ON ORIGINS OF ORBITS AND THE SHADOW OF CHAOS

by

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A thesis submitted to

The University of Birmingham

for the degree of

DOCTOR OF PHILOSOPHY

School of Mathematics

College of Engineering and Physical Sciences

University of Birmingham

August 2020

Abstract

The work presented in this thesis concerns three areas within topological dynamics. The first area is that of chaos: specifically topological equicontinuity, transitivity, and sensitivity. We present two Auslander–Yorke dichotomy type theorems before constructing a chaotic system with an even continuity pair but no equicontinuity point. The second topic is an exploration of the preservation of various notions of shadowing under inverse limits, products, factor maps, and the induced maps for symmetric products and hyperspaces. The third area is that of α-limit sets and ω-limit sets in dynamical systems. These sets may be thought of as the origins and, respectively, destinations of orbit sequences. Many of our results on these sets relate to the aforementioned shadowing property and variations thereof. Included in these results is a characterisation of when the set of α-limit sets, the set of ω-limit sets, and the set of nonempty closed internally chain transitive sets (ICT_f) coincide. Moreover, we demonstrate that shadowing is sufficient to mean that every element of ICT_f can be approximated (to any prescribed accuracy) by both the α-limit set and the ω-limit set of the same full-trajectory.

ACKNOWLEDGEMENTS

I express my gratitude to Chris Good, without whose patience and guidance this work would not have been possible. Thanks must further go to my co-supervisor, Simon Goodwin, and my collaborators: Robert Leek, Sergio Macías, Jonathan Meddaugh, and Joe Thomas. I also pay homage to you, Mr McMillan; a better teacher, there is none.

I owe a debt of gratitude to my family: To my parents, Andy and Fiona; to my brothers, Luke and Thomas; to my sister-in-law, Claire; and to our dog, Archie. Their love has been a blessing beyond words. I am grateful too for a close extended family and, whilst space does not permit me to list them all here, I thank Grandma Lawrie for continuously maintaining, throughout all my mathematical uncertainties, that $2 + 2 = 4$.

Thanks should also go to my friends and colleagues for making my time in Birmingham thoroughly enjoyable. A special mention to my housemates, one of whom put up with me for five years, with whom I shared many laughs, dinners, and caterpillar cakes.

In our chaotic lives, even seemingly small events can have an unknowable impact. Accordingly, I cannot list everyone to whom I owe my thanks (and for fear of forgetting someone, I would not wish to try). However, my gratitude goes out to them all the same; their input is not to be understated.

Finally, I acknowledge that everything I am is a consequence of a vast array of external factors. I thank God, as the origin of all, for any mathematical ability I might possess.

CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

In this thesis, we are concerned with studying discrete dynamical systems. A dynamical system is comprised of a pair (X, f), where $f\colon X \to X$ is a continuous function on the topological space X. The goal of studying dynamical systems abstractly is to uncover and understand the properties they might exhibit. This knowledge might then be applicable for an applied mathematician or scientist working with such systems. For example, one important concept arising in dynamical systems is that of chaos. In popular culture, a system is chaotic if it is sensitive to minute changes. In the film *Sliding Doors* [56], for example, the plot follows the events of a young woman's life in two sets of circumstances; in one, she manages to catch a certain train, in the other, she misses it by a fraction of a second. The consequence of this small change is different friends, a different job, house, and relationship, contrasting amounts of joy; in short, a completely different life. Such sensitivity to initial conditions may be captured mathematically and, whilst there is no universally agreed-upon mathematical definition of chaos, it does form a key component in many of the mathematical variants, for example, *Auslander–Yorke chaos* and *Devaney chaos*. More on this will be seen in Chapter 2.

Usually when one studies dynamical systems, one's attention is restricted to certain spaces — or at least, certain types of spaces. Indeed, maps of the closed unit interval $[0, 1]$ and of the circle S^1 have been extensively studied (e.g. [5, 15, 21, 32, 85, 92]). More generally, a restriction is often made to the compact metric setting (e.g. [7, 10, 24, 42, 73]).

In this thesis, we endeavour to keep our results as general as possible, mainly working in the compact Hausdorff setting but removing the assumption of compactness where possible. Compactness, as we will see, allows us to make use of the fact that there is a unique (up to equivalence) uniformity which generates the topology on X. This allows for natural abstractions of metric definitions which coincide with their metric cousins when the phase space is compact metric. Without the assumption of compactness, one can still choose to take a purely topological approach to dynamics. However, even when applicable, it usually cannot be guaranteed that such topological definitions coincide with their metric or uniform cousins (see section 2.1.1).

This thesis will, for the most part, concern topological equicontinuity, shadowing, and limit sets. Loosely, a dynamical system is equicontinuous if points that are close in the phase space X remain close; here 'closeness' may be in terms of a metric or uniformity on X. Topological equicontinuity, introduced by Royden [87, pp. 362], is a purely topological version of equicontinuity. In Chapter 2, we will define topological equicontinuity and prove some results on its relation to properties inherent in existing notions of chaos, such as transitivity and sensitivity. The other chapters in this thesis largely concern the shadowing property and its variants.[1] Pseudo-orbits (see section 1.1.2 for precise definitions), may be thought of as orbit sequences but with some error. Such sequences arise naturally during the computation of orbits of points in dynamical systems. Indeed, through calculation, one often encounters rounding errors and so the generated sequence of points is not a true orbit of the system, but rather a pseudo-orbit. One question that may then be asked is: to what extent does a pseudo-orbit reflect any of the original dynamics of the system? One line of enquiry is to determine if such sequences are closely followed by true orbits of the system; this leads directly to the notion of shadowing. Informally, a system has shadowing if pseudo-orbits are followed, or *shadowed*, by true orbits.

Some variants of the pseudo-orbit tracing property have been proven to have a bearing on the ω-limit sets of a dynamical system. These sets are an important object of study

[1] NB. The shadowing property is also known as the pseudo-orbit tracing property.

in dynamics and there have been many treatments of their structure in recent literature (e.g. [1, 8, 10, 11, 12, 18, 21, 47, 49, 70]). The ω-limit set of a point $x \in X$ is the set of accumulation points of the orbit sequence $x, f(x), f^2(x) \ldots$. Thus, the ω-limit set of a point may be thought of as its *target* — it is where it *ends up*. Analogously, one may ask where a given point came from — what is its *source*, so to speak: enter the α-limit set. Defining such sets, as we shall see in Chapter 5, is not nearly as straightforward as defining their ω cousins because the map f need not be injective.

The order of the work presented in this thesis largely corresponds to the chronology in which it was carried out. We start, in this chapter, by giving some of the definitions which will be most fundamental throughout this thesis.

In Chapter 2 we study sensitivity, topological equicontinuity, and even continuity in dynamical systems. In doing so, we provide a classification of topologically transitive dynamical systems in terms of equicontinuity pairs, give a generalisation of the Auslander–Yorke dichotomy for minimal systems, and show there exists a transitive system with an even continuity pair but no equicontinuity point. We define what it means for a system to be eventually sensitive and we give a dichotomy for transitive dynamical systems concerning eventual sensitivity. Along the way, we define a property called splitting and discuss its relation to some existing notions of chaos.

In Chapter 3 we examine the preservation of various notions of shadowing under inverse limits, products, factor maps, and the induced maps for symmetric products and hyperspaces.

In Chapter 4 we show that pseudo-orbits trap ω-limit sets in a neighbourhood of prescribed accuracy after a uniform time period. A consequence of this is a generalisation of a result of Pilyugin *et al.*: every compact Hausdorff dynamical system has the second weak shadowing property. By way of further applications, we give a characterisation of minimal systems in terms of pseudo-orbits and show that every minimal system exhibits a shadowing variant introduced by Good and Meddaugh [47], namely the strong orbital shadowing property.

In Chapter 5 we examine when the set of α-limit sets, the set of ω-limit sets, and the set of nonempty closed internally chain transitive sets, denoted α_f, ω_f, and ICT_f respectively, coincide. We show that if the map f has shadowing then every element of ICT_f can be approximated (to any prescribed accuracy) by both the α-limit set and the ω-limit set of a full-trajectory. Furthermore, if f is additionally expansive then every element of ICT_f is equal to both the α-limit set and the ω-limit set of a full-trajectory. In particular, this means that shadowing guarantees that $\overline{\alpha_f} = \overline{\omega_f} = \mathrm{ICT}_f$ (where the closures are taken with respect to the Hausdorff topology on the space of compact sets), while the addition of expansivity entails $\alpha_f = \omega_f = \mathrm{ICT}_f$. Inspired by the work in [47], we introduce novel variants of shadowing which we use to characterise both maps for which $\overline{\alpha_f} = \mathrm{ICT}_f$ and maps for which $\alpha_f = \mathrm{ICT}_f$. Finally, we characterise maps for which every element of ICT_f is equal to (resp. may be approximated by) the α-limit set and the ω-limit set of the same full trajectory.

Much of the content in this thesis has been published in peer-reviewed journals whilst most of the remainder has been submitted for publication (preprints may be found at https://arxiv.org/a/mitchell_j_1.html). For the most part, the content of each chapter corresponds to that of a paper (see below). This allows us the benefit of keeping each chapter largely self-contained and, as such, we permit ourselves minor changes in notation where stylistically convenient. With this in mind, apart from the definitions given in section 1.1, we will define all the relevant terms for each chapter within the said chapter. Although this means that there is some repetition in terms, it has the benefit of ensuring that the important definitions are always close at hand.

Largely, Chapter 2 is [44]; Chapter 3 is [51]; Chapter 4 is [71]; Chapter 5 is both [49] and [72]. Some of these papers are collaborative efforts while some are solo ventures. Accordingly, the author gives credit to his collaborators and makes the following statement:

In addition to some solo work [71, 72]*, the work presented in this thesis involves collaboration with the following authors: Chris Good* [44, 49, 51]*, Robert Leek* [44]*, Jonathan*

Meddaugh [49]*, and Joe Thomas* [51].

1.1 Preliminaries

1.1.1 Notation

Throughout this thesis the following notation will be used:

- The set of natural numbers (excluding 0) will be denoted by $\mathbb{N}$, so that $\mathbb{N} = \{1, 2, 3, \ldots\}$. Meanwhile $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$. For stylistic reasons, we sometimes choose to use ω in place of $\mathbb{N}_0$. Notation is fixed within each chapter.
- The set of integers will be denoted by $\mathbb{Z}$.
- The set of real numbers will be denoted by $\mathbb{R}$.
- The set of rational numbers will be denoted by $\mathbb{Q}$.

1.1.2 Dynamical systems

For those wanting a thorough introduction to topological dynamics, [35] is an excellent resource. Most of the definitions in this section are standard and can be found there.

A *dynamical system* is a pair (X, f) consisting of a topological space X and a continuous function $f \colon X \to X$. For $x \in X$, we say the *orbit* of x under f is the set of points $\{x, f(x), f^2(x), \ldots\}$; we denote this set by $\mathrm{Orb}_f(x)$. We say x is *periodic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$; the least such n is called the *period* of x; if $n = 1$ we say x is a *fixed point*. A point $x \in X$ is *eventually periodic* if there exists $y \in \mathrm{Orb}_f(x)$ such that y is periodic. It immediately follows that $\mathrm{Orb}_f(x)$ is finite if and only if x is eventually periodic. For $x \in X$, we define the *ω-limit set* of x under f, denoted $\omega_f(x)$, or simply $\omega(x)$ where there is no ambiguity, to be the set of limit points of the sequence $(f^n(x))_{n \in \mathbb{N}}$.

Formally

$$\omega_f(x) = \bigcap_{N\in\mathbb{N}} \overline{\{f^n(x) \mid n > N\}}.$$

This means that $y \in \omega_f(x)$ if and only if for every neighbourhood U of y and every $N \in \mathbb{N}$ there exists $n > N$ such that $f^n(x) \in U$. If X is compact $\omega_f(x) \neq \emptyset$ for any $x \in X$ by Cantor's intersection theorem. Notice that $\overline{\mathrm{Orb}_f(x)} = \mathrm{Orb}_f(x) \cup \omega_f(x)$. We say a dynamical system (X, f) is *one-to-one* or *injective* if f is an injection. In a similar fashion, we say the system is *onto* or *surjective* if f is a surjection. We do not assume, unless stated, that a dynamical system is necessarily onto (or, for that matter, one-to-one). However, since surjective dynamical systems are usually the more interesting from a dynamics viewpoint, we ensure that the majority of examples and counterexamples we construct in this paper are surjective. The exception to this loose rule is if it is surjectivity itself which is under examination, such as in Example 3.4.8.

If (X, f) and (Y, g) are dynamical systems, we call a continuous surjection $\varphi\colon X \to Y$ a *factor map* or *semi-conjugacy* if

$$\varphi \circ f = g \circ \varphi.$$

In this case we say that (Y, g) is a *factor* of (X, f). In the case when φ is a homeomorphism, we call it a *conjugacy*, and say that the systems (X, f) and (Y, g) are *conjugate* to each other.

1.1.3 Uniform spaces

Let X be a nonempty set and $A \subseteq X \times X$. Let $A^{-1} = \{(y, x) \mid (x, y) \in A\}$; we call this the *inverse* of A. The set A is said to be *symmetric* if $A = A^{-1}$. For any $A_1, A_2 \subseteq X \times X$ we define the *composite* $A_1 \circ A_2$ of A_1 and A_2 as

$$A_1 \circ A_2 = \{(x, z) \mid \exists y \in X : (x, y) \in A_1, (y, z) \in A_2\}.$$

For any $n \in \mathbb{N}$ and $A \subseteq X \times X$ we denote by nA the n-fold composition of A with itself, i.e.

$$nA = \underbrace{A \circ A \circ A \cdots A}_{n \text{ times}}.$$

The *diagonal* of $X \times X$ is the set $\Delta = \{(x, x) \mid x \in X\}$. A subset $A \subseteq X \times X$ is called an *entourage* if $A \supseteq \Delta$.

Definition 1.1.1. A *uniformity* $\mathscr{U}$ on a set X is a collection of entourages of the diagonal such that the following conditions are satisfied.

a. $E_1, E_2 \in \mathscr{U} \implies E_1 \cap E_2 \in \mathscr{U}$.

b. $E \in \mathscr{U}, E \subseteq D \implies D \in \mathscr{U}$.

c. $E \in \mathscr{U} \implies D \circ D \subseteq E$ for some $D \in \mathscr{U}$.

d. $E \in \mathscr{U} \implies D^{-1} \subseteq E$ for some $D \in \mathscr{U}$.

We call the pair $(X, \mathscr{U})$ a *uniform space*. We say $\mathscr{U}$ is *separating* if $\bigcap_{E \in \mathscr{U}} E = \Delta$; in this case we say X is *separated*. A subcollection $\mathscr{V}$ of $\mathscr{U}$ is said to be a *base* for $\mathscr{U}$ if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{V}$ such that $D \subseteq E$. Clearly any base $\mathscr{V}$ for a uniformity will have the following properties:

1. $E_1, E_2 \in \mathscr{U} \implies$ there exists $D \in \mathscr{V}$ such that $D \subseteq E_1 \cap E_2$.

2. $E \in \mathscr{U} \implies D \circ D \subseteq E$ for some $D \in \mathscr{V}$.

3. $E \in \mathscr{U} \implies D^{-1} \subseteq E$ for some $D \in \mathscr{V}$.

If $\mathscr{U}$ is separating then $\mathscr{V}$ will satisfy $\bigcap_{E \in \mathscr{V}} E = \Delta$. A *subbase* for $\mathscr{D}$ is a subcollection such that the collection of all finite intersections from said subcollection form a base.

Remark 1.1.2. It is easy to see that the symmetric entourages of a uniformity $\mathscr{U}$ form a base for said uniformity.

For an entourage $E \in \mathscr{U}$ and a point $x \in X$ we define the set $B_E(x) = \{y \in X \mid (x, y) \in E\}$; we refer to this set as the *E-ball about x*. This naturally extends to a subset $A \subseteq X$; $B_E(A) = \bigcup_{x \in A} B_E(x)$; in this case we refer to the set $B_E(A)$ as the *E-ball about A*. We emphasise that (see [95, Section 35.6]):

- For all $x \in X$, the collection $\mathscr{B}_x := \{B_E(x) \mid E \in \mathscr{U}\}$ is a neighbourhood base at x, making X a topological space. The same topology is produced if any base $\mathscr{V}$ of $\mathscr{U}$ is used in place of $\mathscr{U}$.

- The topology is Hausdorff if and only if $\mathscr{U}$ is separating.

A topological space is said to be *Tychonoff*, or $T_{3\frac{1}{2}}$, if it is both Hausdorff and *completely regular* (i.e. points and closed sets can be separated by a bounded continuous real-valued function). A topological space is Tychonoff precisely when it admits a separating uniformity. Finally, we remark that for a compact Hausdorff space X there is a unique uniformity $\mathscr{U}$ which induces the topology and the space is metric if the uniformity has a countable base (see [39, Chapter 8]). For a metric space, a natural base for the uniformity would be the $1/2^n$ neighbourhoods of the diagonal.

1.1.4 Hyperspaces

For a uniform space $(X, \mathscr{U})$, we define the *hyperspace of compact sets*

$$2^X := \{A \subseteq X \mid A \text{ is compact and nonempty}\}.$$

Let $\mathscr{B}_{\mathscr{U}}$ be the family of all sets

$$2^V := \{(A, A') \mid A \subseteq B_V(A') \text{ and } A' \subseteq B_V(A)\},\ V \in \mathscr{U}.$$

The uniformity on the set 2^X generated by the base $\mathscr{B}_{\mathscr{U}}$ is denoted $2^{\mathscr{U}}$. If X is a compact Hausdorff space then 2^X forms a compact Hausdorff topological space with the topology, known as the Vietoris topology, induced by this uniformity [61, § 42]. If X is a compact metric space then 2^X is a compact metric space when equipped with the *Hausdorff metric*:

$$d_H(A, A') = \inf\{\varepsilon > 0 \colon A \subseteq B_\varepsilon(A') \text{ and } A' \subseteq B_\varepsilon(A)\}.$$

The topology generated by this metric is the Vietoris topology [68].

1.1.5 Shift spaces

Shift systems form an important class of systems in the theory of topological dynamics and will be important for some of our examples, particularly in chapters 2 and 5. An important class of such systems, called shifts of finite type, are precisely the shift systems which exhibit the shadowing property [93]. Recently, Good and Meddaugh [48] established that if a system (X, f) has shadowing, where X is a totally disconnected compact Hausdorff space, then the system is conjugate to an inverse limit of shifts of finite type, thus demonstrating that shifts of finite type are some of the fundamental objects in the theory of shadowing.

Given a finite set Σ considered with the discrete topology, *the one-sided full shift with alphabet* Σ consists of the set of infinite sequences in Σ, that is $\Sigma^{\mathbb{N}_0}$, which we consider with the product topology. This forms a dynamical system with the *shift map* σ, given by

$$\pi_i(\sigma(\mathbf{x})) = \pi_{i+1}(\mathbf{x}).$$

where π_i is the projection map for each i. (Thus, if $\mathbf{x} = \langle \mathtt{a}_i \rangle_{i \geq 0}$, so that $\pi_i(\mathbf{x}) = \mathtt{a}_i$, then $\sigma(\mathbf{x}) = \langle \mathtt{a}_{i+1} \rangle_{i \geq 0}$.) A *one-sided shift space* is a compact strongly invariant (under σ) subset of a one-sided full shift. A *word* in Σ is a finite sequence $\mathtt{a}_0\mathtt{a}_1 \ldots \mathtt{a}_m$, made up of elements of Σ. Given a word, $\mathtt{a}_0\mathtt{a}_1 \ldots \mathtt{a}_m$, we denote by $[\mathtt{a}_0\mathtt{a}_1 \ldots \mathtt{a}_m]$ the *cylinder set* induced by the word $\mathtt{a}_0\mathtt{a}_1 \ldots \mathtt{a}_m$; this is all points in X which begin with '$\mathtt{a}_0\mathtt{a}_1 \ldots \mathtt{a}_m$'. The collection of all cylinder sets intersected with X forms a base for the induced subspace topology from the Tychonoff product $\Sigma^{\mathbb{N}_0}$. For a symbol $\mathtt{a} \in \Sigma$, we use the notation $\mathtt{a}^n$, for some $n \in \mathbb{N}$, to mean

$$\underbrace{\mathtt{aaa\ldots a}}_{n \text{ times.}}$$

For a word W, we use $|W|$ to denote the length of W. So if $W = \mathtt{w}_0\mathtt{w}_1\mathtt{w}_2\ldots\mathtt{w}_n$, where $n \in \mathbb{N}_0$, then $|W| = n+1$. For the word W, we refer to the set $\{\mathtt{w}_k\mathtt{w}_{k+1}\ldots\mathtt{w}_{k+j} \mid 0 \le k \le n,\ 0 \le j \le n-k\}$ as the *set of all subwords of* W; the elements of this set are called *subwords* of W. We refer to any subword of the form $\mathtt{w}_0\mathtt{w}_1\ldots\mathtt{w}_k$, for some $k \le n$, as an *initial segment* of W. In similar fashion, if $\mathbf{x} = \langle \mathtt{a}_i \rangle_{i \ge 0} \in \Sigma^{\mathbb{N}_0}$ and $n \in \mathbb{N}_0$, we refer to $\mathtt{a}_0\mathtt{a}_1\ldots\mathtt{a}_n$ as an *initial segment* of $\mathbf{x}$ (of length $n+1$).

There are also two-sided forms of shift spaces, which, unlike the one-sided systems, are necessarily injective. The *two-sided full shift with alphabet* Σ consists of the set of bi-infinite sequences in Σ, that is $\Sigma^{\mathbb{Z}}$, which we consider with the product topology. As before, this forms a dynamical system with the *shift map* σ, which we define by saying that, for each $i \in \mathbb{Z}$,

$$\pi_i(\sigma(\mathbf{x})) = \pi_{i+1}(\mathbf{x}).$$

A *two-sided shift space* is some compact strongly invariant (under σ) subset of some two-sided full shift. If (X, σ) is a two-sided shift space and $\mathbf{x} = \langle \mathtt{a}_i \rangle_{i \in \mathbb{Z}} \in X$, so that $\pi_i(\mathbf{x}) = \mathtt{a}_i$, then we refer to the sequences $\langle \mathtt{a}_i \rangle_{i \ge 0}$ and $\langle \mathtt{a}_i \rangle_{i \le 0}$ as the *right-tail* and *left-tail* of $\mathbf{x}$ respectively. For $n \in \mathbb{N}_0$, we refer to the word $\mathtt{a}_{-n}\ldots\mathtt{a}_{-1}\mathtt{a}_0\mathtt{a}_1\ldots\mathtt{a}_n$ as a *central segment* of $\mathbf{x}$. When writing out $\mathbf{x}$ in full, we use a "$\cdot$" to indicate the position of the middle of the central segment:

$$\mathbf{x} = \ldots\mathtt{a}_{-3}\mathtt{a}_{-2}\mathtt{a}_{-1} \cdot \mathtt{a}_0\mathtt{a}_1\mathtt{a}_2\mathtt{a}_3\ldots.$$

Let Σ be a finite alphabet and let $\mathcal{F}$ be a finite set of words in Σ. The *one-sided shift of finite type associated with* $\mathcal{F}$ is the dynamical system $(X_{\mathcal{F}}, \sigma)$ where $X_{\mathcal{F}}$ is the set of all infinite sequences which do not contain any occurrence of any word from $\mathcal{F}$. The *two-sided shift of finite type associated with* $\mathcal{F}$ is the dynamical system $(Z_{\mathcal{F}}, \sigma)$ where $Z_{\mathcal{F}}$

is the set of all bi-infinite sequences which do not contain any occurrence of any word from $\mathcal{F}$. A shift space (X, σ) is said to be a *one-sided (resp. two-sided) shift of finite type* if there exists a finite set of words $\mathcal{F}$ such that $X = X_{\mathcal{F}}$ (resp. $X = Z_{\mathcal{F}}$).

The following theorems concerning ω-limit sets in shift spaces are folklore.

Theorem 1.1.3. *Let (X, σ) be a one-sided shift space. Let $\mathbf{x}, \mathbf{y} \in X$. Then $\mathbf{y} \in \omega(\mathbf{x})$ if and only if every initial segment of $\mathbf{y}$ occurs infinitely often in $\mathbf{x}$.*

Theorem 1.1.4. *Let (X, σ) be a two-sided shift space. Let $\mathbf{x}, \mathbf{y} \in X$. Then $\mathbf{y} \in \omega(\mathbf{x})$ if and only if every central segment of $\mathbf{y}$ occurs infinitely often in the right-tail of $\mathbf{x}$.*

For those wanting more information about shift systems, [35, Chapter 5] provides a thorough introduction to the topic.

1.2 The shadowing property

We close this chapter by introducing the shadowing property, from which this thesis in part takes its name.

Let $f \colon X \to X$ be a continuous map on a (typically compact) metric space X. A sequence (x_i) in X, which might be finite, infinite or bi-infinite, is called a *δ-pseudo-orbit* provided $d(f(x_i), x_{i+1}) < \delta$ for each i. Pseudo-orbits are obviously relevant when calculating an orbit numerically, as rounding errors mean a computed orbit will in fact be a pseudo-orbit. The (finite or infinite) sequence (y_i) in X is said to *ε-shadow* the (x_i) provided $d(y_i, x_i) < \varepsilon$ for all indices i. We then say that the system has *shadowing*, or *pseudo-orbit tracing*, if pseudo-orbits are shadowed by true orbits (see Definition 1.2.1 for the precise definition).

Whilst shadowing is clearly important when modelling a system numerically (for example [29, 80]), it has also been shown to have theoretical importance; for example, Bowen [17] used shadowing implicitly as a key step in his proof that the nonwandering set of an Axiom A diffeomorphism is a factor of a shift of finite type. Since then it has been studied extensively, in the setting of numerical analysis [29, 30, 80], as an important factor in stability theory [83, 86, 93], in understanding the structure of ω-limit sets and Julia sets [10, 11, 12, 18, 49, 70], and as a property in and of itself [31, 48, 63, 76, 81, 83, 88].

Various other notions of shadowing have since been studied including, for example, ergodic, thick, and Ramsey shadowing [19, 20, 34, 40, 78], limit shadowing [9, 52, 84], s-limit shadowing [9, 52, 63], orbital shadowing [47, 82, 84], and inverse shadowing [30, 50, 62]. Some of these properties will be examined in chapters 3, 4, and 5.

Definition 1.2.1. Let (X, f) be a dynamical system where (X, d) is a metric space. A sequence $(x_i)_{i\in\omega}$ in X is said to be a *δ-pseudo-orbit* for some $\delta > 0$ if $d(f(x_i), x_{i+1}) < \delta$ for each $i \in \omega$. A point $z \in X$ is said to *ε-shadow* a sequence $(x_i)_{i\in\omega}$ for some $\varepsilon > 0$ if $d(x_i, f^i(z)) < \varepsilon$ for each $i \in \omega$. The dynamical system (X, f) is said to have *shadowing*, or *pseudo-orbit tracing*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed.

We will see in Chapter 3 that appropriate generalisations of the shadowing property (and its variants) may be given for compact Hausdorff spaces using the unique uniformity (see 1.1.3) generating the topology on X.

CHAPTER 2

EQUICONTINUITY, TRANSITIVITY AND SENSITIVITY: THE AUSLANDER–YORKE DICHOTOMY REVISITED

Recently there has been a move towards studying dynamical systems without assuming the underlying phase space is necessarily metric or compact (see, for example, [4, 25, 45, 51, 55, 71, 75, 96, 97]). This chapter forms part of that project, extending the ideas developed by Good and Macías [45] and addressing a question raised by the referee of that paper asking about the Auslander–Yorke dichotomy from a topological point of view.

Let (X, f) be a discrete dynamical system, so that $f\colon X \to X$ is a (continuous) map on the metric space X. The dynamical system is *equicontinuous at a point* $x \in X$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that the δ-ball around x does not expand to more than diameter ε under iteration of f. The system itself is said to be *equicontinuous* if it is equicontinuous at every point. Compactness of the space X ensures that equicontinuity is equivalent to uniform equicontinuity: for any $\varepsilon > 0$ there is a $\delta > 0$ such that no δ-ball expands to more than diameter ε under iteration of f. Equicontinuity is extremely important in mathematical analysis where it provides the primary condition in the Arzelà–Ascoli theorem (see [39, Theorem 8.2.10]). A related concept to equicontinuity is that of sensitivity. The system (X, f) is *sensitive* there is a real number $\delta > 0$ such that every nonempty open set expands to at least diameter δ under iteration of f. Clearly the properties of sensitivity and equicontinuity are mutually exclusive. Examining the quantifiers one sees that sensitivity is *almost* a negation of equicontinuity. Indeed, negating the property of equicontinuity at a given point gives a localised version of sensitivity. Auslander and Yorke [5] specify a type of system for which sensitivity is precisely the negation of equicontinuity: a dynamical system (X, f) is said to be *minimal* if the forward orbit of every point is dense in the space. The Auslander–Yorke dichotomy states that a compact metric minimal system is either equicontinuous or sensitive. Various analogues of this theorem have since been offered [57].

Topological transitivity, or simply transitivity, is a weakening of minimality. The system (X, f) is said to be *transitive* if for any nonempty open sets U and V there is an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Under certain conditions (compact metric being sufficient) this is equivalent to the existence of a *transitive point* (i.e. a point with a dense orbit) [3]. Transitivity and sensitivity are often cited as two key ingredients for a system to be chaotic (see, for example, [5, 36]). The former prevents the system from being decomposed into multiple invariant open sets (and thereby studied as a collection of subsystems). The latter brings an element of unpredictability to the system; a small error in initial conditions may be exacerbated over time. This is clearly of particular importance in an applied setting where there is almost always going to be an error in one's measurements and computations. In his definition of chaos, along with these two properties, Robert Devaney [36, pp. 50] included the condition that the set of periodic points be dense in whole space, thus providing "an element of regularity" amid seemingly random behaviour. Perhaps surprisingly, this regularity condition together with transitivity proved sufficient in a compact space to entail sensitivity [7, 42]. Since then, the gap between transitivity and sensitivity has been researched extensively (see, for example, [2, 45, 64, 73]); Akin *et al.* [2] gave the following dichotomy: a compact metric transitive system is either sensitive or contains a point of equicontinuity; in 2007, Moothathu [73] generalised results in [7] and [2] by defining stronger notions of sensitivity. These variations on sensitivity have since attracted an array of interest [57, 66, 89, 94].

For a survey on recent developments in the theory of chaotic systems, including results on sensitivity, equicontinuity, and transitivity, see [65].

In [45] the authors introduce what they term *Hausdorff sensitivity* of a system $f\colon X \to X$, where f is a continuous map on a topological space, showing that this coincides with the usual notion of sensitivity if the phase space is compact metric. Topological equicontinuity, introduced by Royden [87, pp. 362], is, in general, weaker than equicontinuity. The concept of even continuity, introduced by Kelley [59, pp. 234], dates back further than topological equicontinuity and is even weaker still, although all three concepts (i.e. equicontinuity, topological equicontinuity, and even continuity) coincide in the presence of compactness (see [59, Theorem 7.23]). In contrast to equicontinuity, which is an inherently uniform concept, neither topological equicontinuity nor even continuity requires the phase space to be anything more than a topological space. Whilst the concepts of topological equicontinuity and even continuity have gained some attention with regard to topological semigroups and families of mappings in a general setting (e.g. [27, 28]), little appears to have been done with regard to dynamical systems.

In this chapter we take a careful look at the Auslander–Yorke dichotomy via a topological approach which leads to some interesting results: After the preliminaries in section 2.1, we build up some theory related to topological equicontinuity in dynamical systems in section 2.2. Two fruits of this theory are Corollary 2.2.25 — a generalisation of the Auslander–Yorke dichotomy — along with an exposition, with regard to topological equicontinuity, of when a system is transitive (Theorem 2.2.10). Section 2.3 starts by building up theory regarding even continuity in dynamical systems. This section culminates in a construction of a compact topologically transitive system with an even continuity pair but no point of even continuity; this provides an element of regularity in a system which is Auslander–Yorke chaotic, densely and strongly Li–Yorke chaotic, but not Devaney chaotic. In section 2.4 we discuss a property we call *splitting* and its relationship to topological equicontinuity, even continuity and existing notions of chaos. Finally, in section 2.6 we give a dichotomy for compact Hausdorff transitive systems (Theorem 2.6.3); they are either equicontinuous or *eventually sensitive*.

Throughout this chapter X is a topological space. Usually it is assumed to be Hausdorff, while some results rely on the additional assumption of compactness. We will always state the relevant assumptions.

2.1 Preliminaries

Let (X, f) be a dynamical system, where X is a Hausdorff space. For any $x \in X$ we denote the set of neighbourhoods of x by $\mathcal{N}_x$; the elements of this set are not assumed to be open. A point x is said to be *recurrent* if $x \in \omega(x)$. It is said to be *non-wandering* if, for any neighbourhood $U \in \mathcal{N}_x$ and any $N \in \mathbb{N}$ there is $n > N$ such that $f^n(U) \cap U \neq \emptyset$. Clearly a recurrent point is non-wandering. We define the *non-wandering set* of x, denoted $\Omega_f(x)$, or simply $\Omega(x)$ where there is no ambiguity, by saying that $y \in \Omega_f(x)$ if and only if for any $V \in \mathcal{N}_y$, any $U \in \mathcal{N}_x$ and any $N \in \mathbb{N}$ there exists $n > N$ such that $f^n(U) \cap V \neq \emptyset$. It follows that, for any $x \in X$, $\omega(x) \subseteq \Omega(x)$.

When X is a compact Hausdorff space we will denote the unique uniformity associated with X by $\mathscr{U}_X$ or usually simply $\mathscr{U}$ if there is no ambiguity. Given $A, B \subseteq X$, we denote by $N(A, B)$ the (forward) hitting times of A on B under f; specifically

$$N(A, B) = \{n \in \mathbb{N} \mid f^n(A) \cap B \neq \emptyset\}. \tag{2.1}$$

If $x \in X$ and $B \subseteq X$, we will abuse notation by writing $N(x, B)$ instead of $N(\{x\}, B)$. A dynamical system (X, f) is *topologically transitive*, or simply *transitive*, when, for any pair of nonempty open sets U and V, $N(U, V) \neq \emptyset$. It is *weakly mixing* if the product system $(X \times X, f \times f)$ is transitive. A point $x \in X$ is said to be a *transitive point* if $\omega(x) = X$. (NB. Some literature defines a transitive point as a point whose orbit is dense in the space. This is, in general, a weaker condition. However these two versions coincide for a point x if either $x \in f(X)$ or x is not isolated in X (see, for example, [35, Corollary 1.3.3]).) A system (X, f) is said to be minimal if $\omega(x) = X$ for all $x \in X$; equivalently, if there are no proper, nonempty, closed, positively-invariant subsets of X. (A subset $A \subseteq X$ is said to be *positively invariant* (under f) if $f(A) \subseteq A$.)

In [3] the authors introduce the concept of a *density basis*; a density basis for a topological space X is a collection $\mathcal{V}$ of nonempty open sets in X such that if $A \subseteq X$ is such that $A \cap V \neq \emptyset$ for any $V \in \mathcal{V}$, then $\overline{A} = X$. They go on to show that if X is of Baire second category (i.e. nonmeagre) and has a countable density basis then topological transitivity is equivalent to the existence of a transitive point. Topologists may be more familiar with the concept of a π-*base* than a density basis.

Definition 2.1.1. A π-*base* for a topological space X is a collection $\mathcal{U}$ of nonempty open sets in X such that if R is any nonempty open set in X then there exists $V \in \mathcal{U}$ such that $V \subseteq R$.

Proposition 2.1.2. *Let X be a topological space. A collection is a π-base if and only if it is a density basis.*

Proof. Note first that both are defined as collections of nonempty open sets.

Suppose $\mathcal{U}$ is a π-base. Suppose $A \subseteq X$ is such that $A \cap U \neq \emptyset$ for all $U \in \mathcal{U}$. Let W be open and nonempty. Then there exists $U \in \mathcal{U}$ such that $U \subseteq W$. Then $A \cap U \neq \emptyset$; therefore $A \cap W \neq \emptyset$ and so $\overline{A} = X$.

Now suppose $\mathcal{U}$ is a density basis. Assume $\mathcal{U}$ is not a π-base. Then there exists a nonempty open set W such that $U \not\subseteq W$ for any $U \in \mathcal{U}$. This means that $U \setminus W \neq \emptyset$ for any $U \in \mathcal{U}$. Take

$$A = \bigcup_{U \in \mathcal{U}} U \setminus W.$$

It follows that $A \cap W = \emptyset$ and, for each $U \in \mathcal{U}$, $A \cap U \neq \emptyset$. Since $\mathcal{U}$ is a density basis the latter entails $\overline{A} = X$, contradicting the fact that $A \cap W = \emptyset$. Hence $\mathcal{U}$ is a π-base. $\square$

The following lemma is folklore (e.g. [3]) and will be useful throughout. We include a proof for completeness.

Lemma 2.1.3. *Let (X, f) be a dynamical system, where X is a topological space. Then (X, f) is topologically transitive if and only if $N(U, V)$ is infinite for any pair of nonempty open sets U and V.*

Proof. Suppose (X, f) is topologically transitive and let U and V be nonempty open sets. Pick $n \in N(U, V)$. We will show that there exists $l \in N(U, V)$ with $l > n$. Let $W := U \cap f^{-n}(V)$. Then $W \neq \emptyset$. Pick $m \in N(U, W)$. Let $a \in U$ be such that $f^m(a) \in W$. Then $f^{m+n}(a) \in V$. Hence $l = m + n \in N(U, V)$.

The converse is immediate from the definition of transitivity. □

Remark 2.1.4. It follows from Lemma 2.1.3 that, for a transitive system (X, f) where X is a Hausdorff space, we have $\Omega(x) = X$ for any $x \in X$.

For the rest of this section $(X, \mathscr{U})$ is a separated uniform space.

We say the system (X, f) is *uniformly rigid* if for any $E \in \mathscr{U}$ there exists $n \in \mathbb{N}$ such that, for any $x \in X$, $(x, f^n(x)) \in E$. Clearly every point in a uniformly rigid system is recurrent.

Let $U \subseteq X$ and let $D \in \mathscr{U}$ be symmetric. Define

$$N_D(U) = \{n \in \mathbb{N} \mid \exists x, y \in U \text{ such that } (f^n(x), f^n(y)) \notin D\}. \tag{2.2}$$

We say the system (X, f) is *sensitive* if there exists a symmetric entourage $D \in \mathscr{U}$ such that $N_D(U) \neq \emptyset$ for any nonempty open $U \subseteq X$. In this case we say D is a sensitivity entourage (X, f). If X is a metric space, for $U \subseteq X$ and $\delta > 0$ we define

$$N_\delta(U) = \{n \in \mathbb{N} \mid \exists x, y \in U \text{ such that } d(f^n(x), f^n(y)) \geq \delta\}. \tag{2.3}$$

In this case we say the system is sensitive if there exists $\delta > 0$ such that $N_\delta(U) \neq \emptyset$ for any nonempty open set U. The definitions for a metric space coincide when it is equipped with the metric uniformity (see [45]). We invite readers unfamiliar with uniformities to notice the similarities in these definitions; it may be helpful for such readers to view the statement, “there exists $D \in \mathscr{U}$ such that $(x, y) \in D$,” as, “there exists $\delta > 0$ such that $d(x, y) < \delta$”. Similarly “$(x, y) \notin D$” may be read as “$d(x, y) \geq \delta$”. In this way, $B_D(x)$ may be thought of as $B_\delta(x)$. The uniform structure of a space can be used to mimic existing metric proofs (see, for example, [45]). In the proof of the following lemma, which is folklore, we invite the reader to observe how entourages have simply replaced the real numbers which would have designated distances for a metric version.

Lemma 2.1.5. *If $(X, \mathscr{U})$ is a separated uniform space and (X, f) is a sensitive dynamical system, with sensitivity $D \in \mathscr{U}$, then for any nonempty open $U \subseteq X$ the set $N_D(U)$ is infinite.*

Proof. Let $U \subseteq X$ be nonempty open and suppose $N_D(U)$ is finite; let $k \in \mathbb{N}$ be an upper bound for this set. Let $E \in \mathscr{U}$ be such that $2E \subseteq D$. Let $x \in U$. By continuity we may choose a symmetric entourage $D_0 \in \mathscr{U}$ such that, for any $y \in X$, if $(x, y) \in D_0$ then $(f^i(x), f^i(y)) \in E$ for all $i \in \{1, \ldots, k\}$. Consider the set $W := U \cap D_0[x]$; W is a neighbourhood of x. Thus $N_D(W) \neq \emptyset$ by sensitivity, but $f^i(W) \subseteq B_E(f^i(x))$ for $i \in \{1, \ldots, k\}$; in particular if $y, z \in W$ then $(f^i(y), f^i(z)) \in D$ for $i \in \{1, \ldots, k\}$. Therefore there exists $n > k$ and $y, z \in W$ such that $(f^n(y), f^n(z)) \notin D$. As $W \subseteq U$ we have a contradiction and the result follows. $\square$

A point $x \in X$ is said to be an *equicontinuity point* of the system (X, f) if

$$\forall E \in \mathscr{U}\ \exists D \in \mathscr{U} : \forall n \in \mathbb{N},\ y \in B_D(x) \implies f^n(y) \in B_E(f^n(x)). \tag{2.4}$$

In this case we say (X, f) is *equicontinuous at x*. If (X, f) is equicontinuous at every $x \in X$ then we say the system itself is *equicontinuous*. When X is compact this is equivalent to the system being *uniformly equicontinuous*, that is

$$\forall E \in \mathscr{U}\ \exists D \in \mathscr{U} : \forall n \in \mathbb{N},\ (x, y) \in D \implies (f^n(x), f^n(y)) \in E. \tag{2.5}$$

We denote the set of all equicontinuity points by $\mathrm{Eq}(X, f)$, so a system is equicontinuous if $\mathrm{Eq}(X, f) = X$.

The following results will be useful; versions for compact metric systems may be found in [2], the proofs of which may be mimicked to give the following more general versions.

Lemma 2.1.6. [2] *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. If $x \in \mathrm{Eq}(X, f)$ then $\omega_f(x) = \Omega_f(x)$.*

Theorem 2.1.7. [2] *Let (X, f) be a transitive dynamical system where X is a compact Hausdorff space. If $\mathrm{Eq}(X, f) \neq \emptyset$ then the set of equicontinuity points coincide with the set of transitive points.*

Corollary 2.1.8. *Let (X, f) be a transitive dynamical system where X is a compact Hausdorff space. If X is not separable then $\mathrm{Eq}(X, f) = \emptyset$.*

Proof. If $\mathrm{Eq}(X, f) \neq \emptyset$ then every equicontinuity point is a transitive point. If the system has a transitive point then it has a countable dense subset and is thereby separable. □

We end this section of the preliminaries with three common notions of chaos. A dynamical system is said to be *Auslander–Yorke chaotic* (see [5]) if it is both transitive and sensitive. If, in addition, it has a dense set of periodic points it is said to be *Devaney chaotic* (see [36]). It is a well-known result in dynamics that, in a compact metric setting, if a system is transitive and has a dense set of periodic points then it is also sensitive [7]. This result also holds in the compact Hausdorff setting [45].

If X is a metric space and (X, f) a dynamical system, then we say a pair $(x, y) \in X \times X$ is *proximal* if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$

and *asymptotic* if

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

The pair (x, y) is said to be a *Li–Yorke pair* if they are proximal but not asymptotic. It is said to be a *strong Li–Yorke pair* if it is both a Li–Yorke pair and recurrent in the product system $(X^2, f \times f)$. A set $S \subseteq X$ is said to be *scrambled* if every pair of distinct points in

S form a Li–Yorke pair; it is said to be *strongly scrambled* if every pair of distinct points in S form a strong Li–Yorke pair. A system (X, f) is said to be *Li–Yorke chaotic* (see [67]) if there exists an uncountable scrambled set S. If S is strongly scrambled we say (X, f) is *strongly Li–Yorke chaotic*. Finally if S is dense in X then we say the system is *densely Li–Yorke chaotic* [35, Section 7.3].

2.1.1 A note on taking a topological approach to dynamical systems

Before we proceed a final remark is in order about taking a topological approach to dynamical systems. When seeking to define appropriate topological versions of metric definitions one cannot always ensure they coincide in an arbitrary metric setting. To take an elementary example, consider the dynamical system $f\colon (0,\infty) \to (0,\infty)\colon x \mapsto 2x$. Equipped with the Euclidean metric, this system is sensitive. However, a topologically equivalent metric is the following $d_0(x,y) := |{}^1/_x - {}^1/_y|$. Equipped with this metric the system is equicontinuous.[1] Since sensitivity and equicontinuity are mutually exclusive in a metric setting, we cannot expect topological variants of these notions to coincide with the metric versions in this instance. As we shall see, however, the presence of compactness is often enough for topological definitions to be equivalent to their metric (resp. uniform) cousins when the space is metrisable (resp. uniformisable).

2.2 Topological equicontinuity and the Auslander–Yorke dichotomy

Previously we defined equicontinuity for dynamical systems where the phase space is separated uniform. More generally [95], if X is any topological space and Y a uniform space, we say that a family $\mathscr{F}$ of continuous functions from X to Y is *equicontinuous at* $x \in X$ if for each $E \in \mathscr{U}_Y$ there exists $U \in \mathcal{N}_x$ such that, for each $f \in \mathscr{F}$, $f(U) \subseteq$

[1] We remark that (X, f) with the Euclidean metric is conjugate to (X, f^{-1}) equipped with d_0 via the homeomorphism given by $x \mapsto 1/x$.

$B_E(f(x))$. We say $\mathscr{F}$ is *equicontinuous* provided it is equicontinuous at each point of X. To generalise this to arbitrary spaces, Royden [87] presents the following concept of *topological equicontinuity.* If X and Y are topological spaces we say a collection of maps $\mathscr{F}$ from X to Y is *topologically equicontinuous* at an ordered pair $(x, y) \in X \times Y$ if for any $O \in \mathcal{N}_y$ there exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that, for any $f \in \mathscr{F}$, if $f(U) \cap V \neq \emptyset$ then $f(U) \subseteq O$; when this is the case we refer to (x, y) as an *equicontinuity pair.* We say $\mathscr{F}$ is *topologically equicontinuous* at a point $x \in X$ if it is topologically equicontinuous at (x, y) for all $y \in Y$. We say the collection is *topologically equicontinuous* if it is topologically equicontinuous at every $x \in X$. If (x, y) is an equicontinuity pair then we will say y is an *equicontinuity partner* of x.

Topological equicontinuity and the usual notion of equicontinuity coincide when Y is a compact Hausdorff space.

Theorem 2.2.1. [87, pp. 364] *Let X and Y be topological spaces, with $\mathscr{F}$ a collection of continuous functions from X to Y. Let $x \in X$. If Y is a separated uniform space and $\mathscr{F}$ is equicontinuous at x with respect to some compatible uniformity then $\mathscr{F}$ is topologically equicontinuous at x. If Y is a compact Hausdorff space then the collection $\mathscr{F}$ is equicontinuous at $x \in X$ if and only if it is topologically equicontinuous at x.*

If (X, f) is a dynamical system, we will denote the set of equicontinuity pairs by $\mathrm{EqP}(X, f)$. Note that in this case, if we consider the above definitions, $Y = X$ and $\mathscr{F} = \{f^n \mid n \in \mathbb{N}\}$. By definition it follows that (X, f) is topologically equicontinuous precisely when $\mathrm{EqP}(X, f) = X \times X$. For $(x, y) \in \mathrm{EqP}(X, f)$, we refer to the condition

$$\forall O \in \mathcal{N}_y\, \exists U \in \mathcal{N}_x\, \exists V \in \mathcal{N}_y : \forall n \in \mathbb{N}, f^n(U) \cap V \neq \emptyset \implies f^n(U) \subseteq O, \tag{2.6}$$

as the *topological equicontinuity condition* for x and y. We say that U and V, as in Equation 2.6, satisfy the topological equicontinuity condition for x, y and O.

The following simple observation relies solely on continuity and will be useful throughout what follows.

Lemma 2.2.2. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Let $x, y \in X$ and $n \in \mathbb{N}$. Pick $O \in \mathcal{N}_y$ and let*

$$S = \{k \in \{1, \dots, n\} \mid f^k(x) = y\}.$$

There exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $N(U, V) \cap \{1, \dots, n\} = S$ and $f^k(U) \subseteq V \subseteq O$ for all $k \in S$.

Proof. Let $S = \{k \in \{1, \dots, n\} \mid f^k(x) = y\}$ (this set may be empty). For all $i \in \{1, \dots, n\} \setminus S$, let $U_i \in \mathcal{N}_{f^i(x)}$ and $V_i \in \mathcal{N}_y$ be such that $U_i \cap V_i = \emptyset$. Define

$$V := \left(\bigcap_{i \in \{1, \dots, n\} \setminus S} V_i \right) \cap O.$$

Then $V \in \mathcal{N}_y$. Now take

$$U := \left(\bigcap_{i \in \{1, \dots, n\} \setminus S} f^{-i}(U_i) \right) \cap \left(\bigcap_{i \in S} f^{-i}(V) \right).$$

Notice $U \in \mathcal{N}_x$.

By construction, $N(U, V) \cap \{1, \dots, n\} = S$ and $f^k(U) \subseteq V \subseteq O$ for all $k \in S$. □

In particular Lemma 2.2.2 shows that any pair $(x, y) \in X \times X$, satisfy the following weakened version of the topological equicontinuity condition (Equation 2.6).

Corollary 2.2.3. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Let $x, y \in X$ and $n \in \mathbb{N}$. Then for any $O \in \mathcal{N}_y$ there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that, for any $k \in \{1, \dots, n\}$,*

$$f^k(U) \cap V \neq \emptyset \implies f^k(U) \subseteq O.$$

Proof. Immediate from Lemma 2.2.2. □

If (X, f) is a Hausdorff dynamical system and the points $x, y \in X$ are such that there exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that, for all $n \in \mathbb{N}$, $f^n(U) \cap V = \emptyset$ then $(x, y) \in \mathrm{EqP}(X, f)$; this is vacuously true. The following result adds to this.

Proposition 2.2.4. *Let (X, f) be a dynamical system where X is Hausdorff space. Let $x, y \in X$ and suppose that $y \notin \Omega(x)$. Then $(x, y) \in \mathrm{EqP}(X, f)$.*

Proof. Let $O \in \mathcal{N}_y$. Take $U \in \mathcal{N}_x$, $V \in \mathcal{N}_y$ and $N \in \mathbb{N}$ such that $f^n(U) \cap V = \emptyset$ for all $n > N$. By Corollary 2.2.3, there exist U' and V' such that, for any $k \in \{1, \dots, N\}$, if $f^k(U') \cap V' \neq \emptyset$ then $f^k(U') \subseteq O$; without loss of generality $U' \subseteq U$ and $V' \subseteq V \cap O$. Then, since $f^n(U') \cap V' = \emptyset$ for all $n > N$, U' and V' satisfy the topological equicontinuity condition for x, y and O. As $O \in \mathcal{N}_y$ was picked arbitrarily the result follows. □

With this in mind we make the following definition.

Definition 2.2.5. If (X, f) is a dynamical system, where X is a Hausdorff space, then we say $(x, y) \in X \times X$ is a *trivial equicontinuity pair* if $y \notin \Omega(x)$.

Remark 2.2.6. Proposition 2.2.4 tells us that a trivial equicontinuity pair is indeed an equicontinuity pair.

Generally, in a non-compact separated uniform space, topologically equicontinuity, whilst clearly necessary for equicontinuity (Theorem 2.2.1), is not sufficient for equicontinuity; it is a strictly weaker property. Example 2.2.7 shows this. First, recall that a metric system (X, f) is said to be *expansive* if there exists $\delta > 0$ such that for any x and y, with $x \neq y$, there exists $k \in \mathbb{N}_0$ such that $d(f^k(x), f^k(y)) \geq \delta$. It is easy to see that if X is perfect (i.e. without isolated points) then expansivity implies sensitivity.

Example 2.2.7. Consider the dynamical system $(X = \mathbb{R} \setminus \{0\}, f)$, where $f(x) = 2x$. Using Proposition 2.2.4 it can be verified that $\mathrm{EqP}(X, f) = X \times X$, hence the system is topologically equicontinuous. However this system is not only sensitive but it is also expansive. Each of these properties (the latter, since X is perfect) are mutually exclusive with the existence of an equicontinuity point, thus $x \notin \mathrm{Eq}(X, f)$ for any $x \in X$.

Lemma 2.2.8. *Let (X, f) be a dynamical system, where X is a Hausdorff space. If $(x, y) \in \mathrm{EqP}(X, f)$ then either (x, y) is a trivial equicontinuity pair or $y \in \omega(x)$.*

Proof. Suppose (x, y) is a non-trivial equicontinuity pair (otherwise we are done). Now suppose $y \notin \omega(x)$; then there exists $O \in \mathcal{N}_y$ and $N \in \mathbb{N}$ such that for all $n > N$ we have $f^n(x) \notin O$. Since (x, y) is a non-trivial pair, for any neighbourhoods U and V of x and y respectively, the set $N(U, V)$ is infinite. Pick $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ and let $n > N$ be such that $f^n(U) \cap V \neq \emptyset$. Then, as $f^n(x) \notin O$, $f^n(U) \nsubseteq O$. As U and V were arbitrary neighbourhoods this contradicts the fact that $(x, y) \in \mathrm{EqP}(X, f)$. □

This means that a pair (x, y) is a non-trivial equicontinuity pair if and only if it is an equicontinuity pair and $y \in \omega(x)$.

The statement $(x, y) \notin \mathrm{EqP}(X, f)$, for $x, y \in X$, means precisely

$$\exists O \in \mathcal{N}_y : \forall U \in \mathcal{N}_x \, \forall V \in \mathcal{N}_y \, \exists n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset \text{ and } f^n(U) \nsubseteq O. \tag{2.7}$$

In particular, for any pair of neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, we have that U meets V after some number of iterations of f. If $N(U, V)$ were finite then $(x, y) \in \mathrm{EqP}(X, f)$ by Proposition 2.2.4 (it would be a trivial equicontinuity pair). Thus $N(U, V)$ is infinite. By definition this means that $y \in \Omega(x)$. (NB. We shall refer to a neighbourhood such as O in Equation (2.7) as a *splitting neighbourhood* of y with regard to x.) This leads us to the following generalisation of Lemma 2.1.6.

Lemma 2.2.9. *Let (X, f) be a dynamical system where X is a Hausdorff space. If (X, f) is topologically equicontinuous at $x \in X$ then $\omega(x) = \Omega(x)$.*

Proof. Since $\omega(x) \subseteq \Omega(x)$ it suffices to show that $\Omega(x) \subseteq \omega(x)$. Pick $y \in \Omega(x)$. It follows that (x, y) is a non-trivial equicontinuity pair. Hence $y \in \omega(x)$ by Lemma 2.2.8. □

We are now in a position to characterise transitive dynamical systems on Hausdorff spaces purely with reference to equicontinuity pairs.

Theorem 2.2.10. *Let X be a Hausdorff space, and let $f\colon X \to X$ be a continuous function. Then (X, f) is a transitive dynamical system if and only if there are no trivial equicontinuity pairs.*

Proof. Suppose first that (X, f) is transitive. Let $(x, y) \in X \times X$ be given and let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$. By transitivity, $N(U, V)$ is infinite (see Lemma 2.1.3). Since U and V were arbitrary neighbourhoods it follows that (x, y) is not a trivial equicontinuity pair.

Now suppose (X, f) has no trivial equicontinuity pairs and let U and V be nonempty open sets. Pick $x \in U$ and $y \in V$; (x, y) is a not a trivial equicontinuity pair. If $(x, y) \in \mathrm{EqP}(X, f)$ then, by Lemma 2.2.8, $y \in \omega(x)$ from which is follows that $N(U, V) \neq \emptyset$. If $(x, y) \notin \mathrm{EqP}(X, f)$ then by Equation (2.7) there exists $n \in N(U, V)$. In every case, $N(U, V) \neq \emptyset$ and we have transitivity. □

The following corollary is a direct consequence of putting Lemma 2.2.8 and Theorem 2.2.10 together.

Corollary 2.2.11. *Let X be a Hausdorff space and (X, f) be a transitive dynamical system. If $(x, y) \in \mathrm{EqP}(X, f)$ then $y \in \omega(x)$.*

We now construct a class of examples which have no isolated points and non-trivial equicontinuity pairs but no points of topological equicontinuity. The information provided on shift spaces in section 1.1.5 will be of relevance here.

Example 2.2.12. Take $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \dots, \mathtt{m}\}$, where $\mathtt{m} \geq 2$. For each $k \in \mathbb{N}$, let W_k represent a word of length k containing only the symbols $\{\mathtt{1}, \mathtt{2}, \dots, \mathtt{m}\}$. Let $\mathcal{W}$ be the collection of all sequences of the form:

$$W_1 \mathtt{0} W_2 \mathtt{0}^2 W_3 \mathtt{0}^3 \dots \mathtt{0}^{n-1} W_n \mathtt{0}^n \dots,$$

Now take

$$Y = \mathcal{W} \cup \{\mathtt{0}^n \mathbf{x} \mid \mathbf{x} \in \mathcal{W}, n \in \mathbb{N}\} \cup \{\mathtt{0}^\infty\},$$

and let

$$X := \overline{\{\sigma^k(\mathbf{y}) \mid \mathbf{y} \in Y, k \in \mathbb{N}_0\}},$$

where the closure is taken with regard to the full shift $\Sigma^{\mathbb{N}_0}$. It is worth observing that the

ω-limit sets of points in $\mathcal{W}$ are points of the following forms:

$$\mathtt{0}^\infty \text{ and } W_k\mathtt{0}^\infty.$$

Notice that, for any $\mathbf{x} \in \mathcal{W}$, $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, $\sigma^k(\mathbf{x}) \in [\mathtt{0}^n]$ if and only if, for all $\mathbf{y} \in \mathcal{W}$, $\sigma^k(\mathbf{y}) \in [\mathtt{0}^n]$. With this observation in mind, we claim that if $\mathbf{x} \in \mathcal{W}$, then $(\mathbf{x}, \mathtt{0}^\infty) \in \mathrm{EqP}(X, \sigma)$. Indeed, pick such an $\mathbf{x}$; write $\mathbf{x} = W_1\mathtt{0}W_2\mathtt{0}^2W_3\mathtt{0}^3\ldots$. Now let $O \ni \mathtt{0}^\infty$ be open. Let $V = [\mathtt{0}^n] \subseteq O$ and take $U = [W_1\mathtt{0}W_2]$. If $\mathbf{y} \in U$ then $\mathbf{y} \in \mathcal{W}$ by construction. But by our observation, if $\sigma^k(\mathbf{y}) \in V$ then $\sigma^k(\mathcal{W}) \subseteq V \subseteq O$. Hence $(\mathbf{x}, \mathtt{0}^\infty) \in \mathrm{EqP}(X, \sigma)$.

It remains to observe that $\mathrm{Eq}(X, \sigma) = \emptyset$, because shift systems, with no isolated points, are sensitive. By Theorem 2.2.1, this means there are no points of topological equicontinuity.

Example 2.2.12 demonstrates that, even in a compact metric setting, a point may have non-trivial equicontinuity partners but not be a point of equicontinuity.

We will now build up some results relating to equicontinuity pairs in dynamical systems, this will culminate in a generalisation of the Auslander–Yorke dichotomy.

Lemma 2.2.13. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Let $x, y \in X$. If $(x, y) \in \mathrm{EqP}(X, f)$, f is open at y, and there is a neighbourhood base for y, $\mathcal{B}_y \subseteq \mathcal{N}_y$, such that $f^{-1}(f(O)) = O$ for all $O \in \mathcal{B}_y$, then $(x, f(y)) \in \mathrm{EqP}(X, f)$.*

Proof. Let $O \in \mathcal{N}_{f(y)}$. Then $f^{-1}(O) \in \mathcal{N}_y$. Let $O' \in \mathcal{B}_y$ be such that $f(O') \subseteq O$. Notice that, since f is open at y, $f(O') \in \mathcal{N}_{f(y)}$. Since $(x, y) \in \mathrm{EqP}(X, f)$ there exist $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ satisfying the topological equicontinuity for x, y and O'; without loss of generality $V \subseteq O'$ and $V \in \mathcal{B}_y$. If $x \neq y$ then, without loss of generality, $U \cap V = \emptyset$. If $x = y$ then, without loss of generality $U = V$. Because f is open at y, $f(V) \in \mathcal{N}_{f(y)}$. For any $n \in \mathbb{N}$, if $f^n(U) \cap f(V) \neq \emptyset$ then $f^{n-1}(U) \cap f^{-1}(f(V)) \neq \emptyset$. Because $V \in \mathcal{B}_y$ we have $f^{-1}(f(V)) = V$, hence $f^{n-1}(U) \cap V \neq \emptyset$. If $n = 1$ then it follows that $U = V$

and so $U \subseteq O'$. This itself implies $f(U) \subseteq f(O') \subseteq O$. If $n > 1$ then $f^{n-1}(U) \subseteq O'$ by topological equicontinuity at x and y. This implies $f^n(U) \subseteq f(O') \subseteq O$. □

Corollary 2.2.14. *Let (X, f) be a dynamical system, where X is a Hausdorff space. If f is a homeomorphism and $(x, y) \in \mathrm{EqP}(X, f)$ then $(x, f(y)) \in \mathrm{EqP}(X, f)$.*

Proof. Immediate from Lemma 2.2.13. □

Lemma 2.2.15. *Let (X, f) be a dynamical system, where X is Hausdorff space. Suppose $(x, y) \notin \mathrm{EqP}(X, f)$ and let O be a splitting neighbourhood of y with regard to x. Then, for any pair of neighbourhoods U and V of x and y respectively, the set of natural numbers n for which $f^n(U) \cap V \neq \emptyset$ and $f^n(U) \not\subseteq O$ is infinite.*

Proof. The proof is similar to that of Proposition 2.2.4.

Let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$. Take

$$A = \{n \in \mathbb{N} \mid f^n(U) \cap V \neq \emptyset \text{ and } f^n(U) \not\subseteq O\}.$$

Suppose that A is finite; note that $A \neq \emptyset$ as $(x, y) \notin \mathrm{EqP}(X, f)$. Let N be the largest element in A. By Corollary 2.2.3, there exist $U' \in \mathcal{N}_x$ and $V' \in \mathcal{N}_y$ such that, for any $k \in \{1, \ldots, N\}$, if $f^k(U') \cap V' \neq \emptyset$ then $f^k(U') \subseteq O$; without loss of generality $U' \subseteq U$ and $V' \subseteq V$. But as $(x, y) \notin \mathrm{EqP}(X, f)$ we have $A' \neq \emptyset$, where

$$A' = \{n \in \mathbb{N} \mid f^n(U') \cap V' \neq \emptyset \text{ and } f^n(U') \not\subseteq O\}.$$

Thus there exists $m > N$ with $m \in A' \subseteq A$. □

Lemma 2.2.16. *Let (X, f) be a dynamical system, where X is Hausdorff space. Let $x, y, z \in X$ and let $z \in \overline{\mathrm{Orb}(x)}$. If $(x, y) \notin \mathrm{EqP}(X, f)$ and O is a splitting neighbourhood of y with regard to x then $(z, y) \notin \mathrm{EqP}(X, f)$ and O is a splitting neighbourhood of y with regard to z.*

Proof. Let $U \in \mathcal{N}_z$ and $V \in \mathcal{N}_y$. Let $n \in \mathbb{N}$ be such that $W = f^{-n}(U) \ni x$. Take $m > n$ such that $f^m(W) \cap V \neq \emptyset$ and $f^m(W) \not\subseteq O$; such an m exists by Lemma 2.2.15. □

Remark 2.2.17. The contrapositive of Lemma 2.2.16 is: If $(z, y) \in \mathrm{EqP}(X, f)$ and $z \in \overline{\mathrm{Orb}(x)}$ then $(x, y) \in \mathrm{EqP}(X, f)$.

Corollary 2.2.18. *Let (X, f) be a Hausdorff dynamical system and suppose $x, y, z \in X$. If (y, z) is a trivial (resp. non-trivial) equicontinuity pair and $y \in \overline{\mathrm{Orb}(x)}$ then (x, z) is a trivial (resp. non-trivial) equicontinuity pair. In particular, if (x, y) and (y, z) are non-trivial equicontinuity pairs then so is (x, z).*

Proof. If (y, z) is a trivial equicontinuity pair then $z \notin \Omega(y)$. Let $U \in \mathcal{N}_y$, $V \in \mathcal{N}_z$ and $N \in \mathbb{N}$ be such that, for any $n > N$, $f^n(U) \cap V = \emptyset$. Now let $m \in \mathbb{N}_0$ be such that $W = f^{-m}(U) \ni x$. Then, for all $n > N + m$, $f^n(W) \cap V = \emptyset$. Thus (x, z) is a trivial equicontinuity pair.

Now suppose that (y, z) is a non-trivial equicontinuity pair. If $y \in \mathrm{Orb}(x)$ then $\omega(x) = \omega(y)$ and so $z \in \omega(x)$. If $y \in \omega(x)$ then, since ω-limit sets are closed and positively invariant, $\overline{\mathrm{Orb}(y)} \subseteq \omega(x)$. Because (y, z) is a non-trivial equicontinuity pair, $z \in \omega(y) \subseteq \overline{\mathrm{Orb}(y)}$. Thus $z \in \omega(x)$. Therefore in each case we have $z \in \omega(x)$. It now suffices to check $(x, z) \in \mathrm{EqP}(X, f)$; but this is just Remark 2.2.17.

Finally, if (x, y) and (y, z) are non-trivial equicontinuity pairs then $y \in \omega(x)$ and the result follows by the above. □

Remark 2.2.19. Corollary 2.2.18 shows that the relation given by 'non-trivial equicontinuity pair' is transitive.

Remark 2.2.20. It follows from Corollary 2.2.18 that if a system has a transitive point x then for any equicontinuity pair (a, b), (x, b) is an equicontinuity pair, i.e. every equicontinuity partner is an equicontinuity partner of the transitive point.

Corollary 2.2.21. *Let X be a Hausdorff space. If (X, f) is minimal then, for any $x, y \in X$,*

$$(x, y) \in \mathrm{EqP}(X, f) \implies \forall z \in X,\ (z, y) \in \mathrm{EqP}(X, f),$$

and

$$(x, y) \notin \mathrm{EqP}(X, f) \implies \forall z \in X,\ (z, y) \notin \mathrm{EqP}(X, f).$$

Proof. The former statement follows from Corollary 2.2.18 (by noting that, as the system is minimal, for any $z \in X$, $x \in \overline{\mathrm{Orb}(z)}$). The latter statement follows from Lemma 2.2.16 (by noting that, as the system is minimal, for any $z \in X$, $z \in \overline{\mathrm{Orb}(x)}$). □

The following theorem is a generalisation of [2, Theorem 2.4] (see Theorem 2.1.7).

Theorem 2.2.22. *Let (X, f) be a transitive dynamical system, where X is a Hausdorff space. Suppose there exists a topological equicontinuity point. Then the set of topological equicontinuity points coincides with the set of transitive points.*

In particular, if (X, f) is a minimal system and there is a topological equicontinuity point then the system is topologically equicontinuous.

Proof. Let $x \in X$ be a point of topological equicontinuity. By Lemma 2.2.9, $\omega(x) = \Omega(x)$; but since (X, f) is a transitive system $\Omega(x) = X$ by Remark 2.1.4. Hence x is a transitive point.

Now suppose x is a transitive point. Let y be a point of topological equicontinuity. Then $y \in \omega(x)$ as x is a transitive point. Now, $(y, z) \in \mathrm{EqP}(X, f)$ for all $z \in X$, and these are all non-trivial equicontinuity pairs by Theorem 2.2.10, therefore, by Corollary 2.2.18, it follows that $(x, z) \in \mathrm{EqP}(X, f)$ for all $z \in X$; i.e. x is a point of topological equicontinuity. □

We are now in a position to present a generalised version of the Auslander–Yorke dichotomy for minimal systems; in [5] the authors show that a compact metric minimal system is either equicontinuous or is sensitive. The following definition was given by Good and Macías in [45] where they show it is equivalent to sensitivity if X is a compact Hausdorff space (and therefore, in particular, if X is compact metric).[1]

Definition 2.2.23. A dynamical system (X, f), where X is a Hausdorff space, is said to be *Hausdorff sensitive* if there exists a finite open cover $\mathcal{U}$ such that for any nonempty

[1] In a non-compact metric setting, a Hausdorff sensitive map need not be sensitive. Consider the example given in section 2.1.1 with the metric d_0. This is not sensitive. However it is Hausdorff sensitive: it can be verified that the finite open cover given by $\{\bigcup_{k \in \mathbb{N}_0}(2k, 2k+5/4), \bigcup_{k \in \mathbb{N}_0}(2k+1, 2k+9/4)\}$ bears witness to this.

open set V there exist $x, y \in V$, $x \neq y$, and $k \in \mathbb{N}$ such that $\{f^k(x), f^k(y)\} \nsubseteq U$ for all $U \in \mathcal{U}$.

In similar fashion to metric and uniform settings, for a subset $U \subseteq X$ and an open cover $\mathcal{U}$ of X we define the set $N_{\mathcal{U}}(U)$ as the set of natural numbers k for which there exist $x, y \in U$, $x \neq y$, such that $\{f^k(x), f^k(y)\} \nsubseteq U$ for all $U \in \mathcal{U}$. Thus, a system is Hausdorff sensitive precisely when there is a finite open cover $\mathcal{U}$ for which $N_{\mathcal{U}}(U) \neq \emptyset$ for any nonempty open set U.

Theorem 2.2.24. *Let (X, f) be a system with a transitive point x, where X is a regular Hausdorff space (i.e. T_3). If there exists $y \in X$ with $(x, y) \notin \mathrm{EqP}(X, f)$ then (X, f) is Hausdorff sensitive.*

Proof. Let x and y be as in the statement. Therefore

$$\exists O \in \mathcal{N}_y : \forall U \in \mathcal{N}_x \forall V \in \mathcal{N}_y \exists n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset \text{ and } f^n(U) \nsubseteq O. \tag{2.8}$$

Let V_1 and V_2 be open neighbourhoods of y such that $\overline{V_1} \subseteq O$ and $\overline{V_2} \subseteq V_1$; these exist as X is regular. Then $\mathcal{U} := \{V_1, X \setminus \overline{V_2}\}$ is a finite open cover. Now let U be an arbitrary nonempty open set. Let $n \in \mathbb{N}$ be such that $W = f^{-n}(U) \ni x$. Take $m > n$ such that $f^m(W) \cap V_2 \neq \emptyset$ and $f^m(W) \nsubseteq O$; such an m exists by Lemma 2.2.15. Then $f^{m-n}(U) \cap V_2 \neq \emptyset$ and $f^{m-n}(U) \nsubseteq O$. In particular there exists $a, b \in U$ such that $f^{m-n}(a) \notin O$ and $f^{m-n}(b) \in V_2$. Then $\{f^{m-n}(a), f^{m-n}(b)\} \cap V_1 = \{f^{m-n}(b)\}$ and $\{f^{m-n}(a), f^{m-n}(b)\} \cap X \setminus \overline{V_2} = \{f^{m-n}(a)\}$. □

Corollary 2.2.25. *(Generalised Auslander–Yorke dichotomy I) Let X be a T_3 space. A minimal system (X, f) is either topologically equicontinuous or Hausdorff sensitive.*

Proof. Suppose it is not equicontinuous. Then there exists $x, y \in X$ with $(x, y) \notin \mathrm{EqP}(X, f)$. Since x is a transitive point the result follows from Theorem 2.2.24. □

The following theorem is a generalisation of the result by Banks *et al.* [7] that the first two ingredients of Devaney chaos (transitivity and dense set of periodic points) entail the third (sensitivity).

Theorem 2.2.26. *Let (X, f) be a transitive system where X is an infinite T_3 space. If the set of eventually periodic points is dense in X then the system is Hausdorff sensitive.*

Proof. Suppose first that the set of periodic points is not dense in X. Let U be a nonempty open set not containing any periodic points. Let V_1 and V_2 be nonempty open sets such that $\overline{V_1} \subseteq U$ and $\overline{V_2} \subseteq V_1$. We claim the finite open cover $\mathcal{U} := \{V_1, X \setminus \overline{V_2}\}$ bears witness to Hausdorff sensitivity. Indeed, let W be a nonempty open set. Since the set of eventually periodic points is dense there is such a point in W. Because there are no periodic points in U it follows that $\mathbb{N} \setminus N(W, X \setminus U)$ is finite.[1] However, by transitivity, $N(W, V_2)$ is infinite. Therefore there exists $k \in N(W, X \setminus U) \cap N(W, V_2)$: i.e. there exist $x, y \in W$ such that $f^k(x) \in X \setminus U$ and $f^k(y) \in V_2$. Notice that $f^k(x) \notin V_1$ and $f^k(y) \notin X \setminus \overline{V_2}$, hence $\{f^k(x), f^k(y)\} \not\subseteq V$ for any V in $\mathcal{U}$. Since W was picked arbitrarily we are done.

Now suppose that the set of periodic points is dense in X. Let v and w be two periodic points, with periods n and m respectively, belonging to distinct orbits and take disjoint open sets V and W such that $V \supseteq \text{Orb}(v)$ and $W \supseteq \text{Orb}(w)$. By regularity, there are open sets V_1, V_2, W_1 and W_2 such that

$$\text{Orb}(v) \subseteq V_2 \subseteq \overline{V_2} \subseteq V_1 \subseteq \overline{V_1} \subseteq V; \text{ and,}$$
$$\text{Orb}(w) \subseteq W_2 \subseteq \overline{W_2} \subseteq W_1 \subseteq \overline{W_1} \subseteq W.$$

Consider the open cover $\mathcal{U} := \{V_1, W_1, X \setminus (\overline{V_2} \cup \overline{W_2})\}$: this bears witness to Hausdorff sensitivity. To see this, let U be a nonempty open set. Let $p \in U$ be periodic with period l. Notice that either $\text{Orb}(p) \not\subseteq V$ or $\text{Orb}(p) \not\subseteq W$: without loss of generality assume the former and take $k = \max\{l, n\}$. Let

$$V' = \bigcap_{i \in \{0, \ldots, k\}} f^{-i}(V_2).$$

[1] I.e. $N(W, X \setminus U)$ is *cofinite* in $\mathbb{N}$.

Notice $v \in V'$ so $V' \neq \emptyset$. By transitivity there exists $r \in \mathbb{N}$ such that $f^r(U) \cap V' \neq \emptyset$. By the definition of V' it follows that $f^{r+i}(U) \cap V_2 \neq \emptyset$ for all $i \in \{0, \ldots, k\}$. However

$$\bigcup_{i \in \{0,\ldots,k\}} f^{r+i}(U) \supseteq \mathrm{Orb}(p) \nsubseteq V.$$

Fix $i \in \{0, \ldots, k\}$ such that $f^{r+i}(p) \notin V$. Then $f^{r+i}(p) \notin V_1$. Furthermore there is $x \in U$ with $f^{r+i}(x) \in V_2$. Thus $\{f^{r+i}(x), f^{r+i}(p)\} \nsubseteq A$ for any $A \in \mathcal{U}$. □

We end this section with the following question.

Question 2.2.27. Does there exist a transitive system (X, f), where X is a Hausdorff space, with a non-trivial equicontinuity pair (x, y) but where x is not a topological equicontinuity point?

The following result may help make some headway with Question 2.2.27.

Proposition 2.2.28. *Suppose (X, f) is a transitive dynamical system where X is an infinite Hausdorff space. If $x \in X$ is an eventually periodic point then $(x, y) \notin \mathrm{EqP}(X, f)$ for any $y \in X$.*

Proof. Write $\mathrm{Orb}(x) = \{x, f(x), \ldots, f^l(x)\}$. Suppose $(x, y) \in \mathrm{EqP}(X, f)$. Then by Corollary 2.2.11 it follows that $y \in \omega(x)$; as x is eventually periodic this means $y \in \mathrm{Orb}(x)$ and y is periodic. Write $y = f^m(x)$ and let n be the period of y (so $n \leq l$). Let $z \in X \setminus \mathrm{Orb}(x)$; for each $i \in \{0, \ldots, n-1\}$ let $W_i \in \mathcal{N}_z$ and $O_i \in \mathcal{N}_{f^i(y)}$ be such that $W_i \cap O_i = \emptyset$. Now let

$$O := \bigcap_{i=0}^{n-1} f^{-i}(O_i),$$

and

$$W := \bigcap_{i=0}^{n-1} W_i.$$

Thus $W \in \mathcal{N}_z$ and $O \in \mathcal{N}_y$. Now let $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ satisfy the equicontinuity condition for x, y and O. Notice that $f^i(O) \cap W = \emptyset$ for all $i \in \{0, \ldots, n-1\}$. Since $f^m(U) \cap V \neq \emptyset$ we have $f^m(U) \subseteq O$. Furthermore, $f^{m+an}(U) \cap V \neq \emptyset$ for all $a \in \mathbb{N}_0$, hence $f^{m+an}(U) \subseteq O$. It follows that $f^k(U) \cap W = \emptyset$ for all $k \geq m$, this contradicts Lemma 2.1.3. □

2.3 Even continuity

Even continuity, as defined by Kelley [59, pp. 234], is a weaker concept than that of topological equicontinuity. If X and Y are topological spaces we say a collection of maps $\mathscr{F}$ from X to Y is *evenly continuous* at an ordered pair $(x, y) \in X \times Y$ if for any $O \in \mathcal{N}_y$ there exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that, for any $f \in \mathscr{F}$, if $f(x) \in V$ then $f(U) \subseteq O$; when this is the case we refer to (x, y) as an *even continuity pair*. We say $\mathscr{F}$ is *evenly continuous* at a point $x \in X$ if it is evenly continuous at (x, y) for all $y \in Y$. We say the collection is *evenly continuous* if it is evenly continuous at every $x \in X$. We remark that when Y is a compact Hausdorff space the notions of topological equicontinuity, even continuity and equicontinuity coincide (see [59, Theorem 7.23]). Finally, we observe that if a family is evenly continuous (resp. topological equicontinuous) then each member of that family is necessarily continuous [39, pp. 162].

Given a dynamical system (X, f), we denote the collection of even continuity pairs and the collection of even continuity points by $\text{EvP}(X, f) \subseteq X \times X$ and $\text{Ev}(X, f) \subseteq X$ respectively. Note that in this case, if we consider the above definitions, we have $Y = X$ and $\mathscr{F} = \{f^n \mid n \in \mathbb{N}\}$. By definition it follows that (X, f) is evenly continuous precisely when $\text{EvP}(X, f) = X \times X$. For $(x, y) \in \text{EvP}(X, f)$, we refer to the condition

$$\forall O \in \mathcal{N}_y \, \exists U \in \mathcal{N}_x \, \exists V \in \mathcal{N}_y : \forall n \in \mathbb{N}, f^n(x) \in V \implies f^n(U) \subseteq O, \tag{2.9}$$

as the *even continuity condition* for x and y. We say that U and V, as in Equation 2.9, satisfy the even continuity condition for x, y and O.

Remark 2.3.1. Clearly every equicontinuity pair is an even continuity pair.

As pointed out by others (e.g. [87]), the converse to Remark 2.3.1 is not true in general.

The following example demonstrates this.

Example 2.3.2. For each $n \in \mathbb{N}$, let X_n be the finite word $\mathtt{1}\mathtt{0}^n$ and take $\mathbf{x} = X_1X_2X_3\ldots$. For each $n \in \mathbb{N}$, let $\mathbf{z}_n = X_1X_2\ldots X_n\mathtt{0}^\infty$. Take

$$Y := \{\mathtt{0}^m\mathbf{z}_n, \mathtt{0}^m\mathbf{x}, \mathtt{0}^\infty \mid n, m \in \mathbb{N}\},$$

and let

$$X := \overline{\{\sigma^k(\mathbf{y}) \mid \mathbf{y} \in Y, k \in \mathbb{N}_0\}},$$

where the closure is taken with regard to the full shift $\Sigma^{\mathbb{N}_0}$.

Note that,

$$\omega(\mathbf{x}) = \{\mathtt{0}^\infty, \mathtt{0}^n\mathtt{1}\mathtt{0}^\infty \mid n \in \mathbb{N}_0\},$$

and for each $i \in \mathbb{N}$,

$$\omega(\mathbf{z}_i) = \{\mathtt{0}^\infty\}.$$

Considering the shift system (X, σ), it is easy to see that $(\mathbf{x}, \mathtt{0}^\infty)$ is a non-trivial even continuity pair in (X, σ) (i.e. it is an even continuity pair and $\mathtt{0}^\infty \in \omega(\mathbf{x})$). Indeed, pick $O \in \mathcal{N}_{\mathtt{0}^\infty}$. Let $n \in \mathbb{N}$ be such that $[\mathtt{0}^n] \subseteq O$. Let $U = [X_1X_2]$ and $V = [\mathtt{0}^n]$. If $\mathbf{a} \in U$ then either $\mathbf{a} = \mathbf{x}$ or $\mathbf{a} = \mathbf{z}_m$ for some $m \geq 2$. Either way, for any $l \in \mathbb{N}$, if $\sigma^l(\mathbf{x}) \in V$ then $\sigma^l(\mathbf{a}) \in V \subseteq O$. Hence, since $\mathtt{0}^\infty \in \omega(\mathbf{x})$, $(\mathbf{x}, \mathtt{0}^\infty)$ is a non-trivial even continuity pair.

In contrast, $(\mathbf{x}, \mathtt{0}^\infty)$ it is not an equicontinuity pair; arbitrarily close to $\mathbf{x}$ are points that map onto $\mathtt{0}^\infty$, which is a fixed point, but $\mathbf{x}$ itself is not pre-periodic. To show this explicitly, take $O = [\mathtt{0}] \in \mathcal{N}_{\mathtt{0}^\infty}$. Picking $U \in \mathcal{N}_x$, there exists $N \in \mathbb{N}$ such that $\sigma^k(U) \ni \mathtt{0}^\infty$ for all $k > N$; in particular, for any $V \in \mathcal{N}_{\mathtt{0}^\infty}$, $\sigma^k(U) \cap V \neq \emptyset$ for all $k \geq N$. But there exists $k \geq N$ such that $\sigma^k(\mathbf{x}) \in [\mathtt{1}]$, hence $(\mathbf{x}, \mathtt{0}^\infty) \notin \mathrm{EqP}(X, \sigma)$.

Proposition 2.3.3. *Let (X, f) be a dynamical system where X is a Hausdorff space. Let $x, y \in X$ and suppose $y \notin \omega(x)$. Then $(x, y) \in \mathrm{EvP}(X, f)$.*

Proof. Let $O \in \mathcal{N}_y$ be given. Since $y \notin \omega(x)$ there exist $V \in \mathcal{N}_y$ and $N \in \mathbb{N}$ such that $f^n(x) \notin V$ for all $n > N$.

By Corollary 2.2.3, there exist $U' \in \mathcal{N}_x$ and $V' \in \mathcal{N}_y$ such that, for any $k \in \{1, \dots, N\}$, if $f^k(U') \cap V' \neq \emptyset$ then $f^k(U') \subseteq O$; without loss of generality $V' \subseteq V \cap O$. In particular this means that, for all $k \in \{1, \dots, N\}$, if $f^k(x) \in V'$ then $f^k(U') \subseteq O$. Then, since $f^n(x) \notin V'$ for all $n > N$, U' and V' satisfy the even continuity condition for x, y and O. As $O \in \mathcal{N}_y$ was picked arbitrarily the result follows. □

Remark 2.3.4. If X is a Hausdorff space, putting together Propositions 2.2.4 and 2.3.3, we have, for a pair $x, y \in X$, the following:

- If $y \notin \omega(x)$ then $(x, y) \in \mathrm{EvP}(X, f)$.
- If $y \notin \Omega(x)$ then $(x, y) \in \mathrm{EqP}(X, f)$.

Definition 2.3.5. If (X, f) is a dynamical system, where X is a Hausdorff space. We say $(x, y) \in X \times X$ is a *trivial even continuity pair* if $y \notin \omega(x)$.

Remark 2.3.6. Proposition 2.3.3 tells us that a trivial even continuity pair is indeed an even continuity pair. We emphasise that, by definition, if $(x, y) \in \mathrm{EvP}(X, f)$ then either they are a trivial even continuity pair or $y \in \omega(x)$. Finally, it is worth observing that, by Lemma 2.2.8 and Remark 2.3.1, a non-trivial equicontinuity pair is also a non-trivial even continuity pair.

The statement $(x, y) \notin \mathrm{EvP}(X, f)$, for $x, y \in X$, means precisely

$$\exists O \in \mathcal{N}_y : \forall U \in \mathcal{N}_x \, \forall V \in \mathcal{N}_y \, \exists n \in \mathbb{N} : f^n(x) \in V \text{ and } f^n(U) \nsubseteq O. \tag{2.10}$$

We shall refer to a neighbourhood such as O in equation 2.10 as an *even-splitting neighbourhood* of y with regard to x. It is straightforward to see that every even-splitting neighbourhood of y with regard to x is also a splitting neighbourhood of y with regard to x. Notice that, by Proposition 2.3.3, if $(x, y) \notin \mathrm{EvP}(X, f)$ then $y \in \omega(x)$, so we have:

Corollary 2.3.7. *Let (X, f) be a dynamical system where X is a Hausdorff space. If x has no even continuity partners, then x is a transitive point.*

The proof of Lemmas 2.3.8 and 2.3.10 are very similar to that of Lemmas 2.2.13 and 2.2.15 respectively and are thereby omitted.

Lemma 2.3.8. *Let (X, f) be a dynamical system, where X is a Hausdorff space. If $(x, y) \in \mathrm{EvP}(X, f)$, f is open at y and there is a neighbourhood base for y, $\mathcal{B}_y \subseteq \mathcal{N}_y$, such that $f^{-1}(f(O \cap \mathrm{Orb}(x))) = O \cap \mathrm{Orb}(x)$ for all $O \in \mathcal{B}_y$, then $(x, f(y)) \in \mathrm{EvP}(X, f)$.*

Corollary 2.3.9. *Let (X, f) be a dynamical system, where X is a Hausdorff space. If f is a homeomorphism and $(x, y) \in \mathrm{EvP}(X, f)$ then $(x, f(y)) \in \mathrm{EvP}(X, f)$.*

Proof. Immediate from Lemma 2.3.8. □

Lemma 2.3.10. *Let (X, f) be a dynamical system, where X is Hausdorff space. Suppose $(x, y) \notin \mathrm{EvP}(X, f)$ and let O be an even-splitting neighbourhood of y with regard to x. Then, for any pair of neighbourhoods U and V of x and y respectively, the set of natural numbers n for which $f^n(x) \in V$ and $f^n(U) \nsubseteq O$ is infinite.*

Lemma 2.3.11. *Let (X, f) be a dynamical system, where X is Hausdorff space. Let $x, y \in X$. If $(x, y) \notin \mathrm{EvP}(X, f)$ and O is an even-splitting neighbourhood of y with regard to x then, for any $n \in \mathbb{N}$, $(f^n(x), y) \notin \mathrm{EvP}(X, f)$ and O is an even-splitting neighbourhood of y with regard to $f^n(x)$.*

Proof. Let $U \in \mathcal{N}_{f^n(x)}$ and $V \in \mathcal{N}_y$. Then $W = f^{-n}(U) \in \mathcal{N}_x$. By Lemma 2.3.10 the set

$$A = \{k \in \mathbb{N} \mid f^k(x) \in V \text{ and } f^k(W) \nsubseteq O\},$$

is infinite. Taking $m > n$ with $m \in A$ gives the result. □

Remark 2.3.12. We emphasise the contrapositive of Lemma 2.3.11: Let (X, f) be a dynamical system, where X is Hausdorff space. Suppose $(x, y) \in \mathrm{EvP}(X, f)$ and $x \in \mathrm{Orb}(z)$. Then $(z, y) \in \mathrm{EvP}(X, f)$.

Proposition 2.3.13. *Let X be a Hausdorff space. If (X, f) is a dynamical system and there exists a point $x \in X$ with no even continuity partners, then x is a transitive point and (X, f) has no equicontinuity pairs.*

Proof. Let $x \in X$ be a point with no even continuity partners. By Corollary 2.3.7, x is a transitive point.

Let $y, z \in X$ be picked arbitrarily. Let O be an even-splitting neighbourhood of z with regard to x. Let $U \in \mathcal{N}_y$ and $V \in \mathcal{N}_z$. As x is transitive there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$. By Lemma 2.3.11, $(f^n(x), z)$ is not an even continuity pair and O is an even-splitting neighbourhood of z with regard to $f^n(x)$. It follows that there exists $m \in \mathbb{N}$ such that $f^m(U) \cap V \neq \emptyset$ and $f^m(U) \not\subseteq O$; hence $(y, z) \notin \mathrm{EqP}(X, f)$. $\square$

At the end of the previous section, we asked, in Question 2.2.27, whether there exists a transitive system with an equicontinuity pair but no point of equicontinuity. We now answer, in the positive, an analogous question with regard to even continuity pairs.

Theorem 2.3.14. *There exists a transitive system (X, f) with a non-trivial even continuity pair but no point of even continuity: Furthermore, there is such a system which is additionally Auslander–Yorke chaotic, densely and strongly Li–Yorke chaotic, but not Devaney chaotic, whilst having no equicontinuity pairs.*

Due to the length and technical nature of the proof of Theorem 2.3.14, we leave it until the end of this chapter (section 2.7).

Remark 2.3.15. Devaney [36, pp. 50] defined chaos as a topologically transitive, sensitive system with a dense set of periodic points. This last property means that, "in the midst of random behaviour, we nevertheless have an element of regularity." The construction in the proof of Theorem 2.3.14 shows that a system which is, in some sense, extremely chaotic (it is not only sensitive but expansive, whilst having only two periodic points) can still exhibit some element of regularity: the even continuity pair $(\mathbf{x}, \mathtt{0}^\infty)$ provides some regularity associated with $\mathbf{x}$. When $\mathbf{x}$ moves close to $\mathtt{0}^\infty$, everything from a certain neighbourhood of $\mathbf{x}$ also moves close to $\mathtt{0}^\infty$.

The following corollary is an immediate consequence of Theorem 2.3.14.

Corollary 2.3.16. *The notions equicontinuity pair and even continuity pair, in general, remain distinct for transitive dynamical systems (even in the compact metric setting).*

The last result in this section is a variation on Proposition 2.2.28; it gives us some information about the types of pairs which cannot be even continuity pairs in transitive systems.

Proposition 2.3.17. *Suppose (X, f) is a transitive dynamical system where X is an infinite Hausdorff space. If $x \in X$ is an eventually periodic point then (x, y) is not a non-trivial even continuity pair for any $y \in X$.*

The proof of Proposition 2.3.17 is very similar to that of Proposition 2.2.28 and is thereby omitted.

2.4 Equicontinuity, transitivity and splitting

A subset $N = \{n_1, n_2, n_3, \ldots\} \subseteq \mathbb{N}$, where $n_1 < n_2 < n_3 \ldots$, is said to be *syndetic* if there exists $l \in \mathbb{N}$ such that $n_{i+1} - n_i \leq l$ for each $i \in \mathbb{N}$; such an l is called a *bound of the gaps*. A subset is called *thick* if it contains arbitrarily long strings without gaps. A subset is called *cofinite* if its complement is finite. Using this, a dynamical system (X, f) is said to be:

1. Syndetically (resp. thickly) transitive if $N(U, V)$ is syndetic (resp. thick) for any nonempty open U and V.

2. Strong mixing if $N(U, V)$ is cofinite for any nonempty open U and V.

If X is a separated uniform space we say the system is syndetically (resp. thickly / resp. cofinitely) sensitive if there exists a symmetric $D \in \mathscr{U}$ such that, for any nonempty open $U \subseteq X$, the set $N_D(U)$ is syndetic (resp. thick / resp. cofinite).

In this section we investigate the link between topological equicontinuity, transitivity, and sensitivity. Trivially, if a dynamical system has an equicontinuity point then it is not sensitive. If we restrict our attention to compact metric systems, adding the condition of transitivity is enough to give a partial converse; a transitive map with no equicontinuity points is sensitive [2]. The proof provided by Akin *et al.* does not rely on the space being metrizable; with only minor adjustments the result generalises to give the following.

Theorem 2.4.1. [2] *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. If there exists a transitive point and $\mathrm{Eq}(X, f) = \emptyset$ then (X, f) is sensitive.*

If X is a compact metric space, and (X, f) is a transitive dynamical system, then there exists a transitive point (since X is nonmeagre and has a countable π-base). By Theorems 2.4.1 and 2.2.10 it follows that, for a compact metric system, no equicontinuity pairs implies both transitivity and sensitivity.

Corollary 2.4.2. *Let X be a compact Hausdorff space which yields a countable π-base. If $\mathrm{EqP}(X, f) = \emptyset$ then the system is both transitive and sensitive.*

Proof. By Theorem 2.2.10 the system is transitive. Since compact Hausdorff spaces are nonmeagre there exists a transitive point. Applying Theorem 2.4.1 completes the result. □

Proposition 2.4.3. *Let X be a Hausdorff space and (X, f) a dynamical system. If X is nonmeagre with a countable π-base, then $\mathrm{EqP}(X, f) = \emptyset$ if and only if there exists a transitive point $x \in X$ with no equicontinuity partners.*

Proof. Assume the latter and let x be such a transitive point. Suppose that $(a, b) \in \mathrm{EqP}(X, f)$, for some $a, b \in X$. Then (a, b) is a non-trivial equicontinuity pair by Theorem 2.2.10. As x is a transitive point $a \in \omega(x)$. It follows from Corollary 2.2.18 that $(x, b) \in \mathrm{EqP}(X, f)$, a contradiction.

Now suppose the former. By Corollary 2.4.2 the system is transitive, which entails the existence of a transitive point as X is nonmeagre with a countable π-base. □

We now turn our attention to examining sufficient conditions for $\mathrm{EqP}(X, f) = \emptyset$. One obvious such condition is the following.

Proposition 2.4.4. *Let (X, f) be a dynamical system, where X is a separated uniform space. Suppose there exists a symmetric $D \in \mathscr{U}$ such that for any nonempty open sets U and V, $N(U, V) \cap N_D(U) \neq \emptyset$, then there are no equicontinuity pairs.*

Note that, when the hypothesis of this proposition occurs, it is equivalent to being able to move the existential quantifier to the front of the statement stating $\mathrm{EqP}(X, f) = \emptyset$. To be clear, in the separated uniform setting $\mathrm{EqP}(X, f) = \emptyset$ means,

$$\forall x, y \in X\, \exists D \in \mathscr{U} : \forall U \in \mathcal{N}_x\, \forall V \in \mathcal{N}_y, N(U, V) \cap N_D(U) \neq \emptyset.$$

whilst the hypothesis states,

$$\exists D \in \mathscr{U} : \forall x, y \in X\, \forall U \in \mathcal{N}_x\, \forall V \in \mathcal{N}_y,\ N(U, V) \cap N_D(U) \neq \emptyset.$$

If (X, f) is a dynamical system, where X is a separated uniform space, then for any pair of sets $U, V \subseteq X$, we define $N_D(U, V) \coloneqq N(U, V) \cap N_D(U)$; if X is a metric space and $\delta > 0$ we similarly define $N_\delta(U, V) \coloneqq N(U, V) \cap N_\delta(U)$. Such a set is extremely relevant in an applied setting, where small rounding errors mean that a different point than the one intended might be tracked. This set tells us precisely when U meets V whilst also expanding to at least diameter δ. The importance of such a set leads us to give the following definition.

Definition 2.4.5. Let (X, f) be a dynamical system, where X is a separated uniform space. We say that (X, f) experiences *splitting* if there is a symmetric $D \in \mathscr{U}$ such that for any pair of nonempty open sets U and V we have $N_D(U, V) \neq \emptyset$. Such a D is called a splitting entourage for (X, f).

In similar fashion, if X is a metric space we say the system (X, f) has *splitting* if there exists $\delta > 0$ such that for any pair of nonempty open sets U and V we have $N_\delta(U, V) \neq \emptyset$. Thus a system has splitting when every nonempty open set 'hits' every other such set whilst it is simultaneously pulled apart to diameter at least δ. Proposition 2.4.4 then states that any splitting system has no equicontinuity pairs. To take a purely topological approach, as we did in the previous section, if X is a Hausdorff space we say the system has *Hausdorff splitting* if there exists a finite open cover $\mathcal{U}$ such that for any pair of nonempty open sets U and V we have $N_{\mathcal{U}}(U,V) := N(U,V) \cap N_{\mathcal{U}}(U) \neq \emptyset$.

Remark 2.4.6. In this section we mainly deal with the case when the phase space is a separated uniform space. We observe, however, that each of the results in this section have analogous versions where the space is T_3. These analogous versions are precisely the 'natural' ones that one might expect: they refer to a finite open cover $\mathcal{U}$ instead of an entourage D and to the sets $N_{\mathcal{U}}(U)$ and $N_{\mathcal{U}}(U,V)$ instead of $N_D(U)$ and $N_D(U,V)$ respectively. For the sake of space we do not include these versions here.

The following lemma is analogous to several previously stated.

Lemma 2.4.7. *If (X,f) is a separated uniform system with splitting, with splitting entourage D, then for any nonempty open pair U and V, $N_D(U,V)$ is infinite.*

Proof. Suppose $N_D(U,V)$ is finite. Since (X,f) has splitting, with splitting entourage D, $N_D(U,V) \neq \emptyset$. Let $k \in \mathbb{N}$ be the greatest element of $N_D(U,V)$. Let $W \subseteq U \cap f^{-k}(V)$ be open such that, for any $i \in \{1,\ldots,k\}$ and any $x,y \in W$, $(f^i(x), f^i(y)) \in D$. As $N_D(W,V) \neq \emptyset$ and $W \subseteq U$ we have a contradiction and the result follows. □

Corollary 2.4.8. *Let X be a separated uniform space with at least two points. If (X,f) is weakly mixing, then (X,f) experiences splitting.*

Proof. Suppose (X,f) exhibits weak mixing. Let $E \in \mathscr{U}$ be a symmetric entourage such that, for any $x \in X$, we have $B_E(x) \neq X$. Let $D \in \mathscr{U}$ be symmetric such that $3D = D \circ D \circ D \subseteq E$. Let U and V be nonempty open sets. Let $x \in V$ and pick $y \in X$ such that $(x,y) \notin E$. By weak mixing, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap \big(B_D(x) \cap V\big) \neq \emptyset$ and $f^n(U) \cap B_D(y) \neq \emptyset$. Let $u \in f^n(U) \cap \big(B_D(x) \cap V\big)$ and $u' \in f^n(U) \cap B_D(y)$; by symmetry $(x,u) \in D$ and $(u',y) \in D$. If $(u,u') \in D$ then $(x,y) \in 3D \subseteq E$, a contradiction. □

We remark that, with regard to a system (X,f), since topologically exact implies strong mixing which implies weak mixing, exactness and strong mixing are each sufficient for a system to have splitting.[1]

[1]The system (X,f) is topologically exact if, for any nonempty open set U there exists $n \in \mathbb{N}_0$ such that $f^n(U) = X$.

Clearly we also have the following result.[1]

Proposition 2.4.9. *Let (X, f) be a dynamical system, with X a separated uniform space. Let $\mathcal{F}_P \subseteq \mathcal{P}(\mathbb{N})$ be a full and proper Furstenberg family (induced by property P). If (X, f) is P-transitive and P^*-sensitive then it experiences splitting.*

For example, if (X, f) is syndetically transitive and thickly sensitive (here P is 'syndetic' and P^* is 'thick') it follows that it has splitting. Also, since transitivity implies $N(U, V)$ is infinite for any nonempty open pair U and V, we have that a transitive system which is cofinitely sensitive has splitting; in particular any transitive map on $[0, 1]$ has splitting.[2]

It turns out that any Devaney chaotic system on a compact space has splitting, and consequently has no equicontinuity pairs. We will see that this follows as a corollary to Theorem 2.4.10.

Theorem 2.4.10. *Let (X, f) a syndetically transitive dynamical system, where X is a compact Hausdorff space. If there are two distinct minimal sets then there exists a symmetric entourage $D \in \mathscr{U}$ such that for any nonempty open pair U and V, $N_D(U, V)$ is syndetic; i.e. the system experiences* syndetic splitting.

(NB. The proof below mimics Moothathu's [73, Theorem 1] proof that a non-minimal syndetically transitive system has syndetic sensitivity for metric systems.)

Proof. Let M_1 and M_2 be distinct minimal sets; it follows that $M_1 \cap M_2 = \emptyset$. Let $x \in M_1$ and $y \in M_2$; so $\overline{\mathrm{Orb}(x)} = M_1$ and $\overline{\mathrm{Orb}(y)} = M_2$. Let $D \in \mathscr{U}$ be symmetric such that, for any $z_1 \in M_1$ and any $z_2 \in M_2$, $(z_1, z_2) \notin 8D$. Now let U and V be nonempty open sets and take $z \in V$; without loss of generality $V \subseteq B_D(z)$. Suppose there is $p \in M_1$ and $q \in M_2$ such that $(p, z) \in 4D$ and $(z, q) \in 4D$; then $(p, q) \in 8D$, contradicting our choice of D. Without loss of generality we may thereby assume $(p, z) \notin 4D$ for any $p \in M_1$. Let l_1 be a bound of the gaps for $N(U, V)$. Let $W \ni x$ be open such

[1]For the definitions of the terms in Proposition 2.4.9 please refer to section 2.5.
[2]Any such map is cofinitely sensitive (see [73]).

that if $w \in W$ then $(f^i(w), f^i(x)) \in D$ for all $i \in \{0, 1, \dots, l_1\}$; W exists by continuity. Suppose there exist $w \in W$, $i \in \{0, 1, \dots, l_1\}$, and $v \in V$ such that $(f^i(w), v) \in 2D$; then since $(f^i(x), f^i(w)) \in D$ and $(v, z) \in D$ we have $(f^i(x), z) \in 4D$, a contradiction since $f^i(x) \in M_1$. Therefore for any $w \in W$, any $v \in V$, and any $i \in \{0, 1, \dots, l_1\}$ we have $(f^i(w), v) \notin 2D$. Let l_2 be a bound of the gaps for $N(U, W)$. Let $k \in N(U, W)$. Then $f^{k+i}(U) \cap B_D(M_1) \neq \emptyset$ for all $i \in \{0, 1, \dots, l_1\}$. However, since l_1 is a bound of the gaps for $N(U, V)$, there exists $j \in \{0, 1, \dots, l_1\}$ such that $f^{k+j}(U) \cap V \neq \emptyset$. Then $k + j \in N(U, V) \cap N(U, B_D(M_1))$. It follows that $N(U, V) \cap N(U, B_D(M_1))$ is itself syndetic, with $l_1 + l_2$ a bound of the gaps. Since $N_D(U, V) \supseteq N(U, V) \cap N(U, B_D(M_1))$, the result follows. □

The following corollaries follow from Theorem 2.4.10 and Proposition 2.4.4.

Corollary 2.4.11. *Let (X, f) be a syndetically transitive dynamical system, where X is a compact Hausdorff space. If there are two distinct minimal sets then there are no equicontinuity pairs.*

Corollary 2.4.12. *Let (X, f) be a non-minimal transitive system with a dense set of minimal points, where X is a compact Hausdorff space. Then (X, f) has syndetic splitting.*

Proof. Moothathu [73] notes that a transitive system with a dense set of minimal points is syndetically transitive. To see this, recall the folklore result that a point x in a compact Hausdorff system is minimal if and only if $N(x, U)$ is syndetic for every $U \in \mathcal{N}_x$. Let U and V be nonempty open sets. Since the set of minimal points is dense in X, U and V both contain minimal points. Hence $N(U, U)$ and $N(V, V)$ are both syndetic. Let $n \in N(U, V)$. Let $W = U \cap f^{-n}(V)$ and let x be a minimal point in W; $N(x, W)$ is syndetic. It follows that, for any $m \in N(x, W)$, $f^{n+m}(x) \in V$. Hence $N(U, V)$ is syndetic. Finally, since (X, f) is non-minimal but the set of minimal points is dense, there exist multiple minimal sets. The result now follows from Theorem 2.4.10. □

Corollary 2.4.13. *Let (X, f) be a Devaney chaotic dynamical system where X is a compact Hausdorff space. Then (X, f) has syndetic splitting.*

Proof. This follows from Corollary 2.4.12. □

Corollary 2.4.14. *Let (X, f) exhibit shadowing and chain transitivity, where X is a compact Hausdorff space. If there are two distinct minimal sets then (X, f) has syndetic splitting.*

Proof. A chain transitive system with shadowing is transitive. Moothathu [74] shows that a transitive system with shadowing has a dense set of minimal points. The result follows from Corollary 2.4.12. □

Question 2.4.15. For a dynamical system (X, f), where X is a separated uniform space, is splitting distinct from Auslander–Yorke chaos?

Clearly splitting implies Auslander–Yorke chaos. Thus, a positive answer to Question 2.4.15 should point to a system which exhibits Auslander–Yorke chaos but not splitting. If there is such a system then a follow-up question might be: is there a compact metric such system? Some of the results in this chapter may help with finding such a system. Corollary 2.4.8, for instance, rules out systems with weak mixing since these all have splitting.

We asked previously (Question 2.2.27), whether or not a transitive system can have an equicontinuity pair (x, y) without the system being equicontinuous at x. A more restrictive question is the following: Is it possible for a transitive point to have an equicontinuity partner but not be an equicontinuity point? This itself is related to Question 2.4.15. Indeed, if there exists a compact Hausdorff system (X, f), with a transitive point $x \notin \text{Eq}(X, f)$ and a point $y \in X$ with $(x, y) \in \text{EqP}(X, f)$, then it would follow that splitting is not equivalent to Auslander–Yorke chaos; such a system would be both transitive and, since there would be no equicontinuity points by Theorem 2.2.22, sensitive (Theorem 2.4.1). However, for any entourage $D \in \mathscr{U}$, there would exist neighbourhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $N_D(U, V) = \emptyset$, hence the system would not have splitting.

2.5 Other results connecting transitivity and sensitivity

For a property P which may be held by a subset of the natural numbers (for example, the property of being infinite), we denote by $\mathcal{F}_P$ the set of all subsets of $\mathbb{N}$ which satisfy P. (Thus $\mathcal{F}_P \subseteq \mathcal{P}(\mathbb{N})$.) For such a property, we say a dynamical system is P-transitive if, for all nonempty open sets U and V, $N(U, V) \in \mathcal{F}_P$. In similar fashion, for a uniform dynamical system with uniformity $\mathscr{U}$, we say the system is P-sensitive if there exists $D \in \mathscr{U}$ such that, for any nonempty open set U, $N_D(U) \in \mathcal{F}_P$.

For the results in this section, let P be a property that may be exhibited by certain subsets of $\mathbb{N}$, such that the following conditions hold:

1. The collection $\mathcal{F}_P$ is *hereditary upward*, i.e. if $A \in \mathcal{F}_P$ and $B \supseteq A$ then $B \in \mathcal{F}_P$. (This means $\mathcal{F}_P$ forms a *Furstenberg family*.)
2. The collection $\mathcal{F}_P$ is *full*, i.e. if $A \in \mathcal{F}_P$ then for any $k \in \mathbb{N}$, $A \setminus \{0, 1, 2, \ldots, k\} \in \mathcal{F}_P$.
3. The collection $\mathcal{F}_P$ is *proper*, i.e. $\emptyset \notin \mathcal{F}_P$.

We denote by P^* the property which induces the *dual family* of $\mathcal{F}_P$, i.e. $\mathcal{F}_{P^*} = \{A \in \mathcal{P}(\mathbb{N}) \mid \forall B \in \mathcal{F}_P,\ A \cap B \neq \emptyset\}$. It is an easy exercise to show that $\mathcal{F}_{P^*}$ is both hereditary upward, full, and proper.

Examples of such pairs of properties are:

1. $P =$ syndetic, $P^* =$ thick.
2. $P =$ infinite, $P^* =$ cofinite.
3. $P = \text{IP}$, $P^* = \text{IP}^*$.[1]

The proof given by Moothathu for [73, Proposition 3] (which links syndetic transitivity with syndetic sensitivity for metric systems) can be mimicked almost exactly to give the following sufficient condition for a system being P-sensitive.

[1]Recall, a set of natural numbers is called an IP-set if it contains all finite sums of some subsequence of itself (when viewing the set as a sequence). An IP^*-set is one which has non-empty intersection with any IP-set. See [14] for more information.

Proposition 2.5.1. *Let (X, f) be a dynamical system where X is a separated uniform space. Suppose that for any nonempty open pair U and V, $N(U,V) \in \mathcal{F}_P$. Suppose there exists a set $A \in \mathcal{F}_{P^*}$ and a closed symmetric $D \in \mathscr{U}$ such that for any $n \in A$ there exists $x \in X$ with $(x, f^n(x)) \notin D$. Then for any nonempty open U, $N_D(U) \in \mathcal{F}_P$; so (X, f) is P-sensitive, with sensitivity entourage D.*

Proof. Let $U \subseteq X$ be nonempty open. Since $N(U,U) \in \mathcal{F}_P$, and $A \in \mathcal{F}_{P^*}$, there exists $n \in N(U,U) \cap A$. Define $W := U \cap f^{-n}(U)$; then W is nonempty open. Since $n \in A$ there exists $x \in X$ with $(x, f^n(x)) \notin D$. Let $V \ni x$ be open such that if $y \in V$ then $(y, f^n(y)) \notin D$; such a set exists because D is closed and f is continuous. Now consider the P-set $N(W,V)$; we will show $N(W,V) \subseteq N_D(U)$, from which it will follow that $N_D(U) \in \mathcal{F}_P$. Let $m \in N(W,V)$ and let $w \in W \subseteq U$ be such that $f^m(w) \in V$. Then $(f^m(w), f^{m+n}(w)) \notin D$ by our choice of V. Let $u = f^n(w) \in U$; then $(f^m(w), f^m(u)) \notin D$, so $m \in N_D(U)$. □

This result yields a corollary relating to recurrence, which generalises [2, Theorem 3.6] and [96, Theorem 7]. (NB. In contrast to these results, we do not assume f to be uniformly continuous.)

Corollary 2.5.2. *Let X be a separated uniform space. A topologically transitive system (X, f) is either sensitive or uniformly rigid.*

Proof. By Lemma 2.1.3, since (X, f) is transitive, $N(U,V)$ is infinite for any pair of nonempty open sets U and V. If (X, f) is not uniformly rigid then there exists $D \in \mathscr{U}$ such that for any $n \in \mathbb{N}$ there exists $x \in X$ such that $(x, f^n(x)) \notin D$. Without loss of generality we may assume D to be closed and symmetric. The result follows by applying Proposition 2.5.1 with P as 'infinite' and P^* as 'cofinite'. □

It follows immediately from Corollary 2.5.2, that a transitive separated uniform system with a non-recurrent point is sensitive. This, in turn, gives us the following corollary.

Corollary 2.5.3. *Let (X, f) be a transitive system where X is a separated uniform space. If $\bigcup_{x \in X} \omega(x) \neq X$ then the system is sensitive.*

Proof. The result follows immediately from Corollary 2.5.2. (Of course, for a system with a transitive point this is vacuous.) □

The following is a corollary to Proposition 2.5.1; it is the natural generalisation of [73, Corollary 2].

Corollary 2.5.4. *Let (X, f) be a P-transitive system where X is a separated uniform space. Suppose that there exists two distinct points $x, y \in X$ and a P^*-set $A = \{n_k \mid k \in \mathbb{N}\}$ such that for any symmetric $D \in \mathscr{U}$, there exists $l \in \mathbb{N}$ such that $(f^{n_k}(x), f^{n_k}(y)) \in D$ for all $k \geq l$. Then f is P-sensitive.*

Proof. Choose $D \in \mathscr{U}$ such that $3D \not\ni (x, y)$. Now let $l \in \mathbb{N}$ be such that, for all $k \geq l$, $(f^{n_k}(x), f^{n_k}(y)) \in D$. Then, for any $k \geq l$, either $(x, f^{n_k}(x)) \notin D$ or $(y, f^{n_k}(y)) \notin D$. Indeed, suppose that both are in D. Then by the triangle inequality, used twice, $(x, y) \in 3D$; a contradiction. Now take $B = \{n_k \mid k \geq l\}$; B is a P^*-set, therefore by Proposition 2.5.1 we are done. □

Corollary 2.5.5. *Let (X, f) be a non-injective dynamical system where X is a separated uniform space.*

1. *If (X, f) is P-transitive then it is P-sensitive.*
2. *If (X, f) is syndetically transitive then it is syndetically sensitive.*
3. *If (X, f) is thickly transitive then it is thickly sensitive.*
4. *If (X, f) is transitive then it is sensitive*[1]*, so that a transitive map that is not sensitive is a homeomorphism (see [35, pp. 335]).*

Corollary 2.5.5 follows from Corollary 2.5.4. The system is not injective so there are two distinct points x and y such that $f(x) = f(y)$. Hence, for all $D \in \mathscr{U}$ and all $n \in \mathbb{N}$, $(f^n(x), f^n(y)) \in D$. Therefore we have a cofinite set A (as in Corollary 2.5.4). A cofinite set is also P^* (for any P as described), thick, syndetic, and, of course, cofinite. These give *1*, *2*, *3*, and *4* respectively.

[1]This result holds because saying for all nonempty open pairs $N(U, V) \neq \emptyset$ is equivalent to saying $N(U, V)$ is infinite for all such pairs.

2.6 Eventual sensitivity

The following definition was motivated by the following thought: sensitivity means that, no matter where you start, there are two points arbitrarily close to each other and to that starting location which will move far apart as time progresses; a universal 'far'. Clearly this is extremely relevant in an applied setting; rounding errors mean a computer will not, generally, track true orbits. But what if every point moves arbitrarily close to another point that it will then move away from? What if a computer starts with a true orbit and tracks it accurately, but then the point moves close to another point which will end up going in completely the other direction? — these two points may be so close together that the computer cannot differentiate between them; it may start tracking the wrong orbit and give an extremely inaccurate prediction of the future.

Definition 2.6.1. We say a metric dynamical system (X, f) is *eventually sensitive* if there exists $\delta > 0$ such that for any $x \in X$ and any $\varepsilon > 0$ there exist $n, k \in \mathbb{N}_0$ and $y \in B_\varepsilon(f^n(x))$ such that $d(f^{n+k}(x), f^k(y)) \geq \delta$. We refer to such a δ as an eventual-sensitivity constant.

If X is a compact Hausdorff space, we say that (X, f) is eventually sensitive if there exists $D \in \mathscr{U}$ such that for any $x \in X$ and any $E \in \mathscr{U}$ there exist $n, k \in \mathbb{N}_0$ and $y \in B_E(f^n(x))$ such that $(f^{n+k}(x), f^k(y)) \notin D$. We refer to such a D as an eventual-sensitivity entourage.

Clearly a system which is sensitive is also eventually sensitive; just take $n = 0$ in the above definition. The variable n is something that needs to be taken into account in an applied setting (and clearly it may depend on one's starting point); if the least such n is large, then the computer may provide an accurate model of the reasonably distant future. However, if the least such n is small, or 0 as in the case of sensitivity, the orbit the computer is attempting to track may quickly diverge from what the computer predicts. The example below is an example of an eventually sensitive but non-sensitive system.

Example 2.6.2. Let $X = [0,1]$. Define a map $f\colon X \to X$ by

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, {}^1\!/\!_4], \\ 1 - 2x & \text{if } x \in ({}^1\!/\!_4, {}^1\!/\!_2], \\ {}^{10x}\!/\!_3 - {}^5\!/\!_3 & \text{if } x \in ({}^1\!/\!_2, {}^3\!/\!_5], \\ {}^1\!/\!_3 & \text{if } x \in ({}^3\!/\!_5, {}^4\!/\!_5], \\ {}^{10x}\!/\!_3 - {}^7\!/\!_3 & \text{if } x \in ({}^4\!/\!_5, 1]. \end{cases}$$

Then $f\colon X \to X$, depicted in Figure 2.1, is a continuous surjection which is eventually sensitive but not sensitive.

The point ${}^3\!/\!_4$ has a neighbourhood on which the map is constant, so that f is not sensitive. However, it is eventually sensitive. To see this, notice that every point in $[0,1)$ is eventually mapped into $[0, {}^1\!/\!_2]$, where the map is simply a copy of the tent map, which is sensitive (indeed, it is cofinitely so). On the other hand, 1 is a fixed point ($f(1) = 1$), which is of a fixed distance ${}^1\!/\!_2$ from the interval $[0, {}^1\!/\!_2]$.

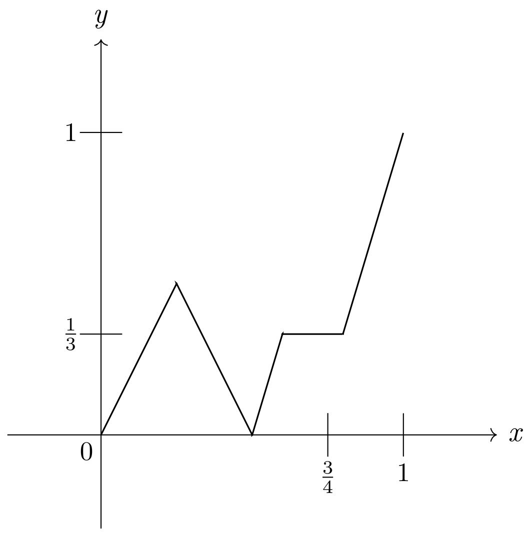

Figure 2.1: A non-sensitive, eventually sensitive system (Example 2.6.2)

For transitive dynamical systems we prove the following dichotomy.

Theorem 2.6.3. *(Generalised Auslander–Yorke dichotomy II) Let X be a compact Hausdorff space. A transitive dynamical system (X, f) is either equicontinuous or eventually* *sensitive. Specifically, it is eventually sensitive if and only if it is not equicontinuous.*

Proof. Suppose first that the system is not equicontinuous. Suppose the system has a dense set of minimal points. If the system is minimal then it is sensitive (see [5, Corollary 2] or Corollary 2.2.25) and the result follows. If it is non-minimal then it is sensitive (see [2, Thereom 2.5]) and therefore eventually sensitive. Now suppose the set of minimal points M is not dense in X. Let $q \in X$ and $D \in \mathscr{U}$ be symmetric such that $B_{3D}(q) \cap \overline{M} = \emptyset$. Let $z \in X$ be picked arbitrarily and let $E \in \mathscr{U}$ be given; without loss of generality $E \subset D$. Let $m \in \omega(z)$ be minimal. Then there exists $n \in \mathbb{N}$ such that $m \in B_E(f^n(z))$. By transitivity, there exists $k \in \mathbb{N}$ such that $f^k\left(B_E(f^n(z))\right) \cap B_D(q) \neq \emptyset$. Let $y \in B_E(f^n(z))$ be such that $f^k(y) \in B_D(q)$. Then $(f^k(y), f^k(m)) \notin 2D$ as $f^k(m)$ is minimal. Then either $(f^{n+k}(z), f^k(y)) \notin D$ or $(f^{n+k}(z), f^k(m)) \notin D$. Therefore (X, f) is eventually sensitive.

Now suppose that the system is eventually sensitive; let $D \in \mathscr{U}$ by an eventual-sensitivity entourage. Assume the system is equicontinuous. Since X is compact the system is uniformly equicontinuous. Let D_0 be such that for any $x, y \in X$ if $(x, y) \in D_0$ then for any $n \in \mathbb{N}$, $(f^n(x), f^n(y)) \in D$. Let $x \in X$ be given. By eventual sensitivity there exist $n, k \in \mathbb{N}_0$ and $y \in D_0[f^n(x)]$ such that $(f^{n+k}(x), f^k(y)) \notin D$; this contradicts our assumption that the system is equicontinuous. □

We conclude this section with a theorem which is simply a collation of Auslander–Yorke type results for transitive dynamical systems on compact spaces. The results it collates are: Corollary 2.5.2, Corollary 2.5.5 (which is [35, Corollary 7.1.12]), Theorem 2.6.3, [2, Theorem 2.4] and [5, Corollary 2].

Theorem 2.6.4. *Let X be a compact Hausdorff space and $f \colon X \to X$ be a continuous function. If f is transitive then exactly one of the following holds:*

1. *f is sensitive.*

2. *f is a non-sensitive, uniformly rigid homeomorphism, and exactly one of the following holds:*

(a) *There exists a transitive point and either*

i. *f is equicontinuous and minimal; or,*

ii. *f is eventually sensitive and* $\mathrm{Eq}(X, f) = \mathrm{Trans}(f) \neq \emptyset$.

(b) *There is no transitive point, so* $\mathrm{Eq}(X, f) = \emptyset$*, and f is eventually sensitive.*

We close with the following question which we believe to be open:

Question 2.6.5. Does there exist a transitive compact Hausdorff system which is neither sensitive nor has any transitive points? Equivalently, is statement 2b) in Theorem 2.6.4 redundant?

2.7 Proof of Theorem 2.3.14

Recursively define the finite words C_n as follows. Let $C_0 \coloneqq \mathtt{10}$ and, for all $n \geq 1$, take

$$C_n \coloneqq \mathtt{1}^{8^n|C_0C_1\dots C_{n-1}|}\mathtt{0}^{2^n|C_0C_1\dots C_{n-1}|}.$$

For each $n \geq 1$ define

$$Q_n \coloneqq \mathtt{0}^{8^n|C_0C_1\dots C_{n-1}|}\mathtt{0}^{2^n|C_0C_1\dots C_{n-1}|}.$$

Let $W_0 \coloneqq C_0Q_1$ and for each $n \geq 1$ let $W_n \coloneqq W_0W_1\dots W_{n-1}C_0C_1\dots C_nQ_{n+1}$ (so $W_1 = W_0C_0C_1Q_2$, $W_2 = W_0W_1C_0C_1C_2Q_3$ and so on).

The first $8^n|C_0C_1\dots C_{n-1}|$ symbols of C_n will be referred to as the $\mathtt{1}$-part of C_n. Similarly, the last $2^n|C_0C_1\dots C_{n-1}|$ symbols of C_n will be referred to as the $\mathtt{0}$-part of C_n. We will refer to the word $C_0\dots C_nQ_{n+1}$ as the *closing segment* of W_n.

Remark 2.7.1. For any $n \in \mathbb{N}, |C_n| = |Q_n|$. We emphasise that Q_n consists solely of $\mathtt{0}$'s.

To prove Theorem 2.3.14 we will first need to prove the following lemma concerning the length of various words in our system.

Lemma 2.7.2. *For any* $n \in \mathbb{N}_0$,

$$6\left(8^{n+1}|C_{n+1}|\right) \geq |W_0W_1\dots W_nC_0\dots C_{n+1}| + 2|W_n|\,. \tag{2.11}$$

Proof. Let $P(n)$ be the statement

$$6\left(8^{n+1}|C_{n+1}|\right) \geq |W_0W_1\dots W_nC_0\dots C_{n+1}| + 2|W_n|\,.$$

Case when $n = 0$. Then $6(8^1|C_1|) = 960$ whilst $|W_0C_0C_1| + 2|W_0| = 88$. Hence $P(0)$ holds.

Assume that $P(n)$ is true for all $n \leq k$ for some $k \in \mathbb{N}_0$. Will will prove $P(k+1)$ holds. For $P(k+1)$:

$$\begin{aligned}
\text{RHS} &= |W_0W_1\dots W_kW_{k+1}C_0\dots C_{k+1}C_{k+2}| + 2|W_{k+1}| \\
&= |W_0W_1\dots W_kC_0\dots C_{k+1}| + 3|W_{k+1}| + |C_{k+2}| \\
&= 4|W_0W_1\dots W_kC_0\dots C_{k+1}| + 4|Q_{k+2}| && \text{as } |Q_{k+2}| = |C_{k+2}| \\
&= 4|W_0W_1\dots W_kC_0\dots C_{k+1}| + 4\left(8^{k+2}|C_0\dots C_{k+1}|\right) \\
&\quad + 4\left(2^{k+2}|C_0\dots C_{k+1}|\right) \\
&\leq 4|W_0W_1\dots W_kC_0\dots C_{k+1}| \\
&\quad + 4\left(8^{k+2}|W_0W_1\dots W_kC_0\dots C_{k+1}|\right) \\
&\quad + 4\left(2^{k+2}|W_0W_1\dots W_kC_0\dots C_{k+1}|\right) \\
&\leq 6\left(8^{k+2}|W_0W_1\dots W_kC_0\dots C_{k+1}|\right) && \text{as } 2(8^{k+2}) \geq 4 + 4(2^{k+2}) \\
&\leq 6\left(8^{k+2}\big(6\big(8^{k+1}|C_{k+1}|\big)\big)\right) && \text{by the induction hypothesis} \\
&\leq 6\left(8^{k+2}\big(8^{k+2}|C_{k+1}|\big)\right) \\
&\leq 6\left(8^{k+2}|C_{k+2}|\right) && \text{by definition} \\
&= \text{LHS}.
\end{aligned}$$

□

Remark 2.7.3. The length of the $\mathtt{1}$-part of C_{n+2} is $8^{n+2}|C_0 \dots C_{n+1}|$. Notice that,

$$8^{n+2}|C_0 \dots C_{n+1}| \geq 6\left(8^{n+1}|C_{n+1}|\right) + 2\left(8^{n+1}|C_{n+1}|\right) > 6\left(8^{n+1}|C_{n+1}|\right).$$

The final line is the LHS of Equation 2.11. By Lemma 2.7.2 this then means that the length of the $\mathtt{1}$-part of C_{n+1} is more than $2\left(8^{n+1}|C_{n+1}|\right)$ greater than the length of $W_0W_1 \dots W_nC_0 \dots C_{n+1}$ plus two times the length of W_n. This observation will prove important later.

Corollary 2.7.4. *For any $n, k \in \mathbb{N}_0$,*

$$6\left(8^{n+1+k}|C_{n+1+k}|\right) \geq |W_0W_1 \dots W_{n+k}C_0 \dots C_{n+1+k}| + \sum_{i=0}^{k-1}|W_{n+1+i}|. \tag{2.12}$$

Proof. Immediate from Lemma 2.7.2. □

We now define a shift system (X, σ) as follows. Let $\mathbf{x} \coloneqq C_0C_1C_2C_3\dots$ and $\mathbf{y} \coloneqq W_0W_1W_2W_3\dots$. Using the shift map σ take

$$X = \overline{\operatorname{Orb}_\sigma(\mathbf{y})},$$

where the closure is taken with regard to the full shift $\Sigma^{\mathbb{N}_0}$.

By construction, the point $\mathbf{y}$ is transitive in the system (X, σ). Furthermore, since $\mathbf{x} \in \omega(\mathbf{y})$, $\mathbf{x} \in X$. Notice that $\mathtt{0}^\infty, \mathtt{1}\mathtt{0}^\infty \in X$ since they are in $\omega(\mathbf{x})$. We will show through a sequence of lemmas that the system (X, σ) satisfies the conditions in the theorem, in particular we will show $(\mathbf{x}, \mathtt{0}^\infty)$ is a non-trivial even continuity pair but $(\mathbf{x}, \mathtt{1}\mathtt{0}^\infty) \notin \operatorname{EvP}(X, \sigma)$.

When working with dynamical systems, it can be helpful to visualise the forward orbit of a point as how it moves through time. In proving our claim we will use language like, 'the first time $\mathbf{x}$ visits $U \subseteq X$' or 'when $\mathbf{x}$ enters U for the first time.' By such statements we mean, the least such $c \in \mathbb{N}_0$ such that $\sigma^c(\mathbf{x}) \in U$. In similar fashion, we may speak of points travelling through words. For example, 'When $\mathbf{x}$ enters the $\mathtt{0}$-part of C_1 for the first time, $\mathbf{y}$ is travelling through W_0 for the first time; more specifically, $\mathbf{y}$ is travelling through the Q_1-part of Q_1C_0 for the first time.' This means that, if t is such that $\sigma^t(\mathbf{x})$ is in the $\mathtt{0}$-part of C_1 (i.e. $[\mathtt{0}^4]$) for the first time, then there exists a unique $a \leq t$ such that $\sigma^a(\mathbf{y}) \in W_0$ and $t-a < |W_0|$. Similarly there exists a unique $b \leq t$ such that $\sigma^b(\mathbf{y}) \in Q_1C_0$ and $t-b < |Q_1|$. In this particular example it can be seen that $t = 18$, $a = 0$ and $b = 2$.

We introduce the following, *first-hitting time*, notation. For $\mathbf{w} \in X$ and $A \subseteq X$ such that $N(\mathbf{w}, A) \neq \emptyset$,

$$\tau(\mathbf{w}, A) := \min N(\mathbf{w}, A).$$

For example, $\tau(\mathbf{x}, [C_2]) = 22$ whilst $\tau(\mathbf{y}, [Q_1C_0]) = 2$. This allows us to translate long-winded sentences such as '$\mathbf{y}$ enters $[Q_1C_0]$ for the first time before $\mathbf{x}$ enters $[C_2]$ for the first time' into an equation, in this example:

$$\tau(\mathbf{y}, [Q_1C_0]) < \tau(\mathbf{x}, [C_2]).$$

Lemma 2.7.5. $(\mathbf{x}, \mathtt{1}\mathtt{0}^\infty) \notin \mathrm{EvP}(X, \sigma)$.

Proof. Let $O = [\mathtt{10}]$; we claim this is an even-splitting neighbourhood of $\mathtt{10}^\infty$ with regard to $\mathbf{x}$. Let U and V be neighbourhoods of $\mathbf{x}$ and $\mathtt{10}^\infty$ respectively. Without loss of generality write $U = [C_0C_1C_2 \dots C_m]$ and $V = [\mathtt{10}^l]$, where $m \geq l \geq 1$. There exists a point $\mathbf{p} \in \mathrm{Orb}(\mathbf{y})$ such that $\mathbf{p} \in [C_0C_1C_2 \dots C_mQ_{m+1}]$. Define $t := |C_0C_1C_2 \dots C_m|$ and note that $\sigma^t(\mathbf{x}) \in [C_{m+1}]$, $\sigma^t(\mathbf{p}) \in [Q_{m+1}]$. Let $k = 8^{m+1}|C_0C_1C_2 \dots C_m|$; this is the length of the $\mathtt{1}$-part of C_{m+1}. It follows that

$$\sigma^{t+k-1}(\mathbf{x}) \in V,$$

and

$$\sigma^{t+k-1}(\mathbf{p}) \in \left[\mathtt{0}^{2^{m+1}}\right].$$

Hence $\sigma^{t+k-1}(\mathbf{p}) \notin O$. Since U and V were picked arbitrarily this means $(\mathbf{x}, 10^\infty) \notin \mathrm{EvP}(X, \sigma)$. In particular $\mathbf{x} \notin \mathrm{Ev}(X, \sigma)$. □

Lemma 2.7.6. *There are no points of even continuity.*

Proof. To see that $\mathrm{Ev}(X, \sigma) = \emptyset$, note that, since X is compact, $\mathrm{Ev}(X, \sigma) = \mathrm{Eq}(X, \sigma)$ (see [59, Theorem 7.23]). But since (X, σ) is a shift space with no isolated points it is sensitive, hence $\mathrm{Eq}(X, \sigma) = \emptyset$. □

We will now set about showing that $(\mathbf{x}, 0^\infty)$ is a non-trivial even continuity pair. To do this we will need the following lemma.

Lemma 2.7.7. *Let $n, a \in \mathbb{N}$ be such that $C_0 \dots C_n Q_{n+1}$ is an initial segment of $\mathbf{z} = \sigma^a(\mathbf{y})$. Then for any $k > n$,*

$$\tau\left(\mathbf{x}, [C_k]\right) \leq \tau\left(\mathbf{z}, [Q_k C_0]\right) \leq 6\left(8^{k-1}|C_{k-1}|\right)$$

In words, the first inequality means that $\mathbf{x}$ enters $[C_k]$ for the first time no later than $\mathbf{z}$ enters $[Q_k C_0]$ for the first time — which itself has happened by *time* "$6\left(8^{k-1}|C_{k-1}|\right)$" by the second inequality.

Proof. Let

$$n_0 = \max\{c \in \mathbb{N} \mid \exists b < a : \sigma^b(\mathbf{y}) \in [W_c]\}.$$

Note that n_0 is well defined and that $n_0 \geq n$. This means that $\mathbf{z}$ is travelling through W_{n_0} for the first time.

Let $k > n$ be given. The first inequality follows immediately from the construction: The word $Q_k C_0$ appears in the sequence of $\mathbf{z}$ for the first time only after the word $C_0 \dots C_{k-1}$. Similarly the word C_k appears in the sequence of $\mathbf{x}$ for the first time exactly after the word $C_0 \dots C_{k-1}$. Observing that $\mathbf{x} = C_0 C_1 \dots C_k C_{k+1} \dots$ now gives the inequality, $\tau(\mathbf{x}, [C_k]) \leq \tau(\mathbf{z}, [Q_k C_0])$. It remains to show that the second inequality holds.

Let $\mathbf{z}' \in \mathrm{Orb}(\mathbf{y})$ be the point at which $\mathbf{y}$ first enters $C_0 \dots C_n Q_{n+1}$; i.e. $\mathbf{z}' = \sigma^m(\mathbf{y})$ where $m = \tau\left(\mathbf{y}, [C_0 \dots C_n Q_{n+1}]\right)$. Note that $\mathbf{z}'$ lies at the start of the closing segment of

W_n. Indeed,

$$\mathbf{z}' = C_0 \dots C_n Q_{n+1} W_{n+1} W_{n+2} W_{n+3} \dots.$$

It is not difficult to see that $\tau\left(\mathbf{z}, [Q_k C_0]\right) \le \tau\left(\mathbf{z}', [Q_k C_0]\right)$; it takes $\mathbf{z}'$ at least as long to enter $[Q_k C_0]$ for the first time as it does for $\mathbf{z}$ to enter $[Q_k C_0]$ for the first time. (Observe that the letters (counting multiplicities) appearing in $\mathbf{z}$ before the first appearance $Q_k C_0$ can be written as a list of words (including multiplicities) which also appear in $\mathbf{z}'$ (with multiplicities) before the first appearance of $Q_k C_0$ there.) Hence the initial segment of $\mathbf{z}'$ up to the first appearance of $Q_k C_0$ is longer than that of the initial segment of $\mathbf{z}$ up to the first appearance of $Q_k C_0$. We know $k > n$. First suppose that $k > n + 1$. Then, by construction,

$$\begin{aligned}
\tau\left(\mathbf{z}', [Q_k C_0]\right) = & |C_0 \dots C_n Q_{n+1}| + \left(\sum_{i=n+1}^{k-2} |W_i|\right) + |W_0 \dots W_{k-2} C_0 \dots C_{k-1}| \\
\le & |W_0 \dots W_{k-2} C_0 \dots C_{k-1}| + 2|W_{k-2}| \\
\le & 6\left(8^{k-1}|C_{k-1}|\right) && \text{by Lemma 2.7.2.}
\end{aligned}$$

Since $\tau\left(\mathbf{z}', [Q_k C_0]\right) \le 6\left(8^{k-1}|C_{k-1}|\right)$, and $\tau(\mathbf{z}, [Q_k C_0]) \le \tau(\mathbf{z}', [Q_k C_0])$, we have that

$$\tau\left(\mathbf{z}, [Q_k C_0]\right) \le 6\left(8^{k-1}|C_{k-1}|\right).$$

Now suppose that $k = n + 1$. Then by Lemma 2.7.2

$$\begin{aligned}
\tau\left(\mathbf{z}', [Q_k C_0]\right) = & \tau\left(\mathbf{x}, [C_k]\right) \\
= & |C_0 \dots C_{k-1}| \\
\le & 6\left(8^{k-1}|C_{k-1}|\right).
\end{aligned}$$

□

Corollary 2.7.8. *Let $n \in \mathbb{N}$ be such that $C_0 \dots C_n Q_{n+1}$ is an initial segment of $\mathbf{z} = \sigma^a(\mathbf{y})$ for some $a \in \mathbb{N}_0$. For any $k > n$, $\mathbf{x}$ enters $[C_k]$ for the first time no later than $\mathbf{z}$ enters $[Q_k C_0]$ for the first time. Additionally, $\mathbf{z}$ enters $[Q_k C_0]$ for the first time before $\mathbf{x}$ enters the $\mathtt{0}$-part of C_k for the first time. In symbols:*

$$\tau\left(\mathbf{x}, [C_k]\right) \leq \tau\left(\mathbf{z}, [Q_k C_0]\right) \leq \tau\left(\mathbf{x}, \left[\mathtt{0}^{2^k |C_0 \dots C_{k-1}|}\right]\right).$$

Proof. By Lemma 2.7.7 it will suffice to show $6\left(8^{k-1}|C_{k-1}|\right) \leq \tau\left(\mathbf{x}, \left[\mathtt{0}^{2^k|C_0 \dots C_{k-1}|}\right]\right)$. Notice that $\mathbf{x}$ has to travel through the $\mathtt{1}$-part of C_k before reaching $\left[\mathtt{0}^{2^k|C_0 \dots C_{k-1}|}\right]$. The length of the $\mathtt{1}$-part of C_k is $8^k|C_0 \dots C_{k-1}| > 6\left(8^{k-1}|C_{k-1}|\right)$. □

We emphasise that Lemma 2.7.7 and Corollary 2.7.8 together mean that $\mathbf{z}$ enters $[Q_k C_0]$ for the first time before $\mathbf{x}$ enters the $\mathtt{0}$-part of C_k for the first time; in particular when $\mathbf{z}$ enters $[Q_k C_0]$ for the first time $\mathbf{x}$ still has to travel through at least $2\left(8^{k-1}|C_{k-1}|\right)$ more $\mathtt{1}$'s in the $\mathtt{1}$-part of C_k before it enters the $\mathtt{0}$-part of C_k.

Corollary 2.7.9. *The ordered pair $(\mathbf{x}, \mathtt{0}^\infty)$ is a non-trivial even continuity pair.*

Proof. Since $\mathtt{0}^\infty \in \omega(\mathbf{x})$, by definition $(\mathbf{x}, \mathtt{0}^\infty)$ is not a trivial even continuity pair. It thus suffices to show that $(\mathbf{x}, \mathtt{0}^\infty) \in \mathrm{EvP}(X, \sigma)$, i.e.

$$\forall O \in \mathcal{N}_{\mathtt{0}^\infty}\, \exists U \in \mathcal{N}_{\mathbf{x}}\, \exists V \in \mathcal{N}_{\mathtt{0}^\infty} : \forall n \in \mathbb{N}, \sigma^n(\mathbf{x}) \in V \implies \sigma^n(U) \subseteq O.$$

Without loss of generality, let O be the basic open neighbourhood $[\mathtt{0}^n]$ of $\mathtt{0}^\infty$. We claim $U = [C_0 C_1 \dots C_n]$ and $V = [\mathtt{0}^n]$ satisfy the even continuity condition. Since $\mathrm{Orb}(\mathbf{y})$ is dense and O, U and V are clopen, it suffices to consider only points in U which are elements of the orbit of $\mathbf{y}$. Let $\mathbf{z} \in \mathrm{Orb}(\mathbf{y}) \cap U$. Then $\mathbf{z} \in [C_0 \dots C_m Q_{m+1}]$ for some $m \geq n$. Suppose $l \in \mathbb{N}$ is such that $\sigma^l(\mathbf{x}) \in V$.

Case 1: $l \geq \tau\left(\mathbf{x}, [C_{m+1}]\right)$. Let $k \geq m+1$ be the greatest integer such that $l \geq \tau\left(\mathbf{x}, [C_k]\right)$. It follows that at time l, $\mathbf{x}$ is travelling through the $\mathtt{0}$-part of C_k for the first time, with at least n $\mathtt{0}$'s left to travel through. Furthermore, since $\tau\left(\mathbf{x},[C_k]\right) \leq \tau\left(\mathbf{z},[Q_kC_0]\right)$, and as $|Q_k| = |C_k|$, we have that $\mathbf{x}$ finishes travelling though C_k before $\mathbf{z}$ finishes travelling through the Q_k-part of Q_kC_0. This means that at time l there are at least as many $\mathtt{0}$'s remaining in Q_k (recall, Q_k consists solely of $\mathtt{0}$'s) for $\mathbf{z}$ to travel through than there are $\mathtt{0}$'s remaining in C_k for $\mathbf{x}$ to travel through. Since there are at least n $\mathtt{0}$'s left in C_k for $\mathbf{x}$ still to travel through (as $\mathbf{x} \in V$), it follows that $\mathbf{z} \in [\mathtt{0}^n] = O$.

Case 2: $l < \tau\left(\mathbf{x},[C_{m+1}]\right)$. The initial segments of $\mathbf{x}$ and $\mathbf{z}$ are identical up to and including the first occurrence of C_{m+1}. The word C_{m+1} begins with a '1', therefore $l \leq \tau\left(\mathbf{x},[C_{m+1}]\right) - n$, because $\sigma^l(\mathbf{x}) \in V = [\mathtt{0}^n]$. In particular, it follows that $\sigma^l(\mathbf{z}) \in [\mathtt{0}^n] = O$. □

We will now set about showing that $\mathrm{EqP}(X,\sigma) = \emptyset$. This will be completed in Lemma 2.7.11. First we show that $\mathbf{y}$ is not topologically equicontinuous with either one of the fixed points.

Lemma 2.7.10. *Neither $(\mathbf{y},\mathtt{0}^\infty)$ nor $(\mathbf{y},\mathtt{1}^\infty)$ is an equicontinuity pair.*

Proof. Recall that, to show that $(\mathbf{y},\mathbf{p}) \notin \mathrm{EqP}(X,\sigma)$, where $\mathbf{p} \in X$, we need to show that:

$$\exists O \in \mathcal{N}_{\mathbf{p}} : \forall U \in \mathcal{N}_{\mathbf{y}}\, \forall V \in \mathcal{N}_{\mathbf{p}}\, \exists n \in \mathbb{N} : \sigma^n(U) \cap V \neq \emptyset \text{ and } \sigma^n(U) \not\subseteq O.$$

Let $O = [\mathtt{0}]$. We claim O is a splitting neighbourhood of $\mathtt{0}^\infty$ with regard to $\mathbf{y}$. Let $U \in \mathcal{N}_{\mathbf{y}}$ and $V \in \mathcal{N}_{\mathtt{0}^\infty}$ be given and let $[W_0 \dots W_n] \subseteq U$ and $[\mathtt{0}^n] \subseteq V$. Let $m \in \mathbb{N}$ be such that $\mathtt{0}^n$ appears as a subword of C_m; notice that it follows that $\mathtt{0}^n$ is a subword of both C_k and W_k for all $k \geq m$. Let $l = \max\{n+2, m+2\}$. Notice that $2(8^{l-1}|C_{l-1}|) > n+1$. Let $t = \tau\left(\mathbf{y},[W_l]\right)$ and write $\mathbf{z} = \sigma^t(\mathbf{y})$. It follows that $\mathbf{z} \in U$. It is worth comparing $\mathbf{z}$ and $\mathbf{y}$ side by side.

$$\mathbf{z} = W_0W_1 \dots W_{l-1}C_0C_1 \dots C_lQ_{l+1}W_{l+1}W_{l+2} \dots,$$

and

$$\mathbf{y} = W_0 W_1 \ldots W_{l-1} W_0 W_1 \ldots W_{l-1} C_0 C_1 \ldots C_l Q_{l+1} W_{l+1} W_{l+2} \ldots .$$

Thus $\mathbf{z}$ and $\mathbf{y}$ share the same initial segment of $W_0 \ldots W_{l-1}$. Following this $\mathbf{z}$ enters $[C_0 C_1 \ldots C_l Q_{l+1}]$ for the first time whilst $\mathbf{y}$ enters $[W_l]$ for the first time. By Lemma 2.7.2,

$$6\left(8^{l-1}|C_{l-1}|\right) \geq |W_0 W_1 \ldots W_{l-2} C_0 \ldots C_{l-1}| . \tag{2.13}$$

In particular the length of the $\mathtt{1}$-part of C_l is greater than

$$|W_0 W_1 \ldots W_{l-2} C_0 \ldots C_{l-1}| + 2\left(8^{l-1}|C_{l-1}|\right).$$

It follows that

$$\tau\left(\mathbf{y}, [Q_l C_0 C_1 \ldots C_l]\right) \leq \tau\left(\mathbf{z}, \left[\mathtt{0}^{2^l|C_0 \ldots C_{l-1}|} Q_{l+1}\right]\right) - 2\left(8^{l-1}|C_{l-1}|\right).$$

That is, $\mathbf{y}$ enters $[Q_l C_0 C_1 \ldots C_l]$ for the first time before $\mathbf{z}$ enters the $\mathtt{0}$-part of $[C_l]$ for the first time; in particular when $\mathbf{y}$ enters $[Q_l C_0 C_1 \ldots C_l]$ for the first time $\mathbf{z}$ still has to travel through at least $2\left(8^{l-1}|C_{l-1}|\right)$ more $\mathtt{1}$'s in the $\mathtt{1}$-part of C_l before it enters the $\mathtt{0}$-part of C_l. Since $\tau\left(\mathbf{z}, [C_l]\right) \leq \tau\left(\mathbf{y}, [Q_l C_0 C_1 \ldots C_l]\right)$ we get that

$$\sigma^{\tau\left(\mathbf{y}, [Q_l C_0 C_1 \ldots C_l]\right)}(\mathbf{y}) \in V$$

but

$$\sigma^{\tau\left(\mathbf{y}, [Q_l C_0 C_1 \ldots C_l]\right)}(\mathbf{z}) \in \left[\mathtt{1}^{2\left(8^{l-1}|C_{l-1}|\right)}\right] \implies \sigma^{\tau\left(\mathbf{y}, [Q_l C_0 C_1 \ldots C_l]\right)}(\mathbf{z}) \notin O.$$

Hence $(\mathbf{y}, 0^\infty) \notin \mathrm{EqP}(X, \sigma)$. (Indeed, we have actually shown the stronger claim that $(\mathbf{y}, \mathtt{0}^\infty) \notin \mathrm{EvP}(X, \sigma)$.)

Now let $O = [\mathtt{1}]$. We claim O is a splitting neighbourhood of $\mathtt{1}^\infty$ with regard to $\mathbf{y}$. Let $U \in \mathcal{N}_{\mathbf{y}}$ and $V \in \mathcal{N}_{\mathtt{1}^\infty}$. Let $[W_0 \ldots W_n] \subseteq U$ and $[\mathtt{1}^n] \subseteq V$. Let $m \in \mathbb{N}$ be such that $\mathtt{1}^n$ appears as a subword of C_m; notice that it follows that $\mathtt{1}^n$ is a subword of both C_k

and W_k for all $k \geq m$. Let $l = \max\{n+2, m+2\}$. Notice that $2(8^{l-1}|C_{l-1}|) > n+1$. Let $t = \tau\left(\mathbf{y}, [W_l]\right)$ and write $\mathbf{z} = \sigma^t(\mathbf{y})$. It follows that $\mathbf{z} \in U$. As before, $\mathbf{z}$ and $\mathbf{y}$ share the same initial segment of $W_0 \ldots W_{l-1}$. After this $\mathbf{z}$ enters $[C_0C_1 \ldots C_lQ_{l+1}]$ for the first time whilst $\mathbf{y}$ enters $[W_l]$ for the first time. By an almost identical argument to the one we used in the previous paragraph, we know that

$$\sigma^{\tau\left(\mathbf{y},[Q_lC_0C_1\ldots C_l]\right)}(\mathbf{z}) \in \left[\mathtt{1}^{2\left(8^{l-1}|C_{l-1}|\right)}\right] \subseteq V.$$

However

$$\sigma^{\tau\left(\mathbf{y},[Q_lC_0C_1\ldots C_l]\right)}(\mathbf{y}) \in [\mathtt{0}] \subseteq X \setminus O.$$

Hence $(\mathbf{y}, \mathtt{1}^\infty) \notin \mathrm{EqP}(X, \sigma)$. $\square$

Lemma 2.7.11. *The system (X, σ) has no equicontinuity pairs.*

Proof. By Remark 2.2.20 it will suffice to show that $(\mathbf{y}, \mathbf{p}) \notin \mathrm{EqP}(X, \sigma)$ for any $\mathbf{p} \in X$. We need to show that, for any $p \in X$,

$$\exists O \in \mathcal{N}_{\mathbf{p}} : \forall U \in \mathcal{N}_{\mathbf{y}}\, \forall V \in \mathcal{N}_{\mathbf{p}}\, \exists n \in \mathbb{N} : \sigma^n(U) \cap V \neq \emptyset \text{ and } \sigma^n(U) \not\subseteq O.$$

Suppose that $(\mathbf{y}, \mathbf{p}) \in \mathrm{EqP}(X, \sigma)$; write $\mathbf{p} = \mathbf{p}_0\mathbf{p}_1\mathbf{p}_2 \ldots$. By Lemma 2.7.10 we have that $\mathbf{p} \notin \{\mathtt{0}^\infty, \mathtt{1}^\infty\}$. This means that there exist $i, j \in \mathbb{N}_0$ such that $\mathbf{p}_i = \mathtt{0}$ and $\mathbf{p}_j = \mathtt{1}$. Fix such an i and a j and take $k \geq \max\{i, j\}$. Let $O = [\mathbf{p}_0\mathbf{p}_1 \ldots \mathbf{p}_k]$. We claim O is a splitting neighbourhood of $\mathbf{p}$ with regard to $\mathbf{y}$. Let $U \in \mathcal{N}_{\mathbf{y}}$ and $V \in \mathcal{N}_{\mathbf{p}}$ be given and let $[W_0 \ldots W_n] \subseteq U$ and $[\mathbf{p}_0\mathbf{p}_1 \ldots \mathbf{p}_n] \subseteq V$; without loss of generality $n \geq k$. Let $m \in \mathbb{N}$ be such that $\mathbf{p}_0\mathbf{p}_1 \ldots \mathbf{p}_n$ appears as a subword of W_m; notice that it follows that $\mathbf{p}_0\mathbf{p}_1 \ldots \mathbf{p}_n$ is a subword of W_a for all $a \geq m$. Let $l \geq \max\{n+2, m+2\}$ be such that $2(8^{l-1}|C_{l-1}|) > n+1$. Let $t = \tau\left(\mathbf{y}, [W_l]\right)$ and write $\mathbf{z} = \sigma^t(\mathbf{y})$. It follows that $\mathbf{z} \in U$. It is worth comparing $\mathbf{z}$ and $\mathbf{y}$ side by side.

$$\mathbf{z} = W_0W_1 \ldots W_{l-1}C_0C_1 \ldots C_lQ_{l+1}W_{l+1}W_{l+2} \ldots,$$

and

$$\mathbf{y} = W_0W_1 \dots W_{l-1}W_0W_1 \dots W_{l-1}C_0C_1 \dots C_lQ_{l+1}W_{l+1}W_{l+2} \dots .$$

Notice $\mathbf{z}$ and $\mathbf{y}$ share the same initial segment given by $W_0 \dots W_{l-1}$. After this $\mathbf{z}$ enters $[C_0C_1 \dots C_lQ_{l+1}]$ for the first time whilst $\mathbf{y}$ enters $[W_l]$ for the first time. Notice that, for all $i \in \mathbb{N}_0, |W_i| \geq |Q_{i+1}| = |C_{i+1}|$. In addition $|W_0| \geq |C_0C_1|$. It follows that

$$\tau\left(\mathbf{z}, [C_l]\right) \leq \tau\left(\mathbf{y}, [W_{l-1}C_0 \dots C_lQ_{l+1}]\right). \tag{2.14}$$

Observe,

$$\tau\left(\mathbf{y}, [Q_lC_0 \dots C_lQ_{l+1}]\right) = |W_0 \dots W_{l-1}| + |W_0 \dots W_{l-2}| + |W_0 \dots W_{l-2}C_0 \dots C_{l-1}|.$$

Similarly observe

$$\tau\left(\mathbf{z}, [C_l]\right) = |W_0 \dots W_{l-1}| + |C_0 \dots C_{l-1}|.$$

Therefore,

$$\begin{aligned}
\tau\left(\mathbf{y}, [Q_lC_0 \dots C_lQ_{l+1}]\right) - \tau\left(\mathbf{z}, [C_l]\right) &= 2|W_0 \dots W_{l-2}| \\
&\leq |W_0W_1 \dots W_{l-2}| + 2|W_{l-2}| \\
&\leq 6\left(8^{l-1}|C_{l-1}|\right) \qquad \text{by Lemma 2.7.2.}
\end{aligned}$$

Thus

$$\tau\left(\mathbf{z}, [C_l]\right) + 6\left(8^{l-1}|C_{l-1}|\right) \geq \tau\left(\mathbf{y}, [Q_lC_0 \dots C_lQ_{l+1}]\right). \tag{2.15}$$

Putting inequalities (2.14) and (2.15) together we obtain:

$$\begin{aligned}\tau\left(\mathbf{z},[C_l]\right) &\leq \tau\left(\mathbf{y},[W_{l-1}C_0\dots C_lQ_{l+1}]\right)\\ &\leq \tau\left(\mathbf{y},[Q_lC_0\dots C_lQ_{l+1}]\right)\\ &\leq \tau\left(\mathbf{z},[C_l]\right)+6\left(8^{l-1}|C_{l-1}|\right)\\ &\leq \tau\left(\mathbf{z},\left[0^{2^l|C_0\dots C_{l-1}|}Q_{l+1}\right]\right)-2\left(8^{l-1}|C_{l-1}|\right).\end{aligned}$$

The final inequality follows because, by definition, the length of the $\mathtt{1}$-part of C_l is more than $8^l|C_{l-1}|$. It follows that, whilst $\mathbf{y}$ enters $W_0W_1\dots W_{l-2}C_0\dots C_{l-1}$ for the second time, $\mathbf{z}$ is travelling through the $\mathtt{1}$-part of C_l. When $\mathbf{y}$ finishes travelling though $W_0W_1\dots W_{l-2}C_0\dots C_{l-1}$ for the second time (and enters $[Q_lC_0\dots C_lQ_{l+1}]$ for the first time), $\mathbf{z}$ still has to travel through at least $2\left(8^{l-1}|C_{l-1}|\right)$ more $\mathtt{1}$'s in the $\mathtt{1}$-part of C_l before it enters the $\mathtt{0}$-part of C_l. Because $\mathbf{p}_0\dots\mathbf{p}_n$ is a subword of W_{l-2}, which is a subword of $W_0W_1\dots W_{l-2}C_0\dots C_{l-1}$, and since $[\mathbf{p}_0\mathbf{p}_1\dots\mathbf{p}_n]\subseteq V$ it follows that $\mathbf{y}$ enters V whilst travelling through $W_0W_1\dots W_{l-2}C_0\dots C_{l-1}$ for the second time. Take $c\in\mathbb{N}_0$ such that $\sigma^c(\mathbf{y})\in V$ where $c>\tau\left(\mathbf{y},[W_l]\right)$ and $c<\tau\left(\mathbf{y},[Q_lC_0\dots C_lQ_{l+1}]\right)$. Since $2\left(8^{l-1}|C_{l-1}|\right)>n+1$ it follows that $\sigma^c(\mathbf{z})\in[1^{n+1}]$. But the word inducing O (i.e. $\mathbf{p}_0\dots\mathbf{p}_k$) contains at least one $\mathtt{0}$ and $n+1\geq k+1$. Hence $\sigma^c(\mathbf{z})\notin O$; in particular $\sigma^c(U)\cap V\neq\emptyset$ and $\sigma^c(U)\not\subseteq O$. □

Lemma 2.7.12. *The dynamical system (X,σ) is Auslander–Yorke chaotic but not Devaney chaotic.*

Proof. The system is both transitive and sensitive, this means it is Auslander–Yorke chaotic. It may be verified that the only periodic points are $\mathtt{0}^\infty$ and $\mathtt{1}^\infty$, hence the system is not Devaney chaotic. □

Lemma 2.7.13. *The system (X,σ) is both strongly and densely Li–Yorke chaotic.*

Proof. Huang and Ye [58] show that a compact metric system without isolated points is both strongly and densely Li–Yorke chaotic if the system is transitive and there is a fixed point. (See also [35, Corollary 7.3.7].) Since our system satisfies these conditions the result follows. □

CHAPTER 3

PRESERVATION OF SHADOWING IN DISCRETE DYNAMICAL SYSTEMS

In the course of showing that systems with shadowing are built up from shifts of finite type, Good and Meddaugh [48] show that an inverse limit of systems with shadowing has shadowing and that factor maps which almost lift pseudo-orbits (see below for a definition) also preserve shadowing. A continuous map on a compact metric space $f\colon X \to X$ induces a continuous map 2^f on the hyperspace of closed subsets of X with the Hausdorff metric. In [41] it is shown that 2^f has shadowing if and only if f has shadowing. It is a natural question, therefore, to ask which notions of shadowing are preserved under operations on dynamical systems. In this chapter, we systematically address this question for various notions of shadowing, namely shadowing, h-shadowing, eventual shadowing, orbital shadowing, first and second weak shadowing, inverse shadowing and various types of limit shadowing. For each of these shadowing types we ask:

- Is it preserved in the induced hyperspatial system?
- Is it preserved in some, or all, induced symmetric product systems?
- Under what conditions is it preserved under semiconjugacy?
- Does an inverse limit system comprised of systems with it also have it?
- Does an arbitrary product of systems exhibiting it also exhibit it?

We provide definitive answers to many of these questions, although we leave some unanswered; particular difficulties seem to arise when dealing with the limit shadowing properties. Clearly, some of these questions have been asked and answered by others. In such cases we provide references.

In keeping with our goals of generality, our setting throughout this chapter will be a compact Hausdorff space X.

The chapter is arranged as follows. We begin with some preliminaries in section 3.1, where amongst other things we give the definitions of symmetric product, inverse limit, and product space. In section 3.2 we provide the definitions of the shadowing types under consideration. We start with the usual metric definitions, before giving the uniform definitions which coincide with the metric ones when the underlying space is compact. Finally, we follow the example of Good and Macías [45] by providing definitions in terms of open covers which coincide with the uniform definitions when the space is compact Hausdorff. We then devote a section to the preservation of each of the aforementioned types of shadowing.

The table below provides a summary of our results.

	2^X	$F_n(X)$	Factor maps	$\varprojlim$	$\prod$
shadowing	✓	✗ (but ✓for $F_2(X)$)	iff ALP	✓	✓
h-shad.	✓	✗ (but ✓for $F_2(X)$)	?	?	✓*
eventual shad.	✗	✗ (but ✓for $F_2(X)$)	iff eALP	✓	✓
orbital shad.	✗	✗	iff oALP	✓	✗
strong orb. shad.	✗	✗	iff soALP	✓	✗
1st weak shad.	✗	✗	iff w1ALP	✓	✗
2nd weak shad.	✓	✓	✓	✓	✓
limit shad.	?	✗ (but ✓for $F_2(X)$)	iff ALAP	?	✓
s-limit shad.	?	✗ (but ✓for $F_2(X)$)	iff ALAεP	?	✓
orb. limit shad.	✗	✗	iff oALAP	?	✗
inverse shad.	✓	✓	?	✓	✓

KEY:

- ✓– "is preserved by."
- ✗– "there is a (surjective) counterexample in which it is not preserved."
- ✓* – "iff all but a finite number of the component systems are surjective."
- ALP – almost lifts pseudo-orbits.[1]

For this chapter we use ω to denote $\mathbb{N} \cup \{0\}$.

[1]The definition of this, and of the other ALP type properties listed in the 'Factor maps' column, will appear in the respective sections in which they are used.

3.1 Preliminaries

3.1.1 Hyperspaces

Recall the content concerning the hyperspace of compact sets 2^X (section 1.1.4). For $n \geq 2$, we denote by $F_n(X)$ the *n-fold symmetric product of* X, i.e.

$$F_n(X) = \{A \in 2^X \mid A \text{ contains at most } n \text{ points.}\}.$$

$F_n(X)$ is a compact Hausdorff space with the subspace topology from 2^X.

If X is compact Hausdorff and $f\colon X \to X$, then the image of a closed set C under f is again a closed subset of X. Therefore, a given dynamical system (X, f) gives rise to an induced system $(2^X, 2^f)$ on the hyperspace by $2^f : C \mapsto f(C) = \{f(x) \mid x \in C\}$. The restriction of 2^f to F_n is denoted f_n.

3.1.2 Products and inverse limits

Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a family of topological spaces. Given the product

$$\prod_{\lambda\in\Lambda} X_\lambda = \left\{(x_\lambda)_{\lambda\in\Lambda} \mid \forall\lambda \in \Lambda, x_\lambda \in X_\lambda\right\},$$

for each $\eta \in \Lambda$ the projection $\pi_\eta\colon \prod_{\lambda\in\Lambda} X_\lambda \to X_\eta$ is defined by $\pi_\eta((x_\lambda)) = x_\eta$. The *Tychonoff product topology* on $\prod_{\lambda\in\Lambda} X_\lambda$ is the topology generated by basic open sets of the form

$$\bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}(U_{\lambda_i}),$$

for some $n \in \mathbb{N}$ and open U_{λ_i} in X_{λ_i}.

If, in the above, each space X_λ is compact Hausdorff with uniformity $\mathscr{U}_\lambda$, then the following is a basic entourage in the uniformity on the product space:

$$\prod_{\lambda\in\Lambda} E_\lambda,$$

where $E_\lambda \in \mathscr{U}_\lambda$ for all λ and $E_\lambda = X_\lambda \times X_\lambda$ for all but finitely many λ. The set of all such entourages forms a base for the uniformity on the product space.

Given a collection of dynamical systems (X_λ, f_λ) we refer to the *product system* $\left(\prod_{\lambda\in\Lambda} X_\lambda, f\right)$, where f is the induced map given by $f((x_\lambda)_{\lambda\in\Lambda}) = (f_\lambda(x_\lambda))_{\lambda\in\Lambda}$. It is straightforward to check that f is continuous (and onto) if and only if each f_λ is continuous (and onto).

Definition 3.1.1. Let $(\Lambda, \leq)$ be a directed set. For each $\lambda \in \Lambda$, let (X_λ, f_λ) be a surjective dynamical system on a compact Hausdorff space and, for each pair λ, η, with $\lambda \leq \eta$, let $g_\lambda^\eta \colon X_\eta \to X_\lambda$ be a continuous (not necessarily surjective) map. Suppose further that

1. g_λ^λ is the identity map for all $\lambda \in \Lambda$, and
2. for all triplets $\lambda \leq \eta \leq \nu$, $g_\lambda^\nu = g_\lambda^\eta \circ g_\eta^\nu$, and
3. for all pairs $\lambda \leq \eta$, $f_\lambda \circ g_\lambda^\eta = g_\lambda^\eta \circ f_\eta$ (i.e. that g_λ^η is a semiconjugacy).

Then the *inverse limit* of $(X_\lambda, g_\lambda^\eta)$ is the compact Hausdorff space[1]

$$\varprojlim\{X_\lambda, g_\lambda^\eta\} = \{(x_\lambda)_{\lambda\in\Lambda} \in \prod X_\lambda \mid \forall \lambda, \eta \text{ with } \lambda \leq \eta, x_\lambda = g_\lambda^\eta(x_\eta)\},$$

with topology inherited as a subspace of the product $\prod X_\lambda$. Moreover, the maps f_λ induce a continuous map

$$\begin{aligned} f \colon \varprojlim\{X_\lambda, g_\lambda^\eta\} &\to \varprojlim\{X_\lambda, g_\lambda^\eta\}, \\ (x_\lambda)_{\lambda\in\Lambda} &\mapsto (f_\lambda(x_\lambda))_{\lambda\in\Lambda}, \end{aligned}$$

resulting in the *inverse system* $\left((X_\lambda, f_\lambda), g_\lambda^\eta\right) = \left(\varprojlim\{X_\lambda, g_\lambda^\eta\}, f\right)$.

[1]See [37, Theorem 3.6].

Given a system (X, f), a frequently studied inverse limit system is that of the shift map σ taking $(x_0, x_1, x_2, \dots)$ to $(x_1, x_2, x_3, \dots)$ acting as a homeomorphism on the inverse limit space $\varprojlim(X, f) = \{(x_i)_{i\in\omega} \mid x_i = f(x_{i+1}),\ i \geq 0\}$. Notice Definition 3.1.1 subsumes this definition. Given a dynamical system (X, f) we will refer to the system $(\varprojlim(X, f), \sigma)$ as the *natural extension* of (X, f). It may also be called the standard inverse limit associated with (X, f). Note that preservation of shadowing properties by the natural extension have been studied by various authors [13, 26, 52].

Definition 3.1.2. We say that an inverse system is *surjective* provided that for any $\lambda \in \Lambda$ and any $\gamma \geq \lambda$, $g^\gamma_\lambda(X_\gamma) = X_\lambda$. We say that the system is *Mittag–Leffler*, provided that for all $\lambda \in \Lambda$ there exists $\gamma \geq \lambda$ such that for all $\eta \geq \gamma$, we have $g^\gamma_\lambda(X_\gamma) = g^\eta_\lambda(X_\eta)$. For such a λ and γ we say γ witnesses the *Mittag–Leffler condition* with respect to λ.

Clearly every surjective inverse system is also Mittag–Leffler. The converse is not true. To take an elementary example, for each $n \in \omega$, let $X_n = \{0, 1\}$ and let $f_n \colon X_n \to X_n$ be the map under which both points in X_n are fixed points. For any $l, m \in \omega$ with $l > m$, let $g^l_m \colon X_l \to X_m$ be given by $g^l_m(x) = 0$ for all $x \in X_l$. By definition, the inverse system is not surjective. However it is straightforward to see that it is Mittag–Leffler.

A useful fact about Mittag–Leffler systems is that if γ witnesses the condition with respect to λ and $x \in g^\gamma_\lambda(X_\gamma) \subseteq X_\lambda$ then $\pi^{-1}_\lambda(x) \cap \varprojlim\{X_\lambda, g^\eta_\lambda\} \neq \emptyset$. See [48] for more on this.

3.2 Shadowing types

3.2.1 Shadowing in metric spaces

Let (X, f) be a dynamical system where X is a metric space.

Definition 3.2.1. A sequence $(x_i)_{i\in\omega}$ in X is said to be a δ-*pseudo-orbit* for some $\delta > 0$ if $d(f(x_i), x_{i+1}) < \delta$ for each $i \in \omega$.

We say $(x_i)_{i\in\omega}$ is an *asymptotic pseudo-orbit* provided that

$$\lim_{i\to\infty} d(f^i(x_i), x_{i+1}) = 0.$$

We say $(x_i)_{i\in\omega}$ is an *asymptotic δ-pseudo-orbit* if it is both a δ-pseudo-orbit and an asymptotic pseudo-orbit.

Definition 3.2.2. A point $z \in X$ is said to *ε-shadow* a sequence $(x_i)_{i\in\omega}$ for some $\varepsilon > 0$ if $d(x_i, f^i(z)) < \varepsilon$ for each $i \in \omega$. It *asymptotically shadows* the sequence if $\lim_{i\to\infty} d(x_i, f^i(z)) = 0$. Finally it *asymptotically ε-shadows* the sequence if it both ε-shadows and asymptotically shadows it.

Definition 3.2.3. The dynamical system (X, f) is said to have *shadowing* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed.

Definition 3.2.4. A system (X, f) has the *eventual shadowing* property provided that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for each δ-pseudo-orbit $(x_i)_{i\in\omega}$, there exists $z \in X$ and $N \in \mathbb{N}$ such that $d(f^i(z), x_i) < \varepsilon$ for all $i \geq N$.

Definition 3.2.5. The system (X, f) is said to have *h-shadowing*, or *shadowing with exact hit*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite δ-pseudo-orbit $\{x_0, x_1, x_2, \ldots, x_m\}$ there exists $y \in X$ such that $d(f^i(y), x_i) < \varepsilon$ for all $i < m$ and $f^m(y) = x_m$.

Definition 3.2.6. The system (X, f) is said to have *limit shadowing* if every asymptotic pseudo-orbit is asymptotically shadowed.

Definition 3.2.7. The system (X, f) is said to have *s-limit shadowing* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following two conditions hold:

1. every δ-pseudo-orbit is ε-shadowed, and

2. every asymptotic δ-pseudo-orbit is asymptotically ε-shadowed.

Definition 3.2.8. The system (X, f) has the *orbital shadowing* property if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point z such that

$$d_H\left(\overline{\{x_i\}_{i\in\omega}}, \overline{\{f^i(z)\}_{i\in\omega}}\right) < \varepsilon.$$

Definition 3.2.9. The system (X, f) has the *strong orbital shadowing* property if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point z such that, for all $N \in \omega$,

$$d_H\left(\overline{\{x_{N+i}\}_{i\in\omega}}, \overline{\{f^{N+i}(z)\}_{i\in\omega}}\right) < \varepsilon.$$

Definition 3.2.10. The system (X, f) has the *asymptotic orbital shadowing* property if for any asymptotic pseudo-orbit $(x_i)_{i\geq 0}$ there exists a point $x \in X$ such that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{N+i}\}_{i\geq 0}}, \overline{\{f^{N+i}(x)\}_{i\geq 0}}) < \varepsilon.$$

This is equivalent (see [47, Theorem 22]) to the following definition of orbital limit shadowing studied by Pilyugin and others [84].

Definition 3.2.11. The system (X, f) has the *orbital limit shadowing* property if given any asymptotic pseudo-orbit $(x_i)_{i\geq 0} \subseteq X$, there exists a point $x \in X$ such that

$$\omega((x_i)_{i\geq 0}) = \omega(x).$$

Definition 3.2.12. The system (X, f) has the *first weak shadowing* property if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point z such that

$$\{x_i\}_{i\in\omega} \subseteq B_\varepsilon\left(\text{Orb}(z)\right).$$

Definition 3.2.13. The system (X, f) has the *second weak shadowing* property if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point z such that

$$\mathrm{Orb}(z) \subseteq B_\varepsilon\left(\{x_i\}_{i\in\omega}\right).$$

Let X be a compact metric space, and let $f\colon X \to X$ be a continuous onto function. Let X^ω denote the product space of all infinite sequences; note that this is compact metric. Then, for any given $\delta > 0$, let $\Phi_f(\delta) \subseteq X^\omega$ be the set of all δ-pseudo-orbits. We call a mapping $\varphi\colon X \to \Phi_f(\delta)$ such that, for each $x \in X$, $\varphi(x)_0 = x$, a δ-method for f where $\varphi(x)_k$ is used to denote the k^{th} term in the sequence $\varphi(x)$. We denote by $\mathcal{T}_0(f, \delta)$ the set of all δ-methods.

Definition 3.2.14. Let $f\colon X \to X$ be a continuous onto function. We say that f experiences *inverse shadowing with respect to the class* $\mathcal{T}_0$ (henceforth simply, *inverse shadowing*) if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in X$ and any $\varphi \in \mathcal{T}_0$ there exists $y \in X$ such that x ε-shadows $\varphi(y)$; i.e.

$$\forall k \in \omega, d(\varphi(y)_k, f^k(x)) < \varepsilon.$$

3.2.2 Shadowing in uniform spaces

Let (X, f) be a dynamical system where X is a uniform space with uniformity $\mathscr{U}$. The definitions below coincide with their corresponding ones in the previous subsection when the underlying space X is compact metric.

Definition 3.2.15. A sequence $(x_i)_{i\in\omega}$ is said to be a *D-pseudo-orbit* for some $D \in \mathscr{U}$ if $(f(x_i), x_{i+1}) \in D$ for each $i \in \omega$.

We say $(x_i)_{i\in\omega}$ is an *asymptotic pseudo-orbit* provided that for each $E \in \mathscr{U}$ there exists $N \in \mathbb{N}$ such that for all $i \geq N$ $(f^i(x_i), x_{i+1}) \in E$.

We say $(x_i)_{i\in\omega}$ is an *asymptotic D-pseudo-orbit* if it is both a D-pseudo-orbit and an asymptotic pseudo-orbit.

Definition 3.2.16. A point $z \in X$ is said to *E-shadow* a sequence $(x_i)_{i\in\omega}$ for some $E \in \mathscr{U}$ if $(x_i, f^i(z)) \in E$ for each $i \in \omega$. It *asymptotically shadows* the sequence if for each $E \in \mathscr{U}$ there exists $N \in \mathbb{N}$ such that for all $i \geq N$ $(x_i, f^i(z)) \in E$. Finally it *asymptotically E-shadows* the sequence if it both E-shadows and asymptotically shadows it.

Definition 3.2.17. The dynamical system (X, f) is said to have *shadowing* if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that every D-pseudo-orbit is E-shadowed.

Definition 3.2.18. A system (X, f) has the *eventual shadowing* property provided that for all $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that for each D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists $z \in X$ and $N \in \mathbb{N}$ such that $(f^i(z), x_i) \in E$ for all $i \geq N$.

Definition 3.2.19. The system (X, f) is said to have *h-shadowing* if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that for every finite D-pseudo-orbit $\{x_0, x_1, x_2, \ldots, x_m\}$ there exists $y \in X$ such that $(f^i(y), x_i) \in E$ for all $i < m$ and $f^m(y) = x_m$.

Definition 3.2.20. The system (X, f) is said to have *limit shadowing* if every asymptotic pseudo-orbit is asymptotically shadowed.

Definition 3.2.21. The system (X, f) is said to have *s-limit shadowing* if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that the following two conditions hold:

1. every D-pseudo-orbit is E-shadowed, and
2. every asymptotic D-pseudo-orbit is asymptotically E-shadowed.

Definition 3.2.22. The system (X, f) has the *orbital shadowing* property if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point $z \in X$ such that

$$\left(\overline{\{x_i\}_{i\in\omega}}, \overline{\{f^i(z)\}_{i\in\omega}}\right) \in 2^E.$$

In this case we say z E-orbital-shadows $(x_i)_{i\in\omega}$.

Definition 3.2.23. The system (X, f) has the *strong orbital shadowing* property if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point $z \in X$ such that, for all $N \in \omega$,

$$\left(\overline{\{x_{N+i}\}_{i\in\omega}}, \overline{\{f^{N+i}(z)\}_{i\in\omega}}\right) \in 2^E.$$

In this case we say z E-strong-orbital-shadows $(x_i)_{i\in\omega}$.

Definition 3.2.24. The system (X, f) has the *asymptotic orbital shadowing* property if for any asymptotic pseudo-orbit $(x_i)_{i\geq 0}$ there exists a point $x \in X$ such that for any $E \in \mathscr{U}$ there exists $N \in \mathbb{N}$ such that

$$(\overline{\{x_{N+i}\}_{i\geq 0}}, \overline{\{f^{N+i}(x)\}_{i\geq 0}}) \in 2^E.$$

Definition 3.2.25. The system (X, f) has the *orbital limit shadowing* property if given any asymptotic pseudo-orbit $(x_i)_{i\geq 0} \subseteq X$, there exists a point $x \in X$ such that

$$\omega((x_i)_{i\geq 0}) = \omega(x).$$

Definition 3.2.26. The system (X, f) has the *first weak shadowing* property if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point z such that

$$\{x_i\}_{i\in\omega} \subseteq B_E\left(\mathrm{Orb}(z)\right).$$

Definition 3.2.27. The system (X, f) has the *second weak shadowing* property if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point

z such that

$$\mathrm{Orb}(z) \subseteq B_E\left(\{x_i\}_{i\in\omega}\right).$$

Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous onto function. Let X^ω denote the product space of all infinite sequences. Then, for any given $D \in \mathscr{U}$, let $\Phi_f(D) \subseteq X^\omega$ be the set of all D-pseudo-orbits. We call a mapping $\varphi\colon X \to \Phi_f(D)$ such that, for each $x \in X$, $\varphi(x)_0 = x$, a D-method for f where $\varphi(x)_k$ is used to denote the k^{th} term in the sequence $\varphi(x)$. We denote by $\mathcal{T}_0(f, D)$ the set of all D-methods.

Definition 3.2.28. Let $f\colon X \to X$ be a continuous onto function. We say that f experiences *inverse shadowing with respect to the class* $\mathcal{T}_0$ (henceforth simply, *inverse shadowing*) if, for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that for any $x \in X$ and any $\varphi \in \mathcal{T}_0$ there exists $y \in X$ such that x E-shadows $\varphi(y)$; i.e.

$$\forall k \in \omega, (\varphi(y)_k, f^k(x)) \in E.$$

Remark 3.2.29. It follows from Remark 1.1.2 that, without loss of generality, we may assume all entourages referred to in the above definitions are symmetric. Throughout what follows we will make this assumption.

3.2.3 Shadowing with open covers

Let X be a topological space and $f\colon X \to X$ a continuous function. The definitions below coincide with their corresponding ones in the previous subsection when the underlying space X is compact Hausdorff.

Definition 3.2.30. A sequence $(x_i)_{i\in\omega}$ is said to be a $\mathcal{U}$*-pseudo-orbit* for some open cover $\mathcal{U}$ if for any $i \in \omega$ there exists $U \in \mathcal{U}$ with $f(x_i), x_{i+1} \in U$.

Definition 3.2.31. A point $z \in X$ is said to $\mathcal{U}$*-shadow* a sequence $(x_i)_{i\in\omega}$ for some open cover $\mathcal{U}$ if for any $i \in \omega$ there exists $U \in \mathcal{U}$ with $x_i, f^i(z) \in U$. We say $z \in X$ *eventually*

$\mathcal{U}$-*shadows* a sequence $(x_i)_{i\in\omega}$ for some open cover $\mathcal{U}$ if there exists $N \in \mathbb{N}$ such that for any $i \geq N$ there exists $U \in \mathcal{U}$ with $x_i, f^i(z) \in U$.

Definition 3.2.32. The dynamical system (X, f) is said to have *shadowing* if for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that every $\mathcal{V}$-pseudo-orbit is $\mathcal{U}$-shadowed.

Definition 3.2.33. The dynamical system (X, f) is said to have *eventual shadowing* if for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that every $\mathcal{V}$-pseudo-orbit is eventually $\mathcal{U}$-shadowed.

Definition 3.2.34. The dynamical system (X, f) is said to have *h-shadowing* if for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that for any finite $\mathcal{V}$-pseudo-orbit $\{x_0, x_1, x_2, \ldots, x_m\}$ there exists $y \in X$ such that for any $i < m$ there exists $U \in \mathscr{U}$ with $f^i(y), x_i \in U$ and $f^m(y) = x_m$.

Definition 3.2.35. The system (X, f) has the *orbital shadowing* property if for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that for any $\mathcal{V}$-pseudo-orbit $(x_i)_{i\in\omega}$ there exists a point $z \in X$ such that

$$\forall y \in \overline{\mathrm{Orb}(z)}\, \exists U \in \mathcal{U}\, \exists y' \in \overline{\{x_i \mid i \in \omega\}} : y, y' \in U,$$

and

$$\forall y' \in \overline{\{x_i \mid i \in \omega\}}\, \exists U \in \mathcal{U}\, \exists y \in \overline{\mathrm{Orb}(z)} : y, y' \in U.$$

Definition 3.2.36. The system (X, f) has the *strong orbital shadowing* property if for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that for any $\mathcal{V}$-pseudo-orbit $(x_i)_{i\in\omega}$ there exists a point $z \in X$ such that for any $N \in \omega$

$$\forall y \in \overline{\mathrm{Orb}(f^N(z))}\, \exists U \in \mathcal{U}\, \exists y' \in \overline{\{x_{N+i} \mid i \in \omega\}} : y, y' \in U,$$

and

$$\forall y' \in \overline{\{x_{N+i} \mid i \in \omega\}}\, \exists U \in \mathcal{U}\, \exists y \in \overline{\mathrm{Orb}(f^N(z))} : y, y' \in U.$$

Definition 3.2.37. The system (X, f) has the *first weak shadowing* property if for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that for any $\mathcal{V}$-pseudo-orbit $(x_i)_{i\in\omega}$ there exists a point $z \in X$ such that

$$\forall y \in \overline{\{x_i \mid i \in \omega\}}\, \exists U \in \mathcal{U}\, \exists y' \in \overline{\mathrm{Orb}(z)} : y, y' \in U.$$

Definition 3.2.38. The system (X, f) has the *second weak shadowing* property if for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that for any $\mathcal{V}$-pseudo-orbit $(x_i)_{i\in\omega}$ there exists a point $z \in X$ such that

$$\forall y \in \overline{\mathrm{Orb}(z)}\, \exists U \in \mathcal{U}\, \exists y' \in \overline{\{x_i \mid i \in \omega\}} : y, y' \in U.$$

Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous onto function. Let X^ω denote the product space of all infinite sequences. Then, for any given finite open cover $\mathcal{U}$, let $\Phi_f(\mathcal{U}) \subseteq X^\omega$ be the set of all $\mathcal{U}$-pseudo-orbits. We call a mapping $\varphi\colon X \to \Phi_f(\mathcal{U})$ such that, for each $x \in X$, $\varphi(x)_0 = x$, a $\mathcal{U}$-method for f where $\varphi(x)_k$ is used to denote the k^{th} term in the sequence $\varphi(x)$. We denote by $\mathcal{T}_0(f, \mathcal{U})$ the set of all $\mathcal{U}$-methods.

Definition 3.2.39. Let $f\colon X \to X$ be a continuous onto function. We say that f experiences *inverse shadowing with respect to the class* $\mathcal{T}_0$ (henceforth simply, *inverse shadowing*) if, for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that for any $x \in X$ and any $\varphi \in \mathcal{T}_0$ there exists $y \in X$ such that $\varphi(y)$ $\mathcal{U}$-shadows x; i.e.

$$\forall k \in \omega\, \exists U \in \mathcal{U} : \varphi(y)_k, f^k(x) \in U.$$

For the rest of this chapter, unless otherwise stated, X is taken to be a compact Hausdorff space and $f\colon X \to X$ a continuous function. Similarly, unless otherwise stated, by "dynamical system", we are assuming the underlying phase space is compact Hausdorff.

3.3 Preservation of shadowing

As mentioned in the introduction, Bowen [17] was one of the first to use the property of shadowing in his study of Axiom A diffeomorphisms and since then it has been both used as a tool and studied extensively in a property in its own right (see, for examples, [10, 18, 29, 30, 31, 48, 63, 70, 76, 80, 81, 83, 86, 88, 93]).

Recall the following definition from the preliminaries: the dynamical system (X, f) is said to have *shadowing* if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that every D-pseudo-orbit is E-shadowed.

3.3.1 Induced map on the hyperspace of compact sets

The following theorem was proved in [41] for compact metric systems. The proof easily generalises to compact Hausdorff systems.

Theorem 3.3.1. [41, Theorem 3.4] *Let X be a compact Hausdorff space and let $f\colon X \to X$ be a continuous function. Then (X, f) has shadowing if and only if $(2^X, 2^f)$ has shadowing.*

3.3.2 Symmetric products

In [43] the authors show that, for any $n \in \mathbb{N}$, if f_n has shadowing then f has shadowing. They also show that if f has shadowing then f_2 has shadowing. However they provide an example ($z \mapsto z^2$ on the unit circle S^1) for which f has shadowing but f_n does not have shadowing for any $n \geq 3$. The following is another such example and will be recalled later.

Example 3.3.2. Let X be the closed unit interval and let $f\colon X \to X$ be the standard tent map, i.e.

$$f(x) = \begin{cases} 2x & \text{if} \quad x \in [0, \frac{1}{2}] \\ 2(1-x) & \text{if} \quad x \in (\frac{1}{2}, 1]\,. \end{cases}$$

Then f has shadowing [10, Example 3.5] but f_n does not have shadowing for any $n \geq 3$.

Fix $n \geq 3$. Let $c = \frac{2}{3}$. Let $\varepsilon = \frac{1}{12}$ and let $\delta > 0$ be given; without loss of generality $\delta < \frac{1}{12}$. Choose $y \in [0, \delta)$ such that there exists $k \in \mathbb{N}$ such that $f^k(y) = c$ and $f^i(y) < \frac{1}{2}$ for all $i < k$. Construct a δ-pseudo-orbit as follows. For any $i \in \omega$, let $A_i = \{0, f^{i \bmod k}(y), c\}$. It is easy to see that $(A_i)_{i \in \omega}$ is a δ-pseudo-orbit. Suppose that $A \in F_n(X)$ ε-shadows this pseudo-orbit. First observe that, since the pseudo-orbit is always a subset of the interval $[0, \frac{2}{3}]$, shadowing entails that $f_n^i(A) \subseteq [0, \frac{3}{4}]$ for any $i \in \omega$. Next notice that, by construction, there exists $k_0 \in \{1, \ldots k-1\}$ such that $A_{mk+k_0} \cap (\varepsilon, 2\varepsilon] \neq \emptyset$ for all $m \in \omega$. By shadowing it follows that for any $m \in \omega$ there exists $a \in A$ such that $f^{mk+k_0}(a) \in (0, 3\varepsilon)$. Notice

$$f_n^{-1}\left((0, 3\varepsilon)\right) = \left(0, \frac{3\varepsilon}{2}\right) \cup \left(1 - \frac{3\varepsilon}{2}, 1\right) \subseteq \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right).$$

Now let z be the least such element of $A \setminus \{0\}$. Let $l \in \omega$ be least such that $f^l(z) > 3\varepsilon$. Let $m \in \omega$ be such that $mk + k_0 > l$. Let $a \in A$ be such that $f^{mk+k_0}(a) \in (0, 3\varepsilon)$; notice $a \neq 0$. The preimage of $(0, 3\varepsilon)$ is a subset of $(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$. However, $f_n^i(A) \cap (\frac{3}{4}, 1] = \emptyset$ for all $i \in \omega$. Therefore $f^{mk+k_0-1}(a) \in (0, \frac{1}{4})$. Indeed, by similar reasoning we find that $f^i(a) \in (0, \frac{1}{4})$ for all $i < mk + k_0$. However, since f is strictly increasing on $[0, \frac{1}{2})$, it follows that $a < z$, contradicting the minimality of z. Therefore f_n does not have the shadowing property.

3.3.3 Factor maps

In [48] the authors introduce the concept of factor maps which *almost lift pseudo-orbits*. For such maps, pseudo-orbits in the codomain system roughly correlate to pseudo-orbits in the domain system — hence they 'almost lift'.

Definition 3.3.3. Suppose X and Y are compact Hausdorff spaces, $f\colon X \to X$ and $g\colon Y \to Y$ are continuous. A factor map $\varphi\colon (X, f) \to (Y, g)$ *almost lifts pseudo-orbits (ALP)* if for every $V \in \mathscr{U}_Y$ and every $D \in \mathscr{U}_X$ there exists $W \in \mathscr{U}_Y$ such that for every W-pseudo-orbit $(y_i)_{i\in\omega}$ in Y, there exists a D-pseudo-orbit $(x_i)_{i\in\omega}$ in X such that $(\varphi(x_i), y_i) \in V$ for all $i \in \omega$.

If X and Y are compact metric spaces, then φ is ALP if and only if for all $\varepsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that if $(y_i)_{i\in\omega}$ is a δ-pseudo-orbit in Y, there exists an η-pseudo-orbit $(x_i)_{i\in\omega}$ in X with $d(\varphi(x_i), y_i) < \varepsilon$.

Theorem 3.3.4. [48] *Let (X, f) and (Y, g) be dynamical systems, where X and Y are compact Hausdorff, and let $\varphi\colon (X, f) \to (Y, g)$ be a factor map. Then the following statements hold:*

1. *If (X, f) has shadowing and φ is an ALP map then (Y, g) has shadowing.*
2. *If (Y, g) has shadowing then φ is an ALP map.*

In particular it follows that a factor map preserves shadowing if and only if it is an ALP map.

3.3.4 Inverse limits

In [48] the authors prove the following theorem.

Theorem 3.3.5. [48] *Let (X, f) be conjugate to a Mittag–Leffler inverse limit system comprised of maps with shadowing on compact Hausdorff spaces. Then (X, f) has shadowing.*

3.3.5 Tychonoff product

The following result is folklore.

Theorem 3.3.6. *Let Λ be an arbitrary index set and let (X_λ, f_λ) be a system with shadowing for each $\lambda \in \Lambda$. Then the product system (X, f), where $X = \prod_{\lambda\in\Lambda} X_\lambda$, has shadowing.*

3.4 Preservation of h-shadowing

The property of h-shadowing was introduced in [10] and was motivated by the fact that certain systems, called shifts of finite type, which are fundamental in the study of shadowing (see [48]) exhibit a stronger form of shadowing, i.e. h-shadowing, which coincides with the usual form for shift systems but is distinct in general (see [9, Example 6.4]).

Recall the definition from section 3.2: The system (X, f) is said to have h-shadowing if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that for any finite D-pseudo-orbit $\{x_0, x_1, \ldots, x_m\}$ there exists $y \in X$ such that $(f^i(y), x_i) \in E$ for all $i < m$ and $f^m(y) = x_m$.

Remark 3.4.1. If X is a perfect space (i.e. it has no isolated points) and (X, f) has h-shadowing then f is a surjection.

3.4.1 Induced map on the hyperspace of compact sets

The following theorem was proved in [41] for compact metric systems. Their proof generalises to give the result for compact Hausdorff systems.

Theorem 3.4.2. [41, Theorem 4.6] *Let X be a compact Hausdorff space and let $f \colon X \to X$ be a continuous function. Then (X, f) has h-shadowing if and only if $(2^X, 2^f)$ has h-shadowing.*

3.4.2 Symmetric products

The following theorem is stated in [41] for compact metric systems. The result generalises to compact Hausdorff systems.

Theorem 3.4.3. [41, Theorem 4.3] *Let X be a compact Hausdorff space and let $f \colon X \to X$ be a continuous function. For any $n \geq 2$, if $(F_n(X), f_n)$ has h-shadowing then (X, f) has h-shadowing.*

Theorem 3.4.4. *Let X be a compact Hausdorff space and let $f \colon X \to X$ be a continuous function. If (X, f) has h-shadowing then $(F_2(X), f_2)$ has h-shadowing.*

Proof. Let $E \in 2^{\mathscr{U}}$ be given. (Recall the standing assumption made in Remark 3.2.29. That is, we assume, without loss of generality, that all entourages we make reference to are symmetric.) Let $E_0 \in \mathscr{U}$ be such that $2^{E_0} \subseteq E$. Let $D \in \mathscr{U}$ correspond to E_0 in h-shadowing for f. We claim 2^D satisfies the h-shadowing condition for E. Suppose that $\{A_0, A_1, \ldots, A_m\}$ is a finite 2^D-pseudo-orbit in $F_2(X)$. Write $A_i = \{x_i, y_i\}$; it is possible that, for some i, $x_i = y_i$. Relabelling the x's and y's where necessary, $\{x_0, \ldots x_m\}$ and $\{y_0, \ldots, y_m\}$ are finite D-pseudo-orbits in X. By h-shadowing there exist $x, y \in X$ such that $f^m(x) = x_m$, $f^m(y) = y_m$ and, for all $i \in \{0, \ldots, m-1\}$, $(f^i(x), x_i) \in E_0$ and $(f^i(y), y_i) \in E_0$. Write $A = \{x, y\} \in F_2(X)$. Notice $f_2^m(A) = A_m$. By the above, for each $i \in \{0, \ldots, m-1\}$, $A_i \subseteq B_{E_0}\left(f_2^i(A)\right)$ and $f_2^i(A) \subseteq B_{E_0}(A_i)$. It follows that $(f_2^i(A), A_i) \in 2^{E_0}$. Since $2^{E_0} \subseteq E$ we get that $(f_2^i(A), A_i) \in E$ for each $i \in \{0, \ldots, m-1\}$. □

Remark 3.4.5. Example 3.3.2 shows that, in general, symmetric products do not preserve h-shadowing for $n \geq 3$. The standard tent map (X, f) has h-shadowing [10, Example 3.5] however $(F_n(X), f_n)$ does not have shadowing for any $n \geq 3$. Since h-shadowing implies shadowing on compact spaces (see [10]) it follows that $(F_n(X), f_n)$ does not possess h-shadowing either.

3.4.3 Factor maps

Clearly it follows from Theorem 3.3.4 that if $\varphi\colon (X, f) \to (Y, g)$ is a factor map and Y has h-shadowing, then φ is ALP. It is unclear, however, whether ALP is strong enough to preserve h-shadowing.

3.4.4 Tychonoff product

Recall Remark 3.4.1: if X is a perfect space and (X, f) has h-shadowing then f must be a surjection. For this reason an arbitrary product of dynamical systems with h-shadowing need not itself have h-shadowing (see Example 3.4.8).

Theorem 3.4.6. *Let Λ be an arbitrary index set and let (X_λ, f_λ) be a surjective compact Hausdorff system with h-shadowing for each $\lambda \in \Lambda$. Then the product system (X, f), where $X = \prod_{\lambda\in\Lambda} X_\lambda$, has h-shadowing.*

Proof. Let $E \in \mathscr{U}$ be given; this entourage is refined by one of the form

$$\prod_{\lambda\in\Lambda} E_\lambda,$$

where $E_\lambda \in \mathscr{U}_\lambda$ for all $\lambda \in \Lambda$ and $E_\lambda = X_\lambda \times X_\lambda$ for all but finitely many of the λ's. Let λ_i, for $1 \leq i \leq k$, be precisely those elements in Λ for which $E_\lambda \neq X_\lambda \times X_\lambda$ (if there are no such elements then we are done). By h-shadowing in each component space, there exist entourages $D_{\lambda_i} \in \mathscr{U}_{\lambda_i}$ such that every D_{λ_i}-pseudo-orbit is E_{λ_i}-h-shadowed. Let

$$D \coloneqq \prod_{\lambda\in\Lambda} D_\lambda$$

where

$$D_\lambda = \begin{cases} X \times X & \text{if} \quad \forall i\, \lambda \neq \lambda_i \\ D_{\lambda_i} & \text{if} \quad \exists i : \lambda = \lambda_i\,. \end{cases}$$

Now let $\{x_0, x_1, \ldots, x_m\}$ be a finite D-pseudo-orbit. Then $\{\pi_{\lambda_i}(x_1), \pi_{\lambda_i}(x_2), \ldots, \pi_{\lambda_i}(x_m)\}$ is a D_{λ_i}-pseudo-orbit in X_{λ_i}, which is E_{λ_i}-h-shadowed by a point $z_i \in X_{\lambda_i}$. Pick a point $z \in X$ such that $\pi_{\lambda_i}(z) = z_i$ for each $1 \leq i \leq k$ and $\pi_\lambda(f^m(z)) = \pi_\lambda(x_m)$ for all $\lambda \in \Lambda$. It follows that z E-h-shadows $\{x_0, x_1, \ldots, x_m\}$. □

Remark 3.4.7. It is easy to see that if only a finite number of the component systems involved in Theorem 3.4.6 were not surjective the result would still hold.

Example 3.4.8. For each $i \in \omega$ let $X_i = \{2\} \cup [0, 1]$ and $f_i \colon X \to X$ be given by

$$f_i(x) = \begin{cases} 2x & \text{if} \quad x \in [0, \frac{1}{2}] \\ 2(1-x) & \text{if} \quad x \in (\frac{1}{2}, 1] \\ 0 & \text{if} \quad x = 2\,. \end{cases}$$

Thus, each system is comprised of the standard tent map together with an isolated point which maps to the fixed point 0. The standard tent map has h-shadowing [10, Example 3.5] and it is obvious that the additional point in these systems does nothing to contradict that. Thus each system (X_i, f_i) has h-shadowing. The product system (X, f), where

$$X = \prod_{i \in \omega} X_i,$$

has no isolated points. However, the point given by $x_i = 2$ for all $i \in \omega$ has no preimage; the system is not onto. Hence, by Remark 3.4.1, the system (X, f) does not have h-shadowing.

3.5 Preservation of eventual shadowing

Eventual shadowing was introduced in [47] in the authors' journey to characterise when the set of ω-limit sets of a system coincides with the set of closed internally chain transitive sets. As remarked upon in [47], the property of eventual shadowing is equivalent with the $(\mathbb{N}, \mathcal{F}_{cf})$-shadowing property of Oprocha [78].

Recall that a system (X, f) has the *eventual shadowing* property provided that for all $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that for each D-pseudo-orbit $(x_i)_{i \in \omega}$, there exists $z \in X$ and $N \in \mathbb{N}$ such that $(f^i(z), x_i) \in E$ for all $i \geq N$.

3.5.1 Induced map on the hyperspace of compact sets

Theorem 3.5.1. *Let X be a compact Hausdorff space and let $f \colon X \to X$ be a continuous function. If the hyperspace system $(2^X, 2^f)$ has eventual shadowing then (X, f) has eventual shadowing.*

Proof. Let $E \in \mathscr{U}$. Let $D \in \mathscr{U}$ be such that 2^D corresponds to 2^E for eventual shadowing for 2^f. Let $(x_i)_{i \in \omega}$ be a D-pseudo-orbit in X. Then $\big(\{x_i\}\big)_{i \in \omega}$ is a 2^D-pseudo-orbit in 2^X. By eventual shadowing there exists $A \in 2^X$ and $N \in \mathbb{N}$ such that $\big((2^f)^i(A), \{x_i\}\big) \in 2^E$

for all $i \geq N$. It follows that, for any $a \in A$, $\big(f^i(a), x_i\big) \in E$ for all $i \geq N$. Since $A \neq \emptyset$ the result holds. □

The following example shows that the converse to Theorem 3.5.1 is not true: the hyperspatial system of a system with eventual shadowing need not have eventual shadowing. The example also demonstrates that symmetric products do not necessarily preserve eventual shadowing for $n \geq 3$.

Example 3.5.2. Let $X = [-1, 1]$ and let $f\colon X \to X$ be given by

$$f(x) = \begin{cases} x^3 & \text{if} \quad x \in [-1, 0] \\ 2x & \text{if} \quad x \in (0, \frac{1}{2}] \\ 2(1-x) & \text{if} \quad x \in (\frac{1}{2}, 1]\,. \end{cases}$$

As observed in [47] f has eventual shadowing but not shadowing. We claim 2^f does not have eventual shadowing. Let $\varepsilon = \frac{1}{12}$ and fix $\delta > 0$; without loss of generality assume $\delta < \varepsilon$. Choose a point $y \in (-1, -1+\delta)$ such that there exists $m \in \omega$ with $f^m(y) = -\frac{1}{2}$; let $k \in \omega$ be such that $f^k(y) \in (-\frac{\delta}{2}, 0]$ (notice that $m < k$). Let $p \in (0, \frac{\delta}{2})$ be periodic with period n and such that there exists $n_0 < n$ with $f^{n_0}(p) \in (1 - \frac{\delta}{2}, 1)$. We may now construct a δ-pseudo-orbit in 2^X as follows. For any $i \in \omega$:

- if $i \mod (k+n) \equiv j \in \{0, 1, \ldots, k-1\}$ then let $A_i = \{-1, f^j(y), 0\}$,
- if $i \mod (k+n) \equiv j \in \{k, k+1, \ldots, k+n-1\}$ then let $A_i = \{-1, f^{j-k}(p)\}$.

We claim $(A_i)_{i\in\omega}$ cannot be eventually ε-shadowed in 2^X. Indeed suppose $A \in 2^X$ eventually ε-shadows this pseudo-orbit. Let $N \in \omega$ be such that

$$d_H\left((2^f)^{N+i}(A), A_{N+i}\right) < \varepsilon$$

for all $i \in \omega$. Let $l > N$ be such that $A_l \ni f^m(y) = -\frac{1}{2}$. Then there exists $a \in A$ such that $f^l(a) \in (-\frac{1}{2} - \varepsilon, -\frac{1}{2} + \varepsilon)$. Now let $l_0 > l$ be such that $l_0 \mod k+n \equiv k+n_0$. Then

$A_{l_0} = \{-1, f^{n_0}(p)\}$ but $f^{l_0}(a) \in (-\frac{1}{2} - \varepsilon, 0)$. Since

$$\left(-\frac{1}{2} - \varepsilon, 0\right) \cap \big(B_\varepsilon(-1) \cup B_\varepsilon(f^{n_0}(p))\big) = \emptyset,$$

we have a contradiction: A does not eventually ε-shadow $(A_i)_{i \in \omega}$.

3.5.2 Symmetric products

Theorem 3.5.3. *Let X be a compact Hausdorff space and let $f \colon X \to X$ be a continuous function. For any $n \geq 2$, if $(F_n(X), f_n)$ has eventual shadowing then (X, f) has eventual shadowing.*

Proof. Let $E \in \mathscr{U}$. Let $D \in \mathscr{U}$ be such that $2^D \cap \big(F_n(X) \times F_n(X)\big)$ corresponds to $2^E \cap \big(F_n(X) \times F_n(X)\big)$ for eventual shadowing for f_n. Let $(x_i)_{i \in \omega}$ be a D-pseudo-orbit in X. Then $\big(\{x_i\}\big)_{i \in \omega}$ is a 2^D-pseudo-orbit in $F_n(X)$. By eventual shadowing there exists $A \in F_n(X)$ and $N \in \mathbb{N}$ such that $\big(f_n^i(A), \{x_i\}\big) \in 2^E$ for all $i \geq N$. It follows that, for any $a \in A$, $\big(f^i(a), x_i\big) \in E$ for all $i \geq N$. Since $A \neq \emptyset$ the result holds. □

Theorem 3.5.4. *Let X be a compact Hausdorff space and let $f \colon X \to X$ be a continuous function. If (X, f) has eventual shadowing then (X, f_2) has eventual shadowing.*

Proof. Let $E \in 2^{\mathscr{U}}$ be given. Let $E_0 \in \mathscr{U}$ be such that $2^{E_0} \subseteq E$. Let $D \in \mathscr{U}$ correspond to E_0 in eventual shadowing for f. We claim $2^D \cap \big(F_2(X) \times F_2(X)\big)$ satisfies the eventual shadowing condition for f_2 and $E \cap \big(F_2(X) \times F_2(X)\big)$. Suppose that $(A_i)_{i \in \omega}$ is a 2^D-pseudo-orbit in $F_2(X)$. Write $A_i = \{x_i, y_i\}$ (it is possible that, for some i, $x_i = y_i$). Relabelling the x's and y's where necessary, $(x_i)_{i \in \omega}$ and $(y_i)_{i \in \omega}$ are D-pseudo-orbits in X. By eventual shadowing for f there exist $x, y \in X$ and $N_1, N_2 \in \mathbb{N}$ such that for all $i \geq N_1$, $(f^i(x), x_i) \in E_0$ and for all $i \geq N_2$ and $(f^i(y), y_i) \in E_0$. Take $N = \max\{N_1, N_2\}$. Then, for all $i \geq N$, $A_i \subseteq B_{E_0}\big(f_2^i(A)\big)$ and $f_2^i(A) \subseteq B_{E_0}(A_i)$. It follows that $(f_2^i(A), A_i) \in 2^{E_0} \cap \big(F_2(X) \times F_2(X)\big)$. Since $2^{E_0} \subseteq E$ we get that $(f_2^i(A), A_i) \in E \cap \big(F_2(X) \times F_2(X)\big)$ for each $i \geq N$. □

Remark 3.5.5. Example 3.5.2 shows that, in general, symmetric products do not preserve eventual shadowing for $n \geq 3$.

3.5.3 Factor maps

Definition 3.5.6. A factor map $\varphi\colon X \to Y$ *eventually almost lifts pseudo-orbits* (eALP) if for every $E \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ there is $V \in \mathscr{U}_Y$ such that for every V-pseudo-orbit $(y_i)_{i\in\omega}$ in Y is a D-pseudo-orbit $(x_i)_{i\in\omega}$ in X such that $(\varphi(x_i))_{i\in\omega}$ eventually E-shadows $(y_i)_{i\in\omega}$.

If X and Y are compact metric spaces, then φ is eALP if and only if for all $\varepsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that if $(y_i)_{i\in\omega}$ is a δ-pseudo-orbit in Y, there exists an η-pseudo-orbit $(x_i)_{i\in\omega}$ in X which eventually ε-shadows $(y_i)_{i\in\omega}$.

Theorem 3.5.7. *Suppose that $\varphi\colon (X, f) \to (Y, g)$ is a factor map.*

1. *If (X, f) has eventual shadowing and φ eALP, then (Y, g) has eventual shadowing.*

2. *If (Y, g) has eventual shadowing, then φ eALP.*

Proof. For (1), let $E \in \mathscr{U}_Y$ be given. Select $E_0 \in \mathscr{U}_Y$ with $2E_0 \subseteq E$. By the uniform continuity of φ there exists $D_1 \in \mathscr{U}_X$ such that for all $a, b \in X$ with $(a, b) \in D_1$ one has $(\varphi(a), \varphi(b)) \in E_0$. Next, let $D_2 \in \mathscr{U}_X$ be chosen so that D_2-pseudo-orbits in X are eventually D_1-shadowed. Extract $W \in \mathscr{U}_Y$ from the definition of eALP using E_0 and D_2, we claim that W-pseudo-orbits of (Y, g) are then eventually E-shadowed in (Y, g). Indeed, given a W-pseudo-orbit $(y_i)_{i\in\omega} \subseteq Y$, there exists a D_2-pseudo-orbit $(x_i)_{i\in\omega} \subseteq X$ and $N \in \mathbb{N}$ such that, for all $i \geq N$, $(y_i, \varphi(x_i)) \in E_0$. Consider $z \in X$ that eventually D_1-shadows $(x_i)_{i\in\omega}$. Let $M \in \mathbb{N}$ be such that $(f^i(z), x_i) \in D_1$ for all $i \geq M$. Take $l = \max\{M, N\}$. Then, using uniform continuity and the triangle inequality, $(g^i(\varphi(z)), y_i) \in E$ for all $i \geq l$. Hence $\varphi(z)$ eventually E-shadows $(y_i)_{i\in\omega}$.

To see (2), fix $E \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ and take $V \in \mathscr{U}_Y$ to correspond to E for eventual shadowing in (Y, g). Let $(y_i)_{i\in\omega}$ be a V-pseudo-orbit in (Y, g) and let $z \in Y$ eventually

E-shadow it. Let $N \in \mathbb{N}$ be such that $g^N(z)$ E-shadows $(y_{N+i})_{i\in\omega}$. Consider $x \in \varphi^{-1}(z)$ and define $x_i = f^i(x)$ for each $i \in \omega$ so that $(x_i)_{i\in\omega}$ is a D-pseudo-orbit in (X, f). In particular, one then has that for all $i \geq N$

$$(\varphi(x_i), y_i) = (g^i(z), y_i) \in E.$$

□

3.5.4 Inverse limits

Theorem 3.5.8. *Let (X, f) be conjugate to a Mittag–Leffler inverse limit system comprised of maps with eventual shadowing on compact Hausdorff spaces. Then (X, f) has eventual shadowing.*

Proof. Let $(\Lambda, \geq)$ be a directed set. For each $\lambda \in \Lambda$, let (X_λ, f_λ) be a dynamical system on a compact Hausdorff space with eventual shadowing and let $((X_\lambda, f_\lambda), g^\eta_\lambda)$ be a Mittag–Leffler inverse system. Without loss of generality $(X, f) = (\varprojlim\{X_\lambda, g^\eta_\lambda\}, f)$.

Let $\mathcal{U}$ be a finite open cover of X. Since $X = \varprojlim\{X_\lambda, g^\eta_\lambda\}$ there exist $\lambda \in \Lambda$ and a finite open cover $\mathcal{W}_\lambda$ of X_λ such that $\mathcal{W} := \{\pi_\lambda^{-1}(W) \cap X \mid W \in \mathcal{W}_\lambda\}$ refines $\mathcal{U}$. Now let $\gamma \in \Lambda$ witness the Mittag–Leffler condition with respect to λ. Let $\mathcal{W}_\gamma := \{g_\lambda^{\gamma(-1)}(W) \mid W \in \mathcal{W}_\lambda\}$. By eventual shadowing for (X_γ, f_γ) there exists a finite open cover $\mathcal{V}_\gamma$ of X_γ such that every $\mathcal{V}_\gamma$-pseudo-orbit in X_γ is eventually $\mathcal{W}_\gamma$-shadowed. Take $\mathcal{V} = \{\pi_\gamma^{-1}(V) \cap X \mid V \in \mathcal{V}_\gamma\}$ and suppose $(x_i)_{i\in\omega}$ is a $\mathcal{V}$-pseudo-orbit in X. It follows that $(\pi_\gamma(x_i))_{i\in\omega}$ is a $\mathcal{V}_\gamma$-pseudo-orbit in X_γ, which means there is a point $z \in X_\gamma$ which eventually $\mathcal{W}_\gamma$-shadows it. By construction, it follows that $g^\gamma_\lambda(z)$ eventually $\mathcal{W}_\lambda$-shadows $(\pi_\lambda(x_i))_{i\in\omega}$. Since the system is Mittag–Leffler there exists $y \in \pi_\lambda^{-1}(g^\gamma_\lambda(z)) \cap X$. It follows that y eventually $\mathcal{W}$-shadows $(x_i)_{i\in\omega}$. Since $\mathcal{W}$ is a refinement of $\mathcal{U}$ the result follows. □

3.5.5 Tychonoff product

Theorem 3.5.9. *Let Λ be an arbitrary index set and let (X_λ, f_λ) be a compact Hausdorff system with eventual shadowing for each $\lambda \in \Lambda$. Then the product system (X, f), where $X = \prod_{\lambda\in\Lambda} X_\lambda$, has eventual shadowing.*

Proof. Let $E \in \mathscr{U}$ be given; this entourage is refined by one of the form

$$\prod_{\lambda\in\Lambda} E_\lambda,$$

where $E_\lambda \in \mathscr{U}_\lambda$ for all $\lambda \in \Lambda$ and $E_\lambda = X_\lambda \times X_\lambda$ for all but finitely many of the λ's. Let λ_i, for $1 \leq i \leq k$, be precisely those elements in Λ for which $E_\lambda \neq X_\lambda \times X_\lambda$ (if there are no such elements then we are done). By eventual shadowing in each component space, there exist entourages $D_{\lambda_i} \in \mathscr{U}_{\lambda_i}$ such that every D_{λ_i}-pseudo-orbit is eventually E_{λ_i}-shadowed. Let

$$D \coloneqq \prod_{\lambda\in\Lambda} D_\lambda$$

where

$$D_\lambda = \begin{cases} X \times X & \text{if} \quad \forall i\, \lambda \neq \lambda_i \\ D_{\lambda_i} & \text{if} \quad \exists i : \lambda = \lambda_i\,. \end{cases}$$

Now let $(x_j)_{j\in\omega}$ be a D-pseudo-orbit. Then $(\pi_{\lambda_i}(x_j))_{j\in\omega}$ is a D_{λ_i}-pseudo-orbit in X_{λ_i}, which is eventually E_{λ_i}-shadowed by a point $z_i \in X_{\lambda_i}$; there exist N_i such that $(\pi_{\lambda_i}(x_j))_{j\geq N_i}$ is E_{λ_i}-shadowed by $f_{\lambda_i}^{N_i}(z_i)$. Pick a point $z \in X$ such that $\pi_{\lambda_i}(z) = z_i$ for each $1 \leq i \leq k$. Take $N = \max_{1\leq i\leq k} N_i$. Then $f^N(z)$ E-shadows $(x_j)_{j\geq N}$. Thus, by definition, z eventually E-shadows $(x_j)_{j\in\omega}$. □

3.6 Preservation of orbital shadowing

The orbital shadowing property was introduced in [82]; the authors studied its relationship to classical stability properties, such as structural stability and Ω-stability. It has since been studied by various other authors (e.g. [47, 84]).

Recall, a system (X, f) has the *orbital shadowing* property if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point z such that

$$\left(\overline{\{x_i\}_{i\in\omega}}, \overline{\{f^i(z)\}_{i\in\omega}}\right) \in 2^E.$$

3.6.1 Induced map on the hyperspace of compact sets

Theorem 3.6.1. *Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous function. If the hyperspace system $(2^X, 2^f)$ witnesses orbital shadowing then the system (X, f) experiences orbital shadowing.*

Proof. Let $E \in \mathscr{U}$ be given and let $E_0 \in \mathscr{U}$ be such that $4E_0 \subseteq E$. Let $D \in \mathscr{U}$ be such that 2^D satisfies the condition for 2^{E_0} in orbital shadowing for the hyperspatial system. Let $(x_i)_{i\in\omega}$ be a D-pseudo-orbit in X. Then $(\{x_i\})_{i\in\omega}$ is a 2^D-pseudo-orbit in 2^X. Then there exists $A \in 2^X$ such that

$$\left(\overline{\{\{x_i\}\}_{i\in\omega}}, \overline{\{(2^f)^i(A)\}_{i\in\omega}}\right) \in 2^{2^{E_0}}.$$

Equivalently

$$\overline{\{\{x_i\}\}_{i\in\omega}} \subseteq B_{2^{E_0}}\left(\overline{\{(2^f)^i(A)\}_{i\in\omega}}\right) \tag{3.1}$$

and

$$\overline{\{(2^f)^i(A)\}_{i\in\omega}} \subseteq B_{2^{E_0}}\left(\overline{\{\{x_i\}\}_{i\in\omega}}\right). \tag{3.2}$$

Pick $z \in A$. It can be verified that

$$\left(\overline{\{x_i\}_{i\in\omega}}, \overline{\{f^i(z)\}_{i\in\omega}}\right) \in 2^{4E_0}.$$

Indeed, suppose not.

Case i). There exists $a \in \overline{\{x_i\}_{i\in\omega}}$ such that for any $b \in \overline{\{f^i(z)\}_{i\in\omega}}$ we have $(a, b) \notin$

$4E_0$. It follows that there exists $k \in \omega$ such that $(x_k, f^i(z)) \notin 2E_0$ for all $i \in \omega$. We have from Equation (3.1) that there exists $l \in \omega$ such that $\left(f^l(A), \{x_k\}\right) \in 2^{E_0}$; in particular, for any $y \in f^l(A)$, $(y, x_k) \in E_0$, a contradiction.

Case ii). There exists $b \in \overline{\{f^i(z)\}_{i\in\omega}}$ such that for any $a \in \overline{\{x_i\}_{i\in\omega}}$ we have $(b, a) \notin 4E_0$. It follows that there exists $k \in \omega$ such that $(f^k(z), x_i) \notin 2E_0$ for all $i \in \omega$. We have from Equation (3.2) that there exists $l \in \omega$ such that $\left(f^k(A), \{x_l\}\right) \in 2^{E_0}$; in particular, for any $y \in f^k(A)$, $(y, x_l) \in E_0$, a contradiction.

It follows that

$$\left(\overline{\{x_i\}_{i\in\omega}}, \overline{\{f^i(z)\}_{i\in\omega}}\right) \in 2^{4E_0} \subseteq 2^E.$$

□

The following example shows that the converse to Theorem 3.6.1 is false.

Example 3.6.2. Let X be the circle $\mathbb{R}/\mathbb{Z}$ and let $f\colon X \to X$ be given by $x \mapsto x + \alpha$, where α is some fixed irrational number. Since (X, f) is minimal, by [71, Corollary 3.7], it has strong orbital shadowing, and thereby orbital shadowing and first weak shadowing. Let x_0 and y_0 be two antipodal points and let $\varepsilon > 0$ be given, with $\varepsilon < \frac{1}{20}$. Suppose $\delta \in \mathbb{Q}$ with $0 < \delta < \varepsilon$. Then construct a δ-pseudo-orbit in 2^X recursively by the following rule: Let $A_0 = \{x_0, y_0\}$ and, for all $i \in \mathbb{N}$, let $A_i = \{x_i, y_i\} := \{f(x_{i-1}) + \frac{\delta}{2}, f(y_{i-1}) + \frac{\delta}{3}\}$. We claim that this is not first weak shadowed. Suppose A ε-first-weak-shadows $(A_i)_{i\in\omega}$; i.e.

$$B_\varepsilon\left(\mathrm{Orb}(A)\right) \supseteq \{A_i\}_{i\in\omega}.$$

Then there exists $n \in \omega$ such that $d_H((2^f)^n(A), \{x_0, y_0\}) < \varepsilon$; thus $(2^f)^n(A) \subseteq B_\varepsilon(x_0) \cup B_\varepsilon(y_0)$, $(2^f)^n(A) \cap B_\varepsilon(x_0) \neq \emptyset$ and $(2^f)^n(A) \cap B_\varepsilon(y_0) \neq \emptyset$. Since x_0 and y_0 are antipodal and f is an isometry, it follows that A is a subset of a union of two antipodal arcs of length ε and that A meets both these arcs; the same holds true of $(2^f)^i(A)$ for all $i \in \omega$. Now let $l \in \omega$ be least such that $d(x_l, y_l) \leq \frac{\delta}{6}$; such an l exists by construction.

We claim $\{x_l, y_l\} \notin B_\varepsilon\left(\text{Orb}(A)\right)$. Suppose not, then there exists $m \in \omega$ such that $d_H\left((2^f)^m(A), \{x_l, y_l\}\right) < \varepsilon$. In particular,

$$(2^f)^m(A) \subseteq B_\varepsilon\left(\{x_l, y_l\}\right).$$

But

$$B_\varepsilon\left(\{x_l, y_l\}\right) \subseteq B_{2\varepsilon}\left(x_l\right),$$

and $B_{2\varepsilon}\left(x_l\right)$ is an arc of length less than $\frac{4}{20}$ by construction, which does not contain any pair of antipodal points, contradicting our analysis of $(2^f)^i(A)$. Hence the hyperspatial system does not have first weak shadowing. Since

$$\text{Strong orbital shadowing} \implies \text{orbital shadowing} \implies \text{first weak shadowing},$$

it also follows that the system has neither strong orbital shadowing nor orbital shadowing.

3.6.2 Symmetric products

The proof of Theorem 3.6.3 is very similar to that of Theorem 3.6.1 and is thereby omitted.

Theorem 3.6.3. *Let X be a compact Hausdorff space, and let $f \colon X \to X$ be a continuous function. For any $n \geq 2$, if the symmetric product system $(F_n(X), f_n)$ witnesses orbital shadowing then the system (X, f) experiences orbital shadowing.*

Proof. Omitted. □

Remark 3.6.4. The converse of Theorem 3.6.3 is false. It is clear that Example 3.6.2 may be suitably adjusted to provide a counterexample. Indeed, with sufficient adjustments, one can see that, for any $n \geq 2$, (X, f) witnessing orbital shadowing does not generally imply that $(F_n(X), f_n)$ has orbital shadowing.

3.6.3 Factor maps

Definition 3.6.5. Let (X, f) and (Y, g) be dynamical systems, where X and Y are compact Hausdorff spaces. A factor map $\varphi\colon X \to Y$ *orbitally almost lifts pseudo-orbits* (oALP) if for every $V \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ there exists $W \in \mathscr{U}_Y$ such that for any W-pseudo-orbit $(y_i)_{i\in\omega} \subseteq Y$, there exists a D-pseudo-orbit $(x_i)_{i\in\omega} \subseteq X$ for which

$$(\varphi(\overline{\{x_i\}_{i\in\omega}}), \overline{\{y_i\}_{i\in\omega}}) \in 2^V.$$

If X and Y are compact metric spaces, then φ is oALP if and only if for all $\varepsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(y_i)_{i\in\omega}$ in Y, there exists an η-pseudo-orbit $(x_i)_{i\in\omega}$ in X such that the Hausdorff distance $d_H(\varphi(\overline{\{x_i\}_{i\in\omega}}), \overline{\{y_i\}_{i\in\omega}}) < \varepsilon$.

Theorem 3.6.6. *Suppose that $\varphi\colon (X, f) \to (Y, g)$ is a factor map.*

1. *If (X, f) exhibits orbital shadowing and φ is oALP, then (Y, g) exhibits orbital shadowing.*
2. *If (Y, g) exhibits orbital shadowing, then φ is oALP.*

Proof. For (1), let $E \in \mathscr{U}_Y$ be given. Select $E_0 \in \mathscr{U}_Y$ with $2E_0 \subseteq E$. By the uniform continuity of φ there exists $D_1 \in \mathscr{U}_X$ such that for all $a, b \in X$ with $(a, b) \in D_1$ one has $(\varphi(a), \varphi(b)) \in E_0$. Next, let $D_2 \in \mathscr{U}_X$ be chosen so that D_2-pseudo-orbits in are D_1 orbital shadowed. Extract $W \in \mathscr{U}_Y$ from the definition of oALP using E_0 and D_2, we claim that W-pseudo-orbits of (Y, g) are then E-orbital shadowed in (Y, g). Indeed, given a W-pseudo-orbit $(y_i)_{i\in\omega} \subseteq Y$, there exists a D_2-pseudo-orbit $(x_i)_{i\in\omega} \subseteq X$ for which

$$(\overline{\{y_i\}_{i\in\omega}}, \varphi(\overline{\{x_i\}_{i\in\omega}})) \in 2^{E_0}.$$

Let $z \in X$ D_1-orbital shadow $(x_i)_{i\in\omega}$. Then, using uniform continuity and the triangle inequality, one may conclude that $\varphi(z)$ E-orbital-shadows $(y_i)_{i\in\omega}$ as required.

For (2), fix $E \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ and take $V \in \mathscr{U}_Y$ to correspond to E for orbital shadowing in (Y, g). Let $(y_i)_{i\in\omega}$ to be a V-pseudo-orbit in (Y, g) and let $z \in Y$ E-orbital shadow it. Consider $x \in \varphi^{-1}(z)$ and define $x_i = f^i(x)$ for each $i \in \omega$ so that $(x_i)_{i\in\omega}$ is a D-pseudo-orbit in (X, f). In particular, one then has that

$$\begin{aligned}(\varphi(\overline{\{x_i\}_{i\in\omega}}), \overline{\{y_i\}_{i\in\omega}}) &= (\overline{\varphi(\{x_i\}_{i\in\omega})}, \overline{\{y_i\}_{i\in\omega}}) \\ &= (\overline{\{g^i(z)\}_{i\in\omega}}, \overline{\{y_i\}_{i\in\omega}}) \in 2^E.\end{aligned}$$

□

3.6.4 Inverse limits

Theorem 3.6.7. *Let (X, f) be conjugate to a Mittag–Leffler inverse limit system comprised of maps with orbital shadowing on compact Hausdorff spaces. Then (X, f) has orbital shadowing.*

Proof. We use the reformulation of orbital shadowing given in Definition 3.2.36.

Let $(\Lambda, \geq)$ be a directed set. For each $\lambda \in \Lambda$, let (X_λ, f_λ) be a dynamical system on a compact Hausdorff space with strong orbital shadowing and let $((X_\lambda, f_\lambda), g^\eta_\lambda)$ be a Mittag–Leffler inverse system. Without loss of generality $(X, f) = (\varprojlim\{X_\lambda, g^\eta_\lambda\}, f)$.

Let $\mathcal{U}$ be a finite open cover of X. Since $X = \varprojlim\{X_\lambda, g^\eta_\lambda\}$ there exist $\lambda \in \Lambda$ and a finite open cover $\mathcal{W}_\lambda$ of X_λ such that $\mathcal{W} := \{\pi_\lambda^{-1}(W) \cap X \mid W \in \mathcal{W}_\lambda\}$ refines $\mathcal{U}$. Now let $\gamma \in \Lambda$ witness the Mittag–Leffler condition with respect to λ. Let $\mathcal{W}_\gamma := \{g_\lambda^{\gamma(-1)}(W) \mid W \in \mathcal{W}_\lambda\}$. By orbital shadowing for (X_γ, f_γ) there exists a finite open cover $\mathcal{V}_\gamma$ of X_γ such that every $\mathcal{V}_\gamma$-pseudo-orbit in X_γ is $\mathcal{W}_\gamma$-orbital-shadowed. Take $\mathcal{V} = \{\pi_\gamma^{-1}(V) \cap X \mid V \in \mathcal{V}_\gamma\}$ and suppose $(x_i)_{i\in\omega}$ is a $\mathcal{V}$-pseudo-orbit in X. It follows that $(\pi_\gamma(x_i))_{i\in\omega}$ is a $\mathcal{V}_\gamma$-pseudo-orbit in X_γ, which means there is a point $z \in X_\gamma$ which $\mathcal{W}_\gamma$-orbital-shadows it. By construction, it follows that $g^\gamma_\lambda(z)$ $\mathcal{W}_\lambda$-orbital-shadows $(\pi_\lambda(x_i))_{i\in\omega}$, i.e.

$$\forall y \in \overline{\mathrm{Orb}(g^\gamma_\lambda(z))} \exists W \in \mathcal{W}_\lambda \exists y' \in \overline{\{\pi_\lambda(x_i)\}_{i\in\omega}} \colon y, y' \in W,$$

and

$$\forall y \in \overline{\{\pi_\lambda(x_i)\}_{i\in\omega}} \exists W \in \mathcal{W}_\lambda \exists y' \in \overline{\mathrm{Orb}(g_\lambda^\gamma(z))} \colon y, y' \in W.$$

Since the system is Mittag–Leffler there exists $z' \in \pi_\lambda^{-1}(g_\lambda^\gamma(z)) \cap X$. We claim z' $\mathcal{U}$-orbital-shadows $(x_i)_{i\in\omega}$.

Let $y' \in \overline{\mathrm{Orb}(z')}$. There exist $W \in W_\lambda$ and $k \in \omega$ such that $\pi_\lambda(y'), \pi_\lambda(x_k) \in W$. Then $y', x_k \in \pi_\gamma^{-1}(W) \cap X \subseteq U$ for some $U \in \mathcal{U}$.

Now suppose $y' \in \overline{\{x_i\}_{i\in\omega}}$. There exist $W \in \mathcal{W}_\lambda$ and $k \in \omega$ such that $\pi_\lambda(y')$ and $f_\lambda^k(g_\lambda^\gamma(z))$ are both in W. Then $y', f^k(z') \in \pi_\gamma^{-1}(W) \cap X \subseteq U$ for some $U \in \mathcal{U}$. □

3.6.5 Tychonoff product

A product of systems with orbital shadowing does not necessarily have orbital shadowing. The following example demonstrates this.

Example 3.6.8. For $i \in \{1, 2\}$ let $X_i = \mathbb{R}/\mathbb{Z}$, d_i be the shortest arc length metric on X_i and $f_i \colon X_i \to X_i \colon x \mapsto x + \alpha \mod 1$, where α is some fixed irrational number. Consider the product space $X = X_1 \times X_2$ with distance d given by $d((a, b), (c, d)) = \sup\{d_1(a, c), d_2(b, d)\}$. Recall that (X_i, f_i) has strong orbital shadowing, and therefore shadowing, by [71, Corollary 2.7]. It will be useful to define

$$|\cdot, \cdot| \colon \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R} \colon (a, b) \mapsto \min\{|a - b|, 1 - |b - a|\}.$$

Now consider the product system (X, f). Let $x_0 = 0$ and $y_0 = \frac{1}{2}$ and let $\varepsilon > 0$ be given, with $\varepsilon < \frac{1}{20}$. Suppose $\delta \in \mathbb{Q}$ with $0 < \delta < \varepsilon$. Then construct a δ-pseudo-orbit in X recursively by the following rule: Let $z_0 = (x_0, y_0)$ and, for all $i \in \mathbb{N}$, let $z_i = (f(x_{i-1}) + \frac{\delta}{2}, f(y_{i-1}) + \frac{\delta}{3})$. We claim that this is not first weak shadowed. Suppose $z = (x, y)$ ε-first-weak-shadows $(z_i)_{i\in\omega}$; i.e.

$$B_\varepsilon(\mathrm{Orb}(z)) \supseteq \{z_i\}_{i\in\omega}.$$

Then there exists $n \in \omega$ such that $d(f^n(z),(x_0,y_0)) < \varepsilon$; that is,

$$d((f_1^n(x), f_2^n(y)),(x_0,y_0)) = \sup\{d_1(f_1^n(x),x_0), d_2(f_2^n(y),y_0)\} < \varepsilon.$$

In particular $d_1(f_1^n(x),x_0) < \varepsilon$ and $d_2(f_2^n(y),y_0) < \varepsilon$; hence $f^n(z) \in B_\varepsilon(z_0) = B_\varepsilon(x_0) \times B_\varepsilon(y_0)$. It follows by the triangle inequality that $|f_1^n(x), f_2^n(y)| \geq \frac{1}{2} - 2\varepsilon > \frac{3}{5}$.

Now let $l \in \omega$ be least such that $|x_l, y_l| \leq \frac{\delta}{6}$; such an l exists by construction. We claim $(x_l,y_l) \notin B_\varepsilon(\text{Orb}(z))$. Suppose not, then there exists $m \in \omega$ such that $d\left(f^m(z),(x_l,y_l)\}\right) < \varepsilon$; thus $d_1(f_1^m(x),x_l) < \varepsilon$ and $d_2(f_2^m(y),y_l) < \varepsilon$. It follows that $|f_1^m(x), f_2^m(y)| \leq 2\varepsilon + \frac{\delta}{6}$. Since f_1 and f_2 are the same isometries

$$|f_1^n(x), f_2^n(y)| \leq 2\varepsilon + \frac{\delta}{6} < \frac{1}{10} + \frac{1}{120} < \frac{1}{5}.$$

But we know $|f_1^n(x), f_2^n(y)| > \frac{3}{5}$, so we have a contradiction. It follows that $(x_l,y_l) \notin B_\varepsilon(\text{Orb}(z))$. Hence the product system does not have first weak shadowing (and thereby nor does it have orbital (resp. strong orbital) shadowing).

3.7 Preservation of strong orbital shadowing

Strong orbital shadowing, a strengthening of orbital shadowing as the name suggests, was introduced in [47] in the authors' pursuit of a characterisation of when the set of ω-limit sets of a system coincides with the set of closed internally chain transitive sets.

The system (X,f) has the *strong orbital shadowing* property if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point $z \in X$ such that, for all $N \in \omega$,

$$\left(\overline{\{x_{N+i}\}_{i\in\omega}}, \overline{\{f^{N+i}(z)\}_{i\in\omega}}\right) \in 2^E.$$

3.7.1 Induced map on the hyperspace of compact sets

Theorem 3.7.1. *Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous function. If the hyperspace system $(2^X, 2^f)$ has strong orbital shadowing then the system (X, f) has strong orbital shadowing.*

Proof. Let $E \in \mathscr{U}$ be given and let $E_0 \in \mathscr{U}$ be such that $4E_0 \subseteq E$. Let $D \in \mathscr{U}$ be such that 2^D satisfies the condition for 2^{E_0} in strong orbital shadowing for the hyperspatial system. Let $(x_i)_{i\in\omega}$ be a D-pseudo-orbit in X. Then $(\{x_i\})_{i\in\omega}$ is a 2^D-pseudo-orbit in 2^X and there exists $A \in 2^X$ such that for any $N \in \omega$, $f^N(A)$ we have

$$\left(\overline{\{\{x_{N+i}\}\}_{i\in\omega}}, \overline{\{(2^f)^{N+i}(A)\}_{i\in\omega}}\right) \in 2^{2^{E_0}}.$$

Equivalently, for any $N \in \omega$

$$\overline{\{\{x_{N+i}\}\}_{i\in\omega}} \subseteq B_{2^{E_0}}\left(\overline{\{(2^f)^{N+i}(A)\}_{i\in\omega}}\right) \tag{3.3}$$

and

$$\overline{\{(2^f)^{N+i}(A)\}_{i\in\omega}} \subseteq B_{2^{E_0}}\left(\overline{\{\{x_{N+i}\}\}_{i\in\omega}}\right). \tag{3.4}$$

Pick $z \in A$. It can be verified that for any $N \in \omega$

$$\left(\overline{\{x_{N+i}\}_{i\in\omega}}, \overline{\{f^{N+i}(z)\}_{i\in\omega}}\right) \in 2^{4E_0}.$$

Indeed, suppose not.

Case i). There exist $N \in \omega$ and $a \in \overline{\{x_{N+i}\}_{i\in\omega}}$ such that for any $b \in \overline{\{f^{N+i}(z)\}_{i\in\omega}}$ we have $(a, b) \notin 4E_0$. It follows that there exists $k \in \omega$ such that $(x_{N+k}, f^{N+i}(z)) \notin 2E_0$ for all $i \in \omega$. We have from Equation (3.3) that there exists $l \in \omega$ such that $\left(f^{N+l}(A), \{x_{N+k}\}\right) \in 2^{E_0}$; in particular, for any $y \in f^{N+l}(A)$, $(y, x_k) \in E_0$, a contradiction.

Case ii). There exist $N \in \omega$ and $b \in \overline{\{f^{N+i}(z)\}_{i\in\omega}}$ such that for any $a \in \overline{\{x_{N+i}\}_{i\in\omega}}$ we have $(b, a) \notin 4E_0$. It follows that there exists $k \in \omega$ such that $(f^{N+k}(z), x_{N+i}) \notin 2E_0$ for all

$i \in \omega$. We have from Equation (3.4) that there exists $l \in \omega$ such that $\left(f^{N+k}(A), \{x_{N+l}\}\right) \in 2^{E_0}$; in particular, for any $y \in f^{N+k}(A)$, $(y, x_{N+l}) \in E_0$, a contradiction.

It follows that for any $N \in \omega$

$$\left(\overline{\{x_{N+i}\}_{i\in\omega}}, \overline{\{f^{N+i}(z)\}_{i\in\omega}}\right) \in 2^{4E_0} \subseteq 2^E.$$

□

Remark 3.7.2. Example 3.6.2 shows that the converse to Theorem 3.7.1 is not true. The hyperspatial system does not have orbital shadowing, therefore nor does it have strong orbital shadowing.

3.7.2 Symmetric products

The proof of Theorem 3.7.3 is very similar to that of Theorem 3.7.1 and is thereby omitted.

Theorem 3.7.3. *Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous function. For any $n \geq 2$, if the symmetric product system $(F_n(X), f_n)$ witnesses orbital shadowing then the system (X, f) experiences orbital shadowing.*

Proof. Omitted. □

Remark 3.7.4. The converse of Theorem 3.7.3 is false. It is clear that Example 3.6.2 may be suitably adjusted to provide a counterexample. Indeed, with sufficient adjustments, one can see that, for any $n \geq 2$, (X, f) witnessing strong orbital shadowing does not generally imply that $(F_n(X), f_n)$ has strong orbital shadowing.

3.7.3 Factor maps

Definition 3.7.5. Let (X, f) and (Y, g) be dynamical systems, where X and Y are compact Hausdorff spaces. A surjective semiconjugacy $\varphi\colon X \to Y$ *strong orbitally almost lifts pseudo-orbits* (soALP) if for every $E \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ there exists $W \in \mathscr{U}_Y$ such that for any W-pseudo-orbit $(y_i)_{i\in\omega} \subseteq Y$, there exists a D-pseudo-orbit $(x_i)_{i\in\omega} \subseteq X$ such that for all $N \in \omega$,

$$(\varphi(\overline{\{x_{N+i}\}_{i\in\omega}}), \overline{\{y_{N+i}\}_{i\in\omega}}) \in 2^E.$$

If X and Y are compact metric spaces, then φ is soALP if and only if for all $\varepsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(y_i)_{i\in\omega}$ in Y, there exists an η-pseudo-orbit $(x_i)_{i\in\omega}$ in X such that for all $N \in \omega$, the Hausdorff distance $d_H(\varphi(\overline{\{x_{N+i}\}_{i\in\omega}}), \overline{\{y_{N+i}\}_{i\in\omega}}) < \varepsilon$.

Theorem 3.7.6. *Suppose that $\varphi\colon (X, f) \to (Y, g)$ is a surjective semiconjugacy.*

1. *If (X, f) exhibits strong orbital shadowing and φ is soALP, then (Y, g) exhibits strong orbital shadowing.*
2. *If (Y, g) exhibits strong orbital shadowing, then φ is soALP.*

Proof. For (1), let $E \in \mathscr{U}_Y$ be given. Select $E_0 \in \mathscr{U}_Y$ with $2E_0 \subseteq E$. By the uniform continuity of φ there exists $D_1 \in \mathscr{U}_X$ such that for all $a, b \in X$ with $(a, b) \in D_1$ one has $(\varphi(a), \varphi(b)) \in E_0$. Next, let $D_2 \in \mathscr{U}_X$ be chosen so that D_2-pseudo-orbits in X are D_1-strong-orbital-shadowed. Using E_0 and D_2 in the definition of soALP then provides $W \in \mathscr{U}_Y$ and we claim that W-pseudo-orbits in (Y, g) are E-strong-orbital-shadowed. Indeed, given a W-pseudo-orbit $(y_i)_{i\in\omega} \subseteq Y$, there exists a D_2-pseudo-orbit $(x_i)_{i\in\omega} \subseteq X$ such that, for all $N \in \omega$,

$$(\overline{\{y_{N+i}\}_{i\in\omega}}, \varphi(\overline{\{x_{N+i}\}_{i\in\omega}})) \in 2^{E_0}.$$

Let $z \in X$ D_1-strong-orbital-shadows $(x_i)_{i\in\omega}$. Then, using uniform continuity and the triangle inequality, one may conclude that $\varphi(z)$ E-strong-orbital-shadows $(y_i)_{i\in\omega}$ as required.

For (2), fix $E \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ and take $V \in \mathscr{U}$ to correspond to E for strong orbital shadowing in (Y, g). Let $(y_i)_{i\in\omega}$ to be an V-pseudo-orbit in (Y, g) and let $z \in Y$

E-strong-orbital shadow it. Consider $x \in \varphi^{-1}(z)$ and define $x_i = f^i(x)$ for each $i \in \omega$ so that $(x_i)_{i\in\omega}$ is a D-pseudo-orbit in (X, f). In particular, for any $N \in \omega$, one then has that

$$\begin{aligned}(\varphi(\overline{\{x_{N+i}\}_{i\in\omega}}), \overline{\{y_{N+i}\}_{i\in\omega}}) &= (\overline{\varphi(\{x_{N+i}\}_{i\in\omega})}, \overline{\{y_{N+i}\}_{i\in\omega}}) \\ &= (\overline{\{g^{N+i}(z)\}_{i\in\omega}}, \overline{\{y_{N+i}\}_{i\in\omega}}) \in 2^E.\end{aligned}$$

□

3.7.4 Inverse limits

Theorem 3.7.7. *Let (X, f) be conjugate to a Mittag–Leffler inverse limit system comprised of maps with strong orbital shadowing on compact Hausdorff spaces. Then (X, f) has strong orbital shadowing.*

Proof. Let $(\Lambda, \geq)$ be a directed set. For each $\lambda \in \Lambda$, let (X_λ, f_λ) be a dynamical system on a compact Hausdorff space with strong orbital shadowing and let $((X_\lambda, f_\lambda), g^\eta_\lambda)$ be a Mittag–Leffler inverse system. Without loss of generality $(X, f) = (\varprojlim\{X_\lambda, g^\eta_\lambda\}, f)$.

Let $\mathcal{U}$ be a finite open cover of X. Since $X = \varprojlim\{X_\lambda, g^\eta_\lambda\}$ there exist $\lambda \in \Lambda$ and a finite open cover $\mathcal{W}_\lambda$ of X_λ such that $\mathcal{W} := \{\pi_\lambda^{-1}(W) \cap X \mid W \in \mathcal{W}_\lambda\}$ refines $\mathcal{U}$. Now let $\gamma \in \Lambda$ witness the Mittag–Leffler condition with respect to λ. Let $\mathcal{W}_\gamma := \{g_\lambda^{\gamma(-1)}(W) \mid W \in \mathcal{W}_\lambda\}$. By strong orbital shadowing for (X_γ, f_γ) there exists a finite open cover $\mathcal{V}_\gamma$ of X_γ such that every $\mathcal{V}_\gamma$-pseudo-orbit in X_γ is $\mathcal{W}_\gamma$-strong-orbital-shadowed. Take $\mathcal{V} = \{\pi_\gamma^{-1}(V) \cap X \mid V \in \mathcal{V}_\gamma\}$ and suppose $(x_i)_{i\in\omega}$ is a $\mathcal{V}$-pseudo-orbit in X. It follows that $(\pi_\gamma(x_i))_{i\in\omega}$ is a $\mathcal{V}_\gamma$-pseudo-orbit in X_γ, which means there is a point $z \in X_\gamma$ which $\mathcal{W}_\gamma$-strong-orbital-shadows it. By construction, it follows that $g^\gamma_\lambda(z)$ $\mathcal{W}_\lambda$-strong-orbital-shadows $(\pi_\lambda(x_i))_{i\in\omega}$. Since the system is Mittag–Leffler there exists $y \in \pi_\lambda^{-1}(g^\gamma_\lambda(z)) \cap X$. It follows that y $\mathcal{W}$-strong-orbital-shadows $(x_i)_{i\in\omega}$. Since $\mathcal{W}$ is a refinement of $\mathcal{U}$ the result follows. □

3.7.5 Tychonoff product

Remark 3.7.8. A product of systems with strong orbital shadowing need not have strong orbital shadowing. The component systems in Example 3.6.8 have strong orbital shadowing however their product does not have strong orbital shadowing (since it does not have orbital shadowing).

3.8 Preservation of first weak shadowing

First weak shadowing was introduced in [30] where it was called weak shadowing. The name was revised to first weak shadowing in [82] to accommodate another similar natural weakening of shadowing, i.e. second weak shadowing.

As stated in section 3.2, a dynamical system (X, f) has the *first weak shadowing* property if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point z such that

$$\{x_i\}_{i\in\omega} \subseteq B_E\left(\text{Orb}(z)\right).$$

3.8.1 Induced map on the hyperspace of compact sets

Theorem 3.8.1. *Let X be a compact Hausdorff space, and let $f \colon X \to X$ be a continuous function. If the hyperspace system $(2^X, 2^f)$ witnesses the first weak shadowing property then the system (X, f) has first weak shadowing.*

Proof. Let $E \in \mathscr{U}$ be given. Let $D \in \mathscr{U}$ be such that 2^D corresponds to 2^E in first weak shadowing for 2^f. Let $(x_i)_{i\in\omega}$ be a D-pseudo-orbit in X. We then have that $\left(\{x_i\}\right)_{i\in\omega}$ is a 2^D-pseudo-orbit in 2^X; let $A \in 2^X$ be such that

$$\left\{\{x_i\}\right\}_{i\in\omega} \subseteq B_{2^E}\left(\text{Orb}(A)\right).$$

Fix $a \in A$. Pick $l \in \omega$ arbitrarily. There exists $m \in \omega$ such that $\{x_l\} \in B_{2^E}\left(f^m(A)\right)$, i.e. $\left(\{x_l\}, f^m(A)\right) \in 2^E$. This implies $(x_l, f^m(a)) \in E$. Since $l \in \omega$ was picked arbitrarily it follows that

$$\{x_i\}_{i\in\omega} \subseteq B_E\left(\mathrm{Orb}(a)\right).$$

□

Remark 3.8.2. Example 3.6.2 shows that the converse to Theorem 3.8.1 is not true; in general hyperspace systems do not preserve the first weak shadowing property.

3.8.2 Symmetric products

The proof of Theorem 3.8.3 is very similar to that of Theorem 3.8.1 and is thereby omitted.

Theorem 3.8.3. *Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous function. For any $n \geq 2$, if the symmetric product system $(F_n(X), f_n)$ has first weak shadowing then (X, f) has first weak shadowing.*

Proof. Omitted. □

Remark 3.8.4. The converse of Theorem 3.8.3 is false. It is clear that Example 3.6.2 may be suitably adjusted to provide a counterexample. Indeed, with sufficient adjustments, one can see that, for any $n \geq 2$, (X, f) witnessing first weak shadowing does not generally imply that $(F_n(X), f_n)$ has first weak shadowing.

3.8.3 Factor maps

Definition 3.8.5. Let (X, f) and (Y, g) be dynamical systems, where X and Y are compact Hausdorff spaces. A factor map $\varphi\colon X \to Y$ between compact Hausdorff spaces X and Y *first weak almost lifts pseudo-orbits* (w1ALP) if for every $V \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ there exists $W \in \mathscr{U}_Y$ such that for any W-pseudo-orbit $(y_i) \subseteq Y$, there exists a D-pseudo-orbit $(x_i) \subseteq X$ for which $\{y_i\}_{i\in\omega} \subseteq B_E(\{\varphi(x_i)\}_{i\in\omega})$.

If X and Y are compact metric spaces, then φ is w1ALP if and only if for every $\varepsilon > 0$ and $\eta > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit in Y, there exists an η-pseudo-orbit in X for which $\{y_i\}_{i\in\omega} \subseteq B_\varepsilon(\{\varphi(x_i)\}_{i\in\omega})$.

Theorem 3.8.6. *Suppose that $\varphi\colon (X, f) \to (Y, g)$ is a factor map.*

1. *If (X, f) exhibits first weak shadowing and φ is w1ALP, then (Y, g) exhibits first weak shadowing.*
2. *If (Y, g) exhibits first weak shadowing, then φ is w1ALP.*

Proof. For (1), let $E \in \mathscr{U}_Y$ be given. Select $E_0 \in \mathscr{U}_Y$ with $2E_0 \subseteq E$. By the uniform continuity of φ there exists $D_1 \in \mathscr{U}_X$ such that for all $a, b \in X$ with $(a, b) \in D_1$ one has $(\varphi(a), \varphi(b)) \in E_0$. Next, let $D_2 \in \mathscr{U}_X$ be chosen so that D_2-pseudo-orbits in X are D_1-first-weak-shadowed. Extract $W \in \mathscr{U}_Y$ from the definition of w1ALP using E_0 and D_2, we claim that W-pseudo-orbits of (Y, g) are then E-first-weak-shadowed in (Y, g). Indeed, let $(y_i)_{i\in\omega} \subseteq Y$ be a W-pseudo-orbit and let $(x_i)_{i\in\omega} \subseteq X$ be a D_2-pseudo-orbit lifted through φ, that is,

$$\{y_i\}_{i\in\omega} \subseteq B_{E_0}(\{\varphi(x_i)\}_{i\in\omega}).$$

Suppose $z \in X$ D_1-first-weak-shadows $(x_i)_{i\in\omega}$ so that for each $i \in \omega$, there exists $j \in \omega$ such that $(x_i, f^j(z)) \in D_1$. Then

$$(\varphi(x_i), \varphi(f^j(z))) = (\varphi(x_i), g^j(\varphi(z))) \in E_0.$$

In turn, this provides

$$\{\varphi(x_i)\}_{i\in\omega} \subseteq B_{E_0}(\mathrm{Orb}(\varphi(z))),$$

and hence,

$$\{y_i\}_{i\in\omega} \subseteq B_{E_0}(\{\varphi(x_i)\}_{i\in\omega}) \subseteq B_E(\mathrm{Orb}(\varphi(z))).$$

For (2), let $E \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ be given and let $V \in \mathscr{U}_Y$ correspond to E-first-weak-shadowing in (Y, g). Consider a V-pseudo-orbit $(y_i)_{i\in\omega} \subseteq Y$ and let $z \in Y$ E-first-weak-shadow it. By surjectivity of φ, there exists $x \in \varphi^{-1}(z)$ so one may construct the orbit $(x_i)_{i\in\omega} = (f^i(x))_{i\in\omega}$ which is trivially a D-pseudo-orbit. Then, for all $i \in \omega$ there exists $j \in \omega$ such that

$$(y_i, g^j(z)) = (y_i, \varphi(x_j)) \in E,$$

and hence,

$$\{y_i\}_{i\in\omega} \subseteq B_E(\{\varphi(x_i)\}_{i\in\omega})$$

so that φ is w1ALP. □

3.8.4 Inverse limits

Theorem 3.8.7. *Let (X, f) be conjugate to a Mittag–Leffler inverse limit system comprised of maps with first weak shadowing on compact Hausdorff spaces. Then (X, f) has first weak shadowing.*

Proof. Let $(\Lambda, \geq)$ be a directed set. For each $\lambda \in \Lambda$, let (X_λ, f_λ) be a dynamical system on a compact Hausdorff space with first weak shadowing and let $((X_\lambda, f_\lambda), g^\eta_\lambda)$ be a Mittag–Leffler inverse system. Without loss of generality $(X, f) = (\varprojlim\{X_\lambda, g^\eta_\lambda\}, f)$.

Let $\mathcal{U}$ be a finite open cover of X. Since $X = \varprojlim\{X_\lambda, g^\eta_\lambda\}$ there exist $\lambda \in \Lambda$ and a finite open cover $\mathcal{W}_\lambda$ of X_λ such that $\mathcal{W} := \{\pi_\lambda^{-1}(W) \cap X \mid W \in \mathcal{W}_\lambda\}$ refines $\mathcal{U}$. Now let $\gamma \in \Lambda$ witness the Mittag–Leffler condition with respect to λ. Let $\mathcal{W}_\gamma := \{g_\lambda^{\gamma(-1)}(W) \mid W \in \mathcal{W}_\lambda\}$.

By first weak shadowing for (X_γ, f_γ) there exists a finite open cover $\mathcal{V}_\gamma$ of X_γ such that every $\mathcal{V}_\gamma$-pseudo-orbit in X_γ is $\mathcal{W}_\gamma$-first-weak-shadowed. Take $\mathcal{V} = \{\pi_\gamma^{-1}(V) \cap X \mid V \in \mathcal{V}_\gamma\}$ and suppose $(x_i)_{i\in\omega}$ is a $\mathcal{V}$-pseudo-orbit in X. It follows that $(\pi_\gamma(x_i))_{i\in\omega}$ is a $\mathcal{V}_\gamma$-pseudo-orbit in X_γ, which means there is a point $z \in X_\gamma$ which $\mathcal{W}_\gamma$-first-weak-shadows it. By construction, it follows that $g^\gamma_\lambda(z)$ $\mathcal{W}_\lambda$-first-weak-shadows $(\pi_\lambda(x_i))_{i\in\omega}$. Since the system is Mittag–Leffler there exists $y \in \pi_\lambda^{-1}(g^\gamma_\lambda(z)) \cap X$. It follows that y $\mathcal{W}$-first-weak-shadows $(x_i)_{i\in\omega}$. Since $\mathcal{W}$ is a refinement of $\mathcal{U}$ the result follows. □

3.8.5 Tychonoff product

Remark 3.8.8. A product of systems with first weak shadowing need not have first weak shadowing. Example 3.6.8 demonstrates this.

3.9 Preservation of second weak shadowing

The compact metric version of second weak shadowing was first introduced in [82]. Recall that a system (X, f) has the second weak shadowing property if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\in\omega}$, there exists a point z such that

$$\text{Orb}(z) \subseteq B_E\left(\{x_i\}_{i\in\omega}\right).$$

Pilyugin *et al.* [82] show that every compact metric system exhibits this property. We will see, in Chapter 4, that this result extends to a compact Hausdorff setting (Theorem 4.2.3). Since the hyperspace, symmetric product, inverse limit, and tychonoff product of (a) compact Hausdorff system(s) are themselves compact Hausdorff it follows that any of these induced systems will also have the second weak shadowing property.

3.10 Preservation of limit shadowing

Limit shadowing was introduced in [38] with reference to hyperbolic sets. Recall that a system (X, f) is said to have *limit shadowing* if every asymptotic pseudo-orbit is asymptotically shadowed.

3.10.1 Induced map on the hyperspace of compact sets

Theorem 3.10.1. *Let X be a compact Hausdorff space and let $f\colon X \to X$ be a continuous function. If $(2^X, 2^f)$ has limit shadowing then (X, f) has limit shadowing.*

Proof. Let $(x_i)_{i\in\omega}$ be an asymptotic pseudo-orbit in X. Then $(\{x_i\})_{i\in\omega}$ is an asymptotic pseudo-orbit in 2^X; this is asymptotically shadowed by a set $A \in 2^X$. Pick $a \in A$. It is easy to verify that a asymptotically shadows $(x_i)_{i\in\omega}$.

□

3.10.2 Symmetric products

The proof of Theorem 3.10.2 is very similar to that of Theorem 3.10.1 and is thereby omitted.

Theorem 3.10.2. *Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous function. For any $n \geq 2$, if the symmetric product system $(F_n(X), f_n)$ has limit shadowing then (X, f) has limit shadowing.*

Proof. Omitted. □

Theorem 3.10.3. *Let X be a compact Hausdorff space and let $f\colon X \to X$ be a continuous function. If (X, f) has limit shadowing then $(F_2(X), f_2)$ has limit shadowing.*

Proof. Suppose that $(A_i)_{i\in\omega}$ is an asymptotic pseudo-orbit in $F_2(X)$. Write $A_i = \{x_i, y_i\}$; it is possible that, for some i, $x_i = y_i$. We may relabel the x's and y's where necessary to give asymptotic pseudo-orbits $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ in X. By limit shadowing there exist

$x, y \in X$ which asymptotically shadow $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ respectively. Write $A = \{x, y\} \in F_2(X)$. It is now straightforward to verify that A asymptotically shadows $(A_i)_{i\in\omega}$. $\square$

Example 3.10.4. Let X be the closed unit interval and let $f\colon X \to X$ be the standard tent map, i.e.

$$f(x) = \begin{cases} 2x & \text{if} \quad x \in [0, \frac{1}{2}] \\ 2(1-x) & \text{if} \quad x \in (\frac{1}{2}, 1]\,. \end{cases}$$

Then f has s-limit shadowing and limit shadowing (see [9]) but f_n does not have limit shadowing (and consequently it does not have s-limit shadowing) for any $n \geq 3$.

Fix $n \geq 3$. Let $c = \frac{2}{3}$. Let $\varepsilon = \frac{1}{12}$ and let $\delta > 0$ be given; without loss of generality $\delta < \frac{1}{12}$. Choose $y \in [0, \delta)$ such that there exists $k \in \mathbb{N}$ such that $f^k(y) = c$ and $f^i(y) < \frac{1}{2}$ for all $i < k$ and set $y_0 \coloneqq y$ and let $y_l = \frac{y_{l-1}}{2}$ for all $l \in \mathbb{N}$. Note that $f^{k+l}(y_l) = c$ for any $l \in \omega$ and $f^{l+i}(y_l) < c$ for all $i < k$.

Construct an asymptotic δ-pseudo-orbit $(A_i)_{i\in\omega}$ as follows. Let $A_0 = \{0, y_0, c\}$, $A_1 = \{0, f(y_0), c\}$, $A_2 = \{0, f^2(y_0), c\}$, ..., $A_k = \{0, c\}$, $A_{k+1} = \{0, y_1, c\}$, $A_{k+2} = \{0, f(y_1), c\}$, $A_{k+3} = \{0, f^2(y_1), c\}$, ..., $A_{2k+2} = \{0, c\}$, $A_{2k+3} = \{0, y_2, c\}$ .... Explicitly, for any $l \in \omega$, and for any $j \in \omega$ such that $0 \leq j \leq k + l$,

$$A_{lk+(\sum_{i=0}^{l} i)+j} = \{0, f^j(y_l), c\}.$$

It is easy to see that $(A_i)_{i\in\omega}$ is an asymptotic δ-pseudo-orbit. Suppose that $A \in F_n(X)$ asymptotically shadows this asymptotic δ-pseudo-orbit. It follows that it eventually $\frac{1}{12}$-shadows $(A_i)_{i\in\omega}$; there exists $N \in \mathbb{N}$ such that $f_n^N(A)$ $\frac{1}{12}$-shadows $(A_{N+i})_{i\in\omega}$.

First observe that, since the pseudo-orbit is always a subset of the interval $[0, \frac{2}{3}]$, shadowing entails that $f_n^{N+i}(A) \subseteq [0, \frac{3}{4})$ for any $i \in \omega$. Finally, every point in $f_n^N(A)$ must either be 0, c or a preimage of c in the interval $[0, \frac{2}{3}]$, otherwise it would enter (or already lie in) $[\frac{3}{4}, 1]$ which would contradict shadowing. Now let z be the least such element of $f_n^N(A) \setminus \{0\}$. Let $p \in \omega$ be least such that $f^p(z) = c$. Then $f_n^{N+p+i}(A) = \{0, c\}$ for all $i \in \omega$; clearly this contradicts the fact that $f_n^N(A)$ is $\frac{1}{12}$-shadowing $(A_{N+i})_{N+i\in\omega}$

since there exists $q > N+p+i$ such that $A_q = \{0, c, \frac{1}{3}\}$. Therefore f_n does not have limit shadowing (resp. s-limit shadowing).

3.10.3 Factor maps

Definition 3.10.5. Suppose X and Y are compact Hausdorff spaces and $f\colon X \to X$, $g\colon Y \to Y$ are continuous. A surjective semiconjugacy $\varphi\colon X \to Y$ *almost lifts asymptotic pseudo-orbits* (ALAP) if for every asymptotic pseudo-orbit $(y_i)_{i\in\omega} \subseteq Y$, there exists an asymptotic pseudo-orbit $(x_i)_{i\in\omega} \subseteq X$ such that $(\varphi(x_i))_{i\in\omega}$ asymptotically shadows $(y_i)_{i\in\omega}$.

The proof of the following is similar to the proofs of Theorems 3.5.7, 3.6.6, 3.7.6, and 3.8.6 and is therefore omitted.

Theorem 3.10.6. *Suppose that $\varphi\colon (X, f) \to (Y, g)$ is a surjective semiconjugacy.*

1. *If (X, f) exhibits limit shadowing and φ is* ALAP*, then (Y, g) exhibits limit shadowing.*

2. *If (Y, g) exhibits limit shadowing, then φ is* ALAP.

3.10.4 Inverse limits

Whilst it remains unclear whether general inverse limit systems preserve limit shadowing, we note the following result proved by Good *et al.* in [52].

Theorem 3.10.7. [52, Theorem 5.1] *Let X be a compact metric space and $f\colon X \to X$ a continuous onto map. Then (X, f) has limit shadowing if and only if $(\varprojlim(X, f), \sigma)$ has limit shadowing.*

3.10.5 Tychonoff product

Theorem 3.10.8. *Let Λ be an arbitrary indexation set and, for each $\lambda \in \Lambda$, let (X_λ, f_λ) be a compact Hausdorff system with limit shadowing. Then the product system (X, f), where $X = \prod_{\lambda\in\Lambda} X_\lambda$, has limit shadowing.*

Proof. Let $(x_i)_{i\in\omega}$ be an asymptotic pseudo-orbit in X. Then, for any $\lambda \in \Lambda$, $\left(\pi_\lambda(x_i)\right)_{i\in\omega}$ is an asymptotic pseudo-orbit in X_λ. By limit shadowing in these component spaces, for each λ there exists $y_\lambda \in X_\lambda$ which limit shadows $\left(\pi_\lambda(x_i)\right)_{i\in\omega}$. Let $y \in X$ be such that $\pi_\lambda(y) = y_\lambda$ for any $\lambda \in \Lambda$. We claim y limit shadows $(x_i)_{i\in\omega}$.

Let $E \in \mathscr{U}$ be given; this entourage is refined by one of the form

$$\prod_{\lambda\in\Lambda} E_\lambda,$$

where $E_\lambda \in \mathscr{U}_\lambda$ for all $\lambda \in \Lambda$ and $E_\lambda = X_\lambda \times X_\lambda$ for all but finitely many of the λ's. Let λ_j, for $1 \leq j \leq k$, be precisely those elements in Λ for which $E_\lambda \neq X_\lambda \times X_\lambda$ (if there are no such elements then we are done). For each such j, let $M_j \in \mathbb{N}$ be such that for any $n \geq M_j$ $(f^n(y_{\lambda_j}), \pi_{\lambda_j}(x_n)) \in E_{\lambda_j}$. Take $M := \max_{1\leq j\leq k} M_j$. It follows that, for any $n \geq M$, $(f^n(y), x_n) \in E$. □

3.11 Preservation of s-limit shadowing

The definition of limit shadowing was extended in [63] to a property the authors called s-limit shadowing. This was done to accommodate the fact that many systems exhibit limit shadowing but not shadowing [60, 83].

Recall that a system (X, f) is said to have *s-limit shadowing* if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that the following two conditions hold:

1. every D-pseudo-orbit is E-shadowed, and

2. every asymptotic D-pseudo-orbit is asymptotically E-shadowed.

Thus, part of what it means for a system to have s-limit shadowing is that it also has shadowing. It is a standard result in the theory of shadowing [83] that a compact metric dynamical system (X, f) has shadowing if and only if for any $\varepsilon > 0$ there is a $\delta > 0$ such that every finite δ-pseudo-orbit $(x_0, \ldots, x_n)$ is ε-shadowed by some $x \in X$ (we call this property *finite shadowing*). This extends to the compact Hausdorff setting: a compact Hausdorff dynamical system (X, f) has shadowing if and only if for any $E \in \mathscr{U}$ there is a $D \in \mathscr{U}$ such that every finite D-pseudo-orbit $(x_0, \ldots, x_n)$ is E-shadowed by some $x \in X$. This fact allows us to make the observation (Theorem 3.11.1) that for a large class of systems, the definition of s-limit shadowing can be simplified.

Theorem 3.11.1. *Suppose X is a compact Hausdorff space. (X, f) has s-limit shadowing if and only if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that every asymptotic D-pseudo-orbit is asymptotically E-shadowed.*

In particular, if X is a compact metric space, then (X, f) has s-limit shadowing if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every asymptotic δ-pseudo-orbit is asymptotically ε-shadowed.

Proof. Condition (1) simply says that part of what it means for a system to have s-limit shadowing is that it has shadowing. Suppose that (X, f) satisfies condition (2). Let $E \in \mathscr{U}$ be given and take a corresponding $D \in \mathscr{U}$. Let $(x_0, x_1, \ldots, x_m)$ be a finite D-pseudo-orbit in X. Then

$$(x_0, x_1, \ldots, x_m, f(x_m), f^2(x_m), \ldots, f^k(x_m), \ldots),$$

is an asymptotic D-pseudo-orbit. By condition (2) this is asymptotically E-shadowed by a point, say x. In particular $(f^i(x), x_i) \in E$ for all $i \in \{0, 1, \ldots, m\}$; hence (X, f) has finite shadowing and thereby shadowing. □

Since our space is compact Hausdorff throughout this chapter, it follows from Theorem 3.11.1 that when checking for s-limit shadowing it suffices to verify whether or not condition (2) in the definition holds.

3.11.1 Induced map on the hyperspace of compact sets

Theorem 3.11.2. *Let X be a compact Hausdorff space and let $f\colon X \to X$ be a continuous function. If $(2^X, 2^f)$ has s-limit shadowing then (X, f) has s-limit shadowing.*

Proof. Let $E \in \mathscr{U}$ be given. Let $D \in 2^{\mathscr{U}}$ correspond to 2^E in condition (2) of s-limit shadowing for 2^f and let $D_0 \in \mathscr{U}$ be such that $2^{D_0} \subseteq D$. Let $(x_i)_{i\in\omega}$ be an asymptotic D_0-pseudo-orbit in X. Then $(\{x_i\})_{i\in\omega}$ is an asymptotic 2^D-pseudo-orbit in 2^X; this is asymptotically 2^E-shadowed by a set $A \in 2^X$. Pick $a \in A$. It is easy to verify that a asymptotically E-shadows $(x_i)_{i\in\omega}$. □

3.11.2 Symmetric products

The proof of Theorem 3.11.3 is very similar to that of Theorem 3.11.2 and is thereby omitted.

Theorem 3.11.3. *Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous function. For any $n \geq 2$, if the symmetric product system $(F_n(X), f_n)$ has s-limit shadowing then (X, f) has s-limit shadowing.*

Proof. Omitted. □

Theorem 3.11.4. *Let X be a compact Hausdorff space and let $f\colon X \to X$ be a continuous function. If (X, f) has s-limit shadowing then $(F_2(X), f_2)$ has s-limit shadowing.*

Proof. Let $E \in 2^{\mathscr{U}}$ be given. Let $E_0 \in \mathscr{U}$ be such that $2^{E_0} \subseteq E$. Let $D \in \mathscr{U}$ correspond to E_0 in s-limit shadowing for f. We claim 2^D satisfies condition (2) of s-limit shadowing for E. Suppose that $(A_i)_{i\in\omega}$ is an asymptotic 2^D-pseudo-orbit in $F_2(X)$. Write $A_i = \{x_i, y_i\}$; it is possible that, for some i, $x_i = y_i$. We may relabel the x's and y's where necessary to give asymptotic D-pseudo-orbits $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ in X. By s-limit shadowing there exist $x, y \in X$ which asymptotically E_0-shadow $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ respectively. Write $A = \{x, y\} \in F_2(X)$. It is now straightforward to verify that A asymptotically E-shadows $(A_i)_{i\in\omega}$. □

Remark 3.11.5. Example 3.10.4 shows that, in general, symmetric products do not preserve s-limit shadowing for $n \geq 3$.

3.11.3 Factor maps

Definition 3.11.6. Suppose X and Y are compact Hausdorff spaces and $f\colon X \to X$, $g\colon Y \to Y$ are continuous. A surjective semiconjugacy $\varphi\colon X \to Y$ *almost lifts asymptotic ε-pseudo-orbits* (ALAεP) if for every $V \in \mathscr{U}_Y$ and $D \in \mathscr{U}_X$ there is $W \in \mathscr{U}_Y$ such that for every asymptotic W-pseudo-orbit $(y_i)_{i\in\omega}$ in Y there is an asymptotic D-pseudo-orbit $(x_i)_{i\in\omega}$ in X such that $(\varphi(x_i))_{i\in\omega}$ asymptotically V-shadows $(y_i)_{i\in\omega}$.

If X and Y are compact metric spaces, then φ is ALAεP if and only if for every $\varepsilon > 0$ and $\eta > 0$ there is $\delta > 0$ such that for every asymptotic δ-pseudo-orbit $(y_i)_{i\in\omega}$ in Y there is an asymptotic η-pseudo-orbit $(x_i)_{i\in\omega}$ in X such that $(\varphi(x_i))_{i\in\omega}$ asymptotically ε-shadows $(y_i)_{i\in\omega}$.

The proof of the following is similar to the proofs of Theorems 3.5.7, 3.6.6, 3.7.6 and 3.8.6 and is therefore omitted

Theorem 3.11.7. *Suppose that $\varphi\colon (X, f) \to (Y, g)$ is a factor map.*

1. *If (X, f) exhibits s-limit shadowing and φ is* ALAεP, *then (Y, g) exhibits s-limit shadowing.*
2. *If (Y, g) exhibits s-limit shadowing, then φ is* ALAεP.

3.11.4 Inverse limits

Whilst it remains unclear whether general inverse limit systems preserve s-limit shadowing, we note the following result proved by the Good *et al.* in [52].

Theorem 3.11.8. [52, Theorem 5.1] *Let X be a compact metric space and $f\colon X \to X$ a continuous onto map. Then (X, f) has s-limit shadowing if and only if $(\varprojlim(X, f), \sigma)$ has s-limit shadowing.*

3.11.5 Tychonoff product

Theorem 3.11.9. *Let Λ be an arbitrary indexation set and, for each $\lambda \in \Lambda$, let (X_λ, f_λ) be a compact Hausdorff system with s-limit shadowing. Then the product system (X, f), where $X = \prod_{\lambda \in \Lambda} X_\lambda$, has s-limit shadowing.*

Proof. Let $E \in \mathscr{U}$ be given; this entourage is refined by one of the form

$$\prod_{\lambda \in \Lambda} E_\lambda,$$

where $E_\lambda \in \mathscr{U}_\lambda$ for all $\lambda \in \Lambda$ and $E_\lambda = X_\lambda \times X_\lambda$ for all but finitely many of the λ's. Let λ_i, for $1 \leq i \leq k$, be precisely those elements in Λ for which $E_\lambda \neq X_\lambda \times X_\lambda$ (without loss of generality this set is nonempty).

By s-limit shadowing in each component space, there exist entourages $D_{\lambda_i} \in \mathscr{U}_{\lambda_i}$ such that every asymptotic D_{λ_i}-pseudo-orbit is asymptotically E_{λ_i}-shadowed. Note that, for every $\lambda \in \Lambda \setminus \{\lambda_i \mid 1 \leq i \leq k\}$ every asymptotic pseudo-orbit is asymptotically E_λ-shadowed. For $\lambda \in \Lambda \setminus \{\lambda_i \mid 1 \leq i \leq k\}$ take $D_\lambda = X \times X$. Let

$$D := \prod_{\lambda \in \Lambda} D_\lambda.$$

Now let $(x_j)_{j \in \omega}$ be an asymptotic D-pseudo-orbit. Then $(\pi_{\lambda_i}(x_j))_{j \in \omega}$ is an asymptotic D_{λ_i}-pseudo-orbit in X_{λ_i}, which is asymptotically E_{λ_i}-shadowed by a point $z_i \in X_{\lambda_i}$. Furthermore $(\pi_\lambda(x_j))_{j \in \omega}$ is an asymptotic pseudo-orbit in X_λ which is asymptotically shadowed by a point z_λ. Let $z \in X$ be such that $\pi_\lambda(z) = z_\lambda$ for all $\lambda \in \Lambda \setminus \{\lambda_i \mid 1 \leq i \leq k\}$ and $\pi_{\lambda_i}(z) = z_{\lambda_i}$ for each i. It is easy to see that z asymptotically E-shadows $(x_j)_{j \in \omega}$. □

3.12 Preservation of orbital limit shadowing

Orbital limit shadowing was introduced by Pilyugin and others in [84] and studied with regard to various types of stability. Good and Meddaugh [47] show that this property is equivalent to one they call *asymptotic orbital shadowing* (see Theorem 3.12.1 and Definition 3.2.10). Recall that a system (X, f) has the *orbital limit shadowing* property if given any asymptotic pseudo-orbit $(x_i)_{i\geq 0} \subseteq X$, there exists a point $x \in X$ such that

$$\omega((x_i)_{i\geq 0}) = \omega(x).$$

Here $\omega((x_i)_{i\geq 0})$ is the set of limit points of the pseudo-orbit.

The following theorem, proved in [47], gives an equivalence between two notions of shadowing that we have defined in section 3.2. It is because of this equivalence that asymptotic orbital shadowing is omitted from the table of results. (NB. The authors [47] prove the theorem below in a compact metric setting. Their result generalises to the case when the underlying space is a compact Hausdorff.)

Theorem 3.12.1. [47, Theorem 22] *Let (X, f) be a compact Hausdorff dynamical system. Then the following are equivalent:*

1. *f has the asymptotic orbital shadowing property;*

2. *f has the orbital limit shadowing property; and*

3. *$\omega_f = \mathrm{ICT}_f$.*

In the above theorem ω_f is the set of ω-limit sets of f, whilst ICT_f is the set of nonempty, closed, *internally chain transitive sets*: a set $A \subseteq X$ is internally chain transitive if for any $D \in \mathscr{U}$ and any $x, y \in A$ there exists a sequence of points in A, called a D-chain, $(x = x_0, x_1, x_2, \ldots, x_n = y)$ such that $(f(x_i), x_{i+1}) \in D$ for every $0 \leq i \leq n-1$.

3.12.1 Induced map on the hyperspace of compact sets

Theorem 3.12.2. *Let X be a compact Hausdorff space, and let $f \colon X \to X$ be a continuous function. If $(2^X, 2^f)$ witnesses orbital limit shadowing then (X, f) experiences orbital limit shadowing.*

We will use the fact that orbital limit shadowing is equivalent to asymptotic orbital shadowing (Theorem 3.12.1). Recall the following definition: The system (X, f) has the *asymptotic orbital shadowing* property if given any asymptotic pseudo-orbit $(x_i)_{i\geq 0} \subseteq X$, there exists a point $x \in X$ such that for any $E \in \mathscr{U}$ there exists $N \in \mathbb{N}$ such that

$$(\overline{\{x_{N+i}\}_{i\geq 0}}, \overline{\{f^{N+i}(x)\}_{i\geq 0}}) \in 2^E.$$

Proof. Let $(x_i)_{i\in\omega}$ be an asymptotic pseudo-orbit in X. Notice that $(\{x_i\})_{i\in\omega}$ is an asymptotic pseudo-orbit in the hyperspace 2^X. Thus, by asymptotic orbital shadowing, there exists $A \in 2^X$ such that for any $E \in 2^{\mathscr{U}}$, there exists $N \in \mathbb{N}$ such that

$$(\overline{\{\{x_{N+i}\}\}_{i\geq 0}}, \overline{\{f^{N+i}(A)\}_{i\geq 0}}) \in 2^E.$$

Pick $z \in A$ and let $D \in \mathscr{U}$, so $2^D \in 2^{\mathscr{U}}$. Let $E \in \mathscr{U}$ be such that $4E \subseteq D$ and let $N \in \mathbb{N}$ be such that

$$(\overline{\{\{x_{N+i}\}\}_{i\geq 0}}, \overline{\{(2^f)^{N+i}(A)\}_{i\geq 0}}) \in 2^{2^E}.$$

Equivalently

$$\overline{\{\{x_{N+i}\}\}_{i\in\omega}} \subseteq B_{2^E}\left(\overline{\{(2^f)^{N+i}(A)\}_{i\in\omega}}\right) \tag{3.5}$$

and

$$\overline{\{(2^f)^{N+i}(A)\}_{i\in\omega}} \subseteq B_{2^E}\left(\overline{\{\{x_{N+i}\}\}_{i\in\omega}}\right). \tag{3.6}$$

We claim

$$(\overline{\{x_{N+i}\}_{i\geq 0}}, \overline{\{f^{N+i}(z)\}_{i\geq 0}}) \in 4E \subseteq 2^D.$$

Indeed, suppose not.

Case i). There exists $a \in \overline{\{x_{N+i}\}_{i\in\omega}}$ such that for any $b \in \overline{\{f^{N+i}(z)\}_{i\in\omega}}$ we have $(a, b) \notin 4E$. It follows that there exists $k \geq N$ such that $(x_k, f^i(z)) \notin 2E$ for all $i \geq N$.

We have from Equation (3.5) that there exists $l \geq N$ such that $\left(f^l(A), \{x_k\}\right) \in 2^E$; in particular, for any $y \in f^l(A)$, $(y, x_k) \in E$, a contradiction.

Case ii). There exists $b \in \overline{\{f^{N+i}(z)\}_{i \in \omega}}$ such that for any $a \in \overline{\{x_i\}_{i \in \omega}}$ we have $(b, a) \notin 4E$. It follows that there exists $k \geq N$ such that $(f^k(z), x_i) \notin 2E_0$ for all $i \geq N$. We have from Equation (3.6) that there exists $l \geq N$ such that $\left(f^k(A), \{x_l\}\right) \in 2^E$; in particular, for any $y \in f^k(A)$, $(y, x_l) \in E$, a contradiction.

It follows that

$$\left(\overline{\{x_{N+i}\}_{i \in \omega}}, \overline{\{f^{N+i}(z)\}_{i \in \omega}}\right) \in 2^{4E} \subseteq 2^D.$$

□

The following example shows that the converse to Theorem 3.12.2 is false.

Example 3.12.3. Let X be the circle $\mathbb{R}/\mathbb{Z}$ and let $f \colon X \to X$ be given by $x \mapsto x + \alpha$, where α is some fixed irrational number. Since (X, f) is minimal it clearly has orbital limit shadowing. (Indeed, this follows as a simple corollary to Theorem 3.12.1 since $\omega_f = \mathrm{ICT}_f$ for minimal systems.) Let x_0 and y_0 be two antipodal points and suppose $\delta \in \mathbb{Q}$ with $0 < \delta < 1$. Then construct an asymptotic pseudo-orbit in 2^X recursively by the following rule: Let $A_0 = \{x_0, y_0\}$ and, for all $i \in \mathbb{N}$, let $A_i = \{x_i, y_i\} \coloneqq \{f(x_{i-1}) + \frac{\delta}{2i}, f(y_{i-1}) + \frac{\delta}{3i}\}$. We claim that this is not orbital limit shadowed. Suppose $A \in 2^X$ orbital limit shadows $(A_i)_{i \in \omega}$; i.e.

$$\omega(A) = \omega\left((A_i)_{i \geq 0}\right).$$

First note that $\omega\left((A_i)_{i \geq 0}\right) = \{\{a, b\} \mid a, b \in X\}$. If A is infinite then there will be infinite sets in its ω-limit set. Therefore A must be finite; let n be its cardinality. If $n \geq 3$ then there will be sets of size larger than 2 in its ω-limit set. It follows that we must have $n = 2$. Write $A = \{x, y\}$ for distinct points $x, y \in X$. Since 2^f is a minimal isometry it follows that

$$\omega(A) = \{\{a, b\} \mid d(a, b) = d(x, y)\}.$$

Pick distinct points $a, b \in X$ with $d(a, b) \neq d(x, y)$. Then $\{a, b\} \in \omega\left((A_i)_{i \geq 0}\right)$ but

$\{a, b\} \notin \omega(A)$, a contradiction.

3.12.2 Symmetric products

The proof of Theorem 3.12.4 is very similar to that of Theorem 3.12.2 and is thereby omitted.

Theorem 3.12.4. *Let X be a compact Hausdorff space, and let $f\colon X \to X$ be a continuous function. For any $n \geq 2$, if the symmetric product system $(F_n(X), f_n)$ witnesses orbital shadowing then the system (X, f) experiences orbital shadowing.*

Proof. Omitted. □

Remark 3.12.5. The converse of Theorem 3.12.4 is false. It is clear that Example 3.12.3 may be suitably adjusted to provide a counterexample. Indeed, with sufficient adjustments, one can see that, for any $n \geq 2$, (X, f) witnessing orbital shadowing does not generally imply that $(F_n(X), f_n)$ has orbital shadowing.

3.12.3 Factor maps

Definition 3.12.6. Let (X, f) and (Y, g) be dynamical systems where X and Y are compact Hausdorff spaces. A surjective semiconjugacy $\varphi\colon X \to Y$ *orbitally almost lifts asymptotic pseudo-orbits* (oALAP) if for every asymptotic pseudo-orbit $(y_i)_{i\in\omega} \subseteq Y$, there exists an asymptotic pseudo-orbit $(x_i)_{i\in\omega} \subseteq X$ such that for any $E \in \mathscr{U}_Y$ there exists $N \in \mathbb{N}$ such that for all $i \geq N$

$$(\varphi(\overline{\{x_i\}_{i\in\omega}}), \overline{\{y_i\}_{i\in\omega}}) \in 2^E.$$

If X and Y are compact metric, then φ is oALAP if and only if for every asymptotic pseudo-orbit $(y_i)_{i\in\omega}$ in Y there exists an asymptotic pseudo-orbit $(x_i)_{i\in\omega}$ in X such that for every $\varepsilon > 0$ there is $N > 0$ for which the Hausdorff distance $d_H(\varphi(\overline{\{x_i\}_{i\in\omega}}), \overline{\{y_i\}_{i\in\omega}}) < \varepsilon$ for any $i \geq N$.

Again the proof of the following theorem is similar to that of Theorems 3.5.7, 3.6.6, 3.7.6, and 3.8.6 bearing in mind the equivalence between orbital limit shadowing and asymptotic orbital shadowing ([47, Theorem 22]).

Theorem 3.12.7. *Suppose that $\varphi\colon (X,f) \to (Y,g)$ is a factor map.*

1. *If (X,f) exhibits orbital limit shadowing and φ is oALAP, then (Y,g) exhibits orbital limit shadowing.*

2. *If (Y,g) exhibits orbital limit shadowing, then φ is oALAP.*

3.12.4 Tychonoff product

A product of systems with orbital limit shadowing does not necessarily have orbital limit shadowing. The following example demonstrates this.

Example 3.12.8. For $i \in \{1,2\}$ let $X_i = \mathbb{R}/\mathbb{Z}$, d_i be the shortest arc length metric on X_i and $f_i\colon X_i \to X_i\colon x \mapsto x+\alpha \mod 1$, where α is some fixed irrational number. Equip the product space $X = X_1 \times X_2$ with the metric d given by $d((a,b),(c,d)) = \sup\{d_1(a,c), d_2(b,d)\}$. Now consider the product system (X,f). Let x_0 and y_0 be two antipodal points and suppose $\delta \in \mathbb{Q}$ with $0 < \delta < 1$. Then construct an asymptotic pseudo-orbit in X recursively by the following rule: Let $z_0 = (x_0, y_0)$ and, for all $i \in \mathbb{N}$, let $z_i = (f(x_{i-1}) + \frac{\delta}{2i}, f(y_{i-1}) + \frac{\delta}{3i})$ where $x_i = f(x_{i-1}) + \frac{\delta}{2i}$ and $y_i = f(y_{i-1}) + \frac{\delta}{3i}$. We claim that this is not orbital limit shadowed. Suppose $z = (x,y) \in X$ orbital limit shadows $(z_i)_{i\in\omega}$; i.e.

$$\omega(z) = \omega\left((z_i)_{i\in\omega}\right).$$

It is easy to see that $\omega\left((z_i)_{i\in\omega}\right) = \{(a,b) \mid a,b \in X\}$.

$$\omega(z) \subseteq \{(a,b) \mid \min\{|a-b|, 1-|b-a|\} = \min\{|x-y|, 1-|y-x|\}\},$$

where equality holds only when $\min\{|x-y|, 1-|y-x|\} \in \{0, \frac{1}{2}\}$. Therefore by picking

$(a, b) \in X$ with $\min\{|a-b|, 1-|b-a|\} \neq \min\{|x-y|, 1-|y-x|\}$ then we get $(a, b) \in \omega\left((z_i)_{i\geq 0}\right)$ but $(a, b) \notin \omega(z)$, a contradiction.

3.13 Preservation of inverse shadowing

The presence (or absence) of shadowing in a dynamical system tells us whether or not any given computed orbit is followed (to within some constant error) by a true trajectory. In a related fashion, one may wonder under what circumstances actual trajectories can be recovered, within a given accuracy, from pseudo-orbits. As observed elsewhere (e.g. [62]), this relates to inverse shadowing. In this chapter we limit our discussion to $\mathcal{T}_0$-inverse shadowing, however we note that weaker formulations of inverse shadowing can be given by restricting one's attention to certain classes of pseudo-orbits (see for example [62, 79]).

Recall that the system (X, f) experiences *inverse shadowing* if, for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that for any $x \in X$ and any $\varphi \in \mathcal{T}_0(f, D)$ there exists $y \in X$ such that $\varphi(y)$ E-shadows x; i.e.

$$\forall k \in \omega, (\varphi(y)_k, f^k(x)) \in E.$$

Compact metric versions of the two results below may be found in [50]; the authors remark that the compact metric versions extend to these two results.

Lemma 3.13.1. [50, Theorem 2.1] *A continuous function $f\colon X \to X$ has inverse shadowing if and only if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that for any $x \in X$ there exists $y \in X$ such that for any $\varphi \in \mathcal{T}_0(f, D)$, $\varphi(y)$ E-shadows x; i.e.*

$$\forall k \in \omega, (\varphi(y)_k, f^k(x)) \in E.$$

Lemma 3.13.2. [50, Corollary 2.2] *A continuous function $f\colon X \to X$ has inverse shadowing if and only if for any $E \in \mathscr{U}$ there exists $D \in \mathscr{U}$ such that for any $x \in X$ there* *exists* $y \in X$ *such that for any* $y' \in B_D(y)$ *and any* $\varphi \in \mathcal{T}_0(f, D)$, $\varphi(y')$ E*-shadows* x*; i.e.*

$$\forall k \in \omega,\ (\varphi(y')_k, f^k(x)) \in E.$$

Recall the open cover formulation of inverse shadowing from section 3.2, which coincides with the uniform definition in the presence of compactness. The system (X, f) experiences *inverse shadowing* if, for any finite open cover $\mathcal{U}$ there exists a finite open cover $\mathcal{V}$ such that for any $x \in X$ and any $\varphi \in \mathcal{T}_0(f, \mathcal{U})$ there exists $y \in X$ such that $\varphi(y)$ $\mathcal{U}$-shadows x; i.e.

$$\forall k \in \omega\, \exists U \in \mathcal{U} : \varphi(y)_k, f^k(x) \in U.$$

The following lemma is an open cover version of Lemma 3.13.1.

Lemma 3.13.3. *A continuous function* $f\colon X \to X$ *has inverse shadowing if and only if for any finite open cover* $\mathcal{U}$ *there exists a finite open cover* $\mathcal{V}$ *such that for any* $x \in X$ *there exists* $y \in X$ *for any* $\varphi \in \mathcal{T}_0(f, \mathcal{V})$ $\varphi(y)$ $\mathcal{U}$*-shadows* x*; i.e.*

$$\forall k \in \omega\, \exists U \in \mathcal{U} : \varphi(y')_k, f^k(x) \in U.$$

3.13.1 Induced map on the hyperspace of compact sets

Theorem 3.13.4. *Let* X *be a compact Hausdorff space and let* $f\colon X \to X$ *be a continuous function. Then* (X, f) *has* $\mathcal{T}_0$*-inverse shadowing if and only if* $(2^X, 2^f)$ *has* $\mathcal{T}_0$*-inverse shadowing.*

Proof. Suppose that f has inverse shadowing. Let $2^E \in \mathscr{B}_{2^\mathscr{U}}$ (so $E \in \mathscr{U}$). Let $E_0 \in \mathscr{U}$ be such that $2E_0 \subseteq E$ and take $D \in \mathscr{U}$ be as in Lemma 3.13.2. That is, for any $x \in X$ there exists $y \in X$ such that for any $y' \in B_D(y)$ and any $\varphi \in \mathcal{T}_0(f, D)$ we have $\varphi(y')$ E_0-shadows x.

Choose $A \in 2^X$ and let $\psi \in \mathcal{T}_0(2^f, 2^D)$. For each $x \in A$ let $y_x \in X$ be such that for any $y' \in B_D(y)$ and any $\varphi \in \mathcal{T}_0(f, D)$ we have $\varphi(y')$ E_0-shadows x. Define

$$C \coloneqq \overline{\bigcup_{x \in A} \{y_x\}}.$$

It is easy to verify that $\psi(C)$ 2^E-shadows A. Indeed, let $k \in \omega$ be given. Choose $a \in (2^f)^k(A)$ and $a' \in A$ such that $f^k(a') = a$. Since ψ is a 2^D-method for 2^f, for each $i \in \{1, \ldots, k\}$ there exists $c_i \in \psi(C)_k$ such that $(f(c_i), c_{i+1}) \in D$ and $(f(y_{a'}), c_1) \in D$. Denote by $c_0 = y_{a'}$ so that by definition of $y_{a'}$, $(c_i, f^i(a')) \in E_0$ for all $i \in \{0, \ldots, k\}$. In particular it follows that $(a, c_k) \in E$. As a was picked arbitrarily from $(2^f)^k(A)$ it follows that $(2^f)^k(A) \subseteq B_E(\psi(C)_k)$. Now choose $c \in \psi(C)_k$. By construction, there exists a sequence $(c_0, c_1, \ldots, c_k)$ with $c_i \in \psi(C)_i$ for each $0 \leq i \leq k$ and $c_k = c$ for which $(f(c_i), c_{i+1}) \in D$. Furthermore, there exist $x \in A$ for which $c_0 \in B_D(y_x)$. It follows that $(c_i, f^i(x)) \in E_0$ for each $0 \leq i \leq k$. In particular it follows that $(c, f^k(x)) \in E$. As c was picked arbitrarily from $\psi(C)_k$ it follows that $\psi(C)_k \subseteq B_E((2^f)^k(A))$.

Now suppose that 2^f has inverse shadowing. Let $E \in \mathscr{U}$ be given and let $E_0 \in \mathscr{U}$ be such that $2E_0 \subseteq E$. Let $D_0, D \in \mathscr{U}$ be such that 2^D satisfies the inverse shadowing condition for 2^{E_0} and $2D_0 \subseteq D$. Now pick $\varphi \in \mathcal{T}_0(f, D_0)$. Construct a method 2^φ as follows: For any $k \in \omega$ and any $A \in 2^X$

$$2^\varphi(A)_k \coloneqq \overline{\bigcup_{x \in A} \{\varphi(x)_k\}}.$$

Clearly $2^\varphi \in \mathcal{T}_0(f, 2^D)$. Let $x \in X$ be given. Then $\{x\} \in 2^X$. By inverse shadowing there exists $A \in 2^X$ such that $2^\varphi(A)$ 2^{E_0}-shadows $\{x\}$. Pick $a \in A$. It is easy to verify that $\varphi(a)$ E-shadows x.

□

3.13.2 Symmetric products

Theorem 3.13.5. *Let X be a compact Hausdorff space and let $f\colon X \to X$ be a continuous function. Then (X, f) has $\mathcal{T}_0$-inverse shadowing if and only if $(F_n(X), f_n)$ has $\mathcal{T}_0$-inverse shadowing for all $n \geq 2$.*

Proof. Suppose that (X, f) has inverse shadowing and fix $n \geq 2$. Let $2^E \in \mathscr{B}_{2^{\mathscr{U}}}$ (so $E \in \mathscr{U}$). Let $E_0 \in \mathscr{U}$ be such that $2E_0 \subseteq E$ and take $D \in \mathscr{U}$ be as in Lemma 3.13.1; that is for any $x \in X$ there exists $y \in X$ such that for any $\varphi \in \mathcal{T}_0(f, D)$ we have $\varphi(y)$ E_0-shadows x.

Pick $A \in F_n(X)$ and let $\psi \in \mathcal{T}_0(f_n, 2^D \cap F_n(X))$. For each $x \in A$ let $y_x \in X$ be such that for any $\varphi \in \mathcal{T}_0(f, D)$, $\varphi(y)$ E_0-shadows x. Define

$$C \coloneqq \bigcup_{x \in A} \{y_x\}.$$

Note the $|C| \leq |A|$ so $C \in F_n(X)$. It is easy to verify that $\psi(C)$ 2^E-shadows A. Indeed, let $k \in \omega$ be given. Pick $a \in f_n^k(A)$ and $a' \in A$ such that $f^k(a') = a$. Since ψ is a $\big(2^D \cap (F_n(X) \times F_n(X))\big)$-method for f_n, for each $i \in \{1, \ldots, k\}$ there exists $c_i \in \psi(C)_k$ such that $(f(c_i), c_{i+1}) \in D$ and $(y_{a'}, c_1) \in D$. Denote $c_0 = y_{a'}$ so that by definition of $y_{a'}$, $(c_i, f^i(a')) \in E_0$ for all $i \in \{0, \ldots, k\}$. In particular, it follows that $(a, c_k) \in E$. As a was picked arbitrarily from $f_n^k(A)$ it follows that $f^k(A) \subseteq B_E(\psi(C)_k)$. Now pick $c \in \psi(C)_k$. By construction, there exists a sequence $(c_0, c_1, \ldots, c_k)$ with $c_i \in \psi(C)_i$ for each $0 \leq i \leq k$ and $c_k = c$ for which $(f(c_i), c_{i+1}) \in D$. Furthermore, there exist $x \in A$ for which $c_0 = y_x$. It follows that $(c_i, f^i(x)) \in E_0$ for each $0 \leq i \leq k$. In particular it follows that $(c, f^k(x)) \in E$. As c was picked arbitrarily from $\psi(C)_k$ it follows that $\psi(C)_k \subseteq B_E(f_n^k(A))$.

Now suppose that f_n has inverse shadowing. Let $E \in \mathscr{U}$ be given and let $E_0 \in \mathscr{U}$ be such that $2E_0 \subseteq E$. Let $D \in \mathscr{U}$ be such that 2^D satisfies the inverse shadowing condition for 2^{E_0}. Now pick $\varphi \in \mathcal{T}_0(f, D)$. Construct method 2^φ as follows: For any $k \in \omega$ and any

$A \in F_n(X)$

$$2^{\varphi}(A)_k := \bigcup_{x \in A} \{\varphi(x)_k\}.$$

Clearly $2^{\varphi} \in \mathcal{T}_0\left(f_n, (2^D \cap (F_n(X) \times F_n(X)))\right)$. Let $x \in X$ be given. Then $\{x\} \in F_n(X)$. By inverse shadowing there exists $A \in F_n(X)$ such that $2^{\varphi}(A)$ 2^{E_0}-shadows $\{x\}$. Pick $a \in A$. It is easy to verify that $\varphi(a)$ E-shadows x. □

3.13.3 Inverse limits

The following theorem generalises part (2) of Theorem 7 in [13], where the author shows that the induced shift space on a system with inverse shadowing also has inverse shadowing.

Theorem 3.13.6. *Let (X, f) be conjugate to a surjective inverse limit system comprised of maps with $\mathcal{T}_0$-inverse shadowing on compact Hausdorff spaces. Then (X, f) has $\mathcal{T}_0$-inverse shadowing.*

Proof. We use the reformulation of inverse shadowing given in Lemma 3.13.3.

Let $(\Lambda, \geq)$ be a directed set. For each $\lambda \in \Lambda$, let (X_λ, f_λ) be a dynamical system on a compact Hausdorff space with inverse shadowing and let $((X_\lambda, f_\lambda), g_\lambda^\eta)$ be a surjective inverse system. Without loss of generality $(X, f) = (\varprojlim\{X_\lambda, g_\lambda^\eta\}, f)$.

Let $\mathcal{U}$ be a finite open cover of X. Since $X = \varprojlim\{X_\lambda, g_\lambda^\eta\}$ there exist $\eta \in \Lambda$ and a finite open cover $\mathcal{W}_\eta$ of X_η such that $\mathcal{W} := \{\pi_\eta^{-1}(W) \cap X \mid W \in \mathcal{W}_\eta\}$ refines $\mathcal{U}$. By inverse shadowing for (X_η, f_η), and by Lemma 3.13.3, there exists a finite open cover $\mathcal{V}_\eta$ of X_η such that for any $x \in X_\eta$ there exists $y \in X_\eta$ such that for any $\varphi \in \mathcal{T}_0(f_\eta, \mathcal{V}_\eta)$, $\varphi(y)$ $\mathcal{W}_\eta$-shadows x.

Take $\mathcal{V} = \{\pi_\eta^{-1}(V) \cap X \mid V \in \mathcal{V}_\lambda\}$. Pick $x \in X$ and take $\phi \in \mathcal{T}_0(f, \mathcal{V})$. Let $y \in X_\eta$ be such that for any $\varphi \in \mathcal{T}_0(f_\eta, \mathcal{V}_\eta)$, $\varphi(y)$ $\mathcal{W}_\eta$-shadows $\pi_\eta(x)$. Notice that, if $z \in \pi_\eta^{-1}(y) \cap X$ then $\left(\pi_\eta(\phi(z)_i)\right)_{i \in \omega}$ is a $\mathcal{V}_\eta$-pseudo-orbit starting from y; hence it $\mathcal{W}_\eta$-shadows $\pi_\eta(x)$. Therefore, taking any such z, we have $\phi(z)$ $\mathcal{W}$-shadows x. Since $\mathcal{W}$ is a refinement of $\mathcal{U}$ the result follows. □

3.13.4 Tychonoff product

Theorem 3.13.7. *Let Λ be an arbitrary index set and, for each $\lambda \in \Lambda$, let (X_λ, f_λ) be a compact Hausdorff system with $\mathcal{T}_0$-inverse shadowing. Then the product system (X, f), where $X = \prod_{\lambda\in\Lambda} X_\lambda$, has $\mathcal{T}_0$-inverse shadowing.*

Proof. Let $E \in \mathscr{U}$ be given; this entourage is refined by one of the form

$$\prod_{\lambda\in\Lambda} E_\lambda,$$

where $E_\lambda \in \mathscr{U}_\lambda$ for all $\lambda \in \Lambda$ and $E_\lambda = X_\lambda \times X_\lambda$ for all but finitely many of the λ's. Let λ_j, for $1 \leq j \leq k$, be precisely those elements in Λ for which $E_\lambda \neq X_\lambda \times X_\lambda$ (if there are no such elements then we are done). By inverse shadowing in each component space, there exist entourages $D_{\lambda_i} \in \mathscr{U}_{\lambda_i}$ such that corresponding to the entourages E_{λ_i} as in Lemma 3.13.1.

Let

$$D := \prod_{\lambda\in\Lambda} D_\lambda$$

where

$$D_\lambda = \begin{cases} X \times X & \text{if} \quad \forall i\, \lambda \neq \lambda_i \\ D_{\lambda_i} & \text{if} \quad \exists i : \lambda = \lambda_i\,. \end{cases}$$

Now pick $x \in X$ and pick $\phi \in \mathcal{T}_0(f, D)$. For each $1 \leq i \leq k$ let $y_i \in X_{\lambda_i}$ be as in Lemma 3.13.1 for E_{λ_i}, D_{λ_i} and $\pi_{\lambda_i}(x)$. Pick a point $y \in X$ such that $\pi_{\lambda_i}(y) = y_i$ for each $1 \leq i \leq k$. It can be seen that $\phi(y)$ E-shadows x. □

CHAPTER 4

ORBITAL SHADOWING, ω-LIMIT SETS AND MINIMALITY

In this chapter we prove (Theorem 4.2.3) that every compact metric dynamical system (X, f) enjoys a weak form of shadowing, specifically: For any $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $\delta > 0$ such that given any δ-pseudo-orbit $(x_i)_{i \geq 0}$ there exists $z \in X$ such that

$$B_\varepsilon\left(\{x_i\}_{i=0}^n\right) \supseteq \omega(z).$$

Thus initial segments of pseudo-orbits trap ω-limit sets, and consequently orbits, in their neighbourhood. As an application of this result, we show that compact minimal systems have the strong orbital shadowing property as introduced in [47]. Our methodology allows us to give a characterisation of minimal systems in terms of pseudo-orbits (Theorem 4.2.4). Along the way we generalise a result of Pilyugin *et al.* [82] by showing that every compact Hausdorff system has the second weak shadowing property.

In order to keep our results as general as possible, we take the phase space throughout this chapter to be compact Hausdorff but not necessarily metric.

4.1 Preliminaries

For a dynamical system (X, f), a subset $A \subseteq X$ is said to be *positively invariant* (under f) if $f(A) \subseteq A$. The system is *minimal* if there are no proper, nonempty, closed, positively-invariant subsets of X. Equivalently, a system is minimal if $\omega(x) = X$ for all $x \in X$.

Definition 4.1.1. Let X be a metric space. The system (X, f) has the *orbital shadowing property* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i\geq 0}$, there exists a point z such that

$$d_H\left(\overline{\{x_i\}_{i\geq 0}}, \overline{\{f^i(z)\}_{i\geq 0}}\right) < \varepsilon.$$

Here d_H denotes the Hausdorff metric, defined on the compact subsets of X, which is given by:

$$d_H(A, A') = \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon(A') \text{ and } A' \subseteq B_\varepsilon(A)\}.$$

The following weakening of orbital shadowing was introduced in [82].

Definition 4.1.2. Let X be a metric space. The system (X, f) has the *second weak shadowing property* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i\geq 0}$, there exists a point z such that

$$\text{Orb}(z) \subseteq B_\varepsilon\left(\{x_i\}_{i\geq 0}\right).$$

The following strengthening of orbital shadowing was introduced in [47]. The authors demonstrate it to be distinct.

Definition 4.1.3. Let X be a metric space. The system (X, f) has the *strong orbital shadowing property* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i\geq 0}$, there exists a point z such that, for all $N \in \mathbb{N}_0$,

$$d_H\left(\overline{\{x_{N+i}\}_{i\geq 0}}, \overline{\{f^{N+i}(z)\}_{i\geq 0}}\right) < \varepsilon.$$

Recall the following definitions from section 3.2.2. First of all, given an entourage $D \in \mathscr{U}$, a sequence $(x_i)_{i\geq 0}$ in X is called a D-pseudo-orbit if $(f(x_i), x_{i+1}) \in D$ for all $i \geq 0$.

Definition 4.1.4. Let X be a uniform space. The system (X, f) has the *orbital shadowing property* if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\geq 0}$, there exists a point z such that

$$\mathrm{Orb}(z) \subseteq B_E\left(\{x_i\}_{i\geq 0}\right)$$

and

$$\{x_i\}_{i\geq 0} \subseteq B_E\left(\mathrm{Orb}(z)\right).$$

Definition 4.1.5. Let X be a uniform space. The system (X, f) has the *second weak shadowing property* if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\geq 0}$, there exists a point z such that

$$\mathrm{Orb}(z) \subseteq B_E\left(\{x_i\}_{i\geq 0}\right).$$

Definition 4.1.6. Let X be a uniform space. The system (X, f) has the *strong orbital shadowing property* if for all $E \in \mathscr{U}$, there exists $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\geq 0}$, there exists a point z such that, for all $N \in \mathbb{N}_0$,

$$\{f^{N+i}(z)\}_{i\geq 0} \subseteq B_E\left(\{x_{N+i}\}_{i\geq 0}\right)$$

and

$$\{x_{N+i}\}_{i\geq 0} \subseteq B_E\left(\{f^{N+i}(z)\}_{i\geq 0}\right).$$

When X is a compact metric space these definitions coincide with the previously given metric versions.

4.2 Main results

Throughout this chapter, as was the case in the previous one, X denotes a compact Hausdorff space and the unique uniformity associated with X is denoted by $\mathscr{U}$. Once again we assume, without loss of generality, that all entourages we refer to are symmetric.

Lemma 4.2.1. *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. Then (X, f) satisfies the following:*

$$\forall E \in \mathscr{U}\, \forall x \in X\, \exists n \in \mathbb{N}\, \exists z \in X \text{ s.t.}$$

$$\bigcup_{i=1}^{n} B_E\left(f^i(x)\right) \supseteq \omega(z).$$

Proof. Take $E \in \mathscr{U}$ and pick $x \in X$. Let $E_0 \in \mathscr{U}$ be such that $2E_0 \subseteq E$. Take a finite subcover of the open cover $\{\text{int}(B_{E_0}(y)) \mid y \in \omega(x)\}$ of $\omega(x)$. For each element of this subcover there exists m such that $f^m(x)$ lies inside it. Pick one such m for each element and then let n be the largest. The result follows by taking $z = x$. □

Lemma 4.2.2. *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. Then (X, f) satisfies the following:*

$$\forall E \in \mathscr{U}\, \exists n \in \mathbb{N} \text{ s.t. } \forall x \in X\, \exists z \in X \text{ s.t.}$$

$$\bigcup_{i=1}^{n} B_E\left(f^i(x)\right) \supseteq \omega(z).$$

Proof. Fix $E \in \mathscr{U}$. Let $E_0 \in \mathscr{U}$ be such that $2E_0 \subseteq E$. For each $x \in X$ let $n_x \in \mathbb{N}$ be as in the condition in Lemma 4.2.1 for E_0 and let $D_x \in \mathscr{U}$ be such that, for any $y \in X$, if $(x, y) \in D_x$ then, for each $i \in \{0, \ldots, n_x\}$, $(f^i(x), f^i(y)) \in E_0$. Without loss of generality, the collection $\{B_{D_x}(x) \mid x \in X\}$ forms an open cover. Let

$$\left\{B_{D_{x_i}}(x_i) \mid i \in \{1, \ldots, k\}\right\},$$

be a finite subcover. Take $n = \max_{i\in\{1,\ldots,k\}} n_{x_i}$. Pick $x \in X$ arbitrarily. There exists $l \in \{1,\ldots,k\}$ such that $x \in B_{D_{x_l}}(x_l)$, which in turn implies $(f^i(x), f^i(x_l)) \in E_0$ for each $i \in \{0,\ldots,n_{x_l}\}$. By Lemma 4.2.1 there exists $z \in X$ such that

$$\bigcup_{i=1}^{n_l} B_{E_0}\left(f^i(x_l)\right) \supseteq \omega(z).$$

Since $2E_0 \subseteq E$, by entourage composition combined with the fact that $n_l \leq n$, it follows that

$$\bigcup_{i=1}^{n} B_{E}\left(f^i(x)\right) \supseteq \omega(z).$$

□

Theorem 4.2.3. *Let (X, f) be a dynamical system, where X is a compact Hausdorff space. Then for any $E \in \mathscr{U}$ there exist $n \in \mathbb{N}$ and $D \in \mathscr{U}$ such that given any D-pseudo-orbit $(x_i)_{i\geq 0}$ there exists $z \in X$ such that*

$$B_E\left(\{x_i\}_{i=0}^n\right) \supseteq \overline{\mathrm{Orb}(z)}.$$

In particular, (X, f) exhibits second weak shadowing.

Proof. Let $E \in \mathscr{U}$ be given and let $E_0 \in \mathscr{U}$ be such that $2E_0 \subseteq E$. Take $n \in \mathbb{N}$ as in the condition in Lemma 4.2.2 with respect to E_0. By uniform continuity we can choose $D \in \mathscr{U}$ such that every D-pseudo-orbit E_0-shadows the first n iterates of its origin. Explicitly: Let $D_1 \subseteq E_0$ be an entourage such that, for any $y, z \in X$, if $(y,z) \in D_1$ then $(f(y), f(z)) \in E_0$. For each $i \in \{2,\ldots,n\}$ let $D_i \in \mathscr{U}$ be such that $2D_i \subseteq f^{-1}(D_{i-1}) \cap D_{i-1}$.

Now take $D := D_n$. Suppose $(x_i)_{i\geq 0}$ is a D-pseudo-orbit. Then $(f^i(x_0), x_i) \in E_0$ for all $i \in \{0,\ldots,n\}$. By Lemma 4.2.2 there exists $y \in X$ such that

$$\bigcup_{i=1}^{n} B_{E_0}\left(f^i(x_0)\right) \supseteq \omega(y).$$

Since $2E_0 \subseteq E$ and, for each $i \in \{0,\ldots,n\}$, $(f^i(x_0), x_i) \in E_0$ it follows by entourage composition that

$$B_E\left(\{x_i\}_{i=0}^n\right) \supseteq \omega(y).$$

Since X is compact, $\omega(y) \neq \emptyset$. Pick $z \in \omega(y)$. Because ω-limit sets are closed and positively invariant $\overline{\mathrm{Orb}(z)} \subseteq \omega(y)$. The result follows. □

Notice that we could replace $\overline{\mathrm{Orb}(z)}$ with either $\omega(z)$ or $\mathrm{Orb}(z)$ in the statement of Theorem 4.2.3. Our methodology in the proof means that each of these would be equivalent statements.

Theorem 4.2.4. *Let X be a compact Hausdorff space and $f\colon X \to X$ be a continuous function. The system (X, f) is minimal if and only if for any $E \in \mathscr{U}$ there exist $n \in \mathbb{N}$ and $D \in \mathscr{U}$ such that for any two D-pseudo-orbits $(x_i)_{i\geq 0}$ and $(y_i)_{i\geq 0}$*

$$\{y_i\}_{i=0}^n \subseteq B_E\left(\{x_i\}_{i=0}^n\right)$$

and

$$\{x_i\}_{i=0}^n \subseteq B_E\left(\{y_i\}_{i=0}^n\right).$$

Proof. First suppose the system is minimal. Let $E \in \mathscr{U}$ be given. Take $n \in \mathbb{N}$ and $D \in \mathscr{U}$ corresponding to E as in Theorem 4.2.3. Now let $(x_i)_{i\geq 0}$ and $(y_i)_{i\geq 0}$ be two D-pseudo-orbits. By Theorem 4.2.3 there exist $z_1, z_2 \in X$ such that $B_E\left(\{x_i\}_{i=0}^n\right) \supseteq \omega(z_1)$ and $B_E\left(\{y_i\}_{i=0}^n\right) \supseteq \omega(z_2)$. As (X, f) is minimal $\omega(z_1) = \omega(z_2) = X$. It follows that $B_E\left(\{x_i\}_{i=0}^n\right) = B_E\left(\{y_i\}_{i=0}^n\right) = X$. Hence

$$\{y_i\}_{i=0}^n \subseteq B_E\left(\{x_i\}_{i=0}^n\right)$$

and

$$\{x_i\}_{i=0}^n \subseteq B_E\left(\{y_i\}_{i=0}^n\right).$$

Now suppose the system is not minimal. Then there exists $x \in X$ such that $\omega(x) \neq X$. Pick $y \in \omega(x)$ and let $z \in X \setminus \omega(x)$. Take $E \in \mathscr{U}$ such that $B_E(z) \cap \omega(x) = \emptyset$. Consider the pseudo-orbits given by the orbit sequences of y and z: these are D-pseudo-orbits for any $D \in \mathscr{U}$. As ω-limit sets are positively invariant, $\mathrm{Orb}(y) \subseteq \omega(x)$. Since $z \notin B_E(\omega(x))$ it also follows that $z \notin B_E(\mathrm{Orb}(y))$. In particular, for any $n \in \mathbb{N}$, $\{f^i(z)\}_{i=0}^n \not\subseteq B_E\left(\{f^i(y)\}_{i=0}^n\right)$. □

For the case when X is a compact metric space Theorem 4.2.4 may be formulated as follows: A dynamical system (X, f) is minimal precisely when for any $\varepsilon > 0$ there exist $\delta > 0$ and $n \in \mathbb{N}$ such that for any two δ-pseudo-orbits $(x_i)_{i\geq 0}$ and $(y_i)_{i\geq 0}$

$$d_H\left(\{x_i\}_{i=0}^n, \{y_i\}_{i=0}^n\right) < \varepsilon.$$

Corollary 4.2.5. *Let X is a compact Hausdorff space and $f\colon X \to X$ be a continuous function. The system (X, f) is minimal if and only if for any $E \in \mathscr{U}$ there exist $n \in \mathbb{N}$ and $D \in \mathscr{U}$ such that for any D-pseudo-orbit $(x_i)_{i\geq 0}$ we have $B_E\left(\{x_i\}_{i=0}^n\right) = X$.*

Proof. Immediate from the proof of Theorem 4.2.4. □

Corollary 4.2.6. *Let X be a compact Hausdorff space. If (X, f) is a minimal dynamical system then it exhibits the strong orbital shadowing property.*

Proof. Let $E \in \mathscr{U}$ be given. Take $n \in \mathbb{N}$ and $D \in \mathscr{U}$ corresponding to E as in Theorem 4.2.4. Now let $(x_i)_{i\geq 0}$ be a D-pseudo-orbit and pick any $z \in X$. Since $(x_{N+i})_{i\geq 0}$ and $(f^{N+i}(z))_{i\geq 0}$ are D-pseudo-orbits for all $N \in \mathbb{N}_0$, by Corollary 4.2.5,

$$B_E\left(\{x_{N+i}\}_{i\geq 0}\right) = X = B_E\left(\{f^{N+i}(z)\}_{i\geq 0}\right).$$

The result follows. □

CHAPTER 5

SHADOWING, INTERNAL CHAIN TRANSITIVITY AND α-LIMIT SETS

For this chapter let X be a compact metric space. As usual, $f\colon X \to X$ is a continuous map so that (X, f) forms a dynamical system. Recall from Preliminaries 1.1.2 that given a point $x \in X$, its *ω-limit set* is the set of accumulation points of the sequence $x, f(x), f^2(x), \ldots$. Calculating the ω-limit set of a given point is often relatively easy. Conversely one may ask if a given set is an ω-limit set: this can be quite difficult to answer. As such, various authors have either studied or attempted to characterise, the set of all ω-limit sets, denoted here by ω_f, in a variety of settings. For example, ω-limit sets of continuous maps of the closed unit interval I have been completely characterised in [1, 21]: the authors show that a nonempty subset E of I is an ω-limit set of some continuous map f if and only if E is either a closed, nowhere dense set, or a union of finitely many non-degenerate closed intervals. Furthermore, it has been shown that ω_f is closed (with respect to the Hausdorff topology) for maps of the circle [85], the interval [16], and other finite graphs [69]. It is known [54] that every ω-limit set is *internally chain transitive*: briefly a set $A \subseteq X$ is internally chain transitive if for any $a, b \in A$ and any $\varepsilon > 0$ there exists a finite sequence $\langle x_0, x_1, \ldots, x_n \rangle$ in A such that $x_0 = a$, $x_n = b$ and $d(f(x_i), x_{i+1}) < \varepsilon$ for each i. We denote the set of nonempty closed internally chain transitive sets by ICT_f.

The shadowing property and its variants have been shown to have a bearing on the structure of ω-limit sets (see, for example, [10, 11, 12, 18, 70]). Of particular importance to us is a result of Meddaugh and Raines [70] who establish that, for maps with shadowing, $\overline{\omega_f} = \mathrm{ICT}_f$. More recently, using novel variants of shadowing, Good and Meddaugh [47] precisely characterised maps for which $\overline{\omega_f} = \mathrm{ICT}_f$ and $\omega_f = \mathrm{ICT}_f$.

Whilst the ω-limit set of a point can be thought of as its *target* — it is where the point *ends up* — an α-limit set concerns where a point came from — its source, so to speak. However, whilst the definition of an ω-limit set is fairly natural, giving an appropriate definition of an α-limit set is less straightforward. This is because a point may have multiple points in its preimage (or indeed, if the map is not surjective, it may have empty preimage). Various approaches to this difficulty have been taken; these will be discussed in more detail in section 5.2. We follow the approach taken in [6] and [54], by refraining from defining such sets for individual points, but rather defining them for *backward trajectories*. Given a point $x \in X$ an infinite sequence $\langle x_i \rangle_{i \leq 0}$ is called a *backward trajectory* of x if $f(x_i) = x_{i+1}$ for all $i \leq -1$ and $x_0 = x$. The α-limit set of $\langle x_i \rangle_{i \leq 0}$ is the set of accumulation points of this sequence. We denote the set of all α-limit sets by α_f. Although α-limit sets have not been studied quite as extensively as their ω counterparts, interest in them has been growing (see, for example, [6, 32, 33, 53, 54]).

As with ω-limit sets, it is known that α-limit sets are internally chain transitive [54]. In this chapter, we seek to provide a characterisation of maps for which α_f and ICT_f coincide. We start with the preliminaries in section 5.1. Section 5.2 is a standalone section in which we briefly explain the various types of α-limit sets that have been studied in the literature. In section 5.3, we show that for maps with shadowing, for any $\varepsilon > 0$ and any $A \in \mathrm{ICT}_f$ there is a full trajectory whose α-limit set and ω-limit set both lie within ε of A (with respect to the Hausdorff distance). Furthermore, we show that the addition of expansivity entails that there is a full trajectory whose limit sets equal A. In particular this means that for maps with shadowing $\overline{\alpha_f} = \overline{\omega_f} = \mathrm{ICT}_f$, whilst the addition of expansivity means that $\alpha_f = \omega_f = \mathrm{ICT}_f$. We progress, in section 5.4, by introducing novel types of shadowing which we use to characterise both maps for which $\overline{\alpha_f} = \mathrm{ICT}_f$ and maps for which $\alpha_f = \mathrm{ICT}_f$, complementing the work of the first and second author in [47]. Motivated by the aforementioned result on shadowing, in section 5.5, we characterise, by introducing novel variants of shadowing, maps for which every element of ICT_f is equal to (resp. may be approximated by) the α-limit set and the ω-limit set of the same full trajectory. We end the chapter, in section 5.6, with two final examples which serve to draw some of the themes discussed throughout the chapter, and in [47], together.

5.1 Preliminaries

For this chapter X is a compact metric space and $f \colon X \to X$ is a continuous map. We say the *positive orbit of x under f* is the set of points $\{x, f(x), f^2(x), \ldots\}$; we denote this set by $\mathrm{Orb}_f^+(x)$. A *backward trajectory* of the point x is a sequence $\langle x_i \rangle_{i \leq 0}$ for which $f(x_i) = x_{i+1}$ for all $i \leq -1$ and $x_0 = x$. We say a bi-infinite sequence $\langle x_i \rangle_{i \in \mathbb{Z}}$ is a *full orbit*, or *full trajectory*, (of each x_i) if $f(x_i) = x_{i+1}$ for each $i \in \mathbb{Z}$. We emphasise that a full orbit of a point need not be unique. Note further that we do not assume that the map f is a surjection. (NB. Because we will be particularly concerned with backward accumulation points of individual trajectories, for clarity we will say that a point which does not have an infinite backward trajectory does not have a full orbit. Whenever we say *full orbit*, we mean a bi-infinite trajectory.)

For a sequence $\langle x_i \rangle_{i > N}$ in X, where $N \geq -\infty$, we define its ω-limit set, denoted $\omega(\langle x_i \rangle_{i > N})$, or simply $\omega(\langle x_i \rangle)$, to be the set of accumulation points of the positive tail of the sequence. Formally:

$$\omega(\langle x_i \rangle) = \bigcap_{M \in \mathbb{N}} \overline{\{x_n \mid n > M\}}.$$

For $x \in X$, we define the *ω-limit set of x*: $\omega(x) \coloneqq \omega(\langle f^n(x) \rangle_{n=0}^{\infty})$. (This coincides with the definition given in the introduction to this thesis (see 1.1.2).) In similar fashion, for a sequence $\langle x_i \rangle_{i < N}$ in X, where $N \leq \infty$, we define its *α-limit set*, denoted $\alpha(\langle x_i \rangle_{i < N})$, or simply $\alpha(\langle x_i \rangle)$, to be the set of accumulation points of the negative tail of the sequence.

Formally:

$$\alpha(\langle x_i \rangle) = \bigcap_{M \in \mathbb{N}} \overline{\{x_n \mid n < -M\}}.$$

We denote by ω_f the set of all ω-limit sets of points in X. We denote by α_f the set of all α-limit sets of full trajectories in (X, f). Note that since X is compact it follows that elements of α_f and ω_f are closed, compact, and nonempty. Note that, as collections of nonempty compact sets, α_f and ω_f are subsets of the hyperspace 2^X.

A set $A \subseteq X$ is said to be *invariant* if $f(A) \subseteq A$. It is *strongly invariant* if $f(A) = A$. A nonempty closed set A is *minimal* if $\omega(x) = A$ for all $x \in A$.

A finite or infinite sequence $\langle x_i \rangle_{i=0}^N$ is said to be an *ε-chain* if $d(f(x_i), x_{i+1}) < \varepsilon$ for all indices $i < N$. If $N = \infty$ then we say the sequence is an *ε-pseudo-orbit*. A set A is *internally chain transitive* if for any pair of points $a, b \in A$ and any $\varepsilon > 0$ there exists a finite ε-chain $\langle x_i \rangle_{i=0}^N$ in A with $x_0 = a$, $x_N = b$ and $N \geq 1$. We denote by ICT_f the set of all nonempty closed internally chain transitive sets. Notice that $\mathrm{ICT}_f \subseteq 2^X$. Meddaugh and Raines [70] establish the following result.

Lemma 5.1.1. [70] *Let (X, f) be a dynamical system. Then ICT_f is closed*[1] *in 2^X.*

Hirsch *et al.* [54] show that the α-limit set (resp. ω-limit set) of any pre-compact backward (resp. forward) trajectory is internally chain transitive. Since our setting is a compact metric space all α- and ω-limit sets are internally chain transitive. We formulate this as Lemma 5.1.2 below.

Lemma 5.1.2. [54] *Let (X, f) be a dynamical system. Then $\alpha_f \subseteq \mathrm{ICT}_f$ and $\omega_f \subseteq \mathrm{ICT}_f$.*

Remark 5.1.3. When one first encounters positive and negative limit sets of trajectories, it is natural to ask (for a surjective map) if every ω-limit set is also an α-limit set, along with the converse. Balibrea et al. [6] answer this question for interval maps by showing that every α-limit set of an interval map is an ω-limit set, while the converse is not necessarily true. In general, systems may have α-limit sets which are not ω-limit sets, as

[1] We emphasise that closedness here refers to the topology of the hyperspace.

well as ω-limit sets which are not α-limit sets. The following is one such example. Take two copies of the interval and embed them side by side in the plane (i.e. one on the left and one on the right). Snake one infinite line between them which has each interval as an accumulation set — akin to how the topologist's sine curve approaches the y-axis. Define a continuous map as follows: Let every point on each of the two intervals be fixed whilst points on the line move continuously along it, away from the left interval and towards the right. It follows that the left interval is the α-limit set of the unique backward trajectory of any point on the line, whilst the right interval is the ω-limit set of any point on the line. However it is clear that the left interval is not an ω-limit set, whilst the right interval is not an α-limit set. The space described in this system may be viewed in Figure 5.1.

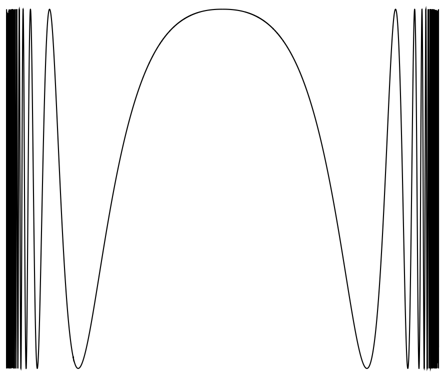

Figure 5.1: The space described in Remark 5.1.3

Remark 5.1.4. As stated in [6], a minimal set is both an ω-limit set and an α-limit set.

Whilst it may be the case that $\alpha_f \neq \omega_f$, it is true that every α-limit set contains the ω-limit set of every one of its points and, similarly, every ω-limit set contains an α-limit set of a backward trajectory of each of its points. To show this we recall the well-known fact that the ω-limit sets in compact systems are strongly invariant (e.g. [35, Theorem 3.1.9]). The same is true of the α-limit sets of backward trajectories (e.g. [6, Lemma 1]).

Proposition 5.1.5. *Let $x, y \in X$ and suppose that $\langle z_i \rangle_{i \leq 0}$ is a backward trajectory of a point $z = z_0 \in X$. Then:*

1. *If $x \in \alpha(\langle z_i \rangle)$ then $\overline{\mathrm{Orb}_f^+(x)} \subseteq \alpha(\langle z_i \rangle)$.*

2. *If $y \in \omega(x)$ then there is a backward trajectory $\langle y_i \rangle_{i \leq 0}$, with $y_0 = y$, which lies in $\omega(x)$ and such that $\alpha(\langle y_i \rangle) \subseteq \omega(x)$.*

Proof. Condition (1) is immediate from the fact that α-limit sets are closed and invariant under f.

Condition (2) is trivial after observing that ω-limit sets are closed and strongly invariant. □

Remark 5.1.6. In [53] the author proves condition (1) in Proposition 5.1.5 holds for interval maps.

Recall, a point x is said to ε-*shadow* a sequence $\langle x_i \rangle_{i=0}^{\infty}$ if $d(f^i(x), x_i) < \varepsilon$ for all $i \in \mathbb{N}_0$. We say the system (X, f) has the *shadowing property*, or simply shadowing, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed.

Definition 5.1.7. Suppose that (X, f) is a dynamical system.

1. The sequence $\langle x_i \rangle_{i \leq 0}$ is a *backward δ-pseudo-orbit* if $d(f(x_i), x_{i+1}) < \delta$ for each $i \leq -1$.

2. The sequence $\langle x_i \rangle_{i \in \mathbb{Z}}$ is a *two-sided δ-pseudo-orbit* if $d(f(x_i), x_{i+1}) < \delta$ for each $i \in \mathbb{Z}$.

3. The system (X, f) has *backward shadowing* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any backward δ-pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ such that $d(x_i, z_i) < \varepsilon$ for all $i \leq 0$.

4. The system (X, f) has *two-sided shadowing* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any two-sided δ-pseudo-orbit $\langle x_i \rangle_{i \in \mathbb{Z}}$ there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ such that $d(x_i, z_i) < \varepsilon$ for all $i \in \mathbb{Z}$.

A sequence $\langle x_i \rangle_{i=0}^{\infty}$ is called an *asymptotic pseudo-orbit* if $d(f(x_i), x_{i+1}) \to 0$ as $i \to \infty$. Similarly a sequence $\langle x_i \rangle_{i \leq 0}$ is a *backward asymptotic pseudo-orbit* if $d(f(x_i), x_{i+1}) \to 0$

as $i \to -\infty$. Finally a sequence $\langle x_i \rangle_{i \in \mathbb{Z}}$ is called a *two-sided asymptotic pseudo-orbit* if $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm\infty$.

The system (X, f) has *s-limit shadowing* if, in addition to having shadowing, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any asymptotic δ-pseudo orbit $\langle x_i \rangle_{i=0}^{\infty}$ there exists $z \in X$ which *asymptotically* ε-shadows $\langle x_i \rangle_{i=0}^{\infty}$ (i.e. $d(f^i(z), x_i) \to 0$ as $i \to \infty$ and $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{N}_0$). The system has *two-sided s-limit shadowing* if, in addition to two-sided shadowing, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any two-sided asymptotic δ-pseudo orbit $\langle x_i \rangle_{i \in \mathbb{Z}}$ there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ which asymptotically ε-shadows $\langle x_i \rangle_{i \in \mathbb{Z}}$ (i.e. $d(f^i(z), x_i) \to 0$ as $i \to \pm\infty$ and $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$).

5.1.1 Shift spaces

Recall the information on shift spaces provided in the introduction to this thesis (1.1.5). The following two theorems concerning limit sets in shift spaces are folklore.

Theorem 5.1.8. *Let (X, σ) be a one-sided shift space. Let $\mathbf{x}, \mathbf{y} \in X$. Then $\mathbf{y} \in \omega(\mathbf{x})$ if and only if every initial segment of $\mathbf{y}$ occurs infinitely often in $\mathbf{x}$. Given a backward trajectory $\langle \mathbf{x}_i \rangle_{i \leq 0}$ consider the backward infinite sequence $\langle \mathsf{a}_i \rangle_{i \leq 0}$ where $\mathsf{a}_i = \pi_0(\mathbf{x}_i)$. Then $\mathbf{y} \in \alpha(\langle \mathbf{x}_i \rangle)$ if and only if every initial segment of $\mathbf{y}$ occurs infinitely often in $\langle \mathsf{a}_i \rangle_{i \leq 0}$.*

Theorem 5.1.9. *Let (X, σ) be a two-sided shift space. Let $\mathbf{x}, \mathbf{y} \in X$. Then $\mathbf{y} \in \omega(\mathbf{x})$ if and only if every central segment of $\mathbf{y}$ occurs infinitely often in the right-tail of $\mathbf{x}$. Given a backward trajectory $\langle \mathbf{x}_i \rangle_{i \leq 0}$ then $\mathbf{y} \in \alpha(\langle \mathbf{x}_i \rangle)$ if and only if every central segment of $\mathbf{y}$ occurs infinitely often in the left-tail of $\mathbf{x}_0$.*

As stated in Lemma 5.1.2, α_f and ω_f are both subsets of ICT_f. Example 5.1.10 gives a surjective shift space (X, σ) where α_σ, ω_σ, and ICT_σ are all distinct, complementing the discussion in Remark 5.1.3.

Example 5.1.10. Let $\mathbf{x} = 1010^2 10^3 \ldots$, and $\mathbf{y} = 2020^2 20^3 \ldots$. Let

$$P(\mathbf{x}) = \{30^n 30^{n-1} \ldots 30\mathbf{x} \mid n \in \mathbb{N}\}.$$

Take

$$X = \overline{\bigcup_{\mathbf{z} \in P(\mathbf{x})} \mathrm{Orb}_\sigma^+(\mathbf{z}) \cup \mathrm{Orb}_\sigma^+(\mathbf{y}) \cup \{0^n \mathbf{y} \mid n \in \mathbb{N}\}},$$

where the closure is taken with regard to the one-sided full shift on the alphabet $\{0, 1, 2, 3\}$. Considering the system (X, σ), $\alpha_\sigma \neq \omega_\sigma \neq \mathrm{ICT}_\sigma \neq \alpha_\sigma$. Furthermore $\alpha_\sigma \not\subseteq \omega_\sigma$ and $\omega_\sigma \not\subseteq \alpha_\sigma$.

In Example 5.1.10, $\omega(\mathbf{x}) = \{0^\infty, 0^n 10^\infty \mid n \geq 0\}$ and $\omega(\mathbf{y}) = \{0^\infty, 0^n 20^\infty \mid n \geq 0\}$. It is easy to see that the only other ω-limit set is $\{0^\infty\}$. Thus

$$\omega_\sigma = \{\{0^\infty\}, \{0^\infty, 0^n 10^\infty \mid n \geq 0\}, \{0^\infty, 0^n 20^\infty \mid n \geq 0\}\}.$$

Meanwhile

$$\alpha_\sigma = \{\{0^\infty\}, \{0^\infty, 0^n 30^\infty \mid n \geq 0\}\}.$$

Finally whilst $\mathrm{ICT}_\sigma \supseteq \alpha_\sigma \cup \omega_\sigma$ it additionally contains

$$\{0^\infty, 0^n 10^\infty, 0^n 20^\infty \mid n \geq 0\},$$

$$\{0^\infty, 0^n 10^\infty, 0^n 30^\infty \mid n \geq 0\},$$

$$\{0^\infty, 0^n 20^\infty, 0^n 30^\infty \mid n \geq 0\},$$

and

$$\{0^\infty, 0^n 10^\infty, 0^n 20^\infty, 0^n 30^\infty \mid n \geq 0\}.$$

Hence $\alpha_\sigma \neq \omega_\sigma \neq \mathrm{ICT}_\sigma \neq \alpha_\sigma$, $\alpha_\sigma \not\subseteq \omega_\sigma$ and $\omega_\sigma \not\subseteq \alpha_\sigma$.

5.2 Various notions of negative limit sets

In the previous section, we defined what we mean by the term α-limit set: it was defined for backward sequences. Meanwhile, the definition of an ω-limit set was extended to individual points. This was done in the only natural way: any given point only has one forward orbit. If one wishes to define the α-limit set of a point, say x, the best way forward is less obvious; there are multiple approaches one might reasonably take when defining negative limit sets of points. In this standalone section, we give a brief outline of several different approaches taken in the literature and give two examples which serve to illustrate their differences.

For homeomorphisms one can define α-limit sets (or negative limit sets) in precisely the same way as ω-limit sets. With non-invertible maps, however, a seemingly natural definition is less obvious. One approach is to take the set of accumulation points of the sequence of sets $f^{-k}(\{x\})$: this is done in [32] and [33]. Call this Approach 1 (A1). Two further approaches are motivated by considering the accumulation points of backward trajectories of the point in question. One might say that y is in the negative limit set of a point x if there exists a sequence $\langle y_i \rangle_{i=0}^{\infty}$ such that $y_i \in \text{Orb}_f^+(y_{i+1})$ for each i, $x = y_0$ and $\lim_{i\to\infty} y_i = y$: that is, the negative limit set of x is the union of all accumulation points of backward trajectories from x. In [53] the author defines this set as the *special α-limit set of x* and examines them for interval maps. These sets are investigated in [91] and [90] for graph maps and dendrites. Call this Approach 2 (A2). The final approach, A3, used in [53], is to say y is in the α-limit set of a point x if there exists a sequence $\langle y_i \rangle_{i=1}^{\infty}$ and a strictly increasing sequence $\langle n_i \rangle_{i=1}^{\infty}$ such that $f^{n_i}(y_i) = x$ for each i and $\lim_{i\to\infty} y_i = y$. Clearly this set contains the one given by A2. The converse is not true (see Example 5.2.2).

By means of demonstrating some of the differences A1–3 yield, we provide the following two examples.

Example 5.2.1. Define a map $f\colon [-1,1] \to [-1,1]$ by

$$f(x) = \begin{cases} 2x+1 & \text{if} \quad x \in [-1, -1/2), \\ 0 & \text{if} \quad x \in [-1/2, 1/2), \\ 2x-1 & \text{if} \quad x \in [1/2, 1]. \end{cases}$$

The graph of this function may be seen in Figure 5.2.

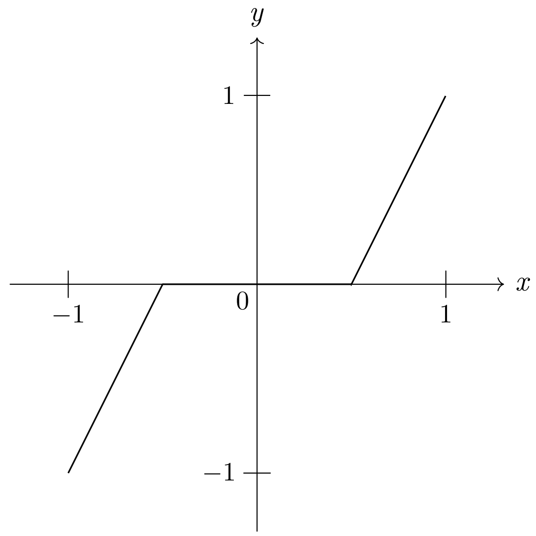

Figure 5.2: The graph of the system in Example 5.2.1

In Example 5.2.1, under A1 the negative limit set of 0 can be seen to be the whole interval $[-1,1]$. Under A2 and A3 the negative limit set of 0 is simply $\{-1,0,1\}$. Notice that the negative limit set of any backward trajectory from 0 will be either $\{-1\}$ or $\{0\}$ or $\{1\}$.

Example 5.2.2. Define a map $f\colon [-1,2] \to [-1,2]$ by

$$f(x) = \begin{cases} 2x+2 & \text{if} \quad x \in [-1, 0), \\ 2-2x & \text{if} \quad x \in [0, 1), \\ 2x-2 & \text{if} \quad x \in [1, 2]. \end{cases}$$

The graph of this function may be seen in Figure 5.3.

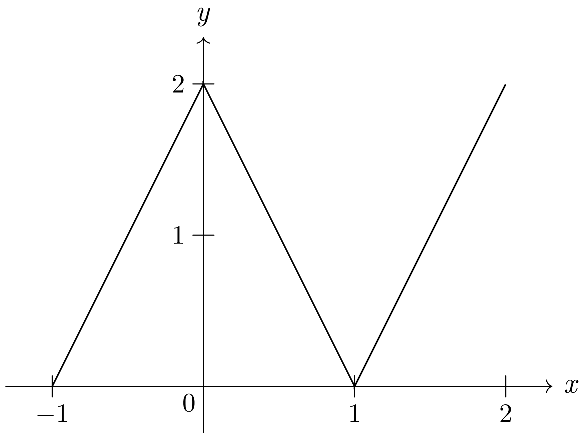

Figure 5.3: The graph of the system in Example 5.2.2

In Example 5.2.2, under A2 the negative limit set of 0 is $\{2/3, 2\}$. Consider the backward trajectory of 0 given by the increasing sequence $\langle x_i \rangle_{i \geq 0}$, where $x_0 = 0$ and $x_1 = 1$, $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4} \ldots$ This sequence approaches 2. However each point x_i in this sequence has a preimage y_i in the interval $[-1, 0)$. Each of these y_i thereby eventually map onto 0 but they do not themselves have preimages. Furthermore, if $f^n(y_i) = 0$ and $f^m(y_{i+1}) = 0$ then by construction $m > n$. This, together with the fact that $\lim_{i \to \infty} y_i = 0$ implies that 0 is in the negative limit set of itself under A3. Under A3 the negative limit set of 0 is $\{0, 2/3, 2\}$. (NB. Hero [53] provides an example illustrating this same difference. For Hero, 0 would be an α-limit point of itself but not a special α-limit point of itself: these would only be 2/3 and 2.)

As stated previously, we will not define α-limit sets of individual points, instead we focus on the accumulation points of individual backward trajectories. Note that this is the approach taken in [6] and [54].

5.3 Shadowing, ICT and α_f

The following lemma is a recent observation of Good *et al.* (see [46]).

Lemma 5.3.1. [46] *Let (X, f) be a dynamical system. If f has shadowing then it has backward shadowing and two-sided shadowing. If f is onto then all three properties are* *equivalent.*

Theorem 5.3.2. *Let (X, f) be a dynamical system with shadowing. Then for any $\varepsilon > 0$ and any $A \in \mathrm{ICT}_f$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that*

1. $d_H(\omega(x_0), A) < \varepsilon$
2. $d_H(\alpha(\langle x_i \rangle), A) < \varepsilon$.

In particular every element of ICT_f is in both $\overline{\alpha_f}$ and $\overline{\omega_f}$.

Remark 5.3.3. Before proving Theorem 5.3.2, we observe that for any $A \in \mathrm{ICT}_f$, for each $\eta > 0$, and for each $a \in A$, there exists a finite η-chain $\langle a = a_0, a_1, \ldots, a_m = a \rangle$ in A which is η*-dense in* A, i.e. for each $i \in \{0, \ldots, m\}$, $a_i \in A$ and $\bigcup_{i=0}^{m} B_\eta(a_i) \supseteq A$.

Proof of Theorem 5.3.2. Let $A \in \mathrm{ICT}_f$ and let $\varepsilon > 0$ be given. By Lemma 5.3.1 there exists $\delta > 0$ such that every two-sided δ-pseudo-orbit is $\varepsilon/2$-shadowed by a full orbit. We will construct a two-sided asymptotic δ-pseudo-orbit in A which is η-dense in A for all $\eta > 0$. To this end, let $l \in \mathbb{N}$ be such that $1/2^l < \delta$. Pick $b \in A$. For each $k \in \mathbb{N}_0$ choose a finite $1/2^{l+k}$-chain $\langle a_{k \cdot 0} = b, a_{k \cdot 1}, a_{k \cdot 2}, \ldots, a_{k \cdot m_k} \rangle$ in A which is $1/2^{l+k}$-dense in A and such that $d(f(a_{k \cdot m_k}), b) < 1/2^{l+k}$. (Here we are simply using the observation in Remark 5.3.3.) Concatenation of these chains now gives us an asymptotic δ-pseudo-orbit in A:

$$\langle a_{0 \cdot 0}, a_{0 \cdot 1}, a_{0 \cdot 2}, \ldots, a_{0 \cdot m_0}, a_{1 \cdot 0}, a_{1 \cdot 1}, a_{1 \cdot 2}, \ldots, a_{1 \cdot m_1}, \ldots, a_{k \cdot 0}, a_{k \cdot 1}, a_{k \cdot 2}, \ldots, a_{k \cdot m_k}, \ldots \rangle.$$

We can now extend this into a two-sided asymptotic δ-pseudo-orbit in A by 'running backwards' through the δ-chains:

$$\langle \ldots, a_{2 \cdot 0}, a_{2 \cdot 1}, \ldots, a_{2 \cdot m_2}, a_{1 \cdot 0}, a_{1 \cdot 1}, \ldots, a_{1 \cdot m_1} \cdot a_{0 \cdot 0}, a_{0 \cdot 1}, \ldots, a_{0 \cdot m_0}, a_{1 \cdot 0}, a_{1 \cdot 1}, \ldots, a_{1 \cdot m_1}, \ldots \rangle.$$

We call this two-sided asymptotic δ-pseudo-orbit φ. In order to simplify notation we now denote the k^{th} coordinate of φ by a_k, so that, for example, $a_0 = a_{0 \cdot 0}$ is the 0^{th} coordinate of φ and $a_{-1} = a_{1 \cdot m_1}$ is the $(-1)^{\text{th}}$ coordinate of φ. With this revised notation $\varphi = \langle a_i \rangle_{i \in \mathbb{Z}}$. From the construction of φ it follows that

$$A = \bigcap_{n \geq 0} \overline{\{a_i \mid i \geq n\}},$$

and

$$A = \bigcap_{n \leq 0} \overline{\{a_i \mid i \leq n\}}.$$

Let $\langle x_i \rangle_{i \in \mathbb{Z}}$ be a full trajectory such that $d(x_i, a_i) < \varepsilon/2$ for all $i \in \mathbb{Z}$. We claim that $d_H(\alpha(\langle x_i \rangle), A) < \varepsilon$. Indeed, pick $a \in A$. Then there is a decreasing sequence $\langle i_n \rangle_{n \in \mathbb{N}}$ of negative integers such that $a = \lim_{n \to \infty} a_{i_n}$. Thus there is $N \in \mathbb{N}$ such that $d(a, a_{i_n}) < \varepsilon/3$ for all $n > N$. Since $d(x_{i_n}, a_{i_n}) < \varepsilon/2$ for all $n \in \mathbb{N}$, it follows that $x_{i_n} \in B_{\frac{5\varepsilon}{6}}(a)$ for $n > N$. By compactness the sequence $\langle x_{i_n} \rangle_{n > N}$ has a limit point $z \in \overline{B_{5\varepsilon/6}(a)}$: in particular $d(z, a) < \varepsilon$. Hence $z \in \alpha(\langle x_i \rangle)$ and

$$A \subseteq \bigcup_{y \in \alpha(\langle x_i \rangle)} B_\varepsilon(y). \tag{5.1}$$

Now take $z \in \alpha(\langle x_i \rangle)$. Then there is a decreasing sequence $\langle i_n \rangle_{n \in \mathbb{N}}$ of negative integers such that $z = \lim_{n \to \infty} x_{i_n}$. Let $k \in \mathbb{N}$ be such that $d(z, x_{i_k}) < \varepsilon/2$. By shadowing $d(a_{i_k}, x_{i_k}) < \varepsilon/2$. By the triangle inequality $d(z, a_{i_k}) < \varepsilon$. Since $a_{i_k} \in A$ it follows that

$$\alpha(\langle x_i \rangle) \subseteq \bigcup_{a \in A} B_\varepsilon(a). \tag{5.2}$$

By Equations (5.1) and (5.2) it follows that $d_H(\alpha(\langle x_i \rangle_{i \in \mathbb{Z}}), A) < \varepsilon$.

The fact that $d_H(\omega(x_0), A) < \varepsilon$ follows by similar argument. □

The following example shows that the converse to Theorem 5.3.2 is false.

Example 5.3.4. Define a map $f\colon [-1,1] \to [-1,1]$ by

$$f(x) = \begin{cases} (x+1)^2 - 1 & \text{if} \quad x \in [-1,0), \\ x^2 & \text{if} \quad x \in [0,1]. \end{cases}$$

Then f does not have shadowing but $\mathrm{ICT}_f = \alpha_f = \omega_f$. The graph of this function may be seen in Figure 5.4.

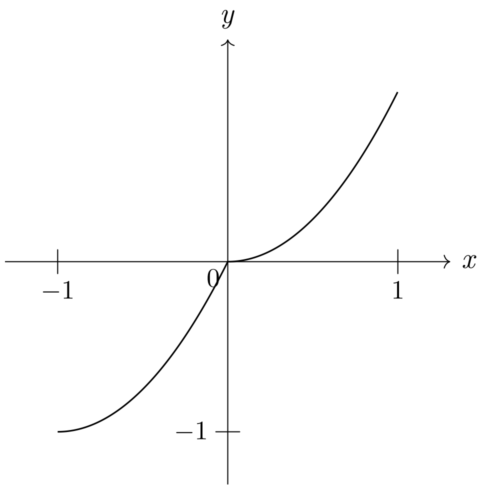

Figure 5.4: The graph of the system in Example 5.3.4

In Example 5.3.4, it is easy to see that $\mathrm{ICT}_f = \alpha_f = \omega_f = \{\{-1\},\{0\},\{1\}\}$. However f does not have shadowing. Let $\varepsilon = 1/3$. For any $\delta > 0$ we can construct a δ-pseudo-orbit which is not ε-shadowed. Indeed, fix $\delta > 0$ and let $n > 1$ be such that $1/n < \delta$. Now pick $z \in (2/3, 1)$ such that $1/n \in \mathrm{Orb}_f^+(z)$. Let $m \in \mathbb{N}$ be such that $f^m(z) = 1/n$. Now let $k \in \mathbb{N}$ be such that $f^k(-1/n) \in (-1, -3/4)$. Then

$$\langle z, f(z), \ldots, f^m(z), 0, -1/n, f(-1/n), \ldots, f^k(-1/n) \rangle$$

is a finite δ-pseudo orbit. Suppose x ε-shadows this pseudo-orbit. Then $x \in B_\varepsilon(z) \subseteq (1/3, 1]$. But $[0,1]$ is strongly invariant under f, hence $\mathrm{Orb}_f^+(x) \subseteq [0,1]$. Since $(-1,-3/4) \cap B_\varepsilon([0,1]) = \emptyset$ this is a contradiction: f does not exhibit shadowing.

Corollary 5.3.5. *Let (X, f) be a dynamical system with shadowing. Then $\overline{\alpha_f} = \overline{\omega_f} = \mathrm{ICT}_f$.*

Proof. By Lemma 5.1.2, $\mathrm{ICT}_f \supseteq \alpha_f$ and $\mathrm{ICT}_f \supseteq \omega_f$. The result now follows immediately from Theorem 5.3.2. □

Remark 5.3.6. The fact that $\overline{\omega_f} = \mathrm{ICT}_f$ for systems with shadowing has been proved previously by Meddaugh and Raines in [70].

Since ICT_f is always closed in the hyperspace 2^X (see Lemma 5.1.1), we also get the following corollary.

Corollary 5.3.7. *Let (X, f) be a dynamical system for which $\alpha_f = \mathrm{ICT}_f$. Then α_f is closed.*

Theorem 5.3.2 suggests the following question: when is it the case that every element of ICT_f is both the α-limit set and the ω-limit set of the same full trajectory? The next result gives a sufficient condition for this to be the case.

Theorem 5.3.8. *Let (X, f) be a dynamical system with two-sided s-limit shadowing. Then for any $A \in \mathrm{ICT}_f$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. In particular $\alpha_f = \omega_f = \mathrm{ICT}_f$.*

Proof. Let $A \in \mathrm{ICT}_f$ and let $\varepsilon > 0$ be given. By two-sided s-limit shadowing there exists $\delta > 0$ such that every two-sided asymptotic δ-pseudo-orbit is asymptotically $\varepsilon/2$-shadowed by a full trajectory (without loss of generality we assume $\delta < \varepsilon/2$).

Now follow the construction of the two-sided asymptotic δ-pseudo orbit $\langle a_i \rangle_{i \in \mathbb{Z}}$ in the proof of Theorem 5.3.2. Recall that

$$A = \bigcap_{n \geq 0} \overline{\{a_i \mid i \geq n\}},$$

and

$$A = \bigcap_{n \leq 0} \overline{\{a_i \mid i \leq n\}}.$$

Let $\langle x_i \rangle_{i \in \mathbb{Z}}$ be a full trajectory such that

1. $d(x_i, a_i) < \varepsilon/2$ for all $i \in \mathbb{Z}$,
2. $\lim_{i \to \pm\infty} d(x_i, a_i) = 0$.

It follows that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. The fact that $\alpha_f = \omega_f = \text{ICT}_f$ now follows from Lemma 5.1.2. $\square$

Remark 5.3.9. We did not use the fact that $\langle x_i \rangle_{i \in \mathbb{Z}}$ $\varepsilon/2$-shadows $\langle a_i \rangle_{i \in \mathbb{Z}}$ in the proof of Theorem 5.3.8. Therefore, we could replace the hypothesis of "two-sided s-limit shadowing" with the weaker condition: "there exists $\delta > 0$ such that for any two-sided asymptotic δ-pseudo-orbit $\langle y_i \rangle_{i \in \mathbb{Z}}$ there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ such that $\lim_{i \to \pm\infty} d(y_i, z_i) = 0$."

A system (X, f) is *expansive* if there exists $\eta > 0$ (referred to as an *expansivity constant*) such that given any two distinct full trajectories $\langle x_i \rangle_{i \in \mathbb{Z}}$ and $\langle y_i \rangle_{i \in \mathbb{Z}}$ there exists $i \in \mathbb{Z}$ such that $d(x_i, y_i) \geq \eta$. In [10] Barwell *et al.* showed that an expansive map has shadowing if and only if it has s-limit shadowing. We extended that result in [46] to show that an expansive map has shadowing if and only if it has two-sided s-limit shadowing. Combining this result with Theorem 5.3.8, we immediately obtain the following.

Theorem 5.3.10. *Let (X, f) be a dynamical system with shadowing. If f is expansive then for any $A \in \text{ICT}_f$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. In particular $\alpha_f = \omega_f = \text{ICT}_f$.*

Corollary 5.3.11. *Let (X, σ) be a shift of finite type (whether one- or two-sided). Then for any $A \in \text{ICT}_\sigma$ there is a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. In particular $\alpha_\sigma = \omega_\sigma = \text{ICT}_\sigma$.*

Proof. Shifts of finite type are precisely the shift systems that exhibit shadowing [93]. By Theorem 5.3.10 it now suffices to note that all shift spaces are expansive. $\square$

Remark 5.3.12. Corollary 5.3.11 enhances a result of Barwell *et al.* [8] who show that $\text{ICT}_\sigma = \omega_\sigma$ for shifts of finite type.

5.3.1 A remark on γ-limit sets

At this point we digress from our main topic to make a brief foray into γ-limit sets. First introduced by Hero [53] who studied them for interval maps, γ-limit sets have since been further examined by Sun *et al.* in [91] and [90] for graph maps and dendrites respectively. The γ-*limit set* of a point x, denoted $\gamma(x)$, is defined by saying that, for any $y \in X$, $y \in \gamma(x)$ if and only if $y \in \omega(x)$ and there exists a sequence $\langle y_i \rangle_{i=1}^{\infty}$ in X and a strictly increasing sequence $\langle n_i \rangle_{i=1}^{\infty}$ in $\mathbb{N}$ such that $f^{n_i}(y_i) = x$ for each i and $\lim_{i \to \infty} y_i = y$. Note that it is possible that $\gamma(x) = \emptyset$. We denote by γ_f the set of all γ-limit sets of (X, f).

Remark 5.3.13. Whilst we have refrained from defining the α-limit set of a point, if one were to use Hero's definition of such (see section 5.2), then it would follow that $\gamma(x) = \alpha(x) \cap \omega(x)$.

Remark 5.3.14. For a dynamical system (X, f), if f is a homeomorphism it is easy to see that, for any $x \in X$, $\gamma(x) = \alpha(\langle x_i \rangle) \cap \omega(x)$, where $\langle x_i \rangle_{i \leq 0}$ is the unique backward trajectory of x.

Unlike α- and ω-limit sets, γ-limit sets are not necessarily internally chain transitive. The example below demonstrates this.

Example 5.3.15. Let (X, σ) be the full two-sided shift with alphabet $\{0, 1, 2\}$. Consider the point $\mathbf{x}$:

$$\mathbf{x} = \ldots 0^n 1^n 0^{n-1} 1^{n-1} \ldots 0^2 1^2 01 \cdot 0^2 21^2 20^3 21^3 \ldots 0^n 21^n \ldots.$$

Then $\gamma(\mathbf{x})$ is not internally chain transitive.

In Example 5.3.15, let $\langle \mathbf{x}_i \rangle_{i \leq 0}$ be the unique backward trajectory of $\mathbf{x}$. By Theorem 5.1.9 we can observe that:

$$\alpha(\langle \mathbf{x}_i \rangle) = \{0^{\infty}, 1^{\infty}, \sigma^n(0^{\infty} \cdot 1^{\infty}) \mid n \in \mathbb{Z}\},$$

$$\omega(\mathbf{x}) = \{0^\infty, 1^\infty, \sigma^n(0^\infty 2 \cdot 1^\infty) \mid n \in \mathbb{Z}\}.$$

Since σ is a homeomorphism, by Remark 5.3.14,

$$\gamma(\mathbf{x}) = \{0^\infty, 1^\infty\}.$$

It is obvious that $\gamma(\mathbf{x})$ is not internally chain transitive.

Example 5.3.15 notwithstanding, every γ-limit set of a dynamical system is closed and contained in a single *chain component* of that system, i.e. for each $\varepsilon > 0$ and for all $a, b \in \gamma(x)$ there is an ε-chain from a to b in X (as opposed to in $\gamma(x)$).

Proposition 5.3.16. *Let (X, f) be a dynamical system. For any $x \in X$, $\gamma(x)$ is closed and contained in a single chain component of (X, f).*

Proof. If $\gamma(x) = \emptyset$ then the closedness holds and chain transitivity is vacuous.

Let $a, b \in \gamma(x)$. Let $\delta > 0$ be given. Let $y \in X$ be such that $d(f(y), f(a)) < \delta$ and there exists $n > 1$ such that $f^n(y) = x$: such a point exists by the continuity of f combined with the fact that $a \in \gamma(x)$. Now let $m \in \mathbb{N}$ be such that $d(f^m(x), b) < \delta$. It follows that $\langle a, f(y), f^2(y) \ldots, f^n(y) = x, f(x), f^2(x), \ldots, f^{m-1}(x), b\rangle$ is a δ-chain from a to b.

Now suppose $z \in \overline{\gamma(x)}$. Then there is a sequence $\langle y_i\rangle_{i=1}^\infty$ in $\gamma(x)$ such that $\lim_{i\to\infty} y_i = z$. Note that, since $\omega(x)$ is closed and $y_i \in \omega(x)$ for each i it follows that $z \in \omega(x)$. Now, for each $i \in \mathbb{N}$, let $z_i \in B_{1/i}(y_i)$ and $n_i \in \mathbb{N}$ be such that $f^{n_i}(z_i) = x$ and $\langle n_i\rangle_{i=1}^\infty$ is an increasing sequence. Then, as $\lim_{i\to\infty} z_i = z$, it follows that $z \in \gamma(x)$. □

Using Theorems 5.3.8 and 5.3.10 we obtain the following corollaries concerning the nonempty closed internally chain transitive sets in systems with two-sided s-limit shadowing.

Corollary 5.3.17. *If (X, f) is a system with two-sided s-limit shadowing then* $\mathrm{ICT}_f \subseteq \gamma_f$.

Proof. Let $A \in \mathrm{ICT}_f$. By Theorem 5.3.8 there is a full trajectory $\langle x_i\rangle_{i\in\mathbb{Z}}$ through $x_0 = x$ such that $\alpha(\langle x_i\rangle) = \omega(x) = A$. Notice that $\gamma(x) \subseteq \omega(x)$ by definition. Since $\alpha(\langle x_i\rangle) =$

$\omega(x)$, and $\langle x_i \rangle_{i \leq 0}$ is a backward trajectory of x, it follows that $\gamma(x) = A$. Hence $\mathrm{ICT}_f \subseteq \gamma_f$. □

Corollary 5.3.18. *If (X, f) is an expansive system with shadowing then* $\mathrm{ICT}_f \subseteq \gamma_f$.

5.4 Characterising $\overline{\alpha_f} = \mathrm{ICT}_f$ and $\alpha_f = \mathrm{ICT}_f$

In [47] the authors characterise systems for which $\overline{\omega_f} = \mathrm{ICT}_f$ and $\omega_f = \mathrm{ICT}_f$ in terms of novel shadowing properties. In this section, we show that the natural backward analogues of these shadowing properties characterise when $\overline{\alpha_f} = \mathrm{ICT}_f$ and $\alpha_f = \mathrm{ICT}_f$. We also demonstrate by way of examples that, in contrast to the shadowing property, there is no general entailment between the backward and forward versions of these types of shadowing.

In [47] it is shown that the property of $\overline{\omega_f} = \mathrm{ICT}_f$ is characterised by a variation on shadowing the authors term *cofinal orbital shadowing*. A system $f\colon X \to X$ has the cofinal orbital shadowing property if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit $\langle x_i \rangle_{i=0}^{\infty}$ there exists a point $z \in X$ such that for any $K \in \mathbb{N}$ there exists $N \geq K$ such that

$$d_H(\overline{\{f^{N+i}(z)\}_{i=0}^{\infty}}, \overline{\{x_{N+i}\}_{i=0}^{\infty}}) < \varepsilon.$$

The authors additionally demonstrate that this form of shadowing is equivalent to one which seems *prima facie* stronger: the *eventual strong orbital shadowing property*. A system $f\colon X \to X$ has the eventual strong orbital shadowing property if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit $\langle x_i \rangle_{i=0}^{\infty}$ there exist $z \in X$ and $K \in \mathbb{N}$ such that

$$d_H(\overline{\{f^{N+i}(z)\}_{i=0}^{\infty}}, \overline{\{x_{N+i}\}_{i=0}^{\infty}}) < \varepsilon$$

for all $N \geq K$.

Definition 5.4.1. A system $f\colon X \to X$ has the *backward cofinal orbital shadowing property* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any backward δ-pseudo-orbit $\langle x_i \rangle_{i \leq 0}$

there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ such that for any $K \in \mathbb{N}$ there exists $N \geq K$ such that

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon.$$

Definition 5.4.2. A system $f\colon X \to X$ has the *backward eventual strong orbital shadowing property* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any backward δ-pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ and there exists $K \in \mathbb{N}$ such that

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon$$

for all $N \geq K$.

Remark 5.4.3. One might wonder what the difference would be if we replaced

$$\text{`}\forall N \geq K, d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon\text{'}$$

in Definition 5.4.2, with '$d_H(\overline{\{z_{i-K}\}_{i \leq 0}}, \overline{\{x_{i-K}\}_{i \leq 0}}) < \varepsilon$'. With such a replacement we would have what might be termed *backward eventual orbital shadowing*. These two notions are not equivalent. Consider the system given by

$$f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}\colon x \mapsto x^2.$$

It is not difficult to verify that this system has backward eventual orbital shadowing but not backward eventual strong orbital shadowing. The key observation here is that for any $\varepsilon > 0$ and any backward trajectory $\langle z_i \rangle_{i \leq 0}$ there exists $K \in \mathbb{N}$ such that $\overline{\{z_{i-K}\}_{i \leq 0}} \subseteq B_\varepsilon(0)$.

Theorem 5.4.4. *Let (X, f) be a dynamical system. The following are equivalent:*

1. *f has the backward cofinal orbital shadowing property;*

2. *f has the backward eventual strong orbital shadowing property;*

3. $\overline{\alpha_f} = \mathrm{ICT}_f$.

Proof. From the definitions it is easy to see that (2) $\implies$ (1). We will show (1) $\implies$ (3) and that (3) $\implies$ (2).

Suppose that f has the backward cofinal orbital shadowing property. Recall that $\overline{\alpha_f} \subseteq \mathrm{ICT}_f$, hence it will suffice to show $\mathrm{ICT}_f \subseteq \overline{\alpha_f}$. Let $A \in \mathrm{ICT}_f$. Let $\varepsilon > 0$ be given. It will suffice to show there exists $B \in \alpha_f$ with $d_H(A, B) < \varepsilon$. Let $\delta > 0$ correspond to $\varepsilon/2$ for backward cofinal orbital shadowing. Now, follow the construction of the sequence $\langle a_i \rangle_{i \in \mathbb{Z}}$ in Theorem 5.3.2 (but for $\varepsilon/2$ and δ as here) and let $x_i = a_i$ for all $i \leq 0$. Recall that this means

$$A = \alpha(\langle x_i \rangle_{i \leq 0}).$$

Let $\langle z_i \rangle_{i \leq 0}$ be given by backward cofinal orbital shadowing so that for any $K \in \mathbb{N}$ there exists $N \geq K$ such that

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon/2.$$

Notice that in particular this means that

$$d_H(\alpha(\langle x_i \rangle), \alpha(\langle z_i \rangle)) < \varepsilon.$$

Since $\alpha(\langle x_i \rangle_{i \leq 0}) = A$ it follows that $A \in \overline{\alpha_f}$.

Now suppose that (X, f) does not have backward eventual strong orbital shadowing and let $\varepsilon > 0$ witness this. (We will show $\mathrm{ICT}_f \neq \overline{\alpha_f}$.) This means that for each $n \in \mathbb{N}$ there is a backward $1/2^n$-pseudo-orbit $\langle x_i^n \rangle_{i \leq 0}$ such that for any backward orbit $\langle z_i \rangle_{i \leq 0}$ and any $K \in \mathbb{N}$ there exists $N \geq K$ with

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) \geq \varepsilon.$$

It follows that, in particular, for each backward orbit $\langle z_i \rangle_{i \leq 0}$ and any $n \in \mathbb{N}$

$$d_H(\alpha(\langle z_i \rangle), \alpha(\langle x_i^n \rangle)) \geq \varepsilon/2. \tag{5.3}$$

For each $n \in \mathbb{N}$ let $A_n = \alpha(\langle x_i^n \rangle_{i \leq 0})$. The sequence of compact sets $\langle A_n \rangle_{n \in \mathbb{N}}$ has a convergent subsequence which converges in the hyperspace 2^X. Without loss of generality we may assume the sequence itself is convergent; let A be its limit. We claim $A \in \mathrm{ICT}_f$ but that $A \notin \overline{\alpha_f}$.

Let $a, b \in A$ and let $\xi > 0$ be arbitrary. By the uniform continuity of f, there exists $\eta > 0$ such that for any $x, y \in X$ if $d(x, y) < \eta$ then $d(f(x), f(y)) < \xi/2$. Without loss of generality take $\eta < \xi/2$. Let $M \in \mathbb{N}$ be such that $1/2^M < \eta/3$ and $d_H(A_M, A) < \eta/3$. Now take $K \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{i-K}^M\}_{i \leq 0}}, A_M) < \eta/3.$$

Thus

$$d_H(\overline{\{x_{i-K}^M\}_{i \leq 0}}, A) < 2\eta/3.$$

Let $m \in \mathbb{N}$ be such that $d(x_{-m-K}^M, b) < 2\eta/3$ and let $l > m$ be such that $d(x_{-l-K}^M, a) < 2\eta/3$. Let $y_0 = a$ and $y_{l-m} = b$. For each $j \in 1, \ldots, l-m-1$ pick $y_j \in A$ with $d(y_j, x_{-l-K+j}^M) < 2\eta/3$. We claim $\langle y_0, y_1, \ldots, y_{l-k} \rangle$ is a ξ-chain from a to b. Indeed, for $j \in \{0, \ldots, l-m-1\}$

$$\begin{aligned} d(f(y_j), y_{j+1}) &\leq d(f(y_j), f(x_{-l-K+j}^M)) + d(f(x_{-l-K+j}^M), x_{-l-K+j+1}^M) \\ &\quad + d(x_{-l-K+j+1}^M, y_{j+1}) \\ &\leq \xi/2 + 1/2^M + 2\eta/3 \\ &\leq \xi/2 + \eta/3 + 2\eta/3 \\ &\leq \xi. \end{aligned}$$

Since a and b were chosen arbitrarily in A we have that A is internally chain transitive. Thus, since A is nonempty and closed, $A \in \mathrm{ICT}_f$.

Suppose for a contradiction that $A \in \overline{\alpha_f}$. Then there exists a backward trajectory $\langle z_i \rangle_{i \leq 0} \in X$ such that $d_H(\alpha(\langle z_i \rangle_{i \leq 0}), A) < \varepsilon/4$. Let $M \in \mathbb{N}$ be such that $d_H(A_M, A) < \varepsilon/4$. Then $d_H(\alpha(\langle z_i \rangle_{i \leq 0}), A_M) < \varepsilon/2$, which contradicts Equation 5.3. Therefore $A \in \mathrm{ICT}_f \setminus \overline{\alpha_f}$. Thus $\overline{\alpha_f} \neq \mathrm{ICT}_f$. □

Remark 5.4.5. Unlike with shadowing (see Lemma 5.3.1), none of the shadowing properties in Theorem 5.4.4 imply their forward analogues (nor vice-versa). To see this, by Theorem 5.4.4 and [47, Theorem 13], it suffices to give an example where $\alpha_f = \text{ICT}_f$ but $\overline{\omega_f} \neq \text{ICT}_f$ and an example where $\omega_f = \text{ICT}_f$ but $\overline{\alpha_f} \neq \text{ICT}_f$. Such examples are provided in Example 5.4.6.

Example 5.4.6. Let $\mathbf{x} = \mathtt{0}^\infty \cdot \mathtt{1010}^2\mathtt{10}^3\ldots$. Take

$$X = \overline{\text{Orb}^+_\sigma(\mathbf{x}) \cup \text{Orb}^+_{\sigma^{-1}}(\mathbf{x})},$$

where the closure is taken with regard to the two-sided full shift on the alphabet $\{\mathtt{0}, \mathtt{1}\}$. For the system (X, σ), $\text{ICT}_\sigma = \omega_\sigma \neq \overline{\alpha_\sigma}$. For the system (X, σ^{-1}), $\text{ICT}_{\sigma^{-1}} = \alpha_{\sigma^{-1}} \neq \overline{\omega_{\sigma^{-1}}}$

In the system (X, σ) in Example 5.4.6, $\omega(\mathbf{x}) = \{\mathtt{0}^\infty, \mathtt{0}^n\mathtt{10}^\infty \mid n \geq 0\}$. It is easy to see that the only other ω-limit set is $\{\mathtt{0}^\infty\}$. Thus

$$\omega_\sigma = \{\{\mathtt{0}^\infty\}, \{\mathtt{0}^\infty, \mathtt{0}^n\mathtt{10}^\infty \mid n \geq 0\}\}.$$

Meanwhile

$$\alpha_\sigma = \{\{\mathtt{0}^\infty\}\}.$$

Observe that $\overline{\alpha_\sigma} = \alpha_\sigma$. Finally $\text{ICT}_\sigma = \omega_\sigma \neq \overline{\alpha_\sigma}$. Hence the system has cofinal orbital shadowing and eventual strong orbital shadowing by [47, Theorem 13] but the system does not have their backward analogues by Theorem 5.4.4.

Since σ is a homeomorphism, in the inverse system (X, σ^{-1}) the α-limit sets (resp. ω-limit sets) of (X, σ) are now the ω-limit sets (resp. α-limit sets). Thus,

$$\omega_{\sigma^{-1}} = \{\{\mathtt{0}^\infty\}\}.$$

Meanwhile

$$\alpha_{\sigma^{-1}} = \{\{\mathtt{0}^\infty\}, \{\mathtt{0}^\infty, \mathtt{0}^n\mathtt{10}^\infty \mid n \geq 0\}\}.$$

Observe that $\overline{\omega_{\sigma^{-1}}} = \omega_{\sigma^{-1}}$. Finally $\mathrm{ICT}_{\sigma^{-1}} = \alpha_{\sigma^{-1}} \neq \overline{\omega_{\sigma^{-1}}}$. Hence the system (X, σ^{-1}) has backward cofinal orbital shadowing and backward eventual strong orbital shadowing by Theorem 5.4.4 but it does not have their forward analogues by [47, Theorem 13].

In [47, Theorem 22], the authors show that the property of $\omega_f = \mathrm{ICT}_f$ is characterised by several equivalent asymptotic variants of shadowing: These are *asymptotic orbital shadowing*, *asymptotic strong orbital shadowing* and *orbital limit shadowing*. The system (X, f) has then has the *asymptotic orbital shadowing* property if for any asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{N+i}\}_{i\geq 0}}, \overline{\{f^{N+i}(z)\}_{i\geq 0}}) < \varepsilon.$$

The system has the *asymptotic strong orbital shadowing* property if for any asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$d_H(\overline{\{x_{N+i}\}_{i\geq 0}}, \overline{\{f^{N+i}(z)\}_{i\geq 0}}) < \varepsilon$$

for all $N \geq K$. Finally, the system has the *orbital limit shadowing* property, as introduced by Pilyugin [84], if for any asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that $\omega(z) = \omega(\langle x_i \rangle)$.

Before characterising $\omega_f = \mathrm{ICT}_f$ by these notions of shadowing, the authors [47] note that *asymptotic shadowing*, also known as limit shadowing, is sufficient but not necessary for $\omega_f = \mathrm{ICT}_f$: a system has asymptotic shadowing if for each asymptotic pseudo-orbit $\langle x_i \rangle_{i \geq 0}$ there exists a point $z \in X$ such that

$$\lim_{i\to\infty} d(f^i(z), x_i) = 0.$$

As with other shadowing variants, asymptotic shadowing has a backward analogue.

Definition 5.4.7. A system $f\colon X \to X$ has the *backward asymptotic shadowing property* if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ there exists a backward trajectory

$\langle z_i \rangle_{i\leq 0}$ such that

$$\lim_{i\to-\infty} d(z_i, x_i) = 0.$$

We shall see (Corollary 5.4.13) that backward asymptotic shadowing is sufficient for $\alpha_f = \mathrm{ICT}_f$, however it is not necessary. The irrational rotation of the circle satisfies $\alpha_f = \mathrm{ICT}_f$ (as a minimal map, both are equal to $\{X\}$) however it fails to have backward asymptotic shadowing. To see this one can observe that for any irrational rotation f of the circle, the inverse function f^{-1} is also an irrational rotation of the circle. It thereby suffices to note that no irrational rotation of the circle has asymptotic shadowing [84].

Definition 5.4.8. A system $f\colon X \to X$ has the *backward asymptotic orbital shadowing property* if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i\leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i\leq 0}$ such that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d_H(\overline{\{z_{i-N}\}_{i\leq 0}}, \overline{\{x_{i-N}\}_{i\leq 0}}) < \varepsilon.$$

Definition 5.4.9. A system $f\colon X \to X$ has the *backward asymptotic strong orbital shadowing property* if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i\leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i\leq 0}$ such that for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$d_H(\overline{\{z_{i-N}\}_{i\leq 0}}, \overline{\{x_{i-N}\}_{i\leq 0}}) < \varepsilon$$

for all $N \geq K$.

The following is a backward version of the orbital limit shadowing property, studied by Pilyugin *et al.* [84].

Definition 5.4.10. A system $f\colon X \to X$ has the *backward orbital limit shadowing property* if for each backward asymptotic pseudo-orbit $\langle x_i \rangle_{i\leq 0}$ there exists a backward trajectory $\langle z_i \rangle_{i\leq 0}$ such that

$$\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle).$$

As mentioned previously, Hirsch *et al.* [54] showed that the α-limit set (resp. ω-limit set) of any backward (resp. forward) pre-compact trajectory is internally chain transitive. In the same paper, the authors show that the ω-limit set of any pre-compact asymptotic pseudo-orbit is internally chain transitive [54, Lemma 2.3]. Whilst we omit the proof, the same is true of pre-compact backward asymptotic pseudo-orbits. We formulate this as Lemma 5.4.11 below.

Lemma 5.4.11. [54] *Let (X, f) be a dynamical system where X is a (not necessarily compact) metric space. The α-limit set (resp. ω-limit set) of any backward (resp. forward) pre-compact asymptotic pseudo-orbit is internally chain transitive. In particular, when X is compact, all such limit sets are in ICT_f.*

Theorem 5.4.12. *Let (X, f) be a dynamical system. The following are equivalent:*

1. $\alpha_f = \mathrm{ICT}_f$;
2. *f has the backward orbital limit shadowing property;*
3. *f has the backward asymptotic orbital shadowing property;*
4. *f has the backward asymptotic strong orbital shadowing property.*

Proof. Clearly (4) $\implies$ (3). It is also easy to see that (2) $\implies$ (4). We will show (3) $\implies$ (1) $\implies$ (2).

To this end, suppose that f has backward asymptotic orbital shadowing. Let $A \in \mathrm{ICT}_f$. Form a backward asymptotic pseudo-orbit $\langle x_i \rangle_{i \leq 0}$ by following the construction as in the proof of Theorem 5.3.2 and taking $x_i = a_i$ for all $i \leq 0$. (We may ignore the ε and δ in the construction, we can simply take $l = 0$.) Recall that this means

$$A = \bigcap_{n \leq 0} \overline{\{x_i \mid i \leq n\}},$$

or equivalently,

$$A = \alpha(\langle x_i \rangle).$$

Let $\langle z_i \rangle_{i \leq 0}$ be given by backward asymptotic orbital shadowing. Now let $\varepsilon > 0$ be given and let $N \in \mathbb{N}$ be such that

$$d_H(\alpha(\langle z_i \rangle), \overline{\{z_{i-N}\}_{i \leq 0}}) < \varepsilon/3,$$

$$d_H(\overline{\{z_{i-N}\}_{i \leq 0}}, \overline{\{x_{i-N}\}_{i \leq 0}}) < \varepsilon/3,$$

and

$$d_H(\overline{\{x_{i-N}\}_{i \leq 0}}, \alpha(\langle x_i \rangle)) < \varepsilon/3.$$

By the triangle inequality it follows that $d_H(\alpha(\langle z_i \rangle), A) < \varepsilon$. Since $\varepsilon > 0$ was picked arbitrarily this implies that $A = \alpha(\langle z_i \rangle)$. Hence $\mathrm{ICT}_f \subseteq \alpha_f$. By Lemma 5.1.2 we have $\alpha_f \subseteq \mathrm{ICT}_f$, thus (1) holds.

Now suppose that $\alpha_f = \mathrm{ICT}_f$. Let $\langle x_i \rangle_{i \leq 0}$ be a backward asymptotic pseudo-orbit. By Lemma 5.4.11 $\alpha(\langle x_i \rangle) \in \mathrm{ICT}_f$. Since $\alpha_f = \mathrm{ICT}_f$ there exists a backward trajectory $\langle z_i \rangle_{i \leq 0}$ with $\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle)$. Hence f has the backward orbital limit shadowing property, i.e. (2) holds. □

Corollary 5.4.13. *If (X, f) has backward asymptotic shadowing then $\alpha_f = \mathrm{ICT}_f$.*

Proof. By Theorem 5.4.12 it suffices to note that backward asymptotic shadowing implies backward orbital limit shadowing. □

Remark 5.4.14. Combining Theorems 5.4.4 and 5.4.12 we have that if α_f is closed then the following are equivalent:

1. f has the backward orbital limit shadowing property;
2. f has the backward eventual strong orbital shadowing property;
3. f has the backward asymptotic (strong) orbital shadowing property;
4. f has the backward cofinal orbital shadowing property.

Remark 5.4.15. Example 5.4.6, together with Theorem 5.4.12 and [47, Theorem 22], show that, unlike shadowing (see Lemma 5.3.1), neither the backward orbital limit shadowing property nor the backward asymptotic orbital shadowing nor the backward asymptotic strong orbital shadowing is equivalent to its forward analogue.

5.5 When is the beginning the end?

The content of Theorems 5.3.2 and 5.3.8 suggest two final questions to us:

Question 5.5.1. When is it the case that every nonempty closed internally chain transitive set is both the α-limit set and ω-limit set of the same full trajectory? I.e. when is it true that for any $A \in \text{ICT}_f$ there exists a full trajectory $\langle x_i \rangle_{i\in\mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$?

Question 5.5.2. When is it the case that every nonempty closed internally chain transitive set may be approximated, to any given accuracy, by both the α-limit set and ω-limit set of the same full trajectory? I.e. when is it true that for any $A \in \text{ICT}_f$ and any $\varepsilon > 0$ there exists a full trajectory $\langle x_i \rangle_{i\in\mathbb{Z}}$ such that $d_H(\alpha(\langle x_i \rangle), A) < \varepsilon$ and $d_H(\omega(\langle x_i \rangle), A) < \varepsilon$?

In answering these questions, this section aims to provide the final chapter in the journey to characterise when limit sets approximate, or are precisely, the elements of ICT_f in terms of shadowing properties. Due to their lengthy statements, we will refer to the properties in Questions 5.5.1 and 5.5.2 as P_e and P_a respectively ('e' for 'equal', 'a' for 'approximate'). Thus:

- Property P_e: 'For any $A \in \text{ICT}_f$ there exists a full trajectory $\langle x_i \rangle_{i\in\mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$.'
- Property P_a: 'For any $A \in \text{ICT}_f$ and any $\varepsilon > 0$ there exists a full trajectory $\langle x_i \rangle_{i\in\mathbb{Z}}$ such that $d_H(\alpha(\langle x_i \rangle), A) < \varepsilon$ and $d_H(\omega(\langle x_i \rangle), A) < \varepsilon$.'

Thus, Theorem 5.3.2 states that shadowing is a sufficient condition for property P_a, whilst Theorem 5.3.8 tells us that two-sided s-limit shadowing is sufficient for property P_e.

Since it will be important for our characterisations of P_a and P_e, we recall the construction of the two-sided asymptotic δ-pseudo-orbit $\langle a_i \rangle_{i\in\mathbb{Z}}$ from Theorem 5.3.2. Rather than continually referring back to it in this manner, we pull out the construction of such a sequence as a proof of the following lemma.

Lemma 5.5.3. *Let (X, f) be a dynamical system. For any $A \in \mathrm{ICT}_f$ and any $\varepsilon > 0$ there exists a two-sided asymptotic ε-pseudo-orbit $\langle a_i \rangle_{i\in\mathbb{Z}}$ in A such that $\alpha(\langle a_i \rangle) = \omega(\langle a_i \rangle) = A$.*

5.5.1 Property P_e

Recall property P_e:

'For any $A \in \mathrm{ICT}_f$ there exists a full trajectory $\langle x_i \rangle_{i\in\mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$.'

It is obvious, given Lemma 5.1.2, that a necessary condition for P_e is $\alpha_f = \omega_f = \mathrm{ICT}_f$. A natural starting point, therefore, is to ask if this is also sufficient. It turns out that this is not the case: in Example 5.5.4 we construct a homeomorphism for which $\alpha_f = \omega_f = \mathrm{ICT}_f$ but for which P_e is not satisfied.

Example 5.5.4. We will build up the points in X and define the map $f\colon X \to X$ on them as we go. Let X consist of the following points in the Cartesian plane. Let $(0, 0)$ be a fixed point, so that $f(0, 0) = (0, 0)$. Let $(1, 1), (-1, 1), (-1, -1)$ and $(1, -1)$ also be fixed points and consider the following subsets of the 2×2 square S with these four points as vertices:

$$A = \left\{ \left(\pm\frac{2^n - 1}{2^n}, 1 \right) \mid n \in \mathbb{N}_0 \right\},$$

$$B = \left\{ \left(-1, \pm\frac{2^n - 1}{2^n} \right) \mid n \in \mathbb{N}_0 \right\},$$

$$C = \left\{ \left(\pm\frac{2^n - 1}{2^n}, -1 \right) \mid n \in \mathbb{N}_0 \right\},$$

$$D = \left\{ \left(1, \pm\frac{2^n - 1}{2^n} \right) \mid n \in \mathbb{N}_0 \right\}.$$

We now define the map f on these four sets so that points move anticlockwise between the two vertices which are the limit points of said set.

For example, for A the vertices which form the limit points of A are $(1,1)$ and $(-1,1)$. For any $z \in A$, with $z = (\frac{2^n-1}{2^n}, 1)$ for some $n \geq 1$, let $f(z) = (\frac{2^{n-1}-1}{2^{n-1}}, 1)$. Let $f(0,1) = (-\frac{1}{2}, 1)$. Finally, for any $z \in A$, with $z = (-\frac{2^n-1}{2^n}, 1)$ for some $n \geq 1$, let $f(z) = (-\frac{2^{n+1}-1}{2^{n+1}}, 1)$. Define f on B, C and D similarly, with this anticlockwise movement. We let $Q = \{(1,1), (-1,1), (-1,-1), (1,-1)\} \cup A \cup B \cup C \cup D$.

Next, we insert the points given by $(0, \frac{1}{2^n})$ for each $n \geq 2$. For each of these, we let $f(0, \frac{1}{2^n}) = (0, \frac{1}{2^{n-1}})$.

We now, for each $n \in \mathbb{N}$ insert a finite subset of a square as follows: Insert the points $(\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}), (-\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}), (-\frac{2^n-1}{2^n}, -\frac{2^n-1}{2^n})$ and $(\frac{2^n-1}{2^n}, -\frac{2^n-1}{2^n})$ (these are the vertices). For each $n \in \mathbb{N}$ we insert the following finite subsets of these squares:

$$A_n = \left\{ \left(\pm\frac{2^m-1}{2^m}, \frac{2^n-1}{2^n} \right) \mid m \in \mathbb{N}_0 \text{ and } |\frac{2^m-1}{2^m}| < \frac{2^n-1}{2^n} \right\},$$

$$B_n = \left\{ \left(-\frac{2^n-1}{2^n}, \pm\frac{2^m-1}{2^m} \right) \mid m \in \mathbb{N}_0 \text{ and } |\frac{2^m-1}{2^m}| < \frac{2^n-1}{2^n} \right\},$$

$$C_n = \left\{ \left(\pm\frac{2^m-1}{2^m}, -\frac{2^n-1}{2^n} \right) \mid m \in \mathbb{N}_0 \text{ and } |\frac{2^m-1}{2^m}| < \frac{2^n-1}{2^n} \right\},$$

$$D_n = \left\{ \left(\frac{2^n-1}{2^n}, \pm\frac{2^m-1}{2^m} \right) \mid m \in \mathbb{N}_0 \text{ and } |\frac{2^m-1}{2^m}| < \frac{2^n-1}{2^n} \right\}.$$

Let $Q_n = \{(\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}), (-\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}), (-\frac{2^n-1}{2^n}, -\frac{2^n-1}{2^n}), (\frac{2^n-1}{2^n}, -\frac{2^n-1}{2^n})\} \cup A_n \cup B_n \cup C_n \cup D_n$.

For each $n \in \mathbb{N}$, let $f(\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}) = (\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}})$. All points in $Q_n \setminus \{(\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n})\}$, as before, move anticlockwise around the finite set Q_n under f. So that, in Q_1 for example, $f(0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}), f(-\frac{1}{2}, \frac{1}{2}) = (-\frac{1}{2}, 0), f(-\frac{1}{2}, 0) = (-\frac{1}{2}, -\frac{1}{2}), \dots f(\frac{1}{2}, 0) = (\frac{1}{2}, \frac{1}{2})$.

It follows that the ω-limit set of every point, apart from $(0,0)$, which lies inside the region bounded by Q in the plane is Q, whilst their α-limit set is $\{(0,0)\}$.

Now input the points $(0,y) \in \mathbb{R}^2$ such that $y = \frac{3}{2} + \frac{2^n-1}{2^{n+1}}$ for some $n \in \mathbb{N}_0$. Let $f(0, \frac{3}{2} + \frac{2^n-1}{2^{n+1}}) = (0, \frac{3}{2} + \frac{2^{n+1}-1}{2^{n+2}})$. Let the limit this sequence, $(0,2)$, be a fixed point under f.

Now, for each $n \in \mathbb{N}$ insert a finite subset of a square as follows: Insert the points $(1+\frac{1}{2^n}, 1+\frac{1}{2^n}), (1-\frac{1}{2^n}, 1+\frac{1}{2^n}), (-1-\frac{1}{2^n}, -1-\frac{1}{2^n})$ and $(1+\frac{1}{2^n}, -1-\frac{1}{2^n})$ (these are the vertices). For each $n \in \mathbb{N}$ we insert the following finite subsets of these squares:

$$E_n = \left\{ \left(\pm\frac{2^m-1}{2^m}, 1+\frac{1}{2^n} \right) \mid m \in \mathbb{N}_0 \text{ and } |\frac{2^m-1}{2^m}| < \frac{2^n-1}{2^n} \right\},$$

$$F_n = \left\{ \left(-1-\frac{1}{2^n}, \pm\frac{2^m-1}{2^m} \right) \mid m \in \mathbb{N}_0 \text{ and } |\frac{2^m-1}{2^m}| < \frac{2^n-1}{2^n} \right\},$$

$$G_n = \left\{ \left(\pm\frac{2^m-1}{2^m}, -1-\frac{1}{2^n} \right) \mid m \in \mathbb{N}_0 \text{ and } |\frac{2^m-1}{2^m}| < \frac{2^n-1}{2^n} \right\},$$

$$H_n = \left\{ \left(1+\frac{1}{2^n}, \pm\frac{2^m-1}{2^m} \right) \mid m \in \mathbb{N}_0 \text{ and } |\frac{2^m-1}{2^m}| < \frac{2^n-1}{2^n} \right\}.$$

Let $R_n = \{(1+\frac{1}{2^n}, 1+\frac{1}{2^n}), (-1-\frac{1}{2^n}, 1+\frac{1}{2^n}), (-1-\frac{1}{2^n}, -1-\frac{1}{2^n}), (1+\frac{1}{2^n}, -1-\frac{1}{2^n})\} \cup E_n \cup F_n \cup G_n \cup H_n$.

For each $n \geq 2$, let $f(-\frac{2^{n-1}-1}{2^{n-1}}, 1+\frac{1}{2^n}) = (-\frac{2^{n-1}-1}{2^{n-1}}, 1+\frac{1}{2^{n-1}})$.

For each $n \in \mathbb{N}$, let all points in $R_n \setminus \{(-\frac{2^{n-1}-1}{2^{n-1}}, 1+\frac{1}{2^n})\}$, as before, move anti-clockwise around the finite set R_n under f. So that, in R_1 for example, $f(-\frac{1}{2}, \frac{3}{2}) = (-\frac{3}{2}, \frac{3}{2}), f(-\frac{3}{2}, \frac{3}{2}) = (-\frac{3}{2}, \frac{1}{2}), f(-\frac{3}{2}, \frac{1}{2}) = (-\frac{3}{2}, 0), \ldots f(\frac{3}{2}, \frac{3}{2}) = (\frac{1}{2}, \frac{3}{2}), f(\frac{1}{2}, \frac{3}{2}) = (0, \frac{3}{2})$.

Then $\alpha_f = \omega_f = \text{ICT}_f$ but property P_e is not satisfied.

The system constructed in Example 5.5.4 is shown in Figure 5.5.1 below. It is easily observed that α_f, ω_f and ICT_f are all equal to

$$\{Q, \{(0,0)\}, \{(1,1)\}, \{(-1,1)\}, \{(-1,-1)\}, \{(1,-1)\}, \{(0,2)\}\}.$$

However it is also clear that no full trajectory has Q as both its α-limit set and ω-limit set.

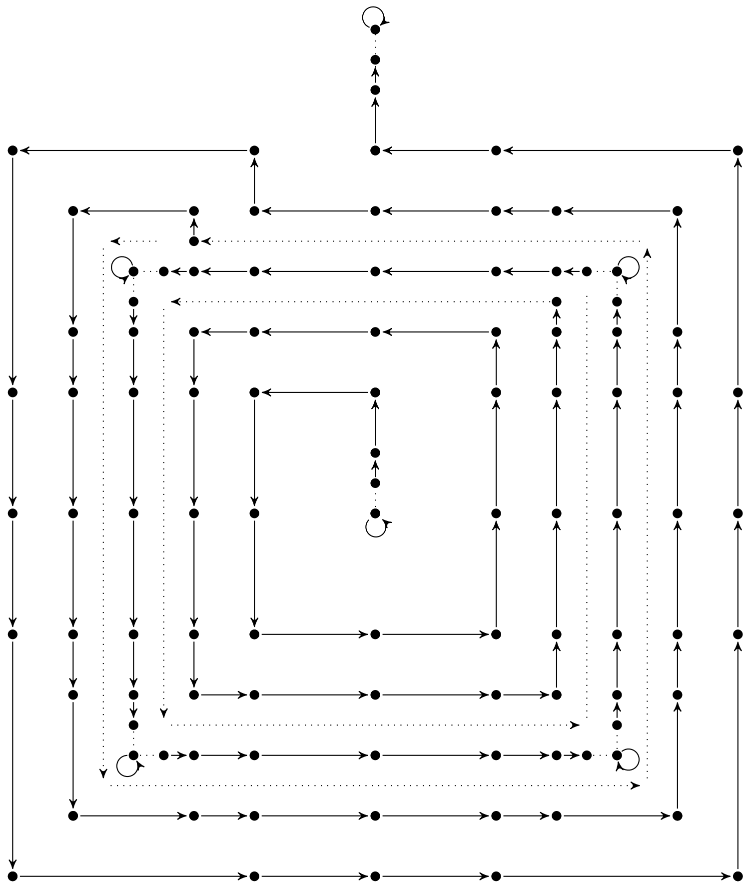

Figure 5.5: A system not satisfying P_e but with $\alpha_f = \omega_f = \text{ICT}_f$ (Example 5.5.4)

In order to characterise P_e, it is will be helpful to consider the characterisations of when $\omega_f = \text{ICT}_f$ and $\alpha_f = \text{ICT}_f$. As previously mentioned, by [47, Theorem 22] the property of $\omega_f = \text{ICT}_f$ is characterised by orbital limit shadowing. We then showed earlier in this chapter (Theorem 5.4.12) that backward orbital limit shadowing characterises when $\alpha_f = \text{ICT}_f$. It follows that the system constructed in Example 5.5.4 has both orbital limit shadowing and backward orbital limit shadowing. These results suggest the following shadowing property.

Definition 5.5.5. A system (X, f) has *two-sided orbital limit shadowing* if for any two-sided asymptotic pseudo-orbit $\langle x_i \rangle_{i \in \mathbb{Z}}$ there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle)$ and $\omega(\langle z_i \rangle) = \omega(\langle x_i \rangle)$.

This property is strictly weaker than the two-sided limit shadowing property studied by various authors (e.g. [22, 23, 24, 46, 77]). An irrational rotation of the circle will have two-sided orbital limit shadowing but not two-sided limit shadowing. We will see in Corollary 5.5.10 that this property is sufficient for P_e. However, it is too strong for our purposes. Indeed, $f\colon [0,1] \to [0,1]\colon x \mapsto x^2$ satisfies property P_e but it does not have two-sided orbital limit shadowing. To see this consider the two-sided asymptotic pseudo-orbit $\langle x_i \rangle_{i\in\mathbb{Z}}$, given by $x_i = 0$ for $i \leq 0$ and $x_i = 1$ for $i > 0$. Then $\omega(\langle x_i \rangle) = \{1\}$ and $\alpha(\langle x_i \rangle) = \{0\}$. However, the only full trajectory whose ω-limit set is $\{1\}$ is given by $z_i = 1$ for all $i \in \mathbb{Z}$, but $\alpha(1) = \{1\} \neq \alpha(\langle x_i \rangle)$. The strength of this shadowing property seems partly to lie in the lack of restriction in where the pseudo-orbit may 'jump'. To overcome this we suggest the following weakening.

Definition 5.5.6. A system (X, f) has *δ-restricted two-sided orbital limit shadowing* if there exists $\delta > 0$ such that for any two-sided asymptotic δ-pseudo-orbit $\langle x_i \rangle_{i\in\mathbb{Z}}$ there exists a full trajectory $\langle z_i \rangle_{i\in\mathbb{Z}}$ such that $\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle)$ and $\omega(\langle z_i \rangle) = \omega(\langle x_i \rangle)$.

We will see (Corollary 5.5.10), that δ-restricted two-sided orbital limit shadowing is indeed sufficient for P_e, however, as Example 5.5.7 shows, it is not necessary.

Example 5.5.7. For each $n \in \mathbb{N}$, let X_n be the circle $\mathbb{R}/\mathbb{Z} \times \{\frac{1}{n}\}$ and let $f_n\colon X_n \to X_n$ be given by $x \mapsto x + \alpha$, where α is some fixed irrational number. Let $X_0 = \mathbb{R}/\mathbb{Z} \times \{0\}$ and $f_0\colon X_0 \to X_0$ also be given by $x \mapsto x + \alpha$. Take $X = \bigcup_{n=0}^{\infty} X_n$ and let $f\colon X \to X$ be defined by saying that, for any $n \in \mathbb{N}_0$ and any $x \in X_n$, $f(x) = f_n(x)$. Then (X, f) has property P_e but not δ-restricted two-sided orbital limit shadowing.

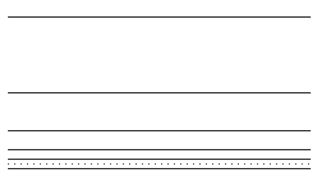

Figure 5.6: The space in Example 5.5.7

To see that (X, f) in Example 5.5.7 has property P_e it suffices to note that it is simply composed of disjoint minimal systems. Since only one system, (X_0, f_0), is a limit of other systems we get that $\mathrm{ICT}_f = \{X_n \mid n \in \mathbb{N}_0\}$. But, for any $n \in \mathbb{N}_0$, the orbit of each point in X_n is dense in X_n. Property P_e now follows. Now suppose that the system has δ-restricted two-sided orbital limit shadowing. Let $\delta > 0$ bear witness to this and let $\frac{1}{n} < \delta$. Let $x = (0,0)$ and $y = (0, \frac{1}{n})$. Then $\langle x_i \rangle_{i \in \mathbb{Z}}$ where $x_i = f^i(x)$ for $i \geq 0$ and $x_i = f^i(y)$ for $i < 0$ is a two-sided asymptotic δ-pseudo-orbit which is not two-sided asymptotically shadowed, a contradiction.

Motivated by the backward and forward orbital limit shadowing properties, we define the following novel variant shadowing which does in fact characterise P_e (Theorem 5.5.9).

Definition 5.5.8. A system (X, f) has *γ-restricted two-sided orbital limit shadowing* if for any two-sided asymptotic pseudo-orbit $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle)$ there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ such that

1. $\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle)$; and,
2. $\omega(\langle z_i \rangle) = \omega(\langle x_i \rangle)$.

Theorem 5.5.9. *A dynamical system (X, f) exhibits property P_e if and only if it has γ-restricted two-sided orbital limit shadowing.*

Proof. First suppose that (X, f) has property P_e. Let $\langle x_i \rangle_{i \in \mathbb{Z}}$ be a two-sided asymptotic pseudo-orbit such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle)$. Let $A = \alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle)$. By Lemma 5.4.11, $A \in \mathrm{ICT}_f$. Therefore, since (X, f) exhibits property P_e, there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ such that $\alpha(\langle z_i \rangle) = \omega(\langle z_i \rangle) = A$. I.e. $\alpha(\langle z_i \rangle) = \alpha(\langle x_i \rangle)$ and $\omega(\langle z_i \rangle) = \omega(\langle x_i \rangle)$. Therefore (X, f) has γ-restricted two-sided orbital limit shadowing.

Now suppose that (X, f) has γ-restricted two-sided orbital limit shadowing. Let $A \in \mathrm{ICT}_f$ be given. By Lemma 5.5.3, there exists a two-sided asymptotic pseudo-orbit $\langle a_i \rangle_{i \in \mathbb{Z}}$ in A such that $\alpha(\langle a_i \rangle) = A = \omega(\langle a_i \rangle)$. Let $\langle z_i \rangle_{i \in \mathbb{Z}}$ be a full trajectory which γ-restricted two-sided orbital limit shadows $\langle a_i \rangle_{i \in \mathbb{Z}}$. Then $\alpha(\langle z_i \rangle) = \alpha(\langle a_i \rangle) = A$ and $\omega(\langle z_i \rangle) = \omega(\langle a_i \rangle) = A$. In particular, $\alpha(\langle z_i \rangle) = \omega(\langle z_i \rangle) = A$. Therefore (X, f) has property P_e. $\square$

Corollary 5.5.10. *If a system exhibits any of the following shadowing properties then it has property P_e:*

1. *two-sided limit shadowing;*

2. *two-sided orbital limit shadowing;*

3. *δ-restricted two-sided orbital limit shadowing.*

Proof. It suffices to note that each property implies γ-restricted two-sided orbital limit shadowing. $\square$

Corollary 5.5.11. *If (X, f) is an expansive system with shadowing then it has γ-restricted two-sided orbital limit shadowing.*

Proof. Shadowing and expansivity together give that P_e is satisfied (Theorem 5.3.10). The result now follows from Theorem 5.5.9. $\square$

Recalling the content on γ-limit sets from section 5.3.1, we now give the following corollary.

Corollary 5.5.12. *If (X, f) is a system with γ-restricted two-sided orbital limit shadowing then* $\text{ICT}_f \subseteq \gamma_f$.

Proof. Let $A \in \text{ICT}_f$. By Theorem 5.5.9, (X, f) exhibits P_e: let $\langle x_i \rangle_{i \in \mathbb{Z}}$ be a full trajectory such that $\alpha(\langle x_i \rangle) = \omega(\langle x_i \rangle) = A$. Let $x = x_0$. Then $\omega(x) = A$. Since $\gamma(x) \subseteq \omega(x)$ by definition it follows that $\gamma(x) \subseteq A$. Furthermore, since $\langle x_i \rangle_{i \leq 0}$ is a backward trajectory from x and $\alpha(\langle x_i \rangle) = A \subseteq \omega(x)$, it follows that $\gamma(x) \supseteq A$. Thus $\gamma(x) = A$. Since $A \in \text{ICT}_f$ was picked arbitrarily it follows that $\text{ICT}_f \subseteq \gamma_f$. $\square$

5.5.2 Property P_a

We now turn our attention to Question 5.5.2 and property P_a, i.e. 'for any $A \in \text{ICT}_f$ and any $\varepsilon > 0$ there exists a full trajectory $\langle x_i \rangle_{i \in \mathbb{Z}}$ such that $d_H(\alpha(\langle x_i \rangle, A) < \varepsilon$ and $d_H(\omega(\langle x_i \rangle), A) < \varepsilon$'.

Whilst for P_a to hold it must be the case that $\overline{\alpha_f} = \overline{\omega_f} = \text{ICT}_f$, this alone is not sufficient. Example 5.5.4 serves to demonstrate this. Recall that, in [47], it is shown that the property of $\overline{\omega_f} = \mathrm{ICT}_f$ is characterised by cofinal orbital shadowing. In similar fashion, backward cofinal orbital shadowing characterises when $\overline{\alpha_f} = \mathrm{ICT}_f$ (see Theorem 5.4.4). These notions motivate the following.

Definition 5.5.13. A system (X, f) has two-sided cofinal orbital shadowing if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any two-sided δ-pseudo-orbit $\langle x_i \rangle_{i\in\mathbb{Z}}$ there exists a full trajectory $z \in X$ such that for any $K \in \mathbb{N}$ there exists $N \geq K$ such that

1. $d_H(\overline{\{z_{N+i}\}_{i\geq 0}}, \overline{\{x_{N+i}\}_{i\geq 0}}) < \varepsilon$;
2. $d_H(\overline{\{z_{i-N}\}_{i\leq 0}}, \overline{\{x_{i-N}\}_{i\leq 0}}) < \varepsilon$.

We will see that this is indeed sufficient for P_a, however it is not necessary.

Example 5.5.14. Let $X = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ and let f be given by $(x, y) \mapsto (x + \alpha, y)$, where α is some fixed irrational number. Then (X, f) satisfies P_a but does not have two-sided cofinal orbital shadowing.

Whilst we omit the construction of such a pseudo-orbit, it can be seen that in Example 5.5.14, given any $\delta > 0$, one can make a two-sided δ-pseudo-orbit whose α-limit set is $\mathbb{R}/\mathbb{Z} \times \{0\}$ and whose ω-limit set is $\mathbb{R}/\mathbb{Z} \times \{\frac{1}{2}\}$. There is no full trajectory which satisfies the conditions in two-sided cofinal orbital shadowing for such a pseudo-orbit with $\varepsilon < \frac{1}{4}$.

As in our search to characterise P_e, a restriction is necessary.

Definition 5.5.15. A system (X, f) has γ-restricted two-sided cofinal orbital shadowing if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any two-sided δ-pseudo-orbit $\langle x_i \rangle_{i\in\mathbb{Z}}$ such that $d_H(\alpha(\langle x_i \rangle), \omega(\langle x_i \rangle)) < \varepsilon$ there exists a full trajectory $z \in X$ such that for any $K \in \mathbb{N}$ there exists $N \geq K$ such that

1. $d_H(\overline{\{z_{N+i}\}_{i\geq 0}}, \overline{\{x_{N+i}\}_{i\geq 0}}) < \varepsilon$;
2. $d_H(\overline{\{z_{i-N}\}_{i\leq 0}}, \overline{\{x_{i-N}\}_{i\leq 0}}) < \varepsilon$.

Remark 5.5.16. It is equivalent to replace conditions (1) and (2) in the definition of γ-restricted two-sided cofinal orbital shadowing with the following:

1. $d_H(\omega(\langle z_i \rangle), \omega(\langle x_i \rangle)) < \varepsilon$;

2. $d_H(\alpha(\langle z_i \rangle), \alpha(\langle x_i \rangle)) < \varepsilon$.

Theorem 5.5.17. *A dynamical system (X, f) exhibits property P_a if and only if it has γ-restricted two-sided cofinal orbital shadowing.*

Proof. First suppose that (X, f) has property P_a. (We will use the content of Remark 5.5.16 to show (X, f) has γ-restricted two-sided cofinal orbital shadowing.) Let $\varepsilon > 0$ be given. Now take $\eta = \frac{\varepsilon}{3}$ and let $\delta = \eta$. Let $\langle x_i \rangle_{i \in \mathbb{Z}}$ be a two-sided δ-pseudo-orbit such that $d_H(\alpha(\langle x_i \rangle), \omega(\langle x_i \rangle)) < \eta$. Let $A = \alpha(\langle x_i \rangle)$ and $B = \omega(\langle x_i \rangle)$. By Lemma 5.4.11, $A, B \in \text{ICT}_f$. Therefore, since (X, f) exhibits property P_a, there exists a full trajectory $\langle z_i \rangle_{i \in \mathbb{Z}}$ such that $d_H(\alpha(\langle z_i \rangle), A) < \eta$ and $d_H(\omega(\langle z_i \rangle), A) < \eta$. It follows by the triangle inequality that $d_H(\omega(\langle z_i \rangle), B) < 2\eta < \varepsilon$. Therefore (X, f) has γ-restricted two-sided cofinal orbital shadowing.

Now suppose that (X, f) has γ-restricted two-sided cofinal orbital shadowing. Let $A \in \text{ICT}_f$ be given and let $\varepsilon > 0$ be given. Take $\delta > 0$ corresponding to ε for the formulation of γ-restricted two-sided orbital limit shadowing given by Remark 5.5.16 (without loss of generality $\delta < \varepsilon$). By Lemma 5.5.3, there exists a two-sided asymptotic δ-pseudo-orbit $\langle a_i \rangle_{i \in \mathbb{Z}}$ in A such that $\alpha(\langle a_i \rangle) = A = \omega(\langle a_i \rangle)$. Let $\langle z_i \rangle_{i \in \mathbb{Z}}$ be a full trajectory for which

1. $d_H(\omega(\langle z_i \rangle), \omega(\langle a_i \rangle)) < \varepsilon$;

2. $d_H(\alpha(\langle z_i \rangle), \alpha(\langle a_i \rangle)) < \varepsilon$.

In particular, $d_H(\alpha(\langle z_i \rangle), A) < \varepsilon$ and $d_H(\omega(\langle z_i \rangle), A) < \varepsilon$. Therefore (X, f) satisfies property P_a. □

Corollary 5.5.18. *If (X, f) has shadowing then it has γ-restricted two-sided cofinal orbital shadowing.*

Proof. By Theorem 5.3.2, a system with shadowing satisfies property P_a. The result follows from Theorem 5.5.17. □

Corollary 5.5.19. *If a system (X, f) has two-sided cofinal orbital shadowing then it has property P_a.*

Corollary 5.5.20. *If (X, f) has γ-restricted two-sided orbital limit shadowing then it has γ-restricted two-sided cofinal orbital shadowing.*

Proof. Since $P_e \implies P_a$, the result follows from Theorems 5.5.9 and 5.5.17. □

5.6 Closing examples

We wrap up this chapter by constructing two further examples. We start, in Example 5.6.1, by constructing a homeomorphism which exhibits P_a but not P_e, thereby demonstrating that γ-restricted two-sided cofinal orbital shadowing does not imply γ-restricted two-sided orbital limit shadowing. We then close by giving one final example (Example 5.6.2) which draws some of the themes from [47] and this chapter, together.

Example 5.6.1. Start with Q as in Example 5.5.4 and let f act on these points in the same manner. Now, for each $n \in \mathbb{N}$, insert the set R_n. However, in contrast to Example 5.5.4, let f act on R_n in a simple anticlockwise manner; so that each R_n consists of a periodic orbit going anticlockwise. Then (X, f) is a homeomorphism which satisfies property P_a but not property P_e.

It is not difficult to see that

$$\alpha_f = \omega_f = \{R_n \mid n \in \mathbb{N}\} \cup \{\{(1,1)\}, \{(-1,1)\}, \{(-1,-1)\}, \{(1,-1)\}\}.$$

Meanwhile ICT_f additionally includes Q. Because, for instance, $\text{ICT}_f \neq \alpha_f$, it follows that P_e is not satisfied by (X, f). However, for any $\varepsilon > 0$ there is a full trajectory whose α-limit set and ω-limit set both lie within ε of Q. To see this observe that the subsystem $(Q, f\restriction_Q))$ is the limit of the subsystems $(R_n, f\restriction_{R_n})$. Let $\varepsilon > 0$ be given and let $n \in \mathbb{N}$ be such that $d_H(R_n, Q) < \varepsilon$. Pick $z \in R_n$ and let $\langle z_i \rangle_{i \in \mathbb{Z}}$ be the unique full trajectory with

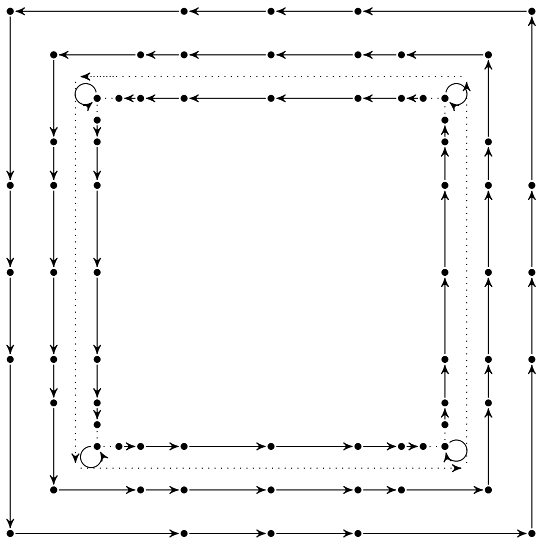

Figure 5.7: A system satisfying P_a but not P_e (Example 5.6.1)

$z_0 = z$; this is a periodic orbit with $\alpha(\langle z_i \rangle) = \omega(\langle z_i \rangle) = R_n$. Therefore $d_H(\alpha(\langle z_i \rangle), Q) < \varepsilon$ and $d_H(\omega(\langle z_i \rangle), Q) < \varepsilon$. Hence P_a holds and, in particular, $\overline{\alpha_f} = \overline{\omega_f} = \text{ICT}_f$.

As stated at the beginning of this section, our final example (Example 5.6.2) serves to draw some of the themes in [47] and this chapter, together. Example 5.6.2 is an informal construction of a homeomorphism which exhibits neither P_e nor P_a and for which

1. $\alpha_f = \omega_f \neq \text{ICT}_f$; and,

2. $\overline{\alpha_f} = \overline{\omega_f} = \text{ICT}_f$.

Furthermore, the only non-singleton elements of ICT_f which may be approximated by the α-limit set and ω-limit set of the same full trajectory are precisely the ones which belong to neither α_f nor ω_f. (Hence the system does not have shadowing.)

Before we give this last example, we recall that the system in Example 5.5.4 has the forward and backward versions of both cofinal orbital shadowing, and orbital limit shadowing. This is because $\alpha_f = \omega_f = \text{ICT}_f$. The system in Example 5.6.2, on the other hand, will have the forward and backward versions of cofinal orbital shadowing (since $\overline{\alpha_f} = \overline{\omega_f} = \text{ICT}_f$), but neither the forward nor backward version of orbital limit shadowing (since $\alpha_f = \omega_f \neq \text{ICT}_f$).

Example 5.6.2. Start with Q as in Example 5.5.4 and let f act on these points in the same manner. For $n \in \mathbb{R}^+$, define the set $nQ := \{(nx, ny) \mid (x, y) \in Q\}$. For each $n \in \mathbb{N}$ insert the sets $\frac{2^{n+1}-1}{2^n}Q$ and $\frac{2^n+1}{2^n}Q$. Also insert the set $2Q$. Let f act on these akin to the way it acts on Q.

Now, for each $n \in \mathbb{N}$, insert a two-sided sequence of points which lies between $\frac{2^{n+1}-1}{2^n}Q$ and $\frac{2^{(n+1)+1}-1}{2^{n+1}}Q$ in the plane such that

1. each point maps onto the next in the sequence;
2. the ω-limit set of every point in the sequence is $\frac{2^{(n+1)+1}-1}{2^{n+1}}Q$; and,
3. the α-limit set of every point in the sequence is $\frac{2^{n+1}-1}{2^n}Q$.

(Combining, and making suitable adjustments to, some of the techniques used in Example 5.5.4 would be one appropriate way to accomplish this.)

Finally, for each $n \in \mathbb{N}$ with $n \geq 2$, insert a two-sided sequence of points which lies between $\frac{2^n+1}{2^n}Q$ and $\frac{2^{n-1}+1}{2^{n-1}}Q$ in the plane such that

1. each point maps onto the next in the sequence;
2. the ω-limit set of every point in the sequence is $\frac{2^{n-1}+1}{2^{n-1}}Q$; and,
3. the α-limit set of every point in the sequence is $\frac{2^n+1}{2^n}Q$.

Whilst we omit a proof of the fact, it is not difficult to see that α_f and ω_f are comprised of $\frac{2^{n+1}-1}{2^n}Q$ and $\frac{2^n+1}{2^n}Q$ (for each $n \in \mathbb{N}$), along with the singleton sets of all fixed points in the system. Meanwhile ICT_f additionally includes Q and $2Q$. This implies that P_e does not hold. Note further that no set of the form $\frac{2^{n+1}-1}{2^n}Q$ may be approximated to any given accuracy by *both* the α-limit set and ω-limit set of the same full trajectory. However, for any $\varepsilon > 0$ there is a full trajectory whose α-limit set and ω-limit set both lie within ε of Q (resp. $2Q$). To see this, observe that the subsystem $(Q, f\restriction_Q))$ is the limit of the sequence of subsystems $(\frac{2^n+1}{2^n}Q, f\restriction_{\frac{2^n+1}{2^n}Q})$ as $n \to \infty$. Let $\varepsilon > 0$ be given and let $n \in \mathbb{N}$ be such that $d_H(\frac{2^n+1}{2^n}Q, Q) < \varepsilon$. Let $\langle z_i \rangle_{i \in \mathbb{Z}}$ be a full trajectory for which $\omega(\langle z_i \rangle) = \frac{2^{n-1}+1}{2^{n-1}}Q$ and

$\alpha(\langle z_i \rangle) = \frac{2^n+1}{2^n}Q$. Then $d_H(\alpha(\langle z_i \rangle), Q) < \varepsilon$ and $d_H(\omega(\langle z_i \rangle), Q) < \varepsilon$. By noting that the subsystem $(2Q, f \restriction_{2Q}))$ is the limit of the sequence of subsystems $(\frac{2^{n+1}-1}{2^n}Q, f \restriction_{\frac{2^{n+1}-1}{2^n}Q})$ as $n \to \infty$, a similar argument may be given with regard to $2Q$. In particular it follows that $\overline{\alpha_f} = \overline{\omega_f} = \mathrm{ICT}_f$.

CHAPTER 6

CONCLUDING REMARKS

We will conclude by summarising some of the key results from each chapter while listing some questions that have arisen along the way. These questions might serve to motivate future research.

In Chapter 2, we kept our phase space as general as possible whilst studying sensitivity, topological equicontinuity, and even continuity in dynamical systems. In doing so, we showed (Theorem 2.2.10) that a system on a Hausdorff space is transitive precisely when there are no trivial equicontinuity pairs. We went on to show (Corollary 2.2.25) that minimal systems are either topologically equicontinuous or Hausdorff sensitive when the phase space is T_3, while (Theorem 2.6.3) transitive systems on a compact Haudorff phase space are either equicontinuous or eventually sensitive. Several questions were raised during this chapter which we recall here:

- Question 2.2.27: Does there exist a transitive system (X, f), where X is a Hausdorff space, with a non-trivial equicontinuity pair (x, y) but where x is not a topological equicontinuity point?
- Question 2.4.15: For a dynamical system (X, f), where X is a Hausdorff uniform space, is splitting distinct from Auslander–Yorke chaos?
- Question 2.6.5: Does there exist a transitive compact Hausdorff system which is neither sensitive nor has any transitive points?

Of particular interest to us is Question 2.2.27 (recall that we answer in the positive the analogous question regarding a non-trivial even continuity pair (Theorem 2.3.14)).

In Chapter 3, in the compact Hausdorff setting, we gave results on the preservation of various notions of shadowing in under inverse limits, products, factor maps, and the induced maps for symmetric products and hyperspaces. We were successful, for the most part, in either proving preservation or else constructing a counterexample. We did, however, have particular difficulty with some of the limit shadowing variants. Consequently, the following questions are still left open:

(1) Is limit shadowing (resp. s-limit shadowing) preserved by the induced hyperspatial system?

(2) Does an inverse limit system comprised of systems with limit shadowing (resp. s-limit shadowing; resp. h-shadowing) itself have limit shadowing (resp. s-limit shadowing; resp. h-shadowing)?

(3) Under what conditions are h-shadowing and inverse shadowing preserved by semi-conjugacy?

In Chapter 4, in the compact Hausdorff setting, we showed that pseudo-orbits trap ω-limit sets in a neighbourhood of prescribed accuracy after a uniform time period. A consequence of this was a generalisation of a result of Pilyugin *et al.*: every compact Hausdorff dynamical system has the second weak shadowing property. We went on to characterise minimal systems in terms of pseudo-orbits and show that every minimal system exhibits a shadowing variant introduced by Good and Meddaugh [47], namely the strong orbital shadowing property.

In Chapter 5, in the compact metric setting, we introduced novel variants of shadowing which we used to characterise maps for which $\overline{\alpha_f} = \text{ICT}_f$, and for which $\alpha_f = \text{ICT}_f$ (Theorems 5.4.4 and 5.4.12 respectively). These were characterised by backward cofinal orbital shadowing and backward orbital limit shadowing respectively. We then characterised (Theorems 5.5.17 and 5.5.9) maps for which every element of ICT_f is equal to, or may be approximated by, the α-limit set and the ω-limit set of the same full trajectory. A particularly nice result was that shadowing is sufficient for the latter whilst shadowing together with expansivity gives the former: in particular, this means that shifts of finite type satisfy property P_e. Although in many ways this chapter completed the journey started by Good and Meddaugh in [47], there are several avenues which would be interesting to explore. For instance, we only touched upon γ-limit sets and it is clear that there are more questions to be answered here. For example, we might ask when is it the case that γ_f equals ICT_f? Or, since γ_f will usually contain the empty set, when does $\gamma_f \setminus \emptyset = \mathrm{ICT}_f$? The novel shadowing variants introduced in this chapter offer another possible avenue for future work. Of particular interest to us is the notion of two-sided orbital limit shadowing. Two-sided limit shadowing is amongst the strongest of the shadowing properties [24]. Whilst weaker, two-sided orbital limit shadowing appears *prima facie* to retain much of its strength. We conjecture, for example, that just as two-sided limit shadowing implies shadowing, so does two-sided orbital limit shadowing imply orbital shadowing. Furthermore, whilst we do not delve into the proofs here, the author has the following three results for compact metric systems.

Theorem 6.0.1. *Two-sided orbital limit shadowing is characterised by the following property: For any $A, B \in \mathrm{ICT}_f$ there exists a full trajectory $\langle x_i \rangle_{i\in\mathbb{Z}}$ such that $\alpha(\langle x_i \rangle) = A$ and $\omega(\langle x_i \rangle) = B$.*

Theorem 6.0.2. *If a surjective system has two-sided orbital limit shadowing then it is topologically transitive.*

Theorem 6.0.3. *There exist transitive systems which do not exhibit two-sided orbital limit shadowing.*

These results mean that two-sided orbital limit shadowing characterises a particularly beautiful subclass of transitive systems — those where all nonempty, closed, internally chain transitive sets are joined to one another pairwise by full trajectories. For such systems, each origin is a destination of every origin; each destination, an origin of every destination. All beginnings are ends of all beginnings; all ends, beginnings of all ends.

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