows from Claims 4.5.9 and 4.5.10. All that remains to prove Case 1 of Lemma 4.4.3 is to prove Lemma 4.5.8.

4.5.3 Proof of Lemma 4.5.8

Let $J \in \mathcal{H}(F, \hat{e}, H_1, H_2) \setminus \mathcal{H}^*(F, \hat{e}, H_1, H_2)$. Choose $J^* \in \mathcal{H}^*(F, \hat{e}, H_1, H_2)$ such that

1) the edge of the copy $H_{\hat{e}}$ of H_2 in J attached at $\hat{e}$ and its orientation when attached are the same as the edge of the copy $H^*_{\hat{e}}$ of H_2 in J^* attached at $\hat{e}$ and its orientation when attached;

2) for each $f \in E_J$, the edge of the copy H_f of H_1 in J attached at f and its orientation when attached are the same as the edge of the copy H^*_f of H_1 in J^* attached at f and its orientation when attached.

Then, recalling definitions from the beginning of Section 4.5.2, we have that $V_J = V_{J^*}$ and $E_J = E_{J^*}$; that is, $H^J_{\hat{e}} = H^{J^*}_{\hat{e}}$. From now on, let $V := V_J$, $E := E_J$ and $H^-_{\hat{e}} := H^J_{\hat{e}}$. Observe for *all* $J' \in \mathcal{H}^*(F, \hat{e}, H_1, H_2)$, that

$$\frac{e^+(J')}{v^+(J')} = \frac{e_1(e_2 - 1)}{(v_1 - 2)(e_2 - 1) + v_2 - 2}. \tag{4.13}$$

Hence, to prove Lemma 4.5.8 it suffices to show

$$\frac{e^+(J)}{v^+(J)} > \frac{e^+(J^*)}{v^+(J^*)}.$$

As in [51], the intuition behind our proof is that J^* can be transformed into J by successively merging the copies H^*_f of H_1 in J^* with each other and vertices in $H^-_{\hat{e}}$. We do this in $e_2 - 1$ steps, fixing carefully a total ordering of the inner edges E. For every edge $f \in E$, we merge the attached outer copy H^*_f of H_1 in J^* with copies of H_1 (attached to edges preceding f in our ordering) and vertices of $H^-_{\hat{e}}$. Throughout, we keep track of the number of edges $\Delta_e(f)$ and the number of vertices $\Delta_v(f)$ vanishing in this process. One could hope that the F-external density of J increases in every step of this process, or, even slightly stronger, that $\Delta_e(f)/\Delta_v(f) < e^+(J^*)/v^+(J^*)$. This does not necessarily hold, but we will show that there exist a collection of edge-disjoint subgraphs A_i of $H_{\hat{e}}^-$ such that, for each i, the edges of $E(A_i)$ are 'collectively good' for this process and every edge not belonging to one of these A_i is also 'good' for this process.

Recalling definitions from the beginning of Section 4.5.2, let $H_f^- := (U_J(f) \,\dot\cup\, f, D_J(f))$ denote the subgraph obtained by removing the edge f from the copy H_f of H_1 in J.

Later, we will carefully define a (total) ordering $\prec$ on the inner edges E.[7] For such an ordering $\prec$ and each $f \in E$, define

$$\Delta_E(f) := D_J(f) \cap \left(\bigcup_{f' \prec f} D_J(f') \right),$$

and

$$\Delta_V(f) := U_J(f) \cap \left(\left(\bigcup_{f' \prec f} U_J(f') \right) \cup V(H_{\hat{e}}^-) \right),$$

and set $\Delta_e(f) := |\Delta_E(f)|$ and $\Delta_v(f) := |\Delta_V(f)|$. We emphasise here that the definition of $\Delta_v(f)$ takes into account how vertices of outer vertex sets can intersect with the inner graph $H_{\hat{e}}^-$. One can see that $\Delta_e(f)$ ($\Delta_v(f)$) is the number of edges (vertices) vanishing from H_f^* when it is merged with preceding attached copies of H_1 and $V(H_{\hat{e}}^-)$.

By our choice of J^*, one can quickly see that

$$e^+(J) = e^+(J^*) - \sum_{f \in E} \Delta_e(f) \tag{4.14}$$

and

[7]For clarity, for any $f \in E$, $f \not\prec f$ in this ordering $\prec$.

$$v^+(J) = v^+(J^*) - \sum_{f \in E} \Delta_v(f). \tag{4.15}$$

By (4.14) and (4.15), we have

$$\frac{e^+(J)}{v^+(J)} = \frac{e^+(J^*) - \sum_{f \in E} \Delta_e(f)}{v^+(J^*) - \sum_{f \in E} \Delta_v(f)}.$$

Then, by Fact 4.3.2, to show that

$$\frac{e^+(J)}{v^+(J)} > \frac{e^+(J^*)}{v^+(J^*)}$$

it suffices to prove that

$$\frac{\sum_{f \in E} \Delta_e(f)}{\sum_{f \in E} \Delta_v(f)} < \frac{e^+(J^*)}{v^+(J^*)}.^{8} \tag{4.16}$$

To show (4.16), we will now carefully order the edges of E using an algorithm Order-Edges (Figure 4.6). The algorithm takes as input the graph $H_{\hat{e}}^- = (V \mathbin{\dot\cup} \hat{e}, E)$ and outputs a stack s containing every edge from E and a collection of edge-disjoint edge sets E_i in E and (not necessarily disjoint) vertex sets V_i in $V \mathbin{\dot\cup} \hat{e}$. We take our total ordering $\prec$ of E to be that induced by the order in which edges of E were placed onto the stack s (that is, $f \prec f'$ if and only if f was placed onto the stack s before f'). Also, for each i, we define $A_i := (V_i, E_i)$ to be the graph on vertex set V_i and edge set E_i and observe that $A_i \subset H_2$. We will utilise this ordering and our choice of E_i and V_i for each i, alongside that H_2 is (strictly) 2-balanced and H_1 is *strictly* balanced w.r.t. $d_2(\cdot, H_2)$, in order to conclude (4.16).

Let us describe algorithm Order-Edges (Figure 4.6) in detail. In lines 2-8, we ini-

[8] Note that $\sum_{f \in E} \Delta_v(f) \geq 1$ as otherwise $J = J^*$.

```
procedure Order-Edges($H_{\hat{e}}^- = (V \dot\cup \hat{e}, E)$)
    $s \leftarrow$ EMPTY-STACK()
    for all $i \in [\lfloor e_2/2 \rfloor]$ do
        $E_i \leftarrow \emptyset$
        $V_i \leftarrow \emptyset$
    end for
    $j \leftarrow 1$
    $E' \leftarrow E$
    while $E' \neq \emptyset$ do
        if $\exists f, f' \in E'$ s.t. $(f \neq f') \wedge (D_J(f) \cap D_J(f') \neq \emptyset)$ then
            $s$.PUSH($f$)
            $E_j$.PUSH($f$)
            $E'$.REMOVE($f$)
            $V_j \leftarrow f \cup (U_J(f) \cap V(H_{\hat{e}}^-))$
            while $\exists\ uw \in E'$ s.t. $(u, w \in V_j) \vee \left(D_J(uw) \cap \bigcup_{f \in E_j} D_J(f) \neq \emptyset\right)$ do
                $s$.PUSH($uw$)
                $E_j$.PUSH($uw$)
                $E'$.REMOVE($uw$)
                $V_j \leftarrow \bigcup_{f \in E_j} \left(f \cup (U_J(f) \cap V(H_{\hat{e}}^-))\right)$
            end while
            $j \leftarrow j + 1$
        else
            for all $f \in E'$ do
                $s$.PUSH($f$)
                $E'$.REMOVE($f$)
            end for
        end if
    end while
    return $s$
    for all $i \in [\lfloor e_2/2 \rfloor]$ s.t. $E_i \neq \emptyset$ do
        return $E_i$
        return $V_i$
    end for
end procedure
```

Figure 4.6: The implementation of algorithm Order-Edges.

tialise several parameters: a stack s, which we will place edges of E on during our algorithm; sets E_i and V_i for each $i \in [\lfloor e_2/2 \rfloor]$,[9] which we will add edges of E and vertices of $V \dot\cup \hat{e}$ into, respectively; an index j, which will correspond to whichever graph A_j we consider constructing next; a set E', which will keep track of those edges of E we have not yet placed onto the stack s. Line 9 ensures the algorithm continues until $E' = \emptyset$, that is, until all the edges of E have been placed onto s.

In line 10, we begin constructing A_j by finding a pair of distinct edges in E' whose outer edge sets (in J) intersect. In lines 11-14, we place one of these edges, f, onto s, into E_j and remove it from E'. We also set V_j to be the two vertices in f alongside any vertices in the outer vertex set $U_J(f)$ that intersect $V(H^-_{\hat{e}})$.

In lines 15-20, we iteratively add onto s, into E_j and remove from E' any edge $uw \in E'$ which either connects two vertices previously added to V_j or has an outer edge set $D_J(uw)$ that intersects the collection of outer edge sets of edges previously added to E_j. We also update V_j in each step of this process.

In line 21, we increment j in preparation for the next check at line 10 (if we still have $E' \neq \emptyset$). If the condition in line 10 fails then in lines 23-26 we arbitrarily place the remaining edges of E' onto the stack s. In line 29, we output the stack s and in lines 30-32 we output each non-empty E_i and V_i.

We will now argue that each proper subgraph $A_i = (V_i, E_i)$ of H_2 and each edge placed onto s in line 24 are 'good', in some sense, for us to conclude (4.16).

For each $f \in E$, define the graph

$$T(f) := (\Delta_V(f) \dot\cup f, \Delta_E(f)) \subseteq H^-_f \subsetneq H_1.$$

[9] Order-Edges can output at most $\lfloor e_2/2 \rfloor$ pairs of sets E_i and V_i.

Observe that one or both vertices of f may be isolated in $T(f)$. This observation will be very useful later.

For each i and $f \in E_i$, define

$$(V(H_{\hat{e}}^-))_f := (V(H_{\hat{e}}^-) \cap U_J(f)) \setminus \left(\bigcup_{\substack{f' \in E_i: \\ f' \prec f}} f' \cup \left(\bigcup_{\substack{f' \in E_i: \\ f' \prec f}} \big(V(H_{\hat{e}}^-) \cap U_J(f')\big) \right) \right) \subseteq \Delta_V(f)$$

since $(V(H_{\hat{e}}^-))_f$ is a subset of $V(H_{\hat{e}}^-) \cap U_J(f)$. One can see that $(V(H_{\hat{e}}^-))_f$ consists of those vertices of $V(H_{\hat{e}}^-)$ which are new to V_i at the point when f is added to E_i but are not contained in f. Importantly for our purposes, every vertex in $(V(H_{\hat{e}}^-))_f$ is isolated in $T(f)$. Indeed, otherwise there exists $k < i$ and $f'' \in E_k$ such that $D_J(f) \cap D_J(f'') \neq \emptyset$ and f would have been previously added to E_k in line 17.

For each $f \in E$, let $T'(f)$ be the graph obtained from $T(f)$ by removing all isolated vertices from $V(T(f))$. Crucially for our proof, since vertices of f may be isolated in $T(f)$, one or more of them may not belong to $V(T'(f))$. Further, no vertex of $(V(H_{\hat{e}}^-))_f$ is contained in $V(T'(f))$.

For all $f \in E$ with $\Delta_e(f) \geq 1$, since $T'(f) \subsetneq H_1$ and H_1 is *strictly* balanced w.r.t. $d_2(\cdot, H_2)$, we have that

$$m_2(H_1, H_2) > d_2(T'(f), H_2) = \frac{|E(T'(f))|}{|V(T'(f))| - 2 + \frac{1}{m_2(H_2)}}. \tag{4.17}$$

Recall (4.12), that is

$$m_2(H_1, H_2) = \frac{e^+(J^*)}{v^+(J^*)}.$$

We now make the key observation of our proof: Since vertices of f may be isolated in $T(f)$, and so not contained in $V(T'(f))$, and no vertex of $(V(H_{\hat{e}}^-))_f$ is contained in $V(T'(f))$, we have that

$$|V(T'(f))| \leq \Delta_v(f) + |f \cap V(T'(f))| - |(V(H_{\hat{e}}))_f|. \tag{4.18}$$

Hence, from (4.17) and (4.18) we have that

$$\begin{aligned}\Delta_e(f) &= |E(T'(f))| \\ &< m_2(H_1, H_2)\left(\Delta_v(f) - (2 - |f \cap V(T'(f))|) - |(V(H_{\hat{e}}^-))_f| + \frac{1}{m_2(H_2)}\right). \end{aligned} \tag{4.19}$$

Edges f such that $|f \cap V(T'(f))| = 2$ will be, in some sense, 'bad' for us when trying to conclude (4.16). Indeed, if $|(V(H_{\hat{e}}^-))_f| = 0$ then we may have that $\frac{\Delta_e(f)}{\Delta_v(f)} \geq m_2(H_1, H_2) = \frac{e^+(J^*)}{v^+(J^*)}$, by (4.12). However, edges f such that $|f \cap V(T'(f))| \in \{0, 1\}$ will be, in some sense, 'good' for us when trying to conclude (4.16). Indeed, since $m_2(H_2) \geq 1$, we have for such edges f that $\frac{\Delta_e(f)}{\Delta_v(f)} < m_2(H_1, H_2) = \frac{e^+(J^*)}{v^+(J^*)}$.

We show in the following claim that our choice of ordering $\prec$, our choice of each A_i and the fact that H_2 is (strictly) 2-balanced ensure that for each i there are enough 'good' edges in A_i to compensate for any 'bad' edges that may appear in A_i.

Claim 4.5.11. *For each i,*

$$\sum_{f \in E_i} \Delta_e(f) < m_2(H_1, H_2) \sum_{f \in E_i} \Delta_v(f).$$

Proof. Fix i. Firstly, as observed before, each A_i is a non-empty subgraph of H_2. Moreover, $|E_i| \geq 2$ (by the condition in line 10). Since H_2 is (strictly) 2-balanced, we have that

$$m_2(H_2) \geq d_2(A_i) = \frac{|E_i| - 1}{|V_i| - 2}. \tag{4.20}$$

Now let us consider $\Delta_e(f)$ for each $f \in E_i$. For the edge f added in line 12, observe that $\Delta_e(f) = 0$. Indeed, otherwise $\Delta_e(f) \geq 1$, and there exists $k < i$ such that $D_J(f) \cap \left(\bigcup_{f' \in E_k} D_J(f')\right) \neq \emptyset$. That is, f would have been added to E_k in line 17 previously in the algorithm. Since $(V(H_{\hat{e}}^-))_f \subseteq \Delta_V(f)$, we have that

$$\Delta_e(f) = 0 \leq m_2(H_1, H_2)\left(\Delta_v(f) - |(V(H_{\hat{e}}^-))_f|\right). \tag{4.21}$$

Now, for each edge added in line 17, either $v, w \in V_i$ when uw was added to E_i, or $D_J(uw) \cap \left(\bigcup_{\substack{f' \in E_i: \\ f' \prec uw}} D_J(f')\right) \neq \emptyset$, that is, $\Delta_e(uw) \geq 1$, and at least one of u, w did not belong to V_i when uw was added to E_i. In the former case, if $\Delta_e(uw) \geq 1$, then by (4.19) and that $-(2 - |f \cap V(T'(f))|) \leq 0$, we have that

$$\Delta_e(uw) < m_2(H_1, H_2)\left(\Delta_v(uw) - |(V(H_{\hat{e}}^-))_{uw}| + \frac{1}{m_2(H_2)}\right). \tag{4.22}$$

Observe that (4.22) also holds when $\Delta_e(uw) = 0$. In the latter case, observe that for all $k < i$, we must have $D_J(uw) \cap \left(\bigcup_{f \in E_k} D_J(f)\right) = \emptyset$. Indeed, otherwise uw would have been added to some E_k in line 17. Combining this with knowing that at least one of u, w did not belong to V_i before uw was added to E_i, we must have that one or both of u, w are isolated in $T(uw)$. That is, one or both do not belong to $T'(uw)$, and so $|uw \cap V(T'(uw))| \in \{0, 1\}$. Thus, since $\Delta_e(uw) \geq 1$, by (4.19) we have that

$$\Delta_e(uw) < m_2(H_1, H_2)\left(\Delta_v(uw) - 1 - |(V(H_{\hat{e}}^-))_{uw}| + \frac{1}{m_2(H_2)}\right) \tag{4.23}$$

if $|uw \cap V(T'(uw))| = 1$, and

$$\Delta_e(uw) < m_2(H_1, H_2)\left(\Delta_v(uw) - 2 - |(V(H_{\hat{e}}^-))_{uw}| + \frac{1}{m_2(H_2)}\right) \tag{4.24}$$

if $|uw \cap V(T'(uw))| = 0$.

In conclusion, except for the two vertices in the edge added in line 12, every time a new vertex x was added to V_i when some edge f was added to E_i, either $x \in (V(H_{\hat{e}}^-))_f$, or $x \in f$ and (4.23) or (4.24) held, dependent on whether one or both of the vertices in f were new to V_i. Indeed, x was isolated in $T(f)$. Moreover, after the two vertices in the edge added in line 12 there are $|V_i| - 2$ vertices added to V_i.

Hence, by (4.21)-(4.24), we have that

$$\sum_{f \in E_i} \Delta_e(f) < m_2(H_1, H_2) \left(\sum_{f \in E_i} \Delta_v(f) - (|V_i| - 2) + \frac{|E_i| - 1}{m_2(H_2)} \right). \tag{4.25}$$

By (4.20),

$$-(|V_i| - 2) + \frac{|E_i| - 1}{m_2(H_2)} \leq 0. \tag{4.26}$$

Thus, by (4.25) and (4.26), we have that

$$\sum_{f \in E_i} \Delta_e(f) < m_2(H_1, H_2) \sum_{f \in E_i} \Delta_v(f)$$

as desired. □

Claim 4.5.12. *For each edge f placed onto s in line 24, we have $\Delta_e(f) = 0$.*

Proof. Assume not. Then $\Delta_e(f) \geq 1$ for some edge f placed onto s in line 24. Observe that $D_J(f) \cap \left(\bigcup_{f'' \in E_i} D_J(f'') \right) = \emptyset$ for any i, otherwise f would have been added to some E_i in line 17 previously.

Thus we must have that $D_J(f) \cap D_J(f') \neq \emptyset$ for some edge $f' \neq f$ where f' was placed onto s also in line 24. But then f and f' satisfy the condition in line 10 and would both be contained in some E_i, contradicting that f was placed onto s in line 24. □

Since $J \in \mathcal{H}(F, \hat{e}, H_1, H_2) \setminus \mathcal{H}^*(F, \hat{e}, H_1, H_2)$, we must have that $\Delta_v(f) \geq 1$ for some $f \in E$. Thus, if $\Delta_e(f) = 0$ for all $f \in E$ then (4.16) holds trivially. If $\Delta_e(f) \geq 1$ for some $f \in E$, then $E_1 \neq \emptyset$ and A_1 is a non-empty subgraph of $H_{\hat{e}}^-$. Then, by (4.12) and Claims 4.5.11 and 4.5.12, we have that

$$\begin{aligned}
\frac{\sum_{f \in E} \Delta_e(f)}{\sum_{f \in E} \Delta_v(f)} &= \frac{\sum_i \sum_{f \in E_i} \Delta_e(f) + \sum_{f \in E \setminus \cup_i E_i} \Delta_e(f)}{\sum_{f \in E} \Delta_v(f)} \\
&< \frac{m_2(H_1, H_2) \sum_i \sum_{f \in E_i} \Delta_v(f)}{\sum_i \sum_{f \in E_i} \Delta_v(f)} \\
&= m_2(H_1, H_2) \\
&= \frac{e^+(J^*)}{v^+(J^*)}.
\end{aligned}$$

Thus (4.16) holds and we are done.

4.6 Case 2: $m_2(H_1, H_2) = m_2(H_2)$.

In this section we prove Lemma 4.4.3 when $m_2(H_1) = m_2(H_2)$. Our proof follows that of Case 1 significantly, but uses a different algorithm Grow-Alt. All definitions and notation are the same as previously unless otherwise stated.

For any graph F and edge $e \in E(F)$, we say that e is *eligible for extension in* Grow-Alt if it satisfies

$$\nexists L \in \mathcal{L}_F, R \in \mathcal{R}_F \text{ s.t. } E(L) \cap E(R) = \{e\}.$$

We note here that this is substantially different to Case 1; indeed, the set $\mathcal{C}^*$ will not feature in what follows. Algorithm Grow-Alt is shown in Figure 4.7. As with Grow, it has input $G' \subseteq G$, the graph that Asym-Edge-Col got stuck on. Grow-Alt operates in a similar way to Grow. In line 14, the function Eligible-Edge-Alt is called which maps F_i to an edge $e \in E(F_i)$ which is eligible for extension in GROW-ALT. As with ELIGIBLE-EDGE in Case 1, this edge e is selected to be unique up to isomorphism. We then apply a new procedure EXTEND which attaches either a graph $L \in \mathcal{L}_{G'}$ or a graph $R \in \mathcal{R}_{G'}$ that contains e to F_i. As in Case 1, because the condition in line 6 of ASYM-EDGE-COL fails, $G' \in \mathcal{C}$.[10]

We now show that the number of edges of F_i increases by at least one and that GROW-ALT operates as desired with a result analogous to Claim 4.5.1.

Claim 4.6.1. *Algorithm* GROW-ALT *terminates without error on any input graph $G' \in \mathcal{C}$ that is no (H_1, H_2)-sparse $\hat{\mathcal{A}}$-graph.*[11] *Moreover, for every iteration i of the while-loop, we have $e(F_{i+1}) > e(F_i)$.*

Proof. The special cases in lines 2-9 and the assignments in lines 10 and 11 operate successfully for the exact same reasons as given in the proof of Claim 4.5.1.

Next, we show that the call to ELIGIBLE-EDGE-ALT in line 14 is always successful. Indeed, suppose for a contradiction that no edge in F_i is eligible for extension in GROW-ALT for some $i \geq 0$. Then for every edge $e \in E(F_i)$ there exist $L \in \mathcal{L}_F$ and $R \in \mathcal{R}_F$ s.t. $E(L) \cap E(R) = \{e\}$, by definition. Hence $F \in \mathcal{C}$. Recall that H_1 and H_2 satisfy the criteria of Conjecture 4.0.4. Hence H_1 and H_2 are strictly 2-balanced and $m_2(H_1) = m_2(H_2) > 1$. Then, by Lemma 4.3.3, we have that H_1 and H_2 are both 2-connected. Hence F_i is 2-connected by construction. However, our choice of F_0 in line 11 guarantees that F_i is not in $\hat{\mathcal{A}}$. Indeed, the edge e selected in line 10 satisfying $|\mathcal{S}_{G'}(e)| = 0$ is an edge of F_0 and $F_0 \subseteq F_i \subseteq G'$. Thus, by the definition of $\hat{\mathcal{A}}$ and that $m_2(H_1) = m_2(H_2)$, we have that $m(F_i) > m_2(H_1, H_2) + \varepsilon$. Thus, there exists a non-empty graph $\tilde{F} \subseteq F_i$ with $d(\tilde{F}) = m(F_i)$ such that

[10]Note that we could also conclude $G' \in \mathcal{C}^*$, however this will not be necessary, as noted earlier.
[11]See page 108 for the definition.

procedure Grow-Alt($G' = (V, E)$)
  if $\forall e \in E : |\mathcal{S}_{G'}(e)| = 1$ **then**
    $T \leftarrow$ any member of $\mathcal{T}_{G'}$
    return $\bigcup_{e \in E(T)} \mathcal{S}_{G'}(e)$
  end if
  if $\exists e \in E : |\mathcal{S}_{G'}(e)| \geq 2$ **then**
    $S_1, S_2 \leftarrow$ any two distinct members of $\mathcal{S}_{G'}(e)$
    return $S_1 \cup S_2$
  end if
  $e \leftarrow$ any $e \in E : |\mathcal{S}_{G'}(e)| = 0$
  $F_0 \leftarrow$ any $R \in \mathcal{R}_{G'} : e \in E(R)$
  $i \leftarrow 0$
  while $(i < \ln(n)) \wedge (\forall \tilde{F} \subseteq F_i : \lambda(\tilde{F}) > -\gamma)$ **do**
    $e \leftarrow$ Eligible-Edge-Alt(F_i)
    $F_{i+1} \leftarrow$ Extend(F_i, e, G')
    $i \leftarrow i + 1$
  end while
  if $i \geq \ln(n)$ **then**
    return F_i
  else
    return Minimising-Subgraph(F_i)
  end if
end procedure

procedure Extend(F, e, G')
  $\{L, R\} \leftarrow$ any pair $\{L, R\}$ such that $L \in \mathcal{L}_{G'}$, $R \in \mathcal{R}_{G'}$ and $E(L) \cap E(R) = e$
  if $L \nsubseteq F$ **then**
    $F' \leftarrow F \cup L$
  else
    $F' \leftarrow F \cup R$
  end if
  return F'
end procedure

Figure 4.7: The implementation of algorithm Grow-Alt.

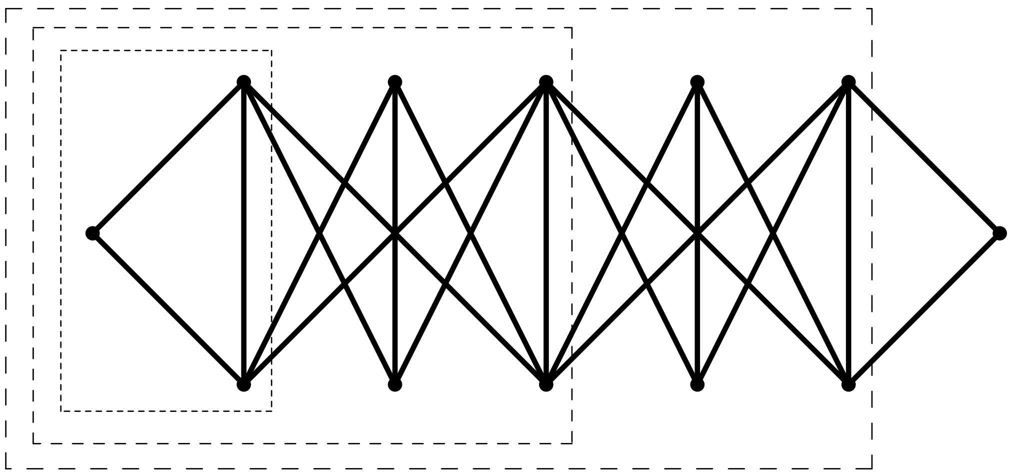

Figure 4.8: A graph F_3 resulting from three non-degenerate iterations for $H_1 = K_3$ and $H_2 = K_{3,3}$.

$$\begin{aligned}\lambda(\tilde{F}) &= v(\tilde{F}) - \frac{e(\tilde{F})}{m_2(H_1, H_2)} \\ &= e(\tilde{F}) \left(\frac{1}{m(F_i)} - \frac{1}{m_2(H_1, H_2)} \right) \\ &< e(\tilde{F}) \left(\frac{1}{m_2(H_1, H_2) + \varepsilon} - \frac{1}{m_2(H_1, H_2)} \right) \\ &= -\gamma e(\tilde{F}) \leq -\gamma.\end{aligned}$$

Thus Grow terminates in line 13 without calling Eligible-Edge-Alt. Thus every call to Eligible-Edge-Alt is successful and returns an edge e. Since $G' \in \mathcal{C}$, the call to Extend(F_i, e, G') is also successful and thus there exist suitable graphs $L \in \mathcal{L}_{G'}$ and $R \in \mathcal{R}_{G'}$ such that $E(L) \cap E(R) = \{e\}$, that is, line 2 is successful.

It remains to show that for every iteration i of the while-loop, we have $e(F_{i+1}) > e(F_i)$. Since e is eligible for extension in Grow-Alt for F_i and $E(L) \cap E(R) = \{e\}$, we must have that either $L \nsubseteq F_i$ or $R \nsubseteq F_i$. Hence Extend outputs $F' := F \cup L$ such that $e(F_{i+1}) = e(F') > e(F_i)$ or $F' := F \cup R$ such that $e(F_{i+1}) = e(F') > e(F_i)$. $\square$

We consider the evolution of F_i now in more detail. We call iteration i of the while-loop in algorithm Grow-Alt *non-degenerate* if the following hold for Extend:

- If $L \not\subseteq F_i$ (that is, line 3 is true), then in line 4 we have $V(F_i) \cap V(L) = e$;
- If $L \subseteq F_i$ (that is, line 3 is false), then in line 6 we have $V(F_i) \cap V(R) = e$.

Otherwise, we call iteration i *degenerate.* Note that, in non-degenerate iterations i, there are only a constant number of graphs that F_{i+1} can be for any given F_i; indeed, Eligible-Edge-Alt determines the exact position where to attach L or R (recall that the edge e found by Eligible-Edge-Alt(F_i) is *unique up to isomorphism of* F_i).

We now prove a result analogous to Claim 4.5.2 for Grow-Alt.

Claim 4.6.2. *If iteration i of the while-loop in procedure* Grow-Alt *is non-degenerate, we have*

$$\lambda(F_{i+1}) = \lambda(F_i).$$

Proof. In a non-degenerate iteration, we either add $v_2 - 2$ vertices and $e_2 - 1$ edges for the copy L of H_2 to F_i or add $v_1 - 2$ vertices and $e_1 - 1$ edges for the copy R of H_1 to F_i. In the former case,

$$\begin{aligned}\lambda(F_{i+1}) - \lambda(F_i) &= v_2 - 2 - \frac{e_2 - 1}{m_2(H_1, H_2)} \\ &= v_2 - 2 - \frac{e_2 - 1}{m_2(H_2)} \\ &= 0,\end{aligned}$$

where the second equality follows from $m_2(H_1, H_2) = m_2(H_2)$ (see Proposition 4.3.1) and the last equality follows from H_2 being (strictly) 2-balanced.

In the latter case,

$$\begin{aligned}\lambda(F_{i+1}) - \lambda(F_i) &= v_1 - 2 - \frac{e_1 - 1}{m_2(H_1, H_2)} \\ &= v_1 - 2 - \frac{e_1 - 1}{m_2(H_1)} \\ &= 0.\end{aligned}$$

where the second equality follows from $m_2(H_1, H_2) = m_2(H_1)$ (see Proposition 4.3.1) and the last equality follows from H_1 being (strictly) 2-balanced. □

As in Case 1, when we have a degenerate iteration i, the structure of F_{i+1} depends not just on F_i but also on the structure of G'. Indeed, if F_i is extended by a copy L of H_2 in line 4 of Extend, then L could intersect F_i in a multitude of ways. Moreover, there may be several copies of H_2 that satisfy the condition in line 2 of Extend. One could say the same for graphs added in line 6 of Extend. Thus, as in Case 1, degenerate iterations cause us difficulties since they enlarge the family of graphs algorithm Grow-Alt can return. However, we will show that at most a constant number of degenerate iterations can happen before algorithm Grow-Alt terminates, allowing us to bound from above sufficiently well the number of non-isomorphic graphs Grow-Alt can return. Pivotal in proving this is the following claim, analogous to Claim 4.5.3.

Claim 4.6.3. *There exists a constant $\kappa = \kappa(H_1, H_2) > 0$ such that if iteration i of the while-loop in procedure* Grow *is degenerate then we have*

$$\lambda(F_{i+1}) \leq \lambda(F_i) - \kappa.$$

Compared to the proof of Claim 4.5.3, the proof of Claim 4.6.3 is relatively straightforward.

Proof. Let $F := F_i$ be the graph before the operation in line 15 (of Grow-Alt) is carried out and let $F' := F_{i+1}$ be the output from line 15. We aim to show there exists a constant $\kappa = \kappa(H_1, H_2) > 0$ such that

$$\lambda(F) - \lambda(F') = v(F) - v(F') - \frac{e(F) - e(F')}{m_2(H_1, H_2)} \geq \kappa$$

whether Extend attached a graph $L \in \mathcal{L}_{G'}$ or $R \in \mathcal{R}_{G'}$ to F. We need only consider the case when $L \in \mathcal{L}_{G'}$ is the graph added by Extend in this non-degenerate iteration i of the while-loop, as the proof of the $R \in \mathcal{R}_{G'}$ case is identical. Let $V_{L'} := V(F) \cap V(L)$ and $E_{L'} := E(F) \cap E(L)$ and set $v_{L'} := |V_{L'}|$ and $e_{L'} := |E_{L'}|$. Let L' be the graph on vertex set $V_{L'}$ and edge set $E_{L'}$. By Claim 4.6.1, we have $e(F') > e(F)$, hence L' is a proper subgraph of H_2. Also, observe that since iteration i was degenerate, we must have that $v_{L'} \geq 3$. Let $F_{\hat{L}}$ be the graph produced by a non-degenerate iteration at e with a copy $\hat{L}$ of H_2, that is, $F_{\hat{L}} := F \cup \hat{L}$ and $V(F) \cap V(\hat{L}) = e$. Our strategy is to compare F' with $F_{\hat{L}}$. By Claim 4.6.2, $\lambda(F) = \lambda(F_{\hat{L}})$. Thus, since $m_2(H_1, H_2) = m_2(H_2)$, we have

$$\begin{aligned}
\lambda(F) - \lambda(F') = \lambda(F_{\hat{L}}) - \lambda(F') &= v_2 - 2 - (v_2 - v_{L'}) - \frac{(e_2 - 1) - (e_2 - e_{L'})}{m_2(H_1, H_2)} \\
&= v_{L'} - 2 - \frac{e_{L'} - 1}{m_2(H_1, H_2)}. \qquad (4.27)
\end{aligned}$$

If $e_{L'} = 1$, then since $v_{L'} \geq 3$ we have $\lambda(F) - \lambda(F') \geq 1$. So assume $e_{L'} \geq 2$. Since H_2 is *strictly* 2-balanced and L' is a proper subgraph of H_2 with $e_{L'} \geq 2$ (that is, L' is not an edge), we have that

$$\frac{v_{L'}-2}{e_{L'}-1} > \frac{1}{m_2(H_2)}. \tag{4.28}$$

Using (4.27), (4.28) and that $e_{L'} \geq 2$ and $m_2(H_1, H_2) = m_2(H)$, we have

$$\lambda(F) - \lambda(F') = (e_{L'}-1)\left(\frac{v_{L'}-2}{e_{L'}-1} - \frac{1}{m_2(H_2)}\right) > 0.$$

Letting $\delta := \frac{1}{2}\min\left\{(e_{L'}-1)\left(\frac{v_{L'}-2}{e_{L'}-1} - \frac{1}{m_2(H_2)}\right) : L' \subset H_2, e_{L'} \geq 2\right\}$, we take

$$\kappa := \min\{1, \delta\}.$$

□

Together, Claims 4.6.2 and 4.6.3 yield the following claim (analogous to Claim 4.5.4).

Claim 4.6.4. *There exists a constant $q_2 = q_2(H_1, H_2)$ such that algorithm* Grow-Alt *performs at most q_2 degenerate iterations before it terminates, regardless of the input instance G'.*

Proof. Analogous to the proof of Claim 4.5.4. □

For $0 \leq d \leq t \leq \lceil \ln(n) \rceil$, let $\mathcal{F}_{\text{Alt}}(t, d)$ denote a family of representatives for the isomorphism classes of all graphs F_t that algorithm Grow-Alt can possibly generate after exactly t iterations of the while-loop with exactly d of those t iterations being degenerate. Let $f_{\text{Alt}}(t, d) := |\mathcal{F}_{\text{Alt}}(t, d)|$.

Claim 4.6.5. *There exist constants $C_0 = C_0(H_1, H_2)$ and $A = A(H_1, H_2)$ such that $f_{\text{Alt}}(t, d) \leq \lceil \ln(n) \rceil^{(C_0+1)d} \cdot A^{t-d}$ for n sufficiently large.*

Proof. Analogous to the proof of Claim 4.5.5. □

Let $\mathcal{F}_{\text{ALT}} = \mathcal{F}_{\text{ALT}}(H_1, H_2, n)$ be a family of representatives for the isomorphism classes of *all* graphs that can be outputted by GROW-ALT (whether GROW-ALT enters the while-loop or not). Note that the proof of the following claim requires Conjecture 4.0.4 to be true; in particular, we need that $\hat{\mathcal{A}}$ is finite when $m_2(H_1) = m_2(H_2)$.

Claim 4.6.6. *There exists a constant $b = b(H_1, H_2) > 0$ such that for all $p \leq bn^{-1/m_2(H_1,H_2)}$, $G_{n,p}$ does not contain any graph from $\mathcal{F}_{\text{ALT}}(H_1, H_2, n)$ a.a.s.*

Proof. Analogous to the proof of Claim 4.5.6. □

Proof of Lemma 4.4.3: Case 2. Suppose that the call to ASYM-EDGE-COL(G) gets stuck for some graph G, and consider $G' \subseteq G$ at this moment. Then GROW-ALT(G') returns a copy of a graph $F \in \mathcal{F}_{\text{ALT}}(H_1, H_2, n)$ that is contained in $G' \subseteq G$. By Claim 4.6.6, this event a.a.s. does not occur in $G = G_{n,p}$ with p as claimed. Thus ASYM-EDGE-COL does not get stuck a.a.s. and, by Lemma 4.4.1, finds a valid colouring for H_1 and H_2 of $G_{n,p}$ with $p \leq bn^{-1/m_2(H_1,H_2)}$ a.a.s.

4.7 Proof of Theorem 4.0.6

Since H_1 and H_2 are regular graphs, let ℓ_1 be the degree of every vertex in H_1 and ℓ_2 be the degree of every vertex in H_2. We begin using an approach employed in [37] and [51]. Let A be a 2-connected graph such that $A \in \mathcal{C}^*(H_1, H_2)$ if $m_2(H_1) > m_2(H_2)$ and $A \in \mathcal{C}(H_1, H_2)$ if $m_2(H_1) = m_2(H_2)$. In both cases, $A \in \mathcal{C}(H_1, H_2)$. Then, since every vertex is contained in a copy of H_1 and a copy of H_2 whose edge-sets intersect in exactly one edge, we must have that $\delta(A) \geq \ell_1 + \ell_2 - 1$. Hence

$$d(A) = \frac{e_A}{v_A} \geq \frac{\ell_1 + \ell_2 - 1}{2}. \tag{4.29}$$

We aim to show $d(A) > m_2(H_1, H_2) + \varepsilon$ for some $\varepsilon = \varepsilon(H_1, H_2) > 0$. Indeed, if there exists $\varepsilon = \varepsilon(H_1, H_2) > 0$ such that for all such graphs A we have $d(A) > m_2(H_1, H_2) + \varepsilon$ then $\hat{\mathcal{A}}(H_1, H_2, \varepsilon) = \emptyset$, and so Conjecture 4.0.4 holds trivially for H_1 and H_2.

Since $m_2(H_1) \geq m_2(H_2) > 1$, we have that H_1 and H_2 cannot be matchings. Hence $\ell_1, \ell_2 \geq 2$. Also, observe that

$$m_2(H_1, H_2) = \frac{e_1}{v_1 - 2 + \frac{1}{m_2(H_2)}} = \frac{\frac{v_1\ell_1}{2}}{v_1 - 2 + \frac{v_2-2}{\frac{v_2\ell_2}{2}-1}} = \frac{v_1\ell_1}{2v_1 - 4 + \frac{4v_2-8}{v_2\ell_2-2}}. \tag{4.30}$$

Furthermore,

$$\begin{aligned} & \frac{\ell_1 + \ell_2 - 1}{2} > m_2(H_1, H_2) \\ \iff & \ell_1 + \ell_2 - 1 > \frac{v_1v_2\ell_1\ell_2 - 2v_1\ell_1}{2v_1v_2\ell_2 - 4v_2\ell_2 - 4v_1 + 4v_2} \\ \iff & \ell_1 + \ell_2 - 1 > \frac{2v_1v_2\ell_1\ell_2 - 4v_1\ell_1}{2v_1v_2\ell_2 - 4v_2\ell_2 - 4v_1 + 4v_2} \\ \iff & 0 < v_1v_2\ell_2(\ell_2 - 1) - 2v_1(\ell_2 - 1) - 2v_2\ell_1(\ell_2 - 1) - 2v_2(\ell_2 - 1)^2 \\ \iff & 0 < v_1v_2\ell_2 - 2v_1 - 2v_2\ell_1 - 2v_2\ell_2 + 2v_2, \end{aligned} \tag{4.31}$$

where in the last implication we used that $\ell_2 \geq 2$.

Let $f(v_1, v_2, \ell_1, \ell_2) := v_1v_2\ell_2 - 2v_1 - 2v_2\ell_1 - 2v_2\ell_2 + 2v_2$. Observe that $\ell_1 \leq v_1 - 1$. Hence $-2v_2\ell_1 \geq -2v_1v_2 + 2v_2$ and we have

$$f(v_1, v_2, \ell_1, \ell_2) \geq v_1v_2(\ell_2 - 2) - 2v_1 + 4v_2 - 2v_2\ell_2.$$

Let $g(v_1, v_2, \ell_2) := v_1v_2(\ell_2 - 2) - 2v_1 + 4v_2 - 2v_2\ell_2$ so $f(v_1, v_2, \ell_1, \ell_2) \geq g(v_1, v_2, \ell_2)$. Observe that, since $\ell_1, \ell_2 \geq 2$, we have that $v_1, v_2 \geq 3$. If $\ell_2 = 2$, then $g(v_1, v_2, 2) = -2v_1 < 0$.

However, if $\ell_2 \geq 3$, then since $v_1, v_2 \geq 3$, we have that

$$\frac{dg}{dv_1} = v_2(\ell_2 - 2) - 2 > 0;$$

$$\frac{dg}{dv_2} = v_1(\ell_2 - 2) + 4 - 2\ell_2 = (v_1 - 2)(\ell_2 - 2) > 0;$$

$$\frac{dg}{d\ell_2} = v_1 v_2 - 2v_2 = v_2(v_1 - 2) > 0.$$

Further,

$$g(4, 5, 3) = 20 - 8 + 20 - 30 = 2 > 0.$$

Thus, for all $v_1 \geq 4, v_2 \geq 5, \ell_2 \geq 3$, we have that

$$f(v_1, v_2, \ell_1, \ell_2) \geq g(v_1, v_2, \ell_2) \geq g(4, 5, 3) = 2 > 0.$$

Hence, by (4.29)-(4.31), we have that there exists a constant $\varepsilon := \varepsilon(H_1, H_2) > 0$ such that

$$d(A) > m_2(H_1, H_2) + \varepsilon.$$

Thus we only have left the cases when $v_1 = 3$, $v_2 \leq 4$ or $\ell_2 = 2$.

Case 1. $\ell_2 = 2$.

Then H_2 is a cycle and

$$f(v_1, v_2, \ell_1, 2) = 2v_1 v_2 - 2v_1 - 2v_2\ell_1 - 2v_2.$$

Observe that $\ell_1 \leq v_1 - 2$, as otherwise H_1 is a clique, contradicting that (H_1, H_2) is not a pair of a clique and a cycle. Thus $-\ell_1 \geq -(v_1 - 2)$. Then, since we excluded considering when H_2 is a cycle and H_1 is a graph with $v_1 = |V(H_1)| \geq |V(H_2)| = v_2$, we have that

$v_2 > v_1$, and so

$$f(v_1, v_2, \ell_1, 2) \geq 2(v_2 - v_1) \geq 2 > 0.$$

Hence by (4.29)-(4.31), we have that there exists a constant $\varepsilon := \varepsilon(H_1, H_2) > 0$ such that $d(A) > m_2(H_1, H_2) + \varepsilon$.

Case 2. $v_1 = 3$.

Then $H_1 = K_3$ and $\ell_1 = 2$. Thus

$$f(3, v_2, 2, \ell_2) = v_2(\ell_2 - 2) - 6.$$

We can assume $\ell_2 \geq 3$, as otherwise we are in Case 1. Observe that we cannot have that $v_2 = 6$ and $\ell_2 \geq 3$. Indeed, when $\ell_2 = 3$, one can check that the only strictly 2-balanced 3-regular graph on 6 vertices is $K_{3,3}$. But then $(H_1, H_2) = (K_3, K_{3,3})$, which is a pair of graphs we excluded from consideration.

When $\ell_2 \geq 4$, we have that $m_2(H_2) > m_2(H_1)$, contradicting that our choice of H_1 and H_2 meet the criteria in Conjecture 4.0.4. If $v_2 \leq 5$, then since also $\ell_2 \geq 3$ we must have that H_2 is a copy of K_4 or K_5.[12] But then $m_2(H_2) > m_2(H_1)$.

Hence $v_2 \geq 7$ and $\ell_2 \geq 3$. Thus $f(3, v_2, 2, \ell_2) > 0$ and, as before, we conclude that there exists a constant $\varepsilon := \varepsilon(H_1, H_2) > 0$ such that $d(A) > m_2(H_1, H_2) + \varepsilon$.

Case 3. $v_2 \leq 4$.

Still assuming $\ell_2 \geq 3$, we have that $H_2 = K_4$, $v_2 = 4$ and $\ell_2 = 3$. If $v_1 \geq \ell_1 + 2$, then

$$f(v_1, 4, \ell_1, 3) = 10v_1 - 8(\ell_1 + 2) \geq 2(\ell_1 + 2) > 0.$$

[12]There exists no graph with both odd regularity and odd order.

If $v_1 = \ell_1 + 1$, then H_1 is a clique. Since H_1 and H_2 meet the criteria in Conjecture 4.0.4, we must have that $v_1 \geq 5$. Therefore

$$f(v_1, 4, \ell_1, 3) = 10v_1 - 8(\ell_1 + 2) = 2v_1 - 8 \geq 2 > 0.$$

Then, as before, there exists a constant $\varepsilon := \varepsilon(H_1, H_2) > 0$ such that $d(A) > m_2(H_1, H_2) + \varepsilon$, as desired.

4.8 Concluding remarks

In [51], the value of ε was set at 0.01 for *every* pair of cliques, that is, the value of ε did not depend explicitly on the graphs H_1 and H_2. It would be interesting to know if there exists a single value of ε satisfying Conjecture 4.0.4 for *all* suitable pairs of graphs H_1 and H_2.

Also, in [51], Marciniszyn et al. note that they do not know whether $\mathcal{C}^*(H_1, H_2) = \mathcal{C}(H_1, H_2)$ or not for any pair of cliques. The author is unaware if this has been resolved for any pair of graphs. According with the definition of $\hat{\mathcal{A}}$, perhaps it is the case that $\mathcal{C}^*(H_1, H_2) = \mathcal{C}(H_1, H_2)$ whenever $m_2(H_1) = m_2(H_2)$?

Note that our method in this chapter does not completely extend to when $m_2(H_2) = 1$. Indeed, in this case H_2 is a forest and so not 2-connected, a property which is specifically used in the proofs of Claims 4.5.1, 4.5.6, 4.6.1 and 4.6.6. In the proofs of Claims 4.5.1 and 4.6.1 we require that F_i is 2-connected for each $i > 0$. However, if the last iteration of the while-loop of either Grow or Grow-Alt was non-degenerate when constructing F_i, then F_i is certainly not 2-connected if H_2 is a tree. A similar problem occurs at the beginning of the proofs of Claims 4.5.6 and 4.6.6 if $T \cong H_2$.

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