INITIAL SEGMENTS AND END-EXTENSIONS OF MODELS OF ARITHMETIC

by

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A thesis submitted to
The University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY

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May 2010

Abstract

This thesis is organized into two independent parts.

In the first part, we extend the recent work on generic cuts by Kaye and the author. The focus here is the properties of the pairs (M, I) where I is a generic cut of a model M. Amongst other results, we characterize the theory of such pairs, and prove that they are existentially closed in a natural category.

In the second part, we construct end-extensions of models of arithmetic that are at least as strong as ATR_0. Two new constructions are presented. The first one uses a variant of Födor's Lemma in ATR_0 to build an internally rather classless model. The second one uses some weak versions of the Galvin–Prikry Theorem in adjoining an ideal set to a model of second-order arithmetic.

ACKNOWLEDGEMENTS

I want to express my foremost thanks to my PhD supervisor Dr. Richard Kaye for his support, guidance, stimulation, and contribution. He has had a great influence in my development as a mathematician. Unless specified otherwise, all results in this thesis should be attributed jointly to my supervisor and I.

I also want to thank a number of colleagues for some very helpful conversations related to this research. They include Andrey Bovykin, Roman Kossak, Alberto Marcone, Antonio Montalbán, Sam Sanders, Steve Simpson, Michiel de Smet, Neil Thapen, and Keita Yokoyama. I especially want to express my gratitude towards Antonio Montalbán for patiently explaining to me the proof of Lemma VI.6.2 in Simpson [72].

Next, I want to thank my wife Celia and my family, whose continuous support and encouragement goes well beyond what is shown in this thesis. Without Celia's constant pressure to clear my desk (and our table), the writing up of this thesis could have taken much longer.

Finally, I want to thank the staff and students at the University of Birmingham for making my stay here pleasant, fruitful, and smooth. In particular, I have benefited a lot from the activities in the combinatorics group and the algebra group in the School of Mathematics, and from those in the theory group in the School of Computer Science. I wish to thank the Overseas Research Students Awards Scheme (ORSAS) and the School of Mathematics for funding my PhD studies.

This thesis is typeset using LaTeX. The packages used include AMS-LaTeX, mathrsfs, setspace, perpage, array, url, textcomp, tipa, and bbm. The diagrams are drawn using PGF and XY-pic.

CONTENTS

CHAPTER 1

INTRODUCTION

> **thesis** (ˈθiːsɪs, ˈθɛsɪs). Pl. theses (ˈθiːsiːz). [a. Gr. $\theta\acute{\epsilon}\sigma\iota\varsigma$ putting, placing; a proposition, affirmation, etc., f. root $\theta\epsilon$- of $\tau\iota$-$\theta\acute{\epsilon}$-$\nu\alpha\iota$ to put, place.]
>
> **I.** In *Prosody*, etc.: opposed to ARSIS. ⟦...⟧
>
> **II.** In *Logic*, *Rhetoric*, etc.
>
> **4.** A proposition laid down or stated, esp. as a theme to be discussed and proved, or to be maintained against attack (in *Logic* sometimes as distinct from HYPOTHESIS 2, in *Rhetoric* from ANTITHESIS 2); a statement, assertion, tenet. ⟦...⟧
>
> **b.** *spec.* distinguished from HYPOTHESIS 1 ⟦...⟧.
>
> **c.** A theme for a school exercise, composition, or essay. ⟦...⟧
>
> **5.** A dissertation to maintain and prove a thesis (in sense 4); esp. one written or delivered by a candidate for a University degree. ⟦...⟧
>
> *The Oxford English Dictionary* [57]

The research reported in this thesis is *not* a finished piece of work. It is a snapshot of my research at the end of my PhD studies. The sole purpose of this thesis is for examination, *not* publication or dissemination. Therefore, the results presented may not be best possible. The threads between chapters are weak, and the mathematics is not written in the most elegant way. I would like to apologize to the reader, especially the examiners, for all these. Nevertheless, this thesis should be a clear and accurate record of the research I did during my PhD years.

This thesis is divided into two independent parts. The picture that the reader should keep in mind throughout is Figure 1.1. If M and N are as shown in this figure, then we say that M is an *initial segment* of N, or N is an *end-extension* of M. In Part I, we study

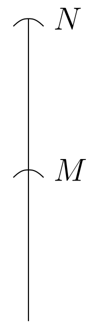

Figure 1.1: An initial segment, or an end-extension

a nice family of initial segments called the *generic cuts*. In Part II, we investigate how to construct end-extensions that are *strong*.

1.1 Generic cuts

In model theory, it is common [1, 2, ...] to study pairs of structures (M_1, M_2) where M_2 is a substructure of M_1. Model theorists are often interested in what properties are and can be preserved in such expansions.

Compared to the amount of work on initial segments of nonstandard models of arithmetic, there is very little research done on pairs (M, I) where I is a cut of a model M. Moreover, these research mostly involve the more arithmetical side of model theory only. For example, Kossak [38, 39], Smoryński [74, 75], and Kossak–Kotlarski [43, 44] considered the theories and the automorphisms of such pairs; Smith [73] studied elementary extensions of these expanded structures; and Kanoveĭ [27] and Kossak–Bamber [42] looked at the elements definable in the expanded language.

Building on the previous work by Kaye and the author [31, 33, 79], we investigate pairs (M, I) where I is a *generic cut* of a countable arithmetically saturated model M of PA. These cuts are 'generic' in the sense that they only satisfy properties that are *forced* on them. As a result, the information content in a generic cut is minimal. This makes the model theory of pairs (M, I) much nicer when I is generic. Since the notion of genericity is intimately connected with model theoretic forcing, the results we obtain are much closer in spirit to mainstream classical model theory.

The main results on generic cuts in the papers cited above are surveyed in Chapter 3. We adopt a slightly different set of terminologies here, so as to facilitate the development in later chapters. In Chapter 4, we investigate the question of whether genericity can be captured by saturation conditions. We state a number of conjectures, and explain why they are of interest to us.

We then move on to look at the theories $\mathrm{Th}(M, I)$ where I is a generic cut. Using the ω-rule, we characterize in Chapter 5 the set of sentences that are true in all such theories. Working towards a quantifier elimination result, a number of conjugacy properties are proved for arbitrary models of these sentences in Chapter 6. This enables one to shift the emphasis from semantical objects (M, I) to syntactical objects $\mathrm{Th}(M, I)$, to which techniques like compactness can be applied.

In Chapter 7, we prove that the pairs (M, I) where I is a generic cut are *existentially closed* in a suitable category. This analysis creates a new point of contact between models of arithmetic and classical model theory that is waiting to be exploited.

1.2 Strong end-extensions

Reverse mathematics is an active programme in the foundations of mathematics that aims to classify theorems according to their logical strength. A striking phenomenon is that a large number of theorems in mathematics fall into just *four* categories. Each of these theorems is equivalent to one of the following theories: WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0.

There must be a reason behind this. As a mathematician, I believe the reason is mathematical, or at least, one should be able to demonstrate it mathematically. There must be some *mathematical* properties of these four theories that make them so special.

For the theories WKL_0 and ACA_0, we already have quite a satisfactory answer. By the work of Scott [68], MacDowell–Specker [50], Paris–Kirby [62], and Tanaka [78], we know that models of these two theories can be characterized in terms of the *end-extensions* they have. In Chapter 8, we survey these results, and discuss how Figure 1.1 can be translated to second-order arithmetic.

End-extension is a natural model theoretic construction that one does to an ordered structure, for example, a model of arithmetic. In fact, it is the only sensible construction on the natural numbers as an arithmetical structure. This justifies our use of end-extensions as a means of analyzing theories of arithmetic.

We would like to generalize the results for WKL_0 and ACA_0 to ones for ATR_0 and Π^1_1-CA_0. There are two steps. Firstly, we need to construct special end-extensions that pertain some characteristics of ATR_0 and Π^1_1-CA_0. As ATR_0 is probably best known for its ability to handle ordinals, we start with model theoretic constructions that uses ordinals. In Chapter 9, we construct *ω_1-like* end-extensions in the sense of a model of ATR_0. With some extra combinatorics on the ordinals, we also prove that the extension constructed is *rather classless*.

Gaifman [17] and Phillips [63] utilized combinatorial principles such as Ramsey's Theorem in constructing special end-extensions of first-order models of arithmetic. They do so by adding to a model an ideal number, whose type is determined by a repeated application of Ramsey's Theorem. We generalize this argument by using versions of the Galvin–Prikry Theorem at the levels of ATR_0 and Π^1_1-CA_0. Instead of adding an ideal number, we add an ideal *set* to a model of second-order arithmetic. Chapter 10 sets up the necessary combinatorics, and Chapter 11 describes the construction.

The second step of our investigation is to isolate notions of end-extensions that are *strong*, i.e., if a model possesses such an extension, then it must satisfy ATR_0, Π^1_1-CA_0 or above. Chapter 11 contains some of our attempts at this problem. However, none of our solutions is satisfactory. Since our aim is to explain the importance of ATR_0 and Π^1_1-CA_0, the notions isolated should be of a model theoretic nature. Our characterizations have the disadvantage of having a combinatorial flavour that originates from reverse mathematics.

We blame this failure on the lack of model theoretic information about ATR_0 and

Π^1_1-CA_0. For WKL_0 and ACA_0, most of this knowledge comes from the study of initial segments of first-order models. We survey these results for ACA_0, and investigate the possibility of extending them to ATR_0 and Π^1_1-CA_0 in Chapter 12.

Initial segments provide a different way to look at models of second-order arithmetic. They are a kind of nonstandard analysis for nonstandard models. From the point of view of nonstandard mathematics, Keisler [34] provided a motivation for each of WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0. Our work can hence be considered as an alternative contribution to nonstandard analysis.

End-extension is not the only kind of model theoretic constructions for nonstandard models of arithmetic. *Cofinal extension* also plays an important role. Chapter 13 contains a brief discussion on how one may use such extensions to study the theories in reverse mathematics.

CHAPTER 2

FIRST- AND SECOND-ORDER ARITHMETIC

AXIOM: DEDEKIND–PEANO
There exist $1 \xrightarrow{0} N \xrightarrow{\sigma} N$ *in* $\mathcal{S}$ *such that for any diagram*

$$1 \xrightarrow{x_0} X \circlearrowleft \xi$$

in $\mathcal{S}$ *there exists a unique sequence* x *for which both* $x0 = x_0$ *and* $x\sigma = \xi x$

$$\begin{array}{ccccc} & \overset{0}{\nearrow} & N & \xrightarrow{\sigma} & N \\ 1 & & \downarrow x & & \downarrow x \\ & \underset{x_0}{\searrow} & X & \xrightarrow[\xi]{} & X \end{array}$$

⟦. . .⟧ In any category, an object N ⟦. . .⟧ satisfying the Dedekind–Peano axiom is called a **natural number object**.

F. William Lawvere and Robert Rosebrugh
Sets for Mathematics [48], Section 9.1

We assume the reader is familiar with first-order arithmetic. Readers of Part II are additionally assumed to have some acquaintance with second-order arithmetic. Important definitions are reproduced either in this chapter or in places where they are first needed. We refer the reader to Kaye [29] and Kossak–Schmerl [46] for background information in first-order arithmetic. For second-order arithmetic, see Simpson [72]. In some places, the language of category theory will be convenient. None of these will require anything deeper than those covered in a typical introductory text such as Herrlich–Strecker [21].

2.1 First-order arithmetic

We denote by $\mathscr{L}_\mathrm{I}$ the usual first-order language $\{0, 1, +, \times, <\}$ for arithmetic. A quantifier is *bounded* if it is of the form

$$\forall v < t \quad \text{or} \quad \exists v < t$$

where t is an $\mathscr{L}_\mathrm{I}$-term not involving v. The smallest class of $\mathscr{L}_\mathrm{I}$-formulas that contains the quantifier-free $\mathscr{L}_\mathrm{I}$-formulas and is closed under bounded quantification is called Δ_0. Set $\Sigma_0 = \Pi_0 = \Delta_0$. For $n \in \mathbb{N}$, define

$$\begin{aligned}\Sigma_{n+1} &= \{\exists \bar{v}\, \varphi(\bar{v}, \bar{z}) : \varphi(\bar{v}, \bar{z}) \in \Pi_n\}, \text{ and}\\ \Pi_{n+1} &= \{\forall \bar{v}\, \varphi(\bar{v}, \bar{z}) : \varphi(\bar{v}, \bar{z}) \in \Sigma_n\}.\end{aligned}$$

Note that up to logical equivalence, every $\mathscr{L}_\mathrm{I}$-formula is in Σ_n for some $n \in \mathbb{N}$, and $\Sigma_n \cup \Pi_n \subset \Sigma_{n+1} \cap \Pi_{n+1}$ for all $n \in \mathbb{N}$.

Our base theory in first-order arithmetic is the theory of the non-negative parts of discretely ordered rings, which we refer to as PA^-. The *induction axiom* on an $\mathscr{L}_\mathrm{I}$-formula $\varphi(v, \bar{z})$ is

$$\forall \bar{z}\big(\varphi(0, \bar{z}) \wedge \forall v(\varphi(v, \bar{z}) \rightarrow \varphi(v + 1, \bar{z})) \rightarrow \forall v\, \varphi(v, \bar{z})\big).$$

The *regularity axiom* on an $\mathscr{L}_\mathrm{I}$-formula $\psi(u, v, \bar{z})$ is

$$\forall \bar{z} \forall a \left(\begin{array}{l}\forall u < a\, \exists b\, \forall v > b\, \psi(u, v, \bar{z})\\ \quad \rightarrow \exists b\, \forall u < a\, \forall v > b\, \psi(u, v, \bar{z})\end{array}\right).$$

The two formulas displayed above are denoted by $\mathrm{I}_v\varphi$ and $\mathrm{R}_{u,v}\psi$ respectively. If Γ is a class of $\mathscr{L}_\mathrm{I}$-formulas, then

$$\begin{aligned}\mathrm{I}\Gamma &= \mathrm{PA}^- + \{\mathrm{I}_v\varphi : \varphi(v, \bar{z}) \in \Gamma\}, \text{ and}\\ \mathrm{B}\Gamma &= \mathrm{I}\Delta_0 + \{\mathrm{R}_{u,v}\psi : \psi(u, v, \bar{z}) \in \Gamma\}.\end{aligned}$$

It is known [62] that the following implications hold for all $n \in \mathbb{N}$.

$$\begin{array}{ccc} \mathrm{I}\Sigma_{n+1} & & \\ \Downarrow & & \\ \mathrm{B}\Sigma_{n+1} & \Leftrightarrow & \mathrm{B}\Pi_n \\ \Downarrow & & \\ \mathrm{I}\Sigma_n & \Leftrightarrow & \mathrm{I}\Pi_n. \end{array}$$

First-order *Peano arithmetic* (PA) is defined to be $\bigcup_{n\in\mathbb{N}} \mathrm{I}\Sigma_n$. We often consider models of $\mathrm{I}\Sigma_1$ as models of finite set theory via the Ackermann interpretation [32].

Let $M \models \mathrm{PA}^-$. We denote by $\mathrm{Th}(M)$ the $\mathscr{L}_\mathrm{I}$-theory of M. An *initial segment* of M is a subset of M that is closed downwards with respect to $<$. A *cut* of M is an initial segment that is closed under successors. We write $I \subseteq_\mathrm{e} M$ to mean 'I is a cut of M'. An *end-extension* of M is an extension K such that

$$\forall y \in K \setminus M\ \forall x \in M\ (x < y).$$

A *cofinal extension* of M is an extension K such that

$$\forall y \in K \setminus M\ \exists x \in M\ (x \geqslant y).$$

We write $M \subseteq_\mathrm{cf} K$ for 'K is a cofinal extension of M'. The combination $\forall y\, \exists x > y \cdots$ is often abbreviated as $\mathsf{Q}x \cdots$.

The language $\mathscr{L}_\mathrm{I}(M)$ is that obtained from $\mathscr{L}_\mathrm{I}$ by adding a new constant symbol for each element of M. The formula class $\Delta_0(M)$, $\Sigma_1(M)$, $\Pi_1(M)$, etc., are defined accordingly. The class of parametrically definable subsets of M is denoted by $\mathrm{Def}(M)$. If $I \subseteq_\mathrm{e} M$, then

$$\mathrm{SSy}_I(M) = \{X \cap I : X \in \mathrm{Def}(M)\}.$$

The *standard system* of M, often denoted by $\mathrm{SSy}(M)$, is defined to be $\mathrm{SSy}_\mathbb{N}(M)$. An end-extension K of M is said to be *conservative* if $\mathrm{SSy}_M(K) = \mathrm{Def}(M)$.

2.2 Second-order arithmetic

The language for second-order arithmetic is denoted by $\mathscr{L}_{\mathrm{II}}$. It is a first-order language with two sorts: a *number sort* and a *set sort*. The number sort has exactly the same symbols as $\mathscr{L}_{\mathrm{I}}$ does. So we can consider $\mathscr{L}_{\mathrm{I}}$ as a sublanguage of $\mathscr{L}_{\mathrm{II}}$. In addition, $\mathscr{L}_{\mathrm{II}}$ has a binary relation symbol $\in$ that relates a term of the number sort to a term of the set sort. There is no other nonlogical symbol in $\mathscr{L}_{\mathrm{II}}$. By convention, we only use lowercase letters for variables of the number sort, and uppercase letters for variables of the set sort.

The smallest class of $\mathscr{L}_{\mathrm{II}}$-formulas that contains the quantifier-free $\mathscr{L}_{\mathrm{II}}$-formulas and is closed under bounded (number) quantification is called Δ^0_0. The classes Σ^0_n and Π^0_n where $n \in \mathbb{N}$ are defined accordingly as in the case of first-order arithmetic. An $\mathscr{L}_{\mathrm{II}}$-formula is *arithmetical* if it is in Σ^0_n for some $n \in \mathbb{N}$. We use Σ^1_0 or Π^1_0 to denote the class of all arithmetical formulas. For $n \in \mathbb{N}$, set

$$\Sigma^1_{n+1} = \{\exists \bar{X}\, \varphi(\bar{X}, \bar{z}, \bar{W}) : \varphi(\bar{X}, \bar{z}.\bar{W}) \in \Pi^1_n\}, \text{ and}$$
$$\Pi^1_{n+1} = \{\forall \bar{X}\, \varphi(\bar{X}, \bar{z}, \bar{W}) : \varphi(\bar{X}, \bar{z}, \bar{W}) \in \Sigma^1_n\}.$$

An $\mathscr{L}_{\mathrm{II}}$-structure will be written in the form $(M, \mathscr{X})$, where M is its number universe and $\mathscr{X}$ is its set universe. A *nonstandard* $\mathscr{L}_{\mathrm{II}}$-structure is one whose number universe is not isomorphic to $\mathbb{N}$. All $\mathscr{L}_{\mathrm{II}}$-structures we consider satisfy the *axiom of extensionality*:

$$\forall X \forall Y \big(\forall v (v \in X \leftrightarrow v \in Y) \to X = Y\big).$$

So it will be assumed throughout that $\mathscr{X} \subseteq \mathcal{P}(M)$ for all $\mathscr{L}_{\mathrm{II}}$-structures $(M, \mathscr{X})$. The *comprehension axiom* on an $\mathscr{L}_{\mathrm{II}}$-formula $\varphi(v, \bar{z}, \bar{W})$ is

$$\forall \bar{z} \forall \bar{W} \exists X \forall v \big(v \in X \leftrightarrow \varphi(v, \bar{z}, \bar{W})\big),$$

and it is denoted by φ_v-CA. If Γ is a class of $\mathscr{L}_{\mathrm{II}}$-formulas, then

$$\Gamma\text{-CA} = \{\varphi_v\text{-CA} : \varphi(v, \bar{z}, \bar{W}) \in \Gamma\}.$$

For $n \in \mathbb{N}$, define Δ^0_n-CA to be the class of $\mathscr{L}_{\mathrm{II}}$-sentences of the form

$$\forall \bar{z} \forall \bar{W} \Big(\forall v \big(\varphi(v, \bar{z}, \bar{W}) \leftrightarrow \psi(v, \bar{z}, \bar{W}) \big) \rightarrow \exists X \forall v \big(v \in X \leftrightarrow \varphi(v, \bar{z}, \bar{W}) \big) \Big),$$

where $\varphi(v, \bar{z}, \bar{W}) \in \Sigma^0_n$ and $\psi(v, \bar{z}, \bar{W}) \in \Pi^0_n$. The axioms $\mathrm{I}_v\varphi$ and $\mathrm{R}_{u,v}\psi$ are defined as in first-order arithmetic for $\mathscr{L}_{\mathrm{II}}$-formulas $\varphi(v, \bar{z}, \bar{W})$ and $\psi(u, v, \bar{z}, \bar{W})$.

The theories RCA_0, ACA_0 and Π^1_1-CA_0 are defined as follows.

$$\begin{aligned} \mathrm{RCA}_0 &= \mathrm{PA}^- + \{\text{axiom of extensionality}\} + \Delta^0_1\text{-CA} \\ &\quad + \{\mathrm{I}_v\varphi : \varphi(v, \bar{z}, \bar{W}) \in \Sigma^0_1\}. \\ \mathrm{ACA}_0 &= \mathrm{RCA}_0 + \Sigma^1_0\text{-CA}. \\ \Pi^1_1\text{-CA}_0 &= \mathrm{RCA}_0 + \Pi^1_1\text{-CA}. \end{aligned}$$

Almost all our second-order models will satisfy RCA_0. It is well-known that many mathematical objects such as pairs, functions, sequences, trees, ordinals, etc., can be coded uniformly in models of RCA_0. Objects that can be coded within our second-order model are called *internal* objects. Exceptionally, we use lowercase Greek letters for internal ordinals, although they are actually objects of the set sort. The theory WKL_0 is RCA_0 together with *Weak König's Lemma*, which informally says that for all internal 0–1 trees T,

$$\begin{aligned} &\forall n\, \exists \text{ path } P \subseteq T \ \big(P \text{ is of length at least } n\big) \\ &\quad \rightarrow \exists \text{ path } P \subseteq T\ \forall n \big(P \text{ is of length at least } n\big). \end{aligned}$$

If H is an object of the set sort and i is a member of an internal ordinal α, then

$$(H)_i = \{x : \langle i, x\rangle \in H\}, \text{ and}$$
$$H{\restriction}_i = \{\langle j, x\rangle \in H : j \prec_\alpha i\}.$$

The theory ATR_0 is ACA_0 together with the *arithmetical transfinite recursion scheme*, which says that for every internal ordinal α and every arithmetical formula $\theta(n, X)$, possibly with parameters,

$$\begin{aligned}&\forall i \in \alpha\ \exists H \Big((H)_i = \big\{n : \theta(n, H{\restriction}_i)\big\}\Big)\\ &\to \exists H\, \forall i \in \alpha \Big((H)_i = \big\{n : \theta(n, H{\restriction}_i)\big\}\Big).\end{aligned}$$

The theories RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0 are known as the *Big Five* in reverse mathematics. It is well-known that

$$\mathrm{RCA}_0 \Leftarrow \mathrm{WKL}_0 \Leftarrow \mathrm{ACA}_0 \Leftarrow \mathrm{ATR}_0 \Leftarrow \Pi^1_1\text{-}\mathrm{CA}_0.$$

None of these implications reverses.

For $n \in \mathbb{N}$, let

$$\mathrm{I}\Sigma^*_n = \mathrm{WKL}_0 + \{\mathrm{I}_v\varphi : \varphi(v, \bar{z}, \bar{W}) \in \Sigma^0_n\}, \text{ and}$$
$$\mathrm{B}\Sigma^*_n = \mathrm{WKL}_0 + \{\mathrm{R}_{u,v}\psi : \psi(u, v, \bar{z}, \bar{W}) \in \Sigma^0_n\}.$$

Define $\mathrm{PA}^* = \bigcup_{n\in\mathbb{N}} \mathrm{I}\Sigma^*_n$. As in first-order arithmetic, we have

$$\mathrm{WKL}_0 \Leftrightarrow \mathrm{I}\Sigma^*_1 \Leftarrow \mathrm{B}\Sigma^*_2 \Leftarrow \mathrm{I}\Sigma^*_2 \Leftarrow \mathrm{B}\Sigma^*_3 \Leftarrow \cdots\ \mathrm{PA}^* \Leftarrow \mathrm{ACA}_0.$$

Note that these induction and regularity schema only tell us how tightly the *numbers* are glued together. They do not tell us how rich the universe of *sets* is. The richness of the universe of sets is controlled by set existence axioms such as comprehension, König's Lemma, and transfinite recursion. For example, any model $(\mathbb{N}, \mathscr{X})$ of WKL_0 satisfies PA^* because $\mathbb{N}$ is a very strong structure. Many of these models do not satisfy ACA_0.

Part I

Generic cuts

CHAPTER 3

GENERICITY

> Thus a must have certain special properties $[\![\dots]\!]$. Rather than describe a directly, it is better to examine the various properties of a and determine which are desirable and which are not. The chief point is that we do not wish a to contain "special" information about M, which can only be seen from the outside $[\![\dots]\!]$. The a which we construct will be referred to as a "generic" set relative to M. The idea is that all the properties of a must be "forced" to hold merely on the basis that a behaves like a "generic" set in M. This concept of deciding when a statement about a is "forced" to hold is the key point of the construction.
>
> Paul J. Cohen
> *Set Theory and the Continuum Hypothesis* [4], Section IV.2

In this chapter, we review the theory of generic cuts developed by Kaye and the author. Some definitions have been reformulated so that they fit better into the theory to be presented. We assume familiarity with Kaye's 'Generic cuts in models of arithmetic' [31], my MPhil (qualifying) thesis 'Generic cuts in a general setting' [79], and our joint paper 'Truth in generic cuts' [33]. These papers will be referred to as GCMA, GCGS, and TiGC respectively in this thesis.

Unless otherwise stated, we follow the notation in GCGS. Unlike the convention there, we write x^g for the image of the element x under the function g. We will often consider several models at a time. So whenever there is a risk of ambiguity, we indicate the model being looked at in the notation. For example, if $a, b \in M \subseteq K \models \mathrm{PA}^-$, then $[a, b]_M$ denotes $\{x \in M : a \leqslant x \leqslant b\}$ while $[a, b]_K$ denotes $\{x \in K : a \leqslant x \leqslant b\}$.

In GCGS, the topology on the space of cuts was generated by an abstract *notion of intervals*. Some of what we talk about in this part still work in this generality, but for the sake of simplicity, we will restrict ourselves to a more concrete special case. This is particularly convenient when the topology needs to be passed from one model to another. As we shall see, a *cut-base* is essentially a notion of intervals that does not depend on any parameter from the model.

Definition. Let M be a nonstandard model of PA. A *cut-base* on M is a recursive set of parameter-free $\mathscr{L}_\mathrm{I}$-formulas $p(x, y)$ such that the following axioms hold in M.

(1) $\exists x \exists y \bigwedge\!\!\!\bigwedge p(x, y)$.

(2) $\forall x \forall y \big(\bigvee\!\!\!\bigvee p(x, y) \rightarrow x < y \big)$.

(3) $\forall x \forall y \big(\bigwedge\!\!\!\bigwedge p(x, y) \rightarrow \exists z \big(\bigwedge\!\!\!\bigwedge p(x, z) \wedge \bigwedge\!\!\!\bigwedge p(z, y) \big) \big)$.

(4) $\forall x \forall y \big(\bigwedge\!\!\!\bigwedge p(x, y) \rightarrow \forall [u, v] \supseteq [x, y] \bigwedge\!\!\!\bigwedge p(u, v) \big)$.

Suppose $p(x, y)$ is a cut-base on M. For a finite semi-interval $[a, b] \subseteq M$, we say that $[a, b]$ is a *p-interval*, and write $a \ll b$, if $M \models \bigwedge\!\!\!\bigwedge p(a, b)$. The double brackets $[\![\cdot, \cdot]\!]$ will only be used for delimiting p-intervals. When there is no risk of ambiguity, the reference to p is often omitted. We say that a set of $\mathscr{L}_\mathrm{I}$-formulas is a *cut-base* if it is a cut-base on some nonstandard model of PA.

Let $p(x, y)$ be a cut-base on a model M of PA. Since $p(x, y)$ is recursive, we can recursively enumerate its elements as $p_0(x, y), p_1(x, y), p_2(x, y), \ldots$ such that p_{i+1} is stronger than p_i for all $i \in \mathbb{N}$. We can also assume without loss that

$$M \models \forall x \forall y \left(\bigwedge\!\!\!\bigwedge p_i(x, y) \rightarrow \forall [u, v] \supseteq [x, y] \bigwedge\!\!\!\bigwedge p_i(u, v) \right)$$

for all $i \in \mathbb{N}$. The notation $p_0(x, y), p_1(x, y), p_2(x, y), \ldots$ will be used throughout. Suppose M is recursively saturated. Then for every element $\partial \in M$, there is a coded function

$Y\colon M_{<\partial} \times M_{<\partial} \to M$ such that for all $x, y < \partial$,

$$Y(x,y) = \begin{cases} (\max n)\big(\forall i < n\; p_i(x,y)\big), & \text{whenever such } n \text{ exists;} \\ \text{a fixed nonstandard element of } M, & \text{otherwise.} \end{cases}$$

Such a function Y is called a *monotone indicator* for $p(x, y)$ below ∂ in M. An indicator Y is said to *cover* a value $c \in M$ if its domain includes $M_{\leqslant c} \times M_{\leqslant c}$. Similarly, an indicator *covers* a cut I if its domain includes $I \times I$.

Remark. Note that although an indicator below a fixed number is actually coded by a number, we usually use the uppercase letter Y to denote it. The tradition of using Y for an indicator is so strong that it seems inappropriate to change it just for the consistency of this thesis. A similar exception is the use of the letter p for types.

Definition. Let $p(x, y)$ be a cut-base on a model M of PA. Then the collection of all p-intervals is called the *notion of intervals associated with* $p(x, y)$ *on* M. We say that a cut I of M is a *p-cut* if

$$\forall n \in \mathbb{N}\; \forall x \in I\; \exists y \in I\; M \models p_n(x, y).$$

It is straightforward to verify that the notion of intervals associated with a cut-base $p(x, y)$ is really a notion of intervals, and the collection of p-cuts is a complete species. Therefore, results in GCGS transfer to the present setting. On the other hand, all the notions of intervals that we mentioned in GCGS are essentially cut-bases. So it seems the additional restrictions posed are not too strict.

Example 3.1. Let $M \models$ PA and Y be a GCMA indicator. Then

$$p^Y(x, y) = \{Y(x, y) > n : n \in \mathbb{N}\}$$

is a cut-base on M.

Example 3.2. Let M be a recursively saturated model of PA. Then

$$p^{\text{elem}}(x,y) = \big\{(\mu v)(\varphi(x,v)) < y : \varphi(x,v) \in \mathscr{L}_{\mathrm{I}}\big\}$$

is a cut-base on M.

Remark. Note, however, the notion of a *neighbourhood system* introduced in TiGC is strictly weaker than that of a notion of intervals. Compare Theorem 4.1 in GCGS with Proposition 3.3 in TiGC.

It was shown in GCMA and in GCGS that arithmetically saturated models of PA contain certain self-similar intervals.

Definition. Let $\bar{c} \in M \models \mathrm{PA}$, and $p(x,y)$ be a cut-base on M. We say that a finite p-interval $[\![a,b]\!]$ is *constant* over $\bar{c}$ if

$$\forall x \in [\![a,b]\!]\ \forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x' \in [\![u,v]\!]\ \ (x,\bar{c}) \equiv (x',\bar{c}).$$

The interval $[\![a,b]\!]$ is *pregeneric* over $\bar{c}$ if

$$\forall x,y \in [\![a,b]\!]\ \forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x',y' \subseteq [\![u,v]\!]\ \ (x,y,\bar{c}) \equiv (x',y',\bar{c}).$$

Theorem 3.3. Fix an arithmetically saturated model M of PA, and a cut-base $p(x,y)$ on M. Let $\bar{c} \in M$. Then every p-interval contains a subinterval that is pregeneric over $\bar{c}$.

Proof. See Section 5 of TiGC. □

It is not hard to express these self-similarity notions in terms of automorphisms, provided the model is countable and recursively saturated. In the other direction, one can finitize these notions further using recursive saturation.

Proposition 3.4. Fix a recursively saturated model M of PA and a cut-base $p(x,y)$ on M. Let $\bar{c} \in M$ and $[\![a,b]\!]$ be a p-interval.

(a) The interval $[\![a,b]\!]$ is constant over $\bar{c}$ if and only if for every $\mathscr{L}_{\mathrm{I}}$-formula $\varphi(x,\bar{z})$, there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} M \models\ & \exists x \in [\![a,b]\!]\ \varphi(x,\bar{c}) \\ & \to \forall [u,v] \subseteq [\![a,b]\!]\ \big(p_n(u,v) \to \exists x \in [u,v]\ \varphi(x,\bar{c})\big). \end{aligned}$$

(b) The interval $[\![a,b]\!]$ is pregeneric over $\bar{c}$ if and only if for every $\mathscr{L}_{\mathrm{I}}$-formula $\varphi(x,y,\bar{z})$, there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} M \models\ & \exists x,y \in [\![a,b]\!]\ \varphi(x,y,\bar{c}) \\ & \to \forall [u,v] \subseteq [\![a,b]\!]\ \big(p_n(u,v) \to \exists x,y \in [u,v]\ \varphi(x,y,\bar{c})\big). \end{aligned}$$ □

Part (a) of this proposition can be reformulated as follows.

Corollary 3.5. Fix a recursively saturated model M of PA. Let $p(x,y)$ be a cut-base on M, and $\bar{c} \in M$. Then a p-interval $[\![a,b]\!]$ is constant over $\bar{c}$ if and only if one *cannot* find $x \in [\![a,b]\!]$ and $\varphi(x,\bar{z}) \in \mathscr{L}_{\mathrm{I}}$ such that for all $n \in \mathbb{N}$,

$$M \models \varphi(x,\bar{c}) \wedge \begin{pmatrix} \forall x' > x\ \big(\neg p_n(x,x') \to \neg\varphi(x',\bar{c})\big) \\ \vee\ \forall x' < x\ \big(\neg p_n(x',x) \to \neg\varphi(x',\bar{c})\big) \end{pmatrix}.$$ □

This corollary tells us that, for instance, the union of two intersecting constant intervals is constant. We cannot find analogous results for pregeneric intervals. So let me repeat an innocent looking question from GCGS that I have been unable to answer.

Question 3.6. Do the notions of constant intervals and pregeneric intervals coincide?

Let $p(x,y)$ be a cut-base on a model M of PA. Recall that the notion of intervals associated with $p(x,y)$ generates a topology on the class of all p-cuts. If M is countable, then this topological space is homeomorphic to the Cantor set. As in GCMA, we borrow a notion from Baire category theory to describe when a class of cuts is large: a class of p-cuts is *enforceable* if it contains a countable intersection of dense open sets.

Definition. Let $M \models \mathrm{PA}$ and $p(x, y)$ be a cut-base on M. A p-cut is *p-generic* if it is an element of each enforceable class of cuts that is closed under the automorphisms of M. The reference to p is sometimes omitted when there is no risk of ambiguity.

Theorem 3.7. Let M be a countable arithmetically saturated model of PA, and let $p(x, y)$ be a cut-base on M. Then the class of p-generic cuts is enforceable.

Proof. See Section 5 of TiGC. □

There is an interesting corollary to this theorem for elementary cuts. The reader may like to compare this with the remark before Theorem 11.10.

Corollary 3.8. Let M be a countable arithmetically saturated model of PA, and let $\mathscr{Z}^{\mathrm{elem}}$ denote the class of all elementary cuts of M. If $P \subseteq \mathscr{Z}^{\mathrm{elem}}$ that is closed under the automorphisms of M, then either P or $\mathscr{Z}^{\mathrm{elem}} \setminus P$ is enforceable in $\mathscr{Z}^{\mathrm{elem}}$.

Proof. Let I be any elementary generic cut in M. If $I \in P$, then by Proposition 7.10 and Theorem 5.19 in TiGC, all elementary generic cuts are in P, and so P is enforceable by Theorem 3.7 above. Similarly, if $I \notin P$, then $\mathscr{Z}^{\mathrm{elem}} \setminus P$ is enforceable in $\mathscr{Z}^{\mathrm{elem}}$. □

Question 3.9. Is arithmetic saturation necessary for this corollary?

The proof of Theorem 3.7 depends on the close relationship between pregeneric intervals and generic cuts.

Theorem 3.10. Let M be a countable arithmetically saturated model of PA, and $p(x, y)$ be a cut-base on M. Then a cut is p-generic if and only if it is contained in a pregeneric interval over $\bar{c}$ for every $\bar{c} \in M$.

Proof. See Section 5 of TiGC. □

Generic cuts behave very nicely with respect to the automorphisms of the model. Recall that two cuts I, J of a model M are *conjugate* over $\bar{c} \in M$ if there is an automorphism $g \in \mathrm{Aut}(M, \bar{c})$ such that $I^g = J$.

Theorem 3.11. Fix a countable arithmetically saturated model M of PA, a cut-base $p(x, y)$ on M, and $\bar{c} \in M$. Then all p-generic cuts within a pregeneric interval over $\bar{c}$ are conjugate over $\bar{c}$.

Proof. See Section 5 of TiGC. □

The point of studying generic cuts is that these cuts do not contain very much additional information about the model. A consequence of this is a weak quantifier elimination result for the pair (M, I) where I is a generic cut of M.

Theorem 3.12. Fix a countable arithmetically saturated model M of PA, a cut-base $p(x, y)$ on M, and a p-generic cut I of M. For two elements $c, c' \in M$, we have $(M, I, c) \cong (M, I, c')$ if and only if for all $\mathscr{L}_{\mathrm{I}}$-formulas $\varphi(x, z)$,

$$M \models \mathsf{Q}x \in I\ \varphi(x, c) \leftrightarrow \mathsf{Q}x \in I\ \varphi(x, c').$$

Proof. See Corollary 6.13 in TiGC. □

Many of these results will be sharpened later.

CHAPTER 4

SATURATION

⟦I⟧s 2 a random number?

Donald E. Knuth
The Art of Computer Programming 2 [37], Section 3.1

The last theorem in the previous chapter tells us that the formulas $\mathsf{Q}x \in I\ \varphi(x, \bar{z})$ where $\varphi(x, \bar{z}) \in \mathscr{L}_{\mathrm{I}}$ are important for generic cuts. These formulas can be rewritten in the form

$$\forall y \in I\ \exists x \in I\ \psi(x, y, \bar{z}),$$

for some $\psi(x, y, \bar{z}) \in \mathscr{L}_{\mathrm{I}}$. Let us look at these formulas in more detail in this chapter.

Recall from GCGS that the language $\mathscr{L}_{\mathrm{Sk}}$ consists of the symbols in $\mathscr{L}_{\mathrm{I}}$ and a function symbol for each $\mathscr{L}_{\mathrm{I}}$-Skolem function. The language $\mathscr{L}^*_{\mathrm{Sk}}$ is obtained from $\mathscr{L}_{\mathrm{Sk}}$ by adding a new unary predicate symbol intended for a cut. We often use I for this new predicate symbol, but sometimes another letter is used depending on the situation.

Definition. The set of all $\mathscr{L}_{\mathrm{Sk}}$-formulas is called $\exists^*_0$ or $\forall^*_0$. For $n \in \mathbb{N}$, define

$$\exists^*_{n+1} = \{\exists \bar{v} \in I\ \varphi(\bar{v}, \bar{z}) : \varphi(\bar{v}, \bar{z}) \in \forall^*_n\}, \text{ and}$$
$$\forall^*_{n+1} = \{\forall \bar{v} \in I\ \varphi(\bar{v}, \bar{z}) : \varphi(\bar{v}, \bar{z}) \in \exists^*_n\}.$$

Theorem 3.12 says that the $\exists^*_2$-type determines the whole $\mathscr{L}^*_{\mathrm{Sk}}$-type for a generic cut. In particular, quantifications over the whole model can be controlled by quantifications over the generic cut.

Corollary 4.1. Fix a countable arithmetically saturated model M of PA and a cut-base $p(x, y)$ on M. With respect to a p-generic cut, if two elements of M have the same $\exists_2^*$-type, then they have the same $\mathscr{L}_{\mathrm{Sk}}^*$-type. □

We expect that $\exists_2^*$ cannot be replaced by $\exists_1^*$ or $\forall_1^*$ in the above corollary. The class $\exists_2^*$ seems to have a very special status in the study of (generic) cuts, the exact reason of which is not clear to us yet.

It is worth noting that the classes $\forall_2$ and $\exists_2$ play a significant role in classical model theory [3, 25] in relation to chains of models, model companions, etc. This line will be pursued in Chapter 7.

Recall that we are interested in $\mathscr{L}_{\mathrm{Sk}}^*$-structures of the form (M, I) where I is a cut of a model of M of PA. Given one such pair (M, I), a natural problem is to find out what $\mathscr{L}_{\mathrm{I}}$-definable functions the cut I is closed under. For instance, such problems have been looked at in the proof theoretic analysis of independence results. As the closure of a cut under an $\mathscr{L}_{\mathrm{I}}$-definable function is expressible by a $\forall_2^*$-formula, this may explain why $\exists_2^*$-formulas are important for the study of cuts.

The class $\exists_2^*$ also appears when one looks at how saturated the pair (M, I) is. In a sense, the following proposition says that generic cuts are closed under as few functions as possible. This matches with our intuition on genericity.

Proposition 4.2. Let M be a countable recursively saturated model of PA, and $p(x, y)$ be a cut-base on M. Then the collection of all p-cuts I such that (M, I) is $\exists_2^*$-recursively saturated is enforceable.

Proof. This is similar to the proof of Theorem 4.6 in GCMA. We show how one can enforce $\exists_2^*$-recursive saturation in a Banach–Mazur game. Suppose we are given an interval $[\![a, b]\!]$ to play in. Consider the recursive set

$$q(v) = \{\exists \bar{x} \in I\ \forall \bar{y} \in I\ \theta_i(v, \bar{x}, \bar{y}, \bar{c}) : i \in \mathbb{N}\},$$

where $\bar{c} \in M$ and $\theta_i(v, \bar{x}, \bar{y}, \bar{z}) \in \mathscr{L}_{\mathrm{I}}$ for each $i \in \mathbb{N}$.

Suppose first that for all $n \in \mathbb{N}$,

$$M \models \exists v \, \exists [r, s] \subseteq [\![a, b]\!] \, \Big(p_n(r, s) \wedge \bigwedge\!\!\!\bigwedge_{i<n} \exists \bar{x} < r \, \forall \bar{y} < s \; \theta_i(v, \bar{x}, \bar{y}, \bar{c}) \Big).$$

Using the recursive saturation of M, pick $v \in M$ and $[\![r, s]\!] \subseteq [\![a, b]\!]$ such that for all $i \in \mathbb{N}$,

$$M \models \exists \bar{x} < r \, \forall \bar{y} < s \; \theta_i(v, \bar{x}, \bar{y}, \bar{c}).$$

We play $[\![r, s]\!]$ in this move. At the end, if I is the outcome of the play, then $r \in I < s$, and so v realizes $q(v)$ in (M, I).

Suppose now that we are not in the previous case. Let $n \in \mathbb{N}$ such that

$$M \models \neg \exists v \, \exists [r, s] \subseteq [\![a, b]\!] \, \Big(p_n(r, s) \wedge \bigwedge\!\!\!\bigwedge_{i<n} \exists \bar{x} < r \, \forall \bar{y} < s \; \theta_i(v, \bar{x}, \bar{y}, \bar{c}) \Big), \qquad (*)$$

and let I be any p-cut in $[\![a, b]\!]$. We show that $q(v)$ is not finitely satisfied in (M, I). Suppose this is not true. Then in particular,

$$M \models \exists v \bigwedge\!\!\!\bigwedge_{i<n} \exists \bar{x} \in I \, \forall \bar{y} \in I \; \theta_i(v, \bar{x}, \bar{y}, \bar{c}).$$

Recall that n is standard. So

$$M \models \exists v \, \exists \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-1} \in I \bigwedge\!\!\!\bigwedge_{i<n} \forall \bar{y} \in I \; \theta_i(v, \bar{x}_i, \bar{y}, \bar{c}).$$

Let $r \in I$ that is an upper bound for the witnesses $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-1}$ and for a. Pick also $s \in I$ such that $M \models p_n(r, s)$, which is possible since I is a p-cut. Then

$$M \models \exists v \bigwedge\!\!\!\bigwedge_{i<n} \exists \bar{x} < r \, \forall \bar{y} < s \; \theta_i(v, \bar{x}, \bar{y}, \bar{c}).$$

This contradicts $(*)$. $\square$

Corollary 4.3. Let M be a countable recursively saturated model of PA, and $p(x, y)$ be a cut-base on M. If I is a p-generic cut, then (M, I) is $\exists_2^*$-recursively saturated. $\square$

We want to understand more deeply what $\exists_2^*$-recursive saturation means for a cut. In particular, we would like to know how much genericity is captured by this property. We cannot hope to get full genericity back because generic cuts are not $\forall_2^*$-recursively saturated. A proof of this fact appeared in GCGS. It is reproduced here for completeness.

Definition. Let $M \models \mathrm{PA}$ and $p(x, y)$ be a cut-base on M. We say that a cut $I \subseteq_{\mathrm{e}} M$ has *index* $\mathbb{N}$ in M if whenever Y is a monotone indicator for $p(x, y)$ that covers I, we have

$$\{n \in M : \forall x \in I\ \exists y \in I\ Y(x, y) > n\} = \mathbb{N}.$$

Lemma 4.4. Let $I \subsetneq_{\mathrm{e}} M \models \mathrm{PA}$ and $p(x, y)$ be a cut-base on M. If we can find a monotone indicator Y for $p(x, y)$ covering I such that

$$\{n \in M : \forall x \in I\ \exists y \in I\ Y(x, y) > n\} = \mathbb{N},$$

then I is of index $\mathbb{N}$ in M.

Proof. Let Y' be another monotone indicator for $p(x, y)$ which covers I. Suppose we can find a nonstandard $n \in M$ that satisfies

$$M \models \forall x \in I\ \exists y \in I\ Y'(x, y) > n.$$

Using overspill, let $b \in M \setminus I$ such that both Y and Y' cover b, and

$$M \models \forall x < b\ \exists y\ Y'(x, y) > n.$$

Consider the set $\left\{Y\big(x, (\mu y)(Y'(x, y) > n)\big) : x < b\right\}$. It is M-finite, and so it must have a minimum element. Let d be its minimum element. By the choices of n and b, we have

$$M \models \forall x \in I \, \exists y \in I \, Y(x, y) \geqslant d.$$

This contradicts our initial assumption on Y because d must be nonstandard. □

Proposition 4.5. Let M be a countable model of PA, and $p(x, y)$ be a cut-base on M. Then the collection of all p-cuts with index $\mathbb{N}$ is enforceable.

Proof. We demonstrate how this can be enforced in a Banach–Mazur game. Suppose we are given $[\![a, b]\!]$ to play in. Let Y be a monotone indicator for $p(x, y)$ that covers b. By the countability of M, it suffices to enforce

$$d \notin \{n \in M : M \models \forall x \in I \, \exists y \in I \, Y(x, y) > n\}$$

for every nonstandard d in M. Since Y is monotone, this can be achieved by choosing a subinterval $[\![u, v]\!] \subseteq [\![a, b]\!]$ where $Y(u, v)$ is less than the nonstandard d that is being considered. The reader can easily verify that such $[\![u, v]\!]$ exists. Lemma 2.15 in TiGC contains the details of the argument. □

Corollary 4.6. Let M be a countable model of PA, and $p(x, y)$ be a cut-base on M. Then the collection of all p-cuts I such that (M, I) is not $\forall_2^*$-recursively saturated is enforceable.

Proof. Let Y be a monotone indicator that covers I. If

$$\{n \in M : M \models \forall x \in I \, \exists y \in I \, Y(x, y) > n\} = \mathbb{N},$$

then the $\forall_2^*$-set

$$q(v) = \{\forall x \in I \, \exists y \in I \, Y(x, y) > v\} \cup \{v > n : n \in \mathbb{N}\}$$

is finitely satisfied but not realized in (M, I). □

Recall that we want to capture genericity using saturation conditions on the pair. Let us pose this as a precise question, after which some partial answers will be presented, and some conjectures will be discussed.

Question 4.7. Let I be a cut of a countable arithmetically saturated model $M \models \mathrm{PA}$ such that (M, I) is $\exists_2^*$-recursively saturated but not $\forall_2^*$-recursively saturated. Can one always find a cut-base on M for which I is generic?

Under these assumptions, we want to seek a pregeneric interval around our cut. This means that 'special' points, for example, those that appeared in Corollary 3.5, cannot be arbitrarily close to the cut. We will use $\exists_1^* \cup \forall_1^*$-recursive saturation to prove something in this direction. Note that as Kaye mentioned in GCMA, Σ_2-recursive saturation is the same as Π_1-recursive saturation for models of $\mathrm{I}\Delta_0 + \exp$.

Proposition 4.8. Fix a recursively saturated $M \models \mathrm{PA}$ and $I \subseteq_{\mathrm{e}} M$. Then (M, I) is $\exists_1^* \cup \forall_1^*$-recursively saturated if and only if for each $\bar{c} \in M$, there exists a finite semi-interval $[a, b]$ containing I such that for all $\varphi(\bar{x}, \bar{z}) \in \mathscr{L}_{\mathrm{I}}$, the following are true in M.

(i) $\exists \bar{x} \in I\ \varphi(\bar{x}, \bar{c}) \rightarrow \exists \bar{x} < a\ \varphi(\bar{x}, \bar{c})$.

(ii) $\forall \bar{x} > I\ \varphi(\bar{x}, \bar{c}) \rightarrow \forall \bar{x} > a\ \varphi(\bar{x}, \bar{c})$.

(iii) $\forall \bar{x} \in I\ \varphi(\bar{x}, \bar{c}) \rightarrow \forall \bar{x} < b\ \varphi(\bar{x}, \bar{c})$.

(iv) $\exists \bar{x} > I\ \varphi(\bar{x}, \bar{c}) \rightarrow \exists \bar{x} > b\ \varphi(\bar{x}, \bar{c})$.

Proof. For the 'only if' direction, realize the recursive type

$$\begin{aligned} q(a, b) = \ & \{\exists u \in I\ (u = a)\} \cup \{\forall u \in I\ (u < b)\} \\ & \cup \{\exists \bar{x} \in I\ \varphi(\bar{x}, \bar{c}) \rightarrow \exists \bar{x} < a\ \varphi(\bar{x}, \bar{c}) : \varphi(\bar{x}, \bar{z}) \in \mathscr{L}_{\mathrm{I}}\} \\ & \cup \{\exists u \in I\ \forall \bar{x} > u\ \varphi(\bar{x}, \bar{c}) \rightarrow \forall \bar{x} > a\ \varphi(\bar{x}, \bar{c}) : \varphi(\bar{x}, \bar{z}) \in \mathscr{L}_{\mathrm{I}}\} \\ & \cup \{\forall \bar{x} \in I\ \varphi(\bar{x}, \bar{c}) \rightarrow \forall \bar{x} < b\ \varphi(\bar{x}, \bar{c}) : \varphi(\bar{x}, \bar{z}) \in \mathscr{L}_{\mathrm{I}}\} \\ & \cup \{\forall u \in I\ \exists \bar{x} > u\ \varphi(\bar{x}, \bar{c}) \rightarrow \exists \bar{x} > b\ \varphi(\bar{x}, \bar{c}) : \varphi(\bar{x}, \bar{z}) \in \mathscr{L}_{\mathrm{I}}\}. \end{aligned}$$

For the 'if' direction, simply note that each recursive $\exists_1^* \cup \forall_1^*$-type

$$\{\forall \bar{x} \in I\ \varphi_i(\bar{x}, \bar{c}) : i \in \mathbb{N}\} \cup \{\exists \bar{x} \in I\ \psi_i(\bar{x}, \bar{c}) : i \in \mathbb{N}\}$$

is equivalent in M to

$$\{\forall \bar{x} < b\ \varphi_i(\bar{x}, \bar{c}) : i \in \mathbb{N}\} \cup \{\exists \bar{x} < a\ \psi_i(\bar{x}, \bar{c}) : i \in \mathbb{N}\}.$$

Actually, only (i) and (iii) are needed. □

We hope to obtain an infinitary version of this proposition. The techniques used in the proof of Theorem 3.10 in TiGC may help.

Conjecture 4.9. Fix a countable recursively saturated model $M \models \mathrm{PA}$, and $I \subsetneq_{\mathrm{e}} M$ such that (M, I) is $\exists_1^* \cup \forall_1^*$-recursively saturated. Then there is an $\mathscr{L}_\mathrm{I}$-elementary embedding $g \colon M \to M$ with the property that

$$\sup I^g < I < \inf(M \setminus I)^g.$$

The following is essentially Proposition 8.2 in GCGS.

Proposition 4.10. Let $p(x, y)$ be a cut-base on a model $M \models \mathrm{PA}$, and I be a p-generic cut of M. If c codes a sequence of length $a + 1 \in M$ such that $(c)_i \ll (c)_{i+1}$ for all $i < a$, then $\{(c)_i \in I : i \leqslant a\}$ cannot be cofinal in I. □

We will need a variant of this in Chapter 7. The proof is similar.

Proposition 4.11. Fix a countable recursively saturated $M \models \mathrm{PA}$, a cut-base $p(x, y)$ on M, and a p-cut I of M. Suppose I is a p-generic cut of M. If F is an $\mathscr{L}_\mathrm{I}(M)$-definable function $M \to M$ under which I is closed, then there is a semi-interval $[a, b]$ around I such that

$$M \models \forall x \in [a, b]\ \neg \bigwedge\!\!\!\bigwedge p(x, F(x)).$$ □

In fact, this is exactly what we will need to show that generic cuts are existentially closed in the proof of Theorem 7.1. It seems this property should follow from our saturation conditions.

Conjecture 4.12. The condition 'I is a p-generic cut' in Proposition 4.11 can be weakened to 'I is of index $\mathbb{N}$ and (M, I) is $\exists_2^*$-recursively saturated.'

Before moving on, I would like to discuss non-$\forall_2^*$-recursive saturation briefly. The only piece of knowledge we have at hand is that if a cut I is of index $\mathbb{N}$ with respect to some definable function in a model M, then the pair (M, I) cannot be $\forall_2^*$-recursively saturated. We expect that the converse is true, provided (M, I) is $\exists_2^*$-recursively saturated.

Conjecture 4.13. Fix a recursively saturated model $M \models \mathrm{PA}$, and a cut $I \subseteq_{\mathrm{e}} M$ that is closed under multiplication. Suppose (M, I) is $\exists_2^*$-recursively saturated but not $\forall_2^*$-recursively saturated. Then there is $\partial \in M \setminus I$ and a coded function $Y\colon M_{<\partial} \times M_{<\partial} \to M$ such that

(1) $\forall a, b < \partial \left(a \in I < b \Rightarrow Y(a, b) > \mathbb{N}\right)$; and

(2) $\{n \in M : \forall x \in I\ \exists y \in I\ Y(x, y) > n\} = \mathbb{N}$.

CHAPTER 5

THE THEORY

> The most important property of the class $\mathscr{E}$ is the existence of a uniform definition of the standard part of members of $\mathscr{E}$. This allows us to have the best of both worlds, for we can look at models of a comparatively weak number theory $[\![\ldots]\!]$ but still have access to full number theory $[\![\ldots]\!]$.
>
> Harold Simmons
> Existentially closed models of basic number theory [69], §2

We saw how the standard cut $\mathbb{N}$ can be described by a generic cut in the previous chapter. In this chapter, we use the standard cut to describe a generic cut in return. More precisely, we characterize the theories $\mathrm{Th}(M, I)$ where I is a generic cut in a model M using the ω-rule.

Definition. The language $\mathscr{L}_{\boldsymbol{\omega}}^{+}$ consists of the symbols in $\mathscr{L}_{\mathrm{Sk}}$ together with two new unary predicate symbols $I, \boldsymbol{\omega}$, one new binary predicate symbol S, and one new ternary predicate symbol p. We usually write $p_i(x, y)$ for $p(i, x, y)$.

The symbols I and p are intended for a cut and a cut-base respectively, and the symbol S is intended for a satisfaction class. The symbol $\boldsymbol{\omega}$ will be interpreted as a cut that behaves like the standard cut.

Definition. The $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-theory Gen consists of the following axioms.

(1) PA is satisfied.

(2) The $\mathscr{L}_{\mathrm{I}}$-Skolem functions are defined as intended.

(3) I and $\boldsymbol{\omega}$ are nonempty proper initial segments.

(4) p is a monotone cut-base with respect to $\boldsymbol{\omega}$:

(a) $$\exists\chi \in \boldsymbol{\omega} \left(\begin{array}{l} \chi \in \Sigma_1 \wedge \forall i \in \boldsymbol{\omega}\ \exists!\varphi \in \boldsymbol{\omega}\ S(\chi, [i, \varphi]) \\ \wedge\ \forall i, \varphi \in \boldsymbol{\omega} \left(\begin{array}{l} S(\chi, [i, \varphi]) \\ \to \varphi \in \mathscr{L}_{\mathrm{I}} \wedge \forall x \forall y \big(S(\varphi, [x, y]) \leftrightarrow p_i(x, y) \big) \end{array} \right) \end{array} \right);$$

(b) $\forall \partial\, \exists \hat{p}\, \forall x, y < \partial\ \forall i \big(p_i(x, y) \leftrightarrow \langle i, x, y \rangle \in \hat{p} \big)$;

(c) $\exists x \exists y \forall i \in \boldsymbol{\omega}\ p_i(x, y)$;

(d) $\forall x \forall y \big(\exists i \in \boldsymbol{\omega}\ p_i(x, y) \to x < y \big)$;

(e) $\forall x \forall y \big(\forall i \in \boldsymbol{\omega}\ p_i(x, y) \to \exists z \big(\forall i \in \boldsymbol{\omega}\ p_i(x, z) \wedge \forall i \in \boldsymbol{\omega}\ p_i(z, y) \big) \big)$;

(f) $\forall x \forall y \forall i \big(p_{i+1}(x, y) \to p_i(x, y) \big)$;

(g) $\forall x \forall y \forall i \big(p_i(x, y) \to \forall [u, v] \supseteq [x, y]\ p_i(u, v) \big)$.

(5) S is an inductive satisfaction class for the $\mathscr{L}_{\mathrm{I}}$-part.

(6) S satisfies Tarski's conditions for truth for all $\mathscr{L}_{\mathrm{I}}$-formulas in $\boldsymbol{\omega}$.

(7) $\boldsymbol{\omega}$ is closed under exponentiation.

(8) $\boldsymbol{\omega}$ is strong.

(9) I is a p-cut of index $\boldsymbol{\omega}$: $\forall i \ \big(i \in \boldsymbol{\omega} \leftrightarrow \forall x \in I\ \exists y \in I\ p_i(x, y) \big)$.

(10) I is p-generic with respect to $\boldsymbol{\omega}$, in the sense that

$$\forall \bar{c}\ \exists [a, b] \ni I\ \forall x, y \in [a, b]\ \forall [u, v] \subseteq [a, b]$$
$$\left(\begin{array}{l} \forall i \in \boldsymbol{\omega}\ p_i(u, v) \\ \to\ \exists x', y' \in [u, v]\ \forall k \in \boldsymbol{\omega}\ \forall \varphi \in \boldsymbol{\omega} \\ \qquad \big(S(\varphi, [x, y, \bar{c}, k]) \leftrightarrow S(\varphi, [x', y', \bar{c}, k]) \big) \end{array} \right).$$

Remark. Axiom (6) is actually not needed for the present chapter. For the next chapter, it will be used to ensure that we can meaningfully talk about $p_i(x, y)$ even when $i \in \boldsymbol{\omega} \setminus \mathbb{N}$.

It is obvious that if I is a p-generic cut in an arithmetically saturated model of PA where $p(x, y)$ is a cut-base on M, and S is a partial inductive satisfaction class on M, then $(M, I, \mathbb{N}, S, p) \models \text{Gen}$. It is also clear that if $(M, I, \mathbb{N}, S, p) \models \text{Gen}$, then p is a cut-base on M and I is a p-generic cut. These enable us to describe the theory of generic cuts using notions from *ω-logic*. The techniques we will use were developed independently by a number of people in the 1950s.

Definition. The *ω-rule* is the deduction rule which says that from the premise

$$\{\theta(n) : n \in \mathbb{N}\}$$

where $\theta(x)$ is some formula in the language $\mathscr{L}^+_{\boldsymbol{\omega}}$, one can derive

$$\forall x \in \boldsymbol{\omega}\ \theta(x).$$

An $\mathscr{L}^+_{\boldsymbol{\omega}}$-theory is said to be *$\omega$-complete* if it is deductively closed in first-order logic with the ω-rule, and it contains the sentence $n \in \boldsymbol{\omega}$ for every $n \in \mathbb{N}$. An *ω-model* of an $\mathscr{L}^+_{\boldsymbol{\omega}}$-theory is a model in which $\boldsymbol{\omega}$ is interpreted as $\mathbb{N}$.

The following was proved in Henkin [20] and in Orey [56], for example.

ω-Completeness Theorem. Every consistent ω-complete $\mathscr{L}^+_{\boldsymbol{\omega}}$-theory T has a countable ω-model.

Proof. This can be proved using the Omitting Types Theorem. By ω-completeness, one sees that the type

$$q(v) = \{v \in \boldsymbol{\omega}\} \cup \{v \neq n : n \in \mathbb{N}\}$$

cannot be isolated over T. This type is therefore omitted in some countable model of T. A model in which $q(v)$ is not realized must be an ω-model. $\square$

In the above proof, the Omitting Types Theorem was used as a black box for producing an ω-complete completion of T. To better understand how this completion can be constructed, one may examine the proof of the Omitting Types Theorem in Marker's book [52], for example. However, we can actually squeeze out more information if we do it directly. The next theorem is essentially Theorem 1 in Leblanc–Roeper–Thau–Weaver [49].

Theorem 5.1. Let T be a consistent ω-complete $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-theory, and σ be an $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-sentence that is consistent with T. Then $T + \{\sigma\}$ has an ω-complete completion.

Proof. Suppose σ is consistent with the ω-complete theory T. We build a sequence of $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-sentences $(\sigma_i)_{i\in\mathbb{N}}$ recursively such that $\sigma_0 = \sigma$ and

$$T + \{\sigma_i : i \in \mathbb{N}\}$$

is complete, consistent, and ω-complete. Fix an enumeration $(\tau_i)_{i\in\mathbb{N}}$ of $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-sentences in which every sentence appears infinitely many times.

Suppose we have already found $\sigma_0, \sigma_1, \ldots, \sigma_n$ such that $T + \{\sigma_i : i \leqslant n\}$ is consistent. Consider τ_n. If τ_n is not $\forall x \in \boldsymbol{\omega}\ \theta(x)$ for any $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-formula $\theta(x)$, then let

$$\sigma_{n+1} = \begin{cases} \tau_n, & \text{if } T + \{\sigma_i : i \leqslant n\} + \{\tau_n\} \text{ is consistent;} \\ \neg\tau_n, & \text{otherwise.} \end{cases}$$

Suppose we can find $\theta(x) \in \mathscr{L}_{\boldsymbol{\omega}}^{+}$ such that τ_n is $\forall x \in \boldsymbol{\omega}\ \theta(x)$. If τ_n is consistent with $T+\{\sigma_i : i \leqslant n\}$, then let $\sigma_{n+1} = \tau_n$. The last case is when $T+\{\sigma_i : i \leqslant n\} \vdash \exists x \in \boldsymbol{\omega}\ \neg\theta(x)$. This is equivalent to

$$T \vdash \bigwedge_{i\leqslant n} \sigma_i \to \exists x \in \boldsymbol{\omega}\ \neg\theta(x)$$

by the Deduction Theorem. In this case, since T is consistent, $T \nvdash \forall x \in \boldsymbol{\omega}\ \big(\bigwedge_{i\leqslant n} \sigma_i \wedge \theta(x)\big)$. Recall that T is closed under the ω-rule. So there is a natural number k_n, say, such that $T \nvdash \bigwedge_{i\leqslant n} \sigma_i \wedge \theta(k_n)$. It follows that $\bigwedge_{i\leqslant n} \sigma_i \to \neg\theta(k_n)$ is consistent with T. So we set $\sigma_{n+1} = \neg\theta(k_n)$.

Let $T^+ = T + \{\sigma_i : i \in \mathbb{N}\}$. Clearly, T^+ is complete and consistent. It remains to prove ω-completeness. Let $\theta(x)$ be an $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-formula such that $T^{+} \nvdash \forall x \in \boldsymbol{\omega}\ \theta(x)$. Since T^{+} is complete, we must have $T^{+} \vdash \exists x \in \boldsymbol{\omega}\ \neg\theta(x)$. Using the Compactness Theorem, let $n \in \mathbb{N}$ such that

$$T + \{\sigma_i : i \leqslant n\} \vdash \exists x \in \boldsymbol{\omega}\ \neg\theta(x).$$

Without loss, suppose τ_n is $\forall x \in \boldsymbol{\omega}\ \theta(x)$. By construction, $\sigma_{n+1} = \neg\theta(k_n)$, so that $T^{+} \vdash \neg\theta(k_n)$. Since T^{+} is consistent, we conclude $T^{+} \nvdash \theta(k_n)$, as required. □

Let us apply these to the theory of generic cuts.

Definition. We say that an $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-theory T *comes from a generic cut* if there exist a cut-base $p(x, y)$ on a countable model $M \models \mathrm{PA}$, a partial inductive satisfaction class S for M, and a p-generic cut $I \subseteq_{\mathrm{e}} M$ such that $T = \mathrm{Th}(M, I, \mathbb{N}, S, p)$.

Theorem 5.2. Let T be an $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-theory. Then T comes from a generic cut if and only if it is an ω-complete completion of Gen.

Proof. This follows directly from the ω-Completeness Theorem. □

Definition. We denote by $\overline{\mathrm{Gen}}$ the closure of Gen under logical deduction and the ω-rule.

Theorem 5.3. $\overline{\mathrm{Gen}} = \bigcap\{T \subseteq \mathscr{L}_{\boldsymbol{\omega}}^{+} : T \text{ comes from a generic cut}\}$.

Proof. It is clear from the previous theorem that if $\sigma \in \overline{\mathrm{Gen}}$, then σ is contained in every $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-theory that comes from a generic cut. Conversely, suppose $\sigma \notin \overline{\mathrm{Gen}}$. Since $\overline{\mathrm{Gen}}$ is closed under logical deduction, we know that $\overline{\mathrm{Gen}} \nvdash \sigma$, and so $\neg\sigma$ is consistent with $\overline{\mathrm{Gen}}$. Using Theorem 5.1, find an ω-complete completion T of $\overline{\mathrm{Gen}} + \{\neg\sigma\}$. By the ω-Completeness Theorem, T has a countable ω-model $(M, I, \mathbb{N}, S, p)$. This I must be a p-generic cut in M. Hence, T is an $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-theory that comes from a generic cut, but it does not contain σ. □

As a corollary, we see that some axioms in Gen are redundant in making the closure $\overline{\mathrm{Gen}}$. For example, Theorem 5.3 tells us that using the ω-rule, the strength of $\boldsymbol{\omega}$ can be proved from the other axioms in Gen. Similarly, the axiom 'I is of index $\boldsymbol{\omega}$' is also redundant with the ω-rule. Nevertheless, it tells us that Gen can actually be axiomatized in the smaller language $\mathscr{L}_{\boldsymbol{\omega}}^{+} \setminus \{\boldsymbol{\omega}\}$.

Remark. It is not hard to see that Theorem 5.2 and Theorem 5.3 can be proved for any class of cuts that is determined by the standard cut in the sense described at the top of page 31.

CHAPTER 6

PSEUDO-GENERIC CUTS

> ⟦E⟧xponentiation is not only a means for denoting "large numbers" but also the means for introducing "nonmathematical" questions into number theory.
>
> Rohit Parikh
> Existence and feasibility in arithmetic [59], §1

Results in model theory tell us that there are many models of Gen which are not ω-models. In particular, given any ω-model of Gen, one can find an elementary extension of it that is recursively saturated as an $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-structure. Such an extension cannot be an ω-model. We are interested in what properties of generic cuts remain true in such models. Although genericity is not $\mathscr{L}_{\boldsymbol{\omega}}^{+}$-definable, we will see that many properties of generic cuts transfer to arbitrary models of Gen. Informally, we say that a cut I of a model M is *pseudo-generic* if (M, I) can be expanded to a model of Gen.

Let us start with some standard results on the automorphisms of models of PA.

Lemma 6.1 (Kotlarski–Smoryński–Vencovská). Fix a recursively saturated model $M \models \mathrm{PA}$. Let $w, \bar{c}, \bar{c}' \in M$ such that

$$\forall k < 2^{2^w}\ (\bar{c}, k) \equiv (\bar{c}', k).$$

Then for each $a \in M$, there exists $a' \in M$ such that

$$\forall k < w\ (\bar{c}, a, k) \equiv (\bar{c}', a', k).$$

Proof. See Lemma 8.4.3 in Kossak–Schmerl [46]. □

Definition. Let $\bar{c} \in M \models \mathrm{PA}$ and $\boldsymbol{\omega} \subseteq M$. Then $\mathrm{Aut}_{\boldsymbol{\omega}}(M, \bar{c})$ denotes the pointwise stabilizer of $\boldsymbol{\omega} \cup \{\bar{c}\}$ in $\mathrm{Aut}(M)$. We abbreviate $\mathrm{Aut}_{\boldsymbol{\omega}}(M, 0)$ as $\mathrm{Aut}_{\boldsymbol{\omega}}(M)$.

Theorem 6.2. Fix a countable recursively saturated model $M \models \mathrm{PA}$, and a strong cut $\boldsymbol{\omega} \subseteq_{\mathrm{e}} M$. If $\bar{c}, \bar{c}' \in M$ such that

$$\forall k \in \boldsymbol{\omega}\ (\bar{c}, k) \equiv (\bar{c}', k),$$

then there is an automorphism in $\mathrm{Aut}_{\boldsymbol{\omega}}(M)$ that maps $\bar{c}$ to $\bar{c}'$.

Proof. This can be shown using a back-and-forth based on the Kotlarski–Smoryński–Vencovská Lemma. For the details, please see Theorem 8.4.5 in Kossak–Schmerl [46]. Simply note that if $\boldsymbol{\omega}$ is a strong cut of M, then $\boldsymbol{\omega}$ is closed under exponentiation and not downward ω-coded. □

Most of what real generic cuts have follow from this for pseudo-generic cuts, modulo an annoying extra condition. Compare (the length of) this proof with that of Theorem 6.1 in GCGS.

Theorem 6.3. Let $(M, I, \boldsymbol{\omega}, S, p)$, $(M, I', \boldsymbol{\omega}, S', p')$ be countable models of Gen, $\partial \in M \setminus (I \cup I')$, and $\hat{p} \in M$ such that

$$\begin{aligned} \hat{p} &= \{\langle i, x, y\rangle : p_i(x, y) \wedge x < \partial \wedge y < \partial\} \\ &= \{\langle i, x, y\rangle : p'_i(x, y) \wedge x < \partial \wedge y < \partial\}. \end{aligned}$$

Take any $\bar{c} \in M$. If there is a finite semi-interval $[a, b] \subseteq M_{<\partial}$ such that $I, I' \in [a, b]$ and

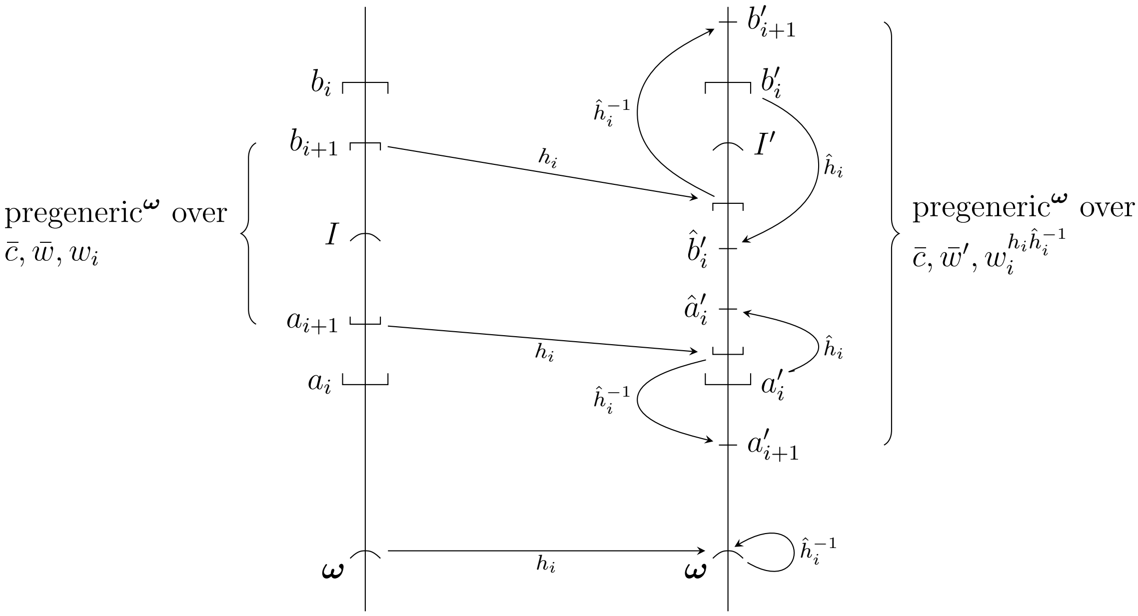

Figure 6.1: The $(i+1)$-th step in the proof of Theorem 6.3

for all $n \in \mathbb{N}$,

$$(M, \boldsymbol{\omega}, p) \models \forall x, y \in [a,b]\ \forall [u,v] \subseteq [a,b]$$
$$\left(\begin{array}{l} \forall i \in \boldsymbol{\omega}\ p_i(u,v) \\ \to \exists x', y' \in [u,v]\ \forall k \in \boldsymbol{\omega}\ \forall \varphi < n \\ \qquad \big(\mathrm{Sat}_{\Sigma_n}(\varphi, [x, y, \bar c, \hat p, \partial, k]) \leftrightarrow \mathrm{Sat}_{\Sigma_n}(\varphi, [x', y', \bar c, \hat p, \partial, k])\big) \end{array} \right),$$

then one can find an automorphism $g \in \mathrm{Aut}_{\boldsymbol{\omega}}(M, \bar c, \hat p, \partial)$ that maps I to I'.

Proof. We prove using a back-and-forth argument. For the purpose of this proof, we say that a semi-interval $[r,s]$ is *pregeneric*$^{\boldsymbol\omega}$ over $\bar w \in M$ if for all $n \in \mathbb{N}$,

$$(M, \boldsymbol{\omega}, p) \models \forall i \in \boldsymbol{\omega}\ p_i(r,s) \wedge \forall x, y \in [r,s]\ \forall [u,v] \subseteq [r,s]$$
$$\left(\begin{array}{l} \forall i \in \boldsymbol{\omega}\ p_i(u,v) \\ \to \exists x', y' \in [u,v]\ \forall k \in \boldsymbol{\omega}\ \forall \varphi < n \\ \qquad \big(\mathrm{Sat}_{\Sigma_n}(\varphi, [x, y, \bar w, \hat p, \partial, k]) \leftrightarrow \mathrm{Sat}_{\Sigma_n}(\varphi, [x', y', \bar w, \hat p, \partial, k])\big) \end{array} \right).$$

Fix an enumeration $(v_i)_{i\in\mathbb{N}}$ of M. At a step i, there are

- elements $w_1, w_2, \ldots, w_{i-1}, w'_1, w'_2, \ldots, w'_{i-1} \in M$;
- a semi-interval $[a_i, b_i]$ containing I that is pregeneric$^{\boldsymbol{\omega}}$ over $\bar{c}, \bar{w}$;
- a semi-interval $[a'_i, b'_i]$ containing I' that is pregeneric$^{\boldsymbol{\omega}}$ over $\bar{c}, \bar{w}'$; and
- an automorphism $h_i \in \mathrm{Aut}_{\boldsymbol{\omega}}(M, \bar{c}, \hat{p}, \partial)$

such that $\langle a_i, b_i, \bar{w}\rangle^{h_i} = \langle a'_i, b'_i, \bar{w}'\rangle$. The automorphism g at the end is given by $w_i \mapsto w'_i$ for each $i \in \mathbb{N}$.

We start with $[a_0, b_0] = [a'_0, b'_0] = [a, b]$. Suppose we have finished step i for some $i \in \mathbb{N}$. We show how to go forth in step $i+1$. Let $w_i = v_i$. Using the fact that $(M, I, \boldsymbol{\omega}, S, p) \models \mathrm{Gen}$, choose a semi-interval $[a_{i+1}, b_{i+1}]$ that contains I and is pregeneric$^{\boldsymbol{\omega}}$ over $a_i, b_i, \bar{c}, w_1, w_2, \ldots, w_i$. It follows that $[a_{i+1}, b_{i+1}]^{h_i} \subseteq [a'_i, b'_i]$. By the pregenericity$^{\boldsymbol{\omega}}$ of $[a'_i, b'_i]$, for every $n \in \mathbb{N}$,

$$(M, \boldsymbol{\omega}) \models \exists x, y \in [a_{i+1}, b_{i+1}]^{h_i}\ \forall k \in \boldsymbol{\omega}\ \forall \varphi < n \begin{pmatrix} \mathrm{Sat}_{\Sigma_n}(\varphi, [a'_i, b'_i, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \\ \leftrightarrow \mathrm{Sat}_{\Sigma_n}(\varphi, [x, y, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \end{pmatrix}.$$

We claim that by overspill, recursive saturation, and the strength of $\boldsymbol{\omega}$, the order of the quantifiers '$\forall n \in \mathbb{N}$' and '$\exists x, y \in [a_{i+1}, b_{i+1}]^{h_i}$' can be changed.

By overspill, one can find for any $n \in \mathbb{N}$ a number $l > \boldsymbol{\omega}$ that satisfies

$$M \models \exists x, y \in [a_{i+1}, b_{i+1}]^{h_i}\ \forall k < l\ \forall \varphi < n \begin{pmatrix} \mathrm{Sat}_{\Sigma_n}(\varphi, [a'_i, b'_i, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \\ \leftrightarrow \mathrm{Sat}_{\Sigma_n}(\varphi, [x, y, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \end{pmatrix}. \qquad (*)$$

For each $n \in \mathbb{N}$, define l_n to be the maximum $l \in M$ that makes $(*)$ true if it exists; otherwise, define l_n to be a fixed number above $\boldsymbol{\omega}$. The sequence $(l_n)_{n\in\mathbb{N}}$ is coded in M

by recursive saturation. Let l be a code for this sequence. Then for all $n \in \mathbb{N}$,

$$M \models \exists x, y \in [a_{i+1}, b_{i+1}]^{h_i}\ \forall k < (l)_n\ \forall \varphi < n \left(\begin{array}{l} \mathrm{Sat}_{\Sigma_n}(\varphi, [a'_i, b'_i, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \\ \leftrightarrow \mathrm{Sat}_{\Sigma_n}(\varphi, [x, y, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \end{array} \right).$$

Using the strength of $\boldsymbol{\omega}$, pick $d > \boldsymbol{\omega}$ such that for all $n \in \mathbb{N}$,

$$(l)_n > \boldsymbol{\omega} \Leftrightarrow (l)_n > d.$$

Our choice of the sequence (l_n) implies that $(l)_n > d$ for each $n \in \mathbb{N}$. Therefore,

$$M \models \exists x, y \in [a_{i+1}, b_{i+1}]^{h_i}\ \forall k < d\ \forall \varphi < n \left(\begin{array}{l} \mathrm{Sat}_{\Sigma_n}(\varphi, [a'_i, b'_i, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \\ \leftrightarrow \mathrm{Sat}_{\Sigma_n}(\varphi, [x, y, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \end{array} \right),$$

for every $n \in \mathbb{N}$. Using recursive saturation again, let $\hat{a}'_i, \hat{b}'_i \in [a_{i+1}, b_{i+1}]^{h_i}$ that satisfy

$$M \models \forall k < d\ \forall \varphi < n \left(\begin{array}{l} \mathrm{Sat}_{\Sigma_n}(\varphi, [a'_i, b'_i, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \\ \leftrightarrow \mathrm{Sat}_{\Sigma_n}(\varphi, [\hat{a}'_i, \hat{b}'_i, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k]) \end{array} \right)$$

for all $n \in \mathbb{N}$. In particular,

$$\forall k \in \boldsymbol{\omega} \quad (a'_i, b'_i, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k) \equiv (\hat{a}'_i, \hat{b}'_i, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial, k),$$

as claimed.

Using Theorem 6.2, let $\hat{h}_i \in \mathrm{Aut}_{\boldsymbol{\omega}}(M, \bar{c}, w'_1, w'_2, \ldots, w'_{i-1}, \hat{p}, \partial)$ which sends $\langle a'_i, b'_i \rangle$ to

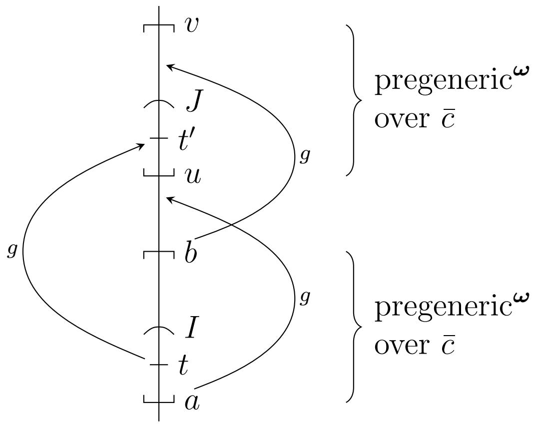

Figure 6.2: The first half of the proof of Theorem 6.4

$\langle \hat{a}'_i, \hat{b}'_i \rangle$. The back-and-forth construction then continues by setting

$$[a'_{i+1}, b'_{i+1}] = [a_{i+1}, b_{i+1}]^{h_i \hat{h}_i^{-1}} \text{ and } w'_i = w_i^{h_i \hat{h}_i^{-1}}.$$

The semi-interval $[a'_{i+1}, b'_{i+1}]$ found is pregeneric$^{\boldsymbol{\omega}}$ over $\bar{c}, w'_1, w'_2, \ldots, w'_i$ because both h_i and $\hat{h}_i$ fix $\hat{p}$, ∂ and $\boldsymbol{\omega}$. $\square$

This annoyance follows through. Fortunately, provided one can put up with the abundance of quantifiers and parameters, nothing goes wrong.

Theorem 6.4. Fix countable models $(M, I, \boldsymbol{\omega}, S, p), (M, J, \boldsymbol{\omega}, T, q) \models \text{Gen}$ and $\bar{c} \in M$. Let $\partial \in M \setminus (I \cup J)$ and $\hat{p} \in M$ such that

$$\begin{aligned} \hat{p} &= \{\langle i, x, y\rangle : p_i(x, y) \wedge x < \partial \wedge y < \partial\} \\ &= \{\langle i, x, y\rangle : q_i(x, y) \wedge x < \partial \wedge y < \partial\}. \end{aligned}$$

Suppose for all $\mathscr{L}_{\mathrm{I}}$-formulas φ,

$$(M, \boldsymbol{\omega}) \models \forall k \in \boldsymbol{\omega} \ \big(\mathsf{Q}x \in I \ \varphi(x, \bar{c}, \hat{p}, \partial, k) \leftrightarrow \mathsf{Q}x \in J \ \varphi(x, \bar{c}, \hat{p}, \partial, k)\big).$$

Then there is an automorphism $g \in \mathrm{Aut}_{\boldsymbol{\omega}}(M, \bar{c}, \hat{p}, \partial)$ that maps I to J.

Proof. Without loss of generality, suppose $I < J$. We use the pregenericity$^{\boldsymbol{\omega}}$ notation from the previous proof. Let $[a, b]$ be a pregeneric$^{\boldsymbol{\omega}}$ interval over $\bar{c}$ that contains I, and $[u, v]$ be a pregeneric$^{\boldsymbol{\omega}}$ interval over $\bar{c}$ that contains J. Pick $t \in [a, b]$ such that $t \in I$ but

$$(M, \boldsymbol{\omega}, p) \models \forall i \in \boldsymbol{\omega}\ p_i(a, t).$$

First, we will show

$$(M, \boldsymbol{\omega}) \models \forall m \in \boldsymbol{\omega}\ \exists x \in [u, v]\ \forall k < m\ \big(\varphi(t, \bar{c}, \hat{p}, \partial, k) \leftrightarrow \varphi(x, \bar{c}, \hat{p}, \partial, k)\big) \tag{†}$$

for each $\varphi \in \mathscr{L}_{\mathrm{I}}$. Fix $\varphi \in \mathscr{L}_{\mathrm{I}}$ and $m \in \boldsymbol{\omega}$. Pick an element $s \in \boldsymbol{\omega}$ that satisfies

$$M \models \forall k < m \begin{pmatrix} (s)_k = 1 \leftrightarrow \varphi(t, \bar{c}, \hat{p}, \partial, k) \\ \wedge\ (s)_k = 0 \leftrightarrow \neg\varphi(t, \bar{c}, \hat{p}, \partial, k) \end{pmatrix}.$$

Such s exists because $\boldsymbol{\omega}$ is closed under exponentiation. Let $\psi(x, \bar{c}, \hat{p}, \partial, m, s)$ be

$$\forall k < m \begin{pmatrix} (s)_k = 1 \leftrightarrow \varphi(x, \bar{c}, \hat{p}, \partial, k) \\ \wedge\ (s)_k = 0 \leftrightarrow \neg\varphi(x, \bar{c}, \hat{p}, \partial, k) \end{pmatrix},$$

so that $M \models \psi(t, \bar{c}, \hat{p}, \partial, m, s)$. Recall that t is an element of $[a, b]$, which is pregeneric$^{\boldsymbol{\omega}}$ over $\bar{c}$. Therefore, $M \models \mathsf{Q}x \in I\ \psi(x, \bar{c}, \hat{p}, \partial, m, s)$. So by hypothesis,

$$M \models \mathsf{Q}x \in J\ \psi(x, \bar{c}, \hat{p}, \partial, m, s).$$

In particular, $M \models \exists x \in [u, v]\ \psi(x, \bar{c}, \hat{p}, \partial, m, s)$, giving (†).

Next, we claim that the quantifier '$\exists x \in [u, v]$' can be pulled out in (†), i.e.,

$$\exists x \in [u, v]\ \forall k \in \boldsymbol{\omega}\ (t, \bar{c}, \hat{p}, \partial, k) \equiv (x, \bar{c}, \hat{p}, \partial, k). \tag{‡}$$

We use the combination of overspill, recursive saturation, and the strength of $\boldsymbol{\omega}$ as in the proof of the previous theorem. By (†) and an overspill, we have for each $\varphi \in \mathscr{L}_{\mathrm{I}}$ a number $l_\varphi > \boldsymbol{\omega}$ that satisfies

$$M \models \forall m < l_\varphi \; \exists x \in [u,v] \; \forall k < m \; \big(\varphi(t,\bar{c},\hat{p},\partial,k) \leftrightarrow \varphi(x,\bar{c},\hat{p},\partial,k)\big).$$

By recursive saturation, the sequence $(l_\varphi)_{\varphi \in \mathscr{L}_{\mathrm{I}}}$ is coded in M, say, by l. Using the strength of $\boldsymbol{\omega}$, let $d > \boldsymbol{\omega}$ such that for all $\varphi \in \mathscr{L}_{\mathrm{I}}$,

$$(l)_\varphi > \boldsymbol{\omega} \Leftrightarrow (l)_\varphi > d.$$

Choose any $m < d$ that is above $\boldsymbol{\omega}$. Then for all $\varphi \in \mathscr{L}_{\mathrm{I}}$,

$$M \models \exists x \in [u,v] \; \forall k < m \; \big(\varphi(t,\bar{c},\hat{p},\partial,k) \leftrightarrow \varphi(x,\bar{c},\hat{p},\partial,k)\big),$$

and so by recursive saturation, there is an $x \in [u,v]$ such that for all $\varphi \in \mathscr{L}_{\mathrm{I}}$,

$$(M,\boldsymbol{\omega}) \models \forall k \in \boldsymbol{\omega} \; \big(\varphi(t,\bar{c},\hat{p},\partial,k) \leftrightarrow \varphi(x,\bar{c},\hat{p},\partial,k)\big).$$

This proves (‡).

Using (‡), let $t' \in [u,v]$ such that $(t,\bar{c},\hat{p},\partial,k) \equiv (t',\bar{c},\hat{p},\partial,k)$ for all $k \in \boldsymbol{\omega}$. We then apply Theorem 6.2 to obtain an automorphism $g \in \mathrm{Aut}_{\boldsymbol{\omega}}(M,\bar{c},\hat{p},\partial)$ that maps t to t'. The rest is similar to the proof of Theorem 7.9 in GCGS. So we only give a sketch here. Note that the intersection of $[a,b]^g$ and $[u,v]$ is 'large' because $(M,\boldsymbol{\omega}) \models \forall i \in \boldsymbol{\omega}\; p_i(a,t)$. Since $[a,b]$ is pregeneric$^{\boldsymbol{\omega}}$ over $\bar{c}$, so is $[a,b]^g$. Using this pregenericity$^{\boldsymbol{\omega}}$, let $h \in \mathrm{Aut}_{\boldsymbol{\omega}}(M,\bar{c},\hat{p},\partial)$ such that

$$(I^g)^h \in [a,b]^g \cap [u,v].$$

Now both I^{gh} and J are pseudo-generic cuts in $[u,v]$, and $[u,v]$ is a pregeneric$^{\boldsymbol{\omega}}$ interval over $\bar{c}$. Hence by Theorem 6.3, there is an automorphism in $\mathrm{Aut}_{\boldsymbol{\omega}}(M,\bar{c},\hat{p},\partial)$ that maps I^{gh} to J, as required. □

As a corollary, the weak quantifier elimination result holds too.

Corollary 6.5. Fix a countable model $(M, I, \boldsymbol{\omega}, S, p) \models \mathrm{Gen}$ and tuples $\bar{c}, \bar{d} \in M$. Let $\partial \in M \setminus I$ and $\hat{p} \in M$ such that

$$\hat{p} = \{\langle i, x, y\rangle : p_i(x, y) \wedge x < \partial \wedge y < \partial\}.$$

Suppose $\forall k \in \boldsymbol{\omega}\ (\bar{c}, \hat{p}, \partial, k) \equiv (\bar{d}, \hat{p}, \partial, k)$, and for all $\mathscr{L}_{\mathrm{I}}$-formulas φ,

$$(M, \boldsymbol{\omega}) \models \forall k \in \boldsymbol{\omega}\ \big(\mathsf{Q}x \in I\ \varphi(x, \bar{c}, \hat{p}, \partial, k) \leftrightarrow \mathsf{Q}x \in I\ \varphi(x, \bar{d}, \hat{p}, \partial, k)\big).$$

Then there is an automorphism $g \in \mathrm{Aut}_{\boldsymbol{\omega}}(M, \hat{p}, \partial)$ that fixes I setwise and maps $\bar{c}$ to $\bar{d}$.

Proof. Apply Theorem 6.2 and Theorem 6.4. □

Since we passed on from a structure to a theory, it seems more likely that we can obtain real quantifier elimination. See Sanders [66] for a recent result of this flavour.

Conjecture 6.6. Let $(M, I, \boldsymbol{\omega}, S, p) \models \mathrm{Gen}$. Take $\partial \in M \setminus I$ and set

$$\hat{p} = \{\langle i, x, y\rangle : p_i(x, y) \wedge x < \partial \wedge y < \partial\}.$$

Let $T = \mathrm{Th}(M, I, \boldsymbol{\omega}, \hat{p}, \partial)$. Then every formula in the language $\mathscr{L}^*_{\mathrm{Sk}} \cup \{\boldsymbol{\omega}\}$ is uniformly equivalent modulo T to a Boolean combination of formulas of the forms

$$\exists v_1 \in \boldsymbol{\omega}\ \forall v_2 \in \boldsymbol{\omega}\ \cdots\ \forall v_{2n} \in \boldsymbol{\omega}\ \theta(\bar{v}, \bar{z}, \hat{p}, \partial)$$

and

$$\forall k \in \boldsymbol{\omega}\ \mathsf{Q}x \in I\ \varphi(x, \bar{z}, k, \hat{p}, \partial),$$

where $\theta(\bar{v}, \bar{z}, \hat{p}, \partial), \varphi(x, \bar{z}, k, \hat{p}, \partial) \in \mathscr{L}_{\mathrm{I}}$.

All of the results presented in this chapter work for recursively saturated models of

Gen in particular. We hope that eventually, these structures will help us answer questions that we were unable to settle in GCGS and in TiGC.

Question 6.7. Is it true that if $(M, I, \boldsymbol{\omega}, S, p)$ is a countable recursively saturated model of Gen, then I is free in M?

CHAPTER 7

EXISTENTIAL CLOSURE

> ⟦W⟧hen we come to apply the detailed techniques of model theory to arithmetic we find that the situation is far less satisfactory. Indeed, while the axiom systems of standard algebra seem to be particularly suitable for the discussion of the model-theoretic properties of the structures satisfied by them, especially as regarding problems of extension and intersection, the contrary is true for arithmetic.
>
> Abraham Robinson
> Model theory and non-standard arithmetic [65], §1

In the previous chapters, we have seen where the ω-models stand relative to the other models of Gen. In this chapter, we study where the generic cuts stand relative to other p-cuts. More specifically, we prove an *existential closure* property of generic cuts.

Existentially closed models of number theories were studied in the 1970s in relation to various notions of forcing and genericity. The theories considered were necessarily quite weak, in order for the model theoretic tools to be applicable. The key references on this line are Goldrei–Macintyre–Simmons [18], Hirschfeld–Wheeler [22], and Simmons [69]. For the general theory of existential closure and forcing, I recommend Hodges's book [25].

Let us first fix the language. It is essentially just $\mathscr{L}_{\boldsymbol{\omega}}^{+}$ without the $\boldsymbol{\omega}$ and the S.

Definition. If $p(x, y)$ is a cut-base, then $\mathscr{L}_p$ denotes the language

$$\mathscr{L}_{\mathrm{I}} + \{I\} + \{S_\theta : \theta(v, \bar{y}) \in \mathscr{L}_{\mathrm{I}}\} + \{F_n : n \in \mathbb{N}\},$$

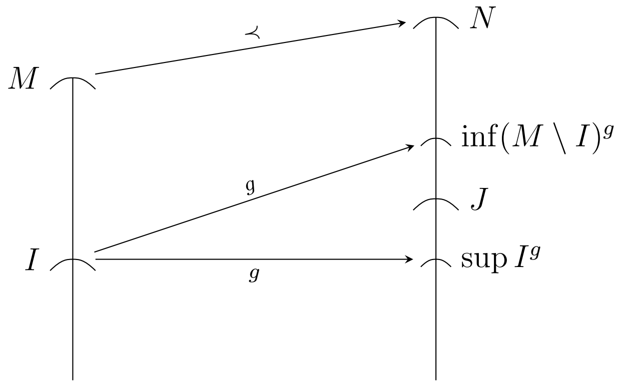

Figure 7.1: A p-arrow $g\colon (M, I) \to (N, J)$

where I is a unary relation symbol, S_θ is a function symbol of arity m for each $\mathscr{L}_\mathrm{I}$-formula $\theta(v, y_1, \ldots, y_m)$, and F_n is a unary function symbol for each $n \in \mathbb{N}$.

The S_θ's are intended for the $\mathscr{L}_\mathrm{I}$-Skolem functions, although we will often write their $\mathscr{L}_\mathrm{I}$-equivalents in practice. The F_n's are used to represent a cut-base $p(x, y)$. Essentially, we will have $F_n(x) = (\mu y)(p_n(x, y)) + 1$.

Cut-bases are a special class of notions of intervals. The advantage of introducing them is that they can be treated uniformly across different models.

Definition. Let $p(x, y)$ be a cut-base. The *category of p-cuts* (***p*-Cuts**) is defined as follows. The objects are pairs of the form (M, I) where $I \subseteq_\mathrm{e} M \models \mathrm{PA}$ such that $p(x, y)$ is a cut-base on M, and I is a p-cut of M. An arrow between two objects (M, I) and (N, J) is an elementary embedding $g\colon M \to N$ with the property that

$$x^g \in J \Leftrightarrow x \in I$$

for all $x \in M$. For convenience, objects and arrows in ***p*-Cuts** will be referred to as *p-objects* and *p-arrows* respectively.

The category ***p*-Cuts** is actually the category of models of an $\mathscr{L}_p$-theory. This theory is shown in Figure 7.2, although it is probably more inspiring to work it out on one's own. It is worth noting that this theory is universal. This observation may help in future work, as universal theories and existential closure have many connections in model theory [25].

$$
\begin{aligned}
&\mathrm{PA}^{-} + \{\forall x\,\forall y \in I\ (x < y \to x \in I)\}\\
&+ \left\{ \begin{matrix} \big(S_{\alpha(x,\bar y)}(\bar y) = 0 \wedge \forall x\,\neg\alpha(x,\bar y)\big) \\ \vee \left(\begin{matrix} S_{\alpha(x,\bar y)}(\bar y) > 0 \wedge \forall w < S_{\alpha(x,\bar y)}(\bar y) \\ \left(\begin{matrix} S_{\alpha(x,\bar y)}(\bar y) = w + 1 \to \\ \left(\begin{matrix} \alpha(w,\bar y) \\ \wedge\, \forall w' < w\ \neg\alpha(w',\bar y) \end{matrix} \right) \end{matrix} \right) \end{matrix} \right) \end{matrix} : \begin{matrix} \varphi(x,v,\bar y) \in \mathscr{L}_{\mathrm{I}} \\ \text{is quantifier-free} \end{matrix} \right\}\\
&+ \left\{ \begin{matrix} \big(S_{\exists v\,\varphi(x,v,\bar y)}(\bar y) = 0 \wedge \forall \hat x\,\forall v\, S_{\neg\varphi(\hat x,v,\bar y)\wedge x=0}(\hat x, v, \bar y) > 0\big) \\ \vee \left(\begin{matrix} S_{\exists v\,\varphi(x,v,\bar y)}(\bar y) > 0 \wedge \forall w < S_{\exists v\,\varphi(x,v,\bar y)}(\bar y) \\ \left(\begin{matrix} S_{\exists v\,\varphi(x,v,\bar y)}(\bar y) = w + 1 \to \\ \left(\begin{matrix} S_{\varphi(w,x,\bar y)}(w,\bar y) > 0 \\ \wedge\, \forall w' < w\ \forall v\, S_{\neg\varphi(w',v,\bar y)\wedge x=0}(w', v, \bar y) > 0 \end{matrix} \right) \end{matrix} \right) \end{matrix} \right) \end{matrix} : \varphi(x,v,\bar y) \in \mathscr{L}_{\mathrm{I}} \right\}\\
&+ \left\{ \begin{matrix} \big(S_{\forall v\,\varphi(x,v,\bar y)}(\bar y) = 0 \wedge \forall \hat x\, S_{\neg\varphi(\hat x,x,\bar y)}(\hat x, \bar y) > 0\big) \\ \vee \left(\begin{matrix} S_{\forall v\,\varphi(x,v,\bar y)}(\bar y) > 0 \wedge \forall w < S_{\forall v\,\varphi(x,v,\bar y)}(\bar y) \\ \left(\begin{matrix} S_{\forall v\,\varphi(x,v,\bar y)}(\bar y) = w + 1 \to \\ \left(\begin{matrix} \forall v\, S_{\varphi(v,w,\bar y)\wedge x=0}(v, w, \bar y) > 0 \\ \wedge\, \forall w' < w\ S_{\neg\varphi(x,w',\bar y)}(w', \bar y) > 0 \end{matrix} \right) \end{matrix} \right) \end{matrix} \right) \end{matrix} : \varphi(x,v,\bar y) \in \mathscr{L}_{\mathrm{I}} \right\}\\
&+ \left\{ \forall \bar y\ \big(S_{\varphi(x,\bar y)}(\bar y) = S_{\psi(x,\bar y)}(\bar y)\big) : \begin{matrix} \vdash \forall x \forall \bar y \big(\varphi(x,\bar y) \leftrightarrow \psi(x,\bar y)\big) \\ \text{where } \varphi(x,\bar y), \psi(x,\bar y) \in \mathscr{L}_{\mathrm{I}} \end{matrix} \right\}\\
&+ \left\{ \forall x \left(\begin{matrix} \big(F_n(x) = 0 \wedge \forall y\, \neg p_n(x,y)\big) \\ \vee \left(\begin{matrix} F_n(x) > 0 \wedge \forall w < F_n(x) \\ \left(\begin{matrix} F_n(x) = w + 1 \to \\ p_n(x,w) \wedge \forall y < w\ \neg p_n(x,y) \end{matrix} \right) \end{matrix} \right) \end{matrix} \right) : n \in \mathbb{N} \right\}\\
&+ \big\{ \forall x \in I\ F_n(x) \in I : n \in \mathbb{N} \big\},
\end{aligned}
$$

Figure 7.2: The theory of p-cuts

Definition. Let $p(x, y)$ be a cut-base. A p-object (M, I) is said to be *existentially closed* if whenever $g\colon (M, I) \to (N, J)$ is a p-arrow, we have

$$(N, J) \models \exists \bar{x}\, \varphi(\bar{x}, \bar{c}^g) \Rightarrow (M, I) \models \exists \bar{x}\, \varphi(\bar{x}, \bar{c})$$

for all quantifier-free $\mathscr{L}_p$-formulas $\varphi(\bar{x}, \bar{y})$ and all $\bar{c} \in M$.

Theorem 7.1. Fix a cut-base $p(x, y)$ on a countable arithmetically saturated model M of PA. If I is a p-generic cut of M, then the p-object (M, I) is existentially closed.

Proof. Let M be countable arithmetically saturated model of PA, and $p(x, y)$ be a cut-base on M. Take a p-generic cut I of M and pick a p-arrow $g\colon (M, I) \to (N, J)$. Let $\bar{c} \in M$ and $\varphi(\bar{x}, \bar{y})$ be a quantifier-free $\mathscr{L}_p$-formula such that

$$(N, J) \models \exists \bar{x}\, \varphi(\bar{x}, \bar{c}^g).$$

After some syntactical manipulations, we can rewrite $\varphi(\bar{x}, \bar{y})$ as

$$\xi(x_1, x_2, \bar{y}) \wedge x_1 \in I \wedge x_2 \notin I,$$

where $\xi(x_1, x_2, \bar{y})$ is an $\mathscr{L}_{\mathrm{I}}$-formula. So we have

$$N \models \exists x_1 \exists x_2 \big(\xi(x_1, x_2, \bar{c}^g) \wedge x_1 \in J \wedge x_2 \notin J\big). \tag{$*$}$$

We split into two cases.

Suppose we can find $a \in I$ such that

$$N \models \exists x_1 < a^g\, \exists x_2\, \big(\xi(x_1, x_2, \bar{c}^g) \wedge x_1 \in J \wedge x_2 \notin J\big). \tag{$\dagger$}$$

If $N \models \mathsf{Q}x_2\, \exists x_1 < a^g\, \xi(x_1, x_2, \bar{c}^g)$, then we are done by the elementarity of g. So suppose not. Then there is a maximum $x_2^* \in N$ such that $N \models \exists x_1 < a^g\, \xi(x_1, x_2^*, \bar{c}^g)$. Note that this x_2^* is above J by (†). Since g is a p-arrow, we have

$$M \models (\max x_2)\big(\exists x_1 < a\ \xi(x_1, x_2, \bar{c})\big) \notin I,$$

which is what we want.

Suppose we cannot find an $a \in I$ that satisfies (†), i.e., for all $a \in I$,

$$N \models \forall x_1 < a^g\ \forall x_2 \big(\xi(x_1, x_2, \bar{c}^g) \wedge x_1 \in J \rightarrow x_2 \in J\big).$$

Then $N \models \forall x_1 < a^g\ \big((\max x_2)(\xi(x_1, x_2, \bar{c}^g)) \in J\big)$ for all $a \in I$, with the convention that if the maximum does not exist, then the value of the 'maximum' is 0. Now, since g is a p-arrow, we have

$$M \models \forall x_1 \in I\ \big((\max x_2)(\xi(x_1, x_2, \bar{c})) \in I\big).$$

In other words, I is closed under the function

$$x_1 \mapsto (\max x_2)(\xi(x_1, x_2, \bar{c})),$$

which we call $\Xi_{\bar{c}}$. Pick an interval $[\![a, b]\!] \subseteq M$ containing I such that

$$\forall x_1 \in [\![a, b]\!]_M\ \neg \bigwedge\!\!\!\bigwedge p(x_1, \Xi_{\bar{c}}(x_1)).$$

Such an interval exists by Proposition 4.11. Using recursive saturation, let $n \in \mathbb{N}$ such that $M \models \forall x_1 \in [\![a, b]\!]\ \neg p_n(x_1, \Xi_{\bar{c}}(x_1))$. Since g is an elementary embedding from M to N, we also have

$$N \models \forall x_1 \in [\![a^g, b^g]\!]\ \neg p_n(x_1, \Xi_{\bar{c}^g}(x_1)).$$

Recall that $a^g \in J < b^g$ and J is a p-cut of M. So J must be closed under $\Xi_{\bar{c}^g}$. By the maximality of $\Xi_{\bar{c}^g}$,

$$N \models \forall x_1 \in J\ \forall x_2 > J\ \neg\xi(x_1, x_2, \bar{c}^g).$$

This contradicts $(*)$. Therefore, this case cannot happen. □

We conjecture there is a partial converse.

Conjecture 7.2. Fix a countable arithmetically saturated model $M \models \mathrm{PA}$, and a cut-base $p(x, y)$ on M. Let I be a p-cut of M that is of index $\mathbb{N}$. If (M, I) is $\exists_1^* \cup \forall_1^*$-recursively saturated and existentially closed, then I is p-generic in M.

Note that if Conjecture 4.9 is true, then so is this one.

Part II

Strong end-extensions

CHAPTER 8

END-EXTENSIONS OF SECOND-ORDER MODELS

> The axiom of infinity is ⟦. . . ⟧ the first step in the progression of ever bolder large cardinal axioms. ⟦. . . *T*⟧*he negation of the axiom of infinity endows* $ZF^{-\infty}$ *with a model theoretic behaviour that* ZF *can only imitate with the help of additional axioms asserting the existence of large cardinals.* This is partially explainable by noting that the negation of the axiom of infinity in $ZF^{-\infty}$ itself can be viewed as a large cardinal axiom, not positing the existence of a large set - indeed denying it - but attributing a large cardinal character to the universe itself.
>
> Ali Enayat
> Analogues of the MacDowell–Specker Theorem for set theory [11]

End-extensions of models of first-order arithmetic are a symbol of *strength*. 'Strong' here means 'similar to the standard model'.

Theorem 8.1 (MacDowell–Specker [50], Paris–Kirby [62]). A model of $\mathrm{I}\Delta_0$ has a proper elementary end-extension if and only if it satisfies PA. □

There are similar results for subsystems of PA.

Theorem 8.2 (Paris–Kirby [62]). Let M be a countable model of $\mathrm{I}\Delta_0$ and $n \in \mathbb{N}$ such that $n \geqslant 2$. Then M has a proper Σ_n-elementary end-extension if and only if it satisfies $\mathrm{B}\Sigma_n$. □

We would like to investigate analogues of these above PA. We want examples of theories that are stronger than PA, and notions of elementarity that are stronger than

$\mathscr{L}_{\mathrm{I}}$-elementarity. The approach we will take is to move up to second-order arithmetic. The first question that one faces is: *what is an end-extension of a model of second-order arithmetic?* In this chapter, we will address this question and survey the related results in the literature.

If we want to have a notion of 'an end-extension of the second-order part', then we need some ordering of the second-order universe to start with. Murawski [55] suggested the ordering analogous to that in Gödel's constructible universe. Dubiel [10] investigated the possibility of end-extending the internal ordinals in a second-order model. None of these seems to help with our problems.

Alternatively, we can start with a notion of end-extension in first-order arithmetic, and see whether it can be modified or strengthened into a second-order notion. Recall that each first-order end-extension N of a model M gives rise to a model of second-order arithmetic $(M, \mathrm{SSy}_M(N))$. This is a point of contact between end-extensions and second-order arithmetic that has been well-exploited.

Theorem 8.3 (Gaifman [17]). Let $(M, \mathscr{X})$ be a countable model of ACA_0. Then M has a proper elementary end-extension N such that $\mathrm{SSy}_M(N) = \mathscr{X}$. □

One would expect similar results to hold for theories weaker than ACA_0.

Conjecture 8.4. Let n be a natural number that is at least 2, and $(M, \mathscr{X})$ be a countable model of $\mathrm{B}\Sigma^*_n$. Then M has a proper Σ_n-elementary end-extension N such that $\mathrm{SSy}_M(N) = \mathscr{X}$.

Recall that there are two theories in the Big Five below ACA_0, namely RCA_0 and WKL_0. A simple overspill argument shows that these two theories are the same for end-extensions.

Proposition 8.5. Let $M \subsetneq_{\mathrm{e}} N \models \mathrm{I}\Sigma_1$. Then $(M, \mathrm{SSy}_M(N)) \models \mathrm{RCA}_0$ if and only if $(M, \mathrm{SSy}_M(N)) \models \mathrm{WKL}_0$. □

There is hence a good reason to take WKL_0 as the base theory for end-extensions. Scott [68] was probably the first to notice this aspect of WKL_0. Surprisingly, it took

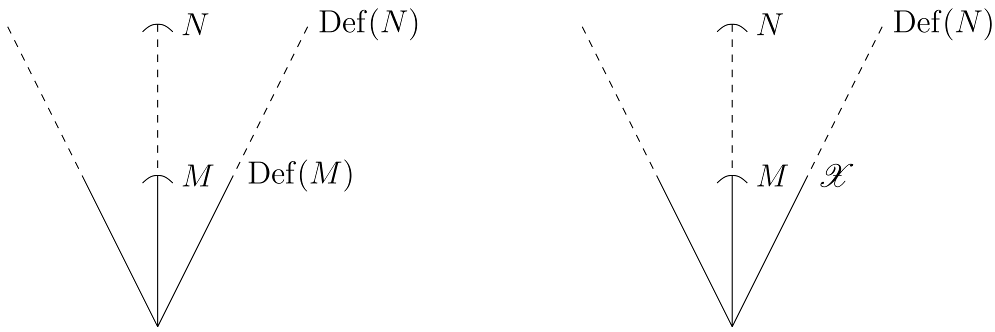

Figure 8.1: A traditional conservative extension (left), and a first-order conservative extension (right)

35 years since Scott's original paper for his theorem to be generalized to arbitrary models of WKL_0.

Theorem 8.6 (Scott [68]; Tanaka [78]). Let $(M, \mathscr{X})$ be a countable model of RCA_0. Then $(M, \mathscr{X}) \models \mathrm{WKL}_0$ if and only if M has a proper end-extension N such that $\mathrm{SSy}_M(N) = \mathscr{X}$. □

The kind of end-extensions that appears in the theorems above is a generalization of the notion of *conservative* extensions for models of first-order arithmetic. So we make the following definition.

Definition. Let $(M, \mathscr{X}) \models \mathrm{WKL}_0$. Then a *first-order conservative extension* of $(M, \mathscr{X})$ is an end-extension N of M such that $\mathrm{SSy}_M(N) = \mathscr{X}$.

A criterion for a good notion of end-extensions is the ability to code strength. As shown above, first-order conservativity did the job in Theorem 8.6, but it is not good enough for Theorem 8.3. For example, there are many nonstandard models of $\mathrm{Th}(\mathbb{N})$ in which $\mathbb{N}$ is not strong. Therefore, we need a better notion of end-extensions for second-order models.

Definition. An *end-extension* of a model $(M, \mathscr{X})$ of RCA_0 is an $\mathscr{L}_{\mathrm{II}}$-structure $(N, \mathscr{Y})$ with an embedding

$$\begin{aligned} \mathscr{X} &\to \mathscr{Y} \\ S &\mapsto S^N \end{aligned}$$

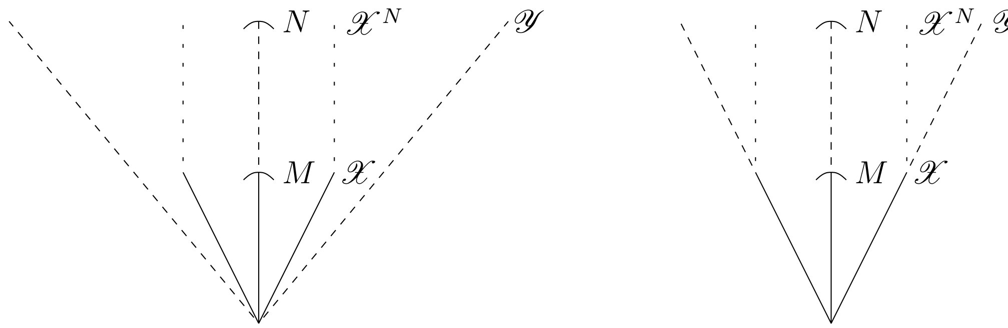

Figure 8.2: A second-order end-extension (left), and a second-order conservative end-extension (right)

such that $N \supseteq_{\mathrm{e}} M$ and $S^N \cap M = S$ for all $S \in \mathscr{X}$. In this case, we define

$$\mathscr{X}^N = \{S^N : S \in \mathscr{X}\}.$$

An end-extension $(N, \mathscr{Y})$ of $(M, \mathscr{X})$ is *proper* if $M \neq N$.

Definition. Let $(M, \mathscr{X}) \models \mathrm{RCA}_0$ and $(N, \mathscr{Y})$ be an end-extension of $(M, \mathscr{X})$. For a class Γ of $\mathscr{L}_{\mathrm{II}}$-formulas, we say that the extension $(N, \mathscr{Y})$ is Γ-*elementary* if whenever $\varphi(\bar{v}, \bar{X}) \in \Gamma$,

$$(M, \mathscr{X}) \models \varphi(\bar{a}, \bar{S}) \Leftrightarrow (N, \mathscr{Y}) \models \varphi(\bar{a}, \bar{S}^N)$$

for all $\bar{a} \in M$ and $\bar{S} \in \mathscr{X}$. The extension $(N, \mathscr{Y})$ is *(second-order) conservative* if

$$\{S \cap M : S \in \mathscr{Y}\} = \mathscr{X}.$$

Remark. Suppose $(N, \mathscr{Y})$ is a proper end-extension of a second-order model $(M, \mathscr{X})$, and $(N, \mathscr{Y}) \models \mathrm{RCA}_0$. Then this extension is conservative if and only if $\mathrm{SSy}_M(N) = \mathscr{X}$. So second-order conservativity implies first-order conservativity in almost all cases.

Example 8.7. Let $I \subsetneq_{\mathrm{e}} M \models \mathrm{PA}$, where M is countable. Recall that an elementary extension K of M is an I-*extension* if $I \subseteq_{\mathrm{e}} K$ and there is $d \in K$ such that $I < d < M \setminus I$.

Notice if K is an I-extension of M and

$$J = \{x \in K : x < b \text{ for all } b \in M \setminus I\},$$

then $(J, \mathrm{SSy}_J(K))$ is an end-extension of $(I, \mathrm{SSy}_I(M))$. Kirby–Paris [36] proved that modulo WKL_0, such an extension exists if and only if $(I, \mathrm{SSy}_I(M)) \models \mathrm{B}\Sigma_2^*$. They also showed that if such extension is required to be conservative, then modulo WKL_0, it is necessary and sufficient to have $(I, \mathrm{SSy}_I(M)) \models \mathrm{ACA}_0$.

Compare this example with the following improvements of Theorems 8.6 and 8.3.

Theorem 8.8 (Scott [68]; Tanaka [78]). Let $M \models \mathrm{I}\Sigma_1$ and $\mathscr{X} \subseteq \mathcal{P}(M)$, where M and $\mathscr{X}$ are both countable. Then $(M, \mathscr{X}) \models \mathrm{WKL}_0$ if and only if $(M, \mathscr{X})$ has a proper conservative end-extension satisfying RCA_0. □

Remark. Tanaka [78] actually showed that if $(M, \mathscr{X})$ is a countable nonstandard model of WKL_0, then it has a proper conservative end-extension that is isomorphic to $(M, \mathscr{X})$.

Theorem 8.9. Let $(M, \mathscr{X})$ be a countable model of RCA_0. Then $(M, \mathscr{X}) \models \mathrm{ACA}_0$ if and only if it has a proper Σ_0^1-elementary conservative end-extension satisfying RCA_0.

Proof. The proof of the 'only if' part is the same as that of Theorem 8.3.

For the 'if' part, let $(N, \mathscr{Y})$ be a proper Σ_0^1-elementary conservative end-extension of $(M, \mathscr{X})$. As in the Paris–Kirby proof of Theorem 8.1, one can show that $(M, \mathscr{X}) \models \mathrm{PA}^*$. It remains to prove arithmetical comprehension.

Let $a \in M$ and $S \in \mathscr{X}$. Take an arithmetical formula $\varphi(v, z, W)$. We want to show

$$\{v \in M : (M, \mathscr{X}) \models \varphi(v, a, S)\} \in \mathscr{X}.$$

Without loss of generality, assume $(M, \mathscr{X}) \models \mathsf{Q}v\, \varphi(v, a, S)$. Take $d \in N \setminus M$. Notice that since $(M, \mathscr{X})$ satisfies induction for all arithmetical formulas, so does $(N, \mathscr{X}^N)$. Inside

$(N, \mathscr{X}^N)$, define the function $F\colon N_{\leqslant d} \to N$ recursively by

$$F(0) = (\mu v)(\varphi(v, a, S^N)),$$

and
$$F(i+1) = (\mu v)(v > F(i) \wedge \varphi(v, a, S^N))$$

for all $i < d$. Now F is coded in N, and $\mathrm{Im}(F) \in \mathrm{Def}(N)$. So by elementarity,

$$\{v \in M : (M, \mathscr{X}) \models \varphi(v, a, S)\} = \mathrm{Im}(F) \cap M \in \mathrm{SSy}_M(N) = \mathscr{X}. \qquad \square$$

Countability is necessary in this theorem. See Mills [53] and Enayat [12]. Conservativity is also necessary.

Theorem 8.10. For a countable model $(M, \mathscr{X}) \models \mathrm{RCA}_0$, the following are equivalent.

(a) $(M, \mathscr{X}) \models \mathrm{PA}^*$.

(b) $(M, \mathscr{X})$ has a proper Σ^1_0-elementary end-extension satisfying RCA_0.

(c) $(M, \mathscr{X})$ has a proper Σ^1_0-elementary end-extension $(N, \mathscr{Y})$ satisfying RCA_0 such that $\mathrm{SSy}_M(N) = \mathrm{Def}(M, \mathscr{X})$.

Proof. Imitate the proof of Theorem 8.1. $\square$

I am not in a position to say how good this notion of end-extension is in this thesis. We are unable to find a better one to use yet.

Our next question is: *what makes an end-extension stronger than another?* In the context of first-order arithmetic, there are two popularly used criteria: elementarity, and the theory of the coded sets.

Theorems 8.1 and 8.2 already showed us how elementarity can be used to characterize strength. Using the theory of the coded sets amounts to very little for us. In view of Theorem 8.6, it is just another way of expressing what second-order models to which a first-order model can expand. For example, consider the following variant of Theorem 8.3.

Corollary 8.11 (Gaifman [17]). Let $M \models \mathrm{I}\Delta_0$. Then $M \models \mathrm{PA}$ if and only if there is a proper end-extension $N \supseteq_{\mathrm{e}} M$ such that $(M, \mathrm{SSy}_M(N)) \models \mathrm{ACA}_0$. □

This essentially just says that every model of PA can be expanded to a model of ACA_0. The next theorem says that every model of $\mathrm{B}\Sigma_n$ is expandable to a model of $\mathrm{B}\Sigma_n^*$ for $n \geqslant 2$.

Theorem 8.12 (Paris [61]). Let $M \models \mathrm{B}\Sigma_2$ and $n \in \mathbb{N}$, where $n \geqslant 2$. Then $M \models \mathrm{B}\Sigma_n$ if and only if there is a proper end-extension $N \supseteq_{\mathrm{e}} M$ such that $(M, \mathrm{SSy}_M(N)) \models \mathrm{B}\Sigma_n^*$. □

Remark. One can additionally require the end-extension to satisfy some specified theory. A more delicate argument is sometimes needed, but it does not seem to increase strength (at the levels we are considering). For instance, it is easy to see that the end-extension constructed in Corollary 8.11 satisfies PA. Paris actually constructed an end-extension satisfying $\mathrm{B}\Sigma_n$ when he proved Theorem 8.12. For a survey and some recent results on this line, see the papers by Dimitracopoulos and Cornaros [5, 9].

Question 8.13. Is it true that for any $\mathscr{L}_{\mathrm{II}}$-theory T extending WKL_0, if M is a countable recursively saturated model of the $\mathscr{L}_{\mathrm{I}}$-consequences of T, then M can be expanded to a model of T?

How these criteria can be useful in the second-order context is not clear. As remarked above, using the theory of the coded sets only gives trivial results. Gaifman's techniques easily give $\Sigma_1^1 \cup \Pi_1^1$-elementary end-extensions. We do not have the right insight to go beyond that yet.

CHAPTER 9

INTERNALLY ω_1-LIKE MODELS

> The problem is not that the uncountable cardinals are complicated — they aren't. Rather it is that we are building up uncountable structures by approximations of smaller size, and this involves climbing up to uncountable cardinals through the ordinals below them. There are just too many different paths from ω to ω_1, and it matters which route we take. In a similar context Georg Kreisel has sometimes quoted the cigarette advertisement: *It's not how long you make it, it's how you make it long.*
>
> Wilfrid Hodges
> *Building Models by Games* [25], Section 5.2

Our first attempt to build strong end-extensions is to consider ω_1*-like models*.

Definition. Let M be an $\mathscr{L}_{\mathrm{I}}$-structure in which $<$ is interpreted as a linear order. Then M is ω_1*-like* if for every $a \in M$, the set

$$\{x \in M : x < a\}$$

is countable, but M itself is of cardinality $\aleph_1$.

The motivation is that the regularity of $\aleph_1$ can be a source of strength.

Proposition 9.1. Let T be a consistent $\mathscr{L}_{\mathrm{I}}$-theory extending $\mathrm{I}\Delta_0$. Then T proves PA if and only if it has an ω_1-like model. □

The obvious disadvantage of ω_1-like models is that they are uncountable. We get round this problem by investigating *internal versions* of ω_1-like models. If we start with a countable model of second-order arithmetic, then internal ω_1-like models can only be countable. I am not able to obtain any strength from such models yet. See Chapter 13 for some ideas of how one may approach this.

The reverse mathematics of model theoretic constructions has recently gained a lot of attention from researchers [6, 7, 19, 23, 40, 47, ...]. The theories involved are in general rather weak, and most existing work seems to belong to recursion theory proper. What we do in this chapter is slightly different. We are not only interested in what can be done internally. We want to understand the interactions between internal and external constructions too.

9.1 Internalizing MacDowell–Specker

The first step of our internalization is to formalize the proof of Theorem 8.1. The usual proof uses a PA-provable version of König's Lemma [29, Section 8.2]. The whole argument is easily formalizable within WKL_0. We will give an outline of this here.

We work inside a fixed model of WKL_0. Let N be an internal model of PA. We internally build a type $P(x)$ using the PA-version of König's Lemma, so that the internal extension K of N by $P(x)$ is *internally conservative*, i.e., whenever X is an internally $\mathscr{L}_\mathrm{I}$-definable subset of K, the set $X \cap N$ is internally $\mathscr{L}_\mathrm{I}$-definable in N. We call this K the *König extension* of M. By conservativity, we must have $K \supseteq_\mathrm{e} M$. Here are some characteristics of this construction that will be important later on.

(A) The internal type $P(x)$ is recursive, parameter-free, and independent of the choice of N.

(B) The König extension K of N is arithmetical in N.

(C) Let N_1 and N_2 be internal models of PA with an internal $\mathscr{L}_\mathrm{I}$-isomorphism $F\colon N_1 \to N_2$. If K_1 and K_2 are respectively König extensions of N_1 and N_2, then there is an internal $\mathscr{L}_\mathrm{I}$-isomorphism $\hat{F}\colon K_1 \to K_2$ extending F that is arithmetical in F.

Just as in Arithmetized Completeness Theorem constructions, internal models of PA are naturally end-extensions of the first-order part of our model $(M, \mathscr{X})$. To obtain an *elementary* end-extension of M using this machinery, we need M to be internal as an $\mathscr{L}_{\mathrm{I}}$-structure. Equivalently, we want an internal *satisfaction class* for M.

Definition. Let $M \models \mathrm{I}\Delta_0 + \exp$. A *full inductive satisfaction class* for M is an inductive subset $S \subseteq M$ with the following properties.

(1) All elements of S are of the form $\langle \varphi(\bar{x}), [\bar{a}] \rangle$, where $\varphi(\bar{x})$ is an $\mathscr{L}_{\mathrm{I}}$-formula in the sense of M and $\bar{a}$ is a tuple that is at least as long as $\bar{x}$.

(2) For all standard $\mathscr{L}_{\mathrm{I}}$-formulas $\varphi(\bar{x})$ and all $\bar{a} \in M$, we have

$$\langle \varphi(\bar{x}), [\bar{a}] \rangle \in S \Leftrightarrow M \models \varphi(\bar{a}).$$

(3) S satisfies Tarski's definition of truth for all $\mathscr{L}_{\mathrm{I}}$-formulas in M.

Not even ACA_0 can guarantee us a full inductive satisfaction class.

Fact 9.2. If a nonstandard model of PA has a full inductive satisfaction class, then it is recursively saturated. In particular, there are models of PA without a full inductive satisfaction class. □

Just a little more suffices.

Definition. The $\mathscr{L}_{\mathrm{II}}$-theory ACA_0^+ is ACA_0 together with an axiom

$$\begin{aligned} &\forall i \, \exists H \left((H)_i = \left\{ n : \theta(n, H{\restriction_i}) \right\} \right) \\ &\to \exists H \, \forall i \left((H)_i = \left\{ n : \theta(n, H{\restriction_i}) \right\} \right) \end{aligned}$$

for every arithmetical formula $\theta(n, X)$ that may contain undisplayed free variables.

Fact 9.3. Let $(M, \mathscr{X}) \models \mathrm{ACA}_0^+$. Then $\mathscr{X}$ contains a unique full inductive satisfaction class for M. In particular, the $\mathscr{L}_{\mathrm{I}}$-structure M is internal in $(M, \mathscr{X})$. □

There is clearly no necessity in having ACA_0^+. All we need is an internal full satisfaction class and WKL_0.

Lemma 9.4. Let $(M, \mathscr{X}) \models \mathrm{RCA}_0$ in which M is internal as an $\mathscr{L}_\mathrm{I}$-structure. Then M is internal model of PA.

Proof. Let S be an internal full satisfaction class for M. Suppose M does not satisfy PA internally. Recall [29] that induction is equivalent to the least number principle over PA^-. Take an internal $\mathscr{L}_\mathrm{I}$-formula $\varphi(v, z)$ and an element $c \in M$ such that

$$M \models \exists v\, \varphi(v, c) \wedge \forall v\big(\varphi(v, c) \rightarrow \exists v' < v\ \varphi(v', c)\big).$$

Then the standard formula

$$\langle \varphi(v, z), [v, c] \rangle \in S$$

defines a nonempty subset of M without a least element, contradicting $\mathrm{I}\Sigma_1^0$ in $(M, \mathscr{X})$. □

Theorem 9.5. Let $(M, \mathscr{X}) \models \mathrm{WKL}_0$ that contains an internal full satisfaction class S. Then M has an internally conservative end-extension that is arithmetical in S.

Proof. As $\mathscr{X}$ contains a full satisfaction class for M, the $\mathscr{L}_\mathrm{I}$-structure M is internal. It follows that M is an internal model of PA by Lemma 9.4. The argument described at the beginning of this section then applies to M, giving the required extension. □

Note that if K is an internally conservative extension of our first-order part M, then $(M, \mathrm{SSy}_M(K)) \not\models \mathrm{ACA}_0^+$. To see this, suppose not. Then in view of Fact 9.3, there is a unique full inductive satisfaction class for M in $\mathrm{SSy}_M(K)$. By conservativity, this satisfaction class is defined by an internal $\mathscr{L}_\mathrm{I}$-formula, contradicting the internal version of Tarski's theorem on the undefinability of truth.

A possible way to get a stronger theory is to consider *coded β-models.* These are natural internal models of ATR_0 whose first-order part is M. More about these models can be found in Chapter VII.2 of Simpson's book [72].

9.2 Ordinals in models of ATR_0

A characteristic feature of ATR_0 is the theory of ordinals it accommodates. This is another reason why we look at ω_1-like models — if we can code the ordinals in our construction, then probably we can get ATR_0 back. We will briefly outline how ordinal theory can be developed in ATR_0 here. For the details, see Hirst's survey article [24].

We make the following definitions within a fixed model $(M, \mathscr{X}) \models \mathrm{RCA}_0$. An *internal ordinal* is a second-order object coding a linear order in which every nonempty internal subset has a least element. This is expressible as a Π^1_1-formula. We use lowercase Greek letters $\alpha, \beta, \lambda, \ldots$ for ordinals, despite our convention that all second-order objects are denoted using uppercase Roman letters. With some abuse of notation, we use $\prec$ for the order associated with *any* internal ordinal.

The class of internal ordinals is denoted by $\omega_1^{\mathscr{X}}$, or simply ω_1. The *zero ordinal* is the internal ordinal with an empty domain. A *sucessor ordinal* is an internal ordinal with a $\prec$-maximum element. A *limit ordinal* is an internal ordinal that is neither zero nor a successor. The class of internal limit ordinals is denoted by $\omega_1^{\mathscr{X}}(\lim)$ or $\omega_1(\lim)$.

For two internal ordinals α and β, we write $\alpha \preceq \beta$ if there is an internal order-preserving injection from α onto a $\prec$-initial segment of β. We simply call such an injection an *embedding*. Similarly, we write $\alpha \cong \beta$ to mean there is an internal order-preserving bijection from α to β. This bijection is called an *order-isomorphism*, and we say that α and β are *isomorphic*. Two internal ordinals α and β are *comparable* if either $\alpha \preceq \beta$ or $\beta \preceq \alpha$. The following theorem is the earliest, and probably the most important, result about ordinal theory in ATR_0.

Theorem 9.6 (H. Friedman). The system ATR_0 is equivalent over RCA_0 to the statement that any two internal ordinals are comparable. □

This, together with arithmetical transfinite recursion, allows us to iterate the König extension $\omega_1^{\mathscr{X}}$-many times in a model $(M, \mathscr{X}) \models \mathrm{ATR}_0$. One point to note is that there is no canonical representation of internal ordinals in a model. As we shall see below, this complicates matters, but fortunately it is not a real obstacle because of the niceties of König's extensions listed in the previous section.

Definition. Fix a model $(M, \mathscr{X}) \models \mathrm{ATR}_0$. Let **On** denote the category of internal ordinals, where the arrows are $\preceq$. Let $\mathbf{Mod}(\mathrm{PA}^-)$ be the category of internal models of PA^-, where an arrow is an internal $\mathscr{L}_\mathrm{I}$-elementary embedding whose image is an initial segment of the codomain. A model $K \models \mathrm{PA}^-$ is said to be *internally ω_1-like* in $(M, \mathscr{X})$ if there is a covariant functor

$$\mathbf{On} \to \mathbf{Mod}(\mathrm{PA}^-)$$

with arithmetical assignments

$$\alpha \mapsto K_\alpha \quad \text{and} \quad F \mapsto \hat{F}$$

for an object α and an arrow F in **On** such that

- an arrow F in **On** is an isomorphism if and only if $\hat{F}$ is an isomorphism in the category $\mathbf{Mod}(\mathrm{PA}^-)$;
- internal direct limits are preserved; and
- $K \cong \bigcup_{\alpha \in \omega_1^{\mathscr{X}}} K_\alpha$.

To help visualize such models, we describe them in more concrete detail. Let K be an internally ω_1-like model in some model $(M, \mathscr{X}) \models \mathrm{ATR}_0$. Then each element of K can be viewed as a disjoint union $\alpha + \{x\}$ where α is an internal ordinal and $x \in M$. The models K_α are thought of as

$$\{x \in M : \alpha + \{x\} \in K\}.$$

We regard two elements $\alpha + \{x\}$ and $\beta + \{y\}$ of K as equal if and only if there exists an arrow $F\colon \alpha \to \beta$ in **On** such that $\hat{F}(x) = y$, or *vice versa*. Note that K does not need to be of cardinality $\aleph_1$ externally.

Since K is the union of an elementary chain of internal models, we can actually define satisfaction for all internal $\mathscr{L}_{\mathrm{I}}$-formulas for K. For example, if $\varphi(v)$ is an internal $\mathscr{L}_{\mathrm{I}}$-formula and $\alpha + \{x\} \in K$, then we can define

$$K \models \varphi(\alpha + \{x\})$$

to mean

$$(M, \mathscr{X}) \models \text{`}K_\alpha \models \varphi(x)\text{'}.$$

This satisfaction relation is Δ^1_1-definable in $(M, \mathscr{X})$. So although K is not internal, our model $(M, \mathscr{X})$ still has a lot of control over it.

Proposition 9.7. Let $(M, \mathscr{X})$ be a nonstandard model of ATR_0. Then all internally ω_1-like models are recursively saturated (externally).

Proof. This is because every internally ω_1-like model is a union of an elementary chain of internal models, and all internal models are recursively saturated. □

We also have an internal version of Proposition 9.1. However, some of its content is lost in the internalization, because a substantial amount of strength is coded in the definition of internally ω_1-like models. As a result, the proof is essentially the same as that of Theorem 8.1. A proof of Proposition 9.1 usually involves a counting argument.

Proposition 9.8. Fix a model $(M, \mathscr{X}) \models \mathrm{ATR}_0$. Let K be a model of $\mathrm{I}\Delta_0$ that is internally ω_1-like in $(M, \mathscr{X})$. Then K is a model of PA. If, moreover, K satisfies the internal version of $\mathrm{I}\Delta_0$, then K satisfies the internal version of PA.

Proof. We only prove the internal version. The external version has essentially the same proof. We show that K satisfies the internal version of $\mathrm{B}\Sigma_n$ for all $n \in M$. This suffices because the statement

$$\forall n\ \mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$$

is easily provable in ACA_0, say.

Let $\psi(u, v, \bar{z})$ be an internal $\mathscr{L}_{\mathrm{I}}$-formula, and $a, \bar{c} \in K$. We want to show

$$K \models \forall u < a\, \exists b\, \forall v > b\, \psi(u, v, \bar{c}) \\ \rightarrow \exists b\, \forall u < a\, \forall v > b\, \psi(u, v, \bar{c}).$$

So suppose $K \models \forall u < a\, \exists b\, \forall v > b\, \psi(u, v, \bar{c})$. Write a as $\alpha + \{x\}$. Without loss of generality, we may assume $\bar{c} \in K_\alpha$. Since the only arrow $F\colon \alpha \to \alpha + 1$ is not an isomorphism, the arrow $\hat{F}\colon K_\alpha \to K_{\alpha+1}$ cannot be an isomorphism. So, $K_\alpha \subsetneq_{\mathrm{e}} K_{\alpha+1}$. Take $b \in K_{\alpha+1} \setminus K_\alpha$. We claim that

$$K \models \forall u < a\, \forall v > b\, \psi(u, v, \bar{c}).$$

Let $u \in K_{<a}$. By elementarity, $K_\alpha \models \exists b'\, \forall v > b'\, \psi(u, v, \bar{c})$. Since any witness b' to this statement has to be less than b, we conclude that $K \models \forall v > b\, \psi(u, v, \bar{c})$. □

Our theory ATR_0 is enough to give us plenty of internally ω_1-like models.

Theorem 9.9. Let $(M, \mathscr{X}) \models \mathrm{ATR}_0$, and N be an internal model of PA. Then there is an internally ω_1-like model in which N is an internally elementary initial segment.

Proof. We start from $K_0 = N$. For each internal ordinal α, we use arithmetical transfinite recursion to iterate the König extension along α, taking unions at limit stages, to get an internal end-extension K_α of N. Every ordinal embedding $F\colon \alpha \to \beta$ naturally induces an internal $\mathscr{L}_{\mathrm{I}}$-embedding $\hat{F}\colon K_\alpha \to K_\beta$. It is straightforward to check that

$$\bigcup_{\alpha \in \omega_1^{\mathscr{X}}} K_\alpha$$

is an internally ω_1-like elementary extension of N. □

Corollary 9.10. Let $(M, \mathscr{X}) \models \mathrm{ATR}_0$. Then there is an internally ω_1-like model in which M is an internally elementary initial segment.

Proof. Just note that $\mathrm{ATR}_0 \vdash \mathrm{ACA}_0^+$. □

We aimed to obtain some kind of reversal. However, it is not even clear how it can be formulated. For example, if a model $(M, \mathscr{X})$ potentially contains incomparable internal ordinals, then how should we define $\bigcup_{\alpha \in \omega_1^{\mathscr{X}}} K_\alpha$? It either gets trivial or messy.

9.3 Rather classless models

Having got an internally ω_1-like model, one may go a step further and ask for an internally *rather classless* model.

Definition. Fix a nonzero ordinal λ. Let $(K_\alpha)_{\alpha<\lambda}$ be a continuous elementary chain of models of PA, and $K = \bigcup_{\alpha<\lambda} K_\alpha$. If $\varphi(x) \in \mathscr{L}_\mathrm{I}(K)$ and $\alpha < \lambda$, then we define

$$\varphi^K(K_\alpha) = \{x \in K_\alpha : K \models \varphi(x)\}.$$

The model K is *rather classless* if for every subset $X \subseteq K$,

$$\begin{aligned} &\forall \alpha < \lambda \ \exists \varphi(x) \in \mathscr{L}_\mathrm{I}(K) \ \ X \cap K_\alpha = \varphi^K(K_\alpha) \\ &\Rightarrow \exists \varphi(x) \in \mathscr{L}_\mathrm{I}(K) \ \forall \alpha < \lambda \ \ X \cap K_\alpha = \varphi^K(K_\alpha). \end{aligned}$$

Remark. Our definition of a rather classless model does not only depend on the model K, but also on the particular elementary chain (K_α) that is chosen.

The usual construction of rather classless ω_1-like models involves a combinatorial lemma known as *Födor's Lemma.*

Födor's Lemma. Let S be a stationary subset of ω_1 and $f \colon S \to \omega_1$ such that $f(\alpha) < \alpha$ for every $\alpha \in S$. Then there exists $\beta \in \omega_1$ whose inverse image under f is stationary in ω_1. □

The notion of stationary sets seems to be outside the realm of second-order arithmetic. So there is very little hope in formalizing the full Födor's Lemma within a model of ATR_0, say. Fortunately, a 'weak' version suffices for our purpose.

Weak Födor's Lemma. Let $f\colon \omega_1(\lim) \to \omega_1$ such that $f(\lambda) < \lambda$ for every $\lambda \in \omega_1(\lim)$. Then there is $\beta \in \omega_1$ such that

$$\{\lambda \in \omega_1(\lim) : f(\lambda) = \beta\} \subseteq_{\mathrm{cf}} \omega_1.$$

Proof. Consider $\tilde{f}\colon \omega_1 \to \omega_1$ defined by

$$\begin{gathered}\tilde{f}(0) = 0;\\ \tilde{f}(\alpha+1) = \alpha;\ \text{and}\\ \tilde{f}(\lambda) = f(\lambda),\end{gathered}$$

for every $\alpha \in \omega_1$ and every $\lambda \in \omega_1(\lim)$. Then for all nonzero $\alpha \in \omega_1$, we have $\tilde{f}(\alpha) < \alpha$. For an ordinal $\alpha \in \omega_1$, let $g(\alpha)$ be the least $n \in \omega$ such that $\tilde{f}^n(\alpha) = 0$. This defines a function $g\colon \omega_1 \to \omega$, because for every $\alpha \in \omega_1$, the sequence

$$\tilde{f}^0(\alpha), \tilde{f}^1(\alpha), \tilde{f}^2(\alpha), \ldots$$

is strictly decreasing before it reaches 0.

Now g partitions ω_1 into countably many pieces. So, there must be a component that is uncountable. Let $n \in \omega$ such that $g^{-1}(n) \subseteq_{\mathrm{cf}} \omega_1$. Set $S = g^{-1}(n)$. Consider the sets

$$\tilde{f}^0(S), \tilde{f}^1(S), \ldots, \tilde{f}^n(S).$$

Since $\tilde{f}^0(S)$ is S, which is uncountable, and $\tilde{f}^n(S)$ is $\{0\}$, which is countable, there must be a natural number, say $m < n$, such that $\tilde{f}^m(S)$ is uncountable but $\tilde{f}^{m+1}(S)$ is countable. Again, $\tilde{f}$ partitions the uncountable set $\tilde{f}^m(S)$ into countably many components. So there must exist an uncountable component. Take $\beta \in \tilde{f}^{m+1}(S)$ with an uncountable inverse image under $\tilde{f}$. By the definition of $\tilde{f}$, at most one successor ordinal can be mapped to β via $\tilde{f}$. The rest must all be limits (or zero). $\square$

Question 9.11. Can we replace $\omega_1(\lim)$ in Weak Födor's Lemma by an arbitrary unbounded subset of ω_1?

Note that we avoided the use of diagonal intersections, which is a key ingredient of the popular proof of Födor's Lemma. While our proof easily formalizes in any second-order model with enough comprehension, it seems very hard to internalize diagonal intersections within second-order arithmetic.

Definition. The scheme Δ^1_1-WFL consists of the axioms

$$\begin{aligned} &\forall \lambda \in \omega_1(\lim)\, \forall \beta \in \omega_1\, \big(\xi(\lambda,\beta) \leftrightarrow \neg\zeta(\lambda,\beta)\big) \\ &\to \left(\begin{aligned} &\forall \lambda \in \omega_1(\lim)\, \exists \beta \in \omega_1\, \big(\beta \prec \lambda \wedge \xi(\lambda,\beta)\big) \\ &\to \exists \beta \in \omega_1\, \forall \alpha \in \omega_1\, \exists \lambda \in \omega_1(\lim)\, \big(\alpha \prec \lambda \wedge \xi(\lambda,\beta)\big) \end{aligned} \right), \end{aligned}$$

where $\xi(\lambda,\beta), \zeta(\lambda,\beta)$ are Σ^1_1-formulas, possibly with undisplayed free variables.

Since the notion of rather classless models actually lies between second- and third-order arithmetic, we need some additional definability conditions in its internal version.

Definition. Fix a model $(M, \mathscr{X}) \models \mathrm{ATR}_0$. Let K be an internally ω_1-like model. We say that a subset $X \subseteq K$ is an *internal (arithmetical) class* if there are arithmetically definable functions

$$\begin{aligned} \omega_1 &\to \mathscr{L}_\mathrm{I} \\ \alpha &\mapsto \varphi_\alpha(x,y) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \omega_1 &\to K \\ \alpha &\mapsto c_\alpha \end{aligned}$$

such that

$$X \cap K_\alpha = \{x \in K_\alpha : K \models \varphi_\alpha(x, c_\alpha)\},$$

for all $\alpha \in \omega_1^{\mathscr{X}}$. The model K is *internally (and arithmetically) rather classless* if every internal (arithmetical) class in K is internally $\mathscr{L}_\mathrm{I}$-definable.

We now have everything ready for the internalization.

Theorem 9.12. Let $(M, \mathscr{X}) \models \mathrm{ATR}_0 + \Delta^1_1$-WFL, and N be an internal model of PA. Then there is an internally rather classless model that is an internally elementary end-extension of N.

Proof. As in the proof of Theorem 9.9, we build an internally ω_1-like model K of PA by iterating the König extension. We check that K is internally rather classless.

Let X be an internal arithmetical class in K, and let

$$\begin{aligned} \omega_1 &\to \mathscr{L}_\mathrm{I} \\ \alpha &\mapsto \varphi_\alpha(x, y) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \omega_1 &\to K \\ \alpha &\mapsto c_\alpha \end{aligned}$$

be a witness to its arithmeticity. Define a function $\Xi\colon \omega_1^{\mathscr{X}} \to \omega_1^{\mathscr{X}}$ by setting $\Xi(\alpha)$ to be the $\preceq$-least internal ordinal β such that

$$(M, \mathscr{X}) \models \exists \psi(x, y) \in \mathscr{L}_\mathrm{I}\ \exists b \in K_\beta\ \forall x \in K_\alpha\ \text{`}K \models \varphi_\alpha(x, c_\alpha) \leftrightarrow \psi(x, b)\text{'}$$

for all $\alpha \in \omega_1^{\mathscr{X}}$. This function is well-defined because $\mathrm{ATR}_0 \vdash \Delta^1_1$-CA. Since the functor $\alpha \mapsto K_\alpha$ preserves internal direct limits, we see that if λ is an internal limit ordinal, then $\Xi(\lambda) \prec \lambda$. Using Δ^1_1-WFL, pick an internal ordinal β with the property that

$$\{\lambda \in \omega_1(\lim) : \Xi(\lambda) = \beta\} \subseteq_{\mathrm{cf}} \omega_1^{\mathscr{X}}. \tag{$*$}$$

Let λ be any internal limit ordinal satisfying $\Xi(\lambda) = \beta$. Pick an internal $\mathscr{L}_\mathrm{I}$-formula $\psi(x, y)$ and an element $b \in K_\beta$ such that for all $x \in K_\lambda$,

$$K \models \varphi_\lambda(x, c_\lambda) \leftrightarrow \psi(x, b).$$

We claim that $\psi(x, b)$ defines X in K.

Take any $a \in K$. Using $(*)$, choose an internal limit ordinal λ' such that $a \in K_{\lambda'}$ and

$\Xi(\lambda') = \beta$. Let $\psi'(x, y)$ be an internal $\mathscr{L}_{\mathrm{I}}$-formula and $b' \in K_\beta$ such that for all $x \in K_{\lambda'}$,

$$K \models \varphi_{\lambda'}(x, c_{\lambda'}) \leftrightarrow \psi'(x, b').$$

First, note that for each $x \in K_\beta$,

$$\begin{aligned}
K_\beta \models \psi(x, b) \quad &\text{iff} \quad K \models \psi(x, b) && \text{by elementarity,}\\
&\text{iff} \quad K \models \varphi_\lambda(x, c_\lambda) && \text{by the choice of } \psi \text{ and } b,\\
&\text{iff} \quad x \in X && \text{by the choice of } \varphi_\lambda \text{ and } c_\lambda,\\
&\text{iff} \quad K \models \varphi_{\lambda'}(x, c_{\lambda'}) && \text{by the choice of } \varphi_{\lambda'} \text{ and } c_{\lambda'},\\
&\text{iff} \quad K \models \psi'(x, b') && \text{by the choice of } \psi' \text{ and } b',\\
&\text{iff} \quad K_\beta \models \psi'(x, b') && \text{by elementarity.}
\end{aligned}$$

Therefore, $K_\beta \models \forall x \big(\psi(x, b) \leftrightarrow \psi'(x, b')\big)$. By elementarity again, we have

$$K \models \forall x \big(\psi(x, b) \leftrightarrow \psi'(x, b')\big). \tag{†}$$

It follows that

$$\begin{aligned}
a \in X \quad &\text{iff} \quad K \models \varphi_{\lambda'}(a, c_{\lambda'}) && \text{by the choice of } \varphi_{\lambda'} \text{ and } c_{\lambda'},\\
&\text{iff} \quad K \models \psi'(a, b') && \text{by the choice of } \psi' \text{ and } b',\\
&\text{iff} \quad K \models \psi(a, b) && \text{by (†),}
\end{aligned}$$

as required. □

Remark. Our definition of (internally) rather classless models hides some conservativity condition on the elementary chain. This is why the notion of (internally) conservative extensions does not appear in the above proof as one may have expected.

Clearly, this theorem is only a base case for a hierarchy of results where internal classes need not be arithmetical. However, this base case is still of independent interest because Δ^1_1-WFL does not seem to be too strong.

Question 9.13. Is it true that $\mathrm{ATR}_0 \vdash \Delta^1_1$-WFL?

CHAPTER 10

THE GALVIN–PRIKRY THEOREM IN SECOND-ORDER ARITHMETIC

> I am speaking here of a theorem that is traditionally used in model theory to build indiscernible sequences. ⟦... I⟧t is a fundamental theorem of combinatorics, which no well-bred person can ignore.
>
> Bruno Poizat, translated by Moses Klein
> *A Course in Model Theory* [64], Section 12.9

The aim of this chapter is to set up the machinery needed for the construction in the next chapter. We have seen in previous chapters how Ramsey's Theorem can be used to build end-extensions at the level of ACA_0. To go higher up, we consider an infinite exponent version of Ramsey's Theorem.

Galvin–Prikry Theorem. Let $S \subseteq_{\mathrm{cf}} \mathbb{N}$, and let $[S]^{\mathbb{N}}$ denote the set of all infinite subsets of S. If θ is a Borel colouring of $[S]^{\mathbb{N}}$ using only finitely many colours, then there is $H \subseteq_{\mathrm{cf}} S$ such that all sets in $[H]^{\mathbb{N}}$ are coloured the same by θ. □

What we need are some weak versions of this, formalized within second-order arithmetic. We make use of the quantifier $\forall^{\mathrm{cf}} S \cdots$, which means

$$\forall S\big(\forall x\, \exists y \in S\ (y > x) \to \cdots\big).$$

Similarly, $\exists^{\mathrm{cf}} S\ \cdots$ means $\exists S\big(\forall x\, \exists y \in S\ (y > x) \wedge \cdots\big)$. For an object S in the set universe, $S_{>n}$ denotes $\{x \in S : x > n\}$.

Definition. Let Γ be a class of $\mathscr{L}_{\mathrm{II}}$-formulas.

(i) The theory Γ-RT_2 consists of all the axioms of the form

$$\begin{aligned} &\forall^{\mathrm{cf}} S\ \exists H \subseteq_{\mathrm{cf}} S \\ &\quad \begin{pmatrix} \exists X \subseteq_{\mathrm{cf}} H\ \chi(X) \\ \to \forall X \subseteq_{\mathrm{cf}} H\ \chi(X) \end{pmatrix}, \end{aligned}$$

where $\chi(X)$ is a formula in Γ that may contain undisplayed free variables.

(ii) The theory Γ-RT consists of all the axioms of the form

$$\begin{aligned} &\forall^{\mathrm{cf}} S\ \forall l\ \exists H \subseteq_{\mathrm{cf}} S \\ &\quad \begin{pmatrix} \forall X \subseteq_{\mathrm{cf}} S\ \exists m < l\ \theta(X, m) \\ \to \exists m < l\ \forall X \subseteq_{\mathrm{cf}} H\ \theta(X, m) \end{pmatrix}, \end{aligned}$$

where $\theta(X, m)$ is a formula in Γ that may contain undisplayed free variables.

(iii) The theory Γ-$\widetilde{\mathrm{RT}}_2$ consists of all the axioms of the form

$$\begin{aligned} &\forall^{\mathrm{cf}} S\ \exists H \subseteq_{\mathrm{cf}} S\ \forall n \\ &\quad \begin{pmatrix} \exists X \subseteq_{\mathrm{cf}} H_{>n}\ \vartheta(X, n) \\ \to \forall X \subseteq_{\mathrm{cf}} H_{>n}\ \vartheta(X, n) \end{pmatrix}, \end{aligned}$$

where $\vartheta(X, n)$ is a formula in Γ that may contain undisplayed free variables.

Clearly, Γ-$\widetilde{\mathrm{RT}}_2 \vdash \Gamma$-$\mathrm{RT}_2$ for all classes Γ of $\mathscr{L}_{\mathrm{II}}$-formulas. Here are some other easy implications between these theories.

Lemma 10.1. Let Γ be a class of $\mathscr{L}_{\mathrm{II}}$-formulas and $\Gamma' = \{\neg\chi : \chi \in \Gamma\}$. Then Γ-RT_2 is equivalent to Γ'-RT_2. $\square$

Lemma 10.2. Let Γ be a class of $\mathscr{L}_{\mathrm{II}}$-formulas that contains all quantifier-free $\mathscr{L}_{\mathrm{I}}$-formulas and is closed under the Boolean operations. Then Γ-RT $\vdash \Gamma$-RT_2.

Proof. Let $\theta(X, m)$ be $(\neg\chi(X) \wedge m = 0) \vee (\chi(X) \wedge m = 1)$. □

Lemma 10.3. For every class Γ of $\mathscr{L}_{\mathrm{II}}$-formulas, $\mathrm{RCA}_0 + \Gamma\text{-}\widetilde{\mathrm{RT}}_2 \vdash \Gamma\text{-RT}$.

Proof. Fix a model $(M, \mathscr{X}) \models \mathrm{RCA}_0 + \Gamma\text{-}\widetilde{\mathrm{RT}}_2$ and $S \subseteq_{\mathrm{cf}} M$. Let $\theta(X, m) \in \Gamma$ and $l \in M$ such that

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S\ \exists m < l\ \theta(X, m). \qquad (*)$$

Using $\Gamma\text{-}\widetilde{\mathrm{RT}}_2$, let $\widetilde{H} \subseteq_{\mathrm{cf}} S$ that satisfies

$$(M, \mathscr{X}) \models \forall n \begin{pmatrix} \exists X \subseteq_{\mathrm{cf}} \widetilde{H}_{>n}\ \theta(X, n) \\ \to \forall X \subseteq_{\mathrm{cf}} \widetilde{H}_{>n}\ \theta(X, n) \end{pmatrix}.$$

Define $H = \tilde{H}_{>l}$. Since $\tilde{H}_{>0} \supseteq_{\mathrm{cf}} \tilde{H}_{>1} \supseteq_{\mathrm{cf}} \tilde{H}_{>2} \supseteq_{\mathrm{cf}} \cdots \supseteq_{\mathrm{cf}} \tilde{H}_{>l} = H$,

$$(M, \mathscr{X}) \models \forall m < l \begin{pmatrix} \exists X \subseteq_{\mathrm{cf}} H\ \theta(X, m) \\ \to \forall X \subseteq_{\mathrm{cf}} H\ \theta(X, m) \end{pmatrix}.$$

This shows H is what we are looking for, because by $(*)$, there is $m < l$ such that $(M, \mathscr{X}) \models \theta(H, m)$, and for this m, we must have $(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} H\ \theta(X, m)$. □

Miniaturizations of the Galvin–Prikry Theorem have been extensively studied in reverse mathematics. So the exact strength of a good number of these schemes is known.

Theorem 10.4 (Friedman–McAloon–Simpson). Over RCA_0, the following theories are equivalent:

(a) ATR_0;

(b) Σ^0_1-RT_2;

(c) Π^0_1-RT.

Proof. See Theorem 3.2 and Corollary 3.12 of Friedman–McAloon–Simpson [14]. □

Theorem 10.5 (Simpson, Shelah–Simpson, Solovay). Over RCA_0, the following theories are equivalent:

(a) Π^1_1-CA_0;

(b) Σ^1_0-RT_2;

(c) Σ^0_1-$\widetilde{\mathrm{RT}}_2$.

Proof. For the equivalence of (a) and (b), see Section VI.6 in Simpson's book [72]. For the equivalence of (a) and (c), see Theorem 3.4 of Simpson [70]. □

We are unable to find the answer to the following.

Question 10.6. What is the exact strength of Σ^1_0-RT and of Σ^1_0-$\widetilde{\mathrm{RT}}_2$?

CHAPTER 11

RAMSEY-TYPE END-EXTENSIONS

> It is, however, interesting to treat the general concept of a local function, especially so since it has natural analogues in other theories, e.g., in number theory with second-order quantifiers and in set theory.
>
> Haim Gaifman
> On local arithmetical functions and their application for constructing types of Peano's arithmetic [15], §0

This whole chapter is devoted to the construction of an end-extension using the Ramsey theory developed in the previous chapter. For all of this chapter, except in the last section, we work with a fixed countable model $(M, \mathscr{X}) \models \Pi^1_1\text{-CA}_0$.

Let us set up some notations. We will see four similar languages. The language $\mathscr{L}_{\mathrm{II}}(M, \mathscr{X})$ is the language obtained from $\mathscr{L}_{\mathrm{II}}$ by adding a new constant symbol of the number sort for each $a \in M$, and a new constant symbol of the set sort for each $S \in \mathscr{X}$. The language $\mathscr{L}_{\mathrm{I}}(M, \mathscr{X})$ is the language obtained from $\mathscr{L}_{\mathrm{I}}$ by adding a new constant symbol for each $a \in M$, and a new unary predicate symbol for each $S \in \mathscr{X}$. The languages $\mathscr{L}^*_{\mathrm{I}}$ and $\mathscr{L}^*_{\mathrm{I}}(M, \mathscr{X})$ only differ from $\mathscr{L}_{\mathrm{I}}$ and $\mathscr{L}_{\mathrm{I}}(M, \mathscr{X})$ in having an extra unary predicate symbol X. The classes $\Sigma_1(M, \mathscr{X})$, $\Sigma^0_1(M, \mathscr{X})$, Σ^*_1, $\Sigma^*_1(M, \mathscr{X})$, etc., are defined accordingly.

The plan of our construction is to first build a sequence

$$M = S_0 \supseteq_{\mathrm{cf}} S_1 \supseteq_{\mathrm{cf}} S_2 \supseteq_{\mathrm{cf}} \cdots$$

of internal sets recursively. Then define

$$p(X) = \{X \subseteq_{\text{cf}} S_i : i \in \mathbb{N}\}.$$

Our extension of $(M, \mathscr{X})$ will be called $(N, \mathscr{Y})$. The first-order part N is Σ_1^**-generated by the type* $p(X)$ *over* $(M, \mathscr{X})$, i.e., N is an $\mathscr{L}_{\text{I}}$-structure in which every element is the realization of

$$(\mu m)(\theta(X, m))$$

for some $\theta(X, m) \in \Sigma_1^0(M, \mathscr{X})$, and truth in N is determined by $p(X)$ and the $\mathscr{L}_{\text{II}}(M, \mathscr{X})$-theory of $(M, \mathscr{X})$. The second-order part $\mathscr{Y}$ is the closure of $\mathscr{X}^N$ and X under arithmetical comprehension.

How the end-product $(N, \mathscr{Y})$ actually looks like depends on how the sequence $(S_i)_{i \in \mathbb{N}}$ is constructed. We say that a property P of $(N, \mathscr{Y})$ is *enforceable* if one can make sure $(N, \mathscr{Y})$ satisfy P just by controlling how $S_1, S_3, S_5, \ldots$ are chosen. This is the same as the notion of enforceability in the Banach–Mazur game on the type space. Clearly, a countable conjunction of enforceable properties is enforceable. So we will split our construction into manageable pieces.

11.1 Elementarity

We give the full definition of $(N, \mathscr{Y})$ in this section. We will actually define an $\mathscr{L}_{\text{I}}^*(M, \mathscr{X})$-structure $(N, \mathscr{X}^N, C)$. The second-order part $\mathscr{Y}$ will then be $\text{Def}(N, \mathscr{X}^N \cup \{C\})$, i.e., the class of subsets of N that are *arithmetically* definable with number parameters from N and set parameters from $\mathscr{X}^N \cup \{C\}$. The $\mathscr{L}_{\text{II}}(M, \mathscr{X})$-theory of $(M, \mathscr{X})$ plus $p(X)$ will be referred to as T. Clearly, T is consistent. For the moment, we will assume T is complete for $\Sigma_1^*(M, \mathscr{X})$-sentences. If it is not, then replace T by one of its completions.

Elements of N are of the form d_θ, where $\theta(X, m) \in \Sigma_1^0(M, \mathscr{X})$ such that $T \vdash \exists! m\, \theta(X, m)$. The operations on $(N, \mathscr{Y})$ are defined as follows for all $d_{\theta_1}, d_{\theta_2}, d_{\theta_3} \in N$.

- $N \models d_{\theta_1} = d_{\theta_2}$ if and only if $T \vdash \exists m\big(\theta_1(X, m) \wedge \theta_2(X, m)\big)$.
- $N \models d_{\theta_1} < d_{\theta_2}$ if and only if $T \vdash \exists m_1 \exists m_2 \big(\theta_1(X, m_1) \wedge \theta_2(X, m_2) \wedge m_1 < m_2\big)$.
- $N \models d_{\theta_1} + d_{\theta_2} = d_{\theta_3}$ if and only if $T \vdash \exists m_1 \exists m_2 \exists m_3 \big(\theta_1(X, m_1) \wedge \theta_2(X, m_2) \wedge \theta_3(X, m_3) \wedge m_1 + m_2 = m_3\big)$.
- $N \models d_{\theta_1} \times d_{\theta_2} = d_{\theta_3}$ if and only if $T \vdash \exists m_1 \exists m_2 \exists m_3 \big(\theta_1(X, m_1) \wedge \theta_2(X, m_2) \wedge \theta_3(X, m_3) \wedge m_1 \times m_2 = m_3\big)$.
- If $a \in M$, then the realization of a in N is $d_{m=a}$.
- If $S \in \mathscr{X}$, then the realization of S in N is $\big\{d_\theta \in N : T \vdash \exists m\big(\theta(X, m) \wedge m \in S\big)\big\}$.
- Similarly, the realization of X in N is $\big\{d_\theta \in N : T \vdash \exists m\big(\theta(X, m) \wedge m \in X\big)\big\}$. We call this set C.

Lemma 11.1. For all $\Sigma_1^*(M, \mathscr{X})$-formulas $\varphi(\bar{v}, X)$ and all $d_{\theta_1}, d_{\theta_2}, \ldots, d_{\theta_k} \in N$,

$$(N, \mathscr{Y}) \models \varphi(d_{\theta_1}, d_{\theta_2}, \ldots, d_{\theta_k}, C)$$
$$\Leftrightarrow T \vdash \exists m_1 \exists m_2 \cdots \exists m_k \left(\bigwedge_{i=1}^{k} \theta_i(X, m_i) \wedge \varphi(m_1, m_2, \ldots, m_k, X)\right).$$

Proof. We first prove the case when $\varphi(\bar{v}, X) \in \Delta_0^0(M, \mathscr{X})$ by an induction on the number of quantifiers in φ. Since T is complete for $\Sigma_1^*(M, \mathscr{X})$-sentences, it is not hard to verify that the claim is true if $\varphi(\bar{v}, X)$ is quantifier-free. Suppose $\psi(u, \bar{v}, X)$ is a $\Delta_0^0(M, \mathscr{X})$-formula with n quantifiers, and t is an $\mathscr{L}_\mathrm{I}(M)$-term. We show that

$$(N, \mathscr{Y}) \models \forall u < t\ \psi(u, d_{\theta_1}, d_{\theta_2}, \ldots, d_{\theta_k}, C)$$
$$\Leftrightarrow T \vdash \exists m_1 \exists m_2 \cdots \exists m_k \left(\bigwedge_{i=1}^{k} \theta_i(X, m_i) \wedge \forall u < t\ \psi(u, m_1, m_2, \ldots, m_k, X)\right).$$

Suppose $(N, \mathscr{Y}) \not\models \forall u < t\ \psi(u, d_{\theta_1}, d_{\theta_2}, \ldots, d_{\theta_k}, C)$. Let $d_{\theta_0} \in N$ such that

$$d_{\theta_0} < t \wedge \neg\psi(d_{\theta_0}, d_{\theta_1}, d_{\theta_2}, \ldots, d_{\theta_k}, C).$$

By the induction hypothesis,

$$T \vdash \exists m_0 \exists m_1 \cdots \exists m_k \left(\bigwedge_{i=0}^{k} \theta_i(X, m_i) \wedge m_0 < t \wedge \neg\psi(m_0, m_1, m_2, \ldots, m_k, X) \right).$$

Since T is consistent, we conclude

$$T \nvdash \exists m_1 \exists m_2 \cdots \exists m_k \left(\bigwedge_{i=1}^{k} \theta_i(X, m_i) \wedge \forall u < t \; \psi(m_0, m_1, m_2, \ldots, m_k, X) \right).$$

Conversely, suppose

$$T \nvdash \exists m_1 \exists m_2 \cdots \exists m_k \left(\bigwedge_{i=1}^{k} \theta_i(X, m_i) \wedge \forall u < t \; \psi(u, m_1, m_2, \ldots, m_k, X) \right).$$

By Σ_1^*-completeness, we must have

$$T \vdash \forall m_1 \forall m_2 \cdots \forall m_k \left(\bigwedge_{i=1}^{k} \theta_i(X, m_i) \rightarrow \exists u < t \; \neg\psi(u, m_1, m_2, \ldots, m_k, X) \right).$$

Let $\theta_0(X, u)$ be the $\Sigma_1^0(M, \mathscr{X})$-formula

$$u < t \wedge \exists v_1 \exists v_2 \cdots \exists v_k \left(\bigwedge_{i=1}^{k} \theta_i(X, v_i) \wedge \neg\psi(u, v_1, v_2, \ldots, v_k, X) \wedge \forall u' < u \; \psi(u', v_1, v_2, \ldots, v_k, X) \right).$$

Then by the induction hypothesis,

$$(N, \mathscr{Y}) \models d_{\theta_0} < t \wedge \neg\psi(d_{\theta_0}, d_{\theta_1}, d_{\theta_2}, \ldots, d_{\theta_k}, C),$$

showing what we want. The case for bounded existential quantification is similar and actually simpler. So we leave this part to the reader. This finishes the induction.

Using a pairing function, one can then add unbounded existential quantifiers to φ by following the inductive step above. Note that it is not immediate that N has a wellbehaved pairing function, but it can be transferred from T using the previous case. $\square$

Since T contains all $\mathscr{L}_{\mathrm{II}}(M,\mathscr{X})$-sentences true in $(M,\mathscr{X})$, this lemma immediately tells us $(M,\mathscr{X})$ is Σ^0_1-elementary in $(N,\mathscr{Y})$. As in many other similar arguments, one can squeeze out a little more.

Corollary 11.2. The extension $(N,\mathscr{Y})$ of $(M,\mathscr{X})$ is Π^0_2-elementary.

Proof. Let $\psi(u)$ be a $\Sigma_1(M,\mathscr{X})$-formula. Consider the $\Pi_2(M,\mathscr{X})$-sentence $\forall u\,\psi(u)$. If $(M,\mathscr{X})\models\forall u\,\psi(u)$, then for every $d_\theta\in N$, we have $T\vdash\exists m\big(\theta(X,m)\wedge\psi(m)\big)$, and so by Lemma 11.1, $(N,\mathscr{Y})\models\forall u\,\psi(u)$. Conversely, if $a\in M$ such that $(M,\mathscr{X})\models\neg\psi(a)$, then $T\nvdash\psi(a)$, and so $(N,\mathscr{Y})\models\neg\psi(d_{m=a})$ by Lemma 11.1. $\square$

Since we are working with a theory that is much stronger than ACA_0, we expect more elementarity than this, at least in the first-order part.

Corollary 11.3. The extension N of M is $\mathscr{L}_{\mathrm{I}}$-elementary.

Proof. Recall from Proposition 9.3 that there is an internal full inductive satisfaction class for M. Let $S\in\mathscr{X}$ such that for all $\mathscr{L}_{\mathrm{I}}$-formulas $\varphi(\bar{x})$,

$$(M,\mathscr{X})\models\forall\bar{a}\big(\varphi(\bar{a})\leftrightarrow\langle\varphi,[\bar{a}]\rangle\in S\big).\tag{$*$}$$

By Δ^0_0-elementarity, it suffices to prove that for all $\mathscr{L}_{\mathrm{I}}$-formulas $\varphi(\bar{x})$,

$$(N,\mathscr{Y})\models\forall\bar{a}\big(\varphi(\bar{a})\leftrightarrow\langle\varphi,[\bar{a}]\rangle\in S\big).$$

We proceed by an induction on the quantifier rank of φ. If $\varphi(\bar{x})\in\Sigma_1$, then the formula in $(*)$ is Π^0_2, and so we are done by Corollary 11.2. Let $\psi(u,\bar{x})$ be an $\mathscr{L}_{\mathrm{I}}$-formula of quantifier rank n. We show that, provided the claim is true for all $\mathscr{L}_{\mathrm{I}}$-formulas of quantifier rank n,

$$(N,\mathscr{Y})\models\forall\bar{a}\big(\exists u\,\psi(u,\bar{a})\leftrightarrow\langle\exists u\,\psi,[\bar{a}]\rangle\in S\big).$$

This is straightforward, because by the induction hypothesis,

$$(N, \mathscr{Y}) \models \forall \bar{a} \big(\exists u \, \psi(u, \bar{a}) \leftrightarrow \exists u \, \langle \psi, [u, \bar{a}] \rangle \in S \big),$$

and by Π^0_2-elementarity,

$$(N, \mathscr{Y}) \models \forall \bar{a} \big(\exists u \, \langle \psi, [u, \bar{a}] \rangle \in S \leftrightarrow \langle \exists u \, \psi, [\bar{a}] \rangle \in S \big).$$

The case for $\forall u \, \psi(u, \bar{a})$ is similar. □

Nothing in the above proof prevents us from adding a fixed set parameter in the formula φ.

Corollary 11.4. The extension $(N, \mathscr{Y})$ of $(M, \mathscr{X})$ is Σ^1_0-elementary. □

Note that we have only used ACA_0^+ in $(M, \mathscr{X})$ so far.

11.2 Completeness

In the previous section, we assumed T is complete for $\Sigma^*_1(M, \mathscr{X})$-sentences. This is necessary for our proof of Lemma 11.1. We will eliminate this assumption by a careful choice of $(S_i)_{i \in \mathbb{N}}$.

Given a $\Sigma^*_1(M, \mathscr{X})$-sentence $\chi(X)$, we need to decide whether $\chi(X)$ or $\neg\chi(X)$ should go into T. Since there are only countably many $\Sigma^*_1(M, \mathscr{X})$-sentences, we only need to show how to deal with one such $\chi(X)$ by the remark at the beginning of this chapter.

Suppose we are given a cofinal subset S_i to play in. Using Σ^0_1-RT_2 in $(M, \mathscr{X})$, choose an internal cofinal subset S_{i+1} of S_i such that

$$(M, \mathscr{X}) \models \exists X \subseteq_{\mathrm{cf}} S_{i+1} \, \chi(X) \\ \rightarrow \forall X \subseteq_{\mathrm{cf}} S_{i+1} \, \chi(X).$$

Then carry on the construction.

Lemma 11.5. The theory T is complete for $\Sigma_1^*(M, \mathscr{X})$-sentences.

Proof. Let $\chi(X)$ be a $\Sigma_1^*(M, \mathscr{X})$-sentence. This sentence is considered at some point of the construction, say, at step i. Recall that S_{i+1} was chosen so that

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_{i+1}\, \chi(X) \vee \forall X \subseteq_{\mathrm{cf}} S_{i+1}\, \neg\chi(X).$$

If $(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_{i+1}\, \chi(X)$, then $T \vdash \chi(X)$. If $(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_{i+1}\, \neg\chi(X)$, then $T \vdash \neg\chi(X)$. □

This is useful in determining where C is relative to M.

Lemma 11.6. $C > M$.

Proof. Suppose $a \in C \cap M$. By Lemma 11.1, we have $T \vdash a \in X$, and so by the Compactness Theorem, there is $i \in \mathbb{N}$ such that

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_i\, (a \in X),$$

which is impossible. □

Note that we have only used ATR_0 in $(M, \mathscr{X})$ so far.

11.3 Enforcing an end-extension

We then make N into an end-extension of M. Similar to the usual Keisler-style end-extension constructions [25, Section 6.1], it is necessary and sufficient to ensure that for all $\theta(X, m) \in \Sigma_1^0(M, \mathscr{X})$ and all $l \in M$, if

$$\exists i \in \mathbb{N} \quad (M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_i\, \exists! m < l\, \theta(X, m),$$

then

$$\exists j \in \mathbb{N} \quad (M, \mathscr{X}) \models \exists! m < l\, \forall X \subseteq_{\mathrm{cf}} S_j\, \theta(X, m).$$

Since there are only countably pairs of the form

$$(\theta(X,m),l),$$

where $\theta(X,m) \in \Sigma^0_1(M,\mathscr{X})$ and $l \in M$, it suffices to demonstrate how one can deal with one such pair.

Suppose we are given $S_i \subseteq_{\mathrm{cf}} M$ to play in. If

$$(M,\mathscr{X}) \not\models \forall X \subseteq_{\mathrm{cf}} S_i\, \exists! m < l\, \theta(X,m),$$

then there is nothing to do at this stage, and we can carry on with $S_{i+1} = S_i$. Suppose not. Then

$$(M,\mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_i\, \exists m < l\, \big(\forall m' < l\, (\theta(X,m') \to m' = m)\big).$$

Using Π^0_1-RT in $(M,\mathscr{X})$, we find an internal cofinal subset S_{i+1} of S_i such that

$$(M,\mathscr{X}) \models \exists m < l\, \forall X \subseteq_{\mathrm{cf}} S_{i+1}\, \big(\forall m' < l\, (\theta(X,m') \to m' = m)\big).$$

Notice we have

$$(M,\mathscr{X}) \models \exists! m < l\, \forall X \subseteq_{\mathrm{cf}} S_{i+1}\, \theta(X,m)$$

with this choice. We repeat this procedure infinitely many times for θ.

Theorem 11.7. The structure N is an end-extension of M.

Proof. Let $d_\theta \in N$ and $l \in M$ such that $(N,\mathscr{Y}) \models d_\theta < l$. By Lemma 11.1,

$$T \vdash \exists m \big(\theta(X,m) \wedge m < l\big).$$

Using the Compactness Theorem, let $i \in \mathbb{N}$ such that

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_i \, \exists! m < l \, \theta(X, m).$$

Without loss of generality, suppose $(\theta(X, m), l)$ is considered in the i-th step. Then by construction, S_{i+1} is chosen so that

$$(M, \mathscr{X}) \models \exists! m < l \, \forall X \subseteq_{\mathrm{cf}} S_{i+1} \, \theta(X, m).$$

Let m_i be the unique element in $M_{<l}$ such that $(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_{i+1} \, \theta(X, m_i)$. Then $T \vdash \theta(X, m_i)$. Since $T \vdash \exists! m \, \theta(X, m)$, we must have

$$T \vdash \exists m \big(\theta(X, m) \wedge m = m_i\big).$$

Therefore, $(N, \mathscr{Y}) \models d_\theta = m_i$ by Lemma 11.1. □

Note that we have only used ATR_0 in $(M, \mathscr{X})$ so far.

Remark. It is probably more natural to use Σ^0_1-RT in place of Π^0_1-RT in this section. However, we cannot find an analogue of Theorem 10.4 for Π^0_1-RT. Results in Section 11.5 may help in obtaining one.

11.4 Enforcing a conservative extension

As in Gaifman's construction of definable types, one needs some extra care when building conservative extensions. What we use here is Σ^0_1-$\widetilde{\mathrm{RT}}_2$. Consider a $\Sigma^0_1(M, \mathscr{X})$-formula

$$\vartheta(X, n).$$

Suppose we are in step i while dealing with $\vartheta(X,n)$. Using Σ^0_1-$\widetilde{\mathrm{RT}}_2$ in $(M,\mathscr{X})$, let S_{i+1} be an internal cofinal subset of S_i such that

$$(M,\mathscr{X}) \models \forall n \begin{pmatrix} \exists X \subseteq_{\mathrm{cf}} (S_{i+1})_{>n}\, \vartheta(X,n) \\ \to \forall X \subseteq_{\mathrm{cf}} (S_{i+1})_{>n}\, \vartheta(X,n) \end{pmatrix}.$$

Theorem 11.8. $\mathrm{SSy}_M(N) = \mathscr{X}$.

Proof. Using Corollary 11.4, it is not hard to see that $\mathscr{X} \subseteq \mathrm{SSy}_M(N)$. So take $A \in \mathrm{SSy}_M(N)$. It is coded by an element d_θ of N, say, so that

$$A = \{n \in M : N \models d_{m=n} \in d_\theta\}.$$

Let $\vartheta(X,n)$ be the $\Sigma^0_1(M,\mathscr{X})$-formula

$$\exists m\big(\theta(X,m) \wedge n \in m\big).$$

Then by Lemma 11.1,

$$(N,\mathscr{Y}) \models d_{m=n} \in d_\theta \quad \text{iff} \quad T \vdash \vartheta(X,n) \tag{†}$$

for all $n \in M$. Suppose $\vartheta(X,n)$ is considered in step i. By construction,

$$(M,\mathscr{X}) \models \forall n \begin{pmatrix} \exists X \subseteq_{\mathrm{cf}} (S_{i+1})_{>n}\, \vartheta(X,n) \\ \to \forall X \subseteq_{\mathrm{cf}} (S_{i+1})_{>n}\, \vartheta(X,n) \end{pmatrix}. \tag{‡}$$

Now, for all $n \in M$,

$$\begin{aligned} n \in A \quad &\text{iff} \quad (N, \mathscr{Y}) \models d_{m=n} \in d_\theta \\ &\text{iff} \quad T \vdash \vartheta(X, n) && \text{by } (\dagger), \\ &\text{iff} \quad \exists j \in \mathbb{N} \quad (M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_j\, \vartheta(X, n) && \text{by Compactness}, \\ &\text{iff} \quad (M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} (S_{i+1})_{>n}\, \vartheta(X, n) && \text{by } (\ddagger), \\ &\text{iff} \quad (M, \mathscr{X}) \models \vartheta((S_{i+1})_{>n}, n) && \text{by } (\ddagger) \text{ again.} \end{aligned}$$

Therefore, by arithmetical comprehension in $(M, \mathscr{X})$,

$$A = \{n \in M : (M, \mathscr{X}) \models \vartheta((S_{i+1})_{>n}, n)\} \in \mathscr{X}. \qquad \square$$

11.5 Reversals

The biggest question in this chapter is about reversals: what properties of $(N, \mathscr{Y})$ can we get ATR_0 or Π^1_1-CA_0 back? I do not have a good answer yet. Ideally, the criteria should be purely model theoretic. At the moment, the best I can get are the following.

Theorem 11.9. For a countable model $(M, \mathscr{X}) \models \mathrm{RCA}_0$, the following are equivalent.

(a) $(M, \mathscr{X}) \models \mathrm{ATR}_0$.

(b) There is an end-extension $(N, \mathscr{Y})$ of $(M, \mathscr{X})$ satisfying RCA_0 with a distinguished set $C \in \mathscr{Y}$ cofinal in N such that for all $\Pi^*_1(M, \mathscr{X})$-sentences $\chi(X)$,

$$(N, \mathscr{Y}) \models \chi(C)$$

if and only if for some cofinal $S \in \mathscr{X}$, we have $C \subseteq_{\mathrm{cf}} S^N$ and

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S\, \chi(X).$$

Proof. For (a) $\Rightarrow$ (b), we will use our construction up to Section 11.3. Note that by Corollary 11.4, $(N, \mathscr{Y}) \models \mathrm{ACA}_0$. Fix a $\Pi_1^*(M, \mathscr{X})$-sentence $\chi(X)$. If $(N, \mathscr{Y}) \models \chi(C)$, then by Lemmas 11.1 and 11.5, $T \vdash \chi(X)$, and so by the Compactness Theorem, we can find $i \in \mathbb{N}$ such that $C \subseteq_{\mathrm{cf}} S_i^N$ and

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_i\, \chi(X).$$

Conversely, suppose S is a cofinal subset of M in $\mathscr{X}$ such that $S^N \supseteq_{\mathrm{cf}} C$ and

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S\, \chi(X).$$

On the one hand, since the sentence $X \subseteq S$ is $\Pi_1^*(M, \mathscr{X})$, we have $T \vdash X \subseteq S$ by Lemmas 11.1 and 11.5. On the other hand, since T includes the $\mathscr{L}_{\mathrm{II}}$-theory of $(M, \mathscr{X})$, we have

$$\begin{aligned} T \vdash\ & \forall W \subseteq_{\mathrm{cf}} S\, \chi(W) \\ & \wedge \forall x\, \exists y \in S\ (y > x) \\ & \wedge \forall x\, \exists y \in X\ (y > x). \end{aligned}$$

It follows that T proves $X \subseteq_{\mathrm{cf}} S$ and hence also $\chi(X)$. Therefore, $(N, \mathscr{X}) \models \chi(C)$ by Lemma 11.1.

Next, we look at (b) $\Rightarrow$ (a). Suppose (b) holds. By Theorem 10.4, it suffices to prove $(M, \mathscr{X}) \models \Pi_1^0$-RT. Take an arbitrary $\Pi_1^0(M, \mathscr{X})$-sentence $\theta(X, m)$ and $l \in M$. We will show how to find a cofinal subset S of M in $\mathscr{X}$ such that

$$\begin{aligned} (M, \mathscr{X}) \models\ & \forall X \subseteq_{\mathrm{cf}} M\, \exists m < l\, \theta(X, m) \\ & \rightarrow \exists m < l\, \forall X \subseteq_{\mathrm{cf}} S\, \theta(X, m). \end{aligned}$$

Suppose $(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} M\, \exists m < l\, \theta(X, m)$. Note that RCA_0 proves the regularity axioms on all Σ_1^0-formulas. So the formula $\exists m < l\, \theta(X, m)$ is equivalent to a $\Pi_1^*(M, \mathscr{X})$-

sentence over RCA_0. Since $C \subseteq_{\mathrm{cf}} N = M^N$, we have by assumption,

$$(N, \mathscr{Y}) \models \exists m < l\, \theta(C, m).$$

Let $m \in N$ such that $(N, \mathscr{Y}) \models m < l \wedge \theta(C, m)$. Since $M \subseteq_{\mathrm{e}} N$, we must have $m \in M$. By our assumption on C again, we can find a cofinal $S \in \mathscr{X}$ such that

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S\, \theta(X, m),$$

as required. □

Remark. One can eliminate the conditions $(N, \mathscr{Y}) \models \mathrm{RCA}_0$ and $M \subseteq_{\mathrm{e}} N$ in the above theorem. In the proof of (b) ⇒ (a), show $(M, \mathscr{X}) \models \Delta^0_1\text{-}\mathrm{RT}_2$ instead. It is known that $\Delta^0_1\text{-}\mathrm{RT}_2$ is equivalent to ATR_0 over RCA_0. See Section V.9 in Simpson's book [72] for more details. Given a $\Delta^*_1(M, \mathscr{X})$-sentence $\chi(X)$, one finds a cofinal subset S of M in $\mathscr{X}$ such that

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S\, \chi(X) \quad \text{or} \quad (M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S\, \neg\chi(X),$$

depending on whether $(N, \mathscr{Y}) \models \chi(C)$. From this, we see that the Galvin–Prikry Theorem can be regarded as a general version of the law of excluded middle.

Theorem 11.10. For a countable model $(M, \mathscr{X}) \models \mathrm{RCA}_0$, the following are equivalent.

(a) $(M, \mathscr{X}) \models \Pi^1_1\text{-}\mathrm{CA}_0$.

(b) There is an end-extension $(N, \mathscr{Y})$ of $(M, \mathscr{X})$ with a distinguished set $C \in \mathscr{Y}$ cofinal in N such that for all $\Pi^0_1(M, \mathscr{X})$-formulas $\vartheta(X, n)$, there exists a cofinal $S \in \mathscr{X}$ satisfying $C \subseteq_{\mathrm{cf}} S^N$ and for all $n \in M$,

$$(N, \mathscr{Y}) \models \vartheta(C, n) \Leftrightarrow (M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_{>n}\, \vartheta(X, n).$$

Proof. Essentially the same as that of Theorem 11.9. □

All of what we did in this chapter can be repeated with a countable model $(M, \mathscr{X}) \models \Sigma^1_0\text{-}\widetilde{\mathrm{RT}}_2$. In this case, the extension N consists of elements of the form d_θ, where $\theta(X, m) \in \Sigma^1_0(M, \mathscr{X})$ satisfying $T \vdash \exists! m\, \theta(X, m)$. We get theorems analogous to Theorem 11.9 and Theorem 11.10, with the same proofs.

Theorem 11.11. For a countable model $(M, \mathscr{X}) \models \mathrm{RCA}_0$, the following are equivalent.

(a) $(M, \mathscr{X}) \models \Pi^1_1\text{-}\mathrm{CA}_0$.

(b) There is an extension $(N, \mathscr{Y})$ of $(M, \mathscr{X})$ with a distinguished set $C \in \mathscr{Y}$ cofinal in N such that for all $\mathscr{L}^*_{\mathrm{I}}(M, \mathscr{X})$-sentences $\chi(X)$,

$$(N, \mathscr{Y}) \models \chi(C)$$

if and only if for some cofinal $S \in \mathscr{X}$, we have $C \subseteq_{\mathrm{cf}} S^N$ and

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S\, \chi(X). \qquad \square$$

Theorem 11.12. For a countable model $(M, \mathscr{X}) \models \mathrm{RCA}_0$, the following are equivalent.

(a) $(M, \mathscr{X}) \models \Sigma^1_0\text{-}\mathrm{RT}$.

(b) There is an end-extension $(N, \mathscr{Y})$ of $(M, \mathscr{X})$ with a distinguished set $C \in \mathscr{Y}$ cofinal in N such that for all $\mathscr{L}^*_{\mathrm{I}}(M, \mathscr{X})$-sentences $\chi(X)$,

$$(N, \mathscr{Y}) \models \chi(C)$$

if and only if for some cofinal $S \in \mathscr{X}$, we have $C \subseteq_{\mathrm{cf}} S^N$ and

$$(M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S\, \chi(X). \qquad \square$$

Theorem 11.13. For a countable model $(M, \mathscr{X}) \models \mathrm{RCA}_0$, the following are equivalent.

(a) $(M, \mathscr{X}) \models \Sigma^1_0\text{-}\widetilde{\mathrm{RT}}_2$.

(b) There is an end-extension $(N, \mathscr{Y})$ of $(M, \mathscr{X})$ with a distinguished set $C \in \mathscr{Y}$ cofinal in N such that for all $\Sigma^1_0(M, \mathscr{X})$-formulas $\vartheta(X, n)$, there exists a cofinal $S \in \mathscr{X}$ satisfying $C \subseteq_{\mathrm{cf}} S^N$ and for all $n \in M$,

$$(N, \mathscr{Y}) \models \vartheta(C, n) \Leftrightarrow (M, \mathscr{X}) \models \forall X \subseteq_{\mathrm{cf}} S_{>n}\, \vartheta(X, n). \qquad \square$$

Question 11.14. Can the countability assumption on $(M, \mathscr{X})$ be eliminated?

CHAPTER 12

STRONGER CUTS

> My paper comes from philosophical reflection on how to respond to the incompleteness results. The Gödel phenomena are so pervasive that I can see no way to overcome them. But I do see one sensible way to respond to the Paris–Kirby–Harrington phenomena. Briefly, one should extend $\mathbb{P}$ to include some general version of the infinite Ramsey theorem, and then one easily derives the finite versions. This should eliminate any second generation of incompleteness phenomena relating to finite combinatorics.
>
> Angus Macintyre
> Ramsey quantifiers in arithmetic [51], Introduction

Our failure in the previous chapter originates partly from the fact that we are unable to effectively tell whether a model of second-order arithmetic satisfies ATR_0 or Π^1_1-CA_0. We investigate this problem in this chapter through the second-order properties of cuts.

Recall that each cut I of a model M of PA gives rise to a model of second-order arithmetic $(I, \mathrm{SSy}_I(M))$. Kirby and Paris [35, 36, 60] were the first to analyze cuts in this respect. They introduced a hierarchy of combinatorial notions of cuts that correspond to the theories $\mathrm{WKL}_0, \mathrm{B}\Sigma^*_2, \mathrm{I}\Sigma^*_2, \ldots \mathrm{ACA}_0$ of second-order arithmetic. To prove the correspondence, one needs a translation between the language of second-order arithmetic and the language of cuts. This translation has been known since the days of Kirby and Paris. For a good survey, see Keisler [34].

Lemma 12.1. Let $I \subseteq_{\mathrm{e}} M \models \mathrm{PA}$ where I is closed under multiplication. Then for every

$\tilde{\varphi}(x) \in \Delta_0^0(I, \mathrm{SSy}_I(M))$, there exists $\varphi(x) \in \Delta_0(M)$ such that for all $x \in I$,

$$(I, \mathrm{SSy}_I(M)) \models \tilde{\varphi}(x) \Leftrightarrow M \models \varphi(x).$$

Proof. Replace the set parameters in $\tilde{\varphi}$ by their codes to get φ. Then the required equivalence holds, because all quantifiers in $\tilde{\varphi}$ are bounded and I is an $\mathscr{L}_\mathrm{I}$-structure on its own. □

As an example, let us see how this lemma helps in establishing the correspondence between $\mathrm{RCA}_0 + \Sigma_1^0\text{-CA}$ and the notion of *strong cuts*.

Definition (Kirby–Paris [36]). Let $I \subsetneq_\mathrm{e} M \models \mathrm{PA}$. We say that I is *strong* in M if for every coded function $F \colon I \to M$, there exists $d > I$ such that for all $x \in I$,

$$F(x) \in I \Leftrightarrow F(x) < d.$$

Theorem 12.2 (Kirby–Paris [36]). Let $I \subsetneq_\mathrm{e} M \models \mathrm{PA}$ where I is itself an $\mathscr{L}_\mathrm{I}$-structure. Then I is strong in M if and only if $(I, \mathrm{SSy}_I(M)) \models \mathrm{RCA}_0 + \Sigma_1^0\text{-CA}$.

Proof. For the 'only if' direction, suppose I is strong in M. By overspill, it is easy to see that I must be semiregular in M. So $(I, \mathrm{SSy}_I(M)) \models \mathrm{RCA}_0$. Next, let $\tilde{\varphi}(x, v)$ be a Δ_0^0-formula with parameters from $(I, \mathrm{SSy}_I(M))$. We want to show

$$\{x \in I : (I, \mathrm{SSy}_I(M)) \models \exists v\, \tilde{\varphi}(v, w)\} \in \mathrm{SSy}_I(M).$$

Using Lemma 12.1, let $\varphi(x, v) \in \Delta_0(M)$ such that for every $x, v \in I$,

$$(I, \mathrm{SSy}_I(M)) \models \tilde{\varphi}(x, v) \Leftrightarrow M \models \varphi(x, v).$$

Choose any $b > I$. Define $F\colon I \to M$ by setting

$$F(x) = \begin{cases} (\mu v)(\varphi(x,v)), & \text{if } M \models \exists v\, \varphi(x,v); \\ b, & \text{otherwise,} \end{cases}$$

for every $x \in I$. Clearly, F is coded in M. Using the strength of I in M, let $d > I$ such that for all $x \in I$,

$$F(x) \in I \Leftrightarrow F(x) < d.$$

Then

$$\{x \in I : (I, \mathrm{SSy}_I(M)) \models \exists v\, \tilde{\varphi}(x,v)\} = \{x \in I : M \models \exists v < d\, \varphi(x,v)\} \in \mathrm{SSy}_I(M),$$

proving what we want.

For the 'if' direction, suppose $(I, \mathrm{SSy}_I(M)) \models \mathrm{RCA}_0 + \Sigma^0_1\text{-CA}$. Let $F\colon I \to M$ be a coded function. By Σ^0_1-comprehension,

$$\{x \in I\colon F(x) \in I\} \in \mathrm{SSy}_I(M).$$

Let $a \in M$ be a code for this set. Then for every $x \in I$,

$$F(x) \in I \Leftrightarrow x \in a.$$

By taking $d = \max\{F(x) : x < b \wedge x \in a\} + b + 1$, for example, one sees that

$$\forall b \in I\, \exists d > b\, \forall x < b\, \big(F(x) < d \Rightarrow x \in a\big).$$

By overspill, there is $b > I$ such that $\exists d > b\, \forall x < b\, \big(F(x) < d \Rightarrow x \in a\big)$. So in particular,

$$\exists d > I\, \forall x \in I\, \big(F(x) < d \Rightarrow x \in a\big).$$

By our choice of a, we have $\exists d > I\ \forall x \in I\ \big(F(x) < d \Rightarrow F(x) \in I\big)$, as required. $\square$

As Kirby–Paris [36] showed, strength is a robust notion with respect to both model theory and reverse mathematics. On the one hand, it can be characterized by a number of model theoretic properties. On the other, it is equivalent to many combinatorial principles.

Another evidence of its robustness is the difficulty one faces when trying to strengthen it. Apparently, there were some attempts by Kirby in strengthening the notion of strong cuts. What he eventually found were a few more notions of cuts that are essentially of the same strength as strength. See, for example, Propositions 7.13, 7.15, 7.16, and 9.2 in his thesis [35]. The following is one of our own naïve attempts, which turns out to have a simple solution.

Proposition 12.3. Let $M \models \mathrm{PA}$ and I be a strong cut in M. Then for every coded family $\langle F_i : i \in I\rangle$ of coded functions from I to M, there exists $d > I$ such that for all $i, x \in I$,

$$F_i(x) \in I \Leftrightarrow F_i(x) < d.$$

Proof. Just consider the family as a single function $F\colon \langle i, x\rangle \mapsto F_i(x)$. $\square$

This phenomenon can probably be explained by the well-known fact that Σ^0_1-CA and ACA_0 are equivalent over RCA_0. So everything that falls between Σ^0_1-CA and ACA_0 is actually the same as ACA_0. To go higher up, it seems we need to mention set quantification in a more essential way. In the next proposition, $[n]$ denotes the set $\{0, 1, \dots, n-1\}$ and $|h|$ denotes the internal cardinality of the coded set h.

Proposition 12.4. Let $I \subsetneq_{\mathrm{e}} M \models \mathrm{PA}$ where I is itself an $\mathscr{L}_{\mathrm{I}}$-structure. Then I is strong if and only if for every coded sequence $\langle P_i : i \in I\rangle$ of coded subsets of I, there exists a coded sequence

$$(c)_0 \geqslant (c)_1 \geqslant (c)_2 \geqslant \cdots > I$$

of length I such that for all $i \in I$,

$$\exists^{\mathrm{cf}} H \in \mathrm{SSy}_I(M)\ \forall u \in H\ (u \in P_i)$$
$$\Leftrightarrow M \models \exists h \subseteq [(c)_i]\ \big(|h| = (c)_{i+1} \wedge \forall u \in h\ (u \in P_i)\big).$$

With some abuse of notation, we identified the coded sequence $\langle P_i : i \in I\rangle$ with its code here, so that it makes sense to talk about the P_i's above I.

Proof. Suppose I is strong in M. Let $\langle P_i : i \in I\rangle$ be a coded sequence of coded subsets of I. Recall that Q means 'there exist cofinally many'. Define

$$A = \big\{i \in I : (I, \mathrm{SSy}_I(M)) \models \mathsf{Q}u\ \big(u \in P_i\big)\big\}.$$

This set is coded by arithmetical comprehension. Pick any $(c)_0 > I$, and let

$$(c)_{i+1} = \begin{cases} (\max x)\big(\exists h \subseteq [(c)_i]\ \big(|h| = x \wedge \forall u \in h\ (u \in P_i)\big)\big), & \text{if } i \in A;\\ (c)_i, & \text{otherwise,}\end{cases}$$

for every $i \in I$. It is not hard to verify that c satisfies our requirements.

Conversely, suppose I satisfies the condition stated in the proposition. Let $F\colon I \to M$ be a coded function. For each $i \in I$, define

$$P_i = \{u \in I : u > F(i)\}.$$

Then $\langle P_i : i \in I\rangle$ is a coded family of coded subsets of I. Using the hypothesis, let $c \in M$ such that

$$\exists^{\mathrm{cf}} H \in \mathrm{SSy}_I(M)\ \forall u \in H\ (u \in P_i)$$
$$\Leftrightarrow M \models \exists h \subseteq [(c)_i]\ \big(|h| = (c)_{i+1} \wedge \forall u \in h\ (u \in P_i)\big)$$

for all $i \in I$. Now

$$\{i \in I : F(i) \in I\} = \big\{i \in I : M \models \exists h \subseteq [(c)_i] \ \big(|h| = (c)_{i+1} \wedge \forall u \in h \ (u \in P_i)\big)\big\},$$

which is coded. Using overspill as in the proof of Theorem 12.2, we obtain $d > I$ such that for all $i \in I$,

$$F(i) \in I \Leftrightarrow F(i) < d. \qquad \square$$

This proposition actually says very little. The complications disappear when one removes the condition

$$\text{‘}(c)_0 \geqslant (c)_1 \geqslant (c)_2 \geqslant \cdots > I\text{’}$$

and replaces

$$\text{‘}M \models \exists h \subseteq [(c)_i] \ \big(|h| = (c)_{i+1} \wedge \forall u \in h \ (u \in P_i)\big)\text{’}$$

by ‘$M \models (c)_i = 0$’. The core of it is just the following.

Definition. $\mathsf{Q}^0_0 = \big\{\mathsf{Q}u \ \gamma(u, \bar{v}, \bar{W}) : \gamma(u, \bar{v}, \bar{W}) \in \Delta^0_0\big\}$.

Fact 12.5. Over RCA_0, the schemes ACA_0 and Q^0_0-CA are equivalent.

Proof. If $\psi(x, z) \in \Delta^0_0$, then $\mathrm{PA}^- \vdash \forall z\big(\exists x \ \psi(x, z) \leftrightarrow \mathsf{Q}u \ \exists x < u \ \psi(x, z)\big)$. $\square$

Remark. In view of the Church–Turing thesis, one would expect $\mathsf{Q}^0_0 \supsetneq \Sigma^0_1$. For related results about Q, see Kaye [30] and Mills–Paris [54].

We state Proposition 12.4 in this way because we hope it will eventually lead to a deeper result. The following theorem was observed by Keita Yokoyama [personal communication]. His idea probably came from Steve Simpson and Keisler [34].

Theorem 12.6 (Yokoyama). Let $M \models \mathrm{PA}$ and I be a semiregular cut in M. Then $(I, \mathrm{SSy}_I(M)) \models \Pi^1_1\text{-}\mathrm{CA}_0$ if and only if for every coded sequence $\langle R_i : i \in I\rangle$ of coded subsets of I^2, there exists a coded sequence

$$(c)_0 \geqslant (c)_1 \geqslant (c)_2 \geqslant \cdots > I$$

of length I such that for all $i \in I$,

$$\exists^{\mathrm{cf}} H \in \mathrm{SSy}_I(M)\, \forall u, v \in H\, \big(u \neq v \Rightarrow \langle u, v\rangle \in R_i\big)$$
$$\Leftrightarrow M \models \exists h \subseteq [(c)_i]\, \big(|h| = (c)_{i+1} \wedge \forall u, v \in h\, \big(u \neq v \rightarrow \langle u, v\rangle \in R_i\big)\big).$$

The proof of this theorem will occupy us until almost the end of the chapter. The main tool we use is the *Ramsey quantifier*, whose relevance to arithmetic was first noticed by Macintyre [51].

Definition. The *Ramsey quantifier* $\mathscr{Q}$ is a quantifier that binds two variables. If $\chi(u, v)$ is an $\mathscr{L}_{\mathrm{II}}$-formula, possibly with some undisplayed variables, then $\mathscr{Q}uv\, \chi(u, v)$ means

$$\exists^{\mathrm{cf}} H\, \forall u, v \in H\, \big(u \neq v \rightarrow \chi(u, v)\big).$$

It is apparent that if $\chi(u, v)$ is arithmetical, then $\mathscr{Q}uv\, \chi(u, v)$ is Σ^1_1. One of the reasons why the Ramsey quantifier is important to arithmetic is that actually, every Σ^1_1-formula can be written in this form. This can be proved via Kleene's Normal Form Theorem for Σ^1_1-formulas. The term $F{\restriction}m$ that appears in this theorem denotes the internally finite set $\{\langle x, y\rangle \in F : x < m\}$.

Theorem 12.7 (Kleene). For all $\rho(w) \in \Sigma^1_1$, there exists $\theta(f, w) \in \Delta^0_0$ such that

$$\mathrm{ACA}_0 \vdash \forall w \Big(\rho(w) \leftrightarrow \exists F \forall m\, \theta(F{\restriction}m, w)\Big).$$

Proof. See Lemma V.1.4 in Simpson's book [72]. □

Theorem 12.8 (Schmerl–Simpson [67]). Over ACA_0, every Σ^1_1-formula is uniformly equivalent to a formula of the form

$$\mathscr{Q}uv\, \chi(u, v, w),$$

where $\chi(u, v, w) \in \Delta^0_0$.

Proof. Let $\theta(f,w)$ be a Δ^0_0-formula. Then over ACA_0, the formula $\exists F \forall m\, \theta(F{\restriction}m, w)$ is equivalent to

$$\mathscr{Q} f_1 f_2 \, \big((f_1 \subseteq f_2 \vee f_2 \subseteq f_1) \wedge \theta(f_1, w) \big)$$

uniformly in w. □

Remark. By this theorem, one need not look at n-variable versions of the Ramsey quantifier for $n \geqslant 3$. In particular, Theorem 12.6 cannot be strengthened in this way.

We are now ready to prove Yokoyama's theorem.

Proof of Theorem 12.6. Suppose $(I, \mathrm{SSy}_I(M)) \models \Pi^1_1\text{-}\mathrm{CA}_0$. We imitate the proof of Proposition 12.4. Let $\langle R_i : i \in I \rangle$ be a coded sequence of coded subsets of I^2. Define

$$A = \{ i \in I : (I, \mathrm{SSy}_I(M)) \models \mathscr{Q} uv \, \big(\langle u, v \rangle \in R_i \big) \}.$$

This set is coded by Π^1_1-comprehension. Pick any $(c)_0 > I$, and let

$$(c)_{i+1} = \begin{cases} (\max x)\big(\exists h \subseteq [(c)_i] \, \big(|h| = x \wedge \forall u, v \in h \; (u \neq v \rightarrow \langle u, v \rangle \in R_i)\big)\big), & \text{if } i \in A; \\ (c)_i, & \text{otherwise,} \end{cases}$$

for every $i \in I$. It is not hard to verify that c satisfies our requirements.

Conversely, suppose I satisfies the conditions in the theorem. Let $\tilde{\chi}(u, v, i) \in \Delta^0_0$, possibly with parameters from $(I, \mathrm{SSy}_I(M))$. Using Lemma 12.1, let $\chi(u, v, i)$ be a $\Delta_0(M)$-formula such that for all $u, v, i \in I$,

$$(I, \mathrm{SSy}_I(M)) \models \tilde{\chi}(u, v, i) \Leftrightarrow M \models \chi(u, v, i).$$

Now for each $i \in I$, define

$$R_i = \{ \langle u, v \rangle \in I : M \models \chi(u, v, i) \}.$$

Notice $\langle R_i : i \in I\rangle$ is coded in M. Using the hypothesis, let $c \in M$ such that $(c)_i \geqslant (c)_{i+1} > I$ and

$$\begin{aligned}&\exists^{\mathrm{cf}} H \in \mathrm{SSy}_I(M)\ \forall u, v \in H\ \big(u \neq v \Rightarrow \langle u, v\rangle \in R_i\big)\\ &\qquad\Leftrightarrow M \models \exists h \subseteq [(c)_i]\ \big(|h| = (c)_{i+1} \wedge \forall u, v \in h\ \big(u \neq v \rightarrow \langle u, v\rangle \in R_i\big)\big)\end{aligned}$$

for all $i \in I$. Then

$$\begin{aligned}&(I, \mathrm{SSy}_I(M)) \models \mathscr{Q}uv\ \tilde{\chi}(u, v, i)\\ &\qquad\Leftrightarrow M \models \exists h \subseteq [(c)_i]\ \big(|h| = (c)_{i+1} \wedge \forall u, v \in h\ \big(u \neq v \rightarrow \langle u, v\rangle \in R_i\big)\big)\end{aligned}$$

for all $i \in I$. Therefore,

$$\begin{aligned}&\big\{i \in I : (I, \mathrm{SSy}_I(M)) \models \mathscr{Q}uv\ \tilde{\chi}(u, v, i)\big\}\\ &\qquad= \big\{i \in I : M \models \exists h \subseteq [(c)_i]\ \big(|h| = (c)_{i+1} \wedge \forall u, v \in h\ \big(u \neq v \rightarrow \langle u, v\rangle \in R_i\big)\big)\big\},\end{aligned}$$

which is coded. □

This theorem is actually only a restatement of Theorem 12.8. Nevertheless, it seems to be a step forward in obtaining a more useful characterization of such cuts.

Remark. Andrey Bovykin and Michiel de Smet [forthcoming] obtained, independently of Yokoyama, other combinatorial and model theoretic characterizations for cuts corresponding to Π^1_1-CA_0. Theorem 12.8 also plays a very important role in their work.

As in Keisler [34], one can extend this result to ATR_0 via the *separation principles*.

Definition. Let Γ be a class of $\mathscr{L}_{\mathrm{II}}$-formulas. Then Γ-Sep denotes the axiom scheme consisting of all axioms of the form

$$\begin{aligned}&\forall x \exists S\big((\varphi(x) \rightarrow x \in S) \wedge (\psi(x) \rightarrow x \notin S)\big)\\ &\rightarrow \exists S \forall x\big((\varphi(x) \rightarrow x \in S) \wedge (\psi(x) \rightarrow x \notin S)\big),\end{aligned}$$

where $\varphi(x)$ and $\psi(x)$ are formulas in Γ that may contain undisplayed free variables.

Theorem 12.9 (Simpson). The scheme Σ^1_1-Sep is equivalent to ATR_0 over RCA_0.

Proof. See Theorem V.5.1 in Simpson's book [72]. □

Theorem 12.10 (Yokoyama). Let $M \models \mathrm{PA}$ and I be a semiregular cut in M. Then the following are equivalent.

(a) $(I, \mathrm{SSy}_I(M)) \models \mathrm{ATR}_0$.

(b) For all coded sequences $\langle R_i : i \in I\rangle$, $\langle S_i : i \in I\rangle$ of coded subsets of I^2, there exists a coded sequence

$$(c)_0 \geqslant (c)_1 \geqslant (c)_2 \geqslant \cdots > I$$

of length I such that if

$$(I, \mathrm{SSy}_I(M)) \models \forall i\, \neg\big(\mathscr{Q}uv\ (\langle u,v\rangle \in R_i) \wedge \mathscr{Q}uv\ (\langle u,v\rangle \in S_i)\big),$$

then for all $i \in I$,

$$\exists^{\mathrm{cf}} H \in \mathrm{SSy}_I(M)\ \forall u,v \in H\ \big(u \neq v \Rightarrow \langle u,v\rangle \in R_i\big)$$
$$\Rightarrow M \models \exists h \subseteq [(c)_i]\ \big(|h| = (c)_{i+1} \wedge \forall u,v \in h\ \big(u \neq v \to \langle u,v\rangle \in R_i\big)\big)$$

and

$$\exists^{\mathrm{cf}} H \in \mathrm{SSy}_I(M)\ \forall u,v \in H\ \big(u \neq v \Rightarrow \langle u,v\rangle \in S_i\big)$$
$$\Rightarrow M \not\models \exists h \subseteq [(c)_i]\ \big(|h| = (c)_{i+1} \wedge \forall u,v \in h\ \big(u \neq v \to \langle u,v\rangle \in R_i\big)\big).$$

Proof. Essentially the same as that of Theorem 12.6. □

Having reached this point, the reader may have already got used to the heuristic

$$\frac{\mathrm{ACA}_0}{\mathrm{WKL}_0} = \frac{\Pi^1_1\text{-}\mathrm{CA}_0}{\mathrm{ATR}_0}$$

$\mathscr{L}_{\mathrm{I}}$	$\mathscr{L}_{\mathrm{II}}$
$\frac{\mathrm{ACA}_0}{\mathrm{WKL}_0}$	$\frac{\Pi^1_1\text{-CA}_0}{\mathrm{ATR}_0}$
$\frac{\text{Theorem 8.3}}{\text{Theorem 8.6}}$	$\frac{\text{Theorem 11.12}}{\text{Theorem 11.9}}$
Example 8.7	??
$\frac{\text{Ramsey's Theorem}}{\text{Pigeonhole Principle}}$	$\frac{\Sigma^1_0\text{-RT}_2}{\Sigma^0_1\text{-RT}_2}$
Fact 12.5	Theorem 12.8
$\frac{\Sigma^0_1\text{-CA}}{\Sigma^0_1\text{-Sep}}$	$\frac{\Sigma^1_1\text{-CA}}{\Sigma^1_1\text{-Sep}}$
$\frac{\text{Proposition 12.4}}{\text{??}}$	$\frac{\text{Theorem 12.6}}{\text{Theorem 12.10}}$

Table 12.1: First- and second-order arithmetic

from Simpson's book [72, Remark I.11.7]. In Table 12.1, we gather the examples that appear in this thesis, together with some questions that we are interested in. In particular, we would like to know whether there is an analogue of Theorem 12.10 for WKL_0. The difficulty comes from Proposition 8.5.

Theorem 12.11. The scheme Σ^0_1-Sep is equivalent to WKL_0 over RCA_0.

Proof. See Lemma IV.4.4 in Simpson's book [72]. □

CHAPTER 13

COFINAL EXTENSIONS

> It illustrates nicely the difference in the rôles of cofinal and end extensions: The former add new definable sets; the latter *can* change truths.
>
> Craig Smoryński
> Lectures on nonstandard models of arithmetic [76], §7

We conclude Part II with a selection of follow-up work that is worth looking into.

The most important question to be investigated is: *what kind of end-extensions entails strength for second-order models?* One way to proceed is to modify the construction in Chapter 11. Instead of using the Galvin–Prikry Theorem, what other combinatorial principles can we use? The wealth of results in reverse mathematics should give us a lot of choices. What notions can one extract from these constructions?

Recall that ω_1-like models of $\mathrm{I}\Delta_0$ satisfy PA. Is there a way to go beyond PA? It is not clear how this can be done within first-order arithmetic. So we pass on to second-order arithmetic. It may be helpful to think in terms of *cofinal extensions* instead of end-extensions. In the rest of this chapter, we shall explore this possibility.

As Gaifman [16] showed, cofinal extensions and end-extensions are the only interesting extensions for models of PA.

Gaifman Splitting Theorem. Let $M \prec N \models \mathrm{PA}$. Then there is a unique $\overline{M}$ such that

$$M \prec_{\mathrm{cf}} \overline{M} \prec_{\mathrm{e}} N.$$

This $\overline{M}$ is $\{x \in N : \exists y \in M\ (x < y)\}$. □

Question 13.1. Enayat and Mohsenipour [13] observed that the regularity scheme alone is enough for this splitting theorem. Kaye [28] has a converse for theories extending $\mathrm{I}\Delta_0$. What happens below $\mathrm{I}\Delta_0$?

The duality between end-extensions and cofinal extensions provides a link between the two groups of results. It would be nice to see more instances of this duality, but they do not seem to exist in the literature yet. For example, what should be the dual of the notion of ω_1-like models? Hopefully, there will be some general theory that can guide us in this kind of 'dualization' in the future. At the moment, we can only make some guesses.

Definition. A nonstandard model $M \models \mathrm{PA}$ is *anti-ω_1-like* if it is of cardinality $\aleph_1$, and for every nonstandard $a \in M$, the initial segment $M_{<a}$ is of cardinality $\aleph_1$.

Definition. Let $I \subsetneq_{\mathrm{e}} M \models \mathrm{PA}$. We say that M is *downward ω_1-like at I* if M is of cardinality $\aleph_1$, and *no* sequence $(c_n)_{n\in\mathbb{N}}$ of elements of $M \setminus I$ satisfies $\{c_n : n \in \mathbb{N}\} \subseteq_{\mathrm{dcf}} M \setminus I$.

It is easy to see that if a model of PA is downward ω_1-like at $\mathbb{N}$, then it is anti-ω_1-like. The converse does not seem to be true.

Proposition 13.2. Every countable nonstandard model of PA has a cofinal extension that is downward ω_1-like at $\mathbb{N}$.

Proof. By the Compactness Theorem and the Gaifman Splitting Theorem, every countable nonstandard model of PA has a countable cofinal extension that adds a new element between $\mathbb{N}$ and the old nonstandard elements. Starting with any countable nonstandard model of PA, we iterate this construction ω_1 times to obtain a continuous elementary chain of length ω_1. It can be verified that the union of this chain is downward ω_1-like. □

Remark. If we start with a recursively saturated model in this proof, then we get a recursively saturated downward ω_1-like model at the end, because cofinal extension preserves recursive saturation [77].

If I is a strong cut of a countable model $M \models \mathrm{PA}$, then we can do this construction at I instead of at $\mathbb{N}$. See Example 8.7. This leads us to thinking that the notion of downward ω_1-like models is the right one. In fact, we have a converse that is dual to Proposition 9.1.

Proposition 13.3. Let I be a countable cut of a model M of PA. If M is downward ω_1-like at I, then $(I, \mathrm{SSy}_I(M)) \models \mathrm{ACA}_0$.

Proof. By cardinality. □

It is then clearer where the next step should be.

Question 13.4. How can we strengthen the notion of downward ω_1-like models to notions that correspond to ATR_0 and Π^1_1-CA_0?

There are many directions in which one can go from this. On the one hand, one can internalize these strengthened notions as in Chapter 9. This time, a reversal seems more probable. On the other hand, one can revert back to end-extensions and see what the corresponding notions are. As suggested above, the paths for cofinal and end-extensions may eventually meet at the study of I-extensions.

Chapters 10 and 11 is another place where the studies of cofinal and end-extensions may meet. Recall that the Infinite Ramsey Theorem can be used to build unbounded indiscernible types, and the Finite Ramsey Theorem can be used to build bounded indiscernible types [46]. Can one use, for example, the finitary version of Π^0_1-RT_2 in Friedman–McAloon–Simpson [14] to build interesting 'bounded' indiscernible types?

Topics in cofinal extensions seem to be much less popular than those in end-extensions. There are many different reasons for this. For example, cofinal extensions of models of PA are always elementary. So they probably cannot be used to find independence results, for example. The only extensions of $\mathbb{N}$ are end-extensions. Therefore, if we are interested in the natural numbers, then end-extensions are the ones to look at. Since cofinal extensions are hard to draw, they are more difficult to visualize, and hence harder to study. Nevertheless, the recent interesting activities in the subject [41, 45] show that relevant research is very much alive and active.

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