CENTRES OF CENTRALIZERS OF NILPOTENT ELEMENTS IN SIMPLE LIE SUPERALGEBRAS

by

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Abstract

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional simple basic classical Lie superalgebra over $\mathbb{C}$. Let G be the reductive algebraic group over $\mathbb{C}$ such that $\mathrm{Lie}(G) = \mathfrak{g}_{\bar{0}}$. Suppose $e \in \mathfrak{g}_{\bar{0}}$ is nilpotent. In this thesis, we calculate the centralizer $\mathfrak{g}^e$ of e in $\mathfrak{g}$ and its centre $\mathfrak{z}(\mathfrak{g}^e)$ especially. We begin by recalling basic notions of Lie algebras and Lie superalgebras, such as root system and Dynkin diagrams. Once this is achieved, we look into further detail about the structure of basic classical Lie superalgebras of type $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $G(3)$ and $F(4)$ to calculate bases for $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$. Note that for Lie superalgebras of type $A(n,n)$, we consider $\mathfrak{sl}(n|n)$ instead of $\mathfrak{psl}(n|n)$. For the above types of Lie superalgebras, we also determine the labelled Dynkin diagram with respect to e. After considering the structure of $\mathfrak{z}(\mathfrak{g}^e)$ under the adjoint action of G^e, we prove theorems relating the dimension of $(\mathfrak{z}(\mathfrak{g}^e))^{G^e}$ and the labelled Dynkin diagram.

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Contents

1 Introduction

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra over $\mathbb{C}$ and let G be the reductive algebraic group over $\mathbb{C}$ given as in Table 1.1. We have that $\mathrm{Lie}(G) = \mathfrak{g}_{\bar{0}}$ and there is a representation $\rho : G \to \mathrm{GL}(\mathfrak{g}_{\bar{1}})$ such that $d_\rho : \mathrm{Lie}(G) \to \mathfrak{gl}(\mathfrak{g}_{\bar{1}})$ determines the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.

Table 1.1: Groups G

Lie superalgebras $\mathfrak{g}$	Groups G
$\mathfrak{sl}(m\|n)$	$\{(A, B) : A \in \mathrm{GL}_m(\mathbb{C}), B \in \mathrm{GL}_n(\mathbb{C}) \text{ and } \det(A) = \det(B)\}$
$\mathfrak{osp}(m\|2n)$	$\mathrm{O}_m(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$
$D(2, 1; \alpha)$	$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$
$G(3)$	$\mathrm{SL}_2(\mathbb{C}) \times G_2$
$F(4)$	$\mathrm{SL}_2(\mathbb{C}) \times \mathrm{Spin}_7(\mathbb{C})$

More generally, if $\mathfrak{g}$ is a direct sum of the above Lie superalgebras, we define G similarly as in Table 1.1.

Let $e \in \mathfrak{g}_{\bar{0}}$ be nilpotent. Recall that the centralizer $\mathfrak{g}^e$ of e in $\mathfrak{g}$ is given by $\mathfrak{g}^e = \{x \in \mathfrak{g} : [x, e] = 0\}$ and the centre $\mathfrak{z}(\mathfrak{g}^e)$ of centralizer of e in $\mathfrak{g}$ is defined to be $\mathfrak{z}(\mathfrak{g}^e) = \{x \in \mathfrak{g}^e : [x, y] = 0 \text{ for all } y \in \mathfrak{g}^e\}$. In this thesis, we investigate the centralizer $\mathfrak{g}^e$ of e in $\mathfrak{g}$, especially its centre $\mathfrak{z}(\mathfrak{g}^e)$. In particular, we determine bases of $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$. Write $G^e = \{g \in G : geg^{-1} = e\}$ for the centralizer of e in G, we also find a basis for $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \{x \in \mathfrak{z}(\mathfrak{g}^e) : gxg^{-1} = x \text{ for all } g \in G^e\}$.

The study of G^u for G a simple algebraic group and $u \in G$ a unipotent element is closely related problem to $\mathfrak{g}^e$. Work on G^u dates back to 1966, when Springer considered the centralizer G^u in [26]. Many mathematicians studied G^u for different types of G

after that, the reader is referred to the introduction of [18] for an overview of the other research of G^u. Seitz [24] pointed out the dimension of $Z(G^u)$ is of considerable interest. In [18], Lawther–Testerman studied the centralizer G^u, especially its centre $Z(G^u)$ and determined the dimension of the Lie algebra of $Z(G^u)$ over a field of characteristic 0 or a good prime. Using a G-equivariant homeomorphism, Lawther–Testerman worked with a nilpotent element $e \in \mathrm{Lie}(G)$ rather than u. In [15], Jantzen gave an explicit account on the structure of $\mathfrak{g}^e$ for classical Lie algebras $\mathfrak{g}$. The study of the centre $\mathfrak{z}(\mathfrak{g}^e)$ for classical Lie algebras $\mathfrak{g}$ over a field of characteristic 0 was undertaken by Yakimova in [29] and Lawther–Testerman [18] made use of work of Yakimova in [29] to deal with classical cases.

To our best knowledge, there is a lot less study in this direction in the case of Lie superalgebras. Finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero were classified by V. G. Kac in [16]. After this classification, a wide range of relevant problems have drawn the attention of mathematicians. Centralizers of nilpotent elements e in Lie superalgebras $\mathfrak{g}$ for the case where $\mathfrak{g} = \mathfrak{gl}(m|n)$ was done in [28] over a field of prime characteristic. In [11], Hoyt claimed that the construction is identical in characteristic zero and further describe the construction of $\mathfrak{g}^e$ for $\mathfrak{g} = \mathfrak{osp}(m|2n)$. However, the dimension of $\mathfrak{z}(\mathfrak{g}^e)$ for basic classical Lie superalgebras has not previously been studied and we attempt to shed some light upon this mystery here.

Our results about $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$ can be viewed as Lie superalgebra versions of those obtained by Lawther and Testerman in [18]. They obtain four theorems about $Z(G^u)$ as a consequence of their work. In this thesis, we aim to obtain analogues of Theorems 2–4 in [18] for Lie superalgebras in Table 1.1. We view $(\mathfrak{z}(\mathfrak{g}^e))^{G^e}$ as the correct replacement for $Z(G^u)$ since $\mathrm{Lie}(Z(G^e)) = (\mathfrak{z}(\mathfrak{g}^e))^{G^e}$ for a field of characteristic zero. Note that for $\mathfrak{g} = \mathfrak{sl}(m|n)$ or $D(2,1;\alpha)$, we have $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \mathfrak{z}(\mathfrak{g}^e)$.

In addition to computing bases for $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$, we also determine the labelled Dynkin diagram with respect to e in this thesis. A full definition of the labelled Dynkin diagram with respect to e is given in Section 4.3. Note that e lies in an $\mathfrak{sl}(2)$-triple $\{e, h, f\} \subseteq \mathfrak{g}_{\bar{0}}$ by Jacobson–Morozov Theorem. We use h to determine the labelled Dynkin diagram with respect to e. In contrary to Lie algebra case, in general e determines more than one labelled Dynkin diagram.

Write $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(j)$ as its ad$h$-eigenspace decomposition, we can decompose $\mathfrak{g}^e$ into the direct sum of ad$h$-eigenspaces, i.e. $\mathfrak{g}^e = \bigoplus_{j\geq 0} \mathfrak{g}^e(j)$. For $\mathfrak{g} = D(2,1;\alpha)$, $G(3)$ or $F(4)$, we also calculate the $\mathfrak{g}^e(0)$-module structure on each $\mathfrak{g}^e(j)$ for $j > 0$.

In the rest of this introduction, we give a more detailed survey of our results.

Fix $\varDelta$ to be a labelled Dynkin diagram with respect to e. Let $n_i(\varDelta)$ be the number of nodes which have labels equal to i in $\varDelta$. As a consequence of our calculations, we observe that the choice of $\varDelta$ does not affect the following theorems and labels in $\varDelta$ can only be $0, 1$ or 2.

We first consider the case where $\varDelta$ only has even labels.

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(m|2n)$, $D(2,1;\alpha)$, $G(3)$ or $F(4)$ and $e \in \mathfrak{g}_{\bar{0}}$ be nilpotent. Let G be the reductive algebraic group over $\mathbb{C}$ defined as in Table 1.1. Assume $\varDelta$ has no label equal to 1, then*

$$\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = n_2(\varDelta) = \dim \mathfrak{z}(\mathfrak{g}^h)$$

except for $\mathfrak{g} = \mathfrak{sl}(m|n)$ and $m = n$, in which case we have $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - 1 = n_2(\varDelta) = \dim \mathfrak{z}(\mathfrak{g}^h)$.

Our next result gives a more general result relating $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e}$ and $\varDelta$. In this statement we use notation for nilpotent elements as introduced later in Subsections 6.2.1, 6.3.1 and 6.4.1 respectively.

Theorem 1.2. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a Lie superalgebra* $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(m|2n)$, $D(2,1;\alpha)$, $G(3)$ *or* $F(4)$ *and* $e \in \mathfrak{g}_{\bar{0}}$ *be nilpotent. Let* G *be the reductive algebraic group over* $\mathbb{C}$ *defined as in Table 1.1. Let* $a_1, \dots, a_l$ *be the labels in* Δ*. Then*

$$\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = \left\lceil \frac{1}{2} \sum_{i=1}^{l} a_i \right\rceil + \varepsilon$$

where the value of ε *is equal to* 0 *with the following exceptions:* $\varepsilon = 1$ *when* $\mathfrak{g} = \mathfrak{sl}(n|n)$; $\varepsilon = -1$ *when* $\mathfrak{g} = D(2,1;\alpha)$, $e = E_1 + E_2 + E_3$ *or* $\mathfrak{g} = F(4)$, $e = E + e_{(7)}$.

Theorem 1.1 is subsumed by a more general result as stated in Theorem 1.3 and the proof of which involves more techniques. In order to state Theorem 1.3, we require some notations first which will be introduced later in Subsections 5.1.6 and 5.2.10. We define the sub-labelled Dynkin diagram Δ_0 to be the 2-*free core of* Δ where Δ_0 obtained by removing all nodes with labels equal to 2 from Δ. For $\mathfrak{g} = \mathfrak{sl}(m|n), m \neq n$ (resp. $\mathfrak{g} = \mathfrak{osp}(m|2n)$), let λ be a partition of $(m|n)$ (resp. $(m|2n)$) and let P be the Dynkin pyramid (resp. ortho-symplectic Dynkin pyramid) of shape λ which will be defined in Subsection 5.1.2 (resp. Subsection 5.2.2). Let r_i (resp. s_i) be the number of boxes on the ith column with parity $\bar{0}$ (resp. $\bar{1}$) in P and $k \geq 0$ be minimal such that the kth column in P contains no boxes. Then we define τ to be:

(1) the number of i such that $r_i = s_i \neq 0$ for all $i > k$ or $i < -k$ when $\mathfrak{g} = \mathfrak{sl}(m|n)$;

(2) the number of i such that $r_i = s_i \neq 0$ for all $i > k$ when $\mathfrak{g} = \mathfrak{osp}(m|2n)$.

We also define

$$\nu_0 = \begin{cases} 0 & \text{if } \mathfrak{g} = \mathfrak{sl}(m|n), \sum_{|i|<k} r_i \neq \sum_{|i|<k} s_i \text{ or } \mathfrak{g} = \mathfrak{osp}(m|2n); \\ 1 & \text{if } \mathfrak{g} = \mathfrak{sl}(m|n), \sum_{|i|<k} r_i = \sum_{|i|<k} s_i. \end{cases}$$

Theorem 1.3. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a Lie superalgebra* $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(m|2n)$, $D(2,1;\alpha)$, $G(3)$ *or* $F(4)$ *and* $e \in \mathfrak{g}_{\bar{0}}$ *be nilpotent. Let* G *be the reductive algebraic group over* $\mathbb{C}$ *defined as in Table 1.1. Let* Δ_0 *be the 2-free core of* Δ*. Denote by* $\mathfrak{g}_0$ *the subalgebra of* $\mathfrak{g}$ *generated by the root vectors corresponding to the simple roots in* Δ_0*. Let* G_0 *be the reductive algebraic group with respect to* $(\mathfrak{g}_0)_{\bar{0}}$*. Then* $\mathfrak{g}_0$ *is a direct sum of Lie superalgebras and there exists a nilpotent orbit in* $(\mathfrak{g}_0)_{\bar{0}}$ *having labelled Dynkin diagram* Δ_0*. Suppose* $e_0 \in (\mathfrak{g}_0)_{\bar{0}}$ *is a representative of this orbit, then*

(1). $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = n_2(\Delta)$ *for all* $\mathfrak{g}$*;*

(2). When $\mathfrak{g} = D(2,1;\alpha)$, $G(3)$ *or* $F(4)$*, we have that* $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = n_2(\Delta)$*;*

(3). When $\mathfrak{g} = \mathfrak{sl}(m|n), m \neq n$ *or* $\mathfrak{osp}(m|2n)$*, then* $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = n_2(\Delta) - \tau - \nu_0$*;*

(4). When $\mathfrak{g} = \mathfrak{sl}(n|n)$*, then* $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = n_2(\Delta) + 1 - \tau - \nu_0$.

The thesis is organized as follows. In Chapter 2, we recall some fundamental concepts of Lie algebras that we require, e.g. homomorphisms, modules and solvable and nilpotent Lie algebras. We also introduce some basic background about linear algebraic groups in Chapter 2. We further recall some basic vocabulary of Lie superalgebras and the classification of simple Lie superalgebras based on the work of V. G. Kac in Chapter 3. Concepts of root space decomposition, labelled Dynkin diagrams and so on are introduced in Chapter 4. In Chapter 5, we describe the structure of Lie superalgebras $\mathfrak{g} = A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$ and explain explicitly the centralizers $\mathfrak{g}^e$ and centres $\mathfrak{z}(\mathfrak{g}^e)$ of centralizers of nilpotent even elements e in $\mathfrak{g}$. Note that for $\mathfrak{g} = A(m, n)$ and $m = n$, we consider $\mathfrak{sl}(n|n)$ instead of $\mathfrak{psl}(n|n)$. In Chapter 6, we recall the structure of exceptional Lie superalgebras and determine $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$. For the above types of Lie superalgebras, we also determine the labelled Dynkin diagrams with respect to e.

2 Lie algebras

We begin by introducing the fundamental definitions of Lie algebras and provide typical examples related to what we shall refer. The concepts of homomorphisms, representations and modules will be introduced after that. We also give the classification of finite-dimensional simple Lie algebras and define nilpotent elements for classical Lie algebras. All vocabulary and detailed proofs mentioned in this chapter could be found in [7] and [25]. Throughout this chapter we work over the field of complex numbers $\mathbb{C}$.

2.1 Basic definitions

2.1.1 Definition of Lie algebras

Definition 2.1. A *Lie algebra* over $\mathbb{C}$ is a $\mathbb{C}$-vector space L, together with a bilinear Lie bracket $L \times L \longrightarrow L$ satisfying the following properties:

(L1) Alternativity: $[x, x] = 0$, for all $x \in L$;

(L2) The Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in L$.

Applying bilinearity to expand $[x + y, x + y]$, together with alternativity we get $[x, y] + [y, x] = 0$, which implies $[x, y] = -[y, x]$ for all $x, y \in L$. Below we give some typical examples of Lie algebras.

Example 2.2. 1. Any vector space L with the Lie bracket $[x, y] = 0$ for all $x, y \in L$ is a Lie algebra and is an *abelian Lie algebra.*

2. The *general Lie algebra* $\mathfrak{gl}(n, \mathbb{C})$ is the vector space of all $n \times n$ matrices over $\mathbb{C}$ along

with the Lie bracket defined by

$$[x, y] := xy - yx, \tag{2.1}$$

where xy is the product of the matrices x and y.

Clearly alternativity holds as $[x, x] = xx - xx = 0$ for all $n \times n$ matrices. Bilinearity holds because matrices form a ring. The Jacobi identity can be shown through simple calculation.

3. Let V be a finite-dimensional vector space over $\mathbb{C}$. Define $\mathrm{End}(V)$ to be the vector space of linear maps from V to V. Note that for $x, y \in \mathrm{End}(V)$, the product xy is defined by $(xy)(v) = x(y(v))$ for $v \in V$. Then $\mathrm{End}(V)$ is a Lie algebra with bracket $[x, y] = xy - yx$ according to Example 2.2.2. It is denoted by $\mathfrak{gl}(V)$. Moreover, when $\dim V = n$, we write $\mathfrak{gl}(n, \mathbb{C})$ for $\mathfrak{gl}(V)$.

4. The subspace $\mathfrak{sl}(n, \mathbb{C})$ of $\mathfrak{gl}(n, \mathbb{C})$ which consisting of all matrices of trace 0 is called the *special linear algebra*. Since $xy - yx$ has trace 0 for any square matrices x and y, the Lie bracket defined in (2.1) also gives a Lie algebra structure on $\mathfrak{sl}(n, \mathbb{C})$. In particular, we know that $\mathfrak{sl}(2, \mathbb{C})$ has a basis $\{e, h, f\}$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2.1.2 Subalgebras and ideals

For a given Lie algebra L, a *Lie subalgebra* of L is defined to be a vector subspace $S \subseteq L$ satisfying $[x, y] \in S$ for any $x, y \in S$. We also define an *ideal* I of L to be a vector subspace $I \subseteq L$ such that $[x, y] \in I$ for any $x \in I, y \in L$.

An ideal is a subalgebra, but the converse is not always true. For example, the vector space $\mathfrak{b}(n, \mathbb{C})$ consisting of all upper triangular matrices is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ but not an ideal when $n \geq 2$. To see this, let e_{ij} be the $n \times n$ matrix with a 1 in the ij-th position and all other entries are 0. Then we have $e_{11} \in \mathfrak{b}(n, \mathbb{C})$ and $e_{21} \in \mathfrak{gl}(n, \mathbb{C})$ but $[e_{21}, e_{11}] = e_{21} \notin \mathfrak{b}(n, \mathbb{C})$.

A Lie algebra L is an ideal of itself and $\{0\}$ is an ideal of L. They are called the *trivial ideals* of L. An important example of ideals is the *centre* of L which is defined by

$$\mathfrak{z}(L) := \{x \in L : [x, y] = 0 \text{ for all } y \in L\}.$$

We can see that $L = \mathfrak{z}(L)$ if and only if L is abelian.

Lemma 2.3. Let I and J be ideals of a Lie algebra L. Define $[I, J] := \text{Span}\{[x, y] : x \in I, y \in J\}$. Then $[I, J]$ is an ideal of L.

Proof. Let $a \in [I, J]$. We know that $a = \sum a_i [x_i, y_i]$ where the a_i are scalars and $x_i \in I$ and $y_i \in J$. For any $b \in L$, we have

$$\begin{aligned} [a, b] &= [\sum a_i [x_i, y_i], b] = \sum a_i [[x_i, y_i], b] \\ &= \sum a_i ([x_i, [y_i, b]] + [[x_i, b], y_i]) \end{aligned}$$

Since $[x_i, [y_i, b]]$,$[[x_i, b], y_i] \in [I, J]$, we deduce $[[x_i, y_i], b] \in [I, J]$ and hence $[a, b] \in [I, J]$. Therefore, $[I, J]$ is an ideal of L. □

In particular, when $I = J = L$, we denote $[L, L]$ by L' and call it the *derived algebra* of L.

2.1.3 Quotient Lie algebras

Let $a \in L$ and let I be an ideal of L, recall that the *coset* of I with respect to a is $a + I := \{a + x : x \in I\}$ and the quotient vector space is $L/I := \{a + I : a \in L\}$. A Lie algebra structure on L/I is defined by $[u + I, v + I] = [u, v] + I$ for any $u, v \in L$. To make sure that the Lie bracket on L/I is well defined, we suppose that $u_1 + I = u_2 + I$ and $v_1 + I = v_2 + I$. Hence $u_2 - u_1 \in I, v_2 - v_1 \in I$. Then

$$\begin{aligned}[u_2, v_2] &= [u_1 + (u_2 - u_1), v_1 + (v_2 - v_1)] \\ &= [u_1, v_1] + [u_2 - u_1, v_1] + [u_1, v_2 - v_1] + [u_2 - u_1, v_2 - v_1].\end{aligned}$$

As $[u_2 - u_1, v_1], [u_1, v_2 - v_1]$ and $[u_2 - u_1, v_2 - v_1]$ all lie in I, we deduce that $[u_2, v_2] + I = [u_1, v_1] + I$. The above proof implies that $[u, v] + I$ depends only on the cosets containing u and v, as required.

2.2 Homomorphisms

Definition 2.4. Let L_1 and L_2 be Lie algebras over $\mathbb{C}$. A *homomorphism* is a linear map $\phi : L_1 \to L_2$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in L_1$.

We say that ϕ is an *isomorphism* if ϕ is a bijection. A significant homomorphism is the *adjoint homomorphism* $\text{ad} : L \to \mathfrak{gl}(L)$, which is defined by $(\text{ad}x)(y) := [x, y]$ for all $x, y \in L$. Note that the kernel of ad is $\mathfrak{z}(L)$.

2.3 Modules and representations

2.3.1 Modules

Let L be a Lie algebra and V be a finite-dimensional vector space over $\mathbb{C}$.

Definition 2.5. An *L-module* is a finite-dimensional vector space V over $\mathbb{C}$ equipped with an operation $\cdot$

$$L \times V \to V : (x, v) \mapsto x\cdot v$$

satisfying the following axioms:

(M1) $(\lambda x + \mu y)\cdot v = \lambda(x\cdot v) + \mu(y\cdot v)$,

(M2) $x\cdot (\lambda v + \mu w) = \lambda(x\cdot v) + \mu(x\cdot w)$,

(M3) $[x, y] \cdot v = x\cdot (y\cdot v) - y\cdot (x\cdot v)$

for all $x, y \in L$, $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$.

We say W is a *submodule* of V if W is a subspace of V and for all $x \in L, w \in W$, we have $x\cdot w \in W$. An L-module V is said to be *irreducible* (or simple) if $V \neq \{0\}$ and its only submodules are $\{0\}$ and V. On the other hand, V is said to be *completely reducible* if it can be expressed as a direct sum of irreducible L-modules, i.e. $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ where V_i for $1 \leq i \leq n$ are irreducible L-modules.

2.3.2 Representations

Definition 2.6. A *representation* of L over $\mathbb{C}$ is a Lie algebra homomorphism $\rho : L \to \mathfrak{gl}(V)$.

Remark 2.7. A representation of a Lie algebra L over $\mathbb{C}$ is equivalent to the module structure on L. Given a representation we can define a module and vice versa.

The representation ρ is said to be *faithful* if it is injective.

Example 2.8. We already seen the adjoint homomorphism

$$\mathrm{ad} : L \to \mathfrak{gl}(L),\ \ (\mathrm{ad} x)\,(y) := [x, y]\,.$$

So ad gives the *adjoint representation* of L on L. Moreover, a Lie algebra L is called *reductive* if the adjoint representation of L is a direct sum of irreducible representations.

2.3.3 Representation theory of $\mathfrak{sl}(2)$

In order to introduce the main theorem of this subsection, we start by constructing irreducible representations of $\mathfrak{sl}(2)$ following [7, Chapter 8]. Given the vector space $\mathbb{C}[X, Y]$ of polynomials in two variables X and Y with complex coefficients. For each non-negative $d \in \mathbb{Z}$, let $V(d)$ be the subspace of homogeneous polynomials in X and Y of degree d. Note that $V(d)$ is the $(d+1)$-dimensional vector space with basis $\{X^d, X^{d-1}Y, ..., XY^{d-1}, Y^d\}$. From Example 2.2.4, we know that $\mathfrak{sl}(2)$ has basis $\{e, h, f\}$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The vector space $V(d)$ is an $\mathfrak{sl}(2)$-module by defining a Lie algebra homomorphism $\phi : \mathfrak{sl}(2) \to \mathfrak{gl}\left(V(d)\right)$ where

$$\phi(e) := X\frac{\partial}{\partial Y}, \phi(f) := Y\frac{\partial}{\partial X}, \text{ and } \phi(h) := X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Clearly ϕ is linear by construction. The detailed process of checking ϕ also preserves Lie brackets is covered in [7, page 68]. Hence ϕ is a representation of $\mathfrak{sl}(2)$ and $V(d)$ is an $\mathfrak{sl}(2)$-module.

With the above concept, we are now able to give the classification of finite-dimensional irreducible $\mathfrak{sl}(2)$-modules in Theorem 2.9.

Theorem 2.9. Any finite-dimensional irreducible $\mathfrak{sl}(2)$-module is isomorphic to $V(d)$ for some $d \in \mathbb{Z}$ and $d \geq 0$.

A fundamental result in the representation theory of Lie algebras is that any finite-dimensional L-module for L a complex semisimple Lie algebra is completely reducible (a semisimple Lie algebra is defined in Section 2.4). This is known as Weyl's Theorem and is covered in [7, Appendix B]. More specifically, any finite-dimensional $\mathfrak{sl}(2)$-module is completely reducible. Then we obtain the following corollary.

Corollary 2.10. Let V be a finite-dimensional representation of $\mathfrak{sl}(2)$ and $v \in V$ is an h-eigenvector of eigenvalue d such that $e \cdot v = 0$. Then the submodule of V generated by v is isomorphic to $V(d)$.

Such a vector $v \in V$ described in above corollary is called a *highest weight vector* and the eigenvalue d is known as a *highest weight*.

2.4 Classification of finite-dimensional simple Lie algebras

Recall that a *simple* Lie algebra is a non-abelian Lie algebra which has no ideals other than 0 and itself. A Lie algebra is *semisimple* if it is a direct sum of simple Lie algebras. The finite-dimensional simple Lie algebras have been classified by Cartan and detailed explanation is covered in [12, Section 11.4]. There are four types of classical Lie algebras and five exceptional Lie algebras. The main result of Cartan can be stated as the following theorem.

Theorem 2.11. Except five exceptional Lie algebras, any finite-dimensional simple Lie algebra over $\mathbb{C}$ is isomorphic to one of the following classical Lie algebras:

$$A_l = \mathfrak{sl}(l+1, \mathbb{C}) \text{ for } l \geq 1,$$

$$B_l = \mathfrak{so}(2l+1, \mathbb{C}) \text{ for } l \geq 1,$$

$$C_l = \mathfrak{sp}(2l, \mathbb{C}) \text{ for } l \geq 1,$$

$$D_l = \mathfrak{so}(2l, \mathbb{C}) \text{ for } l \geq 2.$$

We have seen A_l in Example 2.2.4. In order to define Lie algebras of type B_l, C_l and D_l, we let $\mathfrak{gl}_J(l, \mathbb{C}) = \{x \in \mathfrak{gl}(l, \mathbb{C}) : x^t J = -Jx\}$ where J is an $l \times l$ matrix with entries in $\mathbb{C}$. Then we define

$$\mathfrak{so}(2l+1, \mathbb{C}) = \mathfrak{gl}_J(2l+1, \mathbb{C}) \text{ when } J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

$$\mathfrak{sp}(2l,\mathbb{C}) = \mathfrak{gl}_J(2l,\mathbb{C}) \text{ when } J = \begin{pmatrix} &&&&&1\\ &&&&\cdot^{\cdot^{\cdot}}&\\ &&&1&&\\ &&-1&&&\\ &\cdot^{\cdot^{\cdot}}&&&&\\ -1&&&&& \end{pmatrix},$$

$$\text{and } \mathfrak{so}(2l,\mathbb{C}) = \mathfrak{gl}_J(2l,\mathbb{C}) \text{ when } J = \begin{pmatrix} &&1\\ &\cdot^{\cdot^{\cdot}}&\\ 1&& \end{pmatrix}.$$

The five exceptional Lie algebras are E_6, E_7, E_8, F_4, G_2 which have dimensions 78, 133, 248, 52 and 14 respectively.

2.5 Nilpotent elements in classical Lie algebras

Let V be a vector space over $\mathbb{C}$ and $\dim V = n$. Let L be a Lie algebra of a classical group G. Recall that an element $x \in L$ is called *nilpotent* if x belongs to the derived subalgebra $[L, L]$ and there exists some positive integer n such that $(\mathrm{ad}x)^n = 0$. An element $x \in L$ is called *semisimple* if x is diagonalisable.

According to [15, Theorems 1.1 and 1.4], two elements in L belong to the same G-orbit if and only if they have the same partition. This implies that the nilpotent G-orbits in L are parameterized by the partitions of n. We explain how we get representatives of each orbits below and [15, Theorems 1.1 and 1.4] suggest this gives a one-to-one classification of nilpotent orbits.

We first consider the case where $L = \mathfrak{gl}(V)$. Let J_k be the $(k \times k)$-matrix with an integer $k > 0$ such that the $(j, j+1)$ entries are equal to 1 for $1 \leq j < k$ and all other entries are 0. For example, we have that

$$J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

According to [15, Chapter 1], given a nilpotent element $e \in L$, there exists a basis for V such that the matrix of e with respect to this basis is of the form

$$\begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_r} \end{pmatrix}$$

where $\sum_{i=1}^{r} \lambda_i = n$ and we can assume that $\lambda_1 \geq \cdots \geq \lambda_r$. In fact, there exists $v_1, \ldots, v_r \in V$ such that the vectors $e^j v_i$ with $1 \leq i \leq r$, $0 \leq j \leq \lambda_i - 1$ form a basis for V. Therefore, we can associate a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n to a nilpotent element $e \in L$ which gives the sizes of its Jordan block. We call the partition λ to be the *Jordan type* of e. We define an element $x \in L$ to be *regular* if its Jordan normal form contains a single Jordan block for each eigenvalue.

For $L = \mathfrak{so}(V)$ or $\mathfrak{sp}(V)$, a complete classification of the nilpotent orbits in L can be found in [15, Section 1.6]. More precisely, write $\lambda = (1^{r_1}, 2^{r_2}, 3^{r_3}, \ldots)$ to be a partition of n, then for $L = \mathfrak{so}(V)$ (resp. $L = \mathfrak{sp}(V)$), there exists a nilpotent element in L with partition λ if and only if for all even (resp. odd) i, we have that r_i is even. For example, if $m = 5$, then the partitions occurring in the orthogonal case are

$$\{(5), (3, 1^2), (2^2, 1), (1^5)\};$$

if $2n = 6$ then the partitions occurring in the symplectic case are

$$\{(6), (4, 2), (4, 1^2), (3^2), (2^3), (2^2, 1^2), (2, 1^4), (1^6)\}.$$

2.6 The Jacobson–Morozov theorem

Let L be a semisimple Lie algebra and $e \in L$ be a nilpotent element. Let us take $G = \mathrm{Ad}L$ to be the adjoint group of L. We also define an $\mathfrak{sl}(2)$-*triple* to be a triple of elements $\{e, h, f\}$ of a Lie algebra that satisfies

$$[h, e] = 2e,\ \ [h, f] = -2f \text{ and } \ [e, f] = h.$$

Theorem 2.12. [5, Section 3.3](Jacobson–Morozov) Let L be a semisimple Lie algebra. For any nonzero nilpotent element e in L, there exists $h, f \in L$ such that $\{e, h, f\}$ is an $\mathfrak{sl}(2)$-triple.

Note that this standard $\mathfrak{sl}(2)$-triple is unique up to conjugacy by the centralizer G^e of e in G, which is covered in [17, Section 3.5].

2.7 Linear algebraic groups

In order to consider adjoint action of linear algebraic groups on Lie superalgebras in Chapters 5 and 6, we recall some elementary concepts of linear algebraic groups. In this section, detailed definitions and proofs can be found in the book of Springer [27, Chapters 1–2] and Macdonald [3].

2.7.1 The Zariski topology and irreducibility of topological spaces

Let $\mathbb{C}^n$ be the n-dimensional affine space over $\mathbb{C}$ and let $A = A_n = \mathbb{C}[X_1, \dots, X_n]$, where $X_1, \dots, X_n$ are independent indeterminates over $\mathbb{C}$. Then the polynomial algebra A consists of all polynomials in the X_i with coefficients in $\mathbb{C}$ and can be viewed as $\mathbb{C}$-valued functions on $\mathbb{C}^n$.

Definition 2.13. An *affine algebraic variety* is a subset $\mathfrak{V}(S)$ of $\mathbb{C}^n$ defined by $\mathfrak{V}(S) = \{x \in \mathbb{C}^n : f(x) = 0 \text{ for all } f \in S\}$ where S is a set of polynomials in A.

The operation $\mathfrak{V}$ has the following properties in which I_i are arbitrary subsets of A.
(1) $\mathfrak{V}(A) = \emptyset$ and $\mathfrak{V}(\emptyset) = \mathbb{C}^n$;
(2) $\mathfrak{V}(I_1) \cup \mathfrak{V}(I_2) = \mathfrak{V}(I_1 I_2)$ where $I_1 I_2 = \{f_1 f_2 : f_1 \in I_1, f_2 \in I_2\}$;
(3) $\cap_{i \in J} \mathfrak{V}(I_i) = \mathfrak{V}(\sum_{i \in J} I_i)$ is closed for any index set J.
This system determines a topology in $\mathbb{C}^n$, which is called the *Zariski topology.* The induced topology on a subset $Y \subset \mathbb{C}^n$ is called the *Zariski topology of* Y.

A topological space Y is said to be *irreducible* if it is non-empty and it cannot be written as the union of two proper closed subsets. Otherwise, we say Y is *reducible* if it is not irreducible. In addition, Y is *disconnected* if there exist proper closed subsets U and V of Y such that $Y = U \cup V$ and $U \cap V = \emptyset$. Otherwise, Y is said to be *connected.* Note that irreducible implies connected, but the converse is not always true. For example,

Example 2.14. Let $\mathbb{C}[X_1, X_2]$ be polynomials on $\mathbb{C}^2$. Denote

$$\mathfrak{V}(X_1X_2) = \{(x_1, x_2) : x_1 = 0 \text{ or } x_2 = 0\}.$$

Then we have that $\mathfrak{V}(X_1X_2)$ is connected as it cannot be written as the union of two disjoint closed subsets. However, $\mathfrak{V}(X_1X_2)$ is reducible because $\mathfrak{V}(X_1X_2) = \mathfrak{V}(X_1) \cup \mathfrak{V}(X_2)$.

Definition 2.15. A topological space Y is said to be *Noetherian* if it satisfies the descending chain condition for closed subsets, that is to say, for any decreasing sequence $Y_1 \supseteq Y_2 \supseteq \dots$ of closed subsets of Y, there exists an integer m such that $Y_i = Y_m$ for all $i \geq m$.

The following proposition as in [27, Section 2.2] shows that a Noetherian topological space is the union of finitely many irreducible closed subsets.

Proposition 2.16. Let Y be a Noetherian topological space, then there exists a unique decomposition of Y such that

$$Y = Y_1 \cup \cdots \cup Y_m$$

where Y_i $(i = 1, \dots, m)$ are maximal irreducible closed subsets of Y and $Y_i \not\subset Y_j$ for all $i \neq j$.

The sets Y_i in Proposition 2.16 are called *irreducible components of* Y. The maximal connected subsets of Y are called the *connected components of* Y.

2.7.2 Linear algebraic groups and some basic results

Let V be a finite-dimensional vector space over $\mathbb{C}$. Let $\mathrm{GL}(V)$ be the set of invertible linear transformations from V to V. In particular, we write $\mathrm{GL}_n(\mathbb{C})$ for the set of $n \times n$ invertible matrices with coefficients in $\mathbb{C}$ and call it the *general linear group of rank* n. Denote the space of $n \times n$ matrices over $\mathbb{C}$ by $M_n(\mathbb{C})$. Recall that a function f on $M_n(\mathbb{C})$ is called a polynomial function if

$$f(y) = p\left(x_{11}(y), x_{12}(y), \ldots, x_{nn}(y)\right) \text{ for all } y \in M_n(\mathbb{C})$$

where $p \in \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]$ is a polynomial and x_{ij} are the matrix entry functions on $M_n(\mathbb{C})$.

Definition 2.17. A subgroup G of $\mathrm{GL}_n(\mathbb{C})$ is a *linear algebraic group* if there exists a set S of polynomial functions on $M_n(\mathbb{C})$ such that

$$G = \{g \in \mathrm{GL}_n(\mathbb{C}) : f(g) = 0 \text{ for all } f \in S\}.$$

A basic example of a linear algebraic group is $\mathrm{GL}_n(\mathbb{C})$ where we take the defining set S of relations to consist of the zero polynomial. Below we give more examples of linear algebraic groups.

Example 2.18. 1. The *special linear group* $\mathrm{SL}_n(\mathbb{C})$ is algebraic and defined as $\mathrm{SL}_n(\mathbb{C}) = \{x \subseteq \mathrm{M}_n(\mathbb{C}) : \det(x) = 1\}$.

2. The *orthogonal group* $\mathrm{O}_n(\mathbb{C})$ is algebraic and defined as $\mathrm{O}_n(\mathbb{C}) = \{x \subseteq \mathrm{M}_n(\mathbb{C}) :$

$x^t J_n x = J_n\}$ where x^t represents the transpose of x and

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

3. The *special orthogonal group* is algebraic and defined as $\mathrm{SO}_n(\mathbb{C}) = \mathrm{O}_n(\mathbb{C}) \cap \mathrm{SL}_n(\mathbb{C})$.

4. The *symplectic group* $\mathrm{Sp}_{2n}(\mathbb{C})$ is algebraic and defined as $\mathrm{Sp}_{2n}(\mathbb{C}) = \{x \subseteq \mathrm{M}_{2n}(\mathbb{C}) : x^t K_{2n} x = K_{2n}\}$ where

$$K_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}.$$

5. The group D_n of diagonal matrices where any matrix in D_n has non-zero determinant is an algebraic group.

6. The group of invertible upper triangular matrices $B_n = \{x \in \mathrm{GL}_n(\mathbb{C}) : x_{ij} = 0 \text{ for } i > j\}$ is an algebraic group.

Note that the groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ are connected. The group $\mathrm{O}_n(\mathbb{C})$ is not connected and is the disjoint union of two components, the orthogonal matrices with determinant 1 (i.e. $\mathrm{SO}_n(\mathbb{C})$) and the orthogonal matrices with determinant -1.

The following proposition implies that the notions of irreducibility and connectedness coincide for linear algebraic groups.

Proposition 2.19. Let G be a linear algebraic group. Then

(a) There exist a unique irreducible component G° of G containing the identity element 1_G and G° is a closed normal subgroup;

(b) G° has finite index and the irreducible components of G are cosets with respect to G°;

(c) G° is also the connected component of G that contains the identity element u.

Proof. (a) Let X and Y be irreducible components of G that contain the identity element 1_G. Let $\mu : G \times G \to G$ be multiplication map on G. Then we have that $XY = \mu(X \times Y)$ is irreducible by [27, Lemma 1.2.3]. we know that X is a maximal irreducible subset of G as it is an irreducible components of G. We also know that $X \subseteq XY$ since $u \in Y$. Hence, we have that $X = XY$. Similarly we can deduce that $Y = XY$. Therefore, X is the unique irreducible component of G that contains 1_G. In particular, taking $Y = X$ we get $XY = XX = X$. Thus for any element $x_1, x_2 \in X$, we have $x_1 x_2 \in X$. Hence, X is closed under multiplication.

Let $i : G \to G$ be inversion map. We have that $X^{-1} = i(X) = \{x^{-1} : x \in X\}$ is irreducible by Lemma [27, Lemma 1.2.3]. Furthermore, we claim that X^{-1} is a maximal irreducible subset of G. If there exist an irreducible subset Z of G such that $X^{-1} \subsetneqq Z$, then $X \subsetneqq Z^{-1}$ and Z^{-1} is irreducible. This contradicts to X is an irreducible component of G. Since $1_G \in X$, we know that $1_G \in X^{-1}$. Thus we have that $X^{-1} = X$. Hence, X is closed under inverse. Therefore, we have that X is a subgroup of G.

Now take any $g \in G$. Define the map $\lambda_g : G \to G$ which sends x to gx and $\rho_g : G \to G$ which sends x to xg for any $x \in X$. We have that λ_g and ρ_g are morphisms of affine varieties. Thus $\rho_{g^{-1}} \circ \lambda_g$ is a morphism and $(\rho_{g^{-1}} \circ \lambda_g)(X)$ is irreducible. Note that

$$(\rho_{g^{-1}} \circ \lambda_g)(X) = \rho_{g^{-1}}(\lambda_g(X)) = \rho_{g^{-1}}(gX) = gXg^{-1}.$$

In fact, we have that gXg^{-1} is a maximal irreducible subset of G because if there exist an irreducible subset Z of G such that $gXg^{-1} \subsetneqq Z$, then $X \subsetneqq g^{-1}Zg$ which is a contradiction. Furthermore, we have that $u = gug^{-1} \in gXg^{-1}$. Hence, we deduce that $gXg^{-1} = X$. Therefore, X is a closed normal subgroup of G.

(b) Next let $g \in G$, using the similar argument as above can show that gX is also an irreducible component of G. Thus we can write a decomposition of G such that

$$G = \bigcup_{g \in G} gX$$

where each gX is an irreducible component. Since G has finitely many irreducible components, there exist $g_1, g_2, \ldots, g_m \in G$ such that $G = \bigcup_{i=1}^{m} g_i X$. It follows that X has finite index in G and the cosets of X in G are the irreducible components of G.

(c) Since $g_i X \cap g_j X = \emptyset$ for $i \neq j$, we have that $\{g_i X : i = 1, \ldots, m\}$ are also the connected components of G.

□

3 Lie superalgebras

In this chapter we recall the concept of a $\mathbb{Z}_2$-graded vector superspace and the essential definitions of Lie superalgebras following [19, Chapter 1] and [4, Section 1.1]. We also recall the definition of a basic classical Lie superalgebra as defined in [14, Section 2.1] and give the classification of finite-dimensional simple Lie superalgebras based on [16, Section 4.2]. Let our field be the field of complex numbers $\mathbb{C}$.

3.1 Basic concepts

To start with, let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. We introduce a *$\mathbb{Z}_2$-graded vector superspace* V which is a direct sum of vector spaces $V_{\bar{0}}$ and $V_{\bar{1}}$, i.e. $V = V_{\bar{0}} \oplus V_{\bar{1}}$. An element in $V_{\bar{0}}$ is called *even* and an element in $V_{\bar{1}}$ is called *odd*. Any non-zero element of $V_{\bar{0}} \cup V_{\bar{1}}$ is said to be *homogeneous* and for $v \in V_{\bar{i}}$, we define the *parity* of v by $\bar{v} = \bar{i}$. A *subspace* of V is a vector superspace $U = U_{\bar{0}} \oplus U_{\bar{1}}$ such that $U_{\bar{i}} \subseteq V_{\bar{i}}$ for $\bar{i} \in \mathbb{Z}_2$.

Definition 3.1. A *superalgebra* A over $\mathbb{C}$ is a $\mathbb{Z}_2$-graded algebra with a direct sum decomposition $A = A_{\bar{0}} \oplus A_{\bar{1}}$, together with a bilinear multiplication satisfying $A_{\bar{i}}A_{\bar{j}} \subseteq A_{\bar{i}+\bar{j}}$ for $\bar{i},\ \bar{j} \in \mathbb{Z}_2$. *A supermodule* M over a superalgebra A is an A-module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ such that $A_{\bar{i}}M_{\bar{j}} \subseteq M_{\bar{i}+\bar{j}}$ for $\bar{i}, \bar{j} \in \mathbb{Z}_2$.

Now we are able to provide the definition of a Lie superalgebra.

Definition 3.2. A *Lie superalgebra* is a superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with multiplication given by a Lie superbracket $[\cdot, \cdot]$: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the following axioms:

(1) Graded skew-symmetry: $[x, y] = -(-1)^{\bar{x}\bar{y}} [y, x]$ for all homogeneous $x, y \in \mathfrak{g}$;

(2) Graded Jacobi identity: $(-1)^{\bar{x}\bar{z}} [x, [y, z]] + (-1)^{\bar{x}\bar{y}} [y, [z, x]] + (-1)^{\bar{y}\bar{z}} [z, [x, y]] = 0$ for all homogeneous $x, y, z \in \mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. For subspaces U and W of $\mathfrak{g}$, we denote $[U, W] = \mathrm{Span}\{[u, w] : u \in U, w \in W\}$. Then an *ideal* I of $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded subspace such that $[I, \mathfrak{g}] \subseteq I$. Note that representations and modules are also defined in the $\mathbb{Z}_2$-graded sense.

Example 3.3. 1. Given any associative superalgebra A, we can make A into a Lie superalgebra by letting

$$[a, b] = ab - (-1)^{\bar{a}\bar{b}} ba$$

for homogeneous a, $b \in A$ and extending the bracket by bilinearity to all elements.

2. Given any associative superalgebra A, an endomorphism $D \in \mathrm{End}(A)_{\bar{i}}$ where $\bar{i} \in \mathbb{Z}_2$ satisfying

$$D(xy) = D(x)y + (-1)^{\bar{i}\bar{x}} x D(y), \ x, y \in A$$

is called a *derivation* of degree $\bar{i}$. We denote by $\mathrm{Der}(A)_{\bar{i}}$ the space of derivation on A of degree $\bar{i}$. Then $\mathrm{Der}(A) = \mathrm{Der}(A)_{\bar{0}} \oplus \mathrm{Der}(A)_{\bar{1}}$ is a Lie superalgebra. To see this, it suffices to show that $\mathrm{Der}(A)$ is a subalgebra of $\mathrm{End}(A)$. For $D_1 \in \mathrm{Der}(A)_{\bar{i}}$, $D_2 \in \mathrm{Der}(A)_{\bar{j}}$ where $\bar{i}, \bar{j} \in \mathbb{Z}_2$ and homogeneous $x, y \in A$, we have

$$
\begin{aligned}
[D_1, D_2](xy) &= D_1(D_2(xy)) - (-1)^{\bar{i}\bar{j}} D_2(D_1(xy)) \\
&= D_1 \left(D_2(x)y + (-1)^{\bar{j}\bar{x}} x D_2(y) \right) - (-1)^{\bar{i}\bar{j}} D_2 \left(D_1(x)y + (-1)^{\bar{i}\bar{x}} x D_1(y) \right) \\
&= D_1 D_2(x)y + (-1)^{\bar{i}(\bar{j}+\bar{x})} D_2(x) D_1(y) + (-1)^{\bar{j}\bar{x}} D_1(x) D_2(y) \\
&+ (-1)^{(\bar{i}+\bar{j})\bar{x}} x D_1 D_2(y) - (-1)^{\bar{i}\bar{j}} D_2 D_1(x)y - (-1)^{2\bar{i}\bar{j}+\bar{j}\bar{x}} D_1(x) D_2(y) \\
&- (-1)^{\bar{i}(\bar{j}+\bar{x})} D_2(x) D_1(y) - (-1)^{\bar{i}\bar{j}+(\bar{i}+\bar{j})\bar{x}} x D_2 D_1(y) \\
&= \left(D_1 D_2(x)y - (-1)^{\bar{i}\bar{j}} D_2 D_1(x)y \right) \\
&+ (-1)^{(\bar{i}+\bar{j})\bar{x}} x \left(D_1 D_2(y) - (-1)^{\bar{i}\bar{j}} D_2 D_1(y) \right) \\
&= [D_1, D_2](x)y + (-1)^{(\bar{i}+\bar{j})\bar{x}} x \, [D_1, D_2] \, (y) \in \mathrm{Der}(A)_{\bar{i}+\bar{j}}.
\end{aligned}
$$

3. Let $\mathfrak{g}$ be a Lie superalgebra and define the *adjoint map* ad: $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by

$$\mathrm{ad}(x)(y) := [x, y] \,, \; x, y \in \mathfrak{g}.$$

Then ad is a homomorphism of Lie superalgebras because $\mathrm{End}(\mathfrak{g})$ is an associative superalgebra. Furthermore, the resulting action of $\mathfrak{g}$ on itself is called the *adjoint action*.

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, we recall the following properties as given in [4, Remark 1.5].

- The even part $\mathfrak{g}_{\bar{0}}$ is an ordinary Lie algebra, so if $\mathfrak{g}_{\bar{1}} = 0$, then $\mathfrak{g}$ is simply a Lie algebra.
- $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module under the adjoint action since $\mathrm{ad}|_{\mathfrak{g}_{\bar{0}}} : \mathfrak{g}_{\bar{0}} \to \mathrm{End}(\mathfrak{g}_{\bar{1}})$ is a homomorphism of Lie algebras.

- The Lie superbracket $[\cdot,\cdot]\colon \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ is a symmetric $\mathfrak{g}_{\bar{0}}$-module.

Note that the above three properties are sufficient for $\mathfrak{g}$ to be a Lie superalgebra.

3.2 Solvable, simple and semisimple Lie superalgebras

Let $\mathfrak{g}$ be a Lie superalgebra, the concepts of derived series and lower central series are defined in a similar way to Lie algebra case. We define the *derived series* $\mathfrak{g}^{(i)}$ of $\mathfrak{g}$ to be the sequence of ideals

$$\mathfrak{g}^{(0)} = \mathfrak{g};$$

$$\mathfrak{g}^{(i+1)} = \left[\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}\right] \text{ for } i \geq 0.$$

And the *lower central series* $\mathfrak{g}^i$ of $\mathfrak{g}$ is given by

$$\mathfrak{g}^0 = \mathfrak{g};$$

$$\mathfrak{g}^i = \left[\mathfrak{g}, \mathfrak{g}^{i-1}\right] \text{ for } i \geq 1.$$

Given a Lie superalgebra $\mathfrak{g}$, we say that $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(n)} = 0$ for large n and $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^n = 0$ for large n.

We also recall the definitions of simple and semisimple Lie superalgebras for further application.

Definition 3.4. A Lie superalgebra $\mathfrak{g}$ is *simple* if it is not abelian and the only $\mathbb{Z}_2$-graded ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$. A Lie superalgebra $\mathfrak{g}$ is said to be *semisimple* if it has no non-zero solvable ideals.

3.3 Basic classical Lie superalgebras

In this thesis, we mainly consider basic classical Lie superalgebras. To recall the explicit definition, we need the following concept as given in [14, Definition 2.1].

Definition 3.5. Let $\mathfrak{g}$ be a Lie superalgebra. For a bilinear form $(\cdot\,,\cdot\,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, we say that:

$(\cdot\,,\cdot\,)$ is *even* if $(\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}) = 0$ for $\bar{i}, \bar{j} \in \{\bar{0}, \bar{1}\}$ and $\bar{i} \neq \bar{j}$;

$(\cdot\,,\cdot\,)$ is *supersymmetric* if $(x, y) = (-1)^{\bar{x}\bar{y}}(y, x)$ for all $x, y \in \mathfrak{g}$;

$(\cdot\,,\cdot\,)$ is *invariant* if $([x, y], z) = (x, [y, z])$ for all $x, y, z \in \mathfrak{g}$;

$(\cdot\,,\cdot\,)$ is *non-degenerate* if $(x, y) = 0$ for all $y \in \mathfrak{g}$ implies that $x = 0$.

We now define a basic classical Lie superalgebra as given in [14, Definition 2.2].

Definition 3.6. A finite-dimensional simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called a *basic classical Lie superalgebra* if $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and there exists a non-degenerate even invariant supersymmetric bilinear form $(\cdot\,,\cdot\,)$ on $\mathfrak{g}$.

3.4 Classification of simple Lie superalgebras

Kac proved the following theorem about the classification of finite-dimensional complex simple Lie superalgebras in [16, Theorem 5].

Theorem 3.7. A finite-dimensional simple Lie superalgebra $\mathfrak{g}$ is either a simple Lie algebra or isomorphic to one of the following types:

1. $A(m,n) = \mathfrak{sl}(m+1|n+1)$ *for* $m, n \geq 0, m \neq n$;
2. $A(n,n) = \mathfrak{psl}(n+1|n+1)$ *for* $n \geq 1$;
3. $B(m,n) = \mathfrak{osp}(2m+1|2n)$ *for* $m \geq 0, n \geq 1$;
4. $C(n) = \mathfrak{osp}(2|2n)$ *for* $n \geq 1$;
5. $D(m,n) = \mathfrak{osp}(2m|2n)$ *for* $m \geq 2,\ n \geq 1$;
6. $D(2,1;\alpha), G(3),$ *and* $F(4)$;
7. Two infinite families denoted by $\mathfrak{p}(n)$ *and* $\mathfrak{q}(n)$ *for* $n \geq 2$;
8. *The Lie superalgebras of Cartan type* $\mathrm{W}(n)$, $\mathrm{S}(n)$, $\mathrm{H}(n)$, $\tilde{\mathrm{S}}(n)$.

The Lie superalgebras of types $A(m,n)$, $A(n,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$ will be described in Subsections 5.1.1 and 5.2.1. The Lie superalgebras of types $D(2,1;\alpha)$, $G(3)$ and $F(4)$ are called *the exceptional Lie superalgebras* and the structure of which are given in Subsections 6.2.1, 6.3.1 and 6.4.5 respectively. In particular, the above Lie superalgebras of type $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $G(3)$ and $F(4)$ are basic classical Lie superalgebras.

4 The root space decomposition and labelled Dynkin diagrams

In this chapter we recall the definitions and relevant properties of root space decomposition and root systems of Lie superalgebras as described in [16, Section 2.5.3] and [19, Section 3.2]. Then we illustrate how to construct Dynkin diagrams and labelled Dynkin diagrams of a Lie superalgebra, the formulas and explicit explanation in Sections 4.2–4.3 are mainly based on [9, Section 15] and [8, Chapter 2].

4.1 The root space decomposition and root systems

We first recall that a *Cartan subalgebra* H of a reductive Lie algebra L is a maximal abelian subalgebra of L such that every element $h \in H$ is semisimple. To construct a decomposition of L, let us fix a Cartan subalgebra H of a reductive Lie algebra L. Given $\alpha \in H^*$ where H^* is the dual space, let the corresponding weight space be $L_\alpha := \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}$.

Denote by Φ the set of non-zero $\alpha \in H^*$ for which $L_\alpha \neq 0$, the elements of Φ are called the *roots* of L. Then we can decompose L into weight spaces for H such that

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where $L_0 = \{x \in L : [h, x] = 0 \text{ for all } h \in H\}$ is the zero weight space which is equal to L^H. According to [7, Theorem 10.4], we know that $H = L^H$, this gives a *root space decomposition* $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$.

The definition of a Cartan subalgebra can be extended to the super case according to

[19, Section 3.2]. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional basic classical Lie superalgebra, a *Cartan subalgebra* $\mathfrak{h}$ is a self-centralizing nilpotent subalgebra of $\mathfrak{g}$. According to [9, Section 3], a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ reduces to the Cartan subalgebra of the even part $\mathfrak{g}_{\bar{0}}$. Then for $\alpha \in \mathfrak{h}^*$, we know that $\mathfrak{g}$ has a weight decomposition such that the corresponding weight spaces are:

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Then there is a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{h} = \mathfrak{g}_0 = \{x \in \mathfrak{g} : [h, x] = 0 \text{ for all } h \in \mathfrak{h}\} = \mathfrak{g}^{\mathfrak{h}}$. The set $\Phi = \{\alpha \in h^* : \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$ is called the *root system of* $\mathfrak{g}$. We say that $\mathfrak{g}_\alpha$ is the root space corresponding to root $\alpha \in \Phi$. Define the *even* and *odd roots* to be

$$\Phi_{\bar{0}} =: \{\alpha \in \Phi : \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\bar{0}}\}, \ \Phi_{\bar{1}} =: \{\alpha \in \Phi : \mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\bar{1}}\}.$$

Let $\mathfrak{g}$ be a basic classical Lie superalgebra, then the following conditions hold according to [16, Proposition 2.5.5].

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ if and only if $\alpha, \beta, \alpha + \beta \in \Phi$;

(2) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha + \beta \neq 0$;

(3) If $\alpha \in \Phi$, we have $-\alpha \in \Phi$;

(4) For $\alpha \in \Phi$, we have $2\alpha \in \Phi$ if and only if $\alpha \in \Phi_{\bar{1}}$ and $(\alpha, \alpha) \neq 0$.

Definition 4.1. A subset $\Phi^+ \subseteq \Phi$ is a *system of positive roots* if

(1) for each root $\alpha \in \Phi$ there is exactly one of $\alpha, -\alpha$ contained in Φ^+;

(2) For any two distinct roots $\alpha, \beta \in \Phi^+$, $\alpha + \beta \in \Phi$ implies that $\alpha + \beta \in \Phi^+$.

For the root system of $\mathfrak{g}$, a system of positive roots Φ^+ always exists and elements of $-\Phi^+$ forms a system of negative roots. Note that $\Phi^+ = \Phi^+_{\bar{0}} \cup \Phi^+_{\bar{1}}$. A subset $\Pi = \{\alpha_1, ..., \alpha_l\} \subseteq \Phi^+$ is called *a system of simple roots* if any root $\alpha_i \in \Pi$ cannot be written as the sum of two elements of Φ^+. Note that l does not depend on choice of Π and we call it the *rank* of $\mathfrak{g}$.

In general, a Lie superalgebra $\mathfrak{g}$ is not associated with only one system of positive roots. According to [8, Section 2], there are many inequivalent conjugacy classes of systems of positive roots and every system of positive roots determines a system of simple roots $\Pi = \{\alpha_1, ..., \alpha_l\}$ for some integer $l \geq 1$. For each conjugacy class of systems of positive roots, a simple root system can be transformed into an equivalent one under the transformation of the *Weyl group* W of $\mathfrak{g}$ where W is a subgroup generated by the orthogonal reflection w relative to the even roots such that

$$w_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$

where $\alpha \in \Phi_{\bar{0}}$ and $\beta \in \Phi$. According to [8, Section 2.3], we know that W can be extended to a larger group by adding transformations associated to odd root $\alpha \in \Phi_{\bar{1}}$ where for $(\alpha, \alpha) \neq 0$,

$$w_\alpha(\beta) = \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha;$$

for $(\alpha,\alpha)=0$, $w_\alpha(\alpha)=-\alpha$ and

$$w_\alpha(\beta)=\begin{cases}\beta+\alpha & \text{if } (\alpha,\beta)\neq 0;\\ \beta & \text{if } (\alpha,\beta)=0.\end{cases}$$

4.2 Dynkin diagrams

We next recall the concept of the Dynkin diagram as defined for example in [8, Section 2.2]. Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a finite-dimensional basic classical Lie superalgebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Decompose $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$ as the root space decomposition where $\Phi\subseteq\mathfrak{h}^*$. Fix a set of simple roots $\mathit{\Pi}=\{\alpha_1,...,\alpha_l\}$ and let $\{h_1,...,h_l\}$ be the generators of $\mathfrak{h}$. Recall that there exists a non-degenerate even invariant supersymmetric bilinear form $(\cdot\,,\cdot)$ on $\mathfrak{g}$. Using the invariance of $(\cdot\,,\cdot)$ we can check that $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ when $\alpha\neq-\beta$ and $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{h}$ for $\alpha,\beta\in\Phi$. Let $x_\alpha\in\mathfrak{g}_\alpha$, $x_\beta\in\mathfrak{g}_\beta$ and $h\in\mathfrak{h}$. Then $([x_\alpha,h],x_\beta)=(x_\alpha,[h,x_\beta])$. Note that $([x_\alpha,h],x_\beta)=-\alpha(h)(x_\alpha,x_\beta)$ and $(x_\alpha,[h,x_\beta])=\beta(h)(x_\alpha,x_\beta)$. Thus we get $(x_\alpha,x_\beta)=0$ unless $\alpha=-\beta$. Hence $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ and similarly $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{h}$. This means that $(\cdot\,,\cdot)$ restricts to a non-degenerate symmetric bilinear form on $\mathfrak{h}$. This gives an isomorphism ψ from $\mathfrak{h}$ to $\mathfrak{h}^*$ which provides a symmetric bilinear form on $\mathfrak{h}^*$ that is defined by $(\alpha,\beta)_{\mathfrak{h}^*}=(\psi^{-1}(\alpha),\psi^{-1}(\beta))$ for $\alpha,\beta\in\mathfrak{h}^*$.

The *Dynkin diagram* of a Lie superalgebra $\mathfrak{g}$ with a simple root system $\mathit{\Pi}$ is a graph where the vertices are labelled by $\mathit{\Pi}$ and there are μ_{ij} lines between the vertices labelled by simple roots α_i and α_j such that:

$$\mu_{ij} = \begin{cases} |(\alpha_i, \alpha_j)| & \text{if } (\alpha_i, \alpha_i) = (\alpha_j, \alpha_j) = 0, \\ \frac{2|(\alpha_i,\alpha_j)|}{min\{|(\alpha_i,\alpha_i)|,|(\alpha_j,\alpha_j)|\}} & \text{if } (\alpha_i, \alpha_i)(\alpha_j, \alpha_j) \neq 0, \\ \frac{2|(\alpha_i,\alpha_j)|}{min_{(\alpha_k,\alpha_k)\neq 0}|(\alpha_k,\alpha_k)|} & \text{if } (\alpha_i, \alpha_i) \neq 0,\ (\alpha_j, \alpha_j) = 0 \text{ and } \alpha_k \in \Phi. \end{cases} \tag{4.1}$$

We say a root $\alpha \in \Phi$ is *isotropic* if $(\alpha, \alpha) = 0$ and is *non-isotropic* if $(\alpha, \alpha) \neq 0$. We associate a white node $\bigcirc$ to each even root, a grey node $\otimes$ to each odd isotropic root and a black node $\bullet$ to each odd non-isotropic root. Moreover, when $\mu_{ij} > 1$, we put an arrow pointing from the vertex labelled by α_i to the vertex labelled by α_j if $(\alpha_i, \alpha_i)(\alpha_j, \alpha_j) \neq 0$ and $(\alpha_i, \alpha_i) > (\alpha_j, \alpha_j)$ or if $(\alpha_i, \alpha_i) = 0, (\alpha_j, \alpha_j) \neq 0$ and $|(\alpha_j, \alpha_j)| < 2$, or pointing from the vertex labelled by α_j to the vertex labelled by α_i if $(\alpha_i, \alpha_i) = 0, (\alpha_j, \alpha_j) \neq 0$ and $|(\alpha_j, \alpha_j)| > 2$. If the value of μ_{ij} is not a natural number, then we label the edge between vertices corresponding to roots α and β with μ_{ij} instead of drawing multiple lines between them. Since there exist many inequivalent conjugacy classes of systems of simple roots up to a transformation of the Weyl group W, there is more than one Dynkin diagram for $\mathfrak{g}$.

4.3 Labelled Dynkin diagrams

In order to define a labelled Dynkin diagram, we also recall that for a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$, there exists an $\mathfrak{sl}(2)$-triple $\{e, h, f\} \subseteq \mathfrak{g}_{\bar{0}}$ by the Jacobson–Morozov Theorem, see for example [5, Theorem 3.3.1]. An $\mathfrak{sl}(2)$-triple determines a grading on $\mathfrak{g}$ according to the eigenvalues of $\mathrm{ad}h$, thus we can decompose $\mathfrak{g}$ into its $\mathrm{ad}h$-eigenspaces

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j) \text{ where } \mathfrak{g}(j) = \{x \in \mathfrak{g} : [h, x] = jx\}. \tag{4.2}$$

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a system of positive roots Φ^+ such that $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j \geq 0} \mathfrak{g}(j)$. Then the corresponding system of simple roots $\Pi = \{\alpha_1, ..., \alpha_l\}$ determines a Dynkin diagram of $\mathfrak{g}$. Furthermore, for each $i = 1, ..., l$, note that $\mathfrak{g}_{\alpha_i}$ is the root space corresponding to α_i and $\mathfrak{g}_{\alpha_i} \subseteq \mathfrak{g}(j_i)$ for some $j_i \geq 0$. Hence, we have $\alpha_i(h) \geq 0$ for $i = 1, ..., l$.

Definition 4.2. A *labelled Dynkin diagram* Δ of e is given by taking the Dynkin diagram of $\mathfrak{g}$ with respect to the above choices and labelling each node with $\alpha(h)$.

Remark. There can be different Dynkin diagrams corresponding to non-conjugate systems of simple roots, therefore we can get different labelled Dynkin diagrams for a given nilpotent $e \in \mathfrak{g}_{\bar{0}}$.

4.4 Chevalley basis of Lie algebras

Let L be a semisimple Lie algebra over $\mathbb{C}$ and H be a Cartan subalgebra for L. Suppose Φ is the root system of L with respect to H and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is a base for Φ. Recall that the weight space corresponding to $\alpha \in \Phi$ is $L_\alpha = \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}$. Let $\alpha, \beta \in \Phi$ be linearly independent, then the chain $-p\alpha + \beta, \dots, \beta, \dots, q\alpha + \beta$ is called the *α-string through β*. According to [12, Section 25], it is possible to choose root vectors $e_\alpha \in L_\alpha$ for $\alpha \in \Phi$ such that:

(i) $[e_\alpha, e_{-\alpha}] = h_\alpha$ where $e_{-\alpha} \in L_{-\alpha}$;

(ii) If $\beta \in \Phi$ and $\alpha + \beta \in \Phi$, then we have $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ and $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$;

(iii) $N_{\alpha,\beta}^2 = q(p+1)\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)}$.

A *Chevalley basis* of L is a basis of the form $\mathfrak{B} = \{e_\alpha : \alpha \in \Phi\} \cup \{h_i = h_{\alpha_i} : 1 \leq i \leq l\}$

where $H = \langle h_{\alpha_i} : 1 \le i \le l \rangle$ and e_α satisfy above conditions (i) and (ii). Note that $N_{\alpha,\beta}$ are called the resulting *structure constants* of L with respect to $\mathfrak{B}$ and $N_{\alpha,\beta} \in \mathbb{Z}$. More precisely, we have the following theorem.

Theorem 4.3. Let $\mathfrak{B} = \{e_\alpha : \alpha \in \Phi\} \cup \{h_{\alpha_i} : 1 \le i \le l\}$ be a Chevalley basis of L, then

(i) $[h_{\alpha_i}, h_{\alpha_j}] = 0$ for $1 \le i, j \le l$;

(ii) $[h_{\alpha_i}, e_{\alpha_j}] = \langle \alpha_j, \alpha_i \rangle e_{\alpha_j}$ for $1 \le i, j \le l$;

(iii) Suppose $\alpha \neq \pm\beta$, consider the α-string through β which is given by $-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta$. If $q = 0$, then $[e_\alpha, e_\beta] = 0$; if $\alpha + \beta \in \Phi$, then $[e_\alpha, e_\beta] = \pm(p+1)e_{\alpha+\beta}$.

We know that $\{h_{\alpha_i} : 1 \le i \le l\}$ are completely determined by Π. However, note that e_α are only defined up to scalar multiples thus there are multiple choices of e_α by choosing signs of structure constants.

5 Centres of centralizers of nilpotent elements in Lie superalgebras $\mathfrak{sl}(m|n)$ or $\mathfrak{osp}(m|2n)$

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra $\mathfrak{sl}(m|n)$ or $\mathfrak{osp}(m|2n)$ and let $e \in \mathfrak{g}_{\bar{0}}$ be nilpotent. We start by describing an explicit structure of $\mathfrak{g}$ and recall the concept of Dynkin pyramids in order to determine e and construct the corresponding labelled Dynkin diagram. Then we give a basis for $\mathfrak{g}^e$ and further for $\mathfrak{z}(\mathfrak{g}^e)$. In Subsections 5.1.6 and 5.2.10, we prove Theorems 1.1–1.3 for $\mathfrak{g}$. Throughout this chapter, we let our field be $\mathbb{C}$.

5.1 Lie superalgebras of type $A(m,n)$

In this section, we first recall the definition of the Lie superalgebras of type $A(m,n)$ and the root system for $A(m,n)$ following [19, Section 2.2] and [4, Subsection 1.1.2]. Given a nilpotent even element $e \in A(m,n)$, we recall the Dynkin pyramid as given in [11, Chapter 7] and construct the labelled Dynkin diagram for $e \in A(m,n)$. Note that when $m = n$, we consider the case $\mathfrak{sl}(n|n), n > 1$ instead of $A(n,n) = \mathfrak{psl}(n|n)$. In order to give a basis for $\mathfrak{sl}(m|n)^e$, we first recall a basis of $\mathfrak{gl}(m|n)^e$ and a formula for $\dim \mathfrak{gl}(m|n)^e$ based on the structure given in [29, Section 1] in Subsection 5.1.3. We also obtain an alternative formula for $\dim \mathfrak{gl}(m|n)^e$ which is different from the formula in [11, Section 3.2]. Once this is achieved, we use the above results to obtain a basis for $\mathfrak{sl}(m|n)^e$. A basis for $\mathfrak{z}(\mathfrak{sl}(m|n)^e)$ is given in Subsection 5.1.4. In Subsection 5.1.5, we present a formula for calculating $\dim \mathfrak{psl}(n|n)^e$, it is interesting to further investigate $\dim \mathfrak{z}(\mathfrak{psl}(n|n)^e)$ in order to complete results for Lie superalgebras of type $A(m,n)$, but we haven't covered this in the thesis.

5.1.1 Construction of $A(m,n)$

We start by recalling the definition of the general linear Lie superalgebras. The *general linear Lie superalgebra* is a $\mathbb{Z}_2$-graded associative superalgebra $\mathfrak{gl}(m|n)$ which contains all block matrices of the form

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{5.1}$$

where A, B, C and D are $m \times m, m \times n, n \times m$ and $n \times n$ matrices respectively. Let $\mathfrak{g} = \mathfrak{gl}(m|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, then $\mathfrak{g}_{\bar{0}}$ is the set of all matrices of the above form with $B = C = 0$ and $\mathfrak{g}_{\bar{1}}$ is the set of all matrices of the above form with $A = D = 0$. The Lie bracket is defined by:

$$[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx \text{ for all homogeneous } x, y \in \mathfrak{gl}(m|n). \tag{5.2}$$

Moreover, let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite-dimensional $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$, then $\mathrm{End}(V)$ forms a Lie superalgebra and is denoted by $\mathfrak{gl}(V)$. When $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$, we can identify $\mathfrak{gl}(m|n)$ for $\mathfrak{gl}(V)$ by choosing a homogeneous basis for V.

For each element $x \in \mathfrak{gl}(m|n)$ of the form in (5.1), we define the *supertrace* of x to be $\mathrm{str}(x) := \mathrm{Trace}(A) - \mathrm{Trace}(D)$. For homogeneous elements $x, y \in \mathfrak{gl}(m|n)$, we can check that $\mathrm{str}\,([x, y]) = 0$. Assume

$$x = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, y = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$

Then

$$\mathrm{str}\,([x, y]) = \mathrm{str}\left(xy - (-1)^{\bar{x}\bar{y}} yx\right). \tag{5.3}$$

We have that

$$[x,y] = \begin{cases} \begin{pmatrix} A_1A_2 - A_2A_1 & 0 \\ 0 & D_1D_2 - D_2D_1 \end{pmatrix} & \text{if } \bar{x} = \bar{y} = \bar{0}; \\ \begin{pmatrix} 0 & A_1B_2 - B_2D_1 \\ D_1C_2 - C_2A_1 & 0 \end{pmatrix} & \text{if } \bar{x} = \bar{0}, \bar{y} = \bar{1}; \\ \begin{pmatrix} B_1C_2 + C_1B_2 & 0 \\ 0 & C_2B_1 + B_2C_1 \end{pmatrix} & \text{if } \bar{x} = \bar{y} = \bar{1}. \end{cases}$$

Hence, if $\bar{x} = \bar{y} = \bar{0}$, we have $\operatorname{str}([x,y]) = \operatorname{tr}([A_1, A_2]) - \operatorname{tr}([D_1, D_2]) = 0$ as the trace of the commutators of A_1 and A_2 and D_1 and D_2 vanish. If $\bar{x} = \bar{0}, \bar{y} = \bar{1}$, we have $\operatorname{str}([x,y]) = \operatorname{tr}(0) - \operatorname{tr}(0) = 0$. If $\bar{x} = \bar{y} = \bar{1}$, we have $\operatorname{str}([x,y]) = \operatorname{tr}(B_1C_2) + \operatorname{tr}(C_1B_2) - \operatorname{tr}(C_2B_1) - \operatorname{tr}(B_2C_1) = 0$ as $\operatorname{tr}(B_1C_2) = \operatorname{tr}(C_2B_1)$ and $\operatorname{tr}(C_1B_2) = \operatorname{tr}(B_2C_1)$. Therefore, we deduce that $\operatorname{str}([x,y]) = 0$ for all $x, y \in \mathfrak{gl}(m|n)$ as required. Thus the subspace $\mathfrak{sl}(m|n) = \{x \in \mathfrak{gl}(m|n) : \operatorname{str}(x) = 0\}$ is a subalgebra of $\mathfrak{gl}(m|n)$ and is called the *special linear Lie superalgebra*. Note that when $\mathfrak{g} = \mathfrak{sl}(m|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, then $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_m \oplus \mathfrak{sl}_n \oplus \mathbb{C}\mathfrak{I}_{m,n}$ where $\mathfrak{I}_{m,n}$ is defined to be the matrix of the form in (5.1) such that $B = C = 0$, $A = nI_m$ and $D = mI_n$. Note that $\mathfrak{sl}(m|n) \cong \mathfrak{sl}(n|m)$.

According to [19, Section 2.2], we have that $\mathfrak{sl}(m|n)$ is simple when $m \neq n$ and $m, n > 0$. When $m = n$, the identity matrix $I_{n|n}$ generates a non-trivial centre of $\mathfrak{sl}(n|n)$. Furthermore, we have $\mathfrak{sl}(1|1)/\mathbb{C}I_{1|1}$ is abelian thus is not simple. For $n \geq 2$, $\mathfrak{sl}(n|n)/\mathbb{C}I_{n|n}$ is simple. We define *the Lie superalgebra of type* $A(m,n)$ by

$$A(m,n) = \begin{cases} \mathfrak{sl}(m+1|n+1) & \text{if } m \neq n,\ m,n \geq 0 \\ \mathfrak{sl}(n+1|n+1)/\mathbb{C}I_{n+1|n+1} & \text{if } m = n > 0 \end{cases}. \tag{5.4}$$

We denote $\mathfrak{sl}(n|n)/\mathbb{C}I_{n|n}$ by $\mathfrak{psl}(n|n)$.

5.1.2 Dynkin pyramids and labelled Dynkin diagrams for $A(m,n)$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite-dimensional vector superspace such that $\dim V_{\bar{0}} = m$ and $\dim V_1 = n$. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{sl}(m|n) = \mathfrak{sl}(V)$. Note that the nilpotent orbits in $\mathfrak{g}_{\bar{0}}$ are parameterized by the partitions of $(m|n)$. Let λ be a partition of $(m|n)$ such that

$$\lambda = (p|q) = (p_1, \dots, p_r|q_1, \dots, q_s) \tag{5.5}$$

where p (resp. q) is a partition of m (resp. n) and $p_1 \geq \cdots \geq p_r$, $q_1 \geq \cdots \geq q_s$. By rearranging the order of numbers in $(p|q)$, we write

$$\lambda = (\lambda_1, \dots, \lambda_{r+s}) \tag{5.6}$$

such that $\lambda_1 \geq \cdots \geq \lambda_{r+s}$. For $i = 1, \dots, r+s$, we define $|i| \in \mathbb{Z}_2$ such that for $c \in \mathbb{Z}$, we have that $|\{i : \lambda_i = c, |i| = \bar{0}\}| = |\{j : p_j = c\}|$, $|\{i : \lambda_i = c, |i| = \bar{1}\}| = |\{j : q_j = c\}|$ for some j and if $\lambda_i = \lambda_j$, $|i| = \bar{0}, |j| = \bar{1}$, then $i < j$. i.e. $\sum_{|i|=\bar{0}} \lambda_i = m$ and $\sum_{|i|=\bar{1}} \lambda_i = n$.

For the purpose of determining the dimension of $\mathfrak{g}^e$ and the labelled Dynkin diagram for each nilpotent orbit $e \in \mathfrak{g}_{\bar{0}}$, we recall the concept of a Dynkin pyramid of shape λ which is given in [6, Section 4] and [11, Section 7].

A *Dynkin pyramid* of shape λ is a diagram consisting of a finite collection of boxes of size 2×2 in the xy-plane which are centred at integer coordinates. By the coordinates of a box, we mean the coordinates of its midpoint. Write $\lambda = (\lambda_1, \dots, \lambda_{r+s})$ as defined in (5.6), we number rows of the Dynkin pyramid P from $1, \dots, r+s$ from bottom to top, then the jth row of P has length λ_j and we mark boxes in the row j with parity

$\bar{0}$ or $\bar{1}$ according to $|j|$. We associate a numbering $1, 2, ..., m+n$ for the boxes of the Dynkin pyramid from top to bottom and from left to right. The row number of the ith box is denoted by $\mathrm{row}(i)$ and the parity of $\mathrm{row}(i)$ is defined to be $|\mathrm{row}(i)|$. We say that the column number $\mathrm{col}(i)$ of i is the x-coordinate of the centre of the ith box.

Define a basis

$$\{v_1, \ldots, v_{m+n}\} \tag{5.7}$$

of $V = V_{\bar{0}} \oplus V_{\bar{1}}$ where $v_i \in V_{|\mathrm{row}(i)|}$. According to [11, Section 7], the Dynkin pyramid P determines a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ and a semisimple element $h \in \mathfrak{g}_{\bar{0}}$. The nilpotent element e is defined to be an endomorphism such that e sends v_i to v_j if the box labelled by j is the left neighbour of i and sends v_i to zero if the box labelled by v_i has no left neighbour. Moreover, write e_{ij} for the ij-matrix unit, then

$$e = \sum_{\substack{\mathrm{row}(i)=\mathrm{row}(j) \\ \mathrm{col}(j)=\mathrm{col}(i)-2}} e_{ij}$$

for all $1 \leq i, j \leq m+n$. The semisimple element h is defined to be the $(m+n)$-diagonal matrix where the ith diagonal entry is $-\mathrm{col}(i)$, i.e.

$$h = \sum_{i=1}^{m+n} -\mathrm{col}(i) e_{ii}.$$

Note that $\{e, h\}$ can be extended to an $\mathfrak{sl}(2)$-triple in $\mathfrak{g}_{\bar{0}}$ according to [11, Section 7].

Next we recall a construction of the root system for $\mathfrak{g}$ based on [19, Section 2.2]. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices in $\mathfrak{g}$ and let $a = \mathrm{diag}(a_1, \ldots, a_{m+n}) \in \mathfrak{h}$. We define $\{\varepsilon_1, \ldots, \varepsilon_{m+n}\} \subseteq \mathfrak{h}^*$ such that $\varepsilon_i(a) = a_i$ for $i = 1, \ldots, m+n$ and the parity of ε_i is equal to $|\mathrm{row}(i)|$. Then the root system of $\mathfrak{g}$

with respect to $\mathfrak{h}$ is $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ where

$$\Phi_{\bar{0}} = \{\varepsilon_i - \varepsilon_j : i \neq j, |i| = |j|\}, \Phi_{\bar{1}} = \{\varepsilon_i - \varepsilon_j : i \neq j, |i| \neq |j|\}$$

and $(\varepsilon_i, \varepsilon_j) = (-1)^{|i|}\delta_{ij}$. For $i \neq j, |i| \neq |j|$, by computing $(\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j) = (\varepsilon_i, \varepsilon_i) + (\varepsilon_j, \varepsilon_j) = (-1)^{|i|} + (-1)^{|j|} = 0$, we have that all odd roots are isotropic.

We use the Dynkin pyramid to determine the labelled Dynkin diagram with respect to e. The labelled Dynkin diagram for a nilpotent orbit $e \in \mathfrak{g}_{\bar{0}}$ is constructed as follows. First, draw the Dynkin pyramid of shape λ. Then we start from the box labelled by 1. For $i = 1, ..., m+n-1$, we associate a white node $\bigcirc$ (resp. a grey node $\otimes$) for root $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ if $|\text{row}(i+1)| = |\text{row}(i)|$ (resp. $|\text{row}(i+1)| \neq |\text{row}(i)|$). We label the ith node with the value $\text{col}(i+1) - \text{col}(i)$. Note that labelled Dynkin diagrams are equally well defined for $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(n|n)$.

Example 5.1. For a partition $\lambda = (5, 1|3)$, the Dynkin pyramid of shape λ is

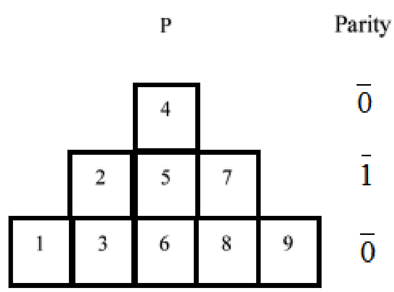


Figure 1: Dynkin Pyramid of $(5, 1|3)$

We calculate that $h = \text{diag}(4, 2, 2, 0, 0, 0, -2, -2, -4)$. Then the corresponding labelled Dynkin diagram is:

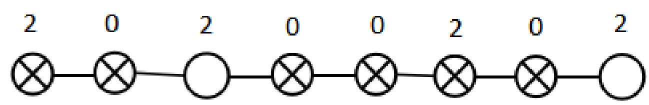


Figure 2: Labelled Dynkin diagram for $\lambda = (5, 1|3)$

Remark 5.2. Different numberings within columns and different choices of parities of rows are possible and would lead to different labelled Dynkin diagrams. In this way one can get all possible labelled Dynkin diagrams. In this thesis, the way we define the partition λ in (5.6) leads to a unique labelled Dynkin diagram.

The following example shows that for different choices of numbering boxes of a given pyramid, we can get different Dynkin diagrams.

Example 5.3. For a partition $\lambda = (3, 2|2, 1)$, the corresponding Dynkin pyramid and labelled Dynkin diagram are shown below:

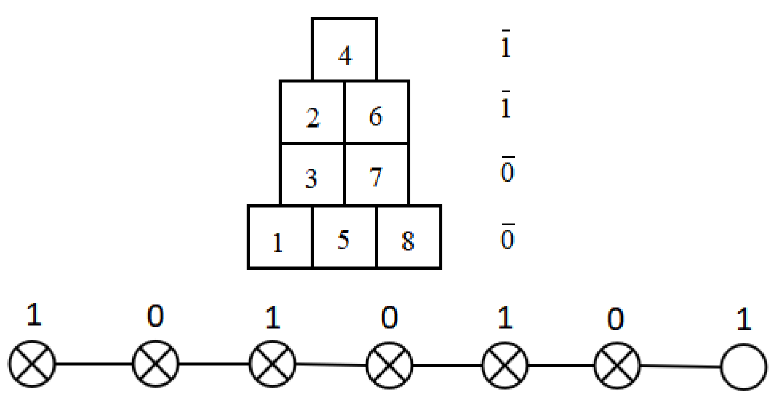


Figure 3: Dynkin pyramid and labelled Dynkin diagram of $\lambda = (3, 2|2, 1)$

However, if we allow different numberings of boxes within columns, then there exists another Dynkin pyramid and therefore a different labelled Dynkin diagram:

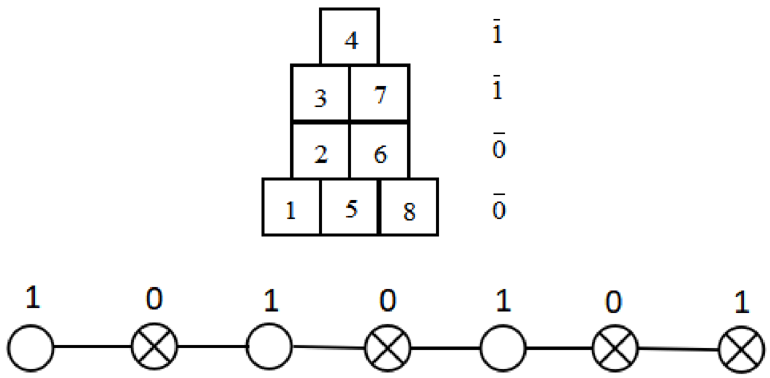


Figure 4: Labelled Dynkin diagram for $\lambda = (3, 2|2, 1)$

5.1.3 Centralizer of nilpotent elements $e \in \mathfrak{sl}(m|n)_{\bar{0}}$

Let $e \in \mathfrak{g}_{\bar{0}}$ be a nilpotent element with Jordan type $\lambda = (\lambda_1, \dots, \lambda_{r+s})$ which is defined as in (5.6). Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{sl}(V)$ and $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{\bar{0}} \oplus \bar{\mathfrak{g}}_{\bar{1}} = \mathfrak{gl}(V)$. In order to calculate the dimension of $\mathfrak{g}^e$, we first recall a basis for $\bar{\mathfrak{g}}^e$ based on [29, Section 1] and [11, Section 3.2].

We know that $\bar{\mathfrak{g}} = \mathrm{End}(V_{\bar{0}} \oplus V_{\bar{1}})$ where $\bar{\mathfrak{g}}_{\bar{0}} = \mathrm{End}(V_{\bar{0}}) \oplus \mathrm{End}(V_{\bar{1}})$ and $\bar{\mathfrak{g}}_{\bar{1}} = \mathrm{Hom}(V_{\bar{0}}, V_{\bar{1}}) \oplus \mathrm{Hom}(V_{\bar{1}}, V_{\bar{0}})$. Let $u_1, \dots, u_{r+s} \in V$ such that $u_i = v_k$ for $\mathrm{row}(k) = i$ and $\mathrm{col}(k) = \lambda_i - 1$ where $\{v_1, \dots, v_{m+n}\}$ is a basis for V defined in (5.7). Then the vectors $e^j u_i$ with $0 \le j \le \lambda_i - 1$, $|i| = \bar{0}$ form a basis for $V_{\bar{0}}$ and the vectors $e^j u_i$ with $0 \le j \le \lambda_i - 1$, $|i| = \bar{1}$ form a basis for $V_{\bar{1}}$. Note that the basis $\{e^j u_i : 0 \le j \le \lambda_i - 1\}$ is the same basis as in (5.7) and we have that $e^{\lambda_i} u_i = 0$ for $1 \le i \le r + s$.

With the above notation, we recall a basis for $\bar{\mathfrak{g}}^e$ as described in [11, Section 3.2.1]. For $\xi \in \bar{\mathfrak{g}}^e$, we have $\xi(e^j u_i) = e^j \xi(u_i)$. Hence, each ξ is determined by $\xi(u_i)$ for $i = 1, \dots, r + s$ and we know that $\xi(u_i)$ is of the form $\xi(u_i) = \sum_{j=1}^{r+s} \sum_{k=0}^{\lambda_j - 1} c_i^{j,k} e^k u_j$ where $c_i^{j,k} \in \mathbb{C}$ are coefficients. Since $e^{\lambda_i} u_i = 0$ for $i = 1, \dots, r + s$, we have that

$\xi(e^{\lambda_i}u_i) = \sum_{j=1}^{r+s}\sum_{k=0}^{\lambda_j-1} c_i^{j,k}e^{\lambda_i+k}u_j = 0$. Hence, we can write

$$\xi(u_i) = \sum_{j=1}^{r+s} \sum_{k=\max\{\lambda_j-\lambda_i,0\}}^{\lambda_j-1} c_i^{j,k}e^k u_j. \tag{5.8}$$

Therefore, in accordance with arguments in [11, Section 3.2] and [29, Section 1], we know that $\bar{\mathfrak{g}}^e$ has a basis

$$\{\xi_i^{j,k} : 1 \leq i,j \leq r+s \text{ and } \max\{\lambda_j-\lambda_i,0\} \leq k \leq \lambda_j - 1\} \tag{5.9}$$

such that $\xi_i^{j,k}(u_t) = \delta_{it}e^k u_j$.

Hence, we have

$$\dim\bar{\mathfrak{g}}_{\bar{0}}^e = \left(m + 2\sum_{i=1}^{r}(i-1)p_i\right) + \left(n + 2\sum_{j=1}^{s}(j-1)q_j\right), \tag{5.10}$$

$$\text{and } \dim\bar{\mathfrak{g}}_{\bar{1}}^e = 2\sum_{i,j=1}^{r,s}\min(p_i,q_j) \tag{5.11}$$

where p_i and q_j are defined in (5.5).

We also want to get a better idea about the structure of $\bar{\mathfrak{g}}^e$ by providing a description of $\bar{\mathfrak{g}}^e(0)$, where $\bar{\mathfrak{g}}^e(l)$ represents the lth ad$h$-eigenspace and h lies in an $\mathfrak{sl}(2)$-triple in $\bar{\mathfrak{g}}_{\bar{0}}$. Define a grading of the vector space V such that $V = \bigoplus_{l\in\mathbb{Z}} V(l)$ and $V(l) = \{v \in V : hv = lv\}$. Note that $e^k u_j \in V(l)$ when $l = 2k+1-\lambda_j$. From Section 4.3 we know that $\bar{\mathfrak{g}}$ has a grading where $\bar{\mathfrak{g}} = \bigoplus_{j\in\mathbb{Z}} \bar{\mathfrak{g}}(j)$ and $\bar{\mathfrak{g}}(j)$ is the jth ad$h$-eigenspace. According to [15, Sections 3.3–3.5], there is another way to define the above grading on $\bar{\mathfrak{g}}$:

$$\bar{\mathfrak{g}}(j) = \{x \in \bar{\mathfrak{g}} : x(V(l)) \subseteq V(l+j) \text{ for all } l \in \mathbb{Z}\}. \tag{5.12}$$

We claim that the above two gradings are equivalent. Let $x \in \bar{\mathfrak{g}}$ and $v \in V(l)$. Suppose $x(V(l)) \subseteq V(l+j)$, then $xv \in V(l+j)$ and thus $h(xv) = (l+j)xv$. Hence, we have that

$$h(xv) = x(hv) + [h,x]v = (l+j)xv,$$

which implies that $[h,x]v = jxv$ for all $v \in V$. Therefore, we have that $[h,x] = jx$. Conversely, suppose $[h,x] = jx$. Then

$$hxv = xhv + [h,x]v = lxv + jxv = (l+j)xv.$$

Hence, we know that $xv \in V(l+j)$ as required.

We also can obtain an ad$h$-eigenvalue decomposition of $\bar{\mathfrak{g}}^e$ such that

$$\bar{\mathfrak{g}}^e = \bigoplus_{l \in \mathbb{Z}} \bar{\mathfrak{g}}^e(l) \tag{5.13}$$

where $\bar{\mathfrak{g}}^e(l) = \bar{\mathfrak{g}}^e \cap \bar{\mathfrak{g}}(l)$. Clearly $e \in \bar{\mathfrak{g}}(2)$. We know that $u_i \in V(-\lambda_i + 1)$ and $\xi_i^{j,k}(u_i) = e^k u_j \in V(-\lambda_j + 2k + 1)$. Hence, we have that

$$\xi_i^{j,k} \in \bar{\mathfrak{g}}(\lambda_i - \lambda_j + 2k). \tag{5.14}$$

Since $k \geq \max\{0, \lambda_j - \lambda_i\}$, then

$$\lambda_i - \lambda_j + 2k \geq \begin{cases} \lambda_i - \lambda_j & \text{if } \lambda_i > \lambda_j; \\ \lambda_j - \lambda_i & \text{if } \lambda_i < \lambda_j; \\ 0 & \text{if } \lambda_i = \lambda_j. \end{cases}$$

Therefore, we have that $\xi_i^{j,k} \in \bar{\mathfrak{g}}^e(0)$ if and only if $\lambda_i = \lambda_j$ and $k = 0$. Thus for an element $\xi_i^{j,k} \in \bar{\mathfrak{g}}^e(0)$, we obtain

$$\xi(u_i) = \sum_{j;\lambda_i=\lambda_j} c_i^{j,0} u_j. \tag{5.15}$$

Rewrite λ as

$$\lambda = (c^{m_c+n_c}, \dots, 1^{m_1+n_1}) \tag{5.16}$$

where $m_i = |\{j : \lambda_j = i, |j| = \bar{0}\}|$ and $n_i = |\{j : \lambda_j = i, |j| = \bar{1}\}|$. Let us define $M^{(j)} = \langle u_i : \lambda_i = j \rangle$ for $1 \leq j \leq c$. Then we can check that $\xi(M^{(j)}) \subseteq M^{(j)}$. Since each element $\xi \in \bar{\mathfrak{g}}^e$ is determined by $\xi(u_i)$ for $1 \leq i \leq r+s$, we have that the map

$$\bar{\mathfrak{g}}^e(0) \to \mathfrak{gl}(M^{(1)}) \times \mathfrak{gl}(M^{(2)}) \times \cdots \times \mathfrak{gl}(M^{(c)})$$

is an injective homomorphism. By [11, Theorem 3.4], we have that $\bar{\mathfrak{g}}^e(0) \cong \mathfrak{gl}(M^{(1)}) \oplus \mathfrak{gl}(M^{(2)}) \oplus \cdots \oplus \mathfrak{gl}(M^{(c)})$. In addition, we have $M^{(j)} = M_{\bar{0}}^{(j)} \oplus M_{\bar{1}}^{(j)}$ where $\dim M_{\bar{0}}^{(j)} = m_j$, $\dim M_{\bar{1}}^{(j)} = n_j$ and thus $\mathfrak{gl}(M^{(j)}) \cong \mathfrak{gl}(m_j|n_j)$ for each j. Therefore, we obtain that $\bar{\mathfrak{g}}^e(0) \cong \bigoplus_{j=1}^{c} \mathfrak{gl}(m_j|n_j)$.

Example 5.4. Suppose $\bar{\mathfrak{g}} = \mathfrak{gl}(11|12)$ and the nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ has Jordan type $(3^2, 2^1, 1^3 | 4^1, 2^3, 1^2)$, then

$$\bar{\mathfrak{g}}^e(0) \cong \mathfrak{gl}(0|1) \times \mathfrak{gl}(2|0) \times \mathfrak{gl}(1|3) \times \mathfrak{gl}(3|2).$$

Instead of using the formula for $\dim \bar{\mathfrak{g}}^e$ in equations (5.10) and (5.11), we obtain an alternative formula for $\dim \bar{\mathfrak{g}}^e$ below. Note that formulas in [11, Section 3.2.1] and Proposition 5.5 are equivalent, but the formula in Proposition 5.5 is more convenient to use in Subsection 5.1.6.

Proposition 5.5. *Let λ be a partition of $(m|n)$ denoted as in (5.6). Denote by P the Dynkin pyramid of λ and $e \in \mathfrak{g}_{\bar{0}}$ be a nilpotent element determined by P. Let c_i be the number of boxes whose column number is i in P. Then* $\dim \bar{\mathfrak{g}}^e = \sum_{i\in\mathbb{Z}} c_i^2 + \sum_{i\in\mathbb{Z}} c_i c_{i+1}$.

Proof. Let $\bar{\mathfrak{g}} = \bigoplus_{j\in\mathbb{Z}} \bar{\mathfrak{g}}(j)$. According to [11, Definition 4.1 and Theorem 7.2], the pair of elements $h, e \in \bar{\mathfrak{g}}_{\bar{0}}$ determined by a Dynkin pyramid P satisfying

$$\text{ad } e : \bar{\mathfrak{g}}(j) \to \bar{\mathfrak{g}}(j+2) \text{ is injective for } j \leq -1, \tag{5.17}$$

$$\text{ad } e : \bar{\mathfrak{g}}(j) \to \bar{\mathfrak{g}}(j+2) \text{ is surjective for } j \geq -1. \tag{5.18}$$

Then we have that the map ad $e : \bar{\mathfrak{g}}(j \geq -1) \to \bar{\mathfrak{g}}(j \geq 1)$ is surjective. Note that

$$\ker(\text{ad } e) = \bar{\mathfrak{g}}^e \subseteq \bar{\mathfrak{g}}(j \geq -1).$$

Thus by the rank–nullity theorem,

$$\dim \ker(\text{ad } e) + \dim \text{im}(\text{ad } e) = \dim \bar{\mathfrak{g}}(j \geq -1).$$

Since $\text{im}(\text{ad } e) = \bar{\mathfrak{g}}(j \geq 1)$, we have that

$$\begin{aligned} \dim \bar{\mathfrak{g}}^e &= \dim \bar{\mathfrak{g}}(j \geq -1) - \dim \bar{\mathfrak{g}}(j \geq 1) \\ &= \dim \bar{\mathfrak{g}}(0) + \dim \bar{\mathfrak{g}}(-1). \end{aligned}$$

We also calculate that $[h, e_{kl}] = [\sum_i -\text{col}(i) e_{ii}, e_{kl}] = (\text{col}(l) - \text{col}(k))\, e_{kl}$, which means

$e_{kl} \in \bar{\mathfrak{g}}(j)$ if $j = \mathrm{col}(l) - \mathrm{col}(k)$. This implies that

$$\bar{\mathfrak{g}}(0) = \langle e_{kl} : \mathrm{col}(l) = \mathrm{col}(k) \rangle \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{End}(\mathbb{C}^{c_i}),$$

and

$$\bar{\mathfrak{g}}(-1) = \langle e_{kl} : \mathrm{col}(l) - \mathrm{col}(k) = -1 \rangle = \bigoplus \mathrm{Hom}(\mathbb{C}^{c_i}, \mathbb{C}^{c_{i+1}}).$$

Therefore, we obtain that

$$\dim \bar{\mathfrak{g}}^e = \sum_{i \in \mathbb{Z}} c_i^2 + \sum_{i \in \mathbb{Z}} c_i c_{i+1}.$$

□

Let $I_m \in \mathfrak{gl}(m)$ be a $m \times m$ identity matrix and $0_{n \times n} \in \mathfrak{gl}(n)$ be a $n \times n$ matrix with all entries equal to 0. Let $I(m|0)$ be the $(m+n) \times (m+n)$ matrix

$$I(m|0) = \begin{pmatrix} I_m & 0_{m \times n} \\ 0_{n \times m} & 0_{n \times n} \end{pmatrix}. \tag{5.19}$$

Next we calculate the dimension of $\mathfrak{g}^e$ in the Corollary below.

Corollary 5.6. *Let $e \in \mathfrak{g}_{\bar{0}}$ be nilpotent, we have that* $\dim \mathfrak{g}^e = \dim \bar{\mathfrak{g}}^e - 1$.

Proof. We first consider a Lie superalgebra homomorphism

$$\mathrm{str} : \bar{\mathfrak{g}} \to \mathbb{C}.$$

Note that $\ker(\mathrm{str}) = \mathfrak{g}$ and $\dim \mathrm{im}(\mathrm{str}) = 1$. By restricting we get $\mathrm{str}^e : \bar{\mathfrak{g}}^e \to \mathbb{C}$ and

$\ker(\mathrm{str}^e) = \mathfrak{g}^e$. Then by the rank-nullity theorem we get

$$\dim\ker(\mathrm{str}^e) + \dim\mathrm{im}(\mathrm{str}^e) = \dim\bar{\mathfrak{g}}^e.$$

Let $I(m|0)$ be the $(m+n)\times(m+n)$ matrix as defined in (5.19). Let

$$e = \begin{pmatrix} e_A & 0 \\ 0 & e_D \end{pmatrix} \in \bar{\mathfrak{g}}_{\bar{0}}$$

where e_A and e_D are $m\times m$ and $n\times n$ matrices respectively. Then we calculate

$$[I(m|0), e] = I(m|0)e - eI(m|0) = \begin{pmatrix} I_m e_A - e_A I_m & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Hence, we have $I(m|0) \in \bar{\mathfrak{g}}^e$. We deduce that

$$\mathrm{str}^e(I(m|0)) = \mathrm{trace}(I_m) = m,$$

thus str^e is non-zero and $\dim\mathrm{im}(\mathrm{str}^e) = 1$. Therefore, we have $\dim\mathfrak{g}^e = \dim\bar{\mathfrak{g}}^e - 1$. □

5.1.4 Centre of centralizer of nilpotent elements $e \in \mathfrak{sl}(m|n)_{\bar{0}}$

In order to give a basis for $\mathfrak{z}(\mathfrak{g}^e)$, we determine a basis for $\mathfrak{z}(\bar{\mathfrak{g}}^e)$ first.

Proposition 5.7. *For a nilpotent element $e \in \bar{\mathfrak{g}}_{\bar{0}}$, we have that $\mathfrak{z}(\bar{\mathfrak{g}}^e) \subseteq \bar{\mathfrak{g}}_{\bar{0}}$.*

Proof. Note that $I(m|0) \in \bar{\mathfrak{g}}^e$ by the proof of Corollary 5.6. Suppose $x \in \mathfrak{z}(\bar{\mathfrak{g}}^e)$, then $x \in \bar{\mathfrak{g}}^e$ and $[x, y] = 0$ for all $y \in \bar{\mathfrak{g}}^e$. Thus $[x, I(m|0)] = 0$ since $I(m|0) \in \bar{\mathfrak{g}}^e$.

Therefore, we have that $\mathfrak{z}(\bar{\mathfrak{g}}^e) \subseteq (\bar{\mathfrak{g}}^e)^{I(m|0)} \subseteq \bar{\mathfrak{g}}^{I(m|0)}$.

Next we turn to look at what is $\bar{\mathfrak{g}}^{I(m|0)}$. Let $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \bar{\mathfrak{g}}$, notice that

$$[I(m|0), x] = \begin{pmatrix} 0 & B \\ -C & 0 \end{pmatrix}.$$

This is equal to zero if and only if $B = C = 0$ if and only if

$$x = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

where $A \in \mathfrak{gl}(m)$ and $D \in \mathfrak{gl}(n)$. Hence, we have $x \in \bar{\mathfrak{g}}_{\bar{0}}$ and it is easy to deduce that $\bar{\mathfrak{g}}^{I(m|0)} = \bar{\mathfrak{g}}_{\bar{0}}$.

Therefore, we conclude that $\mathfrak{z}(\bar{\mathfrak{g}}^e) \subseteq \bar{\mathfrak{g}}_{\bar{0}}$. $\square$

Yakimova shows that $\mathfrak{z}(\mathfrak{gl}(m+n)^e) = \langle I, e, ..., e^l \rangle$ where $l = \lambda_1 - 1$ in [29, Theorem 2]. Now we want to use the above result to work out a basis for $\mathfrak{z}(\bar{\mathfrak{g}}^e)$.

Theorem 5.8. *$\mathfrak{z}(\bar{\mathfrak{g}}^e)$ has a basis $\{I, e, ..., e^l : l = \lambda_1 - 1\}$.*

Proof. Firstly, we denote the Lie algebra $\mathfrak{gl}(m+n)$ by $\mathfrak{g}'$. Note that $\mathfrak{gl}(m+n)$ is isomorphic to $\bar{\mathfrak{g}}$ as a vector space. For $e \in \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) = \bar{\mathfrak{g}}_{\bar{0}}$, on the one hand we can view $e \in \bar{\mathfrak{g}}$ and consider $\bar{\mathfrak{g}}^e$. On the other hand, we also can view $e \in \mathfrak{g}'$ and consider $(\mathfrak{g}')^e$. We deduce that $\bar{\mathfrak{g}}^e = (\mathfrak{g}')^e$ as a vector space because $[e, x] = [e, x']$ for $x \in \bar{\mathfrak{g}}$ and

$x' \in \mathfrak{g}'$. Denote by

$$\mathfrak{g}'_0 = \left\{ \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix} \in \mathfrak{g}' : A' \in \mathfrak{gl}(m) \text{ and } D' \in \mathfrak{gl}(n) \right\}$$

and

$$\mathfrak{g}'_1 = \left\{ \begin{pmatrix} 0 & B' \\ C' & 0 \end{pmatrix} \in \mathfrak{g}' : B' \text{ is a } m \times n \text{ matrix and } C' \text{ is a } n \times m \text{ matrix} \right\}.$$

Then we let $\mathfrak{z}((\mathfrak{g}')^e)_0 = \mathfrak{z}((\mathfrak{g}')^e) \cap \mathfrak{g}'_0$ and $\mathfrak{z}((\mathfrak{g}')^e)_1 = \mathfrak{z}((\mathfrak{g}')^e) \cap \mathfrak{g}'_1$. We further observe that $\mathfrak{z}(\bar{\mathfrak{g}}^e)_{\bar{0}} = \mathfrak{z}((\mathfrak{g}')^e)_0$ by definition. Since we already found that $\mathfrak{z}(\bar{\mathfrak{g}}^e) \subseteq \bar{\mathfrak{g}}_{\bar{0}}$, thus $\mathfrak{z}(\bar{\mathfrak{g}}^e)_{\bar{1}} = 0$. Using the same argument as in Proposition 5.7, we have that $\mathfrak{z}((\mathfrak{g}')^e)_1 = 0$.
Hence, we obtain that $\mathfrak{z}(\bar{\mathfrak{g}}^e) = \mathfrak{z}(\bar{\mathfrak{g}}^e)_{\bar{0}} = \mathfrak{z}((\mathfrak{g}')^e)_0 = \mathfrak{z}((\mathfrak{g}')^e)$.
Therefore, $\mathfrak{z}(\bar{\mathfrak{g}}^e) = \langle I, e, ..., e^l : l = \lambda_1 - 1 \rangle$ and this gives that $\dim \mathfrak{z}(\bar{\mathfrak{g}}^e) = \lambda_1$. □

Now we give a basis for $\mathfrak{z}(\mathfrak{g}^e)$. We divide our analysis into two cases: $m \neq n$ and $m = n > 1$.

Theorem 5.9. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{sl}(m|n)$ and let λ be a partition of $(m|n)$ denoted as* (5.6). *Let $e \in \mathfrak{g}_{\bar{0}}$ be a nilpotent element determined by the Dynkin pyramid of λ, then $\mathfrak{z}(\mathfrak{g}^e) = \langle e, ..., e^l : l = \lambda_1 - 1 \rangle$ except for $m = n, n > 1$, in which case $\mathfrak{z}(\mathfrak{g}^e) = \langle I, e, ..., e^l : l = \lambda_1 - 1 \rangle$.*

Proof. When $m \neq n$, we know that $\bar{\mathfrak{g}}^e = \mathfrak{g}^e \oplus \mathbb{C}I$ and thus $\mathfrak{z}(\bar{\mathfrak{g}}^e) \subseteq \mathfrak{z}(\mathfrak{g}^e) \oplus \mathbb{C}I$. Let $x \in \mathfrak{z}(\mathfrak{g}^e)$, then $[x, y] = 0$ for all $y \in \mathfrak{g}^e$. We also know that $x \in \bar{\mathfrak{g}}^e$ since $\mathfrak{g}^e \subseteq \bar{\mathfrak{g}}^e$. Moreover, we have that $[x, I] = 0$ and thus $[x, y] = 0$ for all $y \in \mathfrak{g}^e \oplus \mathbb{C}I$. Hence, we have that $\mathfrak{z}(\mathfrak{g}^e) \subseteq \mathfrak{z}(\bar{\mathfrak{g}}^e)$. Therefore, a basis for $\mathfrak{z}(\mathfrak{g}^e)$ consists of all basis vectors of $\mathfrak{z}(\bar{\mathfrak{g}}^e)$ except identity matrix I. Hence, we have that

$$\dim \mathfrak{z}(\mathfrak{g}^e) = \dim \mathfrak{z}(\bar{\mathfrak{g}}^e) - 1 = \lambda_1 - 1.$$

When $m = n, n > 1$, for the partition $\lambda = (\lambda_1, \dots, \lambda_{r+s})$, we have elements $\lambda_j \xi_i^{i,0} - (-1)^{\bar{i}} \lambda_i \xi_j^{j,0}$ with $i \neq j$ lie in the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.

Let $S = \langle I, e, ..., e^l : l = \lambda_1 - 1 \rangle$. Clearly $S \subseteq \mathfrak{z}(\mathfrak{g}^e)$. We know that $e^k = \sum_{i=1}^{r+s} \xi_i^{i,k}$ and $e^k \in \mathfrak{g}$ for all $0 \leq k \leq \lambda_1 - 1$. Suppose $x \in \mathfrak{z}(\mathfrak{g}^e)$ is of the form

$$x = \sum_{\substack{1 \leq t,h \leq r+s \\ \max\{\lambda_t - \lambda_h\} \leq k \leq \lambda_t - 1}} c_h^{t,k} \xi_h^{t,k}$$

where $c_h^{t,k} \in \mathbb{C}$ are coefficients. If $r + s \geq 3$, then x commutes with $\lambda_j \xi_i^{i,0} - (-1)^{\bar{i}} \lambda_i \xi_j^{j,0}$ for all i, j. By computing $[\lambda_j \xi_i^{i,0} - (-1)^{\bar{i}} \lambda_i \xi_j^{j,0}, x]$ for $i \neq j$ we have

$$\begin{aligned}
[\lambda_j \xi_i^{i,0} - (-1)^{\bar{i}} \lambda_i \xi_j^{j,0}, x] &= [\lambda_j \xi_i^{i,0} - (-1)^{\bar{i}} \lambda_i \xi_j^{j,0}, \sum_{t,k,h} c_h^{t,k} \xi_h^{t,k}] \\
&= \lambda_j (\sum_{h,k} c_h^{i,k} \xi_h^{i,k} \pm \sum_{t,k} c_i^{t,k} \xi_i^{t,k}) \\
&\quad - (-1)^{\bar{i}} \lambda_i (\sum_{h,k} c_h^{j,k} \xi_h^{j,k} \mp \sum_{t,k} c_j^{t,k} \xi_j^{t,k}).
\end{aligned}$$

This is equal to 0 implies that $c_h^{t,k} = 0$ for all $h \neq t$ and thus $x \in \langle \xi_t^{t,k} \rangle$.

If $r + s = 2$, we have $\lambda_1 = \lambda_2 = n > 1$. In this case we only deal with $\xi_h^{t,k}$ for $h, t \in \{1, 2\}$. Clearly $\xi_1^{1,0} + \xi_2^{2,0} = I_{n|n} \in \mathfrak{z}(\mathfrak{g}^e)$ thus we cannot compute commutators between $\lambda_2 \xi_1^{1,0} + \lambda_1 \xi_2^{2,0}$ and another element to deduce conditions on coefficients. Note that a basis of $\mathfrak{g}^e$ is

$$\{\xi_1^{1,0}+\xi_2^{2,0},\xi_1^{1,j},\xi_2^{2,j},\xi_1^{2,k},\xi_2^{1,k}:j=1,\dots,n-1,k=0,1,\dots,n-1\}.$$

Hence, an element $y\in\mathfrak{g}^e$ is of the form

$$y=\sum_{\substack{1\leq t,h\leq 2\\0\leq k\leq n-1}} c_h^{t,k}\xi_h^{t,k}+c(\xi_1^{1,0}+\xi_2^{2,0})$$

Now suppose $y\in\mathfrak{z}(\mathfrak{g}^e)$, by computing

$$[\xi_1^{1,1},y]=\sum_{k=0}^{n-1}c_2^{1,k}\xi_2^{1,k+1}\pm\sum_{k=0}^{n-1}c_1^{2,k}\xi_1^{2,k+1}$$

we obtain that $c_1^{2,k}=0$ and $c_2^{1,k}=0$ for all $k=0,\dots,n-2$. Then we calculate

$$[\xi_2^{1,0},y]=c_1^{2,n-1}(\xi_1^{1,n-1}\pm\xi_2^{2,n-1}),$$

this implies that $c_1^{2,n-1}=0$. Similarly we have that $c_2^{1,n-1}=0$. Therefore, we obtain that $x\in\langle\xi_1^{1,0}+\xi_2^{2,0},\xi_t^{t,k}:k>0\rangle$.

From above we have that $x=\sum_{t,k}c_t^{t,k}\xi_t^{t,k}$. Adding an element of S we may assume that $c_1^{1,k}=0$ for all k. Suppose $x\notin S$, then there exist some $c_i^{i,k}\neq 0$. Next consider $\xi_1^{i,0}\in\mathfrak{g}^e$, we have

$$\left[x,\xi_1^{i,0}\right]=\sum_{t,k}c_t^{t,k}\left[\xi_t^{t,k},\xi_1^{i,0}\right]=\sum_k c_i^{i,k}\xi_1^{i,k}\neq 0,$$

thus $x\notin\mathfrak{z}(\mathfrak{g}^e)$. Hence $\mathfrak{z}(\mathfrak{g}^e)\subseteq S$. Therefore, we deduce that $\mathfrak{z}(\mathfrak{g}^e)=S$ as required. This implies that $\dim\mathfrak{z}(\mathfrak{g}^e)=\lambda_1$. $\square$

5.1.5 Centralizer of nilpotent elements $e \in \mathfrak{psl}(n|n)_{\bar{0}}$ for $n > 1$

For $\mathfrak{g} = \mathfrak{psl}(n|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and $e \in \mathfrak{g}_{\bar{0}}$ be nilpotent, the dimension of $\mathfrak{g}^e$ can be calculated using the argument below. Let $\phi : \mathfrak{sl}(n|n) \to \mathfrak{g}$ be the quotient map. Suppose $\phi(x) \in \mathfrak{g}^e$, then $[x, e] = 0$ or $aI_{n|n}$. Assume $[x, e] = aI_{n|n} \in \mathfrak{sl}(n|n)_{\bar{0}}$ for some $a \in \mathbb{C}$, we can write $x = x_{\bar{0}} + x_{\bar{1}}$ such that $x_{\bar{0}} \in \mathfrak{sl}(n|n)_{\bar{0}}$ and $x_{\bar{1}} \in \mathfrak{sl}(n|n)_{\bar{1}}$. Then we have that $[x, e] = [x_{\bar{0}}, e] + [x_{\bar{1}}, e]$, thus we know that $[x_{\bar{0}}, e] = aI_{n|n}$. However, this is impossible because $e \in \mathfrak{sl}_n \oplus \mathfrak{sl}_n$ so that $[x_{\bar{0}}, e] \in \mathfrak{sl}_n \oplus \mathfrak{sl}_n$. Hence, we deduce that $\phi^e : \mathfrak{sl}(n|n)^e \to \mathfrak{g}^e$ is surjective and $\ker \phi^e = \langle I_{n|n} \rangle$. Therefore, we have that

$$\dim \mathfrak{g}^e = \dim \mathfrak{sl}(n|n)^e - 1 = \dim \mathfrak{gl}(n|n)^e - 2.$$

Remark 5.10. It is an interesting problem to determine a basis for $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$ but we haven't covered for this thesis.

5.1.6 Analysis of results

Recall that $\bar{\mathfrak{g}} = \mathfrak{gl}(m|n) = \bar{\mathfrak{g}}_{\bar{0}} \oplus \bar{\mathfrak{g}}_{\bar{1}}$ and $\mathfrak{g} = \mathfrak{sl}(m|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. Let $\bar{G} = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ and $G = \{(A, B) : A \in \mathrm{GL}_m(\mathbb{C}), B \in \mathrm{GL}_n(\mathbb{C}) \text{ and } \det(A) = \det(B)\}$. We know that $\bar{G} = GZ(\bar{G}) = \{gg' : g \in G, g' \in Z(\bar{G})\}$ and thus $\bar{G}^e = G^e Z(\bar{G})$. Since $Z(\bar{G})$ centralizes $\mathfrak{g}_{\bar{0}}$ and we have already shown that $\mathfrak{z}(\mathfrak{g}^e) \subseteq \mathfrak{g}_{\bar{0}}$, we deduce that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = (\mathfrak{z}(\mathfrak{g}^e))^{\bar{G}^e}$. We know that $(\mathfrak{z}(\mathfrak{g}^e))^{\bar{G}^e} = \{x \in \mathfrak{z}(\mathfrak{g}^e) : g \cdot x = x \text{ for all } g \in \bar{G}^e\}$ and thus $x \in (\mathfrak{z}(\mathfrak{g}^e))^{\bar{G}^e}$ if and only if $\left(\bar{G}^e\right)^x = \bar{G}^e$. By [13, Theorem 13.2] we deduce that $\mathrm{Lie}\left(\left(\bar{G}^e\right)^x\right) = \left(\bar{\mathfrak{g}}_{\bar{0}}^e\right)^x$ for any $x \in \mathfrak{z}(\mathfrak{g}^e)$, we also know that $\mathrm{Lie}\left(\left(\bar{G}^e\right)^x\right) = \bar{\mathfrak{g}}_{\bar{0}}^e$ if and only if $\left(\bar{G}^e\right)^x = \bar{G}^e$. Moreover, we have that $(\mathfrak{z}(\mathfrak{g}^e))^{\bar{\mathfrak{g}}_{\bar{0}}^e} = \{x \in \mathfrak{z}(\mathfrak{g}^e) : [x, y] = 0 \text{ for all } y \in \bar{\mathfrak{g}}_{\bar{0}}^e\}$ and thus $x \in (\mathfrak{z}(\mathfrak{g}^e))^{\bar{\mathfrak{g}}_{\bar{0}}^e}$ if and only if $\left(\bar{\mathfrak{g}}_{\bar{0}}^e\right)^x = \bar{\mathfrak{g}}_{\bar{0}}^e$. Hence, we obtain that $x \in (\mathfrak{z}(\mathfrak{g}^e))^{\bar{G}^e}$ if and only if $x \in (\mathfrak{z}(\mathfrak{g}^e))^{\bar{\mathfrak{g}}_{\bar{0}}^e}$

and thus $(\mathfrak{z}(\mathfrak{g}^e))^{\bar{G}^e} = (\mathfrak{z}(\mathfrak{g}^e))^{\bar{\mathfrak{g}}^e_{\bar{0}}}$. Moreover, we know that $(\mathfrak{z}(\mathfrak{g}^e))^{\bar{\mathfrak{g}}^e_{\bar{0}}} = (\mathfrak{z}(\mathfrak{g}^e))^{\mathfrak{g}^e_{\bar{0}}} = \mathfrak{z}(\mathfrak{g}^e)$ because $\bar{\mathfrak{g}}^e_{\bar{0}} = \mathfrak{g}^e_{\bar{0}} + \mathfrak{z}(\bar{\mathfrak{g}}_{\bar{0}})$ and $\mathfrak{z}(\bar{\mathfrak{g}}_{\bar{0}})$ centralizes all of $\mathfrak{g}^e_{\bar{0}}$. Therefore, we deduce that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \mathfrak{z}(\mathfrak{g}^e)$.

Now we start to prove Theorems 1.1–1.3. Let λ be a partition of $(m|n)$ as defined in (5.6). Let P be the Dynkin pyramid of shape λ and $e \in \mathfrak{g}_{\bar{0}}$ be a nilpotent element determined by P. Let r_i (resp. s_i) be the number of boxes with parity $\bar{0}$ (resp. $\bar{1}$) in the ith column of P and denote $c_i = r_i + s_i$.

In order to prove Theorem 1.1 for $\mathfrak{sl}(m|n)$, we first look at the case that the corresponding Δ only has even labels. We know that the labels in the labelled Dynkin diagram Δ are the horizontal difference between consecutive boxes in the pyramid. Observe that there is no label equal to 1 in Δ if and only if $\lambda_i - \lambda_{i+1}$ are even for all $i = 1, \dots, r+s$, i.e. λ_i are all even or all odd. Based on the way that labelled Dynkin diagram is constructed, we have that $n_2(\Delta) = \lambda_1 - 1$. Therefore, we have that

$$\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \dim \mathfrak{z}(\mathfrak{g}^e) = \begin{cases} n_2(\Delta) & m \neq n, \\ n_2(\Delta) + 1 & m = n > 1. \end{cases} \tag{5.20}$$

Next we turn to look for $\mathfrak{z}(\mathfrak{g}^h)$. Note that an element in $\mathfrak{g}^h$ is of the form

$$\begin{pmatrix} x_{-\lambda_1+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_{\lambda_1-1} \end{pmatrix}$$

where $x_i \in \mathfrak{gl}(r_i|s_i)$ are block matrices for $i = -\lambda_1+1, -\lambda_1+3 \dots, \lambda_1-3, \lambda_1-1$ such that $\sum_i \mathrm{str}(x_i) = 0$. Thus we have that an element in $\mathfrak{z}(\mathfrak{g}^h)$ is of the form

$$\begin{pmatrix} d_{-\lambda_1+1} I(r_{-\lambda_1+1}|s_{-\lambda_1+1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{\lambda_1-1} I(r_{\lambda_1-1}|s_{\lambda_1-1}) \end{pmatrix}$$

for some $d_i \in \mathbb{Z}$ such that $\sum d_i(r_i - s_i) = 0$. Hence, we deduce that

$$\dim \mathfrak{z}(\mathfrak{g}^h) = \lambda_1 - 1 = n_2(\varDelta). \tag{5.21}$$

This completes the proof of Theorem 1.1 for $\mathfrak{sl}(m|n)$.

Next we prove Theorem 1.2 for Lie superalgebras $\mathfrak{g}$. Based on the way that the labelled Dynkin diagram is constructed, any labelled Dynkin diagram for $e \in \mathfrak{g}_{\bar{0}}$ is the same as the labelled Dynkin diagram for $e \in \mathfrak{gl}(m+n)$ except some of the vertices are $\otimes$, i.e. given a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$, all the labels a_i in the labelled Dynkin diagram with respect to $\mathfrak{g}$ are the same as that in the labelled Dynkin diagram with respect to $\mathfrak{gl}(m+n)$ so that $\sum a_i$ is also the same. We also have $a_i = \mathrm{col}(i+1) - \mathrm{col}(i)$, thus

$$\begin{aligned} \sum a_i &= \sum_{i=1}^{m+n-1} (\mathrm{col}(i+1) - \mathrm{col}(i)) \\ &= \mathrm{col}(m+n) - \mathrm{col}(1) \\ &= 2\lambda_1 - 2 = 2\dim \mathfrak{z}(\mathfrak{g}^e). \end{aligned} \tag{5.22}$$

Therefore, we have that

$$\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = \dim \mathfrak{z}(\mathfrak{g}^e) = \left\lceil \frac{1}{2} \sum a_i \right\rceil + \varepsilon$$

where $\varepsilon = 0$ for $m \neq n$ and $\varepsilon = 1$ for $m = n > 1$.

To prove Theorem 1.3 for $\mathfrak{g}$, we define $\mathfrak{g}_0$ to be the subalgebra generated by the root spaces $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_\alpha$ for α a simple root with label 0 or 1 in Δ. We consider two general cases: the labelled Dynkin diagram Δ has no label equal to 1 and Δ has some labels equal to 1.

1. When Δ has no label equal to 1. We have $\dim \mathfrak{g}^e = \sum_{i\in\mathbb{Z}} c_i^2 - 1$ by Corollary 5.6 and $\dim \mathfrak{z}(\mathfrak{g}^e) = n_2(\Delta)$ for $m \neq n$ and $\dim \mathfrak{z}(\mathfrak{g}^e) = n_2(\Delta) + 1$ for $m = n > 1$ by equation (5.20). Note that $e_0 = 0$ since Δ_0 has all labels equal to 0. We also have $\mathfrak{g}_0^{e_0} = \mathfrak{g}_0 = \bigoplus_{i\in\mathbb{Z}} \mathfrak{sl}(r_i|s_i)$. Then

$$\dim \mathfrak{g}_0^{e_0} = \dim \mathfrak{g}_0 = \sum_{i\in\mathbb{Z}} \dim \mathfrak{sl}(r_i|s_i) = \sum_{i\in\mathbb{Z}:c_i>0} (c_i^2 - 1).$$

Therefore,

$$\begin{aligned}\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} &= \Big(\sum_{i\in\mathbb{Z}:c_i>0} c_i^2 - 1\Big) - \sum_{i\in\mathbb{Z}:c_i>0} (c_i^2 - 1) \\ &= -1 + \sum_{i\in\mathbb{Z}:c_i>0} 1 = \lambda_1 - 1 \\ &= n_2(\Delta)\end{aligned}$$

because there are in total λ_1 columns in P with non-zero boxes. Moreover, $\mathfrak{z}(\mathfrak{g}_0^{e_0}) = \mathfrak{z}(\mathfrak{g}_0) = 0$ when $r_i \neq s_i$ for all i. However, if there exist some column number i such that $r_i = s_i$, then $\dim \mathfrak{z}(\mathfrak{sl}(r_i|s_i)) = 1$ for $r_i = s_i$ and hence $\dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = \tau$

where τ is the number of i such that $r_i = s_i$. Therefore, we have that

$$\dim \mathfrak{z}(\mathfrak{g}^e) - \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = \dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = \begin{cases} n_2(\Delta) - \tau & m \neq n, \\ n_2(\Delta) + 1 - \tau & m = n. \end{cases} \tag{5.23}$$

2. When Δ has some labels equal to 1. There are in total $2\lambda_1 + 1$ columns with labels from $-\lambda_1$ to λ_1 in the Dynkin pyramid P. Let $k > 0$ be minimal such that $c_k = 0$ and thus we know that $n_2(\Delta) = \lambda_1 - k$. Then we have that

$$\mathfrak{g}_0 \cong \mathfrak{sl}(r_{-\lambda_1+1}|s_{-\lambda_1+1}) \oplus \cdots \oplus \mathfrak{sl}(r_{-k-1}|s_{-k-1}) \oplus \mathfrak{sl}\left(\sum_{i=-k+1}^{k-1} r_i | \sum_{i=-k+1}^{k-1} s_i\right) \oplus \mathfrak{sl}(r_{k+1}|s_{k+1}) \cdots \oplus \mathfrak{sl}(r_{\lambda_1-1}|s_{\lambda_1-1}).$$

Note that the projection of e_0 in each $\mathfrak{sl}(r_i|s_i)$ is equal to 0 for $i > k$ and $i < -k$ and so $e_0 \in \mathfrak{sl}\left(\sum_{i=-k+1}^{k-1} r_i | \sum_{i=-k+1}^{k-1} s_i\right)$. We know that

$$\dim \mathfrak{sl}\left(\sum_{i=-k+1}^{k-1} r_i | \sum_{i=-k+1}^{k-1} s_i\right)^{e_0} = \sum_{i=-k+1}^{k-1} c_i^2 + \sum_{i=-k+1}^{k-1} c_i c_{i+1} - 1$$

and $\dim \mathfrak{sl}(r_i|s_i) = c_i^2 - 1$ by Corollary 5.6. We also know that P is symmetric, thus

$$\dim \mathfrak{g}_0^{e_0} = \left(\sum_{i=-k+1}^{k-1} c_i^2 + \sum_{i=-k+1}^{k-1} c_i c_{i+1} - 1\right) + 2(c_{k+1}^2 - 1) + \cdots + 2(c_{\lambda_1-1}^2 - 1).$$

Therefore, we have

$$\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = \lambda_1 - k = n_2(\varDelta). \tag{5.24}$$

Observe that when $r_i \neq s_i$ for all $|i| > k$ and $\sum_{i=-k+1}^{k-1} r_i \neq \sum_{i=-k+1}^{k-1} s_i$, then $\dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = k-1$ as $\mathfrak{z}(\mathfrak{sl}(r_i|s_i)) = 0$. However, when there exist some i for $|i| > k$ such that $r_i = s_i$, then $\dim \mathfrak{z}(\mathfrak{sl}(r_i|r_i)) = 1$. Moreover,

$$\dim \mathfrak{z}\left(\mathfrak{sl}\left(\sum_{i=-k+1}^{k-1} r_i \mid \sum_{i=-k+1}^{k-1} s_i\right)^{e_0}\right) = \begin{cases} k & \text{if } \sum_{i=-k+1}^{k-1} r_i = \sum_{i=-k+1}^{k-1} s_i, \\ k-1 & \text{if } \sum_{i=-k+1}^{k-1} r_i \neq \sum_{i=-k+1}^{k-1} s_i. \end{cases}$$

Let

$$\nu_0 = \begin{cases} 0 & \text{if } \sum_{i=-k+1}^{k-1} r_i \neq \sum_{i=-k+1}^{k-1} s_i; \\ 1 & \text{if } \sum_{i=-k+1}^{k-1} r_i = \sum_{i=-k+1}^{k-1} s_i. \end{cases}$$

Then we have that $\dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = k - 1 + \tau + \nu_0$ where τ is the number of i such that $|i| > k$ and $r_i = s_i$. Therefore, we have

$$\begin{aligned} \dim \mathfrak{z}(\mathfrak{g}^e) - \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) &= \dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} \\ &= \begin{cases} n_2(\varDelta) - \tau - \nu_0 & m \neq n; \\ n_2(\varDelta) + 1 - \tau - \nu_0 & m = n. \end{cases} \end{aligned}$$

5.2 The ortho-symplectic Lie superalgebras

In this section, we first recall the definition of the ortho-symplectic Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{osp}(m|2n)$ following [19, Section 2.3] and [4, Subsection 1.1.3]. We also develop an approach to express any ortho-symplectic Lie superalgebra by matrices. For a partition λ of $(m|2n)$, we recall the ortho-symplectic Dynkin pyramid of shape λ as given in [11, Chapter 8]. Then we construct the labelled Dynkin diagram based on a given ortho-symplectic Dynkin pyramid P.

Let $e \in \mathfrak{g}_{\bar{0}}$ be nilpotent. In Subsection 5.2.5, a basis of $\mathfrak{g}^e$ is given following the structure in [11, Section 3.2.2] and [15, Section 3.2]. Then we deduce an alternative formula to calculate the dimension of $\mathfrak{g}^e$ which is different from the formula in [11, Section 3.2.2]. In order to find out a basis of the centre $\mathfrak{z}(\mathfrak{g}^e)$ of centralizer of e in $\mathfrak{g}$, we split our analysis into two cases: (a) the Jordan type of e has all parts with even multiplicity; and (b) the Jordan type of e has all parts with multiplicity one. After that, we use results in the above two cases to generate a basis for the general case.

Let $G = \mathrm{O}_m(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$. In order to prove our main theorems for $\mathfrak{osp}(m|2n)$, we describe the structure of $\mathfrak{z}(\mathfrak{g}^e)$ under the adjoint action of G^e in Subsection 5.2.9. We start by looking at how the adjoint action of $\mathrm{O}_n(\mathbb{C})$ (resp. $\mathrm{Sp}_{2n}(\mathbb{C})$) acts on the centre of centralizer of e in Lie algebras $\mathfrak{o}(n)$ (resp. $\mathfrak{sp}(2n)$). We describe some basic background about the centralizers in $\mathrm{O}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ and details of proofs are covered in [15, Section 3] and then describe the construction of $(\mathfrak{g}^e)^{G^e}$. Then a basis of $(\mathfrak{z}(\mathfrak{g}^e))^{G^e}$ is given by possibly removing an element from the basis of $\mathfrak{z}(\mathfrak{g}^e)$.

5.2.1 The structure of ortho-symplectic Lie superalgebras

Suppose $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$. Let

$$B : V \times V \to V$$

be a non-degenerate even supersymmetric bilinear form on V, i.e. $B(V_{\bar{i}}, V_{\bar{j}}) = 0$ unless $\bar{i} + \bar{j} = \bar{0}$, the restriction of B to $V_{\bar{0}}$ is symmetric and the restriction of B to $V_{\bar{1}}$ is skew-symmetric. For $\bar{i} \in \{\bar{0}, \bar{1}\}$, we define

$$\mathfrak{osp}(V)_{\bar{i}} := \{x \in \mathfrak{gl}(V)_{\bar{i}} : B(x(v), w) = -(-1)^{\bar{i}\bar{v}} B(v, x(w)) \text{ for homogeneous } v, w \in V\}. \tag{5.25}$$

where $\bar{v}$ is the parity of v. Then $\mathfrak{osp}(V) = \mathfrak{osp}(V)_{\bar{0}} \oplus \mathfrak{osp}(V)_{\bar{1}}$ is a subalgebra of $\mathfrak{gl}(V)$ and is called the *ortho-symplectic Lie superalgebra*. By choosing a homogeneous basis of V as shown later in (5.26), we identify $\mathfrak{osp}(V)$ with a subalgebra of $\mathfrak{gl}(m|2n)$ denoted by $\mathfrak{osp}(m|2n)$, i.e. we write $\mathfrak{osp}(m|2n)$ for $\mathfrak{osp}(V)$ when $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = 2n$. Note that the even part $\mathfrak{g}_{\bar{0}} = \mathfrak{o}(m) \oplus \mathfrak{sp}(2n)$ where $\mathfrak{o}(m)$ and $\mathfrak{sp}(2n)$ are classical Lie algebras.

In order to distinguish the ortho-symplectic Lie superalgebras into several cases, we define

$$B(m, n) = \mathfrak{osp}(2m+1|2n) \text{ for } m \geq 0, n \geq 1,$$

$$D(m, n) = \mathfrak{osp}(2m|2n) \text{ for } m \geq 2, n \geq 1,$$

$$C(n) = \mathfrak{osp}(2|2n) \text{ for } n \geq 1.$$

We next explain how to represent the ortho-symplectic Lie superalgebra $\mathfrak{osp}(m|2n)$ using matrices with respect to certain choices of basis of V. Let $l = \lfloor \frac{m}{2} \rfloor$. Take a sequence $\eta_1, \ldots, \eta_{l+n} \in \{\bar{0}, \bar{1}\}$ such that $|\{i : \eta_i = \bar{0}\}| = l$ and $|\{i : \eta_i = \bar{1}\}| = n$. When m is odd, we define the standard basis $\mathfrak{B}$ of V with respect to η to be

$$\mathfrak{B} = \{v_1^{\eta_1}, \ldots, v_{l+n}^{\eta_{l+n}}, v_0^{\bar{0}}, v_{-(l+n)}^{\eta_{l+n}}, \ldots, v_{-1}^{\eta_1}\}, \tag{5.26}$$

where $\eta_i = \bar{0}$ if $v_i^{\eta_i}, v_{-i}^{\eta_i} \in V_{\bar{0}}$ and $\eta_i = \bar{1}$ if $v_i^{\eta_i}, v_{-i}^{\eta_i} \in V_{\bar{1}}$ for each i. Note that when m is even, $\mathfrak{B} = \{v_1^{\eta_1}, \ldots, v_{l+n}^{\eta_{l+n}}, v_{-(l+n)}^{\eta_{l+n}}, \ldots, v_{-1}^{\eta_1}\}$ is the standard basis of V. With the above basis, the non-degenerate even supersymmetric bilinear form B on V is given by

$$B(v_i^{\eta_i}, v_j^{\eta_j}) = \begin{cases} 0 & \text{if } i \neq -j; \\ 1 & \text{if } i = -j,\ \eta_i = \bar{0} \text{ or } \eta_i = \bar{1} \text{ and } i > 0 \\ -1 & \text{if } i = -j,\ \eta_i = \bar{1},\ i < 0. \end{cases} \tag{5.27}$$

Thus the matrix of B with respect to basis $\mathfrak{B}$ is

$$\begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & (-1)^{\eta_1} & & & \\ & \iddots & & & & \\ (-1)^{\eta_{l+n}} & & & & & \end{pmatrix}$$

Next we explain a basis of $\mathfrak{g}$ corresponding to the above basis $\mathfrak{B}$ of V. Define $e_{j,k}$ to be the linear transformation sending $v_k^{\eta_k}$ to $v_j^{\eta_j}$. Note that $\mathfrak{osp}(V)_{\bar{0}}$ is spanned by elements of the form $e_{j,-j}$ for $\eta_j = \bar{1}$ and $e_{j,k} + \gamma_{-k,-j} e_{-k,-j}$ for $\eta_j = \eta_k$ and $j \neq -k$ and $\mathfrak{osp}(V)_{\bar{1}}$ is spanned by elements of the form $e_{j,k} + \gamma_{-k,-j} e_{-k,-j}$ for $\eta_j \neq \eta_k$ where $\gamma_{-k,-j} = \pm 1$ as specified below. We use equation (5.25) to determine $\gamma_{-k,-j}$. For $i \in \{\bar{0}, \bar{1}\}, j, k \neq 0, j \neq -k$, we have

$$B((e_{j,k} + \gamma_{-k,-j} e_{-k,-j}) v_k^{\eta_k}, v_{-j}^{\eta_j}) = -(-1)^{i\eta_k} B(v_k^{\eta_k}, (e_{j,k} + \gamma_{-k,-j} e_{-k,-j}) v_{-j}^{\eta_j}). \tag{5.28}$$

according to (5.25). Then by (5.27) we have that

$$\text{LHS of (5.28)} = B(v_j^{\eta_j}, v_{-j}^{\eta_j}) = \begin{cases} 1 & \text{if } \eta_j = \bar{0} \text{ or } \eta_j = \bar{1},\ j > 0; \\ -1 & \text{if } \eta_j = \bar{1},\ j < 0. \end{cases}$$

and

$$\begin{aligned} \text{RHS of (5.28)} &= -(-1)^{i\eta_k}\gamma_{-k,-j}B(v_k^{\eta_k}, v_{-k}^{\eta_k}) \\ &= \begin{cases} -(-1)^{i\eta_k}\gamma_{-k,-j} & \text{if } \eta_k = \bar{0} \text{ or } \eta_k = \bar{1},\ k > 0; \\ (-1)^{i\eta_k}\gamma_{-k,-j} & \text{if } \eta_k = \bar{1},\ k < 0. \end{cases} \end{aligned}$$

Hence, for $e_{j,k} + \gamma_{-k,-j}e_{-k,-j} \in \mathfrak{osp}(V)_{\bar{0}}$, we have that

$$\gamma_{-k,-j} = \begin{cases} 1 & \text{if } \eta_j = \eta_k = \bar{1} \text{ and } jk < 0, \\ -1 & \text{if } \eta_j = \eta_k = \bar{0} \text{ or } \eta_j = \eta_k = \bar{1} \text{ and } jk > 0. \end{cases} \tag{5.29}$$

For $e_{j,k} + \gamma_{-k,-j}e_{-k,-j} \in \mathfrak{osp}(V)_{\bar{1}}$, we have that $\eta_j \neq \eta_k$. Hence the value of $\gamma_{-k,-j}$ is as follows.

Case 1: $\eta_j = \bar{0},\ \eta_k = \bar{1}$			Case 2: $\eta_j = \bar{1},\ \eta_k = \bar{0}$		
$\gamma_{-k,-j}$	$j > 0$	$j < 0$	$\gamma_{-k,-j}$	$j > 0$	$j < 0$
$k > 0$	1	1	$k > 0$	-1	1
$k < 0$	-1	-1	$k < 0$	-1	1

Let us denote $\text{sign}(j) = 1$ for $j > 0$ and $\text{sign}(j) = -1$ for $j < 0$. We deduce that

$$\gamma_{-k,-j} = -\eta_j\text{sign}(j) + \eta_k\text{sign}(k) \text{ for all } \eta_j \neq \eta_k. \tag{5.30}$$

We further calculate signs in basis elements $e_{0,k} + \gamma_{-k,0} e_{-k,0}$ for $k > 0$ and $e_{k,0} + \gamma_{0,-k} e_{0,-k}$ for $k > 0$ if they exist. Applying a similar argument we get that for $k > 0$,

$$\gamma_{-k,0} = \begin{cases} -1 & \text{if } \eta_k = \bar{0}, \\ 1 & \text{if } \eta_k = \bar{1}. \end{cases} \quad \text{and } \gamma_{0,-k} = -1. \tag{5.31}$$

Therefore, for the matrix expression of $\mathfrak{osp}(m|2n)$, we fix all entries above the skew diagonal to have positive sign and then use the above rules to determine signs of entries below the skew diagonal. More precisely, a basis of $\mathfrak{osp}(m|2n)$ is

$$\{e_{i,-i}, e_{j,k} + \gamma_{-k,-j} e_{-k,-j} : \eta_i = \bar{1}, 0 < j, k \leq l + n, \text{ or } j = 0, k > 0, \tag{5.32}$$
$$\text{or } j > 0, k = 0 \text{ or } jk < 0, j + k < 0\}$$

where $\gamma_{-k,-j}$ is determined using equations 5.29–5.31.

Example 5.11. For $\mathfrak{g} = \mathfrak{osp}(3|2)$, take the basis to be $\mathfrak{B} = \{v_1^{\bar{0}}, v_2^{\bar{1}}, v_0^{\bar{0}}, v_{-2}^{\bar{1}}, v_{-1}^{\bar{0}}\}$, then we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b & c & d & 0 \\ e & f & g & h & -d \\ k & l & 0 & -g & -c \\ r & s & l & -f & b \\ 0 & r & -k & -e & -a \end{pmatrix} : a, b, c, d, e, f, g, h, k, l, r, s \in \mathbb{C} \right\}.$$

5.2.2 The ortho-symplectic Dynkin pyramid

In this subsection, let $\mathfrak{g} = \mathfrak{osp}(m|2n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. Note that the nilpotent G-orbits in $\mathfrak{g}_{\bar{0}}$ are parameterized by the partitions of $(m|2n)$. Let λ be a partition of $(m|2n)$ such that

$$\lambda = (p|q) = (p_1, \ldots, p_r|q_1, \ldots, q_s) \tag{5.33}$$

where p (resp. q) is a partition of m (resp. $2n$) and $p_1 \geq \cdots \geq p_r, q_1 \geq \cdots \geq q_s$. Write $\lambda = (c^{m_c+n_c}, \ldots, 1^{m_1+n_1})$ such that $m_i = |\{j : p_j = i\}|$ and $n_i = |\{j : q_j = i\}|$.

Recall that there exists a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ with partition λ if and only if m_i is even for all even i, and n_i is even for all odd i.

Similar to $\mathfrak{gl}(m|n)$ case, we recall *the ortho-symplectic Dynkin pyramid* P for λ. It consists of $(m + 2n)$ boxes with size 2×2 in the xy-plane and is centrally symmetric about $(0, 0)$. Define the row number (resp. column number) of a box to be the y-coordinate (resp. x-coordinate) of the centre of the box. Below we describe the rule to place boxes in the upper half plane following [11, Section 8] and the rest of the boxes are added to the lower half plane in a centrally symmetric way.

Firstly, we set the zeroth row to be empty if m is even. If m is odd, there exist some odd parts appearing with odd multiplicity and we let a_1 be the largest such part in λ among all p_i. Then we put a_1 boxes into the zeroth row in the columns $1 - a_1, 3 - a_1, \ldots, a_1 - 1$. Next we remove one part of a_1 from λ.

Then λ becomes a partition that contains an even number of odd parts with odd multiplicity. Denote these representatives by $c_1 > b_1 > \cdots > c_N > b_N$. In the upper half plane, the rest of the boxes are added inductively to the next row following the rules below.

Suppose a_2 is the largest part remaining in λ. When m_{a_2} is odd, then $a_2 = c_k$ for some $k \in \{1, \ldots, N\}$. We add an even skew row of length $\frac{c_k+b_k}{2}$ and put boxes in this row in the columns $1-b_k, 3-b_k, \ldots, c_k-1$ with even parity $\bar{0}$. After that we remove c_k and b_k from the partition. Then add $\left\lfloor \frac{m_{a_2}}{2} \right\rfloor$ rows of length a_2 and boxes in these rows are placed in the columns $1-a_2, 3-a_2, \ldots, a_2-1$ with parity $\bar{0}$. When m_{a_2} is even, we only add $\left\lfloor \frac{m_{a_2}}{2} \right\rfloor$ rows in columns $1-a_2, 3-a_2, \ldots, a_2-1$ with parity $\bar{0}$. When n_{a_2} is odd, then an odd skew row of length $\frac{a_2}{2}$ is added in the columns $1, \ldots, a_2-1$ with boxes labelled by parity $\bar{1}$. We draw a box with a cross through it to represent each missing box in skew rows. Then we add $\left\lfloor \frac{n_{a_2}}{2} \right\rfloor$ rows of length a_2 and put boxes in the columns $1-a_2, 3-a_2, \ldots, a_2-1$ with parity $\bar{1}$. When n_{a_2} is even, we only add $\left\lfloor \frac{n_{a_2}}{2} \right\rfloor$ rows in the columns $1-a_2, 3-a_2, \ldots, a_2-1$ with parity $\bar{1}$. Then remove $a_2^{m_{a_2}+n_{a_2}}$ from λ.

Let $l = \left\lfloor \frac{m}{2} \right\rfloor$. In this paper, we label the Dynkin pyramid down columns from left to right with numbers $1, \ldots, l+n, -(l+n), \ldots, -1$ so that boxes labelled by i and $-i$ are central symmetrically. For the case where m is odd we have an additional even central box which is labelled by 0.

Example 5.12. The ortho-symplectic Dynkin pyramid for a partition $\lambda = (5, 3, 1|3, 3)$ is:

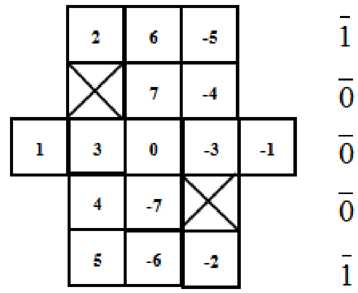


Figure 5: the Dynkin pyramid for $\lambda = (5, 3, 1|3, 3)$

The ortho-symplectic Dynkin pyramid for a partition $\lambda = (3, 3|4)$ is:

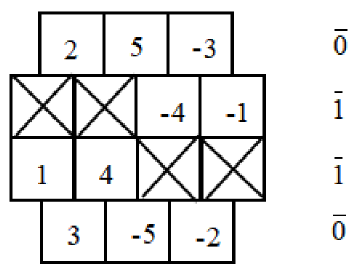


Figure 6: the Dynkin pyramid for $\lambda = (3, 3 \mid 4)$

Let row(i) (resp. col(i)) be the row number (resp. column number) of the ith box. Let $|\text{row}(i)| \in \{\bar{0}, \bar{1}\}$ be the parity of row(i) for $i = \pm 1, \ldots, \pm(l + n)$ (resp. $i = \pm 1, \ldots, \pm(l + n), 0$) if m is even (resp. m is odd). Note that the above numbering of the Dynkin pyramid determines a basis $\mathfrak{B}$ of V which is given in Subsection 5.2.1 with $\eta_i = |\text{row}(i)|$.

According to [11, Section 8], the ortho-symplectic pyramid P determines a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ and $e = \sum \gamma_{i,j} e_{i,j}$ where $\gamma_{i,j}$ is the coefficient of $e_{i,j}$ in the basis of $\mathfrak{g}_{\bar{0}}$ that involves $e_{i,j}$ and the sum is over all i and j such that

1. $\mathrm{row}(i) = \mathrm{row}(j)$ and $\mathrm{col}(j) = \mathrm{col}(i) - 2$;

2. $\mathrm{row}(i) = -\mathrm{row}(j)$ and box labelled by i is in an even skew-row in the upper half plane, $\mathrm{col}(i) = 2$ and $\mathrm{col}(j) = 0$ or $\mathrm{col}(i) = 0$ and $\mathrm{col}(j) = -2$;

3. $\mathrm{row}(i) = -\mathrm{row}(j)$ and box labelled by i is in an odd skew-row in the upper half plane, $\mathrm{col}(i) = 1$ and $\mathrm{col}(j) = -1$.

The pyramid P also defines a semisimple element $h \in \mathfrak{g}_{\bar{0}}$ such that h is the $(m+2n)$-diagonal matrix where the ith entry is $-\mathrm{col}(i)$, i.e. $h = \sum_i -\mathrm{col}(i) e_{i,i}$. Note that $\{e, h\}$ can be extended to an $\mathfrak{sl}(2)$-triple $\{e, h, f\}$ in $\mathfrak{g}_{\bar{0}}$ according to [11, Section 8].

5.2.3 Root system and labelled Dynkin diagrams for $\mathfrak{osp}(m|2n)$

Let $\mathfrak{g} = \mathfrak{osp}(m|2n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and $\mathfrak{h}$ be the set consisting of all diagonal matrices in $\mathfrak{g}$. A basis of $\mathfrak{h}^*$ is given by $\{\varepsilon_1^{\eta_1}, \ldots, \varepsilon_{l+n}^{\eta_{l+n}}\}$ where η_i are the parities as defined in Subsection 5.2.1 and $(\varepsilon_i^{\eta_i}, \varepsilon_j^{\eta_j}) = (-1)^{\eta_i}\delta_{ij}$.

According to [19, Section 2.3], for odd $m \geq 1$, $n \geq 1$, the root system for $\mathfrak{g}$ is given by $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ such that

$$\Phi_{\bar{0}} = \{\pm\varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j} : i \neq j, \eta_i = \eta_j\} \cup \{\pm\varepsilon_i^{\bar{0}}\} \cup \{\pm 2\varepsilon_i^{\bar{1}}\},$$

$$\Phi_{\bar{1}} = \{\pm\varepsilon_i^{\bar{1}}\} \cup \{\pm\varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j} : \eta_i \neq \eta_j\};$$

and a choice of positive roots is $\Phi^+ = \{\varepsilon_k^{\eta_k}, 2\varepsilon_t^{\bar{1}}, \varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j} : i < j\}$.

For $m = 2$, $n \geq 1$, the root system for $\mathfrak{g}$ is given by $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ such that

$$\Phi_{\bar{0}} = \{\pm\varepsilon_i^{\bar{1}} \pm \varepsilon_j^{\bar{1}} : i \neq j\} \cup \{\pm 2\varepsilon_i^{\bar{1}}\}, \Phi_{\bar{1}} = \{\pm\varepsilon_1^{\eta_1} \pm \varepsilon_j^{\eta_j} : \eta_1 \neq \eta_j\};$$

and a choice of positive roots is $\Phi^+ = \{2\varepsilon_t^{\bar{1}}, \varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j} : i < j\}$.

For even $m > 2$, $n \geq 1$, the root system for $\mathfrak{g}$ is given by $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ such that

$$\Phi_{\bar{0}} = \{\pm\varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j} : i \neq j, \eta_i = \eta_j\} \cup \{\pm 2\varepsilon_i^{\bar{1}}\}, \Phi_{\bar{1}} = \{\pm\varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j} : \eta_i \neq \eta_j\};$$

and a choice of positive roots is $\Phi^+ = \{2\varepsilon_t^{\bar{1}}, \varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j} : i < j\}$.

We can check that all odd roots in the Dynkin diagram of Lie superalgebras $\mathfrak{osp}(m|2n)$ are isotropic except roots $\{\pm\varepsilon_j^{\eta_j} : \eta_j = \bar{1}\}$ because for $\eta_i = \bar{0}$ and $\eta_j = \bar{1}$,

$$(\pm\varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j}, \pm\varepsilon_i^{\eta_i} \pm \varepsilon_j^{\eta_j}) = (\varepsilon_i^{\eta_i}, \varepsilon_i^{\eta_i}) + (\varepsilon_j^{\eta_j}, \varepsilon_j^{\eta_j}) = 0,$$

$$\text{and } (\pm\varepsilon_j^{\eta_j}, \pm\varepsilon_j^{\eta_j}) = 1.$$

We describe the corresponding simple roots and draw the corresponding labelled Dynkin diagrams for each case in Table 5.2. Before listing all possible labelled Dynkin diagrams for $\mathfrak{g}$, we explain the general method on determining labelled Dynkin diagrams for each case below.

The labelled Dynkin diagram with respect to $e \in \mathfrak{g}_{\bar{0}}$ is constructed as follows. Firstly, draw the ortho-symplectic Dynkin pyramid P of λ following Subsection 5.2.2. For boxes labelled by $i = 1, \dots, l+n-1$ in P, we know $|\mathrm{row}(i)| = |\mathrm{row}(i+1)|$ (resp. $|\mathrm{row}(i)| \neq |\mathrm{row}(i+1)|$) implies that the simple root $\alpha_i = \varepsilon_i^{\eta_i} - \varepsilon_{i+1}^{\eta_{i+1}}$ is even (resp. odd isotropic) since $|\mathrm{row}(i)|$ is defined to be equal to η_i. Hence, we associate a white node $\bigcirc$ (resp. a grey node $\otimes$) to the root α_i if $|\mathrm{row}(i)| = |\mathrm{row}(i+1)|$ (resp. $|\mathrm{row}(i)| \neq |\mathrm{row}(i+1)|$) and connect the $(i-1)$th and ith node with a single line. We label the ith node with $a_i = \mathrm{col}(i+1) - \mathrm{col}(i)$. For $i = l+n$, we need to consider different cases.

When m is odd, then $|\mathrm{row}(l+n)| = |\mathrm{row}(0)| = \bar{0}$ (resp. $|\mathrm{row}(l+n)| \neq |\mathrm{row}(0)|$) implies that the $(l+n)$th root $\alpha_{l+n} = \varepsilon_{l+n}^{\eta_{l+n}}$ is even (resp. odd non-isotropic). Hence, we associate a white node $\bigcirc$ (resp. a black node $\bullet$) to root α_{l+n} if $|\mathrm{row}(l+n)| =$

$|\text{row}(0)| = \bar{0}$ (resp. $|\text{row}(l+n)| \neq |\text{row}(0)|$). We connect the $(l+n-1)$th and $(l+n)$th node with 2 lines and put an arrow pointing from the $(l+n-1)$th node to the $(l+n)$th node. The $(l+n)$th node is labelled by $a_{l+n} = \text{col}(0) - \text{col}(l+n)$.

When $m = 2$, there are two possibilities for the $(n+1)$th root α_{n+1}.

- If $|\text{row}(n+1)| = \bar{1}$, we have that $\alpha_{n+1} = 2\varepsilon^{\bar{1}}_{n+1}$ is an even root. Hence, we associate a white node $\bigcirc$ to root α_{n+1}. We connect the nth and $(n+1)$th node with 2 lines and put an arrow pointing from the $(n+1)$th node to the nth node. The $(n+1)$th node is labelled by $a_{n+1} = -2\text{col}(n+1)$.

- If $|\text{row}(n+1)| = \bar{0}$, we have that $\alpha_{n+1} = \varepsilon^{\bar{1}}_{n} + \varepsilon^{\bar{0}}_{n+1}$ is odd isotropic. Hence, we associate a grey node $\otimes$ to root α_{n+1}. We connect the nth and $(n+1)$th node with 2 lines and connect the $(n-1)$th and $(n+1)$th node with a single line. The $(n+1)$th node is labelled by $a_{n+1} = -\text{col}(n+1) - \text{col}(n)$.

When $m > 2$ is even, there are three possibilities for the $(l+n)$th root α_{l+n}.

- If $|\text{row}(l+n)| = \bar{1}$, we have that $\alpha_{l+n} = 2\varepsilon^{\bar{1}}_{l+n}$ is even. Hence, we associate a white node $\bigcirc$ to root α_{l+n} and put a single line between the $(l+n)$th and the $(l+n-2)$th node. The $(l+n)$th node is labelled by $a_{l+n} = -2\text{col}(l+n)$.

- If $|\text{row}(l+n)| = \bar{0}$ and $|\text{row}(l+n-1)| = \bar{1}$, we have that $\alpha_{l+n} = \varepsilon^{\bar{1}}_{l+n-1} + \varepsilon^{\bar{0}}_{l+n}$ is an odd isotropic root. Hence, we associate a grey node $\otimes$ to root α_{l+n}. We connect the $(l+n)$th and the $(l+n-1)$th node with 2 lines and connect the $(l+n-2)$th and $(l+n)$th node with a single line. The $(l+n)$th node is labelled by $-\text{col}(l+n) - \text{col}(l+n-1)$.

- If $|\text{row}(l+n)| = |\text{row}(l+n-1)| = \bar{0}$, then $\alpha_{l+n} = \varepsilon^{\bar{0}}_{l+n-1} + \varepsilon^{\bar{0}}_{l+n}$ is even. Hence, we associate a white node $\bigcirc$ to root α_{l+n} and connect the $(l+n)$th and the $(l+n-1)$th node with 2 lines. An arrow is pointing from $(l+n)$th node to the $(l+n-1)$th node. The $(l+n)$th node is labelled by $-\text{col}(l+n) - \text{col}(l+n-1)$.

Note that the symbol $\otimes$ in Table 5.2 represents either a white or grey node can appear.

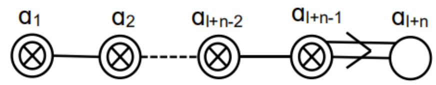
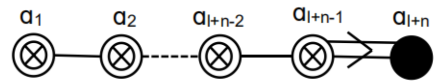
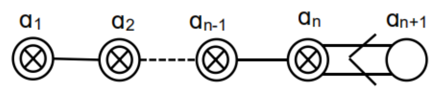
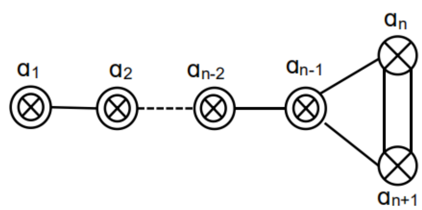
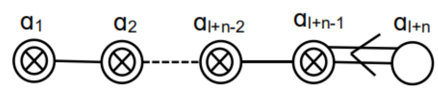
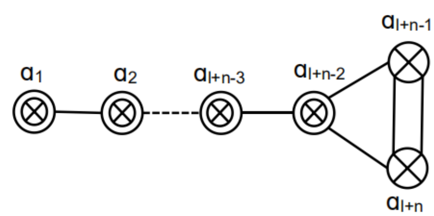
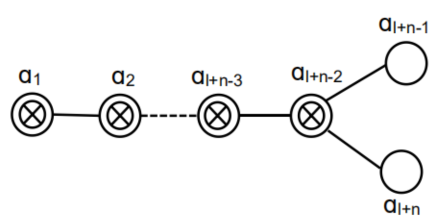
$\mathfrak{osp}(m\|2n)$	sets of simple roots	labelled Dynkin diagram
$m \geq 1$ is odd	$\{\varepsilon_1^{\eta_1} - \varepsilon_2^{\eta_2}, \ldots, \varepsilon_{l+n-1}^{\eta_{l+n-1}} - \varepsilon_{l+n}^{\eta_{l+n}}, \varepsilon_{l+n}^{\eta_{l+n}}\}$	 
$m = 2$	$\{\varepsilon_1^{\eta_1} - \varepsilon_2^{\eta_2}, \ldots, \varepsilon_n^{\eta_n} - \varepsilon_{n+1}^{\bar{1}}, 2\varepsilon_{n+1}^{\bar{1}}\}$	
	$\{\varepsilon_1^{\eta_1} - \varepsilon_2^{\eta_2}, \ldots, \varepsilon_n^{\bar{1}} - \varepsilon_{n+1}^{\bar{0}}, \varepsilon_n^{\bar{1}} + \varepsilon_{n+1}^{\bar{0}}\}$	
$m > 2$ is even	$\{\varepsilon_1^{\eta_1} - \varepsilon_2^{\eta_2}, \ldots, \varepsilon_{l+n-1}^{\eta_{l+n-1}} - \varepsilon_{l+n}^{\bar{1}}, 2\varepsilon_{l+n}^{\bar{1}}\}$	
	$\{\varepsilon_1^{\eta_1} - \varepsilon_2^{\eta_2}, \ldots, \varepsilon_{l+n-1}^{\bar{1}} - \varepsilon_{l+n}^{\bar{0}}, \varepsilon_{l+n-1}^{\bar{1}} + \varepsilon_{l+n}^{\bar{0}}\}$	
	$\{\varepsilon_1^{\eta_1} - \varepsilon_2^{\eta_2}, \ldots, \varepsilon_{l+n-1}^{\bar{0}} - \varepsilon_{l+n}^{\bar{0}}, \varepsilon_{l+n-1}^{\bar{0}} + \varepsilon_{l+n}^{\bar{0}}\}$	

Table 5.2: Labelled Dynkin diagrams for $\mathfrak{osp}(m|2n)$

Example 5.13. For a partition $(5,3,1|3,3)$, the corresponding Dynkin pyramid is shown in Figure 5. Then the corresponding labelled Dynkin diagram is:

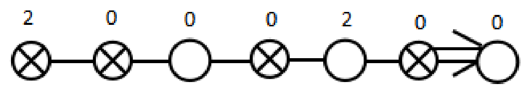


Figure 7: Labelled Dynkin diagram with respect to $(5,3,1|3,3)$

For a partition $(3,3|4)$, the corresponding Dynkin pyramid is shown in Figure 6. Then the corresponding labelled Dynkin diagram is:

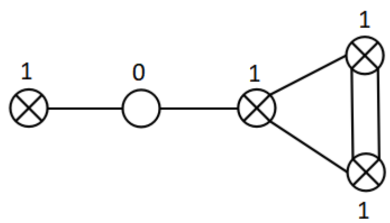


Figure 8: Labelled Dynkin diagram with respect to $(3,3|4)$

Remark 5.14. Similar to Dynkin pyramid, different numberings within columns for a ortho-symplectic Dynkin pyramid are possible and would lead to different labelled Dynkin diagrams. In this way one can get all labelled Dynkin diagrams. The following example shows we get different labelled Dynkin diagram if we allow different numberings within columns.

Example 5.15. For the partition $(5,3,1|3,3)$, if we follow the principle of numbering within columns given in this subsection, the corresponding labelled Dynkin diagram is given in Example 5.13. However, if we choose a different numbering as shown below,

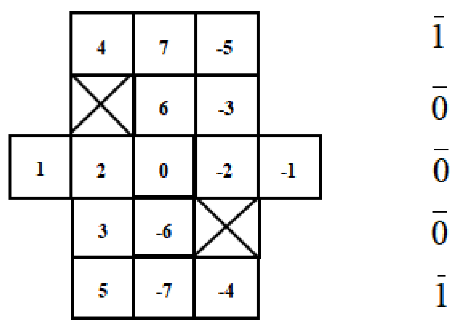


Figure 9: Different Dynkin pyramid of $(5,3,1|3,3)$

The corresponding labelled Dynkin diagram shown below is different from that is in Example 5.13.

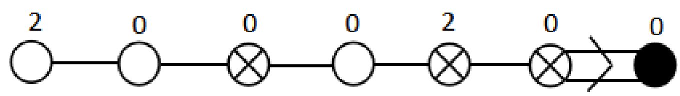


Figure 10: A different labelled Dynkin diagram for $(5,3,1|3,3)$

5.2.4 Alternative Dynkin pyramid for λ

In this subsection, we use an alternative notation for λ and rewrite

$$\lambda = (\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \lambda_{-(a+1)}, \ldots, \lambda_b, \lambda_{-b}) \tag{5.34}$$

where $\lambda_1, \ldots, \lambda_a$ are the parts with odd multiplicity, $\lambda_1 > \lambda_2 > \cdots > \lambda_a$ and $\lambda_{a+1} = \lambda_{-(a+1)} \geq \cdots \geq \lambda_b = \lambda_{-b}$. We define $|i| \in \{\bar{0}, \bar{1}\}$ such that for $c \in \mathbb{Z}$, we have $|\{i : \lambda_i = c, |i| = \bar{0}\}| = |\{j : p_j = c\}|$ and $|\{i : \lambda_i = c, |i| = \bar{1}\}| = |\{j : q_j = c\}|$ for some j. For $i, j > 0$, if $\lambda_i = \lambda_j$ and $|i| = \bar{0}$, $|j| = \bar{1}$, then we assume $i < j$. Next we establish a Dynkin pyramid $\tilde{P}$ which is different from that we used in Subsection 5.2.2. We use

$\tilde{P}$ to determine a basis for $\mathfrak{g}^e$ in Subsection 5.2.5.

The new version of Dynkin pyramid $\tilde{P}$ consists of $(m+2n)$ boxes with size 2×2 in the xy-plane and is centred on $(0,0)$. Firstly, for each λ_i with $i>0$, we put λ_i boxes both into the ith row and $-i$th row in the columns $1-\lambda_i, 3-\lambda_i, \dots, \lambda_i-1$. We start with the parts $\lambda_1,\dots,\lambda_a$. For $1\leq i\leq a$, we cross out $\lfloor \frac{\lambda_i}{2}\rfloor$ boxes in the ith row from left to right and cross out $\lceil \frac{\lambda_i}{2}\rceil$ boxes in the $-i$th row from right to left. If λ_i is odd (resp. even), we label boxes without cross in the ith row from left to right with $i_0, i_2, \dots, i_{\lambda_i-1}$ (resp. $i_1, i_3, \dots, i_{\lambda_i-1}$) and boxes without cross in the $-i$th row from left to right with $i_{-(\lambda_i-1)},\dots,i_{-2}$ (resp. $i_{-(\lambda_i-1)},\dots,i_{-1}$). Then we deal with the parts $\lambda_{a+1},\lambda_{-(a+1)},\dots,\lambda_b,\lambda_{-b}$. For $a+1\leq i\leq b$, we label boxes in the ith row with $i_{1-\lambda_i}, i_{3-\lambda_i},\dots,i_{\lambda_i-1}$ and boxes in the $-i$th row are labelled by $-i_{1-\lambda_i}, -i_{3-\lambda_i},\dots,-i_{\lambda_i-1}$. Let $|\mathrm{row}(i)|$ be the parity of the ith row such that $|\mathrm{row}(i)|=|i|$ and $|i|$ is defined in (5.34).

Note that from the above Dynkin pyramid $\tilde{P}$ we get a basis

$$\{v_{i_j} : i_j \text{ is a box in } \tilde{P}\} \tag{5.35}$$

of V. More precisely, basis elements $\{v_{i_j} : 1\leq i\leq a, 0\leq j\leq \lambda_i-1 \text{ for odd } \lambda_i \text{ and } 1\leq j\leq \lambda_i \text{ for even } \lambda_i\}\cup\{v_{i_j} : a+1\leq i\leq b\}$ (resp. $\{v_{i_j} : 1\leq i\leq a, 1-\lambda_i\leq j< 0 \text{ for odd } \lambda_i \text{ and } -\lambda_i\leq j\leq -1 \text{ for even } \lambda_i\}\cup\{v_{-i_j} : a+1\leq i\leq b\}$) correspond to the boxes in the upper (lower) half of $\tilde{P}$. The bilinear form $B(.,.)$ on V is given by

$$B(v_{i_j}, v_{i_{-j}}) = (-1)^j \text{ for } i=1,\dots,a,$$

$$B(v_{i_j}, v_{-i_{-j}}) = 1 \text{ for } i=\pm(a+1),\dots,\pm b,$$

and all other forms on basis elements are zero. We know that $\tilde{P}$ also gives a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ such that e sends v_{i_j} to $v_{i_{j-2}}$ if there exists a box labelled by $v_{i_{j-2}}$ and sends v_{i_j} to zero if there no such box.

Example 5.16. The alternative Dynkin pyramid for the partition $\lambda = (5, 3|2^2)$ is:

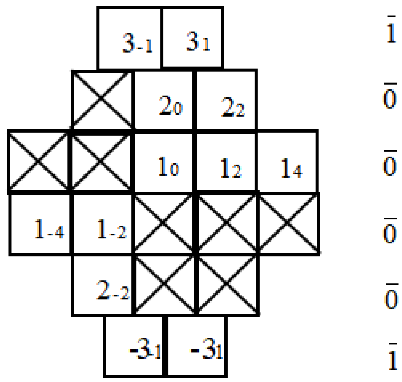


Figure 11: the alternative Dynkin pyramid for the partition $\lambda = (5, 3 \mid 2^2)$

Define $u_i = v_{i_{\lambda_i - 1}} \in V$ and thus $e^j u_i = v_{i_{\lambda_i - 2j-1}}$. Then the vectors $e^j u_i$ with $|i| = \bar{0}$, $0 \leq j \leq \lambda_i - 1$ form a basis for $V_{\bar{0}}$ and $e^j u_i$ with $|i| = \bar{1}$, $0 \leq j \leq \lambda_i - 1$ form a basis for $V_{\bar{1}}$. Moreover, we have $e^{\lambda_i} u_i = 0$ for all i and they satisfy the following conditions:

(1) For $i = 1, \dots, a$, we have

$$B(e^k u_i, e^h u_j) = \begin{cases} (-1)^k & \text{if } i = j \text{ and } k + h = \lambda_i - 1, \\ 0, & \text{otherwise,} \end{cases} \tag{5.36}$$

(2) For $i = \pm(a+1), \dots, \pm b$, then there exists $\theta_i \in \{-1, 1\}$ such that

$$B(e^k u_i, e^h u_j) = \begin{cases} (-1)^k \theta_i & \text{if } -i = j \text{ and } k + h = \lambda_i - 1, \\ 0, & \text{otherwise,} \end{cases} \tag{5.37}$$

5.2.5 Centralizer of nilpotent element $e \in \mathfrak{g}_{\bar{0}}$

In this subsection, we give a basis for $\mathfrak{g}^e$ based on [29, Chapter 1] and then state an alternative formula to the formula given in [11, Subsection 3.2.2] for $\dim \mathfrak{g}^e$. In order to describe a basis for $\mathfrak{g}^e$, we define $i^* = i$ for $i = 1, \dots, a$ and $i^* = -i$ for $i = \pm(a+1), \dots, \pm b$.

Recall that a basis for $\mathfrak{gl}(m|n)^e$ is known in terms of $\xi_i^{j,k}$ such that $\xi_i^{j,k}$ sends u_i to $e^k u_j$ and all other u_t to 0. We know that $\mathfrak{g}^e = \mathfrak{g} \cap \mathfrak{gl}(m|2n)^e$. Therefore, in accordance with arguments in [15, Section 3.2] and [29, Chapter 1], the elements in a basis of $\mathfrak{g}_{\bar{0}}^e$ are of the form:

$$\xi_i^{i^*,\lambda_i-1-k} \text{ for } 0 \leq k \leq \lambda_i - 1,\ k \text{ is odd if } |i| = \bar{0} \text{ and } k \text{ is even if } |i| = \bar{1};$$

$$\xi_i^{j,\lambda_j-1-k} + \varepsilon_i^{j,\lambda_j-1-k} \xi_{j^*}^{i^*,\lambda_i-1-k} \text{ for all } 0 \leq k \leq \min\{\lambda_i, \lambda_j\} - 1, |i| = |j|$$

where $\varepsilon_i^{j,\lambda_j-1-k} \in \{\pm 1\}$ can be determined according to [15, Section 3.2]. More precisely, we have that

$$\varepsilon_i^{j,\lambda_j-1-k} = (-1)^{\lambda_j-k} \theta_j \theta_i \text{ for all } 0 \leq k \leq \min\{\lambda_i, \lambda_j\} - 1, |i| = |j|. \tag{5.38}$$

According to [11, Section 3.2.2], elements in a basis of $\mathfrak{g}_{\bar{1}}^e$ are of the form

$$\xi_i^{j,\lambda_j-1-k} \pm \xi_{j^*}^{i^*,\lambda_i-1-k} \text{ for } 0 \leq k \leq \min\{\lambda_i, \lambda_j\} - 1, |i| \neq |j| \tag{5.39}$$

with appropriate choices of signs. Note that signs in (5.39) can also be determined explicitly.

Let $e = e_{\mathfrak{o}} + e_{\mathfrak{sp}}$ where $e_{\mathfrak{o}} \in \mathfrak{o}(m)$ and $e_{\mathfrak{sp}} \in \mathfrak{sp}(2n)$. Based on [11, Section 3.2.2] and

[15, page 25], we have that

$$\begin{aligned}\dim \mathfrak{g}_{\bar{0}}^e &= \dim \mathfrak{o}(m)^{e_{\mathfrak{o}}} + \dim \mathfrak{sp}(2n)^{e_{\mathfrak{sp}}} \\ &= \frac{1}{2}\dim \mathfrak{gl}(m)^{e_{\mathfrak{o}}} - \frac{1}{2}\left|\{i : \lambda_i \text{ is odd}, |i| = \bar{0}\}\right| \\ &+ \frac{1}{2}\dim \mathfrak{gl}(2n)^{e_{\mathfrak{sp}}} + \frac{1}{2}\left|\{i : \lambda_i \text{ is odd}, |i| = \bar{1}\}\right| \\ &= \frac{1}{2}\dim \mathfrak{gl}(m|2n)_{\bar{0}}^e - \frac{1}{2}\left|\{i : \lambda_i \text{ is odd}, |i| = \bar{0}\}\right| \\ &+ \frac{1}{2}\left|\{i : \lambda_i \text{ is odd}, |i| = \bar{1}\}\right|\end{aligned}$$

In addition, in [11, Subsection 3.2.2] Hoyt argues that

$$\dim \mathfrak{g}_{\bar{1}}^e = \frac{1}{2}\dim \mathfrak{gl}(m|2n)_{\bar{1}}^e = \sum_{|i|=\bar{0},|j|=\bar{1}:i,j} \min(\lambda_i, \lambda_j).$$

Therefore, we have that

$$\begin{aligned}\dim \mathfrak{g}^e &= \frac{1}{2}\dim \mathfrak{gl}(m|2n)^e - \frac{1}{2}\left|\{i : \lambda_i \text{ is odd}, |i| = \bar{0}\}\right| \\ &+ \frac{1}{2}\left|\{i : \lambda_i \text{ is odd}, |i| = \bar{1}\}\right|. \qquad (5.40)\end{aligned}$$

We obtain an alternative formula for $\dim \mathfrak{g}^e$ below.

Proposition 5.17. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{osp}(m|2n)$ and λ be a partition of $(m|2n)$. Denote by P the ortho-symplectic Dynkin pyramid of λ. Then P determines an $\mathfrak{sl}(2)$-triple $\{e, h, f\}$ in $\mathfrak{g}_{\bar{0}}$. Let c_i be the number of boxes in the ith column of P and r_i (resp. s_i) be the number of boxes with parity $\bar{0}$ (resp. $\bar{1}$) in the ith column of P. We have that $\dim \mathfrak{g}^e = \frac{1}{2}(\sum c_i^2 + \sum c_i c_{i+1}) - \frac{r_0}{2} + \frac{s_0}{2}$.*

Proof. Let $\mathfrak{g} = \bigoplus_{l \in \mathbb{Z}} \mathfrak{g}(l)$. Using the same argument as in the proof of Proposition 5.5, the map ad $e : \mathfrak{g}(\geq -1) \to \mathfrak{g}(\geq 1)$ is surjective according to [11, Definition 4.1]. Note that

$$\ker(\text{ad } e) = \mathfrak{g}^e \subseteq \mathfrak{g}(l \geq -1) \text{ and } \text{im}(\text{ad } e) = \mathfrak{g}(l \geq 1).$$

Thus by the rank–nullity theorem, we have $\dim \ker(\text{ad } e) + \dim \text{im}(\text{ad } e) = \dim \mathfrak{g}(l \geq -1)$. Hence,

$$\begin{aligned} \dim \mathfrak{g}^e &= \dim \mathfrak{g}(l \geq -1) - \dim \mathfrak{g}(l \geq 1) \\ &= \dim \mathfrak{g}(0) + \dim \mathfrak{g}(-1). \end{aligned}$$

We also calculate that $[h, e_{j,k} + \gamma_{-k,-j} e_{-k,-j}] = (\text{col}(k) - \text{col}(j))(e_{j,k} + \gamma_{-k,-j} e_{-k,-j})$, this implies that $e_{j,k} + \gamma_{-k,-j} e_{-k,-j} \in \mathfrak{g}(l)$ if $l = \text{col}(k) - \text{col}(j)$. Hence, we have that

$$\mathfrak{g}(0) = \langle e_{j,k} + \gamma_{-k,-j} e_{-k,-j} : \text{col}(k) = \text{col}(j) \rangle \cong \left(\bigoplus_{i<0} \mathfrak{gl}(r_i|s_i) \right) \oplus \mathfrak{osp}(r_0|s_0),$$

Moreover, we know that

$$\dim \mathfrak{osp}(r_0|s_0) = \frac{(r_0 + s_0)^2 - (r_0 + s_0)}{2} + s_0 = \frac{c_0^2 - r_0 + s_0}{2}.$$

Thus we have that

$$\dim \mathfrak{g}(0) = \sum_{i<0} c_i^2 + \frac{c_0^2}{2} - \frac{r_0}{2} + \frac{s_0}{2} = \frac{1}{2} \sum c_i^2 - \frac{r_0}{2} + \frac{s_0}{2}.$$

Observe that

$$\mathfrak{g}(-1) = \langle e_{j,k} + \gamma_{-k,-j} e_{-k,-j} : \text{col}(k) - \text{col}(j) = -1 \rangle \cong \bigoplus_{i<0} \text{Hom}\left(\mathbb{C}^{c_i}, \mathbb{C}^{c_{i+1}} \right).$$

Hence, we have that $\dim \mathfrak{g}(-1) = \sum_{i<0} c_i c_{i+1}$.

Moreover, for each row which corresponds to an odd λ_i, there must exist a box in the 0th column. Thus we have that $r_0 = |\{i : \lambda_i \text{ is odd}, |i| = \bar{0}\}|$ and $s_0 = |\{i : \lambda_i \text{ is odd}, |i| = \bar{1}\}|$. Therefore, we deduce that

$$\dim \mathfrak{g}^e = \frac{1}{2}(\sum c_i^2 + \sum c_i c_{i+1}) - \frac{r_0}{2} + \frac{s_0}{2}.$$

□

5.2.6 Centre of centralizer of nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ with Jordan type λ such that all parts of λ have even multiplicity

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{osp}(m|2n)$ with a Cartan subalgebra $\mathfrak{h}$, we have the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

and $\mathfrak{h}$ is self-centralizing. Construct an alternative Dynkin pyramid $\tilde{P}$ following the rules described in Subsection 5.2.4. We know that $\tilde{P}$ determines a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ and we can embed e into an $\mathfrak{sl}(2)$-triple $\mathfrak{s} = \langle e, h, f \rangle \subseteq \mathfrak{g}_{\bar{0}}$ by the Jacobson–Morozov theorem. Then we have that the centralizer $\mathfrak{h}^e$ of e in $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}^e$ according to [1, Section 3]. Moreover, [1, Lemma 13] shows that the set of weights of $\mathfrak{h}^e$ on $\mathfrak{g}^e$ is equivalent to the set of weights of $\mathfrak{h}^e$ on $\mathfrak{g}$. Then we can obtain the following decomposition for $\mathfrak{g}^e$:

$$\mathfrak{g}^e = (\mathfrak{g}^e)^{\mathfrak{h}^e} \oplus \bigoplus_{\alpha \in \Phi^e} \mathfrak{g}^e_\alpha$$

where $\Phi^e \subseteq (\mathfrak{h}^e)^*$ is defined as the set of non-zero weights of $\mathfrak{h}^e$ on $\mathfrak{g}^e$ and $\mathfrak{g}^e_\alpha = \{x \in \mathfrak{g}^e : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}^e\}$. Hence, we have $\mathfrak{z}(\mathfrak{g}^e) \subseteq (\mathfrak{g}^e)^{\mathfrak{h}^e}$. Note that $\mathfrak{z}(\mathfrak{g}^e) \subseteq (\mathfrak{g}^e)^{\mathfrak{h}^e}$

is a general result for all basic classical Lie superalgebras $\mathfrak{g}$, but in this section we only need this result for the case in which $\mathfrak{g} = \mathfrak{osp}(m|2n)$.

We first consider the case where all parts of the Jordan type $\lambda = (\lambda_1, \lambda_{-1}, \dots, \lambda_b, \lambda_{-b})$ with respect to e have even multiplicity. Then we know that there are $2b$ rows in the Dynkin pyramid $\tilde{P}$ and we label rows in the upper half plane from bottom to top by $1, 2, \dots, b$ and rows in the lower half plane in a symmetric way. Let S be the set spanned by the odd powers of e, i.e. $S = \langle e, e^3, \dots, e^t : t = 2\lfloor \frac{\lambda_1}{2} \rfloor - 1 \rangle$.

Theorem 5.18. *Let $\mathfrak{g} = \mathfrak{osp}(m|2n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and the Jordan type with respect to $e \in \mathfrak{g}_{\bar{0}}$ is $\lambda = (\lambda_1, \lambda_{-1}, \dots, \lambda_b, \lambda_{-b})$ such that $\lambda_1 \geq \cdots \geq \lambda_b$ and $\lambda_i = \lambda_{-i}$. Then $\mathfrak{z}(\mathfrak{g}^e) = S$ except when λ_1 is odd and $\lambda_1 > \lambda_i$ for $i \neq \pm 1$ and $|1| = \bar{0}$. In which case, we have that $\mathfrak{z}(\mathfrak{g}^e) = S \oplus \langle \xi_1^{1,\lambda_1 - 1} - \xi_{-1}^{-1,\lambda_1 - 1} \rangle$.*

Proof. This proof will proceed in steps. It is clear that $S \subseteq \mathfrak{z}(\mathfrak{g}^e)$. We know that $e^l = \sum_{j=1}^{b} (\xi_j^{j,l} \pm \xi_{-j}^{-j,l})$ and $e^l \in \mathfrak{g}$ for all odd l with $0 \leq l < \lambda_1$.

Step 1: Deduce that $\mathfrak{z}(\mathfrak{g}^e) \subseteq \langle \xi_j^{j,\lambda_j - 1 - k} + \varepsilon_j^{j,\lambda_j - 1 - k} \xi_{-j}^{-j,\lambda_j - 1 - k} : 1 \leq j \leq b, 0 \leq k \leq \lambda_j - 1 \rangle$. We first determine $(\mathfrak{g}^e)^{\mathfrak{h}^e}$. Define $h_i = \xi_i^{i,0} - \xi_{-i}^{-i,0}$ for all $0 \leq i \leq b$. Then we have $\mathfrak{h}^e = \langle h_i : 0 \leq i \leq b \rangle$. We define $\beta_i \in (\mathfrak{h}^e)^*$ by

$$\beta_i(h_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j, \end{cases}$$

and $\beta_{-i} = -\beta_i$ for all i. As in the proof of [29, Theorem 2], we can calculate that $[\xi_i^{i,0}, \xi_j^{t,s}] = \delta_{it} \xi_j^{i,s} - \delta_{ij} \xi_i^{t,s}$. Thus for $h \in \mathfrak{h}^e$, by computing the commutator between h and the basis element of $\mathfrak{g}^e$ that is of the form $\xi_l^{j,\lambda_j - 1 - k} + \varepsilon_l^{j,\lambda_j - 1 - k} \xi_{-j}^{-l,\lambda_l - 1 - k}$ for $l, j \geq 0$,

$l \neq j$, we get

$$[h, \xi_l^{j,\lambda_j-1-k} + \varepsilon_l^{j,\lambda_j-1-k}\xi_{-j}^{-l,\lambda_l-1-k}] = (\beta_j - \beta_l)(h)(\xi_l^{j,\lambda_j-1-k} + \varepsilon_l^{j,\lambda_j-1-k}\xi_{-j}^{-l,\lambda_l-1-k}).$$

By computing the commutator between h and the basis element of $\mathfrak{g}^e$ that is of the form
$\xi_l^{-j,\lambda_j-1-k} + \varepsilon_l^{-j,\lambda_j-1-k}\xi_j^{-l,\lambda_l-1-k}$ for $l, j \geq 0$, $l \neq j$, we get

$$[h, \xi_l^{-j,\lambda_j-1-k} + \varepsilon_l^{-j,\lambda_j-1-k}\xi_j^{-l,\lambda_l-1-k}] = (-\beta_j - \beta_l)(h)(\xi_l^{-j,\lambda_j-1-k} + \varepsilon_l^{-j,\lambda_j-1-k}\xi_j^{-l,\lambda_l-1-k})$$

and by computing the commutator between h and the basis element of $\mathfrak{g}^e$ that is of the form $\xi_{-l}^{j,\lambda_j-1-k} + \varepsilon_{-l}^{j,\lambda_j-1-k}\xi_{-j}^{l,\lambda_l-1-k}$ for $l, j \geq 0$, $l \neq j$, we get

$$[h, \xi_{-l}^{j,\lambda_j-1-k} + \varepsilon_{-l}^{j,\lambda_j-1-k}\xi_{-j}^{l,\lambda_l-1-k}] = (\beta_j + \beta_l)(h)(\xi_{-l}^{j,\lambda_j-1-k} + \varepsilon_{-l}^{j,\lambda_j-1-k}\xi_{-j}^{l,\lambda_l-1-k}).$$

Hence, the coefficient of $\xi_l^{j,\lambda_j-1-k} + \varepsilon_l^{j,\lambda_j-1-k}\xi_{-j}^{-l,\lambda_l-1-k}$ in an element of $(\mathfrak{g}^e)^{\mathfrak{h}^e}$ can be nonzero if and only if $(\beta_j - \beta_l)(h) = 0$ for all $h \in \mathfrak{h}^e$, the coefficient of $\xi_l^{-j,\lambda_j-1-k} + \varepsilon_l^{-j,\lambda_j-1-k}\xi_j^{-l,\lambda_l-1-k}$ in an element of $(\mathfrak{g}^e)^{\mathfrak{h}^e}$ can be nonzero if and only if $(-\beta_j - \beta_l)(h) = 0$ for all $h \in \mathfrak{h}^e$ and the coefficient of $\xi_{-l}^{j,\lambda_j-1-k} + \varepsilon_{-l}^{j,\lambda_j-1-k}\xi_{-j}^{l,\lambda_l-1-k}$ in an element of $(\mathfrak{g}^e)^{\mathfrak{h}^e}$ can be nonzero if and only if $(\beta_j + \beta_l)(h) = 0$. Take $h = h_j$, we obtain that $(\beta_j - \beta_l)(h) = 1$ and $(-\beta_j - \beta_l)(h) = -1$ for $l \neq j$. Therefore, we deduce that

$$(\mathfrak{g}^e)^{\mathfrak{h}^e} = \left\langle \xi_j^{j,\lambda_j-1-k} + \varepsilon_j^{j,\lambda_j-1-k}\xi_{-j}^{-j,\lambda_j-1-k} \right\rangle$$

and thus $\mathfrak{z}(\mathfrak{g}^e) \subseteq \left\langle \xi_j^{j,\lambda_j-1-k} + \varepsilon_j^{j,\lambda_j-1-k}\xi_{-j}^{-j,\lambda_j-1-k} \right\rangle$.
We now have that an element x in $\mathfrak{z}(\mathfrak{g}^e)$ is of the form $\sum_{j,k} c_j^{j,k}(\xi_j^{j,\lambda_j-1-k} + \varepsilon_j^{j,l}\xi_{-j}^{-j,\lambda_j-1-k})$ from Step 1.

Fix j and k and let $l = \lambda_j - 1 - k$.

Step 2: Show that $c_j^{j,l} = 0$ whenever l is even except in one special case.

According to Equation (5.38), we have that

$$\varepsilon_j^{j,l} = \begin{cases} 1 & \text{if } l \text{ is odd;} \\ -1 & \text{if } l \text{ is even.} \end{cases}$$

Assume l is even, we consider an element $\xi_i^{-i,0}$. Note that $\xi_i^{-i,0} \in \mathfrak{g}^e$ if λ_i is even and $|i| = \bar{0}$, or λ_i is odd and $|i| = \bar{1}$. Hence, when $\xi_i^{-i,0} \in \mathfrak{g}^e$, the commutator between $\xi_i^{-i,0}$ and x is:

$$[\xi_i^{-i,0}, \sum_{j,l} c_j^{j,l}(\xi_j^{j,l} - \xi_{-j}^{-j,l})] = 2\sum_{j,l} c_j^{j,l}\xi_j^{-j,l}.$$

Hence, we deduce that $c_j^{j,l} = 0$ whenever l is even. When $\xi_i^{-i,0} \notin \mathfrak{g}^e$, i.e. λ_i is odd and $|i| = \bar{0}$, or λ_i is even and $|i| = \bar{1}$. We take the commutator between $\xi_i^{-i,1} \in \mathfrak{g}^e$ and x:

$$[\xi_i^{-i,1}, \sum_{j,l} c_j^{j,l}(\xi_j^{j,l} - \xi_{-j}^{-j,l})] = 2\sum_{j,l} c_j^{j,l}\xi_j^{-j,l+1}.$$

Hence, we deduce that $c_j^{j,l} = 0$ for all l is even except when $l = \lambda_j - 1$, in which case we have that λ_j is odd and $|j| = \bar{0}$.

Next we consider the case when $l = \lambda_j - 1$, λ_j is odd and $|j| = \bar{0}$, suppose that there exist some $i \neq j$, $i > 0$ and $\lambda_i \geq \lambda_j$, then $\xi_i^{j,0} + \varepsilon_i^{j,0}\xi_{-j}^{-i,\lambda_i-\lambda_j} \in \mathfrak{g}^e$, by computing

$$[\xi_i^{j,0} + \varepsilon_i^{j,0}\xi_{-j}^{-i,\lambda_i-\lambda_j}, \sum_j c_j^{j,\lambda_j-1}(\xi_j^{j,\lambda_j-1} - \xi_{-j}^{-j,\lambda_j-1})] = -\sum_j c_j^{j,\lambda_j-1}(\varepsilon_i^{j,0}\xi_{-j}^{-i,\lambda_i-1} + \xi_i^{j,\lambda_j-1}),$$

we obtain that $c_j^{j,\lambda_j-1} = 0$ except there does not exist such $i \neq j$, $i > 0$ and $\lambda_i \geq \lambda_j$, which implies that $j=1$. Hence, we obtain that

$$\mathfrak{z}(\mathfrak{g}^e) \subseteq \left\langle \xi_j^{j,l} + \xi_{-j}^{-j,l} : l \text{ is odd} \right\rangle$$

except for λ_1 is odd and $|1| = \bar{0}$ such that $\lambda_1 > \lambda_i$ for $i \neq \pm 1$, in which case we cannot show that $c_1^{1,\lambda_1-1} = 0$ and $\mathfrak{z}(\mathfrak{g}^e) \subseteq \left\langle \xi_j^{j,l} + \xi_{-j}^{-j,l} : l \text{ is odd} \right\rangle \oplus \left\langle \xi_1^{1,\lambda_1-1} - \xi_{-1}^{-1,\lambda_1-1} \right\rangle$.

Step 3: Show that $c_i^{i,l} = c_t^{t,l}$ for all i,t whenever l is odd.

Now consider $\xi_i^{t,0} + \varepsilon_i^{t,0} \xi_{-t}^{-i,\lambda_i-\lambda_t} \in \mathfrak{g}^e$ with $i < t$ and $i,t > 0$. Then we compute that

$$\begin{aligned}\left[\xi_i^{t,0} + \varepsilon_i^{t,0}\xi_{-t}^{-i,\lambda_i-\lambda_t}, x\right] &= \left[\xi_i^{t,0} + \varepsilon_i^{t,0}\xi_{-t}^{-i,\lambda_i-\lambda_t}, \sum_{j,l} c_j^{j,l}(\xi_j^{j,l} + \xi_{-j}^{-j,l})\right] \\ &= \sum_l (c_i^{i,l} - c_t^{t,l})\xi_i^{t,l} + \sum_l (c_t^{t,l} - c_i^{i,l})\varepsilon_i^{t,0}\xi_{-t}^{-i,\lambda_i-\lambda_t+l}.\end{aligned}$$

This equals to zero if and only if $c_i^{i,l} = c_t^{t,l}$ for all i and t. Hence, $\mathfrak{z}(\mathfrak{g}^e) \subseteq S$ and therefore $\mathfrak{z}(\mathfrak{g}^e) = S$ except for $\lambda_1 > \lambda_i$ for $i \neq \pm 1$ and $|1| = \bar{0}$.

Step 4: Show that $\xi_1^{1,\lambda_1-1} - \xi_{1^*}^{1^*,\lambda_1-1} \in \mathfrak{z}(\mathfrak{g}^e)$ when λ_1 is odd with $|1| = \bar{0}$ such that $\lambda_1 > \lambda_i$ for $i \neq \pm 1$.

Suppose λ_1 is odd with $|1| = \bar{0}$ and $l = \lambda_1 - 1$. Suppose $\lambda_1 > \lambda_i$ for $i \neq \pm 1$. Clearly $\xi_1^{1,\lambda_1-1} - \xi_{-1}^{-1,\lambda_1-1}$ commutes with all basis elements in $\mathfrak{g}^e$ of the form ξ_i^{-i,λ_i-1-k} for $i \neq \pm 1$ and $\xi_i^{j,\lambda_j-1-k} \pm \xi_{-j}^{-i,\lambda_i-1-k}$ for $i,j \neq \pm 1$.

It remains to check whether $\xi_1^{1,\lambda_1-1} - \xi_{-1}^{-1,\lambda_1-1}$ commutes with ξ_1^{-1,λ_1-1-k}, $\xi_{-1}^{1,\lambda_1-1-k}$ for $0 \leq k \leq \lambda_1 - 1$, k is odd, $\xi_1^{j,\lambda_j-1-k} \pm \xi_{-j}^{-1,\lambda_1-1-k}$, $\xi_{-1}^{j,\lambda_j-1-k} \pm \xi_{-j}^{1,\lambda_1-1-k}$ for $0 \leq k \leq \lambda_j - 1$ and $\xi_i^{1,\lambda_1-1-k} \pm \xi_{-1}^{-i,\lambda_i-1-k}$, $\xi_i^{-1,\lambda_1-1-k} \pm \xi_1^{-i,\lambda_i-1-k}$ for $0 \leq k \leq \lambda_i - 1$. Note that

$$[\xi_1^{1,\lambda_1-1} - \xi_{-1}^{-1,\lambda_1-1}, \xi_1^{-1,\lambda_1-1-k}] = -2\xi_1^{-1,2\lambda_1-2-k}$$

and

$$[\xi_1^{1,\lambda_1-1}-\xi_{-1}^{-1,\lambda_1-1},\xi_1^{-1,\lambda_1-1-k}]=2\xi\ {-}_1^{1,2\lambda_1-2-k}\,.$$

We know that $\xi_1^{-1,2\lambda_1-2-k}=0$ and $\xi{-}_1^{1,2\lambda_1-2-k}=0$ because $2\lambda_1-2-k\geq\lambda_1$. Similarly we can compute that $[\xi_1^{1,\lambda_1-1}-\xi_{-1}^{-1,\lambda_1-1},\xi_1^{j,\lambda_j-1-k}\pm\xi_{-j}^{-1,\lambda_1-1-k}]=0$, $[\xi_1^{1,\lambda_1-1}-\xi_{-1}^{-1,\lambda_1-1},\xi_{-1}^{j,\lambda_j-1-k}\pm\xi_{-j}^{1,\lambda_1-1-k}]=0$, $[\xi_1^{1,\lambda_1-1}-\xi_{-1}^{-1,\lambda_1-1},\xi_i^{1,\lambda_1-1-k}\pm\xi_{-1}^{-i,\lambda_i-1-k}]=0$ and $[\xi_1^{1,\lambda_1-1}-\xi_{-1}^{-1,\lambda_1-1},\xi_i^{-1,\lambda_1-1-k}\pm\xi_1^{-i,\lambda_i-1-k}]=0$. Hence, we have that $\xi_1^{1,\lambda_1-1}-\xi_{-1}^{-1,\lambda_1-1}$ commutes with all basis elements in $\mathfrak{g}^e$. Therefore, we have that $\xi_1^{1,\lambda_1-1}-\xi_{-1}^{-1,\lambda_1-1}\in\mathfrak{z}(\mathfrak{g}^e)$ in this case and $\mathfrak{z}(\mathfrak{g}^e)=S\oplus\left\langle\xi_1^{1,\lambda_1-1}-\xi_{-1}^{-1,\lambda_1-1}\right\rangle$. □

5.2.7 Centre of centralizer of nilpotent element $e\in\mathfrak{g}_{\bar{0}}$ with Jordan type λ such that all parts of λ have multiplicity one

Next we consider the case where all parts of the Jordan type $\lambda=(\lambda_1,\dots,\lambda_a)$ of e have multiplicity one, which implies λ_i is odd for $|i|=\bar{0}$ and λ_i is even for $|i|=\bar{1}$. Note that when $m=0$ or $n=0$, then $\mathfrak{g}$ is either an orthogonal or symplectic Lie algebra, a basis of $\mathfrak{z}(\mathfrak{g}^e)$ has been given in [29, Theorem 4]. Recall that $S=\langle e,e^3,\dots,e^t:t=2\lfloor\frac{\lambda_1}{2}\rfloor-1\rangle$. Below we give a general result for $\mathfrak{z}(\mathfrak{g}^e)$.

Theorem 5.19. *Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}=\mathfrak{osp}(m|2n)$ and let $e\in\mathfrak{g}_{\bar{0}}$ be nilpotent with a partition $\lambda=(\lambda_1,\dots,\lambda_a)$ which is defined in (5.34) and $\lambda_1>\dots>\lambda_a$.*

(1) If $m=0$ or $n=0$, we have $\mathfrak{z}(\mathfrak{g}^e)=S$ except when $n=0$, $\lambda_2>\lambda_3$ and both λ_1 and λ_2 are odd, in which case we have $\mathfrak{z}(\mathfrak{g}^e)=S\oplus\langle\xi_1^{2,\lambda_2-1}-\xi_2^{1,\lambda_1-1}\rangle$;

(2) If $m,n\neq0$, we have $\mathfrak{z}(\mathfrak{g}^e)=S$ except when $a\geq3$, $|1|=|2|=\bar{0}$, or $a=2$ and $|1|\neq|2|$, in which cases we have $\mathfrak{z}(\mathfrak{g}^e)=S\oplus\langle\xi_1^{2,\lambda_2-1}-\xi_2^{1,\lambda_1-1}\rangle$.

Proof. For $m = 0$ or $n = 0$, then $\mathfrak{g} = \mathfrak{sp}(2n)$ or $\mathfrak{so}(m)$, the detailed proof can be found in [29, Theorem 4].

For $m, n \neq 0$. It is clear that $S \subseteq \mathfrak{z}(\mathfrak{g}^e)$. We know that $e^l = \sum_{i=1}^{a} \xi_i^{i,l}$ and $e^l \in \mathfrak{g}$ for all odd l and $0 \leq l \leq \lambda_1 - 1$. Note that a basis of $\mathfrak{g}^e$ contains elements of the form:

$$\xi_i^{i,k} \text{ for all } 1 \leq i \leq a \text{ and odd } k \text{ with } 0 < k \leq \lambda_i - 1,$$

$$\xi_i^{j,\lambda_j-1-k} + \varepsilon_i^{j,\lambda_j-1-k} \xi_j^{i,\lambda_i-1-k} \text{ for all } 1 \leq i, j \leq a,\ 0 \leq k \leq \min\{\lambda_i, \lambda_j\} - 1$$

where $\varepsilon_i^{j,\lambda_j-1-k} \in \{\pm 1\}$ can be determined explicitly. Thus an element $x \in \mathfrak{z}(\mathfrak{g}^e)$ is of the form

$$x = \sum_{i,k} c_i^{i,k} \xi_i^{i,k} + \sum_{i,j,k} c_i^{j,k} (\xi_i^{j,\lambda_j-1-k} + \varepsilon_i^{j,\lambda_j-1-k} \xi_j^{i,\lambda_i-1-k})$$

where $c_i^{j,k} \in \mathbb{C}$ are coefficients.

Assume $a \geq 3$. For $1 \leq t \leq a$, we have that $\xi_t^{t,1}$ commutes with $\sum_{i,k} c_i^{i,k} \xi_i^{i,k}$. By taking the commutator between $\xi_t^{t,1}$ and x we obtain that

$$\begin{aligned}
[\xi_t^{t,1}, x] = &\sum_{i<t} \sum_{k=0}^{\lambda_t - 1} c_i^{t,k} (\xi_i^{t,\lambda_t-k} - \varepsilon_i^{t,\lambda_t-1-k} \xi_t^{i,\lambda_i-k}) \\
&+ \sum_{t<i} \sum_{k=0}^{\lambda_i - 1} c_t^{i,k} (\varepsilon_t^{i,\lambda_i-1-k} \xi_i^{t,\lambda_t-k} - \xi_t^{i,\lambda_i-k}).
\end{aligned}$$

This is equal to 0 for all t if and only if $c_i^{j,k} = 0$ for all $0 < k \leq \lambda_j - 1$. Now we have that $x = \sum_{i,k} c_i^{i,k} \xi_i^{i,k} + \sum_{i,j} c_i^{j,0} (\xi_i^{j,\lambda_j-1} + \varepsilon_i^{j,\lambda_j-1} \xi_j^{i,\lambda_i-1})$.

For $1 \leq l < h \leq a$, by taking the commutator between $\xi_l^{h,0} + \varepsilon_l^{h,0} \xi_h^{l,\lambda_l-\lambda_h}$ and x we obtain that

$$
\begin{aligned}
[\xi_l^{h,0}+\varepsilon_l^{h,0}\xi_h^{l,\lambda_l-\lambda_h},x] = &\sum_k (c_l^{l,k}-c_h^{h,k})(\xi_l^{h,k}-\varepsilon_l^{h,0}\xi_h^{l,\lambda_l-\lambda_h+k}) + \sum_i c_i^{l,0}(\xi_i^{h,\lambda_l-1} \\
&\pm \varepsilon_l^{h,0}\varepsilon_i^{j,\lambda_j-1}\xi_h^{i,\lambda_l-\lambda_h+\lambda_i-1}) + \sum_j c_l^{j,0}(\varepsilon_l^{j,\lambda_j-1}\xi_j^{h,\lambda_l-1} \\
&\pm \varepsilon_l^{h,0}\xi_h^{j,\lambda_l-\lambda_h+\lambda_j-1}) + \sum_i c_i^{h,0}(\varepsilon_l^{h,0}\xi_i^{l,\lambda_l-1}\pm\varepsilon_i^{h,\lambda_h-1}\xi_l^{i,\lambda_i-1}) \\
&+ \sum_j c_h^{j,0}(\varepsilon_l^{h,0}\varepsilon_h^{j,\lambda_j-1}\xi_j^{l,\lambda_l-1}\pm\xi_l^{j,\lambda_j-1}).
\end{aligned}
$$

Note that $\xi_i^{h,\lambda_l-1}=\xi_j^{h,\lambda_l-1}=\xi_h^{i,\lambda_l-\lambda_h+\lambda_i-1}=\xi_h^{j,\lambda_l-\lambda_h+\lambda_j-1}=0$ since $\lambda_l>\lambda_h$. Hence, we have that

$$
\begin{aligned}
[\xi_l^{h,0}+\varepsilon_l^{h,0}\xi_h^{l,\lambda_l-\lambda_h},x] = &\sum_k (c_l^{l,k}-c_h^{h,k})(\xi_l^{h,k}-\varepsilon_l^{h,0}\xi_h^{l,\lambda_l-\lambda_h+k}) \\
&+ \sum_i c_i^{h,0}(\varepsilon_l^{h,0}\xi_i^{l,\lambda_l-1}\pm\varepsilon_i^{h,\lambda_h-1}\xi_l^{i,\lambda_i-1}) \\
&+ \sum_j c_h^{j,0}(\varepsilon_l^{h,0}\varepsilon_h^{j,\lambda_j-1}\xi_j^{l,\lambda_l-1}\pm\xi_l^{j,\lambda_j-1}).
\end{aligned} \tag{5.41}
$$

This is equal to 0 if and only if $c_l^{l,k}=c_h^{h,k}$ for all $1\le l<h\le a$ and $c_i^{j,0}=0$ for all $1\le i<j\le a$ except when $(i,j)=(1,2)$ and $(|1|,|2|)=(\bar{0},\bar{0})$. When $|1|=|2|=\bar{0}$, the commutator between $\xi_l^{h,0}+\varepsilon_l^{h,0}\xi_h^{l,\lambda_l-\lambda_h}$ and $\xi_1^{2,\lambda_2-1}-\xi_2^{1,\lambda_1-1}$ gives terms $\varepsilon_l^{2,0}\xi_1^{l,\lambda_l-1}+\xi_l^{1,\lambda_1-1}-\varepsilon_l^{1,0}\xi_2^{l,\lambda_l-1}-\xi_l^{2,\lambda_2-1}$ and we check this is equal to zero for $l<h$. Hence, when $a\ge 3$, we have that $\mathfrak{z}(\mathfrak{g}^e)\subseteq S$ and thus $\mathfrak{z}(\mathfrak{g}^e)=S$ except when $|1|=|2|=\bar{0}$, in which case we have $\mathfrak{z}(\mathfrak{g}^e)\subseteq S\oplus\langle\xi_1^{2,\lambda_2-1}-\xi_2^{1,\lambda_1-1}\rangle$.

Next suppose $a\ge 3$ and $|1|=|2|=\bar{0}$, we know that $\xi_1^{2,\lambda_2-1}-\xi_2^{1,\lambda_1-1}$ commutes with all basis elements in $\mathfrak{g}_{\bar{0}}^e$ by [29, Theorem 4]. Hence, it remains to check that $\xi_1^{2,\lambda_2-1}-\xi_2^{1,\lambda_1-1}$ commutes with basis elements $\xi_i^{j,\lambda_j-1-k}\pm\xi_j^{i,\lambda_i-1-k}$ for $|i|=\bar{0},|j|=\bar{1}$

and $0 \leq k \leq \min\{\lambda_i, \lambda_j\} - 1$. Computing

$$[\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1}, \xi_i^{j,\lambda_j-1-k} \pm \xi_j^{i,\lambda_i-1-k}] = \pm\xi_j^{2,\lambda_2-1+\lambda_1-1-k} \pm\xi_j^{1,\lambda_1-1+\lambda_2-1-k} - \xi_1^{j,\lambda_2-1+\lambda_j-1-k} + \xi_2^{j,\lambda_1-1+\lambda_j-1-k}. \tag{5.42}$$

We have that all terms in (5.42) are equal to 0 for $0 \leq k \leq \min\{\lambda_i, \lambda_j\} - 1$ as $\xi_h^{l,r} = 0$ for all h, l and $r > \lambda_l - 1$. Hence, we have that $\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1} \in \mathfrak{z}(\mathfrak{g}^e)$ and $\mathfrak{z}(\mathfrak{g}^e) = S \oplus \langle \xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1} \rangle$ in this case.

When $a = 2$ and $|1| \neq |2|$, i.e. the Jordan type of e is $(m, 2n)$ such that m is odd. Assume that $m \geq 2n$, in this case a basis of $\mathfrak{g}^e$ only contains elements of the form:

$$\xi_1^{1,k} \text{ for } k \text{ is odd}, 1 \leq k \leq m-1;\ \xi_2^{2,k} \text{ for } k \text{ is odd},\ 1 \leq k \leq 2n-1;$$

$$\text{and } \xi_1^{2,2n-1-k} + \varepsilon_1^{2,2n-1-k}\xi_2^{1,m-1-k} \text{ for } 0 \leq k \leq 2n-1, \varepsilon_1^{2,2n-1-k} \in \{-1, 1\}.$$

By applying the similar argument to that used in the case $r + s \geq 3$, we get that $c_1^{1,k} = c_2^{2,k}$ and $c_1^{2,k} = 0$ for all $1 \leq k \leq 2n-1$. The only remaining element to check is $\xi_1^{2,2n-1} - \xi_2^{1,m-1}$. Note that $\xi_1^{2,2n-1} - \xi_2^{1,m-1}$ commutes with $\xi_1^{1,k}$ and $\xi_2^{2,k}$ for all k is odd. The element $\xi_1^{2,2n-1} - \xi_2^{1,m-1}$ also commutes with $\xi_1^{2,2n-1-k} + \varepsilon_1^{2,2n-1-k}\xi_2^{1,m-1-k}$ for all $k = 0, 1, \ldots, 2n-2$. Thus we only need to show that $[\xi_1^{2,2n-1} - \xi_2^{1,m-1}, \xi_1^{2,0} + \varepsilon_1^{2,0}\xi_2^{1,m-2n}] = 0$. We calculate

$$\begin{aligned}[\xi_1^{2,2n-1} - \xi_2^{1,m-1}, \xi_1^{2,0} + \varepsilon_1^{2,0}\xi_2^{1,m-2n}] &= \varepsilon_1^{2,0}\xi_2^{2,m-1} - \xi_1^{1,m-1} - \xi_2^{2,m-1} + \varepsilon_1^{2,0}\xi_1^{1,m-1} \\ &= (\varepsilon_1^{2,0} - 1)\xi_1^{1,m-1}\end{aligned} \tag{5.43}$$

since $\xi_2^{2,m-1} = 0$ as $\xi_2^{2,k} = 0$ for $k > 2n-1$.

Next we want to know the value of $\varepsilon_1^{2,0}$. Let $\mathfrak{osp}(m|2n) = \mathfrak{osp}(V)$ and $V = V_{\bar{0}} \oplus$

$V_{\bar{1}}$. Then we know that there exist $u_1, u_2 \in V$ such that $u_1, eu_1, \dots, e^{m-1}u_1$ (resp. $u_2, eu_2, \dots, e^{2n-1}u_2$) is a basis for $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) according to Subsection 5.2.5. Moreover, we know that $B(u_1, e^{m-1}u_1) = 1$ and $B(u_2, e^{2n-1}u_2) = 1$ by equations (5.36) and (5.37). By using equation (5.25), we have that

$$B((\xi_1^{2,0} + \varepsilon_1^{2,0}\xi_2^{1,m-2n})u_1, e^{2n-1}u_2) = (u_2, e^{2n-1}u_2) = 1$$

and

$$\begin{aligned} B((\xi_1^{2,0} + \varepsilon_1^{2,0}\xi_2^{1,m-2n})u_1, e^{2n-1}u_2) &= B(u_1, (\xi_1^{2,0} + \varepsilon_1^{2,0}\xi_2^{1,m-2n})e^{2n-1}u_2) \\ &= B(u_1, \varepsilon_1^{2,0}e^{m-1}u_1) = \varepsilon_1^{2,0}. \end{aligned}$$

Therefore, we obtain that $\varepsilon_1^{2,0} = 1$. Therefore, we have that $[\xi_1^{2,2n-1} - \xi_2^{1,m-1}, \xi_1^{2,0} + \varepsilon_1^{2,0}\xi_2^{1,m-2n}] = 0$ and thus $\xi_1^{2,2n-1} - \xi_2^{1,m-1} \in \mathfrak{z}(\mathfrak{g}^e)$.
Therefore, we deduce that $\xi_1^{2,2n-1} - \xi_2^{1,m-1} \in \mathfrak{z}(\mathfrak{g}^e)$ and $\mathfrak{z}(\mathfrak{g}^e) = S \oplus \left\langle \xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1} \right\rangle$ as required.
When $m < 2n$, we obtain same result by applying a similar argument. □

5.2.8 Centre of centralizer of general nilpotent element $e \in \mathfrak{g}_{\bar{0}}$

For Lie superalgebras $\mathfrak{g} = \mathfrak{osp}(m|2n)$, we already know the structure of $\mathfrak{z}(\mathfrak{g}^e)$ if (1) the Jordan type of e has all parts with even multiplicity, see Theorem 5.18; or (2) the Jordan type of e has all parts with multiplicity 1, see Theorem 5.19. Now we want to use results from Theorems 5.18–5.19 to deduce a basis of $\mathfrak{z}(\mathfrak{g}^e)$ for a general nilpotent element e.

In the remaining part of this section, we fix notation as follows: Let $V = \mathbb{C}^{m|2n}$

and $\mathfrak{g} = \mathfrak{osp}(V) \cong \mathfrak{osp}(m|2n)$. Let $e \in \mathfrak{g}_{\bar{0}}$ be a nilpotent element with Jordan type $\lambda = (\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \lambda_{-(a+1)}, \ldots, \lambda_b, \lambda_{-b})$ as defined in (5.34). Given a Dynkin pyramid $\tilde{P}$ following Subsection 5.2.4 and $\{v_{i_j}\}$ in (5.35) form a basis for V with respect to $\tilde{P}$. Now we write $V = V_1 \oplus V_2$ where $\{v_{i_j} : 1 \leq i \leq a\}$ form a basis for V_1 and $\{v_{i_j}, v_{-i_j} : a+1 \leq i \leq b\}$ form a basis for V_2. Define $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1 = \mathfrak{osp}(V_1)$ and $\mathfrak{g}_2 = \mathfrak{osp}(V_2)$. Then the nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ can also be written as $e = e_1 + e_2$ and $e_i \in \mathfrak{osp}(V_i)$ such that the Jordan type of e_1 has all parts with multiplicity 1 and the Jordan type of e_2 has all parts with even multiplicity. That is to say, $(\lambda_1, \ldots, \lambda_a)$ is the Jordan type of e_1 in descending order and $(\lambda_{a+1}, \lambda_{-(a+1)}, \ldots, \lambda_b, \lambda_{-b})$ is the Jordan type of e_2.

We know that a basis for $\mathfrak{h}^e$ is $\{h_i = \xi_i^{i,0} - \xi_{-i}^{-i,0} : i = a+1, \ldots, b\}$. Define $U_i = \mathrm{Span}\{v_{i_j} : i_j \text{ is a box in } \tilde{P}\}$ for $1 \leq i \leq a$ and $U_i = U_i^+ \oplus U_i^-$ for $a+1 \leq i \leq b$ where $U_i^+ = \mathrm{Span}\{v_{i_j} : i_j \text{ is a box in } P\}$ and $U_i^- = \mathrm{Span}\{v_{-i_j} : -i_j \text{ is a box in } \tilde{P}\}$. We also define $U_i^\perp = \{v \in V : (v, u) = 0 \text{ for any } u \in U_i\}$. Note that h_i is of the form $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$, then an element in $\mathfrak{g}^{h_i}$ is of the form

$$\begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix}$$

where A, B, C are block matrices and A and C are symmetric across the skew diagonal. Moreover, all matrices of the form

$$\begin{pmatrix} A & & \\ & 0 & \\ & & C \end{pmatrix}$$

form a subalgebra that is isomorphic to $\mathfrak{gl}\left(U_i^+\right)$ and all matrices of the form

$$\begin{pmatrix} 0 & & \\ & B & \\ & & 0 \end{pmatrix}$$

form a subalgebra that is isomorphic to $\mathfrak{osp}\left(U_i^\perp\right)$. Therefore, we have that $\mathfrak{g}^{h_i} \cong \mathfrak{gl}\left(U_i^+\right) \oplus \mathfrak{osp}\left(U_i^\perp\right)$. Note that elements of $\mathfrak{gl}(U_i^+)$ can be viewed as elements of $\mathfrak{osp}(U_i^+ \oplus U_i^-)$. Let $H = \sum_{i=a+1}^{b} h_i$, then with the above basis of V we have that H is of the form

$$H = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & -1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & -1 \end{pmatrix},$$

Using a similar argument we have that

$$\mathfrak{g}^H \cong \mathfrak{gl}(\bigoplus_{i=a+1}^{b} U_i^+) \oplus \mathfrak{osp}(\bigoplus_{i=1}^{a} U_i). \tag{5.44}$$

Since $\mathfrak{gl}(\bigoplus_{i=a+1}^{b} U_i^+) \subseteq \mathfrak{osp}(V_2)$ and $\mathfrak{osp}(\bigoplus_{i=1}^{a} U_i) = \mathfrak{osp}(V_1)$, we have that $\mathfrak{g}^H \subseteq \mathfrak{g}'$. Therefore, we obtain that $\mathfrak{z}(\mathfrak{g}^e) \subseteq \mathfrak{g}^H \subseteq \mathfrak{g}'$. Let $x \in \mathfrak{z}(\mathfrak{g}^e)$, then $x \in (\mathfrak{g}')^e$ and $[x, y] = 0$ for all $y \in \mathfrak{g}^e$. Since $(\mathfrak{g}')^e \subseteq \mathfrak{g}^e$, we have that $[x, y] = 0$ for all $y \in (\mathfrak{g}')^e$. Therefore, we deduce that $\mathfrak{z}(\mathfrak{g}^e) \subseteq \mathfrak{z}((\mathfrak{g}')^e) = \mathfrak{z}(\mathfrak{g}_1^{e_1}) \oplus \mathfrak{z}(\mathfrak{g}_2^{e_2})$.

Theorem 5.20. *Suppose that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{osp}(m|2n)$ and $e = e_1 + e_2$ is given by a partition $\lambda = (\lambda_1, \dots, \lambda_a, \lambda_{a+1}, \lambda_{-(a+1)}, \dots, \lambda_b, \lambda_{-b})$. Let $S = \langle e^k : k$ is odd and $1 \leq k \leq \max\{\lambda_1, \lambda_{a+1}\} - 1\rangle$. Then $\mathfrak{z}(\mathfrak{g}^e) = S$ except for two special cases:*
Case 1: If $a \geq 2$, $\lambda_2 > \lambda_{a+1}$, $|1| = |2| = \bar{0}$ or $a = 2, |1| \neq |2|$, then we have that $\mathfrak{z}(\mathfrak{g}^e) = S \oplus \langle \xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1}\rangle$;
Case 2: If $\lambda_1 < \lambda_{a+1}$, $\lambda_{a+1} > \lambda_{a+2}$, $|a+1| = \bar{0}$ and λ_{a+1} is odd, we have that $\mathfrak{z}(\mathfrak{g}^e) = S \oplus \langle \xi_{a+1}^{a+1,\mu_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\mu_{a+1}-1}\rangle$.

Proof. It is clear that $S \subseteq \mathfrak{z}(\mathfrak{g}^e)$. We first consider the case that $\mathfrak{z}(\mathfrak{g}_1^{e_1}) = \mathrm{Span}\{e_1^k : k$ is odd and $1 \leq k \leq \lambda_1 - 1\}$ and $\mathfrak{z}(\mathfrak{g}_2^{e_2}) = \mathrm{Span}\{e_2^k : k$ is odd and $1 \leq k \leq \lambda_{a+1} - 1\}$. According to Theorem 5.18 and 5.19, we have that elements in a basis of $\mathfrak{z}((\mathfrak{g}')^e)$ can be written as:

$$\sum_{t=1}^{a} \xi_t^{t,k} \text{ for } k \text{ is odd and } 1 \leq k \leq \lambda_1 - 1;$$

$$\sum_{t=a+1}^{b} (\xi_t^{t,k} + \xi_{-t}^{-t,k}) \text{ for } k \text{ is odd and } 1 \leq k \leq \lambda_{a+1} - 1.$$

Thus an element $x \in \mathfrak{z}(\mathfrak{g}^e)$ is of the form

$$x = \sum_{k \text{ is odd}; k=1}^{\lambda_1-1} a_k \left(\sum_{t=1}^{a} \xi_t^{t,k} \right) + \sum_{k \text{ is odd}; k=1}^{\lambda_{a+1}-1} b_k \left(\sum_{t=a+1}^{b} (\xi_t^{t,k} + \xi_{-t}^{-t,k}) \right) \tag{5.45}$$

for coefficient $a_k, b_k \in \mathbb{C}$. We assume that $\lambda_1 \geq \lambda_{a+1}$, then take the commutator with

$\xi_1^{a+1,0} + \varepsilon_1^{a+1,0}\xi_{-(a+1)}^{1,\lambda_1-\lambda_{a+1}}$:

$$[\xi_1^{a+1,0} + \varepsilon_1^{a+1,0}\xi_{-(a+1)}^{1,\lambda_1-\lambda_{a+1}}, x] = \sum_{k \text{ is odd};k=1}^{\lambda_{a+1}-1} (a_k - b_k)\xi_1^{a+1,k} + \sum_{k \text{ is odd};k=1}^{\lambda_{a+1}-1} (b_k - a_k)\varepsilon_1^{a+1,0}\xi_{-(a+1)}^{1,\lambda_1-\lambda_{a+1}+k}.$$

This is equal to 0 if and only if $a_k = b_k$ for all k is odd and $1 \leq k \leq \lambda_{a+1} - 1$. If $\lambda_1 < \lambda_{a+1}$, then take the commutator between $\xi_1^{a+1,\lambda_{a+1}-\lambda_1} + \varepsilon_1^{a+1,\lambda_{a+1}-\lambda_1}\xi_{-(a+1)}^{1,0}$ and x, we obtain that $a_k = b_k$ for all k is odd and $1 \leq k \leq \lambda_1 - 1$. Hence, we have that $\mathfrak{z}(\mathfrak{g}^e) \subseteq S$ for this case and thus $\mathfrak{z}(\mathfrak{g}^e) = S$. Therefore, we have that $\dim \mathfrak{z}(\mathfrak{g}^e) = \lceil \frac{\max\{\lambda_1,\lambda_{a+1}\}-1}{2} \rceil$ for this case.

Now we look at the special cases:

For the special case when $\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1} \in \mathfrak{z}(\mathfrak{g}_1^{e_1})$ and $\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1} \notin \mathfrak{z}(\mathfrak{g}_2^{e_2})$, then an element $y \in \mathfrak{z}(\mathfrak{g}^e)$ is of the form $y = x + c_1^{2,\lambda_2-1}(\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1})$ where x is defined in (5.45) and $c_1^{2,\lambda_2-1} \in \mathbb{C}$ is the coefficient. By calculating $[y, \xi_1^{a+1,0} + \varepsilon_1^{a+1,0}\xi_{-(a+1)}^{1,\lambda_1-\lambda_{a+1}}] = 0$ for $\lambda_1 > \lambda_{a+1}$ and $[y, \xi_1^{a+1,\lambda_{a+1}-\lambda_1} + \varepsilon_1^{a+1,\lambda_{a+1}-\lambda_1}\xi_{-(a+1)}^{1,0}] = 0$ for $\lambda_1 \leq \lambda_{a+1}$, we obtain that $a_k = b_k$ for all k is odd and $c_1^{2,\lambda_2-1} = 0$ when $\lambda_1 \leq \lambda_{a+1}$. If $\lambda_2 \leq \lambda_{a+1} < \lambda_1$, then computing

$$[x + c_1^{2,\lambda_2-1}(\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1}), \xi_2^{a+1,\lambda_{a+1}-\lambda_2} \pm \xi_{-(a+1)}^{2,0}] = 0$$

implies that $c_1^{2,\lambda_2-1} = 0$. However, if $\lambda_2 > \lambda_{a+1}$, we calculate the commutator between $\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1}$ and basis elements of the form $\xi_i^{j,\lambda_j-1-k} \pm \xi_{-j}^{i,\lambda_i-1-k}$ for $1 \leq i \leq a$,

$a+1 \le j \le b$,$0 \le k \le \min\{\lambda_i, \lambda_j\}$:

$$[\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1}, \xi_i^{j,\lambda_j-1-k} \pm \xi_{-j}^{i,\lambda_i-1-k}] = \pm\xi_{-j}^{2,\lambda_2-1+\lambda_1-1-k} \pm \xi_{-j}^{1,\lambda_1-1+\lambda_2-1-k} \pm \xi_1^{j,\lambda_j-1-k+\lambda_2-1} \pm \xi_2^{j,\lambda_j-1-k+\lambda_1-1}. \tag{5.46}$$

We have that all terms in (5.46) are equal to 0 for $0 \le k \le \min\{\lambda_i, \lambda_j\}$. This implies that $\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1}$ commutes with all other basis elements in $\mathfrak{g}^e$. Therefore, we deduce that $\mathfrak{z}(\mathfrak{g}^e) = S \oplus \left\langle \xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1} \right\rangle$ in this case and $\dim \mathfrak{z}(\mathfrak{g}^e) = \lceil \frac{\max\{\lambda_1, \lambda_{a+1}\}-1}{2} \rceil + 1$. For the special case when $\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1} \notin \mathfrak{z}(\mathfrak{g}_1^{e_1})$ and $\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1} \in \mathfrak{z}(\mathfrak{g}_2^{e_2})$, then an element $z \in \mathfrak{z}(\mathfrak{g}^e)$ is of the form $z = x + c_{a+1}^{a+1,\lambda_{a+1}-1}(\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1})$ where x is defined in (5.45) and $c_{a+1}^{a+1,\lambda_{a+1}-1} \in \mathbb{C}$ is the coefficient. If $\lambda_1 \ge \lambda_{a+1}$, then computing $[x, \xi_1^{a+1,0} + \varepsilon_1^{a+1,0}\xi_{-(a+1)}^{1,\lambda_1-\lambda_{a+1}}] = 0$ gives that $a_k = b_k$ for all k is odd and $1 \le k \le \lambda_{a+1}-1$ and $c_{a+1}^{a+1,\lambda_{a+1}-1} = 0$. However, if $\lambda_1 < \lambda_{a+1}$, by computing $[x, \xi_1^{a+1,\lambda_{a+1}-\lambda_1} + \varepsilon_1^{a+1,\lambda_{a+1}-\lambda_1}\xi_{-(a+1)}^{1,0}] = 0$ gives that $a_k = b_k$ for all k is odd and $1 \le k \le \lambda_1 - 1$. It remains to check that $\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1} \in \mathfrak{z}(\mathfrak{g}^e)$. It is obvious that $\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1}$ commutes with all elements of the form $\xi_i^{j,\lambda_j-1-k} \pm \xi_{-j}^{i,\lambda_i-1-k}$ for $1 \le i,j \le a$ or $a+1 \le i,j \le b$. We now calculate commutators between $\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1}$ and $\xi_i^{j,\lambda_j-1-k} \pm \xi_{-j}^{i,\lambda_i-1-k}$ for $1 \le i \le a$, $a+1 \le j \le b$, $1 \le k \le \min\{\lambda_i, \lambda_j\}$:

$$[\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1}, \xi_i^{j,\lambda_j-1-k} \pm \xi_{-j}^{i,\lambda_i-1-k}] = \xi_i^{a+1,2\lambda_{a+1}-2-k} \pm \xi_{-(a+1)}^{i,\lambda_{a+1}-1+\lambda_i-1-k}. \tag{5.47}$$

We have that all terms in (5.47) are equal to 0. This implies that $\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1}$ commutes with all other basis elements in $\mathfrak{g}^e$. Therefore, we deduce that $\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1} \in \mathfrak{z}(\mathfrak{g}^e)$ in this case and $\dim \mathfrak{z}(\mathfrak{g}^e) = \lceil \frac{\max\{\lambda_1, \lambda_{a+1}\}-1}{2} \rceil + 1$.

Moreover, when $\xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1} \in \mathfrak{z}(\mathfrak{g}_1^{e_1})$ and $\xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1} \in \mathfrak{z}(\mathfrak{g}_2^{e_2})$, applying the similar argument to above we also have that $\mathfrak{z}(\mathfrak{g}^e) = S \oplus \langle \xi_1^{2,\lambda_2-1} - \xi_2^{1,\lambda_1-1} \rangle$ for $\lambda_2 > \lambda_{a+1}$ and $\mathfrak{z}(\mathfrak{g}^e) = S \oplus \langle \xi_{a+1}^{a+1,\lambda_{a+1}-1} - \xi_{-(a+1)}^{-(a+1),\lambda_{a+1}-1} \rangle$ for $\lambda_1 < \lambda_{a+1}$. □

5.2.9 Adjoint action of $\mathrm{O}_m(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$ on $\mathfrak{osp}(m|2n)$

In this subsection, we aim to look at the adjoint action of the algebraic group $G = \mathrm{O}_m(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$ on the centre of centralizer of nilpotent element e in $\mathfrak{g} = \mathfrak{osp}(m|2n)$. We start by recalling some basic properties of the adjoint action of $\mathrm{O}_m(\mathbb{C})$ (resp. $\mathrm{Sp}_{2n}(\mathbb{C})$) acting on the centre of centralizer of e in Lie algebras $\mathfrak{o}(m)$ (resp. $\mathfrak{sp}(2n)$) which is helpful to deal with super case. Then we look for a basis for $(\mathfrak{z}(\mathfrak{g}^e))^{G^e}$ based on results in Subsection 5.2.8.

Centralizer of nilpotent element in $\mathrm{O}(V)$ or $\mathrm{Sp}(V)$

Let V be a finite-dimensional vector space on $\mathbb{C}$. Let $G = \mathrm{O}(V)$ or $\mathrm{Sp}(V)$ and e in $\mathrm{Lie}(G)$ be nilpotent. Let h be the semisimple element in G such that h lies in an $\mathfrak{sl}(2)$-triple that contains e. Recall that we can define a grading $V = \bigoplus_{l \in \mathbb{Z}} V(l)$ of V where $V(l) = \{v \in V : hv = lv\}$. In order to further describe G^e, we recall the group C^e which corresponds to $\mathfrak{g}^e(0)$ and is defined to be

$$C^e = \{g \in G^e : g\,(V(m)) = V(m) \text{ for all } m\} = (G^e)^h.$$

Note that C^e is a closed subgroup of G^e. Now write the Jordan type λ of e in the form $\lambda = (c^{n_c}, \ldots, 2^{n_2}, 1^{n_1})$. Then the following proposition, which is in [15, Section 3.8], describes the structure of C^e.

Proposition 5.21. For $G = \mathrm{O}(V)$, there is an isomorphism of algebraic groups

$$C^e \cong \left(\prod_{i \geq 1; i \text{ is odd}} \mathrm{O}_{n_i}(\mathbb{C}) \right) \times \left(\prod_{i \geq 1; i \text{ is even}} \mathrm{Sp}_{n_i}(\mathbb{C}) \right).$$

For $G = \mathrm{Sp}(V)$, there is an isomorphism of algebraic groups

$$C^e \cong \left(\prod_{i \geq 1; i \textit{ is odd}} \mathrm{Sp}_{n_i}(\mathbb{C}) \right) \times \left(\prod_{i \geq 1; i \textit{ is even}} \mathrm{O}_{n_i}(\mathbb{C}) \right).$$

We know that each $\mathrm{Sp}_{n_i}(\mathbb{C})$ in Proposition 5.21 is connected. Each $\mathrm{O}_{n_i}(\mathbb{C})$ is disconnected and it has $\mathrm{SO}_{n_i}(\mathbb{C})$ as its identity component. Moreover, according to [15, Section 3.12], we have the following theorem.

Theorem 5.22. The group G^e is the semidirect product of the subgroup C^e and the connected normal subgroup R^e, i.e. $G^e \cong C^e \ltimes R^e$.

Moreover, let $V^{(m)} = \bigoplus_{l \geq m} V(l)$ for all $m \in \mathbb{Z}$ and set $U = \{g \in G : (g-1)V^{(m)} \subset V^{(m+1)}$ for all $m \in \mathbb{Z}\}$, then we have that $R^e = G^e \cap U$.

Denote the connected component of G^e (resp. C^e) containing identity by $(G^e)^\circ$ (resp. $(C^e)^\circ$). Since R^e in Theorem 5.22 is connected, we have that $G^e/(G^e)^\circ \cong C^e/(C^e)^\circ$.

Adjoint action of G^e on $\mathfrak{z}(\mathfrak{g}^e)$

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{osp}(m|2n)$ and $G = \mathrm{O}_m(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$. Let $e \in \mathfrak{g}_{\bar{0}}$ be nilpotent. Recall that the adjoint action of G on $\mathfrak{g}$ is given by $g \cdot x = gxg^{-1}$ for all $g \in G$, $x \in \mathfrak{g}$. Then the centralizer G^e of e in G acts on $\mathfrak{g}^e$ by the adjoint action. We claim that $\mathfrak{g}^e$ is stable under the adjoint action of G^e. Take any $x \in \mathfrak{g}^e$ and any $g \in G^e$. Then

$$\begin{aligned}
[g \cdot x, e] &= gxg^{-1}e - egxg^{-1} \\
&= \left(gxg^{-1}\right)\left(geg^{-1}\right) - \left(geg^{-1}\right) gxg^{-1} \\
&= gxeg^{-1} - gexg^{-1} = g\,(xe - ex)\,g^{-1} = 0
\end{aligned} \tag{5.48}$$

as required. Hence, there is a representation $\mathrm{Ad} : G^e \to \mathrm{GL}(\mathfrak{g}^e)$. According to [13, Section 10.4], taking the differential of Ad at the identity element of G gives the adjoint representation $\mathrm{ad} : \mathfrak{g}^e \to \mathfrak{gl}(\mathfrak{g}^e)$. Consider $(\mathfrak{g}^e)^{G^e}$ which is defined to be

$$(\mathfrak{g}^e)^{G^e} = \left\{x \in \mathfrak{g}^e : gxg^{-1} = x \text{ for all } g \in G^e\right\}.$$

Note that the restriction of the adjoint action of G^e on $(\mathfrak{g}^e)^{G^e}$ is the trivial action. Thus $\mathrm{Ad}_{|(\mathfrak{g}^e)^{G^e}} : G^e \to \mathrm{GL}((\mathfrak{g}^e)^{G^e})$ is the trivial map. Then the differential of $\mathrm{Ad}_{|(\mathfrak{g}^e)^{G^e}}$:

$$\mathrm{ad}_{|(\mathfrak{g}^e)^{G^e}} : \mathfrak{g}^e \to \mathfrak{gl}((\mathfrak{g}^e)^{G^e})$$

is the trivial map. Hence, the adjoint action of $\mathfrak{g}^e$ on $(\mathfrak{g}^e)^{G^e}$ is trivial. Therefore, we have that $(\mathfrak{g}^e)^{G^e} \subseteq \mathfrak{z}(\mathfrak{g}^e)$. We claim that $\mathfrak{z}(\mathfrak{g}^e)$ is stable under the adjoint action of G^e. Take any $x \in \mathfrak{z}(\mathfrak{g}^e)$, $g \in G^e$ and $y \in \mathfrak{g}^e$, then $[g\cdot x, y] = [g\cdot x, g\cdot(g^{-1}\cdot y)] = g\cdot[x, g^{-1}\cdot y]$. Since $g^{-1}\cdot y \in \mathfrak{g}^e$ by (5.48), we have that $[g\cdot x, y] = 0$ as required. Therefore, we deduce that $(\mathfrak{g}^e)^{G^e} = (\mathfrak{z}(\mathfrak{g}^e))^{G^e}$.

A basis for $(\mathfrak{z}(\mathfrak{g}^e))^{G^e}$

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{osp}(m|2n)$ where $\mathfrak{g}_{\bar{0}} = \mathfrak{o}(m) \oplus \mathfrak{sp}(2n)$. Given a nilpotent element e in $\mathfrak{g}_{\bar{0}}$, we consider the reductive algebraic group $G = \mathrm{O}_m(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$ and centralizer G^e of e in G. Suppose the Jordan type $\lambda = (\lambda_1, \dots, \lambda_a, \lambda_{a+1}, \lambda_{-(a+1)}, \dots, \lambda_b, \lambda_{-b})$ of e is defined as in (5.34) and the alternative Dynkin pyramid $\tilde{P}$ is given the same way as in Subsection 5.2.4.

Theorem 5.23. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{osp}(m|2n)$ *and* $G = \mathrm{O}_m(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$. *Let* $e = e_1 + e_2 \in \mathfrak{g}_{\bar{0}}$ *be nilpotent with Jordan type* λ *denoted as in (5.34) such that the Jordan type of* e_1 *has all parts with multiplicity one and the Jordan type of* e_2 *has all parts with even multiplicity. Let* $S = \langle e^l : l \text{ is odd and } 0 < l \leq \max\{\lambda_1, \lambda_{a+1}\} - 1\rangle$, *then* $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = S$.

Proof. Recall that $\{v_{i_j}\}$ is a basis of V according to (5.35) and the bilinear form $B(.\,,.)$ on V is given in Subsection 5.2.4. We define an involution ρ on the labels of the basis of V by

$$\rho(i_t) = \begin{cases} i_{-t} & \text{if } i = 1, \dots, a; \\ -i_{-t} & \text{if } i = \pm(a+1), \dots, \pm b. \end{cases}$$

Let e_{i_t,d_l} be the (i_t, d_l)-matrix unit. Then $\{e_{i_t,d_l} \pm e_{\rho(d_l),\rho(i_t)}\}$ is a basis for $\mathfrak{g}$ with appropriate signs and conditions on i_t, d_l.

Let $l = \lfloor \frac{m}{2} \rfloor$. By reordering boxes on the right hand half of $\tilde{P}$ from $1, \dots, l+n$ and boxes on the left hand half of $\tilde{P}$ from $-(l+n), \dots, -1$ (note that there exists a box labelled by 0 if m is odd), we get a basis $\{e_{j,k} + \gamma_{-k,-j} e_{-k,-j}\}$ for $\mathfrak{g}$ as defined in (5.32). There exists an isomorphism between basis element $e_{j,k} + \gamma_{-k,-j} e_{-k,-j}$ and $e_{i_t,d_l} \pm e_{\rho(d_l),\rho(i_t)}$.

It is clear that $S \subseteq (\mathfrak{z}(\mathfrak{g}^e))^{G^e}$. When $\mathfrak{z}(\mathfrak{g}^e) = S$, then $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} \subseteq \mathfrak{z}(\mathfrak{g}^e) = S$ and thus we obtain $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = S$. The following part of this proof deals with two special cases in Theorem 5.20 where $\mathfrak{z}(\mathfrak{g}^e) \neq S$.

Case 1: When $a \geq 2$, $\lambda_2 > \lambda_{a+1}$, $|1| = |2| = \bar{0}$ or $a = 2$, $|1| \neq |2|$.

In this case, the extra basis element x can be written as $e_{1_{-\lambda_1+1}, 2_{\lambda_2-1}} - e_{2_{-\lambda_2+1}, 1_{\lambda_1-1}}$. We consider a matrix Q which sends each v_{1_t} to $-v_{1_t}$ and all other v_{i_t} for $i \neq 1$ to itself. Then we have that

$$B(Qv_{i_t}, Qv_{d_l}) = \begin{cases} (-1)^t \delta_{i_t, \rho(d_l)} & \text{if } 1 \leq i, d \leq a; \\ \delta_{i_t, \rho(d_l)} & \text{if } \pm(a+1) \leq i, d \leq \pm b; \\ 0 & \text{otherwise.} \end{cases}$$

Thus Q preserves the form B on V. Hence, we have that $Q \in G$. For any i_t, d_l, observe that

$$Qe_{i_t, d_l}Q^{-1} = \begin{cases} -e_{i_t, d_l} & \text{if } i \neq d \text{ and } i = 1 \text{ or } d = 1; \\ e_{i_t, d_l} & \text{otherwise.} \end{cases}$$

Denote by $\varepsilon_{i_t, d_l} \in \{\pm 1\}$ the coefficient of e_{i_t, d_l}, $|\mathrm{row}(i)| = |\mathrm{row}(d)|$ in the basis of $\mathfrak{g}_{\bar{0}}$ that involves e_{i_t, d_l}. Thus we can write e_1 to be $\sum_{t;i=1}^{a} \varepsilon_{i_t, i_{t-2}} e_{i_t, i_{t-2}}$. We have that

$$\begin{aligned} Q \cdot e_1 &= Q \cdot \left(\sum_{t;i=1}^{a} \varepsilon_{i_t, i_{t-2}} e_{i_t, i_{t-2}}\right) = \sum_{t;i=1}^{a} \varepsilon_{i_t, i_{t-2}} Q e_{i_t, i_{t-2}} Q^{-1} \\ &= \sum_{t} \varepsilon_{1_t, 1_{t-2}} e_{1_t, 1_{t-2}} + \sum_{t;i \neq 1} \varepsilon_{i_t, i_{t-2}} e_{i_t, i_{t-2}} = \sum_{t;i=1}^{a} \varepsilon_{i_t, i_{t-2}} e_{i_t, i_{t-2}} \\ &= e_1. \end{aligned}$$

Furthermore, based on the way we defined Q, it fixes e_2. This implies that $Q \in G^e$ and thus we know that $Q \cdot e^c = e^c$ for all c is odd. Hence, we have that $S \subseteq (\mathfrak{z}(\mathfrak{g}^e))^{G^e} \subseteq$

$(\mathfrak{z}(\mathfrak{g}^e))^Q$. Next by calculating

$$Q\cdot x = QxQ^{-1} = -e_{1_{-\lambda_1+1},2_{\lambda_2-1}} + e_{2_{-\lambda_2+1},1_{\lambda_1-1}} = -x,$$

we deduce that $x \notin (\mathfrak{z}(\mathfrak{g}^e))^Q$ and $(\mathfrak{z}(\mathfrak{g}^e))^Q \subseteq S$. Therefore, we have that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = (\mathfrak{z}(\mathfrak{g}^e))^Q = S$ and $\dim(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \lceil \frac{\lambda_1-1}{2} \rceil$.

Case 2: When $\lambda_1 < \lambda_{a+1}$, $\lambda_{a+1} > \lambda_{a+2}$, $|a+1| = \bar{0}$ and λ_{a+1} is odd.

In this case, the extra basis element x can be written as

$$x = e_{(a+1)_{-\lambda_{a+1}+1},(a+1)_{\lambda_{a+1}-1}} - e_{-(a+1)_{-\lambda_{a+1}+1},-(a+1)_{\lambda_{a+1}-1}}.$$

We consider a matrix Q which sends $v_{(a+1)_t}$ to $v_{-(a+1)_t}$ and all other v_{i_t} for $i \neq \pm(a+1)$ to itself. Then we have that

$$B(Qv_{i_t}, Qv_{d_l}) = \begin{cases} B(v_{i_t}, v_{d_l}) = (-1)^t \delta_{i_t,\rho(d_l)} & \text{if } 1 \leq i, d \leq a; \\ B(v_{-i_t}, v_{-d_l}) = \delta_{-i_t,\rho(-d_l)} = \delta_{i_t,\rho(d_l)} & \text{if } i, d = \pm(a+1); \\ B(v_{i_t}, v_{d_l}) = \delta_{i_t,\rho(d_l)} & \text{if } \pm(a+2) \leq i, d \leq \pm b; \\ 0 & \text{otherwise.} \end{cases}$$

Thus Q preserves the form on V and $Q \in G$. For any i_t, d_l, observe that

$$Qe_{i_t,d_l}Q^{-1} = \begin{cases} e_{i_t,d_l} & \text{if } i, d \neq \pm(a+1); \\ e_{-i_t,d_l} & \text{if } i = \pm(a+1), d \neq \pm(a+1); \\ e_{i_t,-d_l} & \text{if } i \neq \pm(a+1), d = \pm(a+1); \\ e_{-i_t,-d_l} & \text{if } i, d = \pm(a+1). \end{cases} \tag{5.49}$$

Recall that $\varepsilon_{i_t,d_l} \in \{\pm 1\}$ is the coefficient of e_{i_t,d_l}, $|\text{row}(i)| = |\text{row}(d)|$ in the basis of $\mathfrak{g}_{\bar{0}}$

that involves e_{i_t,d_l}. Thus we can write e_2 to be $\sum_{t;i=\pm(a+1)}^{b} \varepsilon_{i_t,i_{t-2}} e_{i_t,i_{t-2}}$.Then we have that

$$
\begin{aligned}
Q\cdot e_2 = Q\cdot (\sum_{t;i=\pm(a+1)}^{b} \varepsilon_{i_t,i_{t-2}} e_{i_t,i_{t-2}}) &= \sum_{t;i=\pm(a+1)}^{b} \varepsilon_{i_t,i_{t-2}} Q e_{i_t,i_{t-2}} Q^{-1} \\
&= \sum_t \varepsilon_{(a+1)_t,(a+1)_{t-2}} e_{-(a+1)_t,-(a+1)_{t-2}} + \sum_t \varepsilon_{-(a+1)_t,-(a+1)_{t-2}} e_{(a+1)_t,(a+1)_{t-2}} \\
&+ \sum_{t;i\neq\pm(a+1)} \varepsilon_{i_t,i_{t-2}} e_{i_t,i_{t-2}} \\
&= \sum_{t;i=\pm(a+1)}^{b} \varepsilon_{i_t,i_{t-2}} e_{i_t,i_{t-2}} = e_2.
\end{aligned}
$$

Furthermore, based on the way we defined Q, it fixes e_1. This implies that $Q \in G^e$ and thus we know that $Q\cdot e^c = e^c$ for all c is odd. Hence, we have that $S \subseteq (\mathfrak{z}(\mathfrak{g}^e))^{G^e} \subseteq (\mathfrak{z}(\mathfrak{g}^e))^{Q}$. Next by calculating

$$Q\cdot x = QxQ^{-1} = e_{-(a+1)_{-\lambda_{a+1}+1},-(a+1)_{\lambda_{a+1}-1}} - e_{(a+1)_{-\lambda_{a+1}+1},(a+1)_{\lambda_{a+1}-1}} = -x,$$

we know that $x \notin (\mathfrak{z}(\mathfrak{g}^e))^{Q}$ and $(\mathfrak{z}(\mathfrak{g}^e))^{Q} \subseteq S$. Therefore, we have that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = (\mathfrak{z}(\mathfrak{g}^e))^{Q} = S$ and $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \lceil \frac{\lambda_{a+1}-1}{2} \rceil$. □

Remark 5.24. If we choose $G' = \mathrm{SO}_m(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$, applying a similar argument we obtain that $(\mathfrak{z}(\mathfrak{g}^e))^{G'^e} = \mathfrak{z}(\mathfrak{g}^e)$ if $\lambda_2 > \lambda_{a+1}$ and λ_i is even for $3 \leq i \leq b$, or $\lambda_1 < \lambda_{a+1}$, $\lambda_{a+1} \neq \lambda_{a+2}$ and $\lambda_{\pm(a+1)}$ are the only odd parts for $1 \leq i \leq b$. For other cases, we have $(\mathfrak{z}(\mathfrak{g}^e))^{G'^e} = S$.

5.2.10 Analysis of results

We first consider Theorem 1.1 for $\mathfrak{g} = \mathfrak{osp}(m|2n)$. Let λ_1 be the largest part in the Jordan type of e. Based on Subsection 5.2.9, we have that $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = \lceil \frac{\lambda_1-1}{2} \rceil$. Next we calculate the number of labels in the labelled Dynkin diagram $\varDelta$ which are equal to 2. Given a ortho-symplectic Dynkin pyramid P and a partition λ as defined in (5.33), note that all boxes are places in columns $1-i, 3-i, \dots, i-1$ where i is in λ. Hence, we observe that $\varDelta$ has no label equal to 1 if and only if all parts of λ are odd or all parts of λ are even. Now assume that all labels in $\varDelta$ equal to 0 or 2. Then based on the way that labelled Dynkin diagram is constructed, we observe that $n_2(\varDelta) = \lfloor \frac{\lambda_1}{2} \rfloor$. Note that when λ_1 is even, we have that $\lceil \frac{\lambda_1-1}{2} \rceil = \frac{\lambda_1}{2}$ and $n_2(\varDelta) = \lfloor \frac{\lambda_1}{2} \rfloor = \frac{\lambda_1}{2}$, when λ_1 is odd, we have that $\lceil \frac{\lambda_1-1}{2} \rceil = \frac{\lambda_1-1}{2}$ and $n_2(\varDelta) = \frac{\lambda_1-1}{2}$. Therefore, we deduce that $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = n_2(\varDelta)$.

In order to see what is $\mathfrak{z}(\mathfrak{g}^h)$, we first consider the following example.

Example 5.25. For a nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ with Jordan type $(5,3,1|3,3)$, the corresponding Dynkin pyramid is shown in Figure 5. Hence, we have $n_2(\varDelta) = \lfloor \frac{5}{2} \rfloor = 2$. We can calculate that $\mathfrak{g}^h = \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(2|2) \oplus \mathfrak{osp}(3|2)$. Note that $\mathfrak{z}(\mathfrak{gl}(1|0)) = I_{1|0}$, $\mathfrak{z}(\mathfrak{gl}(2|2)) = I_{2|2}$ and $\mathfrak{z}(\mathfrak{osp}(3|2)) = 0$. Therefore, we have that $\dim \mathfrak{z}(\mathfrak{g}^h) = 2$.

Assume that there is no label equal to 1 in $\varDelta$. When m is odd, we observe that

$$\mathfrak{g}^h = \bigoplus_{i>0} \mathfrak{gl}(r_i|s_i) \oplus \mathfrak{osp}(r_0|s_0)$$

where r_i (resp. s_i) denotes the number of boxes with parity $\bar{0}$ (resp. $\bar{1}$) on the ith column. When m is even, then $\mathfrak{g}^h$ is just the direct sum of $\mathfrak{gl}(r_i|s_i)$ for $i > 0$. Note that $\mathfrak{z}(\mathfrak{gl}(r_i|s_i)) = I_{r_i|s_i}$ and $\mathfrak{z}(\mathfrak{osp}(r_0|s_0)) = 0$. Hence, we have that $\mathfrak{z}(\mathfrak{g}^h)$ is the direct sum of $I_{r_i|s_i}$ for $i > 0$. Since there are in total λ_1 columns in P with non-zero boxes, we deduce that $\dim \mathfrak{z}(\mathfrak{g}^h) = \lfloor \frac{\lambda_1}{2} \rfloor = n_2(\Delta)$. The above argument completes the proof for Theorem 1.1 for the case $\mathfrak{g} = \mathfrak{osp}(m|2n)$.

Next we want to find the relation between $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e}$ and the sum of labels $\sum a_i$ in Δ. Note that $a_i = \mathrm{col}(i+1) - \mathrm{col}(i)$ for $i = 1, \dots, l+n-1$. When m is even, we have

$$a_{l+n} = \begin{cases} -2\mathrm{col}(l+n) & \text{if } |l+n| = \bar{1}; \\ \mathrm{col}(-l-n) - \mathrm{col}(l+n-1) & \text{if } |l+n| = \bar{0}. \end{cases}$$

Then

$$\begin{aligned}
\sum a_i &= \sum_{i=1}^{l+n-1} (\mathrm{col}(i+1) - \mathrm{col}(i)) + \begin{cases} -2\mathrm{col}(l+n) & \text{if } |l+n| = \bar{1}; \\ (\mathrm{col}(-l-n) - \mathrm{col}(l+n-1)) & \text{if } |l+n| = \bar{0} \end{cases} \\
&= \begin{cases} \mathrm{col}(-l-n) - \mathrm{col}(1) & \text{if } |l+n| = \bar{1}; \\ -\mathrm{col}(l+n-1) - \mathrm{col}(1) & \text{if } |l+n| = \bar{0}. \end{cases} \\
&= \begin{cases} \lambda_1 - 1 & \text{if } |l+n| = \bar{1}, \mathrm{col}(l+n) = 0 \text{ or } |l+n| = \bar{0}, \mathrm{col}(l+n-1) = 0; \\ \lambda_1 & \text{if } |l+n| = \bar{1}, \mathrm{col}(l+n) = -1 \text{ or } |l+n| = \bar{0}, \mathrm{col}(l+n-1) = -1. \end{cases}
\end{aligned}$$

When m is odd, then there exists a box that is labelled by 0, we have $a_{l+n} = \mathrm{col}(0) - \mathrm{col}(l+n)$. Then

$$\begin{aligned}
\sum a_i &= \sum_{i=1}^{l+n-1} (\mathrm{col}(i+1) - \mathrm{col}(i)) + (\mathrm{col}(0) - \mathrm{col}(l+n)) \\
&= \mathrm{col}(0) - \mathrm{col}(1) = \lambda_1 - 1.
\end{aligned}$$

Therefore, we deduce that

$$\lceil\frac{1}{2}\sum a_i\rceil=\begin{cases}\lceil\frac{\lambda_1}{2}\rceil & \text{if } m \text{ is even and } \lambda_1 \text{ is even;}\\ \lceil\frac{\lambda_1-1}{2}\rceil & \text{otherwise.}\end{cases}$$

Hence, when m is even and λ_1 is even, $\lceil\frac{1}{2}\sum a_i\rceil=\lceil\frac{\lambda_1}{2}\rceil=\lceil\frac{\lambda_1-1}{2}\rceil=\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e}$ and for all other cases $\lceil\frac{1}{2}\sum a_i\rceil=\lceil\frac{\lambda_1-1}{2}\rceil=\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e}$. Therefore, we deduce that

$$\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e}=\lceil\frac{1}{2}\sum a_i\rceil.$$

In order to prove Theorem 1.3 for Lie superalgebras $\mathfrak{osp}(m|2n)$, we need to consider two general cases.

Case 1: When Δ has no label equal to 1, i.e. all labels are equal to 0 or 2. In this case, we know that either all parts of λ are odd or all parts of λ are even. Note that $e_0=0$ since Δ_0 has all labels equal to 0. Thus we have that

$$\mathfrak{g}_0^{e_0}=\mathfrak{g}_0=\left(\bigoplus_{i>0}\mathfrak{sl}(r_i|s_i)\right)\oplus\mathfrak{osp}(r_0|s_0)$$

where r_i (resp. s_i) denotes the number of boxes with parity $\bar{0}$ (resp. $\bar{1}$) on the ith column of P. We denote $c_i=r_i+s_i$. Then

$$\dim\mathfrak{g}_0^{e_0}=\dim\mathfrak{g}_0=\sum_{i>0}\dim\mathfrak{sl}(r_i|s_i)+\dim\mathfrak{osp}(r_0|s_0).$$

Now if all parts of λ are even, then we have $\dim\mathfrak{osp}(r_0|s_0)=0$. This implies that $\dim\mathfrak{g}_0^{e_0}=\sum_{i>0}(c_i^2-1)$. We also have that $|\{i:\lambda_i \text{ is odd}, |i|=\bar{0}\}|=|\{i:\lambda_i \text{ is odd}, |i|=$

$\bar{1}\}\ |= 0$. Thus

$$\dim \mathfrak{g}^e = \frac{1}{2}\dim \mathfrak{gl}(m|2n)^e = \frac{1}{2}(\sum_{i\in\mathbb{Z}} c_i^2)$$

by Proposition 5.17. Hence, we have that

$$\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = \frac{1}{2}(\sum_{i\in\mathbb{Z}} c_i^2) - \sum_{i>0}(c_i^2 - 1) = \sum_{i>0:c_i>0} 1 = n_2(\mathit{\Delta}).$$

Next we consider when all parts of λ are odd, we have that

$$\dim \mathfrak{osp}(r_0|s_0) = \frac{(r_0+s_0)^2 - (r_0+s_0)}{2} + s_0 = \frac{c_0^2 - r_0 + s_0}{2}.$$

Note that in this case $\dim \mathfrak{g}^e = \frac{1}{2}\dim \mathfrak{gl}(m|2n)^e - \frac{r}{2} + \frac{s}{2}$ by Proposition 5.17 where r (resp. s) is the total number of $\lambda_i, |i| = \bar{0}$ (resp. $\lambda_i, |i| = \bar{1}$). Since all parts of λ are odd, then each row which corresponds to a certain part of λ has a box in the 0th column. Hence, the number of λ_i with $|i| = \bar{0}$ (resp. $|i| = \bar{1}$) is equal to the number of even (resp. odd) boxes in the 0th column, i.e. $r = r_0$ and $s = s_0$. Thus we can calculate

$$\dim \mathfrak{g}^e = \frac{1}{2}(\sum_{i\in\mathbb{Z}} c_i^2 + c_0^2) - \frac{r_0}{2} + \frac{s_0}{2}.$$

Therefore, we have that

$$\begin{aligned}\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} &= \left(\frac{1}{2}(\sum_{i\in\mathbb{Z}} c_i^2 + c_0^2) - \frac{r_0}{2} + \frac{s_0}{2}\right) - \left(\sum_{i>0}(c_i^2 - 1) + \frac{c_0^2 - r_0 + s_0}{2}\right) \\ &= \sum_{i>0:c_i>0} 1 = n_2(\mathit{\Delta}).\end{aligned}$$

When $r_i \neq s_i$ for all $i > 0$, we have that $\dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = \dim \mathfrak{z}(\mathfrak{g}_0) = 0$ because $\mathfrak{z}(\mathfrak{sl}(r_i|s_i)) = 0$ for all i and $\mathfrak{z}(\mathfrak{osp}(r_0|s_0)) = 0$. However, if there exists $r_i = s_i$ for some i, we have that $I_{r_i|r_i} \in \mathfrak{z}(\mathfrak{sl}(r_i|r_i))$ and thus $\dim \mathfrak{z}(\mathfrak{g}_0) = \tau$ where τ is the number of $i > 0$

for which $r_i = s_i$. Hence, we have that $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim\left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = n_2(\Delta) - \tau$.

Case 2: When there exist some labels equal to 1 in Δ. Note that there are in total $2\lambda_1 + 1$ columns in P and let $n_2(\Delta) = t$. Observe that Δ has some labels equal to 2 if there exist some $c_{k_j} = 0$ for $j = 1, \dots, t$ and $t = n_2(\Delta)$. Let $k > 0$ be the minimal column number such that $c_k = 0$ and $k + 2t = \lambda_1$. Note that once a label equal to 1 occurs, say a_k, then there is no label equal to 2 for all a_h with $h > k$. Then we have that

$$\mathfrak{g}_0 \cong \left(\bigoplus_{i=1}^{t} \mathfrak{sl}(r_{k+2i-1} \mid s_{k+2i-1})\right) \oplus \mathfrak{osp}\left(\sum_{i=1-k}^{k-1} r_i \mid \sum_{i=1-k}^{k-1} s_i\right).$$

Let $\hat{\mathfrak{g}}_0 = \mathfrak{osp}\left(\sum_{i=1-k}^{k-1} r_i \mid \sum_{i=1-k}^{k-1} s_i\right)$. We observe that the projection of e_0 in each $\mathfrak{sl}(r_{k+2i-1}|s_{k+2i-1})$ is 0 and so $e_0 \in \hat{\mathfrak{g}}_0$. Hence,

$$\dim \mathfrak{g}_0^{e_0} = \sum_{i=k+1}^{\lambda_1} (c_i^2 - 1) + \dim \hat{\mathfrak{g}}_0^{e_0}.$$

Denote the Jordan type of e_0 to be λ^0 which is defined the similar way to (5.6). We also observe that $|\{i : \lambda_i \text{ is odd}, |i| = \bar{0}\}| = |\{i : \lambda_i^0 \text{ is odd}, |i| = \bar{0}\}|$ and $|\{i : \lambda_i \text{ is odd}, |i| = \bar{1}\}| = |\{i : \lambda_i^0 \text{ is odd}, |i| = \bar{1}\}|$. Note that

$$\begin{aligned}\dim \hat{\mathfrak{g}}_0^{e_0} = &\frac{1}{2}\Big(\sum_{i=1-k}^{k-1} (c_i^2 + c_i c_{i+1})\Big) \\ &- \frac{1}{2}\left|\{i : \lambda_i^0 \text{ is odd}, |i| = \bar{0}\}\right| \\ &+ \frac{1}{2}\left|\{i : \lambda_i^0 \text{ is odd}, |i| = \bar{1}\}\right|.\end{aligned}$$

Therefore, we have that $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = \sum_{i=1}^{t} 1 = t = n_2(\Delta)$.

Moreover, we have that

$$\dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = \sum_{i=1}^{t} \dim \mathfrak{z}(\mathfrak{sl}(r_{k+2i-1}|s_{k+2i-1})) + \dim\mathfrak{z}(\hat{\mathfrak{g}}_0^{e_0}).$$

Similar to Case 1, $\mathfrak{z}(\mathfrak{sl}(r_{k+2i-1}|s_{k+2i-1})) = 0$ when $r_{k+2i-1} \neq s_{k+2i-1}$ for all i. If there exist some $i > 0$ such that $r_{k+2i-1} = s_{k+2i-1}$, then $I_{r_{k+2i-1}|r_{k+2i-1}} \in \mathfrak{z}(\mathfrak{sl}(r_{k+2i-1}|r_{k+2i-1}))$. We know that $\dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = \lceil \frac{k_1-1}{2} \rceil + \tau$ where τ is the number of $i > 0$ such that $r_i = s_i$. Hence, we deduce that $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = n_2(\Delta) - \tau$ in this case. The above argument completes the proof for Theorem 1.3 for the case $\mathfrak{g} = \mathfrak{osp}(m|2n)$.

Remark 5.26. Note that although there exist distinguished nilpotent elements in $\mathfrak{g}$, however, there is no analogue of [18, Theorem 1] for ortho-symplectic Lie superalgebras.

6 Centres of centralizers of nilpotent elements in exceptional Lie superalgebras

In this chapter, we describe the explicit constructions for the Lie superalgebras $D(2,1;\alpha)$, $G(3)$ and $F(4)$ following the definition of the exceptional Lie superalgebras given by Scheunert, Nahm, and Rittenberg in [23]. We also give representatives e of nilpotent orbits in the even part of each type of Lie superalgebra and calculate the centralizer $\mathfrak{g}^e$ of e. For each nilpotent even element e, we describe the $\mathfrak{g}^e(0)$-module structure on each $\mathfrak{g}^e(j)$ for $j>0$. The centre $\mathfrak{z}(\mathfrak{g}^e)$ of the centralizer of e is also calculated and the labelled Dynkin diagram Δ with respect to e is drawn afterwards.

6.1 Generalities on $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$

We first give an overview of some general methods on calculating $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$.

Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}=D(2,1;\alpha), G(3)$ or $F(4)$. Note that any element $x\in\mathfrak{g}$ can be written as $x=x_{\bar{0}}+x_{\bar{1}}$ such that $x_{\bar{i}}\in\mathfrak{g}_{\bar{i}}$. For a nilpotent element $e\in\mathfrak{g}_{\bar{0}}$, if $[x,e]=0$ then $[x,e]=[x_{\bar{0}},e]+[x_{\bar{1}},e]=0$. This implies that $[x_{\bar{0}},e]=[x_{\bar{1}},e]=0$ since $[x_{\bar{0}},e]\in\mathfrak{g}_{\bar{0}}$ and $[x_{\bar{1}},e]\in\mathfrak{g}_{\bar{1}}$. Hence $\mathfrak{g}^e=\mathfrak{g}^e_{\bar{0}}\oplus\mathfrak{g}^e_{\bar{1}}$.

For a nilpotent element $e\in\mathfrak{g}_{\bar{0}}$, recall that there exists an $\mathfrak{sl}(2)$-triple $\{e,h,f\}\subseteq\mathfrak{g}_{\bar{0}}$ according to the Jacobson–Morozov Theorem and any two $\mathfrak{sl}(2)$-triples containing e are conjugate under the action of the group G^e, the centralizer of e in the adjoint Lie group G corresponding to the semisimple Lie algebra $\mathfrak{g}_{\bar{0}}$. Hence $\mathfrak{s}=\langle e,h,f\rangle$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$. Then $\mathfrak{g}$ is a module for $\mathfrak{s}$ via the adjoint action. Let $V^{\mathfrak{sl}}(d)$ be the $(d+1)$-dimensional simple $\mathfrak{sl}(2)$-module with highest weight d. By the representation theory of $\mathfrak{sl}(2)$, we can decompose $\mathfrak{g}$ into a direct sum of finite-dimensional $\mathfrak{s}$-submodules $\mathfrak{g}^i$ and each of them is isomorphic to $V^{\mathfrak{sl}}(d_i)$ for some $d_i \in \mathbb{Z}$ and $d_i \geq 0$. The element h of the $\mathfrak{sl}(2)$-triple is semisimple and the eigenvalues of h on $\mathfrak{g}^i$ are $d_i, d_i - 2, ..., -(d_i - 2), -d_i$. The elements e and f move between different eigenspaces of h. More explicitly, the element e increases the eigenvalue by 2 and f decreases the eigenvalue by 2. Thus the only vectors in $\mathfrak{g}^i$ annihilated by e are the multiples of the highest weight vector, i.e. if $\mathfrak{g}^i$ has basis $\{x^i_{d_i}, x^i_{d_i-2}, ..., x^i_{-d_i+2}, x^i_{-d_i}\}$ for $i = 1, 2, ..., r$ then the vectors annihilated by e are $\left\langle x^i_{d_i}\right\rangle$. The vectors in $\mathfrak{g}$ eliminated by e are $\left\langle x^1_{d_1}, x^2_{d_2}, ..., x^r_{d_r}\right\rangle$ and they have ad$h$-eigenvalues $d_1, d_2, ..., d_r$. Hence, from the ad$h$-eigenspace decomposition of $\mathfrak{g}$ we determine the ad$h$ eigenvalues of elements of $\mathfrak{g}^e$.

From now on let us denote $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}^e)$. Given $x = x_{\bar{0}} + x_{\bar{1}} \in \mathfrak{z}$, for any $y = y_{\bar{0}} + y_{\bar{1}} \in \mathfrak{g}^e$, we have $[x, y_{\bar{0}}] = [x_{\bar{0}}, y_{\bar{0}}] + [x_{\bar{1}}, y_{\bar{0}}] = 0$. Since $[x_{\bar{0}}, y_{\bar{0}}] \in \mathfrak{g}^e_{\bar{0}}$ and $[x_{\bar{1}}, y_{\bar{0}}] \in \mathfrak{g}^e_{\bar{1}}$, we have $[x_{\bar{0}}, y_{\bar{0}}] = [x_{\bar{1}}, y_{\bar{0}}] = 0$. Similarly we have $[x_{\bar{0}}, y_{\bar{1}}] = [x_{\bar{1}}, y_{\bar{1}}] = 0$. Therefore, we know that $x_{\bar{0}}, x_{\bar{1}} \in \mathfrak{z}$ and thus $\mathfrak{z} = \mathfrak{z}_{\bar{0}} \oplus \mathfrak{z}_{\bar{1}}$. Moreover, we can decompose $\mathfrak{z}$ into the direct sum of ad$h$-eigenspaces in each case, i.e. $\mathfrak{z} = \bigoplus_j \mathfrak{z}(j)$ for all ad$h$-eigenvalue j.

We consider $\mathfrak{sl}(2)$ frequently in this chapter, so we fix the notation $\mathfrak{sl}(2) = \langle E, H, F\rangle$ where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The commutator relations between basis elements for $\mathfrak{sl}(2)$ are:

$$[H, E] = 2E,\ [H, F] = -2F \text{ and } [E, F] = H.$$

When describing the $\mathfrak{g}^e(0)$-module structure on each $\mathfrak{g}^e(j)$ for $j > 0$, we need the following lemma.

Lemma 6.1. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a Lie superalgebra where $\{u_{-2}, u_0, u_2\}$ is a basis of $A_{\bar{0}}$ and $\{u_{-1}, u_1\}$ is a basis of $A_{\bar{1}}$ such that: (1) $[u_0, u_i] = a_i u_i$ for $i = \pm 1, \pm 2$; (2) $[u_1, u_1] = a u_2$ and $[u_{-1}, u_{-1}] = b u_{-2}$; (3) $[u_2, u_{-2}] = c u_0$ for $a_i, a, b, c \neq 0$. Then A is simple and $A \cong \mathfrak{osp}(1|2)$.

Proof. Let I be a non-zero ideal of A. Then I is a direct sum of $\mathrm{ad} u_0$ eigenspaces, thus $u_i \in I$ for some i. If $i = 0$, then condition (1) implies that $I = A$. If $i = \pm 2$, then condition (3) implies that $u_0 \in I$ and thus $I = A$. If $i = \pm 1$, then condition (2) implies that u_{-2} or u_2 lies in I. Thus $u_0 \in I$ and $I = A$. Therefore, we have that A is simple. According to the classification Theorem of simple Lie superalgebras in [16, Theorem 5], we deduce that $A \cong \mathfrak{osp}(1|2)$. □

We consider the representations of $\mathfrak{osp}(1|2)$ frequently in Sections 6.2–6.4. As shown in [21, Section 2], all finite-dimensional representations of $\mathfrak{osp}(1|2)$ are completely reducible. Also in [21, Section 2] the irreducible representations of $\mathfrak{osp}(1|2)$ are constructed. We recall that the irreducible representations of $\mathfrak{osp}(1|2)$ are parameterized by $l \in \{\frac{a}{2} : a \in \mathbb{Z}_{\geq 0}\}$ and we write $V^{\mathfrak{osp}}(l)$ for the representation corresponding to l. Then $\dim V^{\mathfrak{osp}}(l) = 4l + 1$. We know that $\mathfrak{osp}(1|2)$ is 5-dimensional with basis $\{u_{-2}, u_{-1}, u_0, u_1, u_2\}$. The eigenvalues of u_0 on $V^{\mathfrak{osp}}(l)$ are $l, l - \frac{1}{2}, \ldots, -l$.

6.2 The exceptional Lie superalgebras $D(2, 1; \alpha)$

6.2.1 Structure of the Lie superalgebras $D(2, 1; \alpha)$

According to [19, Section 4.2], the Lie superalgebras $D(2, 1; \alpha)$ with $\alpha \in \mathbb{C}\backslash\{0, 1\}$ form a one-parameter family of superalgebras of dimension 17. Scheunert denotes these algebra by $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ where $\sigma_1, \sigma_2, \sigma_3$ are complex numbers such that $\sigma_1 + \sigma_2 + \sigma_3 = 0$.

According to [19, Section 4.2], the Lie superalgebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ is simple if and only if $\sigma_i \neq 0$ for $i = 1, 2, 3$. If there exists another triple $(\sigma_1^{'}, \sigma_2^{'}, \sigma_3^{'})$ such that $\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong \Gamma(\sigma_1^{'}, \sigma_2^{'}, \sigma_3^{'})$, then there must exist a permutation ρ of $\{1, 2, 3\}$ and a nonzero complex number c such that $\sigma_i^{'}$ can be obtained by $\sigma_i^{'} = c\sigma_{\rho(i)}$. This implies that $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ form a one parameter family and we have that $\Gamma(\sigma_1, \sigma_2, \sigma_3) = D(2, 1; \alpha)$ for a specific choice of $\sigma_1, \sigma_2, \sigma_3$. For any $\alpha \in \mathbb{C} \setminus \{0, -1\}$, we have $D(2, 1; \alpha) \cong \Gamma(1 + \alpha, -1, -\alpha) \cong \Gamma(\frac{1+\alpha}{\alpha}, -1, -\frac{1}{\alpha}) \cong \Gamma(-\alpha, -1, 1 + \alpha)$.

Let V be a two-dimensional vector space with basis $v_1 = (1, 0)^t$ and $v_{-1} = (0, 1)^t$. For $i = 1, 2, 3$, take V_i for $i = 1, 2, 3$ to be a copy of V. Let ψ_i be the non-degenerate skew-symmetric bilinear form on V_i defined by $\psi_i(v_1, v_{-1}) = 1$. We also define a bilinear map $p_i : V_i \times V_i \to \mathfrak{sl}(2)$ by

$$p_i(x, y)(z) = \psi_i(y, z)x - \psi_i(z, x)y$$

for $x, y, z \in V_i$. We can easily calculate that $p_i(v_1, v_1) = 2E$, $p_i(v_1, v_{-1}) = -H$ and $p_i(v_{-1}, v_{-1}) = -2F$.

By definition, $\mathfrak{g} = D(2, 1; \alpha) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where

$$\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$$

and

$$\mathfrak{g}_{\bar{1}} = V_1 \otimes V_2 \otimes V_3.$$

Note that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra thus has Lie bracket $[\cdot, \cdot] : \mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}}$. Let $x = (x_1, x_2, x_3) \in \mathfrak{g}_{\bar{0}}$ and $v = v_i \otimes v_j \otimes v_k \in \mathfrak{g}_{\bar{1}}$, then the bracket $[\cdot, \cdot] : \mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{1}}$ is defined by

$$[x, v] := x\cdot v = x_1 v_i \otimes v_j \otimes v_k + v_i \otimes x_2 v_j \otimes v_k + v_i \otimes v_j \otimes x_3 v_k.$$

According to equation (4.2.1) in [19], the bracket $[\cdot,\cdot] : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ is given by

$$\begin{aligned}[x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3] &= \sigma_1 \psi_2(x_2, y_2)\psi_3(x_3, y_3)p_1(x_1, y_1) \\ &+ \sigma_2 \psi_1(x_1, y_1)\psi_3(x_3, y_3)p_2(x_2, y_2) \\ &+ \sigma_3 \psi_1(x_1, y_1)\psi_2(x_2, y_2)p_3(x_3, y_3),\end{aligned}$$

where $x_i, y_i \in V_i$.

Next we give a basis for $\mathfrak{g}$. We first fix the following notation: let

$$E_1 = (E, 0, 0) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

$$E_2 = (0, E, 0) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

$$\text{and } E_3 = (0, 0, E) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right).$$

Similarly, we denote $F_1 = (F, 0, 0)$, $F_2 = (0, F, 0)$, $F_3 = (0, 0, F)$, $H_1 = (H, 0, 0)$, $H_2 = (0, H, 0)$ and $H_3 = (0, 0, H)$. Clearly, $\mathfrak{g}_{\bar{0}}$ has a basis $\{E_1, H_1, F_1, E_2, H_2, F_2, E_3, H_3, F_3\}$ and $\mathfrak{g}_{\bar{1}}$ has a basis $\{v_i \otimes v_j \otimes v_k : i, j, k = \pm 1\}$.

6.2.2 Root system and Dynkin diagrams for $D(2,1;\alpha)$

According to [14, Appendix A], a Lie superalgebra of type $D(2,1;\alpha)$ has root system

$$\Phi_{\bar{0}} = \{\pm 2\beta_1, \pm 2\beta_2, \pm 2\beta_3\} \text{ and } \Phi_{\bar{1}} = \{i\beta_1 + j\beta_2 + k\beta_3 : i, j, k = \pm 1\},$$

where $\{\beta_1, \beta_2, \beta_3\}$ is an orthogonal basis such that we have the following relations:

$$(\beta_1, \beta_1) = \frac{1}{2},\ (\beta_2, \beta_2) = -\frac{1}{2}\alpha - \frac{1}{2},\ \text{and } (\beta_3, \beta_3) = \frac{1}{2}\alpha.$$

The corresponding root vectors are listed below:

Table 6.1: Root vectors and roots for $D(2,1;\alpha)$

Roots	Root vectors
$2\beta_i$ for $i = 1, 2, 3$	E_i
$-2\beta_i$ for $i = 1, 2, 3$	F_i
$i\beta_1 + j\beta_2 + k\beta_3$ for $i, j, k \in \{\pm 1\}$	$v_i \otimes v_j \otimes v_k$

We can determine whether an odd root $i\beta_1 + j\beta_2 + k\beta_3$ is isotropic by checking whether $(i\beta_1 + j\beta_2 + k\beta_3, i\beta_1 + j\beta_2 + k\beta_3) = 0$. For instance,

$$\begin{aligned}(\beta_1 + \beta_2 - \beta_3, \beta_1 + \beta_2 - \beta_3) &= (\beta_1, \beta_1) + (\beta_2, \beta_2) + (\beta_3, \beta_3) \\ &= \frac{1}{2} + (-\frac{1}{2}\alpha - \frac{1}{2}) + \frac{1}{2}\alpha \\ &= 0.\end{aligned}$$

Therefore, the odd root $\beta_1 + \beta_2 - \beta_3$ is isotropic. We further deduce that all the odd roots in $D(2,1;\alpha)$ are isotropic using a similar argument and we associate a grey node $\otimes$ for each odd simple root in labelled Dynkin diagrams of $D(2,1;\alpha)$. In the remaining part of this subsection, we give all possible Dynkin diagrams with respect to different systems of simple roots based on [9, Section 2.20]. Note that the label of lines between a pair of vertices that corresponds to simple roots are calculated using formula (4.1) in Section 4.2.

1. For the simple system $\Pi = \{\alpha_1 = 2\beta_1, \alpha_2 = -\beta_1 + \beta_2 - \beta_3, \alpha_3 = 2\beta_3\}$, we have the following Dynkin diagram:

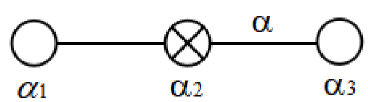


Figure 12: Dynkin diagrams for $D(2, 1; \alpha)$

2. For the simple system $\Pi = \{\alpha_1 = 2\beta_1, \alpha_2 = -\beta_1 - \beta_2 + \beta_3, \alpha_3 = 2\beta_2\}$, we have the following Dynkin diagram:

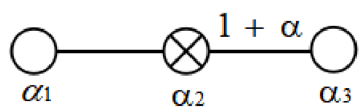


Figure 13: Dynkin diagrams for $D(2, 1; \alpha)$

3. For the simple system $\Pi = \{\alpha_1 = 2\beta_3, \alpha_2 = \beta_1 - \beta_2 - \beta_3, \alpha_3 = 2\beta_2\}$, we have the following Dynkin diagram:

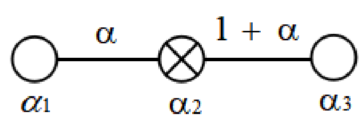


Figure 14: Dynkin diagrams for $D(2, 1; \alpha)$

4. For the simple system $\Pi = \{\alpha_1 = -\beta_1 + \beta_2 + \beta_3, \alpha_2 = \beta_1 - \beta_2 + \beta_3, \alpha_3 = \beta_1 + \beta_2 - \beta_3\}$, we have the following Dynkin diagram:

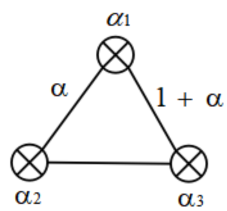


Figure 15: Dynkin diagrams for $D(2,1;\alpha)$

6.2.3 Centres of centralizers of nilpotent elements e in $D(2,1;\alpha)$ and labelled Dynkin diagrams with respect to e

Let $\mathfrak{g} = D(2,1;\alpha) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. A nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ is of the form (e_1, e_2, e_3) where $e_i \in \mathfrak{sl}(2)$ for $i \in \{1,2,3\}$. We know that representatives of nilpotent elements in $\mathfrak{sl}(2)$ upto conjugation by SL(2) are 0 and E. We summarize basis elements for $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$ and labelled Dynkin diagram with respect to $e = 0, E_1, E_1+E_2, E_1+E_2+E_3$ in Table 6.2. Note that the cases $e = E_2$, $e = E_3$ are similar to $e = E_1$ and cases $e = E_2 + E_3$, $e = E_1 + E_3$ are similar to $e = E_1 + E_2$. Hence, any other case is similar to one of above. After Table 6.3, we give explicit explanation on calculating $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$ and obtaining the corresponding labelled Dynkin diagram for cases $e = 0, E_1, E_1+E_2, E_1+E_2+E_3$. Note that numbers in the rows labelled "$\mathit{\Delta}$" represent labels a_i corresponding to α_i for $i = 1,2,3$ in labelled Dynkin diagram with respect to e.

e	$\mathfrak{g}^e$	$\mathfrak{z}(\mathfrak{g}^e)$	Δ
0	$\mathfrak{g}$	$\{0\}$	Figures 12, 13, 14, 15: All labels are zeros.
E_1	$\langle E_1, E_2, H_2, F_2, E_3, H_3, F_3, v_i \otimes v_j \otimes v_k : j, k = \pm 1\rangle$	$\langle e\rangle$	Figure 14: $0, 1, 0$
$E_1 + E_2$	$\langle E_1, E_2, E_3, H_3, F_3, v_1 \otimes v_1 \otimes v_1, v_1 \otimes v_1 \otimes v_{-1}, v_1 \otimes v_{-1} \otimes v_1 - v_{-1} \otimes v_1 \otimes v_1, v_1 \otimes v_{-1} \otimes v_{-1} - v_{-1} \otimes v_1 \otimes v_{-1}\rangle$	$\langle e\rangle$	Figure 12: $2, 0, 0$ Figure 14: $0, 0, 2$ Figure 15: $0, 0, 2$
$E_1 + E_2 + E_3$	$\langle E_1, E_2, E_3, v_1 \otimes v_1 \otimes v_1, v_1 \otimes v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \otimes v_1, v_1 \otimes v_{-1} \otimes v_1 - v_{-1} \otimes v_1 \otimes v_1\rangle$	$\langle e\rangle$	Figure 15: $1, 1, 1$

Table 6.2: $\mathfrak{g}^e$, $\mathfrak{z}(\mathfrak{g}^e)$ and Δ for $\mathfrak{g} = D(2, 1; \alpha)$

Let $V^{\mathfrak{sl}}(j)$ be an $\mathfrak{sl}(2)$-module with highest weight j and $V^{\mathfrak{osp}}(j)$ be an $\mathfrak{osp}(1|2)$-module with highest weight j. We also describe the $\mathfrak{g}^e(0)$-module structure on each $\mathfrak{g}^e(j)$ for $j > 0$ in the following tables.

e	$\mathfrak{g}^e(0)$	$\mathfrak{g}^e(j)$ for $j > 0$
0	$\mathfrak{g}^e$	0
E_1	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\mathfrak{g}^e(1) = V^{\mathfrak{sl}}(1) \otimes V^{\mathfrak{sl}}(1), \mathfrak{g}^e(2) = V^{\mathfrak{sl}}(0) \otimes V^{\mathfrak{sl}}(0)$
$E_1 + E_2$	$\mathfrak{osp}(1\|2)$	$\mathfrak{g}^e(2) = V^{\mathfrak{osp}}(0) \oplus V^{\mathfrak{osp}}(1)$
$E_1 + E_2 + E_3$	$\{0\}$	$\dim \mathfrak{g}^e(1) = 2, \dim \mathfrak{g}^e(2) = 3, \dim \mathfrak{g}^e(3) = 1$

Table 6.3: The $\mathfrak{g}^e(0)$-module structure on $\mathfrak{g}^e(j)$ for $j > 0$

In the remaining part of this subsection, we explain explicit calculations for finding $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$ and obtain the corresponding labelled Dynkin diagram for $e = 0, E_1, E_1 + E_2, E_1 + E_2 + E_3$.

(1) $e = 0$

We have that $\mathfrak{g}^e = \mathfrak{g}$ and $\mathfrak{z}(\mathfrak{g}^e) = 0$. The corresponding labelled Dynkin diagrams are all Dynkin diagrams in Figures 12–15 with labels for all nodes equal to zero.

(2) $e = E_1$

For $x_{\bar{0}} \in \mathfrak{g}_{\bar{0}}$, write $x_{\bar{0}}$ as (x_1, x_2, x_3) such that $x_i \in \mathfrak{sl}(2)$. Note that $[x_{\bar{0}}, E_1] =$

$([x_1, E], 0, 0)$. An easy calculation shows that if $[x_{\bar{0}}, E_1] = 0$ then x_1 is of the form

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \ b \in \mathbb{C},$$

while x_2 and x_3 are arbitrary matrices in $\mathfrak{sl}(2)$. Therefore, we obtain that $\mathfrak{g}^e_{\bar{0}} = \langle E_1, E_2, H_2, F_2, E_3, H_3, F_3 \rangle$.

To calculate $\mathfrak{g}^e_{\bar{1}}$, choose a basis element $v_i \otimes v_j \otimes v_k \in \mathfrak{g}_{\bar{1}}$ where $i, j, k \in \{-1, 1\}$, we have $[E_1, v_i \otimes v_j \otimes v_k] = Ev_i \otimes v_j \otimes v_k$. We know that $Ev_i = 0$ if $i = 1$ and $Ev_i = v_1$ if $i = -1$. Let $x = \sum_{i,j,k} a_{i,j,k} v_i \otimes v_j \otimes v_k \in \mathfrak{g}^e_{\bar{1}}$ where $a_{i,j,k} \in \mathbb{C}$, we compute

$$\begin{aligned} [e, x] = [E_1, \sum_{i,j,k} a_{i,j,k} v_i \otimes v_j \otimes v_k] &= [E_1, \sum_{j,k} a_{-1,j,k} v_{-1} \otimes v_j \otimes v_k] \\ &= \sum_{j,k} a_{-1,j,k} v_1 \otimes v_j \otimes v_k \end{aligned}$$

This is equal to 0 if and only if $a_{-1,j,k} = 0$ for all j and k. Therefore, $\mathfrak{g}^e_{\bar{1}} = \langle v_1 \otimes v_j \otimes v_k : j, k = \pm 1 \rangle$. In conclusion, $\mathfrak{g}^e = \langle E_1, E_2, H_2, F_2, E_3, H_3, F_3 \rangle \oplus \langle v_1 \otimes v_j \otimes v_k : j, k = \pm 1 \rangle$ and $\dim \mathfrak{g}^e = 7 + 4 = 11$.

It is clear that $\mathfrak{g}^e(0) = \langle E_2, H_2, F_2, E_3, H_3, F_3 \rangle = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. By calculating $\mathrm{ad}H_2$-eigenvalues and $\mathrm{ad}H_3$-eigenvalues on $\mathfrak{g}^e(j)$ for $j > 0$, we have that $\mathfrak{g}^e(1) = V^{\mathfrak{sl}}(1) \otimes V^{\mathfrak{sl}}(1)$ and $\mathfrak{g}^e(2) = V^{\mathfrak{sl}}(0) \otimes V^{\mathfrak{sl}}(0)$.

Note that there is no centre in $\mathfrak{sl}(2)$. Thus $\mathfrak{z}(\mathfrak{g}^e_{\bar{0}}) = \langle E_1 \rangle$ and $\mathfrak{z}_{\bar{0}} \subseteq \mathfrak{z}(\mathfrak{g}^e_{\bar{0}})$. This implies that $\mathfrak{z}_{\bar{0}} = \langle E_1 \rangle$. To determine $\mathfrak{z}_{\bar{1}}$, Let $x = \sum_{j,k} a_{1,j,k} v_1 \otimes v_j \otimes v_k \in \mathfrak{z}_{\bar{1}}$. We know that $\mathfrak{z}_{\bar{1}} \subseteq \mathfrak{z}(\mathfrak{g}^e_{\bar{1}})$, then

$$[E_2, x] = \sum_k a_{1,-1,k} v_1 \otimes v_1 \otimes v_k = 0 \text{ for } k = \pm 1;$$

$$[F_2,x] = \sum_k a_{1,1,k} v_1 \otimes v_{-1} \otimes v_k = 0 \text{ for } k = \pm 1.$$

Thus $a_{1,j,k} = 0$ for $j, k = \pm 1$ and we obtain that $x = 0$. Therefore, we have that $\mathfrak{z} = \langle E_1 \rangle$ and $\dim \mathfrak{z} = 1$.

Next we look for the labelled Dynkin diagram with respect to e. Note that we can find an element $h = H_1 = (H, 0, 0)$ such that h belongs to an $\mathfrak{sl}(2)$-triple $\{e, h, f\}$ in $\mathfrak{g}_{\bar{0}}$. Then we calculate the ad$h$-eigenvalue on each root vector in the table below.

root vectors	$e_{2\beta_1}$	$e_{2\beta_2}$	$e_{2\beta_3}$	$e_{-2\beta_1}$	$e_{-2\beta_2}$	$e_{-2\beta_3}$	$e_{i\beta_1+j\beta_2+k\beta_3}$
eigenvalue	2	0	0	-2	0	0	i

Notice that roots in $\mathfrak{g}(>0)$ are $\{2\beta_1, \beta_1+\beta_2+\beta_3, \beta_1+\beta_2-\beta_3, \beta_1-\beta_2+\beta_3, \beta_1-\beta_2-\beta_3\}$ and roots in $\mathfrak{g}(0)$ are $\Phi(0) = \{\pm 2\beta_2, \pm 2\beta_3\}$. Hence, in order to get a system of simple roots Π, we pick a system of simple roots $\Pi(0) = \{2\beta_2, 2\beta_3\}$ in $\Phi(0)$ and extend it to Π. Therefore, there is only one system of positive roots Φ^+ such that $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j \geq 0} \mathfrak{g}(j)$. Hence, we obtain the system of simple roots $\Pi = \{\alpha_1 = 2\beta_3, \alpha_2 = \beta_1 - \beta_2 - \beta_3, \alpha_3 = 2\beta_2\}$ up to conjugacy. We also calculate that

$$\mu_{12} = \frac{2|(\alpha_1, \alpha_2)|}{\min_{|(\alpha_k,\alpha_k)| \neq 0} |(\alpha_k, \alpha_k)|} = \alpha \text{ and } \mu_{23} = \frac{2|(\alpha_2, \alpha_3)|}{\min_{|(\alpha_k,\alpha_k)| \neq 0} |(\alpha_k, \alpha_k)|} = 1 + \alpha$$

for $\alpha_k \in \Phi$. Therefore, the labelled Dynkin diagram for $e = E_1$ is the Dynkin diagram in Figure 14 with labels $0, 1, 0$.

(3) $e = E_1 + E_2$

We easily deduce that $\mathfrak{g}_{\bar{0}}^e = \langle E_1, E_2, E_3, H_3, F_3 \rangle$. To determine $\mathfrak{g}_{\bar{1}}^e$, we again need to look for all $x = \sum_{i,j,k} a_{i,j,k} v_i \otimes v_j \otimes v_k$ with $a_{i,j,k} \in \mathbb{C}$ such that $[E_1 + E_2, x] = 0$. By calculating

$$[E_1+E_2,x]=a_{1,1,1}\cdot 0+a_{1,1,-1}\cdot 0+\sum_k a_{1,-1,k}v_1\otimes v_1\otimes v_k$$
$$+\sum_k a_{-1,1,k}v_1\otimes v_1\otimes v_k+\sum_k a_{-1,-1,k}(v_1\otimes v_{-1}\otimes v_k+v_{-1}\otimes v_1\otimes v_k),$$

we obtain that $a_{1,1,k}$ are arbitrary, $a_{1,-1,k}=-a_{-1,1,k}$ for $k=\pm 1$ and $a_{-1,-1,k}=0$. Hence a basis of $\mathfrak{g}^e_{\bar{1}}$ is $\{v_1\otimes v_1\otimes v_1, v_1\otimes v_1\otimes v_{-1}, v_1\otimes v_{-1}\otimes v_1-v_{-1}\otimes v_1\otimes v_1, v_1\otimes v_{-1}\otimes v_{-1}-v_{-1}\otimes v_1\otimes v_{-1}\}$. Therefore, $\mathfrak{g}^e=\langle E_1,E_2,E_3,H_3,F_3\rangle\oplus\mathfrak{g}^e_{\bar{1}}$ where $\mathfrak{g}^e_{\bar{1}}$ has basis elements as above and $\dim\mathfrak{g}^e=5+4=9$.

By computing commutator relations between basis elements for $\mathfrak{g}^e(0)$, we deduce that $\mathfrak{g}^e(0)\cong\mathfrak{osp}(1|2)$ according to Lemma 6.1 where $F_3, v_1\otimes v_{-1}\otimes v_{-1}-v_{-1}\otimes v_1\otimes v_{-1}, H_3, v_1\otimes v_{-1}\otimes v_1-v_{-1}\otimes v_1\otimes v_1, E_3$ correspond to $u_{-2},u_{-1},u_0,u_1,u_2$ in Lemma 6.1. By calculating adH_3-eigenvalues, we obtain that $\mathfrak{g}^e(2)=V^{\mathfrak{osp}}(0)\oplus V^{\mathfrak{osp}}(1)$. Hence, we have that $\mathfrak{z}=\mathfrak{z}(0)\oplus\mathfrak{z}(2)\subseteq(\mathfrak{g}^e(0))^{\mathfrak{g}^e(0)}\oplus(\mathfrak{g}^e(2))^{\mathfrak{g}^e(0)}=\langle E_1+E_2\rangle$. We also know that $e=E_1+E_2\in\mathfrak{z}$. Therefore, $\mathfrak{z}=\langle e\rangle$ and $\dim\mathfrak{z}=1$.

Next we look for the labelled Dynkin diagram with respect to e. Note that we can find an element $h=H_1+H_2=(H,H,0)$ such that h belongs to an $\mathfrak{sl}(2)$-triple $\{e,h,f\}$ in $\mathfrak{g}_{\bar{0}}$. Then we calculate the ad$h$-eigenvalue on each root vector in the table below.

root vectors	$e_{2\beta_1}$	$e_{2\beta_2}$	$e_{2\beta_3}$	$e_{-2\beta_1}$	$e_{-2\beta_2}$	$e_{-2\beta_3}$	$e_{i\beta_1+j\beta_2+k\beta_3}$
eigenvalue	2	2	0	-2	-2	0	$i+j$

Notice that roots in $\mathfrak{g}(>0)$ are $\{2\beta_1,2\beta_2,\beta_1+\beta_2+\beta_3,\beta_1+\beta_2-\beta_3\}$ and roots in $\mathfrak{g}(0)$ are $\Phi(0)=\{\pm 2\beta_3, i\beta_1-i\beta_2+k\beta_3 : i,k=\pm 1\}$. Hence, there are three systems of simple roots of $\mathfrak{g}(0)$: $\mathit{\Pi}_1(0)=\{-\beta_1+\beta_2-\beta_3,2\beta_3\}$, $\mathit{\Pi}_2(0)=\{2\beta_3,\beta_1-\beta_2-\beta_3\}$ and $\mathit{\Pi}_3(0)=\{-\beta_1+\beta_2+\beta_3,\beta_1-\beta_2+\beta_3\}$ up to conjugacy. By extending $\mathit{\Pi}_i(0)$ to $\mathit{\Pi}$ for $i=1,2,3$, we get three systems of positive roots Φ_i^+ and simple roots $\mathit{\Pi}_i$.

Therefore, there are three conjugacy classes of systems of positive roots Φ^+ such that $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j\geq 0} \mathfrak{g}(j)$. Hence, systems of simple roots are:

- $\Pi_1 = \{\alpha_1 = 2\beta_1, \alpha_2 = -\beta_1 + \beta_2 - \beta_3, \alpha_3 = 2\beta_3\}$. Similarly, we compute $\mu_{12} = 1$ and $\mu_{23} = \alpha$ using Formula (4.1). Therefore, the labelled Dynkin diagram with respect to Π_1 is the Dynkin diagram in Figure 12 with labels $2, 0, 0$.

- $\Pi_2 = \{\alpha_1 = 2\beta_3, \alpha_2 = \beta_1 - \beta_2 - \beta_3, \alpha_3 = 2\beta_2\}$. We compute that $\mu_{12} = \alpha$ and $\mu_{23} = 1 + \alpha$ using Formula (4.1). Therefore, the labelled Dynkin diagram with respect to Π_2 is the Dynkin diagram in Figure 14 with labels $0, 0, 2$.

- $\Pi_3 = \{\alpha_1 = -\beta_1+\beta_2+\beta_3, \alpha_2 = \beta_1-\beta_2+\beta_3, \alpha_3 = \beta_1+\beta_2-\beta_3\}$. We compute that $\mu_{12} = \alpha, \mu_{13} = 1 + \alpha$ and $\mu_{23} = 1$ using Formula (4.1). Therefore, the labelled Dynkin diagram with respect to Π_3 is the Dynkin diagram in Figure 15 with labels $0, 0, 2$.

(4) $e = E_1 + E_2 + E_3$

Similar to cases of E_1 and $E_1 + E_2$, we have $\mathfrak{g}^e_{\bar{0}} = \langle E_1, E_2, E_3\rangle$. To obtain $\mathfrak{g}^e_{\bar{1}}$, let $x = \sum_{i,j,k} a_{i,j,k} v_i \otimes v_j \otimes v_k \in \mathfrak{g}^e_{\bar{1}}$ where $a_{i,j,k} \in \mathbb{C}$. We calculate that

$$\begin{aligned}[E_1 + E_2 + E_3, x] = {} & a_{1,1,1}\cdot 0 + (a_{1,1,-1} + a_{1,-1,1} + a_{-1,1,1})v_1 \otimes v_1 \otimes v_1 \\ & + (a_{1,-1,-1} + a_{-1,1,-1})v_1 \otimes v_1 \otimes v_{-1} + (a_{1,-1,-1} + a_{-1,-1,1})v_1 \otimes v_{-1} \otimes v_1 \\ & + (a_{-1,1,-1} + a_{-1,-1,1})v_{-1} \otimes v_1 \otimes v_1 + a_{-1,-1,-1}(v_1 \otimes v_{-1} \otimes v_{-1} \\ & + v_{-1} \otimes v_1 \otimes v_{-1} + v_{-1} \otimes v_{-1} \otimes v_1)\end{aligned}$$

Therefore, we deduce that $a_{1,1,1}$ is arbitrary, $a_{1,1,-1} + a_{1,-1,1} + a_{-1,1,1} = 0$ and $a_{1,-1,-1} = a_{-1,1,-1} = a_{-1,-1,1} = 0$. From the above relations, we obtain that a basis for $\mathfrak{g}^e_{\bar{1}}$ is

$\{v_1 \otimes v_1 \otimes v_1, v_1 \otimes v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \otimes v_1, v_1 \otimes v_{-1} \otimes v_1 - v_{-1} \otimes v_1 \otimes v_1\}$. Therefore, $\mathfrak{g}^e = \langle E_1, E_2, E_3 \rangle \oplus \mathfrak{g}^e_{\bar{1}}$ where $\mathfrak{g}^e_{\bar{1}}$ has basis elements as above and $\dim \mathfrak{g}^e = 3 + 3 = 6$.

We know that $\mathfrak{z}_{\bar{0}} \subseteq \langle E_1, E_2, E_3 \rangle$ and $\mathfrak{z}_{\bar{0}}$ is at most 3-dimensional. Moreover, we have that $E_1 + E_2 + E_3 \in \mathfrak{z}_{\bar{0}}$ and thus $\mathfrak{z}_{\bar{0}}$ is at least 1-dimensional. If $\mathfrak{z}_{\bar{0}}$ is 2-dimensional, it must contain an element of the form $a_1E_1 + a_2E_2 \in \mathfrak{z}_{\bar{0}}$. Then we have

$$[a_1E_1 + a_2E_2, v_1 \otimes v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \otimes v_1] = -a_1 v_1 \otimes v_1 \otimes v_1 = 0,$$

and

$$[a_2E_2, v_1 \otimes v_{-1} \otimes v_1 - v_{-1} \otimes v_1 \otimes v_1] = a_2 v_1 \otimes v_1 \otimes v_1 = 0.$$

Thus $a_1 = a_2 = 0$ and there is no element of the form $a_1E_1 + a_2E_2$ lies in $\mathfrak{z}_{\bar{0}}$. Hence, $\mathfrak{z}_{\bar{0}} = \langle E_1 + E_2 + E_3 \rangle$.

Again let $x \in \mathfrak{z}_{\bar{1}}$ where x is the linear combination of basis elements of $\mathfrak{g}^e$ with coefficients a_1, a_2 and a_3. Then we have $a_1 = a_2 = a_3 = 0$ by a similar calculation to the above cases. Therefore, $\mathfrak{z} = \langle E_1 + E_2 + E_3 \rangle$ and $\dim \mathfrak{z} = 1$.

Next we look for the labelled Dynkin diagram with respect to e. Note that we can find an element $h = H_1 + H_2 + H_3 = (H, H, H)$ such that h belongs to an $\mathfrak{sl}(2)$-triple $\{e, h, f\}$ in $\mathfrak{g}_{\bar{0}}$. Then we calculate the ad$h$-eigenvalue on each root vector in the table below.

root vectors	$e_{2\beta_1}$	$e_{2\beta_2}$	$e_{2\beta_3}$	$e_{-2\beta_1}$	$e_{-2\beta_2}$	$e_{-2\beta_3}$	$e_{i\beta_1+j\beta_2+k\beta_3}$
eigenvalue	2	2	2	-2	-2	-2	$i+j+k$

Since the ad$h$-eigenvalue of e_α is non-zero for every $\alpha \in \Phi$, the only choice of a system of positive roots Φ^+ such that $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j\geq 0} \mathfrak{g}(j)$ is

$$\Phi^+ = \{2\beta_1, 2\beta_2, 2\beta_3, \beta_1 + \beta_2 + \beta_3, \beta_1 - \beta_2 + \beta_3, \beta_1 + \beta_2 - \beta_3, -\beta_1 + \beta_2 + \beta_3\}.$$

Hence, we get a system of simple roots $\varPi = \{\alpha_1 = -\beta_1 + \beta_2 + \beta_3, \alpha_2 = \beta_1 - \beta_2 + \beta_3, \alpha_3 = \beta_1 + \beta_2 - \beta_3\}$ and the number of lines $\mu_{\alpha\beta}$ between each pair of nodes are easily obtained by calculating $|(\beta_1-\beta_2+\beta_3, -\beta_1+\beta_2+\beta_3)| = \alpha$, $|(\beta_1-\beta_2+\beta_3, \beta_1+\beta_2-\beta_3)| = 1$ and $|(-\beta_1 + \beta_2 + \beta_3, \beta_1 + \beta_2 - \beta_3)| = 1 + \alpha$. Therefore, the labelled Dynkin diagram with respect to $e = E_1 + E_2 + E_3$ is the Dynkin diagram in Figure 15 with labels $1, 1, 1$.

6.2.4 Analysis of results

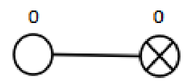
Results obtained in Subsection 6.2.3 can be summarized in the following table:

Table 6.4: $\dim \mathfrak{g}^e$ and $\dim \mathfrak{z}(\mathfrak{g}^e)$

Lie superalgebra	nilpotent element e	$\dim \mathfrak{g}^e$	$\dim \mathfrak{z}(\mathfrak{g}^e)$	$\sum_{i=1}^{l} a_i$
$D(2,1;\alpha)$	0	17	0	0
	E_1	11	1	1
	$E_1 + E_2$	9	1	2
	$E_1 + E_2 + E_3$	6	1	3

Recall that $G = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. Since we have $\mathfrak{z}(\mathfrak{g}^e) = \langle e \rangle$ for all cases, then we have that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \mathfrak{z}(\mathfrak{g}^e)$. Therefore, Theorem 1.2 for $D(2,1;\alpha)$ can be obtained immediately following the above table, i.e. we have that $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \lceil \frac{1}{2} \sum a_i \rceil + \varepsilon$ where $\varepsilon = -1$ for $e = E_1 + E_2 + E_3$ and $\varepsilon = 0$ for all other cases.

In order to prove Theorem 1.1 for $D(2,1;\alpha)$, we only need to consider the case $e = E_1 + E_2$ where the corresponding labelled Dynkin diagram $\varDelta$ has no label equal to 1. When $e = E_1 + E_2$, we have $n_2(\varDelta) = 1$ and $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} = 1$. Next we determine $\mathfrak{g}^h$ and $\mathfrak{z}(\mathfrak{g}^h)$. Note that $\mathfrak{g}^h$ is generated by root vectors $e_{\pm 2\beta_3}$, $e_{\pm(\beta_1-\beta_2+\beta_3)}$, $e_{\pm(\beta_1-\beta_2-\beta_3)}$ and $\mathfrak{h} = \langle H_1, H_2, H_3\rangle$. Hence, we can find a system of positive roots $\Phi_h^+ = \{2\beta_3, \beta_1 - \beta_2 - \beta_3, \beta_1 - \beta_2 + \beta_3\}$ of $\mathfrak{g}^h$ and corresponding simple root system $\Pi_h = \{2\beta_3, \beta_1 - \beta_2 - \beta_3\}$. This is a reductive subalgebra of $D(2,1;\alpha)$ with labelled Dynkin diagram



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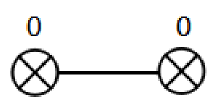
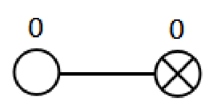
thus this is $\mathfrak{gl}(1|2)$ according to [19, Subsection 3.4.1] and it has centre of dimension 1. Therefore, $\dim \mathfrak{z}(\mathfrak{g}^h) = 1 = n_2(\varDelta) = \dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e}$. The above argument completes the proof for Theorem 1.1 for the case $\mathfrak{g} = D(2,1;\alpha)$.

In order to prove Theorem 1.3 for $D(2,1;\alpha)$, we consider the case $e = E_1 + E_2$ and the remaining cases do not have label equal to 2 so that the 2-free core $\varDelta_0$ of $\varDelta$ is the same as $\varDelta$. Thus equations $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = n_2(\varDelta)$ and $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = n_2(\varDelta)$ are obviously true for the remaining cases as $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0^{e_0}$ and $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}}$. When $e = E_1 + E_2 \in D(2,1;\alpha)$, we have $\dim \mathfrak{g}^e = 9$, $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = 1$ and

$\varDelta_0 =$ 

or 

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Hence, we deduce that the corresponding Lie superalgebra $\mathfrak{g}_0 = \mathfrak{sl}(2|1)$ according to [19, Subsection 3.4.1] and the nilpotent orbit e_0 with respect to $\varDelta_0$ is equal to 0. Therefore, $\dim \mathfrak{g}_0^{e_0} = 8$ and $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = n_2(\varDelta) = 1$. Similarly, we have $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = n_2(\varDelta) = 1$ because $\dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = 0$. The above argument completes the proof for Theorem 1.3 for the case $\mathfrak{g} = D(2,1;\alpha)$.

6.3 The exceptional Lie superalgebra $G(3)$

6.3.1 Structure of the Lie superalgebra $G(3)$

Let $\mathbb{H}$ be the associative quaternion algebra over $\mathbb{R}$. Then $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ where $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$ and $ki = j$. For $x = a + bi + cj + dk \in \mathbb{H}$, we set $\bar{x} = a - bi - cj - dk$. Then we define $t(x) = x + \bar{x}$ to be the *trace* of x. Next we write the octonion algebra to be the nonassociative algebra $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ such that the addition and multiplication are defined by

$$(a, b) + (c, d) = (a + c, b + d),$$

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c})$$

for $a, b, c, d \in \mathbb{H}$. For $x = (a, b) \in \mathbb{O}$, we define $\bar{x} = (\bar{a}, -b)$, then $t(x) = x+\bar{x} = (a+\bar{a}, 0)$. We also set $\mathbb{O}_{(0)} = \{x \in \mathbb{O} : t(x) = 0\}$ and the basis elements of $\mathbb{O}_{(0)}$ are:

$$u_3 = (0, -i),\ u_2 = (0, -j),\ u_1 = (0, -k),\ u_0 = (0, 1),$$

$$u_{-1} = (k, 0),\ u_{-2} = (j, 0),\ \text{and } u_{-3} = (i, 0).$$

Now we define

$$D_{x,z}(y) = [y, [x, z]] - 3\left((xy)\,z - x\,(yz)\right). \tag{6.1}$$

As shown in [19, Proposition 4.3.3], $D_{x,z}$ is a derivation of $\mathbb{O}$. Then we have that $(\mathrm{Der}\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$ is the Lie algebra of type G_2 according to [12, Section 19.3] where $\mathrm{Der}\mathbb{O}$ is spanned by the derivation $D_{x,z} : \mathbb{O} \to \mathbb{O}$ where for $x, y, z \in \mathbb{O}$.

According to [19, Chapter 4], we also know that $\mathbb{O}_{(0)} \otimes_{\mathbb{R}} \mathbb{C}$ is a seven-dimensional simple representation of G_2 which we denote by V_7. A basis of V_7 is given by:

$$e_3 = u_3 \otimes 1 + u_{-3} \otimes i, \ e_2 = u_1 \otimes 1 - u_{-1} \otimes i,$$

$$e_1 = u_2 \otimes 1 - u_{-2} \otimes i, \ e_0 = u_0 \otimes 1, \ e_{-1} = u_2 \otimes 1 + u_{-2} \otimes i,$$

$$e_{-2} = u_1 \otimes 1 + u_{-1} \otimes i, \ e_{-3} = u_3 \otimes 1 - u_{-3} \otimes i.$$

Viewing G_2 as a Lie subalgebra of $\mathfrak{gl}(V_7)$, the elements of G_2 can be written as the elements of $\mathfrak{gl}(7)$ with respect to the above basis. Hence, G_2 has generators x_i, y_i, h_i, for $i = 1, 2$, where

$$x_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$h_1 = \text{diag}(1, -1, 2, 0, -2, 1, -1) \text{ and } h_2 = \text{diag}(0, 1, -1, 0, 1, -1, 0).$$

The above generators are obtained from [19, pages 90–91], but we choose a different normalization of the generators, e.g. our x_1 is obtained by dividing x_1 in [19, page 90] by 4 and our x_2 is obtained by dividing x_2 in [19, page 90] by 12.

The remaining positive root vectors in the basis of G_2 can be generated by x_1 and x_2. For positive root vectors, $x_3 = [x_1, x_2]$, $x_4 = [x_1, x_3]$, $x_5 = [x_1, x_4]$ and $x_6 = [x_5, x_2]$. Negative root vectors can be obtained using a similar way.

Before we start to define the Lie superalgebra $G(3)$, we fix some notation below. Let V_2 be a two-dimensional vector space over $\mathbb{C}$ with basis $\{v_1, v_{-1}\}$, so V_2 is the natural $\mathfrak{sl}(2)$-module. Let ψ_2 be a non-degenerate skew-symmetric bilinear form on V_2 such that $\psi_2(v_1, v_{-1}) = 1$. We define $p_2 : V_2 \times V_2 \to \mathfrak{sl}(2)$ by

$$p_2(x, y)(z) = 4\left(\psi_1(y, z)x - \psi_1(z, x)y\right).$$

We calculate that $p_2(v_1, v_{-1}) = -4H$, $p_2(v_1, v_1) = 8E$ and $p_2(v_{-1}, v_{-1}) = -8F$. We also define $\psi_7 : V_7 \times V_7 \to \mathbb{C}$ by

$$\psi_7(x, y) = \frac{t(x\bar{y})}{2},$$

thus we compute that $\psi_7(e_j, e_{-j}) = 2$, $\psi_7(e_0, e_0) = -1$ and $\psi_7(e_i, e_j) = 0$ if $i \neq -j$. Define $p_7 : V_7 \times V_7 \to G_2$ to be $p_7(x, y) = D_{x,y}$ where the explicit mapping has been calculated from Formula (6.1) and is given in Table 6.5. We will give some explanation about how to calculate p_7 later.

Now we are going to define the Lie superalgebra $G(3)$. By definition, $G(3) = \mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ where

$$\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus G_2$$

and

$$\mathfrak{g}_{\bar{1}} = V_2 \otimes V_7.$$

According to [19, Section 4.7], $G(3)$ has generators x_i, y_i, h_i, for $i = 0, 1, 2$, where x_1, y_1, h_1, x_2, y_2, h_2 are generators of G_2 and

$$x_0 = (v_1 \otimes e_{-3})/2,\ y_0 = (v_{-1} \otimes e_3)/2,\ h_0 = -2H - 2h_1 - h_2,$$

We know that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and the bracket $[\cdot\,,\cdot\,] : \mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{1}}$ is given by

$$[x + y, v_2 \otimes v_7] = xv_2 \otimes v_7 + v_2 \otimes yv_7$$

for $x \in \mathfrak{sl}(2), y \in G_2, v_2 \in V_2$ and $v_7 \in V_7$. The bracket $[\cdot\,,\cdot\,] : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ is given by

$$[v_i \otimes e_k, v_j \otimes e_l] = \psi_2(v_i, v_j)p_7(e_k, e_l) + \psi_7(e_k, e_l)p_2(v_i, v_j). \tag{6.2}$$

for $v_i, v_j \in V_2, e_k, e_l \in V_7$.

Next we work out $p_7 : V_7 \times V_7 \to G_2$ explicitly. We have that

$$p_7(e_{-3}, e_3) = D_{e_{-3},e_3} = D_{u_3 - iu_{-3}, u_3 + iu_{-3}} = 2iD_{u_3,u_{-3}}.$$

In addition, we have that $D_{u_1,u_{-1}} + D_{u_2,u_{-2}} + D_{u_3,u_{-3}} = 0$, $h_1 = \frac{i}{2}D_{u_2,u_{-2}}$ and $h_2 = \frac{i}{6}(D_{u_1,u_{-1}} - D_{u_2,u_{-2}})$ by [19, page 90]. Therefore, we deduce that $p_7(e_{-3}, e_3) = 2iD_{u_3,u_{-3}} = -8h_1 - 12h_2$. According to the graded Jacobi identity, we have that

$$\begin{aligned}[x_1, [v_1 \otimes e_{-3}, v_{-1} \otimes e_3]] &= [v_1 \otimes e_{-3}, [x_1, v_{-1} \otimes e_3]] + [[x_1, v_1 \otimes e_{-3}], v_{-1} \otimes e_3] \\ &= [v_1 \otimes e_{-3}, 0] + [v_1 \otimes e_{-2}, v_{-1} \otimes e_3] = p_7(e_{-2}, e_3).\end{aligned}$$

Moreover, $[x_1, [v_1 \otimes e_{-3}, v_{-1} \otimes e_3]] = [x_1, -8H - 8h_1 - 12h_2] = 4x_1$. Therefore, we deduce that $p_7(e_{-2}, e_3) = 4x_1$. Using similar methods we obtain the explicit mapping

$p_7 : V_7 \times V_7 \to G_2$ which is given in the following table.

Table 6.5: $p_7 : V_7 \times V_7 \to G_2$

	e_3	e_2	e_1	e_0	e_{-1}	e_{-2}	e_{-3}
e_3	0	$-2x_6$	$2x_5$	$2x_4$	$-4x_3$	$-4x_1$	$8h_1+12h_2$
e_2	$2x_6$	0	$-2x_4$	$-4x_3$	$12x_2$	$4h_1+12h_2$	$-4y_1$
e_1	$-2x_5$	$2x_4$	0	$4x_1$	$4h_1$	$12y_2$	$4y_3$
e_0	$-2x_4$	$4x_3$	$-4x_1$	0	$4y_1$	$4y_3$	$2y_4$
e_{-1}	$4x_3$	$-12x_2$	$-4h_1$	$-4y_1$	0	$-2y_4$	$-2y_5$
e_{-2}	$4x_1$	$-4h_1-12h_2$	$-12y_2$	$-4y_3$	$2y_4$	0	$-2y_6$
e_{-3}	$-8h_1 -$ $12h_2$	$4y_1$	$-4y_3$	$-2y_4$	$2y_5$	$2y_6$	0

6.3.2 Root system and Dynkin diagrams for $G(3)$

We follow the description of the root system of $\mathfrak{g} = G(3)$ given in [19, Chapter 4]. Let $\mathfrak{h} = \langle h_0, h_1, h_2 \rangle$ be the Cartan subalgebra of $\mathfrak{g}$. Note that the roots of $G(3)$ can be expressed in terms of $\delta, \varepsilon_{1,}\varepsilon_2, \varepsilon_3 \in \mathfrak{h}^*$ where $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. The root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ is given by

$$\Phi_{\bar{0}} = \{\pm 2\delta, \varepsilon_i - \varepsilon_j, \pm\varepsilon_i : 1 \leq i, j \leq 3\},$$

$$\text{and } \Phi_{\bar{1}} = \{\pm\delta \pm \varepsilon_i, \pm\delta : 1 \leq i \leq 3\},$$

where the bilinear form $(\cdot\,,\cdot\,)$ on $\mathfrak{h}^*$ is defined by

$$(\delta,\delta) = 2,\ (\varepsilon_i,\varepsilon_j) = 1 - 3\delta_{ij},\ (\delta,\varepsilon_i) = 0.$$

The table below lists all roots together with corresponding root vectors.

Roots	2δ	ε_1	$\varepsilon_2-\varepsilon_1$	ε_2	$-\varepsilon_3$	$\varepsilon_1-\varepsilon_3$	$\varepsilon_2-\varepsilon_3$	$i\delta-\varepsilon_3$	$i\delta+\varepsilon_j$
Root vectors	E	x_1	x_2	x_3	x_4	x_5	x_6	$v_i\otimes e_3$	$v_i\otimes e_j$
Roots	-2δ	$-\varepsilon_1$	$\varepsilon_1-\varepsilon_2$	$-\varepsilon_2$	ε_3	$\varepsilon_3-\varepsilon_1$	$\varepsilon_3-\varepsilon_2$	$i\delta+\varepsilon_3$	$i\delta$
Root vectors	F	y_1	y_2	y_3	y_4	y_5	y_6	$v_i\otimes e_{-3}$	$v_i\otimes e_0$

where $i\in\{1,-1\}$, $j\in\{2,1,-1,-2\}$ and we define $\varepsilon_j = -\varepsilon_{-j}$ for $j<0$.

The remaining part of this subsection covers all possible Dynkin diagrams with respect to different systems of simple roots based on [9, Section 2.19]. Note that the number of lines between a pair of vertices that corresponds to simple roots are calculated using Formula (4.1) in Section 4.2.

1. For the simple system $\Pi = \{\alpha_1 = \delta+\varepsilon_3, \alpha_2 = \varepsilon_1, \alpha_3 = \varepsilon_2-\varepsilon_1\}$, we have the Dynkin diagram:

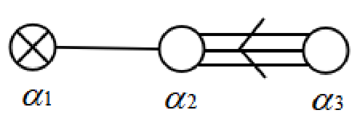


Figure 16: Dynkin diagram for $G(3)$

2. For the simple system $\Pi = \{\alpha_1 = -\delta-\varepsilon_3, \alpha_2 = \delta-\varepsilon_2, \alpha_3 = \varepsilon_2-\varepsilon_1\}$, we have the Dynkin diagram:

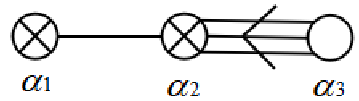


Figure 17: Dynkin diagram for $G(3)$

3. For the simple system $\Pi = \{\alpha_1 = \delta, \alpha_2 = -\delta + \varepsilon_1, \alpha_3 = \varepsilon_2 - \varepsilon_1\}$, we have the Dynkin diagram:

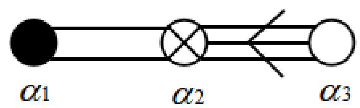


Figure 18: Dynkin diagram for $G(3)$

4. For the simple system $\Pi = \{\alpha_1 = \varepsilon_1, \alpha_2 = -\delta + \varepsilon_2, \alpha_3 = \delta - \varepsilon_1\}$, we have the Dynkin diagram:

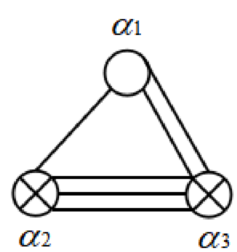


Figure 19: Dynkin diagram for $G(3)$

6.3.3 Centres of centralizers of nilpotent elements e in $G(3)$ and labelled Dynkin diagrams with respect to e

Let $e = e_{\mathfrak{sl}(2)} + e_{G_2} \in \mathfrak{g}_{\bar{0}}$ be nilpotent where $e_{\mathfrak{sl}(2)} \in \mathfrak{sl}(2)$ and $e_{G_2} \in G_2$. According to [18, Section 11], we know that representatives of nilpotent orbits in $\mathfrak{sl}(2)$ are $0, E$ and representatives of nilpotent orbits in G_2 are $0, x_2, x_1, x_2 + x_5, x_1 + x_2$ up to the adjoint action of $G = \mathrm{SL}_2(\mathbb{C}) \times K$ where K is the simple algebraic group of type

G_2. Hence, there are in total 10 possibilities for e. It is clear that $\mathfrak{sl}(2)^E = \langle E\rangle$ and $\mathfrak{sl}(2)^0 = \mathfrak{sl}(2)$. The following tables give basis elements for $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$ and labelled Dynkin diagrams with respect to e. The numbers in the rows labelled "Δ" represent labels a_i corresponding to α_i for $i = 1, 2, 3$ in labelled Dynkin diagram with respect to e. We consider three cases as examples to show the explicit process of finding $\mathfrak{g}^e$, $\mathfrak{z}(\mathfrak{g}^e)$ and labelled Dynkin diagrams after these tables.

For each nilpotent element $e \in \mathfrak{g}_{\bar{0}}$, we find a semisimple element h such that h lies in an $\mathfrak{sl}(2)$-triple in $\mathfrak{g}_{\bar{0}}$ that contains e. We also calculate the $\mathfrak{g}^e(0)$-module structure on each $\mathfrak{g}^e(j)$ for $j > 0$ in the following tables. Let $V^{\mathfrak{sl}}(j)$ be an $\mathfrak{sl}(2)$-module with highest weight j and $V^{\mathfrak{osp}}(j)$ be an $\mathfrak{osp}(1|2)$-module with highest weight j. Note that for $e = x_2$, we have that $\mathfrak{g}^e(0)_{\bar{0}} \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) = \mathfrak{o}(3) \oplus \mathfrak{sp}(2)$, $\dim \mathfrak{g}^e(0)_{\bar{1}} = 6$ and $\mathfrak{g}^e(0)$ has a two-dimensional toral subalgebra $\langle H, 2h_1 + 3h_2\rangle$. Using a similar argument as in Lemma 6.1, we can show that any ideal of $\mathfrak{g}^e(0)$ contains an element of $\langle H, 2h_1 + 3h_2\rangle$ is equal to $\mathfrak{g}^e(0)$. Hence, we deduce that $\mathfrak{g}^e(0)$ is simple and thus $\mathfrak{g}^e(0) \cong \mathfrak{osp}(3|2)$. Note that the $\mathfrak{g}^e(0)$-module structure on $\mathfrak{g}^e(j)$ is not included in this thesis as it requires the construction of $\mathfrak{osp}(3|2)$ representations.

Table 6.6: $e = E + (x_1 + x_2)$

h	$H + \operatorname{diag}(6, 4, 2, 0, -2, -4, -6) = H + (6h_1 + 10h_2)$
$\mathfrak{g}^e$	$\langle E, x_1 + x_2, x_6, v_1 \otimes e_3, v_1 \otimes e_2 + v_{-1} \otimes e_3\rangle$
$\mathfrak{g}^e(0)$	0
$\mathfrak{g}^e(j)$	$\dim \mathfrak{g}^e(10) = \dim \mathfrak{g}^e(7) = \dim \mathfrak{g}^e(5) = 1, \dim \mathfrak{g}^e(2) = 2.$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e, x_6, v_1 \otimes e_3\rangle$
Δ	Figure 18: 1,1,2

Table 6.7: $e = E + x_2$

h	$H + \mathrm{diag}(0, 1, -1, 0, 1, -1, 0) = H + h_2$
$\mathfrak{g}^e$	$\langle E, x_2, y_1, x_3, x_6, y_5, x_4, y_4, 2h_1 + 3h_2, v_1 \otimes e_2, v_1 \otimes e_{-1}, v_1 \otimes e_3, v_1 \otimes e_0, v_1 \otimes e_{-3}, v_1 \otimes e_1 - v_{-1} \otimes e_2, v_1 \otimes e_{-2} + v_{-1} \otimes e_{-1}\rangle$
$\mathfrak{g}^e(0)$	$\mathfrak{osp}(1\|2)$
$\mathfrak{g}^e(j)$	$\mathfrak{g}^e(1) = V^{\mathfrak{osp}}(\frac{3}{2}), \mathfrak{g}^e(2) = V^{\mathfrak{osp}}(0) \oplus V^{\mathfrak{osp}}(\frac{1}{2})$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e \rangle$
Δ	Figure 16: 0,0,1; Figure 17: 0,0,1; Figure 19: 0,0,1

Table 6.8: $e = E + x_1$

h	$H + \mathrm{diag}(1, -1, 2, 0, -2, 1, -1) = H + h_1$
$\mathfrak{g}^e$	$\langle E, x_1, x_5, y_2, x_6, y_6, h_1 + 2h_2, v_1 \otimes e_1, v_1 \otimes e_3, v_1 \otimes e_{-2}, v_1 \otimes e_0 - v_{-1} \otimes e_1, v_1 \otimes e_2 + v_{-1} \otimes e_3, v_1 \otimes e_{-3} - v_{-1} \otimes e_{-2}\rangle$
$\mathfrak{g}^e(0)$	$\mathfrak{osp}(1\|2)$
$\mathfrak{g}^e(j)$	$\mathfrak{g}^e(1) = V^{\mathfrak{osp}}(0), \mathfrak{g}^e(2) = V^{\mathfrak{osp}}(0) \oplus V^{\mathfrak{osp}}(\frac{1}{2}), \mathfrak{g}^e(3) = V^{\mathfrak{osp}}(\frac{1}{2}).$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e \rangle$
Δ	Figure 17: 1,0,0; Figure 19: 1,0,0

Table 6.9: $e = E + (x_2 + x_5)$

h	$H + \mathrm{diag}(2, 2, 0, 0, 0, -2, -2) = H + (2h_1 + 4h_2)$
$\mathfrak{g}^e$	$\langle E, x_6, x_3, x_4, x_2 + x_5, v_1 \otimes e_3, v_1 \otimes e_2, v_1 \otimes e_0, 6v_{-1} \otimes e_3 - v_1 \otimes e_{-1}, v_1 \otimes e_1 - v_{-1} \otimes e_2\rangle$

$\mathfrak{g}^e(0)$	0
$\mathfrak{g}^e(j)$	$\dim \mathfrak{g}^e(4) = 1, \dim \mathfrak{g}^e(3) = 2, \dim \mathfrak{g}^e(2) = 4, \dim \mathfrak{g}^e(1) = 3.$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e, x_6 \rangle$
Δ	Figure 19: 0,1,1

Table 6.10: $e = E$

h	H
$\mathfrak{g}^e$	$\langle E \rangle \oplus \mathrm{G}_2 \oplus \langle v_1 \otimes e_3, v_1 \otimes e_2, v_1 \otimes e_1, v_1 \otimes e_0, v_1 \otimes e_{-1}, v_1 \otimes e_{-2}, v_1 \otimes e_{-3} \rangle$
$\mathfrak{g}^e(0)$	G_2
$\mathfrak{g}^e(j)$	$\mathfrak{g}^e(1) = V_7, \mathfrak{g}^e(2) = \langle e \rangle.$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e \rangle$
Δ	Figure 16: 1,0,0

Table 6.11: $e = x_1 + x_2$

h	$\mathrm{diag}(6, 4, 2, 0, -2, -4, -6) = 6h_1 + 10h_2$
$\mathfrak{g}^e$	$\langle E, H, F, x_1 + x_2, x_6, v_1 \otimes e_3, v_{-1} \otimes e_3 \rangle$
$\mathfrak{g}^e(0)$	$\mathfrak{sl}(2)$
$\mathfrak{g}^e(j)$	$\mathfrak{g}^e(2) = \mathfrak{g}^e(10) = V^{\mathfrak{sl}}(0), \mathfrak{g}^e(6) = V^{\mathfrak{sl}}(1).$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e, x_6 \rangle$
Δ	Figure 18: 0,2,2

Table 6.12: $e = x_2$

h	$\mathrm{diag}(0,1,-1,0,1,-1,0) = h_2$
$\mathfrak{g}^e$	$\langle E, H, F, x_2, y_1, x_3, x_6, y_5, x_4, y_4, 2h_1+3h_2, v_1\otimes e_2, v_{-1}\otimes e_2, v_1\otimes e_{-1}, v_{-1}\otimes e_{-1}, v_1\otimes e_{-3}, v_{-1}\otimes e_{-3}, v_1\otimes e_3, v_{-1}\otimes e_3, v_1\otimes e_0, v_{-1}\otimes e_0\rangle$
$\mathfrak{g}^e(0)$	$\mathfrak{osp}(3\vert 2)$
$\mathfrak{g}^e(j)$	$\dim\mathfrak{g}^e(1) = 8, \dim\mathfrak{g}^e(2) = 1.$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e\rangle$
Δ	Figure 18: 0,0,1

Table 6.13: $e = x_1$

h	$\mathrm{diag}(1,-1,2,0,-2,1,-1) = h_1$
$\mathfrak{g}^e$	$\langle E, H, F, x_1, x_5, y_2, x_6, y_6, h_1+2h_2, v_1\otimes e_1, v_{-1}\otimes e_1, v_1\otimes e_3, v_{-1}\otimes e_3, v_1\otimes e_{-2}, v_{-1}\otimes e_{-2}\rangle$
$\mathfrak{g}^e(0)$	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$
$\mathfrak{g}^e(j)$	$\mathfrak{g}^e(1) = V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{sl}}(1), \mathfrak{g}^e(3) = V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{sl}}(1), \mathfrak{g}^e(2) = \left(V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{sl}}(0)\right)\oplus\left(V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{sl}}(0)\right).$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e\rangle$
Δ	Figure 18: 0,1,0

Table 6.14: $e = x_2 + x_5$

h	$\mathrm{diag}(2,2,0,0,0,-2,-2) = 2h_1+4h_2$

$\mathfrak{g}^e$	$\langle E, H, F, x_6, x_3, x_4, x_2 + x_5, v_1 \otimes e_3, v_{-1} \otimes e_3, v_1 \otimes e_2, v_{-1} \otimes e_2, v_1 \otimes e_0, v_{-1} \otimes e_0 \rangle$
$\mathfrak{g}^e(0)$	$\mathfrak{osp}(1\|2)$
$\mathfrak{g}^e(j)$	$\mathfrak{g}^e(2) = V^{\mathfrak{osp}}(0) \oplus V^{\mathfrak{osp}}(\frac{1}{2}) \oplus V^{\mathfrak{osp}}(\frac{1}{2}), \mathfrak{g}^e(4) = V^{\mathfrak{osp}}(0).$
$\mathfrak{z}(\mathfrak{g}^e)$	$\langle e, x_6 \rangle$
Δ	Figure 18: 0,0,2; Figure 19: 0,2,0

Table 6.15: $e = 0$

h	0
$\mathfrak{g}^e$	$\mathfrak{g}$
$\mathfrak{z}(\mathfrak{g}^e)$	$\{0\}$
Δ	Figures 16, 17, 18, 19: All labels are zeros.

In the remaining part of this section, we give explicit calculation on finding $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$ and obtain the corresponding labelled Dynkin diagram for nilpotent elements $E + (x_1 + x_2)$, $E + x_2$ and $x_2 + x_5$. The results of remaining cases are obtained using the same approach.

(1) $e = E + (x_1 + x_2)$

We have

$$x_1 + x_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We first look for an $\mathfrak{sl}(2)$-triple $\{e_{G_2}, h_{G_2}, f_{G_2}\}$ in G_2 containing $e_{G_2} = x_1 + x_2$. Write $h_{G_2} = \mathrm{diag}(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ such that $[h_{G_2}, e_{G_2}] = 2e_{G_2}$ and $h_{G_2} \in \langle h_1, h_2 \rangle$. An easy calculation shows that $h_{G_2} = \mathrm{diag}(6, 4, 2, 0, -2, -4, -6) = 6h_1 + 10h_2$ and $f_{G_2} = 6y_1 + 10y_2$. Then we work out the $\mathrm{ad}h_{G_2}$-eigenspaces with non-negative eigenvalues of G_2.

Eigenvalues of h_{G_2}	10	8	6	4	2	0
Eigenvectors	x_6	x_5	x_4	x_3	x_2	h_1
					x_1	h_2

Table 6.16: $\mathrm{ad}h_{G_2}$-eigenspace decomposition

Since G_2 is a module for $\mathfrak{sl}(2) \cong \langle e_{G_2}, h_{G_2}, f_{G_2} \rangle$, the above table implies that G_2 is the direct sum of $V^{\mathfrak{sl}}(2)$ and $V^{\mathfrak{sl}}(10)$. Hence $G_2^{e_{G_2}}$ has one basis element with $\mathrm{ad}h_{G_2}$-eigenvalue 10 and it must be x_6. We also deduce that $G_2^{e_{G_2}}$ has one basis element with $\mathrm{ad}h_{G_2}$-eigenvalue 2 and it must be $e_{G_2} = x_1 + x_2$ as e_{G_2} centralizes itself. Hence, $\mathfrak{g}_{\bar{0}}^e = \mathfrak{sl}(2)^E \oplus G_2^{e_{G_2}} = \langle E, x_1 + x_2, x_6 \rangle$.

In order to calculate $\mathfrak{g}_{\bar{1}}^e$, we need to look at $e = E + (x_1 + x_2)$. In fact $\{E + e_{G_2}, H + h_{G_2}, F + f_{G_2}\}$ is an $\mathfrak{sl}(2)$-triple in $\mathfrak{g}_{\bar{0}}$ containing e. Observe that $[h_{G_2}, e_j] = 2je_j$ for $j \in \{3, 2, ..., -2, -3\}$ and $[H, v_i] = iv_i$ for $i \in \{1, -1\}$. Let $h = H + h_{G_2}$, we have $[h, v_i \otimes e_j] = [H + h_{G_2}, v_i \otimes e_j] = Hv_i \otimes e_j + v_i \otimes h_{G_2} e_j = (i + 2j) v_i \otimes e_j$. Then we work out all eigenvalues of $\mathrm{ad}h$ on $\mathfrak{g}_{\bar{1}}$ and $\mathrm{ad}h$-eigenspaces in the table below.

<table>
<tr><td>adh-eigenvalues</td><td>7</td><td>5</td><td>3</td><td>1</td></tr>
<tr><td rowspan="2">adh-eigenspaces</td><td rowspan="2">$v_1 \otimes e_3$</td><td>$v_1 \otimes e_2$</td><td>$v_1 \otimes e_1$</td><td>$v_1 \otimes e_0$</td></tr>
<tr><td>$v_{-1} \otimes e_3$</td><td>$v_{-1} \otimes e_2$</td><td>$v_{-1} \otimes e_1$</td></tr>
<tr><td>adh-eigenvalues</td><td>-1</td><td>-3</td><td>-5</td><td>-7</td></tr>
<tr><td rowspan="2">adh-eigenspaces</td><td>$v_1 \otimes e_{-1}$</td><td>$v_1 \otimes e_{-2}$</td><td>$v_1 \otimes e_{-3}$</td><td rowspan="2">$v_{-1} \otimes e_{-3}$</td></tr>
<tr><td>$v_{-1} \otimes e_0$</td><td>$v_{-1} \otimes e_{-1}$</td><td>$v_{-1} \otimes e_{-2}$</td></tr>
</table>

Table 6.17: $\mathrm{ad}h$-eigenspaces

Therefore, $\mathfrak{g}^e_{\bar{1}}$ has one basis element with $\mathrm{ad}h$-eigenvalue 7, namely $v_1 \otimes e_3$. It also has one basis element with $\mathrm{ad}h$-eigenvalue 5 which is of the form $av_1 \otimes e_2 + bv_{-1} \otimes e_3$. To find this basis element, we compute

$$[e, av_1 \otimes e_2 + bv_{-1} \otimes e_3] = bv_1 \otimes e_3 - av_1 \otimes e_3,$$

this is equal to 0 if and only if $b - a = 0$. Therefore, we can choose the basis element to be $v_1 \otimes e_2 + v_{-1} \otimes e_3$. In conclusion, the centralizer $\mathfrak{g}^e$ has a basis $\{E, x_1 + x_2, x_6, v_1 \otimes e_3, v_1 \otimes e_2 + v_{-1} \otimes e_3\}$ and $\dim \mathfrak{g}^e = 3 + 2 = 5$.

By calculating $\mathrm{ad}h$-eigenvalues of basis elements for $\mathfrak{g}^e$, we have that $\mathfrak{z} = \mathfrak{z}(2) \oplus \mathfrak{z}(5) \oplus \mathfrak{z}(7) \oplus \mathfrak{z}(10)$. We first observe that

$$[E, v_1 \otimes e_2 + v_{-1} \otimes e_3] = v_1 \otimes e_3 \neq 0.$$

Hence $E \notin \mathfrak{z}$ and $\mathfrak{z}(5) = 0$. Thus $\mathfrak{z}(2)$ is 1-dimensional and is spanned by $e = E + (x_1 + x_2)$. Now we consider $\mathfrak{z}(7)$, observe that $\mathfrak{g}^e(7 + j) = 0$ for all j such that $\mathfrak{g}^e(j) \neq 0$. Hence, for any $x \in \mathfrak{g}^e(j) \neq 0$, we have that $[v_1 \otimes e_3, x] = 0$ and thus $\mathfrak{z}(7) = \langle v_1 \otimes e_3 \rangle$. Similarly, we have $\mathfrak{z}(10) = \langle x_6 \rangle$. Therefore, $\mathfrak{z} = \langle e, x_6, v_1 \otimes e_3 \rangle$ and it is 3-dimensional.

Next we look at the labelled Dynkin diagram with respect to e. Note that we already calculated the ad$h$-eigenvalue of each root vector in Table 6.16 and Table 6.17. Thus we obtain that roots in $\mathfrak{g}(>0)$ are $\{2\delta, \varepsilon_1, \varepsilon_2-\varepsilon_1, \varepsilon_2, -\varepsilon_3, \varepsilon_1-\varepsilon_3, \varepsilon_2-\varepsilon_3, \delta-\varepsilon_3, \delta+\varepsilon_2, \delta+\varepsilon_1, \delta, -\delta-\varepsilon_3, -\delta+\varepsilon_2, -\delta+\varepsilon_1\}$ and there is no root in $\mathfrak{g}(0)$. Hence, there is only one choice of Φ^+ such that $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j\geq 0} \mathfrak{g}(j)$ and the corresponding simple roots are $\Pi = \{\delta, -\delta+\varepsilon_1, \varepsilon_2-\varepsilon_1\}$.

We also need to calculate the following relations in order to determine whether an odd root is isotropic and how many lines between each pair of simple roots in the corresponding labelled Dynkin diagram. We have

$$(\delta,\delta)=2,\ (-\delta+\varepsilon_1, -\delta+\varepsilon_1)=0,\ |(\varepsilon_2-\varepsilon_1, \varepsilon_2-\varepsilon_1)|=6,$$

$$\frac{2|(\delta, -\delta+\varepsilon_1)|}{\min_{(\alpha_k,\alpha_k)\neq 0}|(\alpha_k,\alpha_k)|}=2 \text{ and } \frac{2|(-\delta+\varepsilon_1, \varepsilon_2-\varepsilon_1)|}{\min_{(\alpha_k,\alpha_k)\neq 0}|(\alpha_k,\alpha_k)|}=3 \text{ for } \alpha_k\in\Phi.$$

Therefore, the labelled Dynkin diagram for $e=E+(x_1+x_2)$ is:

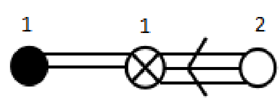


Figure 20: Labelled Dynkin diagram for $e=E+(x_1+x_2)$

(2) $e = E + x_2$

We know that $\mathfrak{sl}(2)^E = \langle E\rangle$, now we are going to work out $\mathrm{G}_2^{x_2}$. Observe that $h_{G_2} = \mathrm{diag}(0,1,-1,0,1,-1,0) = h_2$ belongs to an $\mathfrak{sl}(2)$-triple $\{e_{G_2}, h_{G_2}, f_{G_2}\}$ in G_2 containing $e_{G_2} = x_2$. Then we work out the adh_{G_2}-eigenspaces with non-negative eigenvalues of G_2 in the table below.

Eigenvalues of h'	Eigenvectors
0	$h_1,\ h_2,\ x_4,\ y_4$
1	$y_1,\ x_3,\ x_6,\ y_5$
2	x_2

Table 6.18: $\mathrm{ad}h_{G_2}$-eigenspace decomposition

This demonstrates that $G_2^{e_{G_2}} \cong G_2^{e_{G_2}}(0) \oplus G_2^{e_{G_2}}(1) \oplus G_2^{e_{G_2}}(2)$ where $0, 1, 2$ are the $\mathrm{ad}h_{G_2}$-eigenvalues. Clearly $G_2^{e_{G_2}}(2) = \langle x_2 \rangle$ and $G_2^{e_{G_2}}(1) = \langle y_1, x_3, x_6, y_5 \rangle$. Note that $G_2^{e_{G_2}}(0)$ has dimension 3. Since $[2h_1 + 3h_2, x_2] = 0$ and $[x_4, x_2] = 0 = [y_4, x_2]$, we have that $G_2^{e_{G_2}}(0) = \langle x_4, y_4, 2h_1 + 3h_2 \rangle$. Therefore, we have that $\mathfrak{g}_{\bar{0}}^e$ has a basis $\{E, x_2, y_1, x_3, x_6, y_5, x_4, y_4, 2h_1 + 3h_2\}$.

Now we calculate $\mathfrak{g}_{\bar{1}}^e$. We need to look at the $\mathfrak{sl}(2)$-triple $\{e, h, f\} = \{E + x_2, H + h_{G_2}, F + f_{G_2}\}$ in $\mathfrak{g}_{\bar{0}}$. We work out all eigenvalues of $\mathrm{ad}h$ on basis elements $v_i \otimes e_j \in \mathfrak{g}_{\bar{1}}$ and $\mathrm{ad}h$-eigenspaces in the table below.

$\mathrm{ad}h$-eigenvalues	$\mathrm{ad}h$-eigenvectors
2	$v_1 \otimes e_2,\ v_1 \otimes e_{-1}$
1	$v_1 \otimes e_3,\ v_1 \otimes e_0,\ v_1 \otimes e_{-3}$
0	$v_1 \otimes e_1,\ v_1 \otimes e_{-2},\ v_{-1} \otimes e_2,\ v_{-1} \otimes e_{-1}$
-1	$v_{-1} \otimes e_3,\ v_{-1} \otimes e_0,\ v_{-1} \otimes e_{-3}$
-2	$v_{-1} \otimes e_1,\ v_{-1} \otimes e_{-2}$

Table 6.19: $\mathrm{ad}h$-eigenspaces

Let us write $\mathfrak{g}_{\bar{1}}(j)$ for the j-eigenspace of $\mathrm{ad}h$. Then the above table implies that $\mathfrak{g}_{\bar{1}}^e \cong \mathfrak{g}_{\bar{1}}^e(0) \oplus \mathfrak{g}_{\bar{1}}^e(1) \oplus \mathfrak{g}_{\bar{1}}^e(2)$ where $\mathfrak{g}_{\bar{1}}^e(2)$ has a basis $\{v_1 \otimes e_2, v_1 \otimes e_{-1}\}$ and $\mathfrak{g}_{\bar{1}}^e(1)$ has a basis $\{v_1 \otimes e_3, v_1 \otimes e_0, v_1 \otimes e_{-3}\}$. We also know that $\mathfrak{g}_{\bar{1}}^e(0)$ has dimension 2. To determine $\mathfrak{g}_{\bar{1}}^e(0)$, we need to find elements of the form $x = a_{1,1} v_1 \otimes e_1 + a_{1,-2} v_1 \otimes e_{-2} + a_{-1,2} v_{-1} \otimes e_2 + a_{-1,-1} v_{-1} \otimes e_{-1}$ that are centralized by e. By calculating $[e, x] = 0$, we have that $a_{1,1} = -a_{-1,2}$ and $a_{1,-2} = a_{-1,-1}$. Hence $\mathfrak{g}_{\bar{1}}^e(0)$ has a basis $\{v_1 \otimes e_1 - v_{-1} \otimes e_2, v_1 \otimes e_{-2} + v_{-1} \otimes e_{-1}\}$. Therefore, $\mathfrak{g}_{\bar{1}}^e$ has a basis $\{v_1 \otimes e_2, v_1 \otimes e_{-1}, v_1 \otimes e_3, v_1 \otimes e_0, v_1 \otimes e_{-3}, v_1 \otimes e_1 - v_{-1} \otimes$

$e_2, v_1 \otimes e_{-2} + v_{-1} \otimes e_{-1}\}$. In conclusion, we have that $\dim \mathfrak{g}^e = 9 + 7 = 16$.

By computing commutator relations between basis elements for $\mathfrak{g}^e(0)$, we deduce that $\mathfrak{g}^e(0) \cong \mathfrak{osp}(1|2)$ according to Lemma 6.1 where $y_4, v_1 \otimes e_{-2} + v_{-1} \otimes e_{-1}, 2h_1 + 3h_2, v_1 \otimes e_1 - v_{-1} \otimes e_2, x_4$ correspond to $u_{-2}, u_{-1}, u_0, u_1, u_2$ in Lemma 6.1. Moreover, we obtain that $\mathfrak{g}^e(1) = V^{\mathfrak{osp}}(\frac{3}{2})$ and $\mathfrak{g}^e(2) = V^{\mathfrak{osp}}(0) \oplus V^{\mathfrak{osp}}(\frac{1}{2})$. Hence, we have that

$$\mathfrak{z} = \mathfrak{z}(0) \oplus \mathfrak{z}(1) \oplus \mathfrak{z}(2) \subseteq (\mathfrak{g}^e(0))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(1))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(2))^{\mathfrak{g}^e(0)} = \langle E + x_2 \rangle.$$

Note that $e = E + x_2 \in \mathfrak{z}$, therefore $\mathfrak{z} = \langle e \rangle$ and it has dimension 1.

Next we look at the labelled Dynkin diagrams with respect to e. We obtain that roots in $\mathfrak{g}(>0)$ are $\{2\delta, \varepsilon_2 - \varepsilon_1, -\varepsilon_1, \varepsilon_2, \varepsilon_2 - \varepsilon_{3,} \varepsilon_3 - \varepsilon_1, \delta + \varepsilon_2, \delta - \varepsilon_1, \delta - \varepsilon_3, \delta, \delta + \varepsilon_3\}$ and roots in $\mathfrak{g}(0)$ are $\Phi(0) = \{\pm\varepsilon_3, \pm(\delta + \varepsilon_1), \pm(\delta - \varepsilon_2)\}$ according to Table 6.18 and Table 6.19. There are three systems of simple roots of $\mathfrak{g}(0)$: $\Pi_1(0) = \{-\varepsilon_3, -\delta - \varepsilon_1\}$, $\Pi_2(0) = \{-\delta + \varepsilon_2, \delta + \varepsilon_1\}$ and $\Pi_3(0) = \{\delta - \varepsilon_2, -\varepsilon_3\}$ up to conjugacy. By extending $\Pi_i(0)$ to Π for $i = 1, 2, 3$, we get three systems of positive roots Φ_i^+ and simple roots Π_i and thus there are three conjugacy classes of systems of simple roots Φ^+ satisfying $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j \geq 0} \mathfrak{g}(j)$. We list three systems of simple roots and draw the corresponding labelled Dynkin diagrams below.

- We have $\Pi_1 = \{-\varepsilon_3, -\delta - \varepsilon_1, \delta + \varepsilon_3\}$. We have that $-\delta - \varepsilon_1$ and $\delta + \varepsilon_3$ are odd isotropic roots since $(-\delta - \varepsilon_1, -\delta - \varepsilon_1) = 0$ and $(\delta + \varepsilon_3, \delta + \varepsilon_3) = 0$. Then we calculate the number of lines between each pair of simple roots. Note that

$$\frac{2|(-\varepsilon_3, -\delta - \varepsilon_1)|}{\min_{(\alpha_k, \alpha_k) \neq 0} |(\alpha_k, \alpha_k)|} = 1, \ \frac{2|(-\varepsilon_3, \delta + \varepsilon_3)|}{\min_{(\alpha_k, \alpha_k) \neq 0} |(\alpha_k, \alpha_k)|} = 2 \text{ for } \alpha_k \in \Phi,$$

$$\text{and } |\,(-\delta - \varepsilon_1, \delta + \varepsilon_3)\,| = 3.$$

Therefore, the corresponding labelled Dynkin diagram is:

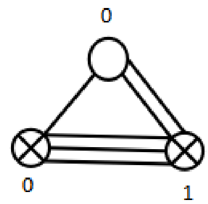


Figure 21: Labelled Dynkin diagram for $e = E + x_2$

- We have $\Pi_2 = \{-\delta + \varepsilon_2, \delta + \varepsilon_1, \varepsilon_3 - \varepsilon_1\}$. Again $-\delta + \varepsilon_2$ and $\delta + \varepsilon_1$ are isotropic and we calculate that

$$|(-\delta + \varepsilon_2, \delta + \varepsilon_1)| = 1, \ \frac{2|(\delta + \varepsilon_1, \varepsilon_3 - \varepsilon_1)|}{\min_{(\alpha_k,\alpha_k)\neq 0} |(\alpha_k, \alpha_k)|} = 3 \text{ for } \alpha_k \in \Phi,$$

$$\text{and} |\,(\varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_1)\,| = 6.$$

Therefore, the corresponding labelled Dynkin diagram is:

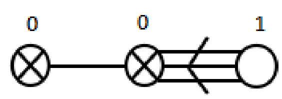


Figure 22: Labelled Dynkin diagram for $e = E + x_2$

- We have that $\Pi_3 = \{\delta - \varepsilon_2, -\varepsilon_3, \varepsilon_3 - \varepsilon_1\}$. Then we calculate that

$$\frac{2|(\delta - \varepsilon_2, -\varepsilon_3)|}{\min_{(\alpha_k,\alpha_k)\neq 0} |(\alpha_k, \alpha_k)|} = 1 \text{ for } \alpha_k \in \Phi$$

$$\text{and } \frac{2|(-\varepsilon_3, \varepsilon_3 - \varepsilon_1)|}{\min\{|(-\varepsilon_3, -\varepsilon_3)|, |(\varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_1)|\}} = 3.$$

Therefore, the corresponding labelled Dynkin diagram is:

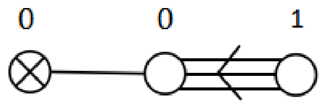


Figure 23: Labelled Dynkin diagram for $e = E + x_2$

(3) $e = x_2 + x_5$

We have

$$x_2 + x_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we are going to work out G_2^e. We first look for an $\mathfrak{sl}(2)$-triple $\{e, h, f\}$ in G_2 containing $e = x_2 + x_5$. One can check that in fact $h = 2h_1 + 4h_2 = \operatorname{diag}(2, 2, 0, 0, 0, -2, -2)$. Then the ad$h$-eigenspaces with non-negative eigenvalues of G_2 are listed below.

Eigenvalues of h	Eigenvectors
0	h_1, h_2, x_1, y_1
2	x_2, x_3, x_4, x_5
4	x_6

Table 6.20: ad$h$-eigenspace decomposition

This implies that G_2^e has one basis vector with ad$h$-eigenvalue 4, that is, x_6. We know that G_2^e also has three basis vectors with ad$h$-eigenvalue 2 and they are of the form $ax_2 + bx_5 + cx_4 + dx_3$. Note that $[e, ax_2 + bx_5 + cx_4 + dx_3] = (b - a)[x_2, x_5] = 0$

if and only if $b = a$. Thus we have that $G_2^e = \langle x_6, x_3, x_4, x_2 + x_5 \rangle$. Hence, $\mathfrak{g}_{\bar{0}}^e = \mathfrak{sl}(2) \oplus \langle x_6, x_3, x_4, x_2 + x_5 \rangle$.

Next we work out all eigenvalues of ad$h$ on basis elements $v_i \otimes e_j \in \mathfrak{g}_{\bar{1}}$ and ad$h$-eigenspaces in the following table.

ad$h$-eigenvalues	ad$h$-eigenspaces
2	$v_1 \otimes e_3$, $v_{-1} \otimes e_3$, $v_1 \otimes e_2$, $v_{-1} \otimes e_2$
0	$v_1 \otimes e_1$, $v_{-1} \otimes e_1$, $v_1 \otimes e_0$, $v_{-1} \otimes e_0$, $v_1 \otimes e_{-1}$, $v_{-1} \otimes e_{-1}$
-2	$v_1 \otimes e_{-2}$, $v_{-1} \otimes e_{-2}$, $v_1 \otimes e_{-3}$, $v_{-1} \otimes e_{-3}$

Table 6.21: ad$h$-eigenspaces

This demonstrates that $\mathfrak{g}_{\bar{1}}^e$ has four basis elements with ad$h$-eigenvalue 2, namely $v_1 \otimes e_3$, $v_{-1} \otimes e_3$, $v_1 \otimes e_2$ and $v_{-1} \otimes e_2$. To find out $\mathfrak{g}_{\bar{1}}^e(0)$, we use the same argument to the previous case. One can check that in fact $\mathfrak{g}_{\bar{1}}^e(0)$ has basis elements $\{v_1 \otimes e_0, v_{-1} \otimes e_0\}$. Hence, we have that $\mathfrak{g}_{\bar{1}}^e$ has basis elements $\{v_1 \otimes e_3, v_{-1} \otimes e_3, v_1 \otimes e_2, v_{-1} \otimes e_2, v_1 \otimes e_0, v_{-1} \otimes e_0\}$. Therefore, $\mathfrak{g}^e$ has dimension $7 + 6 = 13$.

By computing commutator relations between basis elements for $\mathfrak{g}^e(0)$, we deduce that $\mathfrak{g}^e(0) \cong \mathfrak{osp}(1|2)$ according to Lemma 6.1 where $F, v_{-1} \otimes e_0, H, v_1 \otimes e_0, E$ correspond to $u_{-2}, u_{-1}, u_0, u_1, u_2$ in Lemma 6.1. Moreover, we obtain that $\mathfrak{g}^e(2) = V^{\mathfrak{osp}}(0) \oplus V^{\mathfrak{osp}}(\frac{1}{2}) \oplus V^{\mathfrak{osp}}(\frac{1}{2})$ and $\mathfrak{g}^e(4) = V^{\mathfrak{osp}}(0)$. Hence, we have that

$$\mathfrak{z} = \mathfrak{z}(0) \oplus \mathfrak{z}(2) \oplus \mathfrak{z}(4) \subseteq (\mathfrak{g}^e(0))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(2))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(4))^{\mathfrak{g}^e(0)} = \langle x_2 + x_5, x_6 \rangle.$$

We calculate that $e = x_2 + x_5$ and x_6 commute with all basis elements for $\mathfrak{g}^e$, therefore $\mathfrak{z} = \langle e, x_6 \rangle$ and it has dimension 2.

Next we look at the labelled Dynkin diagrams with respect to e. Note that roots in $\mathfrak{g}(>0)$ are $\{\varepsilon_2-\varepsilon_3, \varepsilon_2-\varepsilon_1, \varepsilon_2, -\varepsilon_3, \varepsilon_1-\varepsilon_3, \delta-\varepsilon_3, -\delta-\varepsilon_3, \delta+\varepsilon_2, -\delta+\varepsilon_2\}$ and roots in $\mathfrak{g}(0)$ are $\Phi(0)=\{\pm 2\delta, \pm\varepsilon_1, \pm(\delta+\varepsilon_1), \pm\delta, \pm(\delta-\varepsilon_1)\}$. There are two systems of simple roots of $\mathfrak{g}(0)$: $\Pi_1(0)=\{\delta, -\delta+\varepsilon_1\}$ and $\Pi_2(0)=\{\varepsilon_1, \delta-\varepsilon_1\}$ up to conjugacy. By extending $\Pi_i(0)$ to Π for $i=1,2$, we get two systems of positive roots Φ_i^+ and simple roots Π_i and thus there are two conjugacy classes of systems of positive roots Φ^+ satisfying $\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha \subseteq \bigoplus_{j\geq 0}\mathfrak{g}(j)$. Similar to previous cases, we list two systems of simple roots and draw the corresponding labelled Dynkin diagrams below.

- We have $\Pi_1=\{\delta, -\delta+\varepsilon_1, \varepsilon_2-\varepsilon_1\}$ and the corresponding labelled Dynkin diagram is:

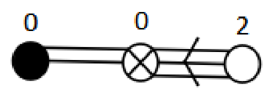


Figure 24: Labelled Dynkin diagram for $e=x_2+x_5$

- We also have $\Pi_2=\{\varepsilon_1, \delta-\varepsilon_1, -\delta+\varepsilon_2\}$ and the corresponding labelled Dynkin diagram is:

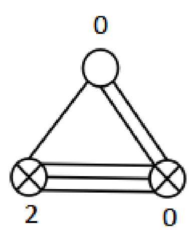


Figure 25: Labelled Dynkin diagram for $e=x_2+x_5$

6.3.4 Adjoint action on $G(3)$

Let K be a simple algebraic group of type G_2, then $\mathrm{Lie}(K)$ is the Lie algebra of type G_2. Write $\mathfrak{g} = G(3) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ where $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathrm{Lie}(K)$ and $G = \mathrm{SL}_2(\mathbb{C}) \times K$. For a nilpotent element $e = e_{\mathfrak{sl}} + e_{G_2} \in \mathfrak{g}_{\bar{0}}$ where $e_{\mathfrak{sl}} \in \mathfrak{sl}(2)$ and $e_{G_2} \in \mathrm{Lie}(K)$, we determine $(\mathfrak{z}(\mathfrak{g}^e))^{G^e}$ in this subsection. Write $(K^{e_{G_2}})^{\circ}$ for the connected component of $K^{e_{G_2}}$ containing the identity and let $K^{e_{G_2}}(0) = K^{h_{G_2}} \cap K^{e_{G_2}}$. An explicit structure of $K^{e_{G_2}}$ has been given in [18, Section 11].

For $e_{\mathfrak{sl}} = 0$, the centralizer in $\mathrm{SL}_2(\mathbb{C})$ is connected. Thus it suffices to only look at the action of K^e on $\mathfrak{z} \subseteq \mathfrak{z}_{\bar{0}}$. In this case, we know that $G^e/(G^e)^{\circ} \cong K^e/(K^e)^{\circ} \cong K^e(0)/(K^e(0))^{\circ}$. Based on [18, Section 11], we know that $K^e(0)/(K^e(0))^{\circ} = 1$ when $e = x_1, x_2$ or $x_1 + x_2$. Hence, we deduce that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = (\mathfrak{z}(\mathfrak{g}^e))^{K^e(0)/(K^e(0))^{\circ}} = \mathfrak{z}(\mathfrak{g}^e)$ for above three cases.

Next we consider $e = x_2 + x_5 \in \mathfrak{g}_{\bar{0}}$. Note that a basis of $(\mathfrak{z}(\mathfrak{g}^e))^{G^e}$ has been given in [18, Section 11]. We include the explicit explanation here as an example.

Let Φ be the root system of $\mathrm{Lie}(K)$ and Π be a base for Φ. We know that $\Pi = \{\alpha_1, \alpha_2\}$ where α_2 is the longer root. The root vectors

$$e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_1+\alpha_2}, e_{2\alpha_1+\alpha_2}, e_{3\alpha_1+\alpha_2}, e_{3\alpha_1+2\alpha_2}$$

are denoted by $x_1, x_2, x_3, x_4, x_5, x_6$ as defined in Subsection 6.3.1. We calculate the structure constants $N_{\alpha_1,\alpha_2} = N_{\alpha_1,\alpha_1+\alpha_2} = N_{\alpha_1,2\alpha_1+\alpha_2} = N_{\alpha_1+\alpha_2,2\alpha_1+\alpha_2} = 1$ and $N_{\alpha_2,3\alpha_1+\alpha_2} = -1$. For each $\alpha \in \Phi$, we write $x_\alpha : \mathbb{C} \to \mathrm{Aut}(\mathrm{Lie}(K))$ such that $x_\alpha(c) = \exp(c \,\mathrm{ad} e_\alpha)$ for $c \in \mathbb{C}$. Define $n_\alpha(c) = x_\alpha(c)x_{-\alpha}(-c^{-1})x_\alpha(c)$, $h_\alpha(c) = n_\alpha(c)n_\alpha(1)^{-1}$ and $n_\alpha = n_\alpha(1)$. For the automorphism $x_\alpha(c)$ on the elements of the Chevalley basis of $\mathrm{Lie}(K)$, we have

$x_\alpha(c)\cdot e_\alpha = e_\alpha$ and

$$x_\alpha(c)\cdot e_\beta = e_\beta + cN_{\alpha,\beta}e_{\alpha+\beta} + \frac{c^2}{2!}N_{\alpha,\beta}N_{\alpha,\alpha+\beta}e_{2\alpha+\beta}+ \\ \cdots + \frac{c^q}{q!}N_{\alpha,\beta}N_{\alpha,\alpha+\beta}\dots N_{\alpha,(q-1)\alpha+\beta}e_{q\alpha+\beta}$$

for $\alpha \neq \pm\beta$ which is given in [2, page 61]. We also have $h_\alpha(c)\cdot e_\beta = c^{\langle\beta,\alpha\rangle}e_\beta$ such that $\langle\beta,\alpha\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$, $n_\alpha\cdot h_\beta = h_{w_\alpha(\beta)}$, $n_\alpha\cdot e_\beta = \eta_{\alpha,\beta}e_{w_\alpha(\beta)}$ where $\eta_{\alpha,\beta} = \pm 1$ and can be calculated using formulas in [2, page 95]. For the purpose of further analysis, we calculate that

$$\eta_{\alpha_1,\alpha_2} = \eta_{\alpha_1,\alpha_1+\alpha_2} = \eta_{\alpha_1,3\alpha_1+2\alpha_2} = 1, \eta_{\alpha_1,2\alpha_1+\alpha_2} = \eta_{\alpha_1,3\alpha_1+\alpha_2} = -1.$$

We know that $\mathfrak{z}(\mathfrak{g}^e) = \langle x_2 + x_5, x_6\rangle$. According to [18, Subsection 9.1.1], we have that $K^e(0)/(K^e(0))^\circ = \langle c_1, c_2\rangle \cong S_3$ where $c_1 = h_1(w)$ and $c_2 = n_{\alpha_1}h_2(-1)$. We check that

$$h_1(w)(x_2 + x_5) = w^{\langle\alpha_2,\alpha_1\rangle}x_2 + w^{\langle\alpha_5,\alpha_1\rangle}x_5 = w^{-3}x_2 + w^3x_5 = x_2 + x_5,$$

$$\text{and } h_1(w)(x_6) = w^0x_6 = x_6,$$

thus c_1 centralizes $x_2 + x_5$ and x_6. Similarly, we check that

$$n_{\alpha_1}h_2(-1)(x_2 + x_5) = n_{\alpha_1}(x_2 - x_5) = \eta_{\alpha_1,\alpha_2}x_5 - \eta_{\alpha_1,\alpha_5}x_2 = x_5 + x_2,$$

$$\text{and } n_{10}h_2(-1)(x_6) = -n_{10}x_6 = -x_6.$$

Thus c_2 centralizes $x_2 + x_5$ but x_6 is not fixed under the action of c_2. Hence, we know that $(\mathfrak{z}(\mathfrak{g}^e))^{c_2} = \langle x_2 + x_5 \rangle = \langle e \rangle$. Note that $\langle e \rangle \subseteq (\mathfrak{z}(\mathfrak{g}^e))^{G^e} \subseteq (\mathfrak{z}(\mathfrak{g}^e))^{c_2} \subseteq \mathfrak{z}(\mathfrak{g}^e)$. Therefore, we deduce that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = (\mathfrak{z}(\mathfrak{g}^e))^{c_2} = \langle e \rangle$.

For $e_{\mathfrak{sl}} = E$, we know that $G^e = (\{\pm 1\} \ltimes R^E) \times K^{e_{G_2}}$ where R^E is a connected normal subgroup of G^e. When $e = E + x_2$, $E + x_1$, $E + (x_2 + x_5)$, we have $\mathfrak{z} \subseteq \mathfrak{z}_{\bar{0}}$. We know that $\pm 1 \in \mathrm{SL}_2(\mathbb{C})^E$ act trivially on $\mathfrak{z}_{\bar{0}}$ and $(\mathfrak{z}(\mathfrak{g}^e))^{K^{e_{G_2}}} = \langle e \rangle$ by [18, Section 11]. Hence, we have that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \mathfrak{z}(\mathfrak{g}^e)$ for $e = E + x_2$, $E + x_1$ and $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \langle e \rangle$ for $e = E + (x_2 + x_5)$.

When $e = E + (x_1 + x_2)$, the component group of $\mathrm{SL}_2(\mathbb{C})^E$ has order 2 and we consider the element $g = -1 \in G^e/(G^e)^\circ$. We know that g acts trivially on $\mathfrak{z}_{\bar{0}}$, thus $e, x_6 \in (\mathfrak{z}(\mathfrak{g}^e))^g$. However, $v_1 \otimes e_3 \notin (\mathfrak{z}(\mathfrak{g}^e))^g$ since the action of g on $v_1 \otimes e_3$ sends it to $-v_1 \otimes e_3$. Hence, we have that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} \subseteq (\mathfrak{z}(\mathfrak{g}^e))^g = \langle e, x_6 \rangle$. Therefore, we have that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \langle e, x_6 \rangle$.

6.3.5 Analysis of results

The results obtained in Subsections 6.3.3–6.3.4 can be summarized in Table 6.22. Note that the summation $\sum a_i$ of the labels in the labelled Dynkin diagram does not depend on the choice of labelled Dynkin diagram.

nilpotent element e	$\dim \mathfrak{g}^e$	$\dim \mathfrak{z}(\mathfrak{g}^e)$	$\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e}$	$\sum_{i=1}^{l} a_i$
$E + (x_1 + x_2)$	5	3	2	4
$E + x_2$	16	1	1	1
$E + x_1$	13	1	1	1
$E + (x_2 + x_5)$	10	2	1	2
E	24	1	1	1
$x_1 + x_2$	7	2	2	4
x_2	21	1	1	1
x_1	15	1	1	1
$x_2 + x_5$	13	2	1	2
0	31	0	0	0

Table 6.22: $\dim \mathfrak{g}^e$ and $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e}$

Theorem 1.2 for $G(3)$ can be obtained immediately from Table 6.22, we have that $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = \left\lceil \frac{1}{2}\sum_{i=1}^{3} a_i \right\rceil$.

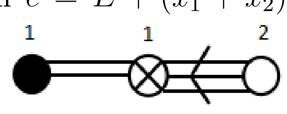
Note that there are two cases that satisfy the condition in Theorem 1.1 such that the corresponding labelled Dynkin diagram has no label equal to 1. They are $e = x_1 + x_2$ and $e = x_2 + x_5$. Let $\mathfrak{h} = \langle H, h_1, h_2 \rangle$ be a Cartan subalgebra of $\mathfrak{g}$.

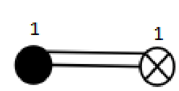
- When $e = x_1 + x_2 \in G(3)$, we have $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = 2$ and $n_2(\Delta) = 2$. Note that $\mathfrak{g}^h$ is generated by root vectors $e_{\pm 2\delta}$, $e_{\pm\delta}$ and $\mathfrak{h}$. Hence, we can find a simple root system $\Pi_h = \{\delta\}$ for $\mathfrak{g}^h$ and this subalgebra is $\mathfrak{osp}(1|2)$ according to [19, Subsection 3.4.1]. Then $\mathfrak{z}(\mathfrak{g}^h) = \{t \in \mathfrak{h} : \delta(t) = 0\}$. Hence, $\dim\mathfrak{z}(\mathfrak{g}^h) = 2 = n_2(\Delta) = \dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e}$.

- When $e = x_2 + x_5 \in G(3)$, we have $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = n_2(\Delta) = 1$. Note that $\mathfrak{g}^h$ is generated by root vectors $e_{\pm 2\delta}, e_{\pm\delta}, e_{\pm\varepsilon_1}, e_{\pm(\delta+\varepsilon_1)}, e_{\pm(\delta-\varepsilon_1)}$ and $\mathfrak{h}$, thus we can find a simple root system $\Pi_h = \{\varepsilon_1, \delta - \varepsilon_1\}$ and this subalgebra is $\mathfrak{osp}(3|2)$ according to [19, Subsection 3.4.1]. Then $\mathfrak{z}(\mathfrak{g}^h) = \{t \in \mathfrak{h} : \varepsilon_1(t) = (\delta - \varepsilon_1)(t) = 0\}$. Hence, $\dim\mathfrak{z}(\mathfrak{g}^h) = 1 = n_2(\Delta)$.

The above argument completes the proof of Theorem 1.1 for $G(3)$.

In order to prove Theorem 1.3 for $G(3)$, we look at three cases below and the remaining cases do not have labels equal to 2 so that the 2-free core Δ_0 of Δ is the same as Δ.

- When $e = E + (x_1 + x_2) \in G(3)$, we have $\dim\mathfrak{g}^e = 5$, $\dim\left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} = 2$ and $\Delta =$ 

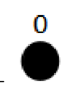
hence $\Delta_0 =$ 

. From [19, Subsection 3.4.1]

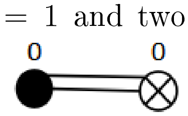
we see that the corresponding Lie superalgebra $\mathfrak{g}_0 = \mathfrak{osp}(3|2)$ and the construction of $\mathfrak{osp}(3|2)$ can also be found in [19, Section 2.3]. Let $\mathfrak{g}_0 = \mathfrak{osp}(3|2)$, we choose the nilpotent orbit $e_0 \in (\mathfrak{g}_0)_{\bar{0}}$ to be

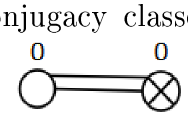
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

thus the Jordan type corresponds to e_0 is $(3|2)$. Hence, we obtain that $\dim \mathfrak{g}_0^{e_0} = 4$ according to Equation (5.40) and $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = n_2(\Delta) = 1$. Furthermore, we know that $G_0 = \mathrm{O}_3(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})$. By considering the element $g' = -1 \in G_0^{e_0}/(G_0^{e_0})^{\circ}$, we obtain that $(\mathfrak{z}(\mathfrak{g}_0^{e_0}))^{G_0^{e_0}} \subseteq (\mathfrak{z}(\mathfrak{g}_0^{e_0}))^{g'} = \langle e_0 \rangle$. Therefore, we further deduce that $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} - \dim (\mathfrak{z}(\mathfrak{g}_0^{e_0}))^{G_0^{e_0}} = n_2(\Delta) = 1$ for this case.

- When $e = x_1 + x_2 \in G(3)$, we have $\dim \mathfrak{g}^e = 7$, $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} = 2$ and $\Delta_0 =$ 

. The corresponding Lie superalgebra $\mathfrak{g}_0 = \mathfrak{osp}(1|2)$ according to [19, Subsection 3.4.1] and $e_0 = 0$. Thus we know that $\mathfrak{g}_0^{e_0} = \mathfrak{g}_0$ and $\mathfrak{z}(\mathfrak{g}_0^{e_0}) = \mathfrak{z}(\mathfrak{g}_0)$. Hence, we have $\dim \mathfrak{g}_0^{e_0} = \dim \mathfrak{g}_0 = 5$ and $\dim (\mathfrak{z}(\mathfrak{g}_0^{e_0}))^{G_0^{e_0}} = \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = \dim \mathfrak{z}(\mathfrak{g}_0) = 0$. Therefore, $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = n_2(\Delta) = 2$ and $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} - \dim (\mathfrak{z}(\mathfrak{g}_0^{e_0}))^{G_0^{e_0}} = n_2(\Delta) = 2$ for this case.

- When $e = x_2 + x_5 \in G(3)$, we have $\dim \mathfrak{g}^e = 13$, $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} = 1$ and two conjugacy classes of positive roots system provide us with $\Delta_0 =$ 

or 

. They are both the corresponding labelled Dynkin diagrams of $\mathfrak{osp}(3|2)$ with respect to nilpotent element 0 by [19, Subsection 3.4.1]. Thus $\mathfrak{g}_0 =$

$\mathfrak{osp}(3|2)$ and $e_0 = 0$. Similar to the above case, we have $\dim \mathfrak{g}_0^{e_0} = \dim \mathfrak{g}_0 = 12$ and $\dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = \dim \mathfrak{z}(\mathfrak{g}_0) = 0$. Therefore, $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = n_2(\varDelta) = 1$ and $\dim \left(\mathfrak{z}(\mathfrak{g}^e)\right)^{G^e} - \dim \left(\mathfrak{z}(\mathfrak{g}_0^{e_0})\right)^{G_0^{e_0}} = 1 = n_2(\varDelta)$ for this case.

The above argument completes the proof of Theorem 1.3 for $G(3)$.

6.4 The exceptional Lie superalgebra $F(4)$

6.4.1 Orthogonal Lie algebra $\mathfrak{so}(7)$

Recall that we have defined orthogonal Lie algebras in Section 2.4. Throughout this section, we use an alternative J such that $\mathfrak{so}(7) = \{x \in \mathfrak{gl}(7) : x^t J + Jx = 0\}$ and

$$J = \begin{pmatrix} & & & & & & 1 \\ & & & & & 1 & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & 1 & & & & & \\ 1 & & & & & & \end{pmatrix}$$

because J fits better with the spin representation and Clifford algebra that defined later. Thus any element $x \in \mathfrak{so}(7)$ is of the form

$$x = \begin{pmatrix} a & b & c & 2d & f & g & 0 \\ h & i & j & 2k & l & 0 & -g \\ m & n & p & 2q & 0 & -l & -f \\ -r & -s & -t & 0 & -q & -k & -d \\ u & v & 0 & 2t & -p & -j & -c \\ w & 0 & -v & 2s & -n & -i & -b \\ 0 & -w & -u & 2r & -m & -h & -a \end{pmatrix}, a, \dots, w \in \mathbb{C}.$$

According to [15, Section 1.6], we know that any nilpotent orbit in $\mathfrak{so}(7)$ has a Jordan type λ such that

$$\lambda \in \{(7), (5, 1^2), (3^2, 1), (3, 2^2), (3, 1^4), (2^2, 1^3), (1^7)\}.$$

By using the orthogonal Dynkin pyramid of λ as defined in [6, Section 6], we are able to give a representative of each nilpotent orbit in the matrix form. We list the corresponding representatives of each of the nilpotent orbits in Table 6.23. Note that the representatives of nilpotent orbits are modified based on the alternative choice of matrix J. For each nilpotent element $e_{\mathfrak{so}} \in \mathfrak{so}(7)$, we also give a semisimple element h such that there is an $\mathfrak{sl}(2)$-triple $\{e, h, f\}$ in $\mathfrak{so}(7)$.

Table 6.23: Matrix form of nilptent orbits in $\mathfrak{so}(7)$

λ	$e_{\mathfrak{so}} \in \mathfrak{so}(7)$	semisimple element h

(7)	$e_{(7)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$h_{(7)} = \text{diag}(6,4,2,0,-2,-4,-6)$
$(5,1^2)$	$e_{(5,1^2)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$h_{(5,1^2)} = \text{diag}(4,2,0,0,0,-2,-4)$
$(3^2,1)$	$e_{(3^2,1)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$h_{(3^2,1)} = \text{diag}(2,2,0,0,0,-2,-2)$
$(3,2^2)$	$e_{(3,2^2)} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$h_{(3,2^2)} = \text{diag}(2,1,1,0,-1,-1,-2)$
$(3,1^4)$	$e_{(3,1^4)} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$h_{(3,1^4)} = \text{diag}(2,0,0,0,0,0,-2)$

$(2^2,1^3)$	$e_{(2^2,1^3)}=\begin{pmatrix}0&0&0&0&0&1&0\\0&0&0&0&0&0&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}$	$h_{(2^2,1^3)}=\mathrm{diag}(1,1,0,0,0,-1,-1)$
(1^7)	$e_{(1^7)}=0$	$h_{(1^7)}=0$

For each nilpotent orbit $e_{\mathfrak{so}} \in \mathfrak{so}(7)$ and any element $x \in \mathfrak{so}(7)$, by calculating $[e_{\mathfrak{so}}, x] = 0$ we obtain centralizers $\mathfrak{so}(7)^{e_{\mathfrak{so}}}$ of each $e_{\mathfrak{so}}$ in $\mathfrak{so}(7)$ in the following table.

Table 6.24: Centralizers $\mathfrak{so}(7)^e$ of nilpotent orbits e in $\mathfrak{so}(7)$

$e_{\mathfrak{so}}$	$\mathfrak{so}(7)^{e_{\mathfrak{so}}}$	$\dim \mathfrak{so}(7)^{e_{\mathfrak{so}}}$
$e_{(7)}$	$\begin{pmatrix}0&b&0&2d&0&g&0\\0&0&b&0&-2d&0&-g\\0&0&0&2b&0&2d&0\\0&0&0&0&-b&0&-d\\0&0&0&0&0&-b&0\\0&0&0&0&0&0&-b\\0&0&0&0&0&0&0\end{pmatrix}$	3
$e_{(5,1^2)}$	$\begin{pmatrix}0&b&c&0&f&g&0\\0&0&0&2b&0&0&-g\\0&0&p&0&0&0&-f\\0&0&0&0&0&-b&0\\0&0&0&0&-p&0&-c\\0&0&0&0&0&0&-b\\0&0&0&0&0&0&0\end{pmatrix}$	5

$e_{(3^2,1)}$	$\begin{pmatrix} a & 0 & c & 2d & f & g & 0 \\ 0 & -a & j & 2k & c & 0 & -g \\ 0 & 0 & a & 0 & 0 & -c & -f \\ 0 & 0 & 0 & 0 & 0 & -k & -d \\ 0 & 0 & 0 & 0 & -a & -j & -c \\ 0 & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a \end{pmatrix}$	7
$e_{(3,2^2)}$	$\begin{pmatrix} 0 & -2q & 2k & 2d & f & g & 0 \\ 0 & i & j & 2k & l & 0 & -g \\ 0 & n & -i & 2q & 0 & -l & -f \\ 0 & 0 & 0 & 0 & -q & -k & -d \\ 0 & 0 & 0 & 0 & i & -j & -2k \\ 0 & 0 & 0 & 0 & -n & -i & 2q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	9
$e_{(3,1^4)}$	$\begin{pmatrix} 0 & b & c & 2d & f & g & 0 \\ 0 & i & j & 0 & l & 0 & -g \\ 0 & n & p & 0 & 0 & -l & -f \\ 0 & 0 & 0 & 0 & 0 & 0 & -d \\ 0 & v & 0 & 0 & -p & -j & -c \\ 0 & 0 & -v & 0 & -n & -i & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$	11
$e_{(2^2,1^3)}$	$\begin{pmatrix} a & b & c & 2d & f & g & 0 \\ h & -a & j & 2k & l & 0 & -g \\ 0 & 0 & p & 2q & 0 & -l & -f \\ 0 & 0 & -t & 0 & -q & -k & -d \\ 0 & 0 & 0 & 2t & -p & -j & -c \\ 0 & 0 & 0 & 0 & 0 & a & -b \\ 0 & 0 & 0 & 0 & 0 & -h & -a \end{pmatrix}.$	13
$e_{(1^7)}$	$\mathfrak{so}(7)$	21

6.4.2 Clifford algebra and $\mathfrak{so}(V, \beta)$ embedded into $\mathrm{C}(V, \beta)$

Let V be a 7-dimensional complex vector space with a symmetric bilinear form β. Let $\{e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3\}$ be a basis for V. Goodman and Wallach give an explicit definition of the *Clifford algebra* $\mathrm{C}(V, \beta)$ for (V, β) in [10, Chapter 6] which we recall below. The Clifford algebra $\mathrm{C}(V,\beta)$ has a basis

$$\{e_1^{\delta_1}e_2^{\delta_2}e_3^{\delta_3}e_0^{\delta_0}e_{-3}^{\delta_{-3}}e_{-2}^{\delta_{-2}}e_{-1}^{\delta_{-1}} : \delta_i = 0 \text{ or } 1\}$$

and for any $x, y \in V$, we have that $\{x, y\} = \beta(x, y)1$ for $x, y \in \mathrm{C}(V,\beta)$ where $\{x, y\} = xy + yx$ is the anticommutator of x, y. Hence the multiplication in $\mathrm{C}(V,\beta)$ satisfies

$$e_0^2 = 1, e_i^2 = 0 \text{ for } i \neq 0,$$

$$e_ie_j = -e_je_i \text{ for } i \neq -j \text{ and } e_ie_{-i} = -e_{-i}e_i + 1.$$

Moreover, there exists a decomposition $\mathrm{C}(V,\beta) = \mathrm{C}^+(V,\beta) \oplus \mathrm{C}^-(V,\beta)$ such that products of an even (resp. odd) number of elements of V form a basis for $\mathrm{C}^+(V,\beta)$ (resp. $\mathrm{C}^-(V,\beta)$).

For $u, w \in V$, we define $R_{u,w} \in \mathrm{End}(V)$ by

$$R_{u,w}(v) = \beta(w, v)u - \beta(u, v)w.$$

For any $x, y \in V$, we can check that

$$\beta(R_{u,w}(x), y) = \beta(w, x)\beta(u, y) - \beta(u, x)\beta(w, y) = -\beta(x, R_{u,w}(y)).$$

Hence, we know that $R_{u,w} \in \mathfrak{so}(V,\beta)$. It has been proved in [10, Lemma 6.2.1] that the linear transformations $R_{u,w}$ for $u, w \in V$ form a basis for $\mathfrak{so}(V,\beta)$. More precisely, there exists a decomposition $V = W \oplus \langle e_0 \rangle \oplus W^*$ where W, W^* is a pair of dual maximal isotropic subspaces of V corresponding to β and $\{e_1, e_2, e_3\}$ (resp. $\{e_{-1}, e_{-2}, e_{-3}\}$) is a basis for W (resp. W^*). The basis is chosen such that $\beta(e_i, e_{-j}) = \delta_{ij}$ for $i, j \in \{1, 2, 3\}$,

$\beta(e_0, e_0) = 2$ and $\beta(e_0, W) = \beta(e_0, W^*) = 0$. Then

$$R_{e_i,e_{-j}} = e_{i,j} - e_{-j,-i}, R_{e_i,e_j} = e_{i,-j} - e_{j,-i} \text{ for } i < j, R_{e_{-i},e_{-j}} = e_{-i,j} - e_{-j,i} \text{ for } i > j,$$

$$\text{and } R_{e_i,e_0} = 2e_{i,0} - e_{0,-i},\ R_{e_{-i},e_0} = 2e_{-i,0} - e_{0,i}$$

form a basis for $\mathfrak{so}(V, \beta)$ where $e_{i,j}$ is the elementary transformation which sends e_i to e_j and all other basis vectors to 0.

According to [10, Lemma 6.2.2], there exists an injective Lie algebra homomorphism $\varphi : \mathfrak{so}(7) \to C(V, \beta)$ such that

$$\varphi(R_{e_i,e_{-j}}) = e_i e_{-j} \text{ for } i \neq j \text{ and } \varphi(R_{e_i,e_{-i}}) = e_i e_{-i} - \frac{1}{2} \text{ for } i \neq 0.$$

6.4.3 A spin representation of Lie algebra $\mathfrak{so}(7)$

Let us consider the subalgebra $D_{0,-}$ of C(V, β) defined by

$$D_{0,-} = \langle e_0^{\delta_0} e_{-3}^{\delta_{-3}} e_{-2}^{\delta_{-2}} e_{-1}^{\delta_{-1}} : \delta_i = 0 \text{ or } 1\rangle.$$

Define $S = C(V, \beta) \otimes_{D_{0,-}} \langle s\rangle$ where $\langle s\rangle$ is the 1-dimensional $D_{0,-}$-module such that $e_{-i}s = 0$ for $i = 1, 2, 3$ and $e_0 s = s$. In fact, S is an 8-dimensional representation for $C(V, \beta)$ with basis

$$\{1 \otimes s, e_1 \otimes s, e_2 \otimes s, e_3 \otimes s, e_1e_2 \otimes s, e_1e_3 \otimes s, e_2e_3 \otimes s, e_1e_2e_3 \otimes s\}.$$

We can restrict S to be a representation of $\mathfrak{so}(7) = \mathfrak{so}(V, \beta) \subseteq C(V, \beta)$ and we call it a *spin representation* for $\mathfrak{so}(7)$ based on [10, Section 6.2.2]. In the remaining sections, we write the basis for S as

$$\{s, e_1s, e_2s, e_3s, e_1e_2s, e_1e_3s, e_2e_3s, e_1e_2e_3s\}. \tag{6.3}$$

We sometimes denote basis elements $s, -e_1s, e_2s, -e_3s, e_1e_2s, e_1e_3s, e_2e_3s, e_1e_2e_3s$ of V_8 by

$$v_{---}, v_{+--}, v_{-+-}, v_{--+}, v_{++-}, v_{+-+}, v_{-++}, v_{+++}$$

respectively. Note that the signs has been modified in order to make sure that β is invariant under the action of $\mathfrak{so}(7)$.

6.4.4 Spin groups

For the Clifford algebra $\mathrm{C}(V, \beta)$, there is an anti-automorphism τ such that

$$\tau(e_{i_1} \cdots e_{i_k}) = e_{i_k} \cdots e_{i_1}$$

for $e_{i_1}, \dots, e_{i_k} \in \mathrm{C}(V, \beta)$. For $e_{i_1}, \dots, e_{i_k} \in \mathrm{C}(V, \beta)$, we define $\alpha : \mathrm{C}(V, \beta) \to \mathrm{C}(V, \beta)$ to be the automomorphism such that $\alpha(e_{i_1}, \dots, e_{i_k}) = (-1)^k e_{i_1}, \dots, e_{i_k}$. Define $u^* = \tau(\alpha(u))$ be the *conjugation* on $\mathrm{C}(V, \beta)$. Thus for $e_{i_1}, \dots, e_{i_k} \in \mathrm{C}(V, \beta)$, we have that $(e_{i_1}, \dots, e_{i_k})^* = (-1)^k e_{i_k} \cdots e_{i_1}$.

Based on the above notation, we define the Pin group and Spin group as below:

Definition 6.2. The *Pin group is* a subgroup of the group of invertible elements of $\mathrm{C}(V, \beta)$ such that

$$\mathrm{Pin}(V, \beta) = \{u \in \mathrm{C}(V, \beta) : u \cdot u^* = 1 \text{ and } \alpha(u) e_i u^* = e_i\}.$$

The *Spin group* is defined to be $\mathrm{Spin}(V,\beta) = \mathrm{Pin}(V,\beta) \cap \mathrm{C}^+(V,\beta)$ for $\dim V \geq 3$.

In fact, the Spin group is the non-trivial double cover of the special orthogonal group $\mathrm{SO}_7(\mathbb{C})$. According to [10, Theorem 6.3.5], we have the following theorem:

Theorem 6.3. There exists a surjective homomorphism $\pi : \mathrm{Spin}(V,\beta) \to \mathrm{SO}(V,\beta)$ such that $\ker \pi = \{\pm 1\}$.

6.4.5 Structure of the Lie superalgebra $F(4)$

In order to fully describe the structure of $F(4)$, we first let V_2 be a two-dimensional vector space with basis $\{v_1, v_{-1}\}$ such that $v_1 = (1,0)^t$ and $v_{-1} = (0,1)^t$, so V_2 is the natural $\mathfrak{sl}(2)$-module. We define $\psi_2 : V_2 \times V_2 \to \mathbb{C}$ to be a non-degenerate skew-symmetric bilinear form such that $\psi_2(v_1, v_{-1}) = 1$. We also define $p_2 : V_2 \times V_2 \to \mathfrak{sl}(2)$ by

$$p_2(x,y)(z) = 3(\psi_2(y,z)x - \psi_2(z,x)y)$$

for $x, y, z \in V_2$. We compute that $p_2(v_1, v_{-1}) = -3H, p_2(v_1, v_1) = 6E$ and $p_2(v_{-1}, v_{-1}) = -6F$.

Next let V_8 be the spin representation of $\mathfrak{so}(7)$ with a basis given in (6.3). Let $\psi_8 : V_8 \times V_8 \to \mathbb{C}$ be the non-degenerate symmetric bilinear form given by

$$\psi_8(v_{\sigma_1,\sigma_2,\sigma_3}, v_{\sigma_1',\sigma_2',\sigma_3'}) = \prod_{i=1}^{3} \delta_{\sigma_i, -\sigma_i'}$$

for $\sigma_i, \sigma_i' \in \{+,-\}$, e.g. we have that $\psi_8(v_{+++}, v_{+--}) = 0$, $\psi_8(v_{+-+}, v_{-+-}) = 1$, etc. Define $p_8 : V_8 \times V_8 \to \mathfrak{so}(7)$ to be the $\mathfrak{so}(7)$-invariant antisymmetric bilinear map such that the explicit mapping is given in Tables 6.25–6.28. Note that values of $p_8(\cdot\,,\cdot)$ are calculated based on the assumption that

$$(v_{+++}, v_{++-}) \longmapsto R_{e_1,e_2}. \tag{6.4}$$

and the fact p_8 is invariant under the action of $\mathfrak{so}(7)$. For example, by applying $R_{e_3,e_{-2}}$ to both sides of (6.4) we get

$$(R_{e_3,e_{-2}} v_{+++}, v_{++-}) + (v_{+++}, R_{e_3,e_{-2}} v_{++-}) \longmapsto [R_{e_3,e_{-2}}, R_{e_1,e_2}],$$

this implies that $(v_{+++}, v_{+-+}) \longmapsto R_{e_1,e_3}$.

	v_{---}	v_{+--}
v_{---}	0	$-R_{e_{-3},e_{-2}}$
v_{+--}	$R_{e_{-3},e_{-2}}$	0
v_{-+-}	$R_{e_{-3},e_{-1}}$	$\frac{1}{2}R_{e_{-3},e_0}$
v_{--+}	$R_{e_{-2},e_{-1}}$	$\frac{1}{2}R_{e_{-2},e_0}$
v_{++-}	$\frac{1}{2}R_{e_{-3},e_0}$	$R_{e_1,e_{-3}}$
v_{+-+}	$-\frac{1}{2}R_{e_{-2},e_0}$	$-R_{e_1,e_{-2}}$
v_{-++}	$\frac{1}{2}R_{e_{-1},e_0}$	$\frac{1}{2}(R_{e_1,e_{-1}} - R_{e_2,e_{-2}} - R_{e_3,e_{-3}})$
v_{+++}	$-\frac{1}{2}(R_{e_1,e_{-1}} + R_{e_2,e_{-2}} + R_{e_3,e_{-3}})$	$-\frac{1}{2}R_{e_1,e_0}$

Table 6.25: $p_8 : V_8 \times V_8 \to \mathfrak{so}(7)$

	v_{-+-}	v_{--+}
v_{---}	$-R_{e_{-3},e_{-1}}$	$-R_{e_{-2},e_{-1}}$
v_{+--}	$-\frac{1}{2}R_{e_{-3},e_0}$	$-\frac{1}{2}R_{e_{-2},e_0}$
v_{-+-}	0	$-\frac{1}{2}R_{e_{-1},e_0}$
v_{--+}	$\frac{1}{2}R_{e_{-1},e_0}$	0
v_{++-}	$-R_{e_2,e_{-3}}$	$\frac{1}{2}(-R_{e_1,e_{-1}} - R_{e_2,e_{-2}} + R_{e_3,e_{-3}})$
v_{+-+}	$\frac{1}{2}(-R_{e_1,e_{-1}} + R_{e_2,e_{-2}} - R_{e_3,e_{-3}})$	$-R_{e_3,e_{-2}}$
v_{-++}	$-R_{e_2,e_{-1}}$	$R_{e_3,e_{-1}}$
v_{+++}	$\frac{1}{2}R_{e_2,e_0}$	$-\frac{1}{2}R_{e_3,e_0}$

Table 6.26: $p_8 : V_8 \times V_8 \to \mathfrak{so}(7)$ (continued)

	v_{++-}	v_{+-+}
v_{---}	$-\frac{1}{2}R_{e_{-3},e_0}$	$\frac{1}{2}R_{e_{-2},e_0}$
v_{+--}	$-R_{e_1,e_{-3}}$	$R_{e_1,e_{-2}}$
v_{-+-}	$R_{e_2,e_{-3}}$	$\frac{1}{2}(R_{e_1,e_{-1}} - R_{e_2,e_{-2}} + R_{e_3,e_{-3}})$
v_{--+}	$\frac{1}{2}(R_{e_1,e_{-1}} + R_{e_2,e_{-2}} - R_{e_3,e_{-3}})$	$R_{e_3,e_{-2}}$
v_{++-}	0	$\frac{1}{2}R_{e_1,e_0}$
v_{+-+}	$-\frac{1}{2}R_{e_1,e_0}$	0
v_{-++}	$-\frac{1}{2}R_{e_2,e_0}$	$-\frac{1}{2}R_{e_3,e_0}$
v_{+++}	R_{e_1,e_2}	R_{e_1,e_3}

Table 6.27: $p_8 : V_8 \times V_8 \to \mathfrak{so}(7)$ (continued)

	v_{-++}	v_{+++}
v_{---}	$-\frac{1}{2}R_{e_{-1},e_0}$	$\frac{1}{2}(R_{e_1,e_{-1}} + R_{e_2,e_{-2}} + R_{e_3,e_{-3}})$
v_{+--}	$\frac{1}{2}(-R_{e_1,e_{-1}} + R_{e_2,e_{-2}} + R_{e_3,e_{-3}})$	$\frac{1}{2}R_{e_1,e_0}$
v_{-+-}	$R_{e_2,e_{-1}}$	$-\frac{1}{2}R_{e_2,e_0}$
v_{--+}	$-R_{e_3,e_{-1}}$	$\frac{1}{2}R_{e_3,e_0}$
v_{++-}	$\frac{1}{2}R_{e_2,e_0}$	$-R_{e_1,e_2}$
v_{+-+}	$\frac{1}{2}R_{e_3,e_0}$	$-R_{e_1,e_3}$
v_{-++}	0	$-R_{e_2,e_3}$
v_{+++}	R_{e_2,e_3}	0

Table 6.28: $p_8 : V_8 \times V_8 \to \mathfrak{so}(7)$ (continued)

For example, we can read from these tables that $p_8(v_{+++}, v_{++-}) = R_{e_1,e_2}$, $p_8(v_{+++}, v_{---}) = -\frac{1}{2}R_{e_1,e_{-1}} - \frac{1}{2}R_{e_2,e_{-2}} - \frac{1}{2}R_{e_3,e_{-3}}$, etc.

Together with the above definitions and notation, we are now able to describe the structure of $F(4)$. By definition, the Lie superalgebra $F(4) = \mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ where

$$\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathfrak{so}(7)$$

and

$$\mathfrak{g}_{\bar{1}} = V_2 \otimes V_8.$$

We know that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and the bracket $[\cdot\,,\cdot] : \mathfrak{g}_{\bar{0}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{1}}$ is given by

$$[x + y, v_2 \otimes v_8] = xv_2 \otimes v_8 + v_2 \otimes yv_8$$

for $x \in \mathfrak{sl}(2), y \in \mathfrak{so}(7)$, $v_2 \in V_2$ and $v_8 \in V_8$. Note that $Ev_1 = 0$, $Ev_{-1} = v_1$ and $Hv_i = iv_i$ for $i \in \{\pm 1\}$. The following table gives the action of $\mathfrak{so}(7)$ on each basis element of V_8.

Table 6.29: The action of $\mathfrak{so}(7)$ on V_8

	s	e_1s	e_2s	e_3s	e_1e_2s	e_1e_3s	e_2e_3s	$e_1e_2e_3s$
$R_{e_1,e_{-1}}$	$-\frac{1}{2}s$	$\frac{1}{2}e_1s$	$-\frac{1}{2}e_2s$	$-\frac{1}{2}e_3s$	$\frac{1}{2}e_1e_2s$	$\frac{1}{2}e_1e_3s$	$-\frac{1}{2}e_2e_3s$	$\frac{1}{2}e_1e_2e_3s$
$R_{e_1,e_{-2}}$	0	0	e_1s	0	0	0	e_1e_3s	0
$R_{e_1,e_{-3}}$	0	0	0	e_1s	0	0	$-e_1e_2s$	0
R_{e_1,e_0}	e_1s	0	$-e_1e_2s$	$-e_1e_3s$	0	0	$e_1e_2e_3s$	0
R_{e_1,e_3}	e_1e_3s	0	$-e_1e_2e_3s$	0	0	0	0	0
R_{e_1,e_2}	e_1e_2s	0	0	$e_1e_2e_3s$	0	0	0	0
$R_{e_2,e_{-1}}$	0	e_2s	0	0	0	e_2e_3s	0	0
$R_{e_2,e_{-2}}$	$-\frac{1}{2}s$	$-\frac{1}{2}e_1s$	$\frac{1}{2}e_2s$	$-\frac{1}{2}e_3s$	$\frac{1}{2}e_1e_2s$	$-\frac{1}{2}e_1e_3s$	$\frac{1}{2}e_2e_3s$	$\frac{1}{2}e_1e_2e_3s$
$R_{e_2,e_{-3}}$	0	0	0	e_2s	0	e_1e_2s	0	0
R_{e_2,e_0}	e_2s	e_1e_2s	0	$-e_2e_3s$	0	$-e_1e_2e_3s$	0	0
R_{e_2,e_3}	e_2e_3s	$e_1e_2e_3s$	0	0	0	0	0	0
$R_{e_3,e_{-1}}$	0	e_3s	0	0	$-e_2e_3s$	0	0	0
$R_{e_3,e_{-2}}$	0	0	e_3s	0	e_1e_3s	0	0	0
$R_{e_3,e_{-3}}$	$-\frac{1}{2}s$	$-\frac{1}{2}e_1s$	$-\frac{1}{2}e_2s$	$\frac{1}{2}e_3s$	$-\frac{1}{2}e_1e_2s$	$\frac{1}{2}e_1e_3s$	$\frac{1}{2}e_2e_3s$	$\frac{1}{2}e_1e_2e_3s$
R_{e_3,e_0}	e_3s	e_1e_3s	e_2e_3s	0	$e_1e_2e_3s$	0	0	0

R_{e_{-1},e_0}	0	$-s$	0	0	e_2s	e_3s	0	$-e_2e_3s$
R_{e_{-2},e_0}	0	0	$-s$	0	$-e_1s$	0	e_3s	e_1e_3s
R_{e_{-3},e_0}	0	0	0	$-s$	0	$-e_1s$	$-e_2s$	$-e_1e_2s$
$R_{e_{-3},e_{-1}}$	0	0	0	0	0	s	0	$-e_2s$
$R_{e_{-3},e_{-2}}$	0	0	0	0	0	0	s	e_1s
$R_{e_{-2},e_{-1}}$	0	0	0	0	s	0	0	e_3s

For $x_2, y_2 \in V_2, x_8, y_8 \in V_8$, we define

$$[x_2 \otimes x_8, y_2 \otimes y_8] = \psi_2(x_2, y_2)p_8(x_8, y_8) + \psi_8(x_8, y_8)p_2(x_2, y_2).$$

6.4.6 Root system and Dynkin diagrams of $F(4)$

In this subsection, we use the structure of the root system of $F(4)$ given in [14, Appendix A]. Note that roots of $F(4)$ are given by $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ where

$$\Phi_{\bar{0}} = \{\pm\delta, \pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i : i, j = 1, 2, 3\} \text{ and } \Phi_{\bar{1}} = \{\frac{1}{2}(\pm\delta \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)\}$$

such that $\{\delta, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is an orthogonal basis and

$$(\delta, \delta) = -6, (\varepsilon_i, \varepsilon_j) = \begin{cases} 2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

We list all roots and the corresponding root vectors in the table below.

Roots	Root vectors
δ	E
$-\delta$	F
$\varepsilon_i + \varepsilon_j (i < j)$	R_{e_i, e_j}
$-\varepsilon_i - \varepsilon_j (i < j)$	$R_{e_{-j}, e_{-i}}$
$\varepsilon_i - \varepsilon_j$	$R_{e_i, e_{-j}}$
$\pm\varepsilon_i$	$R_{e_{\pm i}, e_0}$
$\frac{1}{2}(i\delta + \sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$	$v_i \otimes v_{\sigma_1,\sigma_2,\sigma_3}, i, \sigma_i \in \{\pm\}$

Table 6.30: Roots and root vectors for $F(4)$

The remaining part of this subsection covers all possible Dynkin diagrams with respect to different systems of simple roots based on [9, Section 2.18]. Note that the number of lines between a pair of vertices that corresponds to simple roots are calculated using Formula (4.1) in Section 4.2. Moreover, for an odd root of the form $\frac{1}{2}(i\delta + \sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$, we have that

$$\begin{aligned}
&\left(\frac{1}{2}(i\delta + \sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3), \frac{1}{2}(i\delta + \sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)\right) \\
&= \frac{1}{4}\left(i^2(\delta,\delta) + \sigma_1^2(\varepsilon_1,\varepsilon_1) + \sigma_2^2(\varepsilon_2,\varepsilon_2) + \sigma_2^2(\varepsilon_3,\varepsilon_3)\right) \\
&= \frac{1}{4}(-6 + 2 + 2 + 2) = 0.
\end{aligned}$$

Hence, all odd roots in the root system of the Lie superalgebra $F(4)$ are isotropic.

1. For the simple system $\Pi = \{\alpha_1 = \frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \alpha_2 = \varepsilon_3, \alpha_3 = \varepsilon_2 - \varepsilon_3, \alpha_4 = \varepsilon_1 - \varepsilon_2\}$, we have the following Dynkin diagram:

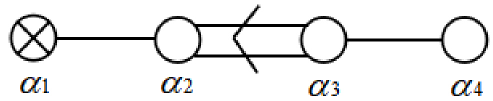


Figure 26: Dynkin diagram for $F(4)$

2. For the simple system $\Pi = \{\alpha_1 = \frac{1}{2}(-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3), \alpha_2 = \frac{1}{2}(\delta-\varepsilon_1-\varepsilon_2+\varepsilon_3), \alpha_3 = \varepsilon_2 - \varepsilon_3, \alpha_4 = \varepsilon_1 - \varepsilon_2\}$, we have the following Dynkin diagram:

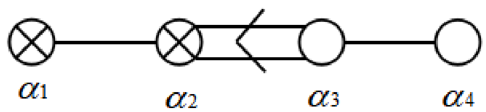


Figure 27: Dynkin diagram for $F(4)$

3. For the simple system $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 - \varepsilon_3), \alpha_3 = \frac{1}{2}(-\delta + \varepsilon_1 + \varepsilon_2 - \varepsilon_3), \alpha_4 = \varepsilon_3\}$, we have the following Dynkin diagram:

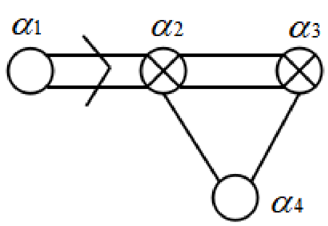


Figure 28: Dynkin diagram for $F(4)$

4. For the simple system $\Pi = \{\alpha_1 = \frac{1}{2}(\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3), \alpha_2 = \frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3), \alpha_3 = \frac{1}{2}(-\delta + \varepsilon_1 - \varepsilon_2 + \varepsilon_3), \alpha_4 = \varepsilon_2 - \varepsilon_3\}$, we have the following Dynkin diagram:

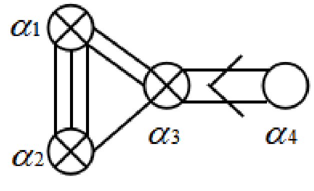


Figure 29: Dynkin diagram for $F(4)$

5. For the simple system $\Pi = \{\alpha_1 = \delta, \alpha_2 = \frac{1}{2}(-\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \alpha_3 = \varepsilon_3, \alpha_4 = \varepsilon_2 - \varepsilon_3\}$, we have the following Dynkin diagram:

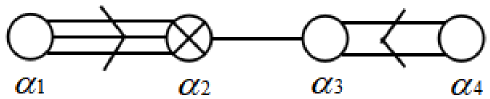


Figure 30: Dynkin diagram for $F(4)$

6. For the simple system $\Pi = \{\alpha_1 = \delta, \alpha_2 = \frac{1}{2}(-\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \alpha_3 = \varepsilon_1 - \varepsilon_2, \alpha_4 = \varepsilon_2 - \varepsilon_3\}$, we have the following Dynkin diagram:

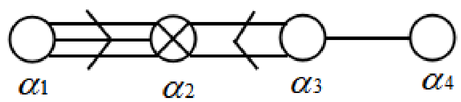


Figure 31: Dynkin diagram for $F(4)$

6.4.7 Centres of centralizers of nilpotent elements $e \in \mathfrak{g}_{\bar{0}}$ and labelled Dynkin diagrams

Let $e = e_{\mathfrak{sl}} + e_{\mathfrak{so}} \in \mathfrak{g}_{\bar{0}}$ be nilpotent where $e_{\mathfrak{sl}} \in \mathfrak{sl}(2)$ and $e_{\mathfrak{so}} \in \mathfrak{so}(7)$. We know that there are two representatives of nilpotent orbits in $\mathfrak{sl}(2)$, i.e. $\{0, E\}$. Based on Table 6.23, there are 7 representatives of nilpotent orbits in $\mathfrak{so}(7)$. Hence, there are in total 14 possibilities for e. We give basis elements for $\mathfrak{g}^e$ in Table 6.31. In Table 6.32, we give a basis for $\mathfrak{z}(\mathfrak{g}^e)$ and list the labelled Dynkin diagrams Δ with respect to e . Note that $\mathfrak{sl}(2)^{e_{\mathfrak{sl}}} = \langle E \rangle$ for $e_{\mathfrak{sl}} = E$, $\mathfrak{sl}(2)^{e_{\mathfrak{sl}}} = \mathfrak{sl}(2)$ for $e_{\mathfrak{sl}} = 0$. The numbers in the column labelled "Δ" represent labels a_i corresponding to α_i for $i = 1, 2, 3, 4$ in Δ. We consider the case $e = E + e_{(7)}$, E and $e_{(7)}$ as examples to show explicit calculation on $\mathfrak{g}^e$, $\mathfrak{z}(\mathfrak{g}^e)$ and labelled Dynkin diagrams with respect to e after these tables.

Table 6.31: $\mathfrak{g}^e$

e	$\mathfrak{g}^e$
$E+e_{(7)}$	$\langle E, e_{(7)}, R_{e_1,e_2}, R_{e_1,e_0}-2R_{e_2,e_3}, v_1\otimes e_1e_2e_3s, v_1\otimes e_1e_2s-v_{-1}\otimes e_1e_2e_3s, v_1\otimes e_1s-v_1\otimes e_2e_3s\rangle$
$E+e_{(5,1^2)}$	$\langle E, e_{(5,1^2)}, R_{e_3,e_{-3}}, R_{e_1,e_{-3}}, R_{e_1,e_3}, R_{e_1,e_2}, v_1\otimes e_1e_2e_3s, v_1\otimes e_1e_2s, v_1\otimes e_1s-v_{-1}\otimes e_1e_2s, v_1\otimes e_1e_3s+v_{-1}\otimes e_1e_2e_3s\rangle$
$E+e_{(3^2,1)}$	$\langle E, e_{(3^2,1)}, R_{e_1,e_{-1}}-R_{e_2,e_{-2}}+R_{e_3,e_{-3}}, R_{e_2,e_{-3}}, R_{e_2,e_0}, R_{e_1,e_0}, R_{e_1,e_3}, R_{e_1,e_2}, v_1\otimes e_1e_2s, v_1\otimes e_1e_2e_3s, v_1\otimes e_1s-v_{-1}\otimes e_1e_2e_3s, v_1\otimes e_2s, v_{-1}\otimes e_1e_2s+v_1\otimes e_2e_3s, v_1\otimes e_1e_3s\rangle$
$E+e_{(3,2^2)}$	$\langle E, R_{e_1,e_0}, R_{e_2,e_3}, R_{e_2,e_{-2}}-R_{e_3,e_{-3}}, R_{e_2,e_{-3}}, R_{e_3,e_{-2}}, 2R_{e_1,e_{-3}}+R_{e_2,e_0}, -2R_{e_1,e_{-2}}+R_{e_3,e_0}, R_{e_1,e_3}, R_{e_1,e_2}, v_1\otimes e_1e_2e_3s, v_1\otimes e_1e_2s, v_1\otimes e_1e_3s, v_1\otimes e_1s-v_{-1}\otimes e_1e_2e_3s, v_1\otimes e_2e_3s-v_{-1}\otimes e_1e_2e_3s, v_1\otimes e_3s+v_{-1}\otimes e_1e_3s, v_1\otimes e_2s+v_{-1}\otimes e_1e_2s\rangle$
$E+e_{(3,1^4)}$	$\langle E, e_{(3,1^4)}, R_{e_1,e_2}, R_{e_2,e_3}, R_{e_2,e_{-3}}, R_{e_2,e_{-2}}, R_{e_1,e_3}, R_{e_3,e_{-3}}, R_{e_{-3},e_{-2}}, R_{e_1,e_{-3}}, v_1\otimes s-v_{-1}\otimes e_1s, v_1\otimes e_2s+v_{-1}\otimes e_1e_2s, v_1\otimes e_3s+v_{-1}\otimes e_1e_3s, v_1\otimes e_1s, v_1\otimes e_2e_3s-v_{-1}\otimes e_1e_2e_3s, v_1\otimes e_1e_2s, v_1\otimes e_1e_3s, v_1\otimes e_1e_2e_3s, R_{e_1,e_{-2}}, R_{e_3,e_{-2}}\rangle$
$E+e_{(2^2,1^3)}$	$\langle E, e_{(2^2,1^3)}, R_{e_1,e_{-2}}, R_{e_1,e_{-1}}-R_{e_2,e_{-2}}, R_{e_3,e_{-3}}, R_{e_2,e_{-1}}, R_{e_{-3},e_0}, R_{e_1,e_3}, R_{e_1,e_0}, R_{e_2,e_{-3}}, R_{e_3,e_0}, R_{e_2,e_3}, R_{e_1,e_{-3}}, R_{e_2,e_0}, v_1\otimes e_1e_2e_3s, v_1\otimes s-v_{-1}\otimes e_1e_2s, v_1\otimes e_3s-v_{-1}\otimes e_1e_2e_3s, v_1\otimes e_1s, v_1\otimes e_2s, v_1\otimes e_1e_3s, v_1\otimes e_2e_3s, v_1\otimes e_1e_2s\rangle$
E	$\langle E\rangle\oplus\mathfrak{so}(7)\oplus\langle v_1\otimes s, v_1\otimes e_1s, v_1\otimes e_2s, v_1\otimes e_3s, v_1\otimes e_1e_2s, v_1\otimes e_1e_3s, v_1\otimes e_2e_3s, v_1\otimes e_1e_2e_3s\rangle$
$e_{(7)}$	$\mathfrak{sl}(2)\oplus\langle e_{(7)}, R_{e_1,e_2}, R_{e_1,e_0}-2R_{e_2,e_3}, v_1\otimes e_1s-v_1\otimes e_2e_3s, v_{-1}\otimes e_1s-v_{-1}\otimes e_2e_3s, v_1\otimes e_1e_2e_3s, v_{-1}\otimes e_1e_2e_3s\rangle$
$e_{(5,1^2)}$	$\mathfrak{sl}(2)\oplus\langle e_{(5,1^2)}, R_{e_1,e_3}, R_{e_1,e_{-3}}, R_{e_1,e_2}, R_{e_3,e_{-3}}, v_1\otimes e_1e_2s, v_{-1}\otimes e_1e_2s, v_1\otimes e_1e_2e_3s, v_{-1}\otimes e_1e_2e_3s\rangle$
$e_{(3^2,1)}$	$\mathfrak{sl}(2)\oplus\langle e_{(3^2,1)}, R_{e_1,e_{-1}}-R_{e_2,e_{-2}}+R_{e_3,e_{-3}}, R_{e_1,e_3}, R_{e_1,e_0}, R_{e_2,e_0}, R_{e_2,e_{-3}}, R_{e_1,e_2}, v_1\otimes e_2s, v_{-1}\otimes e_2s, v_1\otimes e_1e_3s, v_{-1}\otimes e_1e_3s, v_1\otimes e_1e_2s, v_{-1}\otimes e_1e_2s, v_1\otimes e_1e_2e_3s, v_{-1}\otimes e_1e_2e_3s\rangle$
$e_{(3,2^2)}$	$\mathfrak{sl}(2)\oplus\langle R_{e_1,e_0}, R_{e_2,e_3}, R_{e_2,e_{-2}}-R_{e_3,e_{-3}}, R_{e_2,e_{-3}}, 2R_{e_1,e_{-3}}+R_{e_2,e_0}, R_{e_3,e_{-2}}, -2R_{e_1,e_{-2}}+R_{e_3,e_0}, R_{e_1,e_2}, R_{e_1,e_3}, v_1\otimes e_1s-v_1\otimes e_2e_3s, v_{-1}\otimes e_1s-v_{-1}\otimes e_2e_3s, v_1\otimes e_1e_2s, v_{-1}\otimes e_1e_2s, v_1\otimes e_1e_3s, v_{-1}\otimes e_1e_3s, v_1\otimes e_1e_2e_3s, v_{-1}\otimes e_1e_2e_3s\rangle$
$e_{(3,1^4)}$	$\mathfrak{sl}(2)\oplus\langle e_{(3,1^4)}, R_{e_2,e_3}, R_{e_2,e_{-3}}, R_{e_2,e_{-2}}, R_{e_3,e_{-3}}, R_{e_3,e_{-2}}, R_{e_{-3},e_{-2}}, R_{e_1,e_2}, R_{e_1,e_3}, R_{e_1,e_{-3}}, R_{e_1,e_{-2}}, v_1\otimes e_1e_2e_3s, v_{-1}\otimes e_1e_2e_3s, v_1\otimes e_1s, v_{-1}\otimes e_1s, v_1\otimes e_1e_2s, v_{-1}\otimes e_1e_2s, v_1\otimes e_1e_3s, v_{-1}\otimes e_1e_3s\rangle$

$e_{(2^2,1^3)}$	$\mathfrak{sl}(2) \oplus \langle e_{(2^2,1^3)}, R_{e_1,e_{-2}}, R_{e_3,e_0}, R_{e_3,e_{-3}}, R_{e_2,e_{-1}}, R_{e_1,e_{-1}} - R_{e_2,e_{-2}}, R_{e_{-3},e_0}, R_{e_1,e_3}, R_{e_1,e_0}, R_{e_2,e_3}, R_{e_1,e_{-3}}, R_{e_2,e_0}, R_{e_2,e_{-3}}, v_1 \otimes e_1 s, v_{-1} \otimes e_1 s, v_{-1} \otimes e_1e_2e_3 s, v_1 \otimes e_1e_2e_3 s, v_1 \otimes e_1e_2 s, v_{-1} \otimes e_1e_2 s, v_1 \otimes e_2 s, v_{-1} \otimes e_2 s, v_1 \otimes e_1e_3 s, v_{-1} \otimes e_1e_3 s, v_1 \otimes e_2e_3 s, v_{-1} \otimes e_2e_3 s\rangle$
0	$\mathfrak{g}$

Table 6.32: $\mathfrak{z}(\mathfrak{g}^e)$ and Δ

e	$\mathfrak{z}(\mathfrak{g}^e)$	Δ
$E+e_{(7)}$	$\langle e, v_1 \otimes e_1e_2e_3, R_{e_1,e_2}\rangle$	Figure 29: $1,1,1,2$
$E+e_{(5,1^2)}$	$\langle e, R_{e_1,e_2}\rangle$	Figure 28: $2,0,2,0$ Figure 29: $2,0,0,2$ Figure 30: $2,0,0,2$
$E+e_{(3^2,1)}$	$\langle e, R_{e_1,e_2}\rangle$	Figure 28: $0,1,1,0$
$E+e_{(3,2^2)}$	$\langle e\rangle$	Figure 27: $1,0,0,1$ Figure 28: $1,0,0,1$ Figure 29: $1,1,0,0$
$E+e_{(3,1^4)}$	$\langle e\rangle$	Figure 26: $0,0,0,2$ Figure 27: $0,0,0,2$ Figure 28: $2,0,0,0$ Figure 29: $2,0,0,0$
$E+e_{(2^2,1^3)}$	$\langle e\rangle$	Figure 26: $0,0,1,0$ Figure 27: $0,0,1,0$ Figure 28: $0,1,0,0$
E	$\langle e\rangle$	Figure 26: $1,0,0,0$

$e_{(7)}$	$\langle e, R_{e_1,e_2}\rangle$	Figure 29: $0,0,2,2$ Figure 30: $0,0,2,2$ Figure 31: $0,0,2,2$
$e_{(5,1^2)}$	$\langle e, R_{e_1,e_2}\rangle$	Figure 30: $0,1,0,2$
$e_{(3^2,1)}$	$\langle e, R_{e_1,e_2}\rangle$	Figure 28: $0,0,2,0$ Figure 29: $0,0,0,2$ Figure 30: $0,0,0,2$ Figure 31: $0,0,0,2$
$e_{(3,2^2)}$	$\langle e\rangle$	Figure 29: $0,0,1,0$ Figure 30: $0,0,1,0$ Figure 31: $0,0,1,0$
$e_{(3,1^4)}$	$\langle e\rangle$	Figure 30: $0,1,0,0$
$e_{(2^2,1^3)}$	$\langle e\rangle$	Figure 28: $0,0,1,0$ Figure 29: $0,0,0,1$ Figure 30: $0,0,0,1$ Figure 31: $0,0,0,1$
0	$\{0\}$	Figures 26, 27, 28, 29, 30, 31: All labels are zeros

We also calculate the $\mathfrak{g}^e(0)$-module structure on each $\mathfrak{g}^e(j)$ for $j > 0$. Recall that we denote by $V^{\mathfrak{sl}}(j)$ the $\mathfrak{sl}(2)$-module with highest weight j. We also let $V^{\mathfrak{osp}}(j)$ be the $\mathfrak{osp}(1|2)$-module with highest weight j. Let $\mathfrak{t}$ be a 1-dimensional Lie algebra and $\mathfrak{t}_j$ be the $\mathfrak{t}$-module such that $t \cdot a = ja$ for $t \in \mathfrak{t}$, $a \in \mathfrak{t}_j$. For $e = e_{(3^2,1)}$ and $e_{(2^2,1^3)}$, the $\mathfrak{g}^e(0)$-module structure on $\mathfrak{g}^e(j)$ is not included as it requires representations of $\mathfrak{sl}(2|1)$

and $D(2,1;\alpha)$.

Table 6.33: $\mathfrak{g}^e(0)$-module structure

e	$\mathfrak{g}^e(0)$	$\mathfrak{g}^e(j)$ for $j>0$
$E+e_{(7)}$	0	$\dim\mathfrak{g}^e(10)=\dim\mathfrak{g}^e(7)=\dim\mathfrak{g}^e(6)=$ $\dim\mathfrak{g}^e(5)=\dim\mathfrak{g}^e(1)=1,\dim\mathfrak{g}^e(2)=2.$
$E+e_{(5,1^2)}$	$\mathfrak{t}$	$\mathfrak{g}^e(1)=0,\mathfrak{g}^e(2)=\mathfrak{t}_0\oplus\mathfrak{t}_{-1}\oplus\mathfrak{t}_1,\mathfrak{g}^e(4)=$ $\mathfrak{t}_{-2}\oplus\mathfrak{t}_{-1}\oplus\mathfrak{t}_1\oplus\mathfrak{t}_2,\mathfrak{g}^e(6)=\mathfrak{t}_0.$
$E+e_{(3^2,1)}$	$\mathfrak{t}$	$\mathfrak{g}^e(1)=\mathfrak{t}_{-3}\oplus\mathfrak{t}_{-1}\oplus\mathfrak{t}_1\oplus\mathfrak{t}_3;\mathfrak{g}^e(2)=\mathfrak{t}_{-4}\oplus$ $\mathfrak{t}_{-2}\oplus\mathfrak{t}_0\oplus\mathfrak{t}_2\oplus\mathfrak{t}_{-4};\mathfrak{g}^e(3)=\mathfrak{t}_{-1}\oplus\mathfrak{t}_1,\mathfrak{g}^e(4)=0.$
$E+e_{(3,2^2)}$	$\mathfrak{osp}(1\|2)$	$\mathfrak{g}^e(1)=V^{\mathfrak{osp}}(\frac{1}{2})\oplus V^{\mathfrak{osp}}(0),$ $\mathfrak{g}^e(2)=V^{\mathfrak{osp}}(\frac{1}{2})\oplus V^{\mathfrak{osp}}(0)\oplus V^{\mathfrak{osp}}(0),$ $\mathfrak{g}^e(3)=V^{\mathfrak{osp}}(\frac{1}{2}).$
$E+e_{(3,1^4)}$	$\mathfrak{osp}(1\|2)\oplus\mathfrak{osp}(1\|2)$	$\mathfrak{g}^e(2)=$ $\left(V^{\mathfrak{osp}}(\frac{1}{2})\otimes V^{\mathfrak{osp}}(\frac{1}{2})\right)\oplus(V^{\mathfrak{osp}}(0)\otimes V^{\mathfrak{osp}}(0))$
$E+e_{(2^2,1^3)}$	$\mathfrak{sl}(2)\oplus\mathfrak{osp}(1\|2)$	$\mathfrak{g}^e(1)=V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{osp}}(1),\mathfrak{g}^e(2)=$ $\left(V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{osp}}(0)\right)\oplus\left(V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{osp}}(\frac{1}{2})\right).$
E	$\mathfrak{so}(7)$	$\mathfrak{g}^e(1)=V_8,\ \mathfrak{g}^e(2)=\langle e\rangle.$
$e_{(7)}$	$\mathfrak{osp}(1\|2)$	$\mathfrak{g}^e(2)=\mathfrak{g}^e(10)=V^{\mathfrak{osp}}(0),\mathfrak{g}^e(6)=V^{\mathfrak{osp}}(\frac{1}{2}).$
$e_{(5,1^2)}$	$\mathfrak{sl}(2)\oplus\mathfrak{t}$	$\mathfrak{g}^e(2)=V^{\mathfrak{sl}}(0)\otimes\mathfrak{t}_0,\ \mathfrak{g}^e(3)=$ $\left(V^{\mathfrak{sl}}(1)\otimes\mathfrak{t}_{-1}\right)\oplus\left(V^{\mathfrak{sl}}(1)\otimes\mathfrak{t}_1\right),\mathfrak{g}^e(4)=$ $\left(V^{\mathfrak{sl}}(0)\otimes\mathfrak{t}_{-2}\right)\oplus\left(V^{\mathfrak{sl}}(0)\otimes\mathfrak{t}_2\right),$ $\mathfrak{g}^e(6)=V^{\mathfrak{sl}}(0)\otimes\mathfrak{t}_0.$
$e_{(3^2,1)}$	$\mathfrak{sl}(2\|1)$	$\dim\mathfrak{g}^e(2)=9,\dim\mathfrak{g}^e(4)=1.$

$e_{(3,2^2)}$	$\mathfrak{sl}(2)\oplus\mathfrak{osp}(1\|2)$	$\mathfrak{g}^e(1)=V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{osp}}(\frac{1}{2}), \mathfrak{g}^e(3)=V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{osp}}(0),\ \mathfrak{g}^e(2)=\big(V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{osp}}(0)\big)\oplus\big(V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{osp}}(\frac{1}{2})\big)$.
$e_{(3,1^4)}$	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$	$\mathfrak{g}^e(1)=\big(V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{sl}}(1)\big)\oplus\big(V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{sl}}(0)\big),\mathfrak{g}^e(2)=\big(V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{sl}}(1)\otimes V^{\mathfrak{sl}}(1)\big)\oplus\big(V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{sl}}(0)\otimes V^{\mathfrak{sl}}(0)\big)$.
$e_{(2^2,1^3)}$	$D(2,1;2)$	$\dim\mathfrak{g}^e(1)=10,\dim\mathfrak{g}^e(2)=1.$
0	$\mathfrak{g}$	0

In the remaining part of this section, we give explicit calculations on finding $\mathfrak{g}^e$ and $\mathfrak{z}(\mathfrak{g}^e)$ and obtain the corresponding labelled Dynkin diagrams for nilpotent elements $E+e_{(7)}$, E and $e_{(7)}$. The results of all other cases are obtained using the same approach.

(1) $e=E+e_{(7)}$

For $e=E+e_{(7)}$, we have that $\mathfrak{sl}(2)^E=\langle E\rangle$ and $\mathfrak{so}(7)^{e_{(7)}}$ has already been calculated in Table 6.24. Therefore, $\mathfrak{g}^e_{\bar{0}}=\mathfrak{sl}(2)^E\oplus\mathfrak{so}(7)^{e_{(7)}}=\langle E,e_{(7)},R_{e_1,e_2},R_{e_1,e_0}-2R_{e_2,e_3}\rangle$ and it has dimension $1+3=4$.

Now we determine $\mathfrak{g}^e_{\bar{1}}$. We know that $h=H+h_{(7)}$ lies in an $\mathfrak{sl}(2)$-triple $\{e,h,f\}\subseteq\mathfrak{g}_{\bar{0}}$ where $h_{(7)}=6R_{e_1,e_{-1}}+4R_{e_2,e_{-2}}+2R_{e_3,e_{-3}}$ according to Table 6.23. Then we calculate all eigenvalues of ad$h$ on $\mathfrak{g}_{\bar{1}}$ based on Table 6.29. For example,

$$[H+h_{(7)},v_i\otimes s]=Hv_i\otimes s+v_i\otimes h_{(7)}s=(i-6)v_i\otimes s.$$

Thus $v_1\otimes s$ has ad$h$-eigenvalue -5 and $v_{-1}\otimes s$ has ad$h$-eigenvalue -7. The detailed result is shown below:

ad$h$-eigenvalues on $\mathfrak{g}_{\bar{1}}$	basis element of $\mathfrak{g}_{\bar{1}}$
7	$v_1 \otimes e_1e_2e_3s$
5	$v_1 \otimes e_1e_2s,\ v_{-1} \otimes e_1e_2e_3s$
3	$v_1 \otimes e_1e_3s,\ v_{-1} \otimes e_1e_2s$
1	$v_1 \otimes e_1s,\ v_1 \otimes e_2e_3s,\ v_{-1} \otimes e_1e_3s$
-1	$v_{-1} \otimes e_1s, v_1 \otimes e_2s,\ v_{-1} \otimes e_2e_3s$
-3	$v_{-1} \otimes e_2s,\ v_1 \otimes e_3s$
-5	$v_1 \otimes s,\ v_{-1} \otimes e_3s$
-7	$v_{-1} \otimes s$

Hence, $\mathfrak{g}_{\bar{1}}^e$ has one basis element with ad$h$-eigenvalue 7 which is $v_1 \otimes e_1e_2e_3s$. In order to find out the basis element $x = av_1 \otimes e_1e_2s + bv_{-1} \otimes e_1e_2e_3s$ for $a, b \in \mathbb{C}$ with ad$h$-eigenvalue 5, we calculate

$$\begin{aligned}[e, x] &= [E + (R_{e_1,e_{-2}} + R_{e_2,e_{-3}} + R_{e_3,e_0}), av_1 \otimes e_1e_2s + bv_{-1} \otimes e_1e_2e_3s] \\ &= (a + b)v_1 \otimes e_1e_2e_3s.\end{aligned}$$

This is equal to 0 if and only if $b = -a$. Therefore, we have that $v_1 \otimes e_1e_2s - v_{-1} \otimes e_1e_2e_3s \in \mathfrak{g}_{\bar{1}}^e$. Using similar arguments, we obtain that there is no basis element of the form $av_1 \otimes e_1e_3s + bv_{-1} \otimes e_1e_2s$ for $a, b \in \mathbb{C}$ that is centralized by e in $\mathfrak{g}_{\bar{1}}$ and $v_1 \otimes e_1s - v_1 \otimes e_2e_3s \in \mathfrak{g}_{\bar{1}}^e$. Therefore, $\mathfrak{g}_{\bar{1}}^e$ has basis $\{v_1 \otimes e_1e_2e_3s, v_1 \otimes e_1e_2s - v_{-1} \otimes e_1e_2e_3s, v_1 \otimes e_1s - v_1 \otimes e_2e_3s\}$ and thus is 3-dimensional.

By calculating ad$h$-eigenvalues of each basis elements in $\mathfrak{g}^e$, we deduce that $\mathfrak{z} =$

$\mathfrak{z}(1) \oplus \mathfrak{z}(2) \oplus \mathfrak{z}(5) \oplus \mathfrak{z}(6) \oplus \mathfrak{z}(7) \oplus \mathfrak{z}(10)$. We first observe that

$$[R_{e_1,e_0} - 2R_{e_2,e_3}, v_1 \otimes e_1 s - v_1 \otimes e_2 e_3 s] = -3v_1 \otimes e_1 e_2 e_3 \neq 0.$$

Hence $\mathfrak{z}(1) = \mathfrak{z}(6) = 0$. Similarly, we have that

$$[e_{(7)}, v_1 \otimes e_1 e_2 s - v_{-1} \otimes e_1 e_2 e_3 s] = v_1 \otimes e_1 e_2 e_3 \neq 0.$$

Hence, we know that $\mathfrak{z}(5) = 0$ and $e_{(7)} \notin \mathfrak{z}$. Thus $\mathfrak{z}(2)$ has dimension 1 and is spanned by $e = E + e_{(7)}$. Furthermore, we observe that $\mathfrak{g}(7+j)^e = 0$ for all j with $\mathfrak{g}(j)^e \neq 0$. This implies that $\mathfrak{z}(7) = \langle v_1 \otimes e_1 e_2 e_3 s \rangle$. Using a similar argument, we have $\mathfrak{z}(10) = \langle R_{e_1,e_2} \rangle$. Therefore, we deduce that $\mathfrak{z} = \langle e, v_1 \otimes e_1 e_2 e_3 s, R_{e_1,e_2} \rangle$ and it is 3-dimensional.

Next we determine the labelled Dynkin diagram with respect to e. we obtain that roots in $\mathfrak{g}(>0)$ are $\{\delta, \varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_3, \varepsilon_2+\varepsilon_3, \varepsilon_1-\varepsilon_2, \varepsilon_1-\varepsilon_3, \varepsilon_2-\varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \frac{1}{2}(\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3), \frac{1}{2}(-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3), \frac{1}{2}(\delta+\varepsilon_1+\varepsilon_2-\varepsilon_3), \frac{1}{2}(\delta+\varepsilon_1-\varepsilon_2+\varepsilon_3), \frac{1}{2}(-\delta+\varepsilon_1+\varepsilon_2-\varepsilon_3), \frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3), \frac{1}{2}(-\delta+\varepsilon_1-\varepsilon_2+\varepsilon_3), \frac{1}{2}(\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3)\}$ and there are no roots in $\mathfrak{g}(0)$. Hence, there is only one choice of positive roots Φ^+ such that $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j\geq 0} \mathfrak{g}(j)$. The corresponding system of simple roots is $\Pi = \{\alpha_1 = \frac{1}{2}(\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3), \alpha_2 = \frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3), \alpha_3 = \frac{1}{2}(-\delta+\varepsilon_1-\varepsilon_2+\varepsilon_3), \alpha_4 = \varepsilon_2-\varepsilon_3\}$. Note that the ad$h$-eigenvalues with respect to $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are $1, 1, 1, 2$. We also need to calculate the number of lines between each pair of vertices labelled by simple roots in Π in the corresponding labelled Dynkin diagram. Since α_1, α_2 are isotropic and $|(\alpha_1, \alpha_2)| = 3$, there are 3 lines between vertices that are labelled by α_1 and α_2. Similarly, since $|(\alpha_1, \alpha_3)| = 2, |(\alpha_2, \alpha_3)| = 1$, we obtain that there are 2 lines between vertices that are labelled by α_1 and α_3 and there exist a single line between vertices that are labelled by α_2 and α_3. Since $(\alpha_4, \alpha_4) = 4 \neq 0$, we calculate $\frac{2|(\alpha_3,\alpha_4)|}{\min_{(\alpha_k,\alpha_k)\neq 0} |(\alpha_k,\alpha_k)|} = 2$ which is the number of lines between vertices labelled by α_3 and α_4. We also draw an arrow pointing from the vertex labelled by α_4 to the vertex labelled by α_3. Therefore, we obtain the labelled Dynkin diagram for $e = E + e_{(7)}$ is

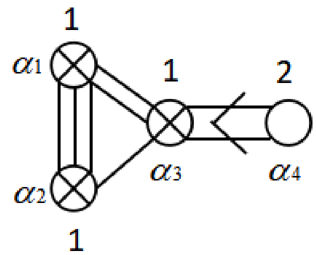


Figure 32: Labelled Dynkin diagram for $e = E + e_{(7)}$

(2) $e = E$

For $e = E$, it is clear that $\mathfrak{g}_{\bar{0}}^e = \langle E \rangle \oplus \mathfrak{so}(7)$ and thus it has dimension $1 + 21 = 22$.

Next we determine $\mathfrak{g}_{\bar{1}}^e$. By calculating $[e, a_1 v_1 \otimes s + a_2 v_1 \otimes e_1 s + a_3 v_1 \otimes e_2 s + a_4 v_1 \otimes e_3 s + a_5 v_1 \otimes e_1 e_2 s + a_6 v_1 \otimes e_1 e_3 s + a_7 v_1 \otimes e_2 e_3 s + a_8 v_1 \otimes e_1 e_2 e_3 s + b_1 v_{-1} \otimes s + b_2 v_{-1} \otimes e_1 s + b_3 v_{-1} \otimes e_2 s + b_4 v_{-1} \otimes e_3 s + b_5 v_{-1} \otimes e_1 e_2 s + b_6 v_{-1} \otimes e_1 e_3 + b_7 v_{-1} \otimes e_2 e_3 + b_8 v_{-1} \otimes e_1 e_2 e_3 s] = 0$ for $a_i, b_i \in \mathbb{C}, i = 1, \dots 8$, we obtain that $b_i = 0$ for $i = 1, \dots, 8$. Therefore, $\mathfrak{g}_{\bar{1}}^e$ has a basis $\{v_1 \otimes s, v_1 \otimes e_1 s, v_1 \otimes e_2 s, v_1 \otimes e_3 s, v_1 \otimes e_1 e_2 s, v_1 \otimes e_1 e_3 s, v_1 \otimes e_2 e_3 s, v_1 \otimes e_1 e_2 e_3 s\}$ and it is 8-dimensional.

Note that H is the semisimple element that lies in an $\mathfrak{sl}(2)$-triple $\{E, H, F\}$ in $\mathfrak{g}_{\bar{0}}$. By calculating the ad$H$-eigenvalues on the basis elements for $\mathfrak{g}^e$, we have that $\mathfrak{g}^e(0) = \mathfrak{so}(7)$, $\mathfrak{g}^e(1) = \langle v_1 \rangle \otimes V_8 \cong V_8$ and $\mathfrak{g}^e(2) = \langle E \rangle$. Thus we have that

$$\mathfrak{z} = \mathfrak{z}(0) \oplus \mathfrak{z}(1) \oplus \mathfrak{z}(2) \subseteq (\mathfrak{g}^e(0))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(1))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(2))^{\mathfrak{g}^e(0)} = \langle E \rangle.$$

Note that $e = E \in \mathfrak{z}$, this implies that $\mathfrak{z} = \langle e \rangle$.

Next we look at the labelled Dynkin diagram with respect to E. We obtain that roots in $\mathfrak{g}(>0)$ are $\{\delta, \frac{1}{2}(\delta + \sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_3 \varepsilon_3) : \sigma_i = + \text{ or } -\}$ and roots in $\mathfrak{g}(0)$ are $\Phi(0) =$

$\{\pm(\varepsilon_1+\varepsilon_2), \pm(\varepsilon_1+\varepsilon_3), \pm(\varepsilon_2+\varepsilon_3), \pm(\varepsilon_1-\varepsilon_2), \pm(\varepsilon_1-\varepsilon_3), \pm(\varepsilon_2-\varepsilon_3), \pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\}$. Hence, there is only one system of simple roots $\Pi(0)$ of $\mathfrak{g}(0)$ up to conjugacy where $\Pi(0) = \{\varepsilon_3, \varepsilon_2-\varepsilon_3, \varepsilon_1-\varepsilon_2\}$. By extending $\Pi(0)$ to a simple system of $\mathfrak{g}$, we get only one conjugacy class of systems of positive roots Φ^+ satisfying $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j\geq 0} \mathfrak{g}(j)$. Therefore, a system of simple roots is $\Pi = \{\alpha_1 = \frac{1}{2}(\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3), \alpha_2 = \varepsilon_3, \alpha_3 = \varepsilon_2-\varepsilon_3, \alpha_4 = \varepsilon_1-\varepsilon_2\}$ and the corresponding ad$h$-eigenvalues $1, 0, 0, 0$. By computing

$$\frac{2|(\alpha_1,\alpha_2)|}{\min_{(\alpha_k,\alpha_k)\neq 0}|(\alpha_k,\alpha_k)|} = 1 \text{ for } \alpha_k \in \Phi \text{ and } |(\alpha_2,\alpha_2)| = 2,$$

we put one line between vertices labelled by α_1 and α_2. We also have that

$$\frac{2|(\alpha_2,\alpha_3)|}{\min\{|(\alpha_2,\alpha_2)|, |(\alpha_3,\alpha_3)|\}} = 2 \text{ and } |(\alpha_3,\alpha_3)| = 4,$$

we put two lines between vertices labelled by α_2 and α_3 and an arrow pointing from vertex labelled by α_3 to α_2. Similarly, there is one line between vertices labelled by α_3 and α_4. Therefore, the labelled Dynkin diagram for $e = E$ is:

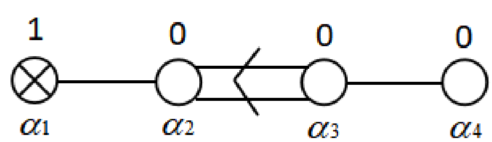


Figure 33: Labelled Dynkin diagram for $e = E$

(3) $e = e_{(7)}$

In this case, a basis of $\mathfrak{so}(7)^{e_{(7)}}$ is given in Table 6.24. Hence, $\mathfrak{g}_{\bar{0}}^e = \mathfrak{sl}(2) \oplus \mathfrak{so}(7)^{e_{(7)}}$ and it has dimension $3+3=6$.

Next we determine $\mathfrak{g}_{\bar{1}}^e$. By calculating $[e, a_1v_1\otimes s + a_2v_1\otimes e_1s + a_3v_1\otimes e_2s + a_4v_1\otimes$

$e_3s+a_5v_1\otimes e_1e_2s+a_6v_1\otimes e_1e_3s+a_7v_1\otimes e_2e_3s+a_8v_1\otimes e_1e_2e_3s+b_1v_{-1}\otimes s+b_2v_{-1}\otimes e_1s+$ $b_3v_{-1}\otimes e_2s+b_4v_{-1}\otimes e_3s+b_5v_{-1}\otimes e_1e_2s+b_6v_{-1}\otimes e_1e_3+b_7v_{-1}\otimes e_2e_3+b_8v_{-1}\otimes e_1e_2e_3s]=0$ for $a_i,b_i\in\mathbb{C}, i=1,\dots 8$, we obtain that

$$a_2+a_7=0, b_2+b_7=0 \text{ and } a_i=b_i=0 \text{ for } i=1,3,4,5,6.$$

Therefore, $\mathfrak{g}^e_{\bar{1}}$ has a basis $\{v_1\otimes e_1s-v_1\otimes e_2e_3s, v_{-1}\otimes e_1s-v_{-1}\otimes e_2e_3s, v_1\otimes e_1e_2e_3s, v_{-1}\otimes e_1e_2e_3s\}$ and it is 4-dimensional.

According to Table 6.23, there is an $\mathfrak{sl}(2)$-triple $\{e,h,f\}$ in $\mathfrak{g}_{\bar{0}}$ such that $h=h_{(7)}=\operatorname{diag}(6,4,2,0,-2,-4,-6)=6R_{e_1,e_{-1}}+4R_{e_2,e_{-2}}+2R_{e_3,e_{-3}}$. Then the ad$h$-eigenvalues of basis elements in $\mathfrak{g}^e$ can be shown in the following table:

ad$h$-eigenvalues	Basis elements in $\mathfrak{g}^e$
0	$E,\ H,\ F,\ v_1\otimes e_1s-v_1\otimes e_2e_3s,\ v_{-1}\otimes e_1s-v_{-1}\otimes e_2e_3s$
2	$e_{(7)}$
6	$R_{e_1,e_0}-2R_{e_2,e_3},\ v_1\otimes e_1e_2e_3s,\ v_{-1}\otimes e_1e_2e_3s$
10	R_{e_1,e_2}

Table 6.34: ad$h$-eigenvalues of basis elements in $\mathfrak{g}^e$

By computing commutator relations in $\mathfrak{g}^e(0)$, we deduce that $\mathfrak{g}^e(0)=\mathfrak{osp}(1|2)$ according to Lemma 6.1 where $F, v_{-1}\otimes e_1s-v_{-1}\otimes e_2e_3s, H, v_1\otimes e_1s-v_1\otimes e_2e_3s, E$ correspond to $u_{-2},u_{-1},u_0,u_1,u_2$ in Lemma 6.1 respectively. Moreover, we have that $\mathfrak{g}^e(2)=\mathfrak{g}^e(10)=$

$V^{\mathfrak{osp}}(0)$ and $\mathfrak{g}^e(6) = V^{\mathfrak{osp}}(1)$. Hence, we deduce that

$$\begin{aligned}\mathfrak{z} &= \mathfrak{z}(0) \oplus \mathfrak{z}(2) \oplus \mathfrak{z}(6) \oplus \mathfrak{z}(10) \\ &\subseteq (\mathfrak{g}^e(0))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(2))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(6))^{\mathfrak{g}^e(0)} \oplus (\mathfrak{g}^e(10))^{\mathfrak{g}^e(0)} \\ &= \langle e, R_{e_1,e_2} \rangle.\end{aligned}$$

Note that e and R_{e_1,e_2} commute with all basis elements in $\mathfrak{g}^e$, therefore $\mathfrak{z}(\mathfrak{g}^e) = \langle e, R_{e_1,e_2} \rangle$.

Next we determine the labelled Dynkin diagram with respect to e. We obtain that roots in $\mathfrak{g}(>0)$ are $\{\varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_3, \varepsilon_2+\varepsilon_3, \varepsilon_1-\varepsilon_3, \varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_1, \varepsilon_2, \varepsilon_3, \frac{1}{2}(\pm\delta+\varepsilon_1+\varepsilon_2-\varepsilon_3), \frac{1}{2}(\pm\delta+\varepsilon_1-\varepsilon_2+\varepsilon_3), \frac{1}{2}(\pm\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3)\}$ and roots in $\mathfrak{g}(0)$ are $\Phi(0) = \{\pm\delta, \frac{1}{2}(\pm\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3), \frac{1}{2}(\pm\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3)\}$. Hence, there are three systems of simple roots of $\mathfrak{g}(0)$ up to conjugacy: $\varPi_1(0) = \{\frac{1}{2}(\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3), \frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3)\}$, $\varPi_2(0) = \{\delta, \frac{1}{2}(-\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3)\}$ and $\varPi_3(0) = \{\delta, \frac{1}{2}(-\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3)\}$. By extending $\varPi_i(0)$ for $i = 1, 2, 3$ to simple root systems of $\mathfrak{g}$, we get three conjugacy classes of systems of positive roots Φ^+satisfying $\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \subseteq \bigoplus_{j\geq 0} \mathfrak{g}(j)$. Hence, the systems of simple roots are:

- We have that $\varPi_1 = \{\alpha_1 = \frac{1}{2}(\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3), \alpha_2 = \frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3), \alpha_3 = \frac{1}{2}(-\delta+\varepsilon_1-\varepsilon_2+\varepsilon_3), \alpha_4 = \varepsilon_2-\varepsilon_3\}$ and the corresponding ad$h$-eigenvalues are $0, 0, 2, 2$. We have already calculated the numbers of lines between each pair of vertices that are labelled by simple roots in $\varPi_1$ when we considered the case $e = E + e_{(7)}$. Therefore, the labelled Dynkin diagram with respect to $\varPi_1$ is:

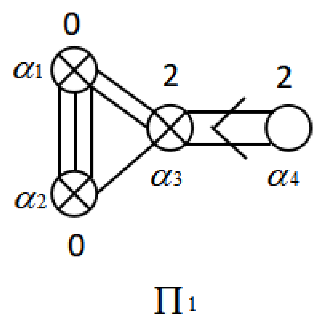


Figure 34: Labelled Dynkin diagram for $e = e_{(7)}$

- We have that $\Pi_2 = \{\alpha_1 = \delta, \alpha_2 = \frac{1}{2}(-\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \alpha_3 = \varepsilon_3, \alpha_4 = \varepsilon_2 - \varepsilon_3\}$ and the corresponding ad$h$-eigenvalues are $0, 0, 2, 2$. By computing

$$|(\alpha_1, \alpha_1)| = 6, \frac{2|(\alpha_1, \alpha_2)|}{\min_{(\alpha_k,\alpha_k)\neq 0} |(\alpha_k, \alpha_k)|} = 3 \text{ for } \alpha_k \in \Phi,$$

we deduce that there are 3 lines between the vertices labelled by α_1 and α_2. Similarly, we have that $|(\alpha_3, \alpha_3)| = 2, |(\alpha_4, \alpha_4)| = 4$ and

$$\frac{2|(\alpha_2, \alpha_3)|}{\min_{(\alpha_k,\alpha_k)\neq 0} |(\alpha_k, \alpha_k)|} = 1, \frac{2|(\alpha_3, \alpha_4)|}{\min_{(\alpha_k,\alpha_k)\neq 0} |(\alpha_k, \alpha_k)|} = 2 \text{ for } \alpha_k \in \Phi.$$

Hence, the number of lines between the vertices labelled by α_2 and α_3 (resp. α_3 and α_4) is 1 (resp. 2). Moreover, we put an arrow pointing from the vertex labelled by α_4 to the vertex labelled by α_3. Therefore, the labelled Dynkin diagram with respect to Π_2 is:

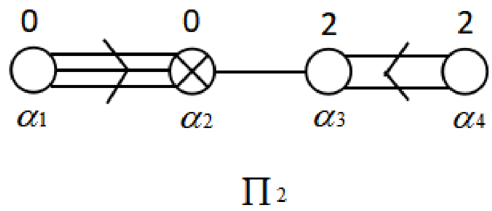


Figure 35: Labelled Dynkin diagram for $e = e_{(7)}$

- We have that $\mathit{\Pi}_3 = \{\alpha_1 = \delta, \alpha_2 = \frac{1}{2}(-\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \alpha_3 = \varepsilon_1 - \varepsilon_2, \alpha_4 = \varepsilon_2 - \varepsilon_3\}$ and the corresponding ad$h$-eigenvalues are $0, 0, 2, 2$. By computing

$$\frac{2|(\alpha_1, \alpha_2)|}{\min_{(\alpha_k,\alpha_k)\neq 0} |(\alpha_k, \alpha_k)|} = 3 \text{ and } |(\alpha_1, \alpha_1)| = 6,$$

  we put three lines between vertices labelled by α_1 and α_2 and an arrow pointing from the vertex labelled by α_1 to the vertex labelled by α_2. We also have that

$$\frac{2|(\alpha_2, \alpha_3)|}{\min_{(\alpha_k,\alpha_k)\neq 0} |(\alpha_k, \alpha_k)|} = 2, |(\alpha_3, \alpha_3)| = |(\alpha_4, \alpha_4)| = 4,$$

$$\text{and } \frac{2|(\alpha_3, \alpha_4)|}{\min\{|(\alpha_3, \alpha_3)|, |(\alpha_4, \alpha_4)|\}} = 1.$$

  Hence, the labelled Dynkin diagram with respect to $\mathit{\Pi}_3$ is:

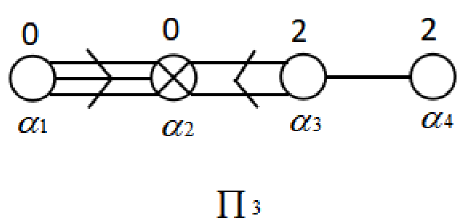


Figure 36: Labelled Dynkin diagram for $e = e_{(7)}$

6.4.8 Analysis of results

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = F(4)$ and $\mathfrak{h} = \langle H, h_1, h_2, h_3 \rangle$ be the Cartan subalgebra of $\mathfrak{g}$. Let Φ be a root system for $\mathfrak{g}$ that is defined in Subsection 6.4.6 and Δ be the corresponding labelled Dynkin diagram. Given an $\mathfrak{sl}(2)$-triple $\{e, h, f\}$, we denote a root system for $\mathfrak{g}^h$ by Φ_h, i.e. $\Phi_h = \{\alpha \in \Phi : \alpha(h) = 0\}$ and a simple root system for $\mathfrak{g}^h$ by Π_h. We also denote the labelled Dynkin diagram for Π_h by Δ_h .

In order to prove Theorem 1.1 for $\mathfrak{g}$, we consider $\mathfrak{g}^h$ for each case such that Δ has no label equal to 1. Note that $\mathfrak{g}^h$ is of the form $\mathfrak{s} \oplus \bigoplus_{\alpha \in \Phi_h} \mathfrak{g}_\alpha$ where $\mathfrak{s}$ is a complement of $\mathfrak{h} \cap \bigoplus_{\alpha \in \Phi_h} \mathfrak{g}_\alpha$ in $\mathfrak{h}$. Then

$$\mathfrak{z}(\mathfrak{g}^h) = \{t \in \mathfrak{h} : \alpha(t) = 0 \text{ for all } \alpha \in \Phi_h\},$$

thus $\mathfrak{z}(\mathfrak{g}^h)$ is a subalgebra of $\mathfrak{h}$ with dimension $\text{rank}\Phi - \text{rank}\Phi_h$. Note that Δ has no label equal to 1 for the nilpotent elements $E + e_{(5,1^2)}$, $E + e_{(3,1^4)}$, $e_{(7)}, e_{(3^2,1)}$.

- When $e = E + e_{(5,1^2)}$, we have that $\Phi_h = \{\pm\varepsilon_3, \pm\frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 - \varepsilon_3), \pm\frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3)\}$. Then a simple root system is $\Pi_h^1 = \{\varepsilon_3, \frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 - \varepsilon_3)\}$ or $\Pi_h^2 = \{\frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), -\frac{1}{2}(\delta + \varepsilon_1 - \varepsilon_2 + \varepsilon_3)\}$. We draw the corresponding labelled Dynkin diagrams below.

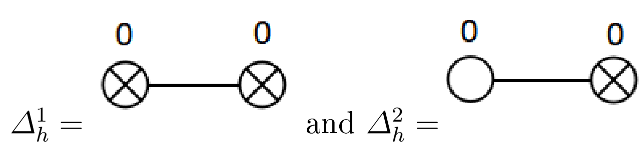


Figure 37: Labelled Dynkin diagrams for Π_h^1 and Π_h^2

Hence Φ_h is of type $\mathfrak{sl}(2|1)$ according to [19, Subsection 3.4.1] and $\mathfrak{g}^h = \mathfrak{s} \oplus \mathfrak{sl}(2|1)$ where $\mathfrak{s}$ is a complement of $\mathfrak{h} \cap \mathfrak{sl}(2|1)$ in $\mathfrak{h}$. Note that $\mathfrak{sl}(2|1)$ has no centre, thus

$\dim \mathfrak{z}(\mathfrak{g}^h) = 4 - \dim(\mathfrak{h} \cap \mathfrak{sl}(2|1)) = 4 - 2 = 2 = n_2(\Delta) = \dim \mathfrak{z}(\mathfrak{g}^e)$.

- When $e = E + e_{(3,1^4)}$, we have that $\Phi_h = \{\pm(\varepsilon_2+\varepsilon_3), \pm(\varepsilon_2-\varepsilon_3), \pm\varepsilon_2, \pm\varepsilon_3, \pm\frac{1}{2}(\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3), \pm\frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3), \pm\frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2-\varepsilon_3), \pm\frac{1}{2}(\delta-\varepsilon_1-\varepsilon_2+\varepsilon_3)\}$. Then the simple root systems are $\Pi_h^1 = \{\frac{1}{2}(\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3), \varepsilon_3, \varepsilon_2-\varepsilon_3\}$, $\Pi_h^2 = \{\frac{1}{2}(-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3), \frac{1}{2}(\delta-\varepsilon_1-\varepsilon_2+\varepsilon_3), \varepsilon_2-\varepsilon_3\}$ and $\Pi_h^3 = \{\frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2-\varepsilon_3), \frac{1}{2}(-\delta+\varepsilon_1+\varepsilon_2-\varepsilon_3), \varepsilon_3\}$. We draw the corresponding labelled Dynkin diagrams below:

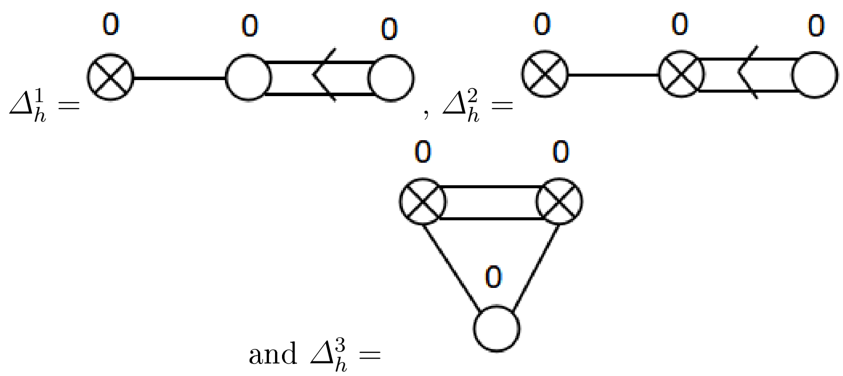
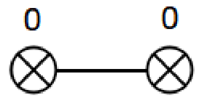


Figure 38: Labelled Dynkin diagrams for Π_h^1, Π_h^2 and Π_h^3

Hence Φ_h is of type $\mathfrak{osp}(2|4)$ according to [19, Subsection 3.4.1] and $\mathfrak{g}^h = \mathfrak{s} \oplus \mathfrak{osp}(2|4)$ where $\mathfrak{s}$ is a complement of $\mathfrak{h} \cap \mathfrak{osp}(2|4)$ in $\mathfrak{h}$. Note that $\mathfrak{osp}(2|4)$ has no centre, thus $\dim \mathfrak{z}(\mathfrak{g}^h) = 4 - \dim(\mathfrak{h} \cap \mathfrak{osp}(2|4)) = 4 - 3 = 1 = n_2(\Delta) = \dim \mathfrak{z}(\mathfrak{g}^e)$.

- When $e = e_{(7)}$, we have that $\Phi_h = \{\pm\delta, \pm\frac{1}{2}(\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3), \pm\frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3)\}$. Then the simple root systems are $\Pi_h^1 = \{\frac{1}{2}(\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3), \frac{1}{2}(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3)\}$ and $\Pi_h^2 = \{\delta, \frac{1}{2}(-\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3)\}$. By scaling the value of $(\cdot\,,\cdot)$ we obtain the corresponding labelled Dynkin diagrams below:

$\Delta_h^1 =$ 

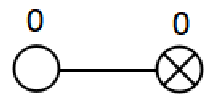
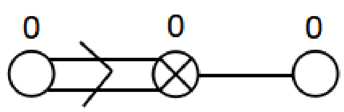
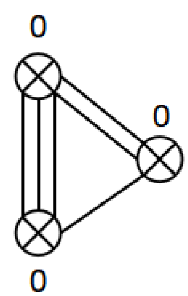
and $\Delta_h^2 =$ 

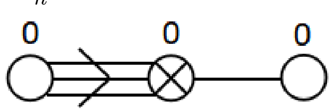
Figure 39: Labelled Dynkin diagrams for Π_h^1 and Π_h^2

Hence Φ_h is of type $\mathfrak{sl}(2|1)$ according to [19, Subsection 3.4.1] and $\mathfrak{g}^h = \mathfrak{s} \oplus \mathfrak{sl}(2|1)$ where $\mathfrak{s}$ is a complement of $\mathfrak{h} \cap \mathfrak{sl}(2|1)$ in $\mathfrak{h}$. Note that $\mathfrak{sl}(2|1)$ has no centre, thus $\dim \mathfrak{z}(\mathfrak{g}^h) = 4 - 2 = 2 = n_2(\Delta) = \dim \mathfrak{z}(\mathfrak{g}^e)$.

- When $e = e_{(3^2,1)}$, we have that $\Phi_h = \{\pm\delta, \pm(\varepsilon_1 - \varepsilon_2), \pm\varepsilon_3, \pm\frac{1}{2}(\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm\frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 - \varepsilon_3), \pm\frac{1}{2}(\delta + \varepsilon_1 - \varepsilon_2 + \varepsilon_3), \pm\frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3)\}$. Then the simple root systems are $\Pi_h^1 = \{\varepsilon_1 - \varepsilon_2, \frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 - \varepsilon_3), \varepsilon_3\}$, $\Pi_h^2 = \{\frac{1}{2}(\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \frac{1}{2}(\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \frac{1}{2}(-\delta + \varepsilon_1 - \varepsilon_2 + \varepsilon_3)\}$, $\Pi_h^3 = \{\delta, \frac{1}{2}(-\delta + \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \varepsilon_3\}$ and $\Pi_h^4 = \{\delta, \frac{1}{2}(-\delta - \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \varepsilon_1 - \varepsilon_2\}$. We draw the corresponding labelled Dynkin diagrams below:

$\Delta_h^1 =$ 

, $\Delta_h^2 =$ 

$\Delta_h^3 =$ 

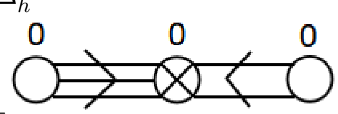
and $\Delta_h^4 =$ 

Figure 40: Labelled Dynkin diagrams for Π_h^1 , Π_h^2, Π_h^3 and Π_h^4

Hence Φ_h is of type $D(2,1;2)$ according to Subsection 6.2.3 and $\mathfrak{g}^h = \mathfrak{s} \oplus D(2,1;2)$ where $\mathfrak{s}$ is a complement of $\mathfrak{h} \cap D(2,1;2)$ in $\mathfrak{h}$. Note that $D(2,1;2)$ has no centre, thus $\dim \mathfrak{z}(\mathfrak{g}^h) = 4 - \dim(\mathfrak{h} \cap D(2,1;2)) = 4 - 3 = 1 = n_2(\Delta)$ but $\dim \mathfrak{z}(\mathfrak{g}^h) \neq \dim \mathfrak{z}(\mathfrak{g}^e)$. We will further consider this case in Subsection 6.4.9 to complete the proof of Theorem 1.

In order to prove Theorem 1.3 for $\mathfrak{g}$, we only need to consider cases such that Δ has labels equal to 2 as for the remaining cases $\mathfrak{g}_0 = \mathfrak{g}$, $e_0 = e$ and $n_2(\Delta) = 0$. For cases that all labels of Δ are equal to 0 and 2, Δ_0 is a Dynkin diagram of $\mathfrak{g}^h$ thus $\mathfrak{g}_0 = \mathfrak{g}^h$. Note that there exist some labels equal to 2 in Δ for the nilpotent elements $E + e_{(7)}, E + e_{(5,1^2)}, E + e_{(3,1^4)}, e_{(7)}, e_{(5,1^2)}, e_{(3^2,1)}$.

- When $e = E + e_{(7)}$, we draw Δ_0 in Figure 41.

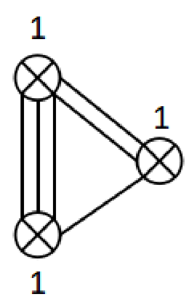
$\Delta_0 =$ 

Figure 41: Δ_0 for $e = E + e_{(7)}$

Hence, we know that $\mathfrak{g}_0 = D(2,1;2)$ and $e_0 = (E, E, E)$ according to Subsection 6.2.3. Therefore, we obtain that $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = 7 - 6 = 1 = n_2(\Delta)$ but $\dim \mathfrak{z}(\mathfrak{g}^e) - \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = 3 - 1 \neq n_2(\Delta)$. We will further discuss this case in Subsection 6.4.9.

- When $e = E + e_{(5,1^2)}$, we know that Δ_0 is the same as Δ_h in Figure 37. Thus we have that $\mathfrak{g}_0 = \mathfrak{sl}(2|1)$ and $e_0 = 0$. Note that $\mathfrak{g}_0^{e_0} = \mathfrak{g}_0$ and $\mathfrak{z}(\mathfrak{g}_0^{e_0}) = \mathfrak{z}(\mathfrak{g}_0) = 0$.

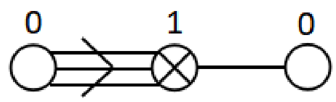
$\varDelta_0 =$ 

Figure 42: $\varDelta_0$ for $e = e_{(5,1^2)}$

Therefore, we obtain that $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = 10 - 8 = 2 = n_2(\varDelta)$ and $\dim \mathfrak{z}(\mathfrak{g}^e) - \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = 2 = n_2(\varDelta)$.

- When $e = E + e_{(3,1^4)}$, we know that $\varDelta_0$ is the same as $\varDelta_h$ in Figure 38. Thus we have that $\mathfrak{g}_0 = \mathfrak{osp}(2|4)$ and $e_0 = 0$. Note that $\dim \mathfrak{g}_0^{e_0} = \dim \mathfrak{g}_0 = 19$ and $\mathfrak{z}(\mathfrak{g}_0^{e_0}) = \mathfrak{z}(\mathfrak{g}_0) = 0$. Therefore, we obtain that $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = 20 - 19 = 1 = n_2(\varDelta)$ and $\dim \mathfrak{z}(\mathfrak{g}^e) - \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = 1 = n_2(\varDelta)$.

- When $e = e_{(7)}$, we know that $\varDelta_0$ is the same as $\varDelta_h$ in Figure 39. Thus we have that $\mathfrak{g}_0 = \mathfrak{sl}(2|1)$ and $e_0 = 0$. Therefore, we obtain that $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = 10 - 8 = 2 = n_2(\varDelta)$ and $\dim \mathfrak{z}(\mathfrak{g}^e) - \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = 2 = n_2(\varDelta)$.

- When $e = e_{(5,1^2)}$, we draw $\varDelta_0$ in Figure 42. Hence, we know that $\mathfrak{g}_0 = D(2,1;2)$ and $e_0 = (0, E, 0)$ according to Subsection 6.2.3. Therefore, we obtain that $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = 12 - 11 = 1 = n_2(\varDelta)$ and $\dim \mathfrak{z}(\mathfrak{g}^e) - \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = 2 - 1 = n_2(\varDelta)$.

- When $e = e_{(3^2,1)}$, we know that $\varDelta_0$ is the same as $\varDelta_h$ in Figure 40. Thus we have that $\mathfrak{g}_0 = D(2,1;2)$ and $e_0 = 0$. Note that $\dim D(2,1;2) = 17$. Therefore, we obtain that $\dim \mathfrak{g}^e - \dim \mathfrak{g}_0^{e_0} = 18 - 17 = 1 = n_2(\varDelta)$ but $\dim \mathfrak{z}(\mathfrak{g}^e) - \dim \mathfrak{z}(\mathfrak{g}_0^{e_0}) = 2 \neq n_2(\varDelta)$. We will further discuss this case in Subsection 6.4.9.

6.4.9 Adjoint action on $F(4)$

Let G be the linear algebraic group $G = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{Spin}_7(\mathbb{C})$. In this subsection, we determine $(\mathfrak{z}(\mathfrak{g}^e))^{G^e}$ in order to complete the proof of Theorems 1.1–1.3 for $F(4)$. Note that we only need to consider cases $e = E + e_{(7)}$, $E + e_{(5,1^2)}$, $E + e_{(3^2,1)}$, $e_{(7)}$, $e_{(5,1^2)}$, $e_{(3^2,1)}$ as for all other cases we have $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \mathfrak{z}(\mathfrak{g}^e) = \langle e \rangle$.

For $e = e_{(7)}, e_{(5,1^2)}, e_{(3^2,1)}$, the results can be obtained via [18, Proposition 4.2] since all calculation take place in $\mathfrak{so}(7)$. In the remaining part of this subsection, we include details for the case $e = e_{(3^2,1)}$ as an example. When $e = e_{(3^2,1)}$, recall that $\mathfrak{z}(\mathfrak{g}^e) = \langle e_{(3^2,1)}, R_{e_1,e_2} \rangle$. According to [10, Theorem 6.3.5], there exists a homomorphism $\mathrm{id} \times \pi : G \to \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SO}_7(\mathbb{C})$ with $\ker \pi = \{1\} \times \{\pm 1\}$. Now let us denote $K = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SO}_7(\mathbb{C})$. We know that $\ker \pi$ acts trivially on $\mathfrak{g}_{\bar{0}}$, thus we obtain an induced action of K on $\mathfrak{g}_{\bar{0}}$. Let $\mathfrak{z}(\mathfrak{g}^e)_{\bar{0}} = \mathfrak{z}(\mathfrak{g}^e) \cap \mathfrak{g}_{\bar{0}}$ and $\mathfrak{z}(\mathfrak{g}^e)_{\bar{1}} = \mathfrak{z}(\mathfrak{g}^e) \cap \mathfrak{g}_{\bar{1}}$, note that

$$(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = (\mathfrak{z}(\mathfrak{g}^e)_{\bar{0}} \oplus \mathfrak{z}(\mathfrak{g}^e)_{\bar{1}})^{G^e} = (\mathfrak{z}(\mathfrak{g}^e)_{\bar{0}})^{G^e} \oplus (\mathfrak{z}(\mathfrak{g}^e)_{\bar{1}})^{G^e}.$$

Furthermore, we have that $(\mathfrak{z}(\mathfrak{g}^e)_{\bar{0}})^{G^e} = (\mathfrak{z}(\mathfrak{g}^e)_{\bar{0}})^{K^e}$. Thus when $\mathfrak{z}(\mathfrak{g}^e)_{\bar{1}} = 0$, it suffices to look at $(\mathfrak{z}(\mathfrak{g}^e)_{\bar{0}})^{K^e}$. It is obvious that $e \subseteq (\mathfrak{z}(\mathfrak{g}^e)_{\bar{0}})^{K^e}$. We have that $K^e = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SO}_7(\mathbb{C})^e$ and $\mathrm{SO}_7(\mathbb{C})^e$ is the semidirect product of the subgroup C^e and the normal subgroup R^e according to [15, Section 3.12]. Furthermore, since $\mathrm{SL}_2(\mathbb{C})$ is connected, we deduce that $K^e/(K^e)^\circ \cong C^e/(C^e)^\circ$. Now we have that $C^e \cong (\mathrm{O}_1(\mathbb{C}) \times \mathrm{O}_2(\mathbb{C})) \cap \mathrm{SO}_7(\mathbb{C})$ where $\mathrm{O}_1(\mathbb{C})$ (resp. $\mathrm{O}_2(\mathbb{C})$) has the connected component $\mathrm{SO}_1(\mathbb{C})$ (resp. $\mathrm{SO}_2(\mathbb{C})$). Let us consider the element $g \in K^e/(K^e)^\circ$ given by

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We calculate that $g \cdot e_{(3^2,1)} = ge_{(3^2,1)}g^{-1} = e$ and $g \cdot R_{e_1,e_2} = gR_{e_1,e_2}g^{-1} = -R_{e_1,e_2}$. Hence, we have that $(\mathfrak{z}(\mathfrak{g}^e))^{K^e} \subseteq (\mathfrak{z}(\mathfrak{g}^e))^g = \langle e \rangle$ and we deduce that $(\mathfrak{z}(\mathfrak{g}^e))^{K^e} = \langle e \rangle$. Therefore, $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} = n_2(\varDelta) = 1$ and $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} - \dim (\mathfrak{z}(\mathfrak{g}_0^{e_0}))^{G_0^{e0}} = n_2(\varDelta) = 1$.

When $e = e_{(7)}, e_{(5,1^2)}$, using a similar analysis, we can verify that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \mathfrak{z}(\mathfrak{g}^e) = \langle e, R_{e_1,e_2} \rangle$.

Next we consider the case $e = E+e_{(7)}$, recall that $\mathfrak{z}(\mathfrak{g}^e) = \langle E+e_{(7)}, v_1 \otimes e_1e_2e_3, R_{e_1,e_2} \rangle$. We know that $G^e = \left(\{\pm 1\} \ltimes R^E \right) \times \mathrm{Spin}_7(\mathbb{C})^{e_{(7)}}$ where R^E is a connected normal subgroup of G^e. Now we take $g = -1 \in \mathrm{SL}_2(\mathbb{C})$ such that $g \in G^e$ and $g \notin (G^e)^\circ$. We know that g acts trivially on $\mathfrak{z}(\mathfrak{g}^e)_{\bar{0}}$, thus $e, R_{e_1,e_2} \in (\mathfrak{z}(\mathfrak{g}^e))^g$. However, $v_1 \otimes e_1e_2e_3 \notin \mathfrak{z}(\mathfrak{g}^e)^g$ since the action of g on $v_1 \otimes e_1e_2e_3$ sends it to $-v_1 \otimes e_1e_2e_3$. Hence, we have that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} \subseteq (\mathfrak{z}(\mathfrak{g}^e))^g = \langle e, R_{e_1,e_2} \rangle$. Therefore, we deduce that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \langle e, R_{e_1,e_2} \rangle$ and $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} - \dim (\mathfrak{z}(\mathfrak{g}_0^{e_0}))^{G_0^{e0}} = n_2(\varDelta) = 1$.

Using similar arguments we obtain that $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \langle e, R_{e_1,e_2} \rangle$ for $e = E + e_{(5,1^2)}$ and $(\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \langle e \rangle$ for $e = E + e_{(3^2,1)}$.

The above argument completes the proof of Theorems 1.1 and 1.3 for $F(4)$.

By combining results in Subsection 6.4.7, we have that $\dim (\mathfrak{z}(\mathfrak{g}^e))^{G^e} = \left\lceil \frac{1}{2} \sum_{i=1}^4 a_i \right\rceil + \varepsilon$ where $\varepsilon = -1$ for $e = E+e_{(7)}$ and $\varepsilon = 0$ for all other cases. This proves the statement of Theorem 1.2 for $F(4)$.

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