The $(\mathfrak{S}_3, \mathfrak{A}_n)$- and $(\mathfrak{S}_3, \mathfrak{S}_n)$-amalgams of characteristic 2 and critical distance 3

by

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Abstract

In this thesis a new characterisation of the $(\mathfrak{S}_3, \mathfrak{A}_n)$- and $(\mathfrak{S}_3, \mathfrak{S}_n)$-amalgams of characteristic 2 and critical distance 3 is obtained. It is shown that such an amalgam exists only in the exceptional cases when $n = 3$, 5 or 8. Let $\mathfrak{X}$ denote the class of all finite groups X of even order and such that $\mathrm{O}^2(X)$ is the unique minimal normal subgroup of X. It is the secondary purpose of this thesis to begin an investigation into the structure of the $(\mathfrak{S}_3, \mathfrak{X})$-amalgams of characteristic 2 and critical distance 3. It is hoped that the results obtained may shed some light on the reason why so few of these amalgams are known to exist.

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Contents

Notational conventions

In this thesis $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denotes the set of non-negative integers and $\mathbb{P}$ denotes the set of prime numbers.

The group and module theoretic notation in this thesis is standard and we ask the reader to consult [2, List of symbols] with the following caveats. We make the conventions that all p-groups are finite and all modules are finite-dimensional. We will use 1 [0] to denote both the trivial [zero] element of the group G [module V] and the trivial subgroup of G [zero submodule of V] depending on the context in which it occurs and $G^\#$ [$V^\#$] denotes the set of all non-trivial [non-zero] elements of G [V]. Let G be a finite group and $p \in \mathbb{P}$. Then $\mathrm{O}_p(G)$ is the largest normal p-subgroup of G, $\mathrm{O}^p(G)$ is the smallest normal subgroup of G with p-power index and $\Omega\,\mathrm{Z}(G) = \{z \in \mathrm{Z}(G) \mid z^p = 1\}$ is a subgroup of G. Let H and K be subgroups of G. We define the group $H_K = \bigcap_{k \in K} H^k$ and if A is a non-empty subset of G, then A^K denotes the set of all K-conjugates of A so that $\langle A^K \rangle = \langle A^k \mid k \in K \rangle$. Hence, H_G and $\langle H^G \rangle$ are the core and normal closure of H in G respectively. Finally, V_G and V^G denote the restricted and induced modules of the module V to G respectively and $\mathbb{F}_p$ denotes the finite field of order p.

We will now describe our notation with regard to the direct and wreath products of permutation groups. Let $H_1, H_2, \ldots, H_n$ be permutation groups on the pairwise disjoint sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ respectively and define the permutation group $H = \langle H_1, H_2, \ldots, H_n \rangle$ on the set $\Delta = \bigcup_{i=1}^n \Delta_i$. Then H is called the *direct product* of the permutation groups $H_1, H_2, \ldots, H_n$ and we write $H = H_1 \times H_2 \times \cdots \times H_n$. If K is a permutation group and $n \in \mathbb{N}_0$, then the n^{th}-*direct power* of K is the permutation group

$$K^{(n)} = \underbrace{K \times K \times \cdots \times K}_{n\text{-factors}}$$

where $K^{(0)}$ is a trivial permutation group of degree one. Now, let K and H be permutation groups on the disjoint sets Δ and Ω respectively and define the permutation group $W =$

$\langle B, \widehat{H} \rangle$ on the set $\Delta \cup \Omega$ where $B = K^{(|\Omega|)}$ and $\widehat{H}$ is isomorphic to H and the elements of $\widehat{H}$ permute the direct factors of B by conjugation in the same way that the corresponding elements of H permute the points of Ω. Then W is called the *wreath product* of K by H with *base group* B.

The notation for the finite groups in this thesis follows the ATLAS OF FINITE GROUPS [5] with the following exceptions. Let $\mathfrak{A}_n$ and $\mathfrak{S}_n$ denote the alternating and symmetric groups of degree n respectively. Let $\mathfrak{C}_n$ denote the cyclic group of order n, $\mathfrak{D}_{2n}$ denote the dihedral group of order $2n$ and $\mathfrak{K}_{2^n}$ denote the elementary abelian group of order 2^n. When in the context of permutation groups we make the conventions that $\mathfrak{C}_n$ and $\mathfrak{K}_{2^n}$ are regular and that $\mathfrak{D}_{2n}$ has degree n.

Chapter 1

Introduction and background

The main theorem that we will prove in this thesis is the following.

Theorem A *Let G be a group generated by a pair P_1, P_2 of proper finite subgroups with $P_1 \cap P_2 = B$. Set $Q_i = \mathrm{O}_2(P_i)$ for $i \in \{1,2\}$ and assume that*

$$P_1/Q_1 \cong \mathfrak{S}_3 \qquad \text{and} \qquad P_2/Q_2 \cong \mathfrak{A}_m \text{ or } \mathfrak{S}_n$$

for some $m, n \in \mathbb{N}$ with $m \neq 8$, together with the following conditions:

(i) no non-trivial subgroup of B is normal in both P_1 and P_2;

(ii) $B \in \mathrm{Syl}_2(P_1) \cap \mathrm{Syl}_2(P_2)$;

(iii) $\mathrm{C}_{P_i}(Q_i) \leq Q_i$ for $i \in \{1,2\}$;

(iv) $Z = \Omega\,\mathrm{Z}(B) \trianglelefteq P_2$; and

(v) $V = \langle \langle Z^{P_1} \rangle^{P_2} \rangle$ is abelian and $\langle V^{P_1} \rangle$ is non-abelian.

Then one of the following two cases hold:

(I) $P_2/Q_2 \cong \mathfrak{S}_3$ or $\mathfrak{S}_5$ and all of the possible shapes of P_1 and P_2 together with an example of a group G in which each configuration exists are given in Table A below.

(II) $P_2/Q_2 \cong \mathfrak{A}_5$ or $\mathfrak{S}_8$, $B = Q_1Q_2$, $|Z| = 2$ and V/Z is isomorphic to either the natural module or a spin module for P_2/Q_2 over $\mathbb{F}_2$.

P_2/Q_2	Shape of P_1	Shape of P_2	Example G
$\mathfrak{S}_3$	$2^{2+2+1}.\mathfrak{S}_3$	$2^{1+2+2}.\mathfrak{S}_3$	M_{12}
$\mathfrak{S}_3$	$2^{2+2+1+1}.\mathfrak{S}_3$	$2^{1+2+2+1}.\mathfrak{S}_3$	$\mathrm{Aut}(\mathrm{M}_{12})$
$\mathfrak{S}_5$	$2^{2+1+2+1+1+2+2+2}.\mathfrak{S}_3$	$2^{1+\bar{4}+1+1+\bar{4}}.\mathfrak{S}_5$	Ru

Table A All of the possible shapes of the groups P_1 and P_2 in Case (I) of Theorem A.

In this thesis we will primarily be concerned with proving that there does not exist a group G which satisfies the hypothesis of Theorem A in the generic $m, n \geq 9$ case. The groups $\mathfrak{A}_5$, $\mathfrak{A}_8$ and $\mathfrak{S}_8$ remain unresolved in Theorem A because they require a radically different approach to the generic case. Perhaps one reason for the exceptional behaviour of these groups follows from the exceptional isomorphisms $\mathfrak{A}_5 \cong \mathrm{SL}_2(4)$, $\mathfrak{A}_8 \cong \mathrm{SL}_4(2)$ and $\mathfrak{S}_8 \cong \mathrm{O}_6^+(2)$ and hence they may be better placed within a study of critical distance 3 amalgams with P_2/Q_2 belonging to the infinite families of linear or orthogonal groups. In the case when $P_2/Q_2 \cong \mathfrak{A}_8$ we will show that Case (II) of Theorem A holds under the additional assumption $|Z| = 2$. In fact, the sporadic simple group $G = \mathrm{Co}_1$ satisfies the hypothesis of Theorem A with $P_2/Q_2 \cong \mathfrak{A}_8$ where P_2 has shape $2^{1+4+6+\bar{4}}.\mathfrak{A}_8$.

This introduction includes a detailed discussion of the conditions and terminology used in the statement of Theorem A. We will now introduce the necessary background material that we will need. The results in this chapter will be used without reference in this thesis and the proofs are left to the reader.

G-sets

Let G be a group and Ω be a non-empty set. We say that G *acts on* Ω (*as a set*) if there is a map $\Omega \times G \longrightarrow \Omega : (\omega, g) \longmapsto \omega \cdot g$ such that, for all $\omega \in \Omega$ and $g, h \in G$,

(i) $\omega \cdot 1 = \omega$.

(ii) $(\omega \cdot g) \cdot h = \omega \cdot gh$.

In this case, Ω is called a *G-set.* A *permutation group* on Ω is a group that acts faithfully on Ω. Let $\omega \cdot G$ denote the G-orbit of a point ω of Ω and G_ω (or $\mathrm{Stab}_G(\omega)$) denote the stabilizer in G of ω. Let Δ be a non-empty subset of Ω. The *(pointwise) stabilizer* in G of Δ is the subgroup of G defined by

$$G_\Delta = \bigcap_{\delta \in \Delta} G_\delta.$$

If Δ is finite, $\Delta = \{\delta_1, \delta_2, \ldots, \delta_r\}$ say, then we will also write $G_{\delta_1 \delta_2 \ldots \delta_r}$ in place of G_Δ. If $\omega \in \Omega$ and $g \in G$, then $G_\omega^g = G_{\omega \cdot g}$ and so $G_\Delta^g = G_{\Delta \cdot g}$.

G-graphs

A *(simple) graph* $\Gamma = (\Gamma, *)$ consists of a non-empty set Γ of *vertices* together with an irreflexive symmetric relation $*$ on Γ called *adjacency.* An *edge* $\{\gamma, \delta\}$ of Γ is an unordered pair of adjacent vertices and the *edge-set* E_Γ is the set of all edges of Γ. The *valency* $\mathrm{val}(\gamma)$ of a vertex γ is the number of vertices adjacent to γ and the graph Γ is *locally finite* if every vertex of Γ has finite valency. The graph Γ is *bipartite* if it can be partitioned into two *vertex-parts* Γ_1, Γ_2 such that each edge has one end in Γ_1 and the other in Γ_2. In a connected graph Γ the map $\mathrm{d} : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$ denotes the usual distance metric on Γ.

Let G be a group. We say that G *acts on* Γ (*as a graph*) if G acts on Γ as a set and the action *preserves adjacency*, that is, for all $\gamma, \delta \in \Gamma$ and $g \in G$,

$$\gamma * \delta \;\Leftrightarrow\; \gamma \cdot g * \delta \cdot g.$$

In this case, Γ is called a *G-graph.* An *automorphism group* on Γ is a group that acts faithfully on Γ. The induced action of G on the edge-set of Γ is defined by the rule: for $\{\gamma, \delta\} \in E_\Gamma$ and $g \in G$,

$$\{\gamma, \delta\} \cdot g = \{\gamma \cdot g, \delta \cdot g\} \in E_\Gamma.$$

We say that G acts *vertex-transitively* on Γ if G acts transitively on Γ and G acts *edge-transitively* on Γ if G acts transitively on the edge-set of Γ. Let γ be a vertex of Γ. Then the *adjacency-set* $\Gamma(\gamma)$ of γ is the set of all vertices adjacent to γ and the *neighbourhood* $\Delta(\gamma)$ of γ is the set $\Gamma(\gamma) \cup \{\gamma\}$. The vertex-stabilizer G_γ acts on $\Gamma(\gamma)$ and $\Delta(\gamma)$ as sets because, for all $g \in G$,

$$\Gamma(\gamma \cdot g) = \Gamma(\gamma) \cdot g \quad \text{and} \quad \Delta(\gamma \cdot g) = \Delta(\gamma) \cdot g.$$

Let Γ be a connected G-graph and $k \in \mathbb{N}_0$. Then the *k-neighbourhood* $\Delta_k(\gamma)$ of γ is the set of all vertices of distance at most k from γ

$$\Delta_k(\gamma) = \{\delta \in \Gamma \mid \mathrm{d}(\delta, \gamma) \le k\}.$$

We have that $\Delta_0(\gamma) = \{\gamma\}$, $\Delta_1(\gamma) = \Delta(\gamma)$ and G_γ acts on $\Delta_k(\gamma)$ because, for all $g \in G$,

$$\Delta_k(\gamma \cdot g) = \Delta_k(\gamma) \cdot g.$$

The distance metric has the property $\mathrm{d}(\gamma \cdot g, \delta \cdot g) = \mathrm{d}(\gamma, \delta)$ for all $\gamma, \delta \in \Gamma$ and $g \in G$. The G-graphs Γ and Λ are *G-isomorphic*, written $\Gamma \cong_G \Lambda$, if there exists a bijection $\theta : \Gamma \longrightarrow \Lambda$ such that $\gamma * \delta \Leftrightarrow \gamma\theta * \delta\theta$ and $(\gamma \cdot g)\theta = (\gamma\theta) \cdot g$ for all $\gamma, \delta \in \Gamma$ and $g \in G$.

G-groups

Let G and X be groups. We say that G *acts on* X (*as a group*) if G acts on X as a set where x^g is written in place of $x \cdot g$, for $x \in X$ and $g \in G$, and the action *preserves the group operation*, that is, for all $x, y \in X$ and $g \in G$,

$$(xy)^g = x^g y^g.$$

In this case, X is called a *G-group*. A subgroup Y of X is a *G-subgroup* of X, written $Y \leq_G X$ (and $Y \trianglelefteq_G X$ when Y is normal in X), if Y is itself a G-group with the action of G on X restricted to Y. If N is a normal G-subgroup of X, then the induced action of G on the quotient group X/N is defined by the rule: for $Nx \in X/N$ and $g \in G$,

$$(Nx)^g = Nx^g \in X/N.$$

Let W be a G-group. A *G-homomorphism* from X to W is a homomorphism $\theta : X \longrightarrow W$ such that $(x^g)^\theta = (x^\theta)^g$ for all $x \in X$ and $g \in G$. A *G-isomorphism* is a bijective G-homomorphism and we say that X and W are *G-isomorphic*, written $X \cong_G W$, if there is a G-isomorphism from X to W.

Lemma (*G-Isomorphism Theorems*) *The following hold:*

(i) If $\theta : X \longrightarrow W$ is a G-homomorphism, then $X/\ker\theta \cong_G \mathrm{im}\,\theta$.

(ii) If $Y \leq_G X$ and $N \trianglelefteq_G X$, then $YN/N \cong_G Y/(Y \cap N)$.

(iii) If $N, M \trianglelefteq_G X$ with $N \leq M$, then $(X/N)/(M/N) \cong_G X/M$.

Throughout this thesis we will frequently make implicit use of the following construction. If K is a normal subgroup of G with $K \leq \mathrm{C}_G(X)$, then the induced action of G/K on the G-group X is defined by the rule: for $x \in X$ and $Kg \in G/K$,

$$x^{Kg} = x^g \in X.$$

Let U be a non-empty subset of X and A be a non-empty subset of G. The *centralizer* in X of A is the subgroup of X defined by

$$\mathrm{C}_X(A) = \{x \in X \mid x^a = x \text{ for all } a \in A\}.$$

The *commutator* of $x \in X$ and $g \in G$ is the element $[x, g] = x^{-1}x^g \in X$. The *commutator subgroup* of U and A is the subgroup of X defined by

$$[U, A] = \langle [u, a] \mid u \in U, a \in A \rangle.$$

The centralizer and commutator subgroup have the following properties:

(i) $\mathrm{C}_X(A)^g = \mathrm{C}_X(A^g)$ and $[U, A]^g = [U^g, A^g]$ for all $g \in G$.

(ii) If $A_1, A_2, \ldots, A_n$ are non-empty subsets of G, then

$$\mathrm{C}_X(\langle A_1, A_2, \ldots, A_n \rangle) = \mathrm{C}_X(A_1) \cap \mathrm{C}_X(A_2) \cap \ldots \cap \mathrm{C}_X(A_n)$$
$$[X, \langle A_1, A_2, \ldots, A_n \rangle] = \langle [X, A_1], [X, A_2], \ldots, [X, A_n] \rangle.$$

(iii) $|X : \mathrm{C}_X(g)| \leq |[X, g]|$ for all $g \in G$. Moreover, if $g \in G$ with $[X, g] \leq \mathrm{Z}(X)$, then $X/\mathrm{C}_X(g) \cong_{\mathrm{C}_G(g)} [X, g]$.

If Y is a subgroup of X and N is a normal G-subgroup of X, then $[YN/N, A] = [Y, A]N/N$. Let $[U, A; 0] = U$ and define the subgroup $[U, A; n]$ of X iteratively by the following rule $[U, A; n] = [[U, A; n-1], A]$ for $n \geq 1$.

Let X be a finite group. A normal G-series $(X_i)_{i=0}^r$ of X is a finite sequence of normal G-subgroups of X with

$$1 = X_0 \trianglelefteq X_1 \trianglelefteq X_2 \trianglelefteq \ldots \trianglelefteq X_{r-1} \trianglelefteq X_r = X.$$

A *chief G-series* of X is a minimal element of the set of all proper normal G-series of X with respect to refinement. As X is a finite group, X has a chief G-series and, by the Jordan-Hölder Theorem for G-groups, any two chief G-series of X are equivalent up to G-isomorphism of the corresponding factors. Thus, the factors of a chief G-series of X are independent of the G-series and so they form a set of invariants of X called the *chief G-factors* of X. A factor $\overline{X}_i$ of a chief G-series $(X_i)_{i=0}^r$ of X is *central* if it is centralized by G and it is *non-central* otherwise. Let $\eta_G(X)$ denote the number of non-central chief G-factors of X. There is a one-to-one correspondence between elementary abelian p, G-groups and $\mathbb{F}_pG$-modules. If V is an $\mathbb{F}_pG$-module, then $\eta_G(V)$ equals the number of non-trivial composition factors of V.

Fix $p \in \mathbb{P}$ and let G be a finite group, X be a non-trivial p-group and N be a normal G-subgroup of X. Let $(X_i)_{i=0}^r$ be a chief G-series of X and $1 \leq i \leq r$. Then the factor $\overline{X}_i$ may be regarded as an irreducible $\mathbb{F}_p(G/Q)$-module where $Q = \mathrm{O}_p(G)$.

Lemma *The following results hold:*

(i) If G is a p-group, then $\mathrm{C}_X(G) \neq 1$ and $[X, G] < X$.

(ii) $[X, \mathrm{O}^p(G), \mathrm{O}^p(G)] = [X, \mathrm{O}^p(G)]$.

(iii) $\eta_G(X/N) = \eta_G(X) - \eta_G(N)$. Moreover, $\eta_G(X/N) = 0$ if and only if $[X, \mathrm{O}^p(G)] \leq N$. In particular, $\eta_G(X) = 0$ if and only if $[X, \mathrm{O}^p(G)] = 1$.

(iv) (Burnside's Lemma) If $\eta_G(X) \geq 1$, then $\eta_G(X/\Phi(X)) \geq 1$. In particular, if $\eta_G(X) = 1$, then $\eta_G(\Phi(X)) = 0$.

Now, let G be any group, K be a subgroup of G and N be a normal subgroup of G. Then K acts on N *by conjugation*, that is, for $n \in N$ and $k \in K$,

$$n^k = k^{-1}nk \in N.$$

In this case, we say that N is a K-group *with respect to action by conjugation.* Throughout this thesis action by conjugation will be implicit whenever we have the above set-up. Let H and L be subgroups of G. The following two properties hold:

(i) If $\emptyset \neq A \subseteq G$, then $\langle [N, A]^K \rangle = [N, \langle A^K \rangle]$.

(ii) If K normalizes H and L, then $[HK, L] = [H, L]\,[K, L] = [KH, L]$.

Lemma (*Three-Subgroup Lemma*) *The following hold:*

(i) If $[H, K, L] = [K, L, H] = 1$*, then* $[L, H, K] = 1$.

(ii) If $[H, K, L] \leq N$ *and* $[K, L, H] \leq N$*, then* $[L, H, K] \leq N$.

(iii) Let $H, K, L \trianglelefteq G$*. If* $[H, K, L] = 1$*, then* $[K, L, H] = [L, H, K]$.

In the semidirect product of X by G, G acts on X by conjugation. In particular, the centralizer, commutator and commutator subgroup take their usual meaning. We refer the reader to [19, Chapter 8] for further details relating to G-groups.

Amalgams of rank 2

Proofs of many of the results in this subsection may be found in [19, Section 10.3]. The following definition of an amalgam of rank 2 comes from [8, page 61].

Let G be a group generated by a pair P_1, P_2 of proper subgroups with $P_1 \cap P_2 = B$ that satisfy the following condition:
($\mathcal{A}1$) no non-trivial subgroup of B is normal in both P_1 and P_2.

In this case, $\mathcal{A}(G, P_1, P_2, B)$ is called an *amalgam (of rank 2)* and we say that G is an *amalgam* of the subgroups P_1 and P_2 over B. The subgroups P_1 and P_2 are called the *parabolic subgroups* and B is called the *Borel subgroup* of the amalgam.

The *(right) coset graph* $\Gamma = \Gamma(G, P_1, P_2)$ of G with respect to P_1 and P_2 is the graph with vertex-set consisting of the right cosets of P_1 and P_2 in G and two vertices $P_i x$ and $P_j y$ are adjacent if $i \neq j$ and $P_i x \cap P_j y \neq \emptyset$. The group G acts on Γ by right multiplication: for $P_i x \in \Gamma$ and $g \in G$,

$$P_i x \cdot g = P_i x g \in \Gamma.$$

The coset graph Γ is a connected bipartite graph with vertex-parts $\mathcal{O}_1, \mathcal{O}_2$ where $\mathcal{O}_1 = P_1 \cdot G$ and $\mathcal{O}_2 = P_2 \cdot G$ are the G-orbits of Γ. The group G is an automorphism group on Γ because the kernel of the action is the core of B in G which is trivial by ($\mathcal{A}1$). Moreover, G acts edge- but not vertex-transitively on Γ. Let $\gamma \in \Gamma$ and $\delta \in \Gamma(\gamma)$. The following two properties of Γ hold:

(i) The vertex-stabilizer G_γ acts transitively on the adjacency-set $\Gamma(\gamma)$.

(ii) There exists $g \in G$ such that $\{G_\gamma, G_\delta\} = \{P_1^g, P_2^g\}$ and $G_{\{\gamma,\delta\}} = G_{\gamma\delta} = B^g$. In particular, the vertex-stabilizers are G-conjugate to either P_1 or P_2 and the edge-stabilizers are G-conjugate to B.

By (ii) above, $\mathcal{A}(G, G_\gamma, G_\delta, G_{\gamma\delta})$ is an amalgam and hence the next result follows from ($\mathcal{A}1$) and the connectivity of Γ.

Lemma ([19, 10.3.3 on page 253]) *If K is a subgroup of $G_{\gamma\delta}$ with*

(i) $K \trianglelefteq G_\gamma$ and $K \trianglelefteq G_\delta$; or

(ii) $\mathrm{N}_{G_\rho}(K)$ acts transitively on $\Gamma(\rho)$ for $\rho \in \{\gamma, \delta\}$,

then $K = 1$.

On the one hand, G is an automorphism group on the connected graph Γ that acts edge- but not vertex-transitively on Γ. On the other hand, we have the lemma below which establishes a one-to-one correspondence between groups that are amalgams and automorphism groups of connected graphs that act edge- but not vertex-transitively.

Lemma ([8, 3.3 on page 72]) *Let Λ be a connected graph and H be an automorphism group on Λ that acts edge- but not vertex-transitively on Λ. Fix $\{\lambda, \mu\} \in E_\Lambda$. Then $\mathcal{A}(H, H_\lambda, H_\mu, H_{\lambda\mu})$ is an amalgam with corresponding coset graph $\Gamma(H, H_\lambda, H_\mu) \cong_H \Lambda$.*

The amalgam method

The following definition is the author's own and is motivated by the hypothesis first considered in [8, Part II, Section 3] and subsequently in many papers.

Fix $p \in \mathbb{P}$ and let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be fixed classes of finite groups X of order divisible by p and such that no non-trivial normal p-subgroup of X exists[1]. Let $\mathcal{A} = \mathcal{A}(G, P_1, P_2, B)$ be an amalgam with P_1 and P_2 finite groups. Set $Q_i = \mathrm{O}_p(P_i)$ for $i \in \{1, 2\}$ and assume

[1]This condition is necessary because if $P_i/Q_i \cong X \in \mathfrak{X}_i$ for $i \in \{1, 2\}$, then, as Q_i is a p-group, $\mathrm{O}_p(X) \cong \mathrm{O}_p(P_i/Q_i) = \mathrm{O}_p(P_i)/Q_i = Q_i/Q_i$.

that the following conditions hold:

($\mathcal{A}2$) $B \in \mathrm{Syl}_p(P_1) \cap \mathrm{Syl}_p(P_2)$;

($\mathcal{A}3$) $\mathrm{C}_{P_i}(Q_i) \le Q_i$ for $i \in \{1, 2\}$; and

($\mathcal{A}4$) $P_1/Q_1 \in \mathfrak{X}_1$ and $P_2/Q_2 \in \mathfrak{X}_2$.

In this case, $\mathcal{A}$ is called a $(\mathfrak{X}_1, \mathfrak{X}_2)$-*amalgam of characteristic* p. The coset graph $\Gamma = \Gamma(G, P_1, P_2)$ is a locally finite graph with, for $i \in \{1, 2\}$ and $\gamma_i \in \mathcal{O}_i$,

$$\mathrm{val}(\gamma_i) = |\Gamma(\gamma_i)| = |P_i\!:\!B| = |P_i/Q_i\!:\!B/Q_i|$$

and, by ($\mathcal{A}2$), B/Q_i is a Sylow p-subgroup of P_i/Q_i. If $\alpha \in \mathcal{O}_1$ and $\beta \in \Gamma(\alpha)$, then $\mathcal{A}(G, G_\alpha, G_\beta, G_{\alpha\beta})$ is a $(\mathfrak{X}_1, \mathfrak{X}_2)$-amalgam of characteristic p.

Observe that the parabolic subgroup P_i is an extension of Q_i by P_i/Q_i. We will use the following notation to describe the structure of the group P_i in terms of this extension. Let us denote each of the irreducible $\mathbb{F}_p(P_i/Q_i)$-modules by their dimension d and use $\overline{d}$, $\overline{\overline{d}}$, ... when more than one module of dimension d exists. We say that P_i has *shape* $p^{d_1+d_2+\cdots+d_r}.X$ and write $P_i \sim p^{d_1+d_2+\cdots+d_r}.X$ to indicate that $P_i/Q_i \cong X$ and, regarding Q_i as a P_i-group with respect to action by conjugation, Q_i has a chief P_i-series $(Q_i^j)_{j=0}^r$ with each $\overline{Q_i^j}$ isomorphic to the $\mathbb{F}_p(P_i/Q_i)$-module d_j. In particular, Q_i has order $p^{d_1+d_2+\cdots+d_r}$.

The aim of the *amalgam method* is to determine all of the possible shapes of the parabolic subgroups P_1 and P_2 and in this way classify all $(\mathfrak{X}_1, \mathfrak{X}_2)$-amalgams of characteristic p. We will use the structure of the quotient groups P_1/Q_1 and P_2/Q_2, specified by the classes $\mathfrak{X}_1$ and $\mathfrak{X}_2$, in order to attempt to construct chief series' of the groups Q_1 and Q_2 in terms of certain geometrically defined subgroups. In a general amalgam problem the analysis is initially divided into cases which make assumptions about the structure of the amalgam. Each case is then analysed and subdivided if necessary. This process of 'divide and conquer' is continued until either a chief P_1-series of the group Q_1 and a chief P_2-series of the group Q_2 is constructed or a contradiction is found. We will make an assumption about the structure of the amalgam at the outset so that we may focus our attention on a particular case of the whole amalgam problem.

We will use the coset graph to analyse the structure of the G-conjugates of Q_1 and Q_2. We have that $\{Q_1^g, Q_2^g \mid g \in G\} = \{\mathrm{O}_p(G_\gamma) \mid \gamma \in \Gamma\}$ and so we will analyse the structure of the p-radical of the vertex-stabilizers. During this analysis we will make use of certain normal p-subgroups K_γ of the vertex-stabilizer G_γ with the property $K_\gamma^g = K_{\gamma \cdot g}$ for all $g \in G$. The coset graph provides a geometric framework in which to keep track of the interaction between these subgroups. Fix $\gamma \in \Gamma$ and $k \in \mathbb{N}_0$. The basic subgroups of G are defined as follows

$$Z_\gamma = \langle \Omega\,\mathrm{Z}(G_{\gamma\delta}) \mid \delta \in \Gamma(\gamma) \rangle \qquad Z_\gamma^{[k]} = \langle Z_\delta \mid \delta \in \Delta_k(\gamma) \rangle$$

$$Q_\gamma = \bigcap_{\delta \in \Gamma(\gamma)} G_{\gamma\delta} \qquad Q_\gamma^{[k]} = \bigcap_{\delta \in \Delta_k(\gamma)} Q_\delta.$$

These subgroups have the following properties:

(i) $Z_\gamma = Z_\gamma^{[0]} \leq Z_\gamma^{[1]} \leq Z_\gamma^{[2]} \leq \dots$ and $Q_\gamma = Q_\gamma^{[0]} \geq Q_\gamma^{[1]} \geq Q_\gamma^{[2]} \dots$.

(ii) $(Z_\gamma^{[k]})^g = Z_{\gamma \cdot g}^{[k]}$ and $(Q_\gamma^{[k]})^g = Q_{\gamma \cdot g}^{[k]}$ for all $g \in G$.

(iii) If $k \geq 1$, then $Z_\gamma^{[k]} = \langle Z_\delta^{[k-1]} \mid \delta \in \Gamma(\gamma) \rangle$ and $Q_\gamma^{[k]} = \bigcap_{\delta \in \Gamma(\gamma)} Q_\delta^{[k-1]}$.

(iv) $Q_\gamma^{[k]} = G_{\Delta_{k+1}(\gamma)}$. In particular, $Q_\gamma = G_{\Delta(\gamma)}$.

By ($\mathcal{A}2$), $\mathrm{Syl}_p(G_\gamma) = \{G_{\gamma\delta} \mid \delta \in \Gamma(\gamma)\}$ and hence $Q_\gamma = \mathrm{O}_p(G_\gamma)$. We also have that $\mathrm{Syl}_p(G_\gamma/Q_\gamma) = \{G_{\gamma\delta}/Q_\gamma \mid \delta \in \Gamma(\gamma)\}$ and $\mathrm{N}_{G_\gamma/Q_\gamma}(G_{\gamma\lambda}/Q_\gamma) = \mathrm{N}_{G_\gamma}(G_{\gamma\lambda})/Q_\gamma$ for any $\lambda \in \Gamma(\gamma)$ giving $|\mathrm{Syl}_p(G_\gamma/Q_\gamma)| = |\mathrm{Syl}_p(G_\gamma)|$. By ($\mathcal{A}3$), $\mathrm{Z}(G_{\gamma\delta}) \leq \mathrm{C}_{G_\gamma}(Q_\gamma) = \mathrm{Z}(Q_\gamma)$ for all $\delta \in \Gamma(\gamma)$ and so $Z_\gamma \leq \Omega\,\mathrm{Z}(Q_\gamma)$. In particular, Z_γ may be regarded as an $\mathbb{F}_p(G_\gamma/Q_\gamma)$-module with respect to action by conjugation. In order to use the subgroups $Z_\gamma^{[k]}$ as the terms of a normal G_γ-series of Q_γ we need to know how many of these subgroups are contained in Q_γ which leads to the next concept. The *critical distance* b is the minimum distance over all pairs of vertices (γ, δ) with $Z_\gamma \not\leq Q_\delta$. A pair (α, β') of vertices with $Z_\alpha \not\leq Q_{\beta'}$ and $\mathrm{d}(\alpha, \beta') = b$ is called a *critical pair* and a path from α to β' is called a *critical path*. The critical distance is a well-defined natural number because we may choose $z \in Z_\gamma^\#$ and, as G acts faithfully on Γ, there exists $\delta \in \Gamma$ such that $z \notin G_\delta$ giving $Z_\gamma \not\leq Q_\delta$. If $\mathrm{d}(\gamma, \delta) < b - k - l$ for $\delta \in \Gamma$ and $l \in \mathbb{N}_0 \cup \{-1\}$, then $Z_\gamma^{[k]}$ is a subgroup of

$Q_\delta^{[l]}$ where $Q_\delta^{[-1]} = G_\delta$. In particular, $Z_\gamma^{[k]}$ is a subgroup of $Q_\gamma^{[b-1-k]}$. Thus,

$$1 \trianglelefteq Z_\gamma \trianglelefteq Z_\gamma^{[1]} \trianglelefteq Z_\gamma^{[2]} \trianglelefteq \ldots \trianglelefteq Z_\gamma^{[b-1]} \trianglelefteq Q_\gamma$$

is a normal G_γ-series of Q_γ. Let $1 \le k \le b-1$. Then, as G_γ acts transitively on $\Gamma(\gamma)$, $\Gamma(\gamma) = \lambda \cdot G_\gamma$ for any $\lambda \in \Gamma(\gamma)$ and so

$$Z_\gamma = \langle \Omega\, \mathrm{Z}(G_{\gamma\lambda})^{G_\gamma} \rangle \qquad Z_\gamma^{[k]} = \langle (Z_\lambda^{[k-1]})^{G_\gamma} \rangle$$

$$Q_\gamma = (G_{\gamma\lambda})_{G_\gamma} \qquad Q_\gamma^{[k]} = (Q_\lambda^{[k-1]})_{G_\gamma}.$$

Define $E_\gamma = \mathrm{O}^p(G_\gamma)$. Then $E_\gamma^g = E_{\gamma\cdot g}$ for all $g \in G$. We have that $G_\gamma = G_{\gamma\lambda}E_\gamma$ for any $\lambda \in \Gamma(\gamma)$ because $G_{\gamma\lambda} \in \mathrm{Syl}_p(G_\gamma)$; and so, as G_γ acts transitively on $\Gamma(\gamma)$, E_γ acts transitively on $\Gamma(\gamma)$. Also, if $K \trianglelefteq G_{\gamma\lambda}$ for any $\lambda \in \Gamma(\gamma)$, then $K_{G_\gamma} = K_{E_\gamma}$ and

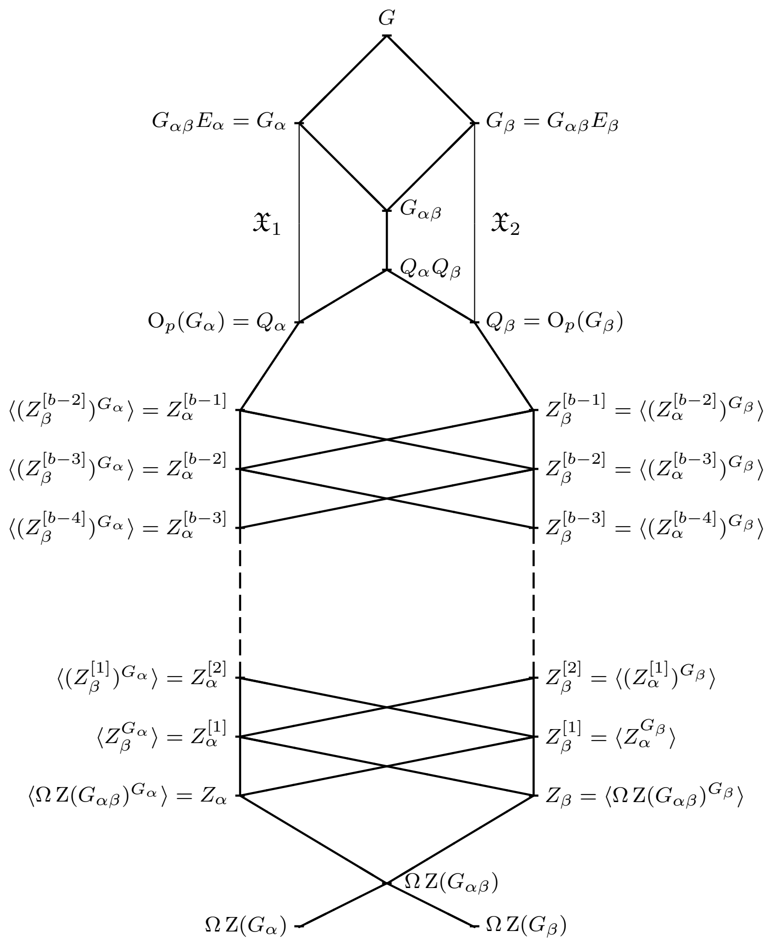


Figure A The subgroup structure of the group G where $\alpha \in \mathcal{O}_1$ and $\beta \in \Gamma(\alpha)$.

$\langle K^{G_\gamma}\rangle = \langle K^{E_\gamma}\rangle$. Hence,

$$Z_\gamma = \langle \Omega\, \mathrm{Z}(G_{\gamma\lambda})^{E_\gamma}\rangle \qquad Z_\gamma^{[k]} = \langle (Z_\lambda^{[k-1]})^{E_\gamma}\rangle$$
$$Q_\gamma = (G_{\gamma\lambda})_{E_\gamma} \qquad Q_\gamma^{[k]} = (Q_\lambda^{[k-1]})_{E_\gamma}$$

for any $\lambda \in \Gamma(\gamma)$.

Theorem A revisited

The theorem below follows from the recent classification of *symplectic amalgams* by Parker and Rowley [27, Theorems 1.10 and 1.11 on pages 12 and 13].

Theorem (Parker and Rowley) *Let $\mathfrak{X}$ denote the class of all finite groups X of even order and such that $\mathrm{O}^2(X)$ is the unique minimal normal subgroup of X. Assume that $\mathcal{A} = \mathcal{A}(G, P_1, P_2, B)$ is a $(\mathfrak{S}_3, \mathfrak{X})$-amalgam of characteristic 2 that satisfies the following conditions:*
(i) $Q_1 \cap Q_2 \ntrianglelefteq P_1$;
(ii) $Z = \Omega\, \mathrm{Z}(B) \trianglelefteq P_2$; and
(iii) $\langle Z^{P_1}\rangle \leq Q_2$ and $\langle\langle Z^{P_1}\rangle^{P_2}\rangle$ is non-abelian.

Then $\mathcal{A}$ is a symplectic amalgam and the shapes of P_1 and P_2 are known.

In Chapter 3 we will see that condition (iii) in the theorem above is equivalent to the critical distance $b = 2$. The authors of [27] were able to classify a large number of critical distance 2 amalgams under a considerably more general hypothesis than the one in the theorem above. Observe that the class $\mathfrak{X}$ contains all of the alternating and symmetric groups of finite degree, excluding $\mathfrak{A}_n$ for $n \in \{1, 2, 3, 4\}$ and $\mathfrak{S}_n$ for $n \in \{1, 2, 4\}$, and, by the Odd-Order Theorem [11], all of the non-abelian finite simple groups. The shapes of the parabolic subgroups in all of the symplectic amalgams with P_2/Q_2 an alternating or symmetric group are listed in Table B. The example .3* given in Table B refers to an amalgam discovered by Chermak [3]. We refer the reader to [27, pages 10–15 and Section 24.1] for more information about the amalgams in Table B.

P_2/Q_2	Shape of P_1	Shape of P_2	Example G	Class
$\mathfrak{S}_3$	$2^{2+2}.\mathfrak{S}_3$	$2^{1+1+2}.\mathfrak{S}_3$	$\mathrm{G}_2(2)'$	$\mathcal{A}_1$
$\mathfrak{S}_3$	$2^{2+1+2}.\mathfrak{S}_3$	$2^{1+1+1+2}.\mathfrak{S}_3$	$\mathrm{G}_2(2)$	$\mathcal{A}_2$
$\mathfrak{A}_5$	$2^{2+2+2}.\mathfrak{S}_3$	$2^{1+4}.\mathfrak{A}_5$	J_2	$\mathcal{A}_{41}$
$\mathfrak{A}_5$	$2^{2+2+2+1+1}.\mathfrak{S}_3$	$2^{1+4+1+1}.\mathfrak{A}_5$	$\mathrm{PSp}_6(3)$	$\mathcal{A}_{42}$
$\mathfrak{S}_5$	$2^{2+2+2+1}.\mathfrak{S}_3$	$2^{1+4}.\mathfrak{S}_5$	$\mathrm{Aut}(\mathrm{J}_2)$	$\mathcal{A}_{41}^1$
$\mathfrak{S}_5$	$2^{2+2+2+1+1+1}.\mathfrak{S}_3$	$2^{1+4+1+1}.\mathfrak{S}_5$	$\mathrm{Aut}(\mathrm{PSp}_6(3))$	$\mathcal{A}_{42}^1$
$\mathfrak{S}_5$	$2^{2+1+2+2+1}.\mathfrak{S}_3$	$2^{1+1+\overline{4}}.\mathfrak{S}_5$	HS	$\mathcal{A}_{43}$
$\mathfrak{S}_5$	$2^{2+1+2+2+1+1}.\mathfrak{S}_3$	$2^{1+1+\overline{4}+1}.\mathfrak{S}_5$	Aut(HS)	$\mathcal{A}_{44}$
$\mathfrak{A}_6$	$2^{2+1+2+2+1+1}.\mathfrak{S}_3$	$2^{1+1+4+1}.\mathfrak{A}_6$	$.3^*$	$\mathcal{A}_{35}$
$\mathfrak{S}_6$	$2^{2+1+2+2+1+1}.\mathfrak{S}_3$	$2^{1+1+4}.\mathfrak{S}_6$	Co_3	$\mathcal{A}_{36}$
$\mathfrak{A}_7$	$2^{2+1+2+2+1+1}.\mathfrak{S}_3$	$2^{1+6}.\mathfrak{A}_7$	$.3^*$	$\mathcal{A}_{47}$
$\mathfrak{A}_9$	$2^{2+1+1+1+2+2+2+1+1+1}.\mathfrak{S}_3$	$2^{1+\overline{8}}.\mathfrak{A}_9$	Th	$\mathcal{A}_{39}$
$\mathfrak{A}_9$	$2^{2+1+1+1+2+2+2+1+1+1}.\mathfrak{S}_3$	$2^{1+8}.\mathfrak{A}_9$	$\mathrm{F}_4(3)$	$\mathcal{A}_{40}$

Table B The shapes of P_1 and P_2 in the symplectic amalgams with $P_2/Q_2 \cong \mathfrak{A}_n$ or $\mathfrak{S}_n$.

We may restate Theorem A in the language developed above as follows.

Theorem A *Let $m, n \in \mathbb{N}$ with $m \neq 8$ and assume that $\mathcal{A}(G, P_1, P_2, B)$ is a $(\mathfrak{S}_3, \mathfrak{A}_m)$- or $(\mathfrak{S}_3, \mathfrak{S}_n)$-amalgam of characteristic 2 that satisfies the following conditions:*

(i) $Z = \Omega\,\mathrm{Z}(B) \trianglelefteq P_2$; and

(ii) $V = \langle\langle Z^{P_1}\rangle^{P_2}\rangle$ is abelian and $\langle V^{P_1}\rangle$ is non-abelian.

Then one of the following two cases hold:

(I) $P_2/Q_2 \cong \mathfrak{S}_3$ or $\mathfrak{S}_5$ and all of the possible shapes of P_1 and P_2 are given in Table A on page 1.

(II) $P_2/Q_2 \cong \mathfrak{A}_5$ or $\mathfrak{S}_8$, $B = Q_1Q_2$, $|Z| = 2$ and V/Z is isomorphic to either the natural module or a spin module for P_2/Q_2 over $\mathbb{F}_2$.

In Chapter 3 we will see that condition (ii) in the theorem above is equivalent to the critical distance $b = 3$ or 4. We may combine the above two theorems in the following manner.

Theorem B *Let $n \in \mathbb{N}$ and assume that $\mathcal{A}(G, P_1, P_2, B)$ is a $(\mathfrak{S}_3, \mathfrak{A}_n)$- or $(\mathfrak{S}_3, \mathfrak{S}_n)$-amalgam of characteristic 2 that satisfies the following conditions:*

(i) $B = Q_1Q_2$;

(ii) $Z = \Omega\, \mathrm{Z}(B) \trianglelefteq P_2$; and

(iii) $\langle Z^{P_1}\rangle \leq Q_2$ and $\langle\langle\langle Z^{P_1}\rangle^{P_2}\rangle^{P_1}\rangle$ is non-abelian.

Then the shapes of P_1 and P_2 are given in Tables A and B on pages 1 and 13 respectively except in the cases when $P_2/Q_2 \cong \mathfrak{A}_5, \mathfrak{A}_8$ or $\mathfrak{S}_8$ and the critical distance $b = 3$.

In Tables A and B we have used the following convention when representing the non-trivial irreducible $\mathbb{F}_2(P_i/Q_i)$-modules in terms of their dimension in the shapes of the parabolic subgroups. A number without a bar represents the natural permutation module for P_i/Q_i over $\mathbb{F}_2$ whereas a number with a bar represents a spin module for P_i/Q_i over $\mathbb{F}_2$. Note that there are two spin modules for $\mathfrak{A}_9$ over $\mathbb{F}_2$.

In this thesis we are secondarily interested in exploring the hypothesis of Theorem A in the case when P_2/Q_2 belongs to the class of groups $\mathfrak{X}$ described above. In each section we will clearly identify which results hold in the general setting. One of the motivations for this work was to begin to understand the reason why so few $(\mathfrak{S}_3, \mathfrak{X})$-amalgams of characteristic 2 and critical distance 3 are known to exist. This is in stark contrast to $(\mathfrak{S}_3, \mathfrak{X})$-amalgams of characteristic 2 and critical distance 2 where there are a wealth of known examples.

The shapes of the parabolic subgroups of $(\mathfrak{S}_3, X)$-amalgams of characteristic 2 have been classified for a few of the alternating and symmetric groups X of small degree. The critical distances of these amalgams are listed in Table C along with the examples of sporadic simple groups that are amalgams in each case. Observe that the critical distance is not equal to 3 in the cases $\mathfrak{A}_6$, $\mathfrak{S}_6$ and $\mathfrak{A}_7$. This information will be used in the proof of Theorem A, however, we will not use the unpublished manuscript [28].

The proof of Theorem A begins with a natural division of the problem into two distinct cases. The approach taken in each case is to initially determine the number of

P_2/Q_2	Critical distance b	Sporadic examples G	References
$\mathfrak{S}_3$	$1, 2, 3$	M_{12}	[12], [8, page 85], [19, Section 10.3]
$\mathfrak{S}_5$	$1, 2, 3$	M_{22}, Ly, HS, Ru	[15], [21]
$\mathfrak{A}_6$	$1, 2$		[20]
$\mathfrak{S}_6$	$1, 2$	HS, Co_3	[22], [23], [24], [32]
$\mathfrak{A}_7$	$1, 2$	McL	[29]
$\mathfrak{A}_9$	2	Th	[28]

Table C The known critical distances of amalgams with $P_2/Q_2 \cong \mathfrak{A}_n$ or $\mathfrak{S}_n$.

non-central chief G_β-factors of the group Q_β. We then focus our attention on analysing the structure of certain modules within Q_β with respect to the action of certain 2-groups. This ultimately leads to the conclusion of Theorem A. The proof is almost entirely self-contained with the following exceptions: the minimal dimension of a faithful module for $\mathfrak{A}_n$ and $\mathfrak{S}_n$ over $\mathbb{F}_2$ by Dickson and Wagner, the classification of the failure-of-factorization modules for $\mathfrak{A}_n$ and $\mathfrak{S}_n$ over $\mathbb{F}_2$ by Aschbacher and the classification of the quadratic modules for $\mathfrak{A}_n$ and $\mathfrak{S}_n$ over $\mathbb{F}_2$ by Meierfrankenfeld and Stroth. We refer the reader to Theorems 2.1.8, 2.3.5 and 2.4.8 on pages 23, 40 and 54 respectively.

This thesis is structured as follows. In Chapter 2 we will develop a good understanding of the structure of the permutation modules, spin modules and Sylow 2-subgroups for the alternating and symmetric groups. We will also describe the failure-of-factorization modules over $\mathbb{F}_2$ for the alternating and symmetric groups and then go on to determine the involutions that induce a transvection on their faithful modules over $\mathbb{F}_2$. Chapter 3 begins by examining amalgams under a general hypothesis and then the proof of Theorem A commences in earnest. In Chapters 4 and 5 we will deal with the two distinct cases that naturally arise at the onset of the proof of Theorem A.

Chapter 2

"Small" modules over $\mathbb{F}_2$ and Sylow 2-subgroups for the alternating and symmetric groups

§2.1 Permutation modules and natural modules

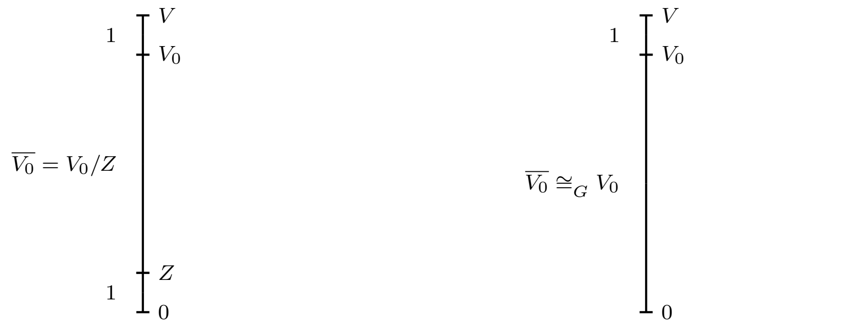
The majority of the results in Sections 2.1 and 2.2 are well-known but essential for our work – see, for example, [2, Exercises 4.5, 4.6, 6.3 and 7.7]. The following notation will hold in this section.

Notation Fix $n \in \mathbb{N}$ and let G be a permutation group on the set $\Delta = \{1, 2, \ldots, n\}$. Let $\mathbb{F}$ be a field and V be an n-dimensional $\mathbb{F}$-vector space with ordered basis $(v_1, v_2, \ldots, v_n)$.

The group G acts on the vector space V by the rule: for $v = \sum_{i=1}^{n} \lambda_i v_i \in V$ and $a \in G$,

$$v^a = \sum_{i=1}^{n} \lambda_i v_{ia} \in V.$$

This action is faithful because if $a \in \mathrm{C}_G(V)$, then $v_{ia} = v_i^a = v_i$ for all $i \in \Delta$ and so $ia = i$ for all $i \in \Delta$ giving $a = 1$. We have constructed a faithful $\mathbb{F}G$-module $V = V(\Delta)$ called the n-dimensional *permutation module* for G over $\mathbb{F}$.



The characteristic of $\mathbb{F}$ divides n — The characteristic of $\mathbb{F}$ does not divide n

Figure B Composition series of the permutation module for $\mathfrak{S}_n$, $n \geq 3$, and $\mathfrak{A}_n$, $n \geq 5$.

The *weight* of a vector $v = \sum_{i=1}^{n} \lambda_i v_i \in V$ is the scalar $\omega(v) = \sum_{i=1}^{n} \lambda_i \in \mathbb{F}$ and the corresponding map $\omega : V \longrightarrow \mathbb{F}$ is called the *weight map*.

Lemma 2.1.1 *The weight map ω is a surjective G-linear map.*

Proof As usual we regard the field $\mathbb{F}$ as a trivial $\mathbb{F}G$-module. Let $\lambda \in \mathbb{F}$ and $u = \sum_{i=1}^{n} \lambda_i v_i, v = \sum_{i=1}^{n} \mu_i v_i \in V$. Then

$$\begin{aligned}\omega(\lambda u + v) &= \omega\Big(\lambda \sum_{i=1}^{n} \lambda_i v_i + \sum_{i=1}^{n} \mu_i v_i\Big) = \omega\Big(\sum_{i=1}^{n} (\lambda\lambda_i + \mu_i) v_i\Big) = \sum_{i=1}^{n} (\lambda\lambda_i + \mu_i) \\ &= \lambda \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \mu_i = \lambda\omega(u) + \omega(v)\end{aligned}$$

and so ω is a linear map. It is visibly surjective and for each $a \in G$, as a is a permutation of Δ,

$$\omega(v^a) = \omega\Big(\sum_{i=1}^{n} \lambda_i v_{ia}\Big) = \sum_{i=1}^{n} \lambda_i = \omega(v) = \omega(v)^a.$$

Thus, ω is a surjective G-linear map. □

The *centre* of the permutation module is the submodule $Z = \langle z \rangle$ where $z = \sum_{i=1}^{n} v_i \in V$ and the *zero weight* submodule of the permutation module is the submodule $V_0 = V_0(\Delta)$ of V consisting of all of the vectors in V of zero weight.

Lemma 2.1.2 *Z is a trivial submodule of V and V_0 is an $(n-1)$-dimensional submodule of V with basis $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n)$. Moreover, the following hold:*

(i) V_0 is a faithful module for $n \geq 3$.

(ii) $z \in V_0$ if and only if the characteristic of $\mathbb{F}$ divides n.

(iii) $V = V_0 \oplus Z$ when the characteristic of $\mathbb{F}$ does not divide n.

Proof As $(v_1, v_2, \ldots, v_n)$ are linearly independent, $z \neq 0$ and so Z is a 1-dimensional subspace of V. For each $a \in G$, as a is a permutation of Δ,

$$z^a = \Big(\sum_{i=1}^{n} v_i\Big)^a = \sum_{i=1}^{n} v_{ia} = \sum_{i=1}^{n} v_i = z.$$

Thus, Z is a trivial submodule of V. By definition, $V_0 = \ker \omega$ and so, by Lemma 2.1.1,

V_0 is a submodule of V. As ω is a surjective map, $\operatorname{im}\omega = \mathbb{F}$ and so, by the Dimension Theorem, $\dim V_0 = \dim\ker\omega = \dim V - \dim\operatorname{im}\omega = n-1$. For each $1 \le i \le n-1$, $\omega(v_i - v_{i+1}) = 1 - 1 = 0$ and so $v_i - v_{i+1} \in \ker\omega = V_0$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{F}$ and assume that

$$\lambda_1(v_1 - v_2) + \lambda_2(v_2 - v_3) + \cdots + \lambda_{n-1}(v_{n-1} - v_n) = 0.$$

Then

$$\lambda_1 v_1 + (\lambda_2 - \lambda_1)v_2 + \cdots + (\lambda_{n-1} - \lambda_{n-2})v_{n-1} - \lambda_{n-1}v_n = 0$$

and so, as $(v_1, v_2, \ldots, v_n)$ are linearly independent, $\lambda_{n-1} = \lambda_{n-2} = \lambda_{n-3} = \cdots = \lambda_1 = 0$. We have shown that the $n-1$ vectors $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n)$ are linearly independent and hence they form a basis for V_0.

(i) Let $n \ge 3$ and $a \in \mathrm{C}_G(V_0)$. Then, for each $1 \le i \le n-1$, $v_i - v_{i+1} = (v_i - v_{i+1})^a = v_{ia} - v_{(i+1)a}$. If $\operatorname{char}\mathbb{F} \ne 2$, then a fixes i and $i+1$ for all $1 \le i \le n-1$. If $\operatorname{char}\mathbb{F} = 2$, then a either fixes or swaps i and $i+1$ for all $1 \le i \le n-1$. However, as a centralizes $v_2 + v_3$, a fixes 1 and 2 and so a fixes $3, 4, \ldots, n$. In both cases $ia = i$ for all $1 \le i \le n$ and so $a = 1$. Thus, $\mathrm{C}_G(V_0) = 1$ and V_0 is a faithful $\mathbb{F}G$-module.

(ii) We have that

$$z \in V_0 \;\Leftrightarrow\; \omega(z) = 0 \;\Leftrightarrow\; n = \sum_{i=1}^{n} 1 = 0 \;\Leftrightarrow\; \operatorname{char}\mathbb{F} \text{ divides } n.$$

(iii) Let n be indivisible by the characteristic of $\mathbb{F}$. Then, by part (ii), $z \notin V_0$ and so $V_0 \cap Z = 0$. We have that $V_0 + Z \le V$ and

$$\dim(V_0 + Z) = \dim V_0 + \dim Z - \dim(V_0 \cap Z) = (n-1) + 1 - 0 = n = \dim V$$

giving $V_0 + Z = V$. Thus, $V = V_0 \oplus Z$. □

The next lemma also holds for $G = \mathfrak{A}_3$ and $\mathfrak{A}_4$ in the case when the field $\mathbb{F} = \mathbb{F}_2$.

Lemma 2.1.3 *Let $G = \mathfrak{S}_n$ for $n \ge 3$ or $G = \mathfrak{A}_n$ for $n \ge 5$. Then Z and V_0 are the unique non-zero proper submodules of V. In particular, Z is the unique non-zero proper submodule of V_0 when the characteristic of $\mathbb{F}$ divides n and V_0 is irreducible otherwise.*

Proof Let U be a non-zero proper submodule of V with $U \neq Z$. Firstly, consider the case when $G = \mathfrak{S}_n$ and $n \geq 3$. Let $u = \sum_{i=1}^{n} \lambda_i v_i \in U \setminus Z$. Then there exists $1 \leq i < j \leq n$ such that $\lambda_i \neq \lambda_j$ and $(\lambda_j - \lambda_i)(v_i - v_j) = u^{(i\,j)} - u = [u, (i\ j)] \in U$ giving $v_i - v_j \in U$. Thus, as G is 2-transitive on Δ, $U = V_0$. Secondly, consider the case when $G = \mathfrak{A}_n$ and $n \geq 5$. Choose $u = \sum_{i=1}^{n} \lambda_i v_i \in U \setminus Z$ with the minimal number m of non-zero scalars λ_i among all of the elements of $U \setminus Z$. Then there exists $1 \leq i < j \leq n$ such that $0 \neq \lambda_i \neq \lambda_j$ and choose $\lambda_j \neq 0$ if possible. We have that $m \geq 2$ because otherwise $u = \lambda_i v_i$ and so $v_i = \lambda_i^{-1} u \in U$ giving, as G is transitive on Δ, $U = V$ which is a contradiction. Suppose, for a contradiction, that $m \geq 4$ and choose $1 \leq k \leq n$ distinct from i and j. Then

$$w = (\lambda_k - \lambda_i)v_i + (\lambda_i - \lambda_j)v_j + (\lambda_j - \lambda_k)v_k = [u, (i\ j\ k)] \in U$$

and so, by the minimality of m, $w \in Z$ so $w = 0$ giving $\lambda_i = \lambda_j$ which is a contradiction. So $m \leq 3 \leq n - 2$ and there exists $1 \leq k < l \leq n$ distinct from i and j such that $\lambda_k = \lambda_l = 0$ unless $(n, m) = (5, 3)$ and $\lambda_j = 0$ in which case, by choice of λ_j, we may choose $\lambda_k = \lambda_l = \lambda_i$. Then $(\lambda_j - \lambda_i)(v_i - v_j) = [u, (i\ j)(k\ l)] \in U$ and so $v_i - v_j \in U$. Thus, as G is 2-transitive on Δ, $U = V_0$. ☐

Define $\overline{V} = V/Z$. The quotient $\mathbb{F}G$-module $\overline{V_0} = \overline{V_0}(\Delta)$ is called the *natural [permutation] module* for G over $\mathbb{F}$.

Lemma 2.1.4 *$\overline{V_0} = V_0/Z$ when the characteristic of $\mathbb{F}$ divides n and $\overline{V_0} \cong_G V_0$ otherwise. In particular, the dimension of $\overline{V_0}$ is $n-2$ when the characteristic of $\mathbb{F}$ divides n and $n-1$ otherwise. Moreover, the following hold:*

(i) $\overline{V_0}$ is a faithful module for $n \geq 5$.

(ii) Both V and $\overline{V_0}$ are self-dual modules.

(iii) If $G = \mathfrak{S}_n$ for $n \geq 3$ or $G = \mathfrak{A}_n$ for $n \geq 5$, then $\overline{V_0}$ is an irreducible module.

(iv) Let $n \geq 3$, fix $k_0 \in \Delta$ and set $K = \mathrm{Stab}_G(k_0)$. Then $(\overline{V_0})_K$ is isomorphic to the natural module for K when the characteristic of $\mathbb{F}$ divides n and the permutation module for K otherwise.

Proof If the characteristic of $\mathbb{F}$ divides n, then $z \in V_0$ and so $\overline{V_0} = (V_0 + Z)/Z = V_0/Z$ and if the characteristic of $\mathbb{F}$ does not divide n, then $z \notin V_0$ and so $\overline{V_0} = (V_0 + Z)/Z \cong_G V_0/(V_0 \cap Z) \cong_G V_0$.

(i) Let $n \geq 5$ and $a \in \mathrm{C}_G(\overline{V_0})$. Then, for each $1 \leq i \leq n-1$, as $\overline{v_i - v_{i+1}} \in \overline{V_0}$,

$$\overline{v_{ia} - v_{(i+1)a}} = \overline{(v_i - v_{i+1})^a} = (\overline{v_i - v_{i+1}})^a = \overline{v_i - v_{i+1}}$$

and so $(v_{ia} - v_{(i+1)a}) - (v_i - v_{i+1}) \in Z$ giving, as $n \geq 5$, $v_{ia} - v_{(i+1)a} = v_i - v_{i+1}$. So, by Lemma 2.1.2(i), $a \in \mathrm{C}_G(V_0) = 1$ and hence $\mathrm{C}_G(\overline{V_0}) = 1$ and $\overline{V_0}$ is a faithful $\mathbb{F}G$-module.

(ii) Define the bilinear form $\langle \cdot , \cdot \rangle$ on V by

$$\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for all $1 \leq i, j \leq n$. The form $\langle \cdot , \cdot \rangle$ is non-degenerate because if $v = \sum_{i=1}^{n} \lambda_i v_i \in V^{\perp}$, then, for each $1 \leq j \leq n$,

$$\lambda_j = \sum_{i=1}^{n} \lambda_i \delta_{ij} = \sum_{i=1}^{n} \lambda_i \langle v_i, v_j \rangle = \left\langle \sum_{i=1}^{n} \lambda_i v_i, v_j \right\rangle = \langle v, v_j \rangle = 0$$

and so $v = \sum_{i=1}^{n} \lambda_i v_i = 0$ giving $V^{\perp} = 0$. Also, for each $1 \leq i, j \leq n$ and $a \in G$,

$$\langle v_i^a, v_j^a \rangle = \langle v_{ia}, v_{ja} \rangle = \delta_{iaja} = \delta_{ij} = \langle v_i, v_j \rangle.$$

We have shown that $\langle \cdot , \cdot \rangle$ is a G-invariant non-degenerate bilinear form on V. So, for each $v \in V$, we may define the linear map $\phi_v : V \longrightarrow \mathbb{F}$ by $\phi_v(u) = \langle v, u \rangle$ for all $u \in V$. Then it is straightforward to show that the map $\phi : V \longrightarrow V^* : v \longmapsto \phi_v$ is an isomorphism between the $\mathbb{F}G$-modules V and V^*. Thus, V is a self-dual module.

Now, the form $\langle \cdot , \cdot \rangle$ on V naturally induces a G-invariant bilinear form $\langle \cdot , \cdot \rangle$ on $\overline{V_0}$ by the rule $\langle \overline{u}, \overline{v} \rangle = \langle u, v \rangle$ for all $\overline{u}, \overline{v} \in \overline{V_0}$. Let $\overline{v} \in \overline{V_0}^{\perp}$ with $v = \sum_{i=1}^{n} \lambda_i v_i \in V_0$. Then, for each $1 \leq i \leq n-1$,

$$\lambda_i - \lambda_{i+1} = \langle v, v_i \rangle - \langle v, v_{i+1} \rangle = \langle v, v_i - v_{i+1} \rangle = \langle \overline{v}, \overline{v_i - v_{i+1}} \rangle = 0$$

and so $\lambda_i = \lambda_{i+1}$ giving $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. So $v = \sum_{i=1}^{n} \lambda_i v_i = \lambda_1 \sum_{i=1}^{n} v_i = \lambda_1 z \in Z$ and so $\overline{v} = 0$ giving $\overline{V_0}^{\perp} = 0$. Hence, $\langle \cdot , \cdot \rangle$ is a G-invariant non-degenerate bilinear form on $\overline{V_0}$ and so we may construct an isomorphism between the $\mathbb{F}G$-modules $\overline{V_0}$ and $\overline{V_0}^*$ in a similar manner to before. Thus, $\overline{V_0}$ is a self-dual module.

(iii) This follows immediately from Lemma 2.1.3.

(iv) Without loss of generality we may take $k_0 = n$. Let U be the permutation module for K over $\mathbb{F}$ with ordered basis $(u_1, u_2, \ldots, u_{n-1})$. In the case when the characteristic of $\mathbb{F}$ divides n, $(\overline{v_1 - v_2}, \overline{v_2 - v_3}, \ldots, \overline{v_{n-2} - v_{n-1}})$ is a basis for $\overline{V_0}$ and so we may define the K-isomorphism $\theta : \overline{V_0} \longrightarrow U_0$ by $\theta(\overline{v_i - v_{i+1}}) = u_i - u_{i+1}$ for all $1 \leq i \leq n-2$. In the case when the characteristic of $\mathbb{F}$ does not divide n, $(\overline{v_1 - v_n}, \overline{v_2 - v_n}, \ldots, \overline{v_{n-1} - v_n})$ is a basis for $\overline{V_0}$ and so we may define the K-isomorphism $\phi : \overline{V_0} \longrightarrow U$ by $\phi(\overline{v_i - v_n}) = u_i$ for all $1 \leq i \leq n-1$. Thus, $(\overline{V_0})_K$ is isomorphic to the natural module for K when the characteristic of $\mathbb{F}$ divides n and the permutation module for K otherwise. $\square$

Corollary 2.1.5 *Let $G = \mathfrak{S}_n$ or $\mathfrak{A}_n$ for $n \geq 3$, $n \neq 4$. Then $\overline{V_0}$ is a faithful irreducible self-dual $\mathbb{F}_2G$-module.*

The next lemma is stated in terms of the binary field $\mathbb{F}_2$, but the proof remains valid for any field of characteristic two.

Lemma 2.1.6 *Let $H \cong \mathfrak{S}_3$ and S and T be distinct Sylow 2-subgroups of H. Let U be a faithful $\mathbb{F}_2H$-module with $\mathrm{C}_U(H) = 0$. Then $U = \mathrm{C}_U(S) \oplus \mathrm{C}_U(T)$ as subspaces of U, $[U, S] = \mathrm{C}_U(S)$ and $\dim \mathrm{C}_U(S) = \frac{1}{2} \dim U$. Moreover, U is isomorphic to a direct sum of natural modules for $\mathfrak{S}_3$ over $\mathbb{F}_2$.*

Proof As $H = \langle S, T \rangle$, $\mathrm{C}_U(S) \cap \mathrm{C}_U(T) = \mathrm{C}_U(\langle S, T \rangle) = \mathrm{C}_U(H) = 0$. Also, as $|S| = 2$, $[U, S] \leq \mathrm{C}_U(S)$ and $U/\mathrm{C}_U(S) \cong [U, S]$ as subspaces of U and so $\dim(U/\mathrm{C}_U(S)) = \dim [U, S] \leq \dim \mathrm{C}_U(S)$ giving $\dim \mathrm{C}_U(S) \geq \frac{1}{2} \dim U$. We have that

$$\begin{aligned}\dim(\mathrm{C}_U(S) + \mathrm{C}_U(T)) &= \dim \mathrm{C}_U(S) + \dim \mathrm{C}_U(T) - \dim(\mathrm{C}_U(S) \cap \mathrm{C}_U(T)) \\ &\geq \tfrac{1}{2} \dim U + \tfrac{1}{2} \dim U = \dim U\end{aligned}$$

and hence $U = \mathrm{C}_U(S) \oplus \mathrm{C}_U(T)$ as subspaces of U. As S and T are conjugate subgroups of H, $\dim U = \dim \mathrm{C}_U(S) + \dim \mathrm{C}_U(T) = 2 \dim \mathrm{C}_U(S)$ and hence $\dim \mathrm{C}_U(S) = \frac{1}{2} \dim U$. Then $[U, S] \leq \mathrm{C}_U(S)$ and $\dim [U, S] = \dim(U/\mathrm{C}_U(S)) = \frac{1}{2} \dim U = \dim \mathrm{C}_U(S)$ and hence

$[U,S] = \mathrm{C}_U(S)$.

Without loss of generality we may assume that $H = \mathfrak{S}_3$, $S = \langle s \rangle$ and $T = \langle t \rangle$ where $s = (1\ 2)$ and $t = (1\ 3)$. Let $G = H = \mathfrak{S}_3$ and $\mathbb{F} = \mathbb{F}_2$ so that, by Lemma 2.1.4, V_0 is isomorphic to the natural module for H over $\mathbb{F}_2$. Fix $u \in \mathrm{C}_U(S)^{\#}$ and set $U_0 = \langle u, u^t \rangle$. Then $u + u^t + u^{ts} \in \mathrm{C}_U(\langle S, T \rangle) = \mathrm{C}_U(H) = 0$ and so $u^{ts} = u + u^t$. Hence, U_0 is a submodule of U. Observe that $(v_1 + v_2)^s = v_1 + v_2$, $(v_1 + v_2)^t = v_2 + v_3$ and $(v_1 + v_2)^{ts} = (v_1 + v_2) + (v_1 + v_2)^t$ and so we may define an H-isomorphism $\phi : U_0 \longrightarrow V_0$ by $\phi(u) = v_1 + v_2$. Thus, U_0 is isomorphic to the natural module for H over $\mathbb{F}_2$. In particular, if U is irreducible, then $U = U_0$ is isomorphic to the natural module. Now, let U be indecomposable. Fix $R \in \mathrm{Syl}_3(H)$. We have that $\mathrm{C}_U(R) = 0$ because otherwise, as $R \trianglelefteq H$, $\mathrm{C}_U(R)$ is a non-zero submodule of U and so, as $H = \langle S, R \rangle$, $\mathrm{C}_U(H) = \mathrm{C}_U(\langle S, R \rangle) = \mathrm{C}_U(S) \cap \mathrm{C}_U(R) = \mathrm{C}_{\mathrm{C}_U(R)}(S) \neq 0$ which is a contradiction. Choose a maximal submodule M of U. Then U/M is an irreducible $\mathbb{F}_2H$-module with, by coprime action, $\mathrm{C}_{U/M}(H) \leq \mathrm{C}_{U/M}(R) = (\mathrm{C}_U(R){+}M)/M = M/M$ and hence U/M is isomorphic to the natural module. As $\dim \mathrm{C}_M(S) = \frac{1}{2} \dim M < \frac{1}{2} \dim U = \dim \mathrm{C}_U(S)$, we may select $u \in \mathrm{C}_U(S) \setminus \mathrm{C}_M(S)$. Then $u \notin M$ and so, as U_0 is irreducible, $M \cap U_0 = 0$. Also, as $\dim U_0 = 2 = \dim (U/M)$, $\dim (M + U_0) = \dim M + \dim (U/M) = \dim U$ and hence $U = M \oplus U_0$. Thus, as U is indecomposable, $M = 0$ and $U = U_0$. We have shown that, up to isomorphism, the natural module is the unique indecomposable $\mathbb{F}_2H$-module with $\mathrm{C}_U(H) = 0$. Therefore, by the Krull-Remak-Schmidt Theorem, U is isomorphic to a direct sum of natural modules for $\mathfrak{S}_3$ over $\mathbb{F}_2$. □

Corollary 2.1.7 *Let $H \cong \mathfrak{S}_3$ and U be an irreducible $\mathbb{F}_2H$-module. Then U is isomorphic to either the trivial module or the natural module for $\mathfrak{S}_3$ over $\mathbb{F}_2$.*

The theorem below follows from the work of Dickson [9, Theorem on page 143] and Wagner [31, Theorem 1.1 on page 151].

Theorem 2.1.8 (Dickson and Wagner) *Let $H \cong \mathfrak{S}_n$ for $n \geq 5$ or $H \cong \mathfrak{A}_n$ for $n \geq 9$. Then the minimal dimension of a faithful $\mathbb{F}_2H$-module is $n-1$ when n is odd and $n-2$ when n is even. Moreover, a faithful $\mathbb{F}_2H$-module of minimal dimension is isomorphic to the natural module for H over $\mathbb{F}_2$ provided that $n \geq 7$ when $H \cong \mathfrak{S}_n$ and $n \geq 10$ when $H \cong \mathfrak{A}_n$.*

In the theorem above the natural module for $\mathfrak{S}_n$ and $\mathfrak{A}_n$ over $\mathbb{F}_2$ is characterised in terms of its dimension provided that n is sufficiently large. This characterisation may be used in order to provide second proofs that the natural module is self-dual and of Lemma 2.2.3 below.

The next lemma will be used frequently when calculating centralizers later on and so it will be known as the *Centralizer Lemma*.

Lemma 2.1.9 (*Centralizer Lemma*) *Let H be a subgroup of G and $\Delta_1, \Delta_2, \ldots, \Delta_r$ be the orbits of H on Δ. Then the vector sums $\left(\sum_{i\in\Delta_1} v_i, \sum_{i\in\Delta_2} v_i, \ldots, \sum_{i\in\Delta_r} v_i\right)$ form a basis for $\mathrm{C}_V(H)$. In particular, $\dim \mathrm{C}_V(H) = r$.*

Proof If $1 \leq k \leq r$, then, for each $a \in H$, as $\{v_{ia} \mid i \in \Delta_k\} = \{v_i \mid i \in \Delta_k\}$,

$$\left(\sum_{i\in\Delta_k} v_i\right)^a = \sum_{i\in\Delta_k} v_i^a = \sum_{i\in\Delta_k} v_{ia} = \sum_{i\in\Delta_k} v_i$$

and so $\sum_{i\in\Delta_k} v_i \in \mathrm{C}_V(H)$. Let $v = \sum_{i=1}^{n} \lambda_i v_i \in \mathrm{C}_V(H)$ and choose a transversal $i_1, i_2, \ldots, i_r$ of the orbits $\Delta_1, \Delta_2, \ldots, \Delta_r$ of H with $i_k \in \Delta_k$ for all $1 \leq k \leq r$. Let $1 \leq k \leq r$. Then, for each $j \in \Delta_k$, there exists $a \in H$ such that $ja = i_k$ and so, as a is a permutation of Δ,

$$\sum_{i=1}^{n} \lambda_i v_{ia} = v^a = v = \sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n} \lambda_{ia} v_{ia}$$

giving, by comparing the coefficients of v_{ja}, $\lambda_j = \lambda_{ja} = \lambda_{i_k}$. So, as $\Delta_1, \Delta_2, \ldots, \Delta_r$ partition the set Δ,

$$v = \sum_{i\in\Delta_1} \lambda_i v_i + \sum_{i\in\Delta_2} \lambda_i v_i + \cdots + \sum_{i\in\Delta_r} \lambda_i v_i$$

$$= \lambda_{i_1}\Big(\sum_{i\in\Delta_1} v_i\Big) + \lambda_{i_2}\Big(\sum_{i\in\Delta_2} v_i\Big) + \cdots + \lambda_{i_k}\Big(\sum_{i\in\Delta_k} v_i\Big).$$

Hence, the vector sums $\big(\sum_{i\in\Delta_1} v_i, \sum_{i\in\Delta_2} v_i, \ldots, \sum_{i\in\Delta_r} v_i\big)$ span $\mathrm{C}_V(H)$. They are also linearly independent because $\Delta_1, \Delta_2, \ldots, \Delta_r$ partition the set Δ and $(v_1, v_2, \ldots, v_n)$ are linearly independent. Thus, the vector sums $\big(\sum_{i\in\Delta_1} v_i, \sum_{i\in\Delta_2} v_i, \ldots, \sum_{i\in\Delta_r} v_i\big)$ form a basis for $\mathrm{C}_V(H)$. □

Corollary 2.1.10 *Assume that G is transitive on Δ. Then $\mathrm{C}_V(G) = Z$ and $[V, G] = V_0$. In particular, $\overline{V_0} \cong_G [V, G]/\mathrm{C}_{[V,G]}(G)$.*

Proof By the Centralizer Lemma above, $\mathrm{C}_V(G) = Z$ and, by Lemma 2.1.4(ii), V is a self-dual module and so $\dim [V, G] = \dim (V/\mathrm{C}_V(G)) = n - 1 = \dim V_0$. For each $v \in V$ and $a \in G$, $\omega([v, a]) = \omega(v^a - v) = \omega(v^a) - \omega(v) = 0$ and so $[v, a] \in V_0$ giving $[V, G] \le V_0$. Thus, $\mathrm{C}_V(G) = Z$ and $[V, G] = V_0$. □

The rule $[\overline{V_0}, H] = \overline{[V_0, H]}$ holds for all subgroups H of G. In the next lemma we will show that when $\mathbb{F} = \mathbb{F}_2$ the rule $\mathrm{C}_{\overline{V_0}}(H) = \overline{\mathrm{C}_{V_0}(H)}$ holds for all subgroups H of G which do not act in a certain way on the set Δ.

Lemma 2.1.11 *Let $\mathbb{F} = \mathbb{F}_2$ and H be a subgroup of G. Then $\mathrm{C}_{\overline{V_0}}(H) \neq \overline{\mathrm{C}_{V_0}(H)}$ if and only if $n \equiv 0\,(4)$ and H has a system of two blocks $\{\Delta_1, \Delta_2\}$ with $|\Delta_1| = n/2 = |\Delta_2|$ and there exists $a \in H$ such that $\Delta_1 a = \Delta_2$. Moreover, in this case, $\dim \mathrm{C}_{\overline{V_0}}(H) = \dim (V_0/[V_0, H]) = \dim \mathrm{C}_{V_0^*}(H)$.*

Proof Firstly, assume that $n \equiv 0\,(4)$ and H has a system of two blocks $\{\Delta_1, \Delta_2\}$ with $|\Delta_1| = n/2 = |\Delta_2|$ and there exists $a \in H$ such that $\Delta_1 a = \Delta_2$. Without loss of generality we may assume that $\Delta_1 = \{1, 2, \ldots, n/2\}$. Define $v = \sum_{i=1}^{n/2} v_i \in V$. Then, as $n \equiv 0\,(4)$, $\omega(v) = n/2 = 0$ so $v \in V_0$ and, as $\Delta_1 a = \Delta_2$, $v^a = v + z$ giving $v \notin \mathrm{C}_{V_0}(H)$. For each $x \in H$, either $\Delta_1^x = \Delta_1$ and $v^x = v$ or $\Delta_1^x = \Delta_2$ and $v^x = v + z$ and so, in either case, $(\overline{v})^x = \overline{v^x} = \overline{v}$. We have shown that $\overline{v} \in \mathrm{C}_{\overline{V_0}}(H) \setminus \overline{\mathrm{C}_{V_0}(H)}$. Thus, $\mathrm{C}_{\overline{V_0}}(H) \neq \overline{\mathrm{C}_{V_0}(H)}$.

Secondly, assume that $\mathrm{C}_{\overline{V_0}}(H) \neq \overline{\mathrm{C}_{V_0}(H)}$. Then there exists $v \in V_0$ such that

$\overline{v} \in \mathrm{C}_{\overline{V_0}}(H) \setminus \overline{\mathrm{C}_{V_0}(H)}$ and we may choose $a \in H$ such that $v^a \neq v$. We have that $\overline{v^a} = (\overline{v})^a = \overline{v}$ and so $v^a - v \in Z = \{0, z\}$ giving $v^a = v + z$. Observe that v has exactly $n/2$ non-zero scalars with respect to $(v_1, v_2, \dots, v_n)$ and so without loss of generality we may assume that $v = \sum_{i=1}^{n/2} v_i$. Then, as $v \in V_0$, $n/2 = \omega(v) = 0$ so $n/2$ is even and hence $n \equiv 0\,(4)$. Set $\Delta_1 = \{1, 2, \dots, n/2\}$ and $\Delta_2 = \Delta \setminus \Delta_1$. Let $x \in H$. Then either $v^x = v$ or $v^x = v + z$. In the case when $v^x = v$,

$$\sum_{i=1}^{n/2} v_{ix} = v^x = v = \sum_{i=1}^{n/2} v_i$$

and so, as x is a permutation of Δ and $(v_1, v_2, \dots, v_n)$ are linearly independent, $\Delta_1 x = \Delta_1$ and $\Delta_2 x = \Delta_2$. In the case when $v^x = v + z$,

$$\sum_{i=1}^{n/2} v_{ix} = v^x = v + z = \sum_{i=n/2}^{n} v_i$$

and so $\Delta_1 x = \Delta_2$ and $\Delta_2 x = \Delta_1$. Thus, $\{\Delta_1, \Delta_2\}$ forms a system of two blocks for H and $\Delta_1 a = \Delta_2$. Now, we have that $z \in [V_0, a]$ because otherwise

$$\begin{aligned}\overline{V_0}/\,\mathrm{C}_{\overline{V_0}}(a) &\cong [\overline{V_0}, a] = \overline{[V_0, a]} = ([V_0, a] + Z)/Z \cong [V_0, a]/([V_0, a] \cap Z) \\ &\cong [V_0, a] \cong V_0/\,\mathrm{C}_{V_0}(a)\end{aligned}$$

and so, as $\dim \overline{V_0} = n - 2 = \dim V_0 - 1$,

$$\begin{aligned}\dim \mathrm{C}_{\overline{V_0}}(a) &= \dim \overline{V_0} - \dim(\overline{V_0}/\,\mathrm{C}_{\overline{V_0}}(a)) = \dim V_0 - \dim(V_0/\,\mathrm{C}_{V_0}(a)) - 1 \\ &= \dim \mathrm{C}_{V_0}(a) - 1 = \dim(\mathrm{C}_{V_0}(a)/Z) = \dim \overline{\mathrm{C}_{V_0}(a)}\end{aligned}$$

giving $\overline{v} \in \mathrm{C}_{\overline{V_0}}(a) = \overline{\mathrm{C}_{V_0}(a)}$ which is a contradiction. So $z \in [V_0, a] \leq [V_0, H]$ and hence $[\overline{V_0}, H] = \overline{[V_0, H]} = [V_0, H]/Z$. Then, by Lemma 2.1.4(ii), $\overline{V_0}$ is a self-dual module and

$$\overline{V_0}/[\overline{V_0}, H] = V_0/Z \Big/ [V_0, H]/Z \cong V_0/[V_0, H]$$

and hence $\dim \mathrm{C}_{\overline{V_0}}(H) = \dim(\overline{V_0}/[\overline{V_0}, H]) = \dim(V_0/[V_0, H]) = \dim \mathrm{C}_{V_0^*}(H)$. □

Corollary 2.1.12 *Let $\mathbb{F} = \mathbb{F}_2$ and H be a subgroup of G. If there exists a generating set A for H in which every element of A has a fixed point on Δ, then $\mathrm{C}_{\overline{V_0}}(H) = \overline{\mathrm{C}_{V_0}(H)}$.*

Lemma 2.1.13 *Let $s \in G$ and assume that $s = s_1 s_2$ where s_1 and s_2 are disjoint non-trivial elements of G. The following hold:*

(i) $\mathrm{C}_V(s) = \mathrm{C}_V(s_1) \cap \mathrm{C}_V(s_2)$ *and, in particular,* $\mathrm{C}_{V_0}(s) = \mathrm{C}_{V_0}(s_1) \cap \mathrm{C}_{V_0}(s_2)$.

(ii) $[V, s] = [V, s_1] + [V, s_2]$ *and if s has a fixed point on Δ, then* $[V_0, s] = [V_0, s_1] + [V_0, s_2]$.

Proof We may choose a partition Δ_1, Δ_2 of the set Δ such that the support of s_i is contained in Δ_i for $i \in \{1, 2\}$. Then $V = V(\Delta_1) \oplus V(\Delta_2)$ as subspaces of V.
(i) As $s \in \langle s_1, s_2 \rangle$, $\mathrm{C}_V(s_1) \cap \mathrm{C}_V(s_2) = \mathrm{C}_V(\langle s_1, s_2 \rangle) \leq \mathrm{C}_V(s)$. Conversely, let $v = u_1 + u_2 \in \mathrm{C}_V(s)$ with $u_i \in V(\Delta_i)$ for $i \in \{1, 2\}$. Then $u_1 + u_2 = v = v^s = u_1^{s_1} + u_2^{s_2}$ and so

$$u_1 - u_1^{s_1} = u_2^{s_2} - u_2 \in V(\Delta_1) \cap V(\Delta_2) = 0$$

giving $u_1^{s_1} = u_1$ and $u_2^{s_2} = u_2$. So, for $i \in \{1, 2\}$, $u_i \in \mathrm{C}_V(s_i) \cap V(\Delta_i) \leq \mathrm{C}_V(s_1) \cap \mathrm{C}_V(s_2)$ and hence $v = u_1 + u_2 \in \mathrm{C}_V(s_1) \cap \mathrm{C}_V(s_2)$. Thus, $\mathrm{C}_V(s) = \mathrm{C}_V(s_1) \cap \mathrm{C}_V(s_2)$.
(ii) As $V = V(\Delta_1) + V(\Delta_2)$,

$$[V, s] = [V(\Delta_1), s] + [V(\Delta_2), s] = [V(\Delta_1), s_1] + [V(\Delta_2), s_2] = [V, s_1] + [V, s_2].$$

Now, as $s \in \langle s_1, s_2 \rangle$, $[V_0, s] \leq [V_0, \langle s_1, s_2 \rangle] = [V_0, s_1] + [V_0, s_2]$. Conversely, let s have a fixed point on Δ, $k_i \in \Delta_i$ for $i \in \{1, 2\}$ say, and choose $k_j \in \Delta_j$ for $j = 3 - i$. Then $V_0 = V_0(\Delta_i) \oplus \langle v_{k_i} - v_{k_j} \rangle \oplus V_0(\Delta_j)$ as subspaces of V_0. As s_i fixes the points k_i and k_j, $[v_{k_i} - v_{k_j}, s_i] = 0$ and so

$$\begin{aligned}[V_0, s_i] &= [V_0(\Delta_i), s_i] + [v_{k_i} - v_{k_j}, s_i] + [V_0(\Delta_j), s_i] = [V_0(\Delta_i), s_i] = [V_0(\Delta_i), s] \\ &\leq [V_0, s].\end{aligned}$$

As s_j and s fix the point k_i, $[v_{k_i} - v_{k_j}, s_j] = [-v_{k_j}, s_j] = [-v_{k_j}, s] = [v_{k_i} - v_{k_j}, s]$ and so

$$\begin{aligned}[V_0, s_j] &= [V_0(\Delta_i), s_j] + [v_{k_i} - v_{k_j}, s_j] + [V_0(\Delta_j), s_j] = [v_{k_i} - v_{k_j}, s_j] + [V_0(\Delta_j), s_j] \\ &= [v_{k_i} - v_{k_j}, s] + [V_0(\Delta_j), s] \leq [V_0, s].\end{aligned}$$

So $[V_0, s_i] + [V_0, s_j] \leq [V_0, s]$ and hence $[V_0, s] = [V_0, s_1] + [V_0, s_2]$. □

Lemma 2.1.14 *Let $\mathbb{F} = \mathbb{F}_2$, n be even and $t = (1\ 2)(3\ 4)\ldots\ (n-1\ n) \in G$. Then $[V_0, t] < [V, t] = \mathrm{C}_V(t) = \mathrm{C}_{V_0}(t)$ with $\dim(\mathrm{C}_{V_0}(t)/[V_0, t]) = 1$ and the following hold:*

(i) $\mathrm{C}_{V_0}(t) = \langle v_1 + v_2, v_3 + v_4, \ldots, v_{n-1} + v_n\rangle$ *and* $\dim \mathrm{C}_{V_0}(t) = n/2$.

(ii) $[V_0, t] = \langle v_1 + v_2 + v_3 + v_4, v_3 + v_4 + v_5 + v_6, \ldots, v_{n-3} + v_{n-2} + v_{n-1} + v_n\rangle$ *and* $\dim [V_0, t] = n/2 - 1$. *In particular, $z \in [V_0, t]$ if and only if $n \equiv 0\,(4)$.*

Proof (i) The orbits of $\langle t\rangle$ are $\{1, 2\}$, $\{3, 4\}$, $\ldots$, $\{n-1, n\}$ and so, by the Centralizer Lemma, the $n/2$ vector sums $(v_1 + v_2, v_3 + v_4, \ldots, v_{n-1} + v_n)$ form a basis for $\mathrm{C}_V(\langle t\rangle) = \mathrm{C}_V(t)$. Thus, as $\mathrm{C}_V(t) \leq V_0$, $\mathrm{C}_{V_0}(t) = \mathrm{C}_V(t) \cap V_0 = \mathrm{C}_V(t)$.
(ii) We have that $[V_0, t] = \langle [v_i + v_{i+1}, t] \mid 1 \leq i \leq n-1\rangle$. Let $1 \leq i \leq n-1$. Then

$$[v_i + v_{i+1}, t] = (v_i + v_{i+1})^t + (v_i + v_{i+1}) = v_{it} + v_{(i+1)t} + v_i + v_{i+1}.$$

If i is odd, then $[v_i + v_{i+1}, t] = 2(v_i + v_{i+1}) = 0$ and if i is even, then $[v_i + v_{i+1}, t] = v_{i-1} + v_i + v_{i+1} + v_{i+2}$. Hence, $[V_0, t]$ is spanned by the vector sums $(v_1 + v_2 + v_3 + v_4, v_3 + v_4 + v_5 + v_6, \ldots, v_{n-3} + v_{n-2} + v_{n-1} + v_n)$. Observe that $[V_0, t]$ is the zero weight submodule of the permutation module with the basis of $\mathrm{C}_{V_0}(t)$ given in part (i). Thus, $\dim [V_0, t] = n/2 - 1$ and, as $z = \sum_{i=1,\, i \text{ odd}}^{n-1} v_i + v_{i+1}$ and by Lemma 2.1.2(ii), $z \in [V_0, t]$ if and only if $n/2$ is even if and only if $n \equiv 0\,(4)$. □

Corollary 2.1.15 *Let $\mathbb{F} = \mathbb{F}_2$ and $t \in G$ be an involution that is fixed-point-free on Δ. Assume that $t = t_1t_2$ where t_1 and t_2 are disjoint non-trivial elements of G. Then $[V_0, t_1] + [V_0, t_2] = \mathrm{C}_{V_0}(t)$.*

Proof Without loss of generality we may assume that $t = (1\ 2)(3\ 4)\ldots(n-1\ n)$. Let Δ_i be the support of t_i for $i \in \{1, 2\}$ and fix $k_i \in \Delta_i$. Let $\{i, j\} = \{1, 2\}$. Then $V_0 = V_0(\Delta_i) \oplus \langle v_{k_i} + v_{k_j}\rangle \oplus V_0(\Delta_j)$ as subspaces of V_0 and so

$$[V_0, t_i] = [V_0(\Delta_i), t_i] + [v_{k_i} + v_{k_j}, t_i] + [V_0(\Delta_j), t_i] = [V_0(\Delta_i), t_i] + [v_{k_i} + v_{k_j}, t_i].$$

We have that, as t is an involution, $[V_0(\Delta_i), t_i] = [V_0(\Delta_i), t] \leq [V_0, t] \leq \mathrm{C}_{V_0}(t)$. Also, $[v_{k_i} + v_{k_j}, t_i] = [v_{k_i}, t_i] = [v_{k_i}, t] \in [V, t]$ and so, by the lemma above, $[v_{k_i} + v_{k_j}, t_i] \in \mathrm{C}_{V_0}(t) \setminus [V_0, t]$. We have shown that $[V_0, t_i] \leq \mathrm{C}_{V_0}(t)$ and $[V_0, t_i] \not\leq [V_0, t]$. Now, as $t \in \langle t_1, t_2\rangle$,

$[V_0, t] \leq [V_0, \langle t_1, t_2\rangle] = [V_0, t_1] + [V_0, t_2] \leq \mathrm{C}_{V_0}(t)$ and hence, as $\dim(\mathrm{C}_{V_0}(t)/[V_0, t]) = 1$, $[V_0, t_1] + [V_0, t_2] = \mathrm{C}_{V_0}(t)$. □

Lemma 2.1.16 *Let $\mathbb{F} = \mathbb{F}_2$, k be even and $t = (1\ 2)(3\ 4)\ldots(k-1\ k) \in G$. The following hold:*

(i) $\dim(V/\mathrm{C}_V(t)) = k/2$.

(ii) $\dim(V_0/\mathrm{C}_{V_0}(t)) = \begin{cases} k/2 & \text{if } k \leq n-1, \\ k/2-1 & \text{if } k = n. \end{cases}$

(iii) $\dim(\overline{V_0}/\mathrm{C}_{\overline{V_0}}(t)) = \begin{cases} k/2 & \text{if } k \leq n-1, \\ k/2-1 & \text{if } k = n \text{ and } n \not\equiv 0\,(4), \\ k/2-2 & \text{if } k = n \text{ and } n \equiv 0\,(4). \end{cases}$

Proof (i) The orbits of $\langle t\rangle$ are $\{1,2\}$, $\{3,4\}$, $\ldots$, $\{k-1,k\}$, $\{k+1\}$, $\{k+2\}$, $\ldots$, $\{n\}$ and so, by the Centralizer Lemma, $\dim \mathrm{C}_V(t) = k/2 + (n-k) = n - k/2$. Hence, $\dim(V/\mathrm{C}_V(t)) = n - (n-k/2) = k/2$.

(ii) In the case when $k \leq n-1$, t has a fixed point on Δ and so, by the Centralizer Lemma, $\mathrm{C}_V(t) \not\leq V_0$ giving, as $\dim(V/V_0) = 1$, $V_0 + \mathrm{C}_V(t) = V$. Then

$$V_0/\mathrm{C}_{V_0}(t) = V_0/(V_0 \cap \mathrm{C}_V(t)) \cong (V_0 + \mathrm{C}_V(t))/\mathrm{C}_V(t) = V/\mathrm{C}_V(t)$$

and hence, by part (i), $\dim(V_0/\mathrm{C}_{V_0}(t)) = \dim(V/\mathrm{C}_V(t)) = k/2$. In the case when $k = n$, $V_0/\mathrm{C}_{V_0}(t) \cong [V_0, t]$ and hence, by Lemma 2.1.14(ii), $\dim(V_0/\mathrm{C}_{V_0}(t)) = \dim[V_0, t] = k/2 - 1$.

(iii) In the case when $k \leq n-1$ or $k = n$ and $n \not\equiv 0\,(4)$, by Lemma 2.1.11,

$$\mathrm{C}_{\overline{V_0}}(t) = \overline{\mathrm{C}_{V_0}(t)} = (\mathrm{C}_{V_0}(t) + Z)/Z \cong \mathrm{C}_{V_0}(t)/(\mathrm{C}_{V_0}(t) \cap Z)$$

and, as $z \in \mathrm{C}_V(t)$, $\mathrm{C}_{V_0}(t) \cap Z = V_0 \cap \mathrm{C}_V(t) \cap Z = V_0 \cap Z$. Then, as $\overline{V_0} \cong V_0/(V_0 \cap Z)$,

$$\begin{aligned} \dim(\overline{V_0}/\mathrm{C}_{\overline{V_0}}(t)) &= \dim(V_0/(V_0 \cap Z)) - \dim \mathrm{C}_{V_0}(t) + \dim(V_0 \cap Z) \\ &= \dim(V_0/\mathrm{C}_{V_0}(t)) \end{aligned}$$

and hence the result follows from part (ii). In the case when $k = n$ and $n \equiv 0\,(4)$, set $\Delta_1 = \{1, 3, \ldots, n-1\}$ and $\Delta_2 = \{2, 4, \ldots, n\}$. Then $\{\Delta_1, \Delta_2\}$ is a block system for

$\langle t \rangle$ with $\Delta_1 t = \Delta_2$ and so, by Lemma 2.1.11, $\dim \mathrm{C}_{\overline{V_0}}(t) = \dim (V_0/[V_0, t])$. Hence, by part (ii),

$$\dim (\overline{V_0}/\mathrm{C}_{\overline{V_0}}(t)) = \dim \overline{V_0} - \dim V_0 + \dim [V_0, t] = \dim (V_0/\mathrm{C}_{V_0}(t)) - 1 = k/2 - 2. \square$$

Corollary 2.1.17 *Let $\mathbb{F} = \mathbb{F}_2$, $n \geq 7$, $n \neq 8$ and let U denote V, V_0 and $\overline{V_0}$. Let $t \in G$ be an involution. The following hold:*

(i) If $\dim (U/\mathrm{C}_U(t)) = 1$, then t is a transposition.

(ii) If $\dim (U/\mathrm{C}_U(t)) = 2$, then t is a double transposition.

§2.2 Extensions of the natural module

The main result in this section is Theorem 2.2.6 where we will show that the only extension by trivial modules of the natural module for $\mathfrak{A}_n$ or $\mathfrak{S}_n$ over $\mathbb{F}_2$ is the zero weight submodule of the permutation module. The following notation will hold in this section.

Notation Let $n \geq 5$ and U be the n-dimensional permutation module for $\mathfrak{A}_n$ over $\mathbb{F}_2$ with ordered basis $(u_1, u_2, \ldots, u_n)$ and centre $Z = \langle z \rangle$.

Recall that $\mathfrak{A}_n = \langle (1\ 2\ 3), (1\ 2\ 4), \ldots, (1\ 2\ n) \rangle$. So $\mathfrak{A}_n = \langle (1\ 2\ 3), (3\ 4\ 5), \ldots, (n{-}2\ n{-}1\ n) \rangle$ when n is odd and $\mathfrak{A}_n = \langle (1\ 2\ 3), (3\ 4\ 5), \ldots, (n-3\ n-2\ n-1), (n-2\ n-1\ n) \rangle$ when n is even. In particular, $\mathfrak{A}_n$ is generated by $\lfloor n/2 \rfloor$ three-cycles.

Lemma 2.2.1 *Set* $K = \mathrm{Stab}_{\mathfrak{A}_n}(\{1,2\})$ *and* $L = \mathrm{Stab}_{\mathfrak{A}_n}(1,2)$. *The following hold:*

(i) $\mathrm{C}_{U_0}(K) = \mathrm{C}_{U_0}(L) = \langle u_1{+}u_2, u_3{+}u_4{+}\cdots{+}u_n \rangle$ *when n is even and* $\mathrm{C}_{U_0}(K) = \langle u_1{+}u_2 \rangle$ *and* $\mathrm{C}_{U_0}(L) = \langle u_2 + u_3 + \cdots + u_n, u_1 + u_3 + \cdots + u_n \rangle$ *when n is odd.*

(ii) $[U_0, K] = \langle u_1 + u_2 \rangle \oplus [U_0, L]$ *and* $[U_0, L] = \langle u_3 + u_4, u_4 + u_5, \ldots, u_{n-1} + u_n \rangle$.

Proof (i) The orbits of K are $\{1, 2\}$ and $\{3, 4, \ldots, n\}$ and so, by the Centralizer Lemma, $\mathrm{C}_U(K) = \langle u_1 + u_2, u_3 + u_4 + \cdots + u_n \rangle$. Thus, as $\mathrm{C}_{U_0}(K) = \mathrm{C}_U(K) \cap U_0$, $\mathrm{C}_{U_0}(K) = \langle u_1 + u_2, u_3 + u_4 + \cdots + u_n \rangle$ when n is even and $\mathrm{C}_{U_0}(K) = \langle u_1 + u_2 \rangle$ when n is odd. Similarly, the orbits of L are $\{1\}$, $\{2\}$ and $\{3, 4, \ldots, n\}$ and so, by the Centralizer Lemma, $\mathrm{C}_U(L) = \langle u_1, u_2, u_3 + u_4 + \cdots + u_n \rangle$. Thus, $\mathrm{C}_{U_0}(L) = \langle u_1 + u_2, u_3 + u_4 + \cdots + u_n \rangle$ when n is even and $\mathrm{C}_{U_0}(L) = \langle u_2 + u_3 + \cdots + u_n, u_1 + u_3 + \cdots + u_n \rangle$ when n is odd.

(ii) As $L = \langle (3\ 4\ 5), (3\ 4\ 6), \ldots, (3\ 4\ n) \rangle$, $[U_0, L] = \langle [u_i + u_{i+1}, (3\ 4\ j)] \mid 1 \leq i \leq n-1 \text{ and } 5 \leq j \leq n \rangle$ and hence $[U_0, L] = \langle u_3 + u_4, u_3 + u_5, \ldots, u_3 + u_n \rangle = \langle u_3 + u_4, u_4 + u_5, \ldots, u_{n-1} + u_n \rangle$. Fix $s = (1\ 2)(3\ 4) \in \mathfrak{A}_n$. Then $[U_0, s] = \langle [u_i + u_{i+1}, s] \mid 1 \leq i \leq n-1 \rangle = \langle u_1 + u_2, u_3 + u_4 \rangle$ and so, as $K = L\langle s \rangle$, $[U_0, K] = [U_0, L\langle s \rangle] = [U_0, L] + [U_0, \langle s \rangle] = [U_0, L] + [U_0, s]$. Thus, $[U_0, K] = \langle u_1 + u_2 \rangle \oplus [U_0, L]$. $\square$

Corollary 2.2.2 *Set* $K = \mathrm{Stab}_{\mathfrak{A}_n}(\{1,2\})$ *and* $L = \mathrm{Stab}_{\mathfrak{A}_n}(1,2)$. *The following hold:*

(i) $\mathrm{C}_{\overline{U_0}}(K) = \langle \overline{u_1 + u_2} \rangle$. $\mathrm{C}_{\overline{U_0}}(L) = \langle \overline{u_1 + u_2} \rangle$ *when* n *is even and* $\mathrm{C}_{\overline{U_0}}(L) = \langle \overline{u_1}, \overline{u_2} \rangle$ *when* n *is odd.*

(ii) $[\overline{U_0}, K] = [\overline{U_0}, L]$ *when* n *is even and* $[\overline{U_0}, K] = \langle \overline{u_1 + u_2} \rangle \oplus [\overline{U_0}, L]$ *when* n *is odd.* $[\overline{U_0}, L] = \langle \overline{u_3 + u_4}, \overline{u_4 + u_5}, \ldots, \overline{u_{n-1} + u_n} \rangle$. *In particular,* $[\overline{U_0}, L]$ *is* L*-isomorphic to the zero weight submodule of the permutation module for* L.

Proof As $K = L\langle (1\ 2)(3\ 4) \rangle$ and by applying Corollary 2.1.12, the calculations follow immediately from the lemma above. Now, as $z \notin [U_0, L]$, $[U_0, L] \cap Z = 0$ and so

$$[\overline{U_0}, L] = \overline{[U_0, L]} = ([U_0, L] + Z)/Z \cong_L [U_0, L]/([U_0, L] \cap Z) \cong_L [U_0, L].$$

Thus, $[\overline{U_0}, L]$ is L-isomorphic to the zero weight submodule of the permutation module for L. □

Lemma 2.2.3 *Let* V *be an* $\mathbb{F}_2\mathfrak{S}_n$*-module and assume that* $V_{\mathfrak{A}_n}$ *is isomorphic to the natural module for* $\mathfrak{A}_n$. *Then* V *is isomorphic to the natural module for* $\mathfrak{S}_n$.

Proof Without loss of generality we may assume that $V_{\mathfrak{A}_n} = \overline{U_0}$. Fix $t = (1\ 2) \in \mathfrak{S}_n$ and set $K = \mathrm{C}_{\mathfrak{A}_n}(t) = \mathrm{Stab}_{\mathfrak{A}_n}(\{1,2\})$ and $L = \mathrm{Stab}_{\mathfrak{A}_n}(1,2)$. Then L is similar to $\mathfrak{A}_{n-2}$ and, by Corollary 2.2.2(ii), $[V, L]$ is L-isomorphic to the zero weight submodule of the permutation module for L. By Lemma 2.1.3 and Corollary 2.2.2, V has the following composition L-series $0 <_1 \mathrm{C}_V(L) <_{n-4} [V, L] <_1 V$ when n is even and $0 <_{n-3} [V, L] <_1 [V, K] <_1 V$ when n is odd where the subscripts denote the dimensions of the respective composition L-factors of V. Observe that, in both cases, $0 <_1 \mathrm{C}_V(K) <_d [V, K] <_1 V$ is also a composition L-series of V where $d \geq 2$. We will proceed via a series of steps.

(1) $\dim [V, t] = 1 = \dim (V / \mathrm{C}_V(t))$.

We have that $[V, t] \neq 0$ because otherwise, as $\mathfrak{S}_n = \langle t^{\mathfrak{S}_n} \rangle$, $[V, \mathfrak{S}_n] = [V, \langle t^{\mathfrak{S}_n} \rangle] = \langle [V, t]^{\mathfrak{S}_n} \rangle = 0$ which is a contradiction. Then, as t is an involution, $0 < [V, t] \leq \mathrm{C}_V(t) < V$ is an L-series of V with $V / \mathrm{C}_V(t) \cong_L [V, t]$. So, as V has three composition L-factors, $[V, t]$ is L-irreducible and hence $\dim (V / \mathrm{C}_V(t)) = \dim [V, t] = 1$.

(2) $[V,t] = \mathrm{C}_V(K)$.

We have that $[V,t] \leq [V,K]$ because otherwise $V = [V,K] + [V,t]$ and so, as $[V,K]$ is $\langle t \rangle$-invariant, $[V,t] = [V,K,t] + [V,t,t] = [V,K,t] \leq [V,K]$ which is a contradiction. So, as $[V,t] + \mathrm{C}_V(K)$ is an L-submodule of $[V,K]$ and $[V,K]/\mathrm{C}_V(K)$ is L-irreducible, either $[V,t] \leq \mathrm{C}_V(K)$ or $[V,t] + \mathrm{C}_V(K) = [V,K]$. In the latter case, $\dim [V,K] \leq \dim [V,t] + \dim \mathrm{C}_V(K) = 2$ and so, by Corollary 2.2.2(ii), $n \leq 5$ when n is even and $n \leq 4$ when n is odd which are both contradictions. So $[V,t] \leq \mathrm{C}_V(K)$ and $\dim [V,t] = 1 = \dim \mathrm{C}_V(K)$. Thus, $[V,t] = \mathrm{C}_V(K)$.

(3) $\mathrm{C}_V(t) = [V,K]$.

Suppose, for a contradiction, that $[V,L] \not\leq \mathrm{C}_V(t)$. Then, by (1), $V = \mathrm{C}_V(t) + [V,L]$ so, by (2) and as $[V,L]$ is $\langle t \rangle$-invariant, $\mathrm{C}_V(K) = [V,t] = [V,L,t] \leq [V,L]$ and hence n is even. Also, by (2), $\mathrm{C}_V(L) = \mathrm{C}_V(K) = [V,t] \leq \mathrm{C}_V(t)$ so that $\mathrm{C}_V(t) \cap [V,L]$ is a proper L-submodule of $[V,L]$ containing $\mathrm{C}_V(L)$ and so, as $[V,L]/\mathrm{C}_V(L)$ is L-irreducible, $\mathrm{C}_V(t) \cap [V,L] = \mathrm{C}_V(L)$. We have that

$$[V,L]/\mathrm{C}_V(L) = [V,L]/([V,L] \cap \mathrm{C}_V(t)) \cong ([V,L] + \mathrm{C}_V(t))/\mathrm{C}_V(t) = V/\mathrm{C}_V(t)$$

and so, by (1), $n - 4 = \dim([V,L]/\mathrm{C}_V(L)) = \dim(V/\mathrm{C}_V(t)) = 1$ giving $n = 5$ which contradicts the assertion that n is even. Hence, $[V,L] \leq \mathrm{C}_V(t)$. So, by (2), $[V,K] = [V,L] + \mathrm{C}_V(K) = [V,L] + [V,t] \leq \mathrm{C}_V(t)$ and $\dim [V,K] = d + 1 = \dim \mathrm{C}_V(t)$. Thus, $\mathrm{C}_V(t) = [V,K]$.

(4) V is isomorphic to the natural module for $\mathfrak{S}_n$.

We have that, by (2),

$$(\overline{u_2 + u_3})^t - (\overline{u_2 + u_3}) = [\overline{u_2 + u_3}, t] \in [V,t] = \mathrm{C}_V(K) = \langle \overline{u_1 + u_2} \rangle = \{0, \overline{u_1 + u_2}\}$$

and, by (3), $\overline{u_2 + u_3} \notin [V,K] = \mathrm{C}_V(t)$ and hence $(\overline{u_2 + u_3})^t = \overline{u_1 + u_3}$. As $V = \mathrm{C}_V(t) \oplus \langle \overline{u_2 + u_3} \rangle$ as subspaces of V and $\mathfrak{S}_n = \mathfrak{A}_n \langle t \rangle$, we have uniquely determined the action of $\mathfrak{S}_n$ on V. Moreover, we have shown that $\mathfrak{S}_n$ acts on V as the natural module for $\mathfrak{S}_n$. Therefore, V is isomorphic to the natural module for $\mathfrak{S}_n$. □

Lemma 2.2.4 *Let V be an $\mathbb{F}_2\mathfrak{S}_n$-module and assume that $V_{\mathfrak{A}_n}$ is isomorphic to the zero weight submodule of the permutation module for $\mathfrak{A}_n$. Then V is isomorphic to the zero weight submodule of the permutation module for $\mathfrak{S}_n$.*

Proof In the case when n is odd, the result follows immediately from Lemma 2.2.3. Assume that n is even. Without loss of generality we may assume that $V_{\mathfrak{A}_n} = U_0$. Fix $t = (1\ 2) \in \mathfrak{S}_n$ and set $K = \mathrm{C}_{\mathfrak{A}_n}(t) = \mathrm{Stab}_{\mathfrak{A}_n}(\{1,2\})$ and $L = \mathrm{Stab}_{\mathfrak{A}_n}(1,2)$. Then L is similar to $\mathfrak{A}_{n-2}$ and, by Lemma 2.2.1(ii), $[V,L]$ is the zero weight submodule of the permutation module for L. By Lemmas 2.1.3 and 2.2.1, V has the following composition L-series $0 <_1 \mathrm{C}_V(L) \cap [V,L] <_{n-4} [V,L] <_1 [V,K] <_1 V$ where the subscripts denote the dimensions of the respective composition L-factors of V. Observe that $0 <_1 Z <_1 \mathrm{C}_V(K) <_{n-4} [V,K] <_1 V$ is also a composition L-series of V. We will proceed via a series of steps.

(1) $\dim [V,t] = 1 = \dim (V/\mathrm{C}_V(t))$.

We have that $[V,t] \neq 0$ because otherwise, as $\mathfrak{S}_n = \langle t^{\mathfrak{S}_n}\rangle$, $[V,\mathfrak{S}_n] = [V,\langle t^{\mathfrak{S}_n}\rangle] = \langle [V,t]^{\mathfrak{S}_n}\rangle = 0$ which is a contradiction. Then, as t is an involution and $\dim V = n-1$ is odd, $0 < [V,t] < \mathrm{C}_V(t) < V$ is an L-series of V with $V/\mathrm{C}_V(t) \cong_L [V,t]$. So, as V has four composition L-factors, $[V,t]$ is L-irreducible and hence $\dim (V/\mathrm{C}_V(t)) = \dim [V,t] = 1$.

(2) $[V,t] = \langle u_1 + u_2\rangle$.

We have that $[V,t] \leq [V,K]$ because otherwise $V = [V,K] + [V,t]$ and so, as $[V,K]$ is $\langle t\rangle$-invariant, $[V,t] = [V,K,t] + [V,t,t] = [V,K,t] \leq [V,K]$ which is a contradiction. So, as $[V,t] + \mathrm{C}_V(K)$ is an L-submodule of $[V,K]$ and $[V,K]/\mathrm{C}_V(K)$ is L-irreducible, either $[V,t] \leq \mathrm{C}_V(K)$ or $[V,t] + \mathrm{C}_V(K) = [V,K]$. In the latter case, by Lemma 2.2.1(ii) and (1), $n-2 = \dim [V,K] \leq \dim [V,t] + \dim \mathrm{C}_V(K) = 3$ and so $n \leq 5$ which is a contradiction because n is even. Hence, by Lemma 2.2.1(i),

$$[V,t] \leq \mathrm{C}_V(K) = \langle u_1 + u_2, u_3 + u_4 + \cdots + u_n\rangle.$$

In the case when $[V,t] = Z$, as $\mathfrak{S}_n = \mathfrak{A}_n\langle t\rangle$, Z is $\mathfrak{S}_n$-invariant and so, as $\mathfrak{S}_n = \langle t^{\mathfrak{S}_n}\rangle$,

$[V, \mathfrak{S}_n] = [V, \langle t^{\mathfrak{S}_n}\rangle] = \langle [V,t]^{\mathfrak{S}_n}\rangle = Z$ which is a contradiction. In the case when $[V,t] = \langle u_3 + u_4 + \cdots + u_n\rangle$, for each $v \in V$,

$$v^t - v = [v,t] \in [V,t] = \langle u_3 + u_4 + \cdots + u_n\rangle = \{0, u_3 + u_4 + \cdots + u_n\}.$$

Assume first that $u_2 + u_3 \in \mathrm{C}_V(t)$. Then, as $t^{(3\,4\,5)} = t$,

$$u_2 + u_4 = (u_2 + u_3)^{(3\,4\,5)} \in \mathrm{C}_V(t)^{(3\,4\,5)} = \mathrm{C}_V(t^{(3\,4\,5)}) = \mathrm{C}_V(t)$$

and so, continuing in this way,

$$\langle u_2 + u_3, u_3 + u_4, \ldots, u_{n-1} + u_n\rangle = \langle u_2 + u_3, u_2 + u_4, \ldots, u_2 + u_n\rangle \leq \mathrm{C}_V(t).$$

Hence, as $\dim \mathrm{C}_V(t) = n-2$, $\mathrm{C}_V(t) = \langle u_2+u_3, u_3+u_4, \ldots, u_{n-1}+u_n\rangle$. Set $s = (1\,3\,4) \in \mathfrak{A}_n$. Then $(2\ 3\ 4) = s^t = tst$ and so, as $u_1 + u_2, z \notin \mathrm{C}_V(t)$,

$$u_1 + u_3 = (u_1 + u_2)^{(2\,3\,4)} = (u_1 + u_2)^{tst} = z^{st} = z^t = u_1 + u_2$$

which is a contradiction. Assume second that $u_2 + u_3 \notin \mathrm{C}_V(t)$. Set $r = (2\ 4\ 3) \in \mathfrak{A}_n$. Then $(1\ 4\ 3) = r^t = trt$ and so, as $u_3 + u_4 + \cdots + u_n \in [V,t] \leq \mathrm{C}_V(t)$,

$$\begin{aligned} u_1 + u_2 &= (u_2 + u_3)^{(1\,4\,3)} = (u_2 + u_3)^{trt} = (u_2 + u_4 + u_5 + \cdots + u_n)^{rt} \\ &= (u_3 + u_4 + \cdots + u_n)^t = u_3 + u_4 + \cdots + u_n \end{aligned}$$

which is also a contradiction. Thus, $[V,t] = \langle u_1 + u_2\rangle$.

(3) $\mathrm{C}_V(t) = [V,K]$.

Suppose, for a contradiction, that $[V,L] \not\leq \mathrm{C}_V(t)$. We have that, by (1), $V = \mathrm{C}_V(t) + [V,L]$ and $\mathrm{C}_V(t) \cap [V,L]$ is a proper L-submodule of $[V,L]$ giving, by Lemma 2.1.3, $\dim(\mathrm{C}_V(t) \cap [V,L]) \leq 1$. Then

$$[V,L]/([V,L] \cap \mathrm{C}_V(t)) \cong ([V,L] + \mathrm{C}_V(t))/\mathrm{C}_V(t) = V/\mathrm{C}_V(t)$$

and so, by (1), $n - 4 \leq \dim([V,L]/(\mathrm{C}_V(t) \cap [V,L])) = \dim(V/\mathrm{C}_V(t)) = 1$ giving $n \leq 5$ which is a contradiction because n is even. Hence, $[V,L] \leq \mathrm{C}_V(t)$. So, by (2), $[V,K] = [V,L] + \langle u_1 + u_2\rangle = [V,L] + [V,t] \leq \mathrm{C}_V(t)$ and $\dim[V,K] = n-2 = \dim \mathrm{C}_V(t)$. Thus, $\mathrm{C}_V(t) = [V,K]$.

(4) V is isomorphic to the zero weight submodule of the permutation module for $\mathfrak{S}_n$.

We have that, by (2),

$$(u_2+u_3)^t-(u_2+u_3)=[u_2+u_3,t]\in[V,t]=\langle u_1+u_2\rangle=\{0,u_1+u_2\}$$

and, by (3), $u_2+u_3\notin[V,K]=\mathrm{C}_V(t)$ and hence $(u_2+u_3)^t=u_1+u_3$. As $V=\mathrm{C}_V(t)\oplus\langle u_2+u_3\rangle$ as subspaces of V and $\mathfrak{S}_n=\mathfrak{A}_n\langle t\rangle$, we have uniquely determined the action of $\mathfrak{S}_n$ on V. Moreover, we have shown that $\mathfrak{S}_n$ acts on V as the zero weight submodule of the permutation module for $\mathfrak{S}_n$. Therefore, V is isomorphic to the zero weight submodule of the permutation module for $\mathfrak{S}_n$. □

Lemma 2.2.5 $\dim[\overline{U_0},(1\ 2\ 3)]=2$ *and* $[\overline{U_0},(1\ 2\ 3)]\cap[\overline{U_0},(2\ 3\ 4)]\neq 0$.

Proof We have that $[U_0,(1\ 2\ 3)]=\langle u_1+u_2,u_2+u_3\rangle$ so $[\overline{U_0},(1\ 2\ 3)]=\langle\overline{u_1+u_2},\overline{u_2+u_3}\rangle$ and $\dim[\overline{U_0},(1\ 2\ 3)]=2$. Also, as $(2\ 3\ 4)=(1\ 2\ 3)^{(1\,4\,5)}$,

$$\overline{u_2+u_3}=(\overline{u_2+u_3})^{(1\,4\,5)}\in[\overline{U_0},(1\ 2\ 3)]^{(1\,4\,5)}=[\overline{U_0},(1\ 2\ 3)^{(1\,4\,5)}]=[\overline{U_0},(2\ 3\ 4)]$$

and hence $\overline{u_2+u_3}\in[\overline{U_0},(1\ 2\ 3)]\cap[\overline{U_0},(2\ 3\ 4)]\neq 0$. □

The theorem below is intimately related to the dimension of the first cohomology group $\mathrm{H}^1(\mathfrak{A}_n,\overline{U_0})$ although we shall not need this concept.

Theorem 2.2.6 *Let $G=\mathfrak{S}_n$ or $\mathfrak{A}_n$ and V be an $\mathbb{F}_2G$-module with $[V,\mathfrak{A}_n]=V$ and $V/\mathrm{C}_V(\mathfrak{A}_n)$ isomorphic to the natural module for G. Then V is isomorphic to either the natural module for G or the zero weight submodule of the permutation module for G.*

Proof Define $\widetilde{V}=V/\mathrm{C}_V(\mathfrak{A}_n)$. By Lemmas 2.2.3 and 2.2.4 we may assume that $G=\mathfrak{A}_n$. In the case when $\mathrm{C}_V(\mathfrak{A}_n)=0$, V is isomorphic to the natural module and we are done. Assume that $\dim\mathrm{C}_V(\mathfrak{A}_n)\geq 1$. There exist three-cycles $t_1=(1\ 2\ 3)$, $t_2=(3\ 4\ 5)$, …, $t_m=(n-2\ n-1\ n)$ with $m=\lfloor n/2\rfloor$ such that $\mathfrak{A}_n=\langle t_1,t_2,\dots,t_m\rangle$. We will proceed via a series of steps.

(1) $\dim[V,t_i]=2$ *for all* $1\leq i\leq m$.

Fix $1 \leq i \leq m$. Then, by coprime action, $V = [V, t_i] \oplus \mathrm{C}_V(t_i)$ and so, as $\mathrm{C}_V(\mathfrak{A}_n) \leq \mathrm{C}_V(t_i)$, $[V, t_i] \cap \mathrm{C}_V(\mathfrak{A}_n) = 0$. So

$$[\widetilde{V}, t_i] = \widetilde{[V, t_i]} \cong [V, t_i]/([V, t_i] \cap \mathrm{C}_V(\mathfrak{A}_n)) \cong [V, t_i]$$

and hence, by Lemma 2.2.5, $\dim [V, t_i] = \dim [\widetilde{V}, t_i] = 2$.

(2) n is even.

Suppose, for a contradiction, that n is odd. Then, as $\mathfrak{A}_n = \langle t_1, t_2, \ldots, t_m \rangle$,

$$V = [V, \mathfrak{A}_n] = [V, t_1] + [V, t_2] + \cdots + [V, t_m]$$

and so, by (1), $\dim V \leq \sum_{i=1}^{m} \dim [V, t_i] = 2m = n - 1$. Then, as $\dim \widetilde{V} = n - 1$, $\dim \mathrm{C}_V(\mathfrak{A}_n) = \dim V - \dim \widetilde{V} \leq 0$ giving $\mathrm{C}_V(\mathfrak{A}_n) = 0$ which is a contradiction. Thus, n is even.

(3) Define $K = \langle t_1, t_2, \ldots, t_{m-1} \rangle$. Then K is similar to $\mathfrak{A}_{n-1}$ and $V = [V, K] \oplus_K \mathrm{C}_V(K)$ where $[V, K]$ is isomorphic to the natural module for K and $\mathrm{C}_V(K) = \mathrm{C}_V(\mathfrak{A}_n)$.

As $K = \mathrm{Stab}_{\mathfrak{A}_n}(n)$, K is similar to $\mathfrak{A}_{n-1}$ and so, by (2) and applying Lemma 2.1.4(iv), $(\widetilde{V})_K$ is isomorphic to the natural module for K. In particular, by Lemma 2.1.4(iii), $\widetilde{V}$ is K-irreducible. As $[\widetilde{V}, K] \neq 0$, $[V, K] \not\leq \mathrm{C}_V(\mathfrak{A}_n)$ so $\mathrm{C}_V(\mathfrak{A}_n) \leq \mathrm{C}_V(K) < V$ and hence $\mathrm{C}_V(\mathfrak{A}_n) = \mathrm{C}_V(K)$ and $V = [V, K] + \mathrm{C}_V(K)$. Then $[V, K, K] = [V, K]$ and

$$[V, K]/\mathrm{C}_{[V,K]}(K) = [V, K]/([V, K] \cap \mathrm{C}_V(K)) \cong_K ([V, K] + \mathrm{C}_V(K))/\mathrm{C}_V(K) = \widetilde{V}$$

and hence $[V, K]/\mathrm{C}_{[V,K]}(K)$ is isomorphic to the natural module for K. So, as $n - 1$ is odd and applying the argument used in (2) to the module $[V, K]$, $[V, K] \cap \mathrm{C}_V(K) = \mathrm{C}_{[V,K]}(K) = 0$. Thus, $V = [V, K] \oplus_K \mathrm{C}_V(K)$ and $[V, K]$ is isomorphic to the natural module for K.

(4) $\dim \mathrm{C}_V(\mathfrak{A}_n) = 1$.

Set $s_1 = (2\ 3\ 4) \in K$ and $g = (1\ n-3)(2\ n-2)(3\ n-1)(4\ n) \in \mathfrak{A}_n$ so that $t_1^g = t_{m-1}$ and $s_1^g = t_m$. Then, by (3) and Lemma 2.2.5, $[V, t_1] \cap [V, s_1] = [[V, K], t_1] \cap [[V, K], s_1] \neq 0$ and so $[V, t_{m-1}] \cap [V, t_m] \neq 0$. Now, as $\mathfrak{A}_n = \langle K, t_m \rangle$, $V = [V, \mathfrak{A}_n] = [V, K] + [V, t_m]$ and so, by

(3) and (1),

$$\dim V = \dim [V, K] + \dim [V, t_m] - \dim([V, K] \cap [V, t_m]) \leq (n-2) + 2 - 1 = n-1$$

giving, as $\dim \widetilde{V} = n-2$, $\dim \mathrm{C}_V(\mathfrak{A}_n) = \dim V - \dim \widetilde{V} \leq 1$. Thus, $\dim \mathrm{C}_V(\mathfrak{A}_n) = 1$.

(5) V is isomorphic to a submodule of the permutation module.

By Lemma 2.1.4(ii), the permutation module U is self-dual and so it suffices to prove that the dual V^* of V is isomorphic to a quotient of U. So, taking duals, $\mathrm{C}_{V^*}(\mathfrak{A}_n) = 0$ and, by Lemma 2.1.4(ii), $[V^*, \mathfrak{A}_n]$ is isomorphic to the natural module for $\mathfrak{A}_n$. We also have that, by dualising (3) and (4), $V^* = [V^*, K] \oplus_K \mathrm{C}_{V^*}(K)$, $[V^*, K] = [V^*, \mathfrak{A}_n]$ and $\dim (V^*/[V^*, \mathfrak{A}_n]) = 1$. We may choose $v \in \mathrm{C}_{V^*}(K)^{\#}$ because otherwise $V^* = [V^*, K] = [V^*, \mathfrak{A}_n]$ which is a contradiction. Then $\langle v^{\mathfrak{A}_n} \rangle \cap [V^*, \mathfrak{A}_n]$ is a submodule of $[V^*, \mathfrak{A}_n]$ and so, as $[V^*, \mathfrak{A}_n]$ is irreducible, either $\langle v^{\mathfrak{A}_n} \rangle \cap [V^*, \mathfrak{A}_n] = 0$ or $[V^*, \mathfrak{A}_n] \leq \langle v^{\mathfrak{A}_n} \rangle$. In the former case,

$$1 \leq \dim \langle v^{\mathfrak{A}_n} \rangle = \dim (\langle v^{\mathfrak{A}_n} \rangle + [V^*, \mathfrak{A}_n]) - \dim [V^*, \mathfrak{A}_n] \leq \dim (V^*/[V^*, \mathfrak{A}_n]) = 1$$

so $\langle v^{\mathfrak{A}_n} \rangle = \langle v \rangle = \{0, v\}$ giving $v \in \mathrm{C}_{V^*}(\mathfrak{A}_n) = 0$ which is a contradiction. Hence, $[V^*, \mathfrak{A}_n] \leq \langle v^{\mathfrak{A}_n} \rangle$ and, as $\dim (V^*/[V^*, \mathfrak{A}_n]) = 1$, either $\langle v^{\mathfrak{A}_n} \rangle = V^*$ or $\langle v^{\mathfrak{A}_n} \rangle = [V^*, \mathfrak{A}_n]$. In the latter case, $v \in \langle v^{\mathfrak{A}_n} \rangle = [V^*, \mathfrak{A}_n] = [V^*, K]$ and so $v \in [V^*, K] \cap \mathrm{C}_{V^*}(K) = 0$ which is a contradiction. Hence, $V^* = \langle v^{\mathfrak{A}_n} \rangle$ and V^* is isomorphic to a quotient of the induced module $\langle v \rangle^{\mathfrak{A}_n}$. Now, there exists $u \in \mathrm{C}_U(K)$ such that $U = \langle u^{\mathfrak{A}_n} \rangle$ and, as $\dim U = n = |\mathfrak{A}_n : K| \dim \langle u \rangle$, U is isomorphic to the induced module $\langle u \rangle^{\mathfrak{A}_n}$. As u and v are centralized by K, $\langle u \rangle$ and $\langle v \rangle$ are trivial $\mathbb{F}_2 K$-modules and so $\langle u \rangle^{\mathfrak{A}_n}$ and $\langle v \rangle^{\mathfrak{A}_n}$ are isomorphic $\mathbb{F}_2 \mathfrak{A}_n$-modules. Thus, V^* is isomorphic to a quotient of U.

(6) V is isomorphic to the zero weight submodule of the permutation module.

By (4), $\dim V = \dim \widetilde{V} + \dim \mathrm{C}_V(\mathfrak{A}_n) = n-1$ and therefore, by (5) and Lemma 2.1.3, V is isomorphic to the zero weight submodule of the permutation module. □

§2.3 FF-modules and transvections

We refer the reader to An atlas of Brauer characters [17] and [27, Chapter 16] for the facts about the $\mathbb{F}_2\mathfrak{A}_n$- and $\mathbb{F}_2\mathfrak{S}_n$-modules for $n \in \{5, 7, 8, 9\}$ which are used in this section and the following one.

Apart from the amalgam method failure-of-factorization modules, or FF-modules for short, arise in the study of certain finite groups in the case when Thompson factorization fails [2, Section 32]. Before defining what FF-modules are we give a brief outline of their history. The FF-modules for the groups of Lie type in characteristic two were classified by Cooperstein and Mason in [6] and [7]. Aschbacher [1] classified the FF-modules for the groups of Lie type in odd characteristic, the alternating groups and the sporadic simple groups. The classification of the finite simple groups allows us to conclude that the FF-modules for all of the non-abelian finite simple groups have been classified. More recently, Stroth [30] has extended the work of Aschbacher, Cooperstein and Mason by applying the classification of quadratic modules by Meierfrankenfeld and Stroth. This work has already been extended by Guralnick and Malle in [13]. We will only require Aschbacher's classification of the FF-modules for the alternating and symmetric groups. We pause to make the following observation.

Lemma 2.3.1 *Let G be a finite group, $p \in \mathbb{P}$ and V be an $\mathbb{F}_pG$-module. Assume that $\mathrm{O}^p(G)$ is the unique minimal normal subgroup of G. Then V is a faithful module if and only if V has a non-trivial composition factor, that is $\eta_G(V) \geq 1$.*

Proof Recall from the introduction that $\eta_G(V)$ equals the number of non-trivial composition factors of V. We have that, as $\mathrm{C}_G(V) \trianglelefteq G$,

$$\mathrm{C}_G(V) = 1 \;\Leftrightarrow\; \mathrm{O}^p(G) \not\leq \mathrm{C}_G(V) \;\Leftrightarrow\; [V, \mathrm{O}^p(G)] \neq 1 \;\Leftrightarrow\; \eta_G(V) \geq 1.$$

Thus, V is a faithful module if and only if V has a non-trivial composition factor. □

Corollary 2.3.2 *Let G be a finite group, $p \in \mathbb{P}$ and V be a faithful $\mathbb{F}_pG$-module. Assume that $\mathrm{O}^p(G)$ is the unique minimal normal subgroup of G. Then every minimal faithful submodule of V has exactly one non-trivial composition factor.*

Let G be a finite group, $p \in \mathbb{P}$ and V be a faithful $\mathbb{F}_pG$-module. We call V a *failure-of-factorization module* (or an *FF-module*) if there exists a non-trivial elementary abelian p-subgroup A of G such that[2] $|V/\mathrm{C}_V(A)| \leq |A|$. In this case, A is called an *offending subgroup* for V (or an *offender* for V). Let $\mathcal{A}^*(G,V)$ denote the set of all offending subgroups A for V with $|B|\,|\mathrm{C}_V(B)| < |A|\,|\mathrm{C}_V(A)|$ for all non-trivial proper subgroups B of A. Choose a minimal offending subgroup A for V. If B is a non-trivial proper subgroup of A, then B is not an offending subgroup for V and so $|B|\,|\mathrm{C}_V(B)| < |V| \leq |A|\,|\mathrm{C}_V(A)|$. Thus, $A \in \mathcal{A}^*(G,V)$.

Recall that a *section* of an $\mathbb{F}G$-module V is a quotient module of the form U/Z where U and Z are submodules of V.

Lemma 2.3.3 *Let G be a finite group, $p \in \mathbb{P}$ and V be an $\mathbb{F}_pG$-module. Let W be a section of V and A be a non-empty subset of G. Then $|W/\mathrm{C}_W(A)| \leq |V/\mathrm{C}_V(A)|$.*

Proof Let $W = U/Z$ where U and Z are submodules of V. Then, as $(\mathrm{C}_U(A)+Z)/Z \leq \mathrm{C}_{U/Z}(A)$,

$$\begin{aligned}|W/\mathrm{C}_W(A)| &= |U/Z/\mathrm{C}_{U/Z}(A)| \leq |U/Z/(\mathrm{C}_U(A)+Z)/Z| = |U/(\mathrm{C}_U(A)+Z)| \\ &\leq |U/\mathrm{C}_U(A)| = |U/(U \cap \mathrm{C}_V(A))| = |(U+\mathrm{C}_V(A))/\mathrm{C}_V(A)| \\ &\leq |V/\mathrm{C}_V(A)|.\end{aligned}$$

□

Corollary 2.3.4 *Let G be a finite group, $p \in \mathbb{P}$ and V be a failure-of-factorization $\mathbb{F}_pG$-module with offending subgroup A. Then all faithful sections of V are failure-of-factorization modules with offending subgroup A.*

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[2] It is conventional to phrase such conditions in terms of order rather than dimension.

The theorem below follows from the classification of the failure-of-factorization modules for the alternating and symmetric groups by Aschbacher [1].

Theorem 2.3.5 (Aschbacher) *Let $G \cong \mathfrak{S}_n$ for $n \geq 7$, $n \neq 8$ or $G \cong \mathfrak{A}_n$ for $n \geq 9$ and set $E = \mathrm{O}^2(G)$. Let V be a failure-of-factorization $\mathbb{F}_2G$-module and set $W = [V, E]$. Then W is isomorphic to either the natural module for G or the zero weight submodule of the permutation module for G.*

Proof As observed above, $\mathcal{A}^*(G, V)$ is non-empty and so we may apply [1, 8.5 on page 721] in order to deduce that $W/\mathrm{C}_W(E)$ is isomorphic to the natural module for E. Now, $[W, E] = [V, E, E] = [V, E] = W$ and therefore, by Lemma 2.2.3 and Theorem 2.2.6, W is isomorphic to either the natural module for G or the zero weight submodule of the permutation module for G. □

Corollary 2.3.6 *Let $G \cong \mathfrak{S}_n$ for $n \geq 7$ or $G \cong \mathfrak{A}_n$ for $n \geq 8$ and set $E = \mathrm{O}^2(G)$. Let V be a failure-of-factorization $\mathbb{F}_2G$-module, set $W = [V, E]$ and assume that $\mathrm{C}_W(E) \neq 0$. Then n is even and W is isomorphic to the zero weight submodule of the permutation module for G.*

Proof By Lemma 2.1.4(iii), the natural module for G is irreducible and hence the result follows immediately from the theorem above provided that $n \neq 8$. In the case when $n = 8$, [1, 8.5 on page 721] gives us that either $W/\mathrm{C}_W(E)$ is isomorphic to the natural module for E or $V = W \oplus \mathrm{C}_V(E)$ in which case $\mathrm{C}_W(E) = W \cap \mathrm{C}_V(E) = 0$ which is a contradiction. Thus, by Lemma 2.2.3 and Theorem 2.2.6, W is isomorphic to the zero weight submodule of the permutation module for G. □

Let G be a finite group, $p \in \mathbb{P}$, V be a faithful $\mathbb{F}_pG$-module and $k \in \mathbb{N}$. We say that an element $t \in G$ of order p *induces a k-transvection on V* if $|V/\mathrm{C}_V(t)| \leq p^k$ and that a non-trivial elementary abelian p-subgroup A of G *induces k-transvections on V* if $|V/\mathrm{C}_V(A)| \leq p^k$. In the case $k = 1$, we will also say that t *induces a transvection on V* and that A *induces transvections on V*. We will use the adjective $[S$-$]$*central* in conjunction with these terms in order to indicate that t is in the centre of S and that A is normal in S where S is a Sylow p-subgroup of G. Observe that if A induces k-transvections on V, then $|V/\mathrm{C}_V(a)| \leq |V/\mathrm{C}_V(A)| \leq p^k$ for all $a \in A$ and so every non-trivial element of A induces a k-transvection on V. It follows from Lemma 2.3.3 that if t or A induce k-transvections on V, then they induce k-transvections on all faithful sections of V. Also, if t induces a k-transvection on V, then $|V^*/\mathrm{C}_{V^*}(t)| = |[V,t]| = |V/\mathrm{C}_V(t)| \leq p^k$ and hence t induces a k-transvection on the dual V^* of V.

Lemma 2.3.7 *Let G be a finite group, $p \in \mathbb{P}$, set $E = \mathrm{O}^p(G)$ and assume that E is the unique minimal normal subgroup of G. Let V be a faithful $\mathbb{F}_pG$-module and set $W = [V,E]$. Assume that $t \in G$ induces a k-transvection on V and let $m = m(t)$ be the minimal number of conjugates of t which generate the group $\langle t^G \rangle$. Then W is a faithful $\mathbb{F}_pG$-module of dimension at most km and t induces a k-transvection on W.*

Proof We have that $\eta_G(V) = \eta_G(V/W) + \eta_G(W) = \eta_G(W)$ and so, by Lemma 2.3.1, W is a faithful $\mathbb{F}_pG$-module. Let $t_1, t_2, \ldots, t_m$ be conjugates of t such that $\langle t^G \rangle = \langle t_1, t_2, \ldots, t_m \rangle$. Then, for each $1 \leq i \leq m$, $|[V,t_i]| = |[V,t]| = |V/\mathrm{C}_V(t)| \leq p^k$. As $\langle t^G \rangle$ is a non-trivial normal subgroup of G, $E \leq \langle t^G \rangle$ and so

$$W = [V,E] \leq [V, \langle t^G \rangle] = [V,t_1] + [V,t_2] + \cdots + [V,t_m]$$

giving $\dim W \leq \sum_{i=1}^{m} \dim[V,t_i] \leq km$. Also, W is a faithful submodule of V and hence, by Lemma 2.3.3, t induces a k-transvection on W. □

If $G = \mathfrak{S}_n$ or $\mathfrak{A}_n$ for $n \geq 7$ and $t \in G$, then, by [14, 6.1], $m(t) = n-1$ when t is a transposition and $m(t) \leq \lfloor n/2 \rfloor$ otherwise. Hence, we may use the lemma above together with Theorem 2.1.8 in order to prove the following result provided that n is sufficiently large. However, we will deduce it from Theorem 2.3.5 together with Corollary 2.1.17(i) instead.

Theorem 2.3.8 *Let $G = \mathfrak{S}_n$ or $\mathfrak{A}_n$ for $n \geq 5$, $n \neq 6$ and set $E = \mathrm{O}^2(G)$. Let V be a faithful $\mathbb{F}_2G$-module and set $W = [V, E]$. The following hold:*

(i) If $G = \mathfrak{S}_n$ and $t \in G$ induces a transvection on V, then t is a transposition and W is isomorphic to either the natural module for G or the zero weight submodule of the permutation module for G.

(ii) If $G = \mathfrak{A}_n$, then no involution in G induces a transvection on V except when $n = 8$, t is a fixed-point-free involution and W is isomorphic to a natural $\mathrm{SL}_4(2)$-module for G.

Proof Assume that $t \in G$ induces a transvection on V. Then t is not similar to $(1\ 2)(3\ 4)$ $[(1\ 2)(3\ 4)(5\ 6)]$ because otherwise, as $H = \mathfrak{A}_5$ $[H = \mathfrak{S}_7]$ is generated by three conjugates of t and by applying Lemma 2.3.7, $\dim [V_H, \mathrm{O}^2(H)] \leq 3$ and so $[V_H, \mathrm{O}^2(H)]$ is not a faithful $\mathbb{F}_2H$-module which is a contradiction. In the case when $G = \mathfrak{S}_5$, we have that t is a transposition and, as G is generated by four conjugates of t and by applying Lemma 2.3.7, $\dim W \leq 4$ and an inspection of the two irreducibles for G over $\mathbb{F}_2$ yields that W is isomorphic to the natural module for G. In the case when $G = \mathfrak{S}_8$ or $\mathfrak{A}_8$ and t is similar to $(1\ 2)(3\ 4)(5\ 6)(7\ 8)$, as $\mathfrak{A}_8$ is generated by four conjugates of t and by applying Lemma 2.3.7, $\dim W \leq 4$ and hence, as $\mathrm{SL}_4(2) \cong \mathfrak{A}_8$, $G = \mathfrak{A}_8$ and W is isomorphic to a natural $\mathrm{SL}_4(2)$-module for G. In the case when $G = \mathfrak{S}_7$ or $\mathfrak{S}_8$, we have that t is a transposition and, as G is generated by seven conjugates of t and by applying Lemma 2.3.7, $\dim W \leq 7$ and hence W is isomorphic to the natural module for G. Now, let $n \geq 9$ and set $A = \langle t \rangle$. Then $|V/\,\mathrm{C}_V(A)| \leq 2 = |A|$ and so V is a failure-of-factorization module giving, by Theorem 2.3.5, W is isomorphic to either the natural module for G or the zero weight submodule of the permutation module for G. By Lemma 2.3.7, t induces a transvection on W and hence, by Corollary 2.1.17(i), $G = \mathfrak{S}_n$ and t is a transposition. □

Corollary 2.3.9 *Let $G = \mathfrak{S}_n$ for $n \geq 5$, $n \neq 6$, A be a non-trivial elementary abelian 2-subgroup of G and V be a faithful $\mathbb{F}_2G$-module. If A induces transvections on V, then $A = \langle t \rangle$ for some transposition $t \in A$.*

Proof Assume that A induces transvections on V and fix $t \in A^{\#}$. Then t induces a transvection on V and so, by the theorem above, t is a transposition. If $s \in A^{\#}$, then s is also a transposition and so, as st is not a transposition, $st = 1$ giving $s = t$. Thus, $A = \langle t \rangle$. □

Corollary 2.3.10 *Let $G = \mathfrak{A}_8$, A be a non-trivial elementary abelian 2-subgroup of G and V be a faithful $\mathbb{F}_2G$-module. If A induces transvections on V, then A is similar to a subgroup of a regular eight-group $\mathfrak{K}_8$.*

Proof Assume that A induces transvections on V. Then, by the theorem above, A is a semiregular elementary abelian 2-group and hence A is similar to a subgroup of $\mathfrak{K}_8$. □

A calculation reveals that each of the two $\mathfrak{A}_8$-conjugacy classes of regular eight-groups $\mathfrak{K}_8$ induces central transvections on exactly one of the natural $\mathrm{SL}_4(2)$-modules for $\mathfrak{A}_8$ depending on the class.

§2.4 Quadratic action and spin modules

Let G be a finite group, $p \in \mathbb{P}$ and V be a faithful $\mathbb{F}_pG$-module. We say that a non-trivial element $t \in G$ acts *quadratically* on V if $[V, t, t] = 0$ and that a non-trivial subgroup A of G acts *quadratically* on V if $[V, A, A] = 0$.

Lemma 2.4.1 *Let G be a finite group, $p \in \mathbb{P}$ and V be a faithful $\mathbb{F}_pG$-module. Let A be a subgroup of G which acts quadratically on V. The following hold:*

(i) For each $v \in V$, the map $\phi_v : A \longrightarrow V : a \longmapsto [v, a]$ is a homomorphism of groups.

(ii) A is an elementary abelian p-group.

(iii) A acts quadratically on all faithful sections of V and on the dual V^* of V.

Proof (i) Fix $v \in V$. Then, as A acts quadratically on V, $[V, A] \leq \mathrm{C}_V(A)$ and so, for each $a, b \in A$,

$$\phi_v(ab) = [v, ab] = [v, a]^b + [v, b] = [v, a] + [v, b] = \phi_v(a) + \phi_v(b).$$

Thus, ϕ_v is a homomorphism of groups.

(ii) We have that $[A, V, A] = [V, A, A] = 0$ and so, by applying the Three-Subgroup Lemma and as V is faithful, $[A, A] \leq \mathrm{C}_A(V) = 1$. Hence, A is abelian. If $a \in A$, then, by part (i), $[v, a^p] = \phi_v(a^p) = p\phi_v(a) = p[v, a] = 0$ for all $v \in V$ and so $a^p \in \mathrm{C}_A(V) = 1$ giving $a^p = 1$. Thus, A is an elementary abelian p-group.

(iii) Let $W = U/Z$ be a faithful section of V where U and Z are submodules of V. Then

$$[W, A, A] = [U/Z, A, A] \leq [V/Z, A, A] = ([V, A, A] + Z)/Z = Z/Z$$

and hence A acts quadratically on W. Recall that $[V^*, A] \cong_A (V/\mathrm{C}_V(A))^*$ and so

$$[V^*, A, A] \cong_A [(V/\mathrm{C}_V(A))^*, A] \cong_A \left(V/\mathrm{C}_V(A) \big/ \mathrm{C}_{V/\mathrm{C}_V(A)}(A)\right)^*.$$

As A acts quadratically on V, $[V, A] \leq \mathrm{C}_V(A)$ and so

$$[V/\mathrm{C}_V(A), A] = ([V, A] + \mathrm{C}_V(A))/\mathrm{C}_V(A) = \mathrm{C}_V(A)/\mathrm{C}_V(A)$$

giving $\mathrm{C}_{V/\mathrm{C}_V(A)}(A) = V/\mathrm{C}_V(A)$. Thus, $[V^*, A, A] = 0$. □

Let G be a finite group, $p \in \mathbb{P}$ and V be a faithful $\mathbb{F}_pG$-module. Let A be a subgroup of G which acts quadratically on V. Then, by part (ii) of the lemma above, we may regard A as an $\mathbb{F}_pA$-module with respect to action by conjugation and the map ϕ_v defined in part (i) of the lemma above is a linear map from A to $[V, A]$. Let $\lambda \in \mathbb{F}_p$, $v, v_1, v_2 \in V$ and $x \in A$. Then, for each $a \in A$,

$$\phi_{\lambda v_1 + v_2}(a) = [\lambda v_1 + v_2, a] = \lambda[v_1, a] + [v_2, a] = \lambda\phi_{v_1}(a) + \phi_{v_2}(a) = (\lambda\phi_{v_1} + \phi_{v_2})(a)$$

and

$$\phi_v^x(a) = \phi_v(a^{x^{-1}})^x = [v, a^{x^{-1}}]^x = [v^x, a] = \phi_{v^x}(a)$$

and hence $\phi_{\lambda v_1 + v_2} = \lambda\phi_{v_1} + \phi_{v_2}$ and $\phi_v^x = \phi_{v^x}$. We have shown that the map $\phi : V \longrightarrow \mathrm{Hom}(A, [V, A]) : v \longmapsto \phi_v$ is an A-linear map of $\mathbb{F}_pA$-modules. Moreover, the kernel of ϕ is $\mathrm{C}_V(A)$ and therefore $V/\,\mathrm{C}_V(A)$ embeds into $\mathrm{Hom}(A, [V, A])$.

Now, if $p = 2$ and $t \in G$ is an involution, then t acts quadratically on V because, for each $v \in V$,

$$[v, t, t] = (v^t + v)^t + (v^t + v) = v^{t^2} + v^t + v^t + v = 2(v^t + v) = 0$$

giving $[V, t, t] = 0$. We have already used this fact several times. If $t \in G$ induces a transvection on V, then t acts quadratically on V because, as $[V/\,\mathrm{C}_V(t), t] < V/\,\mathrm{C}_V(t)$,

$$([V, t] + \mathrm{C}_V(t))/\,\mathrm{C}_V(t) = [V/\,\mathrm{C}_V(t), t] = \mathrm{C}_V(t)/\,\mathrm{C}_V(t)$$

and so $[V, t] \leq \mathrm{C}_V(t)$ giving $[V, t, t] = 0$. In general, if V is a failure-of-factorization module, then we may choose an offending subgroup $A \in \mathcal{A}^*(G, V)$ and, by [19, 9.2.4 on page 209], A acts quadratically on V.

Lemma 2.4.2 *Let G be a finite group and V be a faithful $\mathbb{F}_2G$-module. Let A be an elementary abelian 2-subgroup of G of order at least 2^3 and assume that $t \in A$ induces a 2-transvection on V. Then there exists a subgroup F of A containing t such that $|F| = 4$ and F acts quadratically on V.*

Proof Suppose, for a contradiction, that $\mathrm{C}_A([V,t]) = \langle t\rangle$. Then $|[V,t]| = |V/\mathrm{C}_V(t)| \le 2^2$ and so $[V,t]$ is an $\mathbb{F}_2A$-module of dimension at most 2. So $A/\langle t\rangle = A/\mathrm{C}_A([V,t])$ embeds into $\mathrm{SL}_2(2)$ and hence $|A|$ divides $|\mathrm{SL}_2(2)|\,|\langle t\rangle| = 3.2^2$ giving $|A| \le 2^2$ which is a contradiction. So we may choose $s \in \mathrm{C}_A([V,t]) \setminus \langle t\rangle$. Define $F = \langle s,t\rangle$. Then

$$[V,F,F] = [V,s,s] + [V,s,t] + [V,t,s] + [V,t,t].$$

We have that $[V,s,s] = 0 = [V,t,t]$ and, for each $v \in V$, as s and t commute,

$$\begin{aligned}[v,s,t] &= (v^s+v)^t + (v^s+v) = v^{st} + v^t + v^s + v = v^{ts} + v^s + v^t + v \\ &= (v^t+v)^s + (v^t+v) = [v,t,s]\end{aligned}$$

giving, by choice of s, $[V,s,t] = [V,t,s] = 0$. Thus, $[V,F,F] = 0$. □

Corollary 2.4.3 *Let $G = \mathfrak{S}_n$ for $n \ge 6$ or $G = \mathfrak{A}_n$ for $n \ge 8$ and V be a faithful $\mathbb{F}_2G$-module. If $t \in G$ induces a 2-transvection on V, then there exists a subgroup F of G containing t such that $|F| = 4$ and F acts quadratically on V.*

Proof This result follows immediately from the lemma above because every involution in G is contained in an elementary abelian 2-group of order 2^3. □

We now turn our attention to spin modules for $\mathfrak{S}_n$ and $\mathfrak{A}_n$ over $\mathbb{F}_2$. Let $n \ge 5$ and $G = \mathfrak{S}_n$ or $\mathfrak{A}_n$. Let V be a faithful irreducible $\mathbb{F}_2G$-module and assume that the three-cycles in G act fixed-point-freely on V. Then V is called a *spin module* for G over $\mathbb{F}_2$. If V is a spin module for $\mathfrak{A}_n$ over $\mathbb{F}_2$, then, with respect to the action of the group algebra $\mathbb{F}_2\mathfrak{A}_n$ on V, $V^{(d^2+d+1)} = 0$ for all three-cycles $d \in \mathfrak{A}_n$ because, for each $v \in V$,

$$\left(v^{(d^2+d+1)}\right)^d = v^{(d^2+d+1)d} = v^{(d^3+d^2+d)} = v^{(d^2+d+1)}$$

and so $v^{(d^2+d+1)} \in \mathrm{C}_V(d) = 0$. Thus, it was shown in [25, Theorem 2(i) on page 240] that a spin module for $\mathfrak{A}_n$ over $\mathbb{F}_2$ is indeed a spin module defined in terms of Clifford algebras [4]. In particular, there are, up to isomorphism, one or two non-isomorphic spin modules for $\mathfrak{A}_n$ over $\mathbb{F}_2$. We will assume the existence of spin modules for $\mathfrak{S}_n$ over $\mathbb{F}_2$. Observe that there is a single conjugacy class of three-cycles in G and hence in order to identify a spin module for G over $\mathbb{F}_2$, it suffices to verify that a single three-cycle acts fixed-point-freely on V. By coprime action, a three-cycle $d \in G$ acts fixed-point-freely on V if and only if $V = [V, d]$. In particular, the dual of a spin module for G over $\mathbb{F}_2$ is also a spin module for G over $\mathbb{F}_2$. The dimension of a spin module V for G over $\mathbb{F}_2$ is even because we may choose an involution $t \in G$ which inverts a three-cycle $d \in G$ by conjugation, then $D = \langle t, d \rangle$ is the dihedral group of order 6 with $\mathrm{C}_V(D) \leq \mathrm{C}_V(d) = 0$ and hence, by Lemma 2.1.6, $\dim V$ is even. The next lemma is also useful when identifying the spin modules for $\mathfrak{S}_n$ over $\mathbb{F}_2$.

Lemma 2.4.4 *Let $n \geq 5$ and V be a faithful irreducible $\mathbb{F}_2\mathfrak{S}_n$-module. Then V is a spin module for $\mathfrak{S}_n$ if and only if $V_{\mathfrak{A}_n}$ is either a spin module for $\mathfrak{A}_n$ or a direct sum of two non-isomorphic spin modules for $\mathfrak{A}_n$.*

Proof If V is a spin module for $\mathfrak{S}_n$, then, by Clifford's Theorem [27, 2.37 on page 30], $V_{\mathfrak{A}_n}$ is either a spin module for $\mathfrak{A}_n$ or a direct sum of two non-isomorphic spin modules for $\mathfrak{A}_n$. Conversely, if $V_{\mathfrak{A}_n}$ is a spin module for $\mathfrak{A}_n$, then it is a spin module for $\mathfrak{S}_n$ and if $V_{\mathfrak{A}_n} = U_1 \oplus U_2$ where U_1 and U_2 are non-isomorphic spin modules for $\mathfrak{A}_n$, then $\mathrm{C}_V(d) = \mathrm{C}_{U_1}(d) \oplus \mathrm{C}_{U_2}(d) = 0$ for all three-cycles $d \in \mathfrak{A}_n$ and hence V is a spin module for $\mathfrak{S}_n$. □

We will now give a few examples of spin modules. The $\mathbb{F}_2\mathfrak{A}_5$- and $\mathbb{F}_2\mathfrak{S}_5$-modules arising from the natural $\mathrm{SL}_2(4)$-module for $\mathfrak{A}_5$ are the unique spin modules for $\mathfrak{A}_5$ and $\mathfrak{S}_5$ over $\mathbb{F}_2$ respectively. The natural $\mathrm{SL}_4(2)$-modules for $\mathfrak{A}_8$ are the two spin modules for $\mathfrak{A}_8$ over $\mathbb{F}_2$ and the $\mathbb{F}_2\mathfrak{S}_8$-module induced from either of these two modules is the unique spin module for $\mathfrak{S}_8$ over $\mathbb{F}_2$. Moreover, the spin modules for $\mathfrak{A}_8$ and $\mathfrak{S}_8$ are, by restriction, the spin modules for $\mathfrak{A}_7$ and $\mathfrak{S}_7$ respectively. The two spin modules for $\mathfrak{A}_9$ over $\mathbb{F}_2$ each have dimension eight and the spin module for $\mathfrak{S}_9$ over $\mathbb{F}_2$ has dimension sixteen.

In the theorem below we will show that a spin module for $\mathfrak{S}_n$ or $\mathfrak{A}_n$ over $\mathbb{F}_2$ has no extension by trivial modules.

Theorem 2.4.5 *Let $G = \mathfrak{S}_n$ or $\mathfrak{A}_n$ for $n \geq 7$ and V be an $\mathbb{F}_2G$-module with $[V, \mathfrak{A}_n] = V$ and $V/\operatorname{C}_V(\mathfrak{A}_n)$ a spin module for G. Then $\operatorname{C}_V(\mathfrak{A}_n) = 0$ and V is a spin module for G.*

Proof We will prove the dual statement. Let $\operatorname{C}_V(\mathfrak{A}_n) = 0$ and $[V, \mathfrak{A}_n]$ be a spin module for G and suppose, for a contradiction, that $[V, \mathfrak{A}_n] \neq V$. Then, by refining the series $[V, \mathfrak{A}_n] < V$ to a composition series of V, we may choose a submodule U of V with $[V, \mathfrak{A}_n] \leq U$ and $\dim(U/[V, \mathfrak{A}_n]) = 1$. We have that, as $\mathfrak{A}_n = \operatorname{O}^2(G)$, $[V, \mathfrak{A}_n] = [V, \mathfrak{A}_n, \mathfrak{A}_n] \leq [U, \mathfrak{A}_n] \leq [V, \mathfrak{A}_n]$ and so $[U, \mathfrak{A}_n] = [V, \mathfrak{A}_n]$. Let $d \in G$ be a three-cycle. Then, by coprime action, $U = [U, d] + \operatorname{C}_U(d) = [U, \mathfrak{A}_n] + \operatorname{C}_U(d)$ and, as $[U, \mathfrak{A}_n]$ is a spin module for G, $[U, \mathfrak{A}_n] \cap \operatorname{C}_U(d) = \operatorname{C}_{[U,\mathfrak{A}_n]}(d) = 0$ giving $U = [U, \mathfrak{A}_n] \oplus \operatorname{C}_U(d)$ as subspaces of U. Hence, $\dim \operatorname{C}_U(d) = \dim(U/[U, \mathfrak{A}_n]) = 1$. Then, as $\operatorname{C}_U(d)$ is $\operatorname{C}_{\mathfrak{A}_n}(d)$-invariant and of dimension one, $\operatorname{C}_U(d) \leq \operatorname{C}_U(\operatorname{C}_{\mathfrak{A}_n}(d))$ and, as $d \in \operatorname{C}_{\mathfrak{A}_n}(d)$, $\operatorname{C}_U(\operatorname{C}_{\mathfrak{A}_n}(d)) \leq \operatorname{C}_U(d)$ giving $\operatorname{C}_U(d) = \operatorname{C}_U(\operatorname{C}_{\mathfrak{A}_n}(d))$. Now, consider the group $\langle \operatorname{C}_{\mathfrak{A}_n}(d_1), \operatorname{C}_{\mathfrak{A}_n}(d_2)\rangle$ where $d_1 = (1\ 2\ 3)$ and $d_2 = (n-2\ n-1\ n)$. We have that

$$\operatorname{C}_{\mathfrak{A}_n}(d_2) = \mathfrak{A}_{n-3} \times \langle d_2 \rangle = \langle (1\ 2\ 3), (1\ 2\ 4), \ldots, (1\ 2\ n-3), (n-2\ n-1\ n)\rangle$$

and, as $n \geq 7$, $(n-3\ n-2\ n-1) \in \operatorname{C}_{\mathfrak{A}_n}(d_1)$. Also, $(1\ 2\ n-2) = (1\ 2\ n-3)^{(n-3\,n-2\,n-1)}$, $(1\ 2\ n-1) = (1\ 2\ n-2)^{(n-2\,n-1\,n)}$ and $(1\ 2\ n) = (1\ 2\ n-1)^{(n-2\,n-1\,n)}$ and hence $\mathfrak{A}_n = \langle (1\ 2\ 3), (1\ 2\ 4), \ldots, (1\ 2\ n)\rangle = \langle \operatorname{C}_{\mathfrak{A}_n}(d_1), \operatorname{C}_{\mathfrak{A}_n}(d_2)\rangle$. As d_1 and d_2 commute,

$$\operatorname{C}_U(d_1) = \operatorname{C}_U(\operatorname{C}_{\mathfrak{A}_n}(d_1)) \leq \operatorname{C}_U(d_2) = \operatorname{C}_U(\operatorname{C}_{\mathfrak{A}_n}(d_2)) \leq \operatorname{C}_U(d_1)$$

and so $\operatorname{C}_U(d_1) = \operatorname{C}_U(d_2)$. Then, as $\mathfrak{A}_n = \langle \operatorname{C}_{\mathfrak{A}_n}(d_1), \operatorname{C}_{\mathfrak{A}_n}(d_2)\rangle$, $\operatorname{C}_U(d_1)$ is $\mathfrak{A}_n$-invariant and of dimension one and so $\operatorname{C}_U(d_1) \leq \operatorname{C}_V(\mathfrak{A}_n) = 0$ which is a contradiction. Therefore, $V = [V, \mathfrak{A}_n]$ is a spin module for G. □

According to [16, Table VI on page 167] the theorem above does not hold for the groups $G = \mathfrak{S}_5$ and $G = \mathfrak{A}_5$.

Lemma 2.4.6 *Let $G = \mathfrak{S}_n$ or $\mathfrak{A}_n$ for $n \geq 5$ and V be a spin module for G over $\mathbb{F}_2$. Let A be a subgroup of G which acts quadratically on V and t be an involution in A with a fixed point on the set $\Delta = \{1, 2, \ldots, n\}$. Then $[V, t] = [V, A] = \mathrm{C}_V(A) = \mathrm{C}_V(t)$ and $|V/\mathrm{C}_V(A)| = |V|^{1/2}$.*

Proof As t has a fixed point on Δ, we may choose a three-cycle $d \in G$ which is inverted by t by conjugation and so $D = \langle t, d\rangle$ is the dihedral group of order 6 and $\mathrm{C}_V(D) \leq \mathrm{C}_V(d) = 0$. We have that, as A acts quadratically on V, $[V, t] \leq [V, A] \leq \mathrm{C}_V(A) \leq \mathrm{C}_V(t)$ and, by Lemma 2.1.6, $[V, t] = \mathrm{C}_V(t)$ and hence $[V, t] = [V, A] = \mathrm{C}_V(A) = \mathrm{C}_V(t)$. Also, by Lemma 2.1.6, $|\mathrm{C}_V(t)| = |V|^{1/2}$ and hence $|V/\mathrm{C}_V(A)| = |V/\mathrm{C}_V(t)| = |V|^{1/2}$. □

We will see in the proof of Theorem 2.4.8 that all subgroups of order at least 4 which act quadratically on a spin module for $\mathfrak{S}_n$ or $\mathfrak{A}_n$ over $\mathbb{F}_2$ are contained in $\mathfrak{A}_n$ and hence it suffices to determine these groups.

Lemma 2.4.7 *Let $G = \mathfrak{S}_n$ or $\mathfrak{A}_n$ for $n \geq 5$, $n \neq 6$. Let V be a spin module for G over $\mathbb{F}_2$ and A be a subgroup of $\mathfrak{A}_n$ of order at least 4 which acts quadratically on V. Then one of the following holds:*

(i) A is similar to a Klein four-group $\mathfrak{K}_4$.

(ii) $G \cong \mathfrak{A}_8$ or $\mathfrak{S}_8$ and A is similar to a subgroup of the group $\mathfrak{K}_4 \times \mathfrak{K}_4$.

(iii) $G \cong \mathfrak{A}_8$, $\mathfrak{S}_8$ or $\mathfrak{A}_9$ and A is similar to a subgroup of a regular eight-group $\mathfrak{K}_8$.

Proof It suffices to consider the case when A is maximal among the subgroups of $\mathfrak{A}_n$ which act quadratically on V. By Lemma 2.4.1(ii), A is an elementary abelian 2-group. Choose an element $s \in A^{\#}$ with the largest number of fixed points on the set $\Delta = \{1, 2, \ldots, n\}$ among all of the elements of $A^{\#}$. Then s is an involution that is the product of m disjoint transpositions where, as $s \in \mathfrak{A}_n$, m is even. Without loss of generality we may assume that $s = (1\ 2)(3\ 4)\ldots(2m-1\ 2m)$. Define $H = \mathrm{C}_{\mathfrak{S}_n}(s)$ and $K = \mathrm{C}_{\mathfrak{A}_n}(s)$. Then $K = H \cap \mathfrak{A}_n$ where H is similar to the permutation group $\mathfrak{C}_2 \wr \mathfrak{S}_m \times \mathfrak{S}_{n-2m}$ and, as K is normal in H, $\mathrm{O}_2(K) = \mathrm{O}_2(H) \cap K = \mathrm{O}_2(H) \cap \mathfrak{A}_n$ where $\mathrm{O}_2(H)$ is similar to the permutation group

$\mathfrak{C}_2 \wr \mathrm{O}_2(\mathfrak{S}_m) \times \mathrm{O}_2(\mathfrak{S}_{n-2m})$. Recall that $\mathrm{O}_2(\mathfrak{S}_2) = \mathfrak{S}_2 = \mathfrak{C}_2$, $\mathrm{O}_2(\mathfrak{S}_4) = \mathfrak{K}_4$ and $\mathrm{O}_2(\mathfrak{S}_k) = 1$ otherwise. Define $L = \langle A^K \rangle$, $M = \langle (1\ 2), (3\ 4), \ldots, (2m-1\ 2m) \rangle$ and $M_0 = M \cap \mathfrak{A}_n$. Then, as A is abelian, $A \leq \mathrm{C}_{\mathfrak{A}_n}(s) = K$ and so L is a normal subgroup of K. As M_0 is a normal 2-subgroup of K, $M_0 \leq \mathrm{O}_2(K)$ and, as m is even and by choice of s, $A \cap M_0 = \langle s \rangle$. We will proceed via a series of steps.

(1) $[V, L] \leq \mathrm{C}_V(s)$ *and* $[V, s] \leq \mathrm{C}_V(L)$.

We may regard $[V, \langle s \rangle]$ as a K-submodule of V and so, as $[V, \langle s \rangle, A] \leq [V, A, A] = 0$, $[V, \langle s \rangle, L] = [V, \langle s \rangle, \langle A^K \rangle] = \langle [V, \langle s \rangle, A]^K \rangle = 0$. Also, $[\langle s \rangle, L, V] \leq [\langle s \rangle, K, V] = [1, V] = 0$ and hence, by the Three-Subgroup Lemma, $[V, L, \langle s \rangle] = [L, V, \langle s \rangle] = 0$. Thus, $[V, L] \leq \mathrm{C}_V(s)$ and $[V, s] \leq \mathrm{C}_V(L)$.

(2) Either s *has a fixed point on* Δ *or* $(n, m) = (8, 4)$.

Let $n \neq 8$ and suppose, for a contradiction, that s is fixed-point-free on Δ. Then n is even, $m = n/2$ and, by choice of s, A is semiregular on Δ. In particular, $|A|$ divides $|\Delta| = n$ and so 4 divides n giving $n \geq 12$. Also, as $m \notin \{2, 4\}$, $\mathrm{O}_2(H) = M$ and $\mathrm{O}_2(K) = M_0$.

(2a) $K/\mathrm{O}_2(K) \cong \mathfrak{S}_{n/2}$.

As $m \notin \{2, 4\}$, H is similar to the wreath product $\mathfrak{C}_2 \wr \mathfrak{S}_{n/2}$ with base group $\mathrm{O}_2(H)$ giving $H/\mathrm{O}_2(H) \cong \mathfrak{S}_{n/2}$. Then, as $|\mathfrak{S}_n : \mathfrak{A}_n| = 2$ and $H \not\leq \mathfrak{A}_n$, $\mathfrak{S}_n = H\mathfrak{A}_n$ and so

$$|H : K\,\mathrm{O}_2(H)|\ |K\,\mathrm{O}_2(H) : K| = |H : K| = |H : (H \cap \mathfrak{A}_n)| = |H\mathfrak{A}_n : \mathfrak{A}_n| = |\mathfrak{S}_n : \mathfrak{A}_n| = 2$$

giving, as $\mathrm{O}_2(H) \not\leq K$, $H = K\,\mathrm{O}_2(H)$. Thus,

$$K/\mathrm{O}_2(K) = K/(\mathrm{O}_2(H) \cap K) \cong K\,\mathrm{O}_2(H)/\mathrm{O}_2(H) = H/\mathrm{O}_2(H) \cong \mathfrak{S}_{n/2}.$$

(2b) $\mathrm{O}^2(K) \leq L$. *In particular, all of the elements of* K *of odd order are contained in* L.

We have that $L \not\leq \mathrm{O}_2(K)$ because otherwise $A \leq L \leq \mathrm{O}_2(K) = M_0$ which is a contradiction. Then $1 \neq L\,\mathrm{O}_2(K)/\mathrm{O}_2(K) \trianglelefteq K/\mathrm{O}_2(K)$ and so, by (2a), $\mathrm{O}^2(K/\mathrm{O}_2(K)) \leq L\,\mathrm{O}_2(K)/\mathrm{O}_2(K)$ giving $|K : L\,\mathrm{O}_2(K)| = |K/\mathrm{O}_2(K) : L\,\mathrm{O}_2(K)/\mathrm{O}_2(K)|$ is a 2-power. We also have that $|L\,\mathrm{O}_2(K) : L| = |\mathrm{O}_2(K) : (\mathrm{O}_2(K) \cap L)|$ is a 2-power and hence $|K : L| =$

$|K\!:\!L\,\mathrm{O}_2(K)|\,|L\,\mathrm{O}_2(K)\!:\!L|$ is a 2-power. Thus, $\mathrm{O}^2(K) \leq L$.

(2c) $\dim \mathrm{C}_V(e) \geq \frac{1}{2}\dim V$ *for all elements* e *of* L *of odd order.*

Write $s = (1\ 2)(3\ 4)s_1$ and let $r = (1\ 3)(2\ 4)s_1$ so that $sr = (1\ 4)(2\ 3)$. As $\mathrm{C}_V(s)\cap\mathrm{C}_V(r) = \mathrm{C}_V(\langle s, r\rangle) \leq \mathrm{C}_V(sr)$ and by applying Lemma 2.4.6, $\dim(\mathrm{C}_V(s) \cap \mathrm{C}_V(r)) \leq \dim \mathrm{C}_V(sr) = \frac{1}{2}\dim V$ and, as s and r are $\mathfrak{A}_n$-conjugate, $\dim \mathrm{C}_V(s) = \dim \mathrm{C}_V(r)$. So

$$\begin{aligned}\dim V &\geq \dim(\mathrm{C}_V(s) + \mathrm{C}_V(r)) = \dim \mathrm{C}_V(s) + \dim \mathrm{C}_V(r) - \dim(\mathrm{C}_V(s) \cap \mathrm{C}_V(r))\\ &\geq 2\ \dim \mathrm{C}_V(s) - \tfrac{1}{2}\dim V\end{aligned}$$

and so $\dim \mathrm{C}_V(s) \leq \frac{3}{4}\dim V$ giving $\dim(V/\,\mathrm{C}_V(s)) \geq \frac{1}{4}\dim V$. Now, let $e \in L$ be of odd order. Then, by (1), $[V, e] \leq [V, L] \leq \mathrm{C}_V(s)$ and so, by coprime action, $V = [V, e] + \mathrm{C}_V(e) = \mathrm{C}_V(s) + \mathrm{C}_V(e)$. Also, as $[V, A, A] = 0$, $[V, s] \leq [V, A] \leq \mathrm{C}_V(A) \leq \mathrm{C}_V(s)$ and, by (1), $[V, s] \leq \mathrm{C}_V(L) \leq \mathrm{C}_V(e)$ and so $[V, s] \leq \mathrm{C}_V(s) \cap \mathrm{C}_V(e)$. So

$$\begin{aligned}\dim V &= \dim(\mathrm{C}_V(s) + \mathrm{C}_V(e)) = \dim \mathrm{C}_V(s) + \dim \mathrm{C}_V(e) - \dim(\mathrm{C}_V(s) \cap \mathrm{C}_V(e))\\ &\leq \dim \mathrm{C}_V(s) + \dim \mathrm{C}_V(e) - \dim\,[V, s]\end{aligned}$$

giving, as $\dim(V/\,\mathrm{C}_V(s)) \geq \frac{1}{4}\dim V$,

$$\dim \mathrm{C}_V(e) \geq \dim V - \dim \mathrm{C}_V(s) + \dim\,[V, s] = 2\ \dim(V/\,\mathrm{C}_V(s)) \geq \tfrac{1}{2}\dim V.$$

(2d) $\dim \mathrm{C}_V(e) = \frac{3}{8}\dim V$ *for some element* e *of* L *of odd order.*

Set $e = (1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12) \in \mathfrak{A}_n$. Then, by (2b), $e \in \mathrm{O}^2(K) \leq L$. Define the following three elementary abelian 3-groups:

$$E_1 = \langle(1\ 3\ 5), (2\ 4\ 6)\rangle$$

$$E_2 = \langle(1\ 3\ 5), (2\ 4\ 6)(7\ 9\ 11)\rangle$$

$$E_3 = \langle(1\ 3\ 5), (2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12)\rangle.$$

We have that, by [2, Exercise 4.1(3)], $V = \bigoplus_{B\leq E_i, |B|=3} \mathrm{C}_V(B)$ for $i \in \{1, 2, 3\}$. Recall that $\mathrm{C}_V(d) = 0$ for all three-cycles $d \in \mathfrak{A}_n$ and $\mathrm{C}_V(x) = \mathrm{C}_V(x^{-1})$ for all $x \in \mathfrak{A}_n$. Then

$$V = \mathrm{C}_V((1\ 3\ 5)(2\ 4\ 6)) \oplus \mathrm{C}_V((1\ 3\ 5)(2\ 6\ 4))$$

where (1 3 5)(2 4 6) and (1 3 5)(2 4 6) are $\mathfrak{A}_n$-conjugate giving $\dim \mathrm{C}_V((1\ 3\ 5)(2\ 4\ 6)) = \frac{1}{2}\dim V$. Similarly,

$$V = \mathrm{C}_V((2\ 4\ 6)(7\ 9\ 11)) \oplus \mathrm{C}_V((1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)) \oplus \mathrm{C}_V((1\ 5\ 3)(2\ 4\ 6)(7\ 9\ 11))$$

where (1 3 5)(2 4 6) and (2 4 6)(7 9 11) are $\mathfrak{A}_n$-conjugate and (1 3 5)(2 4 6)(7 9 11) and (1 5 3)(2 4 6)(7 9 11) are $\mathfrak{A}_n$-conjugate giving $\dim \mathrm{C}_V((1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)) = \frac{1}{4}\dim V$. Finally,

$$V = \mathrm{C}_V((2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12)) \oplus \mathrm{C}_V(e) \oplus \mathrm{C}_V((1\ 5\ 3)(2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12))$$

where (1 3 5)(2 4 6)(7 9 11) and (2 4 6)(7 9 11)(8 10 12) are $\mathfrak{A}_n$-conjugate and e and (1 5 3)(2 4 6)(7 9 11)(8 10 12) are $\mathfrak{A}_n$-conjugate giving $\dim \mathrm{C}_V(e) = \frac{3}{8}\dim V$.

(2e) s has a fixed point on Δ.

The statements (2c) and (2d) are contradictory and hence s has a fixed point on Δ.

(3) Either $A = L \leq \mathrm{O}_2(K)$ or $(n, m) = (8, 4)$.

Let $(n, m) \neq (8, 4)$. Then, by (1), (2) and Lemma 2.4.6, $[V, L] \leq \mathrm{C}_V(s) = [V, s] \leq \mathrm{C}_V(L)$ and so L acts quadratically on V. Hence, by the maximality of A, $A = L$ and so L is a normal 2-subgroup of K giving $L \leq \mathrm{O}_2(K)$.

(4) Either $m = 2$ or $(n, m) \in \{(8, 4), (9, 4)\}$.

Suppose, for a contradiction, that $m \notin \{2, 4\}$. In the case when $n - 2m \notin \{2, 4\}$, $\mathrm{O}_2(K) = M_0$ and so, by (3), $A \leq \mathrm{O}_2(K) = M_0$ which is a contradiction. In the case when $n - 2m \in \{2, 4\}$, $\mathrm{O}_2(H) = M \times F$ where F is similar to either $\mathfrak{C}_2$ or $\mathfrak{K}_4$. Fix $t \in A \setminus \langle s \rangle$. Then, by (3), $t \in A \leq \mathrm{O}_2(K) \leq \mathrm{O}_2(H) = M \times F$ and, as $A \cap M = \langle s \rangle$, $t \notin M$ giving $|\mathrm{Fix}_\Delta(st)| = 2m - |\mathrm{Fix}_\Delta(t)|$. So, as $t, st \in A^\#$,

$$2m = |\mathrm{Fix}_\Delta(t)| + |\mathrm{Fix}_\Delta(st)| \leq 2\,|\mathrm{Fix}_\Delta(s)| = 2(n - 2m) \leq 8$$

giving, as m is even, $m \in \{2, 4\}$ which contradicts our supposition. Thus, $m \in \{2, 4\}$. Now, let $m = 4$ and suppose, for a contradiction, that $n \geq 10$. Then $\mathrm{O}_2(H) = X \times F$ where X is similar to $\mathfrak{C}_2 \wr \mathfrak{K}_4$ and $F = 1$ except when $n \in \{10, 12\}$ and F is similar to either $\mathfrak{C}_2$ or $\mathfrak{K}_4$. Fix $t \in A \smallsetminus \langle s \rangle$. Then, by (3), $t \in A \le O_2(K) \le O_2(H) = X \times F$ and so $t = xf$ for some $x \in X$ and $f \in F$. Let $(i_5\ j_5) = (9\ 10)$ except when $n = 12$ and $f = (i_5\ j_5)(i_6\ j_6)$. Consider the following blocks of H: $\Delta_1 = \{1,2\}$, $\Delta_2 = \{3,4\}$, $\Delta_3 = \{5,6\}$ and $\Delta_4 = \{7,8\}$. In the case when t moves a block $\Delta_k = \{i_1, j_1\}$ for $1 \le k \le 4$, we may write $t = (i_1\ i_2)(j_1\ j_2)r$. Then, by (3), A is normal in K and so, as $(i_1\ j_1)(i_5\ j_5) \in K$,

$$(i_1\ j_1)(i_2\ j_2) = [t, (i_1\ j_1)(i_5\ j_5)] \in [A, K] \le A$$

which contradicts our choice of s. In the case when t fixes the blocks Δ_1, Δ_2, Δ_3 and Δ_4, as $t \neq s$, $n \in \{10, 12\}$ and t has fixed points on the set $\{1, 2, \ldots, 8\}$ because otherwise $n = 12$ and $t = sf$ giving $(i_5\ j_5)(i_6\ j_6) = f = st \in A$ which contradicts our choice of s. So there exists $1 \le k \le 4$ such that $\Delta_k = \{i_4, j_4\}$ with i_4 and j_4 fixed by t and we may write $t = (i_1\ j_1)(i_2\ j_2)(i_3\ j_3)f$ when $n = 10$ and $t = (i_1\ j_1)(i_2\ j_2)f$ when $n = 12$. Then, by (3), A is normal in K and so, as $(i_1\ i_4)(j_1\ j_4) \in K$,

$$(i_1\ j_1)(i_4\ j_4) = [t, (i_1\ i_4)(j_1\ j_4)] \in [A, K] \le A$$

which contradicts our choice of s. Thus, $n \in \{8, 9\}$.

(5) One of the following holds:

(i) A is similar to a Klein four-group $\mathfrak{K}_4$.

(ii) $G \cong \mathfrak{A}_8$ or $\mathfrak{S}_8$ and A is similar to the group $\mathfrak{K}_4 \times \mathfrak{K}_4$.

(iii) $G \cong \mathfrak{A}_8$, $\mathfrak{S}_8$ or $\mathfrak{A}_9$ and A is similar to a regular eight-group $\mathfrak{K}_8$.

Assume that $m = 2$. In the case when $n \neq 8$, as $n - 4 \notin \{2, 4\}$, $O_2(H)$ is similar to $\mathfrak{D}_8$ and so, as $O_2(K) = O_2(H) \cap \mathfrak{A}_n$, $O_2(K)$ is similar to $\mathfrak{K}_4$ giving, by (3), A is similar to $\mathfrak{K}_4$ and hence (i) holds. In the case when $n = 8$, $O_2(H)$ is similar to $\mathfrak{D}_8 \times \mathfrak{K}_4$ and so, as $O_2(K) = O_2(H) \cap \mathfrak{A}_8$, $O_2(K)$ is similar to $\mathfrak{K}_4 \times \mathfrak{K}_4$ giving, by (3), A is similar to a subgroup of $\mathfrak{K}_4 \times \mathfrak{K}_4$. A calculation on one of the spin modules for $\mathfrak{A}_8$ together with Lemmas 2.4.1 and 2.4.4 shows that $\mathfrak{K}_4 \times \mathfrak{K}_4$ acts quadratically on V. So, by the maximality of A, A is similar to $\mathfrak{K}_4 \times \mathfrak{K}_4$ and hence (ii) holds. Assume that $m \neq 2$. Then, by (4), $(n,m) \in \{(8,4),(9,4)\}$. If $n = 9$, then H is similar to $\mathfrak{C}_2 \wr \mathfrak{S}_4$ and so in both cases, by choice of s, A is a semiregular elementary abelian 2-subgroup of degree 8 giving A is similar to a subgroup of $\mathfrak{K}_8$. It may be verified that each of the two $\mathfrak{A}_8$-conjugacy classes of regular eight-groups $\mathfrak{K}_8$ acts quadratically on both of the spin modules for $\mathfrak{A}_8$ and so, by Lemma 2.4.4, the single $\mathfrak{S}_8$-conjugacy class of regular eight-groups $\mathfrak{K}_8$ acts quadratically on the spin module for $\mathfrak{S}_8$. By contrast, each of the two $\mathfrak{A}_9$-conjugacy classes of regular eight-groups $\mathfrak{K}_8$ acts quadratically on exactly one of the spin modules for $\mathfrak{A}_9$, depending on the class, and so, by Lemma 2.4.4, the single $\mathfrak{S}_9$-conjugacy class of regular eight-groups $\mathfrak{K}_8$ does not act quadratically on the spin module for $\mathfrak{S}_9$. Thus, $G \cong \mathfrak{A}_8$, $\mathfrak{S}_8$ or $\mathfrak{A}_9$ and, by the maximality of A, A is similar to $\mathfrak{K}_8$ and hence (iii) holds. □

In the proof of the theorem below we will apply the classification of the quadratic modules for the alternating and symmetric groups by Meierfrankenfeld and Stroth [26].

Theorem 2.4.8 *Let $G \cong \mathfrak{S}_n$ or $\mathfrak{A}_n$ for $n \geq 5$, $n \neq 6$. Let V be a faithful irreducible $\mathbb{F}_2G$-module and A be a subgroup of G of order at least 4 which acts quadratically on V. Then one of the following two cases hold:*

(I) V is isomorphic to the natural module for G.

(II) V is isomorphic to a spin module for G. Moreover, one of the following holds:

(i) A is similar to a Klein four-group $\mathfrak{K}_4$.

(ii) $G \cong \mathfrak{A}_8$ or $\mathfrak{S}_8$ and A is similar to a subgroup of the group $\mathfrak{K}_4 \times \mathfrak{K}_4$.

(iii) $G \cong \mathfrak{A}_8$, $\mathfrak{S}_8$ or $\mathfrak{A}_9$ and A is similar to a subgroup of a regular eight-group $\mathfrak{K}_8$.

Proof The result holds for $n = 5$ by inspection of the two irreducibles for G over $\mathbb{F}_2$. Let $n \geq 7$ and set $E = \mathrm{O}^2(G)$. In the case when $\langle A^G \rangle = G$, we may apply [26, Theorem 4(a) on page 2102] and hence V is isomorphic to the natural module for G or $G \cong \mathfrak{A}_n$ and V is isomorphic to a spin module for G. In the case when $\langle A^G \rangle \neq G$, $G \cong \mathfrak{S}_n$ and $\langle A^G \rangle = E$. By Clifford's Theorem [27, 2.37 on page 30], we may write $V_E = \bigoplus_{i=1}^m U_i$ as a direct sum of pairwise non-isomorphic faithful irreducible $\mathbb{F}_2E$-modules of equal dimension. Moreover, A acts quadratically on each of the direct summands U_i of V_E and hence, by [26, Theorem 4(a) on page 2102], each U_i is isomorphic to either the natural module for E or a spin module for E and $m \leq 3$. Assume that one of the direct summands of V_E is isomorphic to the natural module for E. If $n \neq 9$, then, by Theorem 2.1.8, the dimension of a spin module for E is not equal to the dimension of the natural module for E and hence $m = 1$. If $n = 9$, then $m = 1$ as well because otherwise, by applying Lemma 2.4.7, A is similar to either a Klein four-group $\mathfrak{K}_4$ or a subgroup of a regular eight-group $\mathfrak{K}_8$, however, none of these groups act quadratically on the natural module for E which is a contradiction. So, in all cases, V_E is isomorphic to the natural module for E and hence, by Lemma 2.2.3, V is isomorphic to the natural module for G. Alternatively, assume that none of the direct summands of V_E are isomorphic to the natural module for E. Then $m \leq 2$ and V_E is either a spin module for E or a direct sum of two non-isomorphic spin modules for E and so, by Lemma 2.4.4, V is isomorphic to a spin module for G. Now, we have shown that $A \leq E$ when V is isomorphic to a spin module for G and hence, by Lemma 2.4.7, one of (i)–(iii) holds. □

The theorem above may be used in order to provide an alternative proof of Theorem 2.3.5 which we will now outline. Let $G \cong \mathfrak{S}_n$ or $\mathfrak{A}_n$ for $n \geq 9$ and set $E = \mathrm{O}^2(G)$. Let V be a failure-of-factorization $\mathbb{F}_2G$-module, set $W = [V, E]$ and choose $A \in \mathcal{A}^*(G, W)$. Then, by [19, 9.2.4 on page 209], A acts quadratically on W and so, by Corollary 2.4.3, we may apply the theorem above together with Lemma 2.4.6 to show that each of the non-trivial composition factors of W are isomorphic to the natural module for G. Now, by the argument given in the second paragraph of the proof of [1, 8.5 on page 721], W has a unique non-trivial composition factor. Then $W/\mathrm{C}_W(E)$ is irreducible and hence it is isomorphic to the natural module for G. Therefore, by Theorem 2.2.6, W is isomorphic to either the natural module for G or the zero weight submodule of the permutation module for G.

§2.5 Sylow p-subgroups

In this section we will describe the structure of the Sylow p-subgroups of the symmetric and alternating groups with particular emphasis on the $p = 2$ case. Some of these results originate from Kaloujnine [18] and may be found in many textbooks – see, for example, [10, Example 2.6.1 on page 48]. The following notation will hold for the first few results in this section.

Notation Let $n \in \mathbb{N}$ and $p \in \mathbb{P}$. Then, by the division algorithm, we may write n to the base p as follows

$$n = n_0 + n_1 p + \cdots + n_k p^k \quad \text{with each} \quad 0 \leq n_i < p \quad \text{and} \quad n_k \neq 0.$$

This unique expression for n is called the *p-adic decomposition* of n.

Lemma 2.5.1 *A Sylow p-subgroup of* $\mathfrak{S}_n$ *has order* $p^{\nu(n)}$ *where*

$$\nu(n) = n_1 + n_2\left(\frac{p^2-1}{p-1}\right) + \cdots + n_k\left(\frac{p^k-1}{p-1}\right).$$

Proof Observe that, for each $1 \leq i \leq k$, there are $\lfloor \frac{n}{p^i} \rfloor$ integers among $1, 2, \ldots, n$ that are divisible by p^i, namely $p^i, 2p^i, 3p^i, \ldots, \lfloor \frac{n}{p^i} \rfloor p^i$. So, for each $1 \leq i \leq k-1$, there are $\lfloor \frac{n}{p^i} \rfloor - \lfloor \frac{n}{p^{i+1}} \rfloor$ integers among $1, 2, \ldots, n$ that are divisible by p^i, but not divisible by p^{i+1}. Hence, the largest power of p which divides $n!$ is

$$p^{\nu(n)} = p^{(\lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor)} (p^2)^{(\lfloor \frac{n}{p^2} \rfloor - \lfloor \frac{n}{p^3} \rfloor)} \ldots (p^{k-1})^{(\lfloor \frac{n}{p^{k-1}} \rfloor - \lfloor \frac{n}{p^k} \rfloor)} (p^k)^{\lfloor \frac{n}{p^k} \rfloor}$$

where

$$\begin{aligned} \nu(n) &= \sum_{i=1}^{k-1} i \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{n}{p^{i+1}} \right\rfloor \right) + k \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{i=1}^{k} i \left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{i=1}^{k-1} i \left\lfloor \frac{n}{p^{i+1}} \right\rfloor \\ &= \sum_{i=1}^{k} i \left\lfloor \frac{n}{p^i} \right\rfloor - \sum_{i=1}^{k} (i-1) \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_{i=1}^{k} \left\lfloor \frac{n}{p^i} \right\rfloor. \end{aligned}$$

We have that $n = \sum_{j=0}^{k} n_j p^j$ and so, for each $1 \leq i \leq k$,

$$\frac{n}{p^i} = \sum_{j=0}^{k} n_j p^{j-i} = \sum_{j=0}^{i-1} n_j p^{j-i} + \sum_{j=i}^{k} n_j p^{j-i}$$

and, as $\sum_{j=0}^{i-1} n_j p^j < p^i$, $\sum_{j=0}^{i-1} n_j p^{j-i} < 1$ giving $\lfloor \frac{n}{p^i} \rfloor = \sum_{j=i}^{k} n_j p^{j-i}$. Thus, interchanging the order of summation,

$$\nu(n) = \sum_{i=1}^{k} \sum_{j=i}^{k} n_j p^{j-i} = \sum_{j=1}^{k} \sum_{i=1}^{j} n_j p^{j-i} = \sum_{j=1}^{k} n_j \sum_{i=1}^{j} p^{j-i} = \sum_{j=1}^{k} n_j \left(\frac{p^j - 1}{p - 1} \right). \qquad \square$$

Recall our convention that the cyclic group $\mathfrak{C}_p$ of order p is regular when in the context of permutation groups.

Notation Let $\mathfrak{T}_0$ be a trivial permutation group of degree one and define the permutation group $\mathfrak{T}_m$ iteratively by the rule $\mathfrak{T}_m = \mathfrak{T}_{m-1} \wr \mathfrak{C}_p$ for $m \geq 1$. Then $\mathfrak{T}_m$ is similar to a wreath product of the form

$$\underbrace{\mathfrak{C}_p \wr \mathfrak{C}_p \wr \ldots \wr \mathfrak{C}_p}_{m\text{-terms}}$$

where any bracketing may be chosen to evaluate this expression. In particular, $\mathfrak{T}_1$ is similar to $\mathfrak{C}_p$ and $\mathfrak{T}_m$ is similar to $\mathfrak{C}_p \wr \mathfrak{T}_{m-1}$. We have that $\mathfrak{T}_m$ is a transitive permutation group of degree p^m because the wreath product of transitive permutation groups is transitive. Also, $\mathfrak{T}_m$ is a p-group of order $p^{\mu(m)}$ where $\mu(m) = (p^m - 1)/(p-1)$ because, as $\mathfrak{T}_{m+1}$ is similar to $\mathfrak{C}_p \wr \mathfrak{T}_m$,

$$p^{(p\mu(m)+1)} = (p^{\mu(m)})^p p = |\mathfrak{T}_m|^p \, |\mathfrak{C}_p| = |\mathfrak{T}_{m+1}| = |\mathfrak{C}_p|^{p^m} \, |\mathfrak{T}_m| = p^{p^m} p^{\mu(m)} = p^{(p^m + \mu(m))}$$

and so $p\mu(m) + 1 = p^m + \mu(m)$ giving $\mu(m) = (p^m - 1)/(p-1)$. As $\mathfrak{C}_p$ is abelian and $\mathfrak{T}_{m-1}$ is transitive, $\mathrm{Z}(\mathfrak{T}_m)$ is similar to the diagonal subgroup of the base group of $\mathfrak{C}_p \wr \mathfrak{T}_{m-1}$ and hence $\mathrm{Z}(\mathfrak{T}_m) = \langle t \rangle$ where t is a product of p^{m-1} p-cycles.

Theorem 2.5.2 *A Sylow p-subgroup S of $\mathfrak{S}_n$ is similar to a direct product of the form $\mathfrak{T}_1^{(n_1)} \times \mathfrak{T}_2^{(n_2)} \times \cdots \times \mathfrak{T}_k^{(n_k)}$ and $\mathrm{Z}(S)$ is similar to a direct product of the form $\mathrm{Z}(\mathfrak{T}_1)^{(n_1)} \times \mathrm{Z}(\mathfrak{T}_2)^{(n_2)} \times \cdots \times \mathrm{Z}(\mathfrak{T}_k)^{(n_k)}$. In particular, $|\mathrm{Z}(S)| = p^{\zeta(n)}$ where $\zeta(n) = \sum_{i=1}^{k} n_i$.*

Proof By Sylow's Theorem, there is a single conjugacy class of Sylow p-subgroups of $\mathfrak{S}_n$ and so it suffices to consider the group $T = \mathfrak{T}_1^{(n_1)} \times \mathfrak{T}_2^{(n_2)} \times \cdots \times \mathfrak{T}_k^{(n_k)}$. We have that, as

$|\mathfrak{T}_i| = p^{\mu(i)}$ for all $1 \le i \le k$,

$$|T| = \prod_{i=1}^{k} |\mathfrak{T}_i^{(n_i)}| = \prod_{i=1}^{k} |\mathfrak{T}_i|^{n_i} = \prod_{i=1}^{k} (p^{\mu(i)})^{n_i} = \prod_{i=1}^{k} p^{n_i \mu(i)}$$

and so T has order p^{ν} where

$$\nu = \sum_{i=1}^{k} n_i \mu(i) = \sum_{i=1}^{k} n_i \left(\frac{p^i - 1}{p - 1} \right) = \nu(n).$$

Thus, by Lemma 2.5.1, T is a Sylow p-subgroup of $\mathfrak{S}_n$. Now,

$$\begin{aligned} \mathrm{Z}(T) &= \mathrm{Z}(\mathfrak{T}_1^{(n_1)}) \times \mathrm{Z}(\mathfrak{T}_2^{(n_2)}) \times \cdots \times \mathrm{Z}(\mathfrak{T}_k^{(n_k)}) \\ &= \mathrm{Z}(\mathfrak{T}_1)^{(n_1)} \times \mathrm{Z}(\mathfrak{T}_2)^{(n_2)} \times \cdots \times \mathrm{Z}(\mathfrak{T}_k)^{(n_k)}. \end{aligned}$$

So, as $|\mathrm{Z}(\mathfrak{T}_i)| = p$ for all $1 \le i \le k$,

$$|\mathrm{Z}(T)| = \prod_{i=1}^{k} |\mathrm{Z}(\mathfrak{T}_i)^{(n_i)}| = \prod_{i=1}^{k} |\mathrm{Z}(\mathfrak{T}_i)|^{n_i} = \prod_{i=1}^{k} p^{n_i}$$

and hence $\mathrm{Z}(T)$ has order $p^{\zeta(n)}$ where $\zeta(n) = \sum_{i=1}^{k} n_i$. □

As an illustration of the theorem above we will construct a Sylow 2-subgroup of $\mathfrak{S}_{12}$. The 2-adic decomposition of 12 is

$$12 = 0 + 0.2 + 1.2^2 + 1.2^3$$

and so a Sylow 2-subgroup of $\mathfrak{S}_{12}$ is similar to $\mathfrak{T}_2 \times \mathfrak{T}_3$. Now, $\mathfrak{T}_2$ is similar to the group $\langle (1\ 2), (1\ 3)(2\ 4) \rangle$ and $\mathfrak{T}_3$ is similar to the group $\langle (1\ 2), (1\ 3)(2\ 4), (1\ 5)(2\ 6)(3\ 7)(4\ 8) \rangle$. Hence, the group

$$\langle (1\ 2), (1\ 3)(2\ 4), (5\ 6), (5\ 7)(6\ 8), (5\ 9)(6\ 10)(7\ 11)(8\ 12) \rangle$$

is a Sylow 2-subgroup of $\mathfrak{S}_{12}$ and the centre of this group is the group

$$\langle (1\ 2)(3\ 4), (5\ 6)(7\ 8)(9\ 10)(11\ 12) \rangle.$$

Lemma 2.5.3 *The maximal order of an abelian p-subgroup of* $\mathfrak{S}_n$ *is* $p^{\lfloor n/p \rfloor}$.

Proof Let A be an abelian p-subgroup of $\mathfrak{S}_n$ with orbits $\Delta_1, \Delta_2, \ldots, \Delta_r$ on the set $\Delta = \{1, 2, \ldots, n\}$. Fix $1 \le i \le r$. Then, by the Orbit-Stabilizer Theorem, $|\Delta_i|$ divides

$|A|$ and so there exists $m_i \in \mathbb{N}_0$ such that $|\Delta_i| = p^{m_i}$. So, as $\Delta_1, \Delta_2, \ldots, \Delta_r$ form a partition of Δ, $n = |\Delta| = |\Delta_1| + |\Delta_2| + \cdots + |\Delta_r| = p^{m_1} + p^{m_2} + \cdots + p^{m_r}$. The constituent A^{Δ_i} of A on Δ_i is isomorphic to A/A_{Δ_i} and so A^{Δ_i} is a transitive abelian permutation group on Δ_i giving A^{Δ_i} is regular. So $|A^{\Delta_i}| = |\Delta_i| = p^{m_i} \leq p^{\lfloor p^{m_i - 1} \rfloor}$ and hence

$$\begin{aligned}|A| &\leq |A^{\Delta_1}|\,|A^{\Delta_2}| \ldots |A^{\Delta_r}| \leq p^{\lfloor p^{m_1-1} \rfloor} p^{\lfloor p^{m_2-1} \rfloor} \ldots p^{\lfloor p^{m_r-1} \rfloor} \\ &= p^{(\lfloor p^{m_1-1} \rfloor + \lfloor p^{m_2-1} \rfloor + \cdots + \lfloor p^{m_r-1} \rfloor)} \leq p^{\lfloor (p^{m_1} + p^{m_2} + \cdots + p^{m_r})/p \rfloor} = p^{\lfloor n/p \rfloor}.\end{aligned}$$

Now, the group $\mathfrak{C}_p^{(\lfloor n/p \rfloor)}$ is an abelian p-subgroup of $\mathfrak{S}_n$ of order $p^{\lfloor n/p \rfloor}$. Thus, the maximal order of an abelian p-subgroup of $\mathfrak{S}_n$ is $p^{\lfloor n/p \rfloor}$. □

We will now show that permutation modules over $\mathbb{F}_p$ appear as "internal" modules of certain wreath products. Let G be a permutation group of degree n and M be the base group of the wreath product $W = \mathfrak{C}_p \wr G$. Then $M = \langle t_1, t_2, \ldots, t_n \rangle$ where $t_1, t_2, \ldots, t_n$ are pairwise disjoint p-cycles and, as M is normal in W, we may regard M as an n-dimensional $\mathbb{F}_pG$-module. Observe that G acts on M by permuting the subscripts of the basis vectors $t_1, t_2, \ldots, t_n$ and hence M is isomorphic to the permutation module for G over $\mathbb{F}_p$. Moreover, the zero weight submodule of M is $M_0 = M \cap \mathfrak{A}_n = \langle t_1 t_2^{-1}, t_2 t_3^{-1}, \ldots, t_{n-1} t_n^{-1} \rangle$ and the centre of M is $\langle s \rangle$ where $s = t_1 t_2 \ldots t_n$. We encountered such a wreath product in the proof of Lemma 2.4.7.

We end this section by discussing the Sylow 2-subgroups of $\mathfrak{S}_n$ and $\mathfrak{A}_n$ in more detail. The following notation will hold for the next few results in this section.

Notation Let $p = 2$ and $n = 2^m$ with $m \in \mathbb{N}$. Let $T = \mathfrak{T}_m$ be a permutation group on the set $\Delta = \{1, 2, \ldots, n\}$ and set $T_0 = T \cap \mathfrak{A}_n$. Then, by Theorem 2.5.2, T is a Sylow 2-subgroup of $\mathfrak{S}_n$ with $\mathrm{Z}(T) = \langle (1\ 2)(3\ 4) \ldots (n-1\ n) \rangle$ and, as $\mathfrak{A}_n$ is normal in $\mathfrak{S}_n$, T_0 is a Sylow 2-subgroup of $\mathfrak{A}_n$. Let B be the base group of the wreath product $T = \mathfrak{T}_{m-1} \wr \mathfrak{C}_2$ and Δ_1, Δ_2 be the orbits of B on Δ. Then $B = T^1 \times T^2$ where T^i similar to $\mathfrak{T}_{m-1}$ for $i \in \{1, 2\}$ and there exists $s \in T$ such that $(T^1)^s = T^2$ and $(T^2)^s = T^1$. Set

$B_0 = B \cap \mathfrak{A}_n$ and $T_0^i = T^i \cap \mathfrak{A}_n$ for $i \in \{1,2\}$. We will call B_0 the *base group* of T_0. Set $A^* = \langle (1\ 2), (3\ 4), \ldots, (n-1\ n)\rangle$ and $A_0^* = A^* \cap \mathfrak{A}_n$.

Lemma 2.5.4 *The following hold:*

(i) $[A^*, T] = A_0^*$ *and* $[A_0^*, T_0] = (A_0^* \cap \mathfrak{S}_{\Delta_1}) \times (A_0^* \cap \mathfrak{S}_{\Delta_2})$.

(ii) $\mathrm{C}_{\mathfrak{S}_n}(A_0^*) = A^*$ *for* $m \geq 3$, $\mathrm{C}_{\mathfrak{A}_n}([A_0^*, T_0]) = B_0$ *for* $m = 3$ *and* $\mathrm{C}_{\mathfrak{A}_n}([A_0^*, T_0]) = A_0^*$ *for* $m \geq 4$.

Proof (i) We may view T as a wreath product of the form $\mathfrak{C}_2 \wr \mathfrak{T}_{m-1}$ with base group A^*. By the observation made earlier, A^* is isomorphic to the permutation module for $\mathfrak{T}_{m-1}$ over $\mathbb{F}_2$ and A_0^* is the zero weight submodule of A^*. So, as $\mathfrak{T}_{m-1}$ is transitive and by Corollary 2.1.10, $A_0^* = [A^*, \mathfrak{T}_{m-1}] \leq [A^*, T]$. Then, as A^* is normal in T, A^* is a $2, T$-group giving $[A^*, T] < A^*$ and hence, as $|A^* : A_0^*| = 2$, $[A^*, T] = A_0^*$. Similarly, a module calculation gives us that $(A_0^* \cap \mathfrak{S}_{\Delta_1}) \times (A_0^* \cap \mathfrak{S}_{\Delta_2}) = [A_0^*, \mathfrak{T}_{m-1}] \leq [A_0^*, T_0]$. Then, as A_0^* is normal in T_0, A_0^* is a $2, T_0$-group giving $[A_0^*, T_0] < A_0^*$ and hence, as $|A_0^* : [A_0^*, \mathfrak{T}_{m-1}]| = 2$, $[A_0^*, T_0] = [A_0^*, \mathfrak{T}_{m-1}] = (A_0^* \cap \mathfrak{S}_{\Delta_1}) \times (A_0^* \cap \mathfrak{S}_{\Delta_2})$.
(ii) Let $m \geq 3$. Then, as A^* is abelian, $A^* \leq \mathrm{C}_{\mathfrak{S}_n}(A_0^*)$. Conversely, let $x \in \mathrm{C}_{\mathfrak{S}_n}(A_0^*)$ and fix $1 \leq i \leq n-3$ with i odd. Then, as $(i\ i+1)(i+2\ i+3) \in A_0^*$,

$$(ix\ (i+1)x)((i+2)x\ (i+3)x) = \big((i\ i+1)(i+2\ i+3)\big)^x = (i\ i+1)(i+2\ i+3)$$

and so x either fixes or swaps the sets $\{i, i+1\}$ and $\{i+2, i+3\}$. However, as $n = 2^m \geq 8$, x centralizes $(3\ 4)(5\ 6)$ so x fixes $\{1,2\}$ and $\{3,4\}$ and hence x fixes all of the remaining sets $\{5,6\}, \{7,8\}, \ldots, \{n-1, n\}$ giving $x \in A^*$. Thus, $\mathrm{C}_{\mathfrak{S}_n}(A_0^*) = A^*$. In the case when $m = 3$, by part (i), $[A_0^*, T_0] = \langle (1\ 2)(3\ 4), (5\ 6)(7\ 8)\rangle$ and so $\mathrm{C}_{\mathfrak{A}_n}([A_0^*, T_0]) = B_0$. In the case when $m \geq 4$, by part (i), $[A_0^*, T_0] = (A_0^* \cap \mathfrak{S}_{\Delta_1}) \times (A_0^* \cap \mathfrak{S}_{\Delta_2})$ and a similar argument to the one above gives us that $\mathrm{C}_{\mathfrak{S}_n}([A_0^*, T_0]) = A^*$ and hence $\mathrm{C}_{\mathfrak{A}_n}([A_0^*, T_0]) = A_0^*$. □

Lemma 2.5.5 *Let* $m \geq 3$ *and* Q *be a subgroup of* T *of index* 2. *Then either* $Q = B$ *and* $\mathrm{Z}(Q) = \mathrm{Z}(T^1) \times \mathrm{Z}(T^2)$ *or* Q *is transitive on* Δ *and* $\mathrm{Z}(Q) = \mathrm{Z}(T)$.

Proof Assume that $Q \neq B$. Then $\mathrm{Stab}_T(n) \not\leq Q$ because otherwise there exists $i \in \{1,2\}$ such that $T^i \leq \mathrm{Stab}_T(n) \leq Q$ and, as Q is normal in T, $T^j = (T^i)^s \leq Q^s = Q$ for $j = 3-i$ and so $B = T^iT^j \leq Q$ giving, as $|B| = \frac{1}{2}|T| = |Q|$, $Q = B$ which is a contradiction. So, as $|T:Q| = 2$, $T = \mathrm{Stab}_T(n)Q$ and hence, as T is transitive on Δ, Q is transitive on Δ. We will now show that $A_0^* \leq Q$. In the case when $\mathrm{C}_T(A^*) \leq Q$, as A^* is abelian, $A_0^* \leq \mathrm{C}_T(A^*) \leq Q$. In the case when $\mathrm{C}_T(A^*) \not\leq Q$, there exists $r \in \mathrm{C}_T(A^*) \smallsetminus Q$ and, as $|T:Q| = 2$, $T = Q\langle r\rangle$ and so, by Lemma 2.5.4(i) and as A^* and Q are normal in T,

$$A_0^* = [A^*, T] = [A^*, Q\langle r\rangle] = [A^*, Q]\,[A^*, \langle r\rangle] = [A^*, Q] \leq Q.$$

Hence, in both cases, $A_0^* \leq Q$ and so, by Lemma 2.5.4(ii), $\mathrm{Z}(T) \leq \mathrm{Z}(Q) \leq \mathrm{C}_{\mathfrak{S}_n}(A_0^*) = A^*$. Suppose, for a contradiction, that there exists $x \in \mathrm{Z}(Q)^\#$ with a fixed point on Δ, k_0 say. Then, for each $a \in Q$, $x = x^a \in \mathrm{Stab}_Q(k_0)^a = \mathrm{Stab}_Q(k_0 \cdot a)$ and so, as Q is transitive on Δ, $x \in \mathrm{Stab}_Q(\Delta) = 1$ which is a contradiction. Thus, $\mathrm{Z}(Q) = \mathrm{Z}(T)$. □

Corollary 2.5.6 *If $m \geq 3$, then T_0 is transitive on Δ with $\mathrm{Z}(T_0) = \mathrm{Z}(T)$. Moreover, either $\mathrm{C}_T(T_0) = \mathrm{Z}(T)$ or $m = 2$ and $\mathrm{C}_T(T_0) = T_0$.*

Proof We have that $T_0 = 1$ for $m = 1$ and T_0 is similar to $\mathfrak{K}_4$ for $m = 2$ and so these two cases are trivial. Let $m \geq 3$. Then, as $|\mathfrak{S}_n : \mathfrak{A}_n| = 2$ and $T \not\leq \mathfrak{A}_n$, $T\mathfrak{A}_n = \mathfrak{S}_n$ and so $|T:T_0| = |T:(T \cap \mathfrak{A}_n)| = |T\mathfrak{A}_n : \mathfrak{A}_n| = |\mathfrak{S}_n : \mathfrak{A}_n| = 2$ giving T_0 is a subgroup of T of index 2 and, as B contains transpositions, $T_0 \neq B$. Hence, by the lemma above, T_0 is transitive on Δ and $\mathrm{Z}(T_0) = \mathrm{Z}(T)$. Also, $\mathrm{C}_T(T_0) \leq T_0$ because otherwise there exists $r \in \mathrm{C}_T(T_0) \smallsetminus T_0$ and so, as $|T:T_0| = 2$, $T = T_0\langle r\rangle$, but then

$$r \in \mathrm{C}_T(T_0) \cap \mathrm{C}_T(\langle r\rangle) = \mathrm{C}_T(T_0\langle r\rangle) = \mathrm{C}_T(T) = \mathrm{Z}(T) = \mathrm{Z}(T_0) \leq T_0$$

which is a contradiction. Thus, $\mathrm{C}_T(T_0) = \mathrm{Z}(T_0) = \mathrm{Z}(T)$. □

Lemma 2.5.7 *Let $m \geq 3$ and Q be a subgroup of T_0 of index 2. Then either $Q = B_0$ and $\mathrm{Z}(Q) = \mathrm{Z}(B)$ or Q is transitive on Δ and $\mathrm{Z}(Q) = \mathrm{Z}(T)$ except when $m = 3$ and $\mathrm{Z}(Q)$ is similar to the group $\langle (1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3)(2\ 4)(5\ 7)(6\ 8)\rangle$.*

Proof Firstly, assume that $\mathrm{Stab}_{T_0}(n) \leq Q$. Then there exists $i \in \{1, 2\}$ such that $T_0^i\langle r\rangle \leq \mathrm{Stab}_{T_0}(n) \leq Q$ where $r = (n/2-1\ n/2)(n/2+1\ n/2+2)$ and, as Q is normal in T_0, $T_0^j = (T_0^i)^s \leq Q^s = Q$ for $j = 3-i$ and so $B_0 = T_0^i\langle r\rangle T_0^j \leq Q$ giving, as $|B_0| = \frac{1}{2}|T_0| = |Q|$, $Q = B_0$. Secondly, assume that $\mathrm{Stab}_{T_0}(n) \not\leq Q$. Then, as $|T_0{:}Q| = 2$, $T_0 = \mathrm{Stab}_{T_0}(n)Q$ and hence, as T_0 is transitive on Δ, Q is transitive on Δ. We will now show that $[A_0^*, T_0] \leq Q$ for all Q. In the case when $\mathrm{C}_{T_0}(A_0^*) \leq Q$, as A_0^* is abelian, $[A_0^*, T_0] \leq A_0^* \leq \mathrm{C}_{T_0}(A_0^*) \leq Q$. In the case when $\mathrm{C}_{T_0}(A_0^*) \not\leq Q$, there exists $r \in \mathrm{C}_{T_0}(A_0^*) \setminus Q$ and, as $|T_0{:}Q| = 2$, $T_0 = Q\langle r\rangle$ and so, as A_0^* and Q are normal in T_0,

$$[A_0^*, T_0] = [A_0^*, Q\langle r\rangle] = [A_0^*, Q]\,[A_0^*, \langle r\rangle] = [A_0^*, Q] \leq Q.$$

Hence, in both cases, $[A_0^*, T_0] \leq Q$. Now, if $x \in \mathrm{Z}(Q)^\#$ has a fixed point k_0 on Δ, then, for each $a \in Q$, $x = x^a \in \mathrm{Stab}_Q(k_0)^a = \mathrm{Stab}_Q(k_0 \cdot a)$ and so x fixes all of the points in the orbit $k_0 \cdot Q$. In the case when $Q = B_0$, Δ_1, Δ_2 are the orbits of Q on Δ and hence $\mathrm{Z}(Q) = \mathrm{Z}(B)$. In the case when $Q \neq B_0$, as Q is transitive on Δ, $\mathrm{Z}(Q)$ is semiregular on Δ. If $m = 3$, then, by Lemma 2.5.4(ii), $\mathrm{Z}(Q) \leq \mathrm{C}_{\mathfrak{A}_n}([A_0^*, T_0]) = B_0$ and, as B_0 is intransitive, $\mathrm{Z}(Q)$ is not regular and hence either $\mathrm{Z}(Q) = \mathrm{Z}(T)$ or $\mathrm{Z}(Q)$ is similar to the group $\langle(1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3)(2\ 4)(5\ 7)(6\ 8)\rangle$. If $m \geq 4$, then, by Lemma 2.5.4(ii), $\mathrm{Z}(Q) \leq \mathrm{C}_{\mathfrak{A}_n}([A_0^*, T_0]) = A_0^* \leq A^*$ and hence $\mathrm{Z}(Q) = \mathrm{Z}(T)$. □

The following notation will hold for the last few results in this section.

Notation Let $n \in \mathbb{N}$, $n \geq 6$ and denote the 2-adic decomposition of n as follows

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r} \quad \text{with} \quad 0 \leq m_1 < m_2 < \cdots < m_r$$

and assume that $r \geq 2$. Let S be a Sylow 2-subgroup of $\mathfrak{S}_n$ and set $S_0 = S \cap \mathfrak{A}_n$ so that, as $\mathfrak{A}_n$ is normal in $\mathfrak{S}_n$, S_0 is a Sylow 2-subgroup of $\mathfrak{A}_n$. Then, by Theorem 2.5.2, $S = T^1 \times T^2 \times \cdots \times T^r$ where T^i is similar to $\mathfrak{T}_{m_i}$ for $1 \leq i \leq r$. Let the orbits of S be $\Delta_1, \Delta_2, \ldots, \Delta_r$ on the set $\Delta = \{1, 2, \ldots, n\}$ where $|\Delta_i| = 2^{m_i}$ for $1 \leq i \leq r$. Set $T_0^i = T^i \cap \mathfrak{A}_n$ for $1 \leq i \leq r$ and $K = T_0^1 \times T_0^2 \times \cdots \times T_0^r$. Set $n_0 = 0$ when n is even and $n_0 = 1$ when n is odd. Choose transpositions $\tau_i \in T^i$ for all $n_0 + 1 \leq i \leq r$

and set $M_0 = \langle \tau_{n_0+1}\tau_{n_0+2}, \tau_{n_0+2}\tau_{n_0+3}, \ldots, \tau_{r-1}\tau_r\rangle$. Let $Z_i = \langle z_i\rangle$ denote the centre of the permutation module $V(\Delta_i)$ for $\mathfrak{S}_{\Delta_i}$ over $\mathbb{F}_2$ for $1 \leq i \leq r$ and recall that $V_0(\Delta_i)$ denotes the zero weight submodule of $V(\Delta_i)$.

Lemma 2.5.8 *The group $S_0 = K \rtimes M_0$ is the internal semidirect product of K by M_0. Moreover, $\Delta_1, \Delta_2, \ldots, \Delta_r$ are the orbits of S_0 on Δ and $\mathrm{Z}(S_0) = \mathrm{Z}(S) \cap \mathfrak{A}_n$.*

Proof As T^i is normal in S for all $1 \leq i \leq r$, T_0^i is normal in S_0 for all $1 \leq i \leq r$ and so K is normal in S_0. For each $n_0+1 \leq i \leq r$, every non-trivial element of T_0^i is a product of at least two transpositions and so $\tau_j\tau_k \notin K$ for all $n_0+1 \leq j < k \leq r$ giving $K \cap M_0 = 1$. We have that, as $|S/S_0| = 2$ and $|T^i/T_0^i| = 2$ for all $n_0+1 \leq i \leq r$,

$$|S_0/K| = |S/S_0|^{-1}\,|S/K| = \tfrac{1}{2}\prod_{i=n_0+1}^{r} |T^i/T_0^i| = 2^{(r-n_0)-1} = |M_0|$$

and so $|KM_0| = |K|\,|M_0| = |S_0|$ giving $S_0 = KM_0$. Thus, $S_0 = K \rtimes M_0$. Now, the orbits of S_0 are contained in the orbits $\Delta_1, \Delta_2, \ldots, \Delta_r$ of S and, by Corollary 2.5.6, T_0^i is transitive on Δ_i unless $m_i = 1$ in which case M_0 is transitive on Δ_i. Hence, $\Delta_1, \Delta_2, \ldots, \Delta_r$ are the orbits of S_0 on Δ. Let $s = s_1s_2\ldots s_r \in \mathrm{Z}(S_0)$ with $s_i \in T^i$ for $1 \leq i \leq r$. Then, for each $1 \leq i \leq r$, $s_i \in \mathrm{C}_{T^i}(T_0^i)$ and so, by Corollary 2.5.6, $s_i \in \mathrm{Z}(T^i)$ unless $m_i = 2$ in which case $s_i \in T_0^i$ and, as $n \geq 6$ and s centralizes M_0, s_i centralizes the transposition τ_i giving $s_i \in \mathrm{Z}(T^i)$. So $s \in \mathrm{Z}(S) \cap \mathfrak{A}_n$ and hence $\mathrm{Z}(S_0) = \mathrm{Z}(S) \cap \mathfrak{A}_n$. □

Lemma 2.5.9 *The following hold:*

(i) $\mathrm{C}_{V_0}(S) = \mathrm{C}_{V_0}(S_0)$ has basis $(z_1, z_2, \ldots, z_r)$ when n is even and $(z_2, z_3, \ldots, z_r)$ when n is odd. In particular, the dimension of $\mathrm{C}_{V_0}(S) = \mathrm{C}_{V_0}(S_0)$ is r when n is even and $r-1$ when n is odd.

(ii) $\mathrm{C}_{\overline{V_0}}(S) = \mathrm{C}_{\overline{V_0}}(S_0)$ has basis $(\overline{z_2}, \overline{z_3}, \ldots, \overline{z_r})$. In particular, the dimension of $\mathrm{C}_{\overline{V_0}}(S) = \mathrm{C}_{\overline{V_0}}(S_0)$ is $r-1$.

(iii) $[V_0, S] = [V_0, S_0] = \bigoplus_{i=n_0+1}^{r} V_0(\Delta_i)$ as S-submodules of V_0.

Proof (i) By the Centralizer Lemma, $\mathrm{C}_V(S)$ has basis $(z_1, z_2, \ldots, z_r)$. For each $1 \leq i \leq r$, $\omega(z_i) = 2^{m_i} = 0$ unless $m_i = 0$, that is $i = 1$ and n is odd. Thus, as $\mathrm{C}_{V_0}(S) = \mathrm{C}_V(S) \cap V_0$, $\mathrm{C}_{V_0}(S)$ has basis $(z_1, z_2, \ldots, z_r)$ when n is even and $(z_2, z_3, \ldots, z_r)$ when n is odd. We may replace S by S_0 in the above argument because, by Lemma 2.5.8, $\Delta_1, \Delta_2, \ldots, \Delta_r$ are the orbits of S_0 on Δ.

(ii) By part (i), Lemma 2.5.8 and Corollary 2.1.12, $\mathrm{C}_{\overline{V_0}}(S) = \overline{\mathrm{C}_{V_0}(S)} = \overline{\mathrm{C}_{V_0}(S_0)} = \mathrm{C}_{\overline{V_0}}(S_0)$ and, as $z = \sum_{i=1}^{r} z_i$, $\overline{z_1} = \sum_{i=2}^{r} \overline{z_i}$ giving, by part (i), $\mathrm{C}_{\overline{V_0}}(S) = \overline{\mathrm{C}_{V_0}(S)} = \langle \overline{z_2}, \overline{z_3}, \ldots, \overline{z_r} \rangle$. Also,

$$\mathrm{C}_{\overline{V_0}}(S) = \overline{\mathrm{C}_{V_0}(S)} = (\mathrm{C}_{V_0}(S) + Z)/Z \cong \mathrm{C}_{V_0}(S)/(\mathrm{C}_{V_0}(S) \cap Z)$$

and, as $z \in \mathrm{C}_V(S)$, $z \in \mathrm{C}_{V_0}(S)$ if and only if $z \in V_0$ if and only if n is even. So, by part (i), $\dim \mathrm{C}_{\overline{V_0}}(S) = \dim \mathrm{C}_{V_0}(S) - \dim (\mathrm{C}_{V_0}(S) \cap Z) = r - 1$. Thus, $(\overline{z_2}, \overline{z_3}, \ldots, \overline{z_r})$ forms a basis for $\mathrm{C}_{\overline{V_0}}(S) = \mathrm{C}_{\overline{V_0}}(S_0)$.

(iii) For each $n_0 + 1 \leq i \leq r$, as T^i is transitive on Δ_i and fixes $\Delta \setminus \Delta_i$, $[V_0, T^i] = V_0(\Delta_i)$ and so

$$\begin{aligned} [V_0, S] &= [V_0, T^{n_0+1}] + [V_0, T^{n_0+2}] + \cdots + [V_0, T^r] \\ &= V_0(\Delta_{n_0+1}) + V_0(\Delta_{n_0+2}) + \cdots + V_0(\Delta_r) = \bigoplus_{i=n_0+1}^{r} V_0(\Delta_i). \end{aligned}$$

Now, fix $n_0 + 1 \leq i \leq r$. If $m_i \neq 1$, then, by Corollary 2.5.6, T_0^i is transitive on Δ_i and fixes $\Delta \setminus \Delta_i$ and so $V_0(\Delta_i) = [V_0, T_0^i] \leq [V_0, S_0]$. If $m_i = 1$, then, as $n \geq 6$, we may choose $n_0 + 1 \leq j \leq r$, $j \neq i$ with $m_j \geq 2$ and $V_0(\Delta_i) \leq [V_0, \tau_i \tau_j] \leq [V_0, S_0]$. Then

$$\bigoplus_{i=n_0+1}^{r} V_0(\Delta_i) \leq [V_0, S_0] \leq [V_0, S] = \bigoplus_{i=n_0+1}^{r} V_0(\Delta_i)$$

and hence $[V_0, S] = [V_0, S_0] = \bigoplus_{i=n_0+1}^{r} V_0(\Delta_i)$ as S-submodules of V_0. $\square$

Chapter 3

Beginning the proof of Theorem A

§3.1 $(\mathfrak{X}, \mathfrak{X})$-amalgams of characteristic p

The results in this section have largely been abstracted from well-known results. We will assume the following hypothesis in this section although some results hold generally.

Hypothesis Fix $p \in \mathbb{P}$ and let $\mathfrak{X} = \mathfrak{X}(p)$ denote the class of all finite groups X of order divisible by p and such that $\mathrm{O}^p(X)$ is the unique minimal normal subgroup of X. Assume that $\mathcal{A} = \mathcal{A}(G, P_1, P_2, B)$ is a $(\mathfrak{X}, \mathfrak{X})$-amalgam of characteristic p.

Let X be a group in $\mathfrak{X}$. The hypothesis ensures that every non-trivial normal subgroup of X has p-power index. In particular, no non-trivial normal p-subgroup of X exists because otherwise X is a p-group giving $\mathrm{O}^p(X) = 1$ which is a contradiction. Observe that the class $\mathfrak{X}$ contains all of the non-abelian finite simple groups of order divisible by p.

Lemma 3.1.1 *Let $\gamma \in \Gamma$. Then $E_\gamma \leq \langle T^{G_\gamma} \rangle$ for all $T \leq G_\gamma$ with $T \not\leq Q_\gamma$.*

Proof Let $T \leq G_\gamma$ with $T \not\leq Q_\gamma$. Then $1 \neq \langle T^{G_\gamma} \rangle Q_\gamma / Q_\gamma \trianglelefteq G_\gamma / Q_\gamma$ and so, by the hypothesis, $|G_\gamma : \langle T^{G_\gamma} \rangle Q_\gamma| = |G_\gamma / Q_\gamma : \langle T^{G_\gamma} \rangle Q_\gamma / Q_\gamma|$ is a p-power. Also,

$$|\langle T^{G_\gamma} \rangle Q_\gamma : \langle T^{G_\gamma} \rangle| = |Q_\gamma : (Q_\gamma \cap \langle T^{G_\gamma} \rangle)|$$

divides $|Q_\gamma|$ and so $|\langle T^{G_\gamma} \rangle Q_\gamma : \langle T^{G_\gamma} \rangle|$ is a p-power. Thus,

$$|G_\gamma : \langle T^{G_\gamma} \rangle| = |G_\gamma : \langle T^{G_\gamma} \rangle Q_\gamma| \; |\langle T^{G_\gamma} \rangle Q_\gamma : \langle T^{G_\gamma} \rangle|$$

is a p-power giving $E_\gamma \leq \langle T^{G_\gamma} \rangle$. □

Corollary 3.1.2 *Let $\gamma \in \Gamma$ and $K \trianglelefteq G_\gamma$. Then either $K \leq Q_\gamma$ or $E_\gamma \leq K$. In particular, either K acts trivially on $\Gamma(\gamma)$ or K acts transitively on $\Gamma(\gamma)$.*

Lemma 3.1.3 *Let* $\gamma \in \Gamma$. *Then* $G_\gamma = O^{p'}(G_\gamma) = \langle G_{\gamma\delta} \mid \delta \in \Gamma(\gamma)\rangle = \langle G_{\gamma\lambda}^{G_\gamma}\rangle$ *for any* $\lambda \in \Gamma(\gamma)$.

Proof Fix $\lambda \in \Gamma(\gamma)$. Then $Q_\gamma < G_{\gamma\lambda}$ so that, by applying Lemma 3.1.1, $E_\gamma \leq \langle G_{\gamma\lambda}^{G_\gamma}\rangle$ and so, as $\mathrm{Syl}_p(G_\gamma) = \{G_{\gamma\delta} \mid \delta \in \Gamma(\gamma)\}$,

$$G_\gamma = G_{\gamma\lambda}E_\gamma = G_{\gamma\lambda}\langle G_{\gamma\lambda}^{G_\gamma}\rangle = \langle G_{\gamma\lambda}^{G_\gamma}\rangle = \langle G_{\gamma\delta} \mid \delta \in \Gamma(\gamma)\rangle = O^{p'}(G_\gamma). \qquad \square$$

Notation In this section set $b_0 = \left\lfloor \frac{b-1}{2} \right\rfloor$. The next lemma explains its significance.

Lemma 3.1.4 *Let* $\gamma \in \Gamma$ *and* $0 \leq k \leq b_0$. *Then* $Z_\gamma^{[k]}$ *is an elementary abelian p-group. In particular,*

$$1 \trianglelefteq Z_\gamma \trianglelefteq Z_\gamma^{[1]} \trianglelefteq \ldots \trianglelefteq Z_\gamma^{[b_0]} \trianglelefteq C_{Q_\gamma}(Z_\gamma^{[b_0]}) \trianglelefteq C_{Q_\gamma}(Z_\gamma^{[b_0-1]}) \trianglelefteq \ldots \trianglelefteq C_{Q_\gamma}(Z_\gamma) = Q_\gamma$$

is a normal G_γ*-series of* Q_γ.

Proof If $b_0 = 0$, then the result is trivial and so assume that $b_0 \geq 1$. It suffices to show that $Z_\gamma^{[b_0]} = \langle Z_\delta \mid \delta \in \Delta_{b_0}(\gamma)\rangle$ is an elementary abelian p-group. If $\delta, \tau \in \Delta_{b_0}(\gamma)$, then

$$d(\delta,\tau) \leq d(\delta,\gamma) + d(\gamma,\tau) \leq 2b_0 \leq b-1 < b$$

and so $Z_\delta \leq Q_\tau$ giving $[Z_\delta, Z_\tau] = 1$. So $Z_\gamma^{[b_0]}$ is an abelian group generated by elementary abelian p-groups and hence $Z_\gamma^{[b_0]}$ is an elementary abelian p-group. $\square$

Lemma 3.1.5 *Let* $\gamma \in \Gamma$ *and* $1 \leq k \leq b-1$. *Then* $\eta_{G_\gamma}(Z_\gamma^{[k]}/Z_\gamma^{[k-2]}) \geq 1$ *where* $Z_\gamma^{[-1]} = 1$. *In particular,* $\eta_{G_\gamma}(Z_\gamma^{[1]}) \geq 1$.

Proof Suppose, for a contradiction, that $\eta_{G_\gamma}(Z_\gamma^{[k]}/Z_\gamma^{[k-2]}) = 0$. Fix $\lambda \in \Gamma(\gamma)$. Then $Z_\lambda^{[k-1]} \leq Q_\lambda \leq G_{\lambda\gamma}$ and, as $Z_\lambda^{[k-1]} \trianglelefteq G_\lambda$, $N_{G_\lambda}(Z_\lambda^{[k-1]}) = G_\lambda$ acts transitively on $\Gamma(\lambda)$. Also,

$$[Z_\lambda^{[k-1]}, E_\gamma] \leq [Z_\gamma^{[k]}, E_\gamma] \leq Z_\gamma^{[k-2]} \leq Z_\lambda^{[k-1]}$$

and so $E_\gamma \leq N_{G_\gamma}(Z_\lambda^{[k-1]})$ giving $N_{G_\gamma}(Z_\lambda^{[k-1]})$ acts transitively on $\Gamma(\gamma)$. Hence, $Z_\lambda^{[k-1]} = 1$ which is a contradiction. Thus, $\eta_{G_\gamma}(Z_\gamma^{[k]}/Z_\gamma^{[k-2]}) \geq 1$. $\square$

Lemma 3.1.6 *Let* $\gamma \in \Gamma$ *and* $1 \leq k \leq b-1$. *The following hold:*

(i) Either $\mathrm{C}_{G_\gamma}(Z_\gamma) = Q_\gamma$ *or* $\mathrm{C}_{G_\gamma}(Z_\gamma) = G_\gamma$.

(ii) If $K \leq_{G_\gamma} Q_\gamma$ *with* $\eta_{G_\gamma}(K) \geq 1$, *then* $\mathrm{C}_{G_\gamma}(K) \leq Q_\gamma$. *In particular,* $\mathrm{C}_{G_\gamma}(Z_\gamma^{[k]}) \leq Q_\gamma$.

Proof (i) Assume that $\mathrm{C}_{G_\gamma}(Z_\gamma) \neq Q_\gamma$ and fix $\lambda \in \Gamma(\gamma)$. Then $Q_\gamma < \mathrm{C}_{G_\gamma}(Z_\gamma)$ so that, by Corollary 3.1.2, $E_\gamma \leq \mathrm{C}_{G_\gamma}(Z_\gamma)$ and so $Z_\gamma = \langle \Omega\,\mathrm{Z}(G_{\gamma\lambda})^{E_\gamma}\rangle = \Omega\,\mathrm{Z}(G_{\gamma\lambda})$ giving $G_\gamma = G_{\gamma\lambda}E_\gamma \leq \mathrm{C}_{G_\gamma}(Z_\gamma)$. Thus, $\mathrm{C}_{G_\gamma}(Z_\gamma) = G_\gamma$.
(ii) Let K be a G_γ-subgroup of Q_γ with $\eta_{G_\gamma}(K) \geq 1$. Then $E_\gamma \not\leq \mathrm{C}_{G_\gamma}(K)$ and hence, by Corollary 3.1.2, $\mathrm{C}_{G_\gamma}(K) \leq Q_\gamma$. □

Let $\gamma \in \Gamma$ and regard Z_γ as an $\mathbb{F}_p(G_\gamma/Q_\gamma)$-module. Part (i) of the lemma above tells us that either G_γ/Q_γ acts faithfully on Z_γ or G_γ/Q_γ acts trivially on Z_γ.

Lemma 3.1.7 *Let* $\gamma \in \Gamma$ *and* $\lambda \in \Gamma(\gamma)$. *The following three statements are equivalent:*

(i) $\mathrm{C}_{G_\gamma}(Z_\gamma) = G_\gamma$.

(ii) $Z_\gamma = \Omega\,\mathrm{Z}(G_{\gamma\lambda}) = \Omega\,\mathrm{Z}(G_\gamma)$.

(iii) $\Omega\,\mathrm{Z}(G_{\gamma\lambda}) \trianglelefteq G_\gamma$.

Proof (i)$\Rightarrow$(ii) If $\mathrm{C}_{G_\gamma}(Z_\gamma) = G_\gamma$, then $Z_\gamma \leq \Omega\,\mathrm{Z}(G_\gamma) \leq \Omega\,\mathrm{Z}(G_{\gamma\lambda}) \leq Z_\gamma$ and so $Z_\gamma = \Omega\,\mathrm{Z}(G_{\gamma\lambda}) = \Omega\,\mathrm{Z}(G_\gamma)$.
(ii)$\Rightarrow$(iii) This follows from $Z_\gamma \trianglelefteq G_\gamma$.
(iii)$\Rightarrow$(i) If $\Omega\,\mathrm{Z}(G_{\gamma\lambda}) \trianglelefteq G_\gamma$, then $Z_\gamma = \langle \Omega\,\mathrm{Z}(G_{\gamma\lambda})^{G_\gamma}\rangle = \Omega\,\mathrm{Z}(G_{\gamma\lambda})$ and so $Q_\gamma < G_{\gamma\lambda} \leq \mathrm{C}_{G_\gamma}(Z_\gamma)$ giving, by Lemma 3.1.6(i), $\mathrm{C}_{G_\gamma}(Z_\gamma) = G_\gamma$. □

Corollary 3.1.8 *Let* $i \in \{1,2\}$. *The following three statements are equivalent:*

(i) $\mathrm{C}_{G_{\gamma_i}}(Z_{\gamma_i}) = G_{\gamma_i}$ *for all* $\gamma_i \in \mathcal{O}_i$.

(ii) $\Omega\,\mathrm{Z}(B) = \Omega\,\mathrm{Z}(P_i)$.

(iii) $\Omega\,\mathrm{Z}(B) \trianglelefteq P_i$.

By $(\mathcal{A}1)$ in the definition of an amalgam the next two lemmas cover all of the possible cases.

Lemma 3.1.9 *Assume that $\Omega\,\mathrm{Z}(B)$ is not normal in P_1 or P_2. Let $\gamma \in \Gamma$ and $0 \le k \le b-1$. Then $\mathrm{C}_{G_\gamma}(Z_\gamma^{[k]}) = Q_\gamma^{[k]}$.*

Proof By Lemma 3.1.6(i) and Corollary 3.1.8, $\mathrm{C}_{G_\gamma}(Z_\gamma) = Q_\gamma$ and so the result holds for $k = 0$. Let $k \ge 1$ and assume, inductively, that the result holds for $k-1$. Then

$$\mathrm{C}_{Q_\gamma}(Z_\gamma^{[k]}) = \bigcap_{\delta\in\Gamma(\gamma)} \mathrm{C}_{Q_\gamma}(Z_\delta^{[k-1]}) = Q_\gamma \cap \bigcap_{\delta\in\Gamma(\gamma)} \mathrm{C}_{G_\delta}(Z_\delta^{[k-1]}) = Q_\gamma \cap \bigcap_{\delta\in\Gamma(\gamma)} Q_\delta^{[k-1]} = Q_\gamma^{[k]}$$

and so, by Lemma 3.1.6(ii), $\mathrm{C}_{G_\gamma}(Z_\gamma^{[k]}) = \mathrm{C}_{Q_\gamma}(Z_\gamma^{[k]}) = Q_\gamma^{[k]}$. □

Lemma 3.1.10 *Let $\{i, j\} = \{1, 2\}$ and assume that $\Omega\,\mathrm{Z}(B)$ is normal in P_j. Let $\alpha \in \mathcal{O}_i$ and $\beta \in \Gamma(\alpha)$. Then $Z_\beta = \Omega\,\mathrm{Z}(G_{\alpha\beta}) = \Omega\,\mathrm{Z}(G_\beta)$. Moreover, the following hold:*

(i) $\mathrm{C}_{G_\alpha}(Z_\alpha) = Q_\alpha$ and $\mathrm{C}_{G_\beta}(Z_\beta) = G_\beta$.

(ii) If $0 \le k \le b-1$, then $Z_\alpha^{[k]} = Z_\alpha^{[k+1]}$ and $\mathrm{C}_{G_\alpha}(Z_\alpha^{[k]}) = Q_\alpha^{[k]}$ when k is even and $Z_\beta^{[k]} = Z_\beta^{[k+1]}$ and $\mathrm{C}_{G_\beta}(Z_\beta^{[k]}) = Q_\beta^{[k]}$ when k is odd.

(iii) $\mathrm{Z}(G_\alpha) = 1$. In particular, $\mathrm{C}_{Z_\alpha}(G_\alpha) = 1$.

(iv) $Z_\beta = \mathrm{C}_{Z_\alpha}(G_{\alpha\beta}) = \mathrm{C}_{Z_\alpha}(G_\beta)$ and $Z_\beta = \mathrm{C}_{Z_\beta^{[k]}}(G_{\alpha\beta}) = \mathrm{C}_{Z_\beta^{[k]}}(G_\beta)$ for all $0 \le k \le b_0$.

Proof (i) Observe that $\Omega\,\mathrm{Z}(B)$ is not normal in P_i because otherwise, by $(\mathcal{A}1)$ in the definition of an amalgam, $\Omega\,\mathrm{Z}(B) = 1$ which is a contradiction. Thus, by Lemma 3.1.6(i) and Corollary 3.1.8, $\mathrm{C}_{G_\alpha}(Z_\alpha) = Q_\alpha$ and $\mathrm{C}_{G_\beta}(Z_\beta) = G_\beta$.

(ii) We have that $Z_\alpha = \langle \Omega\,\mathrm{Z}(G_{\alpha\beta})^{G_\alpha}\rangle = \langle Z_\beta^{G_\alpha}\rangle = Z_\alpha^{[1]}$, $Z_\beta^{[1]} = \langle Z_\alpha^{G_\beta}\rangle = \langle (Z_\alpha^{[1]})^{G_\beta}\rangle = Z_\beta^{[2]}$, $Z_\alpha^{[2]} = \langle (Z_\beta^{[1]})^{G_\alpha}\rangle = \langle (Z_\beta^{[2]})^{G_\alpha}\rangle = Z_\alpha^{[3]}$, … and so on. By Lemma 3.1.6(ii) and part (i),

$$\mathrm{C}_{G_\beta}(Z_\beta^{[1]}) = \mathrm{C}_{Q_\beta}(Z_\beta^{[1]}) = Q_\beta \cap \bigcap_{\delta\in\Gamma(\beta)} \mathrm{C}_{G_\delta}(Z_\delta) = Q_\beta \cap \bigcap_{\delta\in\Gamma(\beta)} Q_\delta = Q_\beta^{[1]}$$

$$\mathrm{C}_{G_\alpha}(Z_\alpha^{[2]}) = \mathrm{C}_{Q_\alpha}(Z_\alpha^{[2]}) = Q_\alpha \cap \bigcap_{\delta\in\Gamma(\alpha)} \mathrm{C}_{G_\delta}(Z_\delta^{[1]}) = Q_\alpha \cap \bigcap_{\delta\in\Gamma(\alpha)} Q_\delta^{[1]} = Q_\alpha^{[2]}$$

… and so on.

(iii) By part (i), $\mathrm{Z}(G_\alpha) \le \mathrm{C}_{G_\alpha}(Z_\alpha) = Q_\alpha$ so that $\mathrm{Z}(G_\alpha)$ is a p-group and so it suffices to show that $\Omega\,\mathrm{Z}(G_\alpha) = 1$. We have that $\Omega\,\mathrm{Z}(G_\alpha) \le \Omega\,\mathrm{Z}(G_{\alpha\beta}) = Z_\beta \le \mathrm{Z}(G_\beta)$ and so $\Omega\,\mathrm{Z}(G_\alpha)$ is normal in both G_α and G_β giving $\Omega\,\mathrm{Z}(G_\alpha) = 1$. Thus, $\mathrm{Z}(G_\alpha) = 1$.

(iv) We have that $Z_\beta \le C_{Z_\alpha}(G_\beta) \le C_{Z_\alpha}(G_{\alpha\beta}) = \Omega\, Z(G_{\alpha\beta}) = Z_\beta$ and so $Z_\beta = C_{Z_\alpha}(G_{\alpha\beta}) = C_{Z_\alpha}(G_\beta)$. Let $0 \le k \le b_0$. Then, by Lemma 3.1.4, $Z_\beta^{[k]}$ is an elementary abelian p-group and so $Z_\beta \le C_{Z_\beta^{[k]}}(G_\beta) \le C_{Z_\beta^{[k]}}(G_{\alpha\beta}) \le \Omega\, Z(G_{\alpha\beta}) = Z_\beta$. Thus, $Z_\beta = C_{Z_\beta^{[k]}}(G_{\alpha\beta}) = C_{Z_\beta^{[k]}}(G_\beta)$. □

Lemma 3.1.11 *Let (α,β') be a critical pair. Then $C_{G_\alpha}(Z_\alpha) = Q_\alpha$. Moreover, the following three statements are equivalent:*

(i) (β',α) is not a critical pair.

(ii) $[Z_\alpha, Z_{\beta'}] = 1$.

(iii) $C_{G_{\beta'}}(Z_{\beta'}) = G_{\beta'}$.

Proof Let $(\alpha_0,\alpha_1,\ldots,\alpha_b)$ be a critical path from α to β'. Then $C_{G_\alpha}(Z_\alpha) = Q_\alpha$ because otherwise, by Lemma 3.1.6(i), $C_{G_\alpha}(Z_\alpha) = G_\alpha$ and so, by Corollary 3.1.8 and Lemma 3.1.10(ii), $Z_\alpha \le Z_{\alpha_1}^{[1]} = Z_{\alpha_1} \le Q_{\beta'}$ which is a contradiction. By Lemma 3.1.6(i),

$$(\beta',\alpha) \text{ is not a critical pair} \Leftrightarrow Z_{\beta'} \le Q_\alpha \Leftrightarrow Z_{\beta'} \le C_{G_\alpha}(Z_\alpha) \Leftrightarrow [Z_\alpha, Z_{\beta'}] = 1$$
$$\Leftrightarrow Z_\alpha \le C_{G_{\beta'}}(Z_{\beta'}) \Leftrightarrow C_{G_{\beta'}}(Z_{\beta'}) = G_{\beta'}.$$ □

Lemma 3.1.12 *Let (α,β') be a critical pair with critical path $(\alpha_0,\alpha_1,\ldots,\alpha_b)$ and $0 \le k \le b_0$. The following hold:*

(i) $[Z_{\alpha_k}^{[k]}, Z_{\beta'}^{[k]}] \le Z_{\alpha_k}^{[k]} \cap Z_{\beta'}^{[k]}$. In particular, $Z_{\alpha_k}^{[k]}$ and $Z_{\beta'}^{[k]}$ act on each other by conjugation.

(ii) $[Z_{\alpha_k}^{[k]}, Z_{\beta'}^{[k]}] \ne 1$ provided $k \ge 1$.

(iii) $[Z_{\alpha_k}^{[k]}, Z_{\beta'}^{[k]}, Z_{\beta'}^{[k]}] = 1 = [Z_{\beta'}^{[k]}, Z_{\alpha_k}^{[k]}, Z_{\alpha_k}^{[k]}]$.

Proof (i) This holds because $Z_{\alpha_k}^{[k]}$ and $Z_{\beta'}^{[k]}$ are normal in $G_{\alpha_k\beta'}$.

(ii) If $k \ge 1$, then $[Z_{\alpha_k}^{[k]}, Z_{\beta'}^{[k]}] \ne 1$ because otherwise, by Lemma 3.1.6(ii), $Z_\alpha \le Z_{\alpha_k}^{[k]} \le C_{G_{\beta'}}(Z_{\beta'}^{[k]}) \le Q_{\beta'}$ which is a contradiction.

(iii) This follows from part (i) and Lemma 3.1.4. □

Lemma 3.1.13 *Let $k_0 \in \mathbb{N}_0$. The following hold:*

(i) $Z_\gamma^{[b_0]}$ is abelian for all $\gamma \in \Gamma$ and there exists $\gamma_0 \in \Gamma$ such that $Z_{\gamma_0}^{[b_0+1]}$ is non-abelian.

(ii) If $Z_\gamma^{[k_0]}$ is abelian for all $\gamma \in \Gamma$ and there exists $\delta_0 \in \Gamma$ such that $Z_{\delta_0}^{[k_0+1]}$ is non-abelian, then $b \in \{2k_0+1, 2k_0+2\}$.

Proof (i) By Lemma 3.1.4, $Z_\gamma^{[b_0]}$ is abelian for all $\gamma \in \Gamma$. Let (α, β') be a critical pair with critical path $(\alpha_0, \alpha_1, \ldots, \alpha_b)$ and suppose, for a contradiction, that $Z_{\alpha_{b_0+1}}^{[b_0+1]}$ is abelian. Then, as $Z_\alpha, Z_{\beta'} \leq Z_{\alpha_{b_0+1}}^{[b_0+1]}$, $[Z_\alpha, Z_{\beta'}] = 1$ so that, by Lemma 3.1.11, α and β' are in different G-orbits of Γ and so b is odd giving $b_0 = (b-1)/2$. Now, as $Z_\alpha, Z_{\beta'}^{[1]} \leq Z_{\alpha_{b_0+1}}^{[b_0+1]}$, $[Z_\alpha, Z_{\beta'}^{[1]}] = 1$ and so $\mathrm{C}_{G_{\beta'}}(Z_{\beta'}^{[1]}) \not\leq Q_{\beta'}$. Then, by Corollary 3.1.2, $E_{\beta'} \leq \mathrm{C}_{G_{\beta'}}(Z_{\beta'}^{[1]}) \leq \mathrm{N}_{G_{\beta'}}(Z_{\alpha_{b-1}})$ giving $Z_{\alpha_{b-1}} = 1$ which is a contradiction. Thus, $Z_{\alpha_{b_0+1}}^{[b_0+1]}$ is non-abelian.

(ii) Choose $\delta_0 \in \Gamma$ such that $Z_{\delta_0}^{[k_0+1]}$ is non-abelian. Then $b_0 + 1 > k_0$ because otherwise $Z_{\gamma_0}^{[b_0+1]} \leq Z_{\gamma_0}^{[k_0]}$ and so $Z_{\gamma_0}^{[b_0+1]}$ is abelian which contradicts part (i). Also, $k_0+1 > b_0$ because otherwise $Z_{\delta_0}^{[k_0+1]} \leq Z_{\delta_0}^{[b_0]}$ and so, by part (i), $Z_{\delta_0}^{[k_0+1]}$ is abelian which is a contradiction. We have that $k_0 \leq b_0 \leq k_0$ and so $k_0 = b_0$. Thus, $b \in \{2k_0+1, 2k_0+2\}$. ☐

Corollary 3.1.14 *Assume that $Z = \Omega\,\mathrm{Z}(B) \trianglelefteq P_2$. The following hold:*

(i) $b = 2$ if and only if $\langle Z^{P_1}\rangle \leq Q_2$ and $\langle\langle Z^{P_1}\rangle^{P_2}\rangle$ is non-abelian.

(ii) $b \in \{3,4\}$ if and only if $\langle\langle Z^{P_1}\rangle^{P_2}\rangle$ is abelian and $\langle\langle\langle Z^{P_1}\rangle^{P_2}\rangle^{P_1}\rangle$ is non-abelian.

Proof Take $\alpha = P_1$ and $\beta = P_2$ as representatives of the G-orbits of Γ.

(i) By definition of the critical distance b, $b \geq 2$ if and only if $Z_\gamma \leq Q_\delta$ for all $\{\gamma, \delta\} \in E_\Gamma$ and so, as G acts edge-transitively on Γ and $\{\alpha, \beta\} \in E_\Gamma$, $b \geq 2$ if and only if $Z_\alpha \leq Q_\beta$ and $Z_\beta \leq Q_\alpha$. On the other hand, by the lemma above, $b \leq 2$ if and only if $Z_\alpha^{[1]}$ or $Z_\beta^{[1]}$ is non-abelian. We have that, by Lemma 3.1.10(ii), $Z_\beta \leq Z_\alpha \leq Q_\alpha$ and $Z_\alpha^{[1]} = Z_\alpha$ is abelian. Thus, $b = 2$ if and only if $\langle Z^{P_1}\rangle = Z_\alpha \leq Q_\beta = Q_2$ and $\langle\langle Z^{P_1}\rangle^{P_2}\rangle = Z_\beta^{[1]}$ is non-abelian.

(ii) By the lemma above, $b \in \{3,4\}$ if and only if $Z_\alpha^{[1]}$ and $Z_\beta^{[1]}$ are abelian and $Z_\alpha^{[2]}$ or $Z_\beta^{[2]}$ is non-abelian. We have that, by Lemmas 3.1.10(ii) and 3.1.4, $Z_\alpha^{[1]} = Z_\alpha$ is abelian and $Z_\beta^{[2]} = Z_\beta^{[1]}$ is abelian when $b \in \{3,4\}$. Thus, $b \in \{3,4\}$ if and only if $\langle\langle Z^{P_1}\rangle^{P_2}\rangle = Z_\beta^{[1]}$ is abelian and $\langle\langle\langle Z^{P_1}\rangle^{P_2}\rangle^{P_1}\rangle = Z_\alpha^{[2]}$ is non-abelian. ☐

§3.2 The hypothesis of Theorem A

This section marks the beginning of the proof of Theorem A. The notation introduced in Sections 3.2 and 3.3 will hold for the remainder of this thesis. Henceforth we will assume the following global hypothesis.

Hypothesis A Fix $n \in \mathbb{N}$ and assume that $\mathcal{A}(G, P_1, P_2, B)$ is a $(\mathfrak{S}_3, \mathfrak{A}_n)$- or $(\mathfrak{S}_3, \mathfrak{S}_n)$-amalgam of characteristic 2 that satisfies the following two conditions:
(i) $Z = \Omega\,\mathrm{Z}(B) \trianglelefteq P_2$; and
(ii) $\langle\langle Z^{P_1}\rangle^{P_2}\rangle$ is abelian and $\langle\langle\langle Z^{P_1}\rangle^{P_2}\rangle^{P_1}\rangle$ is non-abelian.

Hypothesis A implicitly excludes the cases $P_2/Q_2 \cong \mathfrak{A}_n$ for $n \in \{1, 2, 3, 4\}$ and $P_2/Q_2 \cong \mathfrak{S}_n$ for $n \in \{1, 2, 4\}$. This is because the groups $\mathfrak{S}_2$, $\mathfrak{A}_4$ and $\mathfrak{S}_4$ all have a non-trivial normal 2-subgroup and the order of each of the remaining groups is indivisible by 2. In particular, the groups P_1/Q_1 and P_2/Q_2 are in the class $\mathfrak{X} = \mathfrak{X}(2)$ and so all of the

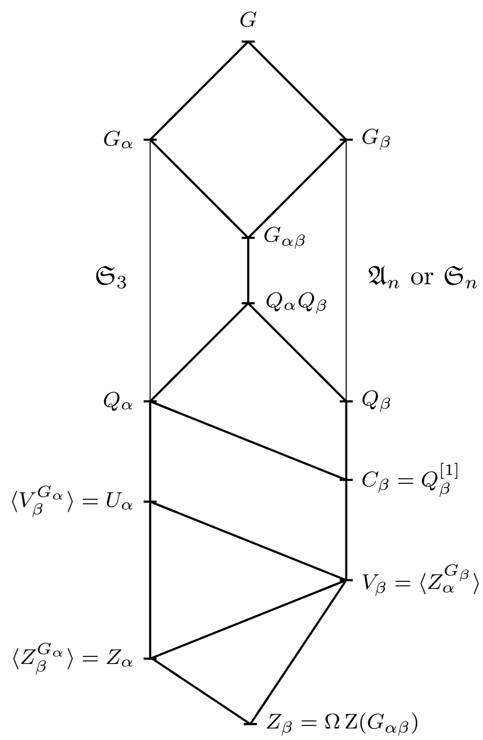

Figure C The subgroup structure of the group G where $\alpha \in \mathcal{O}_1$ and $\beta \in \Gamma(\alpha)$.

results in the previous section are available to us. Observe that, by Corollary 3.1.14(ii), condition (ii) of Hypothesis A is equivalent to the critical distance $b = 3$ or 4. Recall from the introduction that the shapes of the parabolic subgroups are as given in Theorem A when $P_2/Q_2 \cong \mathfrak{S}_3$ and $\mathfrak{S}_5$ and the critical distance is not equal to 3 or 4 when $P_2/Q_2 \cong \mathfrak{A}_6$, $\mathfrak{S}_6$ and $\mathfrak{A}_7$. Henceforth we will assume that $n = 5$ or $n \geq 8$ when $P_2/Q_2 \cong \mathfrak{A}_n$ and $n \geq 7$ when $P_2/Q_2 \cong \mathfrak{S}_n$.

We pause to briefly verify that the known amalgams with parabolic subgroups of one of the three shapes given in Case (I) of Theorem A actually satisfy Hypothesis A. The first two classes of amalgams in Theorem A with $P_2/Q_2 \cong \mathfrak{S}_3$ are the *Goldschmidt amalgams* G_5 and G_5^1 from the pioneering paper [12] which lead to the development of the amalgam method. Recalling that $\mathfrak{S}_3 \cong \mathrm{SL}_2(2)$ we can also find these classes of amalgams in the classification of weak (B, N)-pairs of rank 2 by Delgado and Stellmacher [8, Part II]. According to [8, Part II, Section 4] amalgams in the classes G_5 and G_5^1 are $(\mathfrak{S}_3, \mathfrak{S}_3)$-amalgams of characteristic 2 and critical distance $b = 3$. It remains to show that condition (i) of Hypothesis A holds. If this was not the case, then by symmetry we would have $\Omega\,\mathrm{Z}(B)$ is not normal in P_1 or P_2 which is the situation considered in [19, Section 10.3] and so we can conclude from this reference the contradiction $b = 1$. The third class of amalgams in Theorem A with $P_2/Q_2 \cong \mathfrak{S}_5$ appears in the classification of $(\mathfrak{S}_3, \mathfrak{S}_5)$-amalgams of characteristic 2 by Huang, Stellmacher and Stroth [15]. Referring the reader to [15, 5.4] in this paper we note that an amalgam in this class has critical distance $b = 3$. Condition (i) of Hypothesis A holds in this case by Corollary 3.1.8 and Lemma 3.1.11 because if we choose a critical pair (α, β'), then $\alpha \in \mathcal{O}_1$ and $[Z_\alpha, Z_{\beta'}] = 1$ follows from [15, 5.1(c) and 3.3].

The following notation builds upon the notation established in the introduction on pages 10 and 11.

Notation Let $\gamma \in \Gamma$ and define the following normal G_γ-subgroups of Q_γ

$$V_\gamma = Z_\gamma^{[1]}$$

$$U_\gamma = Z_\gamma^{[2]}$$

$$C_\gamma = \mathrm{C}_{G_\gamma}(V_\gamma)$$

$$W_\gamma = [V_\gamma, E_\gamma]$$

$$Y_\gamma = \bigcap_{\delta \in \Gamma(\gamma)} V_\delta.$$

We begin our study of Hypothesis A by collecting together the results from Section 3.1 that we will need.

Lemma 3.2.1 *Let $\alpha \in \mathcal{O}_1$ and $\beta \in \Gamma(\alpha)$. Then $Z_\beta = \Omega\,\mathrm{Z}(G_{\alpha\beta}) = \Omega\,\mathrm{Z}(G_\beta)$. Moreover, the following hold:*

(i) $\mathrm{C}_{G_\alpha}(Z_\alpha) = Q_\alpha$ *and* $\mathrm{C}_{G_\beta}(Z_\beta) = G_\beta$.

(ii) $Z_\alpha = V_\alpha$, $V_\beta = U_\beta$ *and* $C_\beta = \mathrm{C}_{Q_\beta}(V_\beta) = Q_\beta^{[1]}$.

(iii) $\eta_{G_\alpha}(Z_\alpha) \geq 1$, $\eta_{G_\alpha}(U_\alpha/Z_\alpha) \geq 1$, $\eta_{G_\beta}(Z_\beta) = 0$ *and* $\eta_{G_\beta}(V_\beta/Z_\beta) \geq 1$.

(iv) $\mathrm{Z}(G_\alpha) = 1$. *In particular,* $\mathrm{C}_{Z_\alpha}(G_\alpha) = 1$.

(v) $Z_\beta = \mathrm{C}_{Z_\alpha}(G_{\alpha\beta})$ *and* $Z_\beta = \mathrm{C}_{V_\beta}(G_{\alpha\beta}) = \mathrm{C}_{V_\beta}(G_\beta)$.

(vi) $V_\beta = Z_\alpha W_\beta$, $W_\beta \cap Z_\beta \neq 1$ *and* $\eta_{G_\beta}(W_\beta) = \eta_{G_\beta}(V_\beta)$.

(vii) $V_\beta \leq \mathrm{Z}(C_\beta)$ *and* $Z_\alpha \leq Y_\alpha = (V_\beta)_{G_\alpha} \leq V_\beta \leq \langle V_\beta^{G_\alpha} \rangle = U_\alpha$.

Proof All of these results except for (vi) follow immediately from the definitions above, Lemmas 3.1.10 and 3.1.5 and $b \in \{3, 4\}$.

(vi) The group E_β normalizes $Z_\alpha W_\beta$ because

$$[Z_\alpha W_\beta, E_\beta] \leq [V_\beta, E_\beta] = W_\beta \leq Z_\alpha W_\beta$$

and so $V_\beta = \langle Z_\alpha^{E_\beta} \rangle \leq \langle (Z_\alpha W_\beta)^{E_\beta} \rangle \leq Z_\alpha W_\beta \leq V_\beta$ giving $V_\beta = Z_\alpha W_\beta$. Also, as $\eta_{G_\beta}(V_\beta) \geq 1$, $1 \neq W_\beta \trianglelefteq G_{\alpha\beta}$ and so $W_\beta \cap Z_\beta = W_\beta \cap \Omega\,\mathrm{Z}(G_{\alpha\beta}) \neq 1$ and $\eta_{G_\beta}(V_\beta) = \eta_{G_\beta}(V_\beta/W_\beta) + \eta_{G_\beta}(W_\beta) = \eta_{G_\beta}(W_\beta)$. $\square$

The next lemma exploits the structure of $P_1/Q_1 \cong \mathfrak{S}_3$.

Lemma 3.2.2 *Let $\alpha \in \mathcal{O}_1$ and $\beta, \tau \in \Gamma(\alpha)$ with $\beta \neq \tau$. The following hold:*

(i) $|\Gamma(\alpha)| = 3$ and G_α acts 3-transitively on $\Gamma(\alpha)$.

(ii) $G_\alpha = \langle G_{\alpha\beta}, G_{\alpha\tau} \rangle$ and $|G_{\alpha\beta}/Q_\alpha| = 2$.

(iii) $Z_\alpha = Z_\beta \times Z_\tau$. In particular, $|Z_\alpha| = |Z_\beta|^2$.

(iv) $[Z_\alpha, T] = Z_\beta = \mathrm{C}_{Z_\alpha}(T)$ for all $T \leq G_{\alpha\beta}$ with $T \not\leq Q_\alpha$.

(v) $Z_\alpha = \Omega\,\mathrm{Z}(Q_\alpha)$.

Proof (i) As $B/Q_1 \in \mathrm{Syl}_2(P_1/Q_1)$, $|\Gamma(\alpha)| = |P_1/Q_1 : B/Q_1| = 3$. The action of G_α on $\Gamma(\alpha)$ induces a homomorphism

$$\phi : G_\alpha \longrightarrow \mathfrak{S}_{\Gamma(\alpha)} : g \longmapsto g\phi$$

$$g\phi : \Gamma(\alpha) \longrightarrow \Gamma(\alpha) : \delta \longmapsto \delta \cdot g$$

with $\ker\phi = (G_\alpha)_{\Gamma(\alpha)} = G_{\Delta(\alpha)} = Q_\alpha$. Then, as $G_\alpha/Q_\alpha \cong \mathfrak{S}_3 \cong \mathfrak{S}_{\Gamma(\alpha)}$,

$$|\mathrm{im}\,\phi| = |G_\alpha/\ker\phi| = |G_\alpha/Q_\alpha| = |\mathfrak{S}_{\Gamma(\alpha)}|$$

and so $\mathrm{im}\,\phi = \mathfrak{S}_{\Gamma(\alpha)}$. We have shown that ϕ is surjective: for each $\pi \in \mathfrak{S}_{\Gamma(\alpha)}$, there exists $g \in G_\alpha$ such that $g\phi = \pi$ and so $\delta \cdot g = \delta(g\phi) = \delta\pi$ for all $\delta \in \Gamma(\alpha)$. Thus, as $\mathfrak{S}_{\Gamma(\alpha)}$ acts 3-transitively on $\Gamma(\alpha)$, G_α acts 3-transitively on $\Gamma(\alpha)$.

(ii) By part (i), $|\Gamma(\alpha)| = 3 = |\mathrm{Syl}_2(G_\alpha/Q_\alpha)|$ and so $G_{\alpha\beta}/Q_\alpha$ and $G_{\alpha\tau}/Q_\alpha$ are distinct Sylow 2-subgroups of G_α/Q_α giving $G_\alpha/Q_\alpha = \langle G_{\alpha\beta}/Q_\alpha, G_{\alpha\tau}/Q_\alpha \rangle$. Thus, $G_\alpha = \langle G_{\alpha\beta}, G_{\alpha\tau}, Q_\alpha \rangle = \langle G_{\alpha\beta}, G_{\alpha\tau} \rangle$. Also, $|G_{\alpha\beta}/Q_\alpha| = 2$ because $G_{\alpha\beta}/Q_\alpha \in \mathrm{Syl}_2(G_\alpha/Q_\alpha)$.

(iii) We may regard Z_α as a faithful $\mathbb{F}_2(G_\alpha/Q_\alpha)$-module with respect to action by conjugation and, by Lemma 3.2.1(iv), $\mathrm{C}_{Z_\alpha}(G_\alpha/Q_\alpha) = \mathrm{C}_{Z_\alpha}(G_\alpha) = 1$. So, by Lemma 2.1.6, Z_α may be written as the direct sum of subspaces $Z_\alpha = \mathrm{C}_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha) \oplus \mathrm{C}_{Z_\alpha}(G_{\alpha\tau}/Q_\alpha)$ where, by Lemma 3.2.1(v), $\mathrm{C}_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha) = \mathrm{C}_{Z_\alpha}(G_{\alpha\beta}) = Z_\beta$ and, similarly, $\mathrm{C}_{Z_\alpha}(G_{\alpha\tau}/Q_\alpha) = Z_\tau$. Thus, $Z_\alpha = Z_\beta \times Z_\tau$.

(iv) Let $T \leq G_{\alpha\beta}$ with $T \not\leq Q_\alpha$. Then $1 \neq TQ_\alpha/Q_\alpha \leq G_{\alpha\beta}/Q_\alpha$ and so, as $|G_{\alpha\beta}/Q_\alpha| = 2$,

$TQ_\alpha/Q_\alpha = G_{\alpha\beta}/Q_\alpha$. So, by Lemma 2.1.6,

$$[Z_\alpha, T] = [Z_\alpha, TQ_\alpha/Q_\alpha] = [Z_\alpha, G_{\alpha\beta}/Q_\alpha] = \mathrm{C}_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha)$$
$$= \mathrm{C}_{Z_\alpha}(TQ_\alpha/Q_\alpha) = \mathrm{C}_{Z_\alpha}(T)$$

and, by Lemma 3.2.1(v), $\mathrm{C}_{Z_\alpha}(G_{\alpha\beta}/Q_\alpha) = \mathrm{C}_{Z_\alpha}(G_{\alpha\beta}) = Z_\beta$. Thus, $[Z_\alpha, T] = Z_\beta = \mathrm{C}_{Z_\alpha}(T)$.
(v) Set $Z = \Omega\,\mathrm{Z}(Q_\alpha)$. We may regard Z as a faithful $\mathbb{F}_2(G_\alpha/Q_\alpha)$-module with respect to action by conjugation and, by Lemma 3.2.1(iv), $\mathrm{C}_Z(G_\alpha/Q_\alpha) = \mathrm{C}_Z(G_\alpha) = 1$. So, by Lemma 2.1.6, $Z = \mathrm{C}_Z(G_{\alpha\beta}/Q_\alpha)\,\mathrm{C}_Z(G_{\alpha\tau}/Q_\alpha)$ where $\mathrm{C}_Z(G_{\alpha\beta}/Q_\alpha) = \mathrm{C}_Z(G_{\alpha\beta}) \le \Omega\,\mathrm{Z}(G_{\alpha\beta}) \le Z_\alpha$ and, similarly, $\mathrm{C}_Z(G_{\alpha\tau}/Q_\alpha) \le Z_\alpha$ giving $Z \le Z_\alpha$. Thus, $Z_\alpha = Z = \Omega\,\mathrm{Z}(Q_\alpha)$. □

The following result holds under the weaker hypothesis $P_2/Q_2 \in \mathfrak{X}$.

Corollary 3.2.3 *The critical distance* $b = 3$.

Proof Suppose, for a contradiction, that $b = 4$ and choose a critical pair (α, β') with critical path $(\alpha, \beta, \mu, \lambda, \beta')$. Then, by Lemmas 3.2.1(i) and 3.1.11, $\alpha, \beta' \in \mathcal{O}_1$ and (β', α) is a critical pair. So, by applying Lemma 3.2.2(iv) twice, $Z_\beta = [Z_\alpha, Z_{\beta'}] = [Z_{\beta'}, Z_\alpha] = Z_\lambda$, but, by Lemma 3.2.2(iii), $Z_\mu = Z_\beta \times Z_\lambda$ which is a contradiction. □

§3.3 The local beta pairs

A pair (β, β') of vertices with $V_\beta \not\leq Q_{\beta'}$ and $\mathrm{d}(\beta, \beta') = b - 1$ is called a *beta pair*. Given a critical pair (α, β') we may select $\beta \in \Gamma(\alpha)$ such that (β, β') is a beta pair.

Lemma 3.3.1 *Let (β, β') be a beta pair. The following hold:*

(i) There exists $\alpha = \alpha(\beta, \beta') \in \Gamma(\beta)$ such that (α, β') is a critical pair.

(ii) $(\beta \cdot g, \beta' \cdot g)$ is a beta pair for all $g \in G$.

Proof (i) There exists $\alpha = \alpha(\beta, \beta') \in \Gamma(\beta)$ such that $Z_\alpha \not\leq Q_{\beta'}$ because otherwise $V_\beta = \langle Z_\alpha \mid \alpha \in \Gamma(\beta) \rangle \leq Q_{\beta'}$ which is a contradiction. Also, $\mathrm{d}(\alpha, \beta') \leq \mathrm{d}(\alpha, \beta) + \mathrm{d}(\beta, \beta') = b$ and so, as $Z_\alpha \not\leq Q_{\beta'}$, $\mathrm{d}(\alpha, \beta') = b$. Thus, (α, β') is a critical pair.
(ii) Fix $g \in G$. Then $V_{\beta \cdot g} \not\leq Q_{\beta' \cdot g}$ because otherwise $V_\beta = V_{\beta \cdot g g^{-1}} = V_{\beta \cdot g}^{g^{-1}} \leq Q_{\beta' \cdot g}^{g^{-1}} = Q_{\beta' \cdot g g^{-1}} = Q_{\beta'}$ which is a contradiction. Also, $\mathrm{d}(\beta \cdot g, \beta' \cdot g) = \mathrm{d}(\beta, \beta') = b - 1$. Thus, $(\beta \cdot g, \beta' \cdot g)$ is a beta pair. □

Notation Choose a beta pair (β, β') with medial vertex $\mu \in \Gamma(\beta) \cap \Gamma(\beta')$. Then, by part (i) of the lemma above together with Lemmas 3.2.1(i) and 3.1.11, $\mu \in \mathcal{O}_1$ and so, by Lemma 3.2.2(i), we may assume that $\Gamma(\mu) = \{\beta, \beta', \beta^*\}$. Also, by Lemma 3.2.2(i), G_μ acts 2-transitively on $\Gamma(\mu)$ and so, by part (ii) of the lemma above, any two distinct elements

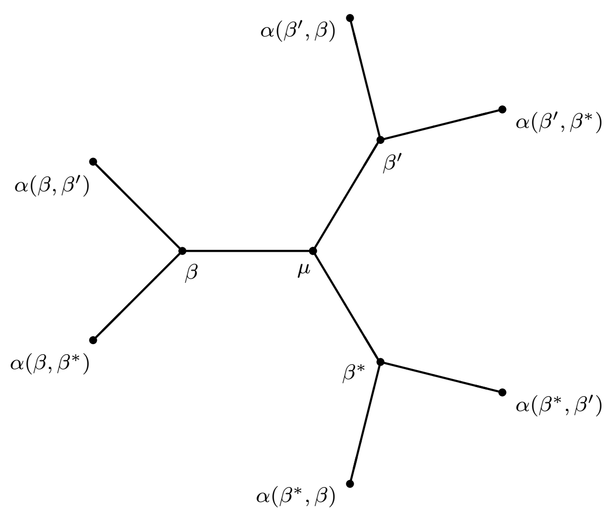

Figure D The critical paths of the local beta pairs.

of $\Gamma(\mu)$ form a beta pair. The six pairs of vertices (β, β'), (β, β^*), (β', β), (β', β^*), (β^*, β) and (β^*, β') are called the *local beta pairs.*

The proof of Theorem A will take place in terms of subgroups defined by the vertices μ, β, β' and β^*. All of the assumptions that we will make, both locally within proofs and globally, which depend on these vertices will symmetrically hold for the corresponding vertices of all of the local beta pairs. So a result that we may prove in terms of subgroups defined by the vertices μ, β, β' and β^* holds for all permutations of the vertices β, β' and β^* by applying the result to the corresponding vertices of each of the local beta pairs. For example, the statement $Y_\mu = V_\beta \cap V_{\beta'}$ may be interpreted as $Y_\mu = V_\beta \cap V_{\beta^*}$ and $Y_\mu = V_{\beta'} \cap V_{\beta^*}$. Alternatively, we could conjugate such statements by the appropriate element of G_μ, however, we have just seen the reason why this is unnecessary. On the other hand, we may still need to use conjugation in order to verify that any assumptions that we make hold for all of the local beta pairs.

Notation Fix the following elements of G_μ

- $g' \in G_{\mu\beta^*}$ such that $(\beta, \beta') \cdot g' = (\beta', \beta)$
- $g^* \in G_{\mu\beta'}$ such that $(\beta, \beta^*) \cdot g^* = (\beta^*, \beta)$
- $g'^* \in G_{\mu\beta}$ such that $(\beta', \beta^*) \cdot g'^* = (\beta^*, \beta')$.

Lemma 3.3.2 *Let K be a normal subgroup of $G_{\mu\beta}$ and define $K_{\mu\beta} = K$, $K_{\mu\beta'} = K^{g'}$ and $K_{\mu\beta^*} = K^{g^*}$. Then $K_{\mu\beta}^g = K_{\mu\beta\cdot g}$ for all $g \in G_\mu$. In particular, $\langle K_{\mu\beta}^{G_\mu} \rangle = \langle K_{\mu\beta}, K_{\mu\beta'}, K_{\mu\beta^*} \rangle$.*

Proof Let $g \in G_\mu$. Then, as G_μ acts on $\Gamma(\mu)$, $\beta \cdot g \in \Gamma(\mu) = \{\beta, \beta', \beta^*\}$. If $\beta \cdot g = \beta$, then $g \in G_{\mu\beta}$ and so, as $K_{\mu\beta} \trianglelefteq G_{\mu\beta}$, $K_{\mu\beta}^g = K_{\mu\beta} = K_{\mu\beta\cdot g}$. If $\beta \cdot g = \beta'$, then

$$\beta \cdot (g'g^{-1}) = (\beta \cdot g') \cdot g^{-1} = \beta' \cdot g^{-1} = \beta$$

so $g'g^{-1} \in G_{\mu\beta}$ and so, as $K_{\mu\beta} \trianglelefteq G_{\mu\beta}$, $K_{\mu\beta}^{g'g^{-1}} = K_{\mu\beta}$ giving $K_{\mu\beta}^g = K_{\mu\beta}^{g'} = K_{\mu\beta'} = K_{\mu\beta\cdot g}$. If $\beta \cdot g = \beta^*$, then, similarly, $K_{\mu\beta}^g = K_{\mu\beta^*} = K_{\mu\beta\cdot g}$. Thus, $K_{\mu\beta}^g = K_{\mu\beta\cdot g}$ for all $g \in G_\mu$. $\square$

Corollary 3.3.3 *Let K_β be a G_β-subgroup of Q_β and define $K_{\beta'} = K_\beta^{g'}$ and $K_{\beta^*} = K_\beta^{g^*}$. Then $K_\beta^g = K_{\beta\cdot g}$ for all $g \in G_\mu$. In particular, $\langle K_\beta^{G_\mu}\rangle = \langle K_\beta, K_{\beta'}, K_{\beta^*}\rangle$.*

Lemma 3.3.4 *Let K_β be a G_β-subgroup of C_β with $K_\beta \not\leq Q_{\beta'}$ and $\eta_{G_\beta}(K_\beta) \geq 1$. Define $K_{\beta'} = K_\beta^{g'}$ and $K_{\beta^*} = K_\beta^{g^*}$. Then the following hold:*

(i) $K_\beta \trianglelefteq Q_\mu$, $\mathrm{C}_{G_\beta}(K_\beta) \leq Q_\beta$ and $E_\beta \leq \langle K_{\beta'}^{G_\beta}\rangle$. In particular, $[K_\beta, K_{\beta'}] \neq 1$.

(ii) $|K_\beta Q_{\beta'}/Q_{\beta'}| = |K_{\beta'}Q_\beta/Q_\beta|$.

Proof (i) As $K_\beta \leq C_\beta \leq Q_\mu$, $Q_\mu \leq G_{\mu\beta} \leq G_\beta$ and K_β is normal in G_β, K_β is normal in Q_μ. As $\eta_{G_\beta}(K_\beta) \geq 1$ and by applying Lemma 3.1.6(ii), $\mathrm{C}_{G_\beta}(K_\beta) \leq Q_\beta$ and, as $K_{\beta'} \leq G_\beta$, $K_{\beta'} \not\leq Q_\beta$ and by applying Lemma 3.1.1, $E_\beta \leq \langle K_{\beta'}^{G_\beta}\rangle$.

(ii) Conjugating by g' we have that

$$\begin{aligned}|K_\beta Q_{\beta'}/Q_{\beta'}| &= |(K_\beta Q_{\beta'})^{g'}/Q_{\beta'}^{g'}| = |K_\beta^{g'} Q_{\beta'}^{g'}/Q_{\beta'}^{g'}| = |K_{\beta\cdot g'}Q_{\beta'\cdot g'}/Q_{\beta'\cdot g'}| \\ &= |K_{\beta'}Q_\beta/Q_\beta|.\end{aligned}$$

□

Corollary 3.3.5 *Let $K_\beta \in \{V_\beta, W_\beta\}$. Then $K_\beta \not\leq Q_{\beta'}$ and $\eta_{G_\beta}(K_\beta) \geq 1$. Moreover, the following hold:*

(i) $K_\beta \trianglelefteq Q_\mu$, $\mathrm{C}_{G_\beta}(K_\beta) \leq Q_\beta$ and $E_\beta \leq \langle K_{\beta'}^{G_\beta}\rangle$. In particular, $[K_\beta, K_{\beta'}] \neq 1$.

(ii) $[K_\beta, K_{\beta'}, C_{\beta'}] = 1 = [K_\beta, C_{\beta'}, K_{\beta'}]$. In particular, $[K_\beta, K_{\beta'}, K_{\beta'}] = 1$.

(iii) $|K_\beta Q_{\beta'}/Q_{\beta'}| = |K_{\beta'}Q_\beta/Q_\beta|$.

Proof By Lemma 3.2.1(vi), $V_\beta = Z_\mu W_\beta$ and $\eta_{G_\beta}(W_\beta) = \eta_{G_\beta}(V_\beta)$ and so, as (β, β') is a beta pair and by Lemma 3.1.5, $K_\beta \not\leq Q_{\beta'}$ and $\eta_{G_\beta}(K_\beta) \geq 1$. Hence, parts (i) and (iii) follow immediately from the lemma above.

(ii) As K_β and $K_{\beta'}$ are normal in Q_μ, $[K_\beta, K_{\beta'}] \leq K_{\beta'} \leq V_{\beta'}$ and so $[K_\beta, K_{\beta'}, C_{\beta'}] \leq [V_{\beta'}, C_{\beta'}] = 1$. Similarly, as K_β and $C_{\beta'}$ are normal in Q_μ, $[K_\beta, C_{\beta'}, K_{\beta'}] \leq [C_{\beta'}, V_{\beta'}] = 1$. Hence, $[K_\beta, K_{\beta'}, C_{\beta'}] = 1 = [K_\beta, C_{\beta'}, K_{\beta'}]$. □

Notation Define $\overline{G_\beta} = G_\beta/Q_\beta$ and identify $\overline{G_\beta}$ with either $\mathfrak{A}_n$ or $\mathfrak{S}_n$. Then $\overline{E_\beta} = E_\beta Q_\beta/Q_\beta$, $\overline{G_{\mu\beta}} = G_{\mu\beta}/Q_\beta$ and $\overline{V_{\beta'}} = V_{\beta'}Q_\beta/Q_\beta$.

Lemma 3.3.6 *Either $Q_\beta \leq Q_\mu$ or $G_{\mu\beta} = Q_\mu Q_\beta$.*

Proof By Lemma 3.2.2(ii),

$$|G_{\mu\beta}\!:\!Q_\mu Q_\beta|\;|Q_\mu Q_\beta\!:\!Q_\mu| = |G_{\mu\beta}\!:\!Q_\mu| = 2$$

and so either $Q_\beta \leq Q_\mu$ or $G_{\mu\beta} = Q_\mu Q_\beta$. □

We will consider separately the cases when $Q_\beta \leq Q_\mu$ and $G_{\mu\beta} = Q_\mu Q_\beta$ in Chapters 4 and 5 of this thesis. At the beginning of Chapters 4 and 5 we will make the assumptions $Q_\beta \leq Q_\mu$ and $G_{\mu\beta} = Q_\mu Q_\beta$ respectively which will automatically hold for all of the local beta pairs because G acts edge-transitively on Γ.

We end this chapter by proving a general lemma which holds under the weaker hypothesis $P_2/Q_2 \in \mathfrak{X}$ and will play a crucial rôle in the $Q_\beta \leq Q_\mu$ case where $C_\beta = Q_\beta$.

Lemma 3.3.7 *Let $\eta_{G_\beta}(C_\beta) = 1$. Then $V_\beta(C_\beta \cap C_{\beta'})$ is an elementary abelian 2-group. Moreover, if $G_\beta = Q_\mu E_\beta$, then $V_\beta(C_\beta \cap C_{\beta'}) = V_\beta(C_\beta \cap C_{\beta^*}) \trianglelefteq G_\beta$.*

Proof Set $Y = V_\beta(C_\beta \cap C_{\beta'}) \leq C_\beta$. Suppose, for a contradiction, that $\Phi(C_\beta \cap C_{\beta'}) \neq 1$ and define $Z_\mu^0 = \langle (\Phi(C_\beta \cap C_{\beta'}) \cap Z_{\beta^*})^{G_\mu} \rangle \leq Z_\mu$. Then, as $C_\beta \cap C_{\beta'} \leq Q_\mu \leq G_{\mu\beta^*}$ and $G_{\mu\beta^*}$ acts on $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$, $1 \neq \Phi(C_\beta \cap C_{\beta'}) \trianglelefteq G_{\mu\beta^*}$ and so, by Lemma 3.2.1,

$$\Phi(C_\beta \cap C_{\beta'}) \cap Z_{\beta^*} = \Phi(C_\beta \cap C_{\beta'}) \cap \Omega\,\mathrm{Z}(G_{\mu\beta^*}) \neq 1$$

giving $Z_\mu^0 \neq 1$. Also, as $\eta_{G_\beta}(C_\beta) = 1$ and by Burnside's Lemma, $\eta_{G_\beta}(\Phi(C_\beta)) = 0$ and so, as $\Phi(C_\beta \cap C_{\beta'}) \leq \Phi(C_\beta)$, we have that $[\Phi(C_\beta \cap C_{\beta'}), E_\beta] = 1$ and $[\Phi(C_\beta \cap C_{\beta'}), E_{\beta'}] = 1$. Then, as $Z_{\beta^*} \leq \mathrm{Z}(G_{\beta^*})$ and G_μ acts on $\Gamma(\mu) = \{\beta, \beta', \beta^*\}$,

$$\Phi(C_\beta \cap C_{\beta'}) \cap Z_{\beta^*} \leq \mathrm{C}_{G_\mu}(E_\beta) \cap \mathrm{C}_{G_\mu}(E_{\beta'}) \cap \mathrm{C}_{G_\mu}(E_{\beta^*}) = \mathrm{C}_{G_\mu}(\langle E_\beta, E_{\beta'}, E_{\beta^*} \rangle) \trianglelefteq G_\mu$$

and so $Z_\mu^0 \leq \mathrm{C}_{G_\mu}(\langle E_\beta, E_{\beta'}, E_{\beta^*} \rangle)$. In particular, $E_\beta \leq \mathrm{C}_{G_\beta}(Z_\mu^0) \leq \mathrm{N}_{G_\beta}(Z_\mu^0)$ giving $Z_\mu^0 = 1$ which is a contradiction. Hence, $\Phi(C_\beta \cap C_{\beta'}) = 1$. Also, as V_β is an elementary abelian 2-group, $\Phi(V_\beta) = 1$. We have that $\Phi(Y) = \Phi(V_\beta)\,[V_\beta, C_\beta \cap C_{\beta'}]\,\Phi(C_\beta \cap C_{\beta'}) = 1$ and therefore Y is an elementary abelian 2-group. Now, let $G_\beta = Q_\mu E_\beta$. Then, as $\eta_{G_\beta}(C_\beta) = 1$ and by Lemma 3.2.1(iii), $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ and so $[Y, E_\beta] \leq [C_\beta, E_\beta] \leq V_\beta \leq Y$ giving, as V_β, C_β

and $C_{\beta'}$ are normal in Q_μ, $Y = V_\beta(C_\beta \cap C_{\beta'}) \trianglelefteq Q_\mu E_\beta = G_\beta$. So, as $Y \trianglelefteq G_{\mu\beta}$ and $g'^* \in G_{\mu\beta}$,

$$Y = Y^{g'^*} = V_\beta^{g'^*}(C_\beta^{g'^*} \cap C_{\beta'}^{g'^*}) = V_{\beta\cdot g'^*}(C_{\beta\cdot g'^*} \cap C_{\beta'\cdot g'^*}) = V_\beta(C_\beta \cap C_{\beta^*}).$$

Thus, $Y = V_\beta(C_\beta \cap C_{\beta^*}) \trianglelefteq G_\beta$. □

Chapter 4

The $Q_\beta \leq Q_\mu$ case

§4.1 Module calculations

We refer the reader to Sections 2.1 and 2.5 for an explanation of the following notation which will hold in this section.

Notation Let $n = 2^m$ with $m \geq 3$ and V be the n-dimensional permutation module for $\mathfrak{S}_n$ over $\mathbb{F}_2$ with ordered basis $(v_1, v_2, \ldots, v_n)$. Recall that V_0 denotes the zero weight submodule of V.

Set $T = \mathfrak{T}_m \in \mathrm{Syl}_2(\mathfrak{S}_n)$ and let $B = T^1 \times T^2$ be the base group of the wreath product $T = \mathfrak{T}_{m-1} \wr \mathfrak{C}_2$ where T^i is similar to $\mathfrak{T}_{m-1}$ for $i \in \{1, 2\}$. Let $\mathrm{Z}(T) = \langle t \rangle$ and assume that $t = (1\ 2)(3\ 4)\ldots(n-1\ n)$. Let $\mathrm{Z}(T^i) = \langle t_i \rangle$ for $i \in \{1, 2\}$ so that $\mathrm{Z}(B) = \langle t_1, t_2 \rangle$. Set $T_0 = T \cap \mathfrak{A}_n \in \mathrm{Syl}_2(\mathfrak{A}_n)$, $B_0 = B \cap \mathfrak{A}_n$ and $T_0^i = T^i \cap \mathfrak{A}_n$ for $i \in \{1, 2\}$. Then $\mathrm{Z}(T_0) = \mathrm{Z}(T)$, $\mathrm{Z}(B_0) = \mathrm{Z}(B)$ and either $\mathrm{Z}(T_0^i) = \mathrm{Z}(T^i)$ for $i \in \{1, 2\}$ or $n = 8$ and T_0^i is similar to $\mathfrak{K}_4$ for $i \in \{1, 2\}$.

Lemma 4.1.1 *Let A be a non-trivial normal abelian subgroup of B with $|V_0/\,\mathrm{C}_{V_0}(A)| \leq |A|$. Then one of the following two cases hold:*

(I) $\mathrm{Z}(B) \leq A$.

(II) Either $A \leq T^1$ or $A \leq T^2$.

Proof Assume that $\mathrm{Z}(B) \not\leq A$. We will show that either $A \leq T^1$ or $A \leq T^2$. We will proceed via a series of steps.

(1) $A \cap T^i \neq 1$ for some $i \in \{1, 2\}$. In particular, $t_i \in A$.

Suppose, for a contradiction, that $A \cap T^1 = 1 = A \cap T^2$. Then with $\{i, j\} = \{1, 2\}$, as T^i and A are normal in B, $[T^i, A] \leq T^i \cap A = 1$ and so $A \leq \mathrm{C}_B(T^i) = \mathrm{Z}(T^i) \times T^j$. We have that

$$A \leq (\mathrm{Z}(T^1) \times T^2) \cap (T^1 \times \mathrm{Z}(T^2)) = \mathrm{Z}(T^1) \times \mathrm{Z}(T^2) = \mathrm{Z}(B) = \langle t_1, t_2 \rangle$$

and so, as $t_1, t_2 \notin A$, $A = \langle t \rangle$. Then, by Lemma 2.1.16(ii),

$$2^{(n/2-1)} = |V_0 / \mathrm{C}_{V_0}(t)| = |V_0 / \mathrm{C}_{V_0}(A)| \leq |A| = 2$$

and so $n/2 - 1 \leq 1$ giving $n \leq 4$ which is a contradiction. Hence, there exists $i \in \{1, 2\}$ such that $A \cap T^i \neq 1$. In particular, as A is normalized by T^i, $A \cap \mathrm{Z}(T^i) \neq 1$ and so $t_i \in A$.

(2) $A \leq T^i \times \mathrm{Z}(T^j)$ *and* $A \cap T^j = 1$ *where* $j = 3 - i$.

We have that $A \cap T^j = 1$ because otherwise $t_j \in A$ and so, by (1), $\mathrm{Z}(B) = \langle t_i, t_j \rangle \leq A$ which is a contradiction. So, as T^i and A are normal in B, $[T^j, A] \leq T^j \cap A = 1$ and hence $A \leq \mathrm{C}_B(T^j) = T^i \times \mathrm{Z}(T^j)$.

(3) $|A| \leq 2^{(n/4)}$.

We have that, by (2),

$$A \cong A/(A \cap T^j) \cong AT^j/T^j \leq (T^i \times T^j)/T^j \cong T^i$$

and so A is isomorphic to an abelian subgroup of T^i. Hence, as $T^i \leq \mathfrak{S}_{n/2}$ and by Lemma 2.5.3, $|A| \leq 2^{(n/4)}$.

(4) $A \leq T^i$.

Suppose, for a contradiction, that $A \not\leq T^i$. Then there exists $a \in A \setminus T^i$ and, by (2), $a = s_i t_j$ for some $s_i \in T^i$. By applying Lemma 2.1.13(i),

$$\mathrm{C}_{V_0}(A) \leq \mathrm{C}_{V_0}(a) = \mathrm{C}_{V_0}(s_i t_j) = \mathrm{C}_{V_0}(s_i) \cap \mathrm{C}_{V_0}(t_j) \leq \mathrm{C}_{V_0}(t_j)$$

and so, by (1) and applying Lemma 2.1.13(i) again,

$$\mathrm{C}_{V_0}(A) \leq \mathrm{C}_{V_0}(t_i) \cap \mathrm{C}_{V_0}(t_j) = \mathrm{C}_{V_0}(t_i t_j) = \mathrm{C}_{V_0}(t).$$

Then, by (3) and Lemma 2.1.16(ii),

$$2^{(n/2-1)} = |V_0/\operatorname{C}_{V_0}(t)| \leq |V_0/\operatorname{C}_{V_0}(A)| \leq |A| \leq 2^{(n/4)}$$

and so $n/2-1 \leq n/4$ giving $n \leq 4$ which is a contradiction. Therefore, $A \leq T^i$. □

Lemma 4.1.2 *Let A be a non-trivial normal abelian subgroup of B_0 with $|V_0/\operatorname{C}_{V_0}(A)| \leq |A|$. Then one of the following two cases hold:*

(I) $\operatorname{Z}(B_0) \leq A$.

(II) Either $A \leq T_0^1$ or $A \leq T_0^2$.

Proof If $m \geq 4$, then, by Corollary 2.5.6, $\operatorname{C}_{T^i}(T_0^i) = \operatorname{Z}(T_0^i)$ for $i \in \{1,2\}$ and so with $\{i,j\} = \{1,2\}$ we have that $\operatorname{C}_B(T_0^i) = \operatorname{C}_{T^i}(T_0^i) \times T^j = \operatorname{Z}(T_0^i) \times T^j$ giving $\operatorname{C}_{B_0}(T_0^i) = \operatorname{C}_B(T_0^i) \cap \mathfrak{A}_n = \operatorname{Z}(T_0^i) \times T_0^j$. Hence, the result holds for $m \geq 4$ by the proof of the previous lemma with B_0, T_0^1 and T_0^2 in place of B, T^1 and T^2 respectively. It remains to prove the result in the case when $m = 3$. In this case we have that T_0^i is similar to $\mathfrak{K}_4$ for $i \in \{1,2\}$ and $\operatorname{C}_{B_0}(T_0^1) = T_0^1 \times T_0^2 = \operatorname{C}_{B_0}(T_0^2)$ because with $\{i,j\} = \{1,2\}$, by Corollary 2.5.6, $\operatorname{C}_B(T_0^i) = \operatorname{C}_{T^i}(T_0^i) \times T^j = T_0^i \times T^j$ and so $\operatorname{C}_{B_0}(T_0^i) = T_0^i \times T_0^j$. Assume that $\operatorname{Z}(B_0) \not\leq A$. We will proceed via a series of steps.

(1) $A \cap T_0^i \neq 1$ for some $i \in \{1,2\}$. In particular, $t_i \in A$.

Suppose, for a contradiction, that $A \cap T_0^1 = 1 = A \cap T_0^2$. Then, as T_0^1 and A are normal in B_0, $[T_0^1, A] \leq T_0^1 \cap A = 1$ and so $A \leq \operatorname{C}_{B_0}(T_0^1) = T_0^1 \times T_0^2$. We have that every element of $A^\#$ is a fixed-point-free involution and so A is semiregular and intransitive giving $|A| \leq 2^2$. Fix $a \in A^\#$. Then, by Lemma 2.1.16(ii),

$$2^3 = |V_0/\operatorname{C}_{V_0}(a)| \leq |V_0/\operatorname{C}_{V_0}(A)| \leq |A| \leq 2^2$$

which is a contradiction. Hence, there exists $i \in \{1,2\}$ such that $A \cap T_0^i \neq 1$. In particular, as $1 \neq A \cap T_0^i \trianglelefteq B_0$, $A \cap T_0^i \cap \operatorname{Z}(B_0) \neq 1$ and so, as $\operatorname{Z}(B_0) = \langle t_1, t_2 \rangle$, $t_i \in A$.

(2) $A \leq T_0^i \times T_0^j$ and $A \cap T_0^j = 1$ where $j = 3 - i$.

We have that $A \cap T_0^j = 1$ because otherwise $t_j \in A$ and so, by (1), $\operatorname{Z}(B_0) = \langle t_i, t_j \rangle \leq A$

which is a contradiction. So, as T_0^j and A are normal in B_0, $[T_0^j, A] \leq T_0^j \cap A = 1$ and hence $A \leq \mathrm{C}_{B_0}(T_0^j) = T_0^i \times T_0^j$.

(3) $|A| \leq 2^2$.

We have that, by (2),

$$A \cong A/(A \cap T_0^j) \cong AT_0^j/T_0^j \leq (T_0^i \times T_0^j)/T_0^j \cong T_0^i$$

and so A is isomorphic to a subgroup of T_0^i. Hence, $|A| \leq |T_0^i| = 2^2$.

(4) $A \leq T_0^i$.

Suppose, for a contradiction, that $A \not\leq T_0^i$. Then there exists $a \in A \setminus T_0^i$ and, by (2), a is a fixed-point-free involution. By Lemma 2.1.16(ii) and (3),

$$2^3 = |V_0/\mathrm{C}_{V_0}(a)| \leq |V_0/\mathrm{C}_{V_0}(A)| \leq |A| \leq 2^2$$

which is a contradiction. Therefore, $A \leq T_0^i$. □

Lemma 4.1.3 *Set* $A^* = \langle (1\ 2), (3\ 4), \ldots, (n-1\ n) \rangle$ *and* $A_0^* = A^* \cap \mathfrak{A}_n$. *The following hold:*

(i) $\mathrm{C}_{\mathfrak{S}_n}([V_0, t]) = A^*$ *and* $\mathrm{C}_{\mathfrak{A}_n}([V_0, t]) = A_0^*$.

(ii) $[V_0, A^*] = \mathrm{C}_{V_0}(A^*) = \mathrm{C}_{V_0}(t)$ *and* $[V_0, A_0^*] = \mathrm{C}_{V_0}(A_0^*) = \mathrm{C}_{V_0}(t)$.

Proof (i) By Lemma 2.1.14(ii), $A^* \leq \mathrm{C}_{\mathfrak{S}_n}([V_0, t])$. Conversely, let $s \in \mathrm{C}_{\mathfrak{S}_n}([V_0, t])$ and fix $1 \leq i \leq n-5$ with i odd. Then, as $v_i + v_{i+1} + v_{i+2} + v_{i+3} \in [V_0, t]$,

$$v_{is} + v_{(i+1)s} + v_{(i+2)s} + v_{(i+3)s} = (v_i + v_{i+1} + v_{i+2} + v_{i+3})^s = v_i + v_{i+1} + v_{i+2} + v_{i+3}$$

and so s fixes the set $\{i, i+1, i+2, i+3\}$. Similarly, s fixes the set $\{i+2, i+3, i+4, i+5\}$ and so s fixes the sets $\{i+2, i+3\}$, $\{i, i+1\}$ and $\{i+4, i+5\}$. We have shown that s either fixes or swaps i and $i+1$ for all odd $1 \leq i \leq n-1$ and hence $s \in A^*$. Thus, $\mathrm{C}_{\mathfrak{S}_n}([V_0, t]) = A^*$ and $\mathrm{C}_{\mathfrak{A}_n}([V_0, t]) = \mathrm{C}_{\mathfrak{S}_n}([V_0, t]) \cap \mathfrak{A}_n = A^* \cap \mathfrak{A}_n = A_0^*$.

(ii) As $t = t_1 t_2$ and by applying Lemma 2.1.13 and Corollary 2.1.15,

$$[V_0, A^*] = [V_0, (1\ 2)] + [V_0, (3\ 4)] + \cdots + [V_0, (n-1\ n)] = [V_0, t_1] + [V_0, t_2] = \mathrm{C}_{V_0}(t)$$

$$= \mathrm{C}_{V_0}((1\ 2)) \cap \mathrm{C}_{V_0}((3\ 4)) \cap \ldots \cap \mathrm{C}_{V_0}((n-1\ n)) = \mathrm{C}_{V_0}(A^*)$$

and so $[V_0, A^*] = \mathrm{C}_{V_0}(A^*) = \mathrm{C}_{V_0}(t)$. Also, as $t_1, t_2 \in A_0^*$ and by applying Corollary 2.1.15,

$$\mathrm{C}_{V_0}(t) = [V_0, t_1] + [V_0, t_2] \leq [V_0, A_0^*] \leq [V_0, A^*] = \mathrm{C}_{V_0}(A^*) \leq \mathrm{C}_{V_0}(A_0^*) \leq \mathrm{C}_{V_0}(t)$$

and so $[V_0, A_0^*] = \mathrm{C}_{V_0}(A_0^*) = \mathrm{C}_{V_0}(t)$. □

Lemma 4.1.4 *Let A be an elementary abelian 2-subgroup of $\mathfrak{S}_n$ with $t \in A$, $[V_0, A, A] = 0$ and $|V_0/\mathrm{C}_{V_0}(A)| \leq |A|$. Then $[V_0, A] = \mathrm{C}_{V_0}(A) = \mathrm{C}_{V_0}(t)$.*

Proof Set $A_0 = A \cap \mathfrak{A}_n$, $A^* = \langle (1\ 2), (3\ 4), \ldots, (n-1\ n) \rangle$ and $A_0^* = A^* \cap \mathfrak{A}_n$. Then, as $t \in A$, $[V_0, t, A] \leq [V_0, A, A] = 0$ and so, by Lemma 4.1.3(i), $A \leq \mathrm{C}_{\mathfrak{S}_n}([V_0, t]) = A^*$. Also, as $t \in A$ and by Lemma 2.1.16(ii),

$$2^{(n/2-1)} = |V_0/\mathrm{C}_{V_0}(t)| \leq |V_0/\mathrm{C}_{V_0}(A)| \leq |A|$$

giving $|A^*/A| \leq 2^{(n/2)}/\,2^{(n/2-1)} = 2$. If $A = A^*$, then the result holds by Lemma 4.1.3(ii) and so we may assume that $|A^*/A| = 2$. If $A \leq \mathfrak{A}_n$, then $A \leq A^* \cap \mathfrak{A}_n = A_0^*$ and $|A| = 2^{(n/2-1)} = |A_0^*|$ so $A = A_0^*$ and the result holds by Lemma 4.1.3(ii). Let $A \not\leq \mathfrak{A}_n$ and suppose, for a contradiction, that $[V_0, t] = [V_0, A]$. We have that

$$2 \leq |A/A_0| = |A/(A \cap \mathfrak{A}_n)| = |A\mathfrak{A}_n/\mathfrak{A}_n| \leq |\mathfrak{S}_n/\mathfrak{A}_n| = 2$$

and so $|A/A_0| = 2$. Then, as A^* is an elementary abelian 2-group, A^*/A_0 is an elementary abelian 2-group of order $|A^*/A_0| = |A^*/A|\,|A/A_0| = 4$ giving $A^*/A_0 \cong \mathfrak{C}_2 \times \mathfrak{C}_2$. In particular, A^*/A_0 has three non-trivial proper subgroups and so there are three proper subgroups of A^* that strictly contain A_0: A, A_0^* and A_1, say. We will now prove the statements (a)–(c) below.

(a) Every transposition in A^* is in either A or A_1.

Let $\tau \in A^*$ be a transposition. Then, as $\tau \notin \mathfrak{A}_n$, $A_0 < A_0\langle\tau\rangle < A^*$ and $A_0\langle\tau\rangle \neq A_0^*$. So either $A_0\langle\tau\rangle = A$ or $A_0\langle\tau\rangle = A_1$ and hence either $\tau \in A$ or $\tau \in A_1$.

(b) A does not contain a product of two disjoint transpositions in A^*. In particular, A contains at most one transposition.

Let $\tau_1 = (i_1\ j_1)$ and $\tau_2 = (i_2\ j_2)$ be disjoint transpositions in A^* and suppose, for a contradiction, that $\tau_1\tau_2 \in A$. Choose $k \in \{1, 2, \ldots, n\} \setminus \{i_1, j_1, i_2, j_2\}$. Then $v_{i_1} + v_{j_1} = [v_{i_1} + v_k, \tau_1\tau_2] \in [V_0, A] = [V_0, t]$ which contradicts Lemma 2.1.14(ii). Hence, $\tau_1\tau_2 \notin A$.

(c) A_1 contains at most one transposition.

Suppose, for a contradiction, that there exist disjoint transpositions τ_1 and τ_2 in A_1. Then, as $|A_1/A_0| = 2$, $\{A_0, A_0\tau_1\} = A_1/A_0 = \{A_0, A_0\tau_2\}$ and so $A_0\tau_1 = A_0\tau_2$ giving $\tau_1\tau_2 \in A_0 \leq A$ which contradicts statement (b). Hence, A_1 contains at most one transposition.

The three statements above show that A^* contains at most two disjoint transpositions giving $n \leq 4$ which is a contradiction and hence $[V_0, t] < [V_0, A]$. Now, as $t \in A$ and $[V_0, A, A] = 0$, $[V_0, t] < [V_0, A] \leq \mathrm{C}_{V_0}(A) \leq \mathrm{C}_{V_0}(t)$ and therefore, by Lemma 2.1.14, $[V_0, A] = \mathrm{C}_{V_0}(A) = \mathrm{C}_{V_0}(t)$. □

Lemma 4.1.5 *Let $n = 8$ and Q be a transitive subgroup of T_0 of index 2. Let A be a non-trivial normal elementary abelian 2-subgroup of Q with $[V_0, A, A] = 0$ and $|V_0/\mathrm{C}_{V_0}(A)| \leq |A|$. Then $\mathrm{Z}(Q) = \mathrm{Z}(T_0)$.*

Proof Suppose, for a contradiction, that $\mathrm{Z}(Q) \neq \mathrm{Z}(T_0)$. Then, by Lemma 2.5.7, $\mathrm{Z}(Q)$ is similar to the group $\langle (1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3)(2\ 4)(5\ 7)(6\ 8)\rangle$. Observe that if an element $x \in Q$ has a fixed point k_0, then x fixes all of the points in the orbit $k_0 \cdot \mathrm{Z}(Q)$. Hence, an involution $x \in Q$ is either fixed-point-free or contained in $\mathfrak{K}_4 \times \mathfrak{K}_4$. Now, as $1 \neq A \trianglelefteq Q$, $A \cap \mathrm{Z}(Q) \neq 1$ and so we may choose a fixed-point-free involution $a \in A$. Then, by Lemma 2.1.16(ii),

$$8 = 2^3 = |V_0/\mathrm{C}_{V_0}(a)| \leq |V_0/\mathrm{C}_{V_0}(A)| \leq |A|.$$

Also, as $[V_0, a, A] \leq [V_0, A, A] = 0$, $A \leq \mathrm{C}_{\mathfrak{A}_n}([V_0, a])$ giving, as $|A| \geq 8 = |\mathrm{C}_{\mathfrak{A}_n}([V_0, a])|$, $A = \mathrm{C}_{\mathfrak{A}_n}([V_0, a])$. In particular, A contains exactly one fixed-point-free element, namely a. By the observation made above, all other elements of A are contained in $\mathfrak{K}_4 \times \mathfrak{K}_4$ and so, as $|A| = 8$, A contains all seven of the elements of $\mathfrak{K}_4 \times \mathfrak{K}_4$ with fixed points. However, A then has more than one fixed-point-free element which is a contradiction. Therefore, $\mathrm{Z}(Q) = \mathrm{Z}(T_0)$. □

Lemma 4.1.6 *Fix $i \in \{1, 2\}$ and let Δ_i be the orbit of T^i. Then $V_0(\Delta_i)$ is a faithful $\mathbb{F}_2\mathfrak{S}_{\Delta_i}$-module that is normalized by B. Moreover, if A and X are subgroups of B with $t_i \in A \leq X$ and $[V_0(\Delta_i), X, A] = 0$, then* $\dim([V_0(\Delta_i), X]/[V_0(\Delta_i), A]) \leq 1$.

Proof As $n/2 \geq 3$ and by applying Lemma 2.1.2(i), $V_0(\Delta_i)$ is a faithful $\mathbb{F}_2\mathfrak{S}_{\Delta_i}$-module. Set $j = 3 - i$. Then $V_0(\Delta_i)$ is normalized by T^i and centralized by T^j and hence $V_0(\Delta_i)$ is normalized by $T^i \times T^j = B$. Now, let A and X be subgroups of B with $t_i \in A \leq X$ and $[V_0(\Delta_i), X, A] = 0$. Then

$$[V_0(\Delta_i), t_i] \leq [V_0(\Delta_i), A] \leq [V_0(\Delta_i), X] \leq \mathrm{C}_{V_0(\Delta_i)}(A) \leq \mathrm{C}_{V_0(\Delta_i)}(t_i)$$

and hence $\dim([V_0(\Delta_i), X]/[V_0(\Delta_i), A]) \leq \dim(\mathrm{C}_{V_0(\Delta_i)}(t_i)/[V_0(\Delta_i), t_i]) = 1$. □

Corollary 4.1.7 *Fix $i \in \{1, 2\}$ and let Δ_i be the orbit of T_0^i. Then $V_0(\Delta_i)$ is a faithful $\mathbb{F}_2\mathfrak{A}_{\Delta_i}$-module that is normalized by B_0. Moreover, if A and X are subgroups of B_0 with $t_i \in A \leq X$ and $[V_0(\Delta_i), X, A] = 0$, then* $\dim([V_0(\Delta_i), X]/[V_0(\Delta_i), A]) \leq 1$.

§4.2 Analysing the structure of the module W_β

We will assume the following hypothesis for the remainder of this chapter.

Hypothesis Assume that $Q_\beta \le Q_\mu$.

The following lemma holds under the weaker hypothesis $P_2/Q_2 \in \mathfrak{X}$.

Lemma 4.2.1 *The following hold:*

(i) $V_\beta \le \Omega\,\mathrm{Z}(Q_\beta)$ *and* $C_\beta = Q_\beta$*. In particular,* V_β *may be regarded as a faithful* $\mathbb{F}_2\overline{G_\beta}$*-module with respect to action by conjugation.*

(ii) $\overline{V_{\beta'}}$ *is a non-trivial elementary abelian 2-subgroup of* $\overline{G_\beta}$ *with* $[V_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = 1$ *and* $|V_\beta/\,\mathrm{C}_{V_\beta}(\overline{V_{\beta'}})| = |\overline{V_{\beta'}}|$*. In particular,* V_β *is an FF-module with offending subgroup* $\overline{V_{\beta'}}$*.*

(iii) $\overline{G_\beta} \not\cong \mathfrak{A}_5$*.*

Proof (i) By hypothesis, $Q_\beta \le Q_\mu$ and so $[Z_\mu, Q_\beta] \le [Z_\mu, Q_\mu] = 1$ giving, as $Q_\beta \trianglelefteq G_\beta$,

$$[V_\beta, Q_\beta] = [\langle Z_\mu^{G_\beta}\rangle, Q_\beta] = \langle [Z_\mu, Q_\beta]^{G_\beta}\rangle = 1$$

and hence, as V_β is an elementary abelian 2-subgroup of Q_β, $V_\beta \le \Omega\,\mathrm{Z}(Q_\beta)$. Thus, by Lemma 3.2.1(ii), $C_\beta = \mathrm{C}_{Q_\beta}(V_\beta) = Q_\beta$.

(ii) As (β', β) is a beta pair, $V_{\beta'} \le Q_\mu \le G_\beta$ and $V_{\beta'} \not\le Q_\beta$ so that $\overline{V_{\beta'}}$ is a non-trivial elementary abelian 2-subgroup of $\overline{G_\beta}$ and, by Corollary 3.3.5(ii), $[V_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = [V_\beta, V_{\beta'}, V_{\beta'}] = 1$. Also, by part (i), $C_{\beta'} = Q_{\beta'}$ and so, as $V_\beta \le Q_\mu \le G_{\beta'}$,

$$\mathrm{C}_{V_\beta}(\overline{V_{\beta'}}) = \mathrm{C}_{V_\beta}(V_{\beta'}) = V_\beta \cap \mathrm{C}_{G_{\beta'}}(V_{\beta'}) = V_\beta \cap C_{\beta'} = V_\beta \cap Q_{\beta'}.$$

Then, by Corollary 3.3.5(iii),

$$|V_\beta/\,\mathrm{C}_{V_\beta}(\overline{V_{\beta'}})| = |V_\beta/(V_\beta \cap Q_{\beta'})| = |V_\beta Q_{\beta'}/Q_{\beta'}| = |V_{\beta'}Q_\beta/Q_\beta| = |\overline{V_{\beta'}}|.$$

(iii) Suppose, for a contradiction, that $\overline{G_\beta} \cong \mathfrak{A}_5$. We have that $1 \ne \overline{V_{\beta'}} \le \overline{Q_\mu} < \overline{G_{\mu\beta}}$ and $|\overline{G_{\mu\beta}}| = 4$ and so $|\overline{V_{\beta'}}| = 2$. Then, by part (ii), $|V_\beta/\,\mathrm{C}_{V_\beta}(\overline{V_{\beta'}})| = 2$ and so $\overline{V_{\beta'}}$ induces a transvection on V_β which contradicts Theorem 2.3.8(ii). □

Corollary 4.2.2 *The submodule $W_\beta = [V_\beta, E_\beta]$ of V_β is isomorphic to the zero weight submodule V_0 of the n-dimensional permutation module for $\overline{G_\beta}$ over $\mathbb{F}_2$. Moreover, $n = 2^m$ with $m \geq 3$.*

Proof We have that, by Lemma 3.2.1(v),

$$W_\beta \cap Z_\beta = W_\beta \cap \mathrm{C}_{V_\beta}(G_{\mu\beta}) = \mathrm{C}_{W_\beta}(G_{\mu\beta}) = \mathrm{C}_{W_\beta}(\overline{G_{\mu\beta}})$$

and, similarly, $W_\beta \cap Z_\beta = \mathrm{C}_{W_\beta}(\overline{G_\beta})$. By Lemma 3.2.1(vi), $W_\beta \cap Z_\beta \neq 1$ and $W_\beta \cap Z_\beta \neq W_\beta$ because otherwise, by Corollary 3.3.5 and Lemma 3.2.1(iii), $1 \leq \eta_{G_\beta}(W_\beta) \leq \eta_{G_\beta}(Z_\beta) = 0$ which is a contradiction. We have shown that $W_\beta \cap Z_\beta = \mathrm{C}_{W_\beta}(\overline{G_{\mu\beta}}) = \mathrm{C}_{W_\beta}(\overline{G_\beta})$ is a non-zero proper submodule of W_β.

By Lemma 4.2.1(ii), V_β is an FF-module with offending subgroup $\overline{V_{\beta'}}$. Also, $W_\beta = [V_\beta, E_\beta] = [V_\beta, \overline{E_\beta}]$ and $1 \neq W_\beta \cap Z_\beta = \mathrm{C}_{W_\beta}(\overline{G_\beta}) \leq \mathrm{C}_{W_\beta}(\overline{E_\beta})$. So with $G = \overline{G_\beta}$, $E = \overline{E_\beta}$, $V = V_\beta$ and $W = W_\beta$ the hypothesis of Corollary 2.3.6 holds and therefore n is even and W_β is isomorphic to the zero weight submodule V_0 of the n-dimensional permutation module for $\overline{G_\beta}$ over $\mathbb{F}_2$. Now, $W_\beta \cap Z_\beta$ is the unique non-zero proper submodule of W_β of dimension one and so, as $\mathrm{C}_{W_\beta}(\overline{G_{\mu\beta}}) = W_\beta \cap Z_\beta$ and $\overline{G_{\mu\beta}} \in \mathrm{Syl}_2(\overline{G_\beta})$, we may apply Lemma 2.5.9(i) to deduce that $n = 2^m$ for some $m \in \mathbb{N}$. Thus, as $n \geq 7$, $n = 2^m$ with $m \geq 3$. □

Lemma 4.2.3 *$\overline{W_{\beta'}}$ is a non-trivial normal elementary abelian 2-subgroup of $\overline{Q_\mu}$ with $[W_\beta, \overline{W_{\beta'}}, \overline{W_{\beta'}}] = 1$, $\mathrm{C}_{W_\beta}(\overline{W_{\beta'}}) = W_\beta \cap Q_{\beta'}$ and $|W_\beta / \mathrm{C}_{W_\beta}(\overline{W_{\beta'}})| = |\overline{W_{\beta'}}|$.*

Proof By Corollary 3.3.5, $W_{\beta'} \not\leq Q_\beta$ and $W_{\beta'} \trianglelefteq Q_\mu$ and so $\overline{W_{\beta'}}$ is a non-trivial normal elementary abelian 2-subgroup of $\overline{Q_\mu}$ and, by Corollary 3.3.5(ii), $[W_\beta, \overline{W_{\beta'}}, \overline{W_{\beta'}}] = [W_\beta, W_{\beta'}, W_{\beta'}] = 1$. As W_β is a faithful $\mathbb{F}_2\overline{G_\beta}$-module, $\mathrm{C}_{G_\beta}(W_\beta) = Q_\beta$ and so

$$\mathrm{C}_{W_\beta}(\overline{W_{\beta'}}) = \mathrm{C}_{W_\beta}(W_{\beta'}) = W_\beta \cap \mathrm{C}_{G_{\beta'}}(W_{\beta'}) = W_\beta \cap Q_{\beta'}.$$

Then, by Corollary 3.3.5(iii),

$$|W_\beta / \mathrm{C}_{W_\beta}(\overline{W_{\beta'}})| = |W_\beta / (W_\beta \cap Q_{\beta'})| = |W_\beta Q_{\beta'} / Q_{\beta'}| = |W_{\beta'} Q_\beta / Q_\beta| = |\overline{W_{\beta'}}|.$$ □

Notation By Theorem 2.5.2, the Sylow 2-subgroup $\overline{G_{\mu\beta}}$ of $\overline{G_\beta}$ is similar to $\mathfrak{T}_m = \mathfrak{T}_{m-1} \wr \mathfrak{C}_2$ when $\overline{G_\beta} \cong \mathfrak{S}_n$ and $\mathfrak{T}_m \cap \mathfrak{A}_n$ when $\overline{G_\beta} \cong \mathfrak{A}_n$. Let $\mathrm{Z}(\overline{G_{\mu\beta}}) = \langle \bar{t} \rangle$. Then $\bar{t}$ is similar to the involution $(1\ 2)(3\ 4)\dots(n-1\ n)$.

Lemma 4.2.4 *Either $\overline{Q_\mu}$ is the base group of $\overline{G_{\mu\beta}}$ or $\overline{Q_\mu}$ is a transitive permutation group and $\mathrm{Z}(\overline{Q_\mu}) = \mathrm{Z}(\overline{G_{\mu\beta}})$.*

Proof We have that, by Lemma 3.2.2(ii),

$$|\overline{G_{\mu\beta}}:\overline{Q_\mu}| = |G_{\mu\beta}/Q_\beta : Q_\mu/Q_\beta| = |G_{\mu\beta}:Q_\mu| = 2$$

and hence we may apply Lemmas 2.5.5, 2.5.7 and 4.1.5 to deduce that either $\overline{Q_\mu}$ is the base group of $\overline{G_{\mu\beta}}$ or $\overline{Q_\mu}$ is a transitive permutation group and $\mathrm{Z}(\overline{Q_\mu}) = \mathrm{Z}(\overline{G_{\mu\beta}})$. □

Lemma 4.2.5 $\mathrm{Z}(\overline{Q_\mu}) \not\leq \overline{W_{\beta'}}$.

Proof Suppose, for a contradiction, that $\mathrm{Z}(\overline{Q_\mu}) \leq \overline{W_{\beta'}}$. By conjugating this statement we have that

$$\mathrm{Z}(Q_\mu/Q_{\beta\cdot g}) \leq W_{\beta'\cdot g}Q_{\beta\cdot g}/Q_{\beta\cdot g} \quad \text{for all } g \in G_\mu$$

and so the symmetrical form of the above supposition holds for all of the local beta pairs. Observe that, by Lemma 4.2.4, $\bar{t} \in \mathrm{Z}(\overline{Q_\mu}) \leq \overline{W_{\beta'}} \cap \overline{W_{\beta^*}}$. We will proceed via a series of steps.

(1) $(W_\beta)_{G_\mu} = W_\beta \cap W_{\beta'} = W_\beta \cap Q_{\beta'} = \mathrm{C}_{W_\beta}(\bar{t})$.

By Lemma 4.2.3 we may apply Lemma 4.1.4 to deduce that $[W_\beta, \overline{W_{\beta'}}] = \mathrm{C}_{W_\beta}(\overline{W_{\beta'}}) = \mathrm{C}_{W_\beta}(\bar{t})$. Then, by Lemma 4.2.3 and as W_β and $W_{\beta'}$ are normal in Q_μ,

$$W_\beta \cap Q_{\beta'} = \mathrm{C}_{W_\beta}(\overline{W_{\beta'}}) = [W_\beta, \overline{W_{\beta'}}] = [W_\beta, W_{\beta'}] \leq W_\beta \cap W_{\beta'} \leq W_\beta \cap Q_{\beta'}$$

and so $W_\beta \cap W_{\beta'} = W_\beta \cap Q_{\beta'} = \mathrm{C}_{W_\beta}(\overline{W_{\beta'}}) = \mathrm{C}_{W_\beta}(\bar{t})$. Now, applying this result to the beta pair (β, β^*) we have that $W_\beta \cap W_{\beta^*} = \mathrm{C}_{W_\beta}(\bar{t}) = W_\beta \cap W_{\beta'}$ and so $(W_\beta)_{G_\mu} = W_\beta \cap W_{\beta'} \cap W_{\beta^*} = W_\beta \cap W_{\beta'}$. Thus, $(W_\beta)_{G_\mu} = W_\beta \cap W_{\beta'} = W_\beta \cap Q_{\beta'} = \mathrm{C}_{W_\beta}(\bar{t})$.

(2) $\eta_{G_\beta}(Q_\beta/W_\beta) = 0$.

As Q_β and $W_{\beta'}$ are normal in Q_μ and by (1), $[Q_\beta, W_{\beta'}] \leq Q_\beta \cap W_{\beta'} \leq W_\beta$ and so, by Corollary 3.3.5(i),

$$[Q_\beta, E_\beta] \leq [Q_\beta, \langle W_{\beta'}^{G_\beta}\rangle] = \langle [Q_\beta, W_{\beta'}]^{G_\beta}\rangle \leq \langle W_\beta^{G_\beta}\rangle = W_\beta.$$

Hence, $\eta_{G_\beta}(Q_\beta/W_\beta) = 0$.

(3) A contradiction.

Set $Y = W_\beta(Q_\beta \cap Q_{\beta'}) \leq Q_\beta$. Then, by (2), $\eta_{G_\beta}(Q_\beta) = \eta_{G_\beta}(Q_\beta/W_\beta) + \eta_{G_\beta}(W_\beta) = 1$ and so, by Lemma 3.3.7, Y is an elementary abelian 2-group. As $\bar{t} \in \overline{W_{\beta'}} \cap \overline{W_{\beta^*}}$, there exists $v_{\beta'} \in W_{\beta'} \smallsetminus Q_\beta$ and $v_{\beta^*} \in W_{\beta^*} \smallsetminus Q_\beta$ such that $\overline{v_{\beta'}} = \bar{t} = \overline{v_{\beta^*}}$. Set $x = v_{\beta'}v_{\beta^*} \in Q_\beta$. As $\overline{v_{\beta^*}} = \bar{t} \in \mathrm{Z}(\overline{Q_\mu})$, $\overline{[\langle v_{\beta^*}\rangle, Q_\mu]} = [\langle \overline{v_{\beta^*}}\rangle, \overline{Q_\mu}] = 1$ and so, as W_{β^*} is normal in Q_μ and by (1), $[\langle v_{\beta^*}\rangle, Q_\mu] \leq W_{\beta^*} \cap Q_\beta = (W_{\beta^*})_{G_\mu} \leq Q_{\beta'}$. Then

$$[\langle v_{\beta^*}\rangle Q_{\beta'}/Q_{\beta'}, Q_\mu/Q_{\beta'}] = [\langle v_{\beta^*}\rangle, Q_\mu]Q_{\beta'}/Q_{\beta'} = Q_{\beta'}/Q_{\beta'}$$

and so $\langle v_{\beta^*}\rangle Q_{\beta'}/Q_{\beta'} \leq \mathrm{Z}(Q_\mu/Q_{\beta'})$. Also, as $v_{\beta'} \in Q_{\beta'}$, $xQ_{\beta'} = v_{\beta'}v_{\beta^*}Q_{\beta'} = v_{\beta^*}Q_{\beta'}$ and so, as $\mathrm{Z}(\overline{Q_\mu}) \leq \overline{W_{\beta'}}$,

$$\langle x\rangle Q_{\beta'}/Q_{\beta'} = \langle xQ_{\beta'}\rangle = \langle v_{\beta^*}Q_{\beta'}\rangle = \langle v_{\beta^*}\rangle Q_{\beta'}/Q_{\beta'} \leq \mathrm{Z}(Q_\mu/Q_{\beta'}) \leq W_\beta Q_{\beta'}/Q_{\beta'}$$

giving $\langle x\rangle \leq W_\beta Q_{\beta'}$. So, by Dedekind's Law, $x \in Q_\beta \cap W_\beta Q_{\beta'} = W_\beta(Q_\beta \cap Q_{\beta'}) = Y$ and so, as Y is an elementary abelian 2-group, $x^2 = 1$. Then, as $v_{\beta^*} = v_{\beta'}x$,

$$[v_{\beta'}, x] = v_{\beta'}^{-1}x^{-1}v_{\beta'}x = (v_{\beta'}x)(v_{\beta'}x) = (v_{\beta^*})^2 = 1$$

and so $x \in \mathrm{C}_{Q_\beta}(v_{\beta'})$. By (1), $\mathrm{C}_{W_\beta}(v_{\beta'}) = \mathrm{C}_{W_\beta}(\overline{v_{\beta'}}) = \mathrm{C}_{W_\beta}(\bar{t}) = W_\beta \cap W_{\beta'}$ and, as $v_{\beta'} \in W_{\beta'} \leq \mathrm{Z}(Q_{\beta'})$, $Q_\beta \cap Q_{\beta'} \leq \mathrm{C}_Y(v_{\beta'})$ giving, by Dedekind's Law,

$$\begin{aligned}\mathrm{C}_Y(v_{\beta'}) &= \mathrm{C}_Y(v_{\beta'}) \cap Y = \mathrm{C}_Y(v_{\beta'}) \cap W_\beta(Q_\beta \cap Q_{\beta'}) = (Q_\beta \cap Q_{\beta'})(\mathrm{C}_Y(v_{\beta'}) \cap W_\beta)\\ &= (Q_\beta \cap Q_{\beta'})\,\mathrm{C}_{W_\beta}(v_{\beta'}) = (Q_\beta \cap Q_{\beta'})(W_\beta \cap W_{\beta'}) = Q_\beta \cap Q_{\beta'}.\end{aligned}$$

Then $x \in \mathrm{C}_{Q_\beta}(v_{\beta'}) \cap Y = \mathrm{C}_Y(v_{\beta'}) = Q_\beta \cap Q_{\beta'} \leq Q_{\beta'}$ and so $v_{\beta^*} = v_{\beta'}x \in Q_{\beta'}$. Therefore, by (1), $v_{\beta^*} \in W_{\beta^*} \cap Q_{\beta'} = (W_{\beta^*})_{G_\mu} \leq Q_\beta$ which is a contradiction. □

Corollary 4.2.6 *$\overline{Q_\mu}$ is the base group of $\overline{G_{\mu\beta}}$. In particular, the following hold:*

(i) There exist subgroups $\overline{T^1}$ and $\overline{T^2}$ of $\overline{Q_\mu}$ such that $\overline{T^1} \times \overline{T^2} \leq \overline{Q_\mu}$ where $\overline{T^i}$ is similar to $\mathfrak{T}_{m-1}$ for $i \in \{1,2\}$ when $\overline{G_\beta} \cong \mathfrak{S}_n$ and $\overline{T^i}$ is similar to $\mathfrak{T}_{m-1} \cap \mathfrak{A}_n$ for $i \in \{1,2\}$ when $\overline{G_\beta} \cong \mathfrak{A}_n$.

(ii) There exists $s \in G_{\mu\beta}$ such that $(\beta', \beta^*) \cdot s = (\beta^*, \beta')$, $(\overline{T^1})^{\overline{s}} = \overline{T^2}$ and $(\overline{T^2})^{\overline{s}} = \overline{T^1}$.

(iii) $\mathrm{Z}(\overline{Q_\mu}) = \langle \overline{t_1}, \overline{t_2} \rangle$ where $\overline{t_i} \in \overline{T^i}$ for $i \in \{1,2\}$.

Proof Suppose, for a contradiction, that $\overline{Q_\mu}$ is a transitive permutation group. Then, as $1 \neq \overline{W_{\beta'}} \trianglelefteq \overline{Q_\mu}$, $\overline{W_{\beta'}} \cap \mathrm{Z}(\overline{Q_\mu}) \neq 1$ giving $\mathrm{Z}(\overline{Q_\mu}) \leq \overline{W_{\beta'}}$ which contradicts the lemma above. Thus, by Lemma 4.2.4, $\overline{Q_\mu}$ is the base group of $\overline{G_{\mu\beta}}$. Now, in the case when $\overline{G_\beta} \cong \mathfrak{S}_n$, $\overline{G_{\mu\beta}}$ is similar to the wreath product $\mathfrak{T}_m = \mathfrak{T}_{m-1} \wr \mathfrak{C}_2$ with base group $\overline{Q_\mu}$ and so there exist subgroups $\overline{T^1}$ and $\overline{T^2}$ of $\overline{Q_\mu}$ such that $\overline{Q_\mu} = \overline{T^1} \times \overline{T^2}$ where $\overline{T^i}$ is similar to $\mathfrak{T}_{m-1}$ for $i \in \{1,2\}$. In the case when $\overline{G_\beta} \cong \mathfrak{A}_n$, $\overline{G_{\mu\beta}}$ is similar to $\mathfrak{T}_m \cap \mathfrak{A}_n$ with base group $\overline{Q_\mu}$ and so there exist subgroups $\overline{T^1}$ and $\overline{T^2}$ of $\overline{Q_\mu}$ such that $|\overline{Q_\mu} : \overline{T^1} \times \overline{T^2}| = 2$ where $\overline{T^i}$ is similar to $\mathfrak{T}_{m-1} \cap \mathfrak{A}_n$ for $i \in \{1,2\}$. In both cases there exists $\overline{s} \in \overline{G_{\mu\beta}} \setminus \overline{Q_\mu}$ such that $(\overline{T^1})^{\overline{s}} = \overline{T^2}$ and $(\overline{T^2})^{\overline{s}} = \overline{T^1}$. Then, as $G_{\mu\beta}$ acts on $\Gamma(\mu) \setminus \{\beta\} = \{\beta', \beta^*\}$ and $s \notin Q_\mu = G_{\Delta(\mu)} = G_{\mu\beta\beta'\beta^*}$, s swaps β' and β^*. Also, $\mathrm{Z}(\overline{Q_\mu}) = \langle \overline{t_1}, \overline{t_2} \rangle$ where $\overline{t_i} \in \overline{T^i}$ for $i \in \{1,2\}$. □

Lemma 4.2.7 *$G_\beta = Q_\mu E_\beta$.*

Proof In the case when $\overline{G_\beta} \cong \mathfrak{A}_n$, $\overline{G_\beta} = \mathrm{O}^2(\overline{G_\beta}) = \overline{E_\beta} = \overline{Q_\mu E_\beta}$ and hence $G_\beta = Q_\mu E_\beta$. In the case when $\overline{G_\beta} \cong \mathfrak{S}_n$, $\overline{E_\beta} = \mathrm{O}^2(\overline{G_\beta}) \cong \mathfrak{A}_n$ so that

$$|\overline{G_\beta} : \overline{Q_\mu E_\beta}|\ |\overline{Q_\mu E_\beta} : \overline{E_\beta}| = |\overline{G_\beta} : \overline{E_\beta}| = 2$$

and so either $\overline{G_\beta} = \overline{Q_\mu E_\beta}$ or $\overline{Q_\mu} \leq \overline{E_\beta}$. In the latter case the base group $\overline{Q_\mu}$ is contained in $\overline{E_\beta} \cong \mathfrak{A}_n$, but $\overline{Q_\mu}$ contains transpositions whereas $\mathfrak{A}_n$ does not which is a contradiction. So $\overline{G_\beta} = \overline{Q_\mu E_\beta}$ and hence $G_\beta = Q_\mu E_\beta$. □

Lemma 4.2.8 *$\overline{W_{\beta'}} \leq \overline{T^i}$ for some $i \in \{1,2\}$.*

Proof By Lemma 4.2.3, $\overline{W_{\beta'}}$ is a non-trivial normal abelian subgroup of $\overline{Q_\mu}$ with $|W_\beta/\mathrm{C}_{W_\beta}(\overline{W_{\beta'}})| = |\overline{W_{\beta'}}|$ and hence, by Lemma 4.2.5 and Corollary 4.2.6, we may apply Lemmas 4.1.1 and 4.1.2 to deduce that $\overline{W_{\beta'}} \le \overline{T^i}$ for some $i \in \{1,2\}$. □

Notation Let Δ_i be the orbit of $\overline{T^i}$ and let W^i_β be the subspace of W_β corresponding to $V_0(\Delta_i)$ in V_0.

Lemma 4.2.9 *The following hold:*

(i) W^i_β is a faithful $\mathbb{F}_2\overline{W_{\beta'}}$-module with $W^i_\beta \le W_\beta \cap Q_{\beta^*}$. In particular, $\mathrm{C}_{W_{\beta'}}(W^i_\beta) \le Q_\beta$.

(ii) W^i_β is a normal subgroup of Q_μ with $W^i_\beta \not\le Q_{\beta'}$ and $|[Q_{\beta'}/W_{\beta'}, W^i_\beta]| \le 2$.

Proof (i) By applying Lemma 4.1.6 and Corollary 4.1.7, W^i_β is a faithful $\mathbb{F}_2\overline{W_{\beta'}}$-module and so $\mathrm{C}_{\overline{W_{\beta'}}}(W^i_\beta) = 1$ giving $\mathrm{C}_{W_{\beta'}}(W^i_\beta) \le \mathrm{C}_{W_{\beta'}Q_\beta}(W^i_\beta) = Q_\beta$. Set $j = 3-i$. Then, as $\overline{W_{\beta'}} \le \overline{T^i}$ and by Corollary 4.2.6(ii),

$$\overline{W_{\beta^*}} = \overline{W_{\beta'\cdot s}} = \overline{W^s_{\beta'}} = (\overline{W_{\beta'}})^{\bar{s}} \le (\overline{T^i})^{\bar{s}} = \overline{T^j}.$$

Also, by Lemma 4.2.3, $\mathrm{C}_{W_\beta}(\overline{W_{\beta'}}) = W_\beta \cap Q_{\beta'}$ and hence $W^i_\beta \le \mathrm{C}_{W_\beta}(\overline{W_{\beta^*}}) = W_\beta \cap Q_{\beta^*}$.

(ii) By applying Lemma 4.1.6 and Corollary 4.1.7, W^i_β is normalized by $\overline{Q_\mu}$ so that W^i_β is a normal subgroup of Q_μ. We have that $W^i_\beta \not\le Q_{\beta'}$ because otherwise, as $W_{\beta'} \le \mathrm{Z}(Q_{\beta'})$ and by part (i), $W_{\beta'} \le \mathrm{C}_{W_{\beta'}}(W^i_\beta) \le Q_\beta$ which contradicts Corollary 3.3.5. Now, as $1 \ne \overline{W_{\beta'}} \trianglelefteq \overline{Q_\mu}$, $\overline{W_{\beta'}} \cap \mathrm{Z}(\overline{Q_\mu}) \ne 1$ and so, as $\overline{W_{\beta'}} \le \overline{T^i}$, $\overline{t_i} \in \overline{W_{\beta'}}$. Also, by Lemma 4.2.1(i) and Corollary 3.3.5(ii), $[W_\beta, \overline{Q_{\beta'}}, \overline{W_{\beta'}}] = [W_\beta, Q_{\beta'}, W_{\beta'}] = 1$. So $\overline{W_{\beta'}}$ and $\overline{Q_{\beta'}}$ are subgroups of $\overline{Q_\mu}$ with $\overline{t_i} \in \overline{W_{\beta'}} \le \overline{Q_{\beta'}}$ and $[W^i_\beta, \overline{Q_{\beta'}}, \overline{W_{\beta'}}] = 1$ giving, by applying Lemma 4.1.6 and Corollary 4.1.7, $|[W^i_\beta, \overline{Q_{\beta'}}]/[W^i_\beta, \overline{W_{\beta'}}]| \le 2$. Then, as W_β and $W_{\beta'}$ are normal in Q_μ, $[W_{\beta'}, W^i_\beta] \le [Q_{\beta'}, W^i_\beta] \cap [W_{\beta'}, W_\beta] \le [Q_{\beta'}, W^i_\beta] \cap W_{\beta'}$ and hence

$$\begin{aligned} |[Q_{\beta'}/W_{\beta'}, W^i_\beta]| &= |[Q_{\beta'}, W^i_\beta]W_{\beta'}/W_{\beta'}| = |[Q_{\beta'}, W^i_\beta]/([Q_{\beta'}, W^i_\beta] \cap W_{\beta'})| \\ &\le |[Q_{\beta'}, W^i_\beta]/[W_{\beta'}, W^i_\beta]| = |[\overline{Q_{\beta'}}, W^i_\beta]/[\overline{W_{\beta'}}, W^i_\beta]| \le 2. \end{aligned}$$ □

Lemma 4.2.10 $\eta_{G_\beta}(Q_\beta/W_\beta) = 0$.

Proof Suppose, for a contradiction, that $\eta_{G_{\beta'}}(Q_{\beta'}/W_{\beta'}) \geq 1$ and let N be a non-central chief $G_{\beta'}$-factor of $Q_{\beta'}/W_{\beta'}$. Then N may be regarded as a faithful $\mathbb{F}_2(G_{\beta'}/Q_{\beta'})$-module. Applying Corollary 4.2.6 to the beta pair (β', β) we have that $Q_\mu/Q_{\beta'}$ is the base group of the Sylow 2-subgroup $G_{\mu\beta'}/Q_{\beta'}$ of $G_{\beta'}/Q_{\beta'}$. By Lemma 4.2.9(ii), $1 \neq W_\beta^i Q_{\beta'}/Q_{\beta'} \trianglelefteq Q_\mu/Q_{\beta'}$ and so $W_\beta^i Q_{\beta'}/Q_{\beta'} \cap \mathrm{Z}(Q_\mu/Q_{\beta'}) \neq 1$. Also, as $\overline{W_{\beta'}} \leq \overline{T^i}$, $\overline{W_{\beta'}} \cap \mathrm{Z}(\overline{G_{\mu\beta}}) = 1$ and so $W_\beta^i Q_{\beta'}/Q_{\beta'} \cap \mathrm{Z}(G_{\mu\beta'}/Q_{\beta'}) = 1$. Hence, there exists an involution $\widehat{x} = xQ_{\beta'} \in W_\beta^i Q_{\beta'}/Q_{\beta'}$ with $x \in W_\beta^i$ that is the product of $n/4$ disjoint transpositions. Then, by Lemma 4.2.9(ii),

$$\begin{aligned} |N/\mathrm{C}_N(\widehat{x})| = |N/\mathrm{C}_N(x)| &\leq |Q_{\beta'}/W_{\beta'} : \mathrm{C}_{Q_{\beta'}/W_{\beta'}}(x)| \leq |[Q_{\beta'}/W_{\beta'}, x]| \\ &\leq |[Q_{\beta'}/W_{\beta'}, W_\beta^i]| \leq 2 \end{aligned}$$

and so $\widehat{x}$ induces a transvection on N. So, by applying Theorem 2.3.8, either $\overline{G_\beta} \cong \mathfrak{S}_n$, $\widehat{x}$ is a transposition and $n/4 = 1$ giving $n = 4$ which is a contradiction or $\overline{G_\beta} \cong \mathfrak{A}_8$, $\widehat{x}$ is a fixed-point-free involution and $n/4 = 4$ giving $8 = n = 16$ which is also a contradiction. Thus, $\eta_{G_{\beta'}}(Q_{\beta'}/W_{\beta'}) = 0$. □

Lemma 4.2.11 *A contradiction.*

Proof By Lemma 4.2.10, $\eta_{G_\beta}(Q_\beta) = \eta_{G_\beta}(Q_\beta/W_\beta) + \eta_{G_\beta}(W_\beta) = 1$ and so, by Lemma 4.2.7, we may apply Lemma 3.3.7 to deduce that $V_{\beta^*}(Q_{\beta^*} \cap Q_\beta) = V_{\beta^*}(Q_{\beta^*} \cap Q_{\beta'})$ is abelian. Then, by Lemma 4.2.9(i),

$$[W_\beta^i, W_{\beta'} \cap Q_{\beta^*}] \leq [W_\beta \cap Q_{\beta^*}, W_{\beta'} \cap Q_{\beta^*}] = 1$$

and so, by Lemma 4.2.9(i), $W_{\beta'} \cap Q_{\beta^*} \leq \mathrm{C}_{W_{\beta'}}(W_\beta^i) \leq Q_\beta$. Therefore, $W_\beta^i \leq W_\beta \cap Q_{\beta^*} \leq Q_{\beta'}$ which contradicts Lemma 4.2.9(ii). □

Chapter 5

The $G_{\mu\beta} = Q_\mu Q_\beta$ case

§5.1 Module calculations

We refer the reader to Sections 2.1 and 2.5 for an explanation of the following notation which will hold in this section.

Notation Let $n \in \mathbb{N}$, $n \geq 7$ and denote the 2-adic decomposition of n as follows

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r} \quad \text{with} \quad 0 \leq m_1 < m_2 < \cdots < m_r$$

and assume that $r \geq 2$. Let V be the n-dimensional permutation module for $\mathfrak{S}_n$ over $\mathbb{F}_2$ with ordered basis $(v_1, v_2, \ldots, v_n)$ and centre $Z = \langle z \rangle$. Recall that V_0 denotes the zero weight submodule of V and $\overline{V_0}$ denotes the natural module for $\mathfrak{S}_n$. We will also denote by U both V_0 and $\overline{V_0}$.

Fix $S \in \mathrm{Syl}_2(\mathfrak{S}_n)$ and let the orbits of S be $\Delta_1, \Delta_2, \ldots, \Delta_r$ on $\Delta = \{1, 2, \ldots, n\}$ where $|\Delta_i| = 2^{m_i}$ for $1 \leq i \leq r$. Then $S = T^1 \times T^2 \times \cdots \times T^r$ where T^i is similar to $\mathfrak{T}_{m_i}$ for $1 \leq i \leq r$. Set $S_0 = S \cap \mathfrak{A}_n \in \mathrm{Syl}_2(\mathfrak{A}_n)$ and $T_0^i = T^i \cap \mathfrak{A}_n$ for $1 \leq i \leq r$. Let $\mathrm{Z}(T^i) = \langle t_i \rangle$ for $1 \leq i \leq r$ so that $\mathrm{Z}(S) = \langle t_1, t_2, \ldots, t_r \rangle$, $\mathrm{Z}(S_0) = \mathrm{Z}(S) \cap \mathfrak{A}_n$ and either t_i is similar to the involution $(1\ 2)(3\ 4)\ldots(2^{m_i} - 1\ 2^{m_i})$ for $1 \leq i \leq r$ or $m_1 = 0$ (n is odd) and $t_1 = 1$. Set $t = t_1 t_2 \ldots t_r$ and assume that $t = (1\ 2)(3\ 4)\ldots(n-1\ n)$ when n is even and $t = (2\ 3)(4\ 5)\ldots(n-1\ n)$ when n is odd. Let $Z_i = \langle z_i \rangle$ denote the centre of the permutation module $V(\Delta_i)$ for $\mathfrak{S}_{\Delta_i}$ over $\mathbb{F}_2$ for $1 \leq i \leq r$ and recall that $V_0(\Delta_i)$ denotes the zero weight submodule of $V(\Delta_i)$.

Lemma 5.1.1 *Assume that* $t_i = (1\ 2)(3\ 4)\ldots(2^{m_i} - 1\ 2^{m_i})$ *for* $1 \leq i \leq r$ *and set* $A^* = \langle (1\ 2), (3\ 4), \ldots, (2^{m_i} - 1\ 2^{m_i}) \rangle$. *Then* $\mathrm{C}_{T^i}([U, t_i]) = A^*$, $[U, t_i] = [U, A^*]$ *and* $\mathrm{C}_U(A^*) = \mathrm{C}_U(t_i)$. *In particular,* $\mathrm{C}_S([U, t_i]) = A^* \times R$ *where* $S = T^i \times R$.

Proof We have that $[V_0, t_i] = \langle [v_k + v_{k+1}, t_i] \mid 1 \leq k \leq n-1 \rangle$. Let $1 \leq k \leq 2^{m_i} - 1$. If k is odd, then $[v_k + v_{k+1}, t_i] = 0$ and if k is even, then $[v_k + v_{k+1}, t_i] = v_{k-1} + v_k + v_{k+1} + v_{k+2}$. If $k = 2^{m_i}$, then $[v_{2^{m_i}} + v_{2^{m_i}+1}, t_i] = v_{2^{m_i}-1} + v_{2^{m_i}}$ and if $2^{m_i} + 1 \leq k \leq n-1$, then $[v_k + v_{k+1}, t_i] = 0$. Hence,

$$[V_0, t_i] = \langle v_1 + v_2 + v_3 + v_4, \ldots, v_{2^{m_i}-3} + v_{2^{m_i}-2} + v_{2^{m_i}-1} + v_{2^{m_i}}, v_{2^{m_i}-1} + v_{2^{m_i}} \rangle$$
$$= \langle v_1 + v_2, v_3 + v_4, \ldots, v_{2^{m_i}-1} + v_{2^{m_i}} \rangle.$$

Clearly, $A^* \leq \mathrm{C}_{T^i}([V_0, t_i])$. Conversely, let $s \in \mathrm{C}_{T^i}([V_0, t_i])$ and fix $1 \leq k \leq 2^{m_i} - 1$ with k odd. Then, as $v_k + v_{k+1} \in [V_0, t_i]$, $v_{ks} + v_{(k+1)s} = (v_k + v_{k+1})^s = v_k + v_{k+1}$ and so s fixes the set $\{k, k+1\}$. We have shown that s either fixes or swaps k and $k+1$ for all odd $1 \leq k \leq 2^{m_i} - 1$ and so $s \in A^*$. Hence, $\mathrm{C}_{T^i}([V_0, t_i]) = A^*$. Also, as $\langle t_i \rangle = \mathrm{Z}(T^i) \trianglelefteq T^i$,

$$[\overline{V_0}, t_i] = \overline{[V_0, t_i]} = ([V_0, t_i] + Z)/Z \cong_{T^i} [V_0, t_i]/([V_0, t_i] \cap Z) \cong_{T^i} [V_0, t_i]$$

and so $\mathrm{C}_{T^i}([\overline{V_0}, t_i]) = \mathrm{C}_{T^i}([V_0, t_i]) = A^*$. Now, as t_i has a fixed point on Δ and by applying Lemma 2.1.13(ii),

$$[V_0, t_i] = [V_0, (1\ 2)] + [V_0, (3\ 4)] + \cdots + [V_0, (2^{m_i} - 1\ 2^{m_i})] = [V_0, A^*]$$

and so $[\overline{V_0}, t_i] = [\overline{V_0}, A^*]$. Also, by applying Lemma 2.1.13(i),

$$\mathrm{C}_{V_0}(A^*) = \mathrm{C}_{V_0}((1\ 2)) \cap \mathrm{C}_{V_0}((3\ 4)) \cap \ldots \cap \mathrm{C}_{V_0}((2^{m_i} - 1\ 2^{m_i})) = \mathrm{C}_{V_0}(t_i)$$

and so, as A^* and t_i have fixed points on Δ, $\mathrm{C}_{\overline{V_0}}(A^*) = \mathrm{C}_{\overline{V_0}}(t_i)$. Thus, $\mathrm{C}_{T^i}([U, t_i]) = A^*$, $[U, t_i] = [U, A^*]$ and $\mathrm{C}_U(A^*) = \mathrm{C}_U(t_i)$. □

Corollary 5.1.2 *Let $A \leq \mathrm{C}_S([U, t_i])$ for $1 \leq i \leq r$. Then $\Phi(A) \cap T^i = 1$.*

Proof Let $S = T^i \times R$ and set $A^* = \mathrm{C}_{T^i}([U, t_i])$. Then, by the lemma above, $A \leq \mathrm{C}_S([U, t_i]) = A^* \times R$. As R is normal in S, $A \cap R$ is a normal subgroup of A with

$$A/(A \cap R) \cong AR/R \leq (A^* \times R)/R \cong A^*$$

and so, as A^* is an elementary abelian 2-group, $\Phi(A) \leq R$. Thus, $\Phi(A) \cap T^i = 1$. □

Lemma 5.1.3 *Assume that $r = 2$. Let A be a non-trivial normal elementary abelian 2-subgroup of S with $[\overline{V_0}, A, A] = 0$, $|\overline{V_0}/\mathrm{C}_{\overline{V_0}}(a)| \leq 2|A|$ for all $a \in A$ and $|A| \leq |\mathrm{C}_{\overline{V_0}}(A)/[\overline{V_0}, S, A]|$. Then $A \leq T^i$ for some $i \in \{1, 2\}$.*

Proof If n is odd, then $A \leq S = T^2$ and so we may assume that n is even. We make the following observation.

($\star$) $[\overline{V_0}, \mathrm{Z}(S)] = \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S))$ and $|\mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S))| = 2^{(n/2-1)}$.

We have that, as $\mathrm{Z}(S) = \langle t_1, t_2 \rangle$ and by applying Corollary 2.1.15 and Lemma 2.1.13(i),

$$[V_0, \mathrm{Z}(S)] = [V_0, t_1] + [V_0, t_2] = \mathrm{C}_{V_0}(t) = \mathrm{C}_{V_0}(t_1) \cap \mathrm{C}_{V_0}(t_2) = \mathrm{C}_{V_0}(\mathrm{Z}(S))$$

and so $[\overline{V_0}, \mathrm{Z}(S)] = \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S)) = \mathrm{C}_{V_0}(t)/Z$ giving, by Lemma 2.1.14(i), $|\mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S))| = 2^{(n/2-1)}$.

We will proceed via a series of steps.

(1) $A \cap T^i \neq 1$ for some $i \in \{1, 2\}$. In particular, $t_i \in A$.

Suppose, for a contradiction, that $A \cap T^1 = 1 = A \cap T^2$. Then with $\{i, j\} = \{1, 2\}$, as T^i and A are normal in S, $[T^i, A] \leq T^i \cap A = 1$ and so $A \leq \mathrm{C}_S(T^i) = \mathrm{Z}(T^i) \times T^j$. We have that

$$A \leq (\mathrm{Z}(T^1) \times T^2) \cap (T^1 \times \mathrm{Z}(T^2)) = \mathrm{Z}(T^1) \times \mathrm{Z}(T^2) = \mathrm{Z}(S) = \langle t_1, t_2 \rangle$$

and so, as $t_1, t_2 \notin A$, $A = \langle t \rangle$. Then, by Lemma 2.1.16(iii), $2^{(n/2-2)} \leq |\overline{V_0}/\mathrm{C}_{\overline{V_0}}(t)| \leq 2|A| = 2^2$ so $n/2 - 2 \leq 2$ and so $n \leq 8$ giving, as $r = 2$ and n is even, $n \leq 6$ which is a contradiction. Hence, there exists $i \in \{1, 2\}$ such that $A \cap T^i \neq 1$. In particular, as A is normalized by T^i, $A \cap \mathrm{Z}(T^i) \neq 1$ and so $t_i \in A$.

(2) $A \cap T^j = 1$ where $j = 3 - i$.

Suppose, for a contradiction, that $A \cap T^j \neq 1$. Then $t_j \in A$ so that, by (1), $\mathrm{Z}(S) = \langle t_i, t_j \rangle \leq A$ and so $[\overline{V_0}, \mathrm{Z}(S)] \leq [\overline{V_0}, A] \leq \mathrm{C}_{\overline{V_0}}(A) \leq \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S))$ giving, by ($\star$), $\mathrm{C}_{\overline{V_0}}(A) = \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S))$. Also, by Lemma 2.5.9(iii), $[V_0, S] = V_0(\Delta_1) + V_0(\Delta_2)$ so that

$$[V_0(\Delta_1), t_1] + [V_0(\Delta_2), t_2] = [V_0, S, t] \leq [V_0, S, A]$$

and so $|[V_0, S, A]| \geq 2^{(2^{m_1-1}-1+2^{m_2-1}-1)} = 2^{(n/2-2)}$ giving $|[\overline{V_0}, S, A]| = |\overline{[V_0, S, A]}| \geq 2^{(n/2-3)}$. Then, by $(\star)$,

$$|A| \leq |\mathrm{C}_{\overline{V_0}}(A)/[\overline{V_0}, S, A]| \leq 2^{(n/2-1)}/2^{(n/2-3)} = 2^2$$

and so $A = \mathrm{Z}(S)$. We have that $2^{(n/2-2)} \leq |\overline{V_0}/\,\mathrm{C}_{\overline{V_0}}(t)| \leq 2\,|A| = 2^3$ so $n/2 - 2 \leq 3$ and so, as $r = 2$ and n is even, $n = 10$, but then $2^4 = |\overline{V_0}/\,\mathrm{C}_{\overline{V_0}}(t)| \leq 2\,|A| = 2^3$ which is a contradiction. Hence, $A \cap T^j = 1$.

(3) Assume that $t_i = (1\,2)(3\,4)\ldots(2^{m_i}{-}1\;2^{m_i})$ and set $A^* = \langle(1\,2), (3\,4), \ldots, (2^{m_i}{-}1\;2^{m_i})\rangle$. Then $A \leq A^* \times \mathrm{Z}(T^j)$. In particular, A has no fixed-point-free elements on Δ.

As T^j and A are normal in S and by (2), $[T^j, A] \leq T^j \cap A = 1$ and so $A \leq \mathrm{C}_S(T^j) = T^i \times \mathrm{Z}(T^j)$. Also, by (1), $[\overline{V_0}, t_i, A] \leq [\overline{V_0}, A, A] = 0$ and so, by Lemma 5.1.1, $A \leq \mathrm{C}_S([\overline{V_0}, t_i]) = A^* \times T^j$. Hence, $A \leq (T^i \times \mathrm{Z}(T^j)) \cap (A^* \times T^j) = A^* \times \mathrm{Z}(T^j)$. In particular, A has no fixed-point-free elements on Δ because otherwise $t \in A$ and so, by (1) and (2), $t_j = t_i t \in A \cap T^j = 1$ which is a contradiction.

(4) $A \leq T^i$.

Suppose, for a contradiction, that $A \not\leq T^i$. Then there exists $a \in A \setminus T^i$ and, by (3), $a = s_i t_j$ for some $s_i \in A^*$ with a fixed point on Δ_i. So, by applying Lemma 2.1.13(ii),

$$[V_0, t_j] \leq [V_0, s_i] + [V_0, t_j] = [V_0, s_i t_j] = [V_0, a] \leq [V_0, A]$$

and so, by (1), $[\overline{V_0}, \mathrm{Z}(S)] = [\overline{V_0}, t_i] + [\overline{V_0}, t_j] \leq [\overline{V_0}, A]$. Also, by applying Lemma 2.1.13(i),

$$\mathrm{C}_{V_0}(A) \leq \mathrm{C}_{V_0}(a) = \mathrm{C}_{V_0}(s_i t_j) = \mathrm{C}_{V_0}(s_i) \cap \mathrm{C}_{V_0}(t_j) \leq \mathrm{C}_{V_0}(t_j)$$

and so, by (1) and (3), $\mathrm{C}_{\overline{V_0}}(A) \leq \mathrm{C}_{\overline{V_0}}(t_i) \cap \mathrm{C}_{\overline{V_0}}(t_j) = \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S))$. We have shown that $[\overline{V_0}, \mathrm{Z}(S)] \leq [\overline{V_0}, A] \leq \mathrm{C}_{\overline{V_0}}(A) \leq \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S))$ and hence, by $(\star)$, $\mathrm{C}_{\overline{V_0}}(A) = \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S))$. Now, $[V_0(\Delta_i), s_i, t_i] \leq [V_0(\Delta_i), A^*, A^*] = 0$ and so, as s_i is a non-trivial element and has a fixed point on Δ_i, $[V_0(\Delta_i), t_i] + [V_0(\Delta_i), s_i] = \mathrm{C}_{V_0(\Delta_i)}(t_i)$. Then, as $[V_0, S] = V_0(\Delta_i) + V_0(\Delta_j)$ and by (1),

$$\mathrm{C}_{V_0(\Delta_i)}(t_i) + [V_0(\Delta_j), t_j] = [V_0(\Delta_i), t_i] + [V_0(\Delta_i), s_i] + [V_0(\Delta_j), t_j]$$

$$= [V_0, S, t_i] + [V_0, S, a] \leq [V_0, S, A]$$

and so $|[V_0, S, A]| \geq 2^{(2^{m_i-1}+2^{m_j-1}-1)} = 2^{(n/2-1)}$ giving $|[\overline{V_0}, S, A]| \geq 2^{(n/2-2)}$. So, by $(\star)$,

$$|A| \leq |\mathrm{C}_{\overline{V_0}}(A)/[\overline{V_0}, S, A]| \leq 2^{(n/2-1)}/2^{(n/2-2)} = 2$$

which is a contradiction. Therefore, $A \leq T^i$. □

Lemma 5.1.4 *Assume that $r = 2$. Let A be a non-trivial normal elementary abelian 2-subgroup of S_0 with $[\overline{V_0}, A, A] = 0$, $|\overline{V_0}/\mathrm{C}_{\overline{V_0}}(a)| \leq 2|A|$ for all $a \in A$ and $|A| \leq |\mathrm{C}_{\overline{V_0}}(A)/[\overline{V_0}, S_0, A]|$. Then $A \leq T_0^i$ for some $i \in \{1, 2\}$.*

Proof If n is odd, then $A \leq S_0 = T_0^2$ and so we may assume that n is even. We have that, by Corollary 2.5.6, either $\mathrm{C}_{T^1}(T_0^1) = \mathrm{Z}(T^1)$ or $m_1 = 2$ and $\mathrm{C}_{T^1}(T_0^1) = T_0^1$ is similar to $\mathfrak{K}_4$ and, in both cases, $\mathrm{C}_{T^2}(T_0^2) = \mathrm{Z}(T^2)$. We make the following observation.

($\star$) Let $t_1' \in \mathrm{C}_{T^1}(T_0^1)^\#$ and set $X = \langle t_1', t_2 \rangle$. Then $[\overline{V_0}, X] = \mathrm{C}_{\overline{V_0}}(X)$ and $|\mathrm{C}_{\overline{V_0}}(X)| = 2^{(n/2-1)}$.

We have that, as $t_1't_2$ is fixed-point-free on Δ and by applying Corollary 2.1.15 and Lemma 2.1.13(i),

$$[V_0, X] = [V_0, t_1'] + [V_0, t_2] = \mathrm{C}_{V_0}(t_1't_2) = \mathrm{C}_{V_0}(t_1') \cap \mathrm{C}_{V_0}(t_2) = \mathrm{C}_{V_0}(X)$$

and so $[\overline{V_0}, X] = \mathrm{C}_{\overline{V_0}}(X) = \mathrm{C}_{V_0}(t_1't_2)/Z$ giving, by Lemma 2.1.14(i), $|\mathrm{C}_{\overline{V_0}}(X)| = 2^{(n/2-1)}$.

We will proceed via a series of steps.

(1) $A \cap T_0^i \neq 1$ for some $i \in \{1, 2\}$. In particular, $t_i \in A$.

Suppose, for a contradiction, that $A \cap T_0^1 = 1 = A \cap T_0^2$. Then with $\{i, j\} = \{1, 2\}$, as T_0^i and A are normal in S_0, $[T_0^i, A] \leq T_0^i \cap A = 1$ and so $A \leq \mathrm{C}_S(T_0^i) = \mathrm{C}_{T^i}(T_0^i) \times T^j$. We have that

$$A \leq (\mathrm{C}_{T^1}(T_0^1) \times T^2) \cap (T^1 \times \mathrm{C}_{T^2}(T_0^2)) = \mathrm{C}_{T^1}(T_0^1) \times \mathrm{C}_{T^2}(T_0^2) = \mathrm{C}_{T^1}(T_0^1) \times \mathrm{Z}(T^2).$$

If $m_1 \neq 2$, then $A \leq \mathrm{Z}(T^1) \times \mathrm{Z}(T^2)$ and if $m_1 = 2$, then $A \leq T_0^1 \times \mathrm{Z}(T^2)$ so that

$$A \cong A/(A \cap T_0^1) \cong AT_0^1/T_0^1 \leq (T_0^1 \times \mathrm{Z}(T^2))/T_0^1 \cong \mathrm{Z}(T^2)$$

giving $|A| \leq |\mathrm{Z}(T^2)| = 2$. So in both cases there exists $t_1' \in \mathrm{C}_{T^1}(T_0^1)^\#$ such that $A = \langle t_1' t_2 \rangle$. Then, by Lemma 2.1.16(iii), $2^{(n/2-2)} \leq |\overline{V_0}/\, \mathrm{C}_{\overline{V_0}}(t_1' t_2)| \leq 2\,|A| = 2^2$ so $n/2 - 2 \leq 2$ and so $n \leq 8$ giving, as $r = 2$ and n is even, $n \leq 6$ which is a contradiction. Hence, there exists $i \in \{1, 2\}$ such that $A \cap T_0^i \neq 1$. In particular, as $1 \neq A \cap T_0^i \trianglelefteq S_0$, $A \cap T_0^i \cap \mathrm{Z}(S_0) \neq 1$ and so $t_i \in A$.

(2) $A \cap T_0^j = 1$ where $j = 3 - i$.

Suppose, for a contradiction, that $A \cap T_0^j \neq 1$. Then, as T_0^i and T_0^j are non-trivial groups, $m_1 \geq 2$ and $t_j \in A$ so that, by (1), $\mathrm{Z}(S_0) = \langle t_i, t_j \rangle \leq A$ and so $[\overline{V_0}, \mathrm{Z}(S_0)] \leq [\overline{V_0}, A] \leq \mathrm{C}_{\overline{V_0}}(A) \leq \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S_0))$ giving, by $(\star)$, $\mathrm{C}_{\overline{V_0}}(A) = \mathrm{C}_{\overline{V_0}}(\mathrm{Z}(S_0))$. Also, by Lemma 2.5.9(iii), $[V_0, S_0] = V_0(\Delta_1) + V_0(\Delta_2)$ so that

$$[V_0(\Delta_1), t_1] + [V_0(\Delta_2), t_2] = [V_0, S_0, t] \leq [V_0, S_0, A]$$

and so $|[V_0, S_0, A]| \geq 2^{(2^{m_1-1}-1+2^{m_2-1}-1)} = 2^{(n/2-2)}$ giving $|[\overline{V_0}, S_0, A]| = |\overline{[V_0, S_0, A]}| \geq 2^{(n/2-3)}$. Then, by $(\star)$,

$$|A| \leq |\mathrm{C}_{\overline{V_0}}(A)/[\overline{V_0}, S_0, A]| \leq 2^{(n/2-1)}/2^{(n/2-3)} = 2^2$$

and so $A = \mathrm{Z}(S_0)$. We have that $2^{(n/2-2)} \leq |\overline{V_0}/\, \mathrm{C}_{\overline{V_0}}(t)| \leq 2\,|A| = 2^3$ so $n/2 - 2 \leq 3$ and so $n \leq 10$ which is a contradiction because $r = 2$, n is even and $m_1 \geq 2$. Hence, $A \cap T_0^j = 1$.

(3) Assume that $t_i = (1\,2)(3\,4)\ldots(2^{m_i}{-}1\;2^{m_i})$ and set $A^* = \langle (1\,2), (3\,4), \ldots, (2^{m_i}{-}1\;2^{m_i}) \rangle$. Then $A \leq A^* \times \mathrm{C}_{T^j}(T_0^j)$. In particular, A has no fixed-point-free elements on Δ.

As T_0^j and A are normal in S_0 and by (2), $[T_0^j, A] \leq T_0^j \cap A = 1$ and so $A \leq \mathrm{C}_S(T_0^j) = T^i \times \mathrm{C}_{T^j}(T_0^j)$. Also, by (1), $[\overline{V_0}, t_i, A] \leq [\overline{V_0}, A, A] = 0$ and so, by Lemma 5.1.1, $A \leq \mathrm{C}_S([\overline{V_0}, t_i]) = A^* \times T^j$. Hence, $A \leq (T^i \times \mathrm{C}_{T^j}(T_0^j)) \cap (A^* \times T^j) = A^* \times \mathrm{C}_{T^j}(T_0^j)$. In particular, A has no fixed-point-free elements on Δ because otherwise there exists $t_j' \in \mathrm{C}_{T^j}(T_0^j)^\#$ such that $a = t_i t_j' \in A$ and, as $a \in A \leq \mathfrak{A}_n$, $m_1 \geq 2$ so that $t_j' \in T_0^j$ and so, by (1) and (2), $t_j' = t_i a \in A \cap T_0^j = 1$ which is a contradiction.

(4) $A \leq T_0^i$.

Suppose, for a contradiction, that $A \not\leq T_0^i$. Then there exists $a \in A \setminus T_0^i$ and, by (3), $a = s_i t_j'$ for some $s_i \in A^*$ with a fixed point on Δ_i and $t_j' \in \mathrm{C}_{T^j}(T_0^j)^\#$. Set $X = \langle t_i, t_j' \rangle$. Then, by applying Lemma 2.1.13(i),

$$[V_0, t_j'] \leq [V_0, s_i] + [V_0, t_j'] = [V_0, s_i t_j'] = [V_0, a] \leq [V_0, A]$$

and so, by (1), $[\overline{V_0}, X] = [\overline{V_0}, t_i] + [\overline{V_0}, t_j'] \leq [\overline{V_0}, A]$. Also, by applying Lemma 2.1.13(i),

$$\mathrm{C}_{V_0}(A) \leq \mathrm{C}_{V_0}(a) = \mathrm{C}_{V_0}(s_i t_j') = \mathrm{C}_{V_0}(s_i) \cap \mathrm{C}_{V_0}(t_j') \leq \mathrm{C}_{V_0}(t_j')$$

and so, by (1) and (3), $\mathrm{C}_{\overline{V_0}}(A) \leq \mathrm{C}_{\overline{V_0}}(t_i) \cap \mathrm{C}_{\overline{V_0}}(t_j') = \mathrm{C}_{\overline{V_0}}(X)$. We have shown that $[\overline{V_0}, X] \leq [\overline{V_0}, A] \leq \mathrm{C}_{\overline{V_0}}(A) \leq \mathrm{C}_{\overline{V_0}}(X)$ and hence, by ($\star$), $\mathrm{C}_{\overline{V_0}}(A) = \mathrm{C}_{\overline{V_0}}(X)$. Now, $[V_0(\Delta_i), s_i, t_i] \leq [V_0(\Delta_i), A^*, A^*] = 0$ and so, as s_i is a non-trivial element and has a fixed point on Δ_i, $[V_0(\Delta_i), t_i] + [V_0(\Delta_i), s_i] = \mathrm{C}_{V_0(\Delta_i)}(t_i)$. Then, as $[V_0, S_0] = V_0(\Delta_i) + V_0(\Delta_j)$ and by (1),

$$\begin{aligned}\mathrm{C}_{V_0(\Delta_i)}(t_i) + [V_0(\Delta_j), t_j'] &= [V_0(\Delta_i), t_i] + [V_0(\Delta_i), s_i] + [V_0(\Delta_j), t_j'] \\ &= [V_0, S_0, t_i] + [V_0, S_0, a] \leq [V_0, S_0, A]\end{aligned}$$

and so $|[V_0, S_0, A]| \geq 2^{(2^{m_i-1}+2^{m_j-1}-1)} = 2^{(n/2-1)}$ giving $|[\overline{V_0}, S_0, A]| \geq 2^{(n/2-2)}$. So, by ($\star$),

$$|A| \leq |\mathrm{C}_{\overline{V_0}}(A)/[\overline{V_0}, S_0, A]| \leq 2^{(n/2-1)}/2^{(n/2-2)} = 2$$

which is a contradiction. Therefore, $A \leq T_0^i$. □

Lemma 5.1.5 *Assume that $r \geq 3$ when $U = \overline{V_0}$. Let A be a non-trivial normal elementary abelian 2-subgroup of S with $[U, A, A] = 0$, $|\mathrm{C}_{[U,A]}(S)| = 2$ and $|U/\mathrm{C}_U(a)| \leq 2\,|A|$ for all $a \in A$. Then $A \leq T^i$ for some $1 \leq i \leq r$.*

Proof We will prove the result for $U = V_0$. In the remaining case $U = \overline{V_0}$ and n even, as $r \geq 3$, $\overline{z_i} \neq \overline{z_j}$ for all $1 \leq i < j \leq r$ and so the proof is identical apart from adding bars to all of the vectors. We make the following observation.

($\star$) *Let $a = t_j x \in A$ where t_j and x are disjoint elements of S, $1 \leq j \leq r$ and $m_j \geq 1$. If $m_j \geq 2$ or a has a fixed point on Δ, then $\mathrm{C}_{[V_0,A]}(S) = \langle z_j \rangle$.*

If $m_j \geq 2$, then, as t_j is fixed-point-free on Δ_j and by applying Lemma 2.1.14(ii),

$$z_j \in [V_0(\Delta_j), t_j] = [V_0(\Delta_j), a] \leq [V_0, A].$$

If a has a fixed point on Δ, then, by applying Lemma 2.1.13(ii),

$$z_j \in [V_0, t_j] \leq [V_0, t_j] + [V_0, x] = [V_0, a] \leq [V_0, A].$$

In both cases, as $|\mathrm{C}_{[V_0,A]}(S)| = 2$ and by Lemma 2.5.9(i), $\mathrm{C}_{[V_0,A]}(S) = [V_0, A] \cap \mathrm{C}_{V_0}(S) = \langle z_j \rangle$.

We will proceed via a series of steps.

(1) $A \cap T^i \neq 1$ for some $1 \leq i \leq r$. In particular, $t_i \in A$ and $\mathrm{C}_{[V_0,A]}(S) = \langle z_i \rangle$.

Suppose, for a contradiction, that $A \cap T^i = 1$ for all $1 \leq i \leq r$. Then, for each $1 \leq i \leq r$, as T^i and A are normal in S, $[T^i, A] \leq T^i \cap A = 1$ and so

$$A \leq \mathrm{C}_S(T^i) = T^1 \times T^2 \times \cdots \times T^{i-1} \times \mathrm{Z}(T^i) \times T^{i+1} \times \cdots \times T^r$$

giving

$$A \leq \bigcap_{1 \leq i \leq r} \mathrm{C}_S(T^i) = \mathrm{Z}(T^1) \times \mathrm{Z}(T^2) \times \cdots \times \mathrm{Z}(T^r).$$

Let $a \in A^{\#}$ so that $a = s_1 s_2 \ldots s_r$ with $s_i \in \mathrm{Z}(T^i) = \langle t_i \rangle$ for all $1 \leq i \leq r$. Then there exists $1 \leq i < j \leq r$ such that s_i and s_j are non-trivial elements and so $s_i = t_i$ and $s_j = t_j$. As $m_j > m_i \geq 1$ and by applying $(\star)$, $\mathrm{C}_{[V_0,A]}(S) = \langle z_j \rangle$ and so $m_i = 1$ and a is fixed-point-free on Δ because otherwise, by applying $(\star)$, $\langle z_i \rangle = \mathrm{C}_{[V_0,A]}(S) = \langle z_j \rangle$ which is a contradiction. Moreover, $r = 2$ because otherwise, as a is fixed-point-free on Δ, there exists $1 \leq k \leq r$ with $k \neq j$ such that $m_k \geq 2$ and $s_k = t_k$ and so, by applying $(\star)$, $\langle z_k \rangle = \mathrm{C}_{[V_0,A]}(S) = \langle z_j \rangle$ which is a contradiction. We have shown that $n = 2 + 2^{m_j}$ and $a = t_i t_j = t$ and so $A = \langle t \rangle$. Then, by Lemma 2.1.16(ii), $2^{(n/2-1)} = |V_0/\mathrm{C}_{V_0}(t)| \leq 2\,|A| = 2^2$ and so $n/2 - 1 \leq 2$ giving $n \leq 6$ which is a contradiction. Hence, there exists $1 \leq i \leq r$ such that $A \cap T^i \neq 1$. In particular, as A is normalized by T^i, $A \cap \mathrm{Z}(T^i) \neq 1$ so $t_i \in A$ and, as $r \geq 2$, we may apply $(\star)$ with $a = t_i$ to deduce that $\mathrm{C}_{[V_0,A]}(S) = \langle z_i \rangle$.

(2) $A \leq \mathrm{Z}(T^1) \times \mathrm{Z}(T^2) \times \cdots \times \mathrm{Z}(T^{i-1}) \times T^i \times \mathrm{Z}(T^{i+1}) \times \cdots \times \mathrm{Z}(T^r)$ *and* $A \cap T^j = 1$ *for all* $1 \leq j \leq r$ *with* $j \neq i$*.*

We have that $A \cap T^j = 1$ for all $1 \leq j \leq r$ with $j \neq i$ because otherwise $t_j \in A$ and so, by applying $(\star)$ with $a = t_j$, $\langle z_j \rangle = \mathrm{C}_{[V_0,A]}(S) = \langle z_i \rangle$ which is a contradiction. So, for each $1 \leq j \leq r$ with $j \neq i$, as T^j and A are normal in S, $[T^j, A] \leq T^j \cap A = 1$ and so

$$A \leq \mathrm{C}_S(T^j) = T^1 \times T^2 \times \cdots \times T^{j-1} \times \mathrm{Z}(T^j) \times T^{j+1} \times \cdots \times T^r$$

giving

$$A \leq \bigcap_{\substack{1 \leq j \leq r \\ j \neq i}} \mathrm{C}_S(T^j) = \mathrm{Z}(T^1) \times \mathrm{Z}(T^2) \times \cdots \times \mathrm{Z}(T^{i-1}) \times T^i \times \mathrm{Z}(T^{i+1}) \times \cdots \times \mathrm{Z}(T^r).$$

(3) $A \leq T^i$*.*

Suppose, for a contradiction, that $A \not\leq T^i$. Then we may choose $a \in A \setminus T^i$ and, by (2), there exists $1 \leq j \leq r$ with $j \neq i$ such that $a = t_j x$ where t_j and x are disjoint elements of S and $m_j \geq 1$. We have that $m_j = 1$ and a is fixed-point-free on Δ because otherwise, by (1) and applying $(\star)$, $\langle z_i \rangle = \mathrm{C}_{[V_0,A]}(S) = \langle z_j \rangle$ which is a contradiction. Moreover, $r = 2$ because otherwise, as a is fixed-point-free on Δ and by (2), there exists $1 \leq k \leq r$ with $k \neq i$ such that $m_k \geq 2$ and $a = t_k y$ giving, by (1) and applying $(\star)$, $\langle z_i \rangle = \mathrm{C}_{[V_0,A]}(S) = \langle z_k \rangle$ which is a contradiction. We have shown that $n = 2 + 2^{m_i}$ and, by (1), $[V_0, t_i, a] \leq [V_0, A, A] = 0$ so that, by Lemma 5.1.1,

$$a \in \mathrm{C}_S([V_0, t_i]) = \langle (1\ 2), (3\ 4), \ldots, (n-1\ n) \rangle$$

and so, as a is fixed-point-free on Δ, $a = (1\ 2)(3\ 4)\ldots(n-1\ n) = t_i t_j$. But then, by (1) and (2), $t_j = t_i a \in A \cap T^j = 1$ which is a contradiction. Therefore, $A \leq T^i$. □

Lemma 5.1.6 *Assume that* $r \geq 3$ *when* $U = \overline{V_0}$*. Let* A *be a non-trivial normal elementary abelian 2-subgroup of* S_0 *with* $[U, A, A] = 0$ *and* $|\mathrm{C}_{[U,A]}(S_0)| = 2$*. Then* $A \leq T_0^i$ *for some* $1 \leq i \leq r$*.*

Proof We will prove the result for $U = V_0$. In the remaining case $U = \overline{V_0}$ and n even, as $r \geq 3$, $\overline{z_i} \neq \overline{z_j}$ for all $1 \leq i < j \leq r$ and so the proof is identical apart from adding bars to all of the vectors. We have that, by Corollary 2.5.6, either $\mathrm{C}_{T^i}(T_0^i) = \mathrm{Z}(T^i)$ for $1 \leq i \leq r$ or $m_i = 2$ and $\mathrm{C}_{T^i}(T_0^i) = T_0^i$ is similar to $\mathfrak{K}_4$. We make the following observation.

($\star$) Let $a = s_j x \in A$ where $s_j \in \mathrm{C}_{T^j}(T_0^j)^\#$ and x are disjoint elements of S and $1 \leq j \leq r$. If $m_j \geq 2$ or a has a fixed point on Δ, then $\mathrm{C}_{[V_0,A]}(S_0) = \langle z_j \rangle$.

If $m_j = 2$, then, as s_j is fixed-point-free on Δ_j and by applying Lemma 2.1.14(ii), $z_j \in [V_0(\Delta_j), s_j]$ and if $m_j \neq 2$, then $s_j = t_j$ and hence in both cases ($\star$) holds as in the proof of the previous lemma with s_j in place of t_j and S_0 in place of S.

We will proceed via a series of steps.

(1) $A \cap T_0^i \neq 1$ for some $1 \leq i \leq r$. In particular, $t_i \in A$ and $\mathrm{C}_{[V_0,A]}(S_0) = \langle z_i \rangle$.

Suppose, for a contradiction, that $A \cap T_0^i = 1$ for all $1 \leq i \leq r$. Then, for each $1 \leq i \leq r$, as T_0^i and A are normal in S_0, $[T_0^i, A] \leq T_0^i \cap A = 1$ and so

$$A \leq \mathrm{C}_S(T_0^i) = T^1 \times T^2 \times \cdots \times T^{i-1} \times \mathrm{C}_{T^i}(T_0^i) \times T^{i+1} \times \cdots \times T^r$$

giving

$$A \leq \bigcap_{1 \leq i \leq r} \mathrm{C}_S(T_0^i) = \mathrm{C}_{T^1}(T_0^1) \times \mathrm{C}_{T^2}(T_0^2) \times \cdots \times \mathrm{C}_{T^r}(T_0^r).$$

Let $a \in A^\#$ so that $a = s_1 s_2 \ldots s_r$ with $s_i \in \mathrm{C}_{T^i}(T_0^i)$ for all $1 \leq i \leq r$. Then there exists $1 \leq i < j \leq r$ such that s_i and s_j are non-trivial elements. As $m_j > m_i \geq 1$ and by applying ($\star$), $\mathrm{C}_{[V_0,A]}(S_0) = \langle z_j \rangle$ and so $m_i = 1$ because otherwise, by applying ($\star$), $\langle z_i \rangle = \mathrm{C}_{[V_0,A]}(S_0) = \langle z_j \rangle$ which is a contradiction. Then, as a and s_k are elements of $\mathfrak{A}_n$ for all $1 \leq k \leq r$ with $k \neq i$, s_i is an element of $\mathfrak{A}_n$ and so, as $m_i = 1$, $s_i \in T^i \cap \mathfrak{A}_n = T_0^i = 1$ which is a contradiction. Hence, there exists $1 \leq i \leq r$ such that $A \cap T_0^i \neq 1$. In particular, as $1 \neq A \cap T_0^i \trianglelefteq S_0$, $A \cap T_0^i \cap \mathrm{Z}(S_0) \neq 1$ so $t_i \in A$ and, as $r \geq 2$, we may apply ($\star$) with $a = t_i$ to deduce that $\mathrm{C}_{[V_0,A]}(S_0) = \langle z_i \rangle$.

(2) $A \leq \mathrm{C}_{T^1}(T_0^1) \times \mathrm{C}_{T^2}(T_0^2) \times \cdots \times \mathrm{C}_{T^{i-1}}(T_0^{i-1}) \times T^i \times \mathrm{C}_{T^{i+1}}(T_0^{i+1}) \times \cdots \times \mathrm{C}_{T^r}(T_0^r)$.

We have that $A \cap T_0^j = 1$ for all $1 \leq j \leq r$ with $j \neq i$ because otherwise $t_j \in A$ and so, by applying $(\star)$ with $a = t_j$, $\langle z_j \rangle = \mathrm{C}_{[V_0,A]}(S_0) = \langle z_i \rangle$ which is a contradiction. So, for each $1 \leq j \leq r$ with $j \neq i$, as T_0^j and A are normal in S_0, $[T_0^j, A] \leq T_0^j \cap A = 1$ and so

$$A \leq \mathrm{C}_S(T_0^j) = T^1 \times T^2 \times \cdots \times T^{j-1} \times \mathrm{C}_{T^j}(T_0^j) \times T^{j+1} \times \cdots \times T^r$$

giving

$$\begin{aligned} A &\leq \bigcap_{\substack{1 \leq j \leq r \\ j \neq i}} \mathrm{C}_S(T_0^j) \\ &= \mathrm{C}_{T^1}(T_0^1) \times \mathrm{C}_{T^2}(T_0^2) \times \cdots \times \mathrm{C}_{T^{i-1}}(T_0^{i-1}) \times T^i \times \mathrm{C}_{T^{i+1}}(T_0^{i+1}) \times \cdots \times \mathrm{C}_{T^r}(T_0^r). \end{aligned}$$

(3) $A \leq T_0^i$.

Suppose, for a contradiction, that $A \not\leq T_0^i$. Then we may choose $a \in A \setminus T_0^i$ and, by (2), there exists $1 \leq j \leq r$ with $j \neq i$ such that $a = s_j x$ where $s_j \in \mathrm{C}_{T^j}(T_0^j)^\#$ and x are disjoint elements of S. We have that $m_j = 1$ and a is fixed-point-free on Δ because otherwise, by (1) and applying $(\star)$, $\langle z_i \rangle = \mathrm{C}_{[V_0,A]}(S_0) = \langle z_j \rangle$ which is a contradiction. Moreover, $r = 2$ because otherwise, as a is fixed-point-free on Δ and by (2), there exists $1 \leq k \leq r$ with $k \neq i$ such that $m_k \geq 2$ and $a = s_k y$ where $s_k \in \mathrm{C}_{T^k}(T_0^k)^\#$ giving, by (1) and applying $(\star)$, $\langle z_i \rangle = \mathrm{C}_{[V_0,A]}(S_0) = \langle z_k \rangle$ which is a contradiction. We have shown that $n = 2 + 2^{m_i}$ and, by (1), $[V_0, t_i, a] \leq [V_0, A, A] = 0$ so that, by Lemma 5.1.1,

$$a \in \mathrm{C}_S([V_0, t_i]) = \langle (1\ 2), (3\ 4), \ldots, (n-1\ n) \rangle$$

and so, as a is fixed-point-free on Δ, $a = (1\ 2)(3\ 4)\ldots(n-1\ n) = t_i t_j$. But then, as $m_j = 1$, $t_j = t_i a \in T^j \cap \mathfrak{A}_n = T_0^j = 1$ which is a contradiction. Therefore, $A \leq T_0^i$. $\square$

Observe that t_i is a transposition when $m_i = 1$ and t_i is a double transposition when $m_i = 2$ for $1 \leq i \leq r$.

Lemma 5.1.7 *Let $1 \le i \le r$. The following hold:*

(i) If t_i is a transposition, then $\mathrm{C}_S([U,S]/[U,t_i]) = \langle t_i \rangle$.

(ii) If t_i is a double transposition, then $\mathrm{C}_S(\mathrm{C}_U(t_i)/[U,t_i]) = T^i$ and, in particular, $\mathrm{C}_{S_0}(\mathrm{C}_U(t_i)/[U,t_i]) = T_0^i$.

Proof (i) Let t_i be a transposition. Then $i = 1$ when n is even and $i = 2$ when n is odd. Firstly, consider the case when $U = V_0$. We have that, by Lemma 2.5.9(iii), $[V_0, S] = \bigoplus_{j=i}^r V_0(\Delta_j)$ as S-submodules of V_0 and so, as $[V_0, t_i] = V_0(\Delta_i)$, $[V_0,S]/[V_0,t_i] \cong_S \bigoplus_{j=i+1}^r V_0(\Delta_j)$. For each $i+1 \le j \le r$, as $|\Delta_j| \ge 3$ and by applying Lemma 2.1.2(i), $V_0(\Delta_j)$ is a faithful $\mathbb{F}_2\mathfrak{S}_{\Delta_j}$-module so $\mathrm{C}_{T^j}(V_0(\Delta_j)) = 1$ and so

$$\mathrm{C}_S(V_0(\Delta_j)) = T^1 \times T^2 \times \cdots \times T^{j-1} \times 1 \times T^{j+1} \times \cdots \times T^r.$$

Hence,

$$\mathrm{C}_S([V_0,S]/[V_0,t_i]) = \mathrm{C}_S\left(\bigoplus_{j=i+1}^r V_0(\Delta_j)\right) = \bigcap_{j=i+1}^r \mathrm{C}_S(V_0(\Delta_j)) = T^i = \langle t_i \rangle.$$

Secondly, consider the case when n is even and $U = \overline{V_0} = V_0/Z$. Set $z_1' = z - z_1$ and $D = \bigoplus_{j=2}^r V_0(\Delta_j)$. Then, as $D \cap ([V_0,t_1] + Z) = D \cap \langle z_1, z_1' \rangle = \langle z_1' \rangle$,

$$\begin{aligned}[\overline{V_0},S]/[\overline{V_0},t_1] &= \overline{[V_0,S]}/\overline{[V_0,t_1]} = ([V_0,S]+Z)/Z \Big/ ([V_0,t_1]+Z)/Z \\ &\cong_S ([V_0,S]+Z)/([V_0,t_1]+Z) = (D + [V_0,t_1] + Z)/([V_0,t_1]+Z) \\ &\cong_S D/(D \cap ([V_0,t_1]+Z)) = D/\langle z_1' \rangle = \sum_{j=2}^r (V_0(\Delta_j) + \langle z_1' \rangle)/\langle z_1' \rangle.\end{aligned}$$

Let $r = 2$. Then $[\overline{V_0},S]/[\overline{V_0},t_1] \cong_S V_0(\Delta_2)/\langle z_2 \rangle$ and $V_0(\Delta_2)/\langle z_2 \rangle$ is the natural module for $\mathfrak{S}_{\Delta_2}$ over $\mathbb{F}_2$. As $|\Delta_2| = n-2 \ge 5$ and by applying Lemma 2.1.4(i), $V_0(\Delta_2)/\langle z_2 \rangle$ is a faithful $\mathbb{F}_2\mathfrak{S}_{\Delta_2}$-module and so $\mathrm{C}_{T^2}(V_0(\Delta_2)/\langle z_2 \rangle) = 1$. Hence,

$$\mathrm{C}_S([\overline{V_0},S]/[\overline{V_0},t_1]) = \mathrm{C}_S(V_0(\Delta_2)/\langle z_2 \rangle) = T^1 \times \mathrm{C}_{T^2}(V_0(\Delta_2)/\langle z_2 \rangle) = T^1 = \langle t_1 \rangle.$$

Let $r \ge 3$. For each $2 \le j \le r$, $V_0(\Delta_j) \cap \langle z_1' \rangle = 0$ and so

$$(V_0(\Delta_j) + \langle z_1' \rangle)/\langle z_1' \rangle \cong_S V_0(\Delta_j)/(V_0(\Delta_j) \cap \langle z_1' \rangle) \cong_S V_0(\Delta_j)$$

giving $\mathrm{C}_S((V_0(\Delta_j)+\langle z_1'\rangle)/\langle z_1'\rangle)=\mathrm{C}_S(V_0(\Delta_j))$. Hence,

$$\mathrm{C}_S([\overline{V_0},S]/[\overline{V_0},t_1])=\mathrm{C}_S\left(\sum_{j=2}^{r}(V_0(\Delta_j)+\langle z_1'\rangle)/\langle z_1'\rangle\right)=\bigcap_{j=2}^{r}\mathrm{C}_S((V_0(\Delta_j)+\langle z_1'\rangle)/\langle z_1'\rangle)$$
$$=\bigcap_{j=2}^{r}\mathrm{C}_S(V_0(\Delta_j))=\mathrm{C}_S([V_0,S]/[V_0,t_1])=\langle t_1\rangle.$$

Thus, $\mathrm{C}_S([U,S]/[U,t_i])=\langle t_i\rangle$.

(ii) Let t_i be a double transposition, $S=T^i\times R$ and set $\Delta_i'=\Delta\setminus\Delta_i$. Firstly, consider the case when $U=V_0$. By the Centralizer Lemma, $\mathrm{C}_{V_0}(t_i)=[V_0,t_i]\oplus_S V_0(\Delta_i')$ and so $\mathrm{C}_{V_0}(t_i)/[V_0,t_i]\cong_S V_0(\Delta_i')$. As $|\Delta_i'|=n-4\geq 3$ and by applying Lemma 2.1.2(i), $V_0(\Delta_i')$ is a faithful $\mathbb{F}_2\mathfrak{S}_{\Delta_i'}$-module and so $\mathrm{C}_R(V_0(\Delta_i'))=1$. Hence,

$$\mathrm{C}_S(\mathrm{C}_{V_0}(t_i)/[V_0,t_i])=\mathrm{C}_S(V_0(\Delta_i'))=T^i\times\mathrm{C}_R(V_0(\Delta_i'))=T^i.$$

Secondly, consider the case when n is even and $U=\overline{V_0}=V_0/Z$. Set $z_i'=z-z_i$. Then, as $V_0(\Delta_i')\cap([V_0,t_i]+Z)=\langle z_i'\rangle$,

$$\mathrm{C}_{\overline{V_0}}(t_i)/[\overline{V_0},t_i]=\overline{\mathrm{C}_{V_0}(t_i)}/\overline{[V_0,t_i]}=(\mathrm{C}_{V_0}(t_i)+Z)/Z\Big/([V_0,t_i]+Z)/Z$$
$$\cong_S(\mathrm{C}_{V_0}(t_i)+Z)/([V_0,t_i]+Z)=(V_0(\Delta_i')+[V_0,t_i]+Z)/([V_0,t_i]+Z)$$
$$\cong_S V_0(\Delta_i')/(V_0(\Delta_i')\cap([V_0,t_i]+Z))=V_0(\Delta_i')/\langle z_i'\rangle$$

and $V_0(\Delta_i')/\langle z_i'\rangle$ is the natural module for $\mathfrak{S}_{\Delta_i'}$ over $\mathbb{F}_2$. As $r\geq 2$ and n is even, $|\Delta_i'|=n-4\geq 5$ and so, by applying Lemma 2.1.4(i), $V_0(\Delta_i')/\langle z_i'\rangle$ is a faithful $\mathbb{F}_2\mathfrak{S}_{\Delta_i'}$-module giving $\mathrm{C}_R(V_0(\Delta_i')/\langle z_i'\rangle)=1$. Hence,

$$\mathrm{C}_S(\mathrm{C}_{\overline{V_0}}(t_i)/[\overline{V_0},t_i])=\mathrm{C}_S(V_0(\Delta_i')/\langle z_i'\rangle)=T^i\times\mathrm{C}_R(V_0(\Delta_i')/\langle z_i'\rangle)=T^i.$$

Thus, $\mathrm{C}_S(\mathrm{C}_U(t_i)/[U,t_i])=T^i$. □

Lemma 5.1.8 *Assume that $r=2$. The following hold:*

(i) If t_1 is a transposition, then $[\overline{V_0},S]=\mathrm{C}_{\overline{V_0}}(t_1)$.

(ii) If t_1 is a double transposition, then $|\mathrm{C}_{\overline{V_0}}(T^1)|=2^{n-5}$.

Proof (i) Let t_1 be a transposition. Then, by Lemma 2.5.9(iii) and the Centralizer Lemma, $[V_0,S]=V_0(\Delta_1)+V_0(\Delta_2)=\mathrm{C}_{V_0}(t_1)$ and so $[\overline{V_0},S]=\mathrm{C}_{\overline{V_0}}(t_1)$.

(ii) Let t_1 be a double transposition. Then, as T^1 is transitive on Δ_1 and applying the Centralizer Lemma, $\mathrm{C}_{V_0}(T^1) = \langle z_1 \rangle + V_0(\Delta_2)$ and, as n is even, $\mathrm{C}_{\overline{V_0}}(T^1) = \mathrm{C}_{V_0}(T^1)/Z$. So $\mathrm{C}_{V_0}(T^1)$ has dimension $1 + ((n-4)-1) = n-4$ and so $\mathrm{C}_{\overline{V_0}}(T^1)$ has dimension $n-5$. Thus, $|\mathrm{C}_{\overline{V_0}}(T^1)| = 2^{n-5}$. □

Lemma 5.1.9 *Assume that $n = 8$ and let A be a semiregular elementary abelian 2-group of order 2^2. Then $|\overline{V_0}/\,\mathrm{C}_{\overline{V_0}}(A)| = 2^3$.*

Proof By similarity we may assume that $A = \langle (1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3)(2\ 4)(5\ 7)(6\ 8) \rangle$. Observe that A has a system of two blocks $\{\Delta_1, \Delta_2\}$ where $\Delta_1 = \{1,3,5,7\}$ and $\Delta_2 = \{2,4,6,8\}$ and so, by Lemma 2.1.11, $|\mathrm{C}_{\overline{V_0}}(A)| = |V_0/[V_0,A]|$. We have that $[V_0, A] = \langle [v_k + v_{k+1}, a] \mid 1 \leq k \leq n-1, a \in A \rangle$ and so

$$[V_0, A] = \langle v_1 + v_2 + v_3 + v_4, v_3 + v_4 + v_5 + v_6, v_5 + v_6 + v_7 + v_8, v_2 + v_4 + v_5 + v_7 \rangle$$

giving $|\mathrm{C}_{\overline{V_0}}(A)| = 2^3$. Thus, $|\overline{V_0}/\,\mathrm{C}_{\overline{V_0}}(A)| = 2^3$. □

§5.2 Initial results and the modules Z_β, Z_μ, V_β/Z_β and Q_β/C_β

We will assume the following hypothesis for the remainder of this chapter.

Hypothesis Assume that $G_{\mu\beta} = Q_\mu Q_\beta$.

We will begin the analysis of the $G_{\mu\beta} = Q_\mu Q_\beta$ case by linking together the number of non-central chief G_β-factors in the groups Q_β/C_β, C_β/V_β and V_β/Z_β to specific structural properties of the amalgam. Using these results we will show that Z_β is a trivial $\mathbb{F}_2\overline{G_\beta}$-module and Z_μ is isomorphic to the natural module for G_μ/Q_μ over $\mathbb{F}_2$. We will also show that V_β/Z_β and Q_β/C_β are faithful $\mathbb{F}_2\overline{G_\beta}$-modules each with a unique non-trivial composition factor and that Q_β/C_β embeds in the dual of V_β/Z_β. Moreover, the non-trivial composition factors in V_β/Z_β and Q_β/C_β are isomorphic to $V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ and $(V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta))^*$ respectively. In this section the results before Lemma 5.2.7 hold under the weaker hypothesis $P_2/Q_2 \in \mathfrak{X}$ and the results after Lemma 5.2.7 hold with the additional assumption that $|Z| = 2$. Observe that, by Lemma 5.2.6, the condition $|Z| = 2$ is equivalent to assuming that Case (I) of Lemma 5.2.7 holds. The proof of Case (I) of Lemma 5.2.7 remains valid when $P_2/Q_2 \in \mathfrak{X}$ provided that P_2/Q_2 does not contain groups of order greater than two which induce central transvections on a faithful $\mathbb{F}_2(P_2/Q_2)$-module.

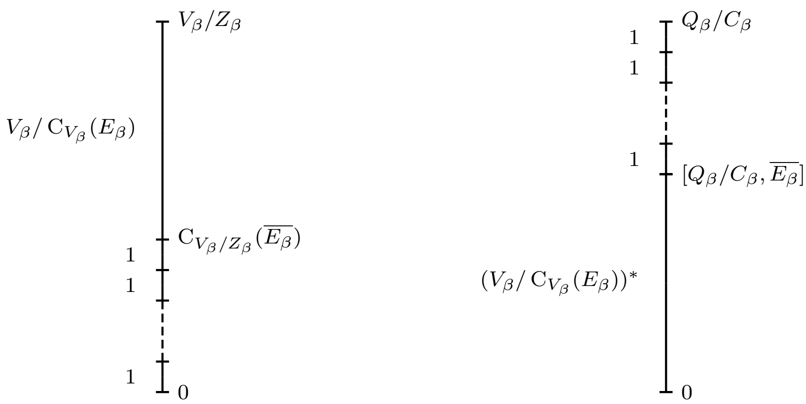

Figure E Composition series of the modules V_β/Z_β and Q_β/C_β with respective factors.

Lemma 5.2.1 *The following hold:*

(i) $G_\mu = Q_\mu\langle Q_\beta, Q_{\beta'}\rangle$ *and* $E_\mu = \mathrm{O}^2(\langle Q_\beta, Q_{\beta'}\rangle)$.

(ii) $|Q_\beta/(Q_\beta \cap Q_\mu)| = 2$ *and* $[V_\beta, Q_\beta] = Z_\beta$.

(iii) $Y_\mu = V_\beta \cap V_{\beta'}$ *and* $\eta_{G_\mu}(Y_\mu/Z_\mu) = 0$.

Proof (i) By hypothesis, $G_{\mu\beta} = Q_\mu Q_\beta$ and $G_{\mu\beta'} = Q_\mu Q_{\beta'}$ and so, by Lemma 3.2.2(ii),

$$G_\mu = \langle G_{\mu\beta}, G_{\mu\beta'}\rangle = \langle Q_\mu Q_\beta, Q_\mu Q_{\beta'}\rangle = \langle Q_\mu, Q_\beta, Q_{\beta'}\rangle = Q_\mu\langle Q_\beta, Q_{\beta'}\rangle.$$

Then, as $Q_\mu = G_{\Delta(\mu)} = G_{\mu\beta\beta'\beta^*}$, $\langle Q_\beta, Q_{\beta'}\rangle \trianglelefteq Q_\mu\langle Q_\beta, Q_{\beta'}\rangle = G_\mu$ and so

$$E_\mu = \mathrm{O}^2(G_\mu) = \mathrm{O}^2(Q_\mu)\,\mathrm{O}^2(\langle Q_\beta, Q_{\beta'}\rangle) = \mathrm{O}^2(\langle Q_\beta, Q_{\beta'}\rangle).$$

(ii) By hypothesis and Lemma 3.2.2(ii), $|Q_\beta/(Q_\beta \cap Q_\mu)| = |Q_\beta Q_\mu/Q_\mu| = |G_{\mu\beta}/Q_\mu| = 2$. Also, as $Q_\beta \not\leq Q_\mu$ and by applying Lemma 3.2.2(iv), $[Z_\mu, Q_\beta] = Z_\beta$ and so, as $Q_\beta \trianglelefteq G_\beta$,

$$[V_\beta, Q_\beta] = [\langle Z_\mu^{G_\beta}\rangle, Q_\beta] = \langle [Z_\mu, Q_\beta]^{G_\beta}\rangle = \langle Z_\beta^{G_\beta}\rangle = Z_\beta.$$

(iii) By part (ii),

$$[V_\beta \cap V_{\beta'}, Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta \leq Z_\mu \leq V_\beta \cap V_{\beta'}$$

and $[V_\beta \cap V_{\beta'}, Q_{\beta'}] \leq V_\beta \cap V_{\beta'}$. So, by part (i), $V_\beta \cap V_{\beta'} \trianglelefteq Q_\mu\langle Q_\beta, Q_{\beta'}\rangle = G_\mu$ and so, by Lemma 3.2.1(vii),

$$V_\beta \cap V_{\beta'} \leq (V_\beta)_{G_\mu} = Y_\mu = V_\beta \cap V_{\beta'} \cap V_{\beta^*} \leq V_\beta \cap V_{\beta'}$$

giving $Y_\mu = V_\beta \cap V_{\beta'}$. Now, by part (i), $E_\mu \leq \langle Q_\beta, Q_{\beta'}\rangle$ and, by part (ii), $[Y_\mu, Q_\beta] \leq Z_\mu$ and $[Y_\mu, Q_{\beta'}] \leq Z_\mu$. So, regarding Y_μ as an abelian G_μ-group,

$$[Y_\mu, E_\mu] \leq [Y_\mu, \langle Q_\beta, Q_{\beta'}\rangle] = [Y_\mu, Q_\beta]\ [Y_\mu, Q_{\beta'}] \leq Z_\mu.$$

Thus, $\eta_{G_\mu}(Y_\mu/Z_\mu) = 0$. □

Lemma 5.2.2 $\eta_{G_\beta}(Q_\beta/C_\beta) = 0$ *if and only if* $Q_\mu \cap Q_\beta \trianglelefteq G_\beta$.

Proof Firstly, assume that $\eta_{G_\beta}(Q_\beta/C_\beta) = 0$. Then, by Lemma 3.2.1(ii),

$$[Q_\mu \cap Q_\beta, E_\beta] \leq [Q_\beta, E_\beta] \leq C_\beta = Q_\beta^{[1]} \leq Q_\mu \cap Q_\beta.$$

Thus, $Q_\mu \cap Q_\beta \trianglelefteq G_{\mu\beta}E_\beta = G_\beta$. Secondly, assume that $Q_\mu \cap Q_\beta \trianglelefteq G_\beta$. Then, as $[Z_\mu, Q_\mu \cap Q_\beta] \leq [Z_\mu, Q_\mu] = 1$,

$$[V_\beta, Q_\mu \cap Q_\beta] = [\langle Z_\mu^{G_\beta}\rangle, Q_\mu \cap Q_\beta] = \langle [Z_\mu, Q_\mu \cap Q_\beta]^{G_\beta}\rangle = 1$$

and so, by Lemma 3.2.1(ii),

$$Q_\mu \cap Q_\beta \leq \mathrm{C}_{G_\beta}(V_\beta) = C_\beta = Q_\beta^{[1]} \leq Q_\mu \cap Q_\beta$$

giving $C_\beta = Q_\mu \cap Q_\beta$. We have that, by Lemma 5.2.1(ii), $Q_\beta/C_\beta \trianglelefteq G_\beta/C_\beta$ and $|Q_\beta/C_\beta| = |Q_\beta/(Q_\beta \cap Q_\mu)| = 2$ and so $Q_\beta/C_\beta \leq \mathrm{Z}(G_\beta/C_\beta)$. Then

$$[Q_\beta, E_\beta]C_\beta/C_\beta \leq [Q_\beta, G_\beta]C_\beta/C_\beta = [Q_\beta/C_\beta, G_\beta/C_\beta] = C_\beta/C_\beta$$

and so $[Q_\beta, E_\beta] \leq C_\beta$. Thus, $\eta_{G_\beta}(Q_\beta/C_\beta) = 0$. □

Lemma 5.2.3 $\eta_{G_\beta}(Q_\beta/C_\beta) \geq 1$.

Proof Suppose, for a contradiction, that $\eta_{G_\beta}(Q_\beta/C_\beta) = 0$. Then, by Lemmas 5.2.1(ii) and 3.2.1(iii), $[V_\beta, Q_\beta, E_\beta] = [Z_\beta, E_\beta] = 1$ and $[Q_\beta, E_\beta, V_\beta] \leq [C_\beta, V_\beta] = 1$ and so, by applying the Three-Subgroup Lemma, $[V_\beta, E_\beta, Q_\beta] = [E_\beta, V_\beta, Q_\beta] = 1$. So $W_\beta = [V_\beta, E_\beta] \leq \mathrm{C}_{V_\beta}(Q_\beta)$ and $W_{\beta'} \leq \mathrm{C}_{V_{\beta'}}(Q_{\beta'})$. Also, by Lemma 3.2.1(vi), $V_\beta = Z_\mu W_\beta$ and $V_{\beta'} = Z_\mu W_{\beta'}$ and so, as Z_μ, W_β and $W_{\beta'}$ are normal in Q_μ,

$$\begin{aligned}[V_\beta, V_{\beta'}] &= [Z_\mu W_\beta, Z_\mu W_{\beta'}] = [Z_\mu, Z_\mu]\ [Z_\mu, W_{\beta'}]\ [W_\beta, Z_\mu]\ [W_\beta, W_{\beta'}] \\ &= [W_\beta, W_{\beta'}] \leq W_\beta \cap W_{\beta'} \leq \mathrm{C}_{V_\beta}(Q_\beta) \cap \mathrm{C}_{V_{\beta'}}(Q_{\beta'}).\end{aligned}$$

Hence, by Lemmas 5.2.1(i) and 3.2.1(iv),

$$\begin{aligned}[V_\beta, V_{\beta'}] \cap Z_\mu &\leq \mathrm{C}_{Z_\mu}(Q_\mu) \cap \mathrm{C}_{Z_\mu}(Q_\beta) \cap \mathrm{C}_{Z_\mu}(Q_{\beta'}) = \mathrm{C}_{Z_\mu}(Q_\mu) \cap \mathrm{C}_{Z_\mu}(\langle Q_\beta, Q_{\beta'}\rangle) \\ &= \mathrm{C}_{Z_\mu}(Q_\mu\langle Q_\beta, Q_{\beta'}\rangle) = \mathrm{C}_{Z_\mu}(G_\mu) = 1.\end{aligned}$$

On the other hand, as V_β and $V_{\beta'}$ are normal in Q_μ and by Corollary 3.3.5(i), $1 \neq [V_\beta, V_{\beta'}] \trianglelefteq Q_\mu$ and so, by Lemma 3.2.2(v), $[V_\beta, V_{\beta'}] \cap Z_\mu = [V_\beta, V_{\beta'}] \cap \Omega\,\mathrm{Z}(Q_\mu) \neq 1$ which is a contradiction. Therefore, $\eta_{G_\beta}(Q_\beta/C_\beta) \geq 1$. □

Corollary 5.2.4 *The following hold:*

(i) $Q_\mu \cap Q_\beta \ntrianglelefteq G_\beta$.

(ii) $G_{\mu\beta} = Q_\mu(G_\mu \cap E_\beta)$ *and* $G_\beta = Q_\mu E_\beta$.

Proof (i) This follows from the lemma above and Lemma 5.2.2.

(ii) We have that, by Lemma 3.2.2(ii),

$$|G_{\mu\beta}:Q_\mu(G_\mu \cap E_\beta)|\ |Q_\mu(G_\mu \cap E_\beta):Q_\mu| = |G_{\mu\beta}:Q_\mu| = 2$$

and so either $G_{\mu\beta} = Q_\mu(G_\mu \cap E_\beta)$ or $G_\mu \cap E_\beta \leq Q_\mu$. In the latter case,

$$[Q_\mu \cap Q_\beta, E_\beta] \leq [Q_\beta, E_\beta] \leq Q_\beta \cap E_\beta = Q_\beta \cap G_\mu \cap E_\beta \leq Q_\mu \cap Q_\beta$$

and so $Q_\mu \cap Q_\beta \trianglelefteq G_{\mu\beta}E_\beta = G_\beta$ which contradicts part (i). Thus, $G_{\mu\beta} = Q_\mu(G_\mu \cap E_\beta)$ and $G_\beta = G_{\mu\beta}E_\beta = Q_\mu(G_\mu \cap E_\beta)E_\beta = Q_\mu E_\beta$. □

Lemma 5.2.5 *V_β/Z_β and Q_β/C_β may be regarded as faithful $\mathbb{F}_2\overline{G_\beta}$-modules.*

Proof The group V_β/Z_β is an elementary abelian $2, G_\beta$-group and, by Lemma 5.2.1(ii), $[V_\beta/Z_\beta, Q_\beta] = [V_\beta, Q_\beta]Z_\beta/Z_\beta = Z_\beta/Z_\beta$ and so $Q_\beta \leq \mathrm{C}_{G_\beta}(V_\beta/Z_\beta)$. So we may regard V_β/Z_β as an $\mathbb{F}_2\overline{G_\beta}$-module and, by Lemma 3.2.1(iii), $\eta_{\overline{G_\beta}}(V_\beta/Z_\beta) = \eta_{G_\beta}(V_\beta/Z_\beta) \geq 1$ and so, by applying Lemma 2.3.1, V_β/Z_β is a faithful module.

We have that, by Lemma 5.2.1(ii), $Q_\beta \cap Q_\mu$ is a normal subgroup of Q_β with $|Q_\beta/(Q_\beta \cap Q_\mu)| = 2$ and so $\Phi(Q_\beta) \leq Q_\mu$. Then, by Lemma 3.2.1(ii), $\Phi(Q_\beta) \leq (Q_\mu)_{G_\beta} = Q_\beta^{[1]} = C_\beta$ and so Q_β/C_β is an elementary abelian $2, G_\beta$-group. Also, $[Q_\beta/C_\beta, Q_\beta] = [Q_\beta, Q_\beta]C_\beta/C_\beta = [Q_\beta/C_\beta, Q_\beta/C_\beta] = C_\beta/C_\beta$ giving $Q_\beta \leq \mathrm{C}_{G_\beta}(Q_\beta/C_\beta)$. So we may regard Q_β/C_β as an $\mathbb{F}_2\overline{G_\beta}$-module and, by Lemma 5.2.3, $\eta_{\overline{G_\beta}}(Q_\beta/C_\beta) = \eta_{G_\beta}(Q_\beta/C_\beta) \geq 1$ and so, by applying Lemma 2.3.1, Q_β/C_β is a faithful module. □

Lemma 5.2.6 $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$ *if and only if* $|Z_\beta| = 2$.

Proof Firstly, assume that $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$. Then, by Lemma 3.2.1(iii), $\eta_{G_\beta}(V_\beta) = \eta_{G_\beta}(V_\beta/Z_\beta) + \eta_{G_\beta}(Z_\beta) = 1$. Set $Q_\beta^0 = [Q_\beta, E_\beta]$. We have that $Q_\beta^0 \nleq Q_\mu$ because otherwise $[Q_\mu \cap Q_\beta, E_\beta] \leq [Q_\beta, E_\beta] = Q_\beta^0 \leq Q_\mu \cap Q_\beta$ and so $Q_\mu \cap Q_\beta \trianglelefteq G_{\mu\beta}E_\beta = G_\beta$ which contradicts Corollary 5.2.4(i). Then, as $Q_\beta^0 \le Q_\beta \le G_{\mu\beta}$ and by applying Lemma 3.2.2(iv), $[Z_\mu, Q_\beta^0] = Z_\beta$ and so, as $Q_\beta^0 \trianglelefteq G_\beta$,

$$[V_\beta, Q_\beta^0] = [\langle Z_\mu^{G_\beta} \rangle, Q_\beta^0] = \langle [Z_\mu, Q_\beta^0]^{G_\beta} \rangle = Z_\beta.$$

In particular, by Lemma 3.2.1(iii), $[V_\beta, Q_\beta^0, E_\beta] = [Z_\beta, E_\beta] = 1$ and so, by applying the Three-Subgroup Lemma, $[Q_\beta^0, E_\beta, V_\beta] = [E_\beta, V_\beta, Q_\beta^0]$. Also, $[Q_\beta^0, E_\beta] = [Q_\beta, E_\beta, E_\beta] = [Q_\beta, E_\beta] = Q_\beta^0$ and so

$$[W_\beta, Q_\beta^0] = [V_\beta, E_\beta, Q_\beta^0] = [E_\beta, V_\beta, Q_\beta^0] = [Q_\beta^0, E_\beta, V_\beta] = [Q_\beta^0, V_\beta] = Z_\beta.$$

Now, as $Z_\mu \not\le \mathrm{C}_{Z_\mu}(E_\beta)$, we may choose $z \in Z_\mu \setminus \mathrm{C}_{Z_\mu}(E_\beta)$. Then $\langle z^{G_\beta} \rangle \le \langle Z_\mu^{G_\beta} \rangle = V_\beta$ and $\eta_{G_\beta}(\langle z^{G_\beta} \rangle) \ge 1$ because otherwise $[z, E_\beta] \le [\langle z^{G_\beta} \rangle, E_\beta] = 1$ giving $z \in \mathrm{C}_{Z_\mu}(E_\beta)$ which is a contradiction. So, as $\eta_{G_\beta}(V_\beta) = 1$, $\eta_{G_\beta}(V_\beta/\langle z^{G_\beta} \rangle) = \eta_{G_\beta}(V_\beta) - \eta_{G_\beta}(\langle z^{G_\beta} \rangle) = 0$ and so $W_\beta = [V_\beta, E_\beta] \le \langle z^{G_\beta} \rangle$. Also, by Lemmas 5.2.1(ii) and 3.2.1, $[z, Q_\beta, G_\beta] \le [V_\beta, Q_\beta, G_\beta] = [Z_\beta, G_\beta] = 1$ and so, as $Q_\beta \trianglelefteq G_\beta$,

$$Z_\beta = [W_\beta, Q_\beta^0] \le [\langle z^{G_\beta} \rangle, Q_\beta] = \langle [z, Q_\beta]^{G_\beta} \rangle = [z, Q_\beta] \le [V_\beta, Q_\beta] = Z_\beta$$

giving $[z, Q_\beta] = Z_\beta$. Then, as $[Q_\beta, z] = Z_\beta \le \mathrm{Z}(Q_\beta)$, $|Q_\beta/\mathrm{C}_{Q_\beta}(z)| = |[Q_\beta, z]| = |Z_\beta|$. We also have that, by Lemma 3.2.1(i), $Q_\beta \cap Q_\mu = \mathrm{C}_{Q_\beta}(Z_\mu) \le \mathrm{C}_{Q_\beta}(z)$ and so, by Lemma 5.2.1(ii), $2 \le |Z_\beta| = |Q_\beta/\mathrm{C}_{Q_\beta}(z)| \le |Q_\beta/(Q_\beta \cap Q_\mu)| = 2$. Thus, $|Z_\beta| = 2$.

Secondly, assume that $|Z_\beta| = 2$. Choose a minimal faithful submodule N_β/Z_β of V_β/Z_β. Then N_β is a G_β-subgroup of V_β containing Z_β and, by Corollary 2.3.2 and Lemma 3.2.1(iii), $\eta_{G_\beta}(N_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) + \eta_{G_\beta}(Z_\beta) = 1$. Define $N_{\beta'} = N_\beta^{g'}$ and $N_{\beta^*} = N_\beta^{g^*}$. Then, by applying Corollary 3.3.3, $N_\beta^g = N_{\beta \cdot g}$ for all $g \in G_\mu$. Assume that $[N_\beta, N_{\beta'}] = 1$. In this case, by applying Lemma 3.1.6(ii), $N_\beta \le \mathrm{C}_{G_{\beta'}}(N_{\beta'}) \le Q_{\beta'}$ and $N_\beta \not\le C_{\beta'}$ because otherwise, by Corollary 3.3.5(i), $[N_\beta, E_\beta] \le [N_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [N_\beta, V_{\beta'}]^{G_\beta} \rangle = 1$ giving $\eta_{G_\beta}(N_\beta) = 0$ which is a contradiction. Also, by Lemma 3.2.1(ii), $C_{\beta'} = Q_{\beta'}^{[1]}$ and so there exists $\rho \in \Gamma(\beta')$ such that $N_\beta \not\le Q_\rho$. Then, as $N_\beta \le Q_{\beta'} \le G_{\rho\beta'}$ and by applying Lemma 3.2.2(iv), $[Z_\rho, N_\beta] = Z_{\beta'}$ and so, as $Z_\rho \le Q_\mu$ and N_β is normal in Q_μ, $Z_{\beta'} = [Z_\rho, N_\beta] \le N_\beta$. So, by Lemma 3.2.2(iii), $Z_\mu = Z_\beta Z_{\beta'} \le N_\beta$ and so

$V_\beta = \langle Z_\mu^{G_\beta}\rangle \le N_\beta \le V_\beta$ giving $V_\beta = N_\beta$. Thus, $\eta_{G_\beta}(V_\beta/Z_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) = 1$. Assume that $[N_\beta, N_{\beta'}] \ne 1$. In this case, as $G_{\mu\beta^*}$ acts on $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$, $1 \ne [N_\beta, N_{\beta'}] \trianglelefteq G_{\mu\beta^*}$ and so, by Lemma 3.2.1, $[N_\beta, N_{\beta'}] \cap Z_{\beta^*} = [N_\beta, N_{\beta'}] \cap \Omega\,\mathrm{Z}(G_{\mu\beta^*}) \ne 1$ giving, as $|Z_{\beta^*}| = 2$, $Z_{\beta^*} \le [N_\beta, N_{\beta'}]$. So, as N_β and $N_{\beta'}$ are normal in Q_μ and by Lemma 3.2.2(iii), $Z_\mu = Z_\beta Z_{\beta^*} \le N_\beta$ and so $V_\beta = \langle Z_\mu^{G_\beta}\rangle \le N_\beta \le V_\beta$ giving $V_\beta = N_\beta$. Thus, $\eta_{G_\beta}(V_\beta/Z_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) = 1$. □

Lemma 5.2.7 *One of the following two cases hold:*

(I) $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$.

(II) $\overline{G_\beta} \cong \mathfrak{A}_8$, $[Q_\beta/C_\beta, \overline{E_\beta}]$ is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$, every minimal faithful submodule of V_β/Z_β is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$ and $\overline{V_{\beta'}}$ is similar to a subgroup of either the group $\mathfrak{K}_4 \times \mathfrak{K}_4$ or a regular eight-group $\mathfrak{K}_8$.

Proof Choose a minimal faithful submodule N_β/Z_β of V_β/Z_β. Then N_β is a G_β-subgroup of V_β containing Z_β and, by Corollary 2.3.2 and Lemma 3.2.1(iii), $\eta_{G_\beta}(N_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) + \eta_{G_\beta}(Z_\beta) = 1$. Define $N_{\beta'} = N_\beta^{g'}$ and $N_{\beta^*} = N_\beta^{g^*}$. Then, by applying Corollary 3.3.3, $N_\beta^g = N_{\beta\cdot g}$ for all $g \in G_\mu$. Assume that $[V_\beta, N_{\beta'} \cap Q_\beta] \ne 1 \ne [V_{\beta'}, N_\beta \cap Q_{\beta'}]$. In this case, by Lemma 3.2.1(ii), $N_\beta \cap Q_{\beta'} \not\le C_{\beta'} = Q_{\beta'}^{[1]}$ and so there exists $\rho \in \Gamma(\beta')$ such that $N_\beta \cap Q_{\beta'} \not\le Q_\rho$. Then, as $N_\beta \cap Q_{\beta'} \le Q_{\beta'} \le G_{\rho\beta'}$ and by applying Lemma 3.2.2(iv), $Z_{\beta'} = [Z_\rho, N_\beta \cap Q_{\beta'}] \le [Z_\rho, N_\beta] \le N_\beta$ so that, by Lemma 3.2.2(iii), $Z_\mu = Z_\beta Z_{\beta'} \le N_\beta$ and so $V_\beta = \langle Z_\mu^{G_\beta}\rangle \le N_\beta \le V_\beta$ giving $V_\beta = N_\beta$. Thus, $\eta_{G_\beta}(V_\beta/Z_\beta) = \eta_{G_\beta}(N_\beta/Z_\beta) = 1$. Assume that $[V_\beta, N_{\beta'} \cap Q_\beta] = 1 = [V_{\beta'}, N_\beta \cap Q_{\beta'}]$. In this case, $N_\beta \not\le Q_{\beta'}$ because otherwise $N_\beta = N_\beta \cap Q_{\beta'} \le C_{\beta'}$ and so, as $N_\beta \trianglelefteq G_\beta$ and by Corollary 3.3.5(i), $[N_\beta, E_\beta] \le [N_\beta, \langle V_{\beta'}^{G_\beta}\rangle] = \langle [N_\beta, V_{\beta'}]^{G_\beta}\rangle = 1$ giving $\eta_{G_\beta}(N_\beta) = 0$ which is a contradiction. By Lemma 5.2.5, Q_β/C_β is a faithful $\mathbb{F}_2\overline{G_\beta}$-module and, as $N_{\beta'} \not\le Q_\beta$ and $N_{\beta'} \trianglelefteq Q_\mu$, $\overline{N_{\beta'}}$ is a non-trivial normal elementary abelian 2-subgroup of $\overline{G_{\mu\beta}}$. As $N_{\beta'}$ and $Q_\beta \cap Q_\mu$ are normal in Q_μ, $[N_{\beta'}, Q_\beta \cap Q_\mu] \le N_{\beta'} \cap (Q_\beta \cap Q_\mu) = N_{\beta'} \cap Q_\beta \le C_\beta$ and so, regarding $(Q_\beta \cap Q_\mu)/C_\beta$ as a $G_{\mu\beta}$-group,

$$[(Q_\beta \cap Q_\mu)/C_\beta, N_{\beta'}] = [Q_\beta \cap Q_\mu, N_{\beta'}]C_\beta/C_\beta = C_\beta/C_\beta$$

giving $(Q_\beta \cap Q_\mu)/C_\beta \leq \mathrm{C}_{Q_\beta/C_\beta}(N_{\beta'})$. Then, by Lemma 5.2.1(ii),

$$\begin{aligned} |Q_\beta/C_\beta \,/\, \mathrm{C}_{Q_\beta/C_\beta}(\overline{N_{\beta'}})| &= |Q_\beta/C_\beta \,/\, \mathrm{C}_{Q_\beta/C_\beta}(N_{\beta'})| \leq |Q_\beta/C_\beta \,/(Q_\beta \cap Q_\mu)/C_\beta| \\ &= |Q_\beta/(Q_\beta \cap Q_\mu)| = 2 \end{aligned}$$

and hence $\overline{N_{\beta'}}$ induces central transvections on Q_β/C_β. Now, as $V_{\beta'} \not\leq Q_\beta$ and $V_{\beta'} \trianglelefteq Q_\mu$, $\overline{V_{\beta'}}$ is a non-trivial normal elementary abelian 2-subgroup of $\overline{G_{\mu\beta}}$ and, as $\mathrm{C}_{N_\beta}(V_{\beta'})/Z_\beta \leq \mathrm{C}_{N_\beta/Z_\beta}(V_{\beta'})$,

$$\begin{aligned} |N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| &= |N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(V_{\beta'})| \leq |N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta}(V_{\beta'})/Z_\beta| \\ &= |N_\beta/\, \mathrm{C}_{N_\beta}(V_{\beta'})|. \end{aligned}$$

Then, as $N_\beta \cap Q_{\beta'} = N_\beta \cap C_{\beta'} = \mathrm{C}_{N_\beta}(V_{\beta'})$ and by applying Lemma 3.3.4(ii),

$$|N_\beta/\, \mathrm{C}_{N_\beta}(V_{\beta'})| = |N_\beta/(N_\beta \cap Q_{\beta'})| = |N_\beta Q_{\beta'}/Q_{\beta'}| = |N_{\beta'} Q_\beta/Q_\beta| = |\overline{N_{\beta'}}|$$

and hence $|N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| \leq |\overline{N_{\beta'}}|$.

Consider the case when $\overline{G_\beta} \not\cong \mathfrak{A}_8$. In this case, $\overline{N_{\beta'}}$ induces central transvections on Q_β/C_β and so, by applying Theorem 2.3.8(ii), $\overline{G_\beta} \cong \mathfrak{S}_n$. Then $|N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| \leq |\overline{N_{\beta'}}| = 2$ and so $\overline{V_{\beta'}}$ induces central transvections on N_β/Z_β giving, by applying Corollary 2.3.9, $\overline{N_{\beta'}} = \overline{V_{\beta'}}$. So $N_{\beta'}Q_\beta = V_{\beta'}Q_\beta$ and hence, by Dedekind's Law,

$$V_{\beta'} = V_{\beta'} \cap V_{\beta'}Q_\beta = V_{\beta'} \cap N_{\beta'}Q_\beta = N_{\beta'}(V_{\beta'} \cap Q_\beta).$$

Define $Z_\mu^0 = \langle ([N_\beta, N_{\beta'}] \cap Z_{\beta^*})^{G_\mu} \rangle \leq Z_\mu$. Then, by applying Lemma 3.3.4(i) and as $G_{\mu\beta^*}$ acts on $\Gamma(\mu) \smallsetminus \{\beta^*\} = \{\beta, \beta'\}$, $1 \neq [N_\beta, N_{\beta'}] \trianglelefteq G_{\mu\beta^*}$ and so, by Lemma 3.2.1, $[N_\beta, N_{\beta'}] \cap Z_{\beta^*} = [N_\beta, N_{\beta'}] \cap \Omega\, \mathrm{Z}(G_{\mu\beta^*}) \neq 1$ giving $Z_\mu^0 \neq 1$. Also, as N_β and $N_{\beta'}$ are normal in Q_μ, $[N_\beta, N_{\beta'}] \leq N_\beta \cap N_{\beta'}$ so that, as G_μ acts on $\Gamma(\mu) = \{\beta, \beta', \beta^*\}$, $[N_\beta, N_{\beta'}] \cap Z_{\beta^*} \leq N_\beta \cap N_{\beta'} \cap N_{\beta^*} \trianglelefteq G_\mu$ and so $Z_\mu^0 \leq N_\beta \cap N_{\beta'} \cap N_{\beta^*}$. Define $N_\beta^0 = \langle (Z_\mu^0)^{G_\beta} \rangle$. We have that, as $Z_\mu^0 \leq N_\beta$, $N_\beta^0 \leq N_\beta$ and $\eta_{G_\beta}(N_\beta^0) \geq 1$ because otherwise $[Z_\mu^0, E_\beta] \leq [N_\beta^0, E_\beta] = 1$ and so Z_μ^0 is normalized by both G_μ and E_β giving $Z_\mu^0 = 1$ which is a contradiction. Then $1 \leq \eta_{G_\beta}(N_\beta^0) \leq \eta_{G_\beta}(N_\beta^0 Z_\beta) \leq \eta_{G_\beta}(N_\beta) = 1$ giving $\eta_{G_\beta}(N_\beta^0 Z_\beta/Z_\beta) = \eta_{G_\beta}(N_\beta^0 Z_\beta) - \eta_{G_\beta}(Z_\beta) = 1$. So, by Lemma 2.3.1, $N_\beta^0 Z_\beta/Z_\beta$ is a faithful submodule of N_β/Z_β and so, by the minimality of N_β/Z_β, $N_\beta^0 Z_\beta/Z_\beta = N_\beta/Z_\beta$ giving $N_\beta = N_\beta^0 Z_\beta$. Then, by Lemmas 5.2.1(ii) and 3.2.1,

$[Z_\mu^0, Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta \leq \mathrm{Z}(G_\beta)$ so, as $Q_\beta \trianglelefteq G_\beta$,

$$\begin{aligned}[N_\beta, Q_\beta] &= [N_\beta^0 Z_\beta, Q_\beta] = [N_\beta^0, Q_\beta]\,[Z_\beta, Q_\beta] = [N_\beta^0, Q_\beta] = [\langle (Z_\mu^0)^{G_\beta}\rangle, Q_\beta] \\ &= \langle [Z_\mu^0, Q_\beta]^{G_\beta}\rangle = [Z_\mu^0, Q_\beta] \leq Z_\mu^0\end{aligned}$$

and $[N_{\beta'}, Q_{\beta'}] \leq Z_\mu^0$. Also, by Lemmas 5.2.1(ii) and 3.2.2(iii),

$$[V_\beta \cap Q_{\beta'}, V_{\beta'} \cap Q_\beta] \leq [V_\beta, Q_\beta] \cap [V_{\beta'}, Q_{\beta'}] = Z_\beta \cap Z_{\beta'} = 1.$$

Now, as $V_\beta = N_\beta(V_\beta \cap Q_{\beta'})$ and $V_{\beta'} = N_{\beta'}(V_{\beta'} \cap Q_\beta)$,

$$\begin{aligned}[V_\beta, V_{\beta'}] &= [N_\beta(V_\beta \cap Q_{\beta'}), N_{\beta'}(V_{\beta'} \cap Q_\beta)] \\ &= [N_\beta, N_{\beta'}]\,[N_\beta, V_{\beta'} \cap Q_\beta]\,[V_\beta \cap Q_{\beta'}, N_{\beta'}]\,[V_\beta \cap Q_{\beta'}, V_{\beta'} \cap Q_\beta] \\ &\leq [N_\beta, N_{\beta'}]\,[N_\beta, Q_\beta]\,[N_{\beta'}, Q_{\beta'}] \leq [N_\beta, N_{\beta'}] Z_\mu^0 \leq N_\beta.\end{aligned}$$

Finally, as $V_\beta \trianglelefteq G_\beta$ and by Corollary 3.3.5(i),

$$[V_\beta, E_\beta] \leq [V_\beta, \langle V_{\beta'}^{G_\beta}\rangle] = \langle [V_\beta, V_{\beta'}]^{G_\beta}\rangle \leq \langle N_\beta^{G_\beta}\rangle = N_\beta$$

and so $\eta_{G_\beta}(V_\beta/N_\beta) = 0$. Therefore, $\eta_{G_\beta}(V_\beta/Z_\beta) = \eta_{G_\beta}(V_\beta/N_\beta) + \eta_{G_\beta}(N_\beta/Z_\beta) = 1$.

It remains to consider the case when $\overline{G_\beta} \cong \mathfrak{A}_8$. In this case, $\overline{N_{\beta'}}$ induces transvections on Q_β/C_β and so, by applying Theorem 2.3.8(ii) and Corollary 2.3.10, $[Q_\beta/C_\beta, \overline{E_\beta}]$ is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$ and $\overline{N_{\beta'}}$ is similar to a subgroup of a regular eight-group $\mathfrak{K}_8$. We have that, by the minimality of N_β/Z_β, $[N_\beta/Z_\beta, \overline{E_\beta}] = N_\beta/Z_\beta$ and $|N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| \leq |\overline{N_{\beta'}}|$. We will consider the three possibilities for $\overline{N_{\beta'}}$ separately. In the case when $|\overline{N_{\beta'}}| = 2$, $\overline{V_{\beta'}}$ induces transvections on N_β/Z_β and hence, by applying Theorem 2.3.8(ii) and Corollary 2.3.10, N_β/Z_β is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$ and $\overline{V_{\beta'}}$ is similar to a subgroup of a regular eight-group $\mathfrak{K}_8$. In the case when $|\overline{N_{\beta'}}| = 2^2$,

$$|N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(\overline{N_{\beta'}})| \leq |N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(\overline{V_{\beta'}})| \leq |\overline{N_{\beta'}}| = 2^2$$

and so $\overline{N_{\beta'}}$ induces 2-transvections on N_β/Z_β giving, by applying Lemma 2.3.7, N_β/Z_β has dimension at most 8. Then $\widetilde{N_\beta/Z_\beta} = N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(\overline{E_\beta})$ is a faithful irreducible $\mathbb{F}_2\overline{G_\beta}$-module of dimension at most 8 and so $\widetilde{N_\beta/Z_\beta}$ is isomorphic to either the natural module for $\overline{G_\beta}$ over $\mathbb{F}_2$ or a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$. In the former case, by applying Lemmas 5.1.9 and 2.3.3,

$$2^3 = |\widetilde{N_\beta/Z_\beta} \,/\, \mathrm{C}_{\widetilde{N_\beta/Z_\beta}}(\overline{N_{\beta'}})| \leq |N_\beta/Z_\beta \,/\, \mathrm{C}_{N_\beta/Z_\beta}(\overline{N_{\beta'}})| \leq 2^2$$

which is a contradiction. So $\widetilde{N_\beta/Z_\beta}$ is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$ and hence, by Theorem 2.4.5, N_β/Z_β is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$. We also have that, by Corollary 3.3.5(ii), $[N_\beta/Z_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = [N_\beta, V_{\beta'}, V_{\beta'}]Z_\beta/Z_\beta = Z_\beta/Z_\beta$ and so, by Lemma 2.4.7, $\overline{V_{\beta'}}$ is similar to a subgroup of either the group $\mathfrak{K}_4 \times \mathfrak{K}_4$ or a regular eight-group $\mathfrak{K}_8$. In the case when $|\overline{N_{\beta'}}| = 2^3$, $\overline{N_{\beta'}}$ is similar to a regular eight-group $\mathfrak{K}_8$ and so $\overline{V_{\beta'}} \leq \mathrm{C}_{\overline{G_\beta}}(\overline{N_{\beta'}}) = \overline{N_{\beta'}} \leq \overline{V_{\beta'}}$ giving $\overline{N_{\beta'}} = \overline{V_{\beta'}}$ and hence $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$ in the same way as before. □

The author expects that Case (II) of Lemma 5.2.7 ultimately leads to $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$. For the remainder of this chapter we will assume that $\eta_{G_\beta}(V_\beta/Z_\beta) = 1$.

Corollary 5.2.8 *The following hold:*

(i) $|Z_\beta| = 2$ *and* $|Z_\mu| = 4$*. In particular,* Z_β *is a trivial* $\mathbb{F}_2\overline{G_\beta}$*-module and* Z_μ *is isomorphic to the natural module for* G_μ/Q_μ *over* $\mathbb{F}_2$*.*

(ii) $V_\beta = W_\beta$*. In particular, if* K *is a proper* G_β*-subgroup of* V_β*, then* $\eta_{G_\beta}(K) = 0$*.*

(iii) $\mathrm{C}_{Y_\mu}(E_\beta) = Z_\beta$*.*

Proof (i) This follows from the lemma above together with Lemmas 5.2.6, 3.2.2(iii) and 2.1.6.

(ii) We have that, by Lemma 3.2.1(vi), $W_\beta \cap Z_\beta \neq 1$ and so, by part (i), $Z_\beta \leq W_\beta$. Also, as $G_{\mu\beta^*}$ acts on $\Gamma(\mu) \smallsetminus \{\beta^*\} = \{\beta, \beta'\}$ and by Corollary 3.3.5(i), $1 \neq [W_\beta, W_{\beta'}] \trianglelefteq G_{\mu\beta^*}$ and so, by Lemma 3.2.1, $[W_\beta, W_{\beta'}] \cap Z_{\beta^*} = [W_\beta, W_{\beta'}] \cap \Omega\,\mathrm{Z}(G_{\mu\beta^*}) \neq 1$ giving, as W_β and $W_{\beta'}$ are normal in Q_μ and by part (i), $Z_{\beta^*} \leq [W_\beta, W_{\beta'}] \leq W_\beta$. Then, by Lemma 3.2.2(iii), $Z_\mu = Z_\beta Z_{\beta^*} \leq W_\beta$ and so $V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq W_\beta \leq V_\beta$ giving $V_\beta = W_\beta$. In particular, if K is a proper G_β-subgroup of V_β, then $\eta_{G_\beta}(K) = 0$ because otherwise $\eta_{G_\beta}(V_\beta/K) = 0$ and so $V_\beta = W_\beta = [V_\beta, E_\beta] \leq K \leq V_\beta$ giving $K = V_\beta$ which is a contradiction.

(iii) By Lemma 5.2.1(iii), $\eta_{G_\mu}(Y_\mu/Z_\mu) = 0$ and so

$$[\mathrm{C}_{Y_\mu}(E_\beta)Z_\mu, E_\mu] \leq [Y_\mu, E_\mu] \leq Z_\mu \leq \mathrm{C}_{Y_\mu}(E_\beta)Z_\mu$$

giving $\mathrm{C}_{Y_\mu}(E_\beta)Z_\mu \trianglelefteq G_{\mu\beta}E_\mu = G_\mu$. Then

$$[\mathrm{C}_{Y_\mu}(E_\beta), Q_\mu] = [\mathrm{C}_{Y_\mu}(E_\beta), Q_\mu]\ [Z_\mu, Q_\mu] = [\mathrm{C}_{Y_\mu}(E_\beta)Z_\mu, Q_\mu] \trianglelefteq G_\mu$$

and, as $\mathrm{C}_{Y_\mu}(E_\beta) \trianglelefteq Q_\mu$, $[\mathrm{C}_{Y_\mu}(E_\beta), Q_\mu] \leq \mathrm{C}_{Y_\mu}(E_\beta)$. So $[\mathrm{C}_{Y_\mu}(E_\beta), Q_\mu]$ is normalized by both G_μ and E_β and so $[\mathrm{C}_{Y_\mu}(E_\beta), Q_\mu] = 1$. Then, by Lemmas 3.2.1(iii) and 3.2.2(v), $Z_\beta \leq \mathrm{C}_{Y_\mu}(E_\beta) \leq \Omega\,\mathrm{Z}(Q_\mu) = Z_\mu$ and hence, as Z_μ is not centralized by E_β and by part (i), $\mathrm{C}_{Y_\mu}(E_\beta) = Z_\beta$. □

Lemma 5.2.9 *Q_β/C_β may be embedded in the dual of V_β/Z_β.*

Proof Fix $q \in Q_\beta$. Then, by Lemma 5.2.1(ii), $[V_\beta, Q_\beta] = Z_\beta$ and so we may define the following map of $\mathbb{F}_2\overline{G_\beta}$-modules

$$\phi_q : V_\beta/Z_\beta \longrightarrow Z_\beta : vZ_\beta \longmapsto [v, q].$$

Let $v_1Z_\beta, v_2Z_\beta \in V_\beta/Z_\beta$. The map ϕ_q is well-defined because if $v_1Z_\beta = v_2Z_\beta$, then, by Lemma 3.2.1, $v_1v_2^{-1} \in Z_\beta \leq \mathrm{Z}(Q_\beta)$ and so $v_1^q(v_2^q)^{-1} = (v_1v_2^{-1})^q = v_1v_2^{-1}$ giving

$$\phi_q(v_1Z_\beta) = [v_1, q] = v_1^{-1}v_1^q = v_2^{-1}v_2^q = [v_2, q] = \phi_q(v_2Z_\beta).$$

We have that

$$\phi_q(v_1Z_\beta v_2Z_\beta) = \phi_q(v_1v_2Z_\beta) = [v_1v_2, q] = [v_1, q][v_2, q] = \phi_q(v_1Z_\beta)\phi_q(v_2Z_\beta)$$

and $\phi_q(Z_\beta) = [1, q] = 1$ and hence ϕ_q is a linear map. We may now define the following map of $\mathbb{F}_2\overline{G_\beta}$-modules

$$\phi : Q_\beta/C_\beta \longrightarrow \mathrm{Hom}(V_\beta/Z_\beta, Z_\beta) : qC_\beta \longmapsto \phi_q.$$

Let $q_1C_\beta, q_2C_\beta \in Q_\beta/C_\beta$. Then, by Lemma 3.2.1(ii), $C_\beta = \mathrm{C}_{Q_\beta}(V_\beta)$ and so

$$\begin{aligned} q_1C_\beta = q_2C_\beta \ &\Leftrightarrow\ q_1q_2^{-1} \in C_\beta \Leftrightarrow\ v^{q_1q_2^{-1}} = v \quad \text{for all } v \in V_\beta \\ &\Leftrightarrow\ [v, q_1] = v^{-1}v^{q_1} = v^{-1}v^{q_2} = [v, q_2] \quad \text{for all } v \in V_\beta \\ &\Leftrightarrow\ \phi_{q_1}(vZ_\beta) = \phi_{q_2}(vZ_\beta) \quad \text{for all } vZ_\beta \in V_\beta/Z_\beta \end{aligned}$$

$$\Leftrightarrow \ \phi(q_1C_\beta) = \phi_{q_1} = \phi_{q_2} = \phi(q_2C_\beta).$$

Hence, ϕ is a well-defined injective map. For each $vZ_\beta \in V_\beta/Z_\beta$, by Lemmas 5.2.1(ii) and 3.2.1, $[V_\beta, Q_\beta, Q_\beta] = [Z_\beta, Q_\beta] = 1$ and so, by Lemma 2.4.1(i),

$$\phi_{q_1q_2}(vZ_\beta) = [v, q_1q_2] = [v, q_1][v, q_2] = \phi_{q_1}(vZ_\beta)\phi_{q_2}(vZ_\beta)$$

and $\phi_1(vZ_\beta) = [v, 1] = 1$ giving $\phi_{q_1q_2} = \phi_{q_1}\phi_{q_2}$ and $\phi_1 = 1$. Then

$$\phi(q_1C_\beta q_2C_\beta) = \phi(q_1q_2C_\beta) = \phi_{q_1q_2} = \phi_{q_1}\phi_{q_2} = \phi(q_1C_\beta)\phi(q_2C_\beta)$$

and $\phi(C_\beta) = \phi_1 = 1$ and hence ϕ is a linear map. Now, let $qC_\beta \in Q_\beta/C_\beta$, $\overline{g} \in \overline{G_\beta}$ and $vZ_\beta \in V_\beta/Z_\beta$. Then, by Lemmas 5.2.1(ii) and 3.2.1, $[V_\beta, Q_\beta, G_\beta] = [Z_\beta, G_\beta] = 1$ so

$$\phi_{q^g}(vZ_\beta) = [v, q^g] = [v^{g^{-1}}, q]^g = [v^{g^{-1}}, q] = \phi_q(v^{g^{-1}}Z_\beta)$$

and so

$$\begin{aligned}\phi((qC_\beta)^{\overline{g}})(vZ_\beta) &= \phi(q^gC_\beta)(vZ_\beta) = \phi_{q^g}(vZ_\beta) = \phi_q(v^{g^{-1}}Z_\beta) = \phi_q((vZ_\beta)^{\overline{g^{-1}}}) \\ &= \phi(qC_\beta)((vZ_\beta)^{(\overline{g})^{-1}}) = \phi(qC_\beta)^{\overline{g}}(vZ_\beta)\end{aligned}$$

giving $\phi((qC_\beta)^{\overline{g}}) = \phi(qC_\beta)^{\overline{g}}$. We have shown that ϕ is a $\overline{G_\beta}$-monomorphism embedding Q_β/C_β in $\mathrm{Hom}(V_\beta/Z_\beta, Z_\beta)$. Also, by Corollary 5.2.8(i), Z_β is a trivial $\mathbb{F}_2\overline{G_\beta}$-module and so Z_β and $\mathbb{F}_2$ are isomorphic as $\mathbb{F}_2\overline{G_\beta}$-modules giving

$$\mathrm{Hom}(V_\beta/Z_\beta, Z_\beta) \cong_{\overline{G_\beta}} \mathrm{Hom}(V_\beta/Z_\beta, \mathbb{F}_2) = (V_\beta/Z_\beta)^*.$$

Therefore, Q_β/C_β may be embedded in the dual $(V_\beta/Z_\beta)^*$ of V_β/Z_β. □

Lemma 5.2.10 *The following hold:*

(i) $[V_\beta/Z_\beta, \overline{E_\beta}] = V_\beta/Z_\beta$ *and* $\mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta}) = \mathrm{C}_{V_\beta}(E_\beta)/Z_\beta$.

(ii) $V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ *may be regarded as a faithful irreducible* $\mathbb{F}_2\overline{G_\beta}$*-module.*

(iii) $V_\beta/Z_\beta\,/\,\mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta})$ *is isomorphic to* $V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ *and* $[Q_\beta/C_\beta, \overline{E_\beta}]$ *is isomorphic to* $(V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta))^*$.

Proof (i) We have that, by Corollary 5.2.8(ii),

$$[V_\beta/Z_\beta, \overline{E_\beta}] = [V_\beta/Z_\beta, E_\beta] = [V_\beta, E_\beta]Z_\beta/Z_\beta = W_\beta Z_\beta/Z_\beta = V_\beta/Z_\beta.$$

Let K_β be the inverse image of $\mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta})$. Then $\mathrm{C}_{V_\beta}(E_\beta)/Z_\beta \leq \mathrm{C}_{V_\beta/Z_\beta}(E_\beta) = K_\beta/Z_\beta$ and so $\mathrm{C}_{V_\beta}(E_\beta) \leq K_\beta$ and

$$[K_\beta, E_\beta]Z_\beta/Z_\beta = [K_\beta/Z_\beta, E_\beta] = [K_\beta/Z_\beta, \overline{E_\beta}] = Z_\beta/Z_\beta$$

and so $[K_\beta, E_\beta] \leq Z_\beta$. We also have that, as K_β/Z_β is a submodule of V_β/Z_β,

$$[K_\beta, G_\beta]Z_\beta/Z_\beta = [K_\beta/Z_\beta, G_\beta] = [K_\beta/Z_\beta, \overline{G_\beta}] \leq K_\beta/Z_\beta$$

so $[K_\beta, G_\beta] \leq K_\beta$ and so we may regard K_β as a G_β-group. Then, by Lemma 3.2.1(iii), $[K_\beta, E_\beta] = [K_\beta, E_\beta, E_\beta] \leq [Z_\beta, E_\beta] = 1$ and so $K_\beta \leq \mathrm{C}_{V_\beta}(E_\beta)$. Thus, $\mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta}) = K_\beta/Z_\beta = \mathrm{C}_{V_\beta}(E_\beta)/Z_\beta$.

(ii) The group $V_\beta/\mathrm{C}_{V_\beta}(E_\beta)$ is an elementary abelian $2, G_\beta$-group and, by Lemmas 5.2.1(ii) and 3.2.1(iii), $[V_\beta, Q_\beta] = Z_\beta \leq \mathrm{C}_{V_\beta}(E_\beta)$ and so $Q_\beta \leq \mathrm{C}_{G_\beta}(V_\beta/\mathrm{C}_{V_\beta}(E_\beta))$. So we may regard $V_\beta/\mathrm{C}_{V_\beta}(E_\beta)$ as an $\mathbb{F}_2\overline{G_\beta}$-module and, by Lemma 5.2.7,

$$\eta_{\overline{G_\beta}}(V_\beta/\mathrm{C}_{V_\beta}(E_\beta)) = \eta_{G_\beta}(V_\beta/\mathrm{C}_{V_\beta}(E_\beta)) = \eta_{G_\beta}(V_\beta/Z_\beta) - \eta_{G_\beta}(\mathrm{C}_{V_\beta}(E_\beta)/Z_\beta) = 1$$

and so, by applying Lemma 2.3.1, $V_\beta/\mathrm{C}_{V_\beta}(E_\beta)$ is a faithful module. Now, let $X_\beta/\mathrm{C}_{V_\beta}(E_\beta)$ be a proper submodule of $V_\beta/\mathrm{C}_{V_\beta}(E_\beta)$. Then X_β is a proper G_β-subgroup of V_β and so, by applying Corollary 5.2.8(ii), $\eta_{G_\beta}(X_\beta) = 0$ giving $X_\beta = \mathrm{C}_{V_\beta}(E_\beta)$. Thus, $V_\beta/\mathrm{C}_{V_\beta}(E_\beta)$ is an irreducible module.

(iii) By part (i), $\mathrm{C}_{V_\beta/Z_\beta}(E_\beta) = \mathrm{C}_{V_\beta}(E_\beta)/Z_\beta$ so

$$V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(E_\beta) = V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta}(E_\beta)/Z_\beta \cong_{G_\beta} V_\beta/\mathrm{C}_{V_\beta}(E_\beta)$$

and hence $V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta}) \cong_{\overline{G_\beta}} V_\beta/\mathrm{C}_{V_\beta}(E_\beta)$. Now, we have that

$$[(V_\beta/Z_\beta)^*, \overline{E_\beta}] \cong_{\overline{G_\beta}} (V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta}))^* \cong_{\overline{G_\beta}} (V_\beta/\mathrm{C}_{V_\beta}(E_\beta))^*$$

and so, by part (ii), $[(V_\beta/Z_\beta)^*, \overline{E_\beta}]$ is irreducible. Also, by Lemmas 5.2.3 and 5.2.9, $[Q_\beta/C_\beta, \overline{E_\beta}]$ is non-zero and embeds in $[(V_\beta/Z_\beta)^*, \overline{E_\beta}]$ and hence

$$[Q_\beta/C_\beta, \overline{E_\beta}] \cong_{\overline{G_\beta}} [(V_\beta/Z_\beta)^*, \overline{E_\beta}] \cong_{\overline{G_\beta}} (V_\beta/\mathrm{C}_{V_\beta}(E_\beta))^*. \qquad \square$$

One consequence of the lemma above is that we now know that $\eta_{G_\beta}(Q_\beta/C_\beta) = 1$.

Lemma 5.2.11 $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ *if and only if* $Y_\mu = V_\beta \cap C_{\beta'}$.

Proof Firstly, assume that $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ and suppose, for a contradiction, that $Y_\mu < V_\beta \cap C_{\beta'}$. By Lemma 5.2.1(ii),

$$[V_\beta \cap C_{\beta'}, Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta \leq Z_\mu \leq Y_\mu \leq V_\beta \cap C_{\beta'}$$

and so $V_\beta \cap C_{\beta'} \trianglelefteq Q_\mu Q_\beta = G_{\mu\beta}$. So $(V_\beta \cap C_{\beta'})/Y_\mu$ is a non-trivial $G_{\mu\beta}$-group and, as $G_{\mu\beta}$ is a 2-group, $\mathrm{C}_{(V_\beta \cap C_{\beta'})/Y_\mu}(G_{\mu\beta}) \neq 1$. Let $T_{\mu\beta}$ be the inverse image of $\mathrm{C}_{(V_\beta \cap C_{\beta'})/Y_\mu}(G_{\mu\beta})$. We have that $[T_{\mu\beta}, G_{\mu\beta}]Y_\mu/Y_\mu = [T_{\mu\beta}/Y_\mu, G_{\mu\beta}] = Y_\mu/Y_\mu$ and so $[T_{\mu\beta}, G_{\mu\beta}] \leq Y_\mu$. In particular, $T_{\mu\beta} \trianglelefteq G_{\mu\beta}$. Define $T_{\mu\beta'} = T_{\mu\beta}^{g'}$ and $T_{\mu\beta^*} = T_{\mu\beta}^{g^*}$. Then, by applying Lemma 3.3.2, $T_{\mu\beta}^{g} = T_{\mu\beta\cdot g}$ for all $g \in G_\mu$. So we may define the groups $F_\mu = \langle T_{\mu\beta}^{G_\mu} \rangle = \langle T_{\mu\beta}, T_{\mu\beta'}, T_{\mu\beta^*} \rangle$ and $H_\beta = \langle F_\mu^{G_\beta} \rangle$. Then $F_\mu = \langle T_{\mu\beta}^{G_\mu} \rangle \leq \langle V_\beta^{G_\mu} \rangle = U_\mu$ and, as $Z_\mu \leq Y_\mu \leq T_{\mu\beta} \leq F_\mu$, $V_\beta = \langle Z_\mu^{G_\beta} \rangle \leq \langle F_\mu^{G_\beta} \rangle = H_\beta$. We have that $T_{\mu\beta} \leq C_\beta \cap C_{\beta'}$ and, as $T_{\mu\beta} \trianglelefteq G_{\mu\beta}$,

$$T_{\mu\beta} = T_{\mu\beta}^{g'^*} \leq C_{\beta'}^{g'^*} = C_{\beta'\cdot g'^*} = C_{\beta^*}$$

and so, as G_μ acts on $\Gamma(\mu) = \{\beta, \beta', \beta^*\}$, $T_{\mu\beta} \leq C_\beta \cap C_{\beta'} \cap C_{\beta^*} \trianglelefteq G_\mu$. Then $F_\mu = \langle T_{\mu\beta}^{G_\mu} \rangle \leq C_\beta \cap C_{\beta'} \cap C_{\beta^*} \leq C_\beta$ and so $H_\beta = \langle F_\mu^{G_\beta} \rangle \leq C_\beta$. Hence, as $V_\beta \leq H_\beta \leq C_\beta$, $\eta_{G_\beta}(H_\beta/V_\beta) \leq \eta_{G_\beta}(C_\beta/V_\beta) = 0$. Then

$$[V_\beta F_\mu, E_\beta] \leq [H_\beta, E_\beta] \leq V_\beta \leq V_\beta F_\mu$$

so that $V_\beta F_\mu \trianglelefteq G_{\mu\beta} E_\beta = G_\beta$ and so

$$H_\beta = \langle F_\mu^{G_\beta} \rangle \leq \langle (V_\beta F_\mu)^{G_\beta} \rangle = V_\beta F_\mu \leq H_\beta$$

giving $H_\beta = V_\beta F_\mu$. Define $K_\beta = [H_\beta, Q_\beta]$. Then, by Lemma 5.2.1(ii) and applying Lemma 3.2.2(iv), $[V_\beta, Q_\beta] = Z_\beta = [Z_\mu, Q_\beta] \leq [F_\mu, Q_\beta]$ and so, as $H_\beta = V_\beta F_\mu$,

$$K_\beta = [V_\beta F_\mu, Q_\beta] = [V_\beta, Q_\beta]\ [F_\mu, Q_\beta] = [F_\mu, Q_\beta].$$

We have that $V_\beta \not\leq K_\beta$ because otherwise $V_\beta \leq K_\beta = [F_\mu, Q_\beta] \leq F_\mu$ so that $H_\beta = V_\beta F_\mu = F_\mu$, but then $H_\beta \trianglelefteq \langle G_\beta, G_\mu \rangle$ giving $H_\beta = 1$ which is a contradiction. So $K_\beta \cap V_\beta$ is a proper G_β-subgroup of V_β and so, by applying Corollary 5.2.8(ii), $\eta_{G_\beta}(K_\beta \cap V_\beta) = 0$. Then, as

$K_\beta/(K_\beta \cap V_\beta) \cong_{G_\beta} K_\beta V_\beta/V_\beta \leq_{G_\beta} H_\beta/V_\beta$,

$$\eta_{G_\beta}(K_\beta/(K_\beta \cap V_\beta)) = \eta_{G_\beta}(K_\beta V_\beta/V_\beta) \leq \eta_{G_\beta}(H_\beta/V_\beta) = 0$$

so that

$$\eta_{G_\beta}(K_\beta) = \eta_{G_\beta}(K_\beta/(K_\beta \cap V_\beta)) + \eta_{G_\beta}(K_\beta \cap V_\beta) = 0$$

and so, by Corollary 5.2.8(iii), $Z_\beta \leq K_\beta \cap Y_\mu \leq \mathrm{C}_{Y_\mu}(E_\beta) = Z_\beta$ giving $K_\beta \cap Y_\mu = Z_\beta$. Now, $[T_{\mu\beta}, Q_\mu] \leq [T_{\mu\beta}, G_{\mu\beta}] \leq Y_\mu$ and so, as $Q_\mu \trianglelefteq G_\mu$,

$$[F_\mu, Q_\mu] = [\langle T_{\mu\beta}^{G_\mu} \rangle, Q_\mu] = \langle [T_{\mu\beta}, Q_\mu]^{G_\mu} \rangle \leq \langle Y_\mu^{G_\mu} \rangle = Y_\mu$$

giving

$$[F_\mu, Q_\mu \cap Q_\beta] \leq [F_\mu, Q_\mu] \cap [F_\mu, Q_\beta] \leq Y_\mu \cap K_\beta = Z_\beta.$$

Then, by Lemma 5.2.1(ii),

$$\begin{aligned} [H_\beta, Q_\mu \cap Q_\beta] &= [V_\beta F_\mu, Q_\mu \cap Q_\beta] = [V_\beta, Q_\mu \cap Q_\beta]\ [F_\mu, Q_\mu \cap Q_\beta] \\ &\leq [V_\beta, Q_\beta]\ [F_\mu, Q_\mu \cap Q_\beta] \leq Z_\beta \end{aligned}$$

and so $Q_\mu \cap Q_\beta \leq \mathrm{C}_{Q_\beta}(H_\beta/Z_\beta)$. We have that, by Lemma 5.2.1(ii),

$$|Q_\beta : \mathrm{C}_{Q_\beta}(H_\beta/Z_\beta)|\ |\mathrm{C}_{Q_\beta}(H_\beta/Z_\beta) : (Q_\mu \cap Q_\beta)| = |Q_\beta : (Q_\mu \cap Q_\beta)| = 2$$

and so either $\mathrm{C}_{Q_\beta}(H_\beta/Z_\beta) = Q_\beta$ or $\mathrm{C}_{Q_\beta}(H_\beta/Z_\beta) = Q_\mu \cap Q_\beta$. In the former case,

$$[F_\mu, Q_\beta] = K_\beta = [H_\beta, Q_\beta] = Z_\beta \leq Z_\mu \leq Y_\mu$$

so $[T_{\mu\beta}, Q_{\beta'}] \leq [F_\mu, Q_{\beta'}] \leq Y_\mu \leq T_{\mu\beta}$ and so, as $T_{\mu\beta} \trianglelefteq G_{\mu\beta} = Q_\mu Q_\beta$ and by Lemma 5.2.1(i), $T_{\mu\beta} \trianglelefteq Q_\mu \langle Q_\beta, Q_{\beta'} \rangle = G_\mu$ giving $F_\mu = \langle T_{\mu\beta}^{G_\mu} \rangle = T_{\mu\beta}$. But then, by Lemma 5.2.1(iii), $T_{\mu\beta} = F_\mu = T_{\mu\beta'} \leq V_\beta \cap V_{\beta'} = Y_\mu$ which is a contradiction. In the latter case, $Q_\mu \cap Q_\beta = \mathrm{C}_{Q_\beta}(H_\beta/Z_\beta) \trianglelefteq G_\beta$ which contradicts Corollary 5.2.4(i). Thus, $Y_\mu = V_\beta \cap C_{\beta'}$.

Secondly, assume that $Y_\mu = V_\beta \cap C_{\beta'}$. Then, as C_β and $V_{\beta'}$ are normal in Q_μ, $[C_\beta, V_{\beta'}] \leq C_\beta \cap V_{\beta'} = Y_\mu \leq V_\beta$ and so, by Corollary 3.3.5(i),

$$[C_\beta, E_\beta] \leq [C_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [C_\beta, V_{\beta'}]^{G_\beta} \rangle \leq \langle V_\beta^{G_\beta} \rangle = V_\beta.$$

Thus, $\eta_{G_\beta}(C_\beta/V_\beta) = 0$. □

§5.3 Analysing the structure of the module V_β/Z_β

In this section we will analyse the extent to which the structure of the module V_β/Z_β is determined by the action of the groups $\overline{V_{\beta'}}$, $\overline{C_{\beta'}}$, $\overline{Q_{\beta'} \cap Q_\mu}$ and $\overline{G_{\mu\beta}}$ on this module. The results in this section hold under the weaker hypothesis $P_2/Q_2 \in \mathfrak{X}$ and $|Z| = 2$.

Lemma 5.3.1 *$\overline{V_{\beta'}}$, $\overline{C_{\beta'}}$ and $\overline{Q_{\beta'} \cap Q_\mu}$ are normal subgroups of $\overline{G_{\mu\beta}}$ with $\overline{V_{\beta'}} \leq \overline{C_{\beta'}} \leq \overline{Q_{\beta'} \cap Q_\mu} \leq \mathrm{C}_{\overline{G_{\mu\beta}}}(\overline{V_{\beta'}})$. Moreover, the following hold:*

(i) $\mathrm{C}_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) = \mathrm{C}_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$ and $\mathrm{C}_{Y_\mu/Z_\beta}(\overline{E_\beta}) = Z_\beta/Z_\beta$.

(ii) If $\overline{V_{\beta'}} = \overline{C_{\beta'}}$, then $\eta_{G_\beta}(C_\beta/V_\beta) = 0$.

(iii) If $\overline{C_{\beta'}} = \overline{Q_{\beta'} \cap Q_\mu}$, then $\overline{V_{\beta'}}$ induces central transvections on Q_β/C_β.

Proof As $V_{\beta'}$ is normal in Q_μ, $\overline{V_{\beta'}} = V_{\beta'}Q_\beta/Q_\beta$ is normal in $Q_\mu Q_\beta/Q_\beta = G_{\mu\beta}/Q_\beta = \overline{G_{\mu\beta}}$ and, similarly, $\overline{C_{\beta'}}$ and $\overline{Q_{\beta'} \cap Q_\mu}$ are normal in $\overline{G_{\mu\beta}}$. We have that, by Lemma 5.2.1(ii), $[V_{\beta'}, Q_{\beta'} \cap Q_\mu] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'} \leq Q_\beta$ so that $[\overline{V_{\beta'}}, \overline{Q_{\beta'} \cap Q_\mu}] = \overline{[V_{\beta'}, Q_{\beta'} \cap Q_\mu]} = 1$ and hence, as $V_{\beta'} \leq C_{\beta'} \leq Q_{\beta'} \cap Q_\mu \leq G_{\mu\beta}$, $\overline{V_{\beta'}} \leq \overline{C_{\beta'}} \leq \overline{Q_{\beta'} \cap Q_\mu} \leq \mathrm{C}_{\overline{G_{\mu\beta}}}(\overline{V_{\beta'}})$.

(i) Let $K_{\mu\beta}$ be the inverse image of $\mathrm{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu})$. Then $[Z_\mu, Q_{\beta'} \cap Q_\mu] \leq [Z_\mu, Q_\mu] = 1$ and so $Z_\mu/Z_\beta \leq \mathrm{C}_{V_\beta}(Q_{\beta'} \cap Q_\mu)/Z_\beta \leq \mathrm{C}_{V_\beta/Z_\beta}(Q_{\beta'} \cap Q_\mu) = K_{\mu\beta}/Z_\beta$ giving $Z_\mu \leq K_{\mu\beta}$. We also have that, as $\overline{Q_{\beta'} \cap Q_\mu}$ is normal in $\overline{G_{\mu\beta}}$, $K_{\mu\beta}/Z_\beta$ is a $\overline{G_{\mu\beta}}$-submodule of V_β/Z_β so $[K_{\mu\beta}, G_{\mu\beta}]Z_\beta/Z_\beta = [K_{\mu\beta}/Z_\beta, \overline{G_{\mu\beta}}] \leq K_{\mu\beta}/Z_\beta$ and so $[K_{\mu\beta}, G_{\mu\beta}] \leq K_{\mu\beta}$ giving $K_{\mu\beta} \trianglelefteq G_{\mu\beta}$. Set $K_\mu = K_{\mu\beta} \cap Y_\mu$. Then, by Lemma 5.2.1(iii), $[K_\mu, E_\mu] \leq [Y_\mu, E_\mu] \leq Z_\mu \leq K_\mu$ and so $K_\mu \trianglelefteq G_{\mu\beta}E_\mu = G_\mu$. By definition of $K_{\mu\beta}$,

$$[K_\mu, Q_{\beta'} \cap Q_\mu]Z_\beta/Z_\beta \leq [K_{\mu\beta}, Q_{\beta'} \cap Q_\mu]Z_\beta/Z_\beta = [K_{\mu\beta}/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] = Z_\beta/Z_\beta$$

so $[K_\mu, Q_{\beta'} \cap Q_\mu] \leq Z_\beta$ and, by Lemma 5.2.1(ii),

$$[K_\mu, Q_{\beta'} \cap Q_\mu] \leq [Y_\mu, Q_{\beta'}] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$$

giving, by Lemma 3.2.2(iii), $[K_\mu, Q_{\beta'} \cap Q_\mu] \leq Z_\beta \cap Z_{\beta'} = 1$. So $Q_{\beta'} \cap Q_\mu \leq \mathrm{C}_{Q_{\beta'}}(K_\mu)$ and, by Lemma 3.2.1(i), $\mathrm{C}_{Q_{\beta'}}(K_\mu) \leq \mathrm{C}_{G_\mu}(Z_\mu) = Q_\mu$ and hence $\mathrm{C}_{Q_{\beta'}}(K_\mu) = Q_{\beta'} \cap Q_\mu$. Now, as $K_\mu \trianglelefteq Q_{\beta'}$ and by Lemma 5.2.1(ii), K_μ is an abelian $(Q_{\beta'}/\mathrm{C}_{Q_{\beta'}}(K_\mu))$-group and $|Q_{\beta'}/\mathrm{C}_{Q_{\beta'}}(K_\mu)| = |Q_{\beta'}/(Q_{\beta'} \cap Q_\mu)| = 2$. Also, by applying Corollary 5.2.8(ii),

$\eta_{G_{\beta'}}(\mathrm{C}_{V_{\beta'}}(Q_{\beta'})) = 0$ and so, by Corollary 5.2.8(iii),

$$Z_{\beta'} \leq \mathrm{C}_{K_\mu}(Q_{\beta'}) \leq Y_\mu \cap \mathrm{C}_{V_{\beta'}}(Q_{\beta'}) \leq Y_\mu \cap \mathrm{C}_{V_{\beta'}}(E_{\beta'}) = \mathrm{C}_{Y_\mu}(E_{\beta'}) = Z_{\beta'}$$

giving $\mathrm{C}_{K_\mu}(Q_{\beta'}) = Z_{\beta'}$. So, as $[K_\mu, Q_{\beta'}] \leq [Y_\mu, Q_{\beta'}] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$,

$$\begin{aligned}|K_\mu/Z_{\beta'}| &= |K_\mu/\mathrm{C}_{K_\mu}(Q_{\beta'})| = |K_\mu/\mathrm{C}_{K_\mu}(Q_{\beta'}/\mathrm{C}_{Q_{\beta'}}(K_\mu))| \\ &= |[K_\mu, Q_{\beta'}/\mathrm{C}_{Q_{\beta'}}(K_\mu)]| = |[K_\mu, Q_{\beta'}]| \leq |Z_{\beta'}|\end{aligned}$$

giving $|K_\mu| \leq |Z_{\beta'}|^2 = |Z_\mu|$. Hence, $K_\mu = Z_\mu$. We have shown that

$$\begin{aligned}\mathrm{C}_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) &= \mathrm{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) \cap Y_\mu/Z_\beta = K_{\mu\beta}/Z_\beta \cap Y_\mu/Z_\beta \\ &= (K_{\mu\beta} \cap Y_\mu)/Z_\beta = K_\mu/Z_\beta = Z_\mu/Z_\beta\end{aligned}$$

and so $Z_\mu/Z_\beta \leq \mathrm{C}_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) \leq \mathrm{C}_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$. Thus, $\mathrm{C}_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) = \mathrm{C}_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$. Also, by Lemma 5.2.10(i) and Corollary 5.2.8(iii),

$$\begin{aligned}\mathrm{C}_{Y_\mu/Z_\beta}(\overline{E_\beta}) &= \mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta}) \cap Y_\mu/Z_\beta = \mathrm{C}_{V_\beta}(E_\beta)/Z_\beta \cap Y_\mu/Z_\beta = (\mathrm{C}_{V_\beta}(E_\beta) \cap Y_\mu)/Z_\beta \\ &= \mathrm{C}_{Y_\mu}(E_\beta)/Z_\beta = Z_\beta/Z_\beta.\end{aligned}$$

(ii) Let $\overline{V_{\beta'}} = \overline{C_{\beta'}}$. Then $C_{\beta'}Q_\beta = V_{\beta'}Q_\beta$ and so, by Dedekind's Law,

$$C_\beta = C_\beta \cap C_\beta Q_{\beta'} = C_\beta \cap V_\beta Q_{\beta'} = V_\beta(C_\beta \cap Q_{\beta'})$$

and, by Lemma 5.2.1(ii), $[V_{\beta'}, C_\beta \cap Q_{\beta'}] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'} \leq Z_\mu \leq V_\beta$. We have that

$$[V_{\beta'}, C_\beta] = [V_{\beta'}, V_\beta(C_\beta \cap Q_{\beta'})] = [V_{\beta'}, V_\beta]\ [V_{\beta'}, C_\beta \cap Q_{\beta'}] \leq V_\beta$$

and so, by Corollary 3.3.5(i),

$$[C_\beta, E_\beta] \leq [C_\beta, \langle V_{\beta'}^{G_\beta}\rangle] = \langle [C_\beta, V_{\beta'}]^{G_\beta}\rangle \leq \langle V_\beta^{G_\beta}\rangle = V_\beta.$$

Thus, $\eta_{G_\beta}(C_\beta/V_\beta) = 0$.

(iii) Let $\overline{C_{\beta'}} = \overline{Q_{\beta'} \cap Q_\mu}$. Then $(Q_{\beta'} \cap Q_\mu)Q_\beta = C_{\beta'}Q_\beta$ and so, by Dedekind's Law,

$$Q_\beta \cap Q_\mu = (Q_\beta \cap Q_\mu) \cap (Q_\beta \cap Q_\mu)Q_{\beta'} = (Q_\beta \cap Q_\mu) \cap C_\beta Q_{\beta'} = C_\beta(Q_\beta \cap Q_\mu \cap Q_{\beta'})$$

and, by Lemma 5.2.1(ii), $[Q_\beta \cap Q_\mu \cap Q_{\beta'}, V_{\beta'}] \leq [Q_{\beta'}, V_{\beta'}] = Z_{\beta'} \leq Z_\mu \leq V_\beta \leq C_\beta$. We have that

$$[Q_\beta \cap Q_\mu, V_{\beta'}] = [C_\beta(Q_\beta \cap Q_\mu \cap Q_{\beta'}), V_{\beta'}] = [C_\beta, V_{\beta'}]\ [Q_\beta \cap Q_\mu \cap Q_{\beta'}, V_{\beta'}] \leq C_\beta$$

and so $(Q_\beta \cap Q_\mu)/C_\beta \leq \mathrm{C}_{Q_\beta/C_\beta}(V_{\beta'})$. Then, by Lemma 5.2.1(ii),

$$|Q_\beta/C_\beta \,/\, \mathrm{C}_{Q_\beta/C_\beta}(\overline{V_{\beta'}})| = |Q_\beta/C_\beta \,/\, \mathrm{C}_{Q_\beta/C_\beta}(V_{\beta'})| \leq |Q_\beta/C_\beta \,/(Q_\beta \cap Q_\mu)/C_\beta| = |Q_\beta/(Q_\beta \cap Q_\mu)| = 2$$

and hence $\overline{V_{\beta'}}$ induces central transvections on Q_β/C_β. □

Lemma 5.3.2 *$\overline{V_{\beta'}}$ acts quadratically on V_β/Z_β. Moreover, the following hold:*

(i) $|\mathrm{C}_{[V_\beta/Z_\beta,\overline{V_{\beta'}}]}(\overline{G_{\mu\beta}})| = 2$ *and* $|\mathrm{C}_{[V_\beta/Z_\beta,\overline{V_{\beta'}}]}(\overline{E_\beta})| = 1$.

(ii) $|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{x})| \leq 2\,|\overline{V_{\beta'}}|$ *for all* $\overline{x} \in \overline{V_{\beta'}}$.

(iii) $\overline{C_{\beta'}} \leq \mathrm{C}_{\overline{G_{\mu\beta}}}([V_\beta/Z_\beta, \overline{V_{\beta'}}])$.

Proof As (β', β) is a beta pair, $\overline{V_{\beta'}}$ is a non-trivial subgroup of $\overline{G_\beta}$ and, by Corollary 3.3.5(ii), $[V_\beta/Z_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = [V_\beta, V_{\beta'}, V_{\beta'}]Z_\beta/Z_\beta = Z_\beta/Z_\beta$. Thus, $\overline{V_{\beta'}}$ acts quadratically on V_β/Z_β.

(i) As $G_{\mu\beta^*}$ acts on $\Gamma(\mu) \setminus \{\beta^*\} = \{\beta, \beta'\}$ and by Corollary 3.3.5(i), $1 \neq [V_\beta, V_{\beta'}] \trianglelefteq G_{\mu\beta^*}$ and so, by Lemma 3.2.1, $[V_\beta, V_{\beta'}] \cap Z_{\beta^*} = [V_\beta, V_{\beta'}] \cap \Omega\, \mathrm{Z}(G_{\mu\beta^*}) \neq 1$ giving, as $|Z_{\beta^*}| = 2$, $Z_{\beta^*} \leq [V_\beta, V_{\beta'}]$. Then, by Lemma 3.2.2(iii), $Z_\mu = Z_{\beta^*}Z_\beta \leq [V_\beta, V_{\beta'}]Z_\beta$ and, as V_β and $V_{\beta'}$ are normal in Q_μ and by Lemma 5.2.1(iii), $[V_\beta, V_{\beta'}]Z_\beta \leq V_\beta \cap V_{\beta'} = Y_\mu$ and so $Z_\mu/Z_\beta \leq [V_\beta, V_{\beta'}]Z_\beta/Z_\beta = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \leq Y_\mu/Z_\beta$. So, by Lemma 5.3.1(i),

$$\mathrm{C}_{[V_\beta/Z_\beta,\overline{V_{\beta'}}]}(\overline{G_{\mu\beta}}) = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \cap \mathrm{C}_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \cap Z_\mu/Z_\beta = Z_\mu/Z_\beta$$

and hence, by Corollary 5.2.8(i), $|\mathrm{C}_{[V_\beta/Z_\beta,\overline{V_{\beta'}}]}(\overline{G_{\mu\beta}})| = 2$. Also, by Lemma 5.3.1(i),

$$\mathrm{C}_{[V_\beta/Z_\beta,\overline{V_{\beta'}}]}(\overline{E_\beta}) = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \cap \mathrm{C}_{Y_\mu/Z_\beta}(\overline{E_\beta}) = [V_\beta/Z_\beta, \overline{V_{\beta'}}] \cap Z_\beta/Z_\beta = Z_\beta/Z_\beta$$

and hence $|\mathrm{C}_{[V_\beta/Z_\beta,\overline{V_{\beta'}}]}(\overline{E_\beta})| = 1$.

(ii) By Lemma 5.2.1(ii), $[V_\beta \cap Q_{\beta'}, V_{\beta'}] \leq [Q_{\beta'}, V_{\beta'}] = Z_{\beta'}$ and so, by Lemma 3.2.2(iii),

$$[(V_\beta \cap Q_{\beta'})/Z_\beta, \overline{V_{\beta'}}] = [V_\beta \cap Q_{\beta'}, V_{\beta'}]Z_\beta/Z_\beta \leq Z_{\beta'}Z_\beta/Z_\beta = Z_\mu/Z_\beta.$$

In particular, $[(V_\beta \cap Q_{\beta'})/Z_\beta, \overline{V_{\beta'}}] \leq Z_\mu/Z_\beta \leq (V_\beta \cap Q_{\beta'})/Z_\beta$ and so $(V_\beta \cap Q_{\beta'})/Z_\beta$ is a $\overline{V_{\beta'}}$-submodule of V_β/Z_β. Fix $\overline{x} \in \overline{V_{\beta'}}$. Then, by Corollary 5.2.8(i),

$$|[(V_\beta \cap Q_{\beta'})/Z_\beta, \overline{x}]| \leq |[(V_\beta \cap Q_{\beta'})/Z_\beta, \overline{V_{\beta'}}]| \leq |Z_\mu/Z_\beta| = 2$$

and so, as $\mathrm{C}_{(V_\beta\cap Q_{\beta'})/Z_\beta}(\overline{x}) \le \mathrm{C}_{V_\beta/Z_\beta}(\overline{x})$ and by Corollary 3.3.5(iii),

$$\begin{aligned}|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{x})| &\le |V_\beta/Z_\beta \,/\, \mathrm{C}_{(V_\beta\cap Q_{\beta'})/Z_\beta}(\overline{x})| \\ &= |V_\beta/Z_\beta \,/(V_\beta\cap Q_{\beta'})/Z_\beta|\; |(V_\beta\cap Q_{\beta'})/Z_\beta \,/\, \mathrm{C}_{(V_\beta\cap Q_{\beta'})/Z_\beta}(\overline{x})| \\ &= |V_\beta/(V_\beta\cap Q_{\beta'})|\; |[(V_\beta\cap Q_{\beta'})/Z_\beta,\overline{x}]| \le 2\,|V_\beta Q_{\beta'}/Q_{\beta'}| \\ &= 2\,|V_{\beta'}Q_\beta/Q_\beta| = 2\,|\overline{V_{\beta'}}|.\end{aligned}$$

Thus, $|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{x})| \le 2\,|\overline{V_{\beta'}}|$ for all $\overline{x}\in\overline{V_{\beta'}}$.

(iii) We have that, by Corollary 3.3.5(ii), $[V_\beta/Z_\beta,\overline{V_{\beta'}},\overline{C_{\beta'}}] = [V_\beta,V_{\beta'},C_{\beta'}]Z_\beta/Z_\beta = Z_\beta/Z_\beta$ and hence $\overline{C_{\beta'}} \le \mathrm{C}_{\overline{G_{\mu\beta}}}([V_\beta/Z_\beta,\overline{V_{\beta'}}])$. □

It follows from V_β/Z_β being a module over $\mathbb{F}_2$ and $\overline{V_{\beta'}}$ being an elementary abelian 2-group that $|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{x})| \le |V_\beta/Z_\beta|^{(1/2)}$ for all $\overline{x}\in\overline{V_{\beta'}}$. One of the consequences of the following lemma is that we will be able to slightly lower this bound.

Lemma 5.3.3 $|V_\beta/Z_\beta| \ge |\overline{V_{\beta'}}|\,|[(Q_\beta\cap Q_\mu)/C_\beta,\overline{V_{\beta'}}]|\,|(V_\beta\cap C_{\beta'})/Z_\beta|$ *where* $(Q_\beta\cap Q_\mu)/C_\beta$ *is a* $\overline{G_{\mu\beta}}$*-submodule of* Q_β/C_β *of codimension one with* $|[(Q_\beta\cap Q_\mu)/C_\beta,\overline{V_{\beta'}}]| \ge \frac{1}{2}|[V_\beta/Z_\beta,\overline{x}]|$ *for all* $\overline{x}\in\overline{V_{\beta'}}$*. In particular,* $|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{x})| \le (4\,|V_\beta/Z_\beta|)^{(1/3)}$ *for all* $\overline{x}\in\overline{V_{\beta'}}$*.*

Proof As $Q_\beta\cap Q_\mu$ and $V_{\beta'}$ are normal in Q_μ, $[Q_\beta\cap Q_\mu,V_{\beta'}] \le (Q_\beta\cap Q_\mu)\cap V_{\beta'} = V_{\beta'}\cap Q_\beta$ and so, regarding $(Q_\beta\cap Q_\mu)/C_\beta$ as a $G_{\mu\beta}$-group,

$$[(Q_\beta\cap Q_\mu)/C_\beta,V_{\beta'}] = [Q_\beta\cap Q_\mu,V_{\beta'}]C_\beta/C_\beta \le (V_{\beta'}\cap Q_\beta)C_\beta/C_\beta$$

giving

$$\begin{aligned}|[(Q_\beta\cap Q_\mu)/C_\beta,\overline{V_{\beta'}}]| &= |[(Q_\beta\cap Q_\mu)/C_\beta,V_{\beta'}]| \le |(V_{\beta'}\cap Q_\beta)C_\beta/C_\beta| \\ &= |(V_{\beta'}\cap Q_\beta)/(V_{\beta'}\cap C_\beta)| = |(V_\beta\cap Q_{\beta'})/(V_\beta\cap C_{\beta'})|.\end{aligned}$$

Also, by Corollary 3.3.5(iii), $|V_\beta/(V_\beta\cap Q_{\beta'})| = |V_\beta Q_{\beta'}/Q_{\beta'}| = |V_{\beta'}Q_\beta/Q_\beta| = |\overline{V_{\beta'}}|$. Hence,

$$\begin{aligned}|V_\beta/Z_\beta| &= |V_\beta/(V_\beta\cap Q_{\beta'})|\; |(V_\beta\cap Q_{\beta'})/(V_\beta\cap C_{\beta'})|\; |(V_\beta\cap C_{\beta'})/Z_\beta| \\ &\ge |\overline{V_{\beta'}}|\; |[(Q_\beta\cap Q_\mu)/C_\beta,\overline{V_{\beta'}}]|\; |(V_\beta\cap C_{\beta'})/Z_\beta|.\end{aligned}$$

Fix $\overline{x}\in\overline{V_{\beta'}}$. Then, by Lemma 5.2.10(iii), $V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta}) \cong_{\overline{G_\beta}} V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ and so, by Lemma 5.3.2(i),

$$|[V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta),\overline{x}]| = |[V_\beta/Z_\beta\,/\,\mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta}),\overline{x}]| = |[V_\beta/Z_\beta,\overline{x}]\,\mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta})/\,\mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta})|$$
$$= |[V_\beta/Z_\beta,\overline{x}]/\,\mathrm{C}_{[V_\beta/Z_\beta,\overline{x}]}(\overline{E_\beta})| = |[V_\beta/Z_\beta,\overline{x}]|$$

giving, by Lemma 5.2.10(iii),

$$|[V_\beta/Z_\beta,\overline{x}]| = |[V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta),\overline{x}]| = |(V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta))^*\,/\,\mathrm{C}_{(V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta))^*}(\overline{x})|$$
$$= |[(V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta))^*,\overline{x}]| \le |[Q_\beta/C_\beta,\overline{x}]|.$$

We have that, by Lemma 5.2.1(ii), $|Q_\beta/C_\beta\,/(Q_\beta\cap Q_\mu)/C_\beta| = |Q_\beta/(Q_\beta\cap Q_\mu)| = 2$ and so $(Q_\beta\cap Q_\mu)/C_\beta$ is a $\overline{G_{\mu\beta}}$-submodule of Q_β/C_β of codimension one giving

$$|[Q_\beta/C_\beta,\overline{x}]| \le 2\,|[(Q_\beta\cap Q_\mu)/C_\beta,\overline{x}]| \le 2\,|[(Q_\beta\cap Q_\mu)/C_\beta,\overline{V_{\beta'}}]|.$$

Thus, $|[(Q_\beta\cap Q_\mu)/C_\beta,\overline{V_{\beta'}}]| \ge \frac{1}{2}|[Q_\beta/C_\beta,\overline{x}]| \ge \frac{1}{2}|[V_\beta/Z_\beta,\overline{x}]|$. We have shown that

$$|V_\beta/Z_\beta| \ge \tfrac{1}{2}|\overline{V_{\beta'}}|\,|[V_\beta/Z_\beta,\overline{x}]|\,|(V_\beta\cap C_{\beta'})/Z_\beta|.$$

Now, by Lemma 5.3.2(ii), $|[V_\beta/Z_\beta,\overline{x}]| = |V_\beta/Z_\beta\,/\,\mathrm{C}_{V_\beta/Z_\beta}(\overline{x})| \le 2\,|\overline{V_{\beta'}}|$ and so $|\overline{V_{\beta'}}| \ge \frac{1}{2}|[V_\beta/Z_\beta,\overline{x}]|$. Also, as V_β and $V_{\beta'}$ are normal in Q_μ, $[V_\beta,V_{\beta'}] \le V_\beta\cap V_{\beta'} \le V_\beta\cap C_{\beta'}$ and so

$$[V_\beta/Z_\beta,\overline{x}] \le [V_\beta/Z_\beta,\overline{V_{\beta'}}] = [V_\beta,V_{\beta'}]Z_\beta/Z_\beta \le (V_\beta\cap C_{\beta'})/Z_\beta$$

giving $|(V_\beta\cap C_{\beta'})/Z_\beta| \ge |[V_\beta/Z_\beta,\overline{x}]|$. So $|V_\beta/Z_\beta| \ge \frac{1}{4}|[V_\beta/Z_\beta,\overline{x}]|^3$ and hence

$$|V_\beta/Z_\beta\,/\,\mathrm{C}_{V_\beta/Z_\beta}(\overline{x})| = |[V_\beta/Z_\beta,\overline{x}]| \le (4\,|V_\beta/Z_\beta|)^{(1/3)}. \qquad \square$$

Corollary 5.3.4 *Assume that V_β/Z_β is a self-dual module. Then V_β/Z_β is irreducible, Q_β/C_β is isomorphic to V_β/Z_β and $|\overline{V_{\beta'}}| \le |\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})/[V_\beta/Z_\beta,\overline{G_{\mu\beta}},\overline{V_{\beta'}}]|$.*

Proof By Lemma 5.2.10(i), $[V_\beta/Z_\beta,\overline{E_\beta}] = V_\beta/Z_\beta$ and so, as V_β/Z_β is self-dual and by Lemma 5.2.10(i), $\mathrm{C}_{V_\beta}(E_\beta)/Z_\beta = \mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta}) = Z_\beta/Z_\beta$ giving, by Lemma 5.2.10(ii), $V_\beta/Z_\beta = V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ is an irreducible module. Hence, by Lemma 5.2.9, $Q_\beta/C_\beta \cong_{\overline{G_\beta}} (V_\beta/Z_\beta)^* \cong_{\overline{G_\beta}} V_\beta/Z_\beta$. Now, by Lemma 5.2.1(ii), $|Q_\beta/(Q_\beta\cap Q_\mu)| = 2$ and so, regarding $Q_\beta/(Q_\beta\cap Q_\mu)$ as a $G_{\mu\beta}$-group, $[Q_\beta/(Q_\beta\cap Q_\mu),G_{\mu\beta}] = 1$ giving $[Q_\beta,G_{\mu\beta}] \le Q_\beta\cap Q_\mu$. Then

$[Q_\beta/C_\beta, \overline{G_{\mu\beta}}] = [Q_\beta, G_{\mu\beta}]C_\beta/C_\beta \leq (Q_\beta \cap Q_\mu)/C_\beta$ and so, as $V_\beta/Z_\beta \cong_{\overline{G_\beta}} Q_\beta/C_\beta$,

$$|[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]| = |[Q_\beta/C_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]| \leq |[(Q_\beta \cap Q_\mu)/C_\beta, \overline{V_{\beta'}}]|.$$

Also, as V_β/Z_β is self-dual, $|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})| = |[V_\beta/Z_\beta, \overline{V_{\beta'}}]| \leq |(V_\beta \cap C_{\beta'})/Z_\beta|$. By the lemma above we have shown that

$$|V_\beta/Z_\beta| \geq |\overline{V_{\beta'}}| \; |[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]| \; |V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})|.$$

Thus, $|\overline{V_{\beta'}}| \leq |\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})/[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{V_{\beta'}}]|$. □

In the next section we will use the following lemma in order to prove that $\eta_{G_\beta}(C_\beta/V_\beta) = 0$.

Lemma 5.3.5 *Assume that $V_\beta \not\leq \Phi(C_\beta)$. Then $\eta_{G_\beta}(\Phi(C_\beta)) = 0$ and, in particular, $[(V_\beta \cap C_{\beta'})/Z_\beta, \overline{C_{\beta'}}] \leq Z_\mu/Z_\beta$.*

Proof Suppose, for a contradiction, that $\eta_{G_\beta}(\Phi(C_\beta)) \geq 1$. Then $1 \neq \Phi(C_\beta) \trianglelefteq G_{\mu\beta}$ and so, by Lemma 3.2.1, $\Phi(C_\beta) \cap Z_\beta = \Phi(C_\beta) \cap \Omega\,\mathrm{Z}(G_{\mu\beta}) \neq 1$ giving, as $|Z_\beta| = 2$, $Z_\beta \leq \Phi(C_\beta)$. Assume that $[\Phi(C_\beta), \Phi(C_{\beta'})] = 1$. In this case, by applying Lemma 3.1.6(ii), $\Phi(C_\beta) \leq \mathrm{C}_{G_{\beta'}}(\Phi(C_{\beta'})) \leq Q_{\beta'}$ and $\Phi(C_\beta) \not\leq C_{\beta'}$ because otherwise, by Corollary 3.3.5(i), $[\Phi(C_\beta), E_\beta] \leq [\Phi(C_\beta), \langle V_{\beta'}^{G_\beta}\rangle] = \langle [\Phi(C_\beta), V_{\beta'}]^{G_\beta}\rangle = 1$ giving $\eta_{G_\beta}(\Phi(C_\beta)) = 0$ which is a contradiction. Also, by Lemma 3.2.1(ii), $C_{\beta'} = Q_{\beta'}^{[1]}$ and so there exists $\rho \in \Gamma(\beta')$ such that $\Phi(C_\beta) \not\leq Q_\rho$. Then, as $\Phi(C_\beta) \leq Q_{\beta'} \leq G_{\rho\beta'}$ and by applying Lemma 3.2.2(iv), $[Z_\rho, \Phi(C_\beta)] = Z_{\beta'}$ and so, as $Z_\rho \leq Q_\mu$ and $\Phi(C_\beta)$ is normal in Q_μ, $Z_{\beta'} = [Z_\rho, \Phi(C_\beta)] \leq \Phi(C_\beta)$. So, by Lemma 3.2.2(iii), $Z_\mu = Z_\beta Z_{\beta'} \leq \Phi(C_\beta)$ and hence $V_\beta = \langle Z_\mu^{G_\beta}\rangle \leq \Phi(C_\beta)$ which is a contradiction. Assume that $[\Phi(C_\beta), \Phi(C_{\beta'})] \neq 1$. In this case, as $G_{\mu\beta^*}$ acts on $\Gamma(\mu) \smallsetminus \{\beta^*\} = \{\beta, \beta'\}$, $1 \neq [\Phi(C_\beta), \Phi(C_{\beta'})] \trianglelefteq G_{\mu\beta^*}$ and so, by Lemma 3.2.1, $[\Phi(C_\beta), \Phi(C_{\beta'})] \cap Z_{\beta^*} = [\Phi(C_\beta), \Phi(C_{\beta'})] \cap \Omega\,\mathrm{Z}(G_{\mu\beta^*}) \neq 1$ giving, as $|Z_{\beta^*}| = 2$, $Z_{\beta^*} \leq [\Phi(C_\beta), \Phi(C_{\beta'})]$. So, as $\Phi(C_\beta)$ and $\Phi(C_{\beta'})$ are normal in Q_μ and by Lemma 3.2.2(iii), $Z_\mu = Z_\beta Z_{\beta^*} \leq \Phi(C_\beta)$ and hence $V_\beta = \langle Z_\mu^{G_\beta}\rangle \leq \Phi(C_\beta)$ which is a contradiction. Thus, $\eta_{G_\beta}(\Phi(C_\beta)) = 0$. Now, as $\eta_{G_{\beta'}}(\Phi(C_{\beta'})) = 0$ and by Corollary 5.2.4(ii), $V_\beta \cap \Phi(C_{\beta'}) \trianglelefteq Q_\mu E_{\beta'} = G_{\beta'}$. Then, as $V_\beta \cap \Phi(C_{\beta'}) \trianglelefteq G_{\mu\beta'}$,

$$V_\beta \cap \Phi(C_{\beta'}) = (V_\beta \cap \Phi(C_{\beta'}))^{g^*} = V_\beta^{g^*} \cap \Phi(C_{\beta'})^{g^*} = V_\beta^{g^*} \cap \Phi(C_{\beta'}^{g^*})$$

$$= V_{\beta\cdot g^*} \cap \Phi(C_{\beta'\cdot g^*}) = V_{\beta^*} \cap \Phi(C_{\beta'})$$

and so, by Lemma 5.2.1(iii), $V_\beta \cap \Phi(C_{\beta'}) \leq V_\beta \cap V_{\beta^*} = Y_\mu$ giving, as $\eta_{G_{\beta'}}(\Phi(C_{\beta'})) = 0$ and by Corollary 5.2.8(iii), $V_\beta \cap \Phi(C_{\beta'}) \leq \mathrm{C}_{Y_\mu}(E_{\beta'}) = Z_{\beta'}$. We have that

$$[V_\beta \cap C_{\beta'}, C_{\beta'}] \leq [V_\beta, C_{\beta'}] \cap [C_{\beta'}, C_{\beta'}] \leq V_\beta \cap \Phi(C_{\beta'}) \leq Z_{\beta'}$$

and hence $[(V_\beta \cap C_{\beta'})/Z_\beta, \overline{C_{\beta'}}] = [V_\beta \cap C_{\beta'}, C_{\beta'}]Z_\beta/Z_\beta \leq Z_\mu/Z_\beta$. □

Lemma 5.3.6 *Assume that* $\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) \neq (V_\beta \cap C_{\beta'})/Z_\beta$. *Then there exists a subgroup* $\overline{K}$ *of* $\overline{V_{\beta'}}$ *such that* $|\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})/(V_\beta \cap C_{\beta'})/Z_\beta| = |\overline{K}|$ *and* $\overline{K}$ *induces* $\overline{G_{\mu\beta}}$*-central transvections on* $(V_\beta/Z_\beta)^*$.

Proof Let $K_{\mu\beta}$ be the inverse image of $\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})$. Then, as $V_\beta \cap C_{\beta'} = \mathrm{C}_{V_\beta}(V_{\beta'})$,

$$(V_\beta \cap C_{\beta'})/Z_\beta = \mathrm{C}_{V_\beta}(V_{\beta'})/Z_\beta \leq \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = K_{\mu\beta}/Z_\beta$$

and so $V_\beta \cap C_{\beta'} \leq K_{\mu\beta}$ and $[K_{\mu\beta}, V_{\beta'}]Z_\beta/Z_\beta = [K_{\mu\beta}/Z_\beta, \overline{V_{\beta'}}] = Z_\beta/Z_\beta$ giving $[K_{\mu\beta}, V_{\beta'}] \leq Z_\beta$. We also have that, as $\overline{V_{\beta'}}$ is normal in $\overline{G_{\mu\beta}}$, $\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})$ is a $\overline{G_{\mu\beta}}$-submodule of V_β/Z_β so $[K_{\mu\beta}, G_{\mu\beta}]Z_\beta/Z_\beta = [K_{\mu\beta}/Z_\beta, \overline{G_{\mu\beta}}] \leq K_{\mu\beta}/Z_\beta$ and so $[K_{\mu\beta}, G_{\mu\beta}] \leq K_{\mu\beta}$ giving $K_{\mu\beta}$ is normal in $G_{\mu\beta}$. Define $K_{\mu\beta'} = K_{\mu\beta}^{g'}$. Then, by Lemma 5.2.1(ii), $[V_{\beta'}, K_{\mu\beta} \cap Q_{\beta'}] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$ so that, by Lemma 3.2.2(iii), $[V_{\beta'}, K_{\mu\beta} \cap Q_{\beta'}] \leq Z_\beta \cap Z_{\beta'} = 1$ and so $K_{\mu\beta} \cap Q_{\beta'} \leq \mathrm{C}_{Q_{\beta'}}(V_{\beta'}) = C_{\beta'}$ giving $K_{\mu\beta} \cap Q_{\beta'} = V_\beta \cap C_{\beta'}$. We have that

$$\begin{aligned}|\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})\,/(V_\beta \cap C_{\beta'})/Z_\beta| &= |K_{\mu\beta}/Z_\beta\,/(V_\beta \cap C_{\beta'})/Z_\beta| = |K_{\mu\beta}/(V_\beta \cap C_{\beta'})| \\ &= |K_{\mu\beta}/(K_{\mu\beta} \cap Q_{\beta'})| = |K_{\mu\beta}Q_{\beta'}/Q_{\beta'}| = |K_{\mu\beta'}Q_\beta/Q_\beta| \\ &= |\overline{K_{\mu\beta'}}|.\end{aligned}$$

Now, $K_{\mu\beta'} \not\leq Q_\beta$ because otherwise $K_{\mu\beta} = K_{\mu\beta} \cap Q_{\beta'} = V_\beta \cap C_{\beta'}$ and so $\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = K_{\mu\beta}/Z_\beta = (V_\beta \cap C_{\beta'})/Z_\beta$ which is a contradiction. So $\overline{K_{\mu\beta'}}$ is a non-trivial normal elementary abelian 2-subgroup of $\overline{G_{\mu\beta}}$ and, by Lemma 3.2.2(iii),

$$[V_\beta/Z_\beta, \overline{K_{\mu\beta'}}] = [V_\beta, K_{\mu\beta'}]Z_\beta/Z_\beta \leq Z_{\beta'}Z_\beta/Z_\beta = Z_\mu/Z_\beta.$$

Then, by Corollary 5.2.8(i), $|(V_\beta/Z_\beta)^*/\operatorname{C}_{(V_\beta/Z_\beta)^*}(\overline{K_{\mu\beta'}})| = |[V_\beta/Z_\beta, \overline{K_{\mu\beta'}}]| \le |Z_\mu/Z_\beta| = 2$ and hence $\overline{K_{\mu\beta'}}$ induces $\overline{G_{\mu\beta}}$-central transvections on $(V_\beta/Z_\beta)^*$. □

Corollary 5.3.7 *Assume that $\eta_{G_\beta}(C_\beta/V_\beta) = 0$ and that no involution in $\overline{V_{\beta'}}$ induces a $\overline{G_{\mu\beta}}$-central transvection on V_β/Z_β. The following hold:*

(i) $Z_\mu/Z_\beta = \operatorname{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = \operatorname{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'}\cap Q_\mu}) \le \operatorname{C}_{V_\beta/Z_\beta}(\overline{C_{\beta'}}) = \operatorname{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = Y_\mu/Z_\beta$ and $[Y_\mu/Z_\beta, \overline{Q_{\beta'}\cap Q_\mu}] \le Z_\mu/Z_\beta$.

(ii) V_β/Z_β is an irreducible module and Q_β/C_β is isomorphic to $(V_\beta/Z_\beta)^*$.

Proof (i) We have that $[Y_\mu, C_{\beta'}] \le [V_{\beta'}, C_{\beta'}] = 1$ and so, by Lemmas 5.3.6 and 5.2.11,

$$\operatorname{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta = Y_\mu/Z_\beta \le \operatorname{C}_{V_\beta/Z_\beta}(\overline{C_{\beta'}}) \le \operatorname{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})$$

giving $\operatorname{C}_{V_\beta/Z_\beta}(\overline{C_{\beta'}}) = \operatorname{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = Y_\mu/Z_\beta$. Then $\operatorname{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'}\cap Q_\mu}) \le \operatorname{C}_{V_\beta/Z_\beta}(\overline{C_{\beta'}}) = Y_\mu/Z_\beta$ and so, by Lemma 5.3.1(i),

$$Z_\mu/Z_\beta \le \operatorname{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) \le \operatorname{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'}\cap Q_\mu}) = \operatorname{C}_{Y_\mu/Z_\beta}(\overline{Q_{\beta'}\cap Q_\mu}) = Z_\mu/Z_\beta$$

giving $\operatorname{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = \operatorname{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'}\cap Q_\mu}) = Z_\mu/Z_\beta$. Now, by Lemma 5.2.1(ii), $[Y_\mu, Q_{\beta'}\cap Q_\mu] \le [V_{\beta'}, Q_{\beta'}] = Z_{\beta'} \le Z_\mu$ and hence $[Y_\mu/Z_\beta, \overline{Q_{\beta'}\cap Q_\mu}] = [Y_\mu, Q_{\beta'}\cap Q_\mu]Z_\beta/Z_\beta \le Z_\mu/Z_\beta$.
(ii) Observe that, as $V_{\beta'}$ and $\operatorname{C}_{V_\beta}(E_\beta)$ are normal in Q_μ and by applying Lemma 5.2.1(iii) and Corollary 5.2.8(iii),

$$[V_{\beta'}, \operatorname{C}_{V_\beta}(E_\beta)] \le V_{\beta'} \cap \operatorname{C}_{V_\beta}(E_\beta) = \operatorname{C}_{(V_{\beta'}\cap V_\beta)}(E_\beta) = \operatorname{C}_{Y_\mu}(E_\beta) = Z_\beta.$$

Suppose, for a contradiction, that $\operatorname{C}_{V_{\beta'}}(E_{\beta'}) \not\le Q_\beta$. Then $\overline{\operatorname{C}_{V_{\beta'}}(E_{\beta'})}$ is a non-trivial normal elementary abelian 2-subgroup of $\overline{G_{\mu\beta}}$ and $[V_\beta/Z_\beta, \overline{\operatorname{C}_{V_{\beta'}}(E_{\beta'})}] = [V_\beta, \operatorname{C}_{V_{\beta'}}(E_{\beta'})]Z_\beta/Z_\beta \le Z_\mu/Z_\beta$. So, by Corollary 5.2.8(i), there exists an involution in $\overline{\operatorname{C}_{V_{\beta'}}(E_{\beta'})}$ which induces a $\overline{G_{\mu\beta}}$-central transvection on V_β/Z_β which is a contradiction. Hence, $\operatorname{C}_{V_{\beta'}}(E_{\beta'}) \le Q_\beta$. Now, by Lemma 5.2.1(ii), $[V_{\beta'}, \operatorname{C}_{V_\beta}(E_\beta)] \le [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$ and so, by Lemma 3.2.2(iii), $[V_{\beta'}, \operatorname{C}_{V_\beta}(E_\beta)] \le Z_\beta \cap Z_{\beta'} = 1$ giving $\operatorname{C}_{V_\beta}(E_\beta) \le \operatorname{C}_{Q_{\beta'}}(V_{\beta'}) = C_{\beta'}$. So $\operatorname{C}_{V_\beta}(E_\beta) \le V_\beta \cap C_{\beta'} = Y_\mu$ and hence, by Corollary 5.2.8(iii), $\operatorname{C}_{V_\beta}(E_\beta) = \operatorname{C}_{Y_\mu}(E_\beta) = Z_\beta$. Thus, by Lemmas 5.2.10(ii) and 5.2.9, $V_\beta/Z_\beta = V_\beta/\operatorname{C}_{V_\beta}(E_\beta)$ is an irreducible module and $Q_\beta/C_\beta \cong_{\overline{G_\beta}} (V_\beta/Z_\beta)^*$. □

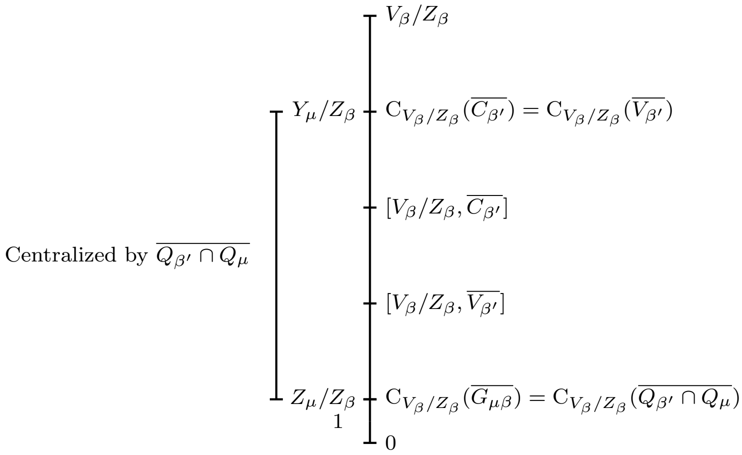

Figure F The structure of the module V_β/Z_β in Corollary 5.3.7.

The corollary above gives rise to the following heuristical argument under the weaker hypothesis $P_2/Q_2 \in \mathfrak{X}$ and $|Z| = 2$. Assume that $\eta_{G_\beta}(C_\beta/V_\beta) = 0$. In the case when there exists an involution in $\overline{V_{\beta'}}$ which induces a $\overline{G_{\mu\beta}}$-central transvection on V_β/Z_β, the group $\overline{G_\beta}$ is unusual and the structure of the module V_β/Z_β is highly restricted. In the alternative case, the corollary above provides the following dilemma when Y_μ/Z_μ has large order. On the one hand, $\mathrm{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu})$ is one-dimensional and so $\overline{Q_{\beta'} \cap Q_\mu}$ is a "large" group. On the other hand, $[Y_\mu/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq Z_\mu/Z_\beta$ and so $\overline{Q_{\beta'} \cap Q_\mu}$ centralizes a large section of the module V_β/Z_β and hence it must be a "small" group. The author conjectures that for the vast majority of groups $\overline{G_\beta}$ the above argument proves that $\eta_{G_\beta}(C_\beta/V_\beta) \geq 1$ or otherwise leads to exceptional configurations. At the end of this chapter we will face the dilemma described above which will lead to a contradiction except in the cases when $\overline{G_\beta} \cong \mathfrak{A}_5$, $\mathfrak{A}_8$ or $\mathfrak{S}_8$.

§5.4 Concluding the proof of Theorem A

In this section we will focus our attention on the case when $\overline{G_\beta} \cong \mathfrak{A}_n$ or $\mathfrak{S}_n$.

Lemma 5.4.1 *Assume that $\overline{G_\beta} \cong \mathfrak{A}_5$. The following hold:*

(i) $\overline{V_{\beta'}} = \overline{C_{\beta'}} < \overline{Q_{\beta'} \cap Q_\mu} = \overline{G_{\mu\beta}}$.

(ii) $\eta_{G_\beta}(C_\beta/V_\beta) = 0$.

(iii) V_β/Z_β is isomorphic to either the natural module for $\overline{G_\beta}$ over $\mathbb{F}_2$ or the spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$ and Q_β/C_β is isomorphic to V_β/Z_β.

Proof (i) By Theorem 2.3.8(ii) and Lemma 5.3.1(iii), $1 \neq \overline{V_{\beta'}} \leq \overline{C_{\beta'}} < \overline{Q_{\beta'} \cap Q_\mu} \leq \overline{G_{\mu\beta}}$ and hence, as $|\overline{G_{\mu\beta}}| = 4$, $\overline{V_{\beta'}} = \overline{C_{\beta'}} < \overline{Q_{\beta'} \cap Q_\mu} = \overline{G_{\mu\beta}}$.
(ii) This follows from part (i) and Lemma 5.3.1(ii).
(iii) By Theorem 2.3.8(ii), we may apply Corollary 5.3.7(ii) to deduce that V_β/Z_β is an irreducible module and Q_β/C_β is isomorphic to $(V_\beta/Z_\beta)^*$. Thus, V_β/Z_β is isomorphic to either the natural module or the spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$ and, as both of these modules are self-dual, Q_β/C_β is isomorphic to V_β/Z_β. □

Henceforth we will assume that $n \geq 8$ when $\overline{G_\beta} \cong \mathfrak{A}_n$ and $n \geq 7$ when $\overline{G_\beta} \cong \mathfrak{S}_n$. Before we proceed we will need to establish the following notation.

Notation Denote the 2-adic decomposition of n as follows

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r} \quad \text{with} \quad 0 \leq m_1 < m_2 < \cdots < m_r.$$

As $\overline{G_{\mu\beta}}$ is a Sylow 2-subgroup of $\overline{G_\beta}$, there exist subgroups $\overline{T^1}$, $\overline{T^2}$, $\ldots$, $\overline{T^r}$ of $\overline{G_{\mu\beta}}$ such that $\overline{T^1} \times \overline{T^2} \times \cdots \times \overline{T^r} \leq \overline{G_{\mu\beta}}$ where $\overline{T^i}$ is similar to $\mathfrak{T}_{m_i}$ for $1 \leq i \leq r$ when $\overline{G_\beta} \cong \mathfrak{S}_n$ and $\overline{T^i}$ is similar to $\mathfrak{T}_{m_i} \cap \mathfrak{A}_n$ for $1 \leq i \leq r$ when $\overline{G_\beta} \cong \mathfrak{A}_n$. Let $\mathrm{Z}(\overline{G_{\mu\beta}}) = \langle \overline{t_1}, \overline{t_2}, \ldots, \overline{t_r} \rangle$ where $\overline{t_i} \in \overline{T^i}$ for $1 \leq i \leq r$ and set $\overline{t} = \overline{t_1}\overline{t_2} \ldots \overline{t_r}$. Then $\overline{t_i}$ is similar to the involution $(1\ 2)(3\ 4) \ldots (2^{m_i} - 1\ 2^{m_i})$ except when either $m_i = 0$ (n is odd) or $\overline{G_\beta} \cong \mathfrak{A}_n$ and $m_i = 1$ in which case $\overline{t_i} = 1$.

Lemma 5.4.2 *$V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ is isomorphic to the natural module for $\overline{G_\beta}$ over $\mathbb{F}_2$ except when $\overline{G_\beta} \cong \mathfrak{A}_8$ or $\mathfrak{S}_8$ and $V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$.*

Proof By Lemma 5.2.10(ii), $V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ is a faithful irreducible $\mathbb{F}_2\overline{G_\beta}$-module and, by Lemmas 5.3.2, 5.2.10(iii), 2.4.1 and 2.3.3, $\overline{V_{\beta'}}$ is a non-trivial elementary abelian 2-subgroup of $\overline{G_\beta}$ with $[V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta), \overline{V_{\beta'}}, \overline{V_{\beta'}}] = 1$ and, for all $\overline{x} \in \overline{V_{\beta'}}$,

$$|V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)\,/\,\mathrm{C}_{V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)}(\overline{x})| \leq |V_\beta/Z_\beta\,/\,\mathrm{C}_{V_\beta/Z_\beta}(\overline{x})| \leq 2\,|\overline{V_{\beta'}}|.$$

In the case when $|\overline{V_{\beta'}}| = 2$, $|V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)\,/\,\mathrm{C}_{V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)}(\overline{V_{\beta'}})| \leq 2^2$ so that $\overline{V_{\beta'}}$ induces a 2-transvection on $V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ and so, by Corollary 2.4.3, there exists $\overline{V_{\beta'}} \leq \overline{F} \leq \overline{G_\beta}$ such that $|\overline{F}| = 4$ and $[V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta), \overline{F}, \overline{F}] = 1$. Therefore, in all cases, we may apply Theorem 2.4.8 to deduce that either $V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ is isomorphic to the natural module or a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$. Assume that we have a counter-example to the lemma. Then $\overline{G_\beta} \cong \mathfrak{S}_7$ or $n \geq 9$ and $V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)$ is isomorphic to a spin module. We have that $|\overline{V_{\beta'}}| \geq 2^3$ because otherwise for $\overline{x} \in (\overline{V_{\beta'}})^{\#}$, by Theorem 2.4.8(II) and Lemma 2.4.6,

$$|V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)|^{1/2} = |V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)\,/\,\mathrm{C}_{V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)}(\overline{x})| \leq 2\,|\overline{V_{\beta'}}| \leq 2^3$$

and so, by Theorem 2.1.8, $2^7 \leq |V_\beta/\,\mathrm{C}_{V_\beta}(E_\beta)| \leq 2^6$ which is a contradiction. So, by Theorem 2.4.8(II), $\overline{G_\beta} \cong \mathfrak{A}_9$ and $\overline{V_{\beta'}}$ is similar to a regular eight-group $\mathfrak{K}_8$. Then, as $\overline{G_{\mu\beta}}$ is similar to a subgroup of $\mathfrak{S}_8$ and by Lemma 5.3.1, $\overline{V_{\beta'}} \leq \overline{Q_{\beta'} \cap Q_\mu} \leq \mathrm{C}_{\overline{G_{\mu\beta}}}(\overline{V_{\beta'}}) = \overline{V_{\beta'}}$ giving $\overline{V_{\beta'}} = \overline{Q_{\beta'} \cap Q_\mu}$. So, by Lemma 5.3.1(iii), $\overline{V_{\beta'}}$ induces transvections on Q_β/C_β which contradicts Theorem 2.3.8(ii). □

Corollary 5.4.3 *One of the following two cases hold:*

(I) $\overline{G_\beta} \cong \mathfrak{A}_8$ or $\mathfrak{S}_8$ and V_β/Z_β is isomorphic to either the natural module for $\overline{G_\beta}$ over $\mathbb{F}_2$ or a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$.

(II) $\overline{G_\beta} \not\cong \mathfrak{A}_8$ or $\mathfrak{S}_8$ and V_β/Z_β is isomorphic to either the zero weight submodule V_0 of the n-dimensional permutation module for $\overline{G_\beta}$ over $\mathbb{F}_2$ or the natural module $\overline{V_0}$ for $\overline{G_\beta}$ over $\mathbb{F}_2$. Moreover, $r \geq 2$ in the 2-adic decomposition of n.

Proof By Lemma 5.2.10(i), $[V_\beta/Z_\beta, \overline{E_\beta}] = V_\beta/Z_\beta$ and, by Lemma 5.2.10(iii) and the lemma above, either $V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta})$ is isomorphic to the natural module for $\overline{G_\beta}$ over $\mathbb{F}_2$ or $\overline{G_\beta} \cong \mathfrak{A}_8$ or $\mathfrak{S}_8$ and $V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta})$ is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$. In the latter case, by Theorem 2.4.5, V_β/Z_β is isomorphic to a spin module for $\overline{G_\beta}$ over $\mathbb{F}_2$. In the former case, by Theorem 2.2.6, V_β/Z_β is isomorphic to either the natural module $\overline{V_0}$ for $\overline{G_\beta}$ over $\mathbb{F}_2$ or the zero weight submodule V_0 of the n-dimensional permutation module for $\overline{G_\beta}$ over $\mathbb{F}_2$. Now, assume that $r = 1$. In the case when $V_\beta/Z_\beta \cong_{\overline{G_\beta}} V_0$, $\overline{G_{\mu\beta}}$ and $\overline{E_\beta}$ are both transitive permutation groups and so $\mathrm{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = \mathrm{C}_{V_\beta/Z_\beta}(\overline{E_\beta})$ giving $\mathrm{C}_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{G_{\mu\beta}}) = \mathrm{C}_{[V_\beta/Z_\beta, \overline{V_{\beta'}}]}(\overline{E_\beta})$ which contradicts Lemma 5.3.2(i). In the case when $V_\beta/Z_\beta \cong_{\overline{G_\beta}} \overline{V_0}$, as $1 \neq \overline{V_{\beta'}} \trianglelefteq \overline{G_{\mu\beta}}$, $\overline{V_{\beta'}} \cap \mathrm{Z}(\overline{G_{\mu\beta}}) \neq 1$ and so $\bar{t} \in \overline{V_{\beta'}}$. Then, by Lemmas 5.3.3 and 2.1.16(iii),

$$2^{(n/2-2)} = |V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\bar{t})| \leq (4\,|V_\beta/Z_\beta|)^{(1/3)} = 2^{(n/3)}$$

so $n/2 - 2 \leq n/3$ and so $n \leq 12$ giving $n = 8$. Thus, $\overline{G_\beta} \cong \mathfrak{A}_8$ or $\mathfrak{S}_8$. □

From now on we will assume that Case (II) of Corollary 5.4.3 holds and aim to find a contradiction. In the next lemma we will show that the group $\overline{V_{\beta'}}$ is similar to either $\langle(1\ 2)\rangle$ or $\langle(1\ 2)(3\ 4)\rangle$. These two cases will often be considered separately in the remainder of this chapter.

Lemma 5.4.4 *$\overline{V_{\beta'}} = \langle \overline{t_i} \rangle$ for some $1 \leq i \leq r$. Moreover, one of the following two cases hold:*

(I) $\overline{t_i}$ is a transposition, $[V_\beta/Z_\beta, \overline{V_{\beta'}}] = Z_\mu/Z_\beta$ and $[V_\beta/Z_\beta, \overline{G_{\mu\beta}}] \leq (V_\beta \cap C_{\beta'})/Z_\beta$.

(II) $\overline{t_i}$ is a double transposition and $\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta$.

Proof By Corollary 5.4.3(II) together with Lemma 5.3.2 and Corollary 5.3.4 we may apply Lemmas 5.1.3–5.1.6 to deduce that $\overline{V_{\beta'}} \leq \overline{T^i}$ for some $1 \leq i \leq r$. In particular, as $1 \neq \overline{V_{\beta'}} \trianglelefteq \overline{G_{\mu\beta}}$, $\overline{V_{\beta'}} \cap \mathrm{Z}(\overline{G_{\mu\beta}}) \neq 1$ and so, as $\overline{V_{\beta'}} \leq \overline{T^i}$, $\overline{t_i} \in \overline{V_{\beta'}}$. Assume first that $\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) \neq (V_\beta \cap C_{\beta'})/Z_\beta$. We have that, as $\overline{V_{\beta'}} \leq \overline{T^i}$, $\overline{V_{\beta'}} \cap \mathrm{Z}(\overline{G_{\mu\beta}}) = \langle \overline{t_i} \rangle$ and so, by Lemma 5.3.6, $\overline{t_i}$ induces a transvection on V_β/Z_β. Hence, by applying Corollary 2.1.17(i),

$\overline{t_i}$ is a transposition and so $\langle\overline{t_i}\rangle \leq \overline{V_{\beta'}} \leq \overline{T^i} = \langle\overline{t_i}\rangle$ giving $\overline{V_{\beta'}} = \langle\overline{t_i}\rangle$. Assume second that $\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta$. Then, by Lemma 5.3.2, $[V_\beta/Z_\beta, \overline{t_i}, \overline{V_{\beta'}}] \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}, \overline{V_{\beta'}}] = 1$ so $\overline{V_{\beta'}} \leq \mathrm{C}_{\overline{T^i}}([V_\beta/Z_\beta, \overline{t_i}])$ giving, by applying Lemma 5.1.1, $(V_\beta \cap C_{\beta'})/Z_\beta = \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = \mathrm{C}_{V_\beta/Z_\beta}(\overline{t_i})$. So, by Lemma 5.3.3,

$$|V_\beta/Z_\beta| \geq \tfrac{1}{2}|\overline{V_{\beta'}}|\; |[V_\beta/Z_\beta, \overline{t_i}]|\; |\mathrm{C}_{V_\beta/Z_\beta}(\overline{t_i})|$$

and so, as $|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{t_i})| = |[V_\beta/Z_\beta, \overline{t_i}]|$,

$$|\overline{V_{\beta'}}| \leq 2\,|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{t_i})|\; |[V_\beta/Z_\beta, \overline{t_i}]|^{-1} = 2$$

giving $\overline{V_{\beta'}} = \langle\overline{t_i}\rangle$. Then, by Lemma 5.3.2(ii), $|V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{t_i})| \leq 2^2$ and so $\overline{t_i}$ induces a 2-transvection on V_β/Z_β. Hence, by applying Corollary 2.1.17, $\overline{t_i}$ is either a transposition or a double transposition.

Now, let $\overline{V_{\beta'}} = \langle\overline{t_i}\rangle$ and $\overline{t_i}$ be a transposition. By Corollary 3.3.5(i), $1 \neq [V_\beta, V_{\beta'}] \trianglelefteq G_{\mu\beta^*}$ and so, by Lemma 3.2.1, $[V_\beta, V_{\beta'}] \cap Z_{\beta^*} = [V_\beta, V_{\beta'}] \cap \Omega\,\mathrm{Z}(G_{\mu\beta^*}) \neq 1$ giving, as $|Z_{\beta^*}| = 2$, $Z_{\beta^*} \leq [V_\beta, V_{\beta'}]$. So, by Lemma 3.2.2(iii), $Z_\mu/Z_\beta = Z_{\beta^*}Z_\beta/Z_\beta \leq [V_\beta, V_{\beta'}]Z_\beta/Z_\beta = [V_\beta/Z_\beta, \overline{V_{\beta'}}]$ and hence, as $|[V_\beta/Z_\beta, \overline{V_{\beta'}}]| = 2 = |Z_\mu/Z_\beta|$, $[V_\beta/Z_\beta, \overline{V_{\beta'}}] = Z_\mu/Z_\beta$. Then $[V_\beta, V_{\beta'}] \leq Z_\mu$ and so $[V_\beta, V_{\beta'}, Q_\mu] \leq [Z_\mu, Q_\mu] = 1$. So, as V_β and $V_{\beta'}$ are normal in Q_μ and by applying the Three-Subgroup Lemma,

$$[V_\beta, Q_\mu, V_{\beta'}] = [Q_\mu, V_\beta, V_{\beta'}] = [V_{\beta'}, Q_\mu, V_\beta].$$

Also, as $\overline{V_{\beta'}} \leq \mathrm{Z}(\overline{G_{\mu\beta}})$, $\overline{[V_{\beta'}, Q_\mu]} = [\overline{V_{\beta'}}, \overline{Q_\mu}] = [\overline{V_{\beta'}}, \overline{G_{\mu\beta}}] = 1$ giving $[V_{\beta'}, Q_\mu] \leq Q_\beta$. Then, by Lemma 5.2.1(ii), $[V_{\beta'}, Q_\mu, V_\beta] \leq [Q_\beta, V_\beta] = Z_\beta$ and so, by Lemma 3.2.2(iii), $[V_\beta, Q_\mu, V_{\beta'}] \leq Z_\beta \cap Z_{\beta'} = 1$ giving $[V_\beta, Q_\mu] \leq \mathrm{C}_{Q_{\beta'}}(V_{\beta'}) = C_{\beta'}$. So $[V_\beta, Q_\mu] \leq V_\beta \cap C_{\beta'}$ and hence $[V_\beta/Z_\beta, \overline{G_{\mu\beta}}] = [V_\beta, Q_\mu]Z_\beta/Z_\beta \leq (V_\beta \cap C_{\beta'})/Z_\beta$. Therefore, one of Case (I) and Case (II) holds. □

Corollary 5.4.5 $\eta_{G_\beta}(C_\beta/V_\beta) = 0$. *In particular,* $Y_\mu = V_\beta \cap C_{\beta'}$.

Proof Suppose, for a contradiction, that $V_\beta \leq \Phi(C_\beta)$. Then $\overline{V_{\beta'}} \leq \overline{\Phi(C_{\beta'})} = \Phi(\overline{C_{\beta'}})$ and so, by Lemma 5.3.2(iii), $\overline{C_{\beta'}} \leq \mathrm{C}_{\overline{G_{\mu\beta}}}([V_\beta/Z_\beta, \overline{V_{\beta'}}])$ giving, by applying Corollary 5.1.2, $\overline{V_{\beta'}} \leq \Phi(\overline{C_{\beta'}}) \cap \overline{T^i} = 1$ which is a contradiction. Hence, $V_\beta \not\leq \Phi(C_\beta)$. Firstly, consider

Case (I) of Lemma 5.4.4. We have that, by Lemma 5.3.5,

$$[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{C_{\beta'}}] \leq [(V_\beta \cap C_{\beta'})/Z_\beta, \overline{C_{\beta'}}] \leq Z_\mu/Z_\beta \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$$

and so, by applying Lemma 5.1.7(i), $\overline{V_{\beta'}} \leq \overline{C_{\beta'}} \leq \overline{V_{\beta'}}$ giving $\overline{V_{\beta'}} = \overline{C_{\beta'}}$. Thus, by Lemma 5.3.1(ii), $\eta_{G_\beta}(C_\beta/V_\beta) = 0$. Secondly, consider Case (II) of Lemma 5.4.4. We have that, by Lemma 5.3.5,

$$[\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}), \overline{C_{\beta'}}] = [(V_\beta \cap C_{\beta'})/Z_\beta, \overline{C_{\beta'}}] \leq Z_\mu/Z_\beta \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$$

and so, by applying Lemma 5.1.7(ii), $\overline{C_{\beta'}} \leq \overline{T^i}$. Then, by Lemma 5.3.2(iii), $\overline{C_{\beta'}} \leq \mathrm{C}_{\overline{T^i}}([V_\beta/Z_\beta, \overline{V_{\beta'}}])$ and so, by applying Lemma 5.1.1, $[V_\beta/Z_\beta, \overline{V_{\beta'}}] = [V_\beta/Z_\beta, \overline{C_{\beta'}}]$. Now, $[V_{\beta'}, V_\beta \cap Q_{\beta'}] \neq 1$ because otherwise $V_\beta \cap C_{\beta'} = V_\beta \cap Q_{\beta'}$ so, by Corollary 3.3.5(iii),

$$\begin{aligned} |V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})| &= |V_\beta/Z_\beta \,/(V_\beta \cap C_{\beta'})/Z_\beta| = |V_\beta/(V_\beta \cap C_{\beta'})| = |V_\beta/(V_\beta \cap Q_{\beta'})| \\ &= |V_\beta Q_{\beta'}/Q_{\beta'}| = |V_{\beta'} Q_\beta/Q_\beta| = |\overline{V_{\beta'}}| = 2 \end{aligned}$$

and so $\overline{V_{\beta'}}$ induces a transvection on V_β/Z_β which contradicts Corollary 2.1.17(i). We have that, by Lemma 5.2.1(ii), $1 \neq [V_\beta, V_{\beta'} \cap Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta$ and so, as $|Z_\beta| = 2$, $Z_\beta = [V_\beta, V_{\beta'} \cap Q_\beta] \leq [V_\beta, V_{\beta'}]$. Then $[V_\beta, V_{\beta'}]/Z_\beta = [V_\beta/Z_\beta, \overline{V_{\beta'}}] = [V_\beta/Z_\beta, \overline{C_{\beta'}}] = [V_\beta, C_{\beta'}]/Z_\beta$ so $[V_\beta, V_{\beta'}] = [V_\beta, C_{\beta'}]$ and so, by Corollary 3.3.5(i),

$$[C_\beta, E_\beta] \leq [C_\beta, \langle V_{\beta'}^{G_\beta} \rangle] = \langle [C_\beta, V_{\beta'}]^{G_\beta} \rangle = \langle [V_\beta, V_{\beta'}]^{G_\beta} \rangle \leq \langle V_\beta^{G_\beta} \rangle = V_\beta$$

giving $\eta_{G_\beta}(C_\beta/V_\beta) = 0$. Therefore, in both cases of Lemma 5.4.4 $\eta_{G_\beta}(C_\beta/V_\beta) = 0$. In particular, by Lemma 5.2.11, $Y_\mu = V_\beta \cap C_{\beta'}$. □

Lemma 5.4.6 *The following hold:*

(i) $\mathrm{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = Z_\mu/Z_\beta$.

(ii) V_β/Z_β *and* Q_β/C_β *are isomorphic to* $\overline{V_0}$ *and* $r = 2$. *In particular,* $n = 2 + 2^m$ *for some* $m \geq 3$ *in Case (I) of Lemma 5.4.4 and* $n = 4 + 2^m$ *for some* $m \geq 3$ *in Case (II) of Lemma 5.4.4.*

(iii) $\mathrm{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$ *and* $[(V_\beta \cap C_{\beta'})/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$.

Proof (i) By Lemmas 2.5.9 and 5.4.4 and Corollary 5.4.5,

$$\mathrm{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) \leq [V_\beta/Z_\beta, \overline{G_{\mu\beta}}] \cap \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) \leq (V_\beta \cap C_{\beta'})/Z_\beta = Y_\mu/Z_\beta$$

and hence, by Lemma 5.3.1(i), $\mathrm{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) = \mathrm{C}_{Y_\mu/Z_\beta}(\overline{G_{\mu\beta}}) = Z_\mu/Z_\beta$.

(ii) We have that V_β/Z_β is isomorphic to $\overline{V_0}$ because otherwise n is even and V_β/Z_β is isomorphic to V_0 and so, by Lemma 2.5.9(i) and part (i), $2^r = |\mathrm{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}})| = |Z_\mu/Z_\beta| = 2$ giving $r = 1$ which contradicts Corollary 5.4.3(II). So V_β/Z_β is self-dual and hence, by Corollary 5.3.4, Q_β/C_β is isomorphic to V_β/Z_β. Then, by Lemma 2.5.9(ii) and part (i), $2^{r-1} = |\mathrm{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}})| = |Z_\mu/Z_\beta| = 2$ and hence $r = 2$.

(iii) In Case (I) of Lemma 5.4.4, by part (ii) and applying Lemma 5.1.8(i),

$$(V_\beta \cap C_{\beta'})/Z_\beta \leq \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = [V_\beta/Z_\beta, \overline{G_{\mu\beta}}] \leq (V_\beta \cap C_{\beta'})/Z_\beta$$

and so $\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta$. So, in both cases of Lemma 5.4.4, by Corollary 5.4.5,

$$\mathrm{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) \leq \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}) = (V_\beta \cap C_{\beta'})/Z_\beta = Y_\mu/Z_\beta$$

and hence, by Lemma 5.3.1(i), $\mathrm{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = \mathrm{C}_{Y_\mu/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$. Also, by Corollary 5.4.5 and Lemma 5.2.1(ii),

$$[V_\beta \cap C_{\beta'}, Q_{\beta'} \cap Q_\mu] = [Y_\mu, Q_{\beta'} \cap Q_\mu] \leq [V_{\beta'}, Q_{\beta'}] = Z_{\beta'}$$

and hence

$$[(V_\beta \cap C_{\beta'})/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] = [V_\beta \cap C_{\beta'}, Q_{\beta'} \cap Q_\mu]Z_\beta/Z_\beta \leq Z_\mu/Z_\beta \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]. \qquad \square$$

Corollary 5.4.7 *A contradiction.*

Proof Firstly, consider Case (I) of Lemma 5.4.4. By Lemma 5.4.6(iii),

$$[V_\beta/Z_\beta, \overline{G_{\mu\beta}}, \overline{Q_{\beta'} \cap Q_\mu}] \leq [(V_\beta \cap C_{\beta'})/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$$

and so, by applying Lemma 5.1.7(i), $\overline{V_{\beta'}} \leq \overline{Q_{\beta'} \cap Q_\mu} \leq \overline{V_{\beta'}}$ giving $\overline{V_{\beta'}} = \overline{Q_{\beta'} \cap Q_\mu}$. Then, by Lemma 5.4.6(iii) and Corollary 5.2.8(i), $|\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})| = |\mathrm{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu})| = |Z_\mu/Z_\beta| = 2$ and so $2^{n-3} = |V_\beta/Z_\beta \,/\, \mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}})| = 2$ giving $n = 4$ which is a contradiction. Secondly, consider Case (II) of Lemma 5.4.4. By Lemma 5.4.6(iii),

$$[\mathrm{C}_{V_\beta/Z_\beta}(\overline{V_{\beta'}}), \overline{Q_{\beta'} \cap Q_\mu}] = [(V_\beta \cap C_{\beta'})/Z_\beta, \overline{Q_{\beta'} \cap Q_\mu}] \leq [V_\beta/Z_\beta, \overline{V_{\beta'}}]$$

and so, by applying Lemma 5.1.7(ii), $\overline{Q_{\beta'} \cap Q_\mu} \leq \overline{T^i}$. Then, by Lemma 5.4.6(i,iii),

$$Z_\mu/Z_\beta = \mathrm{C}_{V_\beta/Z_\beta}(\overline{G_{\mu\beta}}) \leq \mathrm{C}_{V_\beta/Z_\beta}(\overline{T^i}) \leq \mathrm{C}_{V_\beta/Z_\beta}(\overline{Q_{\beta'} \cap Q_\mu}) = Z_\mu/Z_\beta$$

and so, by applying Lemma 5.1.8(ii) and by Corollary 5.2.8(i), $2^{n-5} = |\mathrm{C}_{V_\beta/Z_\beta}(\overline{T^i})| = |Z_\mu/Z_\beta| = 2$ giving $n = 6$ which is a contradiction. □

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