MINIMAL FUSION SYSTEMS

by

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A thesis submitted to
The University of Birmingham
for the degree of

DOCTOR OF PHILOSOPHY

School of Mathematics
The University of Birmingham
July 2010

Abstract

We define minimal fusion systems in a way that every non-solvable fusion system has a section which is minimal. Minimal fusion systems can also be seen as analogs of Thompson's N-groups. In this thesis, we consider a minimal fusion system $\mathcal{F}$ on a finite p-group S that has a unique maximal p-local subsystem containing $N_{\mathcal{F}}(S)$. For an arbitrary prime p, we determine the structure of a certain (explicitely described) p-local subsystem of $\mathcal{F}$. If $p = 2$, this leads to a complete classification of the fusion system $\mathcal{F}$.

Acknowledgements

First of all I would like to thank my supervisor Professor Sergey Shpectorov who gave me the basic idea for this project and at the same time encouraged me to develop my own ideas. I am very grateful to him for many fruitful discussions and for reading some parts of this thesis. The concept of a minimal fusion system is a generalization of a concept suggested to me by him.

I thank Professor Michael Aschbacher for pointing out that my original definition of a minimal fusion system is equivalent to the one given in this thesis.

Also, I am very thankful to Professor Bernd Stellmacher for his encouragement and mathematical support. It was him who suggested to me the application of results from [BHS]. Moreover, some arguments which appear in Chapters 7, 9 and 10 go back to ideas from him. Apart from that, I thank him for his thorough reading of, and suggestions regarding Chapters 7, 8, 9 and 10 of this thesis.

It is also a pleasure to thank Professor Chris Parker for many useful discussions.

Finally, I thankful acknowledge financial support from the EPSRC.

Contents

Chapter 1

Introduction

A pattern for the classification of finite simple groups was set by Thompson in [Th], where he gave a classification of all finite simple *N-groups*. These are non-abelian finite simple groups with the property that every p-local subgroup is solvable, for every prime p. Recall that a *p-local subgroup* of a finite group G is the normalizer of a non-trivial p-subgroup of G. Thompson's work was generalized by Gorenstein and Lyons, Janko and Smith to *(N2)-groups*, that is to non-abelian finite simple groups all of whose 2-local subgroups are solvable. Recall here that, by the Theorem of Feit-Thompson, every non-solvable group has even order.

N-groups play an important role, as every minimal non-solvable finite group is an N-group. Furthermore, every non-solvable group has a section which is an N-group. The respective properties hold also for (N2)-groups.

A new proof for the classification of (N2)-groups was given by Stellmacher in [St2]. It uses the amalgam method, which is a completely local method. Currently, Aschbacher is working on another new proof for the classification of (N2)-groups using *saturated fusion systems*. His plan is to classify all *N-systems*, i.e. all saturated fusion systems $\mathcal{F}$ of characteristic 2-type such that the group $Mor_{\mathcal{F}}(P, P)$ is solvable, for every subgroup P of $\mathcal{F}$. Here the use of the group theoretical concept of solvability fits with the definition of solvable fusion systems as introduced by Puig. However, this concept seems not general enough to ensure that N-systems play the same role in saturated fusion systems as N-groups in groups. Therefore, in our notion of *minimal fusion systems* introduced below, we find it necessary to use a concept of solvable fusion systems as defined by Aschbacher [A1, 15.1].

For the remainder of the introduction let p be a prime and $\mathcal{F}$ be a saturated fusion system on a finite p-group S. For basic definitions and notation regarding fusion systems we refer the reader to Chapter 2. Generic examples of saturated fusion systems are the fusion systems $\mathcal{F}_S(G)$, where G is a finite group containing S as a Sylow p-subgroup, the objects of $\mathcal{F}_S(G)$ are all subgroups of S, and the morphisms in $\mathcal{F}_S(G)$ between two objects are the injective group homomorphisms obtained by conjugation with elements of G. Fusion systems were first studied by Puig, although he called them *Frobenius categories* rather than fusion systems. The now standard terminology (that we also use in this thesis) was introduced by Broto, Levi and Oliver [BLO].

Definition 1.1. *The fusion system $\mathcal{F}$ is called minimal if $O_p(\mathcal{F}) = 1$ and $N_{\mathcal{F}}(U)$ is solvable for every fully normalized subgroup $U \neq 1$ of $\mathcal{F}$.*

Here the fusion system $\mathcal{F}$ is *solvable*, if and only if $O_p(\mathcal{F}/R) \neq 1$, for every strongly closed subgroup $R \neq S$ of $\mathcal{F}$. This implies that indeed every minimal non-solvable fusion system is minimal in the sense defined above. Furthermore, every non-solvable fusion system has a section which is minimal. Therefore, minimal fusion systems play a similar role in saturated fusion systems as N-groups in groups. However, a classification of minimal fusion systems seems a difficult generalization of the original N-group problem. One reason is that in fusion systems the prime 2 does not play such a distinguished role as in groups. Therefore, we would like to treat minimal fusion systems also for odd primes as far as possible. Secondly, the notion of solvability in fusion systems is more general than the group theoretical notion. More precisely, although it turns out that every solvable fusion system is constrained and therefore the fusion system of a finite group, such a group can have certain composition factors that are non-abelian finite simple groups. Aschbacher showed in [A1] that these are all finite simple groups in which fusion is controlled in the normalizer of a Sylow p-subgroup. Furthermore, Aschbacher gives a list of these groups. Generic examples are the finite simple groups of Lie type in characteristic p of Lie rank 1. For odd primes, Aschbacher's proof of these facts requires the complete classification of finite simple groups. For $p = 2$ they follow already from Goldschmidt's Theorem on groups with a strongly closed abelian subgroup.

In this thesis, we use a concept which is an analog to the (abstract) concept of parabolics in finite group theory, where a *parabolic subgroup* is defined to be a p-local subgroup containing a Sylow p-subgroup. This generalizes the definition of parabolics in finite groups of Lie type in characteristic p. Suppose S is a Sylow p-subgroup of a finite group G. It is a common strategy in the classification of finite simple groups and related problems to treat separately the case of a unique maximal (with respect to inclusion) parabolic containing S. In this case, one classifies as a first step a p-local subgroup of G which has the pushing up property as defined in Chapter 5. In the remaining case, two distinct maximal parabolics containing S form an amalgam of two groups that do not have a common normal p-subgroup. This usually allows an elegant treatment using the coset graph, and leads in the generic cases to a group of Lie type and Lie rank at least 2. The main result of this thesis handles the fusion system configuration which loosely corresponds to the pushing up case in the N-group investigation. We next introduce the concept of a parabolic in fusion systems.

Definition 1.2.

- *A subsystem of $\mathcal{F}$ of the form $N_{\mathcal{F}}(R)$ for some non-trivial normal subgroup R of S is called a* ***parabolic subsystem*** *of $\mathcal{F}$, or in short, a parabolic.*

- *A* ***full parabolic*** *is a parabolic containing* $N_{\mathcal{F}}(S)$. *It is called a* ***full maximal parabolic****, if it is not properly contained in any other parabolic subsystem of* $\mathcal{F}$.

Thus, in this thesis, we treat the case of a minimal fusion system having a unique full maximal parabolic. Note that this assumption is slightly more general than just supposing that a minimal fusion system has a unique maximal parabolic. Even more generally, we will in fact assume only that there is a proper saturated subsystem containing every full maximal parabolic. We will use the following notation.

Notation 1.3. *Let* $\mathcal{N}$ *be a subsystem of* $\mathcal{F}$ *on* S. *We write* $\mathcal{F}_{\mathcal{N}}$ *for the set of centric subgroups* Q *of* $\mathcal{F}$ *for which there exists an element of* $Mor_{\mathcal{F}}(Q, Q)$ *that is not a morphism in* $\mathcal{N}$.

Note here that, if $\mathcal{N}$ is a proper subsystem of $\mathcal{F}$, we get as a consequence of Alperin's Fusion Theorem that the set $\mathcal{F}_{\mathcal{N}}$ is non-empty. In our investigation we focus on members of $\mathcal{F}_{\mathcal{N}}$ that are maximal in the sense defined next.

Definition 1.4.

- *For every subgroup* P *of* S *write* $m(P)$ *for the order of an elementary abelian subgroup of* P *of maximal order.*
- *Let* $\mathcal{E}$ *be a set of subgroups of* S. *An element* Q *of* $\mathcal{E}$ *is called* ***Thompson-maximal*** *in* $\mathcal{E}$ *if, for every* $P \in \mathcal{E}$, $m(Q) \geq m(P)$ *and, if* $m(Q) = m(P)$*, then* $|J(Q)| \geq |J(P)|$.

Here, for a finite group G, the *Thompson subgroup* $J(G)$ (for the prime p) is the subgroup of G generated by the elementary abelian p-subgroups of G of maximal order. As a first step in our investigation we show the existence of *Thompson-restricted subgroups*. These are subgroups of S whose normalizer in $\mathcal{F}$ has a very restricted structure and involves $SL_2(q)$ acting on a natural module. More precisely, Thompson-restricted subgroups are defined as follows.

Definition 1.5. *Let $Q \in \mathcal{F}$ be centric and fully normalized. Set $T := N_S(Q)$ and let G be a model for $N_{\mathcal{F}}(Q)$. We call such a subgroup Q **Thompson-restricted** if, for every normal subgroup V of $J(G)T$ with $\Omega(Z(T)) \leq V \leq \Omega(Z(Q))$, the following hold:*

(i) $N_S(J(Q)) = T$ *and* $J(Q)$ *is fully normalized..*

(ii) $C_S(V) = Q$ *and* $C_G(V)/Q$ *is a* p'*-group.*

(iii) $J(G)/C_{J(G)}(V) \cong SL_2(q)$ *for some power* q *of* p*, and* $V/C_V(J(G))$ *is a natural* $SL_2(q)$*-module for* $J(G)/C_{J(G)}(V)$*.*

(iv) $C_T(J(G)/C_{J(G)}(V)) \leq Q$*.*

Here a *model* for $\mathcal{F}$ is a finite group G containing S as a Sylow p-subgroup such that $C_G(O_p(G)) \leq O_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$. By a Theorem of Broto, Castellana, Grodal, Levi and Oliver [BCGLO], there exists a (uniquely determined up to isomorphism) model for $\mathcal{F}$ provided $\mathcal{F}$ is constrained. Here $\mathcal{F}$ is called *constrained* if $\mathcal{F}$ has a normal p-subgroup containing its centralizer in S. For every fully normalized, centric subgroup Q of $\mathcal{F}$, the normalizer $N_{\mathcal{F}}(Q)$ is a constrained saturated subsystem of $\mathcal{F}$. This makes it possible in Definition 1.5 to choose a model for $N_{\mathcal{F}}(Q)$. For the definition of a *natural* $SL_2(q)$*-module* see Definition 4.10.

Crucial in our proof is the following theorem that requires neither the minimality of $\mathcal{F}$, nor the existence of a proper saturated subsystem containing every full maximal parabolic.

Theorem 1. *Let $\mathcal{N}$ be a proper saturated subsystem of $\mathcal{F}$ containing $C_{\mathcal{F}}(\Omega(Z(S)))$ and $N_{\mathcal{F}}(J(S))$. Then there exists a Thompson-maximal subgroup Q of $\mathcal{F}_{\mathcal{N}}$ such that Q is Thompson-restricted.*

The proof of Theorem 1 can be found in Chapter 8. It uses FF-module results of Bundy, Hebbinghaus, Stellmacher [BHS]. Apart from that, the proof is self-contained.

In particular, it is possible to avoid the use of the classification of finite simple groups or any kind of $\mathcal{K}$-group hypothesis in the proof of Theorem 1 and, in fact, in the proof of all the theorems in this thesis.

Note that $N_{\mathcal{F}}(\Omega(Z(S)))$ and $N_{\mathcal{F}}(J(S))$ are full parabolics of $\mathcal{F}$, as $\Omega(Z(S))$ and $J(S)$ are characteristic in S. In particular, if $\mathcal{N}$ is a proper saturated subsystem of $\mathcal{F}$ containing every full parabolic, then $\mathcal{N}$ fulfils the hypothesis of Theorem 1. Hence, there exists a Thompson-maximal subgroup Q of $\mathcal{F}_{\mathcal{N}}$ such that Q is Thompson-restricted. As we show in the next theorem, the fusion system $\mathcal{F}$ being minimal implies for each such Q that $N_{\mathcal{F}}(Q)$ has a very simple structure.

Theorem 2. *Let $\mathcal{F}$ be minimal and let $\mathcal{N}$ be a proper saturated subsystem of $\mathcal{F}$ containing every full parabolic. Let $Q \in \mathcal{F}_{\mathcal{N}}$ such that Q is Thompson-restricted and Thompson-maximal in $\mathcal{F}_{\mathcal{N}}$. Let G be a model for $N_{\mathcal{F}}(Q)$ and $M = J(G)$. Then $N_S(X) = N_S(Q)$ for every non-trivial normal p-subgroup X of M, $M/Q \cong SL_2(q)$, and one of the following holds:*

(I) Q is elementary abelian, and $Q/C_Q(M)$ is a natural $SL_2(q)$-module for M/Q, or

(II) $p = 3$, $S = N_S(Q)$ and $|Q| = q^5$. Moreover, $Q/Z(Q)$ and $Z(Q)/\Phi(Q)$ are natural $SL_2(q)$-modules for M/Q, and $\Phi(Q) = C_Q(M)$.

The proof of Theorem 2 can be found in Chapter 10 and is self-contained. For $p = 2$, Theorem 1 and Theorem 2 lead to a complete classification of the fusion system $\mathcal{F}$. This is a direct consequence of a more general result (Theorem 7.2) on fusion systems of characteristic 2-type that we prove in Chapter 7. This proof relies on a group theoretical result (Theorem 6.3) from Chapter 6. It uses a special case of the classification of weak BN-pairs of rank 2 from [DGS] (see Theorem 6.5), and is apart from that self-contained. However, many of our arguments are similar to the ones in

[A2]. In fact, using the above mentioned theorem of Goldschmidt on groups with a strongly closed 2-group, the following classification for $p = 2$ could also be obtained as a consequence of [A2]. However, we prefer in this thesis to give a proof that does not rely on this theorem.

Theorem 3. *Assume $p = 2$, $\mathcal{F}$ is minimal, and there is a proper saturated subsystem of $\mathcal{F}$ containing every full parabolic of $\mathcal{F}$. Then there is a finite group G containing S as a Sylow 2-subgroup such that $\mathcal{F} \cong \mathcal{F}_S(G)$ and one of the following holds:*

(a) *S is dihedral of order at least 16, and $G \cong L_2(r)$ or $PGL_2(r)$, for some odd prime power r.*

(b) *S is semidihedral, and G is an extension of $L_2(r^2)$ by an automorphism of order 2, for some odd prime power r.*

(c) *S is semidihedral of order 16, and $G \cong L_3(3)$.*

(d) *$|S| = 32$, and $G \cong Aut(A_6)$ or $Aut(L_3(3))$.*

(e) *$|S| = 2^7$ and $G \cong J_3$.*

(f) *$F^*(G) \cong L_3(q)$ or $Sp_4(q)$, $|O^2(G) : F^*(G)|$ is odd and $|G : O^2(G)| = 2$. Moreover, if $F^*(G) \cong Sp_4(q)$ then $q = 2^e$ where e is odd.*

Throughout this thesis, we write mappings on the right side. By p we will always denote a prime. In our notation and terminology regarding fusion systems we mostly follow [BLO]. The reader can find a brief introduction in Chapter 2. We adapt the group theoretic notions from [KS]. In particular, if G is a finite group, we write $O_p(G)$ for the largest normal p-subgroup of G. We define G to be p-closed if it has a normal Sylow p-subgroup. Moreover, for a normal subgroup N of G, we will often make use of the so called "bar"-notation. This means that, after setting $\overline{G} = G/N$, we write $\overline{U}$ (respectively $\overline{g}$) for the image of a subgroup U of G (respectively, an element $g \in G$) in $\overline{G}$. For further notation and terminology regarding groups see Section 3.1.

Chapter 2

Preliminaries on Fusion Systems

2.1 Basic definitions

Let G be a group. For $g \in G$ we write $c_g : G \to G$ for the inner automorphism of G determined by g.

Let P and Q be subgroups of G. For any map $\phi : P \to Q$, $A \leq P$ and $A\phi \leq B \leq G$ we denote by $\phi_{|A,B}$ the map with domain A and range B mapping each element of A to its image under ϕ. In particular for $x \in G$ with $P^x \leq Q$, $c_{x|P,Q}$ is the restriction of c_x to the domain P and the range Q. Set

$$Mor_G(P, Q) = \{c_{g|P,Q} \mid g \in G,\ P^g \leq Q\}.$$

For $P \leq Q \leq G$, $\iota_{P,Q}$ denotes the natural embedding of P into Q, i.e. the map from P to Q which maps each element of P to itself. Throughout this thesis we use the following notation:

Notation 2.1. *For subgroups P and R of G set*

$$R_P := Aut_R(P) := \{c_{g|P,P} : g \in N_R(P)\}.$$

We now start to introduce fusion systems.

Definition 2.2. *Let S be a group. A **fusion system** on S is a category $\mathcal{F}$ whose objects are all subgroups of S and whose morphisms satisfy the following properties for all $P, Q \leq S$.*

(1) $Mor_{\mathcal{F}}(P,Q)$ *is a set of injective group homomorphisms, containing* $Mor_S(P,Q)$.

(2) The composition of morphisms in $\mathcal{F}$ *is the same as the composition of group homomorphisms.*

(3) For each $\phi \in Mor_{\mathcal{F}}(P,Q)$, $\phi_{|P,P\phi} \in Mor_{\mathcal{F}}(P,P\phi)$.

(4) If $\phi \in Mor_{\mathcal{F}}(P,Q)$ *is surjective, then the inverse map* ϕ^{-1} *is an element of* $Mor_{\mathcal{F}}(Q,P)$.

The main class of examples for fusion systems is the following.

Example 2.3. *Let* S *be a subgroup of* G. *Write* $\mathcal{F}_S(G)$ *for the category whose objects are all the subgroups of* S*, and for objects* $P,Q \in \mathcal{F}_S(G)$,

$$Mor_{\mathcal{F}_S(G)}(P,Q) = Mor_G(P,Q).$$

Then $\mathcal{F}_S(G)$ *is a fusion system on* S.

From now on let S be a group and $\mathcal{F}$ a fusion system on S.

By an abuse of notation we will write $\mathcal{F}$ for the set of all objects of $\mathcal{F}$. In particular we write $Q \in \mathcal{F}$ instead of $Q \leq S$.

Note that by axiom (4), an $\mathcal{F}$-morphism is an isomorphism in the sense of category theory if and only if it is a group isomorphism, and the inverse map is then also the inverse in the categorical sense. Moreover we can think of the inclusion maps between appropriate subgroups of S as maps obtained by conjugation with $1 \in S$. Hence, by the first axiom, they all are morphisms in $\mathcal{F}$. Also note that by axiom (3) we can factor every $\mathcal{F}$-morphism as an $\mathcal{F}$-isomorphism followed by inclusion. Thus, usually it is sufficient to consider properties of $\mathcal{F}$-isomorphisms.

Assume now $P,Q \in \mathcal{F}$ and $\phi \in Mor_{\mathcal{F}}(P,Q)$ is an isomorphism. Let $A \leq P$ and $A\phi \leq B \leq S$. Then $\phi_{|A,Q} = \iota_{A,P}\phi \in Mor_{\mathcal{F}}(A,Q)$. Therefore by axiom (3) applied to $\phi_{|A,Q}$ instead of ϕ, $\phi_{|A,A\phi} \in Mor_{\mathcal{F}}(A, A\phi)$. Hence $\phi_{|A,B} = \phi_{|A,A\phi}\iota_{A\phi,B} \in Mor_{\mathcal{F}}(A, B)$. So, for a given $\mathcal{F}$-morphism, we can restrict its source and take any suitable target, and again obtain an $\mathcal{F}$-morphism.

Definition 2.4. • *A **subsystem** of $\mathcal{F}$ is a fusion system $\mathcal{E}$ on a subgroup T of S such that for all $A, B \leq T$, $Mor_{\mathcal{E}}(A, B) \subseteq Mor_{\mathcal{F}}(A, B)$.*

- *Assume now we are given a family $(\mathcal{F}_i : i \in I)$ of fusion systems $\mathcal{F}_i$ on subgroups S_i of S. The fusion system $\mathcal{E}$ is called the **intersection** of all $\mathcal{F}_i$, if $\mathcal{E}$ is a fusion system on $T := \bigcap_{i\in I} S_i$, and $Mor_{\mathcal{E}}(A, B) = \bigcap_{i\in I} Mor_{\mathcal{F}_i}(A, B)$ for all $A, B \leq T$.*

- *The fusion systems $\mathcal{F}$ is said to be **generated by** $(\mathcal{F}_i : i \in I)$, if $\mathcal{F}$ is the intersection of all those fusion systems on S which contain each $\mathcal{F}_i$ as a subsystem. We then write $\mathcal{F} = \langle \mathcal{F}_i : i \in I \rangle$.*

Note here that the intersection of fusion systems as defined above is indeed again a fusion system. Also, if we are given a group S, then we can form the fusion systems on S whose morphisms are all injective group homomorphisms between subgroups of S. This fusion system contains every fusion system on S. Therefore, the generation of fusion systems as above is well defined. Moreover, if $\mathcal{F} = \langle \mathcal{F}_i : i \in I \rangle$ for a family of fusion systems $(\mathcal{F}_i : i \in I)$, then the morphisms in $\mathcal{F}$ are precisely the group homomorphisms between subgroups of S which are the composition of morphisms from the $\mathcal{F}_i$.

Definition 2.5. *Let $\tilde{\mathcal{F}}$ be a fusion system on a group $\tilde{S}$. Then an isomorphism of groups $\alpha : S \to \tilde{S}$ is called an **isomorphism** from $\mathcal{F}$ to $\tilde{\mathcal{F}}$ if*

$$Mor_{\tilde{\mathcal{F}}}(P\alpha, Q\alpha) = \{\alpha^{-1}\phi\alpha : \phi \in Mor_{\mathcal{F}}(P, Q)\} \text{ for all } P, Q \in \mathcal{F}.$$

The fusion systems $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are called isomorphic if there exists an isomorphism between them.

If $\tilde{\mathcal{F}}$ is a fusion system as above and α an isomorphism between $\mathcal{F}$ and $\tilde{\mathcal{F}}$, then for $P, Q \in \mathcal{F}$ the maps

$$\alpha_{P,Q} : Mor_{\mathcal{F}}(P, Q) \to Mor_{\tilde{\mathcal{F}}}(P\alpha, Q\alpha), \phi \mapsto \alpha^{-1}\phi\alpha$$

are bijective. Moreover, together with the map $\mathcal{F} \to \tilde{\mathcal{F}}$ defined by $P \mapsto P\alpha$, they give an isomorphism from the category $\mathcal{F}$ to the category $\tilde{\mathcal{F}}$. Thus, if $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are isomorphic, then they are also isomorphic as categories.

Example 2.6. *Let G, H be groups, $S \leq G$ and let $\phi : G \to H$ be an isomorphism of groups. Then the map from S to $S\phi$ induced by ϕ is an isomorphism of fusion systems from $\mathcal{F}_S(G)$ to $\mathcal{F}_{S\phi}(H)$.*

Definition 2.7. *Let $P, Q \in \mathcal{F}$.*

- *We say P **is fused into** Q if $Mor_{\mathcal{F}}(P, Q) \neq \emptyset$.*
- *Q is said to be **$\mathcal{F}$-conjugate** to P if there exists an isomorphism in $Mor_{\mathcal{F}}(P, Q)$. By $P^{\mathcal{F}}$ we denote the **$\mathcal{F}$-conjugacy class** of P, i.e. the set of all subgroups of S which are $\mathcal{F}$-conjugate to P.*

Note that for two $\mathcal{F}$-conjugate subgroups P, Q of S all the elements of $Mor_{\mathcal{F}}(P, Q)$ are isomorphisms.

Definition 2.8. *Let $P \in \mathcal{F}$.*

- *The subgroup P is called **centric** if for every $Q \in P^{\mathcal{F}}$, $C_S(Q) \leq Q$.*
- *Define P to be **fully centralized (fully normalized)** if for all $Q \in P^{\mathcal{F}}$, $|C_S(P)| \geq |C_S(Q)|$ ($|N_S(P)| \geq |N_S(Q)|$, respectively).*

Remark 2.9. *Assume P is fully centralized and $C_S(P) \leq P$. Then P is centric.*

Proof. Let $Q \in P^{\mathcal{F}}$. Since P is fully centralized, $|Z(Q)| \leq |C_S(Q)| \leq |C_S(P)| = |Z(P)| = |Z(Q)|$. Thus, $C_S(Q) = Z(Q) \leq Q$. □

Remark 2.10. *Let $Q \in \mathcal{F}$ and let U be a characteristic subgroup of Q. Assume U is fully normalized in $\mathcal{F}$ and $N_S(U) = N_S(Q)$. Then Q is fully normalized.*

Proof. Let $P \in Q^{\mathcal{F}}$ and $\phi \in Mor_{\mathcal{F}}(Q, P)$. Then $U\phi \in U^{\mathcal{F}}$ and $U\phi$ is characteristic in $P = Q\phi$. So, as U is fully normalized, we get

$$|N_S(P)| \leq |N_S(U\phi)| \leq |N_S(U)| = |N_S(Q)|.$$

Hence, Q is fully normalized. □

Notation 2.11.

- *For every $P \in \mathcal{F}$ set*

$$Aut_{\mathcal{F}}(P) = Mor_{\mathcal{F}}(P, P).$$

- *For $P, Q \in \mathcal{F}$ and an isomorphism $\phi \in Mor_{\mathcal{F}}(P, Q)$ we write ϕ^* for the map*

$$\phi^* : Aut_{\mathcal{F}}(P) \to Aut_{\mathcal{F}}(Q) \text{ defined by } \alpha \mapsto \phi^{-1}\alpha\phi.$$

- *If $P \leq A \leq S$, $Q \leq B \leq S$ and $\phi \in Mor_{\mathcal{F}}(A, B)$ such that $\phi_{|P,Q}$ is an isomorphism, then we sometimes write ϕ^* instead of $(\phi_{|P,Q})^*$.*

Note that for every $P \in \mathcal{F}$, $Aut_{\mathcal{F}}(P)$ is a group acting on P. Also, $Aut_R(P)$ is a subgroup of $Aut_{\mathcal{F}}(P)$ for each $R \in \mathcal{F}$ containing P. Furthermore, observe that for all $P, Q \in \mathcal{F}$ and every isomorphism $\phi \in Mor_{\mathcal{F}}(P, Q)$, the map $\phi^* : Aut_{\mathcal{F}}(P) \to Aut_{\mathcal{F}}(Q)$ is an isomorphism of groups.

Remark 2.12. *Let $P, R, T \in \mathcal{F}$, $\phi \in Mor_{\mathcal{F}}(R, T)$ and assume $P \leq R \leq N_S(P)$. Then*

$$c_{g|P,P}(\phi_{|P,P\phi})^* = c_{g\phi|P\phi,P\phi} \text{ for every } g \in R.$$

Notation 2.13. *Let $P, Q \in \mathcal{F}$ and $\phi \in Mor_{\mathcal{F}}(P, Q)$ be an isomorphism. Set*

$$N_\phi = \{g \in N_S(P) \mid c_{g|P,P}\phi^* \in Aut_S(Q)\}.$$

Remark 2.14. *Let $P, Q \in \mathcal{F}$ and $\phi \in Mor_{\mathcal{F}}(P, Q)$ be an isomorphism. Then $C_S(P)P \leq N_\phi$.*

Proof. By Remark 2.12 we have $c_{g|P,P}\phi^* = c_{g\phi|Q,Q} \in Aut_S(Q)$ for all $g \in P$. Thus $P \leq N_\phi$. Moreover for $g \in C_S(P)$, $c_{g|P,P}\phi^* = id_P\phi^* = id_Q = c_{1|Q,Q} \in Aut_S(Q)$. □

We are now able to define saturated fusion systems.

Definition 2.15. *$\mathcal{F}$ is said to be **saturated** if S is a finite p-group and the following conditions hold for every $P \in \mathcal{F}$:*

(I) If P is fully normalized, then P is fully centralized and we have $Aut_S(P) \in Syl_p(Aut_{\mathcal{F}}(P))$.

(II) If $Q \in P^{\mathcal{F}}$ is fully centralized and $\phi \in Mor_{\mathcal{F}}(P, Q)$, then ϕ extends to a member of $Mor_{\mathcal{F}}(N_\phi, S)$.

Example 2.16. *Let G be a finite group, $S \in Syl_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$. Then $\mathcal{F}$ is a saturated fusion system on S.*

Moreover the following hold for every $P \in \mathcal{F}$:

(a) P is fully centralized if and only if $C_S(P) \in Syl_p(C_G(P))$.

(b) P is fully normalized if and only if $N_S(P) \in Syl_p(N_G(P))$.

Proof. See for example [Lin, 2.10 and 2.11]. □

2.2 Saturated fusion systems

For the remainder of this chapter let $\mathcal{F}$ be a saturated fusion system on S. Moreover we set

$$A(P) = Aut_{\mathcal{F}}(P)$$

for every $P \in \mathcal{F}$.

Lemma 2.17. *Let $\phi \in A(U)$ and $U \leq X \leq N_S(U)$.*

(a) If ϕ extends to a member of $A(X)$ then $X_U\phi^* = X_U$.

(b) Assume $C_S(U) \leq X$ and U is fully centralized. Then $X_U\phi^* = X_U$ if and only if ϕ extends to a member of $A(X)$.

Proof. Remark 2.12 implies (a). Property (b) is a consequence of (a) and axiom (II) in Definition 2.15. □

Lemma 2.18. *Let $Q \in \mathcal{F}$. Then Q is fully normalized if and only if, for each $P \in Q^{\mathcal{F}}$, there exists a morphism $\phi \in Mor_{\mathcal{F}}(N_S(P), N_S(Q))$ such that $P\phi = Q$.*

Proof. See for example [Lin, 2.6]. □

2.3 Normalizers and centralizers

Definition 2.19 (Puig)**.** *Let $P \in \mathcal{F}$.*

- *We define $N_{\mathcal{F}}(P)$ (the **normalizer in $\mathcal{F}$ of** P) to be the category whose objects are the subgroups of $N_S(P)$ such that for $A, B \leq N_S(P)$, $Mor_{N_{\mathcal{F}}(P)}(A, B)$ is the set of all $\phi \in Mor_{\mathcal{F}}(A, B)$ which extend to an element of $Mor_{\mathcal{F}}(AP, BP)$ taking P to P.*

- *The subgroup P is called **normal** in $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(P)$; that is, $P \trianglelefteq S$ and for all $R, Q \leq S$, each $\phi \in Mor_{\mathcal{F}}(R, Q)$ extends to a member of $Mor_{\mathcal{F}}(RP, QP)$ which normalizes P. We write $P \trianglelefteq \mathcal{F}$ to indicate that P is normal in $\mathcal{F}$.*

- *By $O_p(\mathcal{F})$ we denote the largest subgroup of S which is normal in $\mathcal{F}$*

Note here that $N_{\mathcal{F}}(P)$ is a fusion system on $N_S(P)$. Moreover, $O_p(\mathcal{F})$ is well defined since the product of two normal subgroups of $\mathcal{F}$ is again normal in $\mathcal{F}$.

Definition 2.20 (Puig)**.** *Let $P \in \mathcal{F}$. We define $C_{\mathcal{F}}(P)$ (the **centralizer in $\mathcal{F}$ of** P) to be the category whose objects are the subgroups of $C_S(P)$ such that for $A, B \leq C_S(P)$, $Mor_{C_{\mathcal{F}}(P)}(A, B)$ is the set of all $\phi \in Mor_{\mathcal{F}}(A, B)$ which extend to an element of $Mor_{\mathcal{F}}(AP, BP)$ that is the identity on P.*

Again, note that $C_{\mathcal{F}}(P)$ is a fusion system on $C_S(P)$.

Proposition 2.21 (Puig)**.** *Let P be fully normalized (respectively, fully centralized) in $\mathcal{F}$. Then $N_{\mathcal{F}}(P)$ (respectively $C_{\mathcal{F}}(P)$) is a saturated fusion system on $N_S(P)$.*

Proof. This is a direct consequence of Proposition A.6 in [BLO]. □

Remark 2.22. *Let $P \in \mathcal{F}$ and $P \leq Q \leq N_S(P)$. Then $Aut_{N_{\mathcal{F}(P)}}(Q) = N_{A(Q)}(P)$.*

2.4 Factor systems

Factor systems are defined modulo strongly closed subgroups. Here a strongly closed subgroup is defined as follows.

Definition 2.23 (Puig)**.** *A subgroup $P \in \mathcal{F}$ is called strongly closed if, for all $A, B \leq S$ and all $\phi \in Mor_{\mathcal{F}}(A, B)$, we have $(A \cap P)\phi \leq P$.*

Note that every normal subgroup of $\mathcal{F}$ is strongly closed in $\mathcal{F}$, and that every strongly closed subgroup of F is normal in S.

Definition 2.24 (Puig)**.** *Let $R \leq S$ be a strongly closed subgroup of $\mathcal{F}$. Set $\overline{S} = S/R$. For $P, Q \leq S$ and $\phi \in Mor_{\mathcal{F}}(P, Q)$ define a group isomorphism*

$$\overline{\phi} : \overline{P} \to \overline{Q} \text{ by } \overline{x} \mapsto \overline{(x\phi)}.$$

Note that $\overline{\phi}$ is well defined since R is strongly closed. Now let $\overline{\mathcal{F}}$ be the category whose objects are the subgroups of $\overline{S}$ and, for all $P, Q \leq S$ with $R \leq P \cap Q$,

$$Mor_{\overline{\mathcal{F}}}(\overline{P}, \overline{Q}) = \{\overline{\phi} \mid \phi \in Mor_{\mathcal{F}}(P, Q)\}.$$

Then $\overline{\mathcal{F}}$ is again a fusion system. We denote it by $\mathcal{F}/R$.

Proposition 2.25 (Puig)**.** *Let R be a strongly closed subgroup of $\mathcal{F}$. Then $\mathcal{F}/R$ is saturated.*

Proof. This follows from [Pu, 6.3]. □

2.5 Constrained and solvable fusion systems

Definition 2.26. *We say $\mathcal{F}$ is* ***constrained*** *if there exists a normal centric subgroup in $\mathcal{F}$.*

Example 2.27. *Let G be a finite group, p a prime, $S \in Syl_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$. If G has characteristic p (i.e. $C_G(O_p(G)) \leq O_p(G)$), then $\mathcal{F}$ is constrained.*

Definition 2.28. *A finite group G of characteristic p is called a* ***model*** *for $\mathcal{F}$, if $S \in Syl_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$.*

In fact even the converse of 2.27 is true.

Theorem 2.29 (Broto, Castellana, Grodal, Levi, Oliver)**.** *Let $\mathcal{F}$ be constrained.*

- *There exists a model for $\mathcal{F}$.*
- *If G_1, G_2 are models for $\mathcal{F}$ then there exists an isomorphism $\phi : G_1 \rightarrow G_2$ which is the identity on S.*

Proof. This is Proposition C in [BCGLO]. □

Notation 2.30. *Let $P \in \mathcal{F}$ such that P is fully normalized and $N_{\mathcal{F}}(P)$ is constrained. Then by $G(P)$ we denote a model for $N_{\mathcal{F}}(P)$.*

Note here that, by Theorem 2.29, $G(P)$ exists and is uniquely determined up isomorphism.

In our definition of solvable fusion systems we follow Aschbacher [A1], who defined $\mathcal{F}$ to be solvable if every composition factor of $\mathcal{F}$ is the fusion system of the group of order p. However, as we have not defined normal subsystems and composition factors in this thesis, we prefer to give the definition in the language we introduced. Aschbacher [A1, 15.2,15.3] has shown his definition to be equivalent to the following.

Definition 2.31. *The fusion system $\mathcal{F}$ is* ***solvable*** *if and only if $O_p(\mathcal{F}/R) \neq 1$ for every strongly closed subgroup R of $\mathcal{F}$.*

We will use the following properties of solvable fusion systems.

Theorem 2.32 (Aschbacher)**.** *Let $\mathcal{F}$ be solvable.*

(a) Every saturated subsystem of $\mathcal{F}$ is solvable.

(b) $\mathcal{F}$ is constrained.

In the following definition we follow Aschbacher [A2].

Definition 2.33. *Define $\mathcal{F}$ to be of* ***characteristic p-type*** *if $N_{\mathcal{F}}(P)$ is constrained, for every fully normalized subgroup $P \in \mathcal{F}$.*

Recall from the introduction that we call $\mathcal{F}$ **minimal** if $N_{\mathcal{F}}(P)$ is solvable, for every fully normalized subgroup $P \in \mathcal{F}$. As a consequence of Theorem 2.32(b) we get:

Corollary 2.34. *If $\mathcal{F}$ is minimal then $\mathcal{F}$ is of characteristic p-type.*

2.6 The Alperin–Goldschmidt Fusion Theorem

Definition 2.35. *A subgroup $Q \in \mathcal{F}$ is called* ***essential*** *if Q is centric and*

$$A(Q)/Inn(Q)$$

has a strongly p-embedded subgroup.

Recall that a proper subgroup H of a finite group G is called **strongly p-embedded** if p divides the order of H, and the order of $H \cap H^g$ is not divisible by p for every $g \in G \backslash H$. It is elementary to check that every $\mathcal{F}$-conjugate of an essential subgroup is again essential. This allows to refer to **essential classes** meaning the $\mathcal{F}$-conjugacy classes of essential subgroups.

Theorem 2.36 (The Alperin–Goldschmidt Fusion Theorem, Puig)**.** *Let $\mathcal{C}$ be a set of subgroups of S such that $S \in \mathcal{C}$ and $\mathcal{C}$ intersects non-trivially with every essential class. Then, for all $P, Q \leq S$ and every isomorphism $\phi \in Mor_{\mathcal{F}}(P, Q)$, there exist sequences of subgroups of S*

$$P = P_0, P_1, \ldots, P_n = Q \text{ in } \mathcal{F}, \text{ and } Q_1, \ldots, Q_n \text{ in } \mathcal{C}$$

and elements $\alpha_i \in A(Q_i)$ for $i = 1, \ldots, n$ such that $P_{i-1}, P_i \leq Q_i$, $P_{i-1}\alpha_i = P_i$ and

$$\phi = (\alpha_{1|P_0,P_1})(\alpha_{2|P_1,P_2}) \ldots (\alpha_{n|P_{n-1},P_n}).$$

Proof. This is a direct consequence of [Lin, 5.2] and [DGMP, 2.10]. □

The proof of the following Lemma uses Theorem 2.36.

Lemma 2.37. *Let $N \in \mathcal{F}$, and let $\mathcal{D}$ be a set of representatives of the essential classes of $\mathcal{F}$. Set $\mathcal{C} = \{S\} \cup \mathcal{D}$. Then N is normal in $\mathcal{F}$ if and only if, for every $P \in \mathcal{C}$, $N \leq P$ and N is $A(P)$-invariant.*

Proof. See [H, 2.17]. □

Lemma 2.38. *Let $U \in \mathcal{F}$ such that U is not fully normalized. Then $N_S(U)$ is contained in an essential subgroup of $\mathcal{F}$.*

Proof. By Lemma 2.18, there is $\phi \in Mor_{\mathcal{F}}(N_S(U), S)$ such that $U\phi$ is fully normalized. As U is not fully normalized, ϕ does not extend to an element of $A(S)$. Now by Theorem 2.36, there is $\psi \in A(S)$ such that $N_S(U)\psi$ is contained in an essential subgroup. Since every $\mathcal{F}$-conjugate of an essential subgroup is again essential, this yields the assertion. □

Recall that, given a saturated fusion system $\tilde{\mathcal{F}}$ on a finite p-group $\tilde{S}$, we call a group isomorphism $\alpha : S \to \tilde{S}$ an **isomorphism** (of fusion systems) from $\mathcal{F}$ to $\tilde{\mathcal{F}}$ if for all subgroups A, B of S, $\alpha^{-1}Mor_{\mathcal{F}}(A, B)\alpha = Mor(A\alpha, B\alpha)$. An immediate consequence of Theorem 2.36 is the following remark.

Remark 2.39. *Let $\tilde{\mathcal{F}}$ be a saturated fusion system on a finite p-group $\tilde{S}$. Let $\mathcal{E}$ be a set of representatives of the essential classes of $\mathcal{F}$ and $\mathcal{C} = \mathcal{E} \cup \{S\}$. Then a group isomorphism $\alpha : S \to \tilde{S}$ is an isomorphism between $\mathcal{F}$ and $\tilde{\mathcal{F}}$ if and only if*

$$\{P\alpha : P \in \mathcal{E}\}$$

is a set of representatives of the essential classes of $\tilde{\mathcal{F}}$ and

$$\alpha^{-1}A(P)\alpha = Aut_{\tilde{\mathcal{F}}}(P\alpha)$$

for every $P \in \mathcal{C}$.

We conclude this section with some results that are important in Chapter 8 where we show the existence of Thompson-restricted subgroups. Recall from Notation 1.3 that, for a subsystem $\mathcal{N}$ of $\mathcal{F}$ on S, we write $\mathcal{F}_{\mathcal{N}}$ for the set of centric subgroups Q of $\mathcal{F}$ for which there is an element in $A(Q)$ that is not a morphism in $\mathcal{N}$. Furthermore, we introduce the following notation.

Notation 2.40. *Let $\mathcal{N}$ be a subsystem of $\mathcal{F}$ on S. Then we write $\mathcal{F}^*_{\mathcal{N}}$ for the set of Thompson-maximal members of $\mathcal{F}_{\mathcal{N}}$.*[1]

We get the following three corollaries to Theorem 2.36.

Corollary 2.41. *Let $\mathcal{N}$ be a subsystem of $\mathcal{F}$ on S. Then $\mathcal{F}_{\mathcal{N}} \neq \emptyset$ if and only if $\mathcal{F} \neq \mathcal{N}$.*

[1]Recall Definition 1.4.

Corollary 2.42. *Let $\mathcal{N}$ be a proper subsystem of $\mathcal{F}$ on S. Let $X \in \mathcal{F}_{\mathcal{N}}^*$, $J(X) \leq R \leq S$ and $\phi \in Mor_{\mathcal{F}}(R, S)$. Then $J(R) = J(X)$ or $\phi \in \mathcal{N}$.*

Proof. Assume $\phi \notin \mathcal{N}$. Then by Theorem 2.36, R is fused into an element of $\mathcal{F}_{\mathcal{N}}$. Hence, as $J(X) \leq R$, the subgroup X being Thompson-maximal in $\mathcal{F}_{\mathcal{N}}$ implies $J(R) = J(X)$. □

As a special case of Corollary 2.42 we get

Corollary 2.43. *Let $\mathcal{N}$ be a proper subsystem of $\mathcal{F}$ on S. Let $X \in \mathcal{F}_{\mathcal{N}}^*$ and $Q \in \mathcal{F}$ such that $J(X) \leq Q$ and $A(Q) \not\leq \mathcal{N}$. Then $J(Q) = J(X)$.*

Remark 2.44. *Let $Q \in \mathcal{F}$ such that $J(S) \not\leq Q$. Then $J(N_S(J(Q))) \not\leq Q$.*

Proof. Otherwise $J(N_S(J(Q))) = J(Q)$ and so

$$N_S(N_S(J(Q))) \leq N_S(J(N_S(J(Q)))) = N_S(J(Q)).$$

Then $S = N_S(J(Q))$ and so $J(S) \leq Q$, a contradiction. □

Lemma 2.45. *Let $\mathcal{N}$ be a proper subsystem of $\mathcal{F}$ on S. Then there exists $X \in \mathcal{F}_{\mathcal{N}}^*$ such that $J(X)$ is fully normalized.*

Proof. By Corollary 2.41, we can choose $X_0 \in \mathcal{F}_{\mathcal{N}}^*$. Set $U_0 = J(X_0)$ and let $U \in U_0^{\mathcal{F}}$ be fully normalized. Then by Lemma 2.18, there exists $\phi \in Mor_{\mathcal{F}}(N_S(U_0), N_S(U))$ such that $U_0\phi = U$. Set $X := X_0\phi$. If $J(S) \leq X_0$ then observe that $U_0 = J(S) = U$, so U_0 is fully normalized. Therefore, we may assume that $J(S) \not\leq X_0$. Hence, by Remark 2.44, $J(N_S(U_0)) \not\leq X_0$. It follows now from Corollary 2.42 that ϕ is a morphism in $\mathcal{N}$. Since $A(X_0) = \phi A(X)\phi^{-1}$ and $A(X_0) \not\leq \mathcal{N}$, we get $A(X) \not\leq \mathcal{N}$ and so $X \in \mathcal{F}_{\mathcal{N}}$. As $X_0 \in \mathcal{F}_{\mathcal{N}}^*$ and $X \in X_0^{\mathcal{F}}$, it follows $X \in \mathcal{F}_{\mathcal{N}}^*$. Since $J(X) = U$ is fully normalized, this shows the assertion. □

Lemma 2.46. *Let $\mathcal{N}$ be a proper saturated subsystem of $\mathcal{F}$ containing $C_{\mathcal{F}}(\Omega(Z(S)))$. Let $X \in \mathcal{F}_{\mathcal{N}}$ such that $J(X)$ is fully normalized. Then $A(J(X)) \not\leq \mathcal{N}$.*

Proof. Set $U := J(X)$ and assume $A(U) \leq \mathcal{N}$. Let $\phi \in A(X)$ such that $\phi \notin \mathcal{N}$. Then $\alpha := \phi_{|U,U} \in \mathcal{N}$ and, by Lemma 2.17(a), $X_U\alpha^* = X_U$.[2] In particular, $X \leq N_\alpha$. Observe that U is fully normalized in $\mathcal{N}$. Hence, as $\mathcal{N}$ is saturated, α extends to an element $\psi \in Mor_{\mathcal{N}}(X, S)$. Note that $\phi^{-1}\psi$ is the identity on U. By definition of $\mathcal{F}_{\mathcal{N}}$, X is centric and therefore $\Omega(Z(S)) \leq X$. This yields $\Omega(Z(S)) \leq J(X) = U$. Thus, $\phi^{-1}\psi \in C_{\mathcal{F}}(\Omega(Z(S))) \leq \mathcal{N}$ and so $\phi \in \mathcal{N}$, a contradiction. Hence, $A(U) \not\leq \mathcal{N}$. □

Lemma 2.47. *Let $\mathcal{N}$ be a proper saturated subsystem of $\mathcal{F}$ containing $C_{\mathcal{F}}(\Omega(Z(S)))$. Then there exists $Q \in \mathcal{F}^*_{\mathcal{N}}$ such that $N_S(J(Q)) = N_S(Q)$ and $J(Q)$ is fully normalized.*

Proof. By Lemma 2.45, we can choose $X \in \mathcal{F}^*_{\mathcal{N}}$ such that $U := J(X)$ is fully normalized. Then, by Lemma 2.46, we have $A(U) \not\leq \mathcal{N}$. Set $V := \Omega(Z(U))$ and $Q := C_S(V) \cap N_S(U)$. Then $N_S(U) \leq N_S(Q)$. Since U is fully normalized, $S_U \in Syl_p(A(U))$. Hence, $Q_U \in Syl_p(C_{A(U)}(V))$ and, by a Frattini-Argument,

$$A(U) = C_{A(U)}(V)N_{A(U)}(Q_U).$$

As X is centric, we have $\Omega(Z(S)) \leq J(X) = U$ and thus, $\Omega(Z(S)) \leq V$. Therefore, $C_{A(U)}(V) \leq C_{\mathcal{F}}(\Omega(Z(S))) \leq \mathcal{N}$ and so $N_{A(U)}(Q_U) \not\leq \mathcal{N}$. Since $C_S(U) \leq C_S(V) \cap N_S(U) = Q$, it follows from Lemma 2.17(b) that every element of $N_{A(U)}(Q_U)$ extends to an element of $A(Q)$. Hence, $A(Q) \not\leq \mathcal{N}$. Now Corollary 2.43 implies that $J(Q) = U = J(X)$. Hence, $N_S(U) \leq N_S(Q) \leq N_S(J(Q)) = N_S(U)$ and, by Remark 2.10, Q is fully normalized. In particular, Q is fully centralized and therefore, by Remark 2.9, centric. Thus, $Q \in \mathcal{F}^*_{\mathcal{N}}$. This proves the assertion. □

[2] Recall Notation 2.1 and Notation 2.11.

Chapter 3

Preliminaries on Groups

The aim of this chapter is to give an overview on some basic group theoretical background and to fix some notation as necessary. In that we follow [KS]. We also prove some more specialized results which we will need later on.

In the remainder of this chapter G will always be a finite group, p a prime and T a Sylow p-subgroup of G. Moreover, q is always assumed to be a power of p.

3.1 Notation and basic results

We will write $o(g)$ for the order of an element $g \in G$, and $Syl_p(G)$ for the set of Sylow p-subgroups of G. The group G is called p-closed if T is normal in G. We will make use of the following characteristic subgroups of G:

- $O_p(G)$ is the largest normal p-subgroup of G.
- $O^p(G)$ is the smallest normal subgroup of G whose factor group is a p-group. Equivalently, $O^p(G)$ is the group generated by all p'-elements of G.
- $O^{p'}(G)$ is the smallest normal subgroup of G of index prime to p. Note that $O^{p'}(G) = \langle Syl_p(G) \rangle$.
- If G is a p-group then $\Omega(G)$ is the subgroup generated by all elements of G

of order p. If G is also abelian then $\Omega(G)$ is the largest elementary abelian p-subgroup of G.

- We write $\Phi(G)$ for the Frattini subgroup of G, which is the intersection of all maximal subgroups of G. If G is a p-group then $\Phi(G)$ is the smallest normal subgroup of G with an elementary abelian factor group.

Definition 3.1. *G has* ***characteristic*** *p if $C_G(O_p(G)) \leq O_p(G)$.*

We assume the reader to be familiar with Sylow's Theorem. Moreover we will frequently use that p-groups are nilpotent. In particular, if G is a p-group, then $U < N_G(U)$, for every proper subgroup U of G and $[N, G] < N$ for every normal subgroup N of G.

If A, B and C are subgroups of G such that $G = AB$ and $A \leq C$, then $C = A(C \cap B)$. We will refer to this property as Dedekind's Law. If N is a normal subgroup of G then $G = NN_G(T \cap N)$. We will refer to this property as Frattini Argument.

For subgroups $X, Y, Z \leq G$ we set $[X, Y, Z] := [[X, Y], Z]$. We will apply what is known as the Three-Subgroups Lemma. That is, for subgroups X, Y, Z of G we have

$$[X, Y, Z] = [Y, Z, X] = 1 \implies [Z, X, Y] = 1.$$

In particular, we will apply this Lemma in the semidirect product AG, if A is a finite group acting on G and each of X, Y, Z is a subgroup of either A or G. By the commutators we mean then the commutators in the semidirect product AG.

We conclude this section by stating some results about groups acting on groups that we will frequently use without reference. For the remainder of this section let A be a finite group that acts on the group G. We say A acts **quadratically** on G if $[G, A, A] = 1$. We will use that, if G is an elementary abelian 2-group and $|A| = 2$ then A acts quadratically on G.

We call the action of A on G **coprime** if

(1) $|A|$ and $|G|$ are coprime, and

(2) A or G is solvable.

By a Theorem of Feit-Thompson, every finite group of odd order is solvable. Hence if $|A|$ and $|G|$ are coprime then either A or G is solvable. Therefore assumption (2) is in fact redundant. However we prefer to put it this way since the Theorem of Feit-Thompson is a deep result of finite group theory and in all the situations we are going to apply the results about coprime action, assumption (2) is fulfilled for some other reasons.

Lemma 3.2. *Assume A acts coprimely on G. Then*

(a) $G = [G, A]C_G(A)$*,*

(b) $[G, A] = [G, A, A]$*.*

Proof. See for example [KS, 8.2.7]. □

Lemma 3.3. *Suppose G is abelian and A acts coprimely on G. Then*

$$G = [G, A] \times C_G(A).$$

Proof. See for example [KS, 8.4.2] □

As a generalization of Lemma 3.2(b) one shows that, if A is generated by elements whose order is coprime to $|G|$, then still $[G, A] = [G, A, A]$. In particular, if G is a p-group, then $[G, O^p(A)] = [G, O^p(A), O^p(A)]$.

3.2 The Frattini subgroup

Recall that the **Frattini subgroup** $\Phi(G)$ of G is the intersection of all maximal subgroups of G.

Lemma 3.4. *(a) Let H be a subgroup of G. If $G = H\Phi(G)$ then $G = H$.*

(b) $\Phi(G)$ is nilpotent.

(c) $\Phi(G/\Phi(G)) = 1$.

Proof. For (a) and (b) see 5.2.3 and 5.2.5(a) in [KS]. For every maximal subgroup M of G, we have $\Phi(G) \leq M$ and $M/\Phi(G)$ is maximal in $G/\Phi(G)$. This yields (c). □

Lemma 3.5. *Let N be a normal subgroup of G.*

(a) If $N \leq \Phi(G)$ then $\Phi(G)/N = \Phi(G/N)$.

(b) $\Phi(N) \leq \Phi(G)$.

Proof. If $N \leq \Phi(G)$, then a subgroup M of G is maximal in N if and only if $N \leq M$ and M/N is maximal in G/N. This shows (a). For the proof of (b) assume by contradiction that there is a maximal subgroup M of G such that $\Phi(N) \not\leq M$. Then $G = \Phi(N)M$ and $N = \Phi(N)(M \cap N)$. Hence, by Lemma 3.4(a), $\Phi(N) \leq N = M \cap N \leq M$, a contradiction. □

The main aim of this section is the proof of the following lemma that the author learned from Stellmacher and was probably first proved by Meierfrankenfeld in the case $p = 2$. It will be useful in connection with the pushing up arguments in Chapter 9 and Chapter 10.

Lemma 3.6. *Let G be a finite group with $O_p(G) = 1$, and let N be a normal subgroup of G such that G/N is a p-group. Then $\Phi(G) = \Phi(N)$.*

Proof. Assume the assertion is wrong and let G be a minimal counterexample. Then $O_p(G) = 1$ and we may choose a normal subgroup N of G such that G/N is a p-group and $\Phi(G) \neq \Phi(N)$. We choose this normal subgroup N of maximal order. By Lemma 3.5(b), we have

(1) $\Phi(N) \leq \Phi(G)$.

The assumption $O_p(G) = 1$ implies $O_p(\Phi(G)) = 1$. Hence, Lemma 3.4(b) gives

(2) $\Phi(G)$ has order prime to p.

Consider now $\overline{G} := G/\Phi(N)$. Let X be the full preimage of $O_p(\overline{G})$ in G and $P \in Syl_p(X)$. Then $X = \Phi(N)P$ and the Frattini Argument gives $G = XN_G(P) = \Phi(N)N_G(P)$. Now (1) and Lemma 3.4(a) imply $G = N_G(P)$. Hence, as $O_p(G) = 1$, we have $P = 1$ and so $O_p(\overline{G}) = 1$. Assume now $\Phi(N) \neq 1$. Then $|\overline{G}| < |G|$ and, as G is a minimal counterexample, $\phi(\overline{G}) = \Phi(\overline{N})$. Now by Lemma 3.4(c), $\Phi(\overline{G}) = 1$. Thus, by Lemma 3.5, $\Phi(G) = \Phi(N)$, a contradiction. This shows

(3) $\Phi(N) = 1$.

Set now $G_0 := N\Phi(G)$. Observe that, by Lemma 3.4(a), G_0 is a proper subgroup of G. As $O_p(G) = 1$ and G_0 is normal in G, we have $O_p(G_0) = 1$. Hence, the minimality of G yields $\Phi(G_0) = \Phi(N)$. If $\Phi(G) \not\leq N$, then the maximality of $|N|$ implies $\Phi(G) = \Phi(G_0) = \Phi(N)$, a contradiction. Hence,

(4) $\Phi(G) \leq N$.

Set $V := Z(\Phi(G))$. Observe that, by (1) and Lemma 3.4(b),

(5) $V \neq 1$.

We show next

(6) V has a complement in N.

By (3) and (5), there is a maximal subgroup M_0 of N such that $V \not\leq M_0$. Then $N = VM_0$. Hence, there is a non-empty set $\mathcal{E}$ of maximal subgroups of N such that $N = VU$, for $U := \bigcap \mathcal{E}$. We choose such a set $\mathcal{E}$ of maximal order. If $U \cap V \neq 1$, then (3) implies the existence of a maximal subgroup M of N such that $U \cap V \not\leq M$. Then, in particular, $U \not\leq M$ and so $M \notin \mathcal{E}$. Moreover, $N = (U \cap V)M$, so $U = (U \cap V)(U \cap M)$ and $N = VU = V(U \cap M)$. This is a contradiction to the maximality of $|\mathcal{E}|$. Hence, $U \cap V = 1$ and (6) holds.

We now derive the final contradiction. By (2),(6) and a Theorem of Gaschütz (see e.g. [KS, 3.3.2]), there is a complement K of V in G, i.e. $K \cap V = 1$ and $G = KV = K\Phi(G)$. Now Lemma 3.4(a) implies $G = K$ and so $V = 1$, a contradiction to (5). □

3.3 Minimal parabolics

Definition 3.7. *G is called **minimal parabolic** (with respect to p) if T is not normal in G and there is a unique maximal subgroup of G containing T.*

This concept is originally due to McBride. One of the main properties of minimal parabolic groups is the following.

Lemma 3.8. *Let G be minimal parabolic with respect to p and let N be normal in G. Then $N \cap T \trianglelefteq G$ or $O^p(G) \leq N$.*

Proof. See [PPS, 1.3(b)]. □

Given a group G which is not p-closed, it is easy to obtain minimal parabolic subgroups of G containing a Sylow p-subgroup of G. This is a consequence of the following remark which is elementary to check.

Remark 3.9. *Let H be a subgroup of G such that $N_G(T) \leq H < G$. Assume that P is a subgroup of G which is minimal with the properties $T \leq P$ and $P \not\leq H$. Then P is minimal parabolic and $H \cap P$ is the unique maximal subgroup of P containing T.*

Lemma 3.10. *Let G be minimal parabolic, $O_p(G) = 1$ and $N \trianglelefteq G$. Then $O^p(G) \leq N$ or $N \leq \Phi(G)$.*

Proof. Assume by contradiction, $N \not\leq \Phi(G)$ and $O^p(G) \not\leq N$. Then there is a maximal subgroup M of G such that $N \not\leq M$, and, by Lemma 3.8, $T \cap N = 1$. Hence, $G = MN$ and M contains a Sylow p-subgroup S of G. As $NS \not\leq M$ and S is contained in a unique maximal subgroup of G, this implies $G = NS$ and so $O^p(G) \leq N$. $\square$

Chapter 4

Groups Acting on Modules

Throughout this chapter let G be a finite group, $T \in Syl_p(G)$, and V be a finite dimensional $GF(p)G$-module.

4.1 FF-modules

Definition 4.1.
- *A subgroup A of G is said to be an **offender** on V, if*
  - *(a) $A/C_A(V)$ is a non-trivial elementary abelian p-group,*
  - *(b) $|V/C_V(A)| \leq |A/C_A(V)|$.*
- *A subgroup A of G is called **best offender** if (a) holds and*

  (b') $|A/C_A(V)||C_V(A)| \geq |A^*/C_{A^*}(V)||C_V(A^*)|$ for all subgroups A^* of A.

  We write $\mathcal{O}_G(V)$ for the set of all best offenders in G on V.
- *The module V is called an **FF-module** for G, if there is an offender in G on V.*
- *An offender A on V is called an **over-offender** on V if $|V/C_V(A)| < |A/C_A(V)|$.*

Lemma 4.2. *(a) Every best offender on V is an offender on V.*

(b) The module V is an FF-module if and only if there is a best offender on V.

Proof. See [MS, 2.5(a),(b)]. □

We will use Lemma 4.2 without reference.

Definition 4.3. *Write $\mathcal{A}(G)$ for the set of all elementary abelian p-subgroups of G of maximal order. Recall that the Thompson subgroup $J(G)$ is the subgroup of G generated by $\mathcal{A}(G)$.*

Lemma 4.4. *Let V be an elementary abelian normal p-subgroup of G. Let $A \in \mathcal{A}(G)$ and suppose that A does not centralize V.*

(a) A is a best offender on V.

(b) If A is not an over-offender on V, then $VC_A(V) \in \mathcal{A}(G)$. In particular, we have then $\mathcal{A}(C_G(V)) \subseteq \mathcal{A}(G)$ and $J(C_G(V)) \leq J(G)$.

Proof. For the proof of (a) see [BHS, 2.8(e)]. For the proof of (b) observe that $|VC_A(V)| \leq |A|$, since $VC_A(V)$ is an elementary abelian subgroup of T. Hence,

$$|V/C_V(A)| \leq |V/V \cap A| = |V/V \cap C_A(V)| = |VC_A(V)/C_A(V)| \leq |A/C_A(V)|.$$

If $|V/C_V(A)| = |A/C_A(V)|$, then we have equality above. In particular, $|VC_A(V)| = |A|$. Hence (b) holds. □

Lemma 4.5. *Let V, W be normal elementary abelian p-subgroups of G with $V \leq W$ and $[V, J(G)] \neq 1$. Let $A \in \mathcal{A}(G)$ such that $[V, A] \neq 1$, and $AC_G(W)$ is a minimal with respect to inclusion element of the set*

$$\{BC_G(W) \ : \ B \in \mathcal{A}(G), \ [W, B] \neq 1\}.$$

Assume A is not an over-offender on V. Then we have

$$|W/C_W(A)| = |A/C_A(W)| = |V/C_V(A)| \textit{ and } W = VC_W(A).$$

Proof. It follows from Lemma 4.4 that $|V/C_V(A)| = |A/C_A(V)|$, $B := C_A(V)V \in \mathcal{A}(G)$ and $|W/C_W(A)| \leq |A/C_A(W)|$. Since $[V, A] \neq 1$ and $[V, B] = 1$, $BC_G(W)$ is a proper subgroup of $AC_G(W)$. Hence, the minimality of $AC_G(W)$ yields $[W, B] = 1$. Thus, $C_A(V) = C_A(W)$. It follows that

$$\begin{aligned} |W/C_W(A)| &\leq |A/C_A(W)| = |A/C_A(V)| \\ &= |V/C_V(A)| = |VC_W(A)/C_W(A)| \leq |W/C_W(A)|. \end{aligned}$$

Now equality holds above, i.e $|A/C_A(W)| = |W/C_W(A)| = |V/C_V(A)|$ and $W = VC_W(A)$. □

Lemma 4.6. *Let V be an elementary abelian normal p-subgroup of G such that $[V, J(G)] \neq 1$. Set $\overline{G} = G/C_G(V)$ and assume there is no over-offender in G on V. Then there is $A \in \mathcal{A}(G)$ such that $\overline{A}$ is minimal with respect to inclusion among the offenders in $\overline{G}$ on V.*

Proof. Let $A \in \mathcal{A}(G)$ such that $[V, A] \neq 1$. Choose A so that $AC_G(V)$ is minimal with respect to inclusion. By Lemma 4.4, $\overline{A}$ is an offender on V. Let $B \leq A$ such that $\overline{B}$ is an offender on V. We may assume that $C_A(V) \leq B$. Then $C_A(V) = C_B(V)$ and $A \cap V = B \cap V$. Since there is no over-offender in G on V, we have

$$|A/C_A(V)||C_V(A)| = |V| = |B/C_B(V)||C_V(B)|.$$

Thus, $|A||C_V(A)| = |B||C_V(B)|$ and therefore,

$$|A| = |AC_V(A)| = |A||C_V(A)||A \cap V|^{-1} = |B||C_V(B)||B \cap V|^{-1} = |BC_V(B)|.$$

This yields $BC_V(B) \in \mathcal{A}(G)$. Now by the choice of A, $BC_G(V) = AC_G(V)$. Thus, $\overline{A} = \overline{B}$ and the assertion holds. □

We will also use the following elementary property.

Lemma 4.7. *Assume W is a G-submodule of V such that G acts faithfully on V and $\overline{V} = V/W$. Then the following hold:*

(a) *Every offender on V is an offender on $\overline{V}$.*

(b) *If there is no over-offender in G on $\overline{V}$, then there is no over-offender in G on V.*

Proof. Let A be an offender on V. Then

$$|\overline{V}/C_{\overline{V}}(A)| \leq |\overline{V}/\overline{C_V(A)}| \leq |V/C_V(A)| \leq |A/C_A(V)| = |A| = |A/C_A(\overline{V})|.$$

If there is no over-offender in G on $\overline{V}$, then we have equality above and A is not an over-offender on V. □

Lemma 4.8. *Let W be a G-submodule of V. Then $\{A \in \mathcal{O}_G(V) : [W, A] \neq 1\} \subseteq \mathcal{O}_G(W)$.*

Proof. See [MS, 2.5(c)]. □

Theorem 4.9 (Timmesfeld Replacement Theorem)**.** *Let V be an elementary abelian normal p-subgroup of G. Let $A \in \mathcal{A}(G)$ such that $[V, A] \neq 1$. Then there exists $B \in \mathcal{A}(AC_G(V))$ such that $[V, B] \neq 1$ and $[V, B, B] = 1$.*

Proof. Note that by Lemma 4.4, $A \in \mathcal{O}_G(V)$. Hence, by [KS, 9.2.1],

$$|A/C_A(V)|\,|C_V(A)| = |A^*/C_{A^*}(V)|\,|C_V(A^*)| \text{ for } A^* = C_A([V, A]).$$

Also, $C_{A^*}(V) = C_A(V)$ and hence,

$$|A| = |A^*| \cdot \frac{|C_V(A^*)|}{|C_V(A)|}.$$

Since $V \cap A^* \leq C_V(A)$, it follows now for $B := A^*C_V(A^*)$ that

$$|B| = \frac{|A^*|\,|C_V(A^*)|}{|V \cap A^*|} \geq |A|.$$

Note also that B is elementary abelian. Hence, $B \in \mathcal{A}(AC_G(V))$. By the definition of B, $[V, B, B] = 1$. It follows from [KS, 9.2.3] that $[V, B] = [V, A^*] \neq 1$. □

4.2 Natural $SL_2(q)$-modules

Definition 4.10. *Suppose $G \cong SL_2(q)$ for some power q of p. Then V is called a **natural** $SL_2(q)$**-module** for G if V is irreducible, $F := End_G(V) \cong GF(q)$ and V is a 2-dimensional FG-module.*

The following two lemmas about natural $SL_2(q)$-modules are well known and elementary to check.

Lemma 4.11. *Assume that $G \cong SL_2(q)$ and V is a natural $SL_2(q)$-module for G. Then*

(a) $|C_V(T)| = q$,

(b) $C_V(T) = [V, T] = C_V(a)$ for each $a \in T^{\#}$,

(c) T acts quadratically on V,

(d) $C_G(C_V(T)) = T$.

(e) Every element of G of order coprime to p acts fixed point freely on V.

Lemma 4.12. *Let $G \cong SL_2(q)$ and V be a natural $SL_2(q)$-module for G. Then $\mathcal{O}_G(V) = \{A \leq G \ : \ A$ is an offender on V $\} = Syl_p(G)$. Moreover, there are no over-offenders in G on V.*

Lemma 4.13. *Let $H \trianglelefteq G$ such that $H \cong SL_2(q)$ and $C_T(H) \leq H$. Let V be a natural $SL_2(q)$-module for H and assume $C_G(V) = 1$. Then $C_T(C_V(T \cap H)) \leq H$.*

Proof. Set $Z := C_V(T \cap H)$ and $T_0 := C_T(Z)$. Observe that, by the structure of $Aut(SL_2(q))$, there is an element $x \in H \backslash N_H(T \cap H)$ such that $T_0 = (T \cap H)C_{T_0}(x)$. Then $[Z^x, C_{T_0}(x)] = 1$ and so, $V = ZZ^x$ is centralized by $C_{T_0}(x)$. Hence, $C_{T_0}(x) = 1$ and $T_0 \leq H$. □

Lemma 4.14. *Let $G \cong SL_2(q)$ and $V/C_G(V)$ be a natural $SL_2(q)$-module for G. Let $A \leq G$ be an offender on V. Then*

(a) $|V/C_V(A)| = |A| = q$ *and* $C_V(A) = C_V(a)$ *for every* $a \in A^{\#}$,

(b) $[V, A, A] = 1$.

Proof. As $G \cong SL_2(q)$, for every $a \in A^{\#}$ there exists $g \in G$ such that $G = \langle A, a^g \rangle$. Hence,

$$|V/C_V(G)| \leq |V/C_V(A)||V/C_V(a)| \leq |V/C_V(A)|^2 \leq |A|^2 = q^2 = |V/C_V(G)|.$$

Thus, the inequalities are equalities and (a) holds. Together with Lemma 4.11(a),(b) this implies (b). □

Lemma 4.15. *Let V be an elementary abelian normal subgroup of G. Assume $G/V \cong SL_2(q)$ and $V/C_V(G)$ is a natural $SL_2(q)$-module. Then $V \in \mathcal{A}(T)$. Moreover, the following hold:*

(a) For $R \in \mathcal{A}(T) \backslash \{V\}$, we have $T = VR$, $R \cap V = Z(T)$ and

$$C_{V/C_V(J(G))}(T) = Z(T)/C_V(G).$$

(b) If $p = 2$ and $J(T) \neq V$ then $|\mathcal{A}(T)| = 2$ and every elementary abelian subgroup of T is contained in an element of $\mathcal{A}(T)$.

Proof. Property (a) and $V \in \mathcal{A}(T)$ is a consequence of Lemma 4.4(a), Lemma 4.11(a) and Lemma 4.12. Now (b) is a consequence of (a), Lemma 4.11(b) and the fact that the product of two involutions is an involution if and only if these two involutions commute. □

Lemma 4.16. *Let $p = 2$ and let V be an elementary abelian normal 2-subgroup of G. Suppose S is a 2-group containing T as a subgroup. Assume the following conditions hold:*

(i) $V \leq J(G)$*, and* $J(G)/V \cong SL_2(q)$ *for some power* q *of* 2*.*

(ii) $V/C_V(J(G))$ *is a natural* $SL_2(q)$*-module for* $J(G)/V$*.*

(iii) $S \neq T = N_S(V) = N_S(U)$*, for every* $1 \neq U \leq C_V(J(G))$ *with* $U \trianglelefteq T$*.*

(iv) $C_T(J(G)/V) \leq V$*.*

Then the following hold:

(a) $|N_S(T) : T| = |N_S(J(T)) : T| = 2$ *and* $N_S(T) = N_S(J(T))$*.*

(b) If $J(N_S(J(T))) \not\leq T$ *then* $q = 2$ *and* $|V| = 4$*.*

(c) If $J(N_S(J(T))) \leq T$ *then* $J(T) = J(S)$*, and* $|S : T| = 2$*.*

(d) If $T = J(T)$ *and* $Z(T) = Z(S)$ *then* $C_T(u) = Z(S)$ *for every involution* $u \in N_S(T) \backslash T$*.*

Proof. Since $S \neq T$, there is a conjugate of V in T. Now by Lemma 4.15, $|\mathcal{A}(T)| = 2$ and $V \in \mathcal{A}(T)$. As $S \neq T = N_S(V)$, this implies (a). Assume now there is $R \in \mathcal{A}(N_S(J(T)))$ such that $R \not\leq T$. Observe that by Lemma 4.15(a), $R \cap J(T) \leq V \cap V^x = Z(J(T))$ for $x \in R \backslash T$. Moreover, by (iii), $C_V(J(G)) \cap C_V(J(G))^x = 1$ and so $R \cap J(T) \cap C_V(J(G)) = 1$. Hence, if $R \cap T \leq J(T)$ then $|R \cap T| \leq q$. Since $C_T(J(G)/V)$ embeds into $Aut(J(G)/V)$, we have that $T/J(T)$ is cyclic, so $|R \cap T/R \cap J(T)| \leq 2$. Moreover, by Lemma 4.13, $|C_{Z(J(T))/C_V(J(G))}(t)| < q$ for $t \in T \backslash J(T)$. Hence, if $R \cap T \not\leq J(T)$ then $|R \cap J(T)| < q$ and, again, $|R \cap T| \leq q$. Now by (a), $q^2 \leq |V| \leq |R| \leq 2 \cdot q$, so $q = 2$ and $|V| = q^2 = 4$. This shows (b). Since S is nilpotent, (c) is a consequence of (a).

For the proof of (d) assume now $T = J(T)$ and $Z(T) = Z(S)$. Let $u \in N_S(T) \backslash T$ be an involution and $y \in C_T(u)$. By Lemma 4.15(a), there exist $a, \tilde{a} \in V$ such that $y = a\tilde{a}^u$. Then $a\tilde{a}^u = (a\tilde{a}^u)^u = a^u\tilde{a}$. Now $[V, V^u] \leq V \cap V^u = Z(T)$ implies

$aZ(T) = \tilde{a}Z(T)$. Let $z \in Z(T)$ such that $\tilde{a} = az$. Then, as $Z(T) = Z(S)$, $y = aa^u z$ and $aa^u = yz = (yz)^u = a^u a$. Hence, Lemma 4.11(b) implies $a \in Z(T)$ and so $y \in Z(T) = Z(S)$. □

4.3 Natural S_m-modules

Definition 4.17. *Let $G \cong S_m$ for some $m \geq 3$.*

- *We call a $GF(2)G$-module a* ***permutation module*** *for G, if it has a basis*

$$\{v_1, v_2, \ldots, v_m\}$$

of length m on which G acts faithfully.

- *A $GF(2)G$-module is called a* ***natural*** *S_m**-module*** *for G if it is isomorphic to a non-central irreducible section of the permutation module.*

Natural S_m-modules are by this definition uniquely determined up to isomorphism. Suppose V is a permutation module for S_m with basis $\{v_1, v_2, \ldots, v_m\}$ as above. Set $W = \langle v_i + v_j \ : \ 1 \leq i, j \leq m \rangle$ and $U = \langle v_1 + v_2 + \cdots + v_m \rangle$. If m is odd, then $W \cong V/U$ and the natural module is isomorphic to both W and V/U. In particular, a natural S_m-module has dimension $m - 1$. If m is even, then $U \leq W$ and the natural S_m-module is isomorphic to W/U. Accordingly, it has dimension $m - 2$.

Lemma 4.18. *Assume $p = 2$, $G = S_{2n+1}$ and V is a natural G-module. Then the following conditions hold:*

(a) The elements in $\mathcal{O}_G(V)$ are precisely the subgroups generated by commuting transpositions.

(b) $N_G(J)/J \cong S_n$ for $J := \langle \mathcal{O}_G(V) \rangle$.

(c) If $n > 2$ and J is a subgroup of T generated by elements of $\mathcal{O}_T(V)$, then $N_G(J)$ is not p-closed.

(d) There are no over-offenders in G on V.

Proof. Part (a) follows from [BHS, 2.15]. Note that T contains precisely n transpositions $t_1, \dots, t_n$, which pairwise commute. By (a), $J_T(V)$ is generated by $t_1, \dots, t_n$. Furthermore, $N_G(J_T(V))$ acts on $\{t_1, \dots, t_n\}$ and $C_G(t_1, \dots, t_n) = J_T(V)$. Hence $N_G(J_T(V))/J_T(V)$ is isomorphic to a subgroup of S_n.

If (ij) and (kl) are distinct transpositions in $J_T(V)$, then conjugation with the element $(ik)(jl) \in N_G(J_T(V))$ swaps (ij) and (kl). Hence, $N_G(J_T(V))/J_T(V) \cong S_n$ since S_n is generated by transpositions. This shows (b).

Suppose now $n > 2$ and let J be a subgroup of T generated by elements of $\mathcal{O}_T(V)$. If $J = J_T(V)$, then (c) follows from (b). Thus we may assume that $J \neq J_T(V)$. This means that J is generated by less than n transpositions and $C_G(J)$ contains a subgroup isomorphic to S_3. Since S_3 is not p-closed, this shows (c). $\square$

4.4 The structure of FF-modules

We state here two results of Bundy, Hebbinghaus and Stellmacher [BHS]. They describe the structure of FF-modules under special circumstances. We will need the following notation.

Notation 4.19. *Suppose that $\mathcal{O}_G(V) \neq \emptyset$. Set*

$$m_G(V) := max\{|A/C_A(V)||C_V(A)| \,:\, A \in \mathcal{O}_G(V)\},$$

and define $\mathcal{A}_G(V)$ to be the set of minimal (by inclusion) members of the set

$$\{A \in \mathcal{O}_G(V) \,:\, |A/C_A(V)||C_V(A)| = m_G(V)\}.$$

For a set of subgroups $\mathcal{D}$ of G and $E \leq G$ set

$$\mathcal{D} \cap E = \{A \in \mathcal{D} : A \leq E\}.$$

Note that $m_G(V) \geq |V|$, since every element of $\mathcal{O}_G(V)$ is an offender.

Theorem 4.20. *Assume G acts faithfully on V and A is a non-trivial elementary abelian subgroup of G weakly closed in $T \in Syl_p(G)$ with respect to G, such that*

(i) $[V, A, A] = 1$,

(ii) $|V/C_V(A)| = |A|$ *and* $C_V(A) = C_V(a)$ *for each* $a \in A^{\#}$, *and*

(iii) there is a subgroup M of G with $\langle A^G \rangle \not\leq M$ and $N_G(A) \leq M \geq C_G(C_V(T))$.

Then $L = \langle A^G \rangle \cong SL_2(q)$, where $q = |A|$, and $V/C_V(L)$ is a natural $SL_2(q)$-module for L.

Proof. See [BHS, 4.14]. □

Theorem 4.21. *Suppose G acts faithfully on V and G is minimal parabolic with respect to p. Let $T \in Syl_p(G)$ and $M \leq G$ be the unique maximal subgroup of G containing T. Assume also that*

(i) $O_p(G) = 1$,

(ii) V is an FF-module for G,

(iii) $C_G(C_V(T)) \leq M$.

Then for $\mathcal{D} = \mathcal{A}_G(V)$, there exist subgroups $E_1, \dots, E_r$ of G such that for each $1 \leq i \leq r$ the following hold:

(a) $G = (E_1 \times \dots E_r)T$.

(b) T acts transitively on $\{E_1, \dots, E_r\}$.

(c) $\mathcal{D} = (\mathcal{D} \cap E_1) \cup \dots \cup (\mathcal{D} \cap E_r)$.

(d) $V = C_V(E_1 \times \dots \times E_r) \prod_{i=1}^{r} [V, E_i]$, *with* $[V, E_i, E_j] = 1$ *for* $j \neq i$.

(e) $E_i \cong SL_2(p^n)$*, or* $p = 2$ *and* $E_i \cong S_{2^n+1}$ *for some* $n \in \mathbb{N}$*.*

(f) $[V, E_i]/C_{[V,E_i]}(E_i)$ *is a natural module for* E_i*.*

Proof. This is [BHS, 5.5]. □

Chapter 5

Pushing Up

Throughout this chapter, let G be a finite group, p a prime dividing $|G|$ and $T \in Syl_p(G)$. Let q be a power of p.

5.1 A result by Baumann and Niles

The group G is said to have the **pushing up property** (with respect to p) if the following holds:

(PU) No non-trivial characteristic subgroup of T is normal in G.

Note that this property does not depend on the choice of T since all Sylow p-subgroups of G are conjugate in G. The problem of determining the non-central chief factors of G in $O_p(G)$ under the additional hypothesis

(*) $\overline{G}/\Phi(\overline{G}) \cong L_2(q)$ for $\overline{G} = G/O_p(G)$

was first solved by Baumann [Bau] and Niles [Nil] independently. Later Stellmacher gave in [St1] a simplified proof using the amalgam method. We state here a slight modification of the result.

Hypothesis 5.1. *Let $Q := O_p(G)$ and let $W \leq \Omega(Z(Q))$ be normal in G. Suppose the following conditions hold:*

(1) $G/C_G(W) \cong SL_2(q)$,

(2) $W/C_W(G)$ *is a natural* $SL_2(q)$*-module for* $G/C_G(W)$*,*

(3) G *has the pushing up property (PU), and (*) holds.*

Theorem 5.2. *Suppose Hypothesis 5.1 holds. Then one of the following holds for* $V := [Q, O^p(G)]$.

(I) $V \leq \Omega(Z(O_p(G)))$ *and* $V/C_V(G)$ *is a natural* $SL_2(q)$*-module for* $G/C_G(W)$.

(II) $Z(V) \leq Z(Q)$*,* $p = 3$*, and* $\Phi(V) = C_V(G)$ *has order* q*. Moreover,* $V/Z(V)$ *and* $Z(V)/\Phi(V)$ *are natural* $SL_2(q)$*-modules for* $G/C_G(W)$.

Furthermore, the following hold for every $\phi \in Aut(T)$ *with* $V\phi \not\leq Q$.

(a) $Q = VC_Q(L)$ *for some subgroup* L *of* G *with* $O^p(G) \leq L$ *and* $G = LQ$.

(b) If (II) holds then $Q\phi^2 = Q$.

(c) $\Phi(C_Q(O^p(G)))\phi = \Phi(C_Q(O^p(G)))$.

(d) If (II) holds then T *does not act quadratically on* $V/\Phi(V)$.

(e) If (II) holds then $W\phi \leq Q$ *and* $V \leq W\langle (W\phi)^G \rangle$.

(f) $V \not\leq Q\phi$.

Proof. Theorem 1 in [St1] and [Nil, 3.2] give us the existence of $\psi \in Aut(T)$ such that

$$L/V_0 O_{p'}(L) \cong SL_2(q) \text{ for } L = (V\psi)O^p(G) \text{ and } V_0 = V(L \cap Z(G)),$$

and one of the following hold:

(I') $V \leq \Omega(Z(O_p(G)))$ and $V/C_V(G)$ is a natural $SL_2(q)$-module for $L/V_0 O_{p'}(L)$.

(II') $Z(V) \leq Z(O_p(G))$, $p = 3$, and $\Phi(V) = C_V(G)$ has order q. Moreover, $V/Z(V)$ and $Z(V)/\Phi(V)$ are natural $SL_2(q)$-modules for $L/V_0 O_{p'}(L)$.

Observe that LQ contains $O^p(G)$ and a Sylow p-subgroup of G, so $G = LQ$. Now (a) is a consequence of Theorem 2 in [St1]. Moreover, (b),(c) and (d) follow from 2.4, 3.2, 3.3 and 3.4(b),(c) in [St1]. Clearly (I') implies (I). Moreover, if (I') holds then $C_T(V) = Q$ and $C_T(V\phi) = Q\phi$, so (f) holds in this case.

We assume from now on that (II') holds and show next that $G/C_G(W)$ acts on $V/C_V(G)$. Note that $[V, V] \leq C_V(G)$ and so by (a), $[V, Q] \leq C_V(G)$. Therefore, as $[V, O_{p'}(L)] = 1 = [W, O_{p'}(L)]$ and $[W, O^p(G)] = [Z(V), O^p(G)]$, we have $C_L(W) = O_{p'}(L)V_0$ and $C_G(W) = C_{QL}(W) = QC_L(W) \leq C_G(V/C_V(G))$. So $G/C_G(W)$ acts on $V/C_V(G)$ and (II) holds.

Let now $\phi \in Aut(T)$ such that $V\phi \not\leq Q$. It follows from (2.4) and (3.2) in [St1] that $[W, O^p(G)]\phi \leq Q$. Hence, since $W = [W, O^p(G)]C_W(G) \leq [W, O^p(G)]Z(T)$, we have $W\phi \leq Q$. As $\overline{Q} = Q/C_Q(O^p(G)) \cong V/\Phi(V)$, it follows that $\overline{W}$ and $\overline{Q}/\overline{W}$ are natural $SL_2(q)$-modules. In particular, $[\overline{W}, V\phi] \neq 1$ and so $\overline{W} \neq \overline{W\phi}$. Therefore, $\overline{Q} = \overline{W\langle (W\phi)^G \rangle}$. This implies (e). For the proof of (f) assume $V \leq Q\phi$. Then by (a), $[V, V\phi] \leq [Q\phi, V\phi] \leq C_V(G)\phi \leq Z(T)\phi = Z(T)$. As $V\phi \not\leq Q$ we have $T = \langle (V\phi)^{N_G(T)} \rangle Q$ and so $[V, T] \leq Z(T)$, a contradiction to (d). This proves (f). $\square$

5.2 The Baumann subgroup

A useful subgroup while dealing with pushing up situations is the following:

Definition 5.3. *The subgroup*

$$B(G) = \langle C_P(\Omega(Z(J(P)))) \ : \ P \in Syl_p(G) \rangle$$

is called the ***Baumann subgroup*** *of* G.

Often it is not possible to show immediately that G has the pushing up property. In many of these situations it helps to look at a subgroup X of G such that $B(T) \in$

$Syl_p(X)$ and to show that X has the pushing up property. Here one uses that $B(T)$ is characteristic in T, so a characteristic subgroup of $B(T)$ is also a characteristic subgroup of T. Usually one can then determine the structure of X and thus also of $B(T)$. This often leads to $T = B(T) \leq X$, in which case also the p-structure of G is restricted. When using this method later, we will need the results stated below.

Hypothesis 5.4. *Let $V \leq \Omega(Z(O_p(G)))$ be a normal subgroup of G such that*

- *$G/C_G(V) \cong SL_2(q)$,*
- *$V/C_V(G)$ is a natural $SL_2(q)$-module for $G/C_G(V)$,*
- *$C_G(V)/O_p(G)$ is a p'-group and $[V, J(T)] \neq 1$.*

Lemma 5.5. *Assume Hypothesis 5.4 and suppose there is $d \in G$ such that $G = \langle T, T^d \rangle$. Then $G = C_G(V)B(G)$, the subgroup $\Omega(Z(J(T)))V$ is normal in G, and $B(T) \in Syl_p(B(G))$.*

Proof. Set $Q := O_p(G)$. Let $A \in \mathcal{A}(T)$ such that $[V, A] \neq 1$. Then by Lemma 4.12 and Lemma 4.4, $|V/C_V(A)| = |A/C_A(V)| = q$ and $V(A \cap Q) \leq J(Q) \leq J(T)$. In particular, $T = J(T)Q$ and, since $C_G(V)/Q$ is a p'-group, $\Omega(Z(J(T))) \leq C_T(V) = Q$. Now $W := \Omega(Z(J(T)))V \leq \Omega(Z(J(Q)))$ and

$$|W/C_W(J(T))| = |VC_W(J(T))/C_W(J(T))| = |V/C_V(J(T))| = |V/C_V(T)| = q.$$

By assumption, we may choose $d \in G$ such that $G = \langle T, T^d \rangle$. Then for $X_0 := \langle J(T), J(T)^d \rangle$, we have $G = X_0Q$. Moreover, for $B \in \mathcal{A}(T^d)$, we have $V(B \cap Q) \in \mathcal{A}(Q)$. So $W \leq \Omega(Z(J(Q))) \leq V(B \cap Q)$ and $W = V(B \cap W)$ is normalized by B. Hence, W is normal in $J(T)^d$ and thus also in $G = X_0Q$. We get now

$$\begin{aligned} |VC_W(X_0)/C_W(X_0)| &\leq |W/C_W(X_0)| \leq |W/C_W(J(T))|^2 \\ &= q^2 = |V/C_V(X_0)| = |VC_W(X_0)/C_W(X_0)|. \end{aligned}$$

Hence, we have equality above and therefore $W = VC_W(X_0)$. Set

$$Z_0 := C_{\Omega(Z(J(T)))}(X_0).$$

As $W \leq J(T)$, we have $C_W(X_0) \leq \Omega(Z(J(T)))$. Thus, $C_W(X_0) = Z_0$ and $W = VZ_0$. Now Dedekind's Law implies $\Omega(Z(J(T))) = Z_0(\Omega(Z(J(T))) \cap V)$. So, using $T = J(T)Q$, we get $B(T) = C_T(Z_0)$. Note that Z_0 is normal in $G = QX_0$. Hence, $X := C_G(Z_0) \trianglelefteq G$. This yields $B(T) = T \cap X \in Syl_p(X)$ and $B(G) \leq X$, so $B(T) \in Syl_p(B(G))$. Since $G = QX_0 = QB(G)$, this implies the assertion. □

Lemma 5.6. *Assume Hypothesis 5.4. Then $G = C_G(V)B(G)$ and*

$$B(T) \in Syl_p(B(G)).$$

Proof. Set $W := \Omega(Z(J(T))V$ and $H := O^{p'}(G)$. As $G/C_G(V) \cong SL_2(q)$, there is $d \in G$ such that $G = C_G(V)H_0$, for $H_0 := \langle T, T^d \rangle$. By Lemma 5.5, $W \trianglelefteq H_0$ and $G = C_G(V)B(H_0)$. In particular, $W = V\Omega(Z(J(T^d)))$. So, again by Lemma 5.5 (applied with T^d in place of T), W is normal in $\langle \hat{T}, T^d \rangle$, for every $\hat{T} \in Syl_p(G)$ with $\hat{T}C_G(V) = TC_G(V)$. Now the arbitrary choice of d gives that W is normalized by every Sylow p-subgroup of G and therefore,

$$W \trianglelefteq H.$$

Note that, as $B(H_0) \leq B(G)$, we have

$$B(G) = B(H_0)C_{B(G)}(V) \text{ and } G = C_G(V)B(G).$$

Observe that $B(G) = B(H) = \langle B(T)^H \rangle = \langle B(T)^{B(G)} \rangle$. In particular, $[W, B(G)] \leq V$. Hence, $[W, C_{B(G)}(V), C_{B(G)}(V)] = 1$, and coprime action shows that $C_{B(G)}(V) \leq C_{B(G)}(W)Q$. Therefore, we get $B(G) = B(H_0)C_{B(G)}(V) \leq H_0C_{B(G)}(W)$. So $B(H_0)C_{B(G)}(W)$ is a normal subgroup of $B(G)$ containing $B(T)$ and thus, $B(G) = B(H_0)C_{B(G)}(W)$. By Lemma 5.5, $B(T) \in Syl_p(B(H_0))$. Therefore, we have that $(T \cap B(G))C_{B(G)}(W) = B(T)C_{B(G)}(W)$ and $T \cap B(G) \leq B(T)C_T(W) \leq B(T)$. This shows $B(T) \in Syl_p(B(G))$. □

Chapter 6

Amalgams

An amalgam $\mathcal{A}$ is a tuple $(G_1, G_2, B, \phi_1, \phi_2)$ where G_1, G_2 and B are groups and $\phi_i : B \to G_i$ is a monomorphism for $i = 1, 2$. We write $G_1 *_B G_2$ for the free product of G_1 and G_2 with B amalgamated. Note that we suppress here mention of the monomorphisms ϕ_1 and ϕ_2. We will usually identify G_1, G_2 and B with their images in $G_1 *_B G_2$. Then $G_1 \cap G_2 = B$, and the monomorphisms ϕ_1, ϕ_2 become inclusion maps. We will need the following lemma.

Lemma 6.1. *Let G be a group such that $G = \langle G_1, G_2 \rangle$ for finite subgroups G_1, G_2 of G. Set $B = G_1 \cap G_2$. For $i = 1, 2$, let K_i be a set of right coset representatives of B in G_i and $\iota_i : B \to G_i$ the inclusion map. By $G_1 *_B G_2$ we mean the free amalgamated product with respect to $(G_1, G_2, B, \iota_1, \iota_2)$. Let $g \in G$. Then g can be expressed in the form*

(*) $g = bg_1 \dots g_n$ where $b \in B$, $n \in \mathbb{N}$, $g_1, \dots, g_n \in (K_1 \cup K_2) \backslash B$, and $g_{k+1} \in K_1$ if and only if $g_k \in K_2$, for every $1 \leq k < n$.

This expression is unique if and only if $G \cong G_1 *_B G_2$.

Proof. As $X := G_1 *_B G_2$ is the universal completion of $(G_1, G_2, B, \iota_1, \iota_2)$, the group G is isomorphic to a factor group of X modulo a normal subgroup N of X with $N \cap G_1 = N \cap G_2 = 1$. By (7.9) of Part I in [DGS], every element $g \in X$ can be uniquely expressed in the form (*). This implies the assertion. □

A triple $(\beta_1, \beta_2, \beta)$ of group isomorphisms $\beta : B \to \tilde{B}$ and $\beta_i : G_i \to \tilde{G}_i$, for $i = 1, 2$, is said to be an **isomorphism** from $\mathcal{A}$ to an amalgam $\mathcal{B} = (\tilde{G}_1, \tilde{G}_2, \tilde{G}_{12}, \psi_1, \psi_2)$, if the obvious diagram commutes, i.e. if $\phi_i\beta_i = \beta\psi_i$, for $i = 1, 2$. An **automorphism** of $\mathcal{A}$ is an isomorphism from $\mathcal{A}$ to $\mathcal{A}$. The group of automorphisms of $\mathcal{A}$ will be denoted by $Aut(\mathcal{A})$. If $B = G_1 \cap G_2$ and ϕ_1, ϕ_2 are inclusion maps, then $\alpha_{i|B} = \alpha$, for every automorphism $(\alpha_1, \alpha_2, \alpha)$ of $\mathcal{A}$.

A finite p-subgroup S of a group G is called a **Sylow p-subgroup** of G, if every finite p-subgroup of G is conjugate to a subgroup of S. We write $S \in Syl_p(G)$. We will use the following result, which is stated in this form in [CP]. A similar result was proved first in [Rob].

Theorem 6.2 (Robinson)**.** *Let $(G_1, G_2, B, \phi_1, \phi_2)$ be an amalgam, and let $G = G_1 *_B G_2$ be the corresponding free amalgamated product. Suppose there is $S \in Syl_p(G_1)$ and $T \in Syl_p(G_2) \cap Syl_p(B)$ with $T \leq S$. Then $S \in Syl_p(G)$ and*

$$\mathcal{F}_S(G) = \langle \mathcal{F}_S(G_1), \mathcal{F}_T(G_2) \rangle.$$

Proof. See 3.1 in [CP]. □

When we prove our classification result for $p = 2$ we will apply Theorem 6.2 and the following theorem in order to identify a subsystem of a given saturated fusion system $\mathcal{F}$.

Theorem 6.3. *Let $(G_1, G_2, B, \phi_1, \phi_2)$ be an amalgam of finite groups G_1, G_2 and $G = G_1 *_B G_2$ the corresponding free amalgamated product. Suppose the following hold for $S \in Syl_2(G_1)$, $Q := O_2(G_2)$ and $M := J(G_2)$.*

(i) $N_S(Q) \in Syl_2(G_2)$ *and* $C_{G_2}(Q) \leq Q \leq M$.

(ii) $B = N_{G_1}(Q) = N_{G_2}(J(N_S(Q)))$.

(iii) $|G_1 : B| = 2$.

(iv) $M/Q \cong SL_2(q)$ *where* $q = 2^e > 2$, $\Phi(Q) = 1$, *and* $Q/C_Q(M)$ *is a natural* $SL_2(q)$*-module for* M/Q.

(v) No non-trivial normal p*-subgroup of* $MN_S(Q)$ *is normal in* G_1.

Then there exists a free normal subgroup N *of* G *such that* $H := G/N$ *is finite,* $SN/N \in Syl_2(H)$ *and* $F^*(H) \cong L_3(q)$ *or* $Sp_4(q)$.

Here $F^*(H)$ denotes the generalized Fitting subgroup of H. This is the subgroup of H generated by the Fitting subgroup $F(H)$ and the components of H. Here a component of H is a subnormal subgroup $C \neq 1$ of H such that $[C, C] = C$ and $C/Z(C)$ is simple. We will prove Theorem 6.3 at the end of this chapter. For that we need one more definition and some preliminary results.

Definition 6.4. *Let* G *be a group with finite subgroups* G_1 *and* G_2*. Set* $B := G_1 \cap G_2$.

- *Let* q *be a power of* p*. The pair* (G_1, G_2) *is called a* ***weak BN-pair of*** G ***involving*** $SL_2(q)$ *if, for* $i = 1, 2$*, there are normal subgroups* G_i^* *of* G_i *such that the following properties hold:*
  - $G = \langle G_1, G_2 \rangle$,
  - *no non-trivial normal subgroup of* G *is contained in* B,
  - $C_{G_i}(O_p(G_i)) \leq O_p(G_i) \leq G_i^*$,
  - $G_i = G_i^* B$,
  - $G_i^* \cap B$ *is the normalizer in* G_i^* *of a Sylow* p*-subgroup of* G_i^* *and*

$$G_i^*/O_p(G_i) \cong SL_2(q).$$

- *If* (G_1, G_2) *is a weak BN-pair of* G *and* $\iota_i : B \to G_i$ *is the inclusion map for* $i = 1, 2$*, then we call* $(G_1, G_2, B, \iota_1, \iota_2)$ *the* ***amalgam corresponding to*** (G_1, G_2).

The main tool is the following Theorem.

Theorem 6.5. *Let $(G_1, G_2, B, \phi_1, \phi_2)$ be an amalgam and $G = G_1 *_B G_2$ be the corresponding free amalgamated product. Let q be a power of p. Suppose (G_1, G_2) is a weak BN-pair of G involving $SL_2(q)$, and $O_p(G_i)$ is elementary abelian for $i = 1, 2$. Then there is a free normal subgroup N of G such that $H := G/N$ is finite, and $F^*(H) \cong L_3(q)$ or $p = 2$ and $F^*(H) \cong Sp_4(q)$.*

Proof. This is a consequence of Theorem A in [DGS]. □

Remark 6.6. *Let G be a group and (G_1, G_2) be a weak BN-pair of G involving $SL_2(q)$, for some power q of p. Set $B := G_1 \cap G_2$ and let $S \in Syl_p(B)$.*

(a) Let N be normal in G such that $N \cap G_1 = N \cap G_2 = 1$, and set $\overline{G} = G/N$. Then $(\overline{G_1}, \overline{G_2})$ is a weak BN-pair of $\overline{G}$ involving $SL_2(q)$.

(b) Suppose G is finite and $F^*(G) \cong L_3(q)$ or $p = 2$ and $F^*(G) \cong Sp_4(q)$. Then $S \in Syl_p(G)$ and G embeds into $\Gamma L_3(q)$ respectively $\Gamma Sp_4(q)$. Moreover, setting $G_i^\circ := G_i \cap F^*(G)$ for $i \in \{1, 2\}$, the pair (G_1°, G_2°) is a weak BN-pair of $F^*(G)$ involving $SL_2(q)$, $B = N_G(S \cap F^*(G))$ and $G_i = G_i^\circ B$.

6.7. The weak BN-pairs of $\mathbf{L_3(q)}$. Let q be a power of p and set $G := SL_3(q)$. Let V be the natural 3-dimensional $GF(q)G$-module. Set $\overline{G} := G/Z(G)$. Then $\overline{G} \cong L_3(q)$. Let $\mathcal{V}$ be the set of pairs (V_1, V_2) where V_1, V_2 are non-trivial proper $GF(q)$-subspaces of V such that $V_1 \neq V_2$ and either $V_1 \leq V_2$ or $V_2 \leq V_1$.

(a) A pair (P_1, P_2) of subgroups P_1, P_2 of $\overline{G}$ is a weak BN-pair of $\overline{G}$ involving $SL_2(q)$ if and only if there exists $(V_1, V_2) \in \mathcal{V}$ such that $P_1 = \overline{N_G(V_1)}$ and $P_2 = \overline{N_G(V_2)}$.

(b) Let $(V_1, V_2) \in \mathcal{V}$ such that $V_1 \leq V_2$. Set $P_i := \overline{N_G(V_i)}$, $Q_i := O_p(P_i)$ and $P_i^* := O^{p'}(P_i)$, for $i = 1, 2$. Set $T := Q_1 Q_2$. Then the following hold:

- $Q_1 = \overline{C_G(V/V_1) \cap C_G(V_1)}$ and $P_1^* = \overline{C_G(V_1)}$.

– $Q_2 = \overline{C_G(V/V_2) \cap C_G(V_2)}$ and $P_2^* = \overline{C_G(V/V_2)}$.

– For $i = 1, 2$, we have $P_i = N_G(Q_i)$, $Q_i \le P_i^*$, Q_i is elementary abelian of order q^2, $P_i^*/Q_i \cong SL_2(q)$, and Q_i is a natural $SL_2(q)$-module for P_i^*/Q_i. Furthermore, P_i/Q_i embeds into $GL_2(q)$.

– If $p = 2$ then $\mathcal{A}(T) = \{Q_1, Q_2\}$ and $T = J(T) \in Syl_p(\overline{G}) \cap Syl_p(P_1) \cap Syl_p(P_2)$.

6.8. $\mathbf{Aut(L_3(q))}$. Let q be a power of p and set $G := L_3(q)$. Then the group of automorphisms of G is generated by the automorphisms induced by conjugation with elements of $PGL_3(q)$, the field automorphisms and the contragredient automorphism. In particular, the following properties hold.

(a) Let (P_1, P_2) be a weak BN-pair of G involving $SL_2(q)$. Then there is an automorphism of G of order 2 which swaps P_1 and P_2. In particular, by 6.7, $Aut(G)$ acts transitively on the weak BN-pairs of G involving $SL_2(q)$.

(b) Let $S \in Syl_p(Aut(L_3(q)))$ and identify G with the group of its inner automorphisms. Assume $q \ge 4$. Then $J(S) \in Syl_p(G)$. If $p = 2$ then $|\mathcal{A}(S)| = 2$ and, for $\mathcal{A}(S) = \{Q_1, Q_2\}$, $N_S(Q_1) = N_S(Q_2)$ and the pair $(N_G(Q_1), N_G(Q_2))$ is a weak BN-pair of G. Furthermore, there is an involution $s \in S \backslash N_S(Q_1)$ such that $[Z(J(S)), s] = 1$. If $q = q_0^2$ for some $q_0 \in \mathbb{N}$ then, for every involution $t \in N_S(Q_1) \backslash J(S)$ with $C_G(t) \cong L_3(q_0)$, we can choose such an element s inside $C_S(t)$.

6.9. The weak BN-pairs of $\mathbf{Sp_4(q)}$. Let q be a power of 2 and set $G := Sp_4(q)$. Let V be the natural 4-dimensional $GF(q)G$ module. Let $\mathcal{V}$ be the set of all pairs (V_1, V_2) where V_1 and V_2 are non-trivial isotropic subspaces of V such that $V_1 \ne V_2$ and either $V_1 \le V_2$ or $V_2 \le V_1$. Then V_i has dimension at most 2 for $(V_1, V_2) \in \mathcal{V}$ and $i = 1, 2$. Furthermore, the following properties hold.

(a) A pair (P_1, P_2) of subgroups P_1, P_2 of G is a weak BN-pair of G involving $SL_2(q)$ if and only if there exists $(V_1, V_2) \in \mathcal{V}$ such that $P_1 = N_G(V_1)$ and $P_2 = N_G(V_2)$.

(b) Let $(V_1, V_2) \in \mathcal{V}$ such that $V_1 \leq V_2$. Set $P_i := N_G(V_i)$, $Q_i := O_p(P_i)$ and $P_i^* := O^{p'}(P_i)$ for $i = 1, 2$. Set $T := Q_1Q_2$. Then $T \in Syl_p(G)$ and the following hold:

- $Q_1 = C_G(V/V_1^{\perp}) \cap C_G(V_1^{\perp}/V_1) \cap C_G(V_1)$ and $P_1^* = C_G(V_1) = C_{P_1}(V/V_1^{\perp})$.
- $Q_2 = C_G(V_2) \cap C_G(V/V_2)$.
- For $i = 1, 2$, we have $P_i = N_G(Q_i)$, $Q_i \leq P_i^*$, Q_i is elementary abelian of order q^3, $P_i^*/Q_i \cong SL_2(q)$, P_i/Q_i is is isomorphic to a subgroup of $GL_2(q)$, and Q_i is the 3-dimensional orthogonal $SL_2(q)$-module for P_i^*/Q_i. In particular, $Q_i/C_{Q_i}(P_i^*)$ is a natural $SL_2(q)$-module for P_i^*/Q_i, $Q_i = [Q_i, P_i^*]$ and $Z(T) = [Q_i, T]$.
- If $q \geq 4$ then $Z(P_i) = 1$ for $i = 1, 2$.
- $\mathcal{A}(T) = \{Q_1, Q_2\}$ and $T = J(T) \in Syl_p(G) \cap Syl_p(P_1) \cap Syl_p(P_2)$.

6.10. $\mathbf{Aut(Sp_4(q))}$. Let $q = 2^e$ for some $e \geq 1$ and set $G := Sp_4(q)$. Then $Aut(G)/Inn(G)$ is cyclic of order $2e$, and every automorphism σ of G whose image generates $Aut(G)/Inn(G)$ is a graph automorphism. Furthermore, such an automorphism σ can be chosen such that $\langle \sigma^2 \rangle$ is the group of field automorphisms of G. (See [Car], Section 12.3.) In particular, the following properties hold.

(a) Let (P_1, P_2) be a weak BN-pair of G involving $SL_2(q)$. Then there is an automorphism of G which swaps P_1 and P_2. In particular, by 6.9, $Aut(G)$ acts transitively on the weak BN-pairs of G involving $SL_2(q)$.

(b) Let $S \in Syl_p(Aut(G))$ and identify G with the group of its inner automorphisms. Then we have $J(S) \in Syl_p(G)$, $|\mathcal{A}(S)| = 2$ and, for $\mathcal{A}(S) = \{Q_1, Q_2\}$, the pair $(N_G(Q_1), N_G(Q_2))$ is a weak BN-pair of G.

Lemma 6.11. *Let G be a finite group such that for $M = O^{p'}(G)$, $T \in Syl_p(M)$ and $Q := O_p(G)$ the following hold.*

(i) $M/Q \cong SL_2(q)$ and G/Q is isomorphic to a subgroup of $GL_2(q)$, for some power q of p.

(ii) G/Q acts faithfully on $Q/Z(M)$, Q is elementary abelian, and $Q/Z(M)$ is a natural $SL_2(q)$-module for M/Q. Furthermore, we have $Q = [Q, M]$ and $|Q/C_Q(T)| = q$.

(iii) $Z(G) = 1$.

Set $A := Aut(G)$. Then $C_A(Q) = C_A(Q/Z(M)) \leq C_A(G/Q)$ and $C_A(Q)$ is an elementary abelian p-group. Moreover, $C_A(T) \cong Z(T)$ for $T \in Syl_p(G)$.

Proof. Set $\overline{G} = G/Z(M)$ and $W := C_A(\overline{Q})$. Throughout this proof we will identify G with the group of inner automorphism of G. Note that this is possible by (iii). Observe that by (ii), $[G, W] \leq C_G(\overline{Q}) \leq Q$. Hence, $[M, W, Q] \leq [Q, Q] = 1$. As $[W, Q, M] = [Z(M), M] = 1$ it follows from the Three-Subgroups Lemma that $[Q, W] = [Q, M, W] = 1$. So we have shown that $W = C_A(Q) \leq C_A(G/Q)$. As $[W, G] \leq Q \leq C(W)$ it follows from the Three-Subgroups Lemma that $[W, W, G] = 1$, i.e. $[W, W] = 1$ and W is abelian. Since $[G, W, W] = 1$ and $[G, W] \leq Q$ is elementary abelian, we have $[g, w^p] = [g, w]^p = 1$ for every $g \in G$ and $w \in W$. Hence W is a group of exponent p and thus an elementary abelian p-group.

Set now $C := C_A(M)$. Then $C \leq W$ and so C is elementary abelian. Since G acts coprimely on C, by Maschke's Theorem there is a G-invariant complement C_0 of $Z(M)$ in C. Then $[C_0, G] \leq (C \cap G) \cap C_0 = Z(M) \cap C_0 = 1$. Hence, $C_0 = 1$ and $C = Z(M)$. Set now $W_0 := QC_W(T)$. Note that W_0 is G-invariant as $[W, G] \leq Q$, and that, by (ii), $|W_0/C_W(T)| = |Q/C_Q(T)| = q$. If $C_W(T) \not\leq Q$ then $|W_0/Z(M)| > q^2$ and, for $T \neq S \in Syl_p(G)$, $|Z(M)| < |C_W(T) \cap C_W(S)| = |C_W(M)|$. This contradicts $C = Z(M)$. Hence, $C_A(T) = C_W(T) = C_Q(T) = Z(T)$. □

Lemma 6.12. *Let $q > 2$ be a power of 2 and $G \cong L_3(q)$ or $Sp_4(q)$. Let (G_1, G_2) be a weak BN-pair of G involving $SL_2(q)$. Let $B := G_1 \cap G_2$ and $\mathcal{A}$ be the amalgam corresponding to (G_1, G_2). Then for every $\hat{\alpha} \in Aut(\mathcal{A})$ there exists $\beta \in Aut(G)$ such that $\hat{\alpha} = (\beta_{|G_1}, \beta_{|G_2}, \beta_{|B})$.*

Proof. Let $i \in \{1, 2\}$. Set

$$\begin{aligned} T &= O_p(B), \\ Q_i &= O_p(G_i), \\ M_i &= O^{p'}(G_i), \\ A_i &= \{\alpha_i : (\alpha_1, \alpha_2, \alpha) \in Aut(\mathcal{A})\} \leq Aut(G_i), \\ C_i &= C_{A_i}(Q_i), \\ A_0 &= N_{Aut(G)}(T) \cap N_{Aut(G)}(Q_1). \end{aligned}$$

We will use the following properties of G and $Aut(G)$ that follow from 6.7-6.10:

(1) $M_i/Q_i \cong SL_2(q)$ and G_i/Q_i is isomorphic to a subgroup of $GL_2(q)$.

(2) G_i acts faithfully on $Q_i/Z(M_i)$, $\Phi(Q_i) = 1$, $[Q_i, M_i] = Q_i$, $|Q_i/C_{Q_i}(T)| = q$, and $Q_i/Z(M_i)$ is a natural $SL_2(q)$-module for M_i/Q_i.

(3) $Z(G_i) = 1 = Z(B)$.

(4) $T \in Syl_p(G)$ and $B = N_G(T)$.

(5) $A_0 \leq N(Q_2)$ and $A_0/Q_i \cong N_{Aut(GF(q)) \ltimes GL_2(q)}(\tilde{T})$ for $\tilde{T} \in Syl_2(GL_2(q))$, where we identify Q_i with the inner automorphisms of G induced by Q_i.

In particular, G_i fulfils the hypothesis of Lemma 6.11. Hence we have

(6) $C_i = C_{A_i}(Q_i/Z(M_i)) \leq C(G_i/Q_i)$ and C_i is a 2-group.

(7) $C_{A_i}(T) \cong Z(T)$.

Observe that the map

$$\phi : A_0 \to Aut(\mathcal{A}) \text{ defined by } \alpha \mapsto (\alpha_{|G_1}, \alpha_{|G_2}, \alpha_{|B})$$

is well-defined and a monomorphism of groups. Recall that $\alpha_{1|B} = \alpha_{|B} = \alpha_{2|B}$ for $(\alpha_1, \alpha_2, \alpha) \in Aut(\mathcal{A})$. Moreover, by (3) and (7), $C_{A_1}(B) = 1$ and $C_{A_2}(B) = 1$. Hence, for $i = 1, 2$, the maps

$$\psi_i : Aut(\mathcal{A}) \to A_i \text{ defined by } (\alpha_1, \alpha_2, \alpha) \mapsto \alpha_i$$

are isomorphisms of groups. In particular, it is therefore sufficient to show $|A_1| = |A_0|$. Observe that, by (5),

(8) $A_i/C_{A_i}(M_i/Q_i) \cong N_{Aut(M_i/Q_i)}(T/Q_i)$ for $i = 1, 2$.

Furthermore, since every element in $C_{A_i}(M_i/Q_i)$ acts on $Q_i/Z(M)$ as a scalar from $End_{M_i}(Q_i/Z(M)) \cong GF(q)$, it follows from (2),(5) and (6) that

(9) $C_{A_i}(M_i/Q_i)/C_i \cong C_{q-1}$ for $i = 1, 2$.

Hence, by (5), it is sufficient to show that $|C_1| \leq |Q_1|$. In order to prove that set $C := C_1\psi_1^{-1}\psi_2$. Note that $\alpha_{1|B} = (\alpha_1\psi_1^{-1}\psi_2)_{|B}$ for every $\alpha_1 \in A_1$. Thus, $[Q_1, C] = 1$ and $[T, C] = [Q_1Q_2, C] \leq Q_2$. By (6), $C \cong C_1$ is a 2-group. Hence, by (8), $C \leq TC_{A_2}(M_2/Q_2)$, and by (9), $C_0 := C \cap C(M_2/Q_2) \leq C_2$. Thus, $|C/C_0| \leq q$ and, by (7), $C_0 \leq C_{A_2}(Q_1Q_2) = C_{A_2}(T) \cong Z(T)$. So $|C_1| = |C| \leq q \cdot |Z(T)| = |Q_1|$. As argued above this proves the assertion. □

Lemma 6.13. *Let G be a group, let $q > 2$ be a power of 2, and let (G_1, G_2) be a weak BN-pair of G involving $SL_2(q)$. Let H be a finite group such that $F^*(H) \cong L_3(q)$ or $Sp_4(q)$ for a power $q > 2$ of 2. Let ϕ and ψ be epimorphisms from G to H such that $G_i \cap ker\phi = G_i \cap ker\psi = 1$ for $i = 1, 2$. Then $ker\phi = ker\psi$.*

Proof. Observe that, by Remark 6.6, $(G_1\phi, G_2\phi)$ is a weak BN-pair of H and

$$(F^*(H) \cap G_1\phi, F^*(H) \cap G_2\phi)$$

is a weak BN-pair of $F^*(H)$ involving $SL_2(q)$. Let G_i° be the preimage of $F^*(H) \cap G_i\phi$ in G_i for $i = 1, 2$. Then (G_1°, G_2°) is a weak BN-pair of $G^\circ = \langle G_1^\circ, G_2^\circ \rangle$ involving $SL_2(q)$, and $G^\circ\phi = F^*(H)$. Moreover, for $i = 1, 2$, G_i° is normal in G_i and $G_i = G_i^\circ B$. Thus, G° is normal in $G = \langle G^\circ, B \rangle$. Set now $N := ker\phi$ and $\overline{G} = G/(G^\circ \cap N)$. Then $\overline{G^\circ} \cong F^*(H)$, $\overline{B} \cong B\phi$, and with Dedekind's Law $B(G^\circ \cap N) \cap G^\circ = (B \cap G^\circ)(G^\circ \cap N)$, so $\overline{B} \cap \overline{G^\circ} = \overline{B \cap G^\circ}$. Moreover, $B \cap G^\circ = B \cap G_i \cap G^\circ = B \cap G_i^\circ$ and so $\overline{B} \cap \overline{G^\circ} = \overline{B \cap G_i^\circ} \cong B \cap G_i^\circ \cong (B \cap G_i^\circ)\phi = B\phi \cap G_i^\circ\phi = B\phi \cap G_i\phi \cap F^*(H) = B\phi \cap F^*(H)$, for $i = 1, 2$. Hence,

$$|\overline{G}| = |\overline{G^\circ B}| = |\overline{G^\circ}||\overline{B}/\overline{B} \cap \overline{G^\circ}| = |F^*(H)||B\phi/B\phi \cap F^*(H)| = |F^*(H)(B\phi)| = |H|,$$

and so $N \leq G_0$. The same holds with ψ instead of ϕ. Thus, we may assume without loss of generality that $H = F^*(H) \cong L_3(q)$ or $Sp_4(q)$.

By 6.8 and 6.10, $Aut(H)$ acts transitively on the weak BN-pairs of H. Therefore, we may assume that $G_i\phi = G_i\psi$ for $i = 1, 2$. Then $((\phi_{|G_1})^{-1}\psi, (\phi_{|G_2})^{-1}\psi, (\phi_{|B})^{-1}\psi)$ is an automorphism of the amalgam corresponding to $(G_1\phi, G_2\phi)$. Hence, as a consequence of Lemma 6.12, there is an automorphism α of H such that $(\phi_{|G_i})^{-1}\psi = \alpha_{|G_i}$ for $i = 1, 2$. This implies $\psi = \phi\alpha$ and $ker\psi = ker\phi\alpha = ker\phi$. □

The proof of Theorem 6.3. Let G_1, G_2, B, S, Q, q and M be as in the hypothesis of Theorem 6.3. Set $T := N_S(Q)$. Let $t \in S \backslash T$ and $X := \langle G_2, G_2^t \rangle$. As $G_2^{t^2} = G_2$, X is normal in $G = \langle t, G_2 \rangle$.

Let K be a set of right coset representatives of B in G_2. Then K^t is a set of right coset representatives of $B = B^t$ in G_2^t. So, as $Bt^{-1} = Bt$, the set

$$K^* := \{tkt \ : \ k \in K\}$$

is also a set of right coset representatives of B in G_2^t. Let $g \in G$. By Lemma 6.1, there exists $b \in B$, $n \in \mathbb{N}$ and $g_1, g_2, \ldots, g_n \in (K \cup K^*)\backslash B$ such that

$$g = bg_1 \ldots g_n$$

and $g_{k+1} \in K$ if and only if $g_k \in K^*$ for all $1 \leq k < n$. Since $G = G_1 *_B G_2$ and $\{t, 1\}$ is a set of right coset representatives of B in G_1, it follows from Lemma 6.1 and the definition of K^* that this expression is unique. Hence, again by Lemma 6.1, $X = G_2 *_B G_2^t$.

Assume there is $1 \neq U \leq B$ such that U is normal in X. If U is a p-group then, as U is normal in G_2 and G_2^t, it follows from Lemma 4.15(a) that $U \leq Q \cap Q^t = Z(J(T))$. Hence, as $Q/C_Q(M)$ and $Q^t/C_{Q^t}(M^t)$ are irreducible modules for M respectively M^t, we have $U \leq U_0 := C_Q(M) \cap C_Q(M)^t$. Since U_0 is normal in MT and $G_1 = B\langle t \rangle$, it follows from our assumptions that $U_0 = 1$. Hence, $U = 1$, a contradiction. So U is not a p-group and, as U was arbitrary, also $O_p(U) = 1$. Since $J(T)$ is normal in B, it follows $[U, J(T)] \leq U \cap J(T) \leq O_p(U) = 1$. In particular, $U \leq C_{G_2}(Q) \leq Q$, contradicting U not being a p-group. Hence, no non-trivial normal p-subgroup of X is contained in B. Thus, it is now easy to check that (G_2, G_2^t) is a weak BN-pair of X involving $SL_2(q)$. Hence, by Theorem 6.5, there is a free normal subgroup N of X such that X/N is finite and $F^*(X/N) \cong L_3(q)$ or $Sp_4(q)$. By Remark 6.6, we have $TN/N \in Syl_2(X/N)$.

Define epimorphisms ϕ and ψ from X to X/N via $x\phi = xN$ and $x\psi = x^tN$ for all $x \in X$. Then it follows from Lemma 6.13 that $N = ker\phi = ker\psi = N^{t^{-1}}$. Hence, N is normal in $G = X\langle t \rangle$. Observe now that $H := G/N$ is finite, and X/N has index 2 in H. So $SN/N \in Syl_2(H)$ and $F^*(H) = F^*(X/N) \cong L_3(q)$ or $Sp_4(q)$.

Chapter 7

Classification for $p = 2$

Throughout this chapter let $\mathcal{F}$ be a fusion system on a finite 2-group S.

Hypothesis 7.1. *Assume every parabolic subsystem of $\mathcal{F}$ is constrained. Let $Q \in \mathcal{F}$ such that Q is centric and fully normalized. Set $T := N_S(Q)$ and $M := J(G(Q))$, and assume the following hold:*

(i) $Q \leq M$, $M/Q \cong SL_2(q)$ *for some power q of p, and* $C_T(M/Q) \leq Q$.

(ii) Q *is elementary abelian and* $Q/C_Q(M)$ *is a natural* $SL_2(q)$*-module for* M/Q.

(iii) $T < S$ *and* $N_S(U) = T$ *for every subgroup* $1 \neq U \leq Q$ *with* $U \trianglelefteq MT$.

(iv) *If* $t \in T \backslash J(T)$ *is an involution and* $\langle t \rangle$ *is fully centralized, then* $C_{\mathcal{F}}(\langle t \rangle)$ *is constrained.*

Recall here from Notation 2.30 that, for every fully normalized subgroup $P \in \mathcal{F}$, $G(P)$ denotes a model for $N_{\mathcal{F}}(P)$, provided $N_{\mathcal{F}}(P)$ is constrained. The aim of this chapter is to prove the following theorem.

Theorem 7.2. *Assume Hypothesis 7.1. Then there is a finite group G containing S as a Sylow 2-subgroup such that $\mathcal{F} \cong \mathcal{F}_S(G)$ and one of the following holds:*

(a) S *is dihedral of order at least* 16, $Q \cong C_2 \times C_2$ *and* $G \cong L_2(r)$ *or* $PGL_2(r)$*, for some odd prime power r.*

(b) *S is semidihedral, $Q \cong C_2 \times C_2$ and G is an extension of $L_2(r^2)$ by an automorphism of order 2, for some odd prime power r.*

(c) *S is semidihedral of order 16, $Q \cong C_2 \times C_2$ and $G \cong L_3(3)$.*

(d) *$|S| = 32$, Q has order 8, and $G \cong Aut(A_6)$ or $Aut(L_3(3))$.*

(e) *$|S| = 2^7$ and $G \cong J_3$.*

(f) *$F^*(G) \cong L_3(q)$ or $Sp_4(q)$, $|O^2(G) : F^*(G)|$ is odd and $|G : O^2(G)| = 2$. Moreover, if $F^*(G) \cong Sp_4(q)$ then $q = 2^e$ where e is odd.*

Recall Notation 2.1 and Notation 2.11 which we will use frequently in this chapter. Moreover, to ease notation we set

$$A(P) := Aut_{\mathcal{F}}(P), \text{ for every } P \in \mathcal{F}.$$

7.1 Preliminary results

We start with some group theoretical results. For Lemma 7.3–Lemma 7.7 let G be a finite group.

Lemma 7.3. *Let $S \in Syl_2(G)$, $T := J(S)$ and $t \in S \backslash T$. Assume the following hold.*

(i) *$|S : T| = 2$.*

(ii) *$A \leq Z(S)$ for every elementary abelian subgroup A of T with $C_S(A) \not\leq T$.*

(iii) *$Z(S) \leq T$, $Z(T)$ is elementary abelian, and $|Z(T)/Z(S)| > 2$.*

(iv) *$Z(T)\langle t \rangle \not\leq T^g$ for any $g \in G$.*

Then $t \notin T^g$ for any $g \in G$.

Proof. Set $Z := Z(T)$. Assume there exists $g \in G$ such that $t \in T^g$ and choose this element g such that $Z(S) \cap T^g$ is maximal. We show first

(1) $Z^x \leq T$ for all $x \in G$ with $Z^x \leq S$.

For the proof assume there is $x \in G$ such that $Z^x \leq S$ and $Z^x \not\leq T$. Then by (ii) and (iii), $Z^x \cap T \leq Z(S)$ and so $|Z/Z(S)| \leq |Z/(Z^x \cap T)| = |Z^x/(Z^x \cap T)| = 2$, a contradiction to (iii). This shows (1). Set

$$Z^* := (Z(S) \cap T^g)\langle t \rangle \text{ and } N := N_G(Z^*).$$

Let $h \in G$ such that $Z \cap N \leq N \cap S^h \in Syl_2(N)$. Note that $t \in Z^* \leq O_p(N) \leq S^h$. We show next

(2) $t \in T^h$.

For the proof assume $t \notin T^h$. As $Z^* \leq T^g$, we have $[Z^*, Z^g] = 1$. Therefore, since $C_G(Z^*) \cap S^h \in Syl_2(C_G(Z^*))$, there exists $c \in C_G(Z^*)$ such that $Z^{gc} \leq S^h$. Now by (1), $Z^{gc} \leq T^h$. Note that $[Z^{gc}, t] = 1$, so by (ii), $Z^{gc} \leq Z(S)^h$, a contradiction to (iii). This shows (2).

In particular, by (iv), $Z \not\leq T^h$ and so, by (1), $Z \not\leq S^h$. Thus, the choice of h gives $Z \not\leq N$. By (i), $[Z, S] \leq Z(S)$. Hence, if $Z(S) \leq T^g$ then $Z(S) \leq Z^*$ and so $[Z, Z^*] \leq [Z, S] \leq Z(S) \leq Z^*$, contradicting $Z \not\leq N$. This proves

(3) $Z(S) \not\leq T^g$.

Because of the maximality of $Z(S) \cap T^g$, properties (2) and (3) give now $Z(S) \not\leq T^h$. Note that $Z(S) \leq Z \cap N \leq S^h$ and so $S^h = T^h Z(S)$. Thus, $Z \cap N = Z(S)(Z \cap N \cap T^h)$. Moreover, (ii) and $t \in S^h$ imply $Z \cap N \cap T^h \leq Z(S)^h \leq C_G(t)$. Therefore, $Z \cap N \leq C(t)$, and so $Z \cap N = Z(S)$. Hence, for $z \in Z \backslash Z(S)$, we have $z \notin N$ and so $[z, t] \notin Z(S) \cap T^g$. As $[Z, t] \leq Z(S)$, this gives $[Z, t] \cap T^g = 1$. Hence, $|Z/Z(S)| = |Z/C_Z(t)| = |[Z, t]| = |[Z, t]T^g/T^g| \leq |Z(S)/Z(S) \cap T^g|$. Now the maximality of $Z(S) \cap T^g$ yields $|Z/Z(S)| \leq |Z(S)/Z(S) \cap T^h| = |S^h/T^h| = 2$, a contradiction to (iii). This proves the assertion. $\square$

Corollary 7.4. *Let $S \in Syl_2(G)$. Let T be a subgroup of S such that*

(i) T is elementary abelian and $|S : T| = 2$.

(ii) $|C_T(S)|^2 = |T|$.

Then T is strongly closed in $\mathcal{F}_S(G)$ or $|T| = 4$.

Proof. This is a direct consequence of Lemma 7.3. □

Lemma 7.5. *Let $S \in Syl_2(G)$, $T \leq S$ and $K \leq N_G(T)$ such that for $Z := Z(T)$ the following hold.*

(i) $|S : T| = 2$ and $|Z/C_Z(S)| = 2$.

(ii) $|K|$ is odd and K acts irreducibly on $Z/C_Z(K)$.

(iii) $C_Z(K) \cap C_Z(S) = 1$.

Then $|Z| = 4$.

Proof. Without loss of generality assume $G = N_G(Z)$. Set $\overline{G} = G/C_G(Z)$. Then $|\overline{S}| = 2$ and by Cayley's Theorem there is a normal subgroup U of $\overline{G}$ such that $|U|$ has odd order and $\overline{G} = \overline{S}U$. Set $R := [\overline{S}, U]$. If $R = 1$ then $[\overline{K}, \overline{S}] = 1$ and $C_Z(K)$ is S-invariant. Hence, by (iii), $C_Z(K) = 1$ and by (ii), $[Z, S] = 1$, a contradiction to (i). Thus $1 \neq R \leq R_0 = \langle \overline{S}^U \rangle$. If $O_2(R_0) \neq 1$ then $\overline{S} = O_2(R_0)$ is normal in $\overline{G}$ and $R = 1$, a contradiction. Thus, $O_2(R_0) = 1$. With a Theorem of Glauberman [KS, 9.3.7] it follows from (i) that $R_0 \cong S_3$ and $|Z/C_Z(R_0)| = 4$. In particular, for $D := O^p(R)$, $|D| = 3$, $|[Z, D]| = |[Z, R]| = 4$ and $C_Z(D) = C_Z(R) = C_Z(R_0) \leq C_Z(S)$. Since R is normal in G, $\overline{K}$ acts on R. So, as K has odd order, $[D, \overline{K}] = 1$. Since $C_{[Z,D]}(S) \neq 1$, (iii) yields $[Z, D] \not\leq C_Z(K)$. As D acts irreducibly on $[Z, D]$, we have then $[Z, D] \cap C_Z(K) = 1$ and so $[C_Z(K), D] = 1$. Hence, $C_Z(K) \leq C_Z(D) \leq C_Z(S)$ and by (iii), $C_Z(K) = 1$. So, by (ii), $Z = [Z, D]$ has order 4. □

Lemma 7.6. *Assume one of the following holds:*

(a) $G \cong D_8$, $G \cong C_4 \times C_2$, $G \cong D_8 \times C_2$ *or* $G \cong C_4 * D_8$.

(b) There are subgroups V, K *of* G *such that* $G = K \ltimes V$, $K \neq 1$ *is cyclic of order at most* 4, V *is elementary abelian of order at most* 2^3 *and* $[V, K] \neq 1$.

Then $Aut(G)$ *is a* 2*-group.*

Proof. If $G \cong D_8$, G has a characteristic subgroup that is cyclic of order 4 and therefore $Aut(G)$ is a 2-group. In particular, if $G \cong C_4 * D_8$, then the assertion holds, since then $\Omega(G) \cong D_8$ and $|G : \Omega(G)| = 2$. If $G \cong C_4 \times C_2$ then $Aut(G)$ is a 2-group because of $1 \neq \Phi(G) < Z(G) < G$. Therefore, the assertion holds also in the case $G \cong D_8 \times C_2$, since then $\langle x \in G : o(x) = 4\rangle \cong C_4 \times C_2$. Hence, $Aut(G)$ is a 2-group if (a) holds. Assume now $G = K \ltimes V$ for subgroups K, V of G as in (b). Note that $[V, K] = [G, G]$ is characterstic in G and $|[V, K]| = |V/C_V(K)|$. Suppose first $|K| = 2$. Then $Z(G) = C_V(K)$ and $[V, K] \leq C_V(K)$. This gives $G \cong D_8$ or $G \cong D_8 \times C_2$ and we have shown already that then $Aut(G)$ is a 2-group. Thus, we may assume that $|K| = 4$. Let $s \in K$ be of order 4. Then $\Omega(G) = V\langle s^2\rangle$ has index 2 in G. If $[V, s^2] \neq 1$ then $Aut(\Omega(G))$ is a 2-group by what we have just shown. Thus, we may assume $[V, s^2] = 1$. Then $[V, s] \leq C_V(s)$ and $Z(G) = C_V(s)\langle s^2\rangle$. If $|V| = 4$ then $[G, G] = [V, s] = C_V(s)$, so $1 \neq [G, G] < Z(G) < \Omega(G) < G$ and $Aut(G)$ is a 2-group. If $|V| = 2^3$ then $|C_V(K)| = 4$, $|[G, G]| = |[V, K]| = 2$, $\Phi(G) = [G, G]\langle s^2\rangle$ has order 4 and $Z(G) = C_V(K)\langle s^2\rangle$ has order 2^3. As $|\Omega(G)| = 2^4$ and $|G| = 2^5$, this shows that $Aut(G)$ is a 2-group. □

Lemma 7.7. *Suppose* $G \cong L_3(4)$ *or* $Sp_4(4)$. *Let* $S \in Syl_2(Aut(G))$ *and identify* G *with its group of inner automorphisms. Let* $t \in S \backslash G$ *be a field automorphism of* G *and* $C_S(t) \leq P < S$. *Then* $Aut(P)$ *is a* 2*-group.*

Proof. Let $Q \in \mathcal{A}(S)$. Set $T := N_S(Q)$ and $Z := Z(J(S))$. By 6.8 and 6.10, we have $J(S) \in Syl_p(G)$, $T = J(S)\langle t\rangle$, $|S/J(S)| = 4$ and $S = J(S)C_S(t)$. Furthermore, if $G \cong L_3(4)$, we may choose an involution $s \in C_S(t)\backslash T$ such that $[Z, s] = 1$. If $G \cong Sp_4(4)$, $S/J(S)$ is cyclic and we can pick $s \in C_S(t)\backslash T$ such that $s^2 = t$. In both cases, we set

$$W := C_Q(t) \text{ and } Z_0 := Z \cap P.$$

Lemma 4.15 together with 6.7 and 6.9 gives the following property:

(1) For every $x \in S\backslash T$, we have $\mathcal{A}(S) = \{Q, Q^x\}$, $Z = Q \cap Q^x = [Q, Q^x]$, and every elementary abelian subgroup of $J(S)$ is contained in Q or Q^x.

By 6.7 and 6.9 we have also

(2) $|Z(S)| = 2$ and $Z(S) = [W, W^s]$.

In particular, if $Q = (Q \cap P)Z$ then $Q^s = (Q^s \cap P)Z$ and $Z = [Q, Q^s] \leq P$. Hence, $Q \leq P$ and so $S = (QQ^s)C_S(t) \leq P$, a contradiction. As $|Q : (WZ)| = 2$, this shows

(3) $P \cap Q = WZ_0$ and $P \cap Q^s = W^s Z_0$.

Assume now the assertion is wrong. Pick a non-trivial element $\alpha \in Aut(P)$ of odd order. We show next

(4) $C_S(t) < P$.

Assume $P = C_S(t)$. Then $\Omega(Z(P)) \leq C_S(W) \leq T$ and, by (1), $\Omega(Z(P)) \cap J(S) \leq C_Z(P) = Z(S)$. Hence, $\Omega(Z(P)) = Z(S)\langle t\rangle$. In particular, $P/\Omega(Z(P)) \cong D_8$ if $G \cong L_3(4)$, and $P/\Omega(Z(P)) \cong D_8 \times C_2$ if $G \cong Sp_4(4)$. Hence, Lemma 7.6 gives $[P, \alpha] \leq \Omega(Z(P))$. Moreover, by (2), $|Z(S)| = 2$ and $Z(S) = [W, W^s] \leq P' \leq J(S)$, so $Z(S) = \Omega(Z(P)) \cap P'$. Coprime action shows now $[P, \alpha] = 1$, a contradiction. Thus, (4) holds. We show next

(5) $C_Z(t) < Z_0$.

If $Z_0 = C_Z(t)$ then (3), (4) and $S = J(S)C_S(t)$ imply $J(S) = (P \cap J(S))Q$. Now the module structure of Q as described in 6.7 gives $Z_0 \geq [W, P \cap J(S)] \not\leq C_Z(t)$, a contradiction. Thus, (5) holds. Since $\Omega(Z(P)) \leq C_S(W) \leq T$, (5) gives in particular that $\Omega(Z(P)) \leq J(S)$. Hence, (1) implies $\Omega(Z(P)) \leq C_Z(P) = Z(S)$. So, by (2),

(6) $\Omega(Z(P)) = Z(S)$.

We show next

(7) $Z_0\alpha \neq Z_0$.

Assume $Z_0\alpha = Z_0$ and set $\overline{P} = P/Z_0$. Then $\overline{J(S) \cap P}$ is elementary abelian of order at most 2^3. For $G \cong Sp_4(4)$ we get $[\overline{P}, \alpha] = 1$ as an immediate consequence of Lemma 7.6. For $G \cong L_3(4)$ note that, by (5), $C_P(Z_0) = (J(S) \cap P)\langle s \rangle$ has index 2 in P and, by Lemma 7.6, $[\overline{C_P(Z_0)}, \alpha] = 1$. Hence, in both cases $[\overline{P}, \alpha] = 1$ and so coprime action gives $[Z_0, \alpha] \neq 1$. Now (6) yields $G \cong Sp_4(4)$. Therefore, $C_Z(t) = Z(\Omega(P)) \cap Z_0$ is α-invariant. Now, by (2) and (6), in the series

$$1 \neq Z(S) = \Omega(Z(P)) \leq C_Z(t) \leq C_Z(t)[Z_0, P] \leq Z_0$$

every factor has order at most 2. Hence, $[Z_0, \alpha] = 1$, a contradiction. This shows (7). We prove next

(8) $T = J(S)(Z_0\alpha)$ and $[Z_0, Z_0\alpha] \neq 1$.

Note that $Z_0\alpha$ is an elementary abelian normal subgroup of P. Hence, by (1), $(Z_0\alpha) \cap J(S) \leq Z_0$ and so, by (7), $Z_0\alpha \not\leq J(S)$. Moreover, $[P \cap Q, Z_0\alpha] \leq J(S) \cap (Z_0\alpha) \leq Z_0 \leq P \cap Q$ and so, again by (1), $Z_0\alpha \leq T$ as $Q \cap P \not\leq Z$. This shows $T = J(S)(Z_0\alpha)$ and (8) follows from (5).

Set now $P^* := P$ if $G \cong L_3(4)$, and $P^* := \Omega(P)$ if $G \cong Sp_4(4)$. We show next

(9) $P^* \cap J(S) = C_{J(S)}(t)Z_0$ and $|Z_0 : C_Z(t)| = 2$.

Set $U := C_{P^*}(Z_0)$ and observe that $|P^* : U| = 2$. Hence, also $|P^* : (U\alpha)| = 2$ and $|(P^* \cap J(S)) : (J(S) \cap (U\alpha))| \leq 2$. By 6.8 and 6.10, we have $|C_{J(S)}(u)| \leq |C_{J(S)}(t)|$, for every involution $u \in T \backslash J(S)$. Hence, by (8), $|J(S) \cap (U\alpha)| \leq |C_{J(S)}(t)|$. Now (9) follows from (5). We show next

(10) $G \cong Sp_4(4)$.

Assume $G \cong L_3(4)$. Then, by (9), $P = C_S(t)Z$. By (2) and (6), $[Z(S), \alpha] = 1$. Observe

$$\overline{P} := P/Z(S) = \langle \overline{W}, \overline{s} \rangle \times \langle \overline{t} \rangle \times \overline{Z} \cong D_8 \times C_2 \times C_2,$$

$D := \langle Z, t \rangle \cong D_8$, and $Z(\overline{P}) = \overline{P'D} \cong C_2 \times C_2 \times C_2$. So $P'D$ is characteristic in P. Furthermore, by (2), $Z(S) = [W, W^s] \leq P'$, and so we have $P' \cong C_4$, $D \cap P' = Z(S)$ and $P'D \cong C_4 * D_8$. Hence, by Lemma 7.6, $[P'D, \alpha] = 1$. Moreover, $\overline{P'D} = Z(\overline{P})$ has index 2 in

$$\langle x \in \overline{P} : o(x) = 4 \rangle \cong C_4 \times C_2 \times C_2$$

and hence, $[P, \alpha] = 1$. This shows (10). We show next

(11) $C_Z(t)\alpha = C_Z(t)$.

Note that, by (1) and (8), $Z_0 \cap (Z_0\alpha) = (Z_0\alpha) \cap J(S) \leq C_Z(Z_0\alpha) = C_Z(t)$ and $|Z_0/(Z_0 \cap (Z_0\alpha))| = 2$. Hence, by (5), $Z_0 \cap (Z_0\alpha) = C_Z(t)$. The same holds with α^2 in place of α, so $Z_0 \cap (Z_0\alpha^2) = C_Z(t)$. Hence, $C_Z(t) \leq (Z_0\alpha) \cap (Z_0\alpha^2)$ and, as $|C_Z(t)| = |(Z_0\alpha) \cap (Z_0\alpha^2)|$, we have $C_Z(t) = (Z_0\alpha) \cap (Z_0\alpha^2) = (Z_0 \cap (Z_0\alpha))\alpha = C_Z(t)\alpha$. This shows (11).

We now derive the final contradiction. Set $\hat{P} := P/C_Z(t)$. If $|\widehat{P \cap J(S)}| \leq 2^3$, then by Lemma 7.6, $Aut(\widehat{P})$ is a 2-group and (11) implies $[P, \alpha] = 1$, a contradiction. Therefore, $|\widehat{P \cap J(S)}| \geq 2^4$ and so, by (9), $(P \cap J(S))Q = J(S)$. Hence, $[W, P \cap J(S)] \not\leq C_Z(t)$ and, again by (9), $Z_0 = (P' \cap Z)C_Z(t)$. As $P' \leq J(S)$ it follows from (1) that $\Omega(Z(P')) = Z \cap P'$. Now (11) yields a contradiction to (7). □

Lemma 7.8. *Assume Hypothesis 7.1. Then* $|Q| \leq q^3$.

Proof. By Hypothesis 7.1(iii), we can choose $t \in N_S(T)\backslash T$ such that $t^2 \in T$, and have then $U := C_Q(M) \cap C_Q(M)^t = 1$. So, as $Q/C_Q(M)$ is a natural $SL_2(q)$-module for M/Q,

$$|C_Q(M)| = |C_Q(M)^t/U| = |C_Q(M)^t C_Q(M)/C_Q(M)| \leq |Z(J(T))/C_Q(M)| \leq q$$

and $|Q| \leq q^3$. □

In the next proof and throughout this chapter we will use the well-known and elementary to check fact that a 2-group S is dihedral or semidihedral if it contains a subgroup V such that $V \cong C_2 \times C_2$ and $C_S(V) \leq V$.

Lemma 7.9. *Assume Hypothesis 7.1 and* $T = J(T)$. *Let* $|Q| > 4$ *and let* $P \in \mathcal{F}\backslash(\{T\} \cup Q^{\mathcal{F}})$ *be essential in* $\mathcal{F}$. *Then the following hold.*

(a) $P \not\leq T$ *and* $P \cap T$ *is not* $A(P)$*-invariant.*

(b) P *is not elementary abelian.*

(c) If $Z(S)$ *is* $A(P)$*-invariant then* $Z(T) \leq P$.

Proof. Recall that by Lemma 4.16, $|S : T| = 2$. Assume $P \cap T$ is $A(P)$-invariant. If $P \leq T$ then, as P is centric, $Z(T) < P$. Let $t \in S\backslash T$. Since P is centric and $P \notin Q^{\mathcal{F}}$, we have $P \not\leq Q$ and $P \not\leq Q^t$. So by Lemma 4.15, $\Omega(Z(P)) \leq C_Q(P) = C_{Q^t}(P) = Z(T)$. Thus, $Z(T) = \Omega(Z(P))$ is $A(P)$-invariant. If $P \not\leq T$ then, again by Lemma 4.15, $\Omega(Z(P \cap T)) = Z(T) \cap P$. So in any case, $Z(T) \cap P$ is $A(P)$-invariant. Thus, for $X := \langle T_P^{A(P)} \rangle$, $[P, X] \leq P \cap T$, $[P \cap T, X] \leq Z(T) \cap P$ and $[Z(T) \cap P, X] = 1$. Hence, X is a normal 2-subgroup of $A(P)$. Since P is essential, this yields $T_P \leq X \leq Inn(P)$ and $T \leq P$, a contradiction to $P \neq T$. This shows (a).

For the proof of (c) assume that $Z(S)$ is $A(P)$-invariant. Observe that S acts quadratically on $Z(T)$ and so $[Z(T), P] \leq Z(S)$. Hence, we have $[P, X] \leq Z(S)$

and $[Z(S), X] = 1$, for $X := \langle Z(T)_P{}^{A(P)} \rangle$. Therefore, X is a normal 2-subgroup of $A(P)$ and, as P is essential, we get $X \leq Inn(P)$ and $Z(T) \leq P$. This shows (c).

Assume now P is elementary abelian. Since $|Q| > 4$, S is not dihedral or semidihedral and hence,

(1) $|P| > 4$.

By Lemma 4.15 and (a), $P \cap T = Z(S)$. Hence,

(2) $|P/C_P(S_P)| = |P/Z(S)| = 2$.

Moreover, by Hypothesis 7.1(iii), $P \cap C_Q(M) = 1$. Thus, $|P \cap T| \leq q$ and so $|P| \leq 2 \cdot q$. In particular, by (1),

(3) $q > 2$.

Since P is essential, $A(P)$ has a strongly p-embedded subgroup. So there exists $\phi \in A(P)$ such that $S_P \cap S_P\phi^* = 1$. Set $L = \langle S_P, S_P\phi^* \rangle$. Then, by (2), $\overline{P} := P/C_P(L)$ has order 4 and $L/C_L(\overline{P}) \cong S_3$. Observe that $C_L(\overline{P})$ is a normal 2-subgroup of L and thus contained in $S_P \cap S_P\phi^* = 1$. Hence, $L \cong S_3$ and $|N_S(P) : P| = |S_P| = 2$. As S acts quadratically on $Z(T)$, we have $Z(T) \leq N_S(P)$. Therefore, $|Z(T)/Z(S)| = |Z(T)/Z(T) \cap P| \leq 2$. If $Z(T) = Z(S)$ then $q = |C_{T/Z(S)}(S)| \leq |N_T(P)/Z(S)| \leq 2$, a contradiction to (3). Hence, $|Z(T)/Z(S)| = 2$ and, by (3), $G = G(T)$ fulfils the Hypothesis of Lemma 7.5. Hence, Lemma 7.5 yields $q \leq |Z(T)| = 4$. Thus, $|Z(S)| = 2$ and (2) yields a contradiction to (1). This shows (b). $\square$

7.2 The case $q = 2$

Throughout this section assume Hypothesis 7.1 and $q = 2$. Note that T/Q embeds into $Aut(M/Q) \cong Aut(SL_2(q))$ and, by Lemma 4.15, $J(T) \in Syl_2(M)$. Therefore, $T = J(T) \in Syl_2(M)$.

Lemma 7.10. *Assume* $|Q| = 4$ *and let* P *be essential in* $\mathcal{F}$. *If* P *is not a fours group, then* P *is quaternion of order* 8, $A(P) = Aut(P)$, *and* S *is semidihedral of order* 16.

Proof. It follows from $|Q| = 4$ that S is dihedral or semidihedral. Let $X \leq S$ be cyclic of index 2. As $Aut(P)$ is not a 2-group, $P \not\leq X$ and $P \cap X$ is not characteristic in P. Assume now P is not a fours group. Then $|P \cap X| = 4$, S is semidihedral, P is quaternion of order 8, $Z(P) = Z(S)$ and $A(P) = Aut(P) \cong S_4$. In particular, $N_{\mathcal{F}}(P)$ is a subsystem of $\mathcal{N} := N_{\mathcal{F}}(Z(S))$ and P is essential in $\mathcal{N}$. By Hypothesis 7.1, $\mathcal{N}$ is constrained and so $Z(S) < O_p(\mathcal{N})$. Now it follows from Lemma 2.37 that $P = O_p(\mathcal{N})$. In particular, P is normal in S and thus S has order 16. □

Lemma 7.11. *Assume* $|Q| = 4$. *Then there exists a finite group* G *such that* $S \in Syl_p(G)$, $\mathcal{F} \cong \mathcal{F}_S(G)$ *and one of the following holds.*

(a) S *is dihedral and* $G \cong PGL_2(r)$ *or* $L_2(r)$ *for an odd prime power* r.

(b) S *is semidihedral and, for some odd prime power* r, G *is an extension of* $L_2(r^2)$ *by an automorphism of order* 2.

(c) S *is semidihedral or order* 16 *and* $G \cong L_3(3)$.

Proof. Recall that S is dihedral or semidihedral. Note that $A(S) = Inn(S)$ since S has no automorphisms of odd order. By Lemma 7.10, $A(P) = Aut(P)$ for every essential subgroup P of $\mathcal{F}$, and either every essential subgroup of $\mathcal{F}$ is a fours group, or S is semidihedral of order 16 and the only essential subgroup of S that is not a fours group is the quaternion subgroup of S of order 8. If S is dihedral then there are two conjugacy classes of subgroups of S that are fours groups, and they are conjugate under $Aut(S)$. By Remark 2.39, if $\mathcal{F}$ has only one conjugacy class of essential subgroups then $\mathcal{F}$ is isomorphic to the 2-fusion system of $PGL_2(r)$, and if $\mathcal{F}$ has two conjugacy classes of essential subgroups then $\mathcal{F}$ is isomorphic to the 2-fusion system of $L_2(r)$, in both cases for some odd prime power r. Let now S be semidihedral. Then S has only one conjugacy class of fours groups. Recall that there is always an odd prime power r and an extension H of $L_2(r^2)$ by an automorphism of order 2 that has semidihedral Sylow 2-subgroups of order $|S|$. If the fours groups are the only essential subgroups in $\mathcal{F}$ then it follows from Remark 2.39 that $\mathcal{F}$ is isomorphic to the 2-fusion system of H. Otherwise, it follows from the above and Remark 2.39 that $\mathcal{F}$ is isomorphic to the 2-fusion system of $L_3(3)$. □

Lemma 7.12. *Assume* $|Q| > 4$.

(a) $|Q| = 8$, $M = G(Q) \cong S_4 \times C_2$ *and* $Z(S) = \Phi(T) \leq [Q, M]$.

(b) Let $u \in [Q, M]\backslash Z(S)$ *and* $1 \neq c \in C_Q(M)$. *Then there exists an element* $y \in S\backslash T$ *of order* 8 *such that* $y^u = y^{-1}$, $y^c = y^5$ *and* $S = \langle c, t\rangle \ltimes \langle y\rangle$. *In particular,* S *is uniquely determined up to isomorphism.*

Proof. It follows from Lemma 7.8 that $|Q| = 8$. Hence, $M = G(Q) \cong S_4 \times C_2$. In particular, $T \cong D_8 \times C_2$ and so $|\Phi(T)| = 2$. Now Hypothesis 7.1(iii) implies $Z(S) = \Phi(T)$. In particular, $Z(S) \leq [Q, M]$. This shows (a).

Recall that by Lemma 4.16, $|S : T| = 2$. Set $\overline{S} = S/Z(T)$ and $C = C_Q(M)$. Note that $\overline{T}$ is elementary abelian, and $\overline{S}$ is non-abelian, since Q is not normal in S. In particular, there exists an element $y \in S\backslash T$ such that $\overline{y}$ has order 4. Then $y^4 \in Z(T)$, and $S = T\langle y\rangle$ implies $y^4 \in Z(S)$. If $y^4 = 1$ then $y^2 \in T$ is an involution and, by Lemma 4.15, $y^2 = (y^2)^y \in Q \cap Q^y = Z(T)$, so $\overline{y}$ has order 2, a contradiction. Therefore, y^4 is an involution and y has order 8.

Since $Z(T)$ is normal in S and $[Z(T), y^2] = 1$, y acts quadratically on $Z(T)$. Hence, $[C, y] \leq [Z(T), y] \leq C_{Z(T)}(y) = Z(S) = \langle y^4\rangle$, so $\langle y\rangle$ is normalized by C. By Hypothesis 7.1(iii), $[C, y] \neq 1$. Now $[y^2, C] = 1$ implies $y^c = y^5$. Set $N := \langle y\rangle C$. Observe that $\langle y\rangle$ and $\langle yc\rangle$ are the only cyclic subgroups of N of order 8. Moreover,

$|S : N| = 2$ and so N is normal in S. Hence, u acts on N and either normalizes $\langle y \rangle$ or swaps $\langle y \rangle$ and $\langle yc \rangle$.

Assume first $y^u \in \langle yc \rangle = \langle y^2 \rangle \cup \langle y^2 \rangle yc$. Since $y^u \notin T$, $y^u = y^i c$ for some $i \in \{1, 3, 5, 7\}$. Then $y^2 \neq (y^2)^u = (y^u)^2 = (y^i c)^2 = y^i (y^i)^c = y^i (y^c)^i = y^i y^{5i} = y^{6i}$ implies $i \in \{1, 5\}$. Hence, $[y, u] = y^{-1} y^u \in \langle y^4 \rangle = Z(S) \leq Q$ and Q is normalized by y, a contradiction. Thus, $\langle y \rangle$ is normal in S and S is the semidirect product of $\langle c, u \rangle$ and $\langle y \rangle$. Since $[y^2, u] \neq 1$, $y^u \in \{y^{-1}, y^3\}$. If $y^u = y^3$ then $(yc)^u = y^3 c = (yc)^{-1}$ and $(yc)^c = y^5 c = (yy^5)^2 yc = (yy^c)^2 yc = (yc)^5$, so we may in this case replace y by yc and assume $y^u = y^{-1}$. This shows (b). $\square$

Lemma 7.13. *Assume $|Q| > 4$. Then there exists a finite group G such that $S \in Syl_p(G)$, $\mathcal{F} \cong \mathcal{F}_S(G)$ and $G \cong Aut(A_6)$ or $Aut(L_3(3))$.*

Proof. Let G be a finite group isomorphic to $Aut(A_6)$ or $L_3(3)$, $\hat{S} \in Syl_2(G)$, $\hat{Q} \in \mathcal{A}(\hat{S})$ and $\hat{\mathcal{F}} = \mathcal{F}_{\hat{S}}(G)$. Then by the structure of G, $\hat{Q}$ is essential in $\hat{\mathcal{F}}$ and $\hat{M} := N_G(\hat{Q}) \cong C_2 \times S_4$. By Lemma 7.12, there is a group isomorphism $\alpha : S \to \hat{S}$ such that $Q\alpha = \hat{Q}$ and $[Q, M]\alpha = [\hat{Q}, \hat{M}]$. This implies $\alpha^{-1} A(Q) \alpha = Aut_{\hat{\mathcal{F}}}(Q\alpha)$.

Assume first $G \cong Aut(A_6)$ and observe that $\hat{Q}^{\hat{\mathcal{F}}}$ is the only essential class in $\hat{\mathcal{F}}$. Therefore, if $Q^{\mathcal{F}}$ is the only essential class in $\mathcal{F}$, then it follows from Remark 2.39 that $\mathcal{F} \cong \hat{\mathcal{F}}$.

Therefore, we may assume from now on that there is an essential subgroup $P \in \mathcal{F} \backslash Q^{\mathcal{F}}$. It follows from Lemma 7.9 that $P \not\leq T$ and P is not elementary abelian. We show first

(1) $Z(S) = \Omega(Z(P))$.

If $\Omega(Z(P)) \leq T$ then (1) follows from $PT = S$ and Lemma 4.15. Thus, we may assume that $\Omega(Z(P)) \not\leq T$. As P is not elementary abelian, $P \cap T \not\leq Z(S)$. Since

$Q \cap P \leq C_Q(\Omega(Z(P))) = Z(S)$, we have $|P \cap T| = 4$ and P is dihedral of order 8. Then $A(P)$ is 2-group contradicting P being essential. This shows (1). Now by 7.9(c),

(2) $Z(T) \leq P$.

We show next

(3) $P \cap T = [S, S]Z(T)$.

Since $A(P)$ is not a 2-group, P is not dihedral of order 8 and so $P \cap T \neq Z(T)$. If $R \leq P$ for some $R \in \mathcal{A}(T)$, then $P = S$, a contradiction. Hence, $|P \cap T| = 8$ and there is an element of order 4 in $P \cap T$. Now (3) follows from the structure of S.

(4) $P = [S, S]Z(T)\langle t : t \in S \backslash T,\ t^4 = 1\rangle$

By Lemma 7.9(a), $P \cap T$ is not $A(P)$-invariant. So, as the elements in $P \cap T$ have order at most 4, there is an element $t \in P \backslash T$ of order at most 4. As $|P/(P \cap T)| = 2$ we have $P = (P \cap T)\langle t\rangle$. Note that, by Lemma 4.15, $t^2 \in Z(T)$. Now for $\overline{S} = S/Z(T)$, every involution in $\overline{S} \backslash \overline{T}$ is contained in $C_{\overline{T}}(t)\overline{t} = \overline{[S, S]t}$. Now (4) follows from (3).

(5) $P \cong Q_8 * C_4$, $A(P) = O^2(Aut(P))S_P$ and $A(P)/Inn(P) \cong S_3$.

Let $x \in P \backslash T$ of order at most 4. Then $Z(T)\langle x\rangle$ is dihedral of order 8 and so we may assume $o(x) = 4$. Then, by Lemma 4.15, $\langle x^2\rangle = Z(S)$ and x acts as an involution on T. For $a \in Q \backslash Z(T)$ and $z \in Z(T) \backslash Z(S)$, we have $1 \neq [a^x, a] = [a^x, a]^{-1} = [a, a^x] = [a^x, a]^x \in C_{Z(T)}(x) = Z(S)$ and $1 \neq [z, x] \leq [Z(T), x] = Z(S)$. Therefore, $[a^x, a] = [z, x]$ and thus $y := aa^x z \in C_T(x)$. Now $o(yz) = 4$, $(yz)^2 \in \Phi(T) = Z(S)$ and $1 \neq [yz, x] \in Z(S)$. Hence, $\langle x, yz\rangle \cong Q_8$ and $P = \langle x, yz\rangle\langle y\rangle \cong Q_8 * C_4$. In particular, $Aut(P)/Inn(P) \cong S_3 \times C_2$ and, as $A(P)/Inn(P)$ has a strongly 2-embedded subgroup, (5) follows.

Let now G, $\hat{S}$, $\hat{Q}$, $\hat{\mathcal{F}}$ and α be as above and assume $G \cong Aut(L_3(3))$. Then

$$\hat{P} = Z(J(\hat{S}))[\hat{S}, \hat{S}]\langle t : t \in \hat{S} \backslash J(\hat{S}),\ t^4 = 1\rangle \cong Q_8 * C_4$$

is essential in $\hat{\mathcal{F}}$, $Aut_{\hat{\mathcal{F}}}(\hat{P}) = O^2(Aut(\hat{P}))\hat{S}_{\hat{P}}$, and $\hat{P}$ is the only essential subgroup of $\hat{\mathcal{F}}$ in $\hat{\mathcal{F}}\backslash\hat{Q}^{\hat{\mathcal{F}}}$. It follows from (4) that P is the only essential subgroup of $\mathcal{F}$ in $\mathcal{F}\backslash Q^{\mathcal{F}}$, and that $P\alpha = \hat{P}$. By (5), $\alpha^{-1}A(P)\alpha = Aut_{\hat{\mathcal{F}}}(P\alpha)$ and so by Remark 2.39, α is an isomorphism from $\mathcal{F}$ to $\hat{\mathcal{F}}$. This shows the assertion. □

7.3 The case $q \geq 4$

Throughout this section assume Hypothesis 7.1 and $q \geq 4$.

Set $G_1 = G(J(T))$, $G_2 = G(Q)$, $M = J(G_2)$ and $\mathcal{F}_0 := \langle N_{\mathcal{F}}(J(T)), N_{\mathcal{F}}(Q)\rangle$.

We will use from now on without reference that, by Lemma 4.16(b),(c), $J(S) = J(T) \in Syl_2(M)$ and $|S/T| = 2$. In particular, $N_{\mathcal{F}}(J(T))$ is parabolic, so by Hypothesis 7.1, $N_{\mathcal{F}}(J(T))$ is constrained and G_1 is well-defined. Moreover, $\mathcal{F}_0$ is a fusion system on S, and $S \in Syl_2(G_1)$.

Lemma 7.14. *There is an isomorphism ϕ from $N_{G_1}(Q)$ to $N_{G_2}(J(T))$ such that ϕ is the identity on T. If $X = G_1 *_{N_{G_1}(Q)} G_2$ is the free amalgamated product with respect to $\mathcal{A} = (G_1, G_2, N_{G_1}(Q), id, \phi)$, then $\mathcal{F}_0 = \mathcal{F}_S(X)$.*

Proof. Observe that

$$\mathcal{N} := \mathcal{F}_T(N_{G_1}(Q)) = N_{N_{\mathcal{F}}(J(T))}(Q) = N_{N_{\mathcal{F}}(Q)}(J(T)) = \mathcal{F}_T(N_{G_2}(J(T))).$$

Also note that, as Q is fully normalized in $\mathcal{F}$, Q is fully normalized in $N_{\mathcal{F}}(J(T))$, so $\mathcal{N}$ is saturated. Moreover, $J(T)$ is a normal 2-subgroup of both $N_{G_1}(Q)$ and $N_{G_2}(T)$. It follows from the structure of G_2 as described in Hypothesis 7.1 that $O_p(G_2) = Q \leq J(T)$, so $N_{G_2}(J(T))$ has characteristic 2, since G_2 has characteristic 2. Let $x \in C_{G_1}(J(T))$ be of odd order. As $|S : T| = 2$ and $T/J(T)$ is cyclic, we have $[O_2(G_1), x] = [O_2(G_1), x, x] \leq [J(T) \cap O_2(G_1), x] = 1$. Hence, $x \leq C_{G_1}(O_2(G_1)) \leq O_2(G_1)$ and $x = 1$. So $C_{G_1}(J(T)) \leq O_p(G_1)$, and $N_{G_1}(Q)$ has characteristic 2, since

G_1 has characteristic 2. Therefore, $N_{G_1}(Q)$ and $N_{G_2}(J(T))$ are models for $\mathcal{N}$. Hence, by Theorem 2.29, there exists an isomorphism ϕ between these two groups that is the identity on T. Now the assertion follows from Theorem 6.2 and the definitions of G_1, G_2 and $\mathcal{F}_0$. $\square$

Lemma 7.15. *There exists a finite group G with $S \in Syl_2(G)$ such that $\mathcal{F}_0 = \mathcal{F}_S(G)$ and $F^*(G) \cong L_3(q)$ or $Sp_4(q)$.*

Proof. Let X be as in Lemma 7.14. Observe that $\mathcal{F}_0 \cong \mathcal{F}_{\overline{S}}(\overline{X})$, for every free normal subgroup N of X and $\overline{X} = X/N$. Now the assertion follows from Theorem 6.3. $\square$

Lemma 7.16. $T = J(T) = J(S)$.

Proof. Set $Z := Z(J(S))$ and let G be a finite group such that $S \in Syl_2(G)$, $\mathcal{F}_0 = \mathcal{F}_S(G)$ and $F^*(G) \cong L_3(q)$ or $Sp_4(q)$. Note that G exists by Lemma 7.15. Assume the assertion is wrong. Then, by 6.8 and 6.10 we may choose $q_0 \in \mathbb{N}$ such that $q = q_0^2$. By 6.7, 6.8, 6.9, 6.10 and Lemma 4.15, the following properties hold:

(1) $J(S) = J(T) \in Syl_2(F^*(G))$, and $N_S(P) = T$ for all $P \in \mathcal{A}(S)$.

(2) $\mathcal{A}(S) = \{Q, Q^x\}$ and $Q \cap Q^x = Z$ for all $x \in S \backslash T$.

(3) Every elementary abelian subgroup of $J(T)$ is contained in an element of $\mathcal{A}(S)$.

If $F^*(G) \cong L_3(q)$ then, by 6.8, we can choose an involution $t \in T \backslash J(T)$ such that $S = C_S(t)J(T)$ and $C_{F^*(G)}(t) \cong L_3(q_0)$. If $F^*(G) \cong Sp_4(q)$ then, by 6.10, we can choose $s \in S \backslash T$ and an involution $t \in T \backslash J(S)$ such that $S = J(S)\langle s \rangle$, $J(S) \cap \langle s \rangle = 1$, $t \in \langle s \rangle$ and $C_{F^*(G)}(t) \cong Sp_4(q_0)$. In both cases, set

$$W := C_Q(t),\ L := O^{p'}(N_{F^*(G)}(W) \cap C_{F^*(G)}(t)) \text{ and } L^* := L(C_S(t) \cap N_S(W)).$$

By 6.7, 6.8, 6.9 and 6.10 (also applied with q_0 in place of q), the following properties hold:

(4) $L/W \cong SL_2(q_0)$, and $W/C_W(L)$ is a natural $SL_2(q_0)$-module for L/W.

(5) $C_S(W) \cap C_S(t) = W\langle t \rangle$ and $L^*/C_{L^*}(W)$ embeds into the automorphism group of

$$LC_{L^*}(W)/C_{L^*}(W) \cong L/W \cong SL_2(q_0).$$

In particular, $O_2(L^*/C_{L^*}(W)) = 1$.

(6) $Z(S) \cap C_W(L) = 1$.

(7) $|C_A(x)| \leq |W|$ for every $A \in \mathcal{A}(T)$ and every $x \in T \backslash J(T)$.

(8) For every involution $u \in T \backslash J(T)$ we have $|C_{J(T)}(u)| \leq |C_{J(T)}(t)|$.

Let $R \in \mathcal{A}(T) \backslash \{Q\}$ and set $\hat{W} := C_R(t)$. Note that the situation is symmetric in Q and R. Moreover, $C_{J(S)}(t) = W\hat{W}$ and $\mathcal{A}(C_{J(S)}(t)) = \{W, \hat{W}\}$. Hence, Lemma 4.14, together with 6.7 and 6.9 (applied with q_0 in place of q), gives

(9) $|W/W \cap \hat{W}| = |\hat{W}/W \cap \hat{W}| = q_0$, $W \cap \hat{W} = C_W(\hat{W}) = C_W(a) = C_{\hat{W}}(W) = C_{\hat{W}}(b)$ for all $a \in \hat{W} \backslash W$, $b \in W \backslash \hat{W}$.

We show next

(10) $\langle t \rangle$ is not fully centralized.

Assume (10) is wrong. Then by Hypothesis 7.1(iv), $\mathcal{C} := C_{\mathcal{F}}(\langle t \rangle)$ is constrained. Set $C := O_2(\mathcal{C})$ and $F := N_C(W)$. Observe that every element of $Aut_{\mathcal{C}}(W)$ extends to an element of $Aut_{\mathcal{C}}(WF)$ and hence, by Lemma 2.17(a), $(WF)_W$ is normal in $Aut_{\mathcal{C}}(W)$. In particular, $(WF)_W$ is normal in $Aut_{L^*}(W) \cong L^*/C_{L^*}(W)$. So, by (5), $C \cap T \leq F \leq C_S(W) \cap C_S(t) = \langle t \rangle W$ and $C \cap T = \langle t \rangle (W \cap C)$. Since the situation is symmetric in Q and R, we get also $C \cap T = \langle t \rangle (\hat{W} \cap C)$ and thus, $(W \cap C)C_W(L) < W$. Hence, as $W/C_W(L)$ is an irreducible L-module, $W \cap C \leq C_W(L)$. Therefore, since $\mathcal{C}$ is constrained, $Z(S) \leq C_W(C) \leq W \cap C \leq C_W(L)$, a contradiction to (6).

This proves (10). In particular, by Lemma 2.38, $C_S(t)$ is contained in an essential subgroup of $\mathcal{F}$. Now Lemma 7.7 give

(11) $q > 4$.

In particular, $q_0 > 2$ and hence, 6.7 and 6.9 (applied with q_0 in place of q) give

(12) $[W, \hat{W}] = W \cap \hat{W}$.

By Lemma 2.18, we can choose $\phi \in Mor_{\mathcal{F}}(C_S(t), S)$ such that $\langle t\phi \rangle$ is fully centralized. We set

$$W_1 := W\langle t \rangle \text{ and } \hat{W}_1 := \hat{W}\langle t \rangle.$$

Note that W_1 and $\hat{W}_1$ are elementary abelian. We show next

(13) $|W_1\phi/W_1\phi \cap J(T)| \leq 2$ and $|\hat{W}_1\phi/\hat{W}_1 \cap J(T)| \leq 2$.

If $F^*(G) \cong Sp_4(q)$ then $S/J(T)$ is cyclic and hence (13) holds, as $W_1\phi$ is elementary abelian. Thus, we may assume $F^*(G) \cong L_3(q)$ and $|W_1\phi/W_1\phi \cap J(T)| = 4$. Then, as $T/J(T)$ is cyclic, $t \in J(T)(W_1\phi \cap T)$ and $W_1\phi \not\leq T$.Hence, by (2) and (3), we have $W_1\phi \cap J(T) \leq C_Z(t) = W \cap \hat{W}$. Therefore, $2 \cdot q_0^2 = |W_1\phi| \leq 4 \cdot |W \cap \hat{W}| = 4 \cdot q_0$ and $q_0 \leq 2$, a contradiction to (11). As the situation is symmetric in W and $\hat{W}$, this shows (13).

(14) $W_1\phi \leq T$ and $\hat{W}_1\phi \leq T$.

Assume $W_1\phi \not\leq T$. Then, by Lemma 4.15, $W\phi \cap J(T) \leq Z(J(T))$ and so $[W\phi \cap J(T), \hat{W}\phi \cap J(T)] = 1$. Now (9) and (13) yield $q_0 = 2$, a contradiction to (11). As the situation is symmetric in W and $\hat{W}$, this shows (14).

(15) $t\phi \in Z(J(T))$.

By (14), $t\phi \in T$. Hence, as $S = J(T)C_S(t)$ and $t\phi$ is fully centralized, it follows from (8) and (10) that $t\phi \in J(T)$. Suppose now (15) is wrong. Then (2) and (3) imply $t\phi \in P\backslash Z(J(T))$ for some $P \in \mathcal{A}(T)$ and $C_S(t\phi) = C_T(t\phi)$. Moreover, by Lemma 4.11(b), $C_{J(T)}(t\phi) = P$. Hence, by (13), $|W\phi/W\phi \cap P| \leq 2$ and $|\hat{W}\phi/\hat{W}\phi \cap P| \leq 2$. As $[W\phi \cap P, \hat{W}\phi \cap P] = 1$, it follows now from (9) that $q_0 = 2$, a contradiction to (11). Hence, (15) holds.

(16) $W\phi \leq J(T)$ and $\hat{W}\phi \leq J(T)$.

By (14), $W\phi \leq T$. By (9),(11) and (13), $[W\phi \cap J(T), \hat{W}\phi \cap J(T)] \neq 1$. So, by (3), there are $P_1, P_2 \in \mathcal{A}(T)$ such that $P_1 \neq P_2$, $W\phi \cap J(T) \leq P_1$ and $\hat{W}\phi \cap J(T) \leq P_2$. As, by (12), $W\phi \cap \hat{W}\phi = [W\phi, \hat{W}\phi] \leq J(T)$, this implies $W\phi \cap \hat{W}\phi \leq P_1 \cap P_2 = Z$. Assume $W\phi \not\leq J(T)$. Then $t \in (W\phi)J(T)$ and, by (15), $\langle t\phi \rangle (W\phi \cap \hat{W}\phi) \leq C_Z(t) = W \cap \hat{W}$, a contradiction. Hence, $W\phi \leq J(T)$ and, as the situation is symmetric in W and $\hat{W}$, property (16) holds.

(17) Let $W_1\phi \leq B \in \mathcal{A}(T)$. Then $W_1\phi$ is fully centralized and $C_S(W_1\phi) = B$.

By Lemma 2.18, we can choose $\psi \in Mor_{\mathcal{F}}(N_S(W_1\phi), S)$ such that $\dot{W}_1 := W_1\phi\psi$ is fully normalized. Note $B \leq C_S(W_1\phi)$ and $\dot{W}_1 \leq B\psi \in \mathcal{A}(S) = \mathcal{A}(T)$ and, by (1), $T = N_S(B\psi)$. Let $F := C_S(\dot{W}_1) \cap N_S(B\psi)$. Then $\dot{W}_1 \leq C_{B\psi}(F)$ and so $|C_{B\psi}(F)| \geq |W_1| > |W|$. Thus, by (7), $F \leq T$. Assume now $B\psi < F$. Then $W\phi\psi \leq \dot{W}_1 \leq C_{B\psi}(F) = Z$. Note that $(\hat{W}\phi)B \leq N_S(W_1\phi)$ and, by (1) and (14), $\hat{W}\phi \leq T = N_S(B)$. Hence, we have $\hat{W}\phi\psi \leq N_S(B\psi) = T$. In particular, $|(\hat{W}\phi\psi)/((\hat{W}\phi\psi) \cap J(T))| \leq 2$. As $W\phi\psi \leq Z$, this yields

$$|\hat{W}/C_{\hat{W}}(W)| = |(\hat{W}\phi\psi)/C_{\hat{W}\phi\psi}(W\phi\psi)| \leq 2,$$

a contradiction to (9) and (11). This shows $F = B\psi$. Thus, $C_S(\dot{W}_1) = B\psi$ and (17) holds.

We now derive the final contradiction. Set $L_1 := Aut_L(W_1)$. Then $[t, L_1] = 1$, $L_1 \leq A(W_1)$ and, by (4), $L_1 \cong SL_2(q_0)$. So $L_2 := L_1\phi^* \cong SL_2(q_0)$ and $[t\phi, L_2] = 1$.

By (15), (16) and (3), there are $B, \hat{B} \in \mathcal{A}(T)$ such that $W_1\phi \leq B$ and $\hat{W}_1\phi \leq \hat{B}$. Set $E := O^{p'}(Aut_{F^*(G)}(B))$ and $\hat{E} := O^{p'}(Aut_{F^*(G)}(\hat{B}))$. Note that $B \neq \hat{B}$ and $B, \hat{B}$ are conjugate to Q. Hence, it follows from Hypothesis 7.1(iii) that $C_B(E) \cap C_{\hat{B}}(\hat{E}) = 1$. In particular, either $t\phi \not\leq C_B(E)$ or $t\phi \not\leq C_{\hat{B}}(\hat{E})$. As the situation is symmetric in W and $\hat{W}$, we may assume

$$t\phi \notin C_B(E).$$

By (17), $W_1\phi$ is fully centralized and $C_S(W_1\phi) = B$. Hence, by the saturation properties, every element of L_2 extends to an element of $A(B)$. So, for

$$X := \{\phi \in A(B) : \phi_{|W_1\phi, W_1\phi} \in L_2\},$$

we have $X/C_X(W_1\phi) \cong L_2 \cong SL_2(q_0)$ and $[t\phi, X] = 1$. Note that $E \cong SL_2(q)$, and $B/C_B(E)$ is a natural $SL_2(q)$-module for E. As B is conjugate to Q in S, by Hypothesis 7.1, E is a normal subgroup of $A(B)$ and S_B embeds into $Aut(E)$. By Lemma 4.11(e), every element of E of odd order acts fixed point freely on $B/C_B(E)$. Therefore, as $[t\phi, X] = 1$ and $t\phi \notin C_B(E)$, $X \cap E$ is a normal p-subgroup of X. Hence, as $O_p(SL_2(q_0)) = 1$, we have $X \cap E \leq C_X(W_1\phi)$. Since S_B embeds into $Aut(E)$, $A(B)/E$ has cyclic Sylow 2-subgroups. In particular, $X/C_X(W_1\phi) \cong SL_2(q_0)$ has cyclic Sylow 2-subgroups. This gives $q_0 = 2$, a contradiction to (11). □

Lemma 7.17. *There is a finite group G such that $S \in Syl_p(G)$, $\mathcal{F}_0 = \mathcal{F}_S(G)$, $F^*(G) \cong L_3(q)$ or $Sp_4(q)$, $|O^2(G) : F^*(G)|$ is odd and $|G : O^2(G)| = 2$. Furthermore, if $F^*(G) \cong Sp_4(q)$ then $q = 2^e$ where $e \in \mathbb{N}$ is odd.*

Proof. This is a consequence of 6.8, 6.10, Lemma 7.15 and Lemma 7.16. □

Lemma 7.18. *Let $\mathcal{F} \neq \mathcal{F}_0$. Then $q = |Z(T)| = 4$.*

Proof. We will use throughout the proof that, by Lemma 7.16, $T = J(T)$. Recall also that, by Lemma 4.16, $|S/T| = 2$. Set $Z := Z(T)$ and assume $|Z| > 4$. As $\mathcal{F} \neq \mathcal{F}_0$ it follows from Theorem 2.36 that there is an essential subgroup P of $\mathcal{F}$ such that

$P \notin \{T\} \cup Q^{\mathcal{F}}$. Recall that, by Lemma 7.9(a), $P \not\leq T$ and there is $t \in P \backslash T$ such that $t\phi \in T$ for some $\phi \in A(P)$. By Corollary 7.4, applied to $G(Z)/Z$ in place of G, we have

(1) T is strongly closed in $N_{\mathcal{F}}(Z)$.

We show next

(2) $Z \neq Z(S)$.

For the proof assume $Z = Z(S)$. If $\Omega(Z(P)) \not\leq T$ then Lemma 4.16(d) implies $P \cap T = Z$ and P is elementary abelian, a contradiction to Lemma 7.9(b). Hence, by Lemma 4.15, $\Omega(Z(P)) = Z(S)$, so $Z = Z(S)$ is $A(P)$-invariant. This is a contradiction to Lemma 7.9(a) and (1), so (2) holds. We show next

(3) $|Z/Z(S)| > 2$.

Assume (3) is wrong, then by (2), $|Z/Z(S)| = 2$. Hence, Lemma 7.5, applied with $G(T)$ in place of G, yields $|Z| = 4$. As this contradicts our assumption, (3) holds.

Recall that by Lemma 7.9(b), P is not elementary abelian, and therefore $P_0 := \Omega(C_{\Phi(P)}(P)) \neq 1$. Observe that $\Phi(P) \leq T$ and, by Lemma 4.15, $P_0 \leq Z(S)$, so $N_{\mathcal{F}}(P_0)$ is parabolic and by assumption constrained. Thus, we may set

$$G := G(P_0).$$

As P_0 is characteristic in P, $A(P) \leq N_{\mathcal{F}}(P_0) = \mathcal{F}_S(G)$. Hence, there is $g \in G$ such that $t \in T^g$. Thus, by Lemma 7.3, there exists $h \in G$ such that $Z\langle t\rangle \leq T^h$. By Lemma 4.15, there is $B \in \mathcal{A}(S^h)$ such that $Z \leq B$. Observe that $t \in T^h = N_{S^h}(B)$ and $B\langle t\rangle \leq N_G(Z)$. In particular, there is $x \in N_G(Z)$ such that $B\langle t\rangle \leq S^x$. Then $t \in N_{S^x}(B) = T^x$, a contradiction to (1). This shows the assertion.

□

Lemma 7.19. *Let $\mathcal{F} \neq \mathcal{F}_0$. Then $\mathcal{F} \cong \mathcal{F}_S(G)$ for a group G with $G \cong J_3$ and $S \in Syl_2(G)$.*

Proof. We will use frequently that, by Lemma 7.16, $T = J(T) \in Syl_p(M)$. Set

$$Z := Z(T),\ R := [S,S] \text{ and } \overline{S} := S/Z.$$

By Lemma 7.18 we have $q = |Z| = 4$. It follows from Theorem 2.36 that there is an essential subgroup P of $\mathcal{F}$ such that $P \neq T$ and $P \notin Q^{\mathcal{F}} \cup \{T\}$. Recall that by Lemma 7.9(a), $P \not\leq T$. Let $t \in P \backslash T$ of minimal order. We will use frequently that, by Lemma 4.15, $\mathcal{A}(T) = \{Q, Q^t\}$, every involution in T is contained in Q or Q^t, and $Z = Q \cap Q^t$. We show first

(1) $Z(S)$ is $A(P)$-invariant.

If $\Omega(Z(P)) \leq T$ then $\Omega(Z(P)) = Z(S)$. Thus we may assume $\Omega(Z(P)) \not\leq T$. If $Z = Z(S)$ then, by Lemma 4.16(d), $C_T(z) = Z(S)$ for every involution $z \in S \backslash T$. Hence, $P \cap T = C_T(\Omega(Z(P))) = Z(S)$ and P is elementary abelian, contradicting Lemma 7.9(b). Thus, $Z \neq Z(S)$ and $|Z(S)| = 2$. As P is not elementary abelian and $P = (P \cap T)\Omega(Z(P))$, we have $1 \neq \Phi(P) = \Phi(P \cap T) \leq \Phi(T) \cap C(\Omega(Z(P))) = C_Z(\Omega(Z(P))) = Z(S)$. Hence, $Z(S) = \Phi(P)$ and (1) holds. Hence, by Lemma 7.9(c), we have

(2) $Z \leq P$.

Set now

$$R_0 = O_p(N_{\mathcal{F}}(Z)).$$

We show next

(3) $|(R_0 \cap T)/Z| = 4$ or $T \leq R_0$.

Assume $T \not\leq R_0$. As $N_{\mathcal{F}}(Z)$ is by assumption constrained, we have then $Z < R_0$. So, if $Z = R_0 \cap T$ then $R_0 \not\leq T$ and $[Q,S] \leq [T, TR_0] \leq [T,T](R_0 \cap T) = Z$, a contradiction to Q not being normal in S. Hence, $Z < R_0 \cap T$ and (3) follows from $R_0 \cap T$ being $A(T)$-invariant.

(4) $|S : P| > 2$.

Assume $|S : P| = 2$. Then $|T : (T \cap P)| = 2$. As $T = \langle Q^P \rangle \not\leq P$, we have $Q \not\leq P$ and so $|Q \cap P| = 8$. Observe now

$$J(P \cap T) = \langle (Q \cap P), (Q^t \cap P) \rangle.$$

Assume first $J(P) \not\leq T$. Let $A \in \mathcal{A}(P)$ such that $A \not\leq T$. Then $8 = |Q \cap P| \leq |A| = 2 \cdot |A \cap T|$ and by Lemma 4.15, $A \cap T \leq Z(S)$. Hence, $|Z(S)| \geq 4$ and $Z(S) = Z$. So, by (1), $[Z, O^{2'}(A(P))] = 1$. Moreover, $\overline{J(P)}$ is dihedral of order 8, so $[\overline{J(P)}, O^2(A(P))] = 1$. As $|P/J(P)| = 2$, it follows $[P, O^2(O^{2'}(A(P)))] = 1$ and $A(P)$ is 2-closed, contradicting P being essential. Hence, $J(P) = J(P \cap T)$. Then $Z = Z(J(P))$ is $A(P)$-invariant. In particular, P is essential in $N_{\mathcal{F}}(Z)$. So, by Lemma 2.37, $R_0 \leq P$ and R_0 is $A(P)$-invariant. Observe that $R_0 \cap T$ is $A(T)$-invariant and $J(P) = \langle (Q \cap P), (Q \cap P)^t \rangle$ is not. Hence, by (3), $P \cap T = (R_0 \cap T)J(P)$. As, by Lemma 7.9(a), $P \cap T$ is not $A(P)$-invariant, it follows $R_0 \not\leq T$. Hence, every element of $A(T)$ extends to an element of $A(S)$. In particular, there is $D \leq N_{A(S)}(T)$ such that $D \cong C_3$ and $C_T(D) = 1$. Then D is irreducible on Z and so $Z = Z(S)$. Moreover, $R_0 = [R_0, D]C_{R_0}(D) = (R_0 \cap T)C_{R_0}(D)$ and $|C_{R_0}(D)| = 2$. Hence, $ZC_{R_0}(D) \in \mathcal{A}(P)$ and $J(P) \not\leq T$, a contradiction as above. This shows (4).

(5) Z is not $A(P)$-invariant and $Z \neq Z(S)$.

Assume Z is $A(P)$-invariant. Then P is essential in $N_{\mathcal{F}}(Z)$. So by Lemma 2.37, $R_0 \leq P$ and R_0 is $A(P)$-invariant. Now by (3) and (4), $P \cap T = R_0 \cap T$. By Lemma 7.9(a), $P \cap T$ is not $A(P)$-invariant, so $R_0 = P \not\leq S$. Therefore, every element of $A(T)$ extends to an element of $A(S)$ and $C_{\overline{R_0 \cap T}}(\overline{S}) \neq 1$ is $A(T)$-invariant.

Hence, as $A(T)$ is irreducible on $\overline{R_0 \cap T}$ we have $[\overline{P \cap T}, \overline{S}] = [\overline{R_0 \cap T}, \overline{S}] = 1$. So $\overline{P}$ is abelian and $|\overline{P}/C_{\overline{P}}(S_P)| \leq 2$. As P is essential we may choose $\phi \in A(P)$ such that $S_P \cap S_P\phi^* = Inn(P)$. Set $Y := \langle S_P, S_P\phi^* \rangle$. Then $\hat{P} = \overline{P}/C_{\overline{P}}(Y)$ has order 4 and $Y/C \cong S_3$ for $C = C_Y(\hat{P})$. Since Z is $A(P)$-invariant, it follows from (1) and $[Z, S] \leq Z(S)$ that $[Z, O^p(Y)] = 1$. Hence, $[P, O^p(C)] = 1$ and C is a normal 2-subgroup of Y. Thus, $C \leq S_P \cap S_P^{\phi} = Inn(P)$ and, as $P = R_0$ is normal in S, we get $|S/P| = |S_P/Inn(P)| = 2$, a contradiction to (4). Hence, Z is not $A(P)$-invariant and (5) follows from (1). We show next

(6) $R = P \cap T$.

Set $R_1 = O_p(N_{\mathcal{F}}(Z(S)))$. By (1) and Lemma 2.37, $R_1 \leq P$ and R_1 is $A(P)$-invariant. If $R_1 \leq T$ then by (5) and Lemma 7.9(a), $Z < R_1 < P \cap T$. So, by (4), $|\overline{R_1}| = 2$. Since R_1 is normal in S it follows that R_1 is not elementary abelian. Hence, $\Omega(R_1) = Z$, a contradiction to (5). Therefore, $R_1 \not\leq T$. Thus, $R \leq [T, R_1]Z \leq R_1 Z$ and, by (2), $R \leq P$. Now (6) follows from (4).

(7) t is an involution, $P = R\langle t \rangle = R\langle z : z \in S \backslash T, z^2 = 1 \rangle$, and the essential subgroups of $\mathcal{F}$ are Q, Q^t, T and P.

Observe that every element in T has order at most 4. By Lemma 7.9(a), $P \cap T$ is not $A(P)$-invariant, and so there is an element x in $P \backslash T$ of order at most 4. Then x^2 has order at most 2 and is centralized by $x \in S \backslash T$. Hence, $x^2 \in Z$ and $\langle Z, x \rangle$ has order 8. By (5), $\langle Z, x \rangle$ is non-abelian and thus dihedral. Hence, by (2) there is an involution in $P \backslash T$. Since t has minimal order, t is then an involution as well. Hence, the involutions in $\overline{S} \backslash \overline{T}$ are the elements in $C_{\overline{T}}(t)\overline{t} = \overline{R}\overline{t}$. Thus, by (6), every involution in $S \backslash T$ is contained in P and (7) holds.

(8) $P \cong Q_8 * D_8$, $A(P)/Inn(P) \cong A_5$ and $A(P) = O^p(Aut(P))$.

By (7), t is an involution and $P = R\langle t\rangle$. By (5), $Z\langle t\rangle$ is dihedral of order 8. An elementary calculation shows

$$C_T(t) = \langle qq^t z : q \in Q\backslash Z,\ z \in Z\backslash Z(S)\rangle Z(S) \cong Q_8$$

and $R = P\cap T = C_T(t)Z$. Hence, $P = C_T(t)(Z\langle t\rangle)$. As $C_T(t)\cap(Z\langle t\rangle) = Z(S)$, this shows $P \cong Q_8 * D_8$. In particular, $Aut(P)/Inn(P) \cong S_5$ and $P/Z(P)$ is a natural S_5-module for $Aut(P)/Inn(P)$. Since, by (7), $S_P/Inn(P) \cong S/P \cong C_2 \times C_2$, a Sylow p-subgroup of $A(P)/Inn(P)$ is a fours group. As $A(P)/Inn(P)$ has a strongly 2-embedded subgroup, this implies $A(P)/Inn(P) \cong A_5$ and $A(P) = O^p(Aut(P))$. Hence, (8) holds. We show next:

(9) $\mathcal{F}_0$ is isomorphic to the 2-fusion system of the extension of $PGL_3(4)$ by the automorphism that is the product of the contragredient and the field automorphism.

Recall that by Lemma 7.18, $q = 4$. By (8), we have in particular that $P^{\mathcal{F}} = \{P\}$, so P is fully normalized. Moreover, there is an element of order 3 in $N_{A(P)}(S_P)$, which by Lemma 2.17(b) extends to an element of $A(S)$. Since $J(S) = T$, it follows from Lemma 4.15(b) that every element of $A(S)$ of odd order normalizes Q. Hence, there is an automorphism of Q of order 3 which centralizes $Z(S)$. Therefore, $A(Q) \cong GL_2(4)$. Now Lemma 7.17, (5) and the structure of $Aut(L_3(4))$ imply (9).

We now are able to prove the assertion. Let $G \cong J_3$ and $\hat{S} \in Syl_2(G)$. Set $\hat{\mathcal{F}} = \mathcal{F}_{\hat{S}}(G)$. Let $\hat{Q} \in \mathcal{A}(\hat{S})$, $\hat{T} = N_{\hat{S}}(\hat{Q})$ and $\hat{\mathcal{F}}_0 = \langle N_{\hat{\mathcal{F}}}(\hat{Q}), N_{\hat{\mathcal{F}}}(\hat{T})\rangle$. From the structure of J_3 we will use that Hypothesis 7.1 is fulfilled with $(\hat{\mathcal{F}}, \hat{S}, \hat{Q})$ in place of $(\mathcal{F}, S, Q)$, and that there is an essential subgroup $\hat{P} \in \hat{\mathcal{F}}$ with $\hat{P} \notin \hat{Q}^{\hat{\mathcal{F}}} \cup \{\hat{T}\}$. In particular, the properties we have shown for $\mathcal{F}$ hold for $\hat{\mathcal{F}}$ accordingly. So by (9), we have $\mathcal{F}_0 \cong \hat{\mathcal{F}}_0$, i.e. there is a group isomorphism $\alpha : S \to \hat{S}$ which is an isomorphism of fusion systems from $\mathcal{F}_0$ to $\hat{\mathcal{F}}_0$. By (8), P is the only essential subgroup of $\mathcal{F}$ in $\mathcal{F}\backslash(Q^{\mathcal{F}} \cup \{T\})$, $\hat{P}$ is the only essential subgroup of $\hat{\mathcal{F}}$ in $\hat{\mathcal{F}}\backslash(\hat{Q}^{\hat{\mathcal{F}}} \cup \{\hat{T}\})$, $P\alpha = \hat{P}$

and $Aut_{\hat{\mathcal{F}}}(\hat{P}) = \alpha^{-1}A(P)\alpha$. Now by Remark 2.39, α is also an isomorphism from $\mathcal{F}$ to $\hat{\mathcal{F}}$. This shows the assertion. □

Proof of Theorem 7.2. The assertion follows from Lemma 7.11, Lemma 7.13, Lemma 7.17 and Lemma 7.19.

Chapter 8

Existence of Thompson-restricted Subgroups

Throughout this chapter assume the following hypothesis.

Hypothesis 8.1. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Set*

$$Z := \Omega(Z(S)).$$

Let $\mathcal{N}$ be a proper saturated subsystem of $\mathcal{F}$ containing $C_{\mathcal{F}}(Z)$. By $\mathcal{F}_{\mathcal{N}}^*$ denote the set of Thompson-maximal members of $\mathcal{F}_{\mathcal{N}}$.

Recall here from Notation 1.3 that $\mathcal{F}_{\mathcal{N}}$ is the set of centric subgroups P of $\mathcal{F}$ such that $Aut_{\mathcal{F}}(P) \not\leq \mathcal{N}$. Note that, by Corollary 2.41, $\mathcal{F}_{\mathcal{N}} \neq \emptyset$. Also recall the definition of Thompson-restricted subgroups and Thompson-maximality of Definition 1.5 and Definition 1.4. As introduced in Notation 2.40, we write $\mathcal{F}_{\mathcal{N}}^*$ for the set of Thompson-maximal members of $\mathcal{F}_{\mathcal{N}}$. The aim of this chapter is to prove Theorem 1, i.e. the existence of a Thompson-restricted subgroup of $\mathcal{F}$ in $\mathcal{F}_{\mathcal{N}}^*$, provided $N_{\mathcal{F}}(J(S)) \leq \mathcal{N}$. As before, we set, for every $P \in \mathcal{F}$,

$$A(P) = Aut_{\mathcal{F}}(P).$$

Recall from Notation 2.1 that for $U \in \mathcal{F}$ and $R \leq S$ a subgroup R_U of $A(U)$ is defined by

$$R_U = \{c_{g|U,U} \ : \ g \in N_R(U)\}.$$

Furthermore, set

$$\mathcal{F}_{\mathcal{N}}^{+} := \{Q \in \mathcal{F}_{\mathcal{N}}^{*} : N_S(Q) = N_S(J(Q)) \text{ and } J(Q) \text{ is fully normalized}\}.$$

Lemma 8.2. *We have $\mathcal{F}_{\mathcal{N}}^{+} \neq \emptyset$.*

Proof. This is just a restatement of Lemma 2.47. □

Theorem 8.3. *Let Q be a maximal with respect to inclusion member of $\mathcal{F}_{\mathcal{N}}^{+}$. Then $J(S) = J(Q)$ or Q is Thompson-restricted.*

Proof. Suppose $J(S) \not\leq Q$. As $Q \in \mathcal{F}_{\mathcal{N}}$, Q is centric, and by Remark 2.10, Q is fully normalized. In particular, we may choose a model G of $N_{\mathcal{F}}(Q)$. Set

$$T := N_S(Q) \text{ and } H := \{g \in G : \ c_{g|Q,Q} \in Aut_{\mathcal{N}}(Q)\}.$$

Note that H is a proper subgroup of G, as $A(Q) \not\leq \mathcal{N}$. By assumption, $J(S) \not\leq Q$ and $T = N_S(J(Q))$, so it follows from Remark 2.44 that

(1) $J(T) \not\leq Q$.

Corollary 2.43 yields $A(RQ) \leq \mathcal{N}$ for every subgroup R of T with $J(RQ) \not\leq Q$. Thus, also the restriction of an element of $N_{A(RQ)}(Q)$ to an automorphism of Q is a morphism in $\mathcal{N}$. This yields

(2) $N_G(R) \leq H$ for every subgroup R of T with $J(RQ) \not\leq Q$.

We show next

(3) Let Q_0 be a normal subgroup of T containing Q such that $N_G(Q_0) \not\leq H$. Then $Q = Q_0$.

For the proof of (3) set $M := N_G(Q_0)$. Then every element of $Aut_M(Q)$ extends to an element of $A(Q_0)$. Furthermore, as $M \not\leq H$, we have $Aut_M(Q) \not\leq \mathcal{N}$. Hence,

$A(Q_0) \not\leq \mathcal{N}$. Moreover, Q_0 is centric, since Q is centric. Thus, $Q_0 \in \mathcal{F}_{\mathcal{N}}$, so the Thompson-maximality of Q yields $Q_0 \in \mathcal{F}_{\mathcal{N}}^*$ and $J(Q_0) = J(Q)$. In particular, as $Q \in \mathcal{F}_{\mathcal{N}}^+$, we have $N_S(Q_0) = T = N_S(J(Q_0))$ and $J(Q_0)$ is fully normalized. Therefore, $Q_0 \in \mathcal{F}_{\mathcal{N}}^+$ and the maximality of Q yields $Q = Q_0$. This shows (3). Note that, by (1) and (2), $N_G(T \cap J(G)) \leq N_G(J(T)) \leq H$. By a Frattini Argument, $G = N_G(T \cap J(G))J(G)$ and hence,

(4) $J(G) \not\leq H$.

In particular, $X := J(G)T \not\leq H$. Let $P \leq X$ be minimal with the property $T \leq P$ and $P \not\leq H$. As $N_G(T) \leq N_G(J(T)) \leq H$, it follows from Remark 3.9 that P is minimal parabolic and $P \cap H$ is the unique maximal subgroup of P containing T. Observe that $Q \leq O_p(G) \leq O_p(P)$ and so, by (3),

(5) $O_p(P) = Q = O_p(G)$.

Let now $V \leq \Omega(Z(Q))$ be a normal subgroup of X containing $\Omega(Z(T))$, and $\overline{P} = P/C_P(V)$. Observe that $Q \leq C_S(V)$ and, by a Frattini Argument, $X = C_X(V)N_X(C_T(V))$. Also note $Z \leq \Omega(Z(T)) \leq V$ and so $[C_X(V), Z] = 1$. As $C_{\mathcal{F}}(Z) \leq \mathcal{N}$, this yields $C_X(V) \leq H$ and thus $N_X(C_T(V)) \not\leq H$. Now (3) implies $C_T(V) = Q$, i.e. $N_{C_S(V)}(Q) = Q$ and so, as $C_S(V)$ is nilpotent,

(6) $C_S(V) = Q$ and $C_G(V)/Q$ is a p'-group.

In particular, by (1) and Lemma 4.4(a), $\mathcal{O}_{\overline{P}}(V) \neq \emptyset$. Let now N be the preimage of $O_p(\overline{P})$ in P. Since $P \not\leq H$, we have $[Z, P] \neq 1$ and therefore, $\overline{P}$ is not a p-group. Hence, $O^p(P) \not\leq N$, so by Lemma 3.8 and (5), $N \cap T \leq O_p(P) = Q$. Hence,

(7) $O_p(\overline{P}) = 1$.

Observe that $C_P(V) \leq C_P(Z) \leq P \cap H$ and therefore, $\overline{P}$ is minimal parabolic,

$\overline{H \cap P}$ is the unique maximal subgroup of $\overline{P}$ containing $\overline{T}$, and

$$C_{\overline{P}}(C_V(\overline{T})) \leq C_{\overline{P}}(Z) \leq \overline{H \cap P}.$$

Now, for $\mathcal{D} = \mathcal{A}_{\overline{P}}(V)$,[1] it follows from Theorem 4.21 that we can choose subgroups $E_1, \ldots, E_r$ of P containing $C_P(V)$ such that the following hold.

(i) $\overline{P} = (\overline{E_1} \times \cdots \times \overline{E_r})\overline{T}$ and $\overline{T}$ acts transitively on $\{\overline{E_1}, \ldots, \overline{E_r}\}$,

(ii) $\mathcal{D} = (\mathcal{D} \cap \overline{E_1}) \cup \cdots \cup (\mathcal{D} \cap \overline{E_r})$,[2]

(iii) $V = C_V(E_1 \ldots E_r) \prod_{i=1}^{r} [V, E_i]$, with $[V, E_i, E_j] = 1$ for $j \neq i$,

(iv) $\overline{E_i} \cong SL_2(p^n)$, or $p = 2$ and $\overline{E_i} \cong S_{2^n+1}$, for some $n \in \mathbb{N}$,

(v) $[V, E_i]/C_{[V,E_i]}(E_i)$ is a natural module for $\overline{E_i}$.

This implies together with Lemma 4.7, Lemma 4.12 and Lemma 4.18(d) that we have $|V/C_V(A)| = |A|$, for every $A \in \mathcal{D}$. In particular, by the definition of $\mathcal{D}$, $m_{\overline{P}}(V) = |V|$.[3] Hence, we have

(8) There is no over-offender in P on V, and $\mathcal{D}$ is the set of minimal by inclusion elements of $\mathcal{O}_{\overline{P}}(V)$.

For $B \in \mathcal{A}(T)$, it follows from (2),(6) and a Frattini Argument that $N_{\overline{P}}(\overline{B}) = \overline{N_P(BC_P(V))} = \overline{N_P(BQ)} \leq \overline{H}$. Note that, by Lemma 4.6 and (8), there exists $B \in \mathcal{A}(T)$ such that $\overline{B} \in \mathcal{D}$. Let J be the full preimage of $\mathcal{D} \cap \overline{T}$ in T. Observe that, by Lemma 4.12 and Lemma 4.18(b), $N_P(J)$ acts transitively on $\mathcal{D} \cap \overline{T}$. Therefore, we get the following property.

(9) For every $A \in \mathcal{D} \cap \overline{T}$, there exists $B \in \mathcal{A}(T)$ such that $A = \overline{B}$. In particular, $N_{\overline{P}}(A) \leq \overline{H}$.

[1] Recall Definition 4.19
[2] Recall Definition 4.19
[3] Recall Definition 4.19

Assume $r \neq 1$ or $E_1 \cong S_{2^n+1}$ for some $n > 1$. Then using Lemma 4.18(a) we get

$$\overline{P} = \langle N_{\overline{P}}(A) : A \in \mathcal{D} \cap \overline{T} \rangle \overline{T}.$$

Hence, (9) gives $\overline{P} \leq \overline{H}$. So, as $C_P(V) \leq C_P(Z) \leq H$, we get also $P \leq H$, contradicting the choice of P. Therefore, $r = 1$ and for $E := E_1$, we have

(10) $P = ET$, $\overline{E} \cong SL_2(q)$ for some power q of p, and $V/C_V(E)$ is a natural $SL_2(q)$-module for $\overline{E}$.

Note that $C_{\overline{P}}(\overline{E})/Z(\overline{E}) \cong C_{\overline{P}}(\overline{E})\overline{E}/\overline{E}$ and so $C_{\overline{P}}(\overline{E})/Z(\overline{E})$ is a p-group. Moreover, as $\overline{E} \cong SL_2(q)$, $Z(\overline{E})$ has order prime to p. Hence, for $Y \in Syl_p(C_{\overline{P}}(\overline{E}))$, we have $C_{\overline{P}}(\overline{E}) = Y \times Z(\overline{E})$ and $Y = O_p(C_{\overline{P}}(\overline{E})) \leq O_p(\overline{P})$. So by (7),

(11) $C_{\overline{P}}(\overline{E}) = Z(\overline{E})$.

Let now $A \in \mathcal{A}(T)$. Then by (8) there is $B \in \mathcal{D}$ such that $B \leq \overline{A}$. By Lemma 4.12 we have $B \in Syl_p(\overline{E})$. As $[B, \overline{A}] = 1$, the structure of $Aut(\overline{E})$ yields together with (11) that $\overline{A} \leq \overline{E}C_{\overline{P}}(\overline{E}) = \overline{E}$. Hence, $A \leq E$. Now it follows from (9),(10) and Lemma 4.12 that

(12) $T \cap E = J(T)Q = AQ$, and $E = J(P)C_P(V)$.

Lemma 4.14 gives the following two properties.

(13) $|V/C_V(A)| = |A/C_A(V)| = q$ and $C_V(A) = C_V(a)$ for every $a \in A \backslash C_G(V)$.

(14) $[V, A, A] = 1$.

Set now $\widetilde{N_G(V)} := N_G(V)/C_G(V)$ and $L := J(G)C_G(V)$. Note that, by (12), $\overline{A} = \overline{J(T)}$ and thus also $\widetilde{A} = \widetilde{J(T)}$. Hence, $\widetilde{L} = \langle \widetilde{A}^{\widetilde{L}} \rangle$. Moreover, $\widetilde{A}$ is weakly closed in $\widetilde{T}$ with respect to $\widetilde{N_G(V)}$. In particular, the Frattini argument gives $\widetilde{N_G(V)} = N_{\widetilde{N_G(V)}}(\widetilde{A})\widetilde{L}$. By another application of the Frattini argument and (1), (2), we get

$N_{\widetilde{N_G(V)}}(\widetilde{A}) \leq \widetilde{N_G(V)} \cap \widetilde{N_G(J(T))} \leq \widetilde{N_H(V)}$. Moreover, we have $C_{\widetilde{N_G(V)}}(C_V(\widetilde{T})) \leq C_{\widetilde{N_G(V)}}(Z) \leq \widetilde{N_H(V)}$. By (4), $J(G) \nleq H$ and thus, by (13) and (14), the hypothesis of Theorem 4.20 is fulfilled with $\widetilde{N_G(V)}$, $\widetilde{N_H(V)}$ and $\widetilde{A}$ in place of G, M and A. Hence, we get $\widetilde{L} \cong SL_2(q)$ and $V/C_V(L)$ is a natural $SL_2(q)$-module for $\widetilde{L}$. Observe that, by (11), $C_{\overline{T}}(\overline{E}) = 1$ and so, by (6) and (12), $C_T(J(G)/C_{J(G)}(V)) \leq C_T(\overline{E}) \leq Q$. This completes the proof. □

The proof of Theorem 1. If $N_{\mathcal{F}}(J(S)) \leq \mathcal{N}$ then $A(Q) \leq \mathcal{N}$, for every $Q \in \mathcal{F}$ with $J(S) = J(Q)$. Hence, the assertion follows from Lemma 8.2 and Theorem 8.3.

Chapter 9

Properties of Thompson-restricted Subgroups

In the next chapter we will prove Theorem 2 and Theorem 3. Crucial are the properties of Thompson-restricted subgroups which we will state in this chapter. Throughout this chapter we assume the following hypothesis.

Hypothesis 9.1. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $Q \in \mathcal{F}$ be a Thompson-restricted subgroup. Set $T := N_S(Q)$, $q := |J(T)Q/Q|$ and $A(P) := Aut_{\mathcal{F}}(P)$, for every $P \in \mathcal{F}$.*

Recall from Notation 2.1 that $R_P := Aut_R(P) := \{c_{g_{|P,P}} : g \in N_R(P)\}$ for all $P \leq R \leq S$. Furthermore, recall from Notation 2.30 that, for every fully normalized subgroup $P \in \mathcal{F}$, $G(P)$ denotes a model for $N_{\mathcal{F}}(P)$, provided $N_{\mathcal{F}}(P)$ is constrained.

Notation 9.2. *For every $U \in \mathcal{F}$ set*

$$V(U) := \Omega(Z(U)).$$

Moreover, we set

$$A^{\circ}(Q) := \langle (J(T)_Q)^{A(Q)} \rangle C_{A(Q)}(V(Q)).$$

Remark 9.3. *(a) We have $J(T)_Q Inn(Q) \in Syl_p(A^{\circ}(Q))$ and $J(T)Q = AQ$, for every $A \in \mathcal{A}(T)$ with $A \not\leq Q$.*

(b) $C_{V(Q)}(J(T)) = C_{V(Q)}(A)$ and $[V(Q), J(T), J(T)] = 1$.

(c) *Let* $V \leq V(Q)$ *such that* $[V, A^\circ(Q)] \neq 1$ *and* V *is* $A^\circ(Q)$*-invariant. Then* $V(Q) = VC_{V(Q)}(A^\circ(Q))$, $C_S(V) = Q$, $|V/C_V(A)| = |A/C_A(V)| = q$, $\mathcal{A}(Q) \subseteq \mathcal{A}(T)$ *and* $(A \cap Q)V \in \mathcal{A}(Q)$ *for every* $A \in \mathcal{A}(T)$.

(d) $C_T(J(T)Q/Q) = J(T)Q$

Proof. Since Q is Thompson-restricted, (a) and (b) follow from Lemma 4.4(a) and Lemma 4.14. Property (d) is a consequence of (a) and the structure of $Aut(SL_2(q))$. Let $V \leq V(Q)$ such that $[V, A^\circ(Q)] \neq 1$ and V is $A^\circ(Q)$-invariant. Then, as the module $V(Q)/C_{V(Q)}(A^\circ(Q))$ is irreducible, $V(Q) = VC_{V(Q)}(A^\circ(Q))$. In particular, $C_{J(T)}(V) = C_{J(T)}(V(Q)) \leq C_S(V(Q)) = Q$. Hence, $[C_T(V), J(T)] \leq C_{J(T)}(V) \leq Q$ and, by (d), $C_T(V) = C_{J(T)Q}(V) = Q$. This means $N_{C_S(V)}(Q) = Q$ and so, as $C_S(V)$ is nilpotent, $C_S(V) = Q$. Now property (b) follows from Lemma 4.4 and Lemma 4.14(a). □

Recall the Definition of the Baumann subgroup from Definition 5.3.

Lemma 9.4. $B(T) \leq J(T)Q$.

Proof. Since Q is Thompson-restricted, it follows from Remark 9.3(a), Lemma 4.11(b) and Lemma 4.13 that $C_T([V(Q), J(T)]) \leq J(T)Q$. By Remark 9.3(c), we have $V(Q) \leq J(T)$. So, by Remark 9.3(b), $[V(Q), J(T)] \leq \Omega(Z(J(T)))$. Hence,

$$B(T) \leq C_T([V(Q), J(T)]) \leq J(T)Q.$$

□

Definition 9.5. *We say that* $U \in \mathcal{F}$ *is* ***$\mathcal{F}$-characteristic*** *in* Q *and write*

$$U \ char_{\mathcal{F}} \ Q$$

if $U \leq Q$, $U \trianglelefteq T$ *and* $A^\circ(Q) = C_{A^\circ(Q)}(V(Q))N_{A^\circ(Q)}(U)$.

Lemma 9.6. *Set $G := G(Q)$, $M := J(G)C_G(V(Q))$ and let U $char_{\mathcal{F}}$ Q. Then, for $X := B(N_M(U))$, we have $B(T) \in Syl_p(X)$ and $M = C_G(V(Q))X$.*

Proof. Note that Hypothesis 5.4 is fulfilled with $N_M(U)$ and $V(Q)$ in place of G and V. Therefore, Lemma 5.6 and Lemma 9.4 imply $B(T) \in Syl_p(X)$ and $M = C_G(V(Q))X$. □

Lemma 9.7. *Let $G = G(Q)$, $M = J(G)C_G(V(Q))$ and U $char_{\mathcal{F}}$ Q. Then there is $H \leq N_M(U)$ such that $B(T) \in Syl_p(H)$, H is normalized by T, $M = C_G(V(Q))H$ and, for $\hat{H} := H/O_p(H)$,*

$$\hat{H}/\Phi(\hat{H}) \cong L_2(q).$$

Proof. Set $X := B(N_M(U))$. By Lemma 9.6 we have $T_1 := B(T) \in Syl_p(X)$ and $M = C_G(V(Q))X$. Note that X is normalized by T. Set $X_0 := XT$ and let $H_0 \leq X_0$ be minimal such that $T \leq H_0$ and $H_0 \not\leq N_{X_0}(T_1)C_{X_0}(V(Q))$. Set $H := H_0 \cap X$. Then $H \not\leq N_X(T_1)C_X(V(Q))$, as $H_0 = HT$. Since $X/C_X(V(Q)) \cong SL_2(q)$ is generated by two Sylow p-subgroups, we get $X = C_X(V(Q))H$ and $M = C_G(V(Q))H$. Moreover, $T_1 \in Syl_p(H)$ and H is normal in H_0. Thus, it remains to show that $\hat{H}/\Phi(\hat{H}) \cong L_2(q)$.

Observe that $Q = O_p(H_0)$. Set $\overline{H_0} = H_0/Q$ and $C := C_{H_0}(V(Q))$. By Remark 3.9, H_0 is minimal parabolic, and so $\overline{H_0}$ is minimal parabolic as well. As H_0/C is not a p-group, it follows now from Lemma 3.10 that $\overline{C} \leq \Phi(\overline{H_0})$. Observe that $\overline{H_0} = \overline{TH}$, so by Lemma 3.6, $\Phi(\overline{H_0}) = \Phi(\overline{H})$. Hence, $\overline{C} \leq \Phi(\overline{H})$, so by Lemma 3.5(a), $\Phi(\overline{H}/\overline{C}) = \Phi(\overline{H})/\overline{C}$ and, as $\overline{H}/\overline{C} \cong SL_2(q)$, then

$$\overline{H}/\Phi(\overline{H}) \cong (\overline{H}/\overline{C})/\Phi(\overline{H}/\overline{C}) \cong L_2(q).$$

As $\overline{H} \cong H/(H \cap Q) = H/O_p(H) = \hat{H}$, this implies the assertion. □

Lemma 9.8. *Let U $char_{\mathcal{F}}$ Q such that U is fully normalized. Let $U^* \leq Q$ be invariant under $N_{A^\circ(Q)}(U)$ and $N_{A(S)}(U)$. Set $\mathcal{N} := N_{N_{\mathcal{F}}(U)}(U^*)$, $H := N_{A^\circ(Q)}(U)$ and $X := HS_Q$. Suppose $O^p(H) \not\leq C_X(V(Q))C_X(Q/U^*)$. Then $O_p(\mathcal{N}/U^*) \leq Q/U^*$.*

Proof. Observe first that $N_{\mathcal{F}}(U)$ is saturated as U is fully normalized. Moreover, $U^* \trianglelefteq N_S(U)$ since U^* is $N_{A(S)}(U)$-invariant, so U^* is fully normalized in $N_{\mathcal{F}}(U)$ and $\mathcal{N}$ is saturated. Set

$$\overline{X} := X/C_X(V(Q)).$$

Since U $char_{\mathcal{F}}$ Q we have $X \leq N_{A(Q)}(U)$. In particular, as $C_{A(Q)}(V(Q)) \leq A^\circ(Q)$, we have $C_X(V(Q)) \leq A^\circ(Q) \cap X \leq H$. Since U $char_{\mathcal{F}}$ Q and Q is Thompson-restricted, we have

$$\overline{H} \cong A^\circ(Q)/C_{A^\circ(Q)}(V(Q)) \cong SL_2(q).$$

Moreover, $C_{S_Q}(\overline{H}) \leq Inn(Q)$, so $Z(\overline{X}) = Z(\overline{H})$ and the group $\overline{X}/Z(\overline{X})$ embeds into $Aut(\overline{H}) \cong \Gamma L_2(q)$. This gives the following property.

(1) Let N be a normal subgroup of X containing $C_X(V(Q))$ such that $O^p(H) \not\leq N$. Then $N \leq H$ and $\overline{N} \leq Z(\overline{H})$. In particular, $|N/C_N(V(Q))| \leq 2$ and $N/(N \cap Inn(Q))$ has order prime to p.

Set $C := C_X(Q/U^*)$ and $C_1 := CC_X(V(Q))$. By assumption, $O^p(H) \not\leq C_1$. Hence, by (1),

(2) $\overline{C} = \overline{C_1} \leq Z(\overline{H})$ and $C_S(Q/U^*) \leq Q$.

Set $\mathcal{N}^+ = \mathcal{N}/U^*$, $R^+ = RU^*/U^*$ for every subgroup R of $N_S(U)$, and L^+ for the subgroup of $Aut_{\mathcal{N}^+}(Q^+)$ induced by L, for every subgroup L of X. Then $L^+ \cong LC/C$ for every $L \leq X$.

Observe that Q^+ is fully normalized in $\mathcal{N}^+$ since Q is fully normalized in $\mathcal{F}$. As $\mathcal{N}^+$ is saturated, it follows in particular that Q^+ is fully centralized in $\mathcal{N}^+$. Now (2) and Remark 2.9 yield

(3) Q^+ is centric in $\mathcal{N}^+$.

As already observed above, $\overline{X}/Z(\overline{X})$ embeds into $Aut(\overline{H}) \cong \Gamma L_2(q)$. Hence, there is a subgroup R of T such that $Q \leq R$, $\overline{R_Q}$ is a complement of $\overline{J(T)_Q}$ in $\overline{T_Q}$, and $[\overline{R_Q}, \overline{E}] = 1$ for some subgroup E of H with $\overline{E} \cong SL_2(q_0)$ where $q_0 \neq 1$ is a divisor of q. ($\overline{R_Q}$ corresponds to a group of field automorphisms of $SL_2(q)$.) We may choose E such that $C_H(V(Q)) \leq E$. Note that $C_H(V(Q))/Inn(Q)$ is a p'-group and so $R_Q \in Syl_p(R_Q C_H(V(Q)))$. As E normalizes $R_Q C_H(V(Q))$, it follows from a Frattini Argument that $E = E_0 C_H(V(Q))$ for $E_0 = N_E(R_Q)$. In particular, $\overline{E_0} \cong \overline{E} \cong SL_2(q_0)$.

Set $\mathcal{E} := N_{\mathcal{N}}(J(Q))$. Observe that $J(Q)$ is fully normalized in $\mathcal{N}$, as $J(Q)$ is fully normalized in $\mathcal{F}$, and so $\mathcal{E}$ and $\mathcal{E}^+ := \mathcal{E}/U^*$ are saturated. Moreover, $Aut_{\mathcal{E}}(Q) = Aut_{\mathcal{N}}(Q)$ and so $E_1 := E_0^+ \leq Aut_{\mathcal{E}^+}(Q^+)$. Note that $E_1 \cong E_0C/C$. As $\overline{E_0} \cong \overline{E} \cong SL_2(q_0)$, property (2) implies

$$\overline{E_0 C_1}/\overline{C_1} \cong SL_2(q_0) \text{ or } L_2(q_0).$$

Since $C \leq C_1$, we have $(E_0C_1)/C_1 \cong (E_0C)/((E_0C) \cap C_1)$, and E_1 has a factor group isomorphic to $L_2(q_0)$. In particlar, E_1 is not p-closed. Also observe that E_1 normalizes $(R^+)_{Q^+} = (R_Q)^+$, and Q^+ is fully normalized in $\mathcal{E}^+$, as Q^+ is normal in T^+. Hence, it follows from (2) and Lemma 2.17 that every element of E_1 extends to an element of $Aut_{\mathcal{E}^+}(R^+)$. Thus, $Aut_{\mathcal{E}^+}(R^+)$ is not p-closed. Hence, by Theorem 2.36, there is $P \in \mathcal{E}$ such that $U^* \leq P$, P^+ is essential in $\mathcal{E}^+$, and $(R^+)\phi \leq P^+$ for some element $\phi \in Aut_{\mathcal{E}^+}(T^+)$. Then $R^+ \leq (P^+)\phi^{-1}$ and so, replacing P by the preimage of $(P^+)\phi^{-1}$ in T, we may assume that $R \leq P$.

By the choice of R, we have $Q \leq R \leq P$ and $T = J(T)R = J(T)P$. By Remark 9.3(a),(c), we have $\mathcal{A}(Q) \subseteq \mathcal{A}(T)$ and $J(T)Q = AQ$ for every $A \in \mathcal{A}(T)\backslash\mathcal{A}(Q)$. Hence, if there exists $A \in \mathcal{A}(P)\backslash\mathcal{A}(Q)$ then $J(T) \leq AQ \leq P$ and $P = T$, a contradiction. Thus, $J(P) = J(Q)$.

In particular, $Aut_{\mathcal{E}}(P) = Aut_{\mathcal{N}}(P)$, i.e. $Aut_{\mathcal{E}^+}(P^+) = Aut_{\mathcal{N}^+}(P^+)$ and

$$Aut_{\mathcal{N}^+}(P^+)/Inn(P^+)$$

has a strongly p-embedded subgroup. As $Q^+ \leq R^+ \leq P^+$ it follows from (3) that P^+ is centric in $\mathcal{N}^+$. Therefore, P^+ is essential in $\mathcal{N}^+$ and by Lemma 2.37, $O_p(\mathcal{N}^+) \leq P^+$. In particular,

(4) $O_p(\mathcal{N}^+) \leq T^+$.

Let $U^* \leq Y \leq N_S(U)$ such that $Y^+ = O_p(\mathcal{N}^+)$. Then by (4), $Y \leq T$. Moreover, every element of X^+ extends to an $\mathcal{N}^+$-automorphism of $(YQ)^+$, so by Lemma 2.17(a), $((YQ)_Q)^+ = ((YQ)^+)_{Q^+}$ is normal in X^+. Hence, $(YQ)_Q C$ and thus $Y_Q C_1$ is normal in X. By assumption, $O^p(H) \not\leq C_1$ and so $O^p(H) \not\leq Y_Q C_1$. Therefore, by (1), $Y_Q C_1/Inn(Q)$ is a p'-group. Hence, $Y_Q \leq Inn(Q)$ and so $Y \leq Q$. This proves the assertion. □

Applying Lemma 9.8 with $U^* = 1$ we obtain the following corollary.

Corollary 9.9. *Let $1 \neq U \; char_{\mathcal{F}} \; Q$ such that U is fully normalized. Then*

$$O_p(N_{\mathcal{F}}(U)) \leq Q.$$

Notation 9.10. *Let $1 \neq U \leq Q$ such that $U \trianglelefteq T$. Then we set*

$$\mathcal{D}(Q,U) = \{U_0 \; : \; U_0 \leq Q, \; U_0 \textit{ is invariant under } N_{A(S)}(U) \textit{ and } N_{A(Q)}(U)\}.$$

By $U^*(Q)$ we denote the element of $\mathcal{D}(Q,U)$ which is maximal with respect to inclusion.

Note that here $U^*(Q)$ is well defined since $U \in \mathcal{D}(Q,U)$ and the product of two elements of $\mathcal{D}(Q,U)$ is contained in $\mathcal{D}(Q,U)$. Moreover, if $O_p(N_{\mathcal{F}}(U)) \leq Q$, then $O_p(N_{\mathcal{F}}(U)) \in \mathcal{D}(Q,U)$ and therefore $U \leq O_p(N_{\mathcal{F}}(U)) \leq U^*(Q)$.

Lemma 9.11. *Let $\mathcal{F}$ be minimal, let $1 \neq U$ $char_{\mathcal{F}}$ Q and assume U is fully normalized. Then $O^p(N_{A^\circ(Q)}(U)) \leq C_{A(Q)}(Q/U^*(Q))C_{A(Q)}(V(Q))$.*

Proof. Set

$$H := N_{A^\circ(Q)}(U),\ X := HS_Q,\ U^* := U^*(Q) \text{ and } C := C_X(Q/U^*)C_X(V(Q)).$$

Assume $O^p(H) \not\leq C$. Observe that $N_{\mathcal{F}}(U)$ is saturated and solvable, since U is fully normalized and $\mathcal{F}$ is minimal. Moreover, $U^* \trianglelefteq N_S(U)$ is fully normalized in $N_{\mathcal{F}}(U)$ and so, by Proposition 2.32(a), $\mathcal{N} := N_{N_{\mathcal{F}}(U)}(U^*)$ is saturated and solvable. Therefore, $O_p(\mathcal{N}/U^*) \neq 1$ and so $U^* < U_0$, where U_0 is the full preimage of $O_p(\mathcal{N}/U^*)$ in $N_S(U)$. By Lemma 9.8, $U_0 \leq Q$. Now $U_0 \in \mathcal{D}(Q,U)$ and so $U_0 = U^*(Q) = U$, a contradiction. □

Lemma 9.12. *Let $\mathcal{F}$ be minimal and let $1 \neq U$ $char_{\mathcal{F}}$ Q such that U is fully normalized. Then $U^*(Q)X$ $char_{\mathcal{F}}$ Q for every subgroup X of Q with $X \trianglelefteq T$.*

Proof. Note that $U_0 := U^*(Q)X$ is normal in T. Moreover, by Lemma 9.11, we have

$$\begin{aligned} N_{A^\circ(Q)}(U) &\leq T_Q O^p(N_{A^\circ(Q)}(U)) \leq T_Q C_{A(Q)}(V(Q))C_{A^\circ(Q)}(Q/U^*(Q)) \\ &\leq C_{A(Q)}(V(Q))N_{A(Q)}(U_0). \end{aligned}$$

Hence, we have $A^\circ(Q) = C_{A(Q)}(V(Q))N_{A^\circ(Q)}(U) = C_{A(Q)}(V(Q))N_{A^\circ(Q)}(U_0)$ and U_0 $char_{\mathcal{F}}$ Q. □

Lemma 9.13. *Let $\phi \in Mor_{\mathcal{F}}(N_S(Q), S)$. Then $Q\phi$ is Thompson-restricted.*

Proof. As Q is centric, $\widetilde{Q} := Q\phi$ is centric. Observe that

$$|N_S(J(Q))| = |N_S(Q)| = |N_S(Q)\phi| \leq |N_S(\widetilde{Q})| \leq |N_S(J(\widetilde{Q}))|.$$

So, as Q and $J(Q)$ are fully normalized, $\widetilde{Q}$ and $J(\widetilde{Q}) = J(Q)\phi$ are fully normalized, and $N_S(Q)\phi = N_S(\widetilde{Q}) = N_S(J(\widetilde{Q}))$. Observe that $\phi : N_S(Q) \to N_S(\widetilde{Q})$ is an isomorphism of fusion systems from $N_{\mathcal{F}}(Q)$ to $N_{\mathcal{F}}(\widetilde{Q})$. Moreover, for $V \leq \Omega(Z(\widetilde{Q}))$, we have $C_S(V) = \widetilde{Q}$ if and only if $C_{N_S(\widetilde{Q})}(V) = \widetilde{Q}$. So $\widetilde{Q}$ is Thompson-restricted as Q is Thompson-restricted. $\square$

Chapter 10

Pushing Up in Fusion Systems

10.1 Setup and main results of this chapter

Throughout this chapter, assume the following hypothesis.

Hypothesis 10.1. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Suppose $\mathcal{F}$ is minimal. Let $\mathcal{N}$ be a proper saturated subsystem of $\mathcal{F}$ on S, and let $\mathcal{Q}$ be the set of all Thompson-maximal members of $\mathcal{F}_{\mathcal{N}}$ which are Thompson-restricted.*

Recall here the definition of Thompson-maximality and Thompson-restricted subgroups from Definition 1.4 and Definition 1.5 in the introduction. Furthermore, recall from Notation 1.3 that $\mathcal{F}_{\mathcal{N}}$ is the set of subgroups $P \in \mathcal{F}$ with $Aut_{\mathcal{F}}(P) \not\leq \mathcal{N}$. The aim of this chapter is to prove Theorem 2, which then, together with Theorem 1 and Theorem 7.2, implies Theorem 3. We restate Theorem 2 here for the readers convenience. Recall the Definition of a full maximal parabolic from Definition 1.2.

Hypothesis 10.2. *Assume Hypothesis 10.1 and suppose $\mathcal{N}$ contains every full maximal parabolic of $\mathcal{F}$.*

Theorem 2. *Assume Hypothesis 10.2. Let $Q \in \mathcal{Q}$, $G := G(Q)$ and $M := J(G)$.*[1] *Then $N_S(X) = N_S(Q)$, for every non-trivial normal p-subgroup X of $MN_S(Q)$. Moreover, $Q \leq M$, $M/Q \cong SL_2(q)$ and one of the following holds:*

[1]Recall Notation 2.30.

(I) *Q is elementary abelian, and $Q/C_Q(M)$ is a natural $SL_2(q)$-module for M/Q, or*

(II) *$p = 3$, $S = N_S(Q)$ and $|Q| = q^5$. Moreover, $Q/Z(Q)$ and $Z(Q)/\Phi(Q)$ are natural $SL_2(q)$-modules for M/Q, and $\Phi(Q) = C_Q(M)$ has order q.*

Note here that Theorem 1 yields $\mathcal{Q} \neq \emptyset$ if Hypothesis 10.2 holds. In fact, this is already the case if we assume the following more general hypothesis.

Hypothesis 10.3. *Assume Hypothesis 10.1, and suppose $N_{\mathcal{F}}(C) \leq \mathcal{N}$ for every characteristic subgroup C of S.*

Many arguments in the proof of Theorem 2 require only Hypothesis 10.3. More precisely, we will be able to prove the following Lemma.

Lemma 10.4. *Assume Hypothesis 10.3. Let $Q \in \mathcal{Q}$ and $1 \neq U$ $char_{\mathcal{F}}$ Q. Then $B(N_S(U)) = B(N_S(Q))$.*

Here for a Thompson-restricted subgroup Q of $\mathcal{F}$ recall the definition of $A^\circ(Q)$ and of $\mathcal{F}$-characteristic subgroups from Notation 9.2 and Definition 9.5. For a finite group H, recall the Definition of the Baumann subgroup $B(H)$ from Definition 5.3.

Lemma 10.4 is a major step in the proof of Theorem 2 because, together with Lemma 9.7, it enables us to apply the pushing up result by Baumann and Niles in the form stated in Theorem 5.2.

In the remainder of this chapter we use the following notation: For $P \in \mathcal{F}$ set

$$A(P) := Aut_{\mathcal{F}}(P) \text{ and } V(P) := \Omega(Z(P)).$$

Recall from Notation 2.1 that, for subgroups P and R of S,

$$R_P := Aut_R(P) := \{c_{g|P,P} : g \in N_R(P)\}.$$

10.2 Preliminaries

Throughout Section 10.2 assume Hypothesis 10.3.

Lemma 10.5. *Let $Q \in \mathcal{Q}$. Then $A(YQ) \leq \mathcal{N}$ and $N_{A(Q)}(Y_Q) \leq \mathcal{N}$, for every subgroup Y of T with $J(QY) \not\leq Q$.*

Proof. Set $X := YQ$. By Corollary 2.43, we have $A(X) \leq \mathcal{N}$. Since Q is fully normalized and $C_S(Q) \leq Q \leq X$, Lemma 2.17(b) implies that every element of $N_{A(Q)}(X_Q)$ extends to an element of $A(X)$. As $N_{A(Q)}(Y_Q) \leq N_{A(Q)}(X_Q)$, this shows the assertion. □

Remark 10.6. *Let $Q \in \mathcal{Q}$. Then $A^\circ(Q) \not\leq \mathcal{N}$.*

Proof. Otherwise, by the Frattini Argument and Lemma 10.5,

$$A(Q) = A^\circ(Q) N_{A(Q)}(J(T)_Q) \leq \mathcal{N},$$

contradicting $Q \in \mathcal{F}_{\mathcal{N}}$. □

Lemma 10.7. *Let $Q \in \mathcal{Q}$, let $U \in \mathcal{F}$ be $\mathcal{F}$-characteristic in Q and characteristic in S. Then $U = 1$.*

Proof. Assume $U \neq 1$. As U is characteristic in S, Hypothesis 10.3 implies that $N_{A^\circ(Q)}(U) \leq N_{\mathcal{F}}(U) \leq \mathcal{N}$ and $C_{A^\circ(Q)}(V(Q)) \leq N_{\mathcal{F}}(\Omega(Z(S))) \leq \mathcal{N}$. Hence, as $U \ char_{\mathcal{F}}\ Q$, we have $A^\circ(Q) \leq \mathcal{N}$. This is a contradiction to Remark 10.6. □

For a subgroup U of Q with $1 \neq U \trianglelefteq T$ define $U^*(Q)$ as in Notation 9.10.

Notation 10.8. *Let $Q \in \mathcal{Q}$.*

- *Set*

$$\mathcal{C}(Q) = \{U \leq Q \ : \ U \ char_{\mathcal{F}}\ Q,\ C_S(V(U)) = U = U^*(Q)\}.$$

- *We define $\mathcal{C}^*(Q)$ to be the set of all $1 \neq U$ $char_{\mathcal{F}}$ Q such that U is fully normalized and*

$$|U| = max\{|U^*| : U^* \; char_{\mathcal{F}} \; Q, \; U^* \trianglelefteq N_S(U)\}.$$

Observe that $U = U^*(Q)$, for every $U \in \mathcal{C}^*(Q)$. Also note $V(Q) \leq C_S(V(U)) = U$, for every $U \in \mathcal{C}(Q)$. This implies the following remark.

Remark 10.9. *Let $Q \in \mathcal{Q}$ and $U \in \mathcal{C}(Q)$. Then $V(Q) \leq V(U)$.*

Lemma 10.10. *Let $Q \in \mathcal{Q}$, $U \in \mathcal{C}^*(Q)$ and $X \leq Q$ such that $X \trianglelefteq N_S(U)$. Then $X \leq U$.*

Proof. Since U $char_{\mathcal{F}}$ Q and $U = U^*(Q)$, Lemma 9.12 implies UX $char_{\mathcal{F}}$ Q. Moreover, $UX \trianglelefteq N_S(U)$. Hence, the maximality of $|U|$ yields $X \leq U$. □

Lemma 10.11. *Let $Q \in \mathcal{Q}$, $U \in \mathcal{C}^*(Q)$ and $X \leq Q$ such that $X \not\leq U$. Then there is $t \in N_S(U)$ such that $X^t \not\leq Q$.*

Proof. Otherwise $\langle X^{N_S(U)} \rangle \leq Q$, a contradiction to Lemma 10.10 and $X \not\leq U$. □

Lemma 10.12. *Let $Q \in \mathcal{Q}$. Then $\mathcal{C}^*(Q) \subseteq \mathcal{C}(Q)$.*

Proof. Set $T := N_S(Q)$ and let $U \in \mathcal{C}^*(Q)$. Clearly, $U = U^*(Q)$. By Lemma 10.10, we have $Z := \Omega(Z(S)) \leq U$ and so $Z \leq V(U)$. Hence, as U $char_{\mathcal{F}}$ Q and $Z \leq V(Q)$, we have $V := \langle Z^{A^\circ(Q)} \rangle = \langle Z^{N_{A^\circ(Q)}(U)} \rangle \leq V(U)$. Lemma 10.7 implies $Z \not\leq C(A^\circ(Q))$. Thus, $[V, A^\circ(Q)] \neq 1$ and, by Remark 9.3(c), we have $C_S(V) = Q$. Therefore, $C_S(V(U)) \leq Q$ and so $C_S(V(U)) = C_Q(V(U))$ $char_{\mathcal{F}}$ Q. At the same time, $C_S(V(U)) \trianglelefteq N_S(U)$. Hence, the maximality of $|U|$ yields $U = C_S(V(U))$ and thus $U \in \mathcal{C}(Q)$. □

Notation 10.13. *Let $W \leq S$ be elementary abelian and $W \leq Y \leq N_S(W)$. Then we write $\mathcal{A}_*(Y, W)$ for the set of elements $A \in \mathcal{A}(Y)$ with $[A, W] \neq 1$ for which $AC_Y(W)$ is minimal with respect to inclusion. (In particular, $\mathcal{A}_*(Y) = \emptyset$ if $[W, J(Y)] = 1$.)*

Lemma 10.14. *Let $Q \in \mathcal{Q}$, $U \in \mathcal{C}(Q)$, $W := V(U)$ and $A \in \mathcal{A}_*(T, W)$. Assume $A \not\leq Q$. Then*

$$|W/C_W(A)| = |A/C_A(W)| = q \text{ and } W = V(Q)C_W(A).$$

Proof. As $C_S(V(Q)) = Q$, we have $[V(Q), A] \neq 1$. Remark 9.3(c) implies

$$|V(Q)/C_{V(Q)}(A)| = |A/C_A(V(Q))| = q.$$

Hence, the assertion follows from Lemma 4.5. □

Notation 10.15. *For $Q \in \mathcal{Q}$ set*

$$R(Q) = [V(Q), J(N_S(Q))].$$

Remark 10.16. *Let $Q \in \mathcal{Q}$ and set $T := N_S(Q)$. Then $[R(Q), J(T)Q] = 1$ and $R(Q) = [V(Q), A]$, for every $A \in \mathcal{A}(T)$ with $A \not\leq Q$.*

Proof. This follows from Remark 9.3(a),(b). □

Lemma 10.17. *Let $Q \in \mathcal{Q}$, $U \in \mathcal{C}(Q)$ and $A \in \mathcal{A}_*(T, V(U))$ such that $A \not\leq Q$. Then $R(Q) = [V(U), A]$.*

Proof. By Remark 10.16, we have $R(Q) = [V(Q), A]$. By Lemma 10.14, $V(U) = V(Q)C_{V(U)}(A)$. This implies the assertion. □

Lemma 10.18. *Let $Q \in \mathcal{Q}$ and $\phi \in Mor_{\mathcal{F}}(N_S(Q), S)$. Then $Q\phi \in \mathcal{Q}$, $N_S(Q)\phi = N_S(Q\phi)$, $A^\circ(Q)\phi^* = A^\circ(Q\phi)$,[2] $V(Q)\phi = V(Q\phi)$ and $R(Q)\phi = R(Q\phi)$. Moreover, for every U $char_{\mathcal{F}}$ Q, we have $U\phi$ $char_{\mathcal{F}}$ $Q\phi$.*

Proof. By Lemma 9.13, $Q\phi$ is Thompson-restricted. As $J(N_S(Q)) \not\leq Q$ and Q is Thompson-maximal in $\mathcal{F}_{\mathcal{N}}$, it follows from Corollary 2.42 that ϕ is a morphism in $\mathcal{N}$. Hence, $A(Q\phi) = A(Q)\phi^* \not\leq \mathcal{N}$ as $A(Q) \not\leq \mathcal{N}$. Thus, $Q\phi \in \mathcal{F}_{\mathcal{N}}$ and Thompson-maximal in $\mathcal{F}_{\mathcal{N}}$, since Q is Thompson-maximal in $\mathcal{F}_{\mathcal{N}}$. Hence, $Q\phi \in \mathcal{Q}$. Now the

[2]Recall Notation 2.11

assertion is easy to check as the map $\phi^* : A(Q) \to A(Q\phi)$ is an isomorphism of groups with $J(N_S(Q))_Q\phi^* = J(N_S(Q\phi))_{Q\phi}$. □

Corollary 10.19. *Let $Q \in \mathcal{Q}$ and $1 \neq U$ $char_{\mathcal{F}}$ Q. Then there is*

$$\phi \in Mor_{\mathcal{F}}(N_S(U), S)$$

such that $U\phi$ is fully normalized. For each such ϕ we have $Q\phi \in \mathcal{Q}$, $U\phi$ $char_{\mathcal{F}}$ $Q\phi$, $N_S(Q)\phi = N_S(Q\phi)$, $A^\circ(Q)\phi^* = A^\circ(Q\phi)$, $V(Q)\phi = V(Q\phi)$ and $R(Q)\phi = R(Q\phi)$.

Proof. This is a consequence of Lemma 2.18 and Lemma 10.18. □

10.3 The proof of Lemma 10.4

Throughout Section 10.3 assume Hypothesis 10.3.

Lemma 10.20. *Let $Q \in \mathcal{Q}$, let $U \in \mathcal{C}(Q)$ be fully normalized, and let $R_0 \leq R(Q)$ such that $[R_0, A^\circ(Q)] \neq 1$ and*

$$N_S(U) \cap N_S(R_0) \cap N_S(\langle \mathcal{A}_*(Q, V(U))\rangle) \leq N_S(Q).$$

Then $N_S(U) \cap N_S(R_0) \leq N_S(Q)$.

Proof. Set $W := V(U)$, $T := N_S(Q)$, $T_0 := N_T(R_0)$, $R := R(Q)$ and $\mathcal{A}_*(Y) := \mathcal{A}_*(Y, W)$ for $Y \leq T$. Assume the assertion is wrong. Then $T_0 < N_S(U) \cap N_S(R_0)$. In particular, $T < N_S(U)$ and so $J(Q) \not\leq U$ since $N_S(J(Q)) = T$. Hence, $\mathcal{A}_*(Q) \neq \emptyset$. Moreover, $T_0 < N_S(U) \cap N_S(R_0) \cap N_S(T_0)$, i.e. there is $t \in N_S(U) \cap N_S(R_0) \cap N_S(T_0)$ such that $t \notin T$. Then, by assumption, there is $A \in \mathcal{A}_*(Q)$ such that $A^t \notin \mathcal{A}_*(Q)$. Note that $A^t \leq T_0 \leq T$. Remark 9.3(c) implies

$$\mathcal{A}_*(Q) \subseteq \mathcal{A}_*(T).$$

Therefore, $A^t \in \mathcal{A}_*(T)$ and $A^t \not\leq Q$. Now Lemma 10.17 yields $R = [W, A^t] = [W, A]^t$. Hence, $R^{t^{-1}} = [W, A] = [W, AU]$. So, by Lemma 9.11, $R^{t^{-1}} \cap V(Q) =$

$[W, AU] \cap V(Q)$ is $O^p(N_{A^\circ(Q)}(U))$-invariant. By Remark 9.3(a), we have $J(T) \leq A^t Q \leq T_0$, so $J(T) = J(T_0)$ and $J(T)^t = J(T)$. Remark 10.16 implies $[R, J(T)] = 1$. Therefore, we get $[R^{t^{-1}}, J(T)] = 1$, and it follows from the module structure of $V(Q)$ that $R^{t^{-1}} \cap V(Q) \leq C_{V(Q)}(A^\circ(Q))$. Hence,

$$R_0 = R_0^{t^{-1}} \leq R^{t^{-1}} \cap V(Q) \leq C(A^\circ(Q)),$$

a contradiction. This proves the assertion. □

Lemma 10.21. *Let $Q \in \mathcal{Q}$ and $1 \neq U$ $char_{\mathcal{F}}$ Q. Then $N_S(R_0) \cap N_S(U) \leq N_S(Q)$ for every $R_0 \leq R(Q)$ with $[R_0, A^\circ(Q)] \neq 1$. In particular, $N_S(R(Q)) \cap N_S(U) = N_S(Q)$.*

Proof. Assume the assertion is wrong. Choose $Q, U \in \mathcal{F}$ such that $Q \in \mathcal{Q}$, $1 \neq U$ $char_{\mathcal{F}}$ Q, and there exists $R_0 \leq R(Q)$ with $[R_0, A^\circ(Q)] \neq 1$ and $N_S(R_0) \cap N_S(U) \not\leq N_S(Q)$. We may choose this pair (Q, U) such that $|U|$ is maximal. By Corollary 10.19, there is $\phi \in Mor_{\mathcal{F}}(N_S(U), S)$ such that $Q\phi \in \mathcal{Q}$, $U\phi$ $char_{\mathcal{F}}$ $Q\phi$ and $U\phi$ is fully normalized. Moreover, then $R_0\phi \leq R(Q)\phi = R(Q\phi)$, $[R_0\phi, A^\circ(Q\phi)] \neq 1$ and $N_S(R_0\phi) \geq N_{N_S(U)}(R_0)\phi \not\leq N_S(Q)\phi = N_S(Q\phi)$. So, replacing (Q, U) by $(Q\phi, U\phi)$, we may assume without loss of generality that U is fully normalized. Observe that then $U \in \mathcal{C}^*(Q)$ and thus, by Lemma 10.12, $U \in \mathcal{C}(Q)$.

Note that UQ_* $char_{\mathcal{F}}$ Q for $Q_* = \langle \mathcal{A}_*(Q, V(U)) \rangle$. As $N_S(J(Q)) = N_S(Q) < N_S(U)$, we have $Q^* \not\leq U$. Hence, the maximality of $|U|$ yields $N_S(R_0) \cap N_S(UQ_*) \leq N_S(Q)$ and thus $N_S(R_0) \cap N_S(U) \cap N_S(Q_*) \leq N_S(Q)$. Now Lemma 10.20 yields $N_S(U) \cap N_S(R_0) \leq N_S(Q)$, contradicting the choice of U. □

Lemma 10.22. *Let $Q \in \mathcal{Q}$, $1 \neq U \in \mathcal{C}(Q)$, $A \in \mathcal{A}_*(N_S(Q), V(U))$ and $b \in N_S(U) \backslash N_S(Q)$ such that $A \not\leq Q$ and $A^b \leq N_S(Q)$. Then $A^b \leq Q$.*

Proof. Assume $A^b \not\leq Q$. Then Lemma 10.17 implies

$$R(Q) = [V(U), A^b] = [V(U), A]^b = R(Q)^b.$$

This is a contradiction to Lemma 10.21. □

Lemma 10.23. *Let $Q \in \mathcal{Q}$ and $U \in \mathcal{C}^*(Q)$. Then we have $\mathcal{A}_*(N_S(Q), V(U)) = \mathcal{A}_*(Q, V(U))$ or $J(N_S(U)) \leq N_S(Q)$.*

Proof. Set $T = N_S(Q)$, $T_0 = N_S(U)$, $R := R(Q)$, $W = V(U)$, and $\mathcal{A}_*(Y) = \mathcal{A}_*(Y, W)$ for every $Y \leq T_0$. We will use frequently and without reference that, by Lemma 10.12, $U \in \mathcal{C}(Q)$ and, in particular, by Remark 10.9, $V(Q) \leq W$. Assume $J(T_0) \not\leq T$ and $\mathcal{A}_*(T) \neq \mathcal{A}_*(Q)$. We show first:

(1) $\langle \mathcal{A}_*(T_0) \rangle \not\leq T$.

By assumption, there is $B_* \in \mathcal{A}(T_0)$ with $B_* \not\leq T$. We may choose B_* such that $|B_*U|$ is minimal. Let $B \in \mathcal{A}_*(B_*U)$. Then $B \in \mathcal{A}_*(T_0)$. Let $t \in T_0$ and observe that $B^t \in \mathcal{A}_*(T_0)$. Assume (1) does not hold. Then B and B^t are contained in T.

Suppose $B^t \not\leq Q$. Since $B_*^t U/U$ is elementary abelian, $B^t U$ is normalized by B_*^t. Hence, for every $x \in B_*^t$, $(B^t)^x \leq T$ and $(B^t)^x \not\leq Q$. Hence, by Lemma 10.22, $B_*^t \leq T$. In particular, $\mathcal{A}(T) \subseteq \mathcal{A}(T_0)$ and, by Remark 9.3(a), $B_*^t \leq J(T) \leq B^t Q$. Since $B \leq B_*U$, this gives $B_*^t U = B^t(B_*^t U \cap Q) = B^t U(B_*^t \cap Q)$ and $B_*U = BU(B_* \cap Q^{t^{-1}})$. By Remark 9.3(c), we have $C_* = (B_*^t \cap Q)V(Q) \in \mathcal{A}(T) \subseteq \mathcal{A}(T_0)$. Therefore, $C_*^{t^{-1}} = (B_* \cap Q^{t^{-1}})V(Q)^{t^{-1}} \in \mathcal{A}(T_0)$. Note that, by Remark 10.9, $V(Q)^{t^{-1}} \leq U^{t^{-1}} = U$. In particular, $B_*U = BU(B_* \cap Q^{t^{-1}}) = BUC_*^{t^{-1}}$. As $BU \leq T$ and $B_* \not\leq T$, we get $C_*^{t^{-1}} \not\leq T$. On the other hand, $C_*^{t^{-1}} \leq B_*U$, so the minimality of $|B_*U|$ gives $C_*^{t^{-1}}U = B_*U$. Then $B_* \leq Q^{t^{-1}}$, i.e. $B^t \leq B_*^t \leq Q$ contradicting our assumption. Hence, $B^t \leq Q$.

Since $t \in T_0$ was arbitrary we have shown that $X := \langle B^{T_0} \rangle \leq Q$. Therefore, it follows from Lemma 10.10 that $B \leq X \leq U$, a contradiction to the choice of B. Thus, (1) holds. We show next:

(2) There is $T \leq T_1 \leq N_{T_0}(\langle \mathcal{A}_*(T) \rangle)$ such that $\langle \mathcal{A}_*(T_1) \rangle \not\leq T$.

For the proof let $T \leq Y \leq T_0$ be maximal with respect to inclusion such that $\langle \mathcal{A}_*(Y) \rangle \leq T$. Then, by (1), $Y \neq T_0$ and hence $Y < T_1 := N_{T_0}(Y)$. So the maximality of Y implies $\langle \mathcal{A}_*(T_1) \rangle \not\leq T$. Since $\langle \mathcal{A}_*(Y) \rangle = \langle \mathcal{A}_*(T) \rangle$, we have $T_1 \leq N_{T_0}(\langle \mathcal{A}_*(T) \rangle)$. This shows (2).

So we can choose now T_1 with the properties as in (2). We fix $B_* \in \mathcal{A}_*(T_1)$ such that $B_* \not\leq T$. Note that Theorem 4.9 implies

(3) B_* acts quadratically on W.

We show next:

(4) $|B_*/N_{B_*}(R)| = |B_*/B_* \cap T| = 2 = p$.

By Lemma 10.21, $N_{B_*}(R) = B_* \cap T$. Let $b \in B_* \backslash T$ and assume there is $c \in B_* \backslash ((B_* \cap T) \cup b(B_* \cap T))$. Note that b, c and cb^{-1} are not elements of T. By assumption, $\mathcal{A}_*(T) \neq \mathcal{A}_*(Q)$, i.e. there is $A \in \mathcal{A}_*(T)$ with $A \not\leq Q$. Then by the choice of $B_* \leq T_1$ and Lemma 10.22,

$$A^b \leq Q, \ A^c \leq Q \text{ and } A^{cb^{-1}} \leq Q.$$

This gives

$$A^c \leq Q \cap Q^b \cap Q^{b^{-1}c}.$$

Hence, A^c centralizes $W_0 := V(Q)V(Q)^b V(Q)^{b^{-1}c}$. Note that $W_0 \leq W$ and

$$W_1 := V(Q)[V(Q), b][V(Q), b^{-1}c] \leq W_0.$$

By (3), W_1 is invariant under b and $b^{-1}c$, so $W_0 = W_1$ and $W_0 = W_0^b = W_0^{b^{-1}c}$. Hence, $W_0^c = (W_0^{b^{-1}})^c = W_0^{b^{-1}c} = W_0$. As shown above, $[W_0, A^c] = 1$. So we get $[W_0^c, A^c] = 1$ and thus $[W_0, A] = 1$. In particular, $[V(Q), A] = 1$, a contradiction to $A \not\leq Q$. Hence, (4) holds. We show now

(5) $W = RC_W(B_*)$.

It follows from (4) and Remark 10.16 that

$$|B_*/C_{B_*}(R)| = |B_*/B_* \cap T| \cdot |(B_* \cap T)/C_{B_*}(R)| \leq 2 \cdot |(B_* \cap T)/(B_* \cap J(T))|.$$

Set $\overline{A(Q)} := A(Q)/C_{A(Q)}(V(Q))$. Since Q is Thompson-restricted, we have $\overline{A^\circ(Q)} \cong SL_2(q)$ and $\overline{J(T)_Q} \in Syl_p(\overline{A^\circ(Q)})$. Moreover, $T/Q \cong \overline{T_Q}$ embeds into $Aut(\overline{A^\circ(Q)})$. Hence, $T/J(T)Q \cong Aut(GF(q))$ is cyclic, and $q = 2$ implies $T = J(T)$. Therefore, $|(B_* \cap T)/(B_* \cap J(T))| \leq 2$ and

(*) $|B_*/C_{B_*}(R)| \leq q$.

The module structure of $V(Q)$ implies $|R/C_R(A^\circ(Q))| = q$. By Lemma 10.21, $R \cap B_* \leq C_R(B_*) \leq C_R(A^\circ(Q))$ and hence

$$|R/R \cap B_*| \geq |R/C_R(B_*)| \geq |R/C_R(A^\circ(Q))| = q.$$

Thus, by (*), $|RC_{B_*}(R)| = |R/R \cap B_*| \cdot |C_{B_*}(R)| \geq |B_*|$. Observe that $RC_{B_*}(R)$ is elementary abelian, so $RC_{B_*}(R) \in \mathcal{A}(T_1)$. Since $(RC_{B_*}(R))U = C_{B_*}(R)U$ is a proper subset of B_*U, it follows from the minimality of B_*U that $C_{B_*}(R) \leq U$ and $C_{B_*}(R) = C_{B_*}(W)$. Therefore, by Lemma 4.4 and (*),

$$|W/C_W(B_*)| \leq |B_*/C_{B_*}(W)| = |B_*/C_{B_*}(R)| \leq q.$$

As seen above, $|R/C_R(B_*)| \geq q$. This implies $|RC_W(B_*)| \geq |W|$ and thus (5).

Now choose $t \in A^\circ(Q) \backslash N_{A^\circ(Q)}(T_Q)C_{A^\circ(Q)}(V(Q))$ and $b \in B_* \backslash C_{B_*}(W)$. Set $Y = RR^tR^b$. Note that $Y \leq W$, since $RR^t \leq V(Q) \leq W$. Using (5), we get $[W, b] = [RC_W(B_*), b] = [R, b] \leq RR^b \leq Y$. Hence,

$$Y^b = Y.$$

As before let $A \in \mathcal{A}_*(T)$ with $A \not\leq Q$. Then, by the choice of $B_* \leq T_1$ and Lemma 10.22, we have $A^b \leq Q$. Hence, $[RR^t, A^b] \leq [V(Q), A^b] = 1$. By Remark 10.16, $[R, A] = [R, J(T)] = 1$, so $[R^b, A^b] = 1$ and $[Y, A^b] = 1$. As we have shown above, $Y = Y^b$. So we get $[Y, A]^b = [Y^b, A^b] = [Y, A^b] = 1$ and hence, $[Y, A] = 1$. In particular, $[R^t, A] = 1$ which is a contradiction to the module structure of $V(Q)$. This completes the proof of Lemma 10.23. □

Lemma 10.24. *Let $Q \in \mathcal{Q}$ and $1 \neq U$ $char_{\mathcal{F}}$ Q. Then $J(N_S(U)) \leq N_S(Q)$.*

Proof. Assume the assertion is wrong. Then there is $Q \in \mathcal{Q}$ and U $char_{\mathcal{F}}$ Q such that $J(N_S(U)) \not\leq N_S(Q)$. We can choose the pair (Q, U) such that $|U|$ has maximal order. By Corollary 10.19 we can furthermore choose it such that U is fully normalized. Set $T = N_S(Q)$ and $T_0 := N_S(U)$. Note that $U \in \mathcal{C}^*(Q)$. Thus, by Lemma 10.23, $\mathcal{A}_*(T, V(U)) = \mathcal{A}_*(Q, V(U))$. Set $X := \langle \mathcal{A}_*(Q, V(U)) \rangle$. Observe that $T = N_S(J(Q)) < T_0$, so $J(Q) \not\leq U$ and $X \not\leq U$. Also note $U_1 := XU$ $char_{\mathcal{F}}$ Q. Therefore, by the choice of U, $J(N_S(U_1)) \leq T \leq T_0$. In particular, $J(T) = J(N_S(U_1)) = J(N_{T_0}(U_1))$ and

$$\mathcal{A}_*(Q, V(U)) = \mathcal{A}_*(T, V(U)) = \mathcal{A}_*(N_{T_0}(U_1), V(U)).$$

Hence, $N_{T_0}(N_{T_0}(U_1))$ normalizes $XU = U_1$. It follows that $T_0 \leq N_S(U_1)$, which contradicts $J(N_S(U_1)) \leq T$ and $J(T_0) \not\leq T$. □

The proof of Lemma 10.4. Let $Q \in \mathcal{Q}$ and $1 \neq U$ $char_{\mathcal{F}}$ Q. Set $T := N_S(Q)$ and $T_0 := N_S(U)$. By Lemma 10.24, $J(T_0) = J(T)$ and so $B(T_0) = C_{T_0}(\Omega(Z(J(T))))$. By Remark 10.16 $R(Q) \leq \Omega(Z(J(T)))$. Hence, Lemma 10.21 implies $B(T_0) \leq N_{T_0}(R(Q)) = T$. This shows $B(T_0) = B(T)$ and completes the proof.

10.4 The proofs of Theorem 2 and Theorem 3

From now on assume Hypothesis 10.2. Observe that this implies Hypothesis 10.3. In particular, we can use Lemma 10.4 and the other results from the previous sections.

Lemma 10.25. *Let $Q \in \mathcal{Q}$, let $U \in \mathcal{F}$ be $\mathcal{F}$-characteristic in Q and $A(S)$-invariant. Then $U = 1$.*

Proof. Assume $U \neq 1$. Since U is $A(S)$-invariant, $N_{\mathcal{F}}(U)$ is full parabolic. Hence, $N_{A^\circ(Q)}(U) \leq N_{\mathcal{F}}(U) \leq \mathcal{N}$. Observe also that $C_{A^\circ(Q)}(V(Q)) \leq N_{\mathcal{F}}(\Omega(Z(S))) \leq \mathcal{N}$ and hence, as $U \ char_{\mathcal{F}} \ Q$, $A^\circ(Q) \leq \mathcal{N}$. This is a contradiction to Remark 10.6. □

The proof of Theorem 2. Choose a pair (Q, U) such that $Q \in \mathcal{Q}$, $1 \neq U \ char_{\mathcal{F}} \ Q$ and $|N_S(U)|$ is maximal. Moreover, choose U so that $|U| \geq |U_0|$ for all subgroups $1 \neq U_0 \ char_{\mathcal{F}} \ Q$ with $U_0 \trianglelefteq N_S(U)$. Note that U is fully normalized by Corollary 10.19. So the maximal choice of $|U|$ yields $U \in \mathcal{C}^*(Q)$. Hence, by Lemma 10.12, $Q \in \mathcal{C}(Q)$. Set

$$G := G(Q),\ T := N_S(Q) \text{ and } M^* := C_G(V(Q))J(G).$$

Observe that it is sufficient to show the following properties.

(a) $N_S(Q) = N_S(U)$.

(b) $M^*/Q \cong SL_2(q)$ and one of the following hold:

(I) Q is elementary abelian, $|Q| \leq q^3$ and $Q/C_Q(M^*)$ is a natural $SL_2(q)$-module for M^*/Q.

(II) $p = 3, T = S$, $|Q| = q^5$, $\Phi(Q) = C_Q(M^*)$, and $Q/V(Q)$ and $V(Q)/\Phi(Q)$ are natural $SL_2(q)$-modules for M^*/Q.

For the proof of (a) and (b) set

$$T_1 := B(T),\ T_0 := N_S(U) \text{ and } Q_1 := Q \cap T_1.$$

The maximal choice of $T_0 = N_S(U)$ together with Lemma 10.25 yields the following property.

(1) Let $1 \neq C \leq T_1$ such that $C \ char_{\mathcal{F}} \ Q$. Then C is not $A(T_0)$-invariant and, if $S \neq T_0$, C is not normal in $N_S(T_0)$.

By Lemma 9.7, we can now choose $H \leq N_{M^*}(U)$ such that $T_1 \in Syl_p(H)$, H is normalized by T, $M^* = C_G(V(Q))H$ and $(H/O_p(H))/\Phi(H/O_p(H)) \cong L_2(q)$.

Observe that $Q_1 = O_p(H)$. Note that, by Lemma 10.4, $T_1 = B(T_0)$ and so every characteristic subgroup of T_1 is $A(T_0)$-invariant. Therefore, (1) implies that H fulfils Hypothesis 5.1 with $V(Q)$ in place of W. Thus, by Theorem 5.2 one of the following holds for $V := [Q_1, O^p(H)]$.

(I') $V \leq \Omega(Z(Q_1))$ and $V/C_V(H)$ is a natural $SL_2(q)$-module for $H/C_H(V(Q))$.

(II') $Z(V) \leq Z(Q_1)$, $p = 3$, $\Phi(V) = C_V(H)$ has order q, $V/Z(V)$ and $Z(V)/\Phi(V)$ are natural $SL_2(q)$-modules for $H/C_H(V(Q))$.

Furthermore, the following hold for every $\phi \in Aut(T_1)$ with $V\phi \not\leq Q_1$.

(i) $Q_1 = VC_{Q_1}(L)$ for some subgroup L of H with $O^p(H) \leq L$ and $H = LQ_1$.

(ii) $\Phi(C_{Q_1}(O^p(H)))\phi = \Phi(C_{Q_1}(O^p(H)))$.

(iii) If (II') holds then $V \leq V(Q)\langle (V(Q)\phi)^H \rangle \leq Q_1$.

(iv) If (II') holds then T_1 does not act quadratically on $V/\Phi(V)$.

(v) $V \not\leq Q\phi$.

(vi) If (II') holds then $Q_1\phi^2 = Q_1$.

If $U_0 \leq Q_1$ for $U_0 = \langle V^{A(T_0)} \rangle$ or for $U_0 = \langle V^{N_S(T_0)} \rangle$, then $[U_0, O^p(H)] \leq V \leq U_0$ and $U_0\ char_{\mathcal{F}}\ Q$. Together with (1) this gives the following property.

(2) There is $\phi \in A(T_0)$ such that $V\phi \not\leq Q_1$. If $S \neq T_0$ then we may choose ϕ such that $\phi \in S_{T_0}$.

Let now $\phi \in A(T_0)$ such that $V\phi \not\leq Q_1$. Recall that, by Lemma 10.4, $T_1 = B(T_0)$ and hence $T_1\phi = T_1$. Set

$$D := \Phi(C_{Q_1}(O^p(H))).$$

Note that, as Q_1 and H are T-invariant, D is normal in T and so $\mathcal{F}$-characterstic in Q. By Lemma 10.18 and (ii), $Q\phi \in \mathcal{Q}$ and $D = D\phi\ char_{\mathcal{F}}\ Q\phi$. Assume $D \neq 1$.

By Corollary 10.19, there is $\psi \in Mor_{\mathcal{F}}(N_S(D), S)$ such that $D\psi$ is fully normalized, $Q\psi, Q\phi\psi \in \mathcal{Q}$, and $D\psi$ is $\mathcal{F}$-characteristic in $Q\psi$ and $Q\phi\psi$. Hence, by Corollary 9.9, $D^* := O_p(N_{\mathcal{F}}(D\psi)) \leq Q\psi \cap Q\phi\psi$. As $\mathcal{F}$ is minimal, $N_{\mathcal{F}}(D)$ is solvable and thus constrained. Hence,

$$V(Q\psi)V(Q\phi\psi) \leq C_{N_S(D\psi)}(D^*) \leq D^* \leq (Q\psi) \cap (Q\phi\psi).$$

In particular, $V(Q)\phi \leq Q$. If (I') holds then $V \leq V(Q)C_V(X) = V(Q)Z(T_1)$ and so $V\phi \leq Q$, contradicting the choice of ϕ. Therefore (II') holds. Observe that $H_Q\psi^* \leq N_{\mathcal{F}}(D\psi)$ as $H_Q \leq N_{\mathcal{F}}(D)$.[3] Hence, $H_Q\psi^*$ normalizes D^*, and $V_0 := V(Q\psi)\langle V(Q\phi\psi)^{H_Q\psi^*}\rangle \leq D^* \leq Q\phi\psi$. This implies $V(Q)\langle V(Q\phi)^H\rangle = V_0\psi^{-1} \leq Q\phi$. Then by (iii), $V \leq Q\phi \cap T_1 = Q_1\phi$, a contradiction to (v). This proves $D = 1$ and so we have shown that

(3) $C_{Q_1}(O^p(H))$ is elementary abelian.

We show next that (a) holds. For the proof assume $T < N_S(U)$. Then there is $x \in (N_S(U) \cap N_S(T))\backslash T$. Since $J(Q_1) = J(Q)$ and $N_S(J(Q)) = T$, we have $Q_1^x \neq Q_1$. By (3) and (i), $Q_1 = VC_{Q_1}(H) = VZ(T_1)$ and so $V^x \not\leq Q_1$. On the other hand, $U \in \mathcal{C}(Q)$ and so, by Corollary 9.9, $O_p(N_{\mathcal{F}}(U)) = U^*(Q) = U$. As $U = U^x \leq Q \cap Q^x$ and $N_{\mathcal{F}}(U)$ is constrained, we get $V(Q)V(Q)^x \leq C_{N_S(U)}(U) \cap T_1 \leq U \cap T_1 \leq Q_1$. If (I') holds then $V \leq V(Q)Z(H) \leq V(Q)Z(T_1)$ and so $V^x \leq Q_1$, a contradiction. Hence (II') holds. Then, by (iii), we have $V \leq V(Q)\langle V(Q^x)^H\rangle \leq U$. So $V \leq U \cap T_1 \leq Q^x$, a contradiction to (v). This proves (a). The choice of (Q, U) together with Corollary 10.19 and Lemma 10.25 gives now the following property.

(4) For every $1 \neq U_0 \; char_{\mathcal{F}} \; Q$, we have $T = N_S(U_0)$ and U_0 is fully normalized. In particular, U_0 is not $A(T)$-invariant.

We show next:

[3] Recall Notation 2.11.

(5) Let $\alpha \in A(T)$ such that $Q_1\alpha \neq Q_1$. If (I') holds then $\Phi(Q)\alpha = \Phi(Q)$.

For the proof of (5) assume that (I') holds and α is as in (5). Then $Q_1 = V(Q)Z(T_1)$, so we have $V(Q)\alpha \not\leq Q_1$. By Lemma 4.12, $V(Q\alpha)$ is not an over-offender on $V(Q)$ and vice versa, so $|V(Q)/C_{V(Q)}(V(Q\alpha))| = |V(Q\alpha)/C_{V(Q\alpha)}(V(Q))|$. Hence, $V(Q\alpha)$ is an offender on $V(Q)$ and vice versa. So, again by Lemma 4.12, $J(T)Q = V(Q\alpha)Q$ and $J(T)(Q\alpha) = V(Q)(Q\alpha)$. In particular, $[J(T), Q\alpha] \leq Q$ and, by Remark 9.3(d), $Q\alpha \leq J(T)Q$. Hence, $J(T)Q = J(T)(Q\alpha) = V(Q\alpha)Q = V(Q)(Q\alpha)$. In particular, $Q\alpha = V(Q\alpha)(Q \cap Q\alpha)$ and $Q = V(Q)(Q \cap Q\alpha)$. This yields $\Phi(Q\alpha) = \Phi(Q \cap Q\alpha) = \Phi(Q)$ and proves (5).

From now on let $\alpha \in A(T)$ such that $Q_1\alpha \neq Q_1$. Note that α exists by (4). We show now:

(6) If (I') holds then Q is elementary abelian.

Let $\beta \in A(T)$ such that $Q_1\beta = Q_1$. Then $Q_1\beta\alpha \neq Q_1$. If (I') holds then (5) yields $\Phi(Q)\alpha = \Phi(Q) = \Phi(Q)\beta\alpha$ and hence $\Phi(Q) = \Phi(Q)\beta$. Thus, by (5), $\Phi(Q)$ is $A(T)$-invariant. Now (4) implies $\Phi(Q) = 1$, so (6) holds. We show now:

(7) $C_Q(H) \cap (C_Q(H)\alpha) = 1$.

For the proof of (7) assume $U_1 := C_Q(H) \cap (C_Q(H)\alpha) \neq 1$. By Lemma 10.18, we have $Q\alpha \in \mathcal{Q}$. Note that U_1 $char_{\mathcal{F}}$ Q and U_1 $char_{\mathcal{F}}$ $Q\alpha$. In particular, by (4), U_1 is fully normalized. Moreover, Corollary 9.9 implies $U_1^* := O_p(N_{\mathcal{F}}(U_1)) \leq Q \cap Q\alpha$. By Corollary 2.34, $N_{\mathcal{F}}(U)$ is constrained. Hence,

$$V(Q)V(Q\alpha) \leq C_{N_S(U_1)}(U_1^*) \leq U_1^* \leq Q \cap Q\alpha.$$

If (I') holds then, by (6), $Q = V(Q)$ and so $Q = Q\alpha$, contradicting the choice of α. By (i) and (3), $Q_1 = VZ(T_1)$, so $V\alpha \not\leq Q_1$. Hence, if (II') holds then, by (iii), $V \leq V(Q)\langle V(Q\alpha)^H\rangle \leq U_1^* \leq Q\alpha$, contradicting (v). This shows (7).

It remains to show that (I') implies (I) and (II') implies (II). Assume first (I') holds. By (6), Q is elementary abelian. Hence, $Q = V(Q)$, $C_G(V(Q)) = Q$, $M^*/Q \cong SL_2(q)$ and $Q/C_Q(M^*)$ is a natural $SL_2(q)$-module for M^*/Q. By (7), $C_Q(M^*) \cap (C_Q(M^*)\alpha) = 1$. This implies

$$|C_Q(M^*)| = |(C_Q(M^*)\alpha)C_Q(M^*)/C_Q(M^*)| \leq |Z(J(T))/C_Q(M^*)| \leq q.$$

Hence, $|Q| \leq q^3$ and (I) holds.

Assume from now on that (II') holds. Note that, for $W := Z(Q_1)$, $W/C_W(H)$ is a natural $SL_2(q)$-module for H/Q_1. Hence, $|Z(T_1)/C_{Q_1}(H)| = |C_W(T_1)/C_W(H)| \leq q$. Now (7) yields

$$|C_{Q_1}(H)| = |(C_{Q_1}(H)\alpha)C_{Q_1}(H)/C_{Q_1}(H)| \leq |Z(T_1)/C_{Q_1}(H)| \leq q$$

and so $C_{Q_1}(H) = C_V(H)$. Now by (i) and (3), $Q_1 = VC_{Q_1}(H) = V$. In particular, by (iii), $Q_1 = V = V(Q)\langle(V(Q)\phi)^H\rangle$. So $Q_1 = V$ is generated by elements of order p and $[Q_1, Q_1] = \Phi(Q_1) = C_{Q_1}(H)$. As $Q_1/Z(Q_1)$ is an irreducible module for H, $[Q_1, Q] \leq Z(Q_1)$ and so $[Q_1, Q, Q_1] = 1 = [Q, Q_1, Q_1]$. Now the Three-Subgroups Lemma implies $[C_{Q_1}(H), Q] = [Q_1, Q_1, Q] = 1$. Observe that $Z(Q_1) = V(Q)C_{Q_1}(H)$ and so $[Z(T_1), Q] \leq [Z(Q_1), Q] = 1$. The definition of T_1 gives now $[\Omega(Z(J(T))), Q] = 1$ and $Q = Q_1 = V$. In particular, by (vi), every automorphism of T of odd order normalizes Q. Hence, Q is normal in $N_S(T)$ and so $T = S$. If $[\overline{Q}, C_G(V(Q))] \neq 1$ for $\overline{Q} = Q/C_Q(H)$, then $\overline{Q}$ is the direct sum of two natural $SL_2(q)$-modules for $H/C_H(V(Q))$ and so $[\overline{Q}, T_1, T_1] = 1$, a contradiction to (iv). Thus, $[\overline{Q}, C_G(V(Q))] = 1$ and, if $x \in C_G(V(Q))$ has order prime to p, then $[Q, x] = [Q, x, x] \leq [C_Q(H), x] \leq [V(Q), x] = 1$. Hence, $C_G(V(Q)) = Q$. This shows $M^*/Q \cong SL_2(q)$ and (II) holds. Thus, the proof of Theorem 2 is complete.

The proof of Theorem 3. Theorem 3 follows from Theorem 1, Theorem 2, Corollary 2.34 and Theorem 7.2.

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