BROUÉ'S CONJECTURE FOR FINITE GROUPS

by

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A thesis submitted to
The University of Birmingham
for the degree of
PH.D. IN MATHEMATICS

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April 2018

Abstract

This research project consists of using the theory of perverse equivalences to study Broué's conjecture for the principal block of some finite groups when the defect group is elementary abelian of rank 2. We will look at $G = \Omega_8^+(2)$ and prove the conjecture in characteristic 5, the only open case for this group. We will also run our algorithm for the principal 5-block of $G = {}^2F_4(2)'.2, Sp_8(2)$ and the principal 7-block of ${}^3D_4(2)$; for those groups, it seems that a slight modification of our method is required to complete the proof of the conjecture. Finally, we will see what happens when we apply our method - which is mainly used for groups G of Lie type - to some sporadic groups, namely $G = J_2, He, Suz, Fi_{22}$.

CONTENTS

INTRODUCTION

Let G be a finite group and ℓ a prime number. In representation theory, it is an open problem to determine how and why some aspects of the representation theory of G are somehow controlled by the family of subgroups of the form $N_G(Q)$, where Q is a non-trivial ℓ-subgroup of G; such subgroups are called *ℓ-local*. The amount of information that ℓ-local subgroups carry about the representation theory of G is the content of a family of conjectures - most of them far from being solved - that we usually refer to as "local-global conjectures". One of the oldest and most notable ones, and also very easy to state, is McKay's conjecture (1972), predicting that the number of irreducible representations over $\mathbb{C}$ of dimension not divisible by ℓ is the same for G and $N_G(P)$, where $P \in \mathrm{Syl}_\ell(G)$; namely, the number $\mathrm{Irr}_{\ell'}(G)$ is determined by the ℓ-local subgroup $N_G(P)$. Most of these conjectures are actually taking place at the level of ℓ-blocks of G rather than G itself, and the bridge between blocks of G and blocks of the given ℓ-local subgroup is provided by a crucial result of local-global theory, the Brauer correspondence.

The first version of the Brauer correspondence appeared in 1944 in the article of Richard Brauer [6], and it represented a remarkable result for the future development of Block theory of finite groups and modular representation theory. Briefly, for a fixed subgroup $H \leq G$, under certain conditions there is a bijection between ℓ-blocks of H of a fixed defect group D and ℓ-blocks of G with the same defect group; this will be precisely stated in Theorem 1.1.7. The most relevant and interesting situation occurs when $H = N_G(P)$, where $P \in \mathrm{Syl}_\ell(G)$. Forty-six years later, in 1990, Michel Broué conjectured in [9] that this is not a mere correspondence, and that the ℓ-block b of H and the corresponding ℓ-block B of G are related in terms of their categories $\mathrm{mod}(b)$ and $\mathrm{mod}(B)$, which are expected to be derived equivalent, when D is abelian. This conjecture is the main subject of the dissertation.

The thesis is structured as follows. The basic necessary setting is introduced in Chapter 1. We start by recalling two crucial results in modular representation theory: the Green correspondence and the Brauer correspondence. In the following, we will give some basic notions of equivalences between module categories (stable, derived, etc). In the following, we will introduce the conjecture and the method that we will use to attack some individual groups. Before it, we give a brief description of some related local-global conjectures which are still open and the relevant current ongoing work towards them. The description of our strategy will involve both the theoretical aspect and the computational one.

The last two chapters consist of the actual results of the thesis. We will examine some groups of Lie type first; in particular, we have examined the principal ℓ-blocks of $G = \Omega_8^+(2), {}^2F_4(2)'.2, {}^3D_4(2), Sp_8(2)$. For each of these groups, the prime ℓ to examine is the only one for which the conjecture applies (the Sylow ℓ-subgroup must be abelian) and is still open (essentially, when P is not cyclic); this turns out to be always $\ell = 5$, except for $G = {}^3D_4(2)$ where we have $\ell = 7$. The last chapter will focus on what we manage to find when G is a sporadic group. We looked at: $G = J_2, He, Suz, Fi_{22}, Fi_{23}$, and in each case $\ell = 5$ is the only interesting prime to consider.

As the computational aspect is fundamental for our work, the algorithms that have been used to implement our method are in the appendix. This mainly consists of the algorithm that provides the perverse equivalence between the two blocks, and the algorithm providing the stable equivalence that we aim to lift to our perverse equivalence. More details are given in the appendix.

Notation

Every group considered is intended to be finite; $\ell > 0$ will denote a prime number and k will denote an algebraically closed field of characteristic ℓ. As usual in representation theory, we will denote by $(K, \mathcal{O}, k)$ an ℓ-modular system, i.e. $\mathcal{O}$ is a complete discrete evaluation ring such that $\mathcal{O}/J(\mathcal{O}) \cong k$ and K is the field of quotients of $\mathcal{O}$. The notation p to denote a prime number will be used in the setting of groups of Lie type, when we want to specialise a generic group of Lie type $\mathbb{G}$ to a fixed p-power q, namely $G = \mathbb{G}(q)$. In our particular situation we will always have $\ell \neq p$. This notation matches with the one of [11] and [12]. An ℓ-local subgroup G will often be denoted by H, and in the last two chapters of the thesis we will always have $H = N_G(P)$, where $P \in \mathrm{Syl}_\ell(G)$. In order to denote the opposite of a given group $(G, \cdot)$, we will use G^{opp}, the group having the same elements of G and operation $*$ defined by $g_1 * g_2 := g_2 \cdot g_1$ for each $g_1, g_2 \in G$. In analogy with this, for a given algebra A, the opposite algebra A^{opp} is the same algebra A but with opposite multiplication. For dihedral groups, the notation D_n will denote the dihedral group of order n. For a group G, we denote by $B_0(G)$ the principal ℓ-block of

kG; for a general ℓ-block B of G, the Brauer correspondent block of H will be denoted by b - whereas for principal blocks we will simply use $B_0(H)$.

As for modules, we will always consider left modules. Given an ℓ-block B, we will refer to the set of irreducible B-modules by $\mathcal{S}_B$. The set of *ordinary* representations belonging to the ℓ-block B is $\mathrm{Irr}(B)$. Following the standard notations in the literature, we will write $l(B) := |\mathcal{S}_B|$ and $k(B) := |\mathrm{Irr}(B)|$. Every module is intended to be a left module. For a kG-module M, the restriction of M down to H will be denoted by M_H; for a kH-module V, the induction up to G is V^G. For a module M, the projective cover of M is denoted by $\mathcal{P}(M)$. The set of composition factors (also called constituents) of M is denoted by $\mathrm{cpf}(M)$.

Finally, for a block B, $\mathrm{Mod}(B)$ is the category consisting of all B-modules and $\mathrm{mod}(B)$ the subcategory of all finitely generated B-modules; we will denote by $\mathcal{D}(B)$ and $\mathcal{K}(B)$ the derived category and homotopy category of $\mathrm{mod}(B)$ of *bounded* complexes respectively. The stable category of $\mathrm{mod}(B)$ will be denoted by $\underline{\mathrm{mod}}(B)$. We will say that B and b are derived equivalent if $\mathcal{D}(B)$ and $\mathcal{D}(b)$ are equivalent categories, and stably equivalent if $\underline{\mathrm{mod}}(B)$ and $\underline{\mathrm{mod}}(b)$ are equivalent categories. The idempotent of kG related to a block B is denoted by e_B.

CHAPTER 1

PRELIMINARIES

1.1 Modular representation theory

The book [1] is the source of all the material covered by Sections 1.1.1 and 1.1.2. Definitions and proofs of results which are not provided here and further remarks can be found there. All the modules that we consider are finite-dimensional. As the dissertation focuses on one of the deepest conjectures of local-global type, we immediately introduce the basic results of representation theory describing one fundamental aspect of how the modular representation theory of a finite group G is related to the modular representation theory of its local subgroups $N_G(Q)$, where $Q \leq G$ is an ℓ-group; this leads us to introduce the necessary general background to state and prove the Green correspondence and the Brauer correspondence, for modules and blocks respectively.

1.1.1 Green correspondence

The Green correspondence is a main result of modular representation theory and will be extensively used to describe our results and our algorithm. Here Q is an ℓ-subgroup of G and $H \leq G$ is such that $N_G(Q) \leq H$. The typical situation will be $Q \in \mathrm{Syl}_\ell(G)$ and $H = N_G(Q)$, but the Green correspondence can be stated in more generality. We briefly recall the notation of [1]: we have the family of subgroups defined by $\mathfrak{X} = \{sQs^{-1} \cap Q \mid s \notin H\}$, $\mathfrak{Y} = \{sQs^{-1} \cap H \mid s \notin H\}$, and $\mathfrak{z} = \{R \mid R \leq Q, R \not\subseteq_G \mathfrak{X}\}$. We say that a module U is relatively projective with respect to a family of subgroups if each summand of U is relatively projective with respect to a subgroup in such family.

Theorem 1.1.1. (Green correspondence) *There is a one-to-one correspondence between* *isomorphism classes of indecomposable kG-modules with vertex in $\mathfrak{z}$ and indecomposable kH-modules with vertex in $\mathfrak{z}$. If U and V is such a pair, then they have the same vertex $R \in \mathfrak{z}$ and moreover*

$$
\begin{aligned}
U_L &\cong V \oplus Y, \\
V^G &\cong U \oplus X,
\end{aligned}
\tag{1.1.1}
$$

for some relatively $\mathfrak{Y}$-projective kH-module Y and relatively $\mathfrak{X}$-projective kG-module X.

Most of our computations are carried out at the level of kH-modules. The next result will be constantly used in our Magma computation, as it provides an easy criterion to distinguish the Green correspondent V from the relatively projective part Y; this result shows that we can easily distinguish them by looking at their dimension, with no need to run the dedicated algorithm `IsRelQProj`, which can be slow. For this result, we will assume $N_G(Q) = H$.

Proposition 1.1.2. *Let Y be as in Proposition 1.1.1. Then $\ell \mid \dim Y$.*

We prove first:

Lemma 1.1.3. *Let M be an indecomposable kG-module with vertex Q. Let $P \in \mathrm{Syl}_\ell(G)$ be such that $Q \leq P$. Then $[P : Q] \mid \dim M$.*

Proof. It is sufficient to prove it in the case that G is an ℓ-group. In fact, writing down the decomposition of M_P into indecomposable submodules $M_P = (M_P)_1 \oplus (M_P)_2 \oplus \cdots \oplus (M_P)_r, r \in \mathbb{N}$ and considering that the vertex of each $(M_P)_i$ is a subgroup of a G-conjugate of Q (Mackey's Theorem), we deduce that it suffices to check the statement for these factors, i.e. for ℓ-groups in general. Let us assume $G = P$ and let S be a source of M. Then $M | S^P$. By Green's indecomposability criterion, the module S^P is indecomposable, then necessarily $M = S^P$, and as a consequence $\dim M = [P, Q] \cdot \dim S$. □

Proof. (Proposition 1.1.2): Let Y_1 be an arbitrary indecomposable summand of Y. Since Y is relatively $\mathfrak{Y}$-projective then Y_1 is relatively $N_G(Q) \cap gQg^{-1}$-projective for some $g \notin N_G(Q)$. We distinguish two cases:

- Suppose first that $Q \notin \mathrm{Syl}_\ell(G)$. Let R_1 be a vertex of Y_1 and $P \in \mathrm{Syl}_\ell(G)$ such that $R_1 \leq N_G(Q) \cap gQg^{-1} < P$, where the last inclusion is strict since Q is not a Sylow ℓ-subgroup of G and then, certainly, $N_G(Q) \cap gQg^{-1}$ is not, either. Then $[P, R_1] > 1$ is a power of ℓ dividing $\dim Y_1$ by Lemma 1.1.3, and this happens for all the indecomposable summands of Y, then $\ell \mid \dim Y$.

- Suppose now $Q \in \mathrm{Syl}_\ell(G)$, and then $Q \in \mathrm{Syl}_\ell(N_G(Q))$. We have $N_G(Q) \cap gQg^{-1} \notin \mathrm{Syl}_\ell(N_G(Q))$, since if it was, it would coincide with Q, which is the only Sylow ℓ-subgroup of $N_G(Q)$, and from cardinality we would get $gQg^{-1} = Q$; this would be a contradiction since $g \notin N_G(Q)$. So we can choose again R_1 to be a vertex of Y_1 such that $R_1 \leq N_G(Q) \cap gQg^{-1} < Q$, with the last inclusion being strict. Then again $[Q, R_1] > 1$ is an ℓ-power dividing $\dim Y_1$ by Lemma 1.1.3, therefore ℓ divides the dimension of any indecomposable summand of Y, and then $\ell \mid \dim Y$.

This concludes the proof. □

1.1.2 Brauer correspondence

The notion of Green correspondence can be specialised in the setting of block theory and lead to Brauer's correspondence. We can regard kG as a $k[G \times G]$-module with the natural action

$$(g_1, g_2) \cdot a = g_1 a g_2^{-1}. \tag{1.1.2}$$

Thus being a $k[G \times G]$-submodule is the same as being an ideal of kG, as we can see by considering the action of $(g, 1_G), (1_G, g)$ for any $g \in G$. It follows that the decomposition of kG into ℓ-blocks is the decomposition into the direct sum of indecomposable $k[G \times G]$-submodules. The relevance of Brauer's correspondence lies in the connections (some of them still conjectural) between the ℓ-blocks of G and those of H, where H is a subgroup of G with certain properties. In the following, for any subgroup $D \leq G$, we will denote by $\delta(D) = \{(d, d) \mid d \in D\} \leq G \times G$ the diagonal embedding of D in $G \times G$. This type of subgroup gains relevance in this context due to the following:

Proposition 1.1.4. [See IV.13, [1]] *Let B be an ℓ-block of G. As a $k[G \times G]$-module, B has vertex $\delta(D)$ for some ℓ-subgroup $D \leq G$.*

Definition 1.1.5. A subgroup $D \leq G$ such that $\delta(D)$ is a vertex of B is a *defect group* of B.

A natural way to find a connection between ℓ-blocks of G and H is to restrict kG to $H \times H$.

Definition 1.1.6. Let B be an ℓ-block of G and b an ℓ-block of H. We write $B = b^G$ if $b | B_{H \times H}$ and no other block of kH satisfies the same property.

For a given B, a corresponding ℓ-block b such that $B = b^G$ does not necessarily exist, but it does under some assumptions, for example if $C_G(D) \leq H$; therefore, it is the case when $H = N_G(D)$. This correspondence is a bijection between blocks having the same defect group when some conditions are fulfilled; this is the content of Brauer's theorem. We recall that the principal ℓ-block $B_0(G)$ is the block affording the trivial kG-module k.

Theorem 1.1.7. [Brauer's First and Third Main Theorems][See IV.14, [1]] *Let G be a finite group, H a subgroup and D an ℓ-subgroup such that $N_G(D) \leq H \leq G$; then b^G is defined, and moreover:*

- *there is a one-to-one correspondence between ℓ-blocks of H with defect group D and ℓ-blocks of G with defect group D, and this correspondence is given by associating the block b of H to the block b^G of G;*
- *if $b = B_0(H)$ is the principal ℓ-block of H, then b^G is the principal ℓ-block $B_0(G)$ of G.*

For our purposes, we will always consider the relevant case $H = N_G(D)$ in which this correspondence occurs. In this case, we use the following terminology:

Definition 1.1.8. Let b be an ℓ-block of $N_G(D)$. The block $B = b^G$ of G is defined, and it is called the *Brauer correspondent* of b.

Brauer's result is a remarkable initial fact of a pattern in modular representation theory, namely an example of local-global result: we see that if we want to determine how many blocks of G have a certain ℓ-subgroup D as a defect, we can just count those of $N_G(D)$ with the same property. Moreover, the Brauer correspondence just defined above carries many more results and conjectures - many of them still open - concerning how the representation theory of G is controlled by that of local ℓ-subgroups. The connection between Brauer's correspondence and Green's correspondence that we mentioned before is due to:

Remark 1.1.9. In the setting of Brauer's correspondence, the ℓ-block b^G of G is the Green correspondent of b, regarding the latter as a $k[H \times H]$-module.

1.1.3 Morita equivalence

Let A_1 and A_2 be two k-algebras. In modern representation theory, the common approach to understand how A_1 and A_2 are related consists of looking at their module categories $\mathrm{Mod}(A_1)$

and $\mathrm{Mod}(A_2)$ and eventually find some form of strong or weak equivalences between them. This strategy has benefits as these equivalences will preserve some properties of A_1 and A_2: hence, if we assume that A_2 has a manageable structure, those properties can be checked and they will then hold for A_1 too. We briefly recall the most common equivalences that play a role in modular representation theory.

We express as *Morita equivalence* the classical notion of equivalence of categories: two k-algebras A_1, A_2 are said to be Morita equivalent if $\mathrm{Mod}(A_1)$ and $\mathrm{Mod}(A_2)$ are equivalent as k-linear categories. In practice, we ask that there is an equivalence $F : \mathrm{Mod}(A_1) \to \mathrm{Mod}(A_2)$ which is k-linear, namely F maps between homomorphism spaces $\mathrm{Hom}_A(U, V) \to \mathrm{Hom}_B(F(U), F(V))$ as a linear map of k-modules (k-vector spaces, according to our assumption on k to be a field). Assuming that F is k-linear is enough to get that F preserves the structure of $\mathrm{Mod}(A_1)$ and $\mathrm{Mod}(A_2)$ as abelian categories, such as kernels and cokernels.

The notion of Morita equivalence is often too strong for many purposes: many aspects of the representation theory of A_1 and A_2 are preserved by weaker notions of equivalences, such as derived equivalences. Although being Morita equivalent is a strong requirement, in the study of ℓ-local determination in Block theory it is still open whether the local datum of the defect group is enough to control the Morita theory of a block. This is expressed by the long-standing:

Conjecture 1.1.10. (Donovan - 1975, [2]) *Let ℓ be a prime number and D an ℓ-group. The number of blocks of groups algebras whose defect group is isomorphic to D is finite up to Morita equivalence.*

In other words, for a fixed ℓ-group D, only finitely many blocks up to Morita equivalence should have D as a defect group. The conjecture is known to hold in some cases, for example when D is cyclic, and elementary abelian when $\ell = 2$. We also have the result for D dihedral, semi-dihedral, as well as quaternion and generalised quaternion; it is also known to hold (by Puig) if we restrict to ℓ-blocks of ℓ-solvable groups. It is possible to wonder if Donovan's conjecture might be reversed. The answer is not clear at the moment.

Problem 1.1.11. *Let B_1 and B_2 be ℓ-blocks of groups and let us assume that B_1 and B_2 are Morita equivalent. Are their defect groups isomorphic?*

Indeed, the problem above is a more general formulation of another problem which was also mentioned by Brauer in [8]. This is known as:

Problem 1.1.12. (Modular isomorphism problem) *Let P and Q be ℓ-groups such that $kP \cong kQ$ as algebras. Is it true that $P \cong Q$?*

We carry on this initial chapter by introducing the two types of (weaker) equivalences that will actually play a role in our approach to Broué's conjecture.

1.1.4 Derived Equivalences and Broué's conjecture

A weaker type of equivalence occurring between k-algebras is *derived equivalence.* For two k-algebras A_1 and A_2, an equivalence $F : \mathcal{D}(A_1) \to \mathcal{D}(A_2)$ is called *derived equivalence* between A_1 and A_2. In our setting of Broué's conjecture, we will only consider the case of two Brauer correspondent ℓ-blocks B and b. This equivalence is strictly weaker than Morita equivalence, but some properties of the representation theory of a block which are invariant under a Morita equivalence are invariant under a derived equivalence as well: for example, a derived equivalence between B and b implies $l(B) = l(b)$ and $k(B) = k(b)$; this will be explained better later on, when we will introduce the definition of perfect isometry. A derived equivalence is exactly the structural connection that the conjecture of Broué predicts whenever the defect group D of B is abelian:

Conjecture 1.1.13. (Broué's abelian defect conjecture - 1990, [9]) *Let G be a finite group and let ℓ be a prime number. Let B be an ℓ-block of G with abelian defect group D and b be the ℓ-block of $N_G(D)$ corresponding to B under the Brauer correspondence; then B and b are derived equivalent.*

Actually, the classical version of Broué's conjecture is predicting something more, specifically a derived equivalence at the level of blocks of $\mathcal{O}G$ and $\mathcal{O}N_G(D)$. More precisely, the ℓ-block decomposition $kG = B_1 \oplus \cdots \oplus B_n$ follows from the ℓ-block decomposition of $\mathcal{O}G$, namely $\mathcal{O}G = B'_1 \oplus \cdots \oplus B'_n$, where $B_i = \overline{B'_i}$ and the map $- : \mathcal{O}G \to kG$ reduces the coefficient of each element of $\mathcal{O}G$ modulo $J(\mathcal{O})$. Broué's conjecture predicts that each block B' of $\mathcal{O}G$ is derived equivalent to the Brauer-corresponding block of $\mathcal{O}N_G(D)$. In particular, this would imply a derived equivalence for the corresponding block $B = \overline{B'}$ of kG and its Brauer correspondent block of $kN_G(D)$, whereas a derived equivalence at the level of k does not lift to $\mathcal{O}$ in general. However, we will explain later that the particular type of derived equivalence between B and b that we are aiming for can be always lifted up to $\mathcal{O}$, and therefore, in this dissertation, there is no loss in formulating Broué's conjecture over $\mathcal{O}$ instead of k. This is of great importance, as we will perform all our computations over fields rather than over $\mathcal{O}$.

The conjecture first appeared in [9] and remains open, although it has been verified for a certain number of cases, for example for all blocks with cyclic defect group, for all blocks of ℓ-solvable groups, and for some blocks of some sporadic simple groups (in characteristic 2, 3 or 5). Broué's conjecture is currently known to hold in certain cases; some situations where the conjecture has been verified (the list is not exhaustive) are:

1. G is ℓ-solvable, by Dade, Puig, Harris, Linckelmann;
2. D is cyclic, by Linckelmann, Rouquier, Rickard;
3. $D \cong C_2 \times C_2$, by Erdmann, Linckelmann, Rickard, Rouquier;
4. $D \cong C_3 \times C_3$, B principal block, by Okuyama, Koshitani, Kunugi, Waki;
5. $G = A_n$ by Marcus, $G = S_n$ by Rickard, Chuang, Kessar, Rouquier;
6. D is a 2-group and B principal, by Rouquier, Okuyama, Golan, Marcus;
7. $G = SL_2(q)$, B principal block, where $\ell \mid q$, by Chuang, Kessar, Okuyama;
8. Some other groups of Lie type when $\ell \nmid q$; for example $GL_n(q)$ by Chuang and Rouquier;

A complete list of general and individual cases where the conjecture in known (up to 2008) can be found in [27]. Some more recent results where the conjecture has been checked are:

- $D \cong (C_2)^3, (C_2)^4$ by Eaton [15] and [13]; recently, for $D \cong C_{2^n} \times C_2 \times C_2$ by Eaton and Livesey [14];
- $D \cong C_3 \times C_3$, B block such that its Brauer correspondent b is not nilpotent and has a unique isomorphism class of simple modules (in particular, this is a Morita equivalence), by Kessar [21];
- $G = HN$, every non-principal block B with defect $D \cong C_3 \times C_3$, by Koshitani and Müller [22];
- $G = 2.HS$, B the faithful block of defect $D \cong C_3 \times C_3$, by Koshitani, Müller, Noeske [23] (this completes the proof of the conjecture for each block and each prime when $G = 2.HS$);

- $G = \mathrm{Co}_1$, for the unique block B of defect $D \cong C_3 \times C_3$ (and, as a consequence, this completes the proof for every 3-block), by Koshitani, Müller, Noeske [24].

- $G = \mathrm{Co}_3$, for the 2-block of defect $D \cong C_2 \times C_2 \times C_2$, by Koshitani, Müller, Noeske [25], and this has completed the proof of the conjecture for each block and each prime when $G = \mathrm{Co}_3$; notice that this result for the individual group Co_3 is included in the more recent and general result of Eaton [15] that we have mentioned above.

We will give a very brief overview of other local/global conjectures in the next section. In contrast to other local-global conjectures, a general reduction theorem for Broué's abelian defect group conjecture to simple or quasi-simple groups is currently not known; however, this is known for principal blocks.

We conclude this brief introduction of the conjecture by stating a stronger form of it. Although the existence of a derived equivalence between two such blocks B and b has many consequences, the original formulation of Broué's abelian defect conjecture does not provide any description of the equivalence itself. It is currently believed that we can state a stronger version of Broué's conjecture, for which we now give the concept of *splendid equivalence*, which has been introduced by Rickard in [34].

Definition 1.1.14. (Splendid Equivalence - [34]) Let B and b be Brauer correspondent ℓ-blocks of G and H respectively, of common defect D. A complex C of (b, B)-bimodules being projective as left b-modules and as right B-modules such that

1. $C \otimes_B C^* \cong b \oplus X$, $C^* \otimes_b C \cong B \oplus Y$ where X, Y are complexes of (b, b) and (B, B)-bimodules homotopy equivalent to 0 respectively;

2. each (b, B)-bimodule appearing in C is a summand of a relatively ΔD-projective permutation module,

is said to be a *splendid tilting complex.* If the standard functor $C \otimes_B - : \mathcal{D}(B) \to \mathcal{D}(b)$ is a derived equivalence between B and b, we say that this is a *splendid derived equivalence.*

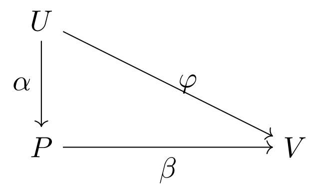
A splendid derived equivalence implies more than a general derived equivalence. A benefit of establishing a splendid derived equivalence is that we can work on the field k, and a splendid derived equivalence at the level of blocks for kG and kH lifts to a derived equivalence between $\mathcal{O}G$ and $\mathcal{O}H$, and this is not automatic for general derived equivalences. This property is especially important for this dissertation, where the computational approach is central; in particular, calculation can be run over fields rather than over $\mathcal{O}$, without any loss. The concept of splendid derived equivalence is introduced as it is believed that the following refinement of Broué's conjecture holds:

Conjecture 1.1.15. (Broué-Rickard) *With the same notation and under the same conditions stated in Broué's abelian defect conjecture, there is a splendid derived equivalence between B and b.*

1.1.5 Stable Equivalences

We finally introduce very briefly the definition of *stable equivalence*: this will be needed to explain how our strategy (due to Rouquier) relies on the construction of such an equivalence between B and b, which is weaker then a derived equivalence, and how we can lift it to a derived equivalence.

Let A be a k-algebra. For any two A-modules U, V we consider the set $\mathrm{Hom}_A^{\mathrm{proj}}(U, V) \subseteq \mathrm{Hom}_A(U, V)$ consisting of all the A-homomorphisms that factor through a projective A-module. In other words, if φ is an A-homomorphism, then $\varphi \in \mathrm{Hom}_A^{\mathrm{proj}}(U, V)$ if there exist A-homomorphisms α, β and a projective A-module P such that this diagram commutes:



As $\mathrm{Hom}_A^{\mathrm{proj}}(U, V)$ is closed under addition and multiplication by scalars, this is a k-subspace of $\mathrm{Hom}_A(U, V)$. The stable category $\underline{\mathrm{mod}}(A)$ is defined by having the same set of objects as $\mathrm{mod}(A)$, and for any two objects U, V we set $\underline{\mathrm{Hom}}_A(U, V) = \mathrm{Hom}_A(U, V)/\mathrm{Hom}_A^{\mathrm{proj}}(U, V)$. Projective objects are annihilated in the stable category, namely U is projective if and only if $U \cong \{0\}$ in $\underline{\mathrm{mod}}(A)$. For two k-algebras A_1 and A_2, an equivalence $F : \underline{\mathrm{mod}}(A_1) \to \underline{\mathrm{mod}}(A_2)$ is said to be a *stable equivalence* between A_1 and A_2. As for the derived equivalence, stable equivalences which are induced by a splendid tilting complex will play a specific role.

Definition 1.1.16. (Splendid stable equivalence) Let B and b be Brauer correspondent ℓ-blocks of G and H respectively, of common defect D. If there is a complex C of (b, B)-bimodules such that

- $C \otimes_B C^* \cong b \oplus X'$ and $C^* \otimes_b C \cong B \oplus Y'$, where X' and Y' are homotopy equivalent to complexes of projective (b, b) and (B, B)-bimodules respectively,
- each (b, B)-bimodule appearing in C is a summand of a relatively ΔD-projective permutation module,

then we say that C induces a *splendid stable equivalence* between b and B.

The standard functor $C \otimes_B - : \mathcal{D}(B) \to \mathcal{D}(b)$ induces an equivalence $\underline{\text{mod}}(B) \to \underline{\text{mod}}(b)$ via the equivalence of categories $\mathcal{D}(B)/\mathcal{K}(\text{proj-}B) \to \underline{\text{mod}}(B)$, see [33]. Again in [33], we see that a derived equivalence implies a stable equivalence. Conversely, a stable equivalence does not imply a derived equivalence in general; in the case when, given a stable equivalence $\bar{F} : \underline{\text{mod}}(B) \to \underline{\text{mod}}(b)$, we can find a derived equivalence $F : \mathcal{D}(B) \to \mathcal{D}(b)$ which induces $\bar{F}$ at the level of stable categories, then we say that F lifts the stable equivalence $\bar{F}$. Splendid stable equivalences will play a role in the development of the future work, in particular we will see that under some condition, it is possible to lift a stable equivalence up to a derived equivalence. This will be explained more clearly in the following.

1.2 Local-global conjectures

As we mentioned in the introduction, many deep and long-standing questions in representation theory of finite groups can be labelled as local-global conjectures. In this section we aim to give an overview of those conjectures, for example how they are related to each other, their current status and how the modular and ordinary representation theory of a group can be related by these conjectured properties.

Broué's abelian defect conjecture is often regarded as a structural conjecture between an ℓ-block B of G and its Brauer correspondent b. Other well-known and open conjectures happen to be related to the character side of B and b. For a recent overview (2015) about the current status of many of these problems and their progress, we mention the survey [26], from which we take some of the following results, with more recent developments. Many of these developments includes the proof of these conjectures in some particular cases or progress in the direction of some reduction results.

Two of the longest-standing conjectures of modular representation theory are due to Brauer. The oldest one, still open, predicts that the number of ordinary characters lying in a given block is controlled by the defect group. We refer to this conjecture by $k(B)$ due to the large use in the literature of the notation $k(B) := |\mathrm{Irr}(B)|$.

Conjecture 1.2.1. (Brauer's $k(B)$-conjecture - 1954, [5]) *Let G be a finite group and B an ℓ-block of G of defect D. We have $|\mathrm{Irr}(B)| \leq |D|$.*

It is still open in general, although singular cases have been covered. This conjecture appeared in [5]; Brauer himself managed to show the bound $|\mathrm{Irr}(B)| \leq \frac{1}{4}|D|^2 + 1$ in a joint work with W. Feit. The case G being ℓ-solvable is also known to hold. Recently (2014), a work of Sambale in form of textbook [37] has proved the conjecture in different cases, for example D abelian of rank at most 3; more open conjectures, including most of the following in this section, are approached and solved in that work in some specific cases.

Given a finite group G and an ℓ-block B of defect D, we recall that the height of a character $\chi \in \mathrm{Irr}(B)$ is defined by the equality $\chi(1)_\ell |D| = \ell^{\mathrm{ht}(\chi)}|G|_\ell$. In [7], Brauer conjectured a relatively simple condition for a block to have an abelian defect group in terms of its characters:

Conjecture 1.2.2. (Brauer's height zero conjecture - 1955, [7]) *Let G be a finite group and B an ℓ-block of G of defect D. All the characters in $\mathrm{Irr}(B)$ have height zero if and only if D is abelian.*

This conjecture is open but important progress have been made. For example, one direction is known: in [20], Kessar and Malle proved that if D is abelian, then $\mathrm{ht}(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$. This result relies on a previous result of reduction type, stating that it is enough to prove that this direction of Brauer's conjecture is true for ℓ-blocks of quasi-simple groups. This reduction has been proved independently in [4] and [30] respectively.

Moreover, the conjecture has been proved in two restricted cases: for ℓ-solvable groups G, and for the case $\ell = 2$, $D \in \mathrm{Syl}_2(G)$. The second result is due to Navarro and Tiep [31], whereas the first was proved by Gluck and Wolf [16].

Another well-known conjecture is the following:

Conjecture 1.2.3. (Alperin-McKay conjecture - 1976, [2]) *Let G be a finite group and ℓ a prime. If B is a ℓ-block of G of defect D, and b the Brauer correspondent block in $N_G(D)$ of B, we have:*

$$|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|, \tag{1.2.1}$$

where $\mathrm{Irr}_0(\cdot)$ is the set of height-zero characters.

Notice that if D is abelian, then the known part of Brauer's Height zero conjecture (Kessar-Malle) gives $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}(B)|$ and $|\mathrm{Irr}_0(b)| = |\mathrm{Irr}(b)|$; hence, Alperin-McKay reduces to say that B and b have the same number of ordinary simple modules. Another conjecture, due to Alperin and known as Alperin's weight conjecture, predicts the same consequence $l(B) = l(b)$ for the number of simple modular representations, again when D is assumed to be abelian.

Let $\mathrm{Irr}_{\ell'}(G)$ be the subset of $\mathrm{Irr}(G)$ consisting of characters whose degree is not divisible by ℓ. Conjecture 1.2.3 is a refinement of a previous conjecture due to McKay in 1972: this states that the equality

$$|\mathrm{Irr}_{\ell'}(G)| = |\mathrm{Irr}_{\ell'}(N_G(P))| \tag{1.2.2}$$

holds for every finite group G and prime ℓ, where $P \in \mathrm{Syl}_\ell(G)$. This descends from the Alperin-McKay conjecture as when $D = P$, then $\mathrm{Irr}_{\ell'}(B) = \mathrm{Irr}_0(B)$ by definition of height. As the sets $\mathrm{Irr}_0(B)$ partition $\mathrm{Irr}_0(G)$, it is enough to sum over them - and over the Brauer correspondent blocks b on the right hand side - to get 1.2.2. The conjecture of McKay is known to hold for all the cases where Alperin-McKay is known to hold, but it has been independently proved in some particular cases, for example for symmetric groups. Moreover, Wilson [39] has proved it for all sporadic groups, and Isaacs for every group of odd order [19].

Further generalisations of this conjectured relation between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(b)$ have been proposed; for instance in [18] (conjecture A), Navarro and Isaacs suggest that having the same number of elements should be somehow consistent with some numerical properties. Precisely, they predict that:

$$|\{\chi \in \mathrm{Irr}_{\ell'}(G) : \chi(1) \equiv \pm h \bmod \ell\}| = |\{\psi \in \mathrm{Irr}_{\ell'}(N_G(P)) : \psi(1) \equiv \pm h \bmod \ell\}| \tag{1.2.3}$$

for each $h = 1, \dots, \ell - 1$. Notice that this statement extends McKay's conjecture; an analogous extension - still based on arithmetic properties - for the more general conjecture 1.2.3 of Alperin-McKay exists (see conjecture B in [18]). Conjecture B reduces to conjecture A in the case that Alperin-McKay's conjecture reduces to McKay's conjecture.

All these conjectures are somehow partially related to each other, and the most immediate connection between them comes out in the special case of an ℓ-block B with abelian defect group D. It is worth mentioning that this is the case where the structural role of Broué's conjecture applies. An example of a connection between two of those conjectures is given by:

Proposition 1.2.4. *In the setting of Alperin-McKay's conjecture and Alperin weight conjecture,* *the two statements are equivalent when D is abelian.*

This equivalence was proved by Knörr and Robinson.

1.2.1 Perfect isometries

The notion of perfect isometry plays a significant role in Block theory and in some local-global conjectures. We now recall the basic definitions and results, as well as how perfect isometries are involved in local-global conjectures. A traditional reference for this subject is [9].

Given a finite group G and the usual ℓ-modular system $(K, \mathcal{O}, k)$, here B is a sum of blocks of $\mathcal{O}G$ (it will often be a single block). We recall that an element $g \in G$ is ℓ-singular if ℓ divides the order of g, and ℓ-regular if not; the set of ℓ-regular elements of G is denoted by G^0. We will use the standard notation used in the literature about isometries: the abelian group generated by $\mathrm{Irr}(B)$ is denoted by $\mathcal{R}(G, B)$; the K-vector space generated by $\mathrm{Irr}(G)$ is the set of K-valued class functions on G, denoted by $CF(G, K)$; in this regard we can embed $\mathcal{R}(G, B)$ into $CF(G, K)$, and $\mathrm{Irr}(B)$ spans the K-subspace $CF(G, B, K)$ inside $CF(G, K)$. Finally we consider two subsets of $CF(G, B, K)$, namely $CF(G, B, \mathcal{O})$ and $CF_{\ell'}(G, B, K)$, the class functions taking values in $\mathcal{O}$, and those vanishing on ℓ-singular elements $g \in G$ respectively. The standard inner product of characters $\langle \cdot, \cdot \rangle$ extends to each of these sets of generalised characters.

We now introduce a second finite group H; we denote by b a block (or a sum of blocks) of $\mathcal{O}H$. Let $\mu \in \mathcal{R}(G \times H^{\mathrm{opp}}, B \otimes_{\mathcal{O}} b^{\mathrm{opp}})$, a generalised character of $G \times H$. We define the linear maps $I_\mu : CF(H, b, K) \to CF(G, B, K)$ and $R_\mu : CF(G, B, K) \to CF(H, b, K)$ by:

$$I_\mu(\alpha)(g) = \langle \mu(g, \cdot), \alpha(\cdot) \rangle_H = \frac{1}{|H|} \sum_{h \in H} \mu(g, h^{-1}) \alpha(h), \quad \alpha \in CF(H, b, K), \tag{1.2.4}$$

and

$$R_\mu(\beta)(h) = \langle \mu(\cdot, h), \beta(\cdot) \rangle_G = \frac{1}{|G|} \sum_{g \in G} \mu(g^{-1}, h) \beta(g), \quad \beta \in CF(G, B, K). \tag{1.2.5}$$

There linear maps are adjoint with respect to the usual scalar product. Vice-versa, if we start from a linear map $I : CF(H, b, K) \to CF(G, B, K)$, it defines a generalised character μ such that $I = I_\mu$ by setting $\mu := \sum_{\theta \in \mathrm{Irr}(H,b)} I(\theta)\theta$.

Proposition 1.2.5. (Broué, [9]) *Let G, H be two finite groups and B, b two blocks of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. Let $I : CF(H, b, K) \to CF(G, B, K)$ be a linear map and $\mu \in \mathcal{R}(G \times$*

$H^{opp}, B \otimes_{\mathcal{O}} b^{opp})$ be such that $I = I_\mu$. The adjoint map is denoted by R_μ. The following two conditions on $I = I_\mu$ together:

- *$I_\mu : CF(H, b, \mathcal{O}) \to CF(G, B, \mathcal{O})$, $R_\mu : CF(G, B, \mathcal{O}) \to CF(H, b, \mathcal{O})$, i.e. I_μ and R_μ send $\mathcal{O}$-valued maps to $\mathcal{O}$-valued maps;*
- *$I_\mu : CF_{\ell'}(H, b, K) \to CF_{\ell'}(G, B, K)$, $R_\mu : CF_{\ell'}(G, B, K) \to CF_{\ell'}(H, b, K)$, i.e. I_μ, R_μ sends maps vanishing on ℓ-singular elements to maps vanishing on ℓ-singular elements;*

are equivalent to the following two conditions on μ together:

- $\dfrac{\mu(g,h)}{|C_G(g)|}, \dfrac{\mu(g,h)}{|C_H(h)|} \in \mathcal{O}$, $\forall (g, h) \in G \times H$;
- *if $\mu(g, h) \neq 0$, then either both g, h are ℓ-singular or are ℓ-regular.*

Introducing these objects makes sense in consideration of the following definition:

Definition 1.2.6. (Perfect Isometry, [9]) Let G, H be two finite groups, B, b blocks of $\mathcal{O}G$ and $\mathcal{O}H$ respectively and $I : CF(H, b, K) \to CF(G, B, K)$ a linear map afforded by the character $\mu \in \mathcal{R}(G \times H^{\text{opp}}, B \otimes_{\mathcal{O}} b^{\text{opp}})$, i.e. $I = I_\mu$. We also assume that I is an isometry. We say that μ is *perfect* and I_μ is a *perfect isometry* if they fulfil the equivalent conditions stated in Proposition 1.2.5.

The notion of perfect isometry is usually regarded as the character implication of a derived equivalence; in fact, Broué has shown that if two blocks are derived equivalent over $\mathcal{O}$, then there is a perfect isometry between them (Theorem 3.1, [9]). Broué also showed that a perfect isometry between two blocks B and b implies $l(B) = l(b)$ and $k(B) = k(b)$ (Theorem 1.5, [9]).

The following lemma follows from the definition of perfect isometry and it will be necessary later to disprove the existence of a perverse equivalence in the case of J_2 in characteristic 5.

Lemma 1.2.7. *Given two finite groups G, H, let $I = I_\mu$ be a perfect isometry between the two blocks B, b of the group rings $\mathcal{O}G$ and $\mathcal{O}H$ respectively, afforded by the perfect character $\mu \in \mathcal{R}(G \times H^{opp}, B \otimes_{\mathcal{O}} b^{opp})$. Let $\theta_1, \theta_2 \in \mathrm{Irr}(b)$ be characters having the same ℓ-reduction, namely $\theta_1(h) = \theta_2(h)$ $\forall h \in H^0$, then $I_\mu(\theta_1)$ and $I_\mu(\theta_2)$ have the same ℓ-reduction as well.*

Proof. We need to check whether $I_\mu(\theta_1)(g) = I_\mu(\theta_2)(g)$, $\forall g \in G^0$. For any fixed $g \in G^0$, we have $\mu(g, h) = 0$ $\forall h \in H \setminus H^0$ as μ is perfect. Then it follows:

$$I_\mu(\theta_1)(g) = \frac{1}{|H|}\sum_{h\in H}\mu(g,h^{-1})\theta_1(h) = \frac{1}{|H|}\sum_{h\in H^0}\mu(g,h^{-1})\theta_1(h) = \frac{1}{|H|}\sum_{h\in H^0}\mu(g,h^{-1})\theta_2(h) =$$
$$\frac{1}{|H|}\sum_{h\in H}\mu(g,h^{-1})\theta_2(h) = I_\mu(\theta_2)(g),$$

as we claimed. □

The notion of perfect isometry for two blocks B and b of G and H can be used to define another (stronger) equivalence between them which was introduced by Broué and is called *isotypy*. For the definition of an isotypy in the case when B and b are principal blocks, we refer to [34], §6. An isotypy between B and b is more than a perfect isometry: roughly speaking, an isotypy carries a family of perfect isometries between blocks of some local ℓ-subgroups (centraliser of subgroups $Q \leq P$ in H and G), and all these isometries fulfil some restrictions of compatibility. Mentioning the concept of isotypy is useful as it represents a stronger version of perfect isometry as much as the notion of splendid derived equivalence represents a stronger version of derived equivalence. The following result, which can be made more general involving non-principal blocks, explains this connection. The property of having the same ℓ-local structure (that is used in the Theorem) is defined in [34]; as an example, when $P \in \mathrm{Syl}_\ell(G)$ is abelian and $H = N_G(P)$, then G and H have the same ℓ-local structure.

Theorem 1.2.8. ([34] Th. 6.3) *Let G and H be two finite groups with the same ℓ-local structure and with isomorphic ℓ-Sylow subgroup. If the principal blocks B, b of the group rings $\mathcal{O}G$ and $\mathcal{O}H$ are splendid derived equivalent, then they are isotypic.*

Moreover, this result uses the fact that a perfect isometry is induced whenever we have a derived equivalence, and it shows indeed that the same implication holds for isotypies and splendid derived equivalences respectively. As the notion of isotypy is the character counterpart of splendid derived equivalence, as perfect isometries are for derived equivalences, this conjecture follows:

Conjecture 1.2.9. (Broué's isotypy conjecture - 1990) *Let B be an ℓ-block of G of abelian defect D. There is an isotypy between B and the Brauer correspondent b of B in $N_G(D)$.*

This was conjectured by Broué in [9]. As we mentioned, it would follow from the strong form of Broué's abelian defect conjecture predicting a splendid derived equivalence between B and b.

CHAPTER 2

STRATEGY AND ALGORITHM

2.1 Perverse equivalences

In recent years, the theory of perverse equivalences has been successfully applied to gain some progress in the study of Broué's conjecture, especially in the case of finite groups of Lie type in non-defining characteristic. In this thesis we will present an algorithm producing perverse equivalences, and then derived equivalences, whenever the blocks are principal and the defect group is abelian of rank 2; these derived equivalences are actually splendid derived equivalences. The algorithmic approach to perverse equivalences has already been used in [11] to produce perverse equivalences for some groups of Lie type as well as some sporadic groups.

Definition 2.1.1. (Perverse equivalence) Let A_1, A_2 be k-algebras and $F : \mathcal{D}(A_1) \to \mathcal{D}(A_2)$ be a derived equivalence. Let us denote by $S_1, \dots S_n$ and $T_1, \dots, T_n$ the sets of isomorphism classes of the simple A_1-modules and A_2-modules respectively and let $\pi : \{1, \dots, n\} \to \mathbb{Z}_{\geq 0}$ be a function. We say that F is a *perverse equivalence* with perversity function π if for every $i \in \{1, \dots, n\}$ the modules occurring as composition factors of $H^{-j}(F(S_i))$ are T_α such that $\pi(\alpha) < j \leq \pi(i)$, and a single copy of T_i if $j = \pi(i)$.

We notice that this definition carries a bijection between the set of simple A_1-modules and A_2-modules, and this is given by indexing those sets from 1 to n. With an abuse of notation, we can often think of π as a function defined on the set $\{T_1, \dots, T_n\}$ rather than $\{1, \dots, n\}$, and therefore write $\pi(T_i)$ instead of $\pi(i)$. This will make the notation easier in some settings.

The goal of this section is to present the two algorithms that this dissertation is based on: `PerverseEq` and `FinalStabEq`. Both those algorithms are integrally included in the appendix, together with the several sub-algorithms having a role inside these two main ones. Here we will explain the theoretical basis of these procedures, as well as providing a criterion to judge when those algorithms are providing a successful output.

2.1.1 The algorithm `PerverseEq`

Given a finite group G and a prime ℓ, we recall our notation: we have $H := N_G(P)$, where $P \in \mathrm{Syl}_\ell(G)$; $B_0(G)$ and $B_0(H)$ are the principal ℓ-blocks of G and H, and $\mathcal{S}_{B_0(G)}, \mathcal{S}_{B_0(H)}$ the set of simple kG and kH-modules which lie in the principal block. As we can see in Definition 2.1.1, finding a suitable perversity function π is necessary in order to find a perverse equivalence. There is no general formula for the perversity function related to perverse equivalences between blocks of kG and kH, but such a formula exists at a conjectural level when G is a group of Lie type. In particular, this formula is computed via the unipotent ordinary characters of G, and then extended to the simple $B_0(H)$-modules.

Let us explain it in more detail. We assume that G is a group of Lie type and consider the set of unipotent characters $\mathrm{Uch}(B_0(G)) := \mathrm{Uch}(G) \cap \mathrm{Irr}(B_0(G))$ of G lying in the principal block $B_0(G)$. Our method consists of finding bijections

$$\mathrm{Uch}(B_0(G)) \xrightarrow{1:1} \mathcal{S}_{B_0(G)} \xrightarrow{1:1} \mathcal{S}_{B_0(H)} \tag{2.1.1}$$

between those three sets, and we want to explain how a perversity function $\pi : \mathcal{S}_{B_0(H)} \to \mathbb{Z}_{\geq 0}$, which will turn out to be valid for each group of Lie type that we consider, can be defined.

- Let us assume that a bijection between $\mathrm{Uch}(B_0(G))$ and $\mathcal{S}_{B_0(H)}$ is defined; this will just be the composition of the two bijections in (2.1.1). Our method consists of defining a map $\pi : \mathrm{Uch}(B_0(G)) \to \mathbb{Z}_{\geq 0}$; using the bijection between $\mathrm{Uch}(B_0(G))$ and $\mathcal{S}_{B_0(H)}$, this automatically defines a map on $\mathcal{S}_{B_0(H)} \to \mathbb{Z}_{\geq 0}$ as well. In Section 2.3 we will mention the geometric side of this method involving perverse equivalences and algebraic geometry, and at the end of this section we will provide an explicit formula to compute $\pi(\chi)$, where $\chi \in \mathrm{Uch}(B_0(G))$. By an abuse of notation, we will refer to π as the perversity function defined either on $\mathrm{Uch}(B_0(G))$ or $\mathcal{S}_{B_0(H)}$, under the assumption that a bijection between $\mathrm{Uch}(B_0(G))$ and $\mathcal{S}_{B_0(H)}$ has been fixed.
- The bijection $\mathrm{Uch}(B_0(G)) \xrightarrow{1:1} \mathcal{S}_{B_0(G)}$ is defined as follows: we order the set $\mathrm{Uch}(B_0(G))$ according to the given perversity function, namely $\chi \leq \chi' \iff \pi(\chi) \leq \pi(\chi')$. If two

or more characters have the same value, we can arbitrarily fix an ordering for them. Permuting the rows of a decomposition matrix with respect to this order, it turns out that there exists a way to permute the list of the simple $B_0(G)$-modules (the columns) to obtain a unitriangular matrix in all the cases that we will consider. This unitriangular structure of the decomposition matrix gives the required bijection between $\mathcal{S}_{B_0(G)}$ and $\mathrm{Uch}(B_0(G))$.

- A more tricky part consists of finding the right bijection between $\mathcal{S}_{B_0(G)}$ and $\mathcal{S}_{B_0(H)}$. This is the bijection which is carried by the definition of perverse equivalence. In [11] we have a way to find the correct bijection in the case of cyclic Sylow subgroup only. Anyway, in the cases treated here, our Sylow ℓ-subgroups are abelian of rank 2, however the number of modules that we consider is limited, therefore we can find the correct bijection using the approach of trial and error (the bijection will be correct if it makes the algorithm work as we will explain). Some additional numerical information will reduce the possibilities a lot; for example, the underlying perfect isometry of the derived equivalence that we aim for would imply that:
$$(-1)^{\pi(T)} \dim(T) \equiv \chi(1) \mod \ell, \tag{2.1.2}$$
where $\chi \in \mathrm{Uch}(B_0(G))$ and $T \in \mathcal{S}_{B_0(H)}$ correspond under the resulting bijection between $\mathrm{Uch}(B_0(G))$ and $\mathcal{S}_{B_0(H)}$. Therefore, if the bijection between $\mathrm{Uch}(B_0(G))$ and $\mathcal{S}_{B_0(G)}$ has already been obtained, the numerical information coming from the relations (2.1.2) restrict the possible choice for the bijection $\mathcal{S}_{B_0(G)} \xrightarrow{1:1} \mathcal{S}_{B_0(H)}$.

In the following, we will explain how the perversity function is built for groups of Lie type.

Let $z = re^{i\theta}$ be a non-zero complex number and κ, d be positive integers such that $(\kappa, d) = 1$. The set $\mathrm{Arg}_{\kappa/d}(z)$ consists of all the positive numbers which are an argument for z and are smaller than $\frac{2\pi\kappa}{d}$, namely

$$\left\{\theta + 2\pi h : h \in \mathbb{Z},\ 0 \leq \theta + 2\pi h \leq \frac{2\pi\kappa}{d}\right\}. \tag{2.1.3}$$

For a polynomial f, we denote by $\mathrm{Arg}_{\kappa/d}(f)$ the multiset produced by the union of all $\mathrm{Arg}_{\kappa/d}(z)$, where z runs over all the roots of f different from 0 and 1, counting their multiplicity. The multiplicity of 0 as a root is denoted by $a(f)$, the degree of the trailing term of f. The root 1 is excluded as we want to count it with half its multiplicity, and we define $\phi_{\kappa/d}(f)$ as the sum of half the multiplicity of 1 as a root of f and $|\mathrm{Arg}_{\kappa/d}(f)|$.

According to the Deligne-Lusztig theory, a group of Lie type G descends from a more general object called a generic group of Lie type, often denoted by $\mathbb{G}$; this is a family of groups of Lie type parametrised by numbers of the form $q = p^s$, where p is prime and $s \geq 1$ is an integer, so we can specialise $\mathbb{G}$ to the prime-power q, and write $G = \mathbb{G}(q)$. We will not focus on any Deligne-Lusztig theory in this thesis, and it is enough to mention that the number of unipotent characters of G are actually determined at the level of $\mathbb{G}$, and in particular a unipotent character of G descends from a more general object called a *generic character* of $\mathbb{G}$, which depends on the type of Dynkin diagram. To a generic unipotent character $\chi \in \mathrm{Uch}(\mathbb{G})$, we can associate a polynomial $f = f_\chi \in \mathbb{Q}[x]$ such that $f(q) = \deg \chi|_q$, and $\chi|_q$ is the character of G descending from the generic χ. We define the *perversity function* as

$$\pi_{\kappa/d}(\chi|_q) := \frac{\kappa}{d}(a(f_\chi) + \deg(f_\chi)) + \phi_{\kappa/d}(f_\chi), \tag{2.1.4}$$

where d is the order of q modulo ℓ, and κ is a positive integer coprime to d. More details about this definition will be given in the section 2.3. Furthermore, the polynomial $f(q)$ is the product of cyclotomic polynomials and a factor of the form aq^N, for $N \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Q}$, and this will make it easier to write an algorithm producing $\pi_{\kappa/d}(\chi|_q)$ given $\chi|_q, \kappa, d$.

As a combination of the map $\pi : \mathrm{Uch}(B_0(G)) \to \mathbb{Z}_{\geq 0}$ that we have just described, and of a fixed bijection between $\mathrm{Uch}(B_0(G))$ and $\mathcal{S}_{B_0(G)}$ we can assume that a perversity function $\pi : \mathcal{S}_{B_0(H)} \to \mathbb{Z}_{\geq 0}$ is now given. Let us see how this is involved in our algorithmic construction. For any $r \in \mathbb{Z}_{\geq 0}$, we define:

$$J_r := \{V \in \mathcal{S}_{B_0(H)} \mid \pi(V) \leq r\}. \tag{2.1.5}$$

Let T be a simple kH-module lying in $B_0(H)$. We now explain how to produce the complex $X_T \in \mathcal{D}(B_0(H))$ which is supposed to be the image of T under a potential perverse derived equivalence.

If $\pi(T) = 0$, then the algorithm will automatically return the complex $X_T : 0 \to T \to 0$. Let us assume now that $n := \pi(T) > 0$. Then we will produce a complex of length $n + 1$ that we will denote by

$$X_T : 0 \to P_n \xrightarrow{\varphi_n} P_{n-1} \xrightarrow{\varphi_{n-1}} \dots \xrightarrow{\varphi_1} P_0 \to 0, \tag{2.1.6}$$

where P_0 is in degree zero, and then P_n in degree $-n$. Before defining each module of the complex, we introduce the notation:

Definition 2.1.2. Let A be an algebra and let $\mathfrak{T}$ be a set of simple A-modules and M an A-module. The $\mathfrak{T}$-*radical* of M is defined as the largest submodule $\mathfrak{T}\text{-rad}(M) \subseteq M$ with composition factors in $\mathfrak{T}$.

Now we can finally build the complex 2.1.6. The first module P_n, of degree $-n$, will be the injective hull of T, so $P_n := I(T)$. We now want to define the map φ_n, and we will start by giving the kernel of it. We define $\ker \varphi_n := M_n$, where M_n is the submodule of P_n such that

$$\frac{M_n}{T} = J_{n-1}\text{-rad}\left(\frac{P_n}{T}\right). \tag{2.1.7}$$

The following term is defined by $P_{n-1} := I\left(\frac{P_n}{M_n}\right)$, with natural map φ_n being the composition of the projection to the quotient and the inclusion in the injective hull. This is just the first step of the inductive process: in general, we set $L_i := \operatorname{Im}(\varphi_i)$ and we get modules M_i such that

$$\frac{M_i}{L_{i+1}} = J_{i-1}\text{-rad}\left(\frac{P_i}{L_{i+1}}\right), \ 2 \leq i \leq n-1. \tag{2.1.8}$$

This allows us to define each module of the complex inductively: we define P_{i-1} and the map φ_i as

$$\begin{aligned} P_{i-1} :=& I\left(\frac{P_i}{M_i}\right), \ i = 3, \dots, n-1, \\ \varphi_i : P_i \twoheadrightarrow & \frac{P_i}{M_i} \hookrightarrow P_{i-1}, \end{aligned} \tag{2.1.9}$$

where the surjective map and the injective map are the natural projection and the natural inclusion in the injective hull respectively. So far, we have defined the construction of our complex X_T up to the degree -2; it remains to define the last two terms P_1 and P_0, namely the final part of X_T:

$$\dots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to 0.$$

The following definition is necessary to describe the condition that P_1 has to satisfy.

Definition 2.1.3. Let M be a kG-module. We say that M is *stacked relatively projective* with respect to a single subgroup Q of G if M admits a filtration by relatively Q-projective modules $\{0\} = M_0 \subseteq \cdots \subseteq M_m = M$ for some $m \in \mathbb{N}$, namely such that M_j/M_{j-1} is relatively Q-projective for each $j = 1, \dots, m$.

In particular, a relatively Q-projective module, and therefore a projective module as well, is a trivial example of stacked relatively projective module. Finally, let us consider a bijection between $\mathcal{S}_{B_0(H)}$ and $\mathcal{S}_{B_0(G)}$, and let S be the $B_0(G)$-module corresponding to T under such bijection. We have to specify that this bijection is usually defined via trial and error, under the criterion that it must fulfil the requirements that we are going to mention. In particular, this will be the bijection which is carried in the definition of the perverse equivalence. Given a bijection, in order to declare the algorithm successful, we request that P_1 and P_0 fulfil the following two conditions:

- P_0 must be a copy of C_S, the Green correspondent of S;
- each indecomposable summand of P_1 must be stacked relatively projective with respect to some proper subgroup $Q \subset P$, which will be a cyclic group of order ℓ, or the trivial subgroup in case of projective summand.

Moreover, the conditions on the cohomology which are imposed by the perverse equivalence, and which are implemented by the relation 2.1.8, must also hold. In order to fulfil this cohomology condition, we build M_1 in the same was as each previous kernel $M_2, M_3, \dots, M_n$, so as a submodule of $I\left(\frac{P_2}{M_2}\right)$; however, rather than defining P_1 as $I\left(\frac{P_2}{M_2}\right)$, we try to build it as an extension of C_S by M_1 whose summands satisfy the second condition above. The last module P_0 is defined as

$$P_0 = \frac{P_1}{M_1} \cong C_S, \tag{2.1.10}$$

where the isomorphism to C_S holds by construction of P_1; this would fulfil the requested condition on P_0. The map $\varphi_1 : P_1 \to P_0$ is the natural projection to the quotient.

Remark 2.1.4. The crucial stage of this algorithm is about the construction of P_1. The construction of all the previous terms $P_2, \dots, P_n$ is determined by an iterative process, whereas the construction of P_1 is subject to the existence of a non-trivial peculiar extension of C_S by M_1. The existence of such an extension is basically determining whether the algorithm is working with the given datum of π and with the chosen bijection between $\mathcal{S}_{B_0(H)}$ and $\mathcal{S}_{B_0(G)}$.

An algorithm to test whether the module P_1 admits a filtration by a given list of modules is provided. In order for P_1 to be filtered by modules with vertex Q, it is always sufficient to consider the list of modules with vertex Q and trivial sources for all the cases considered in the dissertation.

This concludes the technical explanation of what the procedure behind `PerverseEq` consists of. This procedure is based on a result of the theory of perverse equivalences in the setting of Broué's abelian defect group conjecture developed by Rouquier and Chuang; the result in more generality can be found in [11]. For our purposes, we can summarise it as follows:

Proposition 2.1.5. (Rouquier, [11]) *Let G be a finite group, ℓ a prime number, and $H := N_G(P)$, where $P \in \mathrm{Syl}_\ell(G)$. Let C be a bounded complex of $(B_0(H), B_0(G))$-bimodules which are finitely generated and projective when regarded as left $B_0(H)$-modules and right $B_0(G)$-modules. Let us assume that:*

- *the standard functor $L := C \otimes_{B_0(G)} - : \mathcal{D}(B_0(G)) \to \mathcal{D}(B_0(H))$ induces a stable equivalence $\bar{L} : \underline{\mathrm{mod}}(B_0(G)) \xrightarrow{\sim} \underline{\mathrm{mod}}(B_0(H))$;*
- *there is a perversity function π and there is a bijection between $\mathcal{S}_{B_0(G)}$ and $\mathcal{S}_{B_0(H)}$ such that for each $T \in \mathcal{S}_{B_0(H)}$, the complex X_T fulfil the two conditions that make the algorithm* `PerverseEq` *successful;*
- *each X_T is stably isomorphic to $L(S)$, where $T \in \mathcal{S}_{B_0(H)}$ and $S \in \mathcal{S}_{B_0(G)}$ correspond under the bijection introduced above.*

If those three conditions hold, then there is a derived equivalence between $B_0(G)$ and $B_0(H)$, and therefore Broué's conjecture holds for the principal ℓ-block of G. In particular, this derived equivalence between $B_0(G)$ and $B_0(H)$ induces $\bar{L}$ as a stable equivalence, and if we regard S as a complex concentrated in degree zero, X_T is the image of S under such derived equivalence.

We have defined our algorithmic construction of the set of complexes $\{X_T | T \in \mathcal{S}_{B_0(H)}\}$ which are the image of a perverse equivalence (provided that they fulfil the condition of Proposition 2.1.5). We conclude this section by mentioning a property of the cohomology $H(X_i)$ of each complex; this property is explained in [12]. Let us fix a simple $B_0(H)$-module T_i, and let $X_i := X_{T_i}$ be the complex generated by our algorithm. We consider the following virtual module:

$$\bigoplus_{j=0}^{\pi(T_i)} \left(\bigoplus_{T \in \mathrm{cpf}(H^{-j}(X_i))} (-1)^{j-\pi(T)} T \right). \tag{2.1.11}$$

Following [12], this is called the *alternating sum of cohomology*; we explain how this virtual module can be used to reconstruct the unitriangular form of the decomposition matrix that we have been using to define the bijection between a subset of irreducible character of G lying in $B_0(G)$ (if G is of Lie type, this is the set of unipotent characters) and $\mathcal{S}_{B_0(G)}$. For a simple module $T_m \in \mathcal{S}_{B_0(H)}$, we denote by a_m its multiplicity into the alternating sum of cohomology; in particular, we notice that due to the construction of X_i (which is coming from the definition of perverse equivalence), each module T_m appearing in the alternating sum, i.e. $a_m > 0$, is such that $\pi(T_m) \leq \pi(T_i)$, and the equality occurs when $m = i$. In the following we denote by r_j the vector consisting of the j-th row of the fixed unitriangular decomposition matrix, and by v_j the vector consisting of 0 in each entry, except for the j-th entry, which is 1. These vectors have length $|\mathcal{S}_{B_0(H)}|$, as each row of the decomposition matrix. The numbers a_m fulfil the following conditions:

$$v_i = \sum_{\substack{m: \\ \pi(T_m) \leq \pi(T_i)}} a_m \cdot r_m = a_i \cdot r_i + \sum_{\substack{m: \\ \pi(T_m) < \pi(T_i)}} a_m \cdot r_m. \tag{2.1.12}$$

In particular, the rows of the decomposition matrix that we are considering have been ordered according to the π-value of each irreducible character, therefore each row r_m such that $\pi(T_m) < \pi(T_i)$ comes before r_i; for example, the row of the trivial character 1_G, whose π-value is 0, is always at the top of the matrix. The relations 2.1.12 show that we can reconstruct the unitriangular decomposition matrix inductively: assume that we already have each row r_m such that $\pi(T_m) < \pi(T_i)$, then the alternating sum of cohomology would provide the numbers a_m for $m = 1, \ldots, i$ and therefore we can compute the next row r_i of the decomposition matrix.

In all our examples (for instance, see the table 3.1.2 for the case $G = \Omega_8^+(2)$) , we will report the data coming from the alternating sum of cohomology under the label "total", and using the formal expression

$$\sum_{\substack{m: \\ \pi(T_m) \leq \pi(T_i)}} a_m \cdot m$$

instead of the vector notation with r_m. Typically, we will almost always find that our coefficients a_m are 1 or -1.

2.2 Lifting stable equivalences to derived equivalences

This section is meant to explain how the third condition of Proposition 2.1.5 can be checked; therefore, we will explain you the objects $L(S)$, for all $S \in \mathcal{S}_{B_0(G)}$ can be constructed in Magma in order to test the stable isomorphism between them and each X_T previously constructed via

PerverseEq. By Proposition 2.1.5, this will be enough to deduce the existence of a derived equivalence between $B_0(G)$ and $B_0(H)$. We will always assume that our ℓ-block $B_0(G)$ has defect group $P \cong C_\ell \times C_\ell$.

A stable equivalence $\bar{L}$ that will be used to apply Proposition 2.1.5 has been described by Rouquier in [36]. In the following, we will recall this particular stable equivalence and we will implement the construction of each object $L(S)$, for all $S \in \mathcal{S}_{B_0(G)}$.

For an ℓ-subgroup Q of G, we recall that the Brauer map

$$\mathrm{Br}_Q : \mathrm{Mod}(kG) \to \mathrm{Mod}(kN_G(Q))$$

is defined on the objects as $\mathrm{Br}_Q : M \mapsto M^Q/(\sum_{R<Q} \mathrm{Tr}_R^Q M^R)$, where M^R is the set of points fixed by R and the trace map $\mathrm{Tr}_R^Q : M^R \to M^Q$ is defined as $\mathrm{Tr}_R^Q(m) = \sum_{g \in Q/R} gm$, $\forall m \in M^R$. The result connecting a derived equivalence at local level and a stable equivalence at the global level is:

Proposition 2.2.1. (Rouquier, [36]) *Let C be a splendid tilting complex of $(B_0(H), B_0(G))$-bimodules. The following two are equivalent:*

- *$C \otimes_{B_0(G)} -$ is a splendid stable equivalence between $B_0(G)$ and $B_0(H)$;*
- *for every non-trivial subgroup Q of P, then $\mathrm{Br}_{\Delta Q}(C)$ induces a splendid derived equivalence between $B_0(C_G(Q))$ and $B_0(C_H(Q))$,*

where Br *is the Brauer map extended to complexes of modules.*

Still in [36], Rouquier applies this result to build a complex C of $(B_0(H), B_0(G))$-bimodules inducing a stable equivalence whenever $P \cong C_\ell \times C_\ell$. The strategy consists of building complexes of $kN_{H\times G}(\Delta Q)$-modules such that the restriction to $C_H(Q) \times C_G(Q)$, seen as a $(kC_H(Q), kC_G(Q))$-bimodule, induces a derived equivalence between $B_0(C_G(Q))$ and $B_0(C_H(Q))$, for each conjugacy class of non-trivial $Q < P$. In particular, it is crucial here to remind that the construction of such modules relies on the knowledge of the derived equivalence when the defect group is cyclic (Broué's conjecture is known in such a case).

In particular, this strategy applies favourably for $\ell = 2$, and it is used to prove Broué's conjecture in general when $P \cong C_2 \times C_2$ (again, in [36]). When ℓ is odd (in our cases $\ell = 5, 7$), such a general result to lift the stable equivalence induced by C to a derived equivalence does not work, but the construction of the stable equivalence still holds and lifting this particular stable equivalence to a derived equivalence can be done case by case. More details about the actual construction can be found in [36].

In our computational setting, we mostly care about the image in $\mathcal{D}(B_0(H))$ of the stable equivalence induced by C; in particular, we will compute the images of each simple module $L \in \mathcal{S}_{B_0(G)}$ under such stable equivalence and we will compare it with the output of the perverse equivalence algorithm corresponding to L (namely, under the image of L under the supposed perverse equivalence). Together with the construction of C in [36], in [11] we have an explicit construction of what the image $C \otimes_{kG} L$ is as a complex of kH-modules: this is the object that we need to know, and that we will physically build in our algorithm `FinalStabEq`. More precisely, this complex has length 2; the module in degree 0 being the Green correspondent of L (that we already have, in most cases), the algorithm will actually build the module in degree -1. Therefore, now we can focus on how the construction of each image $C \otimes_{kG} L$ is performed; for the details of C as a complex of (kH, kG)-bimodules we refer to [36] and [11].

2.2.1 The algorithm `FinalStabEq`

Let us fix a subgroup $Q < P$ of order ℓ. We are still assuming that $P \cong C_\ell \times C_\ell$. In the following, we will assume that there exist $\bar{N}_G(Q)$ and $\bar{N}_H(Q)$ which denote complements of Q in $N_G(Q)$ and $N_H(Q)$ respectively; those complements exist for each case that we will consider. In particular, we choose them such that $\bar{N}_H(Q) \leq \bar{N}_G(Q)$. Finally, let us consider $\bar{C}_H(Q) = C_H(Q) \cap \bar{N}_H(Q)$ and $\bar{C}_G(Q) = C_G(Q) \cap \bar{N}_G(Q)$; therefore, both $\bar{C}_G(Q)$ and $\bar{C}_H(Q)$ have a cyclic Sylow ℓ-subgroup, and then we have a derived equivalence between their principal blocks (Broué's Conjecture holds when the defect group is cyclic).

We set $N_\Delta := (\bar{C}_H(Q) \times \bar{C}_G(Q)^{\text{opp}})\Delta\bar{N}_H(Q)$; this group acts on $\bar{C}_G(Q)$ and then we can consider $k\bar{C}_G(Q)$ as a kN_Δ-module as well as a $\bar{C}_H(Q) \times \bar{C}_G(Q)^{\text{opp}}$-module. In particular, as kN_Δ-module we have that

$$e_{\bar{C}_H(Q)} k\bar{C}_G(Q) e_{\bar{C}_G(Q)} = M_Q \oplus P, \tag{2.2.1}$$

where M_Q is indecomposable as a $\bar{C}_H(Q) \times \bar{C}_G(Q)^{\text{opp}}$-module and induces a stable equivalence (Rouquier, [36] or [11]), whereas P is projective. We have a precise description of what a projective cover of M_Q is isomorphic to. We consider the map $\gamma : \mathcal{S}_{B_0(\bar{C}_G(Q))} \to \mathcal{S}_{B_0(\bar{C}_H(Q))}$, where $\gamma(V)$ is defined to be the unique simple $B_0(\bar{C}_H(Q))$-module such that

$$\underline{\mathrm{Hom}}(V_{\bar{C}_H(Q)}, \gamma(V)) \neq \{0\}, \quad V \in \mathcal{S}_{B_0(\bar{C}_G(Q))}.$$

A projective cover of M_Q as a $\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}$-module is of the form

$$\mathcal{P}(M_Q) \cong \bigoplus_{V \in \mathcal{S}_{B_0(\bar{C}_G(Q))}} \mathcal{P}(\gamma(V)) \otimes \mathcal{P}(V)^*, \tag{2.2.2}$$

where each summand $\mathcal{P}(\gamma(V)) \otimes \mathcal{P}(V)^*$ gains the natural structure of $\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}$-module. Finally, we define the subset $\mathcal{E} \subseteq \mathcal{S}_{B_0(\bar{C}_G(Q))}$ of all modules whose corresponding edge in the Brauer tree of $B_0(\bar{C}_G(Q))$ has distance $d+1 \pmod 2$ from the exceptional vertex, where d is the distance of the trivial module from the exceptional vertex. In other words, depending on d, we consider the set of edges whose distance from the exceptional node is even or odd (a definition of the Brauer tree can be found in [1]). We now define $U_Q := \bigoplus_{V \in \mathcal{E}} \mathcal{P}(\gamma(V)) \otimes \mathcal{P}(V)^*$; again by [11], it is possible to extend the action of $\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}$ up to N_Δ; with an abuse of notation, we will see U_Q as a N_Δ-module from now on. We define $T_Q := U_Q \oplus P$, where P is the projective N_Δ-module appearing in (2.2.1).

Remark 2.2.2. The module M_Q is what we are building in our algorithm `StableEqSetup`, together with all the necessary groups and subgroups involved, such as $\bar{C}_H(Q), \bar{C}_G(Q), N_\Delta$; the module T_Q is built manually case by case, since the construction depends on the Brauer tree of $\bar{C}_G(Q)$; T_Q will be given as an input to the algorithm `FinalStabEq`.

It remains to explain how to use these objects to get the complex of kH-modules $C \otimes_{kG} L$. The tensor $T_Q \otimes_{k\bar{C}_G(Q)} L_{N_G(Q)}$ gains the structure of $N_\Delta \times N_G(Q)$-module, and in particular we will regard it as a $N_H(Q)$-module: the copy of $N_H(Q)$ inside $N_\Delta \times N_G(Q)$ that we consider is defined by the bijection $h \to ((\bar{h}, \bar{h}^{-1}), h)$, where $\bar{h}$ is the image of h in $\bar{N}_H(Q)$; in our algorithm, $\bar{N}_H(Q)$ is constructed as a subgroup of $N_H(Q)$ such that $N_H(Q) = Q \rtimes \bar{N}_H(Q)$ rather than as a quotient, and therefore $\bar{h}$ will have to be defined as the element such that $h \cdot \bar{h}^{-1} \in Q$. Regarding $T_Q \otimes_{k\bar{C}_G(Q)} L_{N_G(Q)}$ as a $N_H(Q)$-module, we finally have the expression for $C \otimes_{kG} L$:

$$C \otimes_{kG} L \cong (0 \to e_H \bigoplus_{Q<P} (T_Q \otimes_{k\bar{C}_G(Q)} L_{N_G(Q)})^H \to e_H L_H \to 0), \tag{2.2.3}$$

where Q runs over all the H-conjugacy classes of subgroup of order ℓ.

Remark 2.2.3. In the construction of $C \otimes_{kG} L$ above, as an object in the stable category the module in degree 0 consists of the Green correspondent of L together with the relatively projective summands occurring in the correspondence. Again in [11], it is possible to construct a complex C' which is homotopy equivalent to C and such that $C' \otimes_{kG} L$ has the Green correspondent only as a term of degree 0. In particular, the term of degree -1 in C' is constructed as for the one of C, but T_Q is replaced by U_Q.

Remark 2.2.4. The following construction of the objects $C \otimes_{kG} L$ that we will give in 2.2.3 has been implemented in each case considered. The method of constructing $C \otimes_{kG} L$ that we have just explained has proved to be successful as long as each complex X_T for $T \in S_{B_0(H)}$ that is produced by `PerverseEq` has the property that the module X_T^{-1} in degree -1 is a sum of modules that are projective or have vertex Q, for some proper subgroup $Q \leq P$. In general, this is not true: when dealing with individual groups, will see that in general X_T^{-1} is a sum of projective, relatively Q-projective and *stacked* relatively projective modules, and whenever this last type of summand occur, the result for $C \otimes_{kG} L$ given by the algorithm `FinalStabEq` is not the one which would allow us to conclude by applying Proposition 2.1.5. This has not allowed us to complete the proof of Broué's conjecture for $G = {}^2F_4(2).2, Sp_8(2), {}^3D_4(2)$ yet. Work in order to fix this result and make it produce the right stable equivalence is in progress at the time of submission.

2.3 Geometric Broué's conjecture and perversity functions

Some of the groups that we will examine are of Lie type, and a particular remark is necessary. Although we will not provide a deep report of the current theory behind it, we can mention how the search of a perverse equivalence intersects with some underlying geometry of the group, represented by their Deligne-Lusztig varieties. In our setting, this connection (still conjectural in large part) is related to the crucial choice of the perversity function π, for which we have introduced a precise expression 2.1.4 to use in our algorithm. Therefore, in this section $G = \mathbb{G}(q)$ will be a group of Lie type, for some generic group of Lie type $\mathbb{G}$ and some p-power q, where $p \neq \ell$ is a prime; moreover, the facts that we state here are generally valid for *unipotent* ℓ-blocks B of G. We recall that B is a unipotent block if there is a unipotent character lying in B. As the trivial character is unipotent, the principal block $B_0(G)$ is a unipotent block.

Fact 2.3.1. Let B be a unipotent ℓ-block of G, where G is a group of Lie type, $\xi := e^{2\pi\kappa i/d}$ a primitive d-root of unity for $d \in \mathbb{N}$. For each $\kappa \geq 1$ coprime to d, we can define a variety

$Y_\xi = Y_{\kappa/d}$, that we call *Deligne-Lusztig variety*, and then a complex of cohomology $H^\bullet(Y_{\kappa/d}, \bar{\mathbb{Q}}_\ell)$. This variety depends on B.

In this setting, we have to think that an ℓ modular system $(\mathcal{O}, K, k)$ has been fixed and $\mathcal{O}$ is an extension of $\mathbb{Z}_\ell$ large enough such that $K = \bar{\mathbb{Q}}_\ell$ and $k = \bar{\mathbb{F}}_\ell$. Let D be the defect group of the block B. Let d be the multiplicative order of q modulo ℓ, and κ a positive integer prime to d. The complex $H^\bullet(Y_{\kappa/d}, \bar{\mathbb{Q}}_\ell)$ of $\bar{\mathbb{Q}}_\ell G$-modules arises from a complex defined over $\mathcal{O}$, which produces a complex C over k as well, by reducing modulo $J(\mathcal{O})$. The complex C is the central object of the geometric form of Broué's conjecture. What we know is that it carries an action of G on the right, and an action of D on the left; it is conjectured that this action can always be extended to $N_G(D)$, and that as a complex of $(kN_G(D), kG)$-bimodules it is inducing a derived equivalence:

Conjecture 2.3.2. (Geometric Broué's abelian defect conjecture - 1988) *Let d be the order of q modulo ℓ. There exists κ such that the complex $H^\bullet(Y_{\kappa/d}, \bar{\mathbb{Q}}_\ell)$ gives rise to a complex of $(kN_G(D), kG)$-bimodules which induces a derived equivalence between B and its Brauer correspondent b. Moreover, this equivalence is perverse.*

Some cases of this conjecture are known (Dudas, Rouquier). For our computational approach, the object $H^\bullet(Y_{\kappa/d}, \bar{\mathbb{Q}}_\ell)$ is too hard to manage, and so the perverse equivalence that is conjecturally induced must be searched via a different direction. As a complex, $H^\bullet(Y_{\kappa/d}, \bar{\mathbb{Q}}_\ell)$ is predicted to fulfil the following property:

Conjecture 2.3.3. *Let χ a unipotent character of $\bar{\mathbb{Q}}_\ell G$. The complex $H^\bullet(Y_{\kappa/d}, \bar{\mathbb{Q}}_\ell)$ has **unique** degree in which χ appears. This defines a function $\pi_{\kappa/d} : \mathrm{Uch}(B_0(G)) \to \mathbb{Z}_{\geq 0}$, where $\pi_{\kappa/d}(\chi)$ is such degree.*

As we have explained in the previous section, a map $\pi_{\kappa/d} : \mathrm{Uch}(B_0(G)) \to \mathbb{Z}_{\geq 0}$ can be regarded as a map $\pi_{\kappa/d} : \mathcal{S}_{B_0(H)} \to \mathbb{Z}_{\geq 0}$ using a unitriangular form of the decomposition matrix of $B_0(G)$. This is supposed to be the perversity function that characterises the conjectured perverse equivalence:

Conjecture 2.3.4. *The function $\pi_{\kappa/d} : \mathcal{S}_{B_0(H)} \to \mathbb{Z}_{\geq 0}$ descending from the unitriangular form of the decomposition matrix of $B_0(G)$ together with the map of Conjecture 2.3.3 is the perversity function of the perverse equivalence induced by the complex C.*

This conjecture gives the precise source of the perversity function providing the perversity equivalence that we rely on, but still there is no way to find it algorithmically, as we are still supposed to pass through $H^\bullet(Y_{\kappa/d}, \bar{\mathbb{Q}}_\ell)$. The decisive fact is that, conjecturally, we indeed have a relatively simple formula for $\pi_{\kappa/d}$, and this is the same formula that we used to define our algorithm:

Conjecture 2.3.5. (Craven - 2012) *Let $\chi \in \mathrm{Uch}(B_0(G))$ and let $f = f_\chi$ be its degree polynomial. The perversity function from Conjecture 2.3.3 is:*

$$\pi_{\kappa/d}(\chi) = \frac{\kappa}{d}(a(f) + \deg(f)) + \phi_{\kappa/d}(f),$$

where $a(f)$ is the multiplicity of the root $q = 0$, and $\phi_{\kappa/d}(f)$ is a number depending on the remaining roots of f.

This conjectured result would provide a viable way to get our perversity function $\pi_{\kappa/d}$ that our algorithm strongly relies on. Moreover, it is relatively easy to find the list of degree polynomials related to the set of unipotent characters of a fixed block (the principal) of a fixed group of Lie type; finding the value of $a(f)$ and $\phi_{\kappa/d}(f)$ is also easy. It is worth remarking that some of the ground where our algorithm has taken roots is still at a conjectural level; still, there is no reason why we cannot try to use this conjectural data as an input for our algorithm, and as we will see in the following, this choice for our input has always provided the expected result for each group of Lie type that we have considered. In the next section we carry on explaining another computational aspect of this work, namely how we have dealt with the construction of the $B_0(G)$-modules, which are necessary for the implementation of the algorithm.

2.4 Building $B_0(G)$-modules in Magma

Almost all the Magma computations are carried out over the field $\mathbb{F}_\ell$, although on some occasions it could be necessary to extend our modules to $\mathbb{F}_{\ell^2}$: for instance this would happen when a certain module that we need does not exists as a $\mathbb{F}_\ell G$-module, but it does as a $\mathbb{F}_{\ell^2} G$-module. The Magma commands which manage the change of fields are

```
ChangeRing;

IsRealisableOverSmallerField;
```

to extend or reduce the base field respectively.

The group that we examine in each section will be denoted by G, and an ℓ-Sylow subgroup of G will be denoted by P, and its normaliser $N_G(P)$ by H.

In order to carry out the algorithm `PerverseEq`, the objects that we actually need are reasonably small: a copy of each irreducible $B_0(H)$-module and a copy of the Green correspondent of each irreducible $B_0(G)$-module. Although it is almost immediate to get a list of simple modules for the normaliser H, it can be hard to get a copy of the Green correspondent of $B_0(G)$-modules with high dimension: indeed, the Green correspondents are constructed via each simple $B_0(G)$-module, which is restricted down to H, and then decomposed (eventually, after getting rid of all free and projective summands). For high-dimension $B_0(G)$-modules, these steps - especially getting a copy of the $B_0(G)$-module and restricting it - could turn out to be hard and finally not doable. In each section we give a very short description of how a copy of each simple $B_0(G)$-modules was obtained. For each group G that we have considered, a complete list of how the simple kG-modules split among all blocks is available online at The Modular Atlas website [38]. This is a great advantage, since this has allowed us to know which modules we should look for from the beginning, precisely all those lying in the principal block. The Modular Atlas is the source which provided all the decomposition matrices that we will mention in the following. We have also made use of the Atlas of Finite Group Representations - Version 3 [3]. This provides informations such as the generators of our groups, the generators of some small representations, and information about the maximal subgroups of a group.

The most straightforward way to get simple modules consists of applying the command `CompositionFactors` on some modules. Different modules can be easily produced by tensoring small modules, as well as inducing modules from subgroups with reasonable index.

This search can be guided using the ordinary character table together with the decomposition matrix. Using the notation of [32], let us assume that we want to build a simple kG-module U and this is afforded by the irreducible Brauer character φ_U. Our aim is to find two simple kG-modules M, N such that $U \in \mathrm{cpf}(M \otimes N)$. By looking at the decomposition matrix, we can find an ordinary irreducible character (usually we try to consider the smallest) $\alpha_U \in \mathrm{Irr}(G)$ such that α_U^0 contains φ_U in its decomposition. Now assume that we found $\chi, \psi \in \mathrm{Irr}(G)$ such that $\chi \cdot \psi = \alpha_U + \beta$ for some complement β; χ and ψ will be in general much smaller than α_U. If we restrict this expression to the set of ℓ-regular elements G^0, we can write that $\chi^0 \cdot \psi^0 = \varphi_U + \gamma$ for some Brauer character complement γ (which includes β^0 inside). Now we just have to use the decomposition matrix to decompose χ^0, ψ^0 into irreducible Brauer characters: assuming that such decompositions are $\chi^0 = \mu_1 + \cdots + \mu_m$ and $\psi^0 = \nu_1 + \cdots + \nu_n$, for positive integers m, n, then

$$\sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \mu_i \cdot \nu_j = \varphi_U + \gamma. \tag{2.4.1}$$

Under a module-theoretic point of view, this means that U is a composition factor of a tensor $M_i \otimes N_j$ for some i, j, where M_i is the simple kG-module afforded by μ_i and N_j is afforded by ν_j. All in all, this method only needs to find two irreducible characters χ, ϕ such that $\langle \chi \cdot \psi, \alpha_U \rangle \neq 0$, and to check the decomposition matrix.

A different elementary method to build simple kG-modules uses modules induced from subgroups with small index. Let R be a subgroup of G of small index - a maximal one for example. We assume again that we want to build the simple kG-module U, and we aim to find it as a composition factor of some M^G, where M is a simple kR-module. M can be found using a parallel check of the ordinary character table as follows: we keep searching for an ordinary character $\chi \in \mathrm{Irr}(R)$ such that $\chi^G = \alpha_U + \beta$, where α_U is as above and β is some complement. Again, restricting to ℓ-regular elements, $(\chi^G)^0 = \varphi_U + \gamma$ for some γ. On the other hand, χ^0 admits a decomposition of elements in $\mathcal{S}_{kR}$, say $\chi^0 = \mu_1 + \cdots + \mu_m$, $m \geq 1$, and since $(\chi^G)^0 = (\chi^0)^G$ then $\varphi_U + \gamma = \mu_1^G + \cdots + \mu_m^G$. We conclude that U can be found among the composition factors of some M_i^G, where M_i is the kR-module afforded by μ_i. In the end, again we only need the character tables and decomposition matrices of G and R.

2.5 Results

In the next two chapters we present the list of groups that have been examined. For each group, there are two algorithms that we have to run: `PerverseEq` and `FinalStabEq`. The first algorithm is meant to produce suitable complexes $X_i \in \mathcal{D}(B_0(H))$ fulfilling the condition of being the image under a perverse equivalence $\mathcal{D}(B_0(G)) \to \mathcal{D}(B_0(H))$ for each $S_i \in \mathcal{S}_{B_0(G)}$. The second algorithm is necessary to check that this conjectural perverse equivalence is lifting a known stable equivalence between $\underline{\mathrm{mod}}(B_0(G))$ and $\underline{\mathrm{mod}}(B_0(H))$.

It has been possible to run `PerverseEq` successfully for some of them; for others, a "partial perverse equivalence" was achieved, where a partial equivalence stands for a partial list of complexes which actually works, namely we do not have a complex for some simple $B_0(H)$-

module T. We summarise the result of what was achieved with this list (every result is referred to the principal 5-block of each group, except for ${}^3D_4(2)$ where we consider the principal 7-block):

- Both our algorithms work well for $G = \Omega_8^+(2)$, and therefore Broué's conjecture holds in this case;
- For $G = Sp_8(2), {}^2F_4(2)'.2, {}^3D_4(2)$, `PerverseEq` works well and successfully returns a complete lists of complexes. However, the current version of the method in [11] that `FinalStabEq` is based on to provide a stable equivalence to lift to our perverse equivalence is not working in those three cases (this will be explained more clearly in each specific section). A generalisation of this method is the next step that we need in order to claim the conjecture for those three cases;
- We are able to prove that no perverse equivalence exists when we examine the case $G = J_2$;
- For the sporadic He, the method to produce a valid stable equivalence applies and the algorithm has run successfully. As for the perverse equivalence, we currently have a partial result consisting of seven valid complexes out of ten. Completing the perverse equivalence is work in progress at the time of submission;
- Obtaining a full perverse equivalence for Suz, Fi_{22}, Fi_{23} is prevented by the difficulty of getting the Green correspondents of each irreducible $B_0(G)$-module, due to computational reasons. For Suz and Fi_{22}, partial perverse equivalences have been found, and they might potentially be completed to a perverse equivalence as soon as we are able to go beyond the current computational strength of our computers.

CHAPTER 3

PERVERSE EQUIVALENCES FOR GROUPS OF LIE TYPE

In this chapter, we run the algorithm searching for perverse equivalences on the principal block of some groups of Lie type (all simple except for ${}^2F_4(2)'.2$), and then we compare the output with the image of a particular stable equivalence; eventually, this leads to the proof of Broué's conjecture for $G = \Omega_8^+(2)$ and partial results for other three cases (which are work in progress at the time of submission).

3.1 $\Omega_8^+(2)$

Let $G := \Omega_8^+(2)$, a Chevalley simple group of type D_n, for $n = 4$ and $q = 2$. In this section we consider $k := \bar{\mathbb{F}}_5$. We will show that Broué's conjecture holds for the principal block of G by giving a perverse (and therefore derived) equivalence between $B_0(G)$ and $B_0(H)$. We prove that:

Theorem 3.1.1. *Broué's abelian defect group conjecture holds for the principal* 5*-block of* $\Omega_8^+(2)$.

3.1.1 Structure of H

So let $G := \Omega_8^+(2)$; the perversity function $\pi_{\kappa/d}$ will be computed by using $\kappa = 1$, and d must be the order of 2 modulo 5, therefore $d = 4$. We denote by $P \cong C_5 \times C_5$ a Sylow 5-subgroup of G, and $H := N_G(P)$; then $|H| = 400$ and in particular $H \cong P \rtimes S$, where S can be defined via the following presentation:

$$S \cong \{x, y, z | x^4 = y^2 = e, x^2 = z^2, yxy = x^{-1}, xz = zx, yz = zy\}. \tag{3.1.1}$$

There are three conjugacy classes of subgroups of H of order 5, represented by Q_1, Q_2, Q_3.

3.1.2 Irreducible $B_0(H)$ and $B_0(G)$-modules

The following decomposition matrix, as well as the list of simple $B_0(G)$-modules, have been obtained from the Modular Atlas [38]. In this section we will use the chosen labelling for the simple $B_0(G)$-modules S_i:

$$\begin{array}{cc} S_1 = 1_1 & S_6 = 539_2 \\ S_2 = 83_1 & S_7 = 539_3 \\ S_3 = 83_2 & S_8 = 1729_1 \\ S_4 = 83_3 & S_9 = 722_1 \\ S_5 = 539_1 & S_{10} = 28_1 \end{array}$$

The decomposition matrix (in a uni-triangular shape), together with π-values for each unipotent character, is:

$B_0(G)$, $G = \Omega_8^+(2)$, $\ell = 5$											
$\pi_{1/4}$	Unipotent Char	S_1	S_2	S_3	S_4	S_9	S_{10}	S_5	S_6	S_7	S_8
0	1_1	1									
3	84_1	1	1								
3	84_2	1		1							
3	84_3	1			1						
4	972_1	1	1	1	1	1					
5	28_1						1				
5	1344_1		1			1		1			
5	1344_2			1		1			1		
5	1344_3				1	1				1	
6	4096_1					1	1	1	1	1	1

The algebra $kH = B_0(H)$ has 10 simple modules, all absolutely simple, eight of them of dimension 1 and two of dimension 2. We now give a labelling such that $T_1, \ldots, T_8$ have dimension 1 and T_9, T_{10} have dimension 2; T_1 denotes the trivial module. When writing the socle structure of a module, T_i is abbreviated to i.

We denote by T_9 the 2-dimensional simple module appearing as second socle factor of $\mathcal{P}(1)$, the projective cover of T_1, and T_{10} the other 2-dimensional module, which is the dual of T_9.

We define T_8 as the exterior square of T_9 (or of T_{10}, they are isomorphic). The third socle factor of $\mathcal{P}(1)$ is the sum of three simple one-dimensional modules, which will be denoted by T_2, T_3, T_4; these three modules are permuted by $\mathrm{Out}(H) \cong S_3$. It remains to define T_5, T_6, T_7: they appear as composition factors in the fifth socle factor of $\mathcal{P}(1)$, and in particular we set $T_5 := T_4 \otimes T_3$, $T_6 := T_2 \otimes T_4$, $T_7 := T_3 \otimes T_2$; therefore, once we have distinguished T_2, T_3, T_4, we have distinguished T_5, T_6, T_7 as well. Summarising, $\mathcal{P}(1)$ is:

$$\mathcal{P}(1) = \begin{array}{c} 1 \\ 10 \\ 2\ 3\ 4 \\ 9\ 9 \\ 1\ 1\ 5\ 6\ 7 \\ 10\ 10 \\ 2\ 3\ 4 \\ 9 \\ 1 \end{array}$$

The three conjugacy classes of subgroup of order 5 can be labelled by Q_1, Q_2, Q_3 by looking at some relatively projective modules appearing as summands of the term of degree -1 of some complex: in particular there are three modules, that we denote by R_1, R_2, R_3, which are summands of $\mathrm{Ind}_{Q_1}^H k$, $\mathrm{Ind}_{Q_2}^H k$, $\mathrm{Ind}_{Q_3}^H k$ respectively and they appear in the complexes X_2, X_3, X_4 and X_5, X_6, X_7. We have $\dim(R_i) = 10, i = 1, 2, 3$ and their structure is:

$$R_1 = \begin{array}{c} 10 \\ 3\ 4 \\ 9 \\ 1\ 5 \\ 10 \end{array} \qquad R_2 = \begin{array}{c} 10 \\ 2\ 4 \\ 9 \\ 1\ 6 \\ 10 \end{array} \qquad R_3 = \begin{array}{c} 10 \\ 2\ 3 \\ 9 \\ 1\ 7 \\ 10 \end{array}$$

By labelling the simple $B_0(H)$-modules, we are able to distinguish those three modules, and therefore this distinguishes the three conjugacy classes Q_1, Q_2, Q_3.

Also, when running the algorithm, we will see that in the complexes $X_5, \ldots, X_{10}$, the modules U_1, U_2, U_3 (or their duals) appear: each U_i has vertex Q_i and has dimension 30. Their structure is:

$$U_1 = \begin{array}{c} 6\ 7 \\ 10\ 10 \\ 2\ 2\ 3\ 4\ 8\ 8 \\ 9\ 9\ 9 \\ 1\ 5\ 6\ 6\ 7\ 7 \\ 10\ 10 \\ 2\ 8 \end{array} \quad U_2 = \begin{array}{c} 5\ 7 \\ 10\ 10 \\ 2\ 3\ 3\ 4\ 8\ 8 \\ 9\ 9\ 9 \\ 1\ 5\ 5\ 6\ 7\ 7 \\ 10\ 10 \\ 3\ 8 \end{array} \quad U_3 = \begin{array}{c} 5\ 6 \\ 10\ 10 \\ 2\ 3\ 4\ 4\ 8\ 8 \\ 9\ 9\ 9 \\ 1\ 5\ 5\ 6\ 6\ 7 \\ 10\ 10 \\ 4\ 8 \end{array}$$

Each U_i has source of dimension 3.

3.1.3 Green correspondents

The Green correspondent of each S_i is denoted by C_i. We have:

$$C_2 = \begin{array}{c} 5 \\ 10 \\ 3\ 4 \\ 9 \\ 5 \end{array} \quad C_3 = \begin{array}{c} 6 \\ 10 \\ 2\ 4 \\ 9 \\ 6 \end{array} \quad C_4 = \begin{array}{c} 7 \\ 10 \\ 2\ 3 \\ 9 \\ 7 \end{array}$$

$$C_5 = \begin{array}{c} 8\ 9 \\ 5\ 6\ 7\ 9 \\ 1\ 5\ 5\ 6\ 7\ 10\ 10 \\ 2\ 3\ 4\ 8\ 8\ 10\ 10\ 10\ 10 \\ 2\ 3\ 3\ 4\ 4\ 8\ 8\ 9\ 9 \\ 3\ 4\ 5\ 6\ 7\ 9\ 9\ 9 \\ 1\ 5\ 6\ 7\ 9\ 10 \\ 5\ 8\ 10 \end{array} \quad C_6 = \begin{array}{c} 8\ 9 \\ 5\ 6\ 7\ 9 \\ 1\ 5\ 6\ 6\ 7\ 10\ 10 \\ 2\ 3\ 4\ 8\ 8\ 10\ 10\ 10\ 10 \\ 2\ 2\ 3\ 4\ 4\ 8\ 8\ 9\ 9 \\ 2\ 4\ 5\ 6\ 7\ 9\ 9\ 9 \\ 1\ 5\ 6\ 7\ 9\ 10 \\ 6\ 8\ 10 \end{array}$$

$$C_7 = \begin{array}{c} 8\ 9 \\ 5\ 6\ 7\ 9 \\ 1\ 5\ 6\ 7\ 7\ 10\ 10 \\ 2\ 3\ 4\ 8\ 8\ 10\ 10\ 10\ 10 \\ 2\ 2\ 3\ 3\ 4\ 8\ 8\ 9\ 9 \\ 2\ 3\ 5\ 6\ 7\ 9\ 9\ 9 \\ 1\ 5\ 6\ 7\ 9\ 10 \\ 7\ 8\ 10 \end{array}$$

$$C_8 = \begin{array}{c} 2\ 3\ 4\ 8\ 8 \\ 9\ 9\ 9 \\ 1\ 5\ 5\ 6\ 6\ 7\ 7 \\ 10\ 10\ 10 \\ 2\ 3\ 4\ 8\ 8 \end{array} \quad C_9 = \begin{array}{c} 10 \\ 2\ 3\ 4\ 8\ 8 \\ 8\ 9\ 9\ 9\ 9 \\ 1\ 1\ 5\ 5\ 6\ 6\ 7\ 7\ 9 \\ 5\ 6\ 7\ 10\ 10\ 10 \\ 2\ 3\ 4\ 8\ 10\ 10 \\ 8\ 8\ 9 \end{array} \quad C_{10} = \begin{array}{c} 10 \\ 2\ 3\ 4 \\ 9\ 9\ 9 \\ 1\ 1\ 5\ 6\ 7 \\ 10\ 10 \\ 1\ 2\ 3\ 4 \\ 9\ 10 \end{array}$$

We give here a summary of the bijection between ordinary characters, simple $B_0(H)$-modules and simple $B_0(H)$-modules and the perversity function; the polynomial Φ_n denotes the n^{th} cyclotomic polynomial.

$\pi_{1/4}$	χ	Polynomial	kH-mod	$B_0(G)$-mod	dim C_i
0	1_1	1	$T_1 = 1_1$	$S_1 = 1_1$	$\dim(C_1) = 1$
3	84_1	$q^2\Phi_3(q)\Phi_6(q)$	$T_2 = 1_2$	$S_2 = 83_1$	$\dim(C_2) = 8$
3	84_2	$q^2\Phi_3(q)\Phi_6(q)$	$T_3 = 1_3$	$S_3 = 83_2$	$\dim(C_3) = 8$
3	84_3	$q^2\Phi_3(q)\Phi_6(q)$	$T_4 = 1_4$	$S_4 = 83_3$	$\dim(C_4) = 8$
4	972_1	$q^3\Phi_2(q)^4\Phi_6(q)/2$	$T_9 = 2_1$	$S_9 = 722_1$	$\dim(C_9) = 47$
5	28_1	$q^3\Phi_1(q)^4\Phi_3(q)/2$	$T_{10} = 2_2$	$S_{10} = 28_1$	$\dim(C_{10}) = 28$
5	1344_1	$q^6\Phi_3(q)\Phi_6(q)$	$T_5 = 1_5$	$S_5 = 539_1$	$\dim(C_5) = 64$
5	1344_2	$q^6\Phi_3(q)\Phi_6(q)$	$T_6 = 1_6$	$S_6 = 539_2$	$\dim(C_6) = 64$
5	1344_3	$q^6\Phi_3(q)\Phi_6(q)$	$T_7 = 1_7$	$S_7 = 539_3$	$\dim(C_7) = 64$
6	4096_1	q^{12}	$T_8 = 1_8$	$S_8 = 1729_1$	$\dim(C_8) = 29$

No modules with vertex Q_1, Q_2 or Q_3 appear in the Green correspondence for $B_0(G)$.

3.1.4 Perverse equivalence

Complexes X_2, X_3, X_4 with $\pi = 3$.

The complex X_1 produced by T_1 being trivial, we start from those with perversity function $\pi = 3$, namely T_2, T_3 and T_4. We have that the Green correspondents have dimension 8 and the complexes are:

$$X_2 : 0 \to \mathcal{P}(2) \to \mathcal{P}(10) \to \mathcal{P}(5) \oplus R_1 \twoheadrightarrow C_2 \to 0,$$
$$X_3 : 0 \to \mathcal{P}(3) \to \mathcal{P}(10) \to \mathcal{P}(6) \oplus R_2 \twoheadrightarrow C_3 \to 0,$$
$$X_4 : 0 \to \mathcal{P}(4) \to \mathcal{P}(10) \to \mathcal{P}(7) \oplus R_3 \twoheadrightarrow C_4 \to 0.$$

Complex X_9 with $\pi = 4$.

The complex X_9 is:

$$X_9 : \mathcal{P}(9) \to \mathcal{P}(8) \oplus \mathcal{P}(10) \oplus \mathcal{P}(10) \to$$
$$\to \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(7) \oplus \mathcal{P}(8) \to \mathcal{P}(10) \oplus U_1^* \oplus U_2^* \oplus U_3^* \twoheadrightarrow C_9 \to 0.$$

Complexes X_5, X_6, X_7 and X_{10} with $\pi = 5$.

From X_2,X_3,X_4 we can see how $5, 6, 7$ are permuted. Now we move to the triple T_5, T_6, T_7.

$$X_5 : \ \mathcal{P}(5) \to \mathcal{P}(8) \oplus \mathcal{P}(10) \to \mathcal{P}(6) \oplus \mathcal{P}(7) \oplus \mathcal{P}(9) \to \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(7) \oplus \mathcal{P}(10) \to$$
$$\to \mathcal{P}(8) \oplus \mathcal{P}(9) \oplus R_1 \oplus U_2 \oplus U_3 \twoheadrightarrow C_5 \to 0.$$

$$X_6 : \ \mathcal{P}(6) \to \mathcal{P}(8) \oplus \mathcal{P}(10) \to \mathcal{P}(5) \oplus \mathcal{P}(7) \oplus \mathcal{P}(9) \to \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(7) \oplus \mathcal{P}(10) \to$$
$$\to \mathcal{P}(8) \oplus \mathcal{P}(9) \oplus R_2 \oplus U_1 \oplus U_3 \twoheadrightarrow C_6 \to 0.$$

$$X_7 : \ \mathcal{P}(7) \to \mathcal{P}(8) \oplus \mathcal{P}(10) \to \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(9) \to \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(7) \oplus \mathcal{P}(10) \to$$
$$\to \mathcal{P}(8) \oplus \mathcal{P}(9) \oplus R_3 \oplus U_1 \oplus U_2 \twoheadrightarrow C_7 \to 0.$$

The complex X_{10} turns out to be:

$$X_{10} : \ \mathcal{P}(10) \to \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(7) \to \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(7) \oplus \mathcal{P}(8) \to$$
$$\to \mathcal{P}(8) \oplus \mathcal{P}(9) \oplus \mathcal{P}(9) \to \mathcal{P}(10) \oplus R_1^* \oplus R_2^* \oplus R_3^* \twoheadrightarrow C_{10} \to 0.$$

Complex X_8 with $\pi = 6$.

The complex X_8 is:

$$X_8 : \ \mathcal{P}(8) \to \mathcal{P}(8) \oplus \mathcal{P}(8) \to \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(7) \to \mathcal{P}(10) \oplus \mathcal{P}(10) \to$$
$$\to \mathcal{P}(8) \oplus \mathcal{P}(9) \oplus \mathcal{P}(9) \to U_1^* \oplus U_2^* \oplus U_3^* \twoheadrightarrow C_8 \to 0.$$

We can see that the modules U_1, U_2, U_3 appear again. Finally, we give the cohomology of each complex (the meaning of the column "total" has been introduced with the alternating sum of cohomology defined by the expression 2.1.11):

X_i	$\pi_{1/4}$	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	Total
X_2	3				2		1	2-1
X_3	3				3		1	3-1
X_4	3				4		1	4-1
X_9	4			2/3/4/9		$1 \oplus 1$		9-4-3-2+1+1
X_{10}	5		1/10	1				10
X_5	5		3/4/9/5		1			5-9+3+4-1
X_6	5		2/4/9/6		1			6-9+2+4-1
X_7	5		2/3/9/7		1			7-9+2+3-1
X_8	6	A		1				1-2-3-4-5-6-7+8+9+9-10

(3.1.2)

For compactness, we have set $A := 2/3/4/9/9/10/5/6/7/8$ (see complex X_8).

3.1.5 Stable Equivalence

We perform the construction in [11] of the complex determining a stable equivalence between $\mathrm{mod}(B_0(G))$ and $\mathrm{mod}(B_0(H))$.

As we remarked at the beginning of the section, we have three H-conjugacy classes of subgroups of order 5. We can denote by Q a generic subgroup of order 5; the result from the construction of the stable equivalence is the same for each of those three, up to isomorphism.

We recall the notation from [11]: $\bar{N}_G(Q)$ and $\bar{N}_H(Q)$ are complements of Q inside $N_G(Q)$ and $N_H(Q)$ respectively, and they can be chosen such that $\bar{N}_H(Q) \leq \bar{N}_G(Q)$. We need Q-

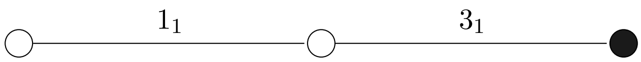
complements of centralisers as well, and we take then $\bar{C}_G(Q) = C_G(Q) \cap \bar{N}_G(Q)$ and $\bar{C}_H(Q) = C_H(Q) \cap \bar{N}_H(Q)$.

For each of the three $Q = Q_1, Q_2, Q_3$, we have $\bar{C}_H(Q) \cong D_{10}$, the dihedral group of order 10, and $\bar{C}_G(Q) \cong A_5$. As a $k[\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}]$-module, we have that $k\bar{C}_G(Q) = M_Q \oplus V$, where M_Q and V are indecomposable, $\dim(M_Q) = 35$, $\dim(V) = 25$, and only M_Q lies in the principal block. So we have $e_{\bar{C}_H(Q)} k\bar{C}_G(Q) e_{\bar{C}_G(Q)} = M_Q$. In particular, no projective summand P appears in this decomposition. As $\bar{N}_H(Q) \leq N_G(Q)$ and in this case $\bar{C}_G(Q) \trianglelefteq N_G(Q)$, then $\bar{N}_H(Q)$ normalises $\bar{C}_H(Q)$ and the action of $\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}$ on $k\bar{C}_G(Q)$ can be extended to a natural action of $N_\Delta = (\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}})\Delta\bar{N}_H(Q)$; it turns out that, as a kN_Δ-module, $k\bar{C}_G(Q)$ does not decompose any further than M_Q and U. So we conclude that

$$e_{\bar{C}_H(Q)} k\bar{C}_G(Q) e_{\bar{C}_G(Q)} = M_Q \tag{3.1.3}$$

as a kN_Δ-module.

The representation theory of $k\bar{C}_H(Q)$ and $k\bar{C}_G(Q)$ is briefly recalled: they decompose into one and two blocks respectively, and $k\bar{C}_H(Q)$ has two simple modules 1_1, 1_2, and $k\bar{C}_G(Q)$ has three simple modules $1_1, 3_1, 5_1$, where the first two of them belong to the principal block. Each simple module can be seen as a simple module for $k[\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}]$, where the original group acts as usual and the other factor acts trivially. The set of irreducible modules for $k[\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}]$ is indeed $1_1 \otimes 1_1, 1_2 \otimes 1_1, 1_1 \otimes 3_1, 1_2 \otimes 3_1, 1_1 \otimes 5_1, 1_2 \otimes 5_1$. The Brauer tree of the principal block of $k\bar{C}_G(Q)$ is:



The map $\gamma : \mathcal{S}_{B_0(\bar{C}_G(Q))} \to \mathcal{S}_{B_0(\bar{C}_H(Q))}$ is defined via the requirement

$$\underline{\mathrm{Hom}}(S_{\bar{C}_H(Q)}, \gamma(S)) \neq \{0\}, \quad S \in \mathcal{S}_{B_0(\bar{C}_G(Q))}.$$

This makes γ send the trivial module to the trivial module and 3_1 to 1_2. As expected according to [11], our computation in Magma confirms that a projective cover of M_Q is in the form

$$\mathcal{P}(1_1 \otimes 1_1) \oplus \mathcal{P}(1_2 \otimes 3_1) \twoheadrightarrow M_Q. \tag{3.1.4}$$

The subset $\mathcal{E}$ of $\mathcal{S}_{B_0(\bar{C}_G(Q))}$ is defined by looking at the Brauer tree of $B_0(\bar{C}_G(Q))$: the distance d between the exceptional vertex and the edge of the trivial module is 1; so the subset $\mathcal{E} \subseteq$

$\mathcal{S}_{B_0(\bar{C}_G(Q))}$ such that the distance of the edge from the exceptional vertex is $1+1=0 \pmod 2$ is formed of 3_1 only. Therefore, the complex X is:

$$X : (0 \to \mathcal{P}(1_2 \otimes 3_1) \xrightarrow{a} M_Q \to 0), \tag{3.1.5}$$

where X is now a complex of $k[\bar{C}_H(Q) \times \bar{C}_G(Q)^{\text{opp}}]$-modules that can be extended to a N_Δ-module. In this situation, this complex coincides with the complex C' in Remark 2.2.3, whereas they are only homotopy equivalent in general.

We can now run the algorithm `FinalStabEq`; this would compute the image of each element in $\mathcal{S}_{B_0(G)}$ under the stable equivalence L in Proposition 2.1.5; if the result matches with the output of the algorithm `PerverseEq`, then by Proposition 2.1.5 we have a splendid derived equivalence between $B_0(G)$ and $B_0(H)$.

As no projective summand appears in the decomposition 3.1.3, we deduce that $T_Q = U_Q = \mathcal{P}(1_2 \otimes 3_1)$, so we have to compute $T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)}$ for every $Q = Q_1, Q_2, Q_3$ and for every simple $B_0(G)$-module $S = S_1, \ldots, S_{10}$. For each $S = S_i$, we must check the following isomorphism in the stable category:

$$\bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong X_i^{-1} \quad in \quad \underline{\text{mod}}(kH), \tag{3.1.6}$$

where X_i^{-1} is the terms in position -1 of the complex X_i which was produced from S_i under the algorithm of the perverse equivalence. Our computations show that conditions 3.1.6 are satisfied, as we get:

$$S = S_1, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};$$

$$S = S_2, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1 \oplus \{0\} \oplus \{0\} \cong R_1;$$

$$S = S_3, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\} \oplus R_2 \oplus \{0\} \cong R_2;$$

$$S = S_4, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\} \oplus \{0\} \oplus R_3 \cong R_3;$$

$$S = S_5, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1 \oplus U_2 \oplus U_3;$$

$$S = S_6, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_1 \oplus R_2 \oplus U_3;$$

$$S = S_7, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_1 \oplus U_2 \oplus R_3;$$

$$S = S_8, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_1^* \oplus U_2^* \oplus U_3^*;$$

$$S = S_9, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_1^* \oplus U_2^* \oplus U_3^*;$$

$$S = S_{10}, \quad \bigoplus_{Q=Q_1,Q_2,Q_3} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1^* \oplus R_2^* \oplus R_3^*.$$

where all these isomorphisms are in the stable category, namely up to projective summands. By comparing these results with our complexes X_i, we have finally concluded the proof of Theorem 3.1.1.

3.2 $^2F_4(2)'.2$

The family of groups $^2F_4(q^2)$ was introduced by Ree; here $q^2 = 2^{2m+1}$, an odd power of 2. These groups are simple apart from the case $q^2 = 2$, that contains a simple subgroup of index 2, namely $^2F_4(2)'$, the simple Tits group. Broué conjecture was already proved by Robbins for both $^2F_4(2)'$ and $^2F_4(2) = {}^2F_4(2)'.2$:

Proposition 3.2.1. ([35]) *Broué's conjecture holds for the principal* 5*-blocks of* $G = {}^2F_4(2)'.2$.

The main purpose of this section is to look for a perverse equivalence between the principal 5-block of $^2F_4(2)'.2$ and the normaliser of a Sylow 5-subgroup. Again, in this section we have $k := \bar{\mathbb{F}}_5$.

3.2.1 ℓ-local subgroups

The normaliser H of the Sylow 5-subgroup $P \cong C_5 \times C_5$ has order 2400 and is of the form $H \cong P \rtimes (S \rtimes S_3)$, where S is the same group that we found for the normaliser H in $\Omega_8^+(2)$.

3.2.2 Irreducible $B_0(H)$ and $B_0(G)$-modules

Here we give a brief description of the representation theory of H and G, and we introduce the labelling for the irreducible $B_0(H)$ and $B_0(G)$-modules that define the desired bijection between the two sets $\mathcal{S}_{B_0(H)}$ and $\mathcal{S}_{B_0(G)}$. The group algebra $kH = B_0(H)$ has 16 simple modules and all of them are absolutely simple. The label for $T_1, \ldots, T_{16}$ will be such that $T_1, \ldots, T_4$ have dimension 1, and in particular T_4 is the only one such that $T_4 \otimes T_4 \cong T_1$; $T_5, \ldots, T_{10}$ have dimension 2, $T_{11}, \ldots, T_{14}$ have dimension 3 and T_{15}, T_{16} have dimension 4. The chosen labelling for our simple $B_0(H)$-modules must be such that the socle factors of $\mathcal{P}(1)$ and $\mathcal{P}(4)$ are:

$$\mathcal{P}(1) = \begin{array}{c} 1 \\ 6 \\ 12 \\ 15 \\ 7\ 14 \\ 16 \\ 11 \\ 5 \\ 1 \end{array} \qquad \mathcal{P}(4) = \begin{array}{c} 4 \\ 9 \\ 11 \\ 15 \\ 7\ 13 \\ 16 \\ 12 \\ 8 \\ 4 \end{array}$$

This uniquely determines the modules of dimension 3 and 4. We should now distinguish T_2 and $T_3 \cong T_2^*$, and it is enough to say that T_3 is a composition factor of $T_5 \otimes T_5$. Finally, the self-dual T_{10} can be defined as the tensor product of the other self-dual T_7 by T_2, and this determines the labelling $\{T_i \mid i = 1, .., 16\} = \mathcal{S}_{B_0(H)}$ completely.

In the following table, we list all the unipotent characters lying in the principal 5-block of G, together with their degree polynomials. The polynomials $\Phi_8', \Phi_8'', \Phi_{24}', \Phi_{24}'' \in \mathbb{R}[x]$ are such that $\Phi_8 = \Phi_8'\Phi_8''$ and $\Phi_{24} = \Phi_{24}'\Phi_{24}''$. In particular:

$$\begin{aligned}
\Phi_8'(q) &= q^2 + \sqrt{2}q + 1 = (q - \psi^3)(q - \psi^5), \\
\Phi_8''(q) &= q^2 - \sqrt{2}q + 1 = (q - \psi)(q - \psi^7), \\
\Phi_{24}'(q) &= q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1 = (q - \alpha^5)(q - \alpha^{11})(q - \alpha^{13})(q - \alpha^{19}), \\
\Phi_{24}''(q) &= q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1 = (q - \alpha)(q - \alpha^7)(q - \alpha^{17})(q - \alpha^{23}),
\end{aligned} \tag{3.2.1}$$

where $\psi = e^{\frac{2\pi i}{8}}$ and $\alpha = e^{\frac{2\pi i}{24}}$. The perversity function $\pi_{\kappa/d}$ is computed by setting $\kappa = 3$ and $d = 8$.

$B_0(G)$, $G = {}^2F_4(2)'.2$, $\ell = 5$			
$\pi_{3/8}$	Ordinary Character	Degree	Degree polynomial
0	$\phi_{1,0}$	1	1
8	${}^2B_2[\psi^3]$	27	$\frac{1}{\sqrt{2}}q\Phi_1\Phi_2\Phi_4^2\Phi_{12}$
8	${}^2B_2[\psi^5]$	27	$\frac{1}{\sqrt{2}}q\Phi_1\Phi_2\Phi_4^2\Phi_{12}$
10	$\phi''_{1,4}$	78	$q^2\Phi_{12}\Phi_{24}$
14	${}^2F_4^{\mathrm{IV}}[-1]$	52	$\frac{1}{3}q^4\Phi_1^2\Phi_2^2\Phi_{12}\Phi_{24}$
14	${}^2F_4^{\mathrm{I}}[-i]$	351	$\frac{1}{4}q^4\Phi_1^2\Phi_2^2\Phi_4^2\Phi_{12}\Phi'_{24}$
15	${}^2F_4^{\mathrm{I}}[-1]$	27	$\frac{1}{12}q^4\Phi_1^2\Phi_2^2\Phi_4^2\Phi_{12}\Phi''_{24}$
15	${}^2F_4^{\mathrm{I}}[-1]$	27	$\frac{1}{12}q^4\Phi_1^2\Phi_2^2\Phi_4^2\Phi_{12}\Phi''_{24}$
15	${}^2F_4^{\mathrm{I}}[-1]$	78	$\frac{1}{6}q^4\Phi_1^2\Phi_2^2\Phi_4^2\Phi_{24}$
15	${}^2F_4^{\mathrm{II}}[-1]$	1	$\frac{1}{12}q^4\Phi_1^2\Phi_2^2(\Phi''_8)^2\Phi_{12}\Phi''_{24}$
15	${}^2F_4^{\mathrm{I}}[i]$	351	$\frac{1}{4}q^4\Phi_1^2\Phi_2^2\Phi_4^2\Phi_{12}\Phi'_{24}$
15	$\phi_{2,3}$	351	$\frac{1}{4}q^4\Phi_4^2(\Phi''_8)^2\Phi_{12}\Phi'_{24}$
16	$\phi'_{1,4}$	1248	$q^{10}\Phi_{12}\Phi_{24}$
17	${}^2B_2[\psi^3];1$	1728	$\frac{1}{\sqrt{2}}q^{13}\Phi_1\Phi_2\Phi_4^2\Phi_{12}$
17	${}^2B_2[\psi^3];1$	1728	$\frac{1}{\sqrt{2}}q^{13}\Phi_1\Phi_2\Phi_4^2\Phi_{12}$
18	$\phi_{1,8}$	4096	q^{24}

As for $\Omega_8^+(q)$, a list of degree polynomials for unipotent characters can be found in [10], although this source has some misprints; a reviewed version of the list can be found for example in [12]. In particular, the polynomial of the two characters of dimension 1728 and of those non-cuspidal of dimension 78 is $\frac{1}{\sqrt{2}}q^{13}\Phi_1\Phi_2\Phi_4^2\Phi_{12}$ and $\frac{1}{\sqrt{2}}q\Phi_1\Phi_2\Phi_4^2\Phi_{12}$ respectively; and the character of dimension 650 denoted by ρ_2 in [10] has polynomial $\frac{1}{2}q^4\Phi_8^2\Phi_{24}$. This last character does not belong to the principal block anyway. Our unitriangular shape of the decomposition matrix of $B_0(G)$ is:

$B_0(G)$, $G = {}^2F_4(2)'.2$, $\ell = 5$																	
$\pi_{3/8}$	χ	1_1	27_1	27_2	78_1	52_1	351_1	351_2	1_2	27_3	27_4	78_2	218_1	920_1	351_3	351_4	1186_1
0	1_1	1															
8	${}^2B_2[\psi^3]; 1$		1														
8	${}^2B_2[\psi^5]; 1$			1													
10	$\phi''_{1,4}$				1												
14	${}^2F_4^{\mathrm{IV}}[-1]$					1											
14	${}^2F_4^{\mathrm{I}}[i]$						1										
14	${}^2F_4^{\mathrm{I}}[-i]$							1									
15	${}^2F_4^{\mathrm{II}}[-1]$								1								
15	${}^2F_4^{\mathrm{II}}[i]$									1							
15	${}^2F_4^{\mathrm{II}}[-i]$										1						
15	${}^2F_4^{\mathrm{I}}[-1]$											1					
15	$\phi_{2,3}$	1	1	1	1								1				
16	$\phi'_{1,4}$	1	1	1					1	1	1		1	1			
17	${}^2B_2[\psi^3]; \epsilon$		1			1		1			1			1	1		
17	${}^2B_2[\psi^5]; \epsilon$			1		1	1			1				1		1	
18	$\phi_{1,8}$		1	1	1	2	1	1		1	1	1	1	1	1	1	1
	351_2								1			1	1				
	351_4							1									
	351_6															1	
	1404_1												1				1

3.2.3 Green correspondents

As usual, the Green correspondent of each S_i is denoted by C_i. We have:

$$C_2 = \begin{array}{c} 12 \\ 8\ 15 \\ 4\ 7\ 13\ 14 \\ 6\ 9\ 16\ 16\ 16 \\ 2\ 10\ 11\ 11\ 12\ 12 \\ 5\ 8\ 15\ 15 \\ 1\ 7\ 13\ 14 \\ 6\ 16 \end{array} \qquad C_3 = \begin{array}{c} 11\ 12 \\ 5\ 8\ 15 \\ 1\ 4\ 13\ 14 \\ 6\ 9\ 16 \\ 11\ 12 \end{array} \qquad C_4 = \begin{array}{c} 5\ 15 \\ 1\ 7\ 13\ 14 \\ 6\ 9\ 16\ 16 \\ 3\ 10\ 11\ 11\ 12\ 12 \\ 5\ 8\ 15\ 15 \\ 4\ 7\ 13\ 14 \\ 9\ 16 \\ 11\ 15 \end{array}$$

$$C_5 = \begin{array}{c} 9 \\ 11 \\ 8\ 15 \\ 7\ 13 \\ 16 \\ 4\ 12 \\ 8\ 9 \end{array} \quad C_6 = \begin{array}{c} 2 \\ 8 \\ 13\ 14 \\ 6\ 16 \\ 10\ 12 \\ 15 \\ 14 \end{array} \quad C_7 = \begin{array}{c} 14 \\ 16 \\ 10\ 11 \\ 15 \\ 13 \\ 5\ 9 \\ 3\ 14 \end{array}$$

$$C_8 = 10 \quad C_9 = \begin{array}{c} 11 \\ 5\ 15 \\ 1\ 7\ 13\ 14 \\ 6\ 9\ 16\ 16\ 16 \\ 3\ 10\ 11\ 11\ 12\ 12 \\ 5\ 8\ 15\ 15 \\ 4\ 7\ 13\ 14 \\ 9\ 16 \end{array} \quad C_{10} = \begin{array}{c} 8\ 15 \\ 4\ 7\ 13\ 14 \\ 6\ 9\ 16\ 16 \\ 2\ 10\ 11\ 11\ 12\ 12 \\ 5\ 8\ 15\ 15 \\ 1\ 7\ 13\ 14 \\ 6\ 16 \\ 12\ 15 \end{array}$$

$$C_{11} = \begin{array}{c} 13 \\ 16 \\ 10\ 12 \\ 15 \\ 14 \\ 6\ 8 \\ 2\ 13 \end{array} \quad C_{12} = \begin{array}{c} 6 \\ 12 \\ 5\ 15 \\ 7\ 14 \\ 16 \\ 1\ 11 \\ 5\ 6 \end{array} \quad C_{13} = \begin{array}{c} 6\ 9\ 16 \\ 2\ 3\ 10\ 11\ 11\ 12\ 12 \\ 5\ 5\ 8\ 8\ 15\ 15\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 13\ 14\ 14\ 14 \\ 6\ 6\ 9\ 9\ 16\ 16\ 16\ 16 \\ 2\ 3\ 10\ 11\ 11\ 12\ 12 \\ 5\ 8\ 15 \end{array}$$

$$C_{14} = \begin{array}{c} 3 \\ 5 \\ 13\ 14 \\ 9\ 16 \\ 10\ 11 \\ 15 \\ 13 \end{array} \quad C_{15} = 4 \quad C_{16} = \begin{array}{c} 7 \\ 16 \\ 11\ 12 \\ 15 \\ 7 \end{array}$$

We give here a summary of the bijection between ordinary characters, simple $B_0(G)$-modules and simple $B_0(H)$-modules and the perversity function.

$\pi_{3/8}$	$B_0(H)$-mod	$B_0(G)$-mod	dim C_i
0	$T_1 = 1_1$	$S_1 = 1_1$	$\dim(C_1) = 1$
8	$T_6 = 2_2$	$S_6 = 27$	$\dim(C_6) = 27$
8	$T_7 = 2_3$	$S_7 = 27$	$\dim(C_7) = 27$
10	$T_{12} = 3_2$	$S_{12} = 78$	$\dim(C_{12}) = 28$
14	$T_2 = 1_2$	$S_2 = 351$	$\dim(C_2) = 76$
14	$T_4 = 1_4$	$S_4 = 351$	$\dim(C_4) = 76$
14	$T_8 = 2_4$	$S_8 = 52_1$	$\dim(C_8) = 2$
15	$T_{11} = 3_1$	$S_{11} = 27$	$\dim(C_{11}) = 27$
15	$T_{14} = 3_4$	$S_{14} = 27$	$\dim(C_{14}) = 27$
15	$T_{15} = 4_1$	$S_{15} = 1_2$	$\dim(C_{15}) = 1$
15	$T_5 = 2_1$	$S_5 = 78$	$\dim(C_5) = 28$
15	$T_{16} = 4_2$	$S_{16} = 218_1$	$\dim(C_{16}) = 18$
16	$T_{13} = 3_3$	$S_{13} = 920$	$\dim(C_{13}) = 120$
17	$T_9 = 2_5$	$S_9 = 351$	$\dim(C_9) = 76$
17	$T_{10} = 2_6$	$S_{10} = 351$	$\dim(C_{10}) = 76$
18	$T_3 = 1_3$	$S_3 = 1186_1$	$\dim(C_3) = 36$

There is a unique conjugacy class of subgroups of order 5 in H, and we denote by Q a representative. No module with vertex Q appears in the restriction $(S_i)_H$ for any $i = 1, \ldots, 16$.

3.2.4 Perverse equivalence

Due to the large values of the perversity function π, our complexes are long and we will only write the final part of each of them, namely the degree 0 and -1. As we are mainly interested in the non-projective part of each module of the complex, and as each term is projective whenever the degree is smaller than -2, we do not have any loss of information. The perversity function has been computed using $\kappa = 3$ and $d = 8$.

Complexes X_6, X_7 with $\pi = 8$.

$$X_6 : 0 \to \mathcal{P}(6) \to \cdots \to M_1 \twoheadrightarrow C_6 \to 0$$

$$X_7 : 0 \to \mathcal{P}(7) \to \cdots \to \mathcal{P}(14) \oplus M_2 \twoheadrightarrow C_7 \to 0$$

where M_1 and M_2 have dimension 150 and are stacked relatively Q-projective as follows:

$$M_1 = \begin{matrix} R_{1,1} \\ R_{1,2} \\ R_{1,3} \\ R_{1,4} \\ R_{1,3}^* \end{matrix} \qquad M_2 = \begin{matrix} R_{2,1} \\ R_{2,2} \\ R_{2,1}^* \\ R_{2,3} \\ R_{2,4} \end{matrix}$$

$$M_1 = \begin{matrix} 2\ 10\ 12 \\ 8\ 8\ 15\ 15 \\ 4\ 7\ 13\ 13\ 13\ 14\ 14 \\ 6\ 6\ 9\ 9\ 13\ 14\ 16\ 16\ 16\ 16 \\ 2\ 3\ 6\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12\ 16 \\ 2\ 5\ 5\ 8\ 10\ 12\ 15\ 15\ 15 \\ 1\ 7\ 8\ 13\ 14\ 14\ 15 \\ 6\ 13\ 14\ 16 \end{matrix} \qquad M_2 = \begin{matrix} 9\ 16 \\ 3\ 10\ 11\ 11\ 12 \\ 5\ 5\ 8\ 15\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 13\ 13\ 14\ 14\ 14\ 14 \\ 6\ 6\ 9\ 9\ 9\ 16\ 16\ 16\ 16\ 16 \\ 2\ 3\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12 \\ 5\ 8\ 8\ 15\ 15\ 15 \\ 4\ 7\ 13\ 13\ 14 \end{matrix}$$

We have:

$$R_{1,1} = \begin{matrix} 2\ 10\ 12 \\ 8\ 15 \\ 13\ 14 \\ 6\ 16 \\ 2\ 10\ 12 \end{matrix} \qquad R_{1,2} = \begin{matrix} 8\ 15 \\ 4\ 7\ 13 \\ 9\ 16 \\ 11\ 12 \\ 8\ 15 \end{matrix} \qquad R_{1,3} = \begin{matrix} 13\ 14 \\ 9\ 16 \\ 3\ 10\ 11 \\ 5\ 15 \\ 13\ 14 \end{matrix} \qquad R_{1,4} = \begin{matrix} 6\ 16 \\ 11\ 12 \\ 5\ 15 \\ 1\ 7\ 14 \\ 6\ 16 \end{matrix}$$

$$R_{2,1} = \begin{matrix} 9\ 16 \\ 3\ 10\ 11 \\ 5\ 15 \\ 13\ 14 \\ 9\ 16 \end{matrix} \qquad R_{2,2} = \begin{matrix} 11\ 12 \\ 5\ 15 \\ 1\ 7\ 14 \\ 6\ 16 \\ 11\ 12 \end{matrix} \qquad R_{2,3} = \begin{matrix} 4\ 7\ 13 \\ 9\ 16 \\ 11\ 12 \\ 8\ 15 \\ 4\ 7\ 13 \end{matrix}$$

Each of these modules has dimension 30 and are they summands of $\mathrm{Ind}_Q^H k$.

Complex X_{12} with $\pi = 10$.

$$X_{12} : 0 \to \mathcal{P}(12) \to \cdots \to \mathcal{P}(6) \oplus M_3 \twoheadrightarrow C_{12} \to 0$$

$$M_3 = \begin{array}{c} 11\ 12 \\ 5\ 8\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 14\ 14 \\ 5\ 6\ 6\ 9\ 9\ 15\ 16\ 16\ 16\ 16 \\ 1\ 2\ 3\ 7\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12\ 14 \\ 5\ 6\ 8\ 8\ 15\ 15\ 15\ 16 \\ 4\ 7\ 11\ 12\ 13\ 13\ 14 \\ 5\ 9\ 15\ 16 \end{array} \qquad M_3 = \begin{array}{c} R_{2,2} \\ R^*_{2,1} \\ R_{2,3} \\ R_{2,1} \\ R^*_{1,4} \end{array}$$

Complexes X_2, X_4, X_8 with $\pi = 14$.

$$X_2 : 0 \to \mathcal{P}(2) \to \cdots \to \mathcal{P}(12) \oplus M_4 \twoheadrightarrow C_2 \to 0$$

The module M_4 at degree -1 is stacked relatively projective and its structure, as well as the filtration, is given by:

$$M_4 = \begin{array}{c} 8\ 15 \\ 4\ 7\ 13\ 13\ 14 \\ 6\ 9\ 9\ 16\ 16\ 16 \\ 2\ 3\ 9\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12\ 16 \\ 5\ 5\ 8\ 8\ 11\ 12\ 15\ 15\ 15\ 15 \\ 1\ 7\ 8\ 13\ 13\ 14\ 14\ 14\ 15 \\ 4\ 6\ 6\ 7\ 13\ 16\ 16 \\ 2\ 9\ 10\ 12\ 16 \end{array} \qquad M_4 = \begin{array}{c} R_{1,2} \\ R_{1,3} \\ R_{1,4} \\ R_{1,1} \\ R^*_{1,2} \end{array}$$

As for X_4, we have:

$$X_4 : 0 \to \mathcal{P}(4) \to \cdots \to \mathcal{P}(5) \oplus \mathcal{P}(15) \oplus M_7 \twoheadrightarrow C_4 \to 0$$

where M_7 is stacked relatively Q-projective and

$$M_7 = \begin{array}{c} 4\ 7\ 13 \\ 9\ 9\ 16\ 16 \\ 3\ 10\ 11\ 11\ 11\ 12\ 12 \\ 5\ 5\ 8\ 8\ 8\ 15\ 15\ 15\ 15\ 15 \\ 1\ 4\ 4\ 7\ 7\ 7\ 13\ 13\ 13\ 13\ 14\ 14\ 14 \\ 6\ 6\ 9\ 9\ 16\ 16\ 16\ 16 \\ 2\ 10\ 11\ 11\ 12\ 12\ 12 \\ 8\ 8\ 15\ 15 \end{array} \qquad M_7 = \begin{array}{c} R_{2,3} \\ R_{2,1} \\ R_{2,2} \\ R^*_{2,1} \\ R_{1,2} \end{array}$$

The complex X_8 is:

$$X_8 : 0 \to \mathcal{P}(8) \to \cdots \to M_5 \oplus M_6 \twoheadrightarrow C_8 \to 0.$$

The two stacked relatively Q-projective modules M_5 and M_6 are:

$$M_5 = \begin{array}{c} 13\ 14 \\ 6\ 9\ 16\ 16 \\ 2\ 3\ 10\ 10\ 11\ 11\ 12\ 12 \\ 3\ 5\ 5\ 8\ 8\ 10\ 11\ 15\ 15\ 15\ 15 \\ 1\ 4\ 5\ 7\ 7\ 13\ 13\ 13\ 14\ 14\ 14\ 15 \\ 6\ 6\ 9\ 13\ 14\ 16\ 16\ 16 \\ 2\ 9\ 10\ 11\ 12\ 12\ 16 \\ 3\ 8\ 10\ 11\ 15 \end{array} \qquad M_5 = \begin{array}{c} R_{1,3} \\ R_{1,4} \\ R_{1,1} \\ R_{1,2} \\ R^*_{1,1} \end{array}$$

$$M_6 = \begin{array}{c} 8\ 15 \\ 4\ 7\ 13\ 13\ 14 \\ 6\ 9\ 9\ 16\ 16\ 16 \\ 2\ 2\ 3\ 10\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12\ 12 \\ 5\ 5\ 8\ 8\ 8\ 15\ 15\ 15\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 13\ 14\ 14\ 14 \\ 6\ 6\ 9\ 16\ 16\ 16 \\ 2\ 10\ 11\ 12\ 12 \end{array} \qquad M_6 = \begin{array}{c} R^*_{2,1} \\ R_{2,3} \\ R_{2,1} \\ R_{2,2} \\ R_{1,1} \end{array}$$

Complexes $X_{11}, X_{14}, X_{15}, X_5, X_{16}$ with $\pi = 15$.

$$X_{11} : 0 \to \mathcal{P}(11) \to \cdots \to \mathcal{P}(13) \oplus U_1 \twoheadrightarrow C_{11} \to 0$$

$$X_{14} : 0 \to \mathcal{P}(14) \to \cdots \to U_2 \twoheadrightarrow C_{14} \to 0$$

where both U_1 and U_2 are stacked relatively Q-projective; in particular:

$$U_1 = \begin{matrix} R^*_{1,4} \\ R_{1,1} \\ R_{1,2} \\ R_{1,3} \\ R_{1,4} \\ R^*_{2,1} \end{matrix} \qquad U_2 = \begin{matrix} R^*_{1,2} \\ R^*_{1,1} \\ R^*_{1,4} \\ R^*_{1,3} \\ R^*_{1,2} \\ R_{1,3} \end{matrix}$$

$$U_1 = \begin{matrix} 2\ 10\ 12 \\ 5\ 8\ 8\ 15\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 13\ 14\ 14\ 14 \\ 6\ 6\ 6\ 8\ 9\ 9\ 15\ 16\ 16\ 16\ 16\ 16 \\ 2\ 3\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12\ 13\ 14 \\ 5\ 5\ 6\ 8\ 15\ 15\ 15\ 16 \\ 1\ 2\ 7\ 10\ 11\ 12\ 12\ 13\ 14\ 14 \\ 5\ 6\ 8\ 15\ 15\ 16 \end{matrix} \qquad U_2 = \begin{matrix} 3\ 10\ 11 \\ 5\ 5\ 15\ 15 \\ 1\ 7\ 13\ 13\ 14\ 14\ 14 \\ 6\ 6\ 9\ 9\ 9\ 13\ 14\ 16\ 16\ 16\ 16\ 16 \\ 2\ 3\ 9\ 10\ 10\ 11\ 11\ 11\ 11\ 12\ 12\ 12\ 12\ 16 \\ 3\ 5\ 8\ 8\ 8\ 10\ 11\ 15\ 15\ 15\ 15 \\ 4\ 4\ 5\ 7\ 7\ 13\ 13\ 13\ 14\ 15 \\ 9\ 9\ 13\ 14\ 16\ 16 \end{matrix}$$

As for X_5 and X_{15}, we have:

$$X_5 : 0 \to \mathcal{P}(5) \to \cdots \to \mathcal{P}(9) \oplus U_2^* \twoheadrightarrow C_5 \to 0$$
$$X_{15} : 0 \to \mathcal{P}(15) \to \cdots \to U_3 \twoheadrightarrow C_{15} \to 0$$

$$U_3 = \begin{matrix} R^*_{1,3} \\ R^*_{1,2} \\ R^*_{1,1} \\ R^*_{1,4} \\ R^*_{1,3} \\ R_{2,3} \end{matrix} \qquad U_3 = \begin{matrix} 9\ 16 \\ 3\ 10\ 11\ 11\ 12 \\ 5\ 5\ 8\ 15\ 15\ 15 \\ 1\ 4\ 4\ 7\ 7\ 7\ 13\ 13\ 13\ 13\ 13\ 14\ 14\ 14\ 14 \\ 6\ 6\ 6\ 9\ 9\ 9\ 16\ 16\ 16\ 16\ 16\ 16 \\ 2\ 2\ 3\ 10\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12\ 12 \\ 5\ 8\ 8\ 8\ 15\ 15\ 15\ 15 \\ 4\ 7\ 13\ 13\ 13\ 14\ 14 \end{matrix}$$

Finally:

$$X_{16} : 0 \to \mathcal{P}(16) \to \cdots \to V_1 \twoheadrightarrow C_{16} \to 0$$

Here V_1 has dimension 330 and is stacked relatively Q-projective as follows:

$$V_1 = \begin{matrix} R_{2,3} \\ R^*_{1,1} \\ R_{1,4} \\ R_{1,1} \\ R^*_{1,2} \\ R_{2,1} \\ R^*_{1,1} \\ R_{1,4} \\ R_{1,1} \\ R_{2,2} \\ R_{1,2} \end{matrix}$$

Complex X_{13} with $\pi = 16$.

$$X_{13} : 0 \to \mathcal{P}(13) \to \cdots \to \mathcal{P}(6) \oplus \mathcal{P}(9) \oplus \mathcal{P}(16) \twoheadrightarrow C_{13} \to 0.$$

As we see, here the module in degree -1 is projective.

Complexes X_9, X_{10} with $\pi = 17$.

We find:

$$X_9 : 0 \to \mathcal{P}(9) \to \cdots \to M_8 \twoheadrightarrow C_9 \to 0,$$

where M_8 is stacked relatively Q-projective of structure:

$$M_8 = \begin{matrix} R_{2,2} \\ R^*_{2,2} \\ R_{2,3} \\ R_{2,1} \\ R_{1,4} \end{matrix} \qquad M_8 = \begin{matrix} 11\ 12 \\ 5\ 8\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 14\ 14 \\ 6\ 6\ 6\ 9\ 9\ 16\ 16\ 16\ 16\ 16 \\ 2\ 3\ 10\ 10\ 11\ 11\ 11\ 11\ 12\ 12\ 12\ 12 \\ 5\ 5\ 8\ 8\ 15\ 15\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 14\ 14 \\ 6\ 9\ 16\ 16 \end{matrix}$$

The complex X_{10} is:

$$X_{10} : 0 \to \mathcal{P}(10) \to \cdots \mathcal{P}(8) \oplus \mathcal{P}(15) \twoheadrightarrow C_{10} \to 0$$

where:

$$M_9 = \begin{matrix} 6\ 16 \\ 2\ 10\ 11\ 12\ 12 \\ 5\ 8\ 8\ 15\ 15\ 15 \\ 1\ 4\ 5\ 7\ 7\ 13\ 13\ 13\ 14\ 14\ 14\ 15 \\ 1\ 6\ 6\ 7\ 9\ 9\ 14\ 16\ 16\ 16\ 16 \\ 2\ 3\ 6\ 10\ 10\ 11\ 11\ 12\ 12\ 16 \\ 5\ 8\ 11\ 12\ 15\ 15 \\ 5\ 13\ 14\ 15 \end{matrix} \qquad M_9 = \begin{matrix} R_{1,4} \\ R_{1,1} \\ R_{1,2} \\ R_{1,3} \\ R^*_{1,4} \end{matrix}$$

Complex X_3 with $\pi = 18$.

$$X_{18} : 0 \to \mathcal{P}(18) \to \cdots \to\to M_{10} \oplus M_{11} \twoheadrightarrow C_{18} \to 0$$

where:

$$M_{10} = \begin{matrix} 5\ 8\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 13\ 14\ 14\ 14 \\ 6\ 6\ 9\ 9\ 9\ 16\ 16\ 16\ 16\ 16 \\ 2\ 3\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12 \\ 5\ 8\ 8\ 11\ 12\ 15\ 15\ 15 \\ 4\ 5\ 7\ 8\ 13\ 13\ 14\ 15\ 15 \\ 9\ 13\ 14\ 16 \end{matrix} \qquad M_{10} = \begin{matrix} R_{1,2} \\ R_{1,3} \\ R^*_{1,4} \\ R^*_{1,3} \\ R^*_{1,2} \end{matrix}$$

$$M_{11} = \begin{matrix} 11\ 12 \\ 5\ 8\ 8\ 15\ 15\ 15 \\ 1\ 4\ 7\ 7\ 13\ 13\ 13\ 14\ 14\ 14 \\ 6\ 6\ 6\ 9\ 9\ 16\ 16\ 16\ 16\ 16 \\ 2\ 3\ 10\ 10\ 11\ 11\ 11\ 12\ 12\ 12 \\ 5\ 5\ 8\ 15\ 15\ 15 \\ 1\ 7\ 11\ 12\ 13\ 14\ 14 \\ 6\ 8\ 15\ 16 \end{matrix} \qquad M_{11} = \begin{matrix} R_{2,2} \\ R^*_{2,1} \\ R_{1,2} \\ R_{1,3} \\ R_{1,4} \end{matrix}$$

The table recording the cohomology of each complex, sorted by the value of π, is the following:

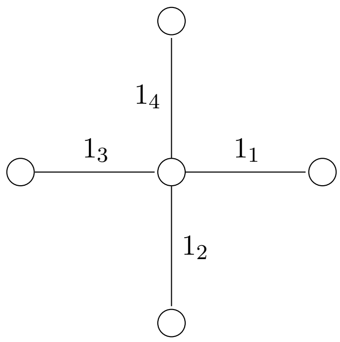
X_i	π	H^{-18}	H^{-17}	H^{-16}	H^{-15}	H^{-14}	H^{-13}	H^{-12}	H^{-11}	H^{-10}	
X_6	8										
X_7	8										
X_{12}	10									1/6/12	
X_2	14					6/2	1/6	1			
X_4	14					4		12	6/12	$6 \oplus 7$	
X_8	14					12/8	12	7	$1 \oplus 7$	1	
X_{14}	15				14		12/6/1	12/6/1			
X_{11}	15				11		7	7			
X_{15}	15				B	12/6/1					
X_5	15				5				7	7	
X_{16}	15				16/7	7			12/6/1		
X_{13}	16			A	C	12/6/1	7				
X_9	17		D	12/8/4	12/8/4	7	7				
X_{10}	17		E	11/5	7	7					
X_3	18	G		I	12/8/4		7	12/6/1		7	
X_i	π	H^{-9}	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	Total
X_6	8		1/6	1							6
X_7	8		7						1	1	7
X_{12}	10	1/6									12
X_2	14				1	1					2
X_4	14	7						1	1		4
X_8	14										8
X_{14}	15										14
X_{11}	15			1	1						11
X_{15}	15										15
X_5	15					1	1				5
X_{16}	15	7	1					1			16-12+1-7-6
X_{13}	16	6/1									13-16-15-14-11+12
X_9	17										16+15+11+9-13-12-4-7-8
X_{10}	17	1	1								F
X_3	18	1					1				J

Here we denote: $A := 16/15/14/13/12/11/8/7/6/2$, $B := 15/12/6/1$, $C := 12/8/6/2$, $D := 16/15/13/12/11/9/8/7/4$, $E := 16/15/14/13/12/11/10/8/6/5/2$, $F := 16+15+14+10-13-12-8-6-2$, $G := 16/15/14/13/11/10/9/5/3$, $I := 16/12/8/7/4$, $J := -16-16-15-$

$14 - 11 - 10 - 9 - 5 - 1 + 13 + 12 + 6 + 3.$

3.2.5 Stable Equivalence

Let Q be a representative of the unique conjugacy class of subgroups of G of order 5. We have $\bar{C}_H(Q) \cong \bar{C}_G(Q) \cong C_5 \rtimes C_4$, and then we have four irreducible $k\bar{C}_G(Q)$-modules $1_1, 1_2, 1_3, 1_4$. As a kN_Δ-module, $k\bar{C}_G(Q)$ is indecomposable of dimension 20, and indeed it is already indecomposable as a $k[\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}]$-module, as it consists of a unique 5-block. The Brauer tree of $B_0(\bar{C}_G(Q))$ is a star:



As we explained in Remark 2.2.4, the method that allowed us to lift a stable equivalence to a perverse equivalence in the previous case $G = \Omega_8^+(2)$ does not work when the algorithm `PerverseEq` returns stacked relatively Q-projective modules in degree -1. This happens to be the case for ${}^2F_4(2)'.2$, as we have observed. At the moment this is preventing us from applying Proposition 2.1.5 and deduce the validity of the conjecture for the principal block $G = {}^2F_4(2)'.2$, therefore this final step of the procedure will be carried out as soon as a generalisation of this method is available.

3.3 $Sp_8(2)$

In this section we will look at is the symplectic $G = Sp_8(2)$, of type C, and again $k := \bar{\mathbb{F}}_5$.

3.3.1 Irreducible $B_0(H)$ and $B_0(G)$-modules

We have fourteen kG-modules lying in the principal block. On the computational level, 51_1 can be found in the atlas online, as well as 35_1 which does not belong to the principal block. It is also easy to build $85_1, 135_1, 510_1, 1055_1, 1785_1$, which do not belong to the principal block but are useful to build $B_0(G)$-modules. In particular: 28238_1 and 3213_1 are constituents of $\Lambda^2(510_1)$; 22932_1 is a constituent of $85_1 \otimes 1055_1$; $866_1, 1274_1, 1939_1$ are constituents of $85_1 \otimes 135_1$; the permutation module of dimension 120 provides 118_1; $238_1, 7015_1$ are constituents of $85_1 \otimes 510_1$ and $35_1 \otimes 1785_1$ respectively; 2738_1 is a constituent of $\Lambda^2(118_1)$; 2534_1 is a constituent of $35_1 \otimes 510_1$; finally, the two last modules 51_1 and 4727_1 can be found by inducing the trivial module of two subgroups of small index, 2295 and 11475 respectively. The two biggest modules (which we cannot store) 22932_1 and 28238_1 can also be found more easily as constituents of $35_1 \otimes 1055$ and of the induction up to G of a simple module of dimension 21 of the maximal subgroup of index 2295, respectively.

As for H, we can check in Magma that its structure is the wreath product $H \cong C_4 \wr C_2 = (C_4 \times C_4) \rtimes C_2$, namely C_2 swaps the coordinates of $C_4 \times C_4$. We have $kH = B_0(H)$ and there are 14 irreducible $B_0(H)$-modules; $T_1, \dots, T_8$ will be the modules of dimension 1, and $T_9, \dots T_{14}$ those of dimension 2. In particular, the fixed labelling makes the projective cover of T_1 be:

$$\mathcal{P}(1) = \begin{array}{c} 1 \\ 10 \\ 7\ 11 \\ 9\ 13 \\ 1\ 2\ 8\ 14 \\ 10\ 12 \\ 6\ 11 \\ 9 \\ 1 \end{array}$$

This identifies each irreducible $B_0(H)$-module except for T_3, T_4, T_5, which are defined to be isomorphic to $T_2 \otimes T_8$, $T_2 \otimes T_6$ and $T_2 \otimes T_7$ respectively. The unitriangular shape of the decomposition matrix realising our perverse equivalence is:

$B_0(G)$, $G = Sp_8(2)$, $\ell = 5$															
$\pi_{1/4}$	χ	S_1	S_7	S_6	S_{10}	S_9	S_{14}	S_{11}	S_2	S_8	S_{12}	S_{13}	S_5	S_3	S_4
0	1_1	1													
3	119_1	1	1												
4	51_1			1											
5	238_1				1										
5	918_1	1		1		1									
6	5712_1	1	1			1	1								
6	1512_1				1			1							
6	2856_1		1						1						
6	2856_2			1		1				1					
7	14688_1		1				1		1		1				
7	3808_1							1				1			
7	30464_1					1	1			1			1		
8	65536_1					1					1	1	1	1	
8	13056_1								1		1				1
	3213_1														1
	3213_2							1		1					
	9639_1										1	1			
	28917_1				1			1		1		1	1		
	38556_1										1			1	1
	51408_1				1								1	1	

3.3.2 Green correspondents

Two Green correspondents are simple: $C_1 \cong T_1$ and $C_6 \cong T_8$. The socle structure of the others are:

$$C_2 = \begin{array}{c} 12 \\ 4\ 5\ 6\ 11 \\ 9\ 9\ 13\ 13 \\ 1\ 2\ 3\ 8\ 14\ 14 \\ 3\ 10\ 10\ 12 \\ 5\ 7\ 11\ 12 \\ 4\ 13 \end{array} \quad C_3 = \begin{array}{c} 10 \\ 5\ 6\ 7\ 11 \\ 9\ 9\ 13\ 13 \\ 1\ 3\ 8\ 14\ 14 \\ 1\ 10\ 12\ 12 \\ 4\ 6\ 10\ 11 \\ 7\ 9 \end{array} \quad C_4 = \begin{array}{c} 10 \\ 7\ 11 \\ 9\ 9\ 13 \\ 1\ 2\ 8\ 14 \\ 10\ 12 \\ 1\ 6\ 11 \\ 9\ 10 \end{array}$$

$$C_5 = \begin{array}{c} 4\ 13 \\ 3\ 9\ 9\ 12\ 14 \\ 1\ 2\ 2\ 4\ 6\ 8\ 10\ 11\ 12\ 14\ 14\ 14 \\ 4\ 5\ 7\ 9\ 9\ 10\ 10\ 10\ 10\ 11\ 12\ 12\ 12\ 12\ 13 \\ 1\ 2\ 4\ 4\ 5\ 5\ 6\ 6\ 7\ 7\ 9\ 11\ 11\ 11\ 11\ 13\ 14\ 14 \\ 2\ 6\ 9\ 9\ 9\ 10\ 10\ 12\ 13\ 13\ 13\ 14 \\ 1\ 2\ 3\ 5\ 7\ 8\ 9\ 10\ 11\ 13\ 14\ 14 \\ 5\ 10\ 12\ 13\ 14 \end{array} \quad C_7 = \begin{array}{c} 2 \\ 10 \\ 11 \\ 9 \\ 2 \end{array} \quad C_8 = \begin{array}{c} 10 \\ 5\ 11 \\ 9\ 9\ 13 \\ 1\ 2\ 3\ 14 \\ 10\ 12 \\ 1\ 4\ 11 \\ 9\ 10 \end{array}$$

$$C_9 = \begin{array}{c} 14 \\ 10\ 12 \\ 6\ 7\ 11 \\ 9\ 13 \\ 14 \end{array} \quad C_{10} = \begin{array}{c} 12 \\ 6\ 11 \\ 9\ 13\ 13 \\ 1\ 3\ 8\ 14 \\ 10\ 12 \\ 7\ 8\ 11 \\ 12\ 13 \end{array} \quad C_{11} = \begin{array}{c} 4\ 13 \\ 3\ 9\ 14 \\ 2\ 3\ 8\ 10\ 12\ 14 \\ 4\ 5\ 7\ 10\ 11\ 12\ 12\ 12 \\ 4\ 5\ 6\ 9\ 11\ 11\ 13 \\ 2\ 9\ 11\ 13\ 13\ 14 \\ 3\ 8\ 10\ 13\ 14 \\ 3\ 5\ 12 \end{array}$$

$$C_{12} = \begin{array}{c} 5\ 7\ 9 \\ 1\ 2\ 13\ 13\ 14\ 14 \\ 2\ 3\ 8\ 10\ 10\ 12\ 12\ 14\ 14 \\ 4\ 4\ 5\ 6\ 6\ 7\ 10\ 10\ 10\ 11\ 11\ 12\ 12 \\ 4\ 5\ 6\ 7\ 9\ 9\ 11\ 11\ 13\ 13 \\ 3\ 5\ 7\ 8\ 9\ 9\ 11\ 13\ 14\ 14 \\ 1\ 2\ 9\ 12\ 12\ 13\ 14 \\ 2\ 4\ 6\ 10\ 14 \end{array} \quad C_{13} = \begin{array}{c} 4\ 5\ 6\ 11 \\ 9\ 9\ 13 \\ 1\ 2\ 3\ 14\ 14 \\ 10\ 10\ 12 \\ 4\ 5\ 7\ 11 \end{array} \quad C_{14} = \begin{array}{c} 10 \\ 4\ 5\ 7\ 11 \\ 5\ 9\ 9\ 13\ 13 \\ 1\ 2\ 3\ 8\ 13\ 14\ 14 \\ 3\ 10\ 12\ 12\ 14 \\ 4\ 6\ 10\ 11\ 12 \\ 4\ 5\ 9 \end{array}$$

A summary of the above informations is given by the following table:

$\pi_{1/4}$	χ	Degree polynomial	$B_0(H)$-mod	$B_0(G)$-mod	dim C_i	Rel. Q-proj
0	1_1	1	$T_1 = 1_1$	$S_1 = 1_1$	$\dim(C_1) = 1$	-
3	119_1	$\frac{1}{2}q\Phi_3\Phi_8$	$T_7 = 1_7$	$S_7 = 118_1$	$\dim(C_7) = 8$	$R_{1,7}$
4	51_1	$\frac{1}{2}q\Phi_6\Phi_8$	$T_6 = 1_6$	$S_6 = 51_1$	$\dim(C_6) = 1$	-
5	238_1	$\frac{1}{2}q^2\Phi_1^2\Phi_3\Phi_8$	$T_{10} = 2_2$	$S_{10} = 238_1$	$\dim(C_{10}) = 28$	$R_{1,2}$
5	918_1	$\frac{1}{2}q^2\Phi_2^2\Phi_6\Phi_8$	$T_9 = 2_1$	$S_9 = 866_1$	$\dim(C_9) = 16$	-
6	5712_1	$q^4\Phi_3\Phi_6\Phi_8$	$T_{14} = 2_8$	$S_{14} = 4727_1$	$\dim(C_{14}) = 47$	$U_{1,4}$
6	1512_1	$\frac{1}{2}q^4\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$T_{11} = 2_3$	$S_{11} = 1274_1$	$\dim(C_{11}) = 64$	$R_{1,8}$
6	2856_1	$\frac{1}{2}q^4\Phi_3\Phi_6\Phi_8$	$T_2 = 1_2$	$S_2 = 1939_1$	$\dim(C_2) = 39$	-
6	2856_2	$\frac{1}{2}q^4\Phi_3\Phi_6\Phi_8$	$T_8 = 1_8$	$S_8 = 2738_1$	$\dim(C_8) = 28$	$R_{1,9}$
7	14688_1	$\frac{1}{2}q^6\Phi_2^2\Phi_6\Phi_8$	$T_{12} = 2_4$	$S_{12} = 7105_1$	$\dim(C_{12}) = 100$	$U_{1,3}$
7	3808_1	$\frac{1}{2}q^6\Phi_1^2\Phi_3\Phi_8$	$T_{13} = 2_5$	$S_{13} = 2534_1$	$\dim(C_{13}) = 29$	$U_{1,6}$
7	30464_1	$\frac{1}{2}q^9\Phi_3\Phi_8$	$T_5 = 1_5$	$S_5 = 22932_1$	$\dim(C_5) = 102$	$U_{1,5}$
8	65536_1	q^{16}	$T_3 = 1_3$	$S_3 = 28238_1$	$\dim(C_3) = 38$	-
8	13056_1	$\frac{1}{2}q^9\Phi_6\Phi_8$	$T_4 = 1_4$	$S_4 = 3213_1$	$\dim(C_4) = 28$	$R_{1,9}$

The modules with vertex Q_1 appearing in the Green correspondence are:

$$R_{1,8} = \begin{array}{c} 11 \\ 13 \\ 3\ 8 \\ 12 \\ 11 \end{array} \qquad R_{1,9} = \begin{array}{c} 1\ 2 \\ 10 \\ 11 \\ 9 \\ 1\ 2 \end{array}$$

$$U_{1,3} = \begin{array}{c} 10 \\ 5\ 7\ 11 \\ 9\ 13\ 13 \\ 3\ 8\ 14\ 14 \\ 10\ 12\ 12 \\ 4\ 6\ 11 \\ 9 \end{array} \quad U_{1,4} = \begin{array}{c} 13 \\ 3\ 8\ 14 \\ 10\ 12\ 12 \\ 4\ 5\ 6\ 7\ 11 \\ 9\ 13\ 13 \\ 3\ 8\ 14 \\ 12 \end{array} \quad U_{1,5} = \begin{array}{c} 12 \\ 4\ 6\ 11 \\ 9\ 9\ 13 \\ 1\ 2\ 14\ 14 \\ 10\ 10\ 12 \\ 5\ 7\ 11 \\ 13 \end{array} \quad U_{1,6} = \begin{array}{c} 9 \\ 1\ 2\ 14 \\ 10\ 10\ 12 \\ 4\ 5\ 6\ 7\ 11 \\ 9\ 9\ 13 \\ 1\ 2\ 14 \\ 10 \end{array}$$

3.3.3 Perverse Equivalence

The complex X_1 being trivial, we write X_i for $i = 2, \dots 14$.

Complex X_7 with $\pi = 3$.

$$X_7 : 0 \to \mathcal{P}(7) \to \mathcal{P}(10) \to \mathcal{P}(2) \oplus R_{1,1} \twoheadrightarrow C_7 \to 0,$$

where

$$R_{1,1} = \begin{matrix} 10 \\ 11 \\ 9 \\ 1\ 2 \\ 10 \end{matrix}$$

has vertex Q_1 and trivial source.

Complex X_6 with $\pi = 4$.

$$X_6 : 0 \to \mathcal{P}(6) \to \mathcal{P}(12) \to \mathcal{P}(3) \oplus \mathcal{P}(12) \to M_{1,1} \twoheadrightarrow C_6 \to 0,$$

where:

$$M_{1,1} = \begin{matrix} 12 \\ 4\ 6\ 11 \\ 9\ 9\ 13 \\ 1\ 2\ 3\ 3\ 8\ 8\ 14\ 14 \\ 10\ 10\ 12\ 12\ 12 \\ 4\ 5\ 6\ 7\ 11\ 11 \\ 9\ 13\ 13 \\ 3\ 8\ 14 \end{matrix} \qquad M_{1,1} = \begin{matrix} R_{1,5} \\ R_{1,4} \\ R^*_{1,1} \\ R_{1,3} \\ R_{1,2} \end{matrix}$$

$$R_{1,2} = \begin{matrix} 3\ 8 \\ 12 \\ 11 \\ 13 \\ 3\ 8 \end{matrix} \quad R_{1,3} = \begin{matrix} 14 \\ 10 \\ 5\ 7 \\ 13 \\ 14 \end{matrix} \quad R_{1,4} = \begin{matrix} 4\ 6 \\ 9 \\ 14 \\ 12 \\ 4\ 6 \end{matrix} \quad R_{1,5} = \begin{matrix} 12 \\ 11 \\ 13 \\ 3\ 8 \\ 12 \end{matrix}$$

Complex X_{10}, X_9 with $\pi = 5$.

$$X_{10}: \ \mathcal{P}(10) \to \mathcal{P}(2) \oplus \mathcal{P}(14) \to \mathcal{P}(2) \oplus \mathcal{P}(4) \oplus \mathcal{P}(14) \to$$
$$\to \mathcal{P}(4) \oplus \mathcal{P}(9) \oplus \mathcal{P}(13) \to \mathcal{P}(12) \oplus R^*_{1,5} \oplus R_{2,1} \twoheadrightarrow C_{10} \to 0,$$

$$X_9: \ \mathcal{P}(9) \to \mathcal{P}(4) \oplus \mathcal{P}(11) \to \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(4) \oplus \mathcal{P}(11) \to$$
$$\to \mathcal{P}(10) \oplus \mathcal{P}(12) \oplus \mathcal{P}(12) \to \oplus M_{1,2} \oplus R^*_{2,1} \twoheadrightarrow C_9 \to 0,$$

$$M_{1,2} = \begin{array}{c} 14 \\ 10\ 12\ 12 \\ 4\ 5\ 6\ 7\ 11\ 11 \\ 9\ 9\ 13\ 13\ 13 \\ 1\ 2\ 3\ 8\ 14\ 14 \\ 10\ 10\ 12 \\ 5\ 7\ 11\ 14 \\ 12\ 13 \end{array} \qquad M_{1,2} = \begin{array}{c} R_{1,3} \\ R_{1,5} \\ R_{1,6} \\ R_{1,7} \\ R^*_{1,6} \end{array}$$

$$R_{2,1} = \begin{array}{c} 9\ 13 \\ 1\ 8\ 14 \\ 10\ 12 \\ 6\ 7\ 11 \\ 9\ 13 \end{array} \qquad R_{1,6} = \begin{array}{c} 12 \\ 4\ 6 \\ 9 \\ 14 \\ 12 \end{array} \qquad R_{1,7} = \begin{array}{c} 11 \\ 9 \\ 1\ 2 \\ 10 \\ 11 \end{array}$$

where $R_{2,1}$ has vertex Q_2 and trivial source, and $R_{1,6}, R_{1,7}$ have vertex Q_1 with trivial source.

Complex X_{14}, X_{11}, X_2, X_8 with $\pi = 6$.

$$X_{14}: \mathcal{P}(14) \to \mathcal{P}(4) \oplus \mathcal{P}(13) \to \mathcal{P}(3) \oplus \mathcal{P}(9) \oplus \mathcal{P}(13) \to \mathcal{P}(5) \oplus \mathcal{P}(10) \oplus \mathcal{P}(12) \oplus \mathcal{P}(11) \to$$
$$\to \mathcal{P}(2) \oplus \mathcal{P}(4) \oplus \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(7) \oplus \mathcal{P}(11) \oplus \mathcal{P}(13) \to \mathcal{P}(10) \oplus M_{1,3} \oplus U_{1,1} \oplus U_{2,1} \twoheadrightarrow C_{14} \to 0,$$

$$U_{2,1} = \begin{array}{c} 4\ 5\ 11 \\ 9\ 9\ 13\ 13 \\ 1\ 2\ 2\ 3\ 3\ 8\ 14\ 14\ 14 \\ 10\ 10\ 10\ 12\ 12\ 12 \\ 4\ 4\ 5\ 5\ 6\ 7\ 11\ 11\ 11 \\ 9\ 9\ 13\ 13 \\ 2\ 3\ 14 \end{array} \qquad U_{1,1} = \begin{array}{c} 5\ 7 \\ 13\ 13 \\ 3\ 8\ 14\ 14 \\ 10\ 12\ 12 \\ 4\ 5\ 6\ 7\ 11 \\ 9\ 13 \\ 14 \end{array} \qquad M_{1,3} = \begin{array}{c} 4\ 6 \\ 9\ 9\ 13 \\ 1\ 2\ 3\ 8\ 14\ 14 \\ 10\ 10\ 12\ 12\ 12 \\ 4\ 5\ 6\ 7\ 11\ 11 \\ 9\ 13\ 13 \\ 3\ 4\ 6\ 8\ 14 \\ 9\ 12 \end{array}$$

$$M_{1,3} = \begin{array}{c} R_{1,4} \\ R^*_{1,1} \\ R^*_{1,6} \\ R_{1,2} \\ R_{1,6} \end{array}$$

where $U_{1,1}$, $U_{2,1}$ have vertex Q_1 and Q_2 respectively, and their source is 3-dimensional.

$$X_{11} : \mathcal{P}(11) \to \mathcal{P}(2) \oplus \mathcal{P}(12) \to \mathcal{P}(4) \oplus \mathcal{P}(10) \oplus \mathcal{P}(12) \to \mathcal{P}(3) \oplus \mathcal{P}(8) \oplus \mathcal{P}(9) \oplus \mathcal{P}(14) \to$$
$$\to \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \mathcal{P}(8) \oplus \mathcal{P}(12) \oplus \mathcal{P}(14) \to \mathcal{P}(4) \oplus \mathcal{P}(10) \oplus M_{1,4} \oplus U^*_{2,1} \twoheadrightarrow C_{11} \to 0,$$

$$M_{1,4} = \begin{array}{c} 13 \\ 3\ 8\ 14 \\ 10\ 12\ 12 \\ 4\ 5\ 6\ 7\ 11\ 11 \\ 9\ 9\ 13\ 13\ 13 \\ 1\ 2\ 3\ 8\ 14\ 14 \\ 10\ 10\ 12 \\ 5\ 7\ 11 \\ 13 \end{array} \qquad M_{1,4} = \begin{array}{c} R^*_{1,5} \\ R^*_{1,3} \\ R_{1,1} \\ R^*_{1,4} \\ R^*_{1,5} \end{array}$$

$$X_2 : \mathcal{P}(2) \to \mathcal{P}(4) \oplus \mathcal{P}(11) \to \mathcal{P}(9) \oplus \mathcal{P}(12 \to \mathcal{P}(10) \oplus \mathcal{P}(12) \oplus \mathcal{P}(12) \to$$
$$\to \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(8) \oplus \mathcal{P}(11) \oplus \mathcal{P}(14) \to M_{1,1} \oplus U_{2,1} \twoheadrightarrow C_2 \to 0,$$

$$X_8: \mathcal{P}(8) \to \mathcal{P}(13) \to \mathcal{P}(5)\oplus\mathcal{P}(13) \to \mathcal{P}(5)\oplus\mathcal{P}(11) \to \mathcal{P}(2)\oplus\mathcal{P}(7)\oplus\mathcal{P}(11) \to \mathcal{P}(10)\oplus M_{1,5} \twoheadrightarrow C_8 \to 0,$$

$$M_{1,5} = \begin{matrix} 11 \\ 9\ 13 \\ 1\ 2\ 3\ 8\ 14 \\ 9\ 10\ 10\ 12\ 12 \\ 1\ 2\ 4\ 5\ 6\ 7\ 11\ 11 \\ 9\ 10\ 13\ 13 \\ 3\ 8\ 11\ 14 \\ 9\ 12 \end{matrix} \qquad M_{1,5} = \begin{matrix} R_{1,7} \\ R^*_{1,6} \\ R_{1,2} \\ R_{1,6} \\ R^*_{1,1} \end{matrix}$$

Complex X_{12}, X_{13}, X_5 with $\pi = 7$.

$$X_{12}: \mathcal{P}(12) \to \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(13) \to \mathcal{P}(3) \oplus \mathcal{P}(3) \oplus \mathcal{P}(5) \oplus \mathcal{P}(8) \oplus \mathcal{P}(14) \to$$

$$\to \mathcal{P}(5) \oplus \mathcal{P}(11) \oplus \mathcal{P}(12) \oplus \mathcal{P}(13) \to \mathcal{P}(2) \oplus \mathcal{P}(9) \oplus \mathcal{P}(13) \oplus \mathcal{P}(13) \oplus \mathcal{P}(13) \to$$

$$\to \mathcal{P}(3) \oplus \mathcal{P}(5) \oplus \mathcal{P}(10) \oplus \mathcal{P}(11) \oplus \mathcal{P}(12) \oplus \mathcal{P}(14) \to$$

$$\to \mathcal{P}(5) \oplus \mathcal{P}(7) \oplus M_{1,6} \oplus M_{1,7} \oplus U^*_{2,1} \twoheadrightarrow C_{12} \to 0,$$

$$M_{1,6} = \begin{matrix} 14 \\ 10\ 12\ 12 \\ 4\ 4\ 5\ 6\ 6\ 7\ 11\ 11 \\ 9\ 9\ 9\ 13\ 13 \\ 1\ 2\ 3\ 8\ 14\ 14\ 14 \\ 10\ 12\ 12\ 12 \\ 4\ 6\ 11 \end{matrix} \qquad M_{1,6} = \begin{matrix} R_{1,6} \\ R_{1,7} \\ R_{1,3} \\ R_{1,5} \\ R_{1,4} \end{matrix}$$

$$M_{1,7} = \begin{array}{c} 9 \\ 1\ 2\ 14 \\ 10\ 10\ 12 \\ 4\ 5\ 6\ 7\ 11\ 11\ 11 \\ 9\ 9\ 9\ 13\ 13 \\ 1\ 2\ 3\ 8\ 14\ 14 \\ 10\ 12\ 12 \\ 4\ 6\ 11 \end{array} \qquad M_{1,7} = \begin{array}{c} R^*_{1,1} \\ R_{1,3} \\ R_{1,5} \\ R_{1,4} \\ R_{1,7} \end{array}$$

$$X_{13} : \mathcal{P}(13) \to \mathcal{P}(5) \oplus \mathcal{P}(12) \to \mathcal{P}(4) \oplus \mathcal{P}(5) \oplus \mathcal{P}(11) \to \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \mathcal{P}(8) \oplus \mathcal{P}(14) \to$$
$$\to \mathcal{P}(10) \oplus \mathcal{P}(12) \oplus \mathcal{P}(12) \to \mathcal{P}(4) \oplus \mathcal{P}(9) \oplus \mathcal{P}(13) \oplus \mathcal{P}(13) \to M_{1,8} \oplus U_{2,1} \twoheadrightarrow C_{13} \to 0,$$

$$M_{1,8} = \begin{array}{c} 4\ 6 \\ 9\ 9\ 13 \\ 1\ 2\ 3\ 8\ 14\ 14\ 14 \\ 10\ 10\ 10\ 12\ 12 \\ 4\ 5\ 5\ 6\ 7\ 7\ 11\ 11 \\ 9\ 13\ 13\ 13 \\ 3\ 8\ 14 \end{array} \qquad M_{1,8} = \begin{array}{c} R^*_{1,6} \\ R_{1,2} \\ R_{1,4} \\ R^*_{1,1} \\ R_{1,3} \end{array}$$

$$X_5 : \mathcal{P}(5) \to \mathcal{P}(4) \oplus \mathcal{P}(13) \to \mathcal{P}(2) \oplus \mathcal{P}(12) \oplus \mathcal{P}(14) \to \mathcal{P}(4) \oplus \mathcal{P}(5) \oplus \mathcal{P}(10) \oplus \mathcal{P}(11) \oplus \mathcal{P}(12) \to$$
$$\to \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(8) \oplus \mathcal{P}(9) \oplus \mathcal{P}(13) \oplus \mathcal{P}(14) \to$$
$$\to \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \mathcal{P}(10) \oplus \mathcal{P}(12) \oplus \mathcal{P}(12) \oplus \mathcal{P}(14) \oplus \mathcal{P}(14) \to$$
$$\to \mathcal{P}(4) \oplus \mathcal{P}(9) \oplus \mathcal{P}(13) \oplus U^*_{1,2} \oplus M_{1,2} \oplus R^*_{2,1} \oplus U^*_{2,1} \twoheadrightarrow C_5 \to 0,$$

where the new module

$$U_{1,2} = \begin{array}{c} 14 \\ 10\ 12 \\ 4\ 5\ 6\ 7\ 11 \\ 9\ 9\ 13 \\ 1\ 2\ 14\ 14 \\ 10\ 10 \\ 5\ 7 \end{array}$$

has vertex Q_1 and its source has dimension 3.

Complex X_3, X_4 with $\pi = 8$.

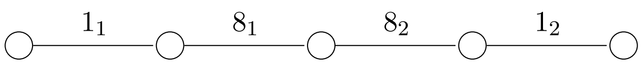
$$X_3 : \mathcal{P}(3) \to \mathcal{P}(3) \oplus \mathcal{P}(4) \to \mathcal{P}(5) \oplus \mathcal{P}(12) \to \mathcal{P}(12) \oplus \mathcal{P}(13) \to \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \mathcal{P}(8) \oplus \mathcal{P}(14) \to$$
$$\to \mathcal{P}(3) \oplus \mathcal{P}(5) \oplus \mathcal{P}(11) \oplus \mathcal{P}(14) \to \mathcal{P}(4) \oplus \mathcal{P}(9) \oplus \mathcal{P}(13) \oplus \mathcal{P}(13) \to$$
$$\to \mathcal{P}(4) \oplus \mathcal{P}(10) \oplus M_{1,3} \oplus R_{2,1} \twoheadrightarrow C_3 \to 0,$$

$$X_4 : \mathcal{P}(4) \to \mathcal{P}(3) \oplus \mathcal{P}(4) \to \mathcal{P}(3) \oplus \mathcal{P}(12) \to \mathcal{P}(12) \oplus \mathcal{P}(13) \to \mathcal{P}(3) \oplus \mathcal{P}(13) \oplus \mathcal{P}(14) \to$$
$$\to \mathcal{P}(3) \oplus \mathcal{P}(5) \oplus \mathcal{P}(11) \oplus \mathcal{P}(14) \to \mathcal{P}(5) \oplus \mathcal{P}(9) \oplus \mathcal{P}(11) \oplus \mathcal{P}(13) \to$$
$$\to \mathcal{P}(10) \oplus M_{1,5} \oplus R_{2,1} \twoheadrightarrow C_4 \to 0.$$

3.3.4 Stable Equivalence

We have two conjugacy classes of subgroups of order 5. We can easily distinguish them by looking at their centralisers: let Q_1 and Q_2 such that $\bar{C}_G(Q_1) \cong S_6$ and $\bar{C}_G(Q_2) \cong A_5$.

Let us look at Q_1. The Brauer tree of the principal block of $\bar{C}_G(Q_1)$ in characteristic 5 is



For Q_2, we have $\bar{C}_G(Q_2) \cong A_5$ and also $\bar{C}_H(Q_2) \cong D_{10}$, exactly like in $\Omega_8^+(2)$. As the candidate complexes for the perverse equivalence do not have any stacked relatively projective module with respect to Q_2, we expect the stable equivalence algorithm to work well when we focus on

Q_2. Indeed, when $Q = Q_2$, we get:

$$
\begin{aligned}
&S = S_1, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};\\
&S = S_2, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_{2,1};\\
&S = S_3, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_{2,1};\\
&S = S_4, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_{2,1};\\
&S = S_5, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R^*_{2,1} \oplus U^*_{2,1};\\
&S = S_6, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};\\
&S = S_7, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};\\
&S = S_8, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};\\
&S = S_9, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R^*_{2,1};\\
&S = S_{10}, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_{2,1};\\
&S = S_{11}, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U^*_{2,1};\\
&S = S_{12}, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U^*_{2,1};\\
&S = S_{13}, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_{2,1};\\
&S = S_{14}, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_{2,1}.
\end{aligned}
$$

This coincides with the contributions of Q_2 in the perverse equivalence in terms of relatively Q_2-projective modules. As for Q_1, we would need to find several stacked relatively Q_1-projective modules, the same that we found when running `PerverseEq`; this will be possible with a more generalised version of the method described in section 2.2.

3.4 $^3D_4(2)$

Let $G = {}^3D_4(2)$. We recall that $|G| = 2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$, hence in this section we fix $k := \overline{\mathbb{F}}_7$. The perversity function $\pi_{\kappa/d}$ that will be used has $\kappa = 1$ and $d = 3$, the order of 2 modulo 7.

3.4.1 Irreducible $B_0(H)$ and $B_0(G)$-modules

We have $kH = B_0(H)$. There are two conjugacy classes of subgroups of order 5 in H, and we denote by Q_1 and Q_2 two representatives. They can be distinguished by looking at their centralisers, in particular we have $\bar{C}_G(Q_2) \cong C_7$ and $\bar{C}_G(Q_1) \cong \mathrm{PSL}(2,7)$.

There are seven kH-modules, of dimensions $1, 1, 1, 2, 2, 2, 3$. The labelling $T_1, \ldots, T_7$ that we choose is the one such that the projective indecomposable module covering the trivial T_1 has the following structure:

$$\mathcal{P}(1) = \begin{array}{c} 1 \\ 6 \\ 7 \\ 5\ 6 \\ 1\ 2\ 7 \\ 4\ 5\ 6 \\ 1\ 7\ 7 \\ 4\ 5\ 6 \\ 1\ 3\ 7 \\ 5\ 6 \\ 7 \\ 5 \\ 1 \end{array}$$

We give the decomposition matrix defining the bijection between simple $B_0(G)$-modules and unipotent characters lying in the principal block. We have:

$B_0(G)$, $G = {}^3D_4(2)$, $\ell = 7$								
$\pi_{1/3}$	Unipotent Character	S_1	S_6	S_2	S_7	S_5	S_4	S_3
0	1_1	1						
3	26_1		1					
5	468_1	1		1				
6	52_1				1			
6	324_1		1			1		
7	1664_1				2	1	1	
8	4096_1			1	1		2	1

3.4.2 Green correspondents

This table summarises the bijection between unipotent characters, $B_0(H)$ and $B_0(G)$-modules.

$\pi_{1/3}$	χ	Polynomial	$B_0(H)$-mod	$B_0(G)$-mod	dim C_i
0	1_1	1	T_1	$S_1 = 1_1$	$\dim(C_1) = 1$
3	26_1	$q(q^4 - q^2 + 1)$	T_6	$S_6 = 26_1$	$\dim(C_6) = 26$
5	468_1	$\frac{1}{2}q^3(q+1)^2(q^4 - q^2 + 1)$	T_2	$S_2 = 467_1$	$\dim(C_2) = 26$
6	324_1	$\frac{1}{2}q^3(q+1)^2(q^2 - q + 1)^2$	T_5	$S_5 = 298_1$	$\dim(C_5) = 102$
6	52_1	$\frac{1}{2}q^3(q-1)^2(q^4 - q^2 + 1)$	T_7	$S_7 = 52_1$	$\dim(C_7) = 52$
7	1664_1	$q^7(q^4 - q^2 + 1)$	T_4	$S_4 = 1262_1$	$\dim(C_4) = 86$
8	4096_1	q^{12}	T_3	$S_3 = 1053_1$	$\dim(C_3) = 24$

There is no summand of vertex Q_1 or Q_2 in each restriction $(S_i)_H$, so each S_i reduces to C_i in $\underline{\mathrm{mod}}(kH)$.

3.4.3 Perverse Equivalence

Complexes X_1 with $\pi = 0$.

The complex related to the trivial module $T_1 \cong C_1$ is the trivial one:

$$X_1 : 0 \to T_1 \cong C_1 \to 0.$$

Complexes X_6 with $\pi = 3$.

For $\pi = 3$, the kH-module T_6 produces the complex:

$$X_6 : 0 \to \mathcal{P}(6) \to \mathcal{P}(7) \to \mathcal{P}(3) \oplus R_1 \twoheadrightarrow C_6 \to 0.$$

Here R_1 has vertex Q_1, and its dimension is 28, with structure:

$$R_1 = \begin{matrix} 2\ 7 \\ 4\ 5 \\ 1\ 7 \\ 4\ 6 \\ 3\ 7 \\ 5\ 6 \\ 2\ 7 \end{matrix}$$

Complexes X_2 with $\pi = 5$.

For $\pi = 3$, the kH-module T_2 produces the complex:

$$X_2 : 0 \to \mathcal{P}(2) \to \mathcal{P}(7) \to \mathcal{P}(5) \oplus P(7) \to \mathcal{P}(4) \oplus P(5) \oplus P(6) \to \mathcal{P}(7) \oplus R_2 \twoheadrightarrow C_2 \to 0.$$

Again, R_2 has dimension 28 and its vertex is Q_2. Its socle structure is:

$$R_2 = \begin{matrix} 4\ 6 \\ 3\ 7 \\ 5\ 6 \\ 2\ 7 \\ 4\ 5 \\ 1\ 7 \\ 4\ 6 \end{matrix}$$

Complexes X_5, X_7 with $\pi = 6$.

$$X_5 : \ \mathcal{P}(5) \to \mathcal{P}(3) \oplus \mathcal{P}(7) \to \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(7) \to \mathcal{P}(4) \oplus \mathcal{P}(4) \oplus \mathcal{P}(5) \to$$
$$\to \mathcal{P}(3) \oplus \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(5) \oplus \mathcal{P}(6) \to \mathcal{P}(2) \oplus \mathcal{P}(7) \oplus M_2 \twoheadrightarrow C_5 \to 0.$$

Here M_2 is a module of dimension 196 filtered by seven modules isomorphic to R_1^*,R_2^*. Its structure is:

$$
M_2 = \begin{array}{c}
4\ 5 \\
1\ 3\ 7\ 7 \\
4\ 4\ 5\ 5\ 6\ 6 \\
1\ 2\ 3\ 3\ 7\ 7\ 7\ 7 \\
4\ 4\ 4\ 5\ 5\ 5\ 5\ 6\ 6\ 6 \\
1\ 1\ 2\ 2\ 3\ 3\ 3\ 7\ 7\ 7\ 7\ 7\ 7\ 7 \\
4\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 6 \\
1\ 2\ 2\ 2\ 3\ 3\ 7\ 7\ 7\ 7\ 7\ 7 \\
4\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 6\ 6 \\
1\ 1\ 2\ 3\ 7\ 7\ 7\ 7 \\
4\ 4\ 4\ 5\ 6\ 6 \\
3\ 3\ 7\ 7
\end{array}
\qquad
M_2 = \begin{array}{c}
R_2^* \\ R_1^* \\ R_2^* \\ R_1^* \\ R_2^* \\ R_1^* \\ R_1^*
\end{array}
$$

$$
\begin{aligned}
X_7 :\ & \mathcal{P}(7) \to \mathcal{P}(4) \oplus \mathcal{P}(5) \to \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(5) \to \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \mathcal{P}(3) \oplus \mathcal{P}(7) \to \\
& \to \mathcal{P}(2) \oplus \mathcal{P}(3) \oplus \mathcal{P}(4) \oplus \mathcal{P}(7) \to \mathcal{P}(6) \oplus M_1 \twoheadrightarrow C_7 \to 0.
\end{aligned}
$$

Here M_1 is a module of dimension 196 filtered by seven modules isomorphic again to R_1^*,R_2^*. Its structure is:

$$
M_1 = \begin{array}{c}
3\ 7 \\
4\ 5\ 5\ 6 \\
1\ 2\ 3\ 7\ 7\ 7 \\
4\ 4\ 4\ 5\ 5\ 5\ 6\ 6 \\
1\ 1\ 2\ 3\ 3\ 7\ 7\ 7\ 7\ 7 \\
4\ 4\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6 \\
1\ 1\ 1\ 2\ 2\ 3\ 3\ 7\ 7\ 7\ 7\ 7\ 7\ 7 \\
4\ 4\ 4\ 4\ 4\ 5\ 5\ 5\ 6\ 6\ 6\ 6 \\
1\ 2\ 3\ 3\ 3\ 7\ 7\ 7\ 7\ 7 \\
4\ 4\ 5\ 5\ 5\ 6\ 6\ 6 \\
2\ 2\ 3\ 7\ 7\ 7 \\
4\ 4\ 5\ 5
\end{array}
\qquad
M_1 = \begin{array}{c}
R_1^* \\ R_2^* \\ R_1^* \\ R_2^* \\ R_1^* \\ R_2^* \\ R_2^*
\end{array}
$$

Complex X_4 with $\pi = 7$.

$$X_4 : \ \mathcal{P}(4) \to \mathcal{P}(3) \oplus \mathcal{P}(4) \to \mathcal{P}(3) \oplus \mathcal{P}(7) \to \mathcal{P}(2) \oplus \mathcal{P}(4) \oplus \mathcal{P}(5) \to$$

$$\to \mathcal{P}(4) \oplus \mathcal{P}(4) \oplus \mathcal{P}(5) \to \mathcal{P}(3) \oplus \mathcal{P}(3) \oplus \mathcal{P}(7) \to \mathcal{P}(2) \oplus M_3 \twoheadrightarrow C_4 \to 0.$$

$$M_3 = \begin{matrix} 4\ 5 \\ 1\ 3\ 3\ 7\ 7\ 7 \\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 6\ 6\ 6 \\ 1\ 1\ 2\ 2\ 3\ 3\ 3\ 7\ 7\ 7\ 7\ 7\ 7\ 7 \\ 4\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 6 \\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 7\ 7\ 7\ 7\ 7\ 7\ 7 \\ 4\ 4\ 4\ 4\ 4\ 5\ 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6 \\ 1\ 1\ 2\ 2\ 3\ 3\ 7\ 7\ 7\ 7\ 7\ 7 \\ 4\ 4\ 4\ 4\ 5\ 5\ 6\ 6 \\ 3\ 3\ 7\ 7 \end{matrix} \qquad M_3 = \begin{matrix} R_2^* \\ R_1^* \\ R_2^* \\ R_1^* \\ R_1^* \\ R_2^* \\ R_1^* \end{matrix}$$

Complexes X_3 with $\pi = 8$.

$$X_3 : \ \mathcal{P}(3) \to \mathcal{P}(3) \oplus \mathcal{P}(3) \to \mathcal{P}(3) \oplus \mathcal{P}(4) \to \mathcal{P}(4) \oplus \mathcal{P}(4) \to \mathcal{P}(4) \oplus \mathcal{P}(7) \to$$

$$\to \mathcal{P}(7) \oplus \mathcal{P}(7) \to \mathcal{P}(4) \oplus \mathcal{P}(5) \oplus \mathcal{P}(7) \to M_1 \twoheadrightarrow C_3 \to 0.$$

The module M_1 in degree -1 is isomorphic to the one that we found in X_7.

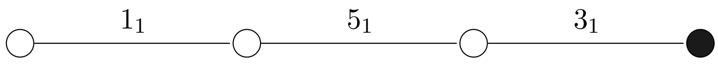
The cohomology table related to the six complexes above is:

X_i	π	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	Total
X_6	3						1/6	1		6
X_2	5				2/6	1/6	1		1	2-1
X_7	6			1/6/7	1/6					7
X_5	6			5		1/6			1	5-6
X_4	7		A	2/6				1		4+6-5-7-7
X_3	8	B								C

In the last two lines, A and B denotes $A := 1/2/4/5/6/6/7/7$ and $B := 1/2/3/4/4/5/5/6/6/7/7/7$, $C := 3+1+5+5+7+7+7-2-4-4-6-6$.

3.4.4 Stable Equivalence

When we consider Q_2, we get $\bar{C}_G(Q_2) \cong C_7$, and then its only irreducible module in characteristic 7 is trivial. In particular, as $\mathcal{E}$ never contains the trivial module, then $\mathcal{E} = \emptyset$ and Q_2 does not give any contribution to Rouquier's construction of the stable equivalence. Indeed, this is in accordance with the complexes coming from the perverse equivalence that we found in the previous section, as we did not find any module of vertex Q_2 in any term. Let us now focus on Q_1; here, we have $\bar{C}_G(Q_1) \cong \mathrm{PSL}(2,7)$. In characteristic 7, the irreducible modules of G are $1_1, 3_1, 5_1, 7_1$, where the first three of them belong to the principal block. The Brauer tree is a line:



The distance between the exceptional vertex and the edge labelled by the trivial module is $d = 2$, then $\mathcal{E} = \{5_1\}$. As $\bar{C}_H(Q_1) \cong C_7 \rtimes C_3$, we have three simple $k\bar{C}_H(Q_1)$-modules $1_1, 1_2, 1_3$, where 1_2 denotes the only module such that $\underline{\mathrm{Hom}}(\mathrm{Res}_{\bar{C}_H(Q)} 5_1, 1_2) \neq \{0\}$. So the map γ sends 5_1 to 1_2. As a N_Δ-module, $e_{\bar{C}_H(Q_1)} k\bar{C}_G(Q_1) e_{\bar{C}_G(Q_1)}$, therefore

$$M_Q = e_{\bar{C}_H(Q_1)} k\bar{C}_G(Q_1) e_{\bar{C}_G(Q_1)}.$$

The restriction of M_Q down to $\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}$ is indecomposable as well, and a projective cover of it is isomorphic to $\bigoplus_{S \in \mathcal{S}_{B_0(\bar{C}_H(Q))}} \mathcal{P}(\gamma(S)) \otimes \mathcal{P}(S)^*$. For $S = S_1, S_2, S_6$, our algorithm computes:

$$S = S_1, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\} \oplus \{0\} \cong \{0\};$$

$$S = S_2, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_2 \oplus \{0\} \cong R_2;$$

$$S = S_6, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1 \oplus \{0\} \cong R_1;$$

As expected, we get the correct module in those three cases where `PerverseEq` provides relatively Q-projective modules in degree -1. As for the previous cases, the completion of the stable equivalence algorithm for the remaining modules S_3, S_4, S_5, S_7 is conditional to a generalisation of the method, as we have explained in Remark 2.2.4.

CHAPTER 4

PERVERSE EQUIVALENCES FOR SPORADIC GROUPS

In the previous chapter, we have combined the use of the algorithm with the theory of perverse equivalences for groups of Lie type; in particular, the theory has provided us with the (conjectural) correct perversity function $\pi_{\kappa/d}$ to use as an input for the algorithm, and a positive output was indeed obtained. When we managed to show that the output of the perverse equivalence algorithm lifted a known stable equivalence to a derived equivalence - namely for $G = \Omega_8^+(2)$ - we have been able to show Broué's conjecture for that case. Although most of the results of this theory are still at a conjectural level, our algorithmic approach allows us to check that the expected perverse equivalences indeed exists. In terms of our algorithm, we have been able to provide a perverse equivalence immediately as the perversity function π was determined by the set of degree polynomials defined by $\mathrm{Uch}(B_0(G))$, together with a suitable bijection between $\mathrm{Uch}(B_0(G))$ and the irreducible $B_0(H)$-modules. For sporadic groups, there is no notion of unipotent character and degree polynomial, and we do not have a formula for the perversity function π as in the Lie type case. However, although we do not have any conjecture about how a derived equivalences between $B_0(G)$ and $B_0(H)$ can be induced when G is not of Lie type, a perverse equivalence could still exist, and our algorithm can be used to try to produce one. It is worth remarking that we "only" need a perversity function $\pi : \mathcal{S}_{B_0(H)} \to \mathbb{Z}_{\geq 0}$ to run that algorithm, and that we can check whether the algorithm is successful provided that we have the set of Green correspondents for each simple $B_0(G)$-module. In the actual algorithmic process, the search of π will mostly be via trial and error.

4.1 $G = J_2$

Broué's conjecture for the principal 5-block of $G = J_2$ has already been proved by Holloway as a result of his PhD dissertation:

Proposition 4.1.1. ([17]) *Broué's Abelian Defect group conjecture holds for the principal* 5-*block* $B_0(G)$ *when* $G = J_2$.

The purpose of this section is to show that no derived equivalence between $B_0(G)$ and $B_0(H)$ is perverse.

4.1.1 Representation theory of H and G

Let $G := J_2$ and $H := N_G(P)$, where $P \in \mathrm{Syl}_5(G), P \cong C_5 \times C_5$. We have that H is maximal in G and $H := P \rtimes D_{12}$. We aim to explain why there is no perverse equivalence between $B_0(H)$ and $B_0(G)$. Here we include a (non-unitriangular) decomposition matrix for the principal block of kG. In order to get a unitriangular form of this matrix, we must choose six ordinary characters χ_i to realise the first six rows and then the bijection with the six simple $B_0(G)$-modules via the unitriangular structure; this is exactly the same procedure used for each group of Lie type treated in the previous chapter, although in that case the modules in the first rows are the unipotent ones, and the modules in the bottom are the non-unipotent. For sporadic groups the split between modules must be worked out by looking at the decomposition matrix.

We see that there are four couples of ordinary characters having the same reduction modulo 5, namely $\{14_1, 14_2\}$, $\{21_1, 21_2\}$, $\{189_1, 189_2\}$, $\{224_1, 224_2\}$; by looking at the decomposition matrix below, it is easy to realise that the only way to achieve a unitriangular form of this matrix must necessarily split at least one of this pair, namely one of them will be among the six in the upper part, and the other one in the bottom. This is roughly the reason why the perverse equivalence approach is not working, and will be formalised properly in Proposition 4.1.2. The decomposition matrices of $B_0(G)$ and $kH = B_0(H)$ in characteristic 5 are the following:

$B_0(G)$, $G = J_2$, $k = \mathbb{F}_5$						
Ord. Character	1_1	14_1	21_1	41_1	85_1	189_1
1_1	1					
14_1		1				
14_2		1				
21_1			1			
21_2			1			
36_1	1	1	1			
63_1	1		1	1		
126_1				1	1	
189_1						1
189_2						1
224_1		1	1			1
224_2		1	1			1
288_1		1			1	1
336_1			1	1	1	1

(4.1.1)

$B_0(H)$, $k = \mathbb{F}_5$						
Ord. Character	1_1	1_2	1_3	1_4	2_1	2_2
1_1	1					
1_2		1				
1_3			1			
1_4				1		
2_1					1	
2_2						1
6_1	1	1			1	1
6_2	1	1			1	1
6_3			1	1	1	1
6_4			1	1	1	1
6_5	1			1	1	1
6_6	1			1	1	1
6_7		1	1		1	1
6_8		1	1		1	1

(4.1.2)

In particular, as ordinary characters we have $\overline{6}_1 = 6_2$, $\overline{6}_3 = 6_4$, $\overline{6}_5 = 6_6$, $\overline{6}_7 = 6_8$.

4.1.2 Perverse equivalence

The information above is enough to conclude that a perverse equivalence between $B_0(H)$ and $B_0(G)$ cannot exist.

Proposition 4.1.2. *Let $G = J_2$, $P \in \mathrm{Syl}_5(G)$ and $H := N_G(P)$. There is no perverse equivalence between $B_0(H)$ and $B_0(G)$.*

Proof. By contradiction, let us suppose that a perverse equivalence $F : \mathcal{D}(B_0(H)) \to \mathcal{D}(B_0(G))$ exists, and in particular the corresponding unitriangular structure for the decomposition matrix of $B_0(G)$ is fixed: this matrix that we consider is a re-arrangement of the decomposition matrix (4.1.1) via permutation of rows and columns. The unitriangular structure for the matrix of $B_0(H)$ is actually diagonal and is given in (4.1.2). In $B_0(G)$, the unitriangular structure of the decomposition matrix determines a subset U of $\mathrm{Irr}(B_0(G))$, where $|U| = 6$ (we can think that U plays the same role that unipotent characters play in the Lie type case). By definition, the perverse equivalence F carries a bijection between $\mathcal{S}_{B_0(H)}$ and $\mathcal{S}_{B_0(G)}$, and the unitriangular structure of the decomposition matrices of $B_0(G)$ and $B_0(H)$ provide natural bijections between $\mathcal{S}_{B_0(H)}$ and $\{1_1, 1_2, 1_3, 1_4, 2_1, 2_2\} \subset \mathrm{Irr}(B_0(H))$, as well as $\mathcal{S}_{B_0(G)}$ and U. As F is a perverse equivalence, then the resulting bijection $\{1_1, 1_2, 1_3, 1_4, 2_1, 2_2\} \to U$ arises from a perfect isometry

$$I_F : CF(H, B_0(H), \mathbb{C}) \to CF(G, B_0(G), \mathbb{C}),$$

namely I_F is preserving the division into upper and lower parts of the triangular decomposition matrix of both $B_0(H)$ and $B_0(G)$. This is all we know about I_F, and it is enough to get a contradiction. In fact, up to sign we have $U = \{I_F(1_1), I_F(1_2), I_F(1_3), I_F(1_4), I_F(2_1), I_F(2_2)\}$. By Lemma 1.2.7, any two of those characters do not have the same reduction modulo ℓ, as $1_1, 1_2, 1_3, 1_4, 2_1, 2_2$ do not either. This implies that the upper part U of our re-arranged unitriangular decomposition matrix of $B_0(G)$ consists of the six ordinary characters which are not pairwise the same when restricted modulo ℓ, namely $U = \{1_1, 36_1, 63_1, 126_1, 288_1, 316_1\}$. But it is now easy to see that this is not compatible with a unitriangular re-arrangement of the decomposition matrix 4.1.1: for example, just notice that the first row can only have 1_1, and it is now impossible to choose a character after 1_1: in order to have a uni-triangular matrix, we should have a character which is decomposing in one Brauer character only, or in two Brauer characters and one of those must be the trivial one. As none of the remaining five characters $36_1, 63_1, 126_1, 288_1, 316_1$ fulfils one of these two requirements, we have a contradiction. □

4.2 $G = He$

4.2.1 Structure of H

Let $G := He$. Let $P \cong C_5 \times C_5$ be a Sylow 5-subgroup of G, and $H := N_G(P)$. From the Atlas, we have that H is maximal in G and $|H| = 1200$; in particular $H \cong P \rtimes S'$, where the ℓ-complement is $S' \cong 4.A_4 \cong (S \rtimes C_3)$, where S is the complement of the Sylow 5-subgroup that we found in the case of $\Omega_8^+(2)$; a presentation (3.1.1) of S was provided.

There is one conjugacy class only of subgroups of H (and, consequently, of G) of order 5, and we denote any of them by Q.

4.2.2 Computational remarks

In the computational setting, we will work over $k := \mathbb{F}_5$. Some modules will turn out to be indecomposable over $\mathbb{F}_5$ but not over $\mathbb{F}_{25}$ (and then not over $\bar{\mathbb{F}}_5$). This is not a problem for us, and indeed working over $\mathbb{F}_5$ rather than $\mathbb{F}_{25}$ (where modules splits as they do over $\bar{\mathbb{F}}_5$) makes computations more efficient. We only have to mention that if a $B_0(H)$-module T is indecomposable over $\mathbb{F}_5$, but splits over $\mathbb{F}_{25}$ as the sum of T' and T'', then the complex X_T over $\mathbb{F}_5$ will split over $\mathbb{F}_{25}$ accordingly, in two complexes $X_{T'}$ and $X_{T''}$.

It is not particularly difficult to get the ten simple kG-modules lying in the principal block. Let us consider a subgroup of index 4116 (there is only one conjugacy classes of these subgroups); this is included in a maximal subgroup of G having index 2058 and isomorphic to $S_4(4):2$. The permutation representation of dimension 4116 provides the simple kG-modules of dimensions $S_3 = 102_1$, 104_1 and 1850_1; we find a copy of 680_1 as well, a simple module lying in the block of defect 1. It turns out that $104_1 \otimes 680_1$ has $S_2 = 10860_1$ as a constituent, as well as $S_{10} = 6394_1$, 4116_1; those two can also be found in $104_1 \otimes 102_1$ and $\Lambda^1(104_1)$ respectively; moreover, $104_1 \otimes 102_1$ also provides $S_9 = 306_1$. The module $S_6 = 6528_1$ is obtained as a constituent of $S_3 \otimes S_9$. Finally, there is a maximal subgroup of index 8330 isomorphic to $2^2.L_3(4).S_3$; the related permutation representation has $S_7 = 4249_1$ among its constituents.

4.2.3 Irreducible $B_0(H)$ and $B_0(G)$-modules

The algebra $kH = B_0(H)$ has ten simple modules, two have dimension 1 (say T_1 and T_2), two have dimension 2 and are not absolutely irreducible (say T_3 and T_4) as they decompose over

$\mathbb{F}_5$, two have dimension 2 and are absolutely irreducible (T_5 and T_6), two have dimension 3 (T_7 and T_8) and finally we have T_9 and T_{10}, being 4-dimensional and are not absolutely irreducible as they split in two 2-dimensional summands on $\mathbb{F}_{25}$. Tensoring by the linear module T_2 swaps T_3 and T_4, T_5 and T_6, T_7 and T_8, T_9 and T_{10}. In the following, T_i is usually abbreviated with i. We can identify them by considering the projective cover $\mathcal{P}(1)$ of the trivial module T_1. The socle factors are:

$$\mathcal{P}(1) = \begin{array}{c} 1 \\ 6 \\ 7 \\ 10 \\ 3\ 8 \\ 9 \\ 7 \\ 5 \\ 1 \end{array}$$

As we mentioned before, T_4 is determined as $T_4 \cong T_3 \otimes T_2$.

In the following list we will state the chosen labelling for the simple kG-modules S_i, and this fixes a bijection between the irreducible kG-modules lying in the principal block and the irreducible modules of kH. As only seven out of ten complexes are available, we cannot state a bijection for T_4, T_5 and T_8, so S_4, S_5 and S_8 are not mentioned now. We set:

$$\begin{array}{c} S_1 = 1_1 \\ S_2 = 10860_1 \\ S_3 = 102_1 \\ S_6 = 6528_1 \end{array} \quad , \quad \begin{array}{c} S_7 = 4249_1 \\ S_9 = 306_1 \\ S_{10} = 6394_1 \end{array}$$

Some of them are not absolutely irreducible, and they are decomposable over $\mathbb{F}_{25}$ as follows: $S_3 = S_{3,1} \oplus S_{3,2}$, $S_9 = S_{9,1} \oplus S_{9,2}$, $S_{10} = S_{10,1} \oplus S_{10,2}$, where

$$\begin{array}{c} S_{3,1} = 51_1 \\ S_{9,1} = 153_1 \\ S_{10,1} = 3197_1 \end{array} \quad , \quad \begin{array}{c} S_{3,2} = 51_2 \\ S_{9,2} = 153_2 \\ S_{10,2} = 3197_2. \end{array}$$

The remaining three $B_0(G)$-modules are not assigned, namely 104_1, 1850_1 and 4116_1. The module 1850_1 splits as a direct sum of 925_1 and 925_2 as an $\mathbb{F}_{25}G$-module, so even if we do not have a π-value, we can think that it corresponds to T_4, which splits over $\mathbb{F}_{25}$ as well. In addition, the requirement $\dim \chi_i \equiv (-1)^{\pi(T_i)} \dim T_i \mod \ell$ would suggest that $\pi(T_i)$ must be odd, as $2058 \equiv (-1) \cdot 2 \mod 5$.

The following table is a (partial) unitriangular decomposition matrix, where the rows are ordered by the π-value, where available. The partial perversity map has been obtained via trial and error. The reducible ordinary characters $102_1, 14994_1, 306_1$ occur as they correspond to S_3, S_9, S_{10}, which are reducible as $\mathbb{F}_{25}G$-modules.

$B_0(G)$, $G = He$, $k = \mathbb{F}_5$											
π	Ord. Character	S_1	S_3	S_7	S_{10}	S_6	S_9	S_2	-	-	-
0	1_1	1									
0	102_1		1								
3	4352_1	1	1	1							
4	14994_1		1	2	1						
5	6528_1					1					
5	306_1						1				
7	21504_1	1		1	1			1			

4.2.4 Green correspondents

The Green correspondent of each S_i is denoted by C_i. We have a full list of the Green correspondents of each simple kG-module lying in the principal block. Two of them are simple, namely C_1 and C_3. Their structures are:

$$C_2 = \begin{array}{c} 2\ 8\ 10 \\ 3\ 5\ 8\ 8\ 9 \\ 4\ 6\ 6\ 6\ 7\ 8\ 9\ 9 \\ 2\ 2\ 7\ 7\ 7\ 7\ 9\ 10 \\ 4\ 5\ 5\ 7\ 8\ 10\ 10 \\ 3\ 5\ 6\ 8\ 8\ 10 \\ 2\ 8\ 9 \end{array} \qquad C_3 = 3 \qquad C_6 = \begin{array}{c} 6 \\ 7 \\ 5\ 10 \\ 3\ 8 \\ 9 \\ 1\ 7 \\ 5\ 6 \end{array}$$

$$C_7 = \begin{array}{c} 8 \\ 6\ 9 \\ 7\ 7 \\ 5\ 10 \\ 8 \end{array} \qquad C_9 = \begin{array}{c} 9 \\ 7\ 7 \\ 5\ 5\ 10\ 10 \\ 1\ 1\ 3\ 8\ 8 \\ 6\ 6\ 9 \\ 3\ 7\ 7 \\ 9\ 10 \end{array} \qquad C_8 = \begin{array}{c} 9 \\ 2\ 2\ 4\ 7\ 7 \\ 4\ 5\ 5\ 5\ 5\ 10\ 10 \\ 1\ 1\ 3\ 8\ 8\ 8\ 8\ 10 \\ 6\ 6\ 8\ 8\ 9\ 9 \\ 4\ 6\ 6\ 7\ 7\ 9 \\ 2\ 2\ 4\ 10 \end{array}$$

The following are the Green correspondents of the non-labelled $B_0(G)$-modules 104_1, 1850_1 and 4116_1, in that order:

$$\begin{array}{c} 4\ 7 \\ 5\ 10 \\ 1\ 8\ 8 \\ 6\ 9 \\ 4\ 7 \end{array} \qquad \begin{array}{c} 4\ 10 \\ 8\ 8\ 10 \\ 3\ 6\ 6\ 8\ 8\ 8\ 8\ 9 \\ 2\ 2\ 4\ 6\ 6\ 6\ 6\ 7\ 7\ 9\ 9\ 9 \\ 2\ 2\ 4\ 5\ 5\ 7\ 7\ 7\ 7\ 10 \\ 2\ 2\ 5\ 5\ 7\ 7\ 8\ 8\ 10\ 10 \\ 3\ 5\ 5\ 8\ 8\ 9\ 10 \\ 4\ 8\ 8\ 9 \end{array} \qquad \begin{array}{c} 9 \\ 4\ 7\ 7 \\ 5\ 5\ 5\ 10\ 10 \\ 1\ 1\ 3\ 8\ 8\ 8 \\ 6\ 6\ 9\ 9 \\ 1\ 4\ 7\ 7 \\ 6\ 10 \end{array}$$

They have dimensions $29, 150$ and 66 respectively.

The partial bijections between the ordinary characters in the decomposition matrix, the $B_0(G)$-modules (given by the triangular structure of the matrix itself) and $\{T_i\}$, as well as the function π, are summarised in the following table:

π	$B_0(H)$-mod	$B_0(G)$-mod	dim C_i
0	T_1	$S_1 = 1_1$	$\dim(C_1) = 1$
0	T_3	$S_3 = 102_1$	$\dim(C_3) = 2$
3	T_7	$S_7 = 4249_1$	$\dim(C_7) = 24$
4	T_{10}	$S_{10} = 6394_1$	$\dim(C_{10}) = 94$
5	T_6	$S_6 = 6528_1$	$\dim(C_6) = 28$
5	T_9	$S_9 = 306_1$	$\dim(C_9) = 56$
7	T_2	$S_2 = 10860_1$	$\dim(C_2) = 110$
?		104_1	29
?		1850_1	150
?		4116_1	66

There is no summand of vertex Q appearing in the decomposition of each $(S_i)_H$, so each module S_i restrict to a sum of its Green correspondent and some projective summands.

4.2.5 Partial perverse equivalence

We now describe the seven (out of ten) complexes that the algorithm has returned. In degree -1 we find the following modules of vertex Q:

$$M_1 = \begin{array}{c} 8\ 8 \\ 6\ 6\ 9\ 9 \\ 2\ 2\ 4\ 4\ 7\ 7\ 7\ 7 \\ 5\ 5\ 5\ 10\ 10\ 10 \\ 1\ 3\ 8\ 8\ 8\ 8\ 8 \\ 6\ 6\ 9\ 9 \\ 2\ 4\ 7 \end{array} \qquad R_1 = \begin{array}{c} 6\ 9 \\ 7\ 7 \\ 5\ 10 \\ 1\ 3\ 8 \\ 6\ 9 \end{array} \qquad R_2 = \begin{array}{c} 5\ 10 \\ 1\ 3\ 8 \\ 6\ 9 \\ 7\ 7 \\ 5\ 10 \end{array}$$

In particular, R_1 and R_2 have trivial source and are 30-dimensional, and M_1 has dimension

90 and source of dimension 3. The list of complexes is the following:

$X_1 : 0 \to T_1 \cong C_1 \to 0,$

$X_3 : 0 \to T_3 \cong C_3 \to 0.$

$X_7 : \mathcal{P}(7) \to \mathcal{P}(6) \oplus \mathcal{P}(9) \to R_1 \oplus \mathcal{P}(8) \twoheadrightarrow C_7 \to 0.$

$X_{10} :\ \mathcal{P}(10) \to \mathcal{P}(4) \oplus \mathcal{P}(6) \oplus \mathcal{P}(6) \oplus \mathcal{P}(9) \to \mathcal{P}(4) \oplus \mathcal{P}(7) \oplus \mathcal{P}(7) \oplus \mathcal{P}(8) \oplus \mathcal{P}(8) \to$

$\to \mathcal{P}(9) \oplus M_1^* \oplus M_1^* \twoheadrightarrow C_{10} \to 0.$

$X_6 : \mathcal{P}(6) \to \mathcal{P}(8) \to \mathcal{P}(2) \oplus \mathcal{P}(8) \to \mathcal{P}(2) \oplus \mathcal{P}(10) \to R_1^* \oplus \mathcal{P}(6) \twoheadrightarrow C_6 \to 0.$

$X_9 :\ \mathcal{P}(9) \to \mathcal{P}(8) \oplus \mathcal{P}(8) \to \mathcal{P}(4) \oplus \mathcal{P}(8) \oplus \mathcal{P}(8) \to \mathcal{P}(4) \oplus \mathcal{P}(5) \oplus \mathcal{P}(5) \oplus \mathcal{P}(10) \to$

$\to \mathcal{P}(9) \oplus R_1^* \oplus R_1^* \twoheadrightarrow C_9 \to 0.$

$X_2 :\ \mathcal{P}(2) \to \mathcal{P}(8) \to \mathcal{P}(5) \oplus \mathcal{P}(8) \to \mathcal{P}(4) \oplus \mathcal{P}(5) \oplus \mathcal{P}(6) \to \mathcal{P}(8) \oplus \mathcal{P}(10) \to$

$\to \mathcal{P}(6) \oplus \mathcal{P}(8) \oplus \mathcal{P}(9) \to \mathcal{P}(2) \oplus \mathcal{P}(10) \oplus R_1 \oplus M_1 \twoheadrightarrow C_2 \to 0.$

The cohomology table related to the complexes that we achieved to get is the following:

X_i	π	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	Total
X_7	3					7		$1 \oplus 3$	7-3-1
X_{10}	4				7/7/10		$1 \oplus 3 \oplus 1$		10-7-7+3+1+1
X_6	5			1/6	1				6
X_9	5			3/9	3				9
X_2	7	6/2	1/6	$1 \oplus (7/10)$		1		1	2-1-1-10+7

4.2.6 Stable Equivalence

We can skip the introductory part as we have $\bar{C}_G(Q) \cong A_5$ and $\bar{C}_H(Q) \cong D_{10}$, so the setting matches the one that we found for $\Omega_8^+(2)$. The isomorphism classes of $C_H(Q), C_G(Q), N_H(Q)$ $N_G(Q), N_\Delta$ as well as the Q-complements of centralisers and normalisers are also the same as in the case $\Omega_8^+(2)$. Hence T_Q is the same kN_Δ-projective module of dimension 50. The images of our simple $B_0(G)$-modules under the usual stable equivalence give the following results:

$$S = S_1, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};$$

$$S = S_2, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1 \oplus M_1;$$

$$S = S_3, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};$$

$$S = S_6, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1^*;$$

$$S = S_7, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1;$$

$$S = S_9, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1^* \oplus R_1^*;$$

$$S = S_{10}, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong M_1^* \oplus M_1^*.$$

This shows that, in the stable category, the images under the partial perverse equivalence that we managed to build coincide with the images of Rouquier's stable equivalence. Unlike the perverse equivalence, we managed to build the images of every simple $B_0(G)$-module. For the remaining three modules we have:

$$S = 104_1, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong M_1^*;$$

$$S = 4116_1, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1^*;$$

$$S = 1850_1, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong M_1 \oplus M_1;$$

In the hypothesis that a perverse equivalence would exist, as the bijection between $\mathcal{S}_{B_0(H)}$ and $\mathcal{S}_{B_0(G)}$ is in general confirmed by the successful outcome of our algorithm, we are now unable to state which of those three modules S_4, S_5 and S_8 are, namely we can only say that $\{104_1, 4116_1, 1850_1\} = \{S_4, S_5, S_8\}$.

4.2.7 Further developments

Question 4.2.1. *Can this partial construction of a perverse equivalence be completed to a full one? Namely, can we:*

- *complete the bijection between $\mathcal{S}_{B_0(H)}$ and $\mathcal{S}_{B_0(G)}$ by finding the remaining unknown correspondence between $\{104_1, 4116_1, 1850_1\}$ and $\{T_4, T_5, T_8\}$ and then defining S_4, S_5, S_8;*
- *complete the perversity function π by assigning $\pi(T_4), \pi(T_5), \pi(T_8)$;*
- *finally produce the three missing complexes X_4, X_5, X_8?*

Computations have given no answer so far. When looking for the information above, it is reasonable to believe that T_4 would correspond to 1850_1, and then $S_4 := 1850_1$; this is supported by the fact that both T_4 and 1850_1 split over $\mathbb{F}_{25}$, and the matching between non-absolutely indecomposable modules is indeed happening for the part of the bijection that we know, i.e. T_3, S_3, T_9, S_9 and T_{10} and S_{10} are all splitting over $\mathbb{F}_{25}$, and the image of the perverse and of the stable equivalence are already decomposable. Under this assumption, the numerical restriction imposed by the perfect isometry would imply that $2058 \equiv (-1)^{\pi(T_4)} \cdot 2 \pmod 5$, and this would imply that $\pi(T_4)$ is odd.

4.3 $G = Suz$

4.3.1 Structure of H

Let $G := Suz$, $k := \bar{\mathbb{F}}_5$, and let $P \cong C_5 \times C_5$ be a Sylow 5-subgroup of G, and $H := N_G(P)$; then $|H| = 600$ and in particular $H \cong P \rtimes (S_3 \times C_4)$; moreover, H is contained in a maximal subgroup isomorphic to $J_2 : 2$, of index 370656 in G. Finally, there are two conjugacy classes of subgroups of H of order 5, denoted by Q_1, Q_2. We distinguish Q_1 and Q_2 by specifying that $|N_G(Q_1)| = 1200$, whereas $|N_G(Q_2)| = 7200$.

4.3.2 Computational remarks

Our computations in Magma show that the Green correspondents of S_1 and S_2 are both simple, T_1 and T_2 respectively, and then we can set $\pi(T_1) = \pi(T_2) = 0$. Using a trail and error approach, the first possible non-zero value for π turns out to be 5. In fact, looking at the decomposition matrix, we can realise that the third choice in the column of the ordinary character must be one among $143_1, 364_1$ and 18954_1. As we must have $\dim \chi_i \equiv (-1)^{\pi(T_i)} \dim T_i \pmod 5$, the π-value for any of these three must be odd, so we test first $\pi = 1$ and $\pi = 3$. By setting $\pi = 1$, testing this choice for the nine remaining $B_0(H)$-modules $T_3, \dots T_{12}$ produced nine candidates complexes. As we have already seen, the module P^{-1} of degree -1 is an extension of the kernel of the previous map $P^{-2} \to P^{-1}$ by the Green correspondent C_i of S_i, and it must by filtered by relatively Q-projective modules, so the dimension must be multiple of 5. The possible Green correspondents have dimensions 13 and 28. The kernels returned by running the algorithm nine times with $\pi(T_1) = \pi(T_2) = 0$, $\pi(T_i) = 1$, $\pi(T_j) >> 0$, $j \neq 1, 2, i$ have dimensions $1, 1, 1, 1, 1, 1, 3, 3, 2, 2$. So the only chance is that P^{-1} is the extension of one of those kernel of dimension 2 (when the algorithm is run on T_{11} and T_{12}) by the Green correspondence of dimension 28, resulting in a module of dimension 30. Unfortunately, it turns out that for T_{11} the only possible extension is the trivial one (direct sum), which does not fulfil the conditions, and in the second case there is a non-trivial extension but not filtered in the desired way. We can now try $\pi = 3$. The kernels that we get for every assignment $\pi(T_i) = 3$, $i = 3, 4, .., 12$ have dimensions: 24,24,24,24,25,25,48,48,48,48. For dimensional reasons we conclude that this choice cannot work, as no extension of any kernel by any of the three Green correspondents of dimensions $13, 13, 28$ would have dimension a multiple of 5. We will see indeed that the choice $\pi(T_4) = \pi(T_6) = \pi(T_{10}) = 5$ works.

The simple $B_0(G)$-modules have been obtained as composition factors of different modules. The four maximal subgroups of smallest index are copies of $G_2(4)$, $3_2.U_4(3).2_{3'}$, $U_5(2)$ and $2^{1+6}.U_4(2)$ of indices 1782, 22880, 32760 and 135135 in G respectively. The permutation representation induced by $G_2(4)$, and so of dimension 1782, provides the simple module $S_2 = 1001$. The permutation representation of dimension 32760 contains $S_{10} = 143_1$, 11869_1 and $S_4 = 363_1$; this last one can be found in the permutation module of dimension 22880 as well. The representation of dimension 135135 is harder to decompose, but still doable and it provides $S_6 = 18953_1$. The small simple modules already available can be used, in particular the exterior square $\Lambda^2(S_4)$ contains 41822_1, and the symmetric square $S^2(S_{10})$ contains 3289_1. Finally, we have two cases where computation did not go through: let us consider the maximal subgroup of index 370656, isomorphic to $J_2 : 2$. The permutation module contains a copy of 75582_1; moreover, $J_2 : 2$ has a non-trivial irreducible representation of dimension 1, which provides a new 370656-dimensional kG-module. It turns out that this module contains 85293_1. Finally, the remaining modules 16785_1 and 116127_1 can be found using modules for $6.G$; in particular, the tensor product of one of the 12-dimensional simple modules and one of the 11076-dimensional ones, chosen such that the tensor product acts trivially on the center C_6, contains both of them as constituents. All these information have been obtained by looking at the Modular Atlas [38] as well as the Atlas of Finite Group Representations [3].

4.3.3 Irreducible $B_0(H)$ and $B_0(G)$-modules

We manage to construct five suitable complexes X_i via our algorithm; therefore, after fixing a labelling on $\mathcal{S}_{B_0(H)}$, we are able to label five simple $B_0(G)$-modules accordingly. We have:

$$S_1 = 1_1,$$
$$S_2 = 1001_1,$$
$$S_4 = 363_1,$$
$$S_6 = 18953_1,$$
$$S_{10} = 143_1,$$

At the current status, we do not have a labelling for the remaining seven $B_0(G)$-modules $3289_1, 11869_1, 16785_1, 41822_1, 75582_1, 85293_1, 116127_1$. The (partial) decomposition matrix in the unitriangular form is:

$B_0(G)$, $G = Suz$, $k = \mathbb{F}_5$						
π	Ord. Character	S_1	S_2	S_4	S_6	S_{10}
0	1_1	1				
0	1001_1		1			
5	364_1	1		1		
5	18954_1	1			1	
5	143_1					1

We have $kH = B_0(H)$ and there are twelve simple $B_0(H)$-modules $T_1, \ldots, T_{12}$; all of them are absolutely simple, eight have dimension one, say $T_1, \ldots, T_8$, and four have dimension 2, say $T_9, \ldots, T_{12}$. The trivial module is T_1. The socle factors of the projective cover of T_1 allows us to fix most of them. The labelling that we fix is such that:

$$\mathcal{P}(1) = \begin{matrix} 1 \\ 10 \\ 2\ 11 \\ 4\ 6\ 9 \\ 1\ 12\ 12 \\ 3\ 5\ 10 \\ 2\ 11 \\ 9 \\ 1 \end{matrix}$$

It remains to define T_7 and T_8. We have $T_8 := T_3 \otimes T_5$ and $T_7 := T_8 \otimes T_2$. In particular, tensoring by T_2 swaps T_3 and T_4, as well as T_5 and T_6, T_7 and T_8, T_9 and T_{10} and finally T_{11} and T_{12}. Moreover, T_3, T_4, T_5, T_6 are such that $T_i \otimes T_i \cong T_2$, $i = 3, 4, 5, 6$, whereas $T_7 \otimes T_7$ and $T_8 \otimes T_8$ are just the trivial module T_1.

By decomposing $\mathrm{Ind}_{Q_i}^H k$ for $i = 1, 2$ we get a list of modules of vertex Q_i, $i = 1, 2$ that could occur in the filtration of the term of degree -1 of our complexes.

4.3.4 Green correspondents

The Green correspondent of each S_i is denoted by C_i. We have:

$$C_2 = 2 \qquad C_4 = \begin{array}{c} 12 \\ 3\ 10 \\ 2\ 11 \\ 4\ 9 \\ 12 \end{array} \qquad C_6 = \begin{array}{c} 12 \\ 5\ 10 \\ 2\ 11 \\ 6\ 9 \\ 12 \end{array} \qquad C_{10} = \begin{array}{c} 10 \\ 2\ 11 \\ 4\ 6\ 9\ 9 \\ 1\ 12\ 12 \\ 3\ 5\ 10 \\ 1\ 2\ 11 \\ 9\ 10 \end{array}$$

In addition, we include the structure of some additional Green correspondents, whose complexes have not been found so far. The following two are the Green correspondents of 3289_1 and 11869_1 respectively.

$$\begin{array}{c} 8\ 9 \\ 6\ 7\ 9\ 12 \\ 1\ 3\ 5\ 7\ 10\ 12\ 12 \\ 3\ 5\ 5\ 5\ 8\ 10\ 10\ 10\ 11\ 11 \\ 2\ 2\ 4\ 6\ 8\ 9\ 11\ 11\ 11\ 11 \\ 4\ 6\ 6\ 7\ 9\ 9\ 9\ 12 \\ 1\ 7\ 10\ 12\ 12 \\ 5\ 6\ 8\ 10 \end{array} \qquad \begin{array}{c} 8\ 9 \\ 4\ 7\ 9\ 12 \\ 1\ 3\ 5\ 7\ 10\ 12\ 12 \\ 3\ 3\ 3\ 5\ 8\ 10\ 10\ 10\ 11\ 11 \\ 2\ 2\ 4\ 6\ 8\ 9\ 11\ 11\ 11\ 11 \\ 4\ 4\ 6\ 7\ 9\ 9\ 9\ 12 \\ 1\ 7\ 10\ 12\ 12 \\ 3\ 4\ 8\ 10 \end{array}$$

Finally, this is the socle structure of the Green correspondent of 41822_1:

$$\begin{array}{c} 3\ 5\ 10 \\ 2\ 8\ 8\ 8\ 11\ 11\ 11 \\ 4\ 4\ 6\ 6\ 9\ 9\ 9\ 9\ 9\ 9 \\ 1\ 1\ 1\ 7\ 7\ 7\ 12\ 12\ 12\ 12\ 12 \\ 3\ 3\ 5\ 5\ 7\ 10\ 10\ 10\ 10 \\ 2\ 8\ 10\ 10\ 11\ 11\ 11 \\ 4\ 6\ 8\ 8\ 9 \end{array}$$

4.3.5 Partial perverse equivalence

Before giving the complexes that we have been able to produce, a brief summary of the current knowledge of the Green correspondent and of the partial perverse equivalence follows:

π	kH-mod	kG-mod	dim C_i	Rel. Q-proj
0	$T_1 = 1_1$	$S_1 = 1_1$	$\dim(C_1) = 1$	-
0	$T_2 = 1_2$	$S_2 = 1001_1$	$\dim(C_2) = 1$	-
5	$T_4 = 1_4$	$S_4 = 363_1$	$\dim(C_4) = 13$	-
5	$T_6 = 1_6$	$S_6 = 18953_1$	$\dim(C_6) = 13$	R_2'
5	$T_{10} = 2_2$	$S_{10} = 143_1$	$\dim(C_{10}) = 28$	R_2''
?		3289_1	74	R_2'
?		11869_1	74	U_2'
?		16785_1	Unknown	Unknown
?		41822_1	77	U_2''
?		75582_1	Unknown	Unknown
?		85293_1	Unknown	Unknown
?		116127_1	Unknown	Unknown

where the modules of vertex Q_2 appearing with the Green correspondents are:

$$R_2' = \begin{array}{c} 1\ 12 \\ 5\ 10 \\ 2\ 11 \\ 6\ 9 \\ 1\ 12 \end{array} \qquad R_2'' = \begin{array}{c} 2\ 11 \\ 6\ 9 \\ 1\ 12 \\ 5\ 10 \\ 2\ 11 \end{array} \qquad U_2' = \begin{array}{c} 4\ 9 \\ 1\ 7\ 12\ 12 \\ 3\ 3\ 5\ 10\ 10\ 10 \\ 2\ 8\ 8\ 11\ 11\ 11 \\ 4\ 4\ 6\ 9\ 9\ 9 \\ 1\ 7\ 12\ 12 \\ 3\ 10 \end{array} \qquad U_2'' = \begin{array}{c} 3\ 10 \\ 2\ 8\ 11\ 11 \\ 4\ 4\ 6\ 9\ 9\ 9 \\ 1\ 7\ 7\ 12\ 12\ 12 \\ 3\ 3\ 5\ 10\ 10\ 10 \\ 2\ 8\ 11\ 11 \\ 4\ 9 \end{array}$$

In particular, $U_2'' \cong U_2' \otimes T_2$.

Complexes X_1, X_2 with $\pi = 0$.

As we said before, $C_1 \cong T_1$ and $C_2 \cong T_2$ are simple and then the complexes X_1, X_2 are

$$X_1 : 0 \to T_1 \cong C_1 \to 0,$$

$$X_2 : 0 \to T_2 \cong C_2 \to 0.$$

Complexes X_4, X_6, X_{10} with $\pi = 5$.

$$X_4 : 0 \to \mathcal{P}(4) \to \mathcal{P}(11) \to \mathcal{P}(5) \oplus \mathcal{P}(11) \to \mathcal{P}(3) \oplus \mathcal{P}(5) \oplus \mathcal{P}(10) \to \mathcal{P}(12) \oplus R_{1,1} \twoheadrightarrow C_4 \to 0$$

$$X_6 : 0 \to \mathcal{P}(6) \to \mathcal{P}(11) \to \mathcal{P}(3) \oplus \mathcal{P}(11) \to \mathcal{P}(3) \oplus \mathcal{P}(5) \oplus \mathcal{P}(10) \to \mathcal{P}(12) \oplus R_{2,1} \twoheadrightarrow C_6 \to 0$$

$$X_{10} : 0 \to \mathcal{P}(10) \to \mathcal{P}(7) \oplus \mathcal{P}(12) \to \mathcal{P}(7) \oplus \mathcal{P}(8) \oplus \mathcal{P}(12) \to$$
$$\to \mathcal{P}(4) \oplus \mathcal{P}(6) \oplus \mathcal{P}(8) \oplus \mathcal{P}(9) \to \mathcal{P}(10) \oplus R^*_{1,1} \oplus R^*_{2,1} \twoheadrightarrow C_{10} \to 0$$

Here we have:

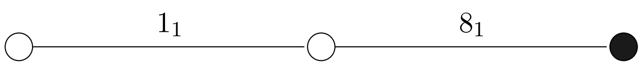
$$R_{1,1} = \begin{array}{c} 3\ 10 \\ 2\ 11 \\ 4\ 9 \\ 1\ 12 \\ 3\ 10 \end{array} \qquad R_{2,1} = \begin{array}{c} 5\ 10 \\ 2\ 11 \\ 6\ 9 \\ 1\ 12 \\ 5\ 10 \end{array}$$

the modules of vertex Q_1 and Q_2 appearing in the above complexes. Here is the cohomology of the complexes that have been shown above:

X_i	π	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	Total
X_4	5		4				1	4-1
X_6	5		6				1	6-1
X_{10}	5		1/10	1				10

4.3.6 Stable Equivalence

Let us consider a representative Q_1 of the conjugacy class of subgroups of order 5 such that $|N_G(Q_1)| = 1200$. We have $\bar{C}_G(Q_1) \cong A_5$ and $\bar{C}_H(Q_1) \cong D_{10}$, so the same setting that occurred in $G = \Omega_8^+(2), He$. We can move on to the conjugacy class represented by Q_2. We have $\bar{C}_G(Q_2) \cong A_6$ and $\bar{C}_H(Q_2) \cong D_{10}$. In characteristic 5, for A_6 we have:



As $d = 1$, we have $\mathcal{E} = \{8_1\}$. Our algorithm returns:

$$S = S_1, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\} \oplus \{0\} \cong \{0\};$$

$$S = S_2, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\} \oplus \{0\} \cong \{0\};$$

$$S = S_4, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_{1,1} \oplus \{0\} \cong R_{1,1};$$

$$S = S_6, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\} \oplus (R_{2,1} \oplus R_2');$$

$$S = S_{10}, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_{1,1}^* \oplus (R_{2,1}^* \oplus R_2'').$$

For S_1, S_2, S_4, the output of the algorithm coincides with the datum from the perverse equivalence. For S_6 and S_{10} we get extra modules of vertex Q_2, namely R_2' and R_2'', and this is what we expected; indeed, R_2' and R_2'' are the non-projective modules appearing together with the Green correspondent C_6 and C_{10}, namely $S_6 \cong R_2' \oplus C_6$ and $S_{10} \cong R_2'' \oplus C_{10}$ in $\underline{\text{mod}}(kH)$. We can also write the image under this stable equivalence of some $B_0(G)$-module for which the perverse equivalence algorithm has not produced a suitable complex yet.

$$S = 3289_1, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_{1,1} \oplus (R_{2,1} \oplus R_2');$$

$$S = 11869_1, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_{1,1} \oplus (U_{2,1} \oplus U_2');$$

$$S = 41822_1, \qquad \bigoplus_{Q=Q_1,Q_2} T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong U_{1,1} \oplus (U_{2,1}^* \oplus U_2'');$$

where $U_{1,1}$, $U_{2,1}$ have vertex Q_1 and Q_2 respectively, have 3-dimensional source, and:

$$U_{1,1} = \begin{matrix} 7\ 12 \\ 3\ 5\ 10\ 10 \\ 2\ 8\ 8\ 11\ 11\ 11 \\ 4\ 6\ 6\ 9\ 9\ 9 \\ 1\ 7\ 7\ 12\ 12\ 12 \\ 3\ 5\ 10\ 10 \\ 8\ 11 \end{matrix} \qquad U_{2,1} = \begin{matrix} 7\ 12 \\ 3\ 5\ 10\ 10 \\ 2\ 8\ 8\ 11\ 11\ 11 \\ 4\ 4\ 6\ 9\ 9\ 9 \\ 1\ 7\ 7\ 12\ 12\ 12 \\ 3\ 5\ 10\ 10 \\ 8\ 11 \end{matrix}$$

4.3.7 Further developments

As the partial perverse equivalence coincides with the (partial) stable equivalence constructed by the algorithm, we can say that:

1. Finding a perverse equivalence could be possible; of course, we need the Green correspondents of each simple $B_0(G)$-module, and at the moment we miss four of them, namely $16785_1, 75582_1, 85293_1, 116127_1$; getting those modules is also necessary to compute the stable equivalence;

2. Some condensation methods could work for these cases; this is a possible direction to look at in order to prove the conjecture for the principal 5-block of $G = Suz$. The theory of condensation is a powerful tool which have been used to deal (computationally) with algebras and modules of high dimension. Some introductory notes about condensation can be found in [28]. As an example, in [29] we can find an application of a condensation methods to find the irreducible modules for Co_2 in characteristic 2.

4.4 $G = Fi_{22}$

We work again in characteristic 5, so let $k := \bar{\mathbb{F}}_5$. Let $P \cong C_5 \times C_5$ be a Sylow 5-subgroup of G, and $H := N_G(P)$; then $|H| = 2400$ and in particular $H \cong P \rtimes S$; the complement S has order 96, and it contains a normal subgroup of order 24 isomorphic to $SL(2,3)$ admitting a cyclic complement of order 4. So $S \cong SL(2,3) \rtimes C_4$. Finally, H is contained in one of the maximal subgroups of G isomorphic to $\Omega_8^+(2) : S_3$, of index 61776. There is a unique conjugacy class of subgroups Q of order 5.

4.4.1 Computational remarks

We are able to get 7 simple modules in the principal block only (out of 16). For standard generators, the matrices generating the simple modules 78_1 and 428_1 are available online on the Atlas of Finite Group Representations. However, in some cases it is convenient to have three generators for G, such that two generate the normaliser H, as this would make the restriction of kG-modules down to H immediate (otherwise, restriction of modules of high dimension, for example 80653_1, could turn out to be computationally difficult). In this latter case, we find 78_1 as a constituent of the permutation module of dimension 142155, arising from the maximal subgroup isomorphic to $2^{10} : M_{22}$ of that index; the maximal subgroup of index 3510 isomorphic to $2.U_6(2)$ provides 428_1. After building 78_1 and 428_1, some others follow quickly: $\Lambda^2(428_1)$ and $78_1 \otimes 428_1$ have 80653_1 and 31954_1 among their constituents respectively. Moreover, $3003_1 = \Lambda^2(78_1)$. It remains to find 1001_1. As a kG-module, 1001_1 is a constituent of $78_1 \otimes 3003_1$, which can be built but it is hard to decompose with current computers; though, 1001_1 can be obtained as a $k[2.G]$-module where the center of order 2 acts trivially. In particular, 1001_1 is a composition factor of $\Lambda^2(352_1)$, where 352_1 is an irreducible $k[2.G]$-module whose matrices can be obtained in the Atlas online.

4.4.2 Irreducible $B_0(H)$ and $B_0(G)$-modules

The algebra $kH = B_0(H)$ has 16 simple modules, all of which are absolutely irreducible. We have that $T_1, \dots, T_4$ are 1-dimensional; $T_5, \dots, T_{10}$ have dimension 2; $T_{11}, \dots, T_{14}$ have dimension 3 and the last two T_{15} and T_{16} have dimension 4. The structure of the projective cover of the trivial module T_1 fixes some of the labelling:

$$
\mathcal{P}(1) = \begin{array}{c} 1 \\ 5 \\ 11 \\ 15 \\ 9\ 13 \\ 16 \\ 12 \\ 6 \\ 1 \end{array}
$$

The module $T_5 \otimes T_5$ determines T_4, as it is the only non-trivial 1-dimensional constituent, and we fix $T_3 := T_4^*$. We set $T_8 := T_4 \otimes T_5$, $T_7 := T_8^*$. This fixes all the simple modules of dimension 2 apart from one, which will be T_{10}; analogously, T_2 is the remaining module of dimension 1. Finally, we set $T_{14} := T_{13} \otimes T_2$.

4.4.3 Green correspondents

Here we write the structure of the Green correspondents that we have been able to get. The last one is the Green correspondent of the simple $B_0(G)$-module of dimension 80653; this is unlabelled as we do not have yet a simple $B_0(H)$-module T_i corresponding to 80653_1 under a (conjectural) perverse equivalence.

$$
C_2 = 2 \qquad C_4 = \begin{array}{c} 9\ 13 \\ 5\ 16 \\ 11\ 12 \\ 6\ 15 \\ 9\ 13 \end{array} \qquad C_5 = \begin{array}{c} 7 \\ 12 \\ 8\ 15 \\ 9\ 14 \\ 16 \\ 2\ 11 \\ 7\ 8 \end{array} \qquad C_7 = \begin{array}{c} 5 \\ 11 \\ 6\ 15 \\ 9\ 13 \\ 16 \\ 1\ 12 \\ 5\ 6 \end{array}
$$

$$C_{12} = \begin{array}{c} 3\ 8 \\ 6\ 14\ 15 \\ 2\ 9\ 13\ 14\ 16 \\ 7\ 7\ 10\ 11\ 16\ 16\ 16 \\ 10\ 11\ 11\ 12\ 12\ 15 \\ 8\ 8\ 13\ 15\ 15 \\ 2\ 5\ 9\ 12\ 14\ 14 \\ 4\ 7\ 15\ 16 \end{array} \quad \begin{array}{c} 5\ 16\ 16 \\ 4\ 10\ 11\ 11\ 12\ 12 \\ 6\ 6\ 6\ 8\ 8\ 15\ 15\ 15\ 15 \\ 1\ 1\ 2\ 9\ 9\ 13\ 13\ 13\ 14\ 14\ 14 \\ 5\ 5\ 7\ 7\ 16\ 16\ 16 \\ 1\ 3\ 10\ 11\ 11\ 12\ 12 \\ 5\ 6\ 15\ 15\ 16 \end{array}$$

4.4.4 Partial perverse equivalence

The partial set of complexes that our algorithm managed to construct is summarised by the following table:

π	kH-mod	kG-mod	dim C_i
0	$T_1 = 1_1$	$S_1 = 1_1$	$\dim(C_1) = 1$
0	$T_2 = 1_2$	$S_2 = 1001_1$	$\dim(C_2) = 1$
5	$T_5 = 2_1$	$S_5 = 78_1$	$\dim(C_5) = 28$
5	$T_7 = 2_2$	$S_7 = 3003_1$	$\dim(C_7) = 28$
9	$T_4 = 1_4$	$S_4 = 428_1$	$\dim(C_4) = 28$
9	$T_{12} = 3_2$	$S_{12} = 31954_1$	$\dim(C_{12}) = 104$

Complexes X_1, X_3 with $\pi = 0$.

We have that $C_1 \cong T_1$ and $C_2 \cong T_2$, so the Green correspondents are simple and two suitable complexes X_1, X_2 are:

$$X_1 : 0 \to T_1 \cong C_1 \to 0,$$

$$X_2 : 0 \to T_2 \cong C_2 \to 0.$$

Complexes X_5, X_7 with $\pi = 5$.

The algorithm finds two suitable complexes attached to the choice $\pi = 5$. These two complexes are produced from T_5 and T_7. We recall that $T_7 \cong T_5 \otimes T_2$.

$$X_5 : \mathcal{P}(5) \to \mathcal{P}(13) \to \mathcal{P}(3) \oplus \mathcal{P}(13) \to \mathcal{P}(3) \oplus \mathcal{P}(15) \to R_1 \oplus \mathcal{P}(7) \twoheadrightarrow C_5.$$

$$X_7 : \mathcal{P}(7) \to \mathcal{P}(14) \to \mathcal{P}(4) \oplus \mathcal{P}(14) \to \mathcal{P}(4) \oplus \mathcal{P}(15) \to R_2 \oplus \mathcal{P}(5) \twoheadrightarrow C_7.$$

$$R_1 = \begin{matrix} 8\ 15 \\ 2\ 9\ 14 \\ 7\ 16 \\ 11\ 12 \\ 8\ 15 \end{matrix} \qquad R_2 = \begin{matrix} 6\ 15 \\ 1\ 9\ 13 \\ 5\ 16 \\ 11\ 12 \\ 6\ 15 \end{matrix}$$

In particular, we notice that $R_2 \cong R_1 \otimes T_2$ and $C_7 \cong C_5 \otimes T_2$. Indeed, we have that both those complexes can be got from the other by tensoring each term by T_2, namely $X_7 \cong X_5 \otimes T_2$ (and vice-versa, as $T_2 \otimes T_2 \cong T_1$).

Complexes X_4, X_{12} with $\pi = 9$.

Here we see that T_4 and T_{12} return suitable complexes when $\pi(T_4) = \pi(T_{12}) = 9$. We have:

$$\begin{aligned} X_4 : \mathcal{P}(4) &\to \mathcal{P}(13) \to \mathcal{P}(6) \oplus \mathcal{P}(13) \to \mathcal{P}(6) \oplus \mathcal{P}(15) \to \mathcal{P}(12) \oplus \mathcal{P}(15) \to \\ &\to \mathcal{P}(10) \oplus \mathcal{P}(11) \oplus \mathcal{P}(12) \to \mathcal{P}(10) \oplus \mathcal{P}(11) \oplus \mathcal{P}(16) \to \\ &\to \mathcal{P}(5) \oplus \mathcal{P}(16) \oplus \mathcal{P}(16) \to \mathcal{P}(9) \oplus \mathcal{P}(13) \oplus R_3 \twoheadrightarrow C_4. \end{aligned}$$

$$\begin{aligned} X_{12} : \mathcal{P}(12) &\to \mathcal{P}(16) \to \mathcal{P}(4) \oplus \mathcal{P}(16) \to \mathcal{P}(4) \oplus \mathcal{P}(9) \oplus \mathcal{P}(13) \to \mathcal{P}(9) \oplus \mathcal{P}(13) \oplus \mathcal{P}(13) \to \\ &\to \mathcal{P}(3) \oplus \mathcal{P}(5) \oplus \mathcal{P}(6) \oplus \mathcal{P}(15) \to \mathcal{P}(6) \oplus \mathcal{P}(15) \oplus \mathcal{P}(15) \to \\ &\to \mathcal{P}(7) \oplus \mathcal{P}(11) \oplus \mathcal{P}(12) \oplus \mathcal{P}(13) \to \mathcal{P}(3) \oplus M \oplus R_4 \twoheadrightarrow C_{12}. \end{aligned}$$

The modules R_3 and R_4 of vertex Q have structure:

$$R_3 = \begin{array}{c} 5\ 16 \\ 11\ 12 \\ 6\ 15 \\ 1\ 9\ 13 \\ 5\ 16 \end{array} \qquad R_4 = \begin{array}{c} 7\ 16 \\ 11\ 12 \\ 8\ 15 \\ 2\ 9\ 14 \\ 7\ 16 \end{array}$$

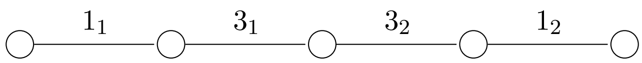
In particular, $R_4 \cong R_3 \otimes T_2$. The module M appearing at degree -1 has dimension 150 and is stacked relatively projective with respect to Q.

We display the cohomology of those complexes here:

X_i	π	H^{-9}	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	Total
X_5	5					5/1	1				5
X_7	5					7/2	2				7
X_4	9	5/4	5/1	1						1	4-1
X_{12}	9	12/7/2	7/2		5/1	1					12-5

4.4.5 Stable Equivalence

We have to deal with one conjugacy class of groups of order 5 only. We have $\bar{C}_G(Q) \cong S_5$ and $\bar{C}_H(Q) \cong C_5 \rtimes C_4$. As a N_Δ-module, we have a decomposition $k\bar{C}_G(Q) = M_Q \oplus P_1 \oplus P_2$, where P_1, P_2 are projective and do not belong to the principal block. The Brauer tree for the principal 5-block of S_5 is:



Let us try to apply the algorithm by considering the far right node as the exceptional vertex. As $d = 3$, we have $\mathcal{E} = \{1_2, 3_1\}$. The algorithm computing the degree -1 of the images under our stable equivalence provides:

$$S = S_1, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};$$

$$S = S_2, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_5;$$

$$S = S_5, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_1;$$

$$S = S_7, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\};$$

$$S = S_4, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_3;$$

$$S = S_{12}, \quad T_Q \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong \{0\}.$$

where

$$R_5 = \begin{array}{c} 2\ 9\ 14 \\ 7\ 16 \\ 11\ 12 \\ 8\ 15 \\ 2\ 9\ 14 \end{array}$$

As we can see, what we get for S_7, S_2, S_{12} under this stable equivalence does not coincide with the datum coming from the perverse equivalence. In particular:

- for S_7, it is worth to mention that a modification of T_Q would provide exactly R_2; in particular, if we include 3_2 in our set $\mathcal{E}$, then the corresponding summand $V := \mathcal{P}_{\gamma(3_2)} \otimes \mathcal{P}_{3_2}$ of T_Q would give the result:

$$S = S_7, \quad V \otimes_{k\bar{C}_G(Q)} S_{N_G(Q)} \cong R_2.$$

  At the moment, we can suppose that the value $\pi(T_7) = 5$ is not the correct one and should then be changed; anyway, due to the very partial status of the potential perverse equivalence, we are not able to explain this fact more accurately;

- it is possible that the perverse equivalence lifting this stable equivalence does not have $\pi(T_2) = 0$, although this choice produces a complex (of length one) satisfying the requirement of the perverse equivalence.

4.4.6 Further development

As for further potential development, the search of a perverse equivalence using this computational approach is out of reach; the same can be said about the stable equivalence, which requires the full set of simple $B_0(G)$-modules. Due to the large dimension of some of those, any attempt to get and store them is not viable at the moment. Although the case of $G = Fi_{22}$ is mostly speculative - due to these computational obstacles - the output consisting of 7 suitable complexes (out of 16 simple $B_0(G)$-modules) shows that the existence of a perverse equivalence proving Broué's conjecture is a reasonable hypothesis, for this group.

4.5 $G = Fi_{23}$

4.5.1 Computational remarks and Green correspondents

Let us consider the Fischer simple group $G = Fi_{23}$, and we set $k := \bar{\mathbb{F}}_5$. In this case there is not much that we can say: apart from the the trivial module, we are only able to produce the simple $B_0(G)$-module of dimension 25806. There is a maximal subgroup of relatively small index, $\Omega_8^+(3) : S_3$, of index 137632. In particular, $\Omega_8^+(3) : S_3$ has two small irreducible representations 1_2 and 2_1; the induced module 1_2^G has a composition series of length 2, whose composition factors are the module of our interest 25806_1 and another simple module of dimension 111862 lying in the non-principal block of defect 2. Moreover, it is maybe possible to get the simple $B_0(G)$-module of dimension 274482: this is a constituent of 2_1^G, the other one being the module of dimension 782 lying in the other block of defect 2. We have not tried it, as we do not expect that we would be able to compute the Green correspondent anyway.

So the only Green correspondent that we can compute is the one of the module of dimension 25806 only. There is only one conjugacy class in H of subgroups of order 5, let Q be one of them. The simple 25806_1 restricts down to H and decomposes into the sum of the Green correspondent, one (non-projective) relatively Q-projective module of dimension 30, and a projective part. The Green correspondent is simple of dimension 1, say T_2. This is easy to identify uniquely: the projective cover of the trivial module $\mathcal{P}(T_1)$ has a module of dimension 4 among its composition factors, say M, and its dual M^*; no more constituents of dimension 4 appear. We have that T_2 is the only non-trivial 1-dimensional modules appearing as a direct summand of $M \otimes M^*$. In summary, at the moment we can just guess that a potential perverse equivalence could start as follows:

π	$B_0(H)$-mod	$B_0(G)$-mod	C_i
0	$T_1 = 1_1$	$S_1 = 1_1$	$C_1 = T_1$
0	$T_2 = 1_2$	$S_2 = 25806_1$	$C_2 = T_2$

The two "small" simple $B_0(G)$-modules, namely 3588_1 and 5083_1, are composition factors of modules which are to large to treat, hence they cannot be obtained and then nothing can be said about their Green correspondents.

Appendices

APPENDIX A

MAGMA

Here we provide the Magma code that has been used, and will be used in the future, to get perverse equivalences and so to test Conjecture 1.1.13. In our MAGMA sessions, we will denote by R the polynomial ring $\mathbb{Q}[x]$.

```
R<x>:=PolynomialRing(Rationals());
```

In this section, as we are in a computational setting, k will necessarily be finite of characteristic ℓ; in each case that we have considered, it is always sufficient to choose the finite field of ℓ elements.

1.1 Perversity function

For a polynomial f and positive coprime integers κ, d, we want to compute $\pi_{\kappa/d}(f)$. For our purposes, f will be a degree polynomial of a unipotent character, which will then be of the shape: $f(q) = aq^m \prod_{j\in J} \Phi_j(q)$ where $m, n \geq 0$, $a \in \mathbb{Q}$, Φ_j is the j-th cyclotomic polynomial and J a set of indices. For a group of Lie type G, the set of unipotent characters and their degree polynomials can be obtained in GAP3 by using `UnipotentCharacters`. For instance, for the group $G = \Omega_8^+(q)$, the list of unipotent characters and their polynomial is given by `Display(UnipotentCharacters(CoxeterGroup( "D", 4)))`. We will start by checking that f has the required shape that we described above, or equivalently that $\frac{f(q)}{aq^m}$ is a product of cyclotomic polynomials. The function $a(f)$ is just the degree of the smallest term of f:

```
function a(f);
return Degree(TrailingTerm(f));
end function;
```

In order to write a test saying if a polynomial is the product of cyclotomic polynomials, we will get a list of enough cyclotomic polynomials and will check whether each irreducible factor of f is in that list. In order to understand how many cyclotomic polynomials we need to produce, we see that if Φ_n is a factor of f for some n, then $\varphi(n) \leq \deg(f)$; in the worst case that f is cyclotomic itself, we could have $\deg(f) = \varphi(n)$. In order to cover all the possible cyclotomic polynomials occurring in f, we need to get a list of the first m cyclotomic polynomials, where m is such that $\deg(f) \leq \varphi(m)$; here f is fixed, and we aim that m is one of the smallest number with this property, not to make the list needlessly long. A reasonable value of m can be obtained by recalling that

$$\frac{m}{e^{\gamma}\log(\log(m)) + \frac{3}{\log(\log(m))}} < \varphi(m),\ \forall m \geq 3, \tag{1.1.1}$$

where γ is the Euler-Mascheroni constant. In particular, we notice that the function in the left-hand side is increasing. So if m is the smallest number such that the left-hand side in 1.1.1 is larger then $\deg(f)$, we can be sure that all the factors of f are contained in the list $\Phi_1, \ldots, \Phi_m$, and such a list is immediately produced by Magma, as we can see in `IsProductOfCyclotomic`. The following computes the left-hand side of Equation 1.1.1.

```
function UpperBoundDegree(n);
g:=EulerGamma(RealField(8));
return n/((Exp(1))^(g)*Log(Log(n))+3/(Log(Log(n))));
end function;
```

Now we can decompose f and check if each factor is a cyclotomic polynomial or not:

```
function IsProductOfCyclotomic(f);
d:=Degree(R!f);
if d eq 0 then
   return false;
end if;
n:=3;
while(UpperBoundDegree(n) le d) do
   n:=n+1;
end while;
L:=[R!CyclotomicPolynomial(i) : i in [1..n]];
F:=Factorization(f);
```

```
for i in [1..#F] do
    if Position(L,R!F[i][1]) eq 0 then
       return false;
    end if;
end for;
return true;
end function;
```

Given a positive integer n and κ, d as above, the following will compute the value of $\phi_{\kappa/d}(\Phi_n)$. We will simply consider each linear factor $(x - e^{\frac{2\pi i t}{n}})$ of $\Phi_n(x) = \prod_{\substack{1 \le t \le n \\ (t,n)=1}} (x - e^{\frac{2\pi i t}{n}})$, and we will increase the number h until the inequality $\theta + 2\pi h \le \frac{2\pi\kappa}{d}$, equivalently $dt + dnh \le kn$, fails. This is iterated for each linear factor of $\Phi_n(x)$.

```
function phiForCyclotomic(k,d,n);
l:=0;
for t in [1..n-1] do
   if Gcd(n,t) eq 1 then
      h:=0;
        while d*t+d*n*h le k*n do
          h:=h+1;
          l:=l+1;
        end while;
   end if;
end for;
return l;
end function;
```

Here we compute the value of $\phi_{\kappa/d}(f)$ where f is a product of cyclotomic polynomial (the first thing will consist of checking this); then we will just have to check that every factor of f appears in the list of cyclotomic polynomials that we create.

```
function phi(k,d,f);
if IsProductOfCyclotomic(f) eq false then
   print "not product of cyclotomic polynomials";
```

```
end if;
c:=0;
n:=3;
while(UpperBoundDegree(n) le Degree(R!f)) do
     n:=n+1;
end while;
L:=[R!CyclotomicPolynomial(i) : i in [1..n]];
F:=Factorization(f);
for i in [1..#F] do
    n:=Position(L,F[i][1]);
    if n eq 1 then
       c:=c+(F[i][2])/2;
    else c:=c+phiForCyclotomic(k,d,n)*F[i][2];
    end if;
end for;
return c;
end function;
```

Finally, we can implement the Perversity function 2.1.4:

```
function PerversityFunction(k,d,f);
if Gcd(k,d) ne 1 then
   return "integers must be coprime";
end if;
g:=f div x^(a(f));
N:=(Degree(f)+a(f))*k/d +phi(k,d,g);
return N;
end function;
```

1.2 Perverse equivalence

For a kG-module U and a list X of simple kG-modules, the following algorithm returns the maximal *semisimple* submodule $V \subseteq U$ with composition factors in the list X; notice that the set of constituents of the zero-module is the empty subset of X, and therefore such a submodule always exists.

```
function SemisimpleXRad(Q,X);
K:=[];
for M in X do
    hom:=AHom(M,Q);
    if Dimension(hom) gt 0 then
       B:=&+[Image(hom.j) : j in [1..Dimension(hom)]];
       K:=Append(K,B);
    end if;
end for;
if #K eq 0 then
   return sub<Q|0>;
else
   T:=&+K; return T;
end if;
end function;
```

The following is a straight application of the previous one. Given a list of simple kG-modules X, a kG-module U and a submodule V, the function returns W such that $V \subseteq W \subseteq U$ and W/V is the X-radical of U/V. This is equivalent to saying that W is the maximal submodule such that $V \subseteq W \subseteq U$ and there is a filtration from V to W whose quotients are in X.

```
function PreImageXRadical(P,M,X);
Q,q:=P/M; N:=M;
_,R:=HasPreimage(SemisimpleXRad(Q,X),q);
/* If N equal R, we do not enter the loop. Indeed, it means that there is nothing */
/* acceptable between M and P, so it returns M itself as M/M is considered to be in X */
while Dimension(N) lt Dimension(R) do
      N:=R; Q,q:=P/N;
      _,R:=HasPreimage(SemisimpleXRad(Q,X),q);
end while;
return R;
```

```
end function;
```

Now let $n \in \mathbb{Z}_{\geq 0}$ and $p : \mathcal{S}_{B_0(H)} \to \mathbb{Z}_{\geq 0}$. Here we get the set J_n in 2.1.5.

```
function J(X,n,p);
I:={};
for M in X do
    if p(M) le n then
       I:=Include(I,M);
    end if;
end for;
return I;
end function;
```

The following returns the injective hull of a kG-module M equipped with an injective map.

```
function InjHull(M);
IM:=Dual(ProjectiveCover(Dual(M))); h:=AHom(M,IM);
repeat f:=Random(h);
until  IsInjective(f);
return IM,f;
end function;
```

We are now finally able to build the algorithm which is indeed returning the complexes X_i, namely the images of $T_i \in \mathcal{S}_{B_0(H)}$ under the perverse equivalence between $\mathcal{D}(B_0(G))$ and $\mathcal{D}(B_0(H))$ that we are trying to construct. Hence, T is a $B_0(H)$-module, X denotes $\mathcal{S}_{B_0(H)}$ and $p : X = \mathcal{S}_{B_0(H)} \to \mathbb{Z}_{\geq 0}$ is a (perversity) function. The following algorithm produces the complex that we need in order to define the desired perverse equivalence between the two principal blocks. Sequences of the kernels, images and cohomologies are also returned.

```
function PerverseEq(T,p,S);
if p(T) eq 0 then
   return "The complex is trivial, T->0";
end if;
n:=p(T); P:=[]; K:=[]; I:=[];
P[1],i:=InjHull(T); T:=Image(i);
```

```
K[1]:=PreImageXRadical(P[1],T,L(S,n-1,p));
for r in [2..n] do
    B:=P[r-1]/K[r-1];
    P[r],i:=InjHull(B);
    I[r-1]:=Image(i);
    Q,q:=P[r]/I[r-1];
    K[r]:=PreImageXRadical(P[r],I[r-1],L(S,n-r,p));
end for;
P[n+1]:=P[n]/K[n]; H:=[K[1]];
for r in [2..n] do
    Append(~H,K[r]/I[r-1]);
end for;
return P,K,H,I;
end function;
```

1.3 Stable equivalence

This algorithm aims to implement the construction of the stable equivalence described in [11]. What we will actually build is the *image* of the simple $B_0(G)$-modules $\mathcal{S}_{B_0(G)}$ under this stable equivalence; the algorithm is then meant to return the complexes of $B_0(H)$-modules described in [11]. We recall the notation of [11] that we have already introduced in 2.2: we have a kN_Δ-module T_Q and a $kN_G(Q)$-module L; previously, L denoted a kG-module, but as we need to restrict it to $N_G(Q)$ even before running the algorithm, we can directly regard it as an $kN_G(Q)$-module. The tensor product $T_Q \otimes_{k\bar{C}_G(Q)} L$ has a natural structure of $N_\Delta \times N_G(Q)$-module, where $N_G(Q)$ acts trivially on T_Q and N_Δ acts trivially on L. Our construction involves $T_Q \otimes_{k\bar{C}_G(Q)} L$ as a $N_H(Q)$-module; this means that we consider the copy of $N_H(Q)$ embedded inside $N_\Delta \times N_G(Q)$ as described in [11], take $(T_Q \otimes L)_{N_H(Q)}$ and build the quotient $(T_Q \otimes L)_{N_H(Q)}/\langle R \rangle_{N_H(Q)}$, where $R = \{ct \otimes l - t \otimes cl \mid c \in \bar{C}_G(Q), t \in T_Q, l \in L\}$; here, the action of $\bar{C}_G(Q)$ on T_Q is meant to be carried by the copy of $\bar{C}_G(Q)$ inside N_Δ, and as for L we have the action of $\bar{C}_G(Q)$ lying inside $N_G(Q)$. With an abuse of notation, we are implicitly using that $\bar{C}_G(Q)$ is fixed at the beginning as a subgroup of G, and then the expression $ct \otimes l - t \otimes cl$ is clear. The main difficulty of this algorithm is about how to build the set R. First, we notice that as we consider the

$N_H(Q)$-span, we do not really need to construct each vector of the shape $ct \otimes l - t \otimes cl$, but we can restrict t to the elements of a basis of T_Q, ℓ to the elements of a basis of L, and c to a set of generators of $\bar{C}_G(Q)$, typically a set of two generators. However, as some L have dimension in the thousands, the tensor $T_Q \otimes L$ would have a prohibitive dimension, but we can skip this problem by remarking two facts:

1. L is the restriction of a simple kG-module down to $N_G(Q)$; then, it is in general decomposable, and it will split in a number of indecomposable non-projective and projective summands: $L = L_1 \oplus \cdots \oplus L_r \oplus P_1 \oplus \cdots \oplus P_s$, where $\{L_i\}_{i=1}^r$ are non-projective and $\{P_j\}_{j=1}^s$ are projective. Decomposing a module of dimension in the thousands can be hard, but in general it is easy to detect all the projective summands - as a projective submodule is a summand - and end up with the non-projective part of L only, which is in general very small. As the tensor product over a subalgebra is linear, we have:

$$T_Q \otimes_{k\bar{C}_G(Q)} L = \left(\bigoplus_{i=1}^{r} T_Q \otimes_{k\bar{C}_G(Q)} L_i \right) \oplus \left(\bigoplus_{j=1}^{s} T_Q \otimes_{k\bar{C}_G(Q)} P_j \right). \tag{1.3.1}$$

   This shows that we can focus on indecomposable modules L only. The module T_Q is in general already indecomposable. Moreover, we realise that it is convenient to compute $T_Q \otimes_{k\bar{C}_G(Q)} P_j$ at the beginning once and for all, so the contribution of the projective part of L to the tensor $T_Q \otimes_{k\bar{C}_G(Q)} L$ is immediately known as soon as we have the decomposition of L.

2. Now we have to find $T_Q \otimes_{k\bar{C}_G(Q)} L'$, where L' is indecomposable. The summands L' of L will often be small enough to proceed with the direct computation, but sometimes not. Although L' is now indecomposable, we notice that in order to get vectors $ct \otimes l - t \otimes cl$, $t \in T_Q, l \in L'$, we only care about the action of $\bar{C}_G(Q)$. So in a computational setting, we can restrict both T_Q and L' further down to $\bar{C}_G(Q)$. For example, if the decomposition of T_Q as a $k\bar{C}_G(Q)$-module is $(T_Q)_{\bar{C}_G(Q)} = T_1 \oplus \cdots \oplus T_m$ for some $m \geq 1$, then a basis of T_Q as a vector space can be chosen as the union of basis for each subspace $T_1, \ldots, T_m$; the massive computational advantage is that an arbitrary element t of the basis of T_Q can now be seen as a vector of some T_j, for $j = 1, \ldots, m$, which is remarkably smaller and so the matrix-vector multiplications $t \cdot c$ is done almost immediately in each case that we considered. As a vector in T_j, then $t \cdot c$ can be easily coerced inside T_Q and tensored with

ℓ; the same argument applies to the $\bar{C}_G(Q)$-summands of L'.

This method allows us to build the term in degree -1 which is supposed to come out from the image of the simple $B_0(G)$-modules under our stable equivalence. The algorithm is mostly based on three parts. First of all, for a given $kN_G(Q)$-module L, we want to detect all the indecomposable summands and their multiplicities - as using the command `IndecomposableSummands()` is not the best option when L has dimension in some thousands. Given L and the list of indecomposable projective $kN_G(Q)$-modules, the following returns a list recording how many times each projective appears as a summand of L, and a module being a copy of L without its projective summands. In the following algorithm, we make use of `RemFree`, that we have not copied here; this take a module M, a positive number n, and for n times it tries to generate a free submodule in M to quotient by. If n is large enough, it quotients M by enough free summand (a free submodule is a summand), we ultimately get the non-free part of M as an output.

```
function SplitL(M,LP);
/* How many times should we try to look for free summands?
The potential number is Dim(M) div #Group(M), the greatest integer
less than or equal to Dim(M)/#G. As RemFree can fail,
we will check two times this number.  */
nf:=Dimension(M) div #Group(M);
if not (nf eq 0) then
   T:=RemFree(M,2*nf);
   else T:=M;
end if;
/* c tells me how many free summands we have removed from M */
c:=(Dimension(M)-Dimension(T)) div #Group(M);
/* LN is a list of integers. It will track how many times each projective is found
inside M, and will be returned in the end. */
if c eq 0 then
   LN:=[0 : x in LP];
   else LN:=[c*Dimension(Socle(x)) : x in LP];
end if;
/* Now we focus on T, to find the remaining projective summands */
```

```
for k in [1..#LP] do
    B,n:=CountProj(T,LP[k]); delete T; T:=B; delete B; LN[k]:=LN[k]+n; delete n;
end for;
return T,LN;
end function;
```

Given the finite group G, the ℓ-local subgroup H (which will always be the normaliser of a Sylow ℓ-subgroup), a cyclic group Q of order ℓ contained in H and its normaliser $N_G(Q)$ - that we denote in the code as NG - the following `StableEqSetup` returns the kN_Δ-module $V = k\bar{C}_G(Q)$, which will provide, as it is described in [11], our module T_Q. Moreover, the code returns the groups denoted as BCG, IBCG, NH, BNH, IBNH, IBCH; they are, respectively, a copy of $\bar{C}_G(Q)$ in $N_G(Q)$, a copy of $\bar{C}_G(Q)$ in N_Δ, a copy of $N_H(Q)$ and $\bar{N}_H(Q)$ inside $N_G(Q)$, a copy of $\bar{N}_H(Q)$ inside N_Δ, and a copy of $\bar{C}_G(Q)$ inside N_Δ. We do not need that the code returns the group N_Δ as well, as it is already carried by V, and it is easily recovered by using the command `Group()`. Each of these group is returned as generated by two elements. Finally, i consists of both the embeddings of $\bar{C}_H(Q)$ and $\bar{C}_G(Q)$ inside N_Δ.

```
function StableEqSetup(G,H,NG,Q);
/* Here we define all groups and subgroups that are involved
in the construction of the stable equivalence.
We make sure that each subgroup is generated by two elements. */
NH:=Normaliser(H,Q);
NH:=GenTwoEl(NH);
CG:=Centraliser(G,Q);
CG:=GenTwoEl(CG);
CH:=Centraliser(H,Q);
CH:=GenTwoEl(CH);
BNH:=Complements(NH,Q)[1];
BNH:=GenTwoEl(BNH);
BNG:=Complements(NG,Q)[1];
BNG:=GenTwoEl(BNG);
/* As requested by the algorithm, BNH must be contained in BNG */
repeat
```

```
      g:=Random(NG); BNG:=Conjugate(BNG,g);
until BNH subset BNG;
BCH:=CH meet BNH;
BCH:=GenTwoEl(BCH);
BCG:=CG meet BNG;
BCG:=GenTwoEl(BCG);
D,i,p:=DirectProduct(NH,NG);
ND:=sub<D|i[1](BCH.1),i[1](BCH.2),i[2](BCG.1),i[2](BCG.2),i[1](BNH.1)*i[2](BNH.1),
i[1](BNH.2)*i[2](BNH.2)>;
DP:=sub<ND|i[1](BCH.1),i[1](BCH.2),i[2](BCG.1),i[2](BCG.2)>;
IBNH:=sub<ND|i[1](BNH.1)*i[2](BNH.1),i[1](BNH.2)*i[2](BNH.2)>;
IBCH:=sub<ND|i[1](BCH.1),i[1](BCH.2)>;
IBCG:=sub<ND|i[2](BCG.1),i[2](BCG.2)>;
/* We can now define the k[BCH]-k[BCG] bimodule k[BCG] */
LG:=[g : g in IBCG];
n:=#IBCG;
k:=GF(#Q);
Zg1:=ZeroMatrix(k,n,n);
for i in LG do Zg1[Position(LG,i),Position(LG,i*IBCG.1)]:=1; end for;
Zg2:=ZeroMatrix(k,n,n);
for i in LG do Zg2[Position(LG,i),Position(LG,i*IBCG.2)]:=1; end for;
Zh1:=ZeroMatrix(k,n,n);
for j in LG do Zh1[Position(LG,j),Position(LG,i[2](BCH.1^(-1))*j)]:=1; end for;
Zh2:=ZeroMatrix(k,n,n);
for j in LG do Zh2[Position(LG,j),Position(LG,i[2](BCH.2^(-1))*j)]:=1; end for;
/* Here we define the action of \bar{N_H(Q)}, so kC_G(Q) is a module
for the whole N_{\Delta} */
Zn1:=ZeroMatrix(k,n,n);
for i in LG do Zn1[Position(LG,i),Position(LG,(IBNH.1)^(-1)*i*IBNH.1)]:=1; end for;
Zn2:=ZeroMatrix(k,n,n);
for i in LG do Zn2[Position(LG,i),Position(LG,(IBNH.2)^(-1)*i*IBNH.2)]:=1; end for;
/* N_{\Delta} has 6 generators: 2 for C_H(Q), 2 for C_G(Q),
```

```
and 2 for the diagonal \bar{N_H(Q)} */
V:=GModule(ND,[Zh1,Zh2,Zg1,Zg2,Zn1,Zn2]);
return V,BCG,IBCG,NH,BNH,IBNH,i,IBCH;
end function;
```

The following algorithm is the main one. This will be used to compute $T_Q \otimes_{k\bar{C}_G(Q)} L'$ when L' is indecomposable as a $kN_G(Q)$-module.

```
function StableEquivalence(Tq,V,H,Q,BCG,IBCG,NH,BNH,IBNH,i);
ND:=Group(Tq);
NG:=Group(V);
Gamma,ii,pp:=DirectProduct(ND,NG);
g:=NH.1;
for x in BNH do
    if x*g^(-1) in Q then y1:=x;
    end if;
end for;
g:=NH.2;
for x in BNH do
    if x*g^(-1) in Q then y2:=x;
    end if;
end for;
s:=hom< NH -> IBNH|i[1](y1)*i[2](y1),i[1](y2)*i[2](y2)>;
/* s is the "quotient" map of NH onto the diagonal copy of BNH inside ND=N_{Delta}. */
/* x1, x2 generate N_H(Q) inside Gamma, and we recall that ii is the embedding
of N_{Delta} and N_G(Q) inside Gamma. */
x1:=ii[1](s(NH.1))*ii[2](NH.1);
x2:=ii[1](s(NH.2))*ii[2](NH.2);
/* Finally, the copy of N_H(Q) which is diagonally embedded inside Gamma: */
NNH:=sub<Gamma|x1,x2>;
k:=Field(Tq);
/* We have V, which is a N_G(Q)-mod, and now we provide it
with the (trivial) action of the other factor of Gamma, i.e. N_{Delta}. */
```

```
d:=Dimension(V);
IdV:=IdentityMatrix(k,d);
a:=ActionGenerators(V);
NewV:=GModule(Gamma,[IdV,IdV,IdV,IdV,IdV,IdV,a[1],a[2]]); delete a;
/* We have T_q now, which is a N_{Delta}-mod, and we give it the (trivial) action
of the other factor of Gamma, i.e. N_G(Q). */
d:=Dimension(Tq);
IdTq:=IdentityMatrix(k,d);
a:=ActionGenerators(Tq);
NewTq:=GModule(Gamma,[a[1],a[2],a[3],a[4],a[5],a[6],IdTq,IdTq]); delete a;
/* Generators of the centraliser. We need them for the relations that we quotient by. */
a1:=ii[1](IBCG.1)^(-1);
b1:=ii[2](BCG.1);
a2:=ii[1](IBCG.2)^(-1);
b2:=ii[2](BCG.2);
Ten:=TensorProduct(NewTq,NewV);
ListT1:=[]; ListT2:=[];
ListV1:=[]; ListV2:=[];
ResTq:=Restriction(NewTq,ii[1](IBCG));
ResV:=Restriction(NewV,ii[2](BCG));
IT:=IndecomposableSummands(ResTq);
print "\nRestricted to the Q-complement of CG(Q), the module Tq decomposes into",
#IT, "summands of dimension:";
l:=[];
for x in IT do Append(~l,Dimension(x));
end for;
l;
IV:=IndecomposableSummands(ResV);
print "\nRestricted to the Q-complement of CG(Q), the module L decomposes into",
#IV, "summands of dimension:";
l:=[];
for x in IV do Append(~l,Dimension(x));
```

```
end for;
l;
"\nNow Tq and L have been decomposed as much as possible, namely the action is
 restricted to the Q-complement of CG(Q).";
NewBasisTq:=[];
/* We want to create vectors of the shape tg*l-t*gl, where * is tensor product.
Here we create two lists, i.e. vectors t*g's and g*l's.   */
for C in IT do
    basC:=Basis(C);
    NewBasisTq:=NewBasisTq cat [NewTq!(ResTq!v) : v in basC];
    LC1:=[NewTq!(ResTq!(v*a1)) : v in basC];
    LC2:=[NewTq!(ResTq!(v*a2)) : v in basC];
    ListT1:=ListT1 cat LC1;
    ListT2:=ListT2 cat LC2;
end for;
"Done with Tq.";
NewBasisV:=[];
for D in IV do
    basD:=Basis(D);
    NewBasisV:=NewBasisV cat [NewV!(ResV!v) : v in basD];
    LD1:=[NewV!(ResV!(v*b1)) : v in basD];
    LD2:=[NewV!(ResV!(v*b2)) : v in basD];
    ListV1:=ListV1 cat LD1;
    ListV2:=ListV2 cat LD2;
end for;
"Done with L, we have our vectors in Tq and L, now we tensor them.";
/* ListT1, ListT2 are coerced vectors in NewTq; ListV1, ListV2 are vectors
of NewV. Now we tensor them, so we get our set of desired vectors in
NewTq x NewV, namely Ten */
ListTen1:=[];
ListTen2:=[];
m:=0;
```

```
for i in [1..#ListT1] do
    for j in [1..#NewBasisV] do
        Append(~ListTen1,Ten!Vector((TensorProduct(ListT1[i],NewBasisV[j])-
        TensorProduct(NewBasisTq[i],ListV1[j]))));
        m:=m+1;
        if (m mod 1000) eq 0 then
           "We have tensored", m, "vectors out of", 2*#ListT1*#NewBasisV;
        end if;
    end for;
end for;
for i in [1..#ListT2] do
    for j in [1..#NewBasisV] do
        Append(~ListTen2,Ten!Vector((TensorProduct(ListT2[i],NewBasisV[j])-
        TensorProduct(NewBasisTq[i],ListV2[j]))));
        m:=m+1;
        if (m mod 1000) eq 0 then
           "We have tensored", m, "vectors out of", 2*#ListT1*#NewBasisV;
        end if;
    end for;
end for;
"\nNow we generate our submodule, quotient, clean off projectives,
and return the final kN(P)-module.";
ListFinal:=ListTen1 cat ListTen2;
"Now we restrict the tensor product to N_H(Q), its dimension is", Dimension(Ten);
Ten:=Restriction(Ten,NNH);
Rel:=sub<Ten|ListFinal>;
Xs:=Ten/Rel;
r:=Representation(Xs);
_,f:=IsIsomorphic(NNH,Group(Xs));
U:=GModule(NH,[r(f(NNH.1)),r(f(NNH.2))]);
p:=#Q;
ProjU:=[ProjectiveCover(x) : x in IrreducibleModules(Group(U),GF(p))];
```

```
n:=Dimension(U) div #Group(U);
U:=RemFree(U,2*n);
U:=RemoveAllProj(U,ProjU);
IV:=Induction(U,H);
return IV;
end function;
```

The final algorithm aims to iterate the previous algorithm `StableEquivalence` over each indecomposable summands of the $kN_G(Q)$-module L. We will use `SplitL` first and we will process the non-projective part of L first, as most of the times the projective summands have been already processed in a previous case and there is no need to redo the calculation. The list of projective indecomposable $kN_G(Q)$-modules is ProjNG. Whether we want to process the projective summands of L as well or not, it is decided by the input "bool".

```
function FinalStabEq(Tq,L,H,Q,BCG,IBCG,NH,BNH,IBNH,i,ProjNG,bool);
/* Here bool decides if we have to compute the tensor of Tq with the projective summands
of L as well. Sometimes, we already know those, as it was done before, and we do not have
to do the same computation again, in this case we set bool=false. */
NG:=Group(L);
T,LN:=SplitL(L,ProjNG);
/* Let us count how many summands L splits into. We will print this result on screen. */
c:=0;
NonZero:=[[Dimension(T),1]];
/* Let us remember that LN is the list of multiplicities of indecomposable projective
inside L. The index h runs across the total number of projectives. */
for h in [1..#ProjNG] do
    if not (LN[h] eq 0) then c:=c+1; Append(~NonZero,[Dimension(ProjNG[h]),LN[h]]);
    end if;
end for;
print "\nThe kN(Q)-module decomposes into summands of dimension (with multiplicities):";
for x in NonZero do x;
end for;
/* first, we find the desired tensor of Tq with the non-projective part of L. We will add
```

```
the "projective" part later. */
print "\nWe work on the tensor no", 1, "out of", #NonZero;
U:=StableEquivalence(Tq,T,H,Q,BCG,IBCG,NH,BNH,IBNH,i);
if not bool then
   return U;
end if;
/* Whenever bool=true, we go on and now we sum the contribution coming from
the projective summands of L. */
num:=2;
for j in [1..#ProjNG] do
    if not (LN[j] eq 0) then
       print "\nWe work on the tensor no", num, "out of", #NonZero;
       StEq:=StableEquivalence(Tq,ProjNG[j],H,Q,BCG,IBCG,NH,BNH,IBNH,i);
       for k in [1..LN[j]] do
           U:=DirectSum(U,StEq);
       end for;
    num:=num+1;
    end if;
end for;
return U;
end function;
```

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