ON SETS, GAMES AND PROCESSES

by

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Abstract

We introduce a two-sided set theory, Amphi-ZF, based on the pure games of Conway et al.; we show Amphi-ZF and ZF are synonymous, with the same result for important subtheories.

An order-theoretic generalisation of Conway games is introduced, and the theory developed. We show the collection of such orders over a poset possesses rich structure, and an analogue of Stone's theorem is proved for posets, using these spaces.

These generalisations are then considered using categories. Compatible set-theoretic notions are introduced, and ideas of regularity axioms with purely game-theoretic motivations are explored; applications to nonstandard arithmetic and multithreaded software are proposed.

We consider topological set theory in a nonstandard model M of Peano arithmetic, and demonstrate that Malitz' original construction works in a finite set theory interpreted by M, with the usual cardinal replaced by a special initial segment. This gives a suitably compact topological model of GPK. Reverse results are also considered: crowdedness of the topological model holds iff the initial segment is strong. A reverse-mathematical principle is investigated, and used it to show that completeness of the topological model is much weaker. Comparisons are made with the standard situation as investigated by Forti et al.

For Bill.

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CHAPTER 1

INTRODUCTION

This thesis reports the more successful research from a period of study between 2009 and 2013; some results here beg for improvement and are unlikely to be optimal. Chapters 2-5 in particular build upon the author's MPhil(Qual) thesis, *Contributions to the theory of combinatorial games* [16]. As such some material should be recognisable to a reader of the MPhil(Qual), but also more complete. Chapter 6 is completely new, although it too has some roots in the predecessor. That chapter is also somewhat unrelated to the rest of the thesis, but certainly reflects an important part of the author's PhD studies.

We begin with a brief survey of relevant literature. This will help us to put the current work into context, and also is an ideal place to introduce necessary concepts and notation.

1.1 Brief survey of relevant literature

1.1.1 Combinatorial game theory

Game theory is a vast subject area, and as such it has been necessary throughout to restrict attention to a very small subset. This was achieved by considering primarily combinatorial games[1], or two-player games of perfect information, in which players alternate in taking

[1] By *combinatorial game* we will always mean a two-player game of perfect information, where the players alternate turns. Often these will come with some notion of strategy, and moving from one position to another. Positions tend to be identified with games (so that each position in a game is considered a game in its own right, and each game may be considered a position in a larger game).

their turns. Although some of the research below will apply outside this restricted area, our focus will only be on such games.

Impartial games, Nim, and the Sprague-Grundy theorem

The first significant result of combinatorial game theory was the Sprague-Grundy theorem, discovered independently by R.P. Sprague [81] and P.M. Grundy [42]. This concerned *impartial* combinatorial games: games between two players, where each move available to one is available to both, and in which the players alternate to take turns and the last to move wins. One particularly distinguished such game is *Nim*, in which players take turns to remove (for instance) counters from a number of stacks. The initial state or *position* of the game is some configuration involving any number $(0, 1, 2, \ldots)$ of heaps of counters. A player must, on their turn, select one heap and remove any positive number of counters from it. Under the *normal play* condition, the last player to remove any counters is the winner (so whichever player cannot make a move when required to loses, and the game terminates).

The outcomes of Nim positions are equivalent to *nimbers*, defined as follows. The empty game (having no moves for either player, and hence a definite win for the second player, II), say $\{\}$, is the nimber ^{*0}. We then set $^*1 = \{^*0\}$, the game in which each player may move only to the position ^{*0} (hence this position is a definite win for the first player). We define

$$\begin{gathered} ^*2 = \{^*0, {}^*1\}, \\ \vdots \\ ^*(n+1) = \{^*0, {}^*2, \ldots, {}^*n\}, \\ \vdots \end{gathered}$$

Assuming that the two players play according to the best possible strategies (in the finite, wellfounded impartial games considered by Sprague and Grundy there is always an optimal strategy), we can talk of the *outcome class* of a position G: say G is an $\mathcal{N}$-position if the next player to move (i.e. I) wins, and a $\mathcal{P}$-position if the previous player to move (i.e. II) wins. For wellfounded such games there is always a winner, and hence every impartial, wellfounded combinatorial game is either an $\mathcal{N}$-position or a $\mathcal{P}$-position. In particular, ^{*0} is a $\mathcal{P}$-position, while every other nimber (allowing the first player to move to ^{*0} and hence to win) is an $\mathcal{N}$-position.

If G, H are games then we may form the *disjunctive sum* $G + H$ by allowing each player, on their turn, to pick exactly one of G, H and proceed by moving in that game as usual. This is an associative operation, and, for the wellfounded combinatorial games, also commutative.

Two games G and H are said to be *equivalent*—in this thesis we write $G \simeq H$—if and only if

$$\forall K \left(G + K \text{ is } \mathcal{N} \Leftrightarrow H + K \text{ is } \mathcal{N}\right)$$

(or the equivalent statement with $\mathcal{N}$ replaced by $\mathcal{P}$). That is, when we play K simultaneously with either H or G, the outcome will be the same. Equivalently, $G \simeq H$ if and only if $G + H$ is a $\mathcal{P}$-position.

The Sprague-Grundy theorem states that every position in every impartial combinatorial game is equivalent to a position in a game of Nim, i.e. to a nimber. It is easily proved by induction, and in fact yields a rule for calculating such equivalence classes: if $G = \{G_1, \dots, G_m\}$, and for each $i \leq n$, $G_i \simeq *n_i$, then

$$G \simeq \text{mex}\{n_1, \dots, n_m\},$$

where the right-hand side denotes the *minimal excludent* of, i.e. the least natural number not among, $n_1, \dots, n_m$. In particular we remark that a single Nim-heap of n counters therefore has the value $^*n$. Much more information on Nim can be easily be found in various locations, but we refer the reader to *On Numbers and Games* [13], hereafter abbreviated as usual as ONAG, and *Winning ways for your mathematical plays* [5], which

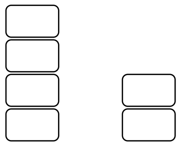

Figure 1.1: Nim heaps ^{*4} and ^{*2} respectively.

we shorten to Winning Ways.

Partisan games

From the 1970s Berlekamp, Conway and Guy contributed a great amount to the theory of combinatorial games. This class of games, a generalisation of the impartial games considered by Sprague and Grundy, contained *partisan* games, i.e. games with similar structure and rules, but which are allowed to offer different moves to players. Significantly these are still games of no chance between two players, who alternate to take turns, and plays (typically) must always end after finitely many turns (implying a certain kind of foundation axiom; see Chapter 2 for more discussion of this). The normal play condition, where the last player to move is the winner, is again predominant because the resulting structure and theory is much richer and more well-behaved (but see the discussion of misère games below).

Above we implicitly regarded impartial games as hereditary containers—sets—of positions. The key abstraction of Conway et al. was to consider partisan games as hereditary two-sided containers: one player, henceforth referred to as *Left*, may move upon his turn to any position from the left side of the container, and the other—*Right*—may choose to move to any position in the right side of the container. The notation

$$x = \{a, b, \ldots \mid c, d, \ldots\}$$

is frequently used, particularly in ONAG. Further, arbitrary positions from the left or right side of the container x are frequently denoted simply x^{L} or x^{R}.[1] The set theory

[1] We develop this notation further by allowing P to denote either L or R, and hence x^{P} denotes an

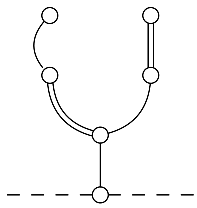

Figure 1.2: A Hackenbush game, after one (poor) move.

of such objects is explored in Chapters 2 (where in particular we are concerned with the synonymy of a particular theory of two-sided sets and ZF, which behaves well with respect to subtheories) and 5 (where we are more interested in the order theoretic consequences of a few basic axioms mixing existence of strategies with players' available positions).

The games receiving most discussion in ONAG and Winning Ways are wellfounded combinatorial games, and as such 'pure' games may be constructed just as a ZF-universe of sets may be constructed by iteration of the power set operator; see ONAG [13, Ch.0] and Chapter 2. However these pure games are not the only games for which we might discuss a meaningful notion of membership. Equally we might consider any Nim-position $^*n$ to be a member (on both sides) of $^*(n+1)$, or the first Hackenbush position of Figure 1.2 to be a member of the position in Figure 1.3.[1] Such membership will often not be extensional, but this does not affect the rich theory of combinatorial games.

There is a natural preorder on any class of such games, defined by [13, p.4]

$$x \leq y \Leftrightarrow \text{all}\, x^{\mathrm{L}} \not\geq y \text{ and } x \not\geq \text{all}\, y^{\mathrm{R}}, \tag{1.1}$$

which can tell us a lot about wellfounded games when played under the normal play

element from the left or right (but not necessarily both) side of x. In fact P may be seen as a variable of a particular sort in a multisorted language, where L and R are constants representing the only values of that sort. We will frequently form logical formulas such as

$$\exists x \bigwedge_{\mathrm{P}}(\forall y(y \notin_{\mathrm{P}} x)) \qquad \text{(there exists an empty game)},$$

and on occasion introduce a second variable, Q, of the same sort.

[1]Hackenbush is typically played on a graph, where players alternate in removing an edge. Some versions admit different edge colourings, and each player may only remove edges of their designated colour. Above we use the convention that the single-lined edges belong to Left, and the double-lined edges belong to Right. Hence the position in Figure 1.2 is a Left-member of that in Figure 1.3.

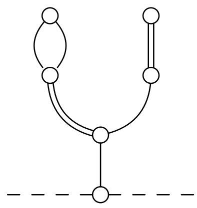

Figure 1.3: A Hackenbush game, before moving.

condition. In particular, $\leq \cap \geq$ is precisely the relation $\simeq$ defined above[1]. A more general version requires the definition of a second order, denoted $\lhd\!\mid$:[2]

$$\begin{aligned} x \leq y \quad &\Leftrightarrow \quad \text{all}\, x^{\mathrm{L}} \lhd\!\mid y \quad \text{and} \quad x \lhd\!\mid \text{all}\, y^{\mathrm{R}} \\ x \lhd\!\mid y \quad &\Leftrightarrow \quad \text{some}\, x^{\mathrm{R}} \leq y \quad \text{or} \quad x \leq \text{some}\, y^{\mathrm{L}} \end{aligned}$$

The relations $\leq$ and $\lhd\!\mid$ correspond to preferability for the second and first players respectively: briefly, $x \leq y$ if and only if Left will fare better in y than in x when playing second (equivalently, Right will prefer to play in x than y when moving first), and $x \lhd\!\mid y$ if and only if Left will fare better in y than x when moving first.

The advantage of this two-ordered approach is that it allows illfounded, nondetermined games to be considered under the same framework; when restricted to wellfounded games, the two notions of $\leq$ coincide and $\lhd\!\mid$ becomes $\ngeq$.

The outcome classes discussed earlier extend to the set $\{\mathcal{N}, \mathcal{P}, \mathcal{L}, \mathcal{R}\}$, where $\mathcal{L}$, $\mathcal{R}$

[1] See ONAG [13, Theorem 54].

[2] See ONAG [13, p.78], where the same definition is given. Notice the use of 'all' and 'some'. We will make use of this notation frequently, as it makes many expressions compact but also much more readable. Informally we define an expression of the form

$$P(\overline{x}, \text{`some}\, y')$$

to mean

$$\exists y \; P(\overline{x}, y),$$

and similarly the occurrence of 'all y' in an expression P should be regarded as shorthand for $\forall y\, P$. We will mix this notation with quantifiers, but only when the scope of the 'all' or 'some' quantification is clear. Occasionally we will use brackets to contain these quantifications (so for example $\forall x\,(x \leq \text{all}\, y)$ is valid shorthand for $\forall x\,(\forall y\,(x \leq y)))$, but we stipulate that 'some' and 'all' take precedence over conjunction, disjunction and implication, so that (1.1) means

$$x \leq y \Leftrightarrow ((\text{all}\, x^{\mathrm{L}} \ngeq y) \wedge (x \ngeq \text{all}\, y^{\mathrm{R}})).$$

indicate a win for Left or Right respectively (regardless of who plays first). Conway et al. [13, 5] prove several results relating the orders $\leq$ and $\lhd\!\vert$, these outcome classes and the disjunctive sum. In particular from this analysis a notion of value—informally, the equivalence classes modulo $\simeq$—arises. If $\mathrm{Values}(\mathcal{G})$ denotes a class of such values (or of representatives for the equivalence classes if we are concerned with foundational issues here) of the wellfounded pure partisan games, then $\mathrm{Values}(\mathcal{G})$ is an abelian group with the disjunctive sum as the group operation. (Moreover, it has been proved to be a universally embedding abelian group by Lurie [64].) The theory of wellfounded partisan games is a powerful generalisation of the theory developed by Sprague and Grundy.

There has been much research into the surreal numbers, a distinguished subclass of the pure combinatorial games. Such a game x is a *surreal number* if $\mathrm{all}\, x^{\mathrm{L}} < \mathrm{all}\, x^{\mathrm{R}}$ (that is, $x^{\mathrm{L}} \leq x^{\mathrm{R}}$ and $x^{\mathrm{L}} \not\geq x^{\mathrm{R}}$ for all $x^{\mathrm{L}}, x^{\mathrm{R}}$). The surreal numbers constitute a universally embedding field of characteristic 0, and there has been much interest in the model theory of, and analysis within, this class. We will not be interested in the specifics of surreal numbers, however.

Another interesting and developing field of study in combinatorial game theory is misère play: where the normal play condition is replaced by the stipulation that the last player to move loses. Recently there have been fruitful developments in this field—in particular the use of misère quotients [2, 74, 79, 73]. We do not explicitly exclude the case of misère games below, but certain order-theoretic properties we are interested in will generally exclude them[1]. Therefore we have not considered the literature on misère games in detail.

Combinatorial games and categories

The generalisations of combinatorial games to category theory are highly relevant to the discussion of Chapter 4. Such discussion began with a paper of Joyal [51], where he observed that Conway games could be made into a symmetric monoidal closed and self-

[1] See in particular Definition 3.1.1.

dual category, with monoidal product the disjunctive sum, and negation the dualising functor. The objects in this category are standard combinatorial games; a morphism $f\colon G \to H$ is a strategy for LII[1]in the game $H - G$. This allows one to use the copycat strategy as an identity arrow, and the swivel chair strategy for composition[2].

This has been developed in the context of combinatorial games by Cockett et al. [12], where the ideas behind Joyal's category are extended by *combinatorial game categories*, or CGCs. These are categories C equipped with a module[3] M (sometimes denoted $M\colon C \nrightarrow C$) and a *diproduct* functor

$$\{-\,|\,-\}\colon \mathcal{P}^{<\omega}(C) \times \mathcal{P}^{<\omega}(C) \to C$$

(here $\mathcal{P}^{<\omega}$ denotes the finite powerset operator, $C \mapsto \{A \subseteq C\colon |A| < \omega\}$), with the following operations of arrow formation.

- If for each $g \in A$ we have $g \nrightarrow \{C\,|\,D\}$, and for each $h \in D$ we have $\{A\,|\,B\} \nrightarrow h$, then $\{A\,|\,B\} \to \{C\,|\,D\}$ (*diproduct*).
- If $x \in B$ and $x \to G$, then $\{A\,|\,B\} \nrightarrow G$ (*injection*).
- If $G \to x$ and $x \in A$, then $G \nrightarrow \{A\,|\,B\}$ (*projection*).

In Chapter 4 we consider the *constructive* property, which is analogous to the conjunction of all these properties. We also consider the reverse implications, as the *instructive* property. Notice that these properties are all derivatives of Conway's recursive definition of $\leq$ and $\lhd\!\mid$, and merely represent the sensible construction of strategies from the availability of moves, and the deduction of possible moves from the existence of strategies. In the category **CGC** of CGCs [12], the wellfounded combinatorial games constitute an initial object.

[1]Throughout we will use L, R, I, II to denote Left, Right, the first and the second player to move respectively. By LII (for example), we denote Left, playing second.

[2]A thorough explanation of these strategies is given by Joyal [51]

[3]Also known as a bimodule, profunctor or distributor. These structures are essentially a second collection of arrows which are not required to compose among themselves, but which must compose with the arrows of the associated category to form module arrows; further, this composition is associative in the obvious ways. These arrows are typically denoted $\nrightarrow$. See Chapter 4 for a more detailed explanation.

Another interesting abstraction of combinatorial games to category theory takes a more coalgebraic approach.

Combinatorial games and coalgebra

Recently there has been productive work in a coalgebraic approach to generalisation of Conway games [46, 47, 48, 49]. This began with a study of *hypergames*, i.e. the final coalgebra for the functor

$$F\colon A \mapsto \mathcal{P}(A) \times \mathcal{P}(A)$$

defined on possibly illfounded classes A [46, 47]. In particular this includes the Conway games (being hereditary two-sided, wellfounded sets), which form an initial algebra for F. Hence there is a nice duality, and many techniques and results from ONAG [13] have analogues for hypergames.

While Conway et al. [13, 5] considered mainly winning strategies, Honsell et al. consider primarily non-losing strategies, which are easier to deal with, and much more abundant, in illfounded games. The corresponding order relations, denoted there by $(\trianglelefteq, \blacktriangleright\!\!\!\!\_)$[1] [47], are the greatest fixpoint of the operator $+\colon (R_1, R_2) \mapsto (R_1^+, R_2^+)$, defined by

$$\begin{array}{rcl} x\ R_1^+\ y & \Leftrightarrow & \text{all}\, x^{\mathrm{L}}\ R_2\ y \quad \text{and} \quad x\ R_2\ \text{all}\, y^{\mathrm{R}} \\ x\ R_2^+\ y & \Leftrightarrow & \text{some}\, x^{\mathrm{R}}\ R_1\ y \quad \text{or} \quad x\ R_1\ \text{some}\, y^{\mathrm{L}}. \end{array} \tag{1.2}$$

This operation has been considered in various publications [12, 46, 46] and also by the current author [16]. Again, this is a simple abstraction of the definitions of $\leq, \lhd\!\!\mid$ presented in ONAG [13, p.78]. The pair of relations $(\trianglelefteq, \blacktriangleright\!\!\!\!\_)$ is an important example and we will discuss it further in Chapters 3 and 5 where we consider a distinguished class of games—and the resultant order relations $(\leq, \lhd\!\!\mid)$—from a purely order-theoretic perspective.

[1] Note that we have inverted their relation $\trianglelefteq$; we will continue to do this in later chapters. Further notice that, although their weak order relation $\blacktriangleleft\!\!\!\!\_$ appears to indicate that the expression to the left is in some sense greater than the expression on the right, it is analogous to the relation $\lhd\!\!\mid$ as defined by Conway, and $x \lhd\!\!\mid y$ asserts that y is preferable to x for LI. The convention of this thesis is to always use the symbols $\leq$ and $\lhd\!\!\mid$, or obvious derivatives such as $\leq'$, to indicate the intended interpretation.

Although the relations $(\trianglelefteq, \blacktriangleright\!\!\!\!\underline{\ \ })$ are shown to satisfy many desirable properties and analogues of results true for Conway games, transitivity of the first order inevitably breaks down [47, p.13]: as is often the case, techniques for transferring strategies—for example to disjunctive compounds—which work in the wellfounded case fail in more general settings. Sums and negation have also been discussed in this context, defined as appropriate final morphisms [47].

Equivalences between Conway games are also of interest in this setting. Honsell and Lenisa [46, 47] consider *equideterminacy*[1] and *contextual equivalence*, a comparison using the behaviour of sums of two games x, y with other games.

This work has been extended to include categories with strategies as morphisms [49]. The same coalgebraic construction is used, with sums and negation defined as final morphisms. Two categories in particular have been introduced by Honsell et al. [49]: one of *fixed* games (i.e. games in which all infinite plays result in a win for one of the players) and one of *mixed* games (i.e. games in which an infinite play can result in a victory for either player or a draw). The former will be rather intuitive to any reader familiar with, for example, bisimulation games or Joyal's category (see above). The latter has as objects pairs $x = (x_1, x_2)$ of fixed games, and as morphisms pairs of winning strategies. Both categories are $*$-autonomous; the latter contains Joyal's category as a full subcategory, and also captures the loopy equivalence of Berlekamp et al. [5].

Aside from the developments listed above, much literature in this area focuses on a fixed combinatorial game and the strategies available to players when playing certain positions. In particular this restricts such work to finite or *short* games. We are not interested in specific games here (though we will consider various examples for the sake of illustration, they will be familiar to readers of ONAG [13] and Winning Ways [5]), and in fact our focus will generally be on collections which include illfounded or infinite games.

[1]Two hypergames x, y are *equidetermined* [46, 47] if they possess the same outcomes, i.e. if x has an LI, LII, RI or RII-strategy precisely when y has an LI, LII, RI or RII-strategy respectively.

1.1.2 Topological set theory

Topological set theory began in earnest with Isaac Malitz' Ph.D. thesis [67], completed in 1976. Malitz considered the operator + on binary relations in a universe of sets (specifically, a model of ZF), defined by

$$x \sim^+ y \leftrightarrow \forall u \in x\ \exists v \in y\, (u \sim v) \wedge \forall v \in y\ \exists u \in x\, (u \sim v).$$

He then defined the equivalence relations $\sim_\alpha$ recursively by

$$\begin{aligned} \sim_0 &= V \times V \\ \sim_{\alpha+1} &= {\sim_\alpha}^+ \\ \sim_\lambda &= \bigcap_{\alpha<\lambda} \sim_\alpha . \end{aligned}$$

The $\sim_\alpha$-equivalence class of a set x, here denoted $x/{\sim_\alpha}$, was defined using Scott's trick to be the set of least representatives. Then the classes

$$\mathcal{M}_\alpha = \{x/{\sim_\alpha} \colon x \in V\}$$

were shown to be a set for all α. On these sets Malitz defined a sequence of extensional membership relations $\in_\alpha$ by

$$u/{\sim_\alpha} \in_\alpha x/{\sim_\alpha} \leftrightarrow \exists v \in x\, (u \sim_\alpha x),$$

and naturally was interested in the structures $(\mathcal{M}_\alpha, \in_\alpha)$ as models of set theory.

Significant to this investigation was the behaviour of α-sequences (i.e. sequences $(x_\beta)_{\beta<\alpha}$) of elements of V (equivalently, one can consider such sequences with elements from $\mathcal{M}_\alpha$). Of particular interest was the convergence of sequences. Such a sequence $(x_\beta)_{\beta<\alpha}$ is said to *converge* to a set y if

$$\forall \varepsilon < \alpha \; \exists \beta < \alpha \; \forall \gamma < \alpha \, (\gamma \geq \beta \rightarrow x_\gamma \sim_\varepsilon y).$$

Also of interest were Cauchy sequences, i.e. sequences $(x_\beta)_{\beta<\alpha}$ such that

$$\forall \varepsilon < \alpha \; \exists \beta < \alpha \; \forall \gamma, \delta < \alpha \, (\gamma, \delta \geq \beta \rightarrow x_\gamma \sim_\varepsilon x_\delta).$$

The natural topology associated with these notions has as basic open sets the $\sim_\beta$-equivalence classes $x/{\sim_\beta}$ for all sets x. This forms a (metrisable) uniform space, where convergence is exactly as defined above.

Malitz proved that if α is ω or a measurable cardinal, $\mathcal{M}_\alpha$ is crowded (that is, every sequence has a Cauchy subsequence). However he was not able to prove completeness of $\mathcal{M}_\alpha$ for any α (in particular he proved that $\mathcal{M}_\omega$ is certainly not complete [67, p.57]). Completeness of $\mathcal{M}_\alpha$ would have yielded many desirable properties such as closure under internal union and replacement [67, p.60]

Various authors have adopted and augmented Malitz' approach [30, 37, 43]. The most significant addition of Forti et al. [30, 37, 40, 38, 39] has been to work within an illfounded set theory, rather than ZF like Malitz. Specifically, they chose a model of ZFC without foundation, plus a *free construction axiom*[1] [36], similar to the antifoundation axiom of Aczel [1]. This allows a neat definition of a kind of quotient structure N_α which is compact precisely when α is a strongly inaccessible, weakly compact cardinal[2]. A consequence of

[1]The simplest form of free construction axiom is the statement that for every function f, there is a unique f-inductive function g, i.e. a function g satisfying

$$\forall x \; g(x) = \{g(u) \colon u \in f(x)\}.$$

See Forti [34] and Honsell for a detailed survey of such principles.

[2]Throughout, we consider ω to be both strongly inaccessible and weakly compact.

compactness in these models is that the space N_α models the positive set theory[1]

$$\mathrm{GPK} = \mathrm{Ext} + \mathrm{Comp(GPF)}.$$

In fact, if $\alpha > \omega$, then N_α is a model of $\mathrm{GPK} + \mathrm{Inf}$[2]. Forti and Honsell [37] provide a detailed account of these constructions. *Models of Self-Descriptive Set Theories* [37] is of particular relevance to this thesis; there the compactness result is broken down into completeness and κ-boundedness, and necessary requirements discussed to prove each of these properties. This analysis is invaluable to the work in Chapter 6, where we consider an analogous construction within a nonstandard model of arithmetic, and consider the properties necessary for similar sequential properties in such a setting. We remark that the structure N_ω is essentially the Cauchy-completion of Malitz' M_ω; this has been generalised in an interesting paper of Hinnion [43].

Extensions of the theory GPK have been considered, and of particular interest is the theory GPK^+_∞ of Esser [23, 24]. Although less relevant to the research in this thesis, Esser's axiom scheme CL encapsulates an important observation about the nature of sets in these topological models. The scheme essentially states that each class A has a unique smallest set $B \supseteq A$; in other words, each class has a *closure* which is a set. We remark that this property was satisfied by Malitz' own structures $(\mathcal{M}_\alpha, \in_\alpha)$ and also in the set theory of Skala [80], which was further investigated by Manakos [68][3] . Esser's theory GPK^+ is defined by

$$\mathrm{GPK}^+ = \mathrm{Ext} + \mathrm{Zero} + \mathrm{Comp(BPF)} + \mathrm{CL},$$

where Zero postulates the existence of an empty set, and BPF is the collection of bounded

[1] The class BPF of formulas is generated by the usual rules for creating propositional formulas, except negation, plus allowing bounded universal quantification. The larger class GPF is formed by using these rules and also admitting $\forall x\,(\theta(x) \to \phi)$, where θ is any arbitrary formula with single free variable and ϕ is already in GPF.

[2] When $\alpha = \omega$, $\omega \cup \{\omega\}$ is the smallest set containing all ordinals.

[3] Skala's set theory is actually rather weak, and attempts to add certain 'sensible' axioms (such as singleton comprehension) result in a contradiction; see Libert and Esser [63].

positive formulas from above. Arguably, this is a more natural *topological* set theory. Esser proved [23] that GPK^+ implies GPK, and also provided a substitute scheme CL' (equivalent to CL when we assume the theory $\mathrm{Comp(BPF)} + \mathrm{Ext} + \mathrm{Zero}$) for CL, which clarifies the link between GPK and GPK^+. The theory GPK^+ has been shown to interpret GPK, ZF and the Morse-Kelley set theory MK. Notions of closure and approximation schemes have been considered further by, for example, Libert and Esser [63].

1.2 This thesis

Now we are in a position to discuss the work presented here.

1.2.1 Amphi-ZF

We define a two-sided set theory, called Amphi-ZF, intended to represent the pure combinatorial games of Conway et al. [13]. The real use of such a theory is to aid discussion of issues in combinatorial game theory—particularly regularity issues such as candidates for a foundation axiom. We demonstrate a synonymy (in the sense of Visser [83]) between Amphi-ZF and ZF, which behaves well with respect to subtheories. The construction essentially relies on the use of Quine pairs in order to make every object a pair, and every pair an object.

We conclude with some discussion of wellfoundedness in Amphi-ZF. In particular we are able to deduce some results regarding the strength of candidate foundation axioms using Rieger-Bernays permutations, and the consequences in combinatorial game theory.

1.2.2 Order theory of combinatorial games

In Chapter 3 we discuss games from a purely order-theoretic perspective. We identify a notion of *two-order* generalising Conway's pairs $(\leq, \lhd\!\!\mid)$ which is of particular interest to us, and discuss properties of general two-orders, such as a well-defined notion of logical duality and its relationship with game-theoretic determinacy. This discussion of duality leads to the consideration of a second kind of morphism, called an *amphimorphism*, which preserves the first order and reverses the second, and has a particular semantic interpretation in terms of strategy preservation for the second player (rather than for Left, which is the case for the more intuitive notion of morphism). As such these may be of interest in combinatorial game theory. Also automorphism spaces are considered for both types of morphism, and towards the end of Chapter 3 we show that any group having compatible two-order structure arises as an automorphism group—a simple extension of Cayley's theorem.

We consider dual spaces for posets arising from two-orders, and derive a generalisation of Stone's theorem. These spaces are essentially collections of 'strategy notions', with a rich and natural structure and which form complete bounded distributive lattices; consequently these structures are Heyting algebras, and models of intuitionistic propositional logic.

1.2.3 Extensions in category theory

Chapter 4 focuses on developing the theory of Chapter 3 further using categories. Many of the results generalise—in particular the rich spaces of two-orders relate to spaces of modules over a category, having similarly rich structure.

We define a value map which generalises the notion of value, discussed by Conway et al. [13, 5] and which originated with Sprague and Grundy, to an adjunction between appropriate categories of 'games'[1] and the two-ordered structures of Chapter 3.

We conclude Chapter 4 by considering the addition of monoidal and set-theoretic structure to such categories to form *gamuts*, and demonstrate that the wellfounded combinatorial games of ONAG and Winning Ways form gamuts.

Chapter 5 considers a distinguished collection of game categories, called 2-*architectures*.

[1] We use the term *game category*, though it should be noted that the term has been used previously with different meanings; see for example Honsell and Lenisa [47].

These are essentially two-ordered structures with membership relations $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$, and such that the structure is both constructive and instructive. We show that for any class of amphisets without self-members (but potentially containing loops of $n \geq 2$ elements) there is a least two-order T^{m} making the structure a 2-architecture. Moreover the operation of adorning such a class of amphisets with this structure is part of an adjunction between an appropriate category of two-ordered structures and one of amphiset collections. In addition to considering construction of an architecture over a collection of amphisets, we discuss the much harder problem of extending an existing architecture by iteration of the powerset functor, which has a less satisfactory answer.

Some discussion of the dual of T^{m} is also given in Chapter 5; we argue that several regularity axioms arising from purely game-theoretic considerations can be expressed using these objects. Further, we relate the relationship of the architecture two-orders on a collection of amphisets to collections of two-orders in general. In particular much of the rich structure of two-order collections fails to transfer, and the collection of architectural two-orders will always be much smaller; we suspect this can be made precise by considering topological notions, perhaps with the topologies discussed in Chapter 3.

Chapter 5 closes with some discussion of applications for architectures in particular. The first is an attempt to make some sense of the arithmetic of cuts in a model of PA. In particular we argue that, by considering a slightly large collection of objects called *rifts*, we obtain more sensible addition and multiplication operations, which leads to an improved algebraic structure. This comes at the cost of the order's totality, however we do maintain a suitable two-ordered structure.

The second suggested application is directed at multithreaded software, or more generally in any area where multiple objects must exist concurrently in space or time. In particular the existence of a least two-order shows that in any collection of processes where no object is trivially self-dependent, there is a least two-order dictating, for instance, relative execution times as well as obstructive dependencies).

1.2.4 Topological set theory in nonstandard arithmetic

Our final subject is topological set theory within a nonstandard model of Peano arithmetic. We begin Chapter 6 by emulating the construction of Malitz within (finite set theory interpreted by) a model of arithmetic; here cuts (that is, initial segments closed under the successor operation) take the place of ordinals and cardinals, and we consider the analogous properties required to derive properties such as crowdedness, completeness and compactness in our quotient model.

This lays the foundation for us to consider the reverse problem, of determining which properties are necessary to prove conditions such as completeness and crowdedness. We manage to determine the exact strength of crowdedness (ACA_0), and are able to demonstrate that completeness in this context is significantly weaker.

The discussion of completeness in particular leads to some interesting concepts. We introduce a notion of *witnessing principle* (essentially a kind of Skolemisation scheme for a cut), which turns out to be closely related to very useful topological and sequential properties. This in particular enables us to demonstrate that completeness holds whenever our cut (analogous to the cardinals α [30] and κ [37] used by Forti et al.) is coded by a strong cut, a property analogous to an ordinal having strongly inaccessible, weakly compact cofinality.

We suspect the latter part of Chapter 6 may have interesting applications in reverse mathematics. In particular, our results transfer easily to the realm of second-order arithmetic, and the witnessing principle we introduce is independent of the 'big theories' of traditional interest when the base theory is sufficiently weak.

CHAPTER 2

A TWO-SIDED SET THEORY FOR COMBINATORIAL GAMES

The material presented in this chapter has been printed in various forms. Firstly, as part of the author's MPhil(Qual) thesis, the theory Amphi-ZF was introduced and shown to be synonymous with ZF. That synonymy result essentially consisted of one interpretation (of AZF in ZF) being 'reflected' in a uniformly defined, proper subclass of models of AZF, allowing the recursive definition of an inverse. While appealing and intuitive, the disadvantage of this approach is a heavy (and, it turns out, unnecessary) dependence on membership-induction.

By analysing the actual behaviour of this recursively defined inverse in models of AZF, it was possible to carefully reconstruct the same interpretation[1] assuming only *ordinal*-induction. This is a significant weakening, as it does not rely on the key theorems of transitive closure and $\in$-induction, allowing us to drop the Infinity and Foundation axioms from the hypothesis. This also removes dependence on the power set axiom required to prove the appropriate Foundation axioms. This much improved version of the main theorem, plus the discussion of Amphi-NBG and Rieger–Bernays permutations, was published under the title *Amphi-*ZF*: axioms for Conway games*, in the *Archive for Mathematical Logic*. We have made only minor alterations to the discussion here.

[1] See Definitions 2.4.4, 2.4.5 and 2.4.6.

2.1 Preliminaries

In a passage on pages 64–67 of ONAG, Conway discusses options for the formalisation of his theory of combinatorial games formed as two-sided sets of *options* for the players Left and Right, these options themselves being two-sided sets of options. He points out that, although his theory could be formalised in ZF, it would be more natural to formulate a theory of two-sided sets and formalise the theory of games in this. More radically, he proposes a 'Mathematicians' Liberation Movement' [13, p.66] in which some general foundational principle that should allow all reasonable ('permissible') constructions and inductions to be considered grounded on a foundation essentially equivalent to that of ZF without requiring further investigation.

In this passage, it is clear that Conway has the idea of a two-sided set theory based on a principle of induction, and also the idea for an interpretation of it in usual ZF. Indeed he refers to Kuratowsi ordered pairs and using the Scott trick of equivalence classes of sets of minimal rank to hint at how this interpretation might be carried out[13, p.65]. Completing this programme of devising the two-sided set theory and interpreting it in ZF as indicated is straightforward. We believe however that it is much more natural to use Quine's ordered pairs [76]—not so much because of the 'typing' advantages of the pairing that was Quine's original motivation, but because every set can be considered as a pair of sets using Quine's pairing. This raises the possibility that ZF and its two-sided version are actually essentially the same theory, a conjecture that is verified here.

By 'interpretation' we shall mean a *relative interpretation*, as defined by Visser [83]. We briefly describe such objects and their category-theoretic framework, which makes discussion of the interpretations much simpler, here. The category **INT** has as objects logical theories. All theories are assumed to have only relations as non-logical symbols; among these relations included we assume there is a unary relation δ, indicating the domain. (Note that equality is also included as a logical symbol). We assume full first-order logic, including the equality rules and the logical axiom $\forall x\ \delta(x)$. By a *relative translation* $\mathfrak{f}\colon T_2 \to T_1$ of an $\mathscr{L}_2$-theory T_2 into an $\mathscr{L}_1$-theory T_1 we mean a mapping $\mathfrak{f}$

of atomic formulas $R(x_0,\ldots,x_{n-1})$ of $\mathscr{L}_2$ to formulas $R(x_0,\ldots,x_{n-1})^{\mathfrak{f}}$ of $\mathscr{L}_1$, in the same free variables. In particular $\delta(x)^{\mathfrak{f}}$ is some domain formula $\delta^{\mathfrak{f}}(x)$, and $(x=y)^{\mathfrak{f}}$ is in our case required to be simply $(x=y)$. This mapping is extended to all $\mathscr{L}_2$-formulas by taking $(\neg\theta(\overline{x}))^{\mathfrak{f}}$ to be $\neg\,\theta(\overline{x})^{\mathfrak{f}}$, $(\phi(\overline{x})\to\psi(\overline{x}))^{\mathfrak{f}}$ to be $\phi(\overline{x})^{\mathfrak{f}}\to\psi(\overline{x})^{\mathfrak{f}}$, and $(\forall\overline{x}\,\phi(\overline{x}))^{\mathfrak{f}}$ to be $\forall\overline{x}\ (\bigwedge_i\delta^{\mathfrak{f}}(x_i)\to\phi(\overline{x})^{\mathfrak{f}})$. A *relative interpretation* $\mathfrak{f}\colon T_2\to T_1$ is a relative translation satisfying $T_1\vdash\exists x\,\delta(x)^{\mathfrak{f}}$ and also $T_2\vdash\phi\Rightarrow T_1\vdash\phi^{\mathfrak{f}}$ for all statements ϕ in the language of T_2. Following Visser we require that, for theories U,V,W in **INT**:

- the interpretation $\mathrm{id}_U\colon U\to U$ leaves relations unchanged;
- if $\mathfrak{f}\colon U\to V$ and $\mathfrak{g}\colon V\to W$ then $R(\overline{x})^{\mathfrak{fg}}$ is $(R(\overline{x})^{\mathfrak{f}})^{\mathfrak{g}}$.

Further, two interpretations $\mathfrak{f},\mathfrak{g}\colon U\to V$ are considered *equivalent* if

- $V\vdash\forall x\big(\delta(x)^{\mathfrak{f}}\leftrightarrow\delta(x)^{\mathfrak{g}}\big)$, and
- $V\vdash\forall\overline{x}\ \big(\bigwedge_i\delta(x_i)^{\mathfrak{f}}\to(\phi(\overline{x})^{\mathfrak{f}}\leftrightarrow\phi(\overline{x})^{\mathfrak{g}})\big)$ for all formulas ϕ in the language of U.

Finally, the morphisms in **INT** are these interpretations, modulo equivalence, though generally we shall refer here to specific interpretations. The interpretations $\mathfrak{f}\colon U\to V$ and $\mathfrak{g}\colon V\to U$ are said to be *inverse* to each other if the corresponding morphisms are; that is, if $\mathfrak{fg}=\mathrm{id}_U$ and $\mathfrak{gf}=\mathrm{id}_V$ in this category.

Having introduced AZF, we shall show that there are interpretations $\mathfrak{f}\colon$ AZF $\to$ ZF and $\mathfrak{g}\colon$ ZF $\to$ AZF which are inverse to each other. In Visser's terminology, AZF and ZF are *synonymous*; less formally, they are essentially the same theory.

We also briefly look at the state of von Neumann–Bernays–Gödel set theory (NBG) and its two sided analogue, NBG_2, and report the expected result that these are also synonymous.

Even though ZF and AZF (or NBG and NBG_2) are equal, there are sometimes strong psychological reasons to prefer the two-sided theory. Conway's games provide one example: there is an elegance in studying combinatorial games and their 'one line proofs' in a system which shows that not only are these games a generalisation of number, but they are also the only notion of 'set' that is required. We found, for example, these two-sided theories to be very useful aids in the discussion of regularity axioms below. That Rieger–Bernays permutations work so easily in the new context hints at the real use of these concepts: being synonymous, our theory remains unchanged. The language, however, is distinct and—for our purposes—much more expressive.

2.2 Amphi-ZF

In this section we shall present the axioms for Amphi-ZF or AZF. Let $\mathscr{L}_2$ denote the first-order language with non-logical symbols $\in_{\mathrm{L}}, \in_{\mathrm{R}}$, both denoting binary relations.

Throughout we shall use '=' to denote the usual logical *identity* in the first-order language $\mathscr{L}_2$. So for example, for us $\{\,0, 1 \mid\,\} \neq \{\,1 \mid\,\}$. The axiom of extensionality (below) will identify = with the notion of two games having the same Left and Right options or members. In other words our = is the notion which Conway calls identity and notates as $\equiv$ [13, p.15]. We believe adhering to standard usage in first-order logic is more important here than adhering to Conway's usage. We will hardly need Conway's notion of equality here; when we do need it we use the symbol $\approx$ for it.

An object of an $\mathscr{L}_2$-structure will be called an *amphiset*, or less formally, a *game*. The relations $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$ are intended to represent the usual Left and Right membership relations of Conway et al. There is also a symmetric membership relation defined by

- $x \in^{\mathrm{L}}_{\mathrm{R}} y \Leftrightarrow (x \in_{\mathrm{L}} y \vee x \in_{\mathrm{R}} y)$.

We define appropriate subset relations as follows. Let x, y be amphisets. Then

- $x \subseteq_{\mathrm{L}} y \Leftrightarrow \forall z \in_{\mathrm{L}} x\ (z \in_{\mathrm{L}} y)$;
- $x \subseteq_{\mathrm{R}} y \Leftrightarrow \forall z \in_{\mathrm{R}} x\ (z \in_{\mathrm{R}} y)$;
- $x \subseteq y \Leftrightarrow (x \subseteq_{\mathrm{L}} y \wedge x \subseteq_{\mathrm{R}} y)$.

Due to the symmetric nature of the system of axioms to be presented, it is useful to adopt the P-notation described in the introduction. Recall that we described a separate sort with exactly two objects, represented by the constant symbols L and R; P and occasionally Q will denote variables of this sort. We therefore allow statements of the form $\forall x \subseteq_{\mathrm{P}} z\ \exists y\,(y \in_{\mathrm{P}} x)$, referring to either $\forall x \subseteq_{\mathrm{L}} z\ \exists y\,(y \in_{\mathrm{L}} x)$ or $\forall x \subseteq_{\mathrm{R}} z\ \exists y\,(y \in_{\mathrm{R}} x)$. If $\phi_{\mathrm{L}}, \phi_{\mathrm{R}}$ are first-order sentences we define $\bigwedge_{\mathrm{P}} \phi_{\mathrm{P}}$ to be $\phi_{\mathrm{L}} \wedge \phi_{\mathrm{R}}$, and so on.

Axiom 0 (Zero Game Axiom)**.** There exists a zero game, i.e.

$$\exists x\ \forall y\ \big(y \notin_{\mathrm{L}} x \wedge y \notin_{\mathrm{R}} x\big).$$

Axiom 1 (Extensionality)**.** Two games are equal if and only if their respective options are equal;

$$\forall x\ \forall y\ \Big(\bigwedge_{\mathrm{P}} \big(\forall z\,(z \in_{\mathrm{P}} x \leftrightarrow z \in_{\mathrm{P}} y)\big) \rightarrow x = y\Big).$$

Extensionality justifies the use of the familiar notation from ONAG, for example using $\{\, u, v \mid x, y \,\}$ to denote the game with Left and Right options u, v and x, y respectively.

Axiom 2 (Pair-game Axiom)**.** If x, y are games there is a game with these games as left options;

$$\forall x\ \forall y\ \exists z\,(x \in_{\mathrm{L}} z \wedge y \in_{\mathrm{L}} z).$$

The replacement axiom will imply that there is a game with only these options, and will also guarantee that a similar game with right options x, y and no left options exists.

Axiom 3 (Replacement)**.** If $\phi_{\mathrm{L}}(x, y, \bar{a})$ and $\phi_{\mathrm{R}}(x, y, \bar{a})$ are first-order formulas in the free variables shown, then

$$\forall \bar{a}\ \forall I\ \Big(\bigwedge_{\mathrm{P}} \big(\forall x \in^{\mathrm{L}}_{\mathrm{R}} I\ \exists! y\, \phi_{\mathrm{P}}(x, y, \bar{a})\big) \rightarrow \exists A\ \forall z\ \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} A \leftrightarrow \exists x \in^{\mathrm{L}}_{\mathrm{R}} I\ \phi_{\mathrm{P}}(x, z, \bar{a})\big)\Big).$$

The next axiom is not really required; as in ordinary ZF it can be deduced from the Replacement axiom. Still, we shall refer to this theorem as Separation when working in

Amphi-ZF.

Axiom 4 (Separation)**.** For all first-order formulas $\phi_{\mathrm{L}}(\bar{u}, v)$, $\phi_{\mathrm{R}}(\bar{u}, v)$ in free variables shown we have

$$\forall \bar{x}\ \exists y\ \forall z \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} y \leftrightarrow z \in_{\mathrm{P}} x \wedge \phi_{\mathrm{P}}(\bar{x}, z)\big).$$

If x is a game for which $\bigwedge_{\mathrm{P}}(y \in_{\mathrm{P}} x \leftrightarrow \phi_{\mathrm{P}}(y))$, we write

$$x = \{\, y^{\mathrm{L}} \colon \phi_{\mathrm{L}}(y) \mid y^{\mathrm{R}} \colon \phi_{\mathrm{R}}(y) \,\}$$

or even (extending our useful shorthand notation using the P) $x = \{\, y^{\mathrm{P}} \colon \phi_{\mathrm{P}}(y^{\mathrm{P}}) \,\}_{\mathrm{P}}$. Separation guarantees the existence of any amphiset of the form $\{\, y^{\mathrm{P}} : y^{\mathrm{P}} \in_{\mathrm{P}} a \wedge \phi_{\mathrm{P}}(y^{\mathrm{P}}) \,\}_{\mathrm{P}}$. If x is a game, i.e. an object of our theory, Conway uses x^{L} as a variable ranging over Left-elements of x, i.e. over z such that $z \in_{\mathrm{L}} x$. Similarly x^{R} ranges over Right-elements of x, i.e. z such that $z \in_{\mathrm{R}} x$. For example $\{\, x^{\mathrm{L}} \mid x^{\mathrm{R}} \,\}$ is an abbreviation for $\{\, z : z \in_{\mathrm{L}} x \mid z : z \in_{\mathrm{R}} x \,\}$, which is of course x itself. We shall frequently use such notation too.

We still require an axiom of Union. There are several different types of union we may wish to use, and so for clarity we designate a symbol for each, as follows.

$$\begin{aligned}
\mathop{\textstyle\bigsqcup\!\!\!\!\!+}\, x &= \{\, z \colon \exists y \in^{\mathrm{L}}_{\mathrm{R}} x\ (z \in^{\mathrm{L}}_{\mathrm{R}} y) \mid z \colon \exists y \in^{\mathrm{L}}_{\mathrm{R}} x\ (z \in^{\mathrm{L}}_{\mathrm{R}} y) \,\};\\
\textstyle\bigsqcup x &= \{\, z \colon \exists y \in_{\mathrm{L}} x\ (z \in^{\mathrm{L}}_{\mathrm{R}} y) \mid z \colon \exists y \in_{\mathrm{R}} x\ (z \in^{\mathrm{L}}_{\mathrm{R}} y) \,\};\\
\textstyle\biguplus x &= \{\, z \colon \exists y \in^{\mathrm{L}}_{\mathrm{R}} x\ (z \in_{\mathrm{L}} y) \mid z \colon \exists y \in^{\mathrm{L}}_{\mathrm{R}} x\ (z \in_{\mathrm{R}} y) \,\};\\
\textstyle\bigcup x &= \{\, z \colon \exists y \in_{\mathrm{L}} x\ (z \in_{\mathrm{L}} y) \mid z \colon \exists y \in_{\mathrm{R}} x\ (z \in_{\mathrm{R}} y) \,\}.
\end{aligned}$$

Our Union axiom simply states that the largest of these, $\mathop{\textstyle\bigsqcup\!\!\!\!\!+}\, x$, exists for all games x. From this the existence of each other type follows simply by Separation.

Axiom 5 (Union)**.**

$$\forall x\ \exists y\ \forall z \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} y \leftrightarrow \exists w \in^{\mathrm{L}}_{\mathrm{R}} x\ (z \in^{\mathrm{L}}_{\mathrm{R}} w)\big).$$

We can also define binary operations of unions and intersections of games, along these lines. If x, y are games then $\{\, x, y \mid \,\}$ exists, and so the game $\{\, x^{\mathrm{L}}, y^{\mathrm{L}} \mid x^{\mathrm{R}}, y^{\mathrm{R}} \,\}$ (which is the Conway notation for $\{\, z \colon z \in_{\mathrm{L}} x \vee z \in_{\mathrm{L}} y \mid z \colon z \in_{\mathrm{R}} x \vee z \in_{\mathrm{R}} y \,\}$) does by Union and Separation. We call this game $x \cup y$. Analogously we define

$$x \cap y = \{\, z : z \in_{\mathrm{L}} x \wedge z \in_{\mathrm{L}} y \mid z : z \in_{\mathrm{R}} x \wedge z \in_{\mathrm{R}} y \,\},$$

and $x \setminus y = \{\, z : z \in_{\mathrm{L}} x \wedge z \notin_{\mathrm{L}} y \mid z : z \in_{\mathrm{R}} x \wedge z \notin_{\mathrm{R}} y \,\}$.

Notice that we may also form successor games $s_{\mathrm{L}}(x) = \{\, x^{\mathrm{L}}, x \mid x^{\mathrm{R}} \,\} = x \cup \{\, x \mid \,\}$ and $s_{\mathrm{R}}(x) = x \cup \{\, \mid x \,\}$, and define $1 = s_{\mathrm{L}}(0),\ 2 = s_{\mathrm{L}}(1)$ and $-1 = s_{\mathrm{R}}(0)$, etc. From this we can state that there exists a Left- and Right-inductive game.

Axiom 6 (Infinity)**.**

$$\exists x \bigwedge_{\mathrm{P}} \big(0 \in_{\mathrm{P}} x \wedge \forall y \in_{\mathrm{P}} x (s_{\mathrm{P}}(y) \in_{\mathrm{P}} x)\big).$$

We choose the following—the $\in^{\mathrm{L}}_{\mathrm{R}}$-induction principle [13, p.64] is derived from it in the usual way.

Axiom 7 (Foundation)**.**

$$\forall x \neq 0 \; \exists y \in^{\mathrm{L}}_{\mathrm{R}} x \; \forall z \in^{\mathrm{L}}_{\mathrm{R}} x \, (z \notin^{\mathrm{L}}_{\mathrm{R}} y).$$

Remark 2.2.1. There are other natural choices for a foundation axiom. Let $\mathrm{wf}(R)$ be the statement that R is a wellfounded relation, i.e.

$$\forall x \,\big(\, \exists y \, (y \; R \; x) \to \exists y \; R \; x \; \forall z \; R \; x \, (z \not R \; y)\big).$$

Then our foundation axiom is simply $\mathrm{wf}(\in^{\mathrm{L}}_{\mathrm{R}})$. We might instead choose to posit that each membership is wellfounded; that is, $\bigwedge_{\mathrm{P}} \mathrm{wf}(\in_{\mathrm{P}})$. This is a clear consequence of $\mathrm{wf}(\in^{\mathrm{L}}_{\mathrm{R}})$, but the converse is not obvious. In fact we will see in Section 2.6 that this second statement is strictly weaker than the first.

For the final axiom, we use the symmetric subset relation $\subseteq$ defined above.

Axiom 8 (Power Game)**.**

$$\forall x\ \exists y\ \forall z\,\big(\bigwedge_{\mathrm{P}} z \in_{\mathrm{P}} y \leftrightarrow z \subseteq x\big).$$

By the Power Game and Separation axioms

$$y = \{\, u : u \subseteq x \text{ and } u \text{ is inductive} \mid u : u \subseteq x \text{ and } u \text{ is inductive}\,\}$$

is a game, as are $\{\, y^{\mathrm{L}} \mid \,\}$ and $\{\, \mid y^{\mathrm{R}}\,\}$. Defining the operator $\bigcap$ in the obvious way, i.e.

$$\bigcap u = \{\, w : \forall v \in_{\mathrm{L}} u\,(w \in_{\mathrm{L}} v) \mid w : \forall v \in_{\mathrm{R}} u\,(w \in_{\mathrm{R}} v)\,\},$$

we may define the game $\omega = \bigcap\{\, y^{\mathrm{L}} \mid \,\}$. Finally here, we issue a word of warning. Taking '$\approx$' to be equality as defined by Conway and $+$ to be Conway's addition of games, while is is the case that $s_{\mathrm{L}}(n) \approx n+1$ for all $n \in_{\mathrm{L}} \omega$, we do not in general have that they are *identical*; for example,

$$2 = s_{\mathrm{L}}(1) = \{\, 0, 1 \mid \,\} \neq \{\, 1 \mid \,\} = 1 + 1.$$

Nor, in fact, do we have the equality $s_{\mathrm{L}}(x) \approx x + 1$ for all games.

There are many ways to define ordered pairs, but it is convenient to say that an ordered pair (u, v) is simply the amphiset $\{\, u \mid v\,\}$; a function is an amphiset f with no Right options and only ordered pairs for Left options, subject to the condition that $\forall x\ \forall y\ \forall z\ \big((x, y) \in_{\mathrm{L}} f \wedge (x, z) \in_{\mathrm{L}} f \to y = z\big)$. We write $f(x) = y$ if $(x, y) \in_{\mathrm{L}} f$, and remark that $\biguplus x$ is the object whose left elements are those in the domain of f, and whose right members are those amphisets in the image of f.

It is easily seen that ω is left-inductive. Using this we can define a transitive closure of an amphiset, and proceed to prove an appropriate $\in^{\mathrm{L}}_{\mathrm{R}}$-induction principle, i.e.

$$\forall x \left(\forall y \in^{\mathrm{L}}_{\mathrm{R}} x \; \phi(y) \to \phi(x) \right) \to \forall x \phi(x)$$

for all formulas $\phi(u)$. In fact, for each union type $\mathbb{U}$ of $\bigsqcup, \biguplus, \bigcup, \biguplus$ the transitive closure $\mathrm{TC}(x, \mathbb{U})$ of an amphiset x is defined recursively by setting $\mathrm{TC}(x, 0, \mathbb{U}) = x$, and $\mathrm{TC}(x, s_{\mathrm{L}}(n), \mathbb{U}) = \mathbb{U}\,\mathrm{TC}(x, n, \mathbb{U})$ for $n \in_{\mathrm{L}} \omega$; then $\mathrm{TC}(x, \mathbb{U})$ is the object

$$\{ \, z\colon \; \exists n \in_{\mathrm{L}} \omega \;\; \exists y \in_{\mathrm{L}} \mathrm{TC}(x, \mathbb{U}, n) \; (z \in_{\mathrm{L}} y) \mid z\colon \; \exists n \in_{\mathrm{L}} \omega \;\; \exists y \in_{\mathrm{R}} \mathrm{TC}(x, \mathbb{U}, n) \; (z \in_{\mathrm{R}} y) \, \},$$

i.e. $\mathrm{TC}(x, \mathbb{U}) = \bigcup A$, where $A = \{ \, \mathrm{TC}(x, \mathbb{U}, n) : n \in_{\mathrm{L}} \omega \mid \mathrm{TC}(x, \mathbb{U}, n) : n \in_{\mathrm{L}} \omega \, \}$. In particular $\mathrm{TC}(x, \biguplus)$ is transitive in the relations $\in_{\mathrm{L}}, \in_{\mathrm{R}}, \in^{\mathrm{L}}_{\mathrm{R}}$, while $\mathrm{TC}(x, \bigcup)$ is transitive in $\in_{\mathrm{L}}, \in_{\mathrm{R}}$, but not necessarily in $\in^{\mathrm{L}}_{\mathrm{R}}$.

By $\in^{\mathrm{L}}_{\mathrm{R}}$-recursion we can define the operations of negation, addition, multiplication, etc. We can also repeat Conway's definitions of what it means for a game to be less than, greater than, equal to, etc., another game, exactly as in ONAG.

2.3 Interpreting Amphi-ZF in ZF

Working in ordinary ZF now, and basing the following on Quine's notion of ordered pairs [76], we make the following definitions.

Definition 2.3.1. For all sets x we define

$$f_{\mathrm{L}}(x) = \{ \, s(u) : u \in x \cap \omega \, \} \cup (x \setminus \omega),$$
$$f_{\mathrm{R}}(x) = \{ \, 0 \, \} \cup \{ \, s(u) : u \in x \cap \omega \, \} \cup (x \setminus \omega).$$

(Here $s(u)$ denotes the set successor of u, $\{ \, u \, \} \cup u$.)

It is useful to note that $f_{\mathrm{L}}, f_{\mathrm{R}}$ have a mutual left-inverse, defined by

$$g(x) = \{\, u \in \omega : s(u) \in x \,\} \cup (x \setminus \omega).$$

We can use f_{L} and f_{R} to define relations $\in_{\mathrm{L}}, \in_{\mathrm{R}}$ by $x \in_{\mathrm{P}} y \Leftrightarrow f_{\mathrm{P}}(x) \in y$. Formally, this is as follows.

Definition 2.3.2. We define a translation $\mathfrak{v}\colon$ AZF $\to$ ZF by setting $(x \in_{\mathrm{P}} y)^{\mathfrak{v}}$ to be equivalent to $(f_{\mathrm{P}}(x) \in y)$.

We argue that $\mathfrak{v}$ is an interpretation, i.e. that for all axioms A of Amphi-ZF, ZF $\vdash A^{\mathfrak{v}}$. Most of these are easy to see. We shall prove here the most difficult case that, if Fnd_2 denotes the amphi-foundation axiom, then ZF $\vdash \mathrm{Fnd}_2^{\mathfrak{v}}$; the other axioms are left to the reader. First define a cumulative hierarchy of games in ZF as follows.

$$\mathscr{G}_0 = 0;$$
$$\mathscr{G}_{\alpha+1} = \{\, f_{\mathrm{L}}(z)\colon z \subseteq \mathscr{G}_\alpha \,\} \cup \{\, f_{\mathrm{R}}(z)\colon z \subseteq \mathscr{G}_\alpha \,\}; \text{ and}$$
$$\mathscr{G}_\lambda = \bigcup_{\delta<\lambda} \mathscr{G}_\lambda \text{ for limit ordinals } \lambda.$$

Notice that this is exactly the interpretation in ZF of an obvious cumulative hierarchy of games, since for all sets x, z we have $z \subseteq x \Leftrightarrow z \sqsubseteq^{\mathfrak{v}} x$. By showing that every set is a member of this hierarchy, we can deduce the translation of Amphi-foundation in ZF.

Proposition 2.3.3. In ZF, for ordinals α, β we have the following.

- If $\alpha < \beta$ then $\mathscr{G}_\alpha \subseteq \mathscr{G}_\beta$.
- $\mathscr{G}_\alpha$ is $\in$-transitive.
- Every set is in some $\mathscr{G}_\alpha$.

Proof. Each claim is proved by induction. Assume that whenever $\gamma < \alpha < \beta$ we have $\mathscr{G}_\gamma \subseteq \mathscr{G}_\alpha$. If $\beta = \alpha + 1$, say, and $x \in \mathscr{G}_\alpha$ then for some $\gamma < \alpha$ there is $z \subseteq \mathscr{G}_\gamma$ such that

$x = f_{\mathrm{P}}(z)$. As $z \subseteq \mathscr{G}_\gamma$ we have $z \subseteq \mathscr{G}_\alpha$ and so $x \in \mathscr{G}_{\alpha+1}$; thus $\mathscr{G}_\alpha \subseteq \mathscr{G}_{\alpha+1}$. The claim is clear when β is a limit.

To see the second claim suppose $\mathscr{G}_\alpha$ is transitive (in $\in$) for all $\alpha < \beta$. If $\beta = \alpha + 1$ and $y \in x \in \mathscr{G}_{\alpha+1}$ then $x = f_{\mathrm{P}}(z)$ for some $z \subseteq \mathscr{G}_\alpha$, and $y \in f_{\mathrm{P}}(z)$. If $y \notin \omega$ then $y \in z \subseteq \mathscr{G}_\alpha$, so by monotonicity $y \in \mathscr{G}_\beta$.

If instead $y \in \omega$ then $y = 0$ (in which case $0 = f_{\mathrm{L}}(0) \in \mathscr{G}_1 \subseteq \mathscr{G}_\beta$) or $y = s(u)$ for some $u \in z \cap \omega$. Assuming the second case, $u \in \mathscr{G}_\alpha$, and so by transitivity $u \subseteq \mathscr{G}_\alpha$. Since the successor map and f_{R} coincide on ω, $y = f_{\mathrm{R}}(u) \in \mathscr{G}_\beta$. If instead β is a limit ordinal then $\mathscr{G}_\beta$ is a union of transitive sets, and hence the claim.

For the final claim, suppose $x \subseteq \mathscr{G}_\alpha$, but $x \notin \mathscr{G}_{\alpha+1}$. Then $g(x) \not\subseteq \mathscr{G}_\alpha$. Let $y \in g(x) \backslash \mathscr{G}_\alpha$. If $y \notin \omega$ then $y \in x \subseteq \mathscr{G}_\alpha$, a contradiction. Therefore $y \in \omega$, so $y \in s(y) \in x \subseteq \mathscr{G}_\alpha$. By transitivity of $\mathscr{G}_\alpha$ we have $y \in \mathscr{G}_\alpha$, a contradiction. Therefore $x \subseteq \mathscr{G}_\alpha \Rightarrow x \in \mathscr{G}_{\alpha+1}$, i.e. $\mathscr{P}(\mathscr{G}_\alpha) \subseteq \mathscr{G}_{\alpha+1}$ for all α. In particular as $V_0 = \mathscr{G}_0$ we have that $V_\alpha \subseteq \mathscr{G}_\alpha$ for all α. The claim follows. $\square$

Theorem 2.3.4. $\mathfrak{v}$: AZF $\to$ ZF.

Proof. We show here that $\mathrm{Fnd}_2^{\mathfrak{v}}$ follows from ZF; the remaining axioms are easily verified. Define the 'game rank' $\mathrm{grank}(y)$ of a set y to be the least ordinal α such that $y \subseteq \mathscr{G}_\alpha$. Let x be an arbitrary set, and pick $y \in^{\mathrm{L}}_{\mathrm{R}} x$ of minimal game rank α. Supposing $z \in^{\mathrm{L}}_{\mathrm{R}} x \wedge z \in^{\mathrm{L}}_{\mathrm{R}} y$, some $f_{\mathrm{P}}(z) \in y$ and so $z \subseteq \mathscr{G}_\beta$ for some $\beta < \alpha$. Hence $\mathrm{grank}(z) \leq \beta < \mathrm{grank}(y)$, contradicting our choice of y. $\square$

Everything we have done here works in a similar way for weaker theories. In particular we consider the 1-sided set theory EST, with axioms of extensionality, empty set, pair set, sum set, separation and replacement.

It is important to note that we can define ω in EST and use the principle of induction and recursion on it (a proof of this is given by, for example, Cox [16]). This is because either the axiom of Infinity holds and ω is a set as usual, or else ω is a definable class—the class of all ordinals. The functions $f_{\mathrm{L}}, f_{\mathrm{R}}$ and g defined above may then be defined in

EST.

AST is the two-sided version of EST with the two-sided versions of all these axioms as described above.

The following is immediate from the above discussion.

Theorem 2.3.5. $\mathfrak{v}\colon$ AST $\to$ EST.

2.4 Interpreting ZF in Amphi-ZF

Firstly we observe that ZF can be interpreted easily as a class in any model $\mathscr{G}$ of Amphi-ZF. This allows us to define a set-membership relation E in $\mathscr{G}$, essentially copying $\in$. More precisely, we may define $\mathscr{G}_{\mathrm{L}}$ to be the subclass of games which are hereditarily Right-empty; formally this is an interpretation $\mathfrak{l}\colon$ ZF $\to$ AZF where we let $\delta^{\mathfrak{l}}(x)$ be the formula $\mathrm{TC}(x, \biguplus) \subseteq_{\mathrm{R}} 0$, and $(x \in y)^{\mathfrak{l}}$ is $(x \in_{\mathrm{L}} y)$.

Proposition 2.4.1. The subclass $(\mathscr{G}_{\mathrm{L}}, \in_{\mathrm{L}})$ satisfies the axioms of ZF; hence $\mathfrak{l}\colon$ ZF $\to$ AZF.

In order to find an interpretation $\mathfrak{g}\colon$ ZF $\to$ AZF whose domain contains *all* games we may construct a (definable) bijection $F\colon \mathscr{G}_{\mathrm{L}} \to \mathscr{G}$, and mimic the behaviour of $\in$ in $\mathscr{G}$. This bijection can be defined in such a way that $\mathfrak{g}$ is inverse to $\mathfrak{v}$.

Working in Amphi-ZF, functions $f^{\mathfrak{l}}_{\mathrm{L}}, f^{\mathfrak{l}}_{\mathrm{R}}$ are determined (uniquely) by

$$f^{\mathfrak{l}}_{\mathrm{L}}(x) = \{\, s^{\mathfrak{l}}(u) : u \in_{\mathrm{L}} x \cap \omega \mid \,\} \cup (x \setminus \omega),$$
$$f^{\mathfrak{l}}_{\mathrm{R}}(x) = \{\, 0 \,\} \cup \{\, s^{\mathfrak{l}}(u) : u \in_{\mathrm{L}} x \cap \omega \mid \,\} \cup (x \setminus \omega).$$

(Here we use ω to denote the game $\{\, 0, 1, \dots \mid \,\}$ in $\mathscr{G}_{\mathrm{L}}$). The appropriate definition of F is then rather straightforward: if x is a set then we interpret it as the game $\{\, y : f_{\mathrm{L}}(y) \in x \mid y : f_{\mathrm{R}}(y) \in x \,\}$, where each set y is already interpreted as a game. Thus we define

$$F(x) = \{\, F(y) : f^{\mathfrak{l}}_{\mathrm{L}}(y) \in_{\mathrm{L}} x \mid F(y) : f^{\mathfrak{l}}_{\mathrm{R}}(y) \in_{\mathrm{L}} x \,\},$$

and notice that F has an inverse according to the rule

$$F^{-1}(x) = \{\, f_{\mathrm{P}}^{\mathfrak{l}}(y) : F(y) \in_{\mathrm{P}} x \mid \}.$$

We define $\in$ by the rule $\forall x \, \forall y \big(y \in x \leftrightarrow F^{-1}(y) \in_{\mathrm{L}} F^{-1}(x) \big)$, i.e. we define a mapping of $\mathscr{L}_{\in}$-formulas by taking $(y \in x)^{\mathfrak{g}}$ to be $(F^{-1}(y) \in_{\mathrm{L}} F^{-1}(x))$. Then the following are not difficult (for full proofs, see Cox [16]).

Theorem 2.4.2. $\mathfrak{g}$: ZF $\to$ AZF.

Theorem 2.4.3. The morphisms $\mathfrak{v}$ and $\mathfrak{g}$ are inverse to one another in **INT**. That is, ZF and AZF are synonymous.

The problem with this approach, natural as it is, is that it appeals to induction and recursion in a strong way, so is not available in models of AST. We must define $\mathfrak{g}$ in AST directly. As the definitions are technical and perhaps not obvious we will spend a little time motivating them.

Start by considering $0, 1, 2, \ldots$ in $\mathscr{V} \vDash$ EST. These are of course given by $n = \{0, 1, \ldots, n-1\}$ but each corresponds to a set $n^{\mathfrak{v}}$ in $\mathscr{G} = \mathscr{V}^{\mathfrak{v}}$. We calculate its notation in $\mathscr{G}$. As 0 is empty it has no members, so no $f_{\mathrm{P}}(x)$ is in 0 so $0^{\mathfrak{v}} = \{|\}$. Similarly $1 = \{0\}$ has single member $0 = f_{\mathrm{L}}(0)$, so $0^{\mathfrak{v}} = \{0^{\mathfrak{v}} \; |\} = \{0 \; |\}$. $2 = \{0, 1\}$ has members $0 = f_{\mathrm{L}}(0)$ and $1 = \{0\} = f_{\mathrm{R}}(0)$ so $2^{\mathfrak{v}} = \{0^{\mathfrak{v}} \mid 0^{\mathfrak{v}}\} = \{0 \mid 0\}$. Similarly, $3 = \{0, 1, 2\} = \{f_{\mathrm{L}}(0), f_{\mathrm{R}}(0), f_{\mathrm{R}}(1)\}$ so $3^{\mathfrak{v}} = \{0 \mid 0, \{0 \; |\}\}$, $4 = \{0, 1, 2, 3\} = \{f_{\mathrm{L}}(0), f_{\mathrm{R}}(0), f_{\mathrm{R}}(1), f_{\mathrm{R}}(2)\}$ so $4^{\mathfrak{v}} = \{0 \mid 0, \{0 \; |\}, \{0 \mid 0\}\}$, and so on.

This motivates the following curious notation for the integers.

Definition 2.4.4 (AST)**.** For $n \in \omega$, define $\nu(n)$ as follows. Let $\nu(0) = \{\,|\,\}$ and $\nu(1) = \{0\,|\,\}$. For $n \in \mathbb{N}$, we define

$$\begin{aligned} \nu(n+2) &= \nu(n+1) \cup \{\,|\,\nu(n)\} \\ &= \{\nu(0) \,|\, \nu(0), \ldots, \nu(n)\}. \end{aligned}$$

We also define $\nu(\omega) = \{\nu(0) \mid \nu(0), \nu(1), \ldots\}$. Notice that $\nu(\omega)$ may not exist as a set in a model of AST (its existence requires an infinity axiom), but is a definable 'amphi-class'.

Now, using the new notions, we may repeat Definition 2.3.1.

Definition 2.4.5 (AST)**.** Define

$$\begin{aligned}
\tilde{f}_{\mathrm{L}}(x) &= \{\mid \nu(n+1) : \nu(n) \in_{\mathrm{R}} x\} \cup (x \setminus \nu(\omega)) && \text{if } 0 \notin_{\mathrm{L}} x \\
&= \{\mid 0\} \cup \{\mid \nu(n+1) : \nu(n) \in_{\mathrm{R}} x\} \cup (x \setminus \nu(\omega)) && \text{otherwise} \\
\tilde{f}_{\mathrm{R}}(x) &= \{0 \mid\} \cup f_{\mathrm{L}}(x)
\end{aligned}$$

Also,

$$\begin{aligned}
\tilde{g}(x) &= \{\mid \nu(n) : \nu(n+1) \in_{\mathrm{R}} x\} \cup (x \setminus \nu(\omega)) && \text{if } 0 \notin_{\mathrm{R}} x \\
&= \{0 \mid\} \cup \{\mid \nu(n) : \nu(n+1) \in_{\mathrm{R}} x\} \cup (x \setminus \nu(\omega)) && \text{otherwise.}
\end{aligned}$$

Now we can define our interpretation.

Definition 2.4.6 (AST)**.** For all x, y,

$$(x \in y)^{\mathfrak{g}} \leftrightarrow (0 \notin_{\mathrm{L}} x \wedge \tilde{g}(x) \in_{\mathrm{L}} y) \vee (0 \in_{\mathrm{L}} x \wedge \tilde{g}(x) \in_{\mathrm{R}} y).$$

The following are now straightforward.

Lemma 2.4.7 (AST)**.** For all $n \in \omega$ and all x, $(\nu(n) \in x)^{\mathfrak{g}}$ if and only if $(\nu(n+1) \in \tilde{f}_{\mathrm{L}}(x))^{\mathfrak{g}}$. Consequently, $\tilde{f}_{\mathrm{P}}(x) = f^{\mathfrak{g}}_{\mathrm{P}}(x)$ and $\tilde{g}(x) = g^{\mathfrak{g}}(x)$.

Proposition 2.4.8. Let $\mathscr{G} \models$ AST. Then $\mathscr{G} \models \mathrm{EST}^{\mathfrak{g}}$.

Finally we show that the interpretations are inverse to one another, i.e. $\mathfrak{g}\mathfrak{v} = 1$ and $\mathfrak{v}\mathfrak{g} = 1$. First, we require a preparatory lemma within a model of EST.

Lemma 2.4.9. In any model of EST, $\tilde{f}^{\mathfrak{v}}_{\mathrm{P}} = f_{\mathrm{P}}$; $\tilde{g}^{\mathfrak{v}} = g$.

Proof. Induction on ω. $\square$

This immediately gives the following.

Proposition 2.4.10. (a) In AST, for all x, y, $x \in_{\mathrm{P}} y$ if and only if $(x \in_{\mathrm{P}} y)^{\mathfrak{gv}}$.

(b) In EST, for all x, y, $x \in y$ if and only if $(x \in y)^{\mathfrak{vg}}$.

Corollary 2.4.11. The theories EST and AST are synonymous in the sense of Visser.

It may be of interest to develop a catalogue of equivalent subtheories of ZF and AZF extending EST and AST respectively, equivalent via the interpretations just defined. This involves showing $\mathrm{EST} + A \vdash A_2^{\mathfrak{v}}$ and $\mathrm{AST} + A_2 \vdash A^{\mathfrak{g}}$ for various axioms A where A_2 is the two-sided version of A. Of course a great many such results may be given, and we give a very small sample here.

The proof of the following is straightforward.

Proposition 2.4.12. For sentences $A \in \{\mathrm{Inf}, \mathrm{Pow}, \mathrm{Inf} \wedge \mathrm{Pow}\}$, $\mathrm{EST} + A \cong \mathrm{AST} + A_2$.

The case for foundation is less straightforward. It is not immediately obvious whether $\mathrm{EST} + \mathrm{Fnd} \cong \mathrm{AST} + \mathrm{Fnd}_2$, although we can say that, by constructing a cumulative hierarchy for each theory within the other (as in Proposition 2.3.3), we can show that $\mathrm{EST} + \mathrm{Pow} + \mathrm{Fnd}$ is synonymous with its amphi-equivalent. However there is good reason to avoid considering foundation alone. Foundation's role is essentially to provide us with $\in$-induction (denoted $\mathrm{Ind}(\in)$), or its amphi-equivalent, $\mathrm{Ind}(\in^{\mathrm{L}}_{\mathrm{R}})$. However this does not follow from $\mathrm{EST} + \mathrm{Fnd}$ (or $\mathrm{AST} + \mathrm{Fnd}_2$) alone: we require the additional axiom that every object has a transitive closure, which is normally provided by the infinity axiom (see Kaye and Wong's article [54] or Mancini and Zambella [69]).

Proposition 2.4.13. The theories $\mathrm{EST} + \mathrm{Ind}(\in)$ and $\mathrm{AST} + \mathrm{Ind}(\in^{\mathrm{L}}_{\mathrm{R}})$ are synonymous, via the interpretations $\mathfrak{g}$ and $\mathfrak{v}$.

Proof. Working in $\mathrm{EST} + \mathrm{Ind}(\in)$, define a 'game rank' inductively, by

$$\mathrm{grank}(x) = \sup\{\mathrm{grank}(u) \colon u \in g(x) + 1\}.$$

If $\phi(x, \bar{a})$ is any formula such that

$$\forall x \big(\forall y \in_{\mathrm{R}}^{\mathrm{L}\,\mathfrak{g}} x \phi(y, \bar{a}) \to \phi(x, \bar{a}) \big).$$

Then $\forall x\, \phi(x)$ can be proved by considering some x of minimal game rank for which $\neg\phi(x)$.

Analogously, if we work within $\mathrm{AST} + \mathrm{Ind}(\in_{\mathrm{R}}^{\mathrm{L}})$ we can define a set rank by

$$\mathrm{rank}(x) = \sup\{\mathrm{rank}(u) \colon \tilde{g}(u) \in_{\mathrm{R}}^{\mathrm{L}} x\} + 1,$$

and proceed as above. □

Of particular interest in combinatorial game theory are the so-called short (hereditarily finite) games. We obtain a suitable theory for such games by negating our infinity axiom, Inf, and ensuring that full induction is available as in the last proposition. By $\mathrm{ZF}_{\mathrm{Inf}}$ and $\mathrm{AZF}_{\mathrm{Inf}}$ we denote the theories of ZF and AZF minus their respective infinity axioms. By $\mathrm{ZF}^*_{\mathrm{Inf}}$ and $\mathrm{AZF}^*_{\mathrm{Inf}}$ we denote these theories plus an appropriate axiom, TC, of transitive containment. (In ZF we take $\forall x\, \exists y (x \subseteq y \wedge \forall u \forall v (u \in v \wedge v \in y \to u \in y))$; in amphi-ZF we take the same, but with $\in$ replaced by $\in_{\mathrm{R}}^{\mathrm{L}}$.) Notice that $\mathrm{ZF}^*_{\mathrm{Inf}}$ is equivalent to the theory $\mathrm{EST} + \mathrm{Ind}(\in) + \mathrm{Pow}$ (and analogously for the appropriate amphi-variants). This immediately gives us the following.

Theorem 2.4.14. $\mathfrak{g} \colon \mathrm{ZF}^*_{\mathrm{Inf}} \to \mathrm{AZF}^*_{\mathrm{Inf}}$ and $\mathfrak{v} \colon \mathrm{AZF}^*_{\mathrm{Inf}} \to \mathrm{ZF}^*_{\mathrm{Inf}}$ are inverse to one another in **INT**.

Notice that by a result of Kaye and Wong [54] this implies $\mathrm{AZF}^*_{\mathrm{Inf}} \cong \mathrm{PA}$ in **INT**, where PA is the theory of Peano Arithmetic.

2.5 Amphi-NBG

For the sake of completeness, in particular as Conway [13, p.67] mentions it, we briefly describe the two-sided version NBG_2 of von-Neumann–Bernays–Gödel set theory, NBG.

Following one of the popular formulations of NBG without choice, we take a two-sorted language with variables for Class-like Games $A, B, C, \ldots$ and (set-like) games $a, b, c, \ldots$. The well-formed atomic formulas are of the form $A = B$ (identity of Class-like Games), $a = b$ (identity of games), $a \in_{\mathrm{L}} B$, $a \in_{\mathrm{R}} B$, $a \in_{\mathrm{L}} b$, and $a \in_{\mathrm{R}} b$ (membership). We use $a = B$ as an abbreviation for $\bigwedge_{\mathrm{P}} \forall x\, (x \in_{\mathrm{P}} a \leftrightarrow x \in_{\mathrm{P}} B)$.

Axiom 9 (Extensionality)**.**

$$\forall x\ \forall y \Big(\bigwedge_{\mathrm{P}} \big(\forall z\, (z \in_{\mathrm{P}} x \leftrightarrow z \in_{\mathrm{P}} y)\big) \to x = y \Big)$$

and

$$\forall X\ \forall Y \Big(\bigwedge_{\mathrm{P}} \big(\forall z\, (z \in_{\mathrm{P}} X \leftrightarrow z \in_{\mathrm{P}} Y)\big) \to X = Y \Big).$$

Axiom 10 (Pair)**.**

$$\forall x\ \forall y\ \exists z\, (x \in_{\mathrm{L}} z \wedge y \in_{\mathrm{L}} z).$$

Axiom 11 (Union)**.**

$$\forall x\ \exists y\ \forall z \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} y \leftrightarrow \exists w \in^{\mathrm{L}}_{\mathrm{R}} x\ (z \in^{\mathrm{L}}_{\mathrm{R}} w)\big).$$

Axiom 12 (Power)**.**

$$\forall x\ \exists y\ \forall z\, \big(\bigwedge_{\mathrm{P}} z \in_{\mathrm{P}} y \leftrightarrow z \subseteq x \big).$$

Axiom 13 (Infinity)**.**

$$\exists x \bigwedge_{\mathrm{P}} \big(0 \in_{\mathrm{P}} x \wedge \forall y \in_{\mathrm{P}} x (s_{\mathrm{P}}(y) \in_{\mathrm{P}} x)\big).$$

Axiom 14 (Foundation)**.**

$$\forall X \neq 0\ \exists y \in^{\mathrm{L}}_{\mathrm{R}} X\ \forall z \in^{\mathrm{L}}_{\mathrm{R}} X\, (z \notin^{\mathrm{L}}_{\mathrm{R}} y).$$

Axiom 15 (Comprehension)**.** For all first-order formulas $\phi_{\mathrm{L}}(\bar{U}, \bar{u}, v)$, $\phi_{\mathrm{R}}(\bar{U}, \bar{u}, v)$ in free variables shown we have

$$\forall X\ \forall \bar{x}\ \exists Y\ \forall z \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} Y \leftrightarrow \phi_{\mathrm{P}}(\bar{X}, \bar{x}, z)\big).$$

Axiom 16 (Replacement)**.**

$$\forall F_{\mathrm{L}}, F_{\mathrm{R}}\ \forall x\ \big(\bigwedge_{\mathrm{P}} (\forall u\ \forall v\ \forall w\ \big((u,v) \in_{\mathrm{L}} F_{\mathrm{P}} \wedge (u,w) \in_{\mathrm{L}} F_{\mathrm{P}} \to v = w\big) \\ \to \exists y \bigwedge_{\mathrm{P}} \forall z\, (z \in_{\mathrm{P}} y \leftrightarrow \exists u\, (u \in_{\mathrm{P}} x \wedge (u,z) \in_{\mathrm{L}} F_{\mathrm{P}})\big)$$

Our formulation of NBG is the obvious one-sided version of these axioms. One could if one wished add to either NBG or NBG_2 any of the usual forms of the axiom of global choice.

Working in NBG, Definition 2.3.1 applies to both sets and classes, sending sets to sets and classes to classes. This gives an interpretation $\mathfrak{v}\colon \mathrm{NBG}_2 \to \mathrm{NBG}$ formally similar to the one in Section 2.3 where $(x \in_{\mathrm{P}} y)^{\mathfrak{v}}$ is $(f_{\mathrm{P}}(x) \in y)$, equality is preserved and the property of being set-like is preserved. Without any difficulty the arguments above show this is indeed an interpretation. Working in the other direction, the interpretation $\mathfrak{l}$ of Section 2.4 is extended to preserve the 'set-like' predicate on objects, and gives $\mathfrak{l}\colon \mathrm{NBG} \to \mathrm{NBG}_2$, restricting to hereditarily right-empty sets and right-empty classes of such sets. The class $\mathscr{G}_{\mathrm{L}}$ of hereditarily right-empty sets is in 1–1 correspondence with the class of all set-like games, for the same reason as in Section 2.4, and this yields also a bijection between the subclasses of $\mathscr{G}_{\mathrm{L}}$ and the class-like games. Thus the technique of mimicking $\in$ in the whole collection of games goes through too, giving an interpretation $\mathfrak{g}\colon \mathrm{NBG} \to \mathrm{NBG}_2$ which is inverse to $\mathfrak{v}$. The straightforward details are omitted.

2.6 Rieger–Bernays permutation models

In defining AZF we have chosen certain axioms almost arbitrarily, where other obvious axioms might have been equally intuitive. For instance, our choice of a Pair-set axiom is simple, though it could be argued that a symmetric version would be more fitting. More interestingly, various different types of union are available—none of which is obviously more appropriate than the rest—and any axiom positing the existence of one such union would suffice. In these cases each choice of axiom is equivalent, modulo the other axioms of AZF, to the axioms it has been chosen above. In the case of union axioms it may be interesting to consider much smaller fragments of AZF (perhaps obtained by weakening the replacement scheme) which do not necessarily provide this equivalence, though we will not do so here.

Of more importance to us is the weakening of foundation. Foundation is of particular interest in the theory of games through potential applications of illfounded games in the semantics of computer processes.

The Rieger–Bernays permutation construction (see for example Forster [28]) can be used to obtain models of AZF in which the full foundation axiom fails but some form of foundation remains, perhaps just enough to preserve certain structure present in Conway games, while also allowing one to consider illfounded games. (See also the questions at the end of this chapter.)

As usual, we let $\mathrm{Sym}(\mathscr{G})$ denote the collection of permutations of $\mathscr{G}$.

Definition 2.6.1. Let $\mathscr{G} \models \mathrm{AZF}$, and suppose $\pi \in \mathrm{Sym}(\mathscr{G})$. For $x, y \in \mathscr{G}$ we write $x \in^{\pi}_{\mathrm{L}} y$ for $x \in_{\mathrm{L}} \pi y$ and $x \in^{\pi}_{\mathrm{R}} y$ for $x \in_{\mathrm{R}} \pi y$. By G^{π} we denote the first-order structure $(\mathscr{G}, \in^{\pi}_{\mathrm{L}}, \in^{\pi}_{\mathrm{R}})$.

It will be useful to use AZF^{-} to refer to the theory with all the axioms given above for AZF except for the foundation axiom A7.

The following is an easy translation of the usual result in ZF.

Theorem 2.6.2. Suppose $\mathscr{G} \models \mathrm{AZF}$. If $\pi \in \mathrm{Sym}(\mathscr{G})$ is definable, then $G^{\pi} \models \mathrm{AZF}^{-}$.

In Remark 2.2.1 we discussed a variant (denoted $\bigwedge_{\mathrm{P}} \mathrm{wf}(\in_{\mathrm{P}})$) of our amphi-foundation axiom ($\mathrm{wf}(\in^{\mathrm{L}}_{\mathrm{R}})$). We can use Rieger-Bernays permutation models to prove that the variant is strictly weaker.

Theorem 2.6.3. Let $\mathscr{G} \vDash$ AZF, with as usual $1 = \{0\,|\,\}$, $-1 = \{\,|\,0\}$, and (for this theorem only[1]) use Conway's definition of $2 = \{1\ |\}$ and $-2 = \{|\ -1\}$. Let π be the permutation

$$(1\ \ -2)\cdot(-1\ 2).$$

Then G^π is a model of

$$\mathrm{AZF}^- + \bigwedge_{\mathrm{P}} \mathrm{wf}(\in_{\mathrm{P}}) + \neg\,\mathrm{wf}(\in^{\mathrm{L}}_{\mathrm{R}}).$$

Proof. Let $\mathrm{wf}_{\mathrm{P}}(x)$ denote the formula

$$\exists y\,(y \in_{\mathrm{P}} x) \to \exists y \in_{\mathrm{P}} x\ \forall z \in_{\mathrm{P}} x\,(z \notin_{\mathrm{P}} y), \tag{2.1}$$

the statement that x is wellfounded. Note that if we can prove $\forall x\ \mathrm{wf}_{\mathrm{L}}(x)$ then $\forall x\ \mathrm{wf}_{\mathrm{R}}(x)$ follows by the symmetry of π and AZF.

Fix an amphiset x and assume that $\mathscr{G}^\pi \vDash \neg\,\mathrm{wf}_{\mathrm{L}}(x)$. Suppose first that $x \in X = \{\,\pm 1,\,\pm 2\,\}$. Since 1 and 2 are $\in^\pi_{\mathrm{L}}$-empty, they satisfy wf_{L}; therefore $x = -1$ or $x = -2$. These contain sole $\in^\pi_{\mathrm{L}}$-members 1 and 0 respectively, which are $\in^\pi_{\mathrm{L}}$-empty. Hence $\mathrm{wf}_{\mathrm{L}}(x)$, a contradiction.

Now suppose $x \notin X$. Let

$$x' = \{\,u \in_{\mathrm{L}} x\colon u \notin X \,|\, u \in_{\mathrm{R}} x\,\}.$$

As $\mathscr{G} \vDash \mathrm{wf}_{\mathrm{L}}(x')$, we can pick $u \in_{\mathrm{L}} x'$ such that $\forall v \in_{\mathrm{L}} x'\,(v \notin_{\mathrm{L}} u)$. As $x' \subseteq_{\mathrm{L}} x$, $u \in_{\mathrm{L}} x$; since $x \notin X$, $u \in^\pi_{\mathrm{L}} x$. Since x is not $\in^\pi_{\mathrm{L}}$-wellfounded, there is $v \in^\pi_{\mathrm{L}} x$ such that $v \in^\pi_{\mathrm{L}} u$. Since $v \in^\pi_{\mathrm{L}} x$, $v \in_{\mathrm{L}} x$, so that $v \notin_{\mathrm{L}} u$. As $v \in^\pi_{\mathrm{L}} u \wedge u \notin_{\mathrm{L}} u$, $u \in X$. This contradicts the choice of u.

Finally, $1 \in_{\mathrm{L}}{}^\pi -1 \in_{\mathrm{R}}{}^\pi\ 1$ showing $\neg\,\mathrm{wf}(\in^{\mathrm{L}}_{\mathrm{R}})$. ∎

Given $\mathscr{G} \vDash$ AZF there are two 'obvious' permutations to look at. The first swaps the left and right members, $x \mapsto x^* = \{u : u \in_{\mathrm{R}} x \mid v : v \in_{\mathrm{L}} x\}$. Then it is easy to see that

[1] See the discussion towards the end of Section 2.2.

$x \in_{\mathrm{L}}{}^* y$ if and only if $x \in_{\mathrm{L}} y^*$, i.e. $x \in_{\mathrm{R}} y$ and similarly for R, so the permutation model $(\mathscr{G}^*, \in_{\mathrm{L}}{}^*, \in_{\mathrm{R}}{}^*)$ is just $(\mathscr{G}, \in_{\mathrm{R}}, \in_{\mathrm{L}})$ with $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$ swapped. Another way of saying this is that the map $x \mapsto -x$ given as usual by

$$-x = \{-u : u \in_{\mathrm{R}} x \mid -v : v \in_{\mathrm{L}} x\}$$

is an isomorphism $(\mathscr{G}, \in_{\mathrm{L}}, \in_{\mathrm{R}}) \to (\mathscr{G}^*, \in_{\mathrm{L}}{}^*, \in_{\mathrm{R}}{}^*)$.

The second 'obvious' permutation is the additive inverse $x \mapsto -x$ itself. It is easy to check that $x \in_{\mathrm{L}}{}^- y$ if and only if $x \in_{\mathrm{L}} -y$, i.e. $-x \in_{\mathrm{R}} y$ and similarly for R. Since the rank of $-x$ is the same as that of x it follows that $\mathscr{G}^-$ satisfies full foundation, i.e. $\mathscr{G}^- \models \mathrm{AZF}$. In fact, $(\mathscr{G}^-, \in_{\mathrm{L}}{}^-, \in_{\mathrm{R}}{}^-)$ is actually isomorphic to $(\mathscr{G}, \in_{\mathrm{L}}, \in_{\mathrm{R}})$ via the isomorphism defined recursively in $\mathscr{G}$ by

$$\phi(y) = \{\phi(u) : -u \in_{\mathrm{R}} y \mid \phi(v) : -v \in_{\mathrm{L}} (y)\}.$$

It is unclear what this ϕ operation is, except that it too is a permutation and may be used to give a further permutation model also isomorphic to the original, via yet another somewhat obscure map. We are not sure if this is an interesting or profitable line of enquiry and have left it here.

This concludes our outline of what might be called the 'traditional' permutation model construction. But since we are working in a two-sided set theory, we may consider a two-sided analogue of these permutations.

In the following we use relations $\equiv_{\mathrm{L}}$ and $\equiv_{\mathrm{R}}$ defined by by $x \equiv_{\mathrm{P}} y \leftrightarrow x \subseteq_{\mathrm{P}} y \subseteq_{\mathrm{P}} x$.

Definition 2.6.4. Assume $\mathscr{G} \models \mathrm{AZF}$. Suppose $\pi = (\pi_{\mathrm{L}}, \pi_{\mathrm{R}})$ is a pair of permutations from $\mathscr{G}$. For $x, y \in \mathscr{G}$, we write

$$x \in_{\mathrm{L}}^{\pi} y \leftrightarrow x \in_{\mathrm{L}} \pi_{\mathrm{L}} y;$$

$$x \in_{\mathrm{R}}^{\pi} y \leftrightarrow x \in_{\mathrm{R}} \pi_{\mathrm{R}} y.$$

By G^π we denote the structure $(\mathscr{G}, \in^\pi_{\mathrm{L}}, \in^\pi_{\mathrm{R}})$. If, in addition,

$$\forall x, y \left(\left(\bigwedge_{\mathrm{P}} \pi_{\mathrm{P}} x \equiv_{\mathrm{P}} \pi_{\mathrm{P}} y \right) \to x = y \right), \tag{2.2}$$

then we call π an *amphi-permutation*. We say that π is a *proper* amphi-permutation if additionally it is not the case that

$$\forall x \; \exists y \, (x \equiv_{\mathrm{L}} \pi_{\mathrm{L}} y \wedge x \equiv_{\mathrm{R}} \pi_{\mathrm{R}} y).$$

Remark 2.6.5. It is interesting to note that our interpretation in Section 2.3, $(x \in_{\mathrm{P}} y)^{\mathfrak{v}}$, takes a form that is 'dual' to the amphi-permutation model, with $x \in_{\mathrm{P}} y$ if and only if $(f_{\mathrm{P}}(x) \in y)$, for 1–1 (but not bijective) functions $f_{\mathrm{L}}, f_{\mathrm{R}}$. In the same way, amphi-permutations may be used to build models with two-sided membership from single-sided models: $x \in_{\mathrm{P}} y$ if and only if $x \in \pi_{\mathrm{P}}(y)$.

Remark 2.6.6. The reason for the condition (2.2) is that we would like G^π to satisfy extensionality. It is easily checked that this condition is equivalent to extensionality in G^π. In the case of a definable amphi-permutation π (meaning, of course, that each π_{P} is definable) in a model of AZF, condition (2.2) is also equivalent to the assertion that the map

$$\hat{\pi}(x) = \{u : u \in_{\mathrm{L}} \pi_{\mathrm{L}} x \mid v : v \in_{\mathrm{R}} \pi_{\mathrm{R}} x\}$$

is 1–1. If this map were also onto our G^π would be the same as $\mathscr{G}^{\hat{\pi}}$ and this reduces to the case of the single permutation. The condition that the map $\hat{\pi}$ is onto is equivalent to the assertion that π is improper; this explains the choice of terminology.

It follows also, by a pigeonhole argument, that if $\pi = (\pi_{\mathrm{L}}, \pi_{\mathrm{R}})$ is an amphi-permutation and $\hat{\pi}$ has finite support then π is improper. For if $S = \mathrm{supp}(\hat{\pi})$ then $\hat{\pi}$ maps S into S since it is 1–1. Note too that $\mathrm{supp}(\hat{\pi}) \subseteq \mathrm{supp}\, \pi_{\mathrm{L}} \cup \mathrm{supp}\, \pi_{\mathrm{R}}$.

Example 2.6.7. We give an example of an definable amphi-permutation $\pi = (\pi_{\mathrm{L}}, \pi_{\mathrm{R}})$ of $\mathscr{G} \models$ AZF such that $\mathscr{G}^\pi$ satisfies extensionality but does not contain an empty set. This shows that proper amphi-permutations exist and that the amphi-permutation construction does not preserve stratified formulas. For simplicity, our π will have $\pi_{\mathrm{R}} = \pi_{\mathrm{L}}^{-1}$.

As usual, let $0 = \{|\}$, $n+1 = \{0, 1, \ldots, n \mid\}$ and $-(n+1) = \{\mid -n, \ldots, -1, 0\}$ for $n \in \omega$. Now define copies of these amphisets by

$$n_k = \{0, 1, \ldots, (n+k-1) \mid 0, -1, \ldots, -(k-1)\}$$

and

$$(-n)_k = \{0, 1, \ldots, (k-1) \mid 0, -1, \ldots, -(n+k-1)\}$$

for all $n, k \in \omega$. (By the only sensible convention for the meaning of '$0, 1, \ldots, -1$' we have $n = n_0$ for all n.) Note that the amphisets n_i are all distinct.)

We define

$$\pi_L \colon z_k \mapsto (z+1)_k$$

for $z = n$ or $-n$, and $n, k \in \omega$, and $\pi_R = \pi_L^{-1}$.

To check the properties of $\mathscr{G}^\pi$ it suffices to check that $\hat{\pi} \upharpoonright S$ maps into S and is 1–1, where $S = \operatorname{supp} \pi_L = \operatorname{supp} \pi_R = \{n_k : n, k \in \omega\}$. A simple calculation shows that $\hat{\pi}(n_k) = (n+1)_k$, $\hat{\pi}((-n)_k) = -(n+1)_k$ for $k \in \omega$ and $n > 0$ and that $\hat{\pi}(0_k) = 0_{k+1}$ for $k \in \omega$. So $\hat{\pi}$ is 1–1 and $0 = 0_0$ is the only amphiset not in its image.

We know of no easily stated conditions on an improper amphi-permutation that ensures $G^\pi \models \mathrm{AZF}^-$. The following proposition is the unsatisfactory result of our investigation into this question.

Proposition 2.6.8. Suppose $\mathscr{G} \models \mathrm{AZF}$ and $\pi = (\pi_{\mathrm{L}}, \pi_{\mathrm{R}})$ is a definable amphi-permutation with the stronger property that

$$\bigwedge_{\mathrm{P}} \forall x, y\, (x \equiv_{\mathrm{P}} y \leftrightarrow \pi_{\mathrm{P}} x \equiv_{\mathrm{P}} \pi_{\mathrm{P}} y).$$

Then π is improper.

Proof. Given x, let u, v satisfy $x = \pi_{\mathrm{L}} u$ and $x = \pi_{\mathrm{R}} v$, and define y so that $y \equiv_{\mathrm{L}} u$ and $y \equiv_{\mathrm{R}} v$. Then by the condition $\pi_{\mathrm{L}} y \equiv_{\mathrm{L}} \pi_{\mathrm{L}} u \equiv_{\mathrm{L}} x$ and similarly for R, so π is improper. $\square$

2.7 Open questions and suggestions for future research

Our intuition about AZF is based on that of ZF but the theory AZF is finer-structured with respect to its subtheories. A full investigation of subtheories of AZF and in particular the effect of weakening the union, power set and foundation axioms should be given. Theorem 2.6.3 presents a small start in this direction.

We have shown that ZF and its amphi version AZF are the same theory via two inverse interpretations $\mathfrak{v}$ and $\mathfrak{g}$. These interpretations are natural but may not be the only possibilities. The main question concerns exactly what axioms are required to define these interpretations and to give a catalogue of equivalent subtheories of ZF and AZF. A start was made to this programme in Section 2.4.

Given a model $\mathscr{G} \models \mathrm{AZF}$, which we regard as a collection of combinatorial games, the operations of addition and additive inverse are definable using recursion as usual,

$$x + y = \{u + y : u \in_{\mathrm{L}} x \mid v + y : v \in_{\mathrm{R}} x\} \cup \{x + u : u \in_{\mathrm{L}} y \mid x + v : v \in_{\mathrm{R}} y\}$$

and

$$-x = \{-u : u \in_{\mathrm{R}} x \mid -v : v \in_{\mathrm{L}} x\}.$$

These operations are central to the theory of games, as are

$$0 \leq x \leftrightarrow \forall u \in_{\mathrm{R}} x \;\; \exists v \in_{\mathrm{L}} u \; (0 \leq v)$$

and

$$0 \lhd\!\!\mid x \leftrightarrow \exists u \in_{\mathrm{L}} x \;\; \forall v \in_{\mathrm{R}} u \; (0 \lhd\!\!\mid v)$$

with $x \leq y \leftrightarrow 0 \leq y - x$ and $x \lhd\!\!\mid y \leftrightarrow 0 \lhd\!\!\mid y - x$. Obviously foundation is required for all these definitions, but how much? Is the full axiom of foundation required to make sense of these notions?

Given $\mathscr{G} \vDash \mathrm{AZF}^-$, we might not be able to define $+, -, \leq, \lhd\!\!\mid$ internally, but we can at least regard each $x \in \mathscr{G}$ as an (external) game (in the metatheory) with three outcomes: either L or R wins, or there is a draw—meaning that after standardly many (i.e. $\mathbb{N}$ in the metatheory) turns there is no winner. Then $+, -, \leq, \lhd\!\!\mid$ can all be defined on these games in the metatheory (with the proviso that we need to account for games that go on for infinitely many turns). It would seem to make sense to study these notions for various models $\mathscr{G} \vDash \mathrm{AZF}^-$.

The amphi-permutation construction for AZF is richer than that of single-sided membership, and this needs a thorough investigation. What sentences are necessarily preserved by such a construction? Also, the amphi-permutation construction (and its 'dual', see Remark 2.6.5) also enables other models of two-sided theories to be obtained from one-sided memberships. This should also be investigated.

We will revisit some of these themes—particularly illfounded amphisets—in chapter 5, where we consider the addition of two-orders (see the next chapter), representing the existence of strategies. In Chapter 5 we show that, assuming no amphisets are self-membered (but allowing loops of length greater than 1), there exists a possibly external notion of strategy which coincides with the interpretation of members as available moves. Some further discussion of foundation principles is given there, focusing on regularity imposed by more game-theoretic assumptions.

CHAPTER 3

TWO-ORDERED STRUCTURES

In this chapter we generalise the order-theoretic aspects of Conway games. We consider the addition of a second order $\lhd\!|$ (as used by Conway et al.[13, 5]) to a preordered structure $(X, \leq)$, but with a basic set of axioms instead of a recursive construction in a wellfounded universe of amphisets. In particular we do not require that $\lhd\!|$ be the complement of $\geq$; rather we view this as a special consequence of wellfoundedness in Conway games.

We discuss the problem of augmenting these *two-orders*, and show that the collection of such objects on a poset $(X, \leq)$ has the rich structure of a Heyting algebra. These spaces are then used in a poset representation result generalising Stone's theorem for boolean algebras.

Finally we consider the addition of group structure. We prove homomorphism theorems for two types of morphism, and demonstrate that—by a simple extension of Cayley's theorem—such groups arise as precisely the automorphism groups of two-ordered structures. We conclude with a discussion of determinacy and duality of two-orders.

3.1 Two-ordered structures

3.1.1 Definitions and examples

Definition 3.1.1. A *pre-two-order* on a nonempty set X is a pair $T = (\leq, \lhd\!\!\mid)$ of binary relations satisfying the following.

- The relation $\leq$ is a preorder:

$$\forall x \in X\,(x \leq x);$$
$$\forall x, y, z \in X\;\;(x \leq y \wedge y \leq z \to x \leq z);$$

- the relation $\lhd\!\!\mid$ is a module over $\leq$:

$$\forall x, y, z \in X\,(x \leq y \wedge y \lhd\!\!\mid z \to x \lhd\!\!\mid z);$$
$$\forall x, y, z \in X\,(x \lhd\!\!\mid y \wedge y \leq z \to x \lhd\!\!\mid z);$$

- and the relation $\lhd\!\!\mid$ is antireflexive:

$$\forall x \in X\,(x \not\lhd\!\!\mid x).$$

When $(X, \leq, \lhd\!\!\mid, \ldots)$ is some structure with $T = (\leq, \lhd\!\!\mid)$ a pre-two-order, we call $(X, \ldots)$ a pre-two-ordered structure.

If the relation $\leq$ is a partial order (i.e. $\leq$ satisfies

$$\forall x, y\,(x \leq y \wedge y \leq x \to x = y)$$

in addition to the above), then we call T a *two-order*, and $(X, \ldots)$ a *two-ordered structure.*

When $(X, \leq)$ is a preordered structure, we write $x \geq y$ for $y \leq x$, $x < y$ for $x \leq y \wedge y \not\leq x$, etc.

Whenever a structure $(X, \leq, \ldots)$ has a preorder, it can be equipped with at least one additional order $\lhd\!\!\mid$ such that $(X, \leq, \lhd\!\!\mid, \ldots)$ is a two-ordered structure.

Example 3.1.2. Let $(X, \leq)$ be any preordered set. Then the following definitions give relations $\lhd\!\!\mid$ for which $(X, \leq, \lhd\!\!\mid)$ is a pre-two-ordered structure. If $\leq$ is a partial order, these make X a two-ordered structure.

Minimal $\forall x, y\, (x \lhd\!\!\mid y \leftrightarrow \perp)$;

Strict $\forall x, y\, (x \lhd\!\!\mid y \leftrightarrow x < y)$;

Maximal $\forall x, y\, (x \lhd\!\!\mid y \leftrightarrow y \not\leq x)$.

Example 3.1.3. Dually, if X is a nonempty set and $\lhd\!\!\mid$ is any irreflexive relation on X, then the trivial partial order $\leq$ on X defined by $x \leq y \Leftrightarrow x = y$ makes $(X, \leq, \lhd\!\!\mid)$ a two-order (and is the minimal such relation). There is no canonical maximum $\leq$ that can be chosen in this case, though by Zorn's lemma there are certainly maximal partial orders $\leq$ so that $(X, \leq, \lhd\!\!\mid)$ is a two-order.

Clearly the relation $\simeq$, defined by $x \simeq y \leftrightarrow x \leq y \wedge y \leq x$, is an equivalence relation. Since the quotient $X/{\simeq}$ has two-ordered structure given by

$$x/{\simeq}\ \leq\ y/{\simeq} \leftrightarrow x \leq y$$

$$x/{\simeq}\ \lhd\!\!\mid\ y/{\simeq} \leftrightarrow x \lhd\!\!\mid y$$

in most cases we can restrict attention to consideration of two-ordered structures without losing any information. The relation $\simeq$ represents the equivalence of Sprague-Grundy and Conway et al.

Example 3.1.4. Let $\mathcal{G}$ be any model of amphi-ZF from the previous chapter, i.e. a class of two-sided sets as seen in ONAG and Winning Ways. There the orders are defined by:

$$g \leq h \leftrightarrow \forall g^L \in_L g\, (g^L \lhd\!\!\mid h) \text{ and } \forall h^R \in_R h\, (g \lhd\!\!\mid h^R);$$

$$g \lhd\!\!\mid h \Leftrightarrow \exists h^L \in_L h\, (g \leq h^L) \text{ or } \exists g^R \in_R g\, (g^R \leq h).$$

The induced relations on the quotient make Values($\mathcal{G}$) $= \mathcal{G}/\simeq$ into a two-ordered structure.

Under the normal play condition[1], this has the following intuitive description. Given two games g and h, $g \leq h$ means h is better than g for Left playing second, and $g \lhd\!| \, h$ means h is better than g for Left playing first. One rapidly verifies the axioms for a pre-two-order by induction, and that $g \lhd\!| \, h$ holds iff $h \not\leq g$, i.e. these games are determined.

Definition 3.1.5. A two-ordered structure $(X, \leq, \lhd\!|, \ldots)$ is said to be *determined* if

$$\forall x, y \in X \, (x \lhd\!| \, y \leftrightarrow x \not\geq y);$$

that is, if $\lhd\!|$ is the complement of the reversed order $\geq$.

It is easily shown by induction that in Example 3.1.4, the collection of games is determined.

Examples from poset morphisms

Given a morphism of posets $\varphi \colon (X, \leq) \to (Y, \leq)$, we can derive a second order $\lhd\!|$ on X such that $(X, \leq, \lhd\!|)$ is a two-ordered structure.

Proposition 3.1.6. Each of the following defines a compatible second order.

Maximal $u \lhd\!|^{\varphi} \, v \leftrightarrow \varphi u \not\geq \varphi v$

Strict $u \lhd\!|_{\varphi} \, v \leftrightarrow \varphi u < \varphi v$

Notice that the above orders are direct generalisations of those in Example 3.1.2, and of course we could add a third for the trivial, empty relation. In each case, if we take the corresponding relation from 3.1.2 in Y, then φ becomes an amphimorphism.

[1]Under the normal play condition, the players take alternating moves in a game; the first player unable to move is the loser, and his opponent the winner. Thus the empty game 0 is won by the second player, regardless.

The Full method in particular can be seen as encoding the image of a homomorphism from X: If $\varphi\colon X \to Y$ is a homomorphism of posets and $\lhd\!| = \lhd\!|^{\varphi}$ then knowing the behaviour of $\lhd\!|$ on X tells us exactly the behaviour of $\leq$ in $\operatorname{im}\varphi$.

Proposition 3.1.7. A weak order $\lhd\!|$ on $(X, \leq)$ encodes a morphism by the Full method if and only if $\not\lhd\!|$ is transitive.

Proof. If $\lhd\!| = \lhd\!|^{\varphi}$ and $x \not\lhd\!| \, y \not\lhd\!| \, z$ then $\varphi x \geq \varphi y \geq \varphi z$, so $\varphi x \geq \varphi z$ and $x \lhd\!| \, z$. Conversely if $\not\lhd\!|$ is transitive, let $\leq' = |\!\not\rhd$ (which is also transitive). It is easily shown that $(X, \leq', \lhd\!|)$ is a pre-two-ordered structure. Notice that $\leq' \supseteq \leq$, and so the identity map $\varphi\colon X \to X$ is a pre-order homomorphism from $(X, \leq)$ to $(X, \leq')$. If required we can always factor by the equivalence relation $\leq' \cap \geq'$ to obtain a poset and the desired result. $\square$

Not every weak order encodes the image of a morphism in this way, however.

Example 3.1.8. Let $X = \{x, y\}$ with $x \lhd\!| \, y \lhd\!| \, x$ and the trivial strong order $=$. If $\lhd\!| = \lhd\!|^{\varphi}$ then some φ satisfies $\varphi x \leq \varphi y \leq \varphi x$, hence $\varphi x \geq \varphi y$, contradicting the assumption.

If $\lhd\!| = \lhd\!|_{\varphi}$ then instead, $\varphi x < \varphi y < \varphi x$, again a contradiction.

3.1.2 Morphisms of two-ordered structures

There are several different types of 'morphism' we might consider:

Definition 3.1.9. If $(X, \leq, \lhd\!|)$ and $(Y, \leq, \lhd\!|)$ are (pre-)two-ordered sets a map $f\colon X \to Y$ is a

promorphism if $x \leq y \Rightarrow f(x) \leq f(y)$ and $x \lhd\!| \, y \Rightarrow f(x) \lhd\!| \, f(y)$

amphimorphism if $x \leq y \Rightarrow f(x) \leq f(y)$ and $x \not\lhd\!| \, y \Rightarrow f(x) \not\lhd\!| \, f(y)$

co-promorphism if $x \not\leq y \Rightarrow f(x) \not\leq f(y)$ and $x \not\lhd\!| \, y \Rightarrow f(x) \not\lhd\!| \, f(y)$

co-amphimorphism if $x \not\leq y \Rightarrow f(x) \not\leq f(y)$ and $x \lhd\!\!| \, y \Rightarrow f(x) \lhd\!\!| \, f(y)$

for all $x, y \in X$.

For a given X, natural objects for study include the various semigroups of promorphisms, amphimorphisms, co-promorphisms, or co-amphimorphisms, $f\colon (X, \leq, \lhd\!\!|) \to (X, \leq, \lhd\!\!|)$ under composition, and the category of all (pre-)two-orders $\leq, \lhd\!\!|$ on X as objects with promorphisms (amphimorphisms, co-promorphisms, co-amphimorphisms) as arrows.

If $(X, \leq, \lhd\!\!|, \ldots)$ is a two-ordered structure, in many problems it is particularly interesting or important to investigate the different ways to change or extend the orders to make the structure determined. This can be described as a modal logic, in which the possible structures $(X', \leq', \lhd\!\!|', \ldots)$ that are images of $(X, \leq, \lhd\!\!|, \ldots)$ under one of the notions of morphisms are the 'possible worlds' of the logic, with the existence of the morphisms as the accessibility relation. This is investigated further in Cox and Kaye [15].

For various reasons, it appears that promorphisms and amphimorphisms are the most natural of these four, with amphimorphisms having the more useful algebraic properties. (See below.)

Example 3.1.10. In the space of pure Conway game values, say Values($\mathcal{G}$), addition by a fixed element G, i.e.

$$G + \bullet\colon g \mapsto G + g,$$

is a promorphism.

Example 3.1.11. In Values($\mathcal{G}$), given $G \in$ Values($\mathcal{G}$) we may define

$$\{G \mid \bullet\}\colon g \mapsto \{G \mid g\}.$$

Then $\{G \mid \bullet\}$ is an amphimorphism. So also is $\{\bullet \mid G\}$.

Proof. Given $g \leq h$, we must show $\{G \mid g\} \leq \{G \mid h\}$. But clearly $G \leq G$, and this shows $G \lhd\!\!| \, \{G \mid h\}$ and hence $\{G \mid g\} \leq \{G \mid h\}$, as required. Also, given $\{G \mid g\} \lhd\!\!| \, \{G \mid h\}$

we must show $g \lhd\!| \, h$. But the assumption implies $\{G \mid g\} \leq G$ or $g \leq \{G \mid h\}$. The first of these is clearly false as $G \in_L \{G \mid g\}$ and $G \not\lhd\!| \, G$. Therefore $g \leq \{G \mid h\}$ and hence $g \lhd\!| \, h$. $\square$

Example 3.1.12. Continuing on from Example 3.1.3, categories of irreflexive relations $\lhd\!|$ on nonempty sets X may be defined taking either as arrows *promorphisms* $f\colon X \to Y$ such that $x \lhd\!| \, y \Rightarrow f(x) \lhd\!| \, f(y)$ or *amphimorphisms* $f\colon X \to Y$ such that $f(x) \lhd\!| \, f(y) \Rightarrow x \lhd\!| \, y$.[1]

3.1.3 Two-orders and boolean algebras

Orders, two-orders, and the interplay between $\leq$ and $\lhd\!|$ in particular is full of ideas of dualism. The following extended example shows how two-orders develop the dualism in boolean algebras.

Example 3.1.13. Let $B = (B, \leq, \wedge, \vee, \neg, \top, \bot)$ be a boolean algebra and $F \subseteq B$ satisfy

$$\bot \notin F \wedge \forall x, y \in B (x \geq y \in F \to x \in F).$$

Then we define $\lhd\!| = \lhd\!|(F)$ by

$$b \lhd\!| \, a \Leftrightarrow a \wedge \neg b \in F.$$

This defines a two-order $(B, \leq, \lhd\!|)$ such that

$$b \lhd\!| \, a \Leftrightarrow \bot \lhd\!| \, a \wedge \neg b \tag{3.1}$$

for all $a, b \in B$. Note that it follows from (3.1) that $a \not\lhd\!| \, \bot$ and $\top \not\lhd\!| \, b$ for all a, b. We

[1]The prefix 'amphi' does not really apply here as such objects are one-sided, but it is easier to overload this word that to introduce a new word to remember.

can also note a useful duality law for $\triangleleft\!\mid$.

$$\forall a, b \in B(b \triangleleft\!\mid a \Leftrightarrow \neg a \triangleleft\!\mid \neg b),$$

hence $b \triangleleft\!\mid a \Leftrightarrow \neg a \vee b \triangleleft\!\mid \top$. The proofs are easy. In the case $F = \{\bot\}^c$ we have $b \triangleleft\!\mid a \Leftrightarrow a \not\leq b$. All two-orders satisfying (3.1) arise in this way from some F. This is seen by defining

$$F(\triangleleft\!\mid) = \{a : \bot \triangleleft\!\mid a\}$$

and observing that $F(\triangleleft\!\mid)$ and $\triangleleft\!\mid(F)$ are inverse operations.

Note that (3.1) implies the cancellation laws

$$a \wedge b \triangleleft\!\mid a \wedge c \Rightarrow b \triangleleft\!\mid c$$

and

$$a \vee b \triangleleft\!\mid a \vee c \Rightarrow b \triangleleft\!\mid c.$$

The set F is nonempty if and only if $\top \in F$ and this corresponds to the relationship

$$\bot \triangleleft\!\mid \top. \tag{3.2}$$

Such F are filters if they are additionally closed under $\wedge$: $\forall a, b \in F a \wedge b \in F$. Equivalently, $F(\triangleleft\!\mid)$ is a filter if (3.2) and

$$\bot \triangleleft\!\mid a \text{ and } \bot \triangleleft\!\mid b \Rightarrow \bot \triangleleft\!\mid a \wedge b \tag{3.3}$$

for all $a, b \in B$. One can check that this is equivalent to

$$b \triangleleft\!\mid c \text{ and } d \triangleleft\!\mid e \Rightarrow b \vee d \triangleleft\!\mid c \wedge e.$$

Note also the familiar dual to this,

$$b \leq c \text{ and } d \leq e \Rightarrow b \wedge d \leq c \vee e.$$

It is interesting to note that the quotient B/F of the boolean algebra B by a filter can be obtained directly from $\mathrel{\lhd\!\!\mid} = \mathrel{\lhd\!\!\mid}(F)$ by redefining $\leq$ by using a rule that is in some sense dual to (3.1): we define the new $\leq'$ by

$$x \leq' y \Leftrightarrow \bot \mathrel{\lhd\!\!\mid} \neg x \vee y.$$

Then $\leq'$ is evidently reflexive and transitive, and also if $x \leq y$ then $\neg x \vee y = \top \mathrel{\mid\!\!\rhd} \bot$ by (3.2), so $x \leq' y$. When we factor out by the equivalence relation $x \leq' y \wedge y \leq' x$ we obtain the quotient algebra B/F.

If relations such as $\mathrel{\lhd\!\!\mid}$ correspond to filters, ultrafilters correspond to two-orders $\leq, \mathrel{\lhd\!\!\mid}$ satisfying the two laws above, (3.1), (3.2) and (3.3), such that

$$\bot \mathrel{\lhd\!\!\mid} a \text{ or } \bot \mathrel{\lhd\!\!\mid} \neg a \tag{3.4}$$

for all a, or, equivalently,

$$\bot \mathrel{\lhd\!\!\mid} a \text{ or } a \mathrel{\lhd\!\!\mid} \top \tag{3.5}$$

for all $a, b \in B$. This in turn is equivalent (using (3.1), (3.2) and (3.3)) to the law

$$a \mathrel{\lhd\!\!\mid} b \text{ or } a \mathrel{\lhd\!\!\mid} \neg b \text{ or } b \mathrel{\lhd\!\!\mid} a \text{ or } \neg b \mathrel{\lhd\!\!\mid} a$$

for all $a, b \in B$, and its easy to check (using (3.1), (3.2) and (3.3) again) that the four possibilities here are mutually exclusive.

Thus two-orders for boolean algebras encode and generalise the idea of (ultra)filters.

This discussion leads us to consider the structure of spaces of weak orders and of two-orders.

3.2 Structures of two-orders

First we show that any space of weak orders compatible with a preorder has a rich structure.

3.2.1 Extending and reducing two-orders

There is a familiar construction that 'linearises' a partial order $\leq$ on X, by defining $x \leq' y \Leftrightarrow x \leq y \vee (x \leq a \wedge b \leq y)$ so that $\leq'$ is a new partial order extending $\leq$ so that $a \leq' b$. (Here, and below, 'extending' and 'reducing' mean as sets of ordered pairs.) This works provided the obvious necessary condition $b \not\leq a$ holds. Several variations of this result are possible, including analogous results for (pre-)two-orders that are useful lemmas towards making a two-order determined. We now present several such methods available for extending and reducing two-orders; these will be useful below, both in application of two-orders and in understanding the rich structure in spaces of two-orders over a given set or poset.

Proposition 3.2.1. Let $(X, \leq, \lhd\!\!\mid)$ be a pre-two-order and $a, b \in X$. If $a \not\leq b$ then there is a unique minimal pre-two-order $\leq', \lhd\!\!\mid'$ on X extending $\leq, \lhd\!\!\mid$ such that $b \lhd\!\!\mid' a$. In fact we may take $\leq' = \leq$ and

$$x \lhd\!\!\mid' y \Leftrightarrow x \lhd\!\!\mid y \vee (x \leq b \wedge a \leq y).$$

Proof. Given $a \not\leq b$, define $\leq', \lhd\!\!\mid'$ to be as in the statement of the result. It is easy to check axioms (c), (d) and (e).

Note that, by the axioms, if $\leq'', \lhd\!\!\mid''$ extend $\leq, \lhd\!\!\mid$ with $b \lhd\!\!\mid'' a$ then $x \lhd\!\!\mid'' y$ holds whenever $x \lhd\!\!\mid' y$, i.e. whenever $x \lhd\!\!\mid y$ or both $x \leq b$ and $a \leq y$, so the above is necessary, and thus the one given is the unique minimal such extension. □

Proposition 3.2.2. Let $(X, \leq, \lhd\!\!\mid)$ be a pre-two-order and $a, b \in X$. If $a \not\leq b$ and $a \not\lhd\!\!\mid b$ then there is a unique minimal pre-two-order $\leq', \lhd\!\!\mid'$ on X with a canonical definition extending $\leq, \mathrel{\lhd\!|}$ such that $b \leq' a$. The relation $\leq'$ is antisymmetric if $\leq$ was.

Proof. Given $a \not\leq b$ and $a \not\mathrel{\lhd\!|} b$, define

$$x \leq' y \Leftrightarrow x \leq y \vee (x \leq b \wedge a \leq y)$$

and

$$x \mathrel{\lhd\!|}' y \Leftrightarrow x \mathrel{\lhd\!|} y \vee (x \leq b \wedge a \mathrel{\lhd\!|} y) \vee (x \mathrel{\lhd\!|} b \wedge a \leq y).$$

Again, the cases given above are clearly necessary for $b \leq' a$. Checking the axioms is somewhat lengthy, but easy. $\square$

The $\mathrel{\lhd\!|}$ relation acts in many ways as a dual to $\leq$. Given this it is not surprising that it is sometimes easy to *reduce* it as well as to expand it.

Proposition 3.2.3. Let $(X, \leq, \mathrel{\lhd\!|})$ be an (antisymmetric) two-order and $a, b \in X$. Then there is a unique maximal two-order $\leq', \mathrel{\lhd\!|}'$ on X with a canonical definition such that $\leq' = \leq$ and $\mathrel{\lhd\!|}' \subseteq \mathrel{\lhd\!|}$ such that $b \not\mathrel{\lhd\!|}' a$.

Proof. Given $a, b \in X$, define $\leq' = \leq$ and

$$x \mathrel{\lhd\!|}' y \Leftrightarrow x \mathrel{\lhd\!|} y \wedge (x \not\geq b \vee a \not\geq y).$$

It's easy to check $b \not\mathrel{\lhd\!|}' a$, and the cases given above are clearly necessary for $b \not\mathrel{\lhd\!|}' a$. Checking the axioms is again easy. $\square$

For a somewhat more interesting example, we now make $a \leq b$ true in some two-order in which $b \not\leq a$ by reducing $\mathrel{\lhd\!|}$ to ensure $b \not\mathrel{\lhd\!|} a$ and expanding $\leq$. The actual definition of $\leq', \mathrel{\lhd\!|}'$ took some time to find as this is not simply a matter of combining previous propositions.

Proposition 3.2.4. Let $(X, \leq, \mathrel{\lhd\!|})$ be a two-order and $a, b \in X$ with $b \not\leq a$. Then there is a unique minimal/maximal two-order $\leq', \mathrel{\lhd\!|}'$ on X with a canonical definition such that $\leq' \supseteq \leq$ and $\mathrel{\lhd\!|}' \subseteq \mathrel{\lhd\!|}$ such that $a \leq' b$ and $b \not\mathrel{\lhd\!|}' a$.

Proof. The correct definition for $\leq', \lhd\!|'$ is

$$x \leq' y \Leftrightarrow x \leq y \vee (x \leq a \wedge b \leq y)$$

(as before) and

$$x \lhd\!|' \, y \Leftrightarrow x \lhd\!| \, y \wedge (y \not\leq a \vee x \lhd\!| \, b) \wedge (b \not\leq x \vee a \lhd\!| \, y).$$

The check that these conditions are necessary is straightforward (though considering $\not\lhd\!|'$ rather than $\lhd\!|'$ may be easier). Axiom checking is also straightforward, but lengthy. $\square$

This completes the list of such 'simple' extensions or reductions of two-orders, for if $X = \{0, 1, 2, 3\}$ with $0 \leq 1 \leq 3$ and $0 \leq 2 \leq 3$ then there is no canonical $\leq' \subseteq \leq$ with $0 \not\leq' 2$.

These propositions show that there may be a number of ways of extending a two-order to a maximal (i.e. determined) one, not just by adding information of the type '$b \lhd\!|' \, a$'.

We can also construct two-orders from chains of such orders, as the following proposition shows.

Proposition 3.2.5.

1. If $\lhd\!|$ is a binary relation on X and $(\leq_i : i \in I)$ a chain of strong orders (ordered by $\subseteq$ or $\supseteq$), such that for each $i \in I$, $(\leq_i, \lhd\!|)$ is a (pre-)two-order, then $(\lim_i \leq_i, \lhd\!|)$ is also a (pre-)two-order.

2. If $\leq$ is a (pre-)order and W a set of weak orders over $(X, \leq)$ then $(\leq, \bigcup W)$ and $(\leq, \bigcap W)$ are (pre-)two-orders.

3. If $(T_i \colon i \in I)$ is a chain of (pre-)two-orders, ordered by any of the relations

$$T \leq S \leftrightarrow \leq^T \subseteq \leq^S \wedge \lhd\!|^T \subseteq \lhd\!|^S$$

$$T \leq S \leftrightarrow \leq^T \supseteq \leq^S \wedge \lhd\!|^T \subseteq \lhd\!|^S$$

$$T \leq S \leftrightarrow \leq^T \subseteq \leq^S \wedge \lhd\!|^T \supseteq \lhd\!|^S$$

$$T \leq S \leftrightarrow \leq^T \supseteq \leq^S \wedge \lhd\!|^T \supseteq \lhd\!|^S$$

then $\lim_i T_i$ is also a (pre-)two-order.

Proof. Notice that in each case, the limit is obtained by taking unions, intersections, or a combination of the two in an obvious fashion. For the first point, clearly the transitivity and reflexivity axioms hold at limits of chains. If each order is antisymmetric and $\leq = \bigcup_i \leq_i$, and $x \leq y \leq x$, then for some i, j, $x \leq_i y \leq_j x$; therefore if $m = \max(i, j)$, $x \leq_m y \leq_m x$, so $x = y$. If $\leq = \bigcap_i \leq_i$, then for all i, $x \leq_i y \leq_i x$, so $x = y$ if any of the $\leq_i$ are antisymmetric.

For (2), notice that since $x \not\lhd\!| x$ for all $\lhd\!| \in W$, the same holds for unions or intersections of weak orders. Therefore it remains to check amphitransitivity. If $x \bigcup W\ y \leq z$ then for some $\lhd\!| \in W$, $x \lhd\!| z$—hence $x \bigcup W\ z$. If $x \bigcap W\ y \leq z$ then for all $\lhd\!| \in W$ we have $x \lhd\!| z$, so $x \bigcap W\ z$. The cases where $x \leq y \lhd\!| z$ are similarly checked.

For the final point, in each case the strong order $\leq$ is a preorder/partial order accordingly, by (1). By (2), the limit weak order $\lhd\!|$ satisfies $\forall x\, (x \not\lhd\!| x)$. This leaves amphitransitivity: we check the case where $x \lhd\!| y \leq z$, and the other case is similarly proved. Let $(\leq, \lhd\!|) = \lim_i T_i$. In the first case, for some i we have $x \lhd\!|_i y \leq_i z$, whence $x \lhd\!| z$. In the second case, $y \leq_i z$ for all i and $x \leq_j y$ for some j, so $x \lhd\!|_j z$. Since $\lhd\!|$ is the union of all the $\lhd\!|_i$, $x \lhd\!| z$. For the third case, $x \lhd\!|_i y$ for all i and $y \leq_i z$ for all i greater than some j, so for all $i > j$ we have $x \lhd\!|_i z$. Therefore $x \lhd\!|_i z$ for all i, and in particular $x \lhd\!| z$. For the final case, we have $x \lhd\!|_i y \leq_i z$, hence $x \lhd\!|_i z$, for all i; therefore $x \lhd\!| z$. □

Remark 3.2.6. While Proposition 3.2.5 can be used to prove, for example, the existence of a minimal two-order satisfying some property, in general that two-order will not be definable from the old in the language $\mathscr{L}_{\text{tos}}$, having nonlogical binary relation symbols $\leq$ and $\lhd$. The preceding propositions, however, show that in certain cases we can define useful new orders (using a Δ_0 formula), and as such we can better tell their effects on the whole of X.

These results give us a lot of information about the space of orders on X, which will be useful in proving a representation theorem below.

Definition 3.2.7. If $(X, \leq)$ is a preordered set then

$$\mathrm{WOrd}(X, \leq) = \{\lhd \colon (\leq, \lhd) \text{ is a two-order}\}.$$

Corollary 3.2.8. If $(X, \leq)$ is a preordered set then $\mathrm{WOrd}(X, \leq)$ is a complete, bounded, distributive lattice—and hence a Heyting algebra.

Since $\mathrm{WOrd}(X, \leq)$ is a Heyting algebra, it is natural to enquire whether the implication objects, $\lhd_1 \to \lhd_2$, are definable in the same sense that the atomic extensions and reductions of Propositions 3.2.1- 3.2.4 are:

Question 1. Supposing $\lhd_1$ and $\lhd_2$ are weak orders compatible with $(X, \leq)$, is it possible to give a first-order definition of $\lhd_1 \to \lhd_2$ in terms of the relations $\leq$, $\lhd_1$ and $\lhd_2$? In particular, when is $\neg\lhd$ definable in terms of $\lhd$?

3.2.2 A representation theorem for posets

We are now in a position to obtain a representation theorem for posets, using these spaces of two-orders. Specifically, we mimic Stone's theorem for boolean algebras; in this setting orders $\lhd$ compatible with an original partial order take the place of filters, much as we discussed above. We define a 'dual space' using such orders, analogous to the Stone space of ultrafilters over a boolean algebra. Then we prove that each nontrivial poset X embeds into the boolean algebra of clopen subsets of its dual space. This is in some sense a direct generalisation of Stone's theorem.

Throughout this section we assume $(X, \leq)$ is a nontrivial (i.e. nonempty, non-singleton) partial order. As mentioned above, over a boolean algebra the filers correspond to a particular kind of order $\lhd$. Here we define several separate notions of dual space, which are distinct in general but which coincide when our structure is a boolean algebra, or when we require compatibility with the additional structure ($\wedge$, $\vee$, $\neg$) there.

Definition 3.2.9. The *hom-space*, and *order space* of X are, respectively, the sets

$$\mathrm{Hom}(X, 2) = \{\varphi\colon \varphi \text{ is a homomorphism } X \to 2\};$$
$$\mathrm{WOrd}(X) = \{\lhd\colon (X, \leq, \lhd) \text{ is a two-order}\};$$

As usual, 2 denotes the 2-point boolean algebra $\{\bot, \top\}$, and a homomorphism is a function φ such that $x \leq y \to \varphi x \leq \varphi y$ for all x, y in its domain.

Notice that in the case where X is a boolean algebra, the maximal filters are precisely the ultrafilters (assuming some amount of choice). In this more general construction, there is no requirement of compatibility with lattice operations (for example, filters are closed under meet and join). The removal of this restriction has the effect that there is precisely one maximal order, namely $\lhd = \not\geq$ (in fact this is the 'strict' order from the previous section).

Remark 3.2.10. The hom-space embeds into the order space, but the reverse is not always true. Given $\varphi \in \mathrm{Hom}(X, 2)$, define an order $\lhd^{\varphi}$ by

$$x \lhd^{\varphi} y \leftrightarrow \varphi x = \bot \wedge \varphi y = \top.$$

It is easy to see that this order composes with the partial order on x, and further that no x satisfies $x \lhd^{\varphi} x$.

To see that the reverse does not always hold, consider any two-ordered structure with points x, y, z such that

$$x \lhd\!\mid y \lhd\!\mid z.$$

Then $\lhd\!\mid$ cannot arise from any $\varphi\colon X \to 2$, since orders derived from homomorphisms split X into a bipartite graph, the nodes of which are split by the preimages of $\top$ and $\bot$, with an edge $x \to y$ precisely when $x \lhd\!\mid y$.

Above we described how a particular order $\lhd\!\mid$ corresponds to a filter, and what those orders corresponding to ultrafilters look like. Call an order $\lhd\!\mid$ *ultra* if

$$\forall x\, \exists y (y \lhd\!\mid x \vee x \lhd\!\mid y).$$

We let

$$\mathrm{UOrd}(X) = \{\lhd\!\mid\colon \ \lhd\!\mid \text{ is an ultra-order on } X\} \cup \{\varnothing\}.$$

Remark 3.2.11. If X is a boolean algebra, this condition is equivalent to both of

$$\forall x(\bot \lhd\!\mid x \vee x \lhd\!\mid \top);$$

$$\forall x(\bot \lhd\!\mid x \vee \bot \lhd\!\mid \neg x).$$

Ignoring closure under meets and joins, this is what makes a filter an ultrafilter: $\forall x\ (x \in F \vee \neg x \in F)$.

There is also a topology on the dual spaces corresponding to that used in Stone's theorem.

Definition 3.2.12. The topology on $\mathrm{WOrd}(X)$ is that generated by the basic open sets

$$U_x = \{\lhd\!\mid\colon\ \exists y (y \lhd\!\mid x)\},$$

$$V_x = \{\lhd\!\mid\colon\ \exists y (x \lhd\!\mid y)\},$$

and their complements. We endow the spaces $\mathrm{Hom}(X, 2)$ and $\mathrm{UOrd}(X)$ with the subspace topology (we identify $\mathrm{Hom}(X, 2)$ with its image under the above-mentioned embedding).

Remark 3.2.13. In Stone's theorem the Stone space topology is often described as being generated by the sets

$$B_x = \{F : x \in F\},$$

where x is an element of the boolean algebra, and the F are ultrafilters. These B_x correspond to the collection of ultra-orders $\lhd\!\mid$ for which $\perp \lhd\!\mid x$ (as outlined above (3.4)[1]). Since in general we do not have a least element $\perp$, we generalise this by requiring only that some element y satisfies $y \lhd\!\mid x$; therefore we may consider the U_x an abstraction of the basic open sets B_x. Furthermore, in a boolean algebra we have

$$B_{\neg x} = \{F : \neg x \in F\},$$

corresponding to the collection of orders $\lhd\!\mid$ such that $x \lhd\!\mid \top$, or equivalently $x \lhd\!\mid y$ for some y; that is, $B_{\neg x}$ corresponds to V_x above. Since for an ultrafilter F, $x \in F \leftrightarrow \neg x \notin F$, $U_x^c = V_x$. In general, without a dual operator $\neg$ this does not follow, so we include the sets U_x^c and V_x^c.

Lemma 3.2.14. The dual spaces $\mathrm{Hom}(X, 2)$ and $\mathrm{UOrd}(X)$ are totally disconnected.

Proof. This is an immediate consequence of our choice of basic open sets. □

Theorem 3.2.15. The map $U : x \mapsto U_x$ is an embedding of X into the collection of open sets in $\mathrm{WOrd}(X)$. Moreover, the induced maps onto the subspaces $\mathrm{Hom}(X, 2)$ and $\mathrm{UOrd}(X)$ are embeddings onto the collections of clopen sets.

Proof. As mentioned above, U_x is open in $\mathrm{WOrd}(X)$, clopen in $\mathrm{Hom}(X, 2)$ and $\mathrm{UOrd}(X)$. If $x \leq y$ then clearly $U_x \subseteq U_y$; suppose now that $x \not\leq y$. Define a map $\varphi : X \to 2$ by setting $\varphi z = \top$ when $z \geq x$, and $\varphi z = \perp$ otherwise. If $u \leq v$ and $\varphi v = \perp$, then $v \not\geq x$. Therefore

[1]Notice that we have do not, however, require closure under meets, since this extra structure is not present in general.

$u \not\geq x$, and so $\varphi u = \bot$. Therefore φ is a homomorphism, and the derived relation $\lhd\!\!|^{\varphi}$ is an ultra-order which makes $y \lhd\!\!| x$; hence $U_x \not\subseteq U_y$. Thus U is an embedding. $\square$

3.3 Two-ordered groups

We will now discuss the addition of compatible group structure.

Definition 3.3.1. A *right two-ordered group* is a group G equipped with a two-order $T = (\leq, \lhd\!\!|)$, such that

TR1 $\forall x, y, z \in G\ (x \leq y \rightarrow zx \leq zy)$;

TR2 $\forall x, y, z \in G\ (x \lhd\!\!| y \rightarrow zx \lhd\!\!| zy)$;

We also define *left two-ordered groups* to be groups having two-orders satisfying the obvious left-sided analogues of TR1 and TR2, and *two-ordered groups* to be groups which satisfy all four axioms.

A *pre-*(right/left) two-ordered group is a group G equipped with a pre-two-order which satisfies the above axioms as appropriate.

The meaning of these additional axioms should be clear: they merely ensure compatibility with the group multiplication. For simplicity we consider only two-ordered groups here.

Remarks 3.3.2.

- As above, any partially ordered group can be given a compatible two-order, by taking $\lhd\!\!|$ to be any of $\not\geq$, $<$ or $\varnothing$. It is easily checked that these are compatible with the group multiplication. In particular, every group has a compatible two-order, and every partially ordered group admits a compatible weak order.

- Notice that any subgroup H of the two-ordered group G will automatically be a two-ordered group under the inherited relations.

Positive cones

It is useful to have specific notation for the classes $\{x\colon x \geq 1\}$ and $\{x\colon x \mathrel{|\!\rhd} 1\}$. We define, for any two-ordered group G, the *positive cones*

$$
\begin{aligned}
P = P_G &= \{x \in G\colon x \geq 1\};\\
Q = Q_G &= \{x \in G\colon x \mathrel{|\!\rhd} 1\}.
\end{aligned}
$$

It is well known that for a group G, the compatible preorders on G correspond to the collection of normal submonoids. We can show the following.

Proposition 3.3.3. Let $P, Q \subseteq G$ and define binary relation $\leq, \mathrel{\lhd\!|}$ on G by

$$
\begin{aligned}
x \leq y &\leftrightarrow yx^{-1} \in P;\\
x \mathrel{\lhd\!|} y &\leftrightarrow yx^{-1} \in Q.
\end{aligned}
$$

Then $(G, \leq, \mathrel{\lhd\!|})$ is a two-ordered group if and only if

- P is a normal submonoid of G;
- Q is a normal subset of G not containing 1;
- $PQ = QP = Q$.

Proof. The normality of P and Q is equivalent to the compatibility of the corresponding order with the multiplication in G. The transitivity of $\leq$ and the closure of P under multiplication are equivalent, and $1 \leq 1$ if and only if $1 \in P$. If G is a two-ordered group then the first two conditions are true, and so the axiom T3 implies $PQ = QP = Q$ as $1 \in P$. Conversely if the three conditions are satisfied then $x \leq y \mathrel{\lhd\!|} z$ if and only if $zy^{-1} \in Q$ and $yx^{-1} \in P$, hence $zx^{-1} \in Q$, and $x \mathrel{\lhd\!|} z$. Analogously, if $x \mathrel{\lhd\!|} y \leq z$ then $x \mathrel{\lhd\!|} z$, and therefore G is a two-ordered group. $\square$

Definable closures

Let $\mathscr{L}_1$ be the language $\mathscr{L}_{\text{tos}}$ plus the constant symbol 1.

Proposition 3.3.4. Suppose G is a two-ordered group and for some $\mathscr{L}_1$-formula $\phi(x)$ with free variable x, $S = \{x \in G\colon \phi(x)\}$. Then $S^G = S$ and $\langle S \rangle \trianglelefteq G$.

Proof. Suppose $\psi(x, \overline{y})$ is a quantifier-free $\mathscr{L}_1$-formula. We prove by induction on the number of logical connectives in ψ that for all $x, \overline{y}, g$ in G, $\psi(x, \overline{y}) \leftrightarrow \psi(x^g, \overline{y}^g)$. An atomic such formula will be of the form uRv, where R is a binary relation among $\leq$, $\lhd\!\!\shortmid$, $=$; clearly the claim holds for these cases. If the claim is true of the formulas ψ_0 and ψ_1, then clearly $\psi_0(x, \overline{y}) \wedge \psi_1(x, \overline{y})$, $\psi_0(x, \overline{y}) \vee \psi_1(x, \overline{y})$, $\neg\psi_0(x, \overline{y})$ also satisfy. This proves the claim.

Now suppose that $\phi(x)$ defines the class S, and find a logically equivalent formula $\theta(x)$ which is in prenex normal form. Rename the bound variables so $\theta(x)$ becomes

$$\mathsf{Q}_0 y_0 \ \mathsf{Q}_1 y_0 \dots \mathsf{Q}_n y_n \ \psi(x, \overline{y}),$$

where the Q_i are quantifiers and ψ is quantifier free. If $g \in G$ is fixed then $\psi(x, y_0, \dots, y_n)$ is true if and only if $\psi(x^g, y_0^g, \dots, y_n^g)$. By considering the universal and existential cases separately, we see that, since conjugation by g is a bijection $G \to G$, $\mathsf{Q}_n y_n \ \psi(x, \overline{y})$ is equivalent to $\mathsf{Q}_n y_n \ \psi(x^g, y_0^g, \dots, y_{n-1}^g, y_n)$. Proceeding in this way we can prove, by induction on the number of quantifiers in θ, that $\theta(x)$ is true if and only if $\theta(x^g)$. Since g was arbitrary and θ logically equivalent to ϕ, S is closed under conjugation. Since conjugation by g is a homomorphism $G \to G$ for each $g \in G$, it follows that $\langle S \rangle \trianglelefteq G$. $\square$

The same proof can apply to a similar proposition. Suppose we fix a two-ordered group G, and take as our language $\mathscr{L}_{\text{rm}}$ the relations $\leq, \lhd\!\!\shortmid$, along with unary function symbols r_g, representing right multiplication by g, for each $g \in G$, and constant symbol 1. Then any subset S of G which is definable by a unary $\mathscr{L}_{\text{rm}}$-formula is normal in G (i.e. closed under conjugation).

Aside from the cases $\mathscr{L}_1, \mathscr{L}_{\mathrm{rm}}$ there are still useful extensions to $\mathscr{L}_1$ for which proposition 3.3.4 still holds. The obvious choices, however, fail. For instance we might consider the addition of an inversion or (binary) multiplication function symbol, or the scalar multiplication functions above with an identity. The following example demonstrates that in these cases proposition 3.3.4 fails.

Example 3.3.5. Let $G = D_6 = \langle x, y \colon x^3 = y^2;\ yxy = x^2\rangle$, and consider $H = \langle y \rangle$. Since $xyx^{-1} = yx \notin H$, H is not normal in G. However, we can define H using the equalities

$$\begin{aligned} H &= \{g \colon g = g^{-1}\} \\ &= \{g \colon r_y(g) = 1 \vee g = 1\}, \end{aligned}$$

in the appropriate language $\mathscr{L}$. Furthermore, the set $\{g \colon \forall h(ggh = h)\} = \{1, y, xy, x^2y\}$ is not normal in G.

Although sets definable using multiplication and inversion are not necessarily normal, we can generalise proposition 3.3.4 as follows.

Proposition 3.3.6. Let $\mathscr{L}_{\mathrm{S}}$ be the language $\mathscr{L}_1$ with additional unary relation symbol S. Assume $X \subseteq G$ is normal, and that $X = \{g \in G \colon S(g)\}$. Then for any $\mathscr{L}_{\mathrm{S}}$-formula $\phi(x)$, $\{g \in G \colon \phi(g)\}$ is normal in G.

Proof. We proceed as above. If $\psi(x, \overline{y})$ we prove that $G \vDash \psi(x, \overline{y}) \leftrightarrow \psi(x, \overline{y}^g)$ for all $g \in G$. Suppose $\psi(x, \overline{y})$ is an atomic $\mathscr{L}_{\mathrm{S}}$-formula. In proposition 3.3.4 we covered all possible cases where ψ is an $\mathscr{L}_1$-formula; we are left with the cases where ψ is $S(x)$ or $S(y)$. Since X is normal, the claim also holds in this case. The remainder of the proof is identical. $\square$

Using this result we can consider a general notion of closure. Suppose $\phi(x)$ is an $\mathscr{L}_{\mathrm{S}}$-formula with single free variable, and define $\phi_S(x)$ to be the formula obtained by replacing all occurrences of $S(v)$ in $\phi(x)$ by $\phi(v)$. So, for example, if $\phi(x)$ is $\exists y\,(S(y) \wedge y \leq x \wedge x \leq y)$, then ϕ^S is the formula $\exists y\,(\phi(y) \wedge y \leq x \wedge x \leq y)$. We will call ϕ a *closure formula* on G if, whenever $\{x\colon S(x)\}$ is normal,

$$G \vDash \ \forall g\ (S(g) \to \phi(g)),\ \text{and}$$
$$G \vDash \ \forall g\ ((\phi(g) \leftrightarrow \phi^S(g))).$$

If $C\colon \mathcal{P}(G) \to \mathcal{P}(G)$ is any function then we say C is a *definable closure* if and only if there is a closure formula ϕ such that whenever $X = \{x \in G\colon S(x)\}$ is normal, $C(X) = \{g \in G\colon \phi(g)\}$.

Notice that, by definition, definable closures are closures, in that $C(C(X)) = C(X)$ and $C(X) \supseteq X$ for all X.

In particular we define

$$\overline{X} = \{y \in G\colon\ \exists x \in X\ y \simeq x\},$$

and

$$\widetilde{X} = \{y \in G\colon\ \exists x, x' \in X\ (x \leq y \leq x')\},$$

(the convex closure) are definable closures, and so in particular each preserves normality of subsets of G. If $g \in G$ we define $C(g)$, for a closure C, to be $C(\{g\})$.

3.3.1 Morphisms and quotients

We define morphisms as above: a *promorphism* of (pre-)two-ordered groups is a group homomorphism which preserves both the strong order and the weak order; an *amphimorphism* of (pre-)two-ordered groups is a group homomorphism which preserves the strong order and reflects the weak order. We also call a function φ a *preembedding* if it is both preserving and reflecting of both orders. A bijective preembedding is an *isomorphism.*

If we are interested in quotients of two-ordered groups we need to consider the related concepts of equivalence, morphism and normal substructure. There are various ways in which a quotient might be defined; we opt for a simple definition which is highlights the dual nature of $\leq$ and $\lhd$, but also makes every group quotient a quotient of pre-two-ordered groups, and behaves well with respect to amphimorphisms. Let K be a normal subgroup of the pre-two-ordered group G (clearly this is a necessary requirement), and attach the inherited orders $\leq, \lhd$. We know that these groups will factor nicely, so it remains to check that the quotient G/K inherits compatible orderings from G which preserve some two-order structure. For all $x, y \in G$, set

$$xK \leq yK \leftrightarrow \exists k \in K \ xk \leq y; \tag{3.6}$$

$$xK \lhd yK \leftrightarrow \forall k \in K \ xk \lhd y. \tag{3.7}$$

Since K is normal in G the analogous conditions with xk replaced by kx are equivalent. If $xK \leq yK$ then some $k \in K$ satisfies $xk \leq y$, and so $xk \ntriangleright y$, implying $xK \ntriangleleft yK$; that is, G/K satisfies T4. Since K is a group $\leq$ is a preorder on G/K. It should also be clear that the compatibility axioms (T1 and T2) are satisfied. If $xK \leq yK \lhd zK$ then for some fixed k_1 and all k_2 in K, $xk_1 \leq y$ and $yk_2 \lhd z$. Therefore $xk_1k_2 \leq yk_2 \lhd z$, implying $xk_1k_2 \lhd z$, for all $k_2 \in K$. If we are given $k \in K$, fix $k_2 = k - k_1 \in K$ so $xk = xk_1k_2 \lhd z$, and hence $xk \lhd z$. Since k was arbitrary, $xK \lhd zK$. The case $xK \lhd yK \leq zK$ is essentially the same.

Since a quotient can be defined whenever $K \trianglelefteq G$ (as groups), it makes sense to call any normal subgroup K endowed with the inherited orders *normal*, and write this simply as $K \trianglelefteq G$, using the standard notation from group theory.

Example 3.3.7. Let $G = \mathbb{Z}$ with the usual addition and ordering $\leq$, but with an empty relation $\lhd$. The only subgroups of $\mathbb{Z}$ are of the form $k\mathbb{Z}$. With the construction above, $\mathbb{Z}/k\mathbb{Z}$ inherits the trivial ordering: $x \simeq y$ for all x, y. Notice that if we were to reverse the quantifiers in our definitions of the inherited orders, we would have that no x satisfies $x \leq x$, and so $\leq$ would not be a preorder, and nor would $\mathbb{Z}/k\mathbb{Z}$ be a two-ordered group.

Proposition 3.3.8. Let G be a pre-two-ordered group, and $K \trianglelefteq G$. Then

- $G/\overline{1}$ is a two-ordered group;

- G/K is a two-ordered group if and only if K is convex.

Proof. If $x\overline{1} \simeq \overline{1}$ there are $k_1, k_2 \simeq 1$ such that $1 \simeq k_1 \leq x \leq k_2 \simeq 1$; so $x \simeq 1$. For the second statement, notice that G/K is partially ordered if and only if for all x,

$$xK \simeq K \Leftarrow x \in K. \tag{$\star$}$$

Since $xK \simeq K$ if and only if there exist $k_1, k_2 \in K$ such that $k_1 \leq x \leq k_2$, the implication ($\star$) is equivalent to the statement that K is convex. □

3.3.2 Homomorphism theorems

A morphism theorem states that, given a morphism $\varphi\colon G \to H$, $G/\ker\varphi$ and $\operatorname{im}\varphi$ are isomorphic via a canonical isomorphism ψ, defined in terms of φ. We present homomorphism theorems for both kinds of morphism here.

Promorphisms

We can prove a similar, weaker result for promorphisms.

Theorem 3.3.9 (Promorphism theorem)**.** Suppose G, H are pre-two-ordered groups, and $\varphi\colon G \to H$ is a promorphism. Define $\psi\colon G/\ker\varphi \to \operatorname{im}\varphi$ by $\psi(x\ker\varphi) = \varphi x$ as usual. Then ψ is a (bijective) promorphism.

Moreover, φ is a preembedding if and only if both ψ is an isomorphism and $\ker\varphi = \overline{1}$.

Proof. We know that ψ is well-defined, and a group isomorphism; it remains to show ψ is a promorphism. Let $K = \ker\varphi$ and suppose $xK \leq yK$. Then for some $k \in K$, $xk \leq y$, so $\varphi x = \varphi(xk) \leq \varphi y$; that is, $\psi(xK) \leq \psi(yK)$. If $xK \lhd\!\shortmid yK$ then in particular $x \lhd\!\shortmid y$, so $\varphi x \lhd\!\shortmid \varphi y$, and $\psi(xK) \lhd\!\shortmid \psi(yK)$.

Suppose in addition φ is a preembedding. Then $\varphi x = 1$ implies $x \simeq 1$, so $\ker\varphi = \overline{1}$. If $\psi(xK) \leq \psi(yK)$, i.e. $\varphi x \leq \varphi y$, then $x \leq y$, so $xK \leq yK$. If $\psi(xK) \lhd\!\shortmid \psi(yK)$ then

$\varphi x \mathrel{\lhd\!\mid} \varphi y$, so $x \mathrel{\lhd\!\mid} y$. If $k \in K$ then $k \simeq 1$, so $xk \mathrel{\lhd\!\mid} y$. Therefore $xK \mathrel{\lhd\!\mid} yK$. This shows ψ is an isomorphism.

Conversely, assume ψ is an isomorphism and $\ker \varphi = \overline{1}$. If $\varphi x \leq \varphi y$ then $\psi(xK) \leq \psi(yK)$, so $xK \leq yK$. Since $K = \overline{1}$, this is equivalent to $x \leq y$. The same argument shows that $\varphi x \mathrel{\lhd\!\mid} \varphi y$ implies $x \mathrel{\lhd\!\mid} y$. Therefore φ is a preembedding. $\square$

It may be that some reasonable condition equivalent to ψ being a preembedding can be found, weakening the hypothesis of theorem 3.3.9. However this definition of morphism also creates problems for projections onto quotients. We make the following definitions to clarify the situation.

Definition 3.3.10. A game k is called

- *neutral* or I-neutral if $\forall x \in G\ (x \mathrel{\mid\!\rhd} 1 \leftrightarrow x \mathrel{\mid\!\rhd} k)$;
- *null* (or perhaps II-neutral) if $\forall x \in G\ (x \geq 1 \leftrightarrow x \geq k)$;

If all the elements in a subclass H are null/neutral in G then we say H is null/neutral in G respectively.

A game k is T-neutral in G if and only if we cannot affect the T-strategies in any game when multiplying by k. Notice that the null objects represent the zero games of Conway et al., and the smaller class of neutral games have a similar interpretation, namely that they can be composed with other games without affecting the first player's strategies.

Proposition 3.3.11. Suppose G is a pre-two-ordered group. Let

$$\mathrm{Null}(G) = \{x \colon x \text{ is null}\};$$

$$\mathrm{Neu}(G) = \{x \colon x \text{ is neutral}\}.$$

Then $\mathrm{Null}(G)$ and $\mathrm{Neu}(G)$ are normal subgroups, with $\mathrm{Null}(G) = \overline{1}$.

Proof. Suppose k is null. Then $k \geq k$, so $k \geq 1$. If $x \geq k^{-1}$ implies $kx \geq 1$, and so $kx \geq k$ by nullity of k; hence $x \geq 1$. As $k \geq 1$, if $x \geq 1$ then $x \geq k^{-1}$. So k^{-1} is also null. In particular $k^{-1} \geq 1$ also, so $k \simeq 1$. Clearly every $k' \in \overline{1}$ is null; therefore $\mathrm{Null}(G) = \overline{1} \trianglelefteq G$.

If k is neutral, then $x \mathrel{|\!\rhd} k^{-1}$ is equivalent to $kx \mathrel{|\!\rhd} 1$, so to $kx \mathrel{|\!\rhd} k$, and hence to $x \mathrel{|\!\rhd} 1$. Therefore k is neutral if and only if k^{-1} is neutral. Since $\operatorname{Neu} G$ is clearly $\mathscr{L}_1$-definable, it is normal in G. Further, since this class is also closed under inverses and clearly contains 1, it is a normal subgroup of G. □

Proposition 3.3.12. Let K be a subgroup of the pre-two-ordered group G. Then K is normal if and only if K is the kernel of a preordered group morphism $(G, \leq) \to (H, \leq)$. If we know K is normal in G, then

- K is neutral if and only if the canonical projection $\pi\colon G \to G/K$ is a promorphism;
- K is null if and only if the canonical projection $\pi\colon G \to G/K$ is a preembedding.

Proof. If $K \trianglelefteq G$ as groups then K is the kernel of the projection $\pi\colon G \to G/K$. By our definition of quotient, this projection is also a morphism of preordered groups. Clearly the converse is true: if $\varphi\colon G \to H$ is a preordered group morphism, then $\ker \varphi$ is normal.

For the next point, we show that π is $\mathrel{\lhd\!|}$-monotonic if and only if K is neutral. If π is $\mathrel{\lhd\!|}$-monotonic and $1 \mathrel{\lhd\!|} x$ then $K \mathrel{\lhd\!|} xK$, whence $k \mathrel{\lhd\!|} x$ for all $k \in K$. If instead $x \mathrel{|\!\rhd} k \in K$ then $xK \mathrel{|\!\rhd} K$, whence $x \mathrel{|\!\rhd} 1$. Conversely, if K is neutral, then for $k \in K$, $k \mathrel{\lhd\!|} x$ is equivalent to $1 \mathrel{\lhd\!|} x$, and so π is monotonic.

For the final point, if $K \trianglelefteq G$ is null then $x \mathrel{\lhd\!|} y$ if and only if $\pi x \mathrel{\lhd\!|} \pi y$, for all $x, y \in G$. Since nullity implies neutrality, π is a promorphism. If $xK \leq yK$ then for some $k \in K$, $xk \leq y$; but then $x \leq y$ since $k \simeq 1$. Hence π is a preembedding. If, conversely, π is a preembedding and $k \in K$, then $\pi(k) = 1$ so $k \simeq 1$. □

This result, along with the weak nature of our morphism theorem, suggests that an alternative may be preferable. We choose to look at morphisms which exploit the apparent duality between $\leq$ and $\mathrel{\lhd\!|}$, but are also fairer to Right. In keeping with the theory of normal play in ONAG, we prioritise favourability for the second player.

Amphimorphisms

Definition 3.3.13. Let G, H be two-ordered groups, and φ a homomorphism $G \to H$. We call φ an *amphimorphism* if both

$$\text{whenever } x \leq y \text{ in } G,\ \varphi x \leq \varphi y; \tag{3.8}$$

$$\text{whenever } \varphi x \mathrel{\lhd\!\!|} \varphi y \text{ in } H,\ x \mathrel{\lhd\!\!|} y. \tag{3.9}$$

Notice that if II prefers y to x, i.e. $y \geq x$ (if Left plays second) or $y \leq x$ (if Right plays second) then this preference is preserved by every amphimorphism φ. Further, if I does not prefer y to x, i.e. $y \not\mathrel{|\!\!\rhd} x$ or $x \not\mathrel{|\!\!\rhd} y$, then I has no preference in the image of φ. So favourability for II is preserved, while that for I is at worst the same. That is, these morphisms favour the second player. Notice that amphimorphisms can be composed to form amphimorphisms.

For any function φ and any class S we let $\varphi[S]$ denote the class $\{\varphi(s) \colon s \in S\}$.

Proposition 3.3.14. If $\varphi \colon G \to H$ is a group homomorphism then φ is an amphimorphism if and only if

1. $\varphi[P_G] \subseteq P_H$;

2. $\varphi^{-1}[Q_H] \subseteq Q_G$.

As with promorphisms, given an amphimorphism $\varphi \colon G \to H$ we can prove that the induced group isomorphism $\psi \colon G/\ker\varphi \to \operatorname{im}\varphi$ is an amphimorphism. Later we will prove a more detailed result.

Theorem 3.3.15 (Amphimorphism theorem, part 1)**.** Suppose $\varphi \colon G \to H$ is an amphimorphism. Then the induced map $\psi \colon G/\ker\varphi \to \operatorname{im}\varphi$ is an amphimorphism. Furthermore φ is a preembedding if and only if ψ is an isomorphism and $\ker\varphi$ is null.

Proof. In light of the analogous theorem for promorphisms, we need only prove ψ preserves $\lhd\!\!|$ in the backwards direction. Let $K = \ker\varphi$, and suppose $xK \not\mathrel{\lhd\!\!|} yK$. Then there exists

$k \in K$ such that $xk \not\mathrel{\lhd\mid} y$. Since φ is an amphimorphism $\varphi x = \varphi(xk) \not\mathrel{\lhd\mid} \varphi y$, i.e. $\psi(xK) \not\mathrel{\lhd\mid} \psi(yK)$. □

We can also show that projections are amphimorphisms, a significant improvement on the analogous result for promorphisms.

Proposition 3.3.16. Suppose $K \trianglelefteqslant G$ and $\pi\colon G \to G/K$ is the canonical projection. Then π is an amphimorphism.

Proof. We have seen that π necessarily preserves $\leq$ in the forwards direction. If $xK \mathrel{\lhd\mid} yK$ then in particular $x \mathrel{\lhd\mid} y$, so π is an amphimorphism. □

We aim to extend theorem 3.3.15 by identifying precisely when ψ is an isomorphism. With this in mind we make the following definitions.

Definition 3.3.17. Let $\varphi\colon G \to H$ be an amphimorphism. We call φ

- $\leq$-*good* if for all $h \in \operatorname{im}\varphi$, if $h \geq 1$ there is $g \in G$ such that $g \geq 1$ and $\varphi g = h$;
- $\mathrel{\lhd\mid}$-*good* if for all $h \in \operatorname{im}\varphi$, if $h \not\mathrel{\mid\rhd} 1$ there is $g \in G$ such that $g \not\mathrel{\mid\rhd} 1$ and $\varphi g = h$;
- *good* if φ is $\leq$-good and $\mathrel{\lhd\mid}$-good.

We can concisely state these definitions as

- φ is $\leq$-good if $P_{\operatorname{im}\varphi} \subseteq \varphi[P_G]$;
- φ is $\mathrel{\lhd\mid}$-good if $Q^{\mathrm{c}}_{\operatorname{im}\varphi} \subseteq \varphi[Q^{\mathrm{c}}_G]$.

Proposition 3.3.18. Suppose $\varphi\colon G \to H$ is an amphimorphism, and $K = \ker\varphi$. Then φ is

1. $\leq$-good if and only if

$$\forall x, y \in G\ (\varphi x \leq \varphi y \to xK \leq yK); \tag{†}$$

2. $\lhd\!\shortmid$-good if and only if

$$\forall x, y \in G\ (xK \lhd\!\shortmid yK \to \varphi x \lhd\!\shortmid \varphi y). \tag{‡}$$

Proof. Assume φ is $\leq$-good, and $\varphi x \leq \varphi y$; then $1 \leq \varphi(yx^{-1})$, so there is $g \in P_G$ such that $\varphi g = \varphi(yx^{-1})$. Hence $g^{-1}yx^{-1} \in K$, and

$$xK = (g^{-1}yx^{-1})K\, xK = g^{-1}yK.$$

As $g^{-1} \leq 1$, $g^{-1}K \leq K$ and so $xK = g^{-1}yK \leq yK$. This proves (†).

Conversely, suppose (†) holds and $\varphi u \geq 1$. Then by (†) $uK \geq K$, i.e. there is a $k \in K$ such that $u \geq k$. Therefore $uk^{-1} \geq 1$, and $\varphi(uk^{-1}) = \varphi u$.

To see the second equivalence, proceed similarly. If φ is $\lhd\!\shortmid$-good and $\varphi x \not\lhd\!\shortmid \varphi y$ in H, $\varphi(yx^{-1}) \not\shortmid\!\rhd 1$, so there is $g \in G$ such that $g \not\shortmid\!\rhd 1$ and $\varphi g = \varphi(yx^{-1})$, proving (‡).

Conversely if (‡) is true and $\varphi u \not\shortmid\!\rhd 1$, then $uK \not\shortmid\!\rhd K$, so for some $k \in K$, $u \not\shortmid\!\rhd k$. Therefore $uk^{-1} \not\shortmid\!\rhd 1$, and $\varphi(uk^{-1}) = \varphi u$. □

Immediately we can extend our morphism theorem.

Theorem 3.3.19. Suppose $\varphi\colon G \to H$ is an amphimorphism. Then the induced group isomorphism $\psi\colon G/\ker\varphi$ is an isomorphism of two-ordered groups if and only if φ is good.

There are various ways of closing the kernel $\ker\varphi$ for a morphism $\varphi\colon G \to H$. A relatively useful closure, which is not generally definable without adding a function symbol to our language, is given by

$$\ker_{\simeq}\varphi = \varphi^{-1}\text{“}\overline{1}.$$

Notice that

$$\ker\varphi \leq \overline{\ker\varphi} \leq \widetilde{\ker\varphi} \leq \ker_{\simeq}\varphi,$$

and that, for example, if φ is a preembedding then $\overline{\ker\varphi} = \ker_{\simeq}\varphi$.

Another direction we might take in extending our morphism theorem is to alter the induced morphism ψ. Suppose that G, H are groups and $\varphi\colon G \to H$, $K = \ker \varphi$. If C denotes any kind of closure which preserves normality, then $CK \trianglelefteq G$ and $CK/K \trianglelefteq G/K$. Therefore CK/K has an isomorphic image $N \trianglelefteq \operatorname{im}\varphi$. By the isomorphism theorems, there is an isomorphism

$$\psi_C\colon G/CK \cong (G/K)/(CK/K) \cong \operatorname{im}\varphi/N.$$

It may be interesting to consider whether, under appropriate restrictions, such a construction might provide a two-ordered group isomorphism, for some C.

Question 2. Suppose $\mathscr{L}'$ is an appropriate 2-sorted language with constant symbol 1, unary function symbol f, unary relation symbol S, and binary relation symbols $\leq, \lhd\!\!|$. Let T be the theory stating that the elements of each sort form two-ordered groups, and that f is an amphimorphism between them. Is there a nontrivial, or minimal, formula ϕ in $\mathscr{L}'$ such that T proves both

- ϕ is a closure formula;
- the group isomorphism ψ_C described above is necessarily an isomorphism of two-ordered groups?

3.3.3 Automorphisms of two-ordered structures

Groups of (pro- or amphi-) morphisms are also equipped with compatible two-orders. Clearly the composition of two morphisms is also a morphism. Let $\operatorname{Aut} G$ denote the class of two-ordered group automorphisms of G (that is, the bijective preembeddings $G \to G$). We now consider general two-ordered groups as collections of automorphisms.

If $\varphi, \psi \colon S \to T$ are morphisms of two-ordered structures, we write

$$\varphi \leq \psi \Leftrightarrow \forall s \in S\,(\varphi s \leq \psi s);$$

$$\varphi \mathrel{\lhd\!|} \psi \Leftrightarrow \exists s \in S\,(\varphi s \mathrel{\lhd\!|} \psi s).$$

Proposition 3.3.20. If S is a two-ordered structure, then $\operatorname{Aut} S$ is a two-ordered group with these relations.

Proof. Clearly $\operatorname{Aut} S$ is a group; we show that the composition operation is compatible with the orders $\leq, \mathrel{\lhd\!|}$, and that these define a two-ordered structure.

Suppose that $\varphi, \psi, \vartheta \in \operatorname{Aut} S$ with $\varphi \leq \psi$. If $s \in S$, we have $\varphi(\vartheta s) \leq \psi(\vartheta s)$, so $\varphi \circ \vartheta \leq \psi \circ \vartheta$. Also $\varphi s \leq \psi s$ so $\vartheta\varphi s \leq \vartheta\psi s$; hence $\operatorname{Aut} S$ satisfies axiom T1.

If $\varphi \mathrel{\lhd\!|} \psi$ there is $s \in S$ such that $\varphi s \mathrel{\lhd\!|} \psi s$; hence $\vartheta \circ \varphi(s) \mathrel{\lhd\!|} \vartheta \circ \psi(s)$. Also, if $t = \vartheta^{-1}s$, then $\varphi \circ \vartheta(t) = \varphi s \mathrel{\lhd\!|} \psi s = \psi \circ \vartheta(t)$. Therefore $\operatorname{Aut} S$ satisfies axiom T2.

If $\varphi \mathrel{\lhd\!|} \psi \leq \vartheta$, take $s \in S$ such that $\varphi s \mathrel{\lhd\!|} \psi s$; then since $\psi s \leq \vartheta s$, $\varphi s \mathrel{\lhd\!|} \vartheta s$ and $\varphi \mathrel{\lhd\!|} \vartheta$. Similarly we can prove that $\varphi \mathrel{\lhd\!|} \psi \leq \vartheta$ implies $\varphi \mathrel{\lhd\!|} \vartheta$, and so $\operatorname{Aut} S$ satisfies T3.

Finally, if $\varphi \leq \vartheta$ then given any $s \in S$, $\varphi s \leq \vartheta s$ and so $\varphi s \mathrel{|\!\ntriangleright} \vartheta s$, implying T4. $\square$

Remark 3.3.21. Here we could replace the existential quantifier with a universal one, without affecting the above result. For simplicity here we keep a single definition; however, in future we may prefer to (for example) use the $\forall$-variant for promorphisms.

Notice that our choice preserves the duality between $\leq$ and $\mathrel{\lhd\!|}$, which has been a productive observation up to this point.

Theorem 3.3.22. Let G be a two-ordered group. Then the usual embedding, ϑ, which takes $g \in G$ to the map defined by $\vartheta g(x) = xg$, is an embedding of two-ordered groups. In particular every two-ordered group arises as an automorphism group for some two-ordered structure.

Proof. Indeed, $\vartheta g \leq \vartheta h$ if and only if $xg \leq xh$ for all $x \in G$, a condition equivalent to $g \leq h$. Similarly $\vartheta g \mathrel{\lhd\!|} \vartheta h$ is equivalent to the existence of some $x \in G$ such that $xg \mathrel{\lhd\!|} xh$, and so to $g \lhd\!\mid h$. $\square$

3.4 Duality and determinacy

Much of the material covered in this chapter hints at some form of duality between the orders $\leq$ and $\lhd\!\mid$. In this section we show that this is indeed the case, and explicitly describe how the duality works and can be used.

Many statements above regarding one order are accompanied by a similar statement regarding the other. The duality is not always clear, since at times (for example) quantifiers are reversed, and at others they remain the same. This is because the dual $\phi^\star$ of a first-order formula ϕ is obtained by swapping and negating relations as follows. Assume $\mathscr{L}$ is any first-order language containing the binary relation symbols $\leq$ and $\lhd\!\mid$, and $\phi(\overline{v})$ an $\mathscr{L}$-formula. Then $\phi^\star(\overline{v})$ is obtained by swapping occurrences of $\leq$ or $\lhd\!\mid$ with $\not\lhd\!\mid$ or $\nleq$ respectively. The following points make this explicit.

- If $\phi(\overline{v})$ is $t_0(\overline{v}) \leq t_1(\overline{v})$ or $t_0(\overline{v}) \lhd\!\mid t_1(\overline{v})$, where the t_i represent terms, then $\phi^\star(\overline{v})$ is $t_0(\overline{v}) \not\lhd\!\mid t_1(\overline{v})$ or $t_0(\overline{v}) \nleq t_1(\overline{v})$ respectively. If $\phi(\overline{v})$ is $t_0(\overline{v}) = t_1(\overline{v})$ then $\phi^\star$ is ϕ.
- If $\phi(\overline{v})$ is $\psi_0(\overline{v}) \wedge \psi_1(\overline{v})$ or $\neg\psi_0(\overline{v})$, then $\phi^\star(\overline{v})$ is $\psi_0^\star(\overline{v}) \wedge \psi_1^\star(\overline{v})$ or $\neg\psi_0(\overline{v})$ respectively.
- If $\phi(\overline{v})$ is $\exists w\ \psi(\overline{v}, w)$, then $\phi^\star(\overline{v})$ is $\exists w\ \psi^\star(\overline{v}, w)$.

Examples 3.4.1.

- Suppose G is a two-ordered group and $K \trianglelefteq G$. We work in an appropriately expanded language $\mathscr{L}$. Recall $xK \leq yK$ if and only if $\exists k \in K\ (xk \leq y)$; the dual of this statement is

$$\exists k \in K\ (xk \not\lhd\!\mid y); \tag{$\dagger$}$$

  if we define $\lhd\!\mid$ on G/K by declaring that $xK \not\lhd\!\mid yK$ if and only if ($\dagger$) holds, then we get the original definition of $\lhd\!\mid$, since ($\dagger$) is equivalent to the negation of the usual definition.

- Suppose G, H are two-ordered groups with $\varphi\colon G \to H$ any function, and we work in an appropriate 2-sorted language, with a function symbol which we interpret as φ. For φ to be an amphimorphism, we require both

$$\forall x, y \in G\ (x \leq y \to \varphi x \leq \varphi y); \tag{3.10}$$

$$\forall x, y \in G\ (\varphi x \lhd\!\mid \varphi y \to x \lhd\!\mid y). \tag{3.11}$$

The second statement is clearly equivalent to the sentence

$$\forall x, y \in G\ (x \not\lhd\!\mid y \to \varphi x \not\lhd\!\mid y),$$

which is dual to statement (3.10) above.

- The formula $\phi(k)$ which states that k is neutral is clearly equivalent to the formula

$$\forall x \in G\ (1 \not\lhd\!\mid x \leftrightarrow k \not\lhd\!\mid x),$$

say $\psi(k)$. Notice that $\psi^{\star}(k)$ is the statement that k is null.

- It should be clear that the notions of $\leq$-good and $\lhd\!\mid$-good are also dual.

Notice that even the axioms of two-ordered groups exhibit some duality. To illustrate this point further we give an alternative axiomatisation to that given above, which is equivalent when G is a group:

T1 $\forall x, y, z \in G\ \big((x \leq y \leftrightarrow zx \leq zy) \land (x \leq y \leftrightarrow xz \leq yz)\big)$;

T2 $\forall x, y, z \in G\ \big((x \not\lhd\!\mid y \leftrightarrow zx \not\lhd\!\mid zy) \land (x \not\lhd\!\mid y \leftrightarrow xz \not\lhd\!\mid yz)\big)$;

T3 $\forall x, y, z \in G\ \big((x \lhd\!\mid y \land y \leq z) \lor (x \leq y \land y \lhd\!\mid z) \to x \lhd\!\mid z\big)$;

T4 $\forall x \in G\ (x \not\lhd\!\mid x)$;

T5 $\forall x, y, z \in G\ (x \leq y \land y \leq z \to x \leq z)$;

T6 $\forall x\ (x \leq x)$.

It should be clear that these axioms are equivalent to those for pre-two-ordered groups. We remark that T1 and T2 are dual to one another, and also T4 is dual to T6, the statement that $\leq$ is reflexive.

We aim to show that the remaining axioms (T3 and T5) are also dual to statements worth consideration. First, however, we require the following definition.

Definition 3.4.2. Let S be a two-ordered group. An element g of G is *determined* if we have

$$(g \leq 1 \vee 1 \mathrel{\lhd\!\!\mid} g) \wedge (g \mathrel{\lhd\!\!\mid} 1 \vee 1 \leq g).$$

We denote the formula above by $\det(g)$.

Given our usual interpretation of $\leq$ and $\mathrel{\lhd\!\!\mid}$, a determined game x is one for which one player will always be favoured, regardless of who moves first (although precisely which player is favoured may depend on who moves first). Clearly such objects are of interest in game theory, and in particular we remark that the wellfounded games considered in ONAG and Winning Ways are determined. In fact, as we aim to show, this kind of determinacy is a natural concept to study in the logical and algebraic theory of two-ordered structures. In particular, we can prove the following. Let Det denote the statement $\forall x\ \det(x)$.

Proposition 3.4.3. Let G be a two-ordered group. Then $G \vDash \mathrm{Det}$ if and only if $G \vDash \mathrm{T3}^\star$.

Proof. Suppose G is determined. If $x \not\mathrel{\lhd\!\!\mid} y \not\leq z$, i.e. $x \geq y \mathrel{\mid\!\!\rhd} z$, then $x \mathrel{\mid\!\!\rhd} z$ so $x \not\leq z$. Similarly $x \not\leq y \not\mathrel{\lhd\!\!\mid} z$ implies $x \not\leq z$, proving $\mathrm{T3}^\star$.

Conversely if $G \vDash \mathrm{T3}^\star$ and $x \not\leq 1 \not\mathrel{\lhd\!\!\mid} x$ in G, then $x \not\leq x$, a contradiction. Thus $x \leq 1$ or $1 \mathrel{\lhd\!\!\mid} x$. Similarly we prove $x \mathrel{\lhd\!\!\mid} 1 \vee 1 \leq x$, and so x is determined. $\square$

Notice that the dual of T5 above is the statement that $\mathrel{\lhd\!\!\mid}$ is the complement of a transitive relation on G, which is implied by determinacy. An immediate consequence of this is that, if Th_{TOG} is the theory of two-ordered groups, then $\mathrm{Th}_{TOG} + \mathrm{Det}$ is self-dual, i.e. for all $\mathscr{L}_{\mathrm{tog}}$-sentences σ, $\sigma \in \mathrm{Th}_{TOG} \cup \{\forall x\ \det(x)\}$ if and only if $\sigma^\star \in \mathrm{Th}_{TOG} \cup \{\forall x\ \det(x)\}$.

Question 3. Let $\mathscr{L}_{\text{tog}}$ be the language of two-ordered groups, i.e. $\mathscr{L}_1$ with additional binary function symbol $\cdot$ and unary function symbol $^{-1}$. Assuming G is a determined two-ordered group, is $\mathrm{Th}(G; \mathscr{L}_{\text{tog}})$, the theory of G in the language $\mathscr{L}_{\text{tog}}$, self-dual?

If the answer is positive, then the implication is an equivalence, by proposition 3.4.3.

Measuring determinacy in two-ordered structures

Given the applications of two-ordered structures to game theory, it would be very useful to have some measure of determinacy in arbitrary structures, which works similarly to the commutator subgroup (for example) in group theory.

Question 4. Is there a definable 'determinator' (sub-) structure which indicates in some sense how determined a two-ordered structure/group is?

One such possibility comes from our representation theorem for posets. Recall that $\mathrm{WOrd}(X, \leq)$ denotes the class of relations $\lhd\!\shortmid$ such that $(X, \leq, \lhd\!\shortmid)$ is a two-ordered structure. For $\lhd\!\shortmid \in \mathrm{WOrd}(X, \leq)$, define

$$\mathrm{WOrd}(X; \lhd\!\shortmid) = \{\lhd\!\shortmid' \in \mathrm{WOrd}(X, \lhd\!\shortmid) \colon \lhd\!\shortmid' \supseteq \lhd\!\shortmid\}.$$

If $(X, \leq, \lhd\!\shortmid^X)$ is a two-ordered structure, then $\mathrm{WOrd}(X; \lhd\!\shortmid)$ is trivial (i.e. equal to $\{\lhd\!\shortmid^X\}$) if and only if X is determined. Therefore $\mathrm{WOrd}(X; \lhd\!\shortmid)$ itself gives an indication of how determined X is, but we might also consider two-ordered groups built on top of this structure, such as

$$X^* = \mathrm{Hom}(X, \mathrm{WOrd}(X; \lhd\!\shortmid))$$

(admitting some choice of poset, pro- or amphimorphism as appropriate). Propositions 3.2.1-3.2.4 may be useful here.

CHAPTER 4

GAME CATEGORIES

In this chapter we discuss the generalisation of Conway games using categories; this is common practice since category theory provides an excellent framework for studying games (in particular, arrows are often useful for describing strategies, which can be of more interest than the games themselves). As such, much of the preliminary work has been done; see in particular Joyal's article [51], where it was first demonstrated that Conway games could be realised as a monoidal category, and the more recent work of Cockett et al. [12], where this is extended to accommodate first-player strategies using module categories.[1]

Once we have introduced our notation we will focus on generalising the work of Chapter 3 and also laying the foundation for Chapter 5.

4.1 Preliminaries

If A is a category then $\mathrm{Obj}(A)$ and $\mathrm{Arr}(A)$ will denote the object and arrow classes respectively. Our categories will be two-sorted structures satisfying an appropriate first-order theory in a language providing the relevant symbols (in particular the domain and

[1]We use 'module' in the sense of Cockett et al. [12]. Other terminology for these objects (or equivalent notions) includes 'bimodule', 'profunctor' and 'distributor'; see in particular the nlab page on profunctors [78]. Below we have chosen to define these objects explicitly for the sake of clarity. Further, in our definitions there is no reliance on the category of sets; since we frequently consider proper classes, we will also benefit by avoiding this limitation.

$$
\begin{tikzcd}
w \arrow[r, "m", "\shortmid"{marking}] & x \arrow[d, "h"] \\
v \arrow[u, "g"] \arrow[r, "hmg", "\shortmid"{marking}] & y \arrow[d, "k"] \\
u \arrow[u, "f"] \arrow[r, "khmgf", "\shortmid"{marking}] & z
\end{tikzcd}
$$

Figure 4.1: Composition with module arrows.

codomain functions). Occasionally this theory will be assumed to provide an appropriate choice axiom. We will also admit numerous categories of sets; typically these categories may be assumed to be ZF-like in nature, though we do not rule out other kinds of sets (in particular ill-founded sets will be of interest). Typically a category A will be denoted as a pair $(\mathrm{Obj}, \mathrm{Arr})$, where Obj is the object class and Arr is the arrow class.

Definition 4.1.1. If A is a category, a module M is a class of arrows m (written $m\colon x \nrightarrow y$ or $x \overset{m}{\nrightarrow} y$) such that the following axioms hold.

- If $f\colon a \to b$ and $g\colon b \nrightarrow c$ there is a composite arrow $g \circ f\colon a \nrightarrow c$ in M;
- if $g\colon a \nrightarrow b$ and $f\colon b \to c$ there is a composite arrow $f \circ g\colon a \nrightarrow c$ in M;

and this composition makes the diagram in Figure 4.1 commute.

Since the composition of arrows in a module category is associative, we will avoid bracketing where the meaning of an expression is clear.

Module categories will typically be denoted as triples $(\mathrm{Obj}, \mathrm{Arr}, \mathrm{Mod})$, where the first two items form the underlying category, and the final item is the module. Given a module category A, we will use $\mathrm{Mod}(A)$ for the module of A. If $a, b \in \mathrm{Obj}(A)$, we use $A[a, b]$ and $A(a, b)$ to denote the classes of arrows $a \to b$ and arrows $a \nrightarrow b$ respectively.

4.2 Game categories

The notion of 'game category' is often used. The objects we reserve this term for are merely those required in order to generalise the work of Chapter 3.

Definition 4.2.1. A *game category* is a module category A such that

$$\forall a \in A\ \neg(a \nrightarrow a).$$

Notice that our game categories form a strict subclass of the collection of module categories discussed by Cockett et al. [12]. We will, however, be discussing additional notions of functor for this class which correspond to the amphimorphisms of Chapter 3.

Since normal arrows (that is, elements of $\mathrm{Arr}(A)$) are intended to represent second-player strategies between games (in the sense of Joyal [51], when there is sufficient structure), and module arrows represent first-player strategies (in the sense of Cockett et al. [12]), we will often refer to them as such. We will also refer to the elements of $\mathrm{Arr}(A)$ as *strong* arrows and elements of $\mathrm{Mod}(A)$ as *weak* arrows, using the terminology of Chapter 3.

The definition we have given for game categories has various equivalent characterisations. One such definition is that A is enriched over some category S of sets and that there is a profunctor or distributor $\Phi\colon A^{\mathrm{op}} \times A \to S$, such that $\Phi(a, a)$ is always empty. In fact, this is essentially the definition of an enriched game category given below.

4.2.1 Functors of game categories

As in chapter 3 we define two different types of morphism. In this case, however, the theory of amphimorphisms breaks down with the discussion of natural transformations. We begin with the analogue of promorphisms, which are precisely module functors.

Promorphisms

Definition 4.2.2. Let A, B be game categories. A *module functor* from A to B is a functor $F\colon A \to B$ of the underlying categories which also assigns to each module arrow

$g\colon x \nrightarrow y$ in A a module arrow $Fg\colon Fx \nrightarrow Fy$ in B such that whenever

$$w \xrightarrow{f} x \overset{g}{\nrightarrow} y \xrightarrow{h} z$$

in A, we have

$$F(h \circ g \circ f) = Fh \circ Fg \circ Ff$$

in B.

We let **GC** denote the collection of game categories, and consider promorphisms $F\colon A \to B$ as arrows between objects A, B in **GC**. It is easily checked that this forms a category.

Definition 4.2.3. If $A, B \in \mathbf{GC}$ and $F, G\colon A \to B$ are module functors then a natural transformation $\tau\colon F \to G$ of the underlying functors is a *strong transformation* in **GC** if, whenever $m\colon x \nrightarrow y$ in A, the diagram in Figure 4.2 commutes.

Strong transformations are simply the appropriate 2-cells for **GC**; hence we will occasionally call them natural transformations, when it is clear that the extra module structure is being used.

$$\begin{tikzcd} Fx \arrow[r, "\tau_x"] \arrow[d, "Fm"', "\shortmid" marking] & Gx \arrow[d, "Gm", "\shortmid" marking] \\ Fy \arrow[r, "\tau_y"'] & Gy \end{tikzcd}$$

Figure 4.2: A natural transformation of module functors.

It is routine to show that the composition of strong transformations is also a strong transformation; moreover this composition is associative, since the underlying functions remain the same. Therefore for game categories A and B the homset $\mathbf{GC}[A, B]$ is also a category.

There are many reasons for us to study a corresponding notion of module arrow in the category $\mathbf{GC}[A, B]$; for instance, if we wish to embed a game category A onto some endomorphism space, this can only be done if there is a sensible notion of module transformation. This will become more apparent when we consider additional monoidal structure. As with two-ordered structures there are (at least) two sensible notions of module transformation. For the sake of simplicity here we choose the existential analogue. Notice this is 'dual' to the notion of strong transformation, in the sense of Chapter 3.

Definition 4.2.4. Suppose $F, G\colon A \to B$ are promorphisms of game categories. A *weak transformation* $\mu\colon F \nrightarrow G$ is a module arrow $\mu\colon Fx \nrightarrow Gx$, for some $x \in A$.

Composition of weak and strong transformations is performed in the obvious way: if

$$F \xrightarrow{\tau} G \xrightarrow{\mu} H \xrightarrow{\eta} K,$$

then μ is a weak arrow $Gx \to Hx$ for some x, and we define $\eta \cdot \mu \cdot \tau$ to be the weak arrow $\eta_x \mu \tau_x$.

Proposition 4.2.5. Let $A, B \in \mathbf{GC}$. Then $\mathbf{GC}[A, B]$ is a game category, with strong transformations as arrows and module arrows as above.

Proof. To see $\mathbf{GC}[A, B]$ is a category we must prove that transformations compose appropriately. Suppose

$$F \xrightarrow{\tau} G \xrightarrow{\eta} H,$$

and let $\eta \cdot \tau$ be the usual composite transformation. If $g\colon x \nrightarrow y$ in A then both squares of the rectangle in figure 4.3 commute, and hence the entire diagram commutes. Thus

$$\begin{tikzcd} Fx \arrow[r, "\tau_x"] \arrow[d, "Fg"'] & Gx \arrow[r, "\eta_x"] \arrow[d, "Fg"'] & Hx \arrow[d, "Hg"] \\ Fy \arrow[r, "\tau_y"'] & Gy \arrow[r, "\eta_y"'] & Hy \end{tikzcd}$$

Figure 4.3: Composition of natural transformations in **GC**.

$\eta \cdot \tau$ is a natural transformation $F \to H$. It should be clear that the composition of natural transformations with weak transformations results in a weak transformation, and moreover that this composition is associative. $\square$

Amphifunctors

We now address the issue of generalising amphimorphisms from chapter 3. This is not quite so simply done as with promorphisms, and while we can give a useful notion of natural transformation, it is not immediately clear that our definition of weak transformation cannot be improved.

Definition 4.2.6. Let A, B be game categories, and suppose $F\colon A \to B$ is a functor of the underlying categories. Assume there is a map $\overleftarrow{F}$ such that whenever arrows

$$w \xrightarrow{f} x \qquad\qquad y \xrightarrow{h} z$$

$$\begin{tikzcd} Fx \arrow[r, "g", "\shortmid" marking] & Fy \end{tikzcd}$$

exist in A and B, $\overleftarrow{F}_{x,y}g$ is an arrow $x \nrightarrow y$ in A satisfying

$$\overleftarrow{F}_{w,z}(Fh \circ g \circ Ff) = h \circ \overleftarrow{F}_{x,y}(g) \circ f \tag{4.1}$$

(see Figure 4.5). Then we call $(F, \overleftarrow{F})$ an *amphifunctor*.

An amphifunctor F as above ensures the existence of an arrow $\overleftarrow{F}_{x,y}g$, as in Figure 4.4. Moreover, the function $\overleftarrow{F}$ (denoted in both figures by $\Rightarrow$ arrows) commutes with the functor F; see Figure 4.5.

$$\begin{tikzcd} x \arrow[r, "\overleftarrow{F}_{x,y}(g)", "\shortmid" marking] & y \\ Fx \arrow[r, "g", "\shortmid" marking] \arrow[ur, phantom, "\Uparrow"] & Fy \end{tikzcd}$$

Figure 4.4: An amphifunctor acting on a weak arrow.

In practice we will sometimes denote both maps simply by F. Since the respective images are in separate categories and the domains distinct, this should not cause confusion. We will also occasionally omit the subscript x, y when x and y are clear from the context. Notice that in particular if A, B are two-ordered structures then the amphifunctors $A \to B$ are precisely the amphimorphisms from A to B.

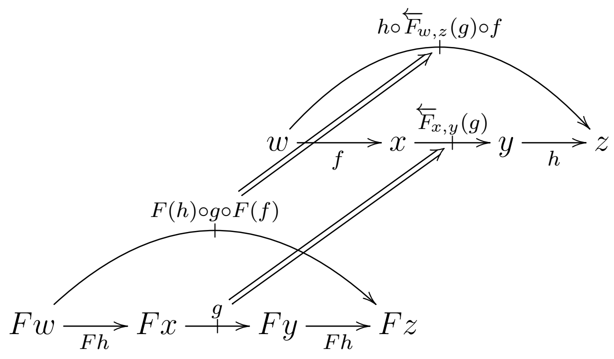

Figure 4.5: Commuting diagram for an amphifunctor

Proposition 4.2.7. The collection $\mathbf{GC}^{\mathrm{am}}$, with object class $\mathbf{GC}$ and amphifunctors as appropriate arrows, forms a category.

Proof. We must prove that the composition of two amphifunctors is also an amphifunctor, and that this composition is associative. Assume

$$A \xrightarrow{F} B \xrightarrow{G} C$$

in $\mathbf{GC}^{\mathrm{am}}$. If we have $w, x, y, z \in A$ satisfying

$$w \xrightarrow{f} x \qquad\qquad y \xrightarrow{h} z$$

$$GFx \overset{g}{\longrightarrow\!\!\!\!\!\!\!+\;\;} GFy$$

then

$$\overleftarrow{G}_{w,z}(GFh \circ g \circ GFf) = Fh \circ \overleftarrow{G}_{x,y} g \circ Ff,$$

so that

$$\begin{aligned}
\overleftarrow{GF}_{w,z}(GFh \circ g \circ GFf) &= \overleftarrow{F}_{w,z}(Fh \circ \overleftarrow{G}_{x,y} g \circ Ff) \\
&= h \circ \overleftarrow{F}_{x,y} \circ \overleftarrow{G}_{x,y} g \circ f \\
&= h \circ \overleftarrow{GF}_{x,y} g \circ f.
\end{aligned}$$

Therefore GF is an amphifunctor.

Now suppose

$$A \xrightarrow{F} B \xrightarrow{G} C \xrightarrow{H} D$$

in $\mathbf{GC}^{\mathrm{am}}$. We know that the underlying functor composition is associative. Further, for $x, y \in A$,

$$\begin{aligned}
\overleftarrow{((HG)F)}_{x,y} &= \overleftarrow{F}_{x,y} \circ \overleftarrow{(HG)}_{Fx,Fy} \\
&= \overleftarrow{F}_{x,y} \circ \left(\overleftarrow{G}_{Fx,Fy} \circ \overleftarrow{H}_{GFx,GFy}\right) \\
&= \left(\overleftarrow{F}_{x,y} \circ \overleftarrow{G}_{Fx,Fy}\right) \circ \overleftarrow{H}_{GFx,GFy} \\
&= \overleftarrow{GF}_{x,y} \circ \overleftarrow{H}_{GFx,GFy} \\
&= \overleftarrow{H(GF)}_{x,y}.
\end{aligned}$$

Therefore the composition of amphifunctors is associative. $\square$

Henceforth we shall occasionally omit the subscript x, y for the map $\overleftarrow{F}$. Otherwise the notation can be cumbersome, and in all cases it should be clear from the context which objects x and y are intended.

Since $\mathbf{GC}^{\mathrm{am}}$ is a category, we should expect that $\mathbf{GC}^{\mathrm{am}}[A, B]$ is a category for all objects A, B in $\mathbf{GC}^{\mathrm{am}}$. With this in mind we make the following definition of appropriate natural transformation.

Definition 4.2.8. Suppose $F, G \colon A \to G$ in $\mathbf{GC}^{\mathrm{am}}$, and that $\tau \colon F \to G$ is a natural transformation of the underlying functors. Assume also that whenever $g \colon Gx \nrightarrow Gy$ in $\operatorname{im} G$, there is an arrow $\overleftarrow{\tau}_{x,y}(g) \colon Fx \nrightarrow Fy$, satisfying

$$\overleftarrow{G}_{x,y}(g) = \overleftarrow{F}_{x,y}(\overleftarrow{\tau}_{x,y}(g)).$$

Then $(\tau, \overleftarrow{\tau})$ is called an *amphitransformation.*

Since this definition is much less conventional than the analogue for promorphisms, some explanation is required. Consider τ as above in terms of advantage for the second player. If II has a strategy in F, τ allows him to transfer this strategy to $\operatorname{im} G$, in such a way that, regardless of when (i.e. for which game x) the transfer takes place from Fx to Gx, composition of strategies will always be the same. Dually, if I has a strategy in the image $\operatorname{im} G$, then he has a strategy in $\operatorname{im} F$ which corresponds to the same root strategy in A; that is, τ ensures that Player I will fare no better playing in $\operatorname{im} G$ than in $\operatorname{im} F$. Amphitransformations benefit the second player just as amphimorphisms do

The following is easily proved.

Proposition 4.2.9. Let A, B be game categories. Then $\mathbf{GC}^{\mathrm{am}}[A, B]$, the collection of amphifunctors $A \to B$, is a category with amphitransformations as arrows.

4.3 Extending and reducing game categories

Here we generalise the problem of extending and reducing from Chapter 3. The following is proved similarly to Proposition 3.2.5.

Proposition 4.3.1.

1. If M is collection of weak arrows on A and $(\mathrm{Arr}_i(A) \colon i \in I)$ a chain of arrow classes making A a category, such that for each $i \in I$, M is a module on $(A, \mathrm{Arr}_i(A))$, then M is a module on $\lim_i (A, \mathrm{Arr}_i(A))$.

2. If A is a category and $\mathcal{M}$ a class of modules on A, then $\bigcup A$, $\bigcap A$ are modules on A.[1]

3. If $(A_i)_{i \in I}$ is a chain of module categories on the same object space, ordered by inclusion of arrow classes and modules (any combination; see Proposition 3.2.5), then $\lim_i A_i$ is also a module category.

Again, this proposition is useful but its value is limited by the lack of definability. We can also generalise the results in Chapter 3 where two-orders were extended and reduced

[1]We assume that, whenever $M_1, M_2 \in \mathcal{M}$ intersect, their compositions on the intersection coincide; otherwise the arrows must be considered distinct.

by Δ_0-formulas. We can extend these results to the case of categories, but notice that in this case we sometimes must quantify over arrows—obtaining Π_1-, rather than Δ_0-, definable modules and arrow classes.[1] If M is a module, we write $M\colon a \nvrightarrow b$ to indicate that there exists $m \in M$ such that $m\colon a \nvrightarrow b$.

Proposition 4.3.2. Let A be a game category, such that $b \nrightarrow a$ and $b \not\nvrightarrow a$. Then there is a least module $M \supseteq \mathrm{Mod}(A)$ over the category $(\mathrm{Obj}(A), \mathrm{Arr}(A))$ such that $M\colon b \nvrightarrow a$. Further, no a will have a module arrow $a \nvrightarrow a$ in M.

Proof. If

$$v \xrightarrow{f} b \text{ and } a \xrightarrow{g} w,$$

define a new weak arrow $(f : g)\colon u \nvrightarrow v$. For strong arrows h, k in A such that

$$u \xrightarrow{k} v \xrightarrow{f} b \text{ and } a \xrightarrow{g} w \xrightarrow{h} x,$$

define $h \circ (f : g) \circ k = (f \circ k) : (h \circ g)$. Clearly this composition is associative, and so

$$M = \mathrm{Mod}(A) \cup \{(f : g)\colon\ \mathrm{cod}(f) = b \wedge \mathrm{dom}(g) = a\}$$

is a module over $(\mathrm{Obj}(A), \mathrm{Arr}(A))$. If $M\colon u \nvrightarrow u$, then for some strong arrows f, g we have $(f : g)\colon u \nvrightarrow u$, i.e.

$$u \xrightarrow{f} b \text{ and } a \xrightarrow{g} u,$$

hence $f \circ g\colon a \to b$, a contradiction. Hence $(\mathrm{Obj}(A), \mathrm{Arr}(A), M)$ is a game category.

Finally, if N is a module also extending A in this way, select any $n\colon b \nvrightarrow a$ in N. Define F to be the identity functor on the underlying category of A, and $F(f : g) = g \circ n \circ f$ for appropriate arrows f, g; clearly F embeds M onto N. □

Proposition 4.3.3. Suppose A is a game category with no arrows $a \to b$ or $a \nvrightarrow b$. Then there is a least pair L, M such that $(\mathrm{Obj}(A), L, M)$ is a game category with $L\colon b \to a$,

[1] If we augment the language by adding operators $A[-,-]$ and $A(-,-)$ for the arrow classes and an appropriate enrichment, however, these definitions will all be Δ_0.

$L \supseteq \mathrm{Arr}(A)$, and $M \supseteq \mathrm{Mod}(A)$.

Proof. If $f\colon u \to b$ and $g\colon a \to v$, let $f : g$ be a new arrow $u \to v$. If either

$$u \xrightarrow{f} b \text{ and } a \overset{g}{\nrightarrow} v$$

or

$$u \overset{f}{\nrightarrow} b \text{ and } a \xrightarrow{g} v$$

we let $f : g$ evaluate to a new, unique weak arrow $u \nrightarrow v$. As above, composition is defined by

$$h \circ (f : g) \circ k = (f \circ k) : (h \circ g),$$

and is associative. If $(f : g)\colon u \nrightarrow u$, then $f \circ g\colon a \nrightarrow a$, a contradiction. Hence $(\mathrm{Obj}(A), L, M)$ is a game category.

If L', M' also extend A in this way, pick any L'-arrow $k\colon b \to a$, and define an embedding F by setting $Fu = u$ for objects u; $Ff = f$ for A-arrows; and $F(f : g) = g \circ k \circ f$. Clearly F is a faithful functor. □

Proposition 4.3.4. Suppose A is a category and M a module on A, with $m\colon a \nrightarrow b$ in M. Then there is a greatest, Π_1-definable module $N \subseteq M$ such that $\neg(N\colon a \nrightarrow b)$.

Proof. Let

$$N = \{g \in M\colon A[a, \mathrm{dom}(g)] = \varnothing \vee A[\mathrm{cod}(g), b] = \varnothing\}.$$

We need only check amphi-transitivity, since clearly $\neg(N\colon x \nrightarrow x)$ for all $x \in A$. Suppose

$$t \xrightarrow{f} u \overset{g}{\nrightarrow} v \xrightarrow{h} w.$$

If $k\colon a \to t$, then $fk\colon a \to u$; if $k\colon w \to b$ then $kh\colon v \to b$. So $A[a, t] = \varnothing$ or $A[w, b] = \varnothing$. So N composes with $\mathrm{Arr}(A)$. Further, any module $N' \subseteq M$ for which $\neg N'\colon a \nrightarrow b$ must clearly be contained in N. □

Under certain conditions it will be possible to remove an individual arrow $g\colon a \nrightarrow b$

while leaving other arrows $a \nvrightarrow b$ intact. However the above Proposition will suffice for our needs.

Proposition 4.3.5. Let A be a game category and $a, b \in A$ such that $b \not\nvrightarrow a$ and $b \nrightarrow a$. There are $L \supseteq \mathrm{Arr}(A)$ and $M \subseteq \mathrm{Mod}(A)$ such that $L\colon a \to b$ and $(\mathrm{Obj}(A), L, M)$ is a game category.

Proof. If $f\colon x \to a$ and $g\colon b \to y$, define $f : g$ as usual. Let

$$L = \mathrm{Arr}(A) \cup \{(f : g)\colon\ \mathrm{cod}(f) = a \wedge \mathrm{dom}(g) = b\},$$

and

$$M = \left\{ f \in \mathrm{Mod}(A)\colon \begin{array}{l} (A[\mathrm{cod}(f), a] = \varnothing \vee A(\mathrm{dom}(f), b) \neq \varnothing) \wedge \\ (A[b, \mathrm{dom}(f)] = \varnothing \vee A(a, \mathrm{cod}(f)) \neq \varnothing) \end{array} \right\}.$$

As above, $(\mathrm{Obj}(A), L)$ is a category. Suppose $f \in M$, and $g, h \in \mathrm{Arr}(A)$. Then $\mathrm{cod}(hfg) = \mathrm{cod}(h)$ and $\mathrm{dom}(hfg) = \mathrm{dom}(g)$. If $A[\mathrm{cod}(h), a] \neq \varnothing$ then $A[\mathrm{cod}(f), a] \neq \varnothing$, hence $A(\mathrm{dom}(f), b) \neq \varnothing$, which implies $A(\mathrm{dom}(g), b) \neq \varnothing$. Similarly $A[b, \mathrm{dom}(f)] = \varnothing \Rightarrow A[b, \mathrm{dom}(g)] = \varnothing$ and $A(a, \mathrm{cod}(f)) = \varnothing \Rightarrow A(a, \mathrm{cod}(h)) = \varnothing$, so that $h \circ f \circ g \in M$. If $f \in L \setminus \mathrm{Arr}(A)$ then we can prove $h \circ f \circ g \in M$ similarly. Therefore $(\mathrm{Obj}(A), L, M)$ is a game category. $\square$

For a category A, let $\mathrm{Modules}(A)$ denote the class of modules over A. As in chapter 3 we equip this space with a topology generated by the sets

$$U_x = \{M \in \mathrm{Modules}(A)\colon\ \exists y\ (M\colon y \nvrightarrow x)\},$$

$$V_x = \{M \in \mathrm{Modules}(A)\colon\ \exists y\ (M\colon x \nvrightarrow y)\}$$

and their complements. Let $\mathrm{ClopMod}(A)$ be the category of clopen subsets of $\mathrm{Modules}(A)$, with functions as arrows.

In Chapter 3 we showed the map $U\colon x \mapsto U_x$ is an embedding from a pre-ordered structure X to the space of clopen sets of weak orders on X. We can turn U into a functor as follows. Suppose $f\colon a \to b$ in A, and M is a module on A. We define the module fM by

$$fM = N \cup \{f \circ m \colon m \in M,\ \mathrm{cod}(m) = a\}, \tag{4.2}$$

where N is the greatest module contained in M for which $\neg(a \nrightarrow b)$.

Proposition 4.3.6. If $f\colon a \to b$ in A and M is a module over A, then fM is also a module over A.

Proof. Clearly no $g \in fM$ has $\mathrm{dom}(g) = \mathrm{cod}(g)$. Let N be as in (4.2). To see amphitransitivity, suppose

$$u \xrightarrow{g} v \xrightarrow{fm} b \xrightarrow{h} w,$$

where $m \in M$, i.e.

$$u \xrightarrow{g} v \xrightarrow{m} a \xrightarrow{f} b \xrightarrow{h} w.$$

Since $\neg(M\colon a \nrightarrow a)$, there cannot be an arrow $a \to u$, hence $A[a, \mathrm{dom}(hmfg)] = \varnothing$, and $hmfg \in N$. Therefore fM is closed under composition with strong arrows in A. $\square$

If f is an arrow $a \to b$ in A, define Uf to be the function $M \mapsto fM$ for modules M over A. It is easily shown that $U(g \circ f) = Ug \circ Uf$ for arrows f, g in A. Hence U is an injective functor $A \to \mathrm{ClopMod}(A)$. In general U will not be faithful, however.

Theorem 4.3.7. If A is a category there is an injective functor from $A \to \mathrm{ClopMod}(A)$.

4.4 Products of game categories

4.4.1 Products in GC

Products of game categories deserve discussion, since different constructions are required for different purposes. In some algebraic contexts, it is correct to assume that the obvious categorical product of two game categories A and B will suffice. That is, to use the usual category with pairs of arrows as morphisms, and taking pairs of module arrows as the product module arrows. However, as the next example demonstrates, this notion of product does not always give the most appropriate first-player structure.

Example 4.4.1. In a monoidal category M the tensor product $\otimes$ is taken to be a bifunctor from $M \times M$ to M, subject to certain constraints. This method does not achieve the desired result when applied to game categories. Consider, for example, a space $\mathscr{G}$ of partisan games containing 0 and 1 (cf. ONAG [13], Winning Ways [5]). If we view $\mathscr{G}$ as a monoidal category equipped with a module for first-player strategies, then a bifunctor

$$F\colon \mathscr{G} \times \mathscr{G} \to \mathscr{G}$$

will have $F(0,0) \nrightarrow F(*,*)$ (this follows from the fact that $0 \nrightarrow *$). Therefore $F(x,y)$ cannot represent $x+y$.

We address this problem by introducing an alternative module structure on the product category $A \times B$.

Definition 4.4.2. Let A, B be game categories. The *exclusive-or product*, or *xor-product* for short, is the category A xor B with objects and normal arrows as in $A \times B$, but with module arrows as follows.

- If $f\colon x_1 \to y_1$ and $g\colon x_2 \nrightarrow y_2$ then the pair $(f,g)\colon (x_1,x_2) \nrightarrow (y_1,y_2)$.
- If $f\colon x_1 \nrightarrow y_1$ and $g\colon x_2 \to y_2$ then the pair $(f,g)\colon (x_1,x_2) \nrightarrow (y_1,y_2)$.

The *or-product* or $\vee$-product, denoted $A \vee B$, has objects and arrows as in $A \times B$ and A xor B, but includes all module arrows from both $A \times B$ and A xor B.

In each product the composition of module arrows and standard arrows is defined pointwise, in the obvious way (see Proposition 4.4.3 below).

Proposition 4.4.3. Each of the above notions of product forms a game category.

Proof. Assume A, B are game categories. We know that $A \times B$ is a category, and so it remains to show that the additional module structure is compatible in each case. It should be clear that we never have $x \nrightarrow x$. Suppose that

$$w \xrightarrow{f} u \overset{g}{\nrightarrow} v \xrightarrow{h} x \tag{4.3}$$

in $A \times B$, where each object and arrow is a pair consisting of something from A and something from B (so, for instance, $w = (w^A, w^B)$ and $f = (f^A, f^B)$). Then we have

$$\begin{aligned} &w^A \xrightarrow{f^A} u^A \overset{g^A}{\nrightarrow} v^A \xrightarrow{h^A} x^A; \\ &w^B \xrightarrow{f^B} u^B \overset{g^B}{\nrightarrow} v^B \xrightarrow{h^B} x^B. \end{aligned}$$

Hence the only sensible definition for $h \circ g \circ f$ is to take the pair $(h^A \circ g^A \circ f^A,\ h^B \circ g^B \circ f^B)$. Since this composition is associative in each component, it too is associative.

In the case of A xor B, if equation 4.3 holds then we have either

$$\begin{aligned} &w^A \xrightarrow{f^A} u^A \xrightarrow{h^A} v^A \xrightarrow{h^A} x^A; \\ &w^B \xrightarrow{f^B} u^B \overset{h^B}{\nrightarrow} v^B \xrightarrow{h^B} x^B; \end{aligned}$$

or the variant where h^A is a module arrow and h^B normal. In the former case, the obvious definition takes $h^A \circ h^A \circ f^A$ as the (normal) arrow in A, and $h^B \circ h^B \circ f^B$ as the (module) arrow in B. Analogously we define the composite for the case where h^A is the module arrow. Again, since this operation is component-wise associative, A xor B forms a game category.

Finally, in the case of $A \vee B$, we already have enough information to demonstrate that the module is compatible, since the two cases inherited from $A \times B$ and A xor B are disjoint. $\square$

These three constructions easily generalise to products of arbitrary classes $\mathscr{A} \subseteq \mathbf{GC}$. In particular $\mathbf{GC}$ has full products.

It is important to note that the or- and exclusive-or-products are not products in the category-theoretic sense: they do not in general even admit projections, as shown in the following example.

Example 4.4.4. Let $A = \{1_A, a\}$ and $B = \{1_B, b\}$ be two-element game categories, with game category structure as depicted in figure 4.6. The true product $A \times B$ is then the game category of four objects, with no non-identity arrows, while the or- and exclusive-or-products are equal to the game category depicted in figure 4.7. Notice that there is no possibility of projections from A xor B to A and B.

Figure 4.6: Two-element game categories.

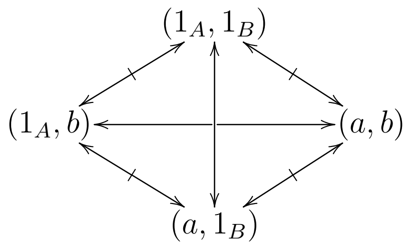

Figure 4.7: The product A xor B.

4.4.2 Products in $\mathbf{GC}^{\mathrm{am}}$

Definition 4.4.5. Let $\mathscr{A}$ be a class of game categories. We define a product $\prod^{\mathrm{AM}}\mathscr{A}$ as follows. Take

- $\mathrm{Obj}(\prod^{\mathrm{AM}}\mathscr{A}) = \prod\mathscr{A}$;
- $\mathrm{Arr}(\prod^{\mathrm{AM}}\mathscr{A}) = \prod_{A\in\mathscr{A}}\mathrm{Arr}(A)$;
- $\mathrm{Mod}(\prod^{\mathrm{AM}}\mathscr{A}) = \sum_{A\in\mathscr{A}}\mathrm{Mod}(A)$,

where $\sum_{A\in\mathscr{A}}\mathrm{Mod}(A)$ denotes the disjoint union of modules. The composition of normal arrows is defined in the obvious way, with $(g \circ f)(A) = g(A) \circ f(A)$. If $g\colon s \nrightarrow t$ in

$\prod^{\mathrm{AM}}\mathscr{A}$ then g corresponds to a (unique) arrow $g_A\colon s(A) \nrightarrow t(A)$ in some A, and we define $f \circ g = f(A) \circ g_A$, $g_A \circ f = g \circ f(A)$.

In practice we will avoid writing g_A to distinguish the module arrow in A from that in $\prod^{\mathrm{AM}}\mathscr{A}$.

Suppose P denotes the projection in **GC** from $\prod^{\mathrm{AM}}\mathscr{A}$ to some $A \in \mathscr{A}$. If $g\colon a \nrightarrow b$ in A, then whenever $x, y \in \mathscr{A}$ with $x(A) = a$ and $y(A) = b$, we have $g\colon x \nrightarrow y$ in $\prod^{\mathrm{AM}}\mathscr{A}$; therefore we can set $\overleftarrow{P}_{x,y}(g)$ as the arrow g in $\prod^{\mathrm{AM}}\mathscr{A}$. It is easily checked that this makes P into an amphimorphism.

Proposition 4.4.6. If $\mathscr{A} \subseteq \mathbf{GC}$ then $\prod^{\mathrm{AM}}\mathscr{A}$ is the product of $\mathscr{A}$ in $\mathbf{GC}^{\mathrm{am}}$.

Proof. Assume $F_A\colon S \to A$ for all $A \in \mathscr{A}$, and let $M\colon S \to \prod^{\mathrm{AM}}\mathscr{A}$ be the mediating arrow for the product in the underlying category **GC**; that is, for $s \in S$, $Ms(A) = F_A s$, and for $f\colon s \to t$ in S, $Mf(A) = F_A f$. We show M can be equipped with a map $\overleftarrow{M}$ such that $(M, \overleftarrow{M})$ is an amphifunctor which acts as a mediator in $\mathbf{GC}^{\mathrm{am}}$.

If $g\colon Ms_1 \nrightarrow Ms_2$ in $\prod^{\mathrm{AM}}\mathscr{A}$ then g is an arrow $Ms_1(A) \nrightarrow Ms_2(A)$ in some (unique) $A \in \mathscr{A}$; therefore $g\colon F_A s_1 \nrightarrow F_A s_2$, and we can define $\overleftarrow{M}g = \overleftarrow{F}_A g\colon s_1 \nrightarrow s_2$. To see this makes M into an amphifunctor, suppose $f\colon s_0 \to s_1$ and $h\colon s_2 \to s_3$ in S. Then

$$\begin{aligned}
\overleftarrow{M}(Mh \circ g \circ Mf) &= \overleftarrow{M}(F_A h \circ g \circ F_A f) \\
&= \overleftarrow{F}_A(F_A h \circ g \circ F_A f) \\
&= h \circ \overleftarrow{F}_A g \circ f \\
&= h \circ \overleftarrow{M} g \circ f,
\end{aligned}$$

and so M is an amphifunctor.

We now show that for each projection $P_A\colon \prod^{\mathrm{AM}}\mathscr{A} \to A$, we have $P_A \circ M = F_A$. This is already evident for the underlying functors. If $g\colon F_A s \nrightarrow F_A t$ in A then

$$\begin{aligned}\overleftarrow{P_A \circ M} g &= \overleftarrow{M}(\overleftarrow{P_A} g)\\ &= \overleftarrow{M} g\\ &= \overleftarrow{F_A} g;\end{aligned}$$

hence $\overleftarrow{P_A \circ M} = \overleftarrow{F_A}$.

Finally if some amphifunctor $N\colon S \to \prod^{\mathrm{AM}} \mathscr{A}$ also satisfies this property then the object and normal arrow assignment of N must be equal to that of M; further, if $g\colon Ns_1 \nrightarrow Ns_2$ in $\prod^{\mathrm{AM}} \mathscr{A}$, $g\colon F_A s_1 \nrightarrow F_A s_2$ in some A; so $\overleftarrow{N} g = \overleftarrow{N}(\overleftarrow{P_A} g) = \overleftarrow{F_A} g = \overleftarrow{M} g$, so that $N = M$. $\square$

4.5 Enriched game categories and the value map

Briefly we concern ourselves with enriched game categories. Such discussion will be of use here and in section 4.7. To clarify notation we recall the definition of an enriched category.

Definition 4.5.1. Assume $(M, \otimes, e_M)$ is a monoidal category. An M-enriched category, or a category enriched over M, is a tuple

$$A = (\mathrm{Obj}(A), A[-,-], \mathrm{id}, \circ),$$

where

- $\mathrm{Obj}(A)$ is a class of objects;
- $A[-,-]$ is a function $\mathrm{Obj}(A) \times \mathrm{Obj}(A) \to M$;
- for all $a \in A$, id_a is an M-morphism $e^M \to A[a, a]$;

- $\circ$ is a partial function on $A \times A \times A$ such that for all $a, b, c \in A$,

$$\circ_{a,b,c} \colon A[b,c] \otimes A[a,b] \to A[a,c]$$

  is a morphism in M;

and furthermore the diagrams in figures 4.8 and 4.9 commute.

$$\begin{tikzcd}
(A[c,d] \otimes A[b,c]) \otimes A[a,b] \arrow[rr, "\alpha^M"] \arrow[d, "\circ\otimes 1"'] & & A[c,d] \otimes (A[b,c] \otimes A[a,b]) \arrow[d, "1\otimes\circ"] \\
A[b,d] \otimes A[a,b] \arrow[dr, "\circ"'] & & A[c,d] \otimes A[a,c] \arrow[dl, "\circ"] \\
& A[a,d] &
\end{tikzcd}$$

Figure 4.8: Associativity of composition.

$$\begin{tikzcd}
e^M \otimes A[a,b] \arrow[dr, "\lambda"] \arrow[dd, bend right, "\mathrm{id}_a \otimes 1"'] & & A[a,b] \otimes e^M \arrow[dl, "\rho"'] \arrow[dd, bend left, "1\otimes \mathrm{id}_a"] \\
& A[a,b] & \\
A[a,a] \otimes A[a,b] \arrow[ur, "\circ"'] & & A[a,b] \otimes A[a,a] \arrow[ul, "\circ"]
\end{tikzcd}$$

Figure 4.9: Left and right-identity rules of composition.

Extending this definition to game categories is fairly straightforward. Suppose we have an additional function, $A(-,-) \colon A \times A \to M$, characterising the first-player strategies in A. In order to simulate the axioms of definition 4.2.1 we require that the composition map is also defined for all pairs of the form $(A[b,c], A(a,b)), (A(b,c), A[a,b])$, and that the diagrams in figures 4.10, 4.11 and 4.12 commute.

To simulate the axiom $\forall x\,(x \not\leftrightarrow y)$, we can require that for each $x \in A$, $A(x,x)$ is initial; in practice we will only be concerned with 'set-like' categories[1], where an initial

[1]By 'set-like' category we mean any category which can be interpreted as a universe of sets, with functions between sets as arrows. In particular this includes the category $2 = \{0 \to 1\}$, where 0 is an emptyset and 1 a nonempty set.

$$\begin{tikzcd}
(A[c,d] \otimes A(b,c)) \otimes A[a,b] \arrow[rr, "\alpha"] \arrow[d, "\circ\otimes 1"'] & & A[c,d] \otimes (A(b,c) \otimes A[a,b]) \arrow[d, "1\otimes\circ"] \\
A(b,d) \otimes A[a,b] \arrow[dr, "\circ"'] & & A[c,d] \otimes A(a,c) \arrow[dl, "\circ"] \\
& A(a,d) &
\end{tikzcd}$$

Figure 4.10: Central associativity for mixed composition.

$$\begin{tikzcd}
(A[c,d] \otimes A[b,c]) \otimes A(a,b) \arrow[rr, "\alpha"] \arrow[d, "\circ\otimes 1"'] & & A[c,d] \otimes (A[b,c] \otimes A(a,b)) \arrow[d, "1\otimes\mathrm{id}_a"] \\
A[b,d] \otimes A(a,b) \arrow[dr, "\circ"'] & & A[c,d] \otimes A(a,c) \arrow[dl, "\circ"] \\
& A(a,d) &
\end{tikzcd}$$

Figure 4.11: Left associativity for mixed composition.

object will always be the empty set. If

$$A = (\mathrm{Obj}(A), A[-,-], A(-,-), \mathrm{id}, \circ)$$

satisfies these criteria then we call A an *M-enriched game category* or a *game category* over M.

The value map

Of particular interest is enrichment over 2, the category with non-identity arrow $0 \to 1$. If A is any game category then following Joyal [51] and Conway et al. [13, 5] we define relations $\leq, \lhd\!\!\mid$ by $x \leq y$ if and only if $A[x,y] \neq \varnothing$ and $x \lhd\!\!\mid y$ if and only if $A(x,y) \neq \varnothing$. We call the two-ordered structure obtained in this way the *value space* of A, and denote it Values(A). If $a \in A$, then the element $a/\simeq$ is denoted val(a). Notice that this defines an isomorphism between the category of 2-enriched game categories and the category of two-ordered structures. In the discussion that follows we will identify a two-ordered

$$
\begin{tikzcd}
(A(c,d) \otimes A[b,c]) \otimes A[a,b] \arrow[rr, "\alpha"] \arrow[d, "\circ\otimes 1"'] & & A(c,d) \otimes (A[b,c] \otimes A[a,b]) \arrow[d, "1\otimes \mathrm{id}_a"] \\
A(b,d) \otimes A[a,b] \arrow[dr, "\circ"'] & & A(c,d) \otimes A[a,c] \arrow[dl, "\circ"] \\
& A(a,d) &
\end{tikzcd}
$$

Figure 4.12: Right associativity for mixed composition.

structure A with the equivalent 2-enriched game category.

In ONAG and Winning Ways the value of a game x is vaguely associated with the equivalence class of x modulo the relation $\simeq$—that is, $\mathrm{val}(x)$. Our value space generalises this construction to general module categories. We can easily extend Values to a functor from **GC** to the category **Tos** of two-ordered structures with promorphisms for arrows.[1] Indeed, if $F\colon A \to B$ is a module functor between game categories, define $\mathrm{Values}(F)$ to have the same object map (that is, $\mathrm{Values}(F)(a) = Fa$), and to send each arrow f to the appropriate relation (strong arrows to $\leq$, and weak arrows to $\lhd\!\mid$). Reversely, suppose B is a 2-enriched game category and $G\colon \mathrm{Values}(A) \to B$ is a functor of 2-enriched game categories. Let emb be the embedding functor from the category of 2-enriched game categories to **GC**. The functor G determines a unique functor $F\colon A \to \mathrm{emb}(B) = B$, with the same object map, and which sends an arrow f to the appropriate relation. Since 2-enriched game categories are essentially two-ordered structures, this proves the following.

Proposition 4.5.2. The value map $\mathrm{Values}\colon \mathbf{GC} \to \mathbf{Tos}$ is a functor, and is left-adjoint to the 'embedding' functor $\mathbf{Tos} \to \mathbf{GC}$.

Although not particularly inspiring this result does help to explain why so much of the theory of Conway games can be deduced by considering only games' values. Later on we will see that the adjunction is preserved when we add extra structure, such as a monoidal product.

[1]This construction will not work in general when we use amphimorphisms as arrows, since we cannot always guarantee a compatible method for assigning module arrows in a game category.

Remark 4.5.3. For games x in any game category A, let $\mathrm{val}(x)$ denote the quotient $x/\simeq$ in $\mathrm{Values}(A)$ (the *value of* x). Then the map $\mathrm{val}\colon A \to \mathrm{Values}(A)$ is easily shown to be a promorphism. We cannot make this map into an amphimorphism, however, since in general there will be no canonical choice of first-player strategies in A.

Remark 4.5.4. Two-ordered structures are essentially the spaces of values of games, and two-ordered groups the automorphism groups of these values.

4.6 Monoidal game categories

Now we look at game categories with additional monoidal structure. Recall that Joyal [51] showed the collection of wellfounded partisan games from ONAG and Winning Ways forms a compact closed monoidal category; we extend this to include first-player strategies. Our choice of functor (amphi- or module) greatly affects the behaviour of any monoidal product, and in particular amphifunctors introduce a new level of complexity. For simplicity we will concern ourselves only with module functors. We can view these functors, and the transformations between them, as the 1-cells and 2-cells respectively in the category **GC**.

Definition 4.6.1. Assume M is a game category, and $\otimes\colon M \text{ xor } M \to M$ is a module functor. Suppose that

$$\begin{aligned} \alpha_{x,y,z}&\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z), \\ \lambda_x&\colon e^M \otimes x \to x, \\ \rho_x&\colon x \otimes e^M \to x \end{aligned}$$

are isomorphisms, natural in x, y, z (that is, natural transformations of module functors which are also isomorphisms), such that the underlying category structure of

$$(M, \otimes, e^M, \alpha, \lambda, \rho)$$

is a monoidal category (cf. Mac Lane [65, p.162]). Then $(M, \ldots)$ is a *monoidal game category.*

We have already discussed the justification for requiring that $\otimes$ be a functor with domain M xor M: this ensures that whenever $f\colon x_0 \to y_0$ and $g\colon x_1 \twoheadrightarrow y_1$, both $f \otimes g\colon x_0 \otimes x_1 \twoheadrightarrow y_0 \otimes y_1$ and $g \otimes f\colon x_1 \otimes x_0 \twoheadrightarrow y_1 \otimes y_0$. This is true of, for example, the disjunctive sum under the normal play condition.

Definition 4.6.2. Assume M, N are monoidal game categories. A monoidal game functor $F\colon M \to N$ is a tuple (F_0, μ^F, ι^F) such that

- $F_0\colon M \to N$ is a game functor;
- (F_0, μ^F, ι^F) is a strong monoidal functor in the sense of Mac Lane [65, p.255], ignoring the module structure.

If $F, G\colon M \to N$ are monoidal game functors then a monoidal (module) transformation $\tau\colon F \to G$ is simply a monoidal transformation (see Mac Lane [65, p.256]) of the underlying functors which also makes the diagram in figure 4.13 commute for all $x, y \in M$.

$$\begin{tikzcd}
Fx \otimes Fy \arrow[r, "\mu^F_{x,y}"] \arrow[d, "\tau_x \otimes \tau_y"'] & F(x \otimes y) \arrow[d, "\tau_{x \otimes y}"] \\
Gx \otimes Gy \arrow[r, "\mu^G_{x,y}"'] & G(x \otimes y)
\end{tikzcd}$$

Figure 4.13: Monoidal natural transformations of game functors.

By **MGC** we denote the collection of monoidal game categories with monoidal module functors as arrows. This can be seen as a 2-category, where the module 2-cells of **MGC** are precisely the module 2-cells of **GC**. With these transformations of functors, the following is easily shown.

Proposition 4.6.3. If A is a game category then $\mathrm{End}(A) = \mathbf{GC}[A, A]$ is a (strict) monoidal game category.

4.6.1 Monoidal game categories and the value map

Recall that the value map $\mathrm{Values}\colon \mathbf{GC} \to \mathbf{Tos}$ is a functor and part of an adjunction. It is natural to consider the restriction of this map to $\mathbf{MGC}$, $\mathrm{Values}\restriction\mathbf{MGC}$. If $M \in \mathbf{MGC}$, however, $\mathrm{Values}(M)$ does not necessarily have compatible monoidal structure. Notice that Values is the composite of two functors: firstly, the map $\mathrm{ptos}\colon \mathbf{GC} \to \mathbf{PreTos}$, which takes a game category to the corresponding 2-enriched game category, or equivalently to a pre-two-ordered space; and secondly the map $\mathrm{Q}_{\simeq}\colon \mathbf{PreTos} \to \mathbf{Tos}$, which factors by the relation $\simeq$. The first map does have suitable structure for a tensor product, since the object class remains the same (hence the product is unchanged by Tos, except for a simplification of the arrow structure). The quotient map $\mathrm{Q}_{\simeq}$, however, does fail to preserve the monoidal structure, since there may exist x, y, z with $z \simeq x \otimes y$ but such that z is not the tensor product of any two elements of M.

This can be avoided by requiring that M is *rigid*. Recall that a pair of dual objects in a monoidal category $(M, \otimes, e)$ is a pair (x, y) with morphisms

$$\eta\colon 1 \to y \otimes x,$$
$$\varepsilon\colon x \otimes y \to 1,$$

such that

$$\lambda_x \circ (\varepsilon \otimes \mathrm{id}_x) \circ \alpha^{-1}_{x,y,x} \circ (\mathrm{id}_x \otimes \eta) \otimes \rho^{-1}_x = \mathrm{id}_x;$$
$$\rho_y \circ (\mathrm{id}_y \otimes \varepsilon) \circ \alpha_{y,x,y} \circ (\eta \otimes \mathrm{id}_y) \circ \lambda^{-1}_y = \mathrm{id}_y\,.$$

Definition 4.6.4. Let M be a monoidal game category. We call M a *rigid game category* if for each $x \in M$ there is $y \in M$ such that (x, y) is a dual pair.

It is easily seen that in the value space of a rigid monoidal category, the product is compatible with the relations $\leq$ and $\lhd\!\shortmid$.

Proposition 4.6.5. If M is a rigid game category then the product $\otimes$ on $\mathrm{Values}(M)$ is well-defined, and makes $(\mathrm{Values}(M), \leq, \lhd\!\!\mid)$ into a two-ordered group. Moreover, $\mathrm{val}\colon M \to \mathrm{Values}(M)$ is a monoidal game functor.

Example 4.6.6. Let $\mathscr{G}$ denote a class of partisan games (cf. ONAG [13], Winning Ways [5]) containing $0 = \{\,|\,\}$, and suppose $\mathscr{G}$ is closed under addition and negation. Then $\mathscr{G}$ is a (strict) rigid game category, where $(x, -x)$ is a dual pair for each x. The strategies η and ε in this case are different realisations of the copycat strategy $1_x\colon x \to x$, as arrows $0 \to -x + x$ and $x - x \to 0$.

4.7 Architectures

In any theory of games a notion of membership can help to connect the notions of strategy, option, position and game. In particular viewing each game as a two-sided container, whose elements are respective players' options, allows us to view these as equivalent notions. We formulate the following definition based primarily on discussion by Joyal [51], and Cockett et al. [12].

Definition 4.7.1. Assume $(V, \otimes, e^V, \ldots)$ is a monoidal category with full products and coproducts. Let A be a game category enriched over V with binary relations $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$. We say A is *instructive* if for all $x, y \in A$ there are S-arrows

$$\sigma_0(x,y)\colon A[x,y] \longrightarrow \prod_{u \in_{\mathrm{L}} x} A(u,y) \times \prod_{v \in_{\mathrm{R}} y} A(x,v), \tag{4.4}$$

$$\sigma_1(x,y)\colon A(x,y) \longrightarrow \sum_{u \in_{\mathrm{R}} x} A[u,y] \uplus \sum_{v \in_{\mathrm{L}} y} A[x,v], \tag{4.5}$$

natural in x, y.

Dually we call $(A, \in_{\mathrm{L}}, \in_{\mathrm{R}})$ *constructive* if for all $x, y \in A$ there are **Set**-arrows

$$\tau_0(x,y)\colon A[x,y] \longleftarrow \prod_{u \in_{\mathrm{L}} x} A(u,y) \times \prod_{v \in_{\mathrm{R}} y} A(x,v), \tag{4.6}$$

$$\tau_1(x,y)\colon A(x,y) \longleftarrow \sum_{u \in_{\mathrm{R}} x} A[u,y] \uplus \sum_{v \in_{\mathrm{L}} y} A[x,v], \tag{4.7}$$

natural in x, y.

If A is both instructive and constructive, and further if $\tau(x,y), \sigma(x,y)$ form a pair of mutual inverses for each x, y, then we call A (or more properly, $(A, \in_{\mathrm{L}}, \in_{\mathrm{R}}, \tau_0, \ldots)$) an *architecture* over S or an S-*architecture*.

The instructive property ensures that, given a second player strategy $f \in A[x,y]$, we may find first-player strategies regardless of Right's move. Dually, given a first-player strategy $g \in A(x,y)$ we may find a second-player strategy in at least one Left option. The constructive property allows us to build new strategies in more complex games, given an appropriate collection of strategies for their options. Notice that an architecture does not require monoidal structure; indeed, there may be a method for combining games which is not monoidal, or we may not have any such structure.

Architectures are closely related to the combinatorial game categories of Cockett et al. [12]: each represents an attempt to capture the common addition of set structure to collections of games. Most obviously, architectures lack a diproduct operation, allowing them to model a greater variety of important combinatorial game classes.

Example 4.7.2. Consider the set $\{P, N, L, R\}$ (analogous to the set $\{0, *, 1, -1\}$ of Conway games, in that order), with arrows $x \to y$ when $x \leq y$ (in Conway's sense) and $x \twoheadrightarrow y$ when $x \lhd\!\shortmid y$, discussed by Cockett et al. [12, Example 5.5, p.18]. As a result of the diproduct, projection and injection rules, there are additional arrows which do not appear in the typical structure $\{0, *, -1, 1\}$: for instance the diproduct $\{P|L\}$ is equal to L, and so by injection $L \twoheadrightarrow L$. Thus the insistence of closure under diproducts introduces unwanted arrows. This structure is better represented as an architecture.

Example 4.7.3. In many versions of Hackenbush there is certainly no obvious definition of diproduct. In, for example, restrained Hackenbush [13, p.86] it is impossible to define a diproduct since no such game is confused with 0. If this collection were a combinatorial game category then a game of the form $\{0 \mid 0\}$, which is confused with 0, would exist.

This leads to the following question.

Question 5. Which versions of Hackenbush have a compatible diproduct?

Our condition $\forall a\, a \not\leftrightarrow a$ is a further difference between architectures and combinatorial game categories, and imposes a weak form of regularity on the games in an architecture. It is easy to see the following.

Proposition 4.7.4. If A is a game category which is instructive or constructive, then A contains no self-membered games, i.e. for all $a \in A$, $a \notin^{\mathrm{L}}_{\mathrm{R}} a$.

In particular, architectures cannot have self-membered elements. We will see in Chapter 5 that the converse is also true: any collection of amphisets without self-members admits the structure of an architecture. Hence there still exist plenty of illfounded architectures, however including many nontrivial examples with elements satisfying $a \in_{\mathrm{L}} b \in_{\mathrm{R}} a$. This answers a question of Cox and Kaye [17], on how much regularity ('wellfoundedness') is required to allow the familiar relations of ONAG and Winning Ways.

For our final example, we will consider the related problem of adding set structure to a two-ordered structure. In particular this will give us a large class of architecture examples.

Definition 4.7.5. Let $(X, \leq, \lhd\!\!\mid)$ be a two-ordered structure satisfying

$$\forall u\, \big((u \lhd\!\!\mid x \to u \lhd\!\!\mid y) \land (y \lhd\!\!\mid y \to x \lhd\!\!\mid u)\big) \to x \leq y \tag{4.8}$$

for all $x, y \in X$. Then we call X a *pinched* two-ordered structure.

Given such a structure X, we can define membership relations $\in_{\mathrm{L}}$, $\in_{\mathrm{R}}$ on X by

$$\in_{\mathrm{L}} = \mathrel{\lhd\!|} \text{ and } \in_{\mathrm{R}} = \mathrel{|\!\rhd}. \tag{4.9}$$

Let **Pinch** be the category of pinched two-ordered structures.

Definition 4.7.6. The functor $\mathrm{Gmm}\colon \mathbf{Pinch} \to \mathbf{Arch}$ (for Greatest Membership Method) takes a pinched two-ordered structure $(X, \leq, \mathrel{\lhd\!|})$ to the architecture having the same two-order, and membership given in (4.9).

Proposition 4.7.7. The map Gmm defines a functor $\mathbf{Pinch} \to \mathbf{Arch}$, and is left-adjoint to the forgetful functor $\mathbf{Arch} \to \mathbf{Pinch}$.

Proof. Because X satisfies (4.8), it is easily shown that the two memberships make X into an architecture. □

We remark further that every 2-architecture arises from a pinched two-ordered structure in this way, since if

$$\forall u\, \big((u \mathrel{\lhd\!|} x \to u \mathrel{\lhd\!|} y) \wedge (y \mathrel{\lhd\!|} y \to x \mathrel{\lhd\!|} u)\big),$$

then in particular all $x^{\mathrm{L}} \mathrel{\lhd\!|} y$ and $x \mathrel{\lhd\!|}$ all y^{R}, so $x \leq y$.

4.7.1 Architectures and the value map

Briefly we consider architectures and the value map of section 4.5. If A is an architecture then the most reasonable definition of membership within the value space Values(A) is

$$\mathrm{val}(u) \in_{\mathrm{P}} \mathrm{val}(x) \Leftrightarrow \exists u' \simeq u\ \exists x' \simeq x\, (u \in_{\mathrm{P}} x).$$

This definition is in part a mathematical convenience; however it is in keeping with the view of games as Dedekind cuts.

Proposition 4.7.8. If $(A, \ldots)$ is an architecture then $\mathrm{Values}(A)$, with the usual relations and above-defined memberships, is an architecture over 2.

Let **Arch** denote the collection of architectures. A functor from the U-architecture A to the V-architecture B is best defined to be an enriched functor of the underlying categories which also preserves both the module structure (making it a game functor) and the membership relations. From the game-theoretic perspective this is sensible: such a map preserves playability. These functors make the collection **Arch** into a category. It is easily seen that the assignment of values on an architecture is such a functor. That is, for $A \in \mathbf{Arch}$, $\mathrm{val}\colon A \to \mathrm{Values}(A)$ is an architecture functor. As above, we can also show that the map Values, which takes a game category to its value space, is part of an adjunction.

4.8 Gamuts

In this section we show that the wellfounded games of ONAG and Winning Ways are examples of the structures discussed above. Notice in particular that this applies, not just to the "pure" partisan games (that is, two-sided containers which behave like sets; see chapter 2), but also to the distinct games, such as Hackenbush, Domineering, Go.

Definition 4.8.1. Suppose G is a monoidal game category enriched over the monoidal category V. If $\in_{\mathrm{L}}, \in_{\mathrm{R}}$ are binary relations on G, and V has natural isomorphisms τ_0, τ_1 which also make $(G, \in_{\mathrm{L}}, \in_{\mathrm{R}}, \ldots)$ an architecture, then G is called a *gamut*.

We define gamut functors similarly, as maps with monoidal and set-theoretic structure.

Definition 4.8.2. Suppose G, H are gamuts, and that $F\colon G \to H$ is an architecture functor. If there exist natural transformations ι^F and μ_F such that (F, μ^F, ι^F) is a monoidal game functor, then F is a *gamut functor*.

By **Gamuts** we denote the category of gamuts and their functors.

4.8.1 Examples of gamuts

We now argue, with informal proofs, that the wellfounded games of ONAG [13] and Winning Ways [5] exhibit the properties discussed above. This discussion includes specific classes of games (for instance Hackenbush, Domineering and Go), as well as the "pure" partisan games (that is, two-sided containers which behave like sets; see chapter 2). Moreover the results which follow are independent of any representation. For example, Hackenbush games can be realised as (for instance) trees, tuples of positions or intuitively as diagrams on the plane; each representation is valid. For clarity we make the following definitions.

Definition 4.8.3. Let A be an architecture. For $x, y \in A$ we write $x \in^{\mathrm{L}}_{\mathrm{R}} y$ if and only if $x \in_{\mathrm{L}} y$ or $x \in_{\mathrm{R}} y$. We call A *wellfounded* if the relation $\in^{\mathrm{L}}_{\mathrm{R}}$ is wellfounded, i.e.

$$\forall x\big(\exists y(y \in^{\mathrm{L}}_{\mathrm{R}} x) \to \exists y \in^{\mathrm{L}}_{\mathrm{R}} x \,\forall z \in^{\mathrm{L}}_{\mathrm{R}} x(z \notin^{\mathrm{L}}_{\mathrm{R}} y)\big).$$

Our claim can now be stated more concisely, as follows.

Theorem 4.8.4. Each wellfounded structure $\mathcal{H}$ of games considered in ONAG and Winning Ways is a gamut, where the disjunctive sum $+$ and negation $-$ determined the rigid monoidal structure.

Before we discuss the proof of this theorem it will be helpful to introduce some new notation and terminology. If $x \in_{\mathrm{P}} y$ where $\mathrm{P} \in \{\mathrm{L}, \mathrm{R}\}$, we will refer to x as a P-option of y, and as in the literature, we will use (for example) x^{L} to stand for an arbitrary element $u \in_{\mathrm{L}} x$. As in chapter 2 we denote by x^{P} an arbitrary P-option of x.

Suppose A is any instructive game category, in which $f : x \to y$. Then for any x^{L} and any y^{R} there exist strategies induced by f. Explicitly, since $f \in A[x, y]$ and

$$A[x, y] \cong \prod_{x^{\mathrm{L}}} A(x^{\mathrm{L}}, y) \times \prod_{y^{\mathrm{R}}} A(x, y^{\mathrm{R}}),$$

we can pick particular first-player strategies $f \downarrow x^{\mathrm{L}} : x^{\mathrm{L}} \nrightarrow y$ and $f \downarrow y^{\mathrm{R}} : x \nrightarrow y^{\mathrm{R}}$ for all

x^{L} and y^{R}. In cases where ambiguity is likely (for instance, if some $u \in A$ satisfies $u \in_{\mathrm{L}} x$ and $u \in_{\mathrm{R}} y$), we write, for example, $f{\downarrow}(u \in_{\mathrm{L}} x)$ or $f{\downarrow}(u \in_{\mathrm{R}} y)$ to distinguish the separate meanings.

In the case of first-player strategies, we can also define an induced strategy. If $f\colon x \twoheadrightarrow y$, then since A is an architecture f corresponds to a second player strategy, either $x^{\mathrm{R}} \to y$ or $x \to y^{\mathrm{L}}$. Again we will write $f{\downarrow}x^{\mathrm{R}}$ or $f{\downarrow}y^{\mathrm{L}}$ respectively, and in cases where ambiguity is possible we instead write $f{\downarrow}(u \in_{\mathrm{R}} x)$ or $f{\downarrow}(u \in_{\mathrm{L}} x)$; notice, however, that in this case $f{\downarrow}$ is only a partial function, applying to a single argument.

In some cases—depending upon the particular definition of "strategy" in A—f will be a function, and we will have $f(x^{\mathrm{P}}) = f \downarrow x^{\mathrm{P}}$. However it is useful to fix additional notation for the next position advocated by f, in order to distinguish this position from the induced strategy. If f is a first-player strategy in x, say, then $f[x]$ will be the next position advocated by a first-player strategy f. Hence $f \downarrow f[x]$ determines the remaining play dictated by f.

When A is an architecture, we can define a new strategy $f\colon x \to y$ by first describing the strategies $f{\downarrow}x^{\mathrm{L}}$ and $f{\downarrow}y^{\mathrm{R}}$. Similarly we can describe $g\colon x \twoheadrightarrow y$ by defining any $g{\downarrow}x^{\mathrm{R}}$ or $g{\downarrow}y^{\mathrm{L}}$.

Henceforth, we fix a class $\mathcal{H}$ of games, and assume

- $\in^{\mathrm{L}}_{\mathrm{R}}$ is wellfounded on $\mathcal{H}$;
- $\mathcal{H}$ is closed under disjunctive sums and negation (however they may be defined);
- $\mathcal{H}$ contains a zero game.

The following is obvious. Indeed, if we work within a monoidal category $(\mathbf{Set}, \times)$ of sets then the instructive and constructive rules of definition 4.7.1 dictate an appropriate definition of strategies, which makes the isomorphisms τ_0, τ_1 identities.

Proposition 4.8.5. With the (usually obvious) definition of strategy, $\mathcal{H}$ is an architecture.

We can also show that the existence of a disjunctive sum leads to a monoidal product, as indicated by Joyal [51]. We assume the existence of a map $+\colon \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ (where here $\mathcal{H} \times \mathcal{H}$ denotes a product of underlying categories) satisfying

$$\forall x, y, z \ \bigwedge_{\mathrm{P}} \big(z \in_{\mathrm{P}} x + y \leftrightarrow \exists w \in_{\mathrm{P}} x\,(z = w + y) \vee \exists w \in_{\mathrm{P}} y\,(z = x + w)\big).$$

That is, $x + y$ has the form $\{x^{\mathrm{P}} + y, x + y^{\mathrm{P}}\}_{\mathrm{P}}$, although $x + y$ may not be the unique such game: in many cases the relations $\in_{\mathrm{L}}, \in_{\mathrm{R}}$ may not satisfy an appropriate extensionality axiom[1]. We also assume that a negation functor $-\colon \mathcal{H}^{\mathrm{op}} \to \mathcal{H}$, satisfying

$$\forall x \ \forall y \ \left(\bigwedge_{\mathrm{P}} (y \in_{\mathrm{P}} -x \leftrightarrow \exists z \in_{\neg\mathrm{P}} x\,(y = -z)\right),$$

exists; that is, $-x$ has the form $\{-y\colon y \in_{\mathrm{R}} x \,|\, -y\colon y \in_{\mathrm{L}} x\}$.

Proposition 4.8.6. Under these assumptions, $\mathcal{H}$ is a rigid game category.

Sketch proof. We merely provide appropriate definitions and some guidance here, as these ideas have long been understood. If $f\colon w \to x$ and $g\colon y \to z$ in A, we recursively define the sum by taking

$$\begin{aligned}
(f+g)\!\downarrow\!(w^{\mathrm{L}} + y) &= (f\!\downarrow\! w^{\mathrm{L}}) + g;\\
(f+g)\!\downarrow\!(w + y^{\mathrm{L}}) &= f + (g\!\downarrow\! y^{\mathrm{L}});\\
(f+g)\!\downarrow\!(x^{\mathrm{R}} + z) &= (f\!\downarrow\! x^{\mathrm{R}}) + g;\\
(f+g)\!\downarrow\!(x + z^{\mathrm{R}}) &= f + (g\!\downarrow\! z^{\mathrm{R}}).
\end{aligned}$$

If $f\colon w \to x$ and $g\colon y \nrightarrow z$, either some $g\!\downarrow\! y^{\mathrm{R}}$ exists, and so we define $(f+g)\!\downarrow\!(w+y^{\mathrm{R}})$ to be $f + (g\!\downarrow\! y^{\mathrm{R}})$; or some $g\!\downarrow\! z^{\mathrm{L}}$ exists, and we define $(f+g)\!\downarrow\! z^{\mathrm{L}}$ to be $f + (g\!\downarrow\! z^{\mathrm{L}})$. Notice

[1] The obvious axiom here is

$$\forall x \ \forall y \ \left(\left(\bigwedge_{\mathrm{P}} \forall z\,(z \in_{\mathrm{P}} x \leftrightarrow z \in_{\mathrm{P}} y)\right) \to x = y\right);$$

see chapter 2.

that since A is an architecture, the strategy g corresponds to exactly one such possibility, so the sum strategy is well defined (and requires no element of choice). The treatment for $g + f$ is analogous.

In this way we define all sums $f + g$ of strategies f, g. It is now easily proved by induction (which is possible since $\in^{\mathrm{L}}_{\mathrm{R}}$ is wellfounded) that the function $+$, along with this arrow assignment, defines a functor of monoidal game categories. $\square$

From Propositions 4.8.5 and 4.8.6 we deduce the following.

Theorem 4.8.7. The structure $\mathcal{H}$ is a gamut.

We can also demonstrate that every wellfounded gamut maps onto a gamut of pure games via a gamut functor, which is part of an adjunction. Such a statement deserves caution, however. We have avoided making reference to "the collection of partisan games", often denoted **Pg**, of ONAG [13, ch.7] and Winning Ways [5]. Rather, we consider any class of games which contains non-impartial games to be such a collection. Moreover, we take the view that any model of amphi-ZF (cf. chapter 2), or possibly of a weaker theory, is a viable candidate for **Pg**. While the distinction is unnecessary for the study of finite games, different models will produce important variations on the theory of infinite games. In the case of wellfounded gamuts (which also contain an empty object), we can state the following.

Proposition 4.8.8. For each wellfounded gamut $\mathcal{H}$ there is a wellfounded gamut of pure games, $\mathcal{H}$, and a full gamut functor $F\colon \mathcal{H} \to \mathcal{H}$.

In stating this result we have ignored particular foundational issues: in particular the proof of proposition 4.8.8 involves an assumption that we can factor $\mathcal{H}$ to obtain an extensional (but categorically equivalent) category $\mathcal{H}$ (such a quotient would likely be formed using Scott's trick). Depending on our definition of category we might also prove that $\mathcal{H}$ is equivalent to some model $\mathcal{H}$ of amphi-ZF, using a Mostowski collapse.

Proposition 4.8.9. Ignoring foundational issues and assuming global choice, we can prove the existence of an equivalence $F\colon \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a gamut of pure two-sided containers (i.e. a model of amphi-ZF).

We remark that such assumptions are consistent with those made earlier, while discussing the value map.

CHAPTER 5

2-ARCHITECTURES

In this chapter we focus on (2-)architectures. First we address the problem of adding game-theoretic structure to an appropriate collection of amphisets. In particular we are able to give a necessary and sufficient condition for such a collection to admit the structure of an architecture. This answers a curiosity of Cox and Kaye [17, s.6] regarding the level of wellfoundedness required to admit Conway game-like structure.[1] We conclude that, where we are only concerned with two-orders compatible with the set structure (i.e. 2-architectures; significantly this does not include discussion of monoidal products), then such structure exists precisely when there are no self-membered objects.

We then turn our attention to the more difficult problem of extending an existing architecture by adding new games constructed from subsets of the original structure.

Finally we consider two possible applications of architectures: first, as a means to generalise the notion of cuts in nonstandard arithmetic; and secondly, as abstract models of concurrent or otherwise partially time-independent processes (such as asynchronous requests and callbacks in general).

2-architectures as two-ordered structures

As with regular game categories, problems concerning architectures are greatly simplified when the category is enriched over the two-point category 2, where we model the existence

[1] Notice that, by the nature of the value map, this answers the seemingly more general question of existence of architectures enriched over categories other than 2.

or nonexistence of arrows[1]using a pair of binary relations. Notice that a 2-architecture is essentially a two-ordered structure

$$(A, \leq, \lhd\!\mid, \in_{\mathrm{L}}, \in_{\mathrm{R}}, \ldots),$$

where $(\leq, \lhd\!\mid)$ is a two-order on A and $\in_{\mathrm{L}}, \in_{\mathrm{R}}$ are binary relations indicating membership, such that for all $x, y \in A$,

- $x \leq y \leftrightarrow \text{all}\, x^{\mathrm{L}} \lhd\!\mid y \wedge x \lhd\!\mid \text{all}\, y^{\mathrm{R}}$;
- $x \lhd\!\mid y \leftrightarrow \text{some}\, x^{\mathrm{R}} \leq y \vee x \leq \text{some}\, y^{\mathrm{L}}$.

When dealing with 2-architectures we will always refer to this simpler definition, rather than using the category-theoretic version required for more complex enrichments; this convention will simplify arguments and remove redundant notation.

Until now our discussion of architectures has been largely restricted to familiar classes of partisan games, as defined in ONAG and Winning Ways. The only other examples we have seen are either trivial, or obtained by the greatest membership method applied to a two-ordered structure. Now we turn our attention to the subject of constructing and extending them from other objects. This will provide a useful framework for later applications.

5.1 Building 2-architectures from amphisets

Just as Conway et al. constructed two-orders on wellfounded classes of partisan games we can construct them on collections of amphisets. In general we will not be assuming wellfoundedness, though; therefore there may be multiple two-orders on the same class of amphisets.

Let **AmphiSet** denote the category of $\mathscr{L}_2$-structures (recall $\mathscr{L}_2$ is the language with

[1]More correctly, the values of the operators $A[-,-]$ and $A(-,-)$ in the category 2.

nonlogical symbols $\in_{\mathrm{L}}, \in_{\mathrm{R}}$) satisfying:

$$\forall a\, \neg(a \in^{\mathrm{L}}_{\mathrm{R}} a).$$

We will often refer to such a structure simply by the domain A. For $A \in$ **AmphiSet**, we discuss possible two-orders $T = (\leq, \lhd\!\mid)$ which are compatible with the membership relations; that is, such that A equipped with $\leq$ and $\lhd\!\mid$ is a 2-architecture. Such a two-order T will be called an *architecture two-order*, or ATO for short. In particular we define a method for obtaining the least ATO, $T^{\mathrm{m}}(A)$, on A. Further, we show that every ATO on A is contained within the dual of $T^{\mathrm{m}}(A)$ (which may or may not be a two-order, depending on the set structure of A). Henceforth we fix an arbitrary object $(A, \in_{\mathrm{L}}, \in_{\mathrm{R}})$ in **AmphiSet**.

For a pair $T = (\leq, \lhd\!\mid)$ of relations on A, define the pair

$$T^{+} = (\leq^{+}, \lhd\!\mid^{+})$$

of relations on A by setting

$$x \leq^{+} y \leftrightarrow \text{all}\, x^{\mathrm{L}} \lhd\!\mid y \wedge x \lhd\!\mid \text{all}\, y^{\mathrm{R}};$$
$$x \lhd\!\mid^{+} y \leftrightarrow \text{some}\, x^{\mathrm{R}} \leq y \vee x \leq \text{some}\, y^{\mathrm{L}}.$$

Lemma 5.1.1. Suppose T_0 is a pair of relations on A. Define $T_{n+1} = T_n^{+}$ for all $n < \omega$. For $R \in \{\subseteq, \supseteq\}$, we have the following.

- If $\leq_n R \leq_{n+1}$ then $\lhd\!\mid_{n+1} R \lhd\!\mid_{n+2}$;
- if $\lhd\!\mid_n R \lhd\!\mid_{n+1}$ then $\leq_{n+1} R \leq_{n+2}$.

If A satisfies sufficient comprehension, then the above implications reverse.

Proof. Suppose $\leq_n \subseteq \leq_{n+1}$. If $x \lhd\!\mid_{n+1} y$ then some $x^{\mathrm{R}} \leq_n y$ or $x \leq_n$ some y^{L}, so some $x^{\mathrm{R}} \leq_{n+1} y$ or $x \leq_{n+1}$ some y^{L}, hence $x \lhd\!\mid_{n+2} y$. The other implications are similarly dealt with. If $\leq_n \not\subseteq \leq_{n+1}$, take x, y such that $x \leq_n y$ but $x \not\leq_{n+1} y$. Provided A satisfies some basic comprehension scheme, we have $\{\,|\,x\} \lhd\!|_{n+1}\, y$ but $\{|x\} \not\lhd\!|_{n+2}\, y$ (for example). The remaining cases are similar. $\square$

For pairs $S = (\leq_S, \lhd\!|_S)$ and $T = (\leq_T, \lhd\!|_T)$ of relations on A we write $S \leq T$ if $\leq_S \subseteq \leq_T$ and $\lhd\!|_S \subseteq \lhd\!|_T$. Notice that the collection of such pairs is a complete lattice, with the obvious definitions of $\vee$, $\wedge$, $\bigvee$ and $\bigwedge$, containing the complete bounded lattices of Chapter 3 (see in particular Proposition 3.2.5).

The arguments from the proof of the above Lemma can be used to prove the following.

Lemma 5.1.2. If T, S are pairs of relations, then

$$T \leq S \Rightarrow T^+ \leq S^+.$$

Definition 5.1.3. The pairs $T^{\mathrm{m}}_\alpha = (\leq^{\mathrm{m}}_\alpha, \lhd\!|^{\mathrm{m}}_\alpha)$ are defined as follows. First, we let

$$\begin{aligned} \leq^{\mathrm{m}}_0 &= \{(x, x) \colon x \in A\}; \\ \lhd\!|^{\mathrm{m}}_0 &= \{(x^{\mathrm{L}}, x) \colon x \in A\} \cup \{(x, x^{\mathrm{R}}) \colon x \in A\}. \end{aligned}$$

For each ordinal α, $T^{\mathrm{m}}_{\alpha+1} = (T^{\mathrm{m}}_\alpha)^+$. Finally, for limit ordinals λ, we define

$$T^{\mathrm{m}}_\lambda = \bigvee_{\alpha<\lambda} T^{\mathrm{m}}_\alpha.$$

Notice that for each x, $x \leq^{\mathrm{m}}_1 x$, as all $x^{\mathrm{L}} \lhd\!|^{\mathrm{m}}_0\, x$ and $x \lhd\!|^{\mathrm{m}}_0$ all y^{R}. Therefore $\leq^{\mathrm{m}}_0 \subseteq \leq^{\mathrm{m}}_1$. Further, as every x satisfies $x \leq^{\mathrm{m}}_0 x$, $\lhd\!|^{\mathrm{m}}_0 \subseteq \lhd\!|^{\mathrm{m}}_1$. Hence, by Lemmas 5.1.1 and 5.1.2, we have that $T^{\mathrm{m}}_\alpha \leq T^{\mathrm{m}}_\beta$ whenever $\alpha \leq \beta$.

In the definition of $(T^{\mathrm{m}}_\alpha)_\alpha$ we began with the following bare minimum rules.

$$\forall x\, (x \leq x); \tag{5.1}$$

$$\forall x\, (\text{all}\, x^{\mathrm{L}} \lhd\!|\, x \lhd\!|\, \text{all}\, x^{\mathrm{R}}). \tag{5.2}$$

However the dual[1] rules

$$\forall x\,(x \not\lhd| \, x); \tag{5.3}$$

$$\forall x\,(\text{all}\, x^{\mathrm{L}} \not\geq x \not\geq \text{all}\, x^{\mathrm{R}}) \tag{5.4}$$

must also hold in a 2-architecture.

Lemma 5.1.4. For each ordinal α, T^{m}_{α} satisfies rules (5.1)-(5.4).

Proof. That (5.1) and (5.2) hold is an immediate consequence of the facts that they are true for the case $\alpha = 0$ and that $(T^{\mathrm{m}}_{\alpha})_{\alpha}$ is nondecreasing. To see that (5.3) and (5.4) also hold, suppose otherwise, and let α be the least ordinal such that

$$(A, T^{\mathrm{m}}_{\alpha}) \not\models (5.3) \wedge (5.4).$$

Since T^{m}_{λ} is defined by taking unions at limits λ, and as (5.3) and (5.4) are clearly satisfied by T^{m}_{0}[2], α is a successor; say $\alpha = \beta + 1$. If for some x, $x \lhd|^{\mathrm{m}}_{\alpha}\, x$, then T^{m}_{β} clearly does not satisfy (5.4), contradicting leastness of α. Therefore some x has some $x^{\mathrm{L}} \geq x$ or $x \geq$ some x^{R}; and T^{m}_{β} violates (5.3), again contradicting the leastness of α. The claim follows. □

Although in general T^{m}_{α} will not be transitive, it will be for special ordinals α. For pairs $T = (\leq_T, \lhd|_T)$, $S = (\leq_S, \lhd|_S)$, we define the composition TS by

$$x \leq_{TS} z \leftrightarrow \exists y\,(x \leq_T y \leq_S z);$$

$$x \lhd|_{TS}\, z \leftrightarrow \exists y\,(x \leq_T y \lhd|_S\, z \vee x \lhd|_T\, y \leq_S z).$$

Note that by transitivity and reflexivity, we will always have $TT = T$ for a two-order T.

[1] See Chapter 3.

[2] Recall that each $(A, \in_{\mathrm{L}}, \in_{\mathrm{R}})$ in **AmphiSet** satisfies $\forall x\,(x \notin^{\mathrm{L}}_{\mathrm{R}} x)$.

Lemma 5.1.5. For all ordinals α, β, we have

$$T^{\mathrm{m}}_{\alpha} T^{\mathrm{m}}_{\beta} \leq T^{\mathrm{m}}_{\alpha+\beta+1},$$

where $+$ is the disjunctive sum.[1] In particular, if λ is closed under $+$, then T^{m}_{λ} is a two-order.

Proof. We prove by induction that for all α, β, we have

$$x \leq^{\mathrm{m}}_{\alpha} y \leq^{\mathrm{m}}_{\beta} z \to x \leq^{\mathrm{m}}_{\alpha+\beta+1} z;$$

$$x \leq^{\mathrm{m}}_{\alpha} y \lhd\!|^{\mathrm{m}}_{\beta} z \to x \lhd\!|^{\mathrm{m}}_{\alpha+\beta+1} z;$$

$$x \lhd\!|^{\mathrm{m}}_{\alpha} y \leq^{\mathrm{m}}_{\beta} z \to x \lhd\!|^{\mathrm{m}}_{\alpha+\beta+1} z.$$

Notice that, since $\leq_0$ is equality, the cases where $\alpha = 0 \vee \beta = 0$ are trivial. Therefore we assume $\alpha, \beta > 0$.

If $x \leq^{\mathrm{m}}_{\alpha} y \leq^{\mathrm{m}}_{\beta} z$ then for some $\gamma < \alpha$ and $\delta < \beta$, all $x^{\mathrm{L}} \lhd\!|^{\mathrm{m}}_{\gamma} y \leq^{\mathrm{m}}_{\beta} z$ (so all $x^{\mathrm{L}} \lhd\!|^{\mathrm{m}}_{\gamma+\beta+1} z$) and $x \leq^{\mathrm{m}}_{\alpha} y \lhd\!|^{\mathrm{m}}_{\delta}$ all z^{R} (so $x \lhd\!|^{\mathrm{m}}_{\alpha+\delta+1}$ all z^{R}). Therefore, all $x^{\mathrm{L}} \lhd\!|^{\mathrm{m}}_{\varepsilon+1} z$ and $x \lhd\!|^{\mathrm{m}}_{\varepsilon+1}$ all z^{R}, where $\varepsilon = \max(\gamma+\beta, \alpha+\delta)$. By definition of disjunctive sum, $\varepsilon < \alpha+\beta$, hence $x \leq^{\mathrm{m}}_{\alpha+\beta+1} z$.

If $x \lhd\!|^{\mathrm{m}}_{\alpha} y \leq^{\mathrm{m}}_{\beta} z$, then either $x^{\mathrm{R}} \leq^{\mathrm{m}}_{\gamma} y \leq^{\mathrm{m}}_{\beta} z$ for some $\beta < \alpha, x^{\mathrm{R}} \in_{\mathrm{R}} x$—hence $x^{\mathrm{R}} \leq_{\alpha+\beta+1} z$, and so $x \lhd\!|^{\mathrm{m}}_{\alpha+\beta+1} z$—or $x \leq^{\mathrm{m}}_{\gamma} y^{\mathrm{L}}$ for some $\gamma < \beta$ and $y^{\mathrm{L}} \in_{\mathrm{L}} y$. In the latter case, $y^{\mathrm{L}} \lhd\!|^{\mathrm{m}}_{\delta} z$ for some $\delta < \beta$, whence $x \lhd\!|^{\mathrm{m}}_{\gamma+\delta+1} z$; as $\gamma+\delta+1 < \alpha+\beta+1$, $x \lhd\!|^{\mathrm{m}}_{\alpha+\beta+1} z$. The case $x \leq^{\mathrm{m}}_{\alpha} y \lhd\!|^{\mathrm{m}}_{\beta} z$ is similarly dealt with.

If λ is closed under disjunctive sum, then since T^{m}_{λ} is defined by taking the unions of previous relations, it follows easily that T^{m}_{λ} is transitive. $\square$

Corollary 5.1.6. If κ is a cardinal, then T^{m}_{κ} is a two-order.

[1] Since the ordinals in the sense of ONAG [13] are all right-empty, it is easy to transfer this definition to a universe of pure sets, if one wishes: define

$$\alpha + \beta = \{\gamma + \beta \colon \gamma < \alpha\} \cup \{\alpha + \gamma \colon \gamma < \beta\}.$$

Proof. Given bijections $f\colon \alpha \to |\alpha|$ and $g\colon \beta \to |\beta|$, define a function

$$h\colon \big((\alpha+1)\times(\beta+1)\big) \setminus \{(\alpha,\beta)\} \to \alpha+\beta$$

by

$$h(\gamma,\delta) = \gamma+\delta.$$

Clearly h is a surjection onto the left- (and only) members of $\alpha+\beta$. Therefore, it is also a surjection onto the members of $\alpha+\beta$ when seen as a set. Hence

$$\begin{aligned} |\alpha+\beta| &\leq |(\alpha+1)\times(\beta+1)| \\ &= |\alpha+1|\cdot|\beta+1|. \end{aligned}$$

In particular, if $\alpha,\beta<\kappa$, then $\alpha+\beta<\kappa$. Therefore T^{m}_{κ} is a two-order. ☐

Consequently $(A, T^{\mathrm{m}}_{\kappa})$ is a two-ordered structure for each cardinal κ. Notice also that if $x \ntriangleleft|^{\mathrm{m}}_{\kappa}\, y$ then $x \ntriangleleft|^{\mathrm{m}}_{\alpha}\, y$ for all $\alpha<\kappa$, and so all $x^{\mathrm{R}} \nleq^{\mathrm{m}}_{\alpha} y$ and $x \nleq^{\mathrm{m}}_{\alpha}$ all y^{L}, for all $\alpha<\kappa$. Therefore all $x^{\mathrm{R}} \nleq^{\mathrm{m}}_{\kappa} y$ and $x \nleq^{\mathrm{m}}_{\kappa}$ all y^{L}, so that $x \ntriangleleft|^{\mathrm{m}}_{\kappa+1}\, y$. That is, $\triangleleft|^{\mathrm{m}}_{\kappa} = \triangleleft|^{\mathrm{m}}_{\kappa+1}$ (in fact this will hold at all limit ordinals).

However, T^{m}_{κ} is not always an ATO: it is possible that for some $x,y\in A$, $x \nleq^{\mathrm{m}}_{\kappa} y$ but $x \leq^{\mathrm{m}}_{\kappa+1} y$. Indeed, suppose that for each $\alpha<\kappa$, we have

$$u_\alpha \ntriangleleft|^{\mathrm{m}}_{\alpha}\, y \wedge u_\alpha \triangleleft|^{\mathrm{m}}_{\alpha+1}\, y.$$

Then $x = \{u_\alpha \colon \alpha<\kappa \,|\,\}$ is such that $x \nleq^{\mathrm{m}}_{\alpha} y$ for all $\alpha<\kappa$ and hence $x \nleq^{\mathrm{m}}_{\kappa} y$, yet $x \leq^{\mathrm{m}}_{\kappa+1} y$.

Lemma 5.1.7. Suppose $\kappa \geq \aleph_0$ is a regular cardinal such that $\kappa > |x|$ for all $x\in A$. Then the sequence $(T^{\mathrm{m}}_{\alpha})_\alpha$ terminates by κ, i.e. $T^{\mathrm{m}}_{\alpha} = T^{\mathrm{m}}_{\kappa}$ for all $\alpha>\kappa$.

Proof. By the above discussion, we need only show that for each $x, y \in A$,

$$x \leq^{\mathrm{m}}_{\kappa+1} y \to x \leq^{\mathrm{m}}_{\kappa} y$$

(here $\kappa + 1$ denotes the first ordinal after κ). Suppose, then, that $x \leq^{\mathrm{m}}_{\kappa+1} y$ but $x \not\leq^{\mathrm{m}}_{\kappa} y$. Each $x^{\mathrm{L}} \mathrel{\lhd\!\mid}^{\mathrm{m}}_{\kappa} y$ and $x \mathrel{\lhd\!\mid}^{\mathrm{m}}_{\kappa}$ all y^{R}. So, as κ is a limit ordinal, for each x^{L} and each y_{R} there is an ordinal $\alpha(x^{\mathrm{L}})$ or $\beta(y^{\mathrm{R}})$ strictly less than κ such that

$$x^{\mathrm{L}} \mathrel{\lhd\!\mid}^{\mathrm{m}}_{\alpha(x^{\mathrm{L}})} y;$$
$$x \mathrel{\lhd\!\mid}^{\mathrm{m}}_{\beta(y^{\mathrm{R}})} y^{\mathrm{R}}$$

respectively. But then, since $x \not\leq^{\mathrm{m}}_{\kappa} y$,

$$\kappa = \bigcup_{x^{\mathrm{L}}} \alpha(x^{\mathrm{L}}) \cup \bigcup_{y^{\mathrm{R}}} \beta(y^{\mathrm{R}}),$$

contradicting the regularity of κ. □

Definition 5.1.8. The *least order* on A, denoted by T^{m} or $T^{\mathrm{m}}(A)$, is given by $T^{\mathrm{m}} = T^{\mathrm{m}}_{\kappa}$ for any regular κ greater than the cardinality of the elements of A.

Corollary 5.1.9. The pair T^{m} is an ATO on A.

Finally we are left to prove the leastness of T^{m}.

Lemma 5.1.10. If T is an ATO on A, then $T^{\mathrm{m}} \leq T$.

Proof. Clearly $T^{\mathrm{m}}_0 \leq T$, and if $S \leq T$ for some pair S of relations, then $S^+ \leq T^+$ by Lemma 5.1.1. Therefore if $T^{\mathrm{m}}_{\alpha} \leq T$, $T^{\mathrm{m}}_{\alpha+1} \leq T^+ = T$; hence the claim follows by an easy induction. □

We summarise these results in the following.

Theorem 5.1.11. Let A be a collection of amphisets, none of which are self-members. Then there is an ATO T^{m} on A which is contained in every other ATO on A.

Corollary 5.1.12. Let A be a collection of amphisets. Then A admits an architecture if and only if A has no self-members.

5.1.1 The least order method as a functor

We can define a functor Lom (for Least Order Method) from the category **AmphiSet** to **Arch**, the category of architectures, which appends to a space A of amphisets the least ATO on A. This gives a left-inverse for the functor

$$\mathrm{Gmm}\colon \mathbf{Pinch} \to \mathbf{Arch}$$

(see Proposition 4.7.7), where **Pinch** is the category of 'pinched' two-ordered structures, i.e. two-ordered structures $(X, \leq, \lhd\!\mid)$ satisfying

$$\forall u\, \big((u \lhd\!\mid x \to u \lhd\!\mid y) \wedge (y \lhd\!\mid y \to x \lhd\!\mid u)\big) \to x \leq y$$

for all x, y.

Let $U_1\colon \mathbf{Arch} \to \mathbf{AmphiSet}$ and $U_2\colon \mathbf{Arch} \to \mathbf{AmphiSet}$ be forgetful functors, dropping the two-order and membership relations respectively.

Proposition 5.1.13. Let $(A, \leq_A, \lhd\!\mid_A) \in \mathbf{Pinch}$, and let B be the category of amphisets $U_1 \circ \mathrm{Gmm}(A)$ with membership relations $\in_{\mathrm{L}} = \lhd\!\mid_A$ and $\in_{\mathrm{R}} = \mid\!\rhd_A$. Then the minimal ATO $T^{\mathrm{m}}(B)$ is equal to $(\leq_A, \lhd\!\mid_A)$. Hence $U_2 \circ \mathrm{Lom}$ is a left-inverse for $U_1 \circ \mathrm{Gmm}$:

$$U_2 \circ \mathrm{Lom} \circ U_1 \circ \mathrm{Gmm} = 1_{\mathbf{Pinch}}.$$

Proof. By minimality of $T^{\mathrm{m}}(B)$, $T^{\mathrm{m}}(B) \leq T_A = (\leq_A, \lhd\!\mid_A)$. If $x \lhd\!\mid_A y$ then $B \vDash x \in_{\mathrm{L}} y \vee y \in_{\mathrm{R}} x$, so $x \lhd\!\mid^{\mathrm{m}} y$. If $x \leq_A y$ then since B is an architecture, all $x^{\mathrm{L}} \lhd\!\mid_A y$ and $x \lhd\!\mid_A$ all y^{R}, hence all $x \lhd\!\mid^{\mathrm{m}} y \wedge x \lhd\!\mid^{\mathrm{m}}$ all y^{R}, giving $x \leq^{\mathrm{m}} y$. Therefore $T_A = T^{\mathrm{m}}(B)$. $\square$

5.1.2 The dual of T^{m}

Recall the notion of duality from Chapter 3: for a pair $T = (\leq, \lhd\!\mid)$, the dual T^* is given by

$$T^* = (\leq^*, \lhd\!\mid^*) = (\not\!\!\mid\!\!\rhd, \not\geq).$$

In general, T^* may not be a two-order, but under the same ordering as above, $T^{\mathrm{M}} = (T^{\mathrm{m}})^*$ is an upper bound on all the ATOs on a collection A of amphisets (this follows easily from the fact that $T \geq T^{\mathrm{m}}$ for each ATO T on A).

Question 6. Under what conditions is T^{M} an ATO on A?

Notice that when A is wellfounded, A is determined, or equivalently $T^{\mathrm{m}} = T^{\mathrm{M}}$. Therefore

$$A \vDash \mathrm{wf} \Rightarrow \begin{array}{c} \forall T\ (A, T) \vDash \mathrm{Det} \\ T^{\mathrm{m}} = T^{\mathrm{M}} \end{array} \Rightarrow A \text{ admits exactly one ATO.}$$

We can show that first implication $A \vDash \mathrm{wf} \Rightarrow T^{\mathrm{m}} = T^{\mathrm{M}}$ does not reverse.

Example 5.1.14. Let $A = \{x, y\}$, where $x \in_{\mathrm{L}} y \in_{\mathrm{R}} x$. Then $x \leq y \wedge x \lhd\!\mid y$, hence A is determined, while clearly being illfounded.

However, it is not clear whether the last implication reverses.

Question 7. If $T^{\mathrm{m}}(A)$ is the unique two-order on A, is $T^{\mathrm{m}}(A) = T^{\mathrm{M}}(A)$?

These considerations affect the regularity and wellfoundedness of collections of games, and are naturally important from a game-theoretic perspective, since they affect how available moves relate to available strategies. Therefore, it is also useful to consider how these relate to traditional set-theoretic regularity principles.

Question 8. Which, if any, set-theoretic properties are the following regularity principles equivalent to?

- $T^{\mathrm{m}} = T^{\mathrm{M}}$, i.e. $\forall T\ (A, T) \vDash \mathrm{Det}$;
- A admits exactly one ATO.

5.2 Extending existing architectures

A natural construction is to extend an existing architecture A by adding amphi-subsets of A; that is, by adding objects x where x is both a left- and right-subset of A. However this is not a trivial operation, if one wishes to obtain another architecture. Even if we drop all restrictions other than the constructive and instructive axioms, we must be very careful to avoid circular definitions.

Example 5.2.1. Let $A = \{a, b\}$ where $a = \{b \,|\, b\}$ and $b = \{a \,|\, a\}$. We must have $a \leq a$, $b \leq b$ and $a \parallel b$ for A to be an architecture. Furthermore, we can prove that $a \not\leq b \not\leq a$. We cannot immediately determine an architecture by simply adding amphi-subsets of A, however. For instance, we cannot determine whether $\{a \,|\,\} \leq b$. We cannot exclude such examples by adding algebraic restrictions, either: we can make A into a two-ordered group by specifying either element as the identity and the other as an involution.

Suppose A is an architecture; we will try to mimic the least-order method on a collection of sets constructed above A as follows. Let $\mathscr{V}_0(A) = A$ and for nonzero ordinals α, we take $\mathscr{V}_\alpha(A)$ to be the set of amphisets x such that

$$\bigwedge_{\mathrm{P}} \forall u \in_{\mathrm{P}} x \; \exists \beta < \alpha \, (u \in \mathscr{V}_\beta(A)).$$

That is, at each successor ordinal $\alpha + 1$ we add the amphisets whose elements are in $\mathscr{V}_\alpha(A)$, and for limits we take the union.

We can attempt to generalise the least-order construction to this context.

Definition 5.2.2. Let A be an architecture, and $\mathscr{V}_\gamma(A)$ some class of amphisets built on top of A as outlined above. We define $T^{\mathrm{m}}_\alpha(\mathscr{V}_\gamma(A)) = T^{\mathrm{m}}_\alpha$ as follows. Define $T_0 = (\leq^{\mathrm{m}}_0, \lhd\!\!\mid^{\mathrm{m}}_0)$ by

$$\leq^{\mathrm{m}}_0 \, = \, \leq_A \cup =;$$

$$\lhd\!\!\mid^{\mathrm{m}}_0 \, = \, \lhd\!\!\mid_A \cup \in_{\mathrm{L}} \cup {\in_{\mathrm{R}}}^{-1};$$

and take

$$T^{\mathrm{m}}_{\alpha+1} = (T^{\mathrm{m}}_{\alpha})^{+};$$
$$T^{\mathrm{m}}_{\lambda} = \bigvee_{\alpha<\lambda} T^{\mathrm{m}}_{\alpha} \quad \text{(for limits } \lambda).$$

Except when ambiguity is possible, we denote T^{m}_{α} by T_α below.

Lemma 5.2.3. If $\alpha \le \beta$, then $T_\alpha(\mathscr{V}_\gamma(A)) \le T_\beta(\mathscr{V}_\gamma(A))$.

Proof. Since we take unions at limits, it will suffice to show that $T_\alpha \le T_{\alpha+1}$ for all α. Denote the two-order in A simply by $T_A = (\le, \lhd\!\shortmid)$.

If $x \le^{\mathrm{m}}_0 y$ then either $x \le y$ in A (so all $x^{\mathrm{L}} \lhd\!\shortmid y$ and $x \lhd\!\shortmid$ all y^{R}, and therefore $x \le^{\mathrm{m}}_1 y$) or $x = y$ (so $x \le^{\mathrm{m}}_1 y$ by monotonicity). If $x \lhd\!\shortmid^{\mathrm{m}}_0 y$ then $x \lhd\!\shortmid y$ in A (so some $x^{\mathrm{R}} \le y$ or $x \le$ some y^{L}), $x \in_{\mathrm{L}} y$, or $y \in_{\mathrm{R}} x$. In either case, $x \lhd\!\shortmid^{\mathrm{m}}_1 y$. The result now follows from Lemma 5.1.2 and an easy induction argument. □

Lemma 5.2.4. Assume A has maximal membership, i.e. $A = \mathrm{Gmm}(A)$. Then $T^{\mathrm{m}}(\mathscr{V}(A))$ is a two-order.

Proof. That the rules (5.1)-(5.4) hold is proved using exactly the arguments of Lemma 5.1.4. It remains, therefore, to prove transitivity. We prove this by induction on the sum of elements' ranks. First we must eliminate some laborious base cases. Throughout we will implicitly use the above fact, that $(T_\alpha)_\alpha$ is nondecreasing.

Assume $x \le_0 y \le_0 z$. If $x, y, z \in A$ then trivially $x \le_0 z$. If one or more of x, y, z are not in A, then at least two of x, y, z are equal, and trivially $x \le_0 z$.

Suppose now that $x \le_0 y \lhd\!\shortmid_0 z$. If $y \in A$, then $x \le^A y$ and so either $z \in A \wedge x \lhd\!\shortmid^A z$ or $z \notin A \wedge x \le^A y \in_{\mathrm{L}} z$; in either case, $x \lhd\!\shortmid_1 z$. If instead $y \notin A$, then $x = y$ so trivially $x \lhd\!\shortmid_0 z$. Similarly if $x \lhd\!\shortmid_0 y \le_0 z$, then $x \lhd\!\shortmid_1 z$.

Assume $x \le_0 y \le_0 z$. If $y \notin G$, then $x = y$, and easily $x \lhd\!\shortmid_0 z$. Assume, then, that $y \in G$. We trivially have all $x^{\mathrm{L}} \lhd\!\shortmid_0 y \le_0 z$, whence all $x^{\mathrm{L}} \lhd\!\shortmid_1 z$ by the above argument. Further, $x \le_0 y \lhd\!\shortmid_0$ all z^{R}. As $x \le_0 y \in G$, $x \in G$ also. If $z \in G$, trivially we have $x \lhd\!\shortmid_0 z$. If instead $z \notin G$, then $x \le^A y \in_{\mathrm{L}} z$, and so $x \lhd\!\shortmid_1 z$.

Next we suppose $x \leq_0 y \lhd\!|_1 z$. If $y^{\mathrm{R}} \leq_0 z$, then $x \lhd\!|_0 y^{\mathrm{R}} \leq_0 z$, so $x \lhd\!|_1 z$. If $y \leq_0 z^{\mathrm{L}}$, then $x \leq_0 y \leq_0 z^{\mathrm{L}}$, so $x \leq_0 z^{\mathrm{L}}$, whence $x \lhd\!|_1 z^{\mathrm{L}}$. The proof that $x \lhd\!|_1 y \leq_0 z$ implies $x \lhd\!|_2 z$ is similar.

The most difficult case is where $x \lhd\!|_0 y \leq_1 z$ (or the analogous case $x \leq_1 y \lhd\!|_0 z$); here we make use of the maximal membership in A. Assume $x \lhd\!|_0 y \leq_1 z$. If $x, y \in A$ then $x \in_{\mathrm{L}} y$ by maximal membership, whence $x \lhd\!|_0 z$ by definition of $\leq_1$. Otherwise, either $x \in_{\mathrm{L}} y$ or $y \in_{\mathrm{R}} x$. In the former case, $x \lhd\!|_0 z$; in the latter, $x \lhd\!|_2 z$.

This concludes the discussion of base cases. The argument for the inductive step is exactly as in Lemma 5.1.5. Hence $T^{\mathrm{m}}(\mathscr{V}(A))$ is a two-order, which clearly coincides with the original two-order when restricted to A. $\square$

In general, the minimal ATO on $\mathscr{V}_\alpha(\mathrm{Gmm}(A))$ will be —somewhat surprisingly—too small. We can, however, prove that it is contained in any ATO extending that of A to $\mathscr{V}_\alpha(A)$.

Proposition 5.2.5. Suppose A is an architecture, and $B = \mathrm{Gmm}(A)$. If S is any two-order on $\mathscr{V}(A)$ which contains the two-order on A, then S also contains $T^{\mathrm{m}}(B)$.

Proof. Let $T^{\mathrm{m}}_\alpha = (\leq^{\mathrm{m}}_\alpha, \lhd\!|^{\mathrm{m}}_\alpha)$ denote the α-th element of the least two-order sequence constructed in $\mathscr{V}(B)$. If $x \leq^{\mathrm{m}}_0 y$, then either $x \leq^B y \vee x = y$, so $x \leq^A y \vee x = y$. Therefore $x \leq_S y$. If $x \lhd\!|^{\mathrm{m}}_0 y$, then $x \lhd\!|^B y \vee x \in_{\mathrm{L}} y \vee y \in_{\mathrm{R}} x$, whence $x \lhd\!|^A y \vee x \in_{\mathrm{L}} y \vee y \in_{\mathrm{R}} x$; so $x \lhd\!|_S y$.

Now suppose $T^{\mathrm{m}}_\beta \leq S$ for all $\beta < \alpha$, and that $x \leq^{\mathrm{m}}_\alpha y$. If $u \in_{\mathrm{L}}{}^A x$ then $u \in_{\mathrm{L}}{}^B x$, so $u \lhd\!|^{\mathrm{m}}_\beta y$ for some $\beta < \alpha$. Therefore $u \lhd\!|_S y$. Similarly if $v \in_{\mathrm{R}}{}^A y$, then $x \lhd\!|_S v$, and so $x \leq_S y$.

If instead $x \lhd\!|^{\mathrm{m}}_\alpha y$, then some $u \in_{\mathrm{R}}{}^B x$ satisfies $u \leq^{\mathrm{m}}_\beta x$, or some $v \in_{\mathrm{L}}{}^A y$ satisfies $x \leq^{\mathrm{m}}_\beta v$, for some $\beta < \alpha$. In the first case, since $u \in_{\mathrm{R}}{}^B x$, $x \lhd\!|^A u \leq^{\mathrm{m}}_\beta y$, so by induction and the fact that S contains the two-order on A, $x \lhd\!|_S u \leq_S y$, whence $x \lhd\!|_S y$. The case where some $v \in_{\mathrm{L}}{}^A y$ satisfies $x \leq^{\mathrm{m}}_\beta v$ is dealt with similarly. $\square$

The above construction will not generalise easily to arbitrary architectures. That is, given an architecture A, it is not always possible to apply the modified least order method to obtain a two-order in $\mathscr{V}(A)$ which coincides with the original ATO on A. The only problem is in proving the base cases for transitivity; even the inductive argument for transitivity can be adapted, provided we can show that T_0^{m} and T_1^{m} compose to form a subset of some following T_α^{m}.

5.3 Cuts in Peano arithmetic as surreal numbers over a model

Proposition 5.3.1. Let $(R, \leq)$ be an ordered ring, and set $T = (\leq^R, <^R)$. Then (R, T) is a two-ordered ring.

Proof. Clearly the two orders are compatible with addition and multiplication in R, and transitive. Further we cannot have $x \leq^R y$ and $y <^R x$. $\square$

Proposition 5.3.2. Let R be an ordered ring. Define T on R as above, and define memberships $\in_{\mathrm{L}}, \in_{\mathrm{R}}$ by

$$x \in_{\mathrm{L}} y \leftrightarrow 0 \leq x < y;$$
$$x \in_{\mathrm{R}} y \leftrightarrow x < y \leq 0.$$

Then $(R, T, \in_{\mathrm{L}}, \in_{\mathrm{R}})$ is a gamut.

We denote this construction by $\mathrm{Gam}(R)$.

Suppose R is an ordered ring, and define left- and right-membership as above. Let $\mathscr{V}(R)$ denote $\mathscr{V}_\lambda(R)$ for some sufficiently large limit λ. In such a case we can show that the least order method produces an ATO on $\mathscr{V}(R) = \mathscr{V}_\lambda$.

Lemma 5.3.3. Suppose R is as above. Then the least order method makes $\mathscr{V}(R)$ into an architecture.

Proof. Recall that the least order method always produces a pair $T = (\leq, \lhd\!\!\mid)$ of relations which satisfy each axiom of an ATO except those for transitivity. Recall also that the construction will result in a nondecreasing sequence T_α of order pairs. Therefore it suffices to show the following, for all ordinals α, β and all $x, y, z \in \mathscr{V}(R)$;

$$x \leq_\alpha y \leq_\beta z \to x \leq_{\alpha+\beta+2} z;$$

$$(x \leq_\alpha y \lhd\!\!\mid_\beta z \vee x \lhd\!\!\mid_\alpha y \leq_\beta z) \to x \lhd\!\!\mid_{\alpha+\beta+2} z.$$

As before, the inductive step is relatively straightforward. For the base cases, we make nontrivial use of the fact that R is linear.

As discussed above, the case $x \leq_0 y \leq_0 z$ is trivial: either $x = y$ or $x \leq^R y$, and in the latter case we must also have $y \leq^R z$. If $x \leq_0 y \lhd\!\!\mid_0 z$, then $x = y$ (again, trivial), or $x \leq^R y$. If $x \leq^R y$, we have $y \in_{\mathrm{L}} z$ (so $x \lhd\!\!\mid_1 z$), $z \in_{\mathrm{R}} y$ (so $x \lhd\!\!\mid^R z$), or $y \lhd\!\!\mid^R z$ (so $x \lhd\!\!\mid_0 z$). Therefore $x \lhd\!\!\mid_1 z$. Similarly, $\lhd\!\!\mid_0 \leq_0 \subseteq \lhd\!\!\mid_1$.

If $x \leq_1 y \leq_0 z$, then all $x^{\mathrm{L}} \lhd\!\!\mid_0 y \leq_0 z$ (so all $x^{\mathrm{L}} \lhd\!\!\mid_1 z$). If z is right-empty then trivially $x \lhd\!\!\mid_1$ all z^{R}, so assume otherwise. Then either $y = z$ (trivial) or $y \leq^R z$. In the latter case, $y, z \in R$ and by the definition of membership we have $0 \geq z^{\mathrm{R}} > z \geq y$, hence every z^{R} is also a right-member of y, and so all $z^{\mathrm{R}} \mid\!\!\rhd_0 x$. Therefore $x \leq_2 z$. Similarly, if $x \leq_0 y \leq_1 z$, we have $x \leq_2 z$.

If $x \leq_1 y \lhd\!\!\mid_0 z$, either $z \in_{\mathrm{R}} y$ (so $x \lhd\!\!\mid_0 z$), $y \in_{\mathrm{L}} z$ (so $x \lhd\!\!\mid_2 z$), or $y \lhd\!\!\mid^R z$. In this last case, either some $y^{\mathrm{R}} \leq^R z$ (so $x \lhd\!\!\mid_0 y^{\mathrm{R}} \leq^R z$, whence $x \lhd\!\!\mid_1 z$), or $y \leq^R$ some z^{L}. Thus $x \leq_2 z^{\mathrm{L}}$, and so $x \lhd\!\!\mid_3 z$ in either case.

The case $x \leq_0 y \lhd\!\!\mid_1 z$ is trivial if $x = y$; otherwise, $x \leq^R y$, so either $y \lhd\!\!\mid^R z$ (so $x \lhd\!\!\mid^R z$) or $y \in_{\mathrm{L}} z$ (so $x \lhd\!\!\mid_1 z$).

Finally, the cases $x \lhd\!\!\mid_0 y \leq_1 z$ and $x \lhd\!\!\mid_1 y \leq_0 z$ are similar to cases given above. This concludes the discussion of base cases. By the same techniques as above, the least order method makes $\mathscr{V}(R)$ into an architecture. $\square$

Corollary 5.3.4. Suppose R is as above. Then R, equipped with the least two-order, is a gamut.

Our aim is to generalise the notion of a cut in models of Peano arithmetic[1], in the hope that certain constructions may be dealt with more easily and uniformly. We argue that this generalised notion allows a preferable theory of addition and multiplication of cuts, as well as allowing one to measure notions such as the size of an external set (studied, for example, by Kaye and Reading [52]) more effectively. This has its own cost, namely that we sacrifice the obvious linear ordering apparent in the traditional study of cuts in arithmetic. This, however, seems a necessity given the nature of problems relating even to such fundamental operations as addition and multiplication. Moreover, architectures give us a natural and convenient setting in which to consider this expanded, nonlinear collection of objects, since they allow greater depth of comparison in structures which are pre- or partially, but not linearly, ordered.

Using architectures to achieve this goal turns out to be quite heavy handed (see Proposition 5.3.12 and Figure 5.1). However we are merely constructing a sensible notion of 'surreal number' on top of the original model M; hence rifts are a natural object to study. That their structure is so easily characterised is a pleasant coincidence, and suggests they may be worth further study.

Definition 5.3.5. Let R be an ordered ring. A *rift* in R is an amphiset A in $\mathscr{V}(R)$ such that

- each side of A is nonempty and contains only elements of R;
- all $A^{\mathrm{L}} <$ all A^{R};
- $\forall u \in_{\mathrm{L}} A\,(u + 1 \in_{\mathrm{L}} A) \wedge \forall v \in_{\mathrm{R}} A\,(v - 1 \in_{\mathrm{R}} A)$.

Notice that a rift is essentially defined by a cut and a 'reverse cut': it is the (potentially empty) space between an initial segment and an end segment. Accordingly, every cut corresponds to a rift.

[1]Throughout, a *cut* in a model M will be an initial segment of M which is closed under the successor operation $n \mapsto n + 1$.

Definition 5.3.6. Let I be a proper cut. Then

$$\mathrm{Ded}(I) = \{i \in I \,|\, i' \colon i' \notin I\}.$$

Clearly, $\mathrm{Ded}(I)$ is a rift.

There are natural notions of addition and multiplication for rifts. Notice that here we are no longer adhering to the disjunctive sum, hence breaking both the ring and gamut structure; this is necessary if we are to make sense of cuts, since (for example) we cannot add a cut I with an element m of M sensibly without losing additive inverses (since, for instance, $I + (m + 1) = I + m$).

Definition 5.3.7. Suppose A, B are rifts, and $r \in R$. We define

- $r + A = A + r = \{A^{\mathrm{P}} + r\}_{\mathrm{P}}$;
- $rA = Ar = \{A^{\mathrm{P}} r\}_{\mathrm{P}}$;
- $A + B = \{A^{\mathrm{P}} + B^{\mathrm{P}}\}_{\mathrm{P}}$;
- $AB = \{A^{\mathrm{P}}\, B^{\mathrm{P}}\}_{\mathrm{P}}$.

The following is easily proved.

Proposition 5.3.8. The above operations commute, distribute, and are associative: let $r, s \in R$ and suppose A, B, C are rifts in R. Then

- $A + B = B + A$ and $AB = BA$;
- $(r + s)A = (rA) + (sA)$ and $r(A + B) = (rA) + (rB)$;
- $(A + B) + C = A + (B + C)$, $r + (A + B) = (r + A) + B$, $(r + s) + A = r + (s + A)$ and $(AB)C = A(BC)$, $r(AB) = (rA)B$, $(rs)A = r(sA)$.

For now, we will restrict attention to positive rifts, i.e. rifts in the structure

$$M = \{r \in R \colon r \geq 0\}.$$

With the definition of membership above, M is itself an architecture (in particular, M is $\in^{\mathrm{L}}_{\mathrm{R}}$-transitive). Therefore the architecture on R allows us to compare positive rifts with the two-order relations. Importantly, the following results demonstrate that the (linearly ordered) set of cuts in M embeds onto the set of rifts in M.

Proposition 5.3.9. Let $x \in M$, and assume A is a rift in M. Then

$$\begin{aligned} x < A &\Leftrightarrow x \leq A \\ &\Leftrightarrow x \lhd\!\mid A \end{aligned}$$

and similarly with the relations reversed. In particular, if I is a cut then

$$\begin{aligned} x < I &\Leftrightarrow x \leq \mathrm{Ded}(I) \\ &\Leftrightarrow x \lhd\!\mid \mathrm{Ded}(I) \\ &\Leftrightarrow x < \mathrm{Ded}(I), \end{aligned}$$

and similarly with the relations reversed.

Proof. If $x = 0$, the above statements are trivially true; therefore we may assume x is positive. Suppose $x \lhd\!\mid A$. Then, since x is right-empty, we must have that $x \leq \text{some}\, A^{\mathrm{L}}$. Therefore, $x < \text{all}\, A^{\mathrm{R}}$ and $\text{all}\, x^{\mathrm{L}} \lhd\!\mid A$, whence $x \leq A$.

Now, $x \leq A$ if and only if $\text{all}\, x^{\mathrm{L}} \lhd\!\mid A$ and $x \lhd\!\mid \text{all}\, A^{\mathrm{R}}$, or equivalently $x - 1 \lhd\!\mid A$ and $x < \text{all}\, A^{\mathrm{R}}$. Since $x - 1$ is right-empty, this implies that $x - 1 \leq \text{some}\, A^{\mathrm{L}}$; further, as the left side of A is closed under successor, this implies $x \leq \text{some}\, A^{\mathrm{L}}$, hence $x \lhd\!\mid A$. If $x < A$, then $x \leq A$; reversely, if $x \leq A$, then $x \lhd\!\mid A$ so $x < A$.

Now suppose $A = \mathrm{Ded}(I)$. Then $x < I$ implies $x \in_{\mathrm{L}} A$, hence $x \lhd\!\mid A$. If $x \lhd\!\mid A$, then $x \leq \text{some}\, A^{\mathrm{L}}$, whence $x \in_{\mathrm{L}} A$. $\square$

Lemma 5.3.10. Let I, J be cuts in M. Then

$$\operatorname{Ded}(I) \leq \operatorname{Ded}(J) \Leftrightarrow I \leq J;$$

$$\operatorname{Ded}(I) \mathrel{\lhd\mid} \operatorname{Ded}(J) \Leftrightarrow I < J.$$

(And consequently, $\operatorname{Ded}(I) < \operatorname{Ded}(J) \Leftrightarrow I < J$.)

Proof. Suppose $I \leq J$. Then every $i \in I$ satisfies $i \mathrel{\lhd\mid} \operatorname{Ded}(J)$ and every $j > J$ satisfies $\operatorname{Ded}(I) \mathrel{\lhd\mid} j$, by Proposition 5.3.9; whence $\operatorname{Ded}(I) \leq \operatorname{Ded}(J)$. If $\operatorname{Ded}(I) \leq \operatorname{Ded}(J)$ then whenever $i \in I$, $i < J$; hence $I \subseteq J$, i.e. $I \leq J$.

If $\operatorname{Ded}(I) \mathrel{\lhd\mid} \operatorname{Ded}(J)$ then some $i' > I$ satisfies $\operatorname{Ded}(i) \leq J$, or some $j < J$ satisfies $\operatorname{Ded}(I) < j$; in each case $I < J$. If $I < J$, there is $j \in J$ such that $I < j$¡ hence $\operatorname{Ded}(I) \leq j$ and $\operatorname{Ded}(I) \mathrel{\lhd\mid} \operatorname{Ded}(J)$. □

The next result is an immediate consequence of the previous propositions.

Corollary 5.3.11. The map Ded defines an order embedding

$$M \cup \{I \subseteq M \colon I \text{ is a cut}\} \to M \cup \{A \colon A \text{ is a rift}\}.$$

Therefore we may consider rifts a generalisation of cuts.

The set of rifts over M is certainly not linearly ordered (see Figure 5.1). However, we can show that the set $M \cup \{A \colon A \text{ is a rift}\}$ is partially ordered.

Proposition 5.3.12. Let A, B be rifts over M. Then $A \leq B$ if and only if

$$A \subseteq_{\mathrm{L}} B \text{ and } B \subseteq_{\mathrm{R}} A.$$

Consequently, $A \simeq B$ if and only if $A = B$.

Proof. One direction is always true in an architecture: if $A \subseteq_{\mathrm{L}} B$ and $B \subseteq_{\mathrm{R}} A$, then all $A^{\mathrm{L}} \mathrel{\lhd\mid} B$ and $A \mathrel{\lhd\mid}$ all B^{R}, so $A \leq B$. If we have $A \leq B$, then by the architecture

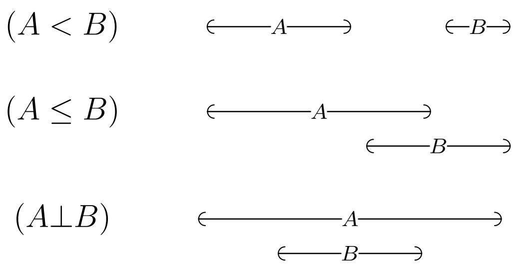

Figure 5.1: Possible comparisons of rifts.

axioms, all $A^{\mathrm{L}} \mathrel{\lhd\!\shortmid} B$ and $A \mathrel{\lhd\!\shortmid}$ all B^{R}; but then, by Proposition 5.3.9, all $A^{\mathrm{L}} < B$ and $A <$ all B^{R}, so that $A \subseteq_{\mathrm{L}} B \subseteq_{\mathrm{R}} A$. $\square$

We can now identify the kinds of comparison available for rifts. There are exactly three cases, which are as in Figure 5.1.

One of the core problems which motivates our introduction of rifts is the addition of cuts. If I, J are cuts then a sensible definition of $I + J$ is

$$I + J = \{i + j \colon i \in I \text{ and } j \in J\},$$

which is also a cut in M. However, there are examples where the operation

$$I \oplus J = \{i' + j' \colon i' > I \text{ and } j' > J\}$$

is more appropriate. Mixing these definitions together, we also conceive of various other definitions, such as

$$(I, J) \mapsto \inf_{j' > J} \sup_{i < I} i + j';$$

$$(I, J) \mapsto \sup_{i < I} \inf_{j' > J} i + j';$$

and so on. Each of these cuts is always at least $I + J$, and no greater than $I \oplus J$. Furthermore, there are always examples where $I + J < I \oplus J$; therefore none of these definitions will be universally satisfactory. Multiplication suffers the same fate: defining

$$I \cdot J = \{ij \colon i \in I \text{ and } j \in J\};$$
$$I \odot J = \{i'j' \colon i' > I \text{ and } j' > J\},$$

we can guarantee only that $I \cdot J \leq I \odot J$, and that the other two definitions lie somewhere in between.

The following tells us that $\mathrm{Ded}(I) + \mathrm{Ded}(J)$ will also lie somewhere in between.

Proposition 5.3.13. Let I, J be cuts in M. Then

$$\mathrm{Ded}(I + J) \leq \mathrm{Ded}(I) + \mathrm{Ded}(J) \leq \mathrm{Ded}(I \oplus J).$$

In particular, when $I + J = I \oplus J$, we have $\mathrm{Ded}(I + J) = \mathrm{Ded}(I) + \mathrm{Ded}(J)$.

Proof. We prove the left inequality; the other is easily proved in the same way. Each left-element of $\mathrm{Ded}(I + J)$ is also a left-element of $\mathrm{Ded}(I) + \mathrm{Ded}(J)$; therefore in particular, all $\mathrm{Ded}(I + J)^{\mathrm{L}} \lhd\!\mid \mathrm{Ded}(I) + \mathrm{Ded}(J)$. If $x \in_{\mathrm{R}} \mathrm{Ded}(I) + \mathrm{Ded}(J)$, then $x = i' + j'$ for some $i' > I$ and $j' > J$; therefore, $x > I + J$, i.e. $x \in_{\mathrm{R}} \mathrm{Ded}(I + J)$. As $\mathscr{V}(R)$ is an architecture, the left hand equality follows. $\square$

Similarly we have the following.

Proposition 5.3.14. Let I, J be cuts in M. Then

$$\mathrm{Ded}(I \cdot J) \leq \mathrm{Ded}(I) \cdot \mathrm{Ded}(J) \leq \mathrm{Ded}(I \odot J).$$

In particular, when $I \cdot J = I \odot J$, we have $\mathrm{Ded}(I \cdot J) = \mathrm{Ded}(I) \cdot \mathrm{Ded}(J)$.

Since we are arguing that rifts make a useful addition to the arithmetic of cuts, it is natural to ask the following questions.

Question 9. Is there a converse to either of the statements

$$I + J = I \oplus J \Rightarrow \mathrm{Ded}(I + J) = \mathrm{Ded}(I) + \mathrm{Ded}(J);$$

$$I \cdot J = I \odot J \Rightarrow \mathrm{Ded}(I \cdot J) = \mathrm{Ded}(I) \cdot \mathrm{Ded}(J)?$$

Question 10. Does every rift arise as a sum $\mathrm{Ded}(I)+\mathrm{Ded}(J)$ or a product $\mathrm{Ded}(I)\cdot\mathrm{Ded}(J)$, where I, J are cuts?

We can give a full answer to the first question.

Proposition 5.3.15. Let I, J be cuts in M. Then

- $\mathrm{Ded}(I) + \mathrm{Ded}(J)$ is equal to $\mathrm{Ded}(K)$ for some cut K if and only if $I + J = I \oplus J$;
- $\mathrm{Ded}(I) \cdot \mathrm{Ded}(J)$ is equal to $\mathrm{Ded}(K)$ for some cut K if and only if $I \cdot J = I \odot J$.

Proof. We have seen above that if $I + J = I \oplus J$ then $\mathrm{Ded}(I) + \mathrm{Ded}(J) = \mathrm{Ded}(I + J)$. If $\mathrm{Ded}(I) + \mathrm{Ded}(J) = \mathrm{Ded}(K)$, then for all $x \in M$,

$$\begin{aligned} x \in I + J &\Leftrightarrow x \in_{\mathrm{L}} \mathrm{Ded}(I) + \mathrm{Ded}(J) \\ &\Leftrightarrow x \in_{\mathrm{L}} \mathrm{Ded}(K) \\ &\Leftrightarrow x \notin_{\mathrm{R}} \mathrm{Ded}(K) \\ &\Leftrightarrow x \notin_{\mathrm{R}} \mathrm{Ded}(I) + \mathrm{Ded}(J) \\ &\Leftrightarrow x \in I \oplus J. \end{aligned}$$

The proof of the second point is similar, although one has to be careful since in general not every element of $\mathrm{Ded}(I) \cdot \mathrm{Ded}(J)$ or $I \cdot J$ is a product. □

Corollary 5.3.16. Let I, J be cuts. Then

$$\begin{aligned} I + J = I \oplus J &\Leftrightarrow \mathrm{Ded}(I) + \mathrm{Ded}(J) = \mathrm{Ded}(I + J) \\ &\Leftrightarrow \mathrm{Ded}(I) + \mathrm{Ded}(J) = \mathrm{Ded}(I \oplus J) \\ &\Leftrightarrow \mathrm{Ded}(I) + \mathrm{Ded}(J) = \mathrm{Ded}(K) \text{ for some cut } K \end{aligned}$$

and

$$
\begin{aligned}
I \cdot J = I \odot J &\Leftrightarrow \mathrm{Ded}(I) \cdot \mathrm{Ded}(J) = \mathrm{Ded}(I \cdot J) \\
&\Leftrightarrow \mathrm{Ded}(I) \cdot \mathrm{Ded}(J) = \mathrm{Ded}(I \odot J) \\
&\Leftrightarrow \mathrm{Ded}(I) \cdot \mathrm{Ded}(J) = \mathrm{Ded}(K) \text{ for some cut } K.
\end{aligned}
$$

We can also give a partial answer to the second question: if we restrict ourselves to the study of positive cuts and rifts as above, then there exist rifts which are not expressible as a sum of cuts (i.e. a sum $\mathrm{Ded}(I) + \mathrm{Ded}(J)$ where I, J are cuts) or a product of cuts.

Example 5.3.17. Let $A = \{n \colon n \in \mathbb{N} \,|\, i' \colon i' > I\}$, where I is a cut strictly larger than $\mathbb{N}$. If $\mathrm{Ded}(J) + \mathrm{Ded}(K) \simeq A$, then in particular, $J + K = \mathbb{N}$, implying that $J = K = \mathbb{N}$. But then $J \oplus K = \mathbb{N}$ as well. The rift A cannot be expressed as a product of cuts for the same reasons.

The above example does not, however, tell us that we cannot express each rift as a sum or product of a positive cut with a *negative* cut. In particular Example 5.3.17 makes implicit but essential use of the fact that $\mathbb{N}$ is the smallest cut in a model of arithmetic.

5.3.1 Rifts as a measure of size

Kaye and Reading [52] have defined the size of an external set by setting

$$
\begin{aligned}
\underline{\mathrm{card}}(X) &= \sup\{\mathrm{card}(A) \colon A \subseteq X \text{ is } M-\text{finite}\}; \\
\overline{\mathrm{card}}(X) &= \inf\{\mathrm{card}(A) \colon A \supseteq X \text{ is } M-\text{finite}\}.
\end{aligned}
$$

Then X is said to be *M-countable* when the two values coincide, in which case we take this value to be the size of X.

There are examples of sets X for which $\underline{\mathrm{card}}(X) < \overline{\mathrm{card}}(X)$; in fact, Kaye has proved the following.

Proposition 5.3.18. Let $I < J$ be cuts in the countable model M. Then there is a set X such that $\underline{\mathrm{card}}(I) = I$ and $\overline{\mathrm{card}}(J) = J$.

Suppose we define the size of a set X by

$$\mathrm{size}(X) = \{m \in \underline{\mathrm{card}}(X) \mid m > \overline{\mathrm{card}}(X)\};$$

then the following is immediate.

Corollary 5.3.19. If M is countable, then every rift is the size of a set.

5.4 Architectures as a threading model

Consider a set of interdependent objects, where each object exists over a given range (perhaps of time, or space), but can only do so following the objects it depends on. An instructive example is of threads in multithreaded software: a thread exists in time (and, perhaps, significantly in memory), but may depend on other objects to complete before it begins processing any information. Even in single-threaded software, we may not know the exact time at which, for instance, a callback may be executed. In either case there is often a fuzzy notion of execution time: we cannot say precisely when a process is executed, but we can specify a window of time in which it will. Posets with weak orders may be very useful for modelling this.

In transition theory, the transitions in a computation are often modelled using ill-founded sets, via a Mostowski collapse: if p is a transition, let $f\colon p \mapsto \{f(q)\colon p \to q\}$, where $p \to q$ denotes the fact that a transition from p to q is possible. This model is very effective when processes operate with single threads, though may fail to fully capture the interdependencies of multiple threads which can operate concurrently.

Example 5.4.1. Let w, x, y, z be threads and suppose Figure 5.2 describes their immediate dependencies, where $a \twoheadrightarrow b$ when b must wait for a to terminate before executing. When studying the execution of the program, we are interested not only in when a thread

$$\begin{tikzcd}[column sep=small, row sep=small]
 & x \arrow[dr, "\shortmid" marking] & \\
w \arrow[ur, "\shortmid" marking] \arrow[dr, "\shortmid" marking] & & z \\
 & y \arrow[ur, "\shortmid" marking] &
\end{tikzcd}$$

Figure 5.2: A race condition

t might run, but also in the state of other threads at times when t might execute. Race conditions are common examples of a situation where we need such information. In Figure 5.2, supposing the behaviour of z is dependent on the order in which x and y complete, we will have a race condition, and must find some way of asserting that x, y complete in the correct order. That is, we must augment the arrow system to ensure, say, $x \nrightarrow y$.

Let A be a set of such interdependent objects. Suppose that, for some subset B of A, the object $x \in A$ will start precisely when every $b \in B$ has terminated. For each $b \in B$, write $b \in_{\mathrm{L}} x$, and $x \in_{\mathrm{R}} b$. Repeating this for every $x \in A$, we obtain two membership relations, $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$. The left-members of any x are the necessary conditions for x, and the right-elements those objects which will wait for x.

If for some—perhaps less direct—reason y must wait for x to complete before running, write $x \lhd\!\mid y$. Write $x \leq y$ when x may exist at least as late as y, or equivalently when y cannot start any earlier than x.

We might decide that $x \in_{\mathrm{L}} y$ only when y directly depends on x, and further that this kind of dependency is obvious and easily—automatically—checkable. If we are considering a potential race condition, then we are likely to be interested in the relation $\leq$ on A.

We can easily prove several properties about the pair of relations, $T = (\leq, \lhd\!\mid)$. First, note that for each x, $x \leq x$. Provided we do not have a immediate self-dependency (which in many cases we will not, and furthermore, this assertion should be easily checked since membership describes a simple and obvious relationship), then no x will have $x \in_{\mathrm{L}} x$, $x \in_{\mathrm{R}} x$, or $x \lhd\!\mid x$. If $x \leq y \leq z$, then z cannot start earlier than y, which cannot start earlier than x; hence $x \leq z$. Similarly, if we have $x \lhd\!\mid y \leq z$ or $x \leq y \lhd\!\mid z$, then $x \lhd\!\mid z$.

Therefore T is a two-order, as defined in Chapter 3.

Furthermore, if x^{R} is a right-element of x, and $x^{\mathrm{R}} \leq y$, then $x \lhd\!\shortmid y$; similarly if $y^{\mathrm{L}} \in_{\mathrm{L}} y$ and $x \leq y^{\mathrm{L}}$, then $x \lhd\!\shortmid y$. Inversely, if $x \lhd\!\shortmid y$, then y must wait for some object which cannot start before x has started, or some object which depends on x cannot start later than y. In symbols,

$$x \lhd\!\shortmid y \Leftrightarrow \exists x^{\mathrm{R}}\,(x^{\mathrm{R}} \leq y) \vee \exists y^{\mathrm{L}}\,(x \leq y^{\mathrm{L}}). \tag{5.5}$$

Dually we can describe $\leq$ in terms of $\lhd\!\shortmid$ and membership. If $x \leq y$, then for every object x^{L} upon which x depends, $x^{\mathrm{L}} \lhd\!\shortmid y$ (otherwise it is possible that x would wait for some x^{L}, while y was in execution/existence); and for every object y^{R} which depends directly on y, we have that y^{R} also depends (indirectly) on x, i.e. $x \lhd\!\shortmid y^{\mathrm{R}}$. Inversely, if every $x^{\mathrm{L}} \lhd\!\shortmid y$ and every y^{R} depends on x, then certainly y cannot start before x. Thus,

$$x \leq y \Leftrightarrow \forall x^{\mathrm{L}}\,(x^{\mathrm{L}} \lhd\!\shortmid y) \wedge \forall y^{\mathrm{R}}\,(x \lhd\!\shortmid y^{\mathrm{R}}).$$

The relationships 5.5 and 5.4 prove that $(A, T, \in_{\mathrm{L}}, \in_{\mathrm{R}})$ is an architecture, as defined above.

Theorem 5.1.11 demonstrates that any collection of such processes and dependencies admits the structure of an architecture, and moreover that this is the minimum time restriction based on the constraints $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$.

CHAPTER 6

TOPOLOGICAL SET THEORY AND NONSTANDARD ARITHMETIC

In this chapter our we first mimic the constructions of Forti et al. in topological set theory, but in a nonstandard set-theoretic context. Specifically our constructions take place in a set theory as interpreted by a nonstandard model of arithmetic. A cut in this model will replace the large cardinal (sometimes $\aleph_0$) often used in topological set theory to guarantee the existence of models, and as such we will be concerned with properties of cuts—and their analogues in standard set theory—throughout.

Once this task has been completed, we begin to uncover which properties are necessary for the above construction. In particular this leads to some interesting results in the reverse mathematics of sequence properties such as convergence (and hence completeness). We show that completeness is certainly much weaker than the assumption of strength in a cut, and give some estimates regarding its precise strength.

6.1 Preliminaries

For the remainder of the chapter we fix a nonstandard model M of Peano arithmetic, and assume that some interpretation $\mathfrak{fst}$ of finite set theory has been specified.[1]By $(\mathcal{V}, \in)$ we refer to the model $(M, \in^{\mathfrak{fst}})$.

[1]For example the Ackermann interpretation $\mathfrak{ack}$, which stipulates that $x \in^{\mathfrak{ack}} y$ if and only if the xth digit in the binary representation of y is set to 1.

6.1.1 Properties of cuts

The following definitions can be found in Kirby's thesis [55].

Definition 6.1.1. A *cut* in M is an initial segment closed under the successor operation.

Definition 6.1.2. A subset A of M is said to be *coded* in M if there exists an element $a \in M$ such that $m \in A \Leftrightarrow M \vDash m \in^{\mathsf{fst}} a$ for all $m \in M$.

If I is a cut, B is coded in M and $A = B \cap I$, then we occasionally abuse the terminology and say A is coded also.

Below we list three possible cut properties. We will be particularly interested in the strength of properties in our model relative to these. All of these definitions (or slight variations thereof) may be found in Kirby's thesis [55], along with the equivalent properties given.

Definition 6.1.3. The cut $I \subseteq_e M$ is said to be *semiregular* in M if, whenever f is a function coded in M and $a < I$, $f``I \cap I$ is not cofinal in I.

A cut I is semiregular if and only if it satisfies $\mathrm{I}\,\Sigma_1^0$, or equivalently I, with its coded subsets, forms a model of the second-order theory RCA_0. In particular semiregular cuts are closed under exponentiation and are also equal to their own cofinality.

Definition 6.1.4. A cut J is *cofinal* in a cut I (written J cf I) if and only if there is a function f coded in M such that $\mathrm{dom}(f) \supseteq J$ and $f``J \cap I$ is cofinal in I.

The cut J *codes* a cut I (written J cd I) if and only if there is a nondecreasing function f coded in M such that $\mathrm{dom}(f) \supseteq J$ and $\sup(f``J) = I$.

The *cofinality* of I is the cut

$$\begin{aligned}\mathrm{cf}(I) &= \bigcap\{J \colon J \text{ cf } I\}\\ &= \bigcap\{J \colon J \text{ cd } I\}.\end{aligned}$$

Experience with such notions in standard set theory would suggest that J cf $I \Leftrightarrow J$ cd I, however this is not the case. Certainly J cd $I \Rightarrow J$ cf I, but the reverse is not true.

Moreover, it is generally not true that $\mathrm{cf}(I)$ is cofinal in I. These are our first examples where analogous properties in standard and nonstandard set theory fail to yield the same results.

As a semiregular cut is equal to its own cofinality (and clearly cuts are our analogues of limit ordinals), they correspond to particular regular limit ordinals (and hence to regular cardinals). As semiregular cuts are also closed under exponentiation, they may be regarded as analogous to strongly inaccessible cardinals.

A cut is I also semiregular if and only if I is inductive in I.

Definition 6.1.5. The cut J is *inductive* in I if and only if, whenever f is a function coded in M with $\mathrm{dom}(f) \supseteq J$ such that $f(0) \in I$ and

$$\forall j \in J\,(f(j) \in I \to f(j+1) \in I),$$

we have $f``J \subseteq I$.

We in fact have that

$$\mathrm{cf}(I) = \bigcup\{J \colon J \text{ is inductive in } I\},$$

and that $\mathrm{cf}(I)$ is always inductive in I.

Definition 6.1.6. The cut I is *regular* if, whenever $e < I$ and $(A_i)_{i<e}$ is a coded sequence of sets in M such that $\bigcup_{i<e} A_i \supseteq I$, there is some $i < e$ such that $A_i \cap I$ is cofinal in I.

Regularity is another familiar principle from standard set theory. Given the axiom of choice, the principle above—that is, that some limit cannot be constructed as a strictly smaller union of strictly smaller entities—is equivalent to the property of being a regular cardinal. While regularity implies semiregularity, the reverse is not true. This is a second example of analogues in nonstandard set theory failing to correspond exactly to behaviour in a standard universe.

Definition 6.1.7. We call I *strong* if, whenever f is a function coded in M, there exists $c \in M$ such that

$$f(i) < c \Leftrightarrow f(i) < I$$

for all $i \in I$.

Strong cuts correspond to measurable cardinals, but also to strongly inaccessible, weakly compact cardinals. The latter follows from the fact that the strong cuts are precisely those which are semiregular and satisfy a suitable tree property (see Kirby [55, ch.7]). The analogy with measurable cardinals is also suggested there, though we will not be concerned with these properties.

6.2 The construction

We first consider the problem of replicating the results of Forti et al. in a nonstandard context. Throughout this section we will be assuming our cut is strong in M.

In order to define the relations $\sim_i$, we first define the $+$ operator for equivalence relations considered by Malitz [67] and Aczel [1], among others. If $\sim$ is an equivalence relation on sets, then $\sim^+$ is the equivalence relation defined by

$$x \sim^+ y \leftrightarrow \forall u \in x \; \exists v \in y \, (u \sim v) \wedge \forall v \in y \; \exists u \in x \, (u \sim v).$$

Definition 6.2.1 (Malitz)**.** The sequence $(\sim_i)_{i \in M}$ is defined by setting $\sim_0 = M \times M$, and

$$\sim_{i+1} = \sim_i.$$

At limit ordinals λ, Malitz defines $\sim_\lambda = \bigcap_{\alpha<\lambda} \sim_\alpha$. We take the same definition at cuts; notice, however, that this definition is not internal.

Definition 6.2.2. For a cut I, $\sim_I = \bigcap_{i \in I} \sim_i$.

6.2.1 Sequences

Definition 6.2.3. An I-sequence is a sequence $(x_i)_{i\in I}$, coded in M.

All sequences we consider will be I-sequences. We will require analogues of convergence and the Cauchy property, as well as stronger notions.

Definition 6.2.4. Let (x_i) be an I-sequence. We say (x_i) *converges* to $y \in M$ if

$$\forall i \in I\ \exists j \in I\ \forall k \in I\, (k \geq j \rightarrow x_k \sim_i y).$$

We say (x_i) converges *strongly*[1]to y if

$$\forall i \in I\, (x_i \sim_i y).$$

Definition 6.2.5. Let (x_i) be an I-sequence. We say (x_i) is *(I-) Cauchy* if

$$\forall i \in I\ \exists j \in I\ \forall k, m \in I\, (m, k \geq j \rightarrow x_m \sim_i x_k).$$

We say (x_i) is *strongly* Cauchy if

$$\forall i \in I\ \forall j \in I\, (j \geq i \rightarrow x_i \sim_i x_j).$$

We will call $\mathcal{V}$

- *I-complete* if every I-Cauchy sequence is convergent;
- *I-crowded* if every I-sequence has an I-Cauchy subsequence;
- *I-compact* if it is both complete and crowded.

Lemma 6.2.6. A strongly Cauchy I-sequence always converges strongly.

[1]This notion was used by Malitz in his thesis [67], as was the corresponding notion for the Cauchy property of sequences below. We remark that it has no immediate relation to the cut property 'strength'.

Proof. Suppose $\forall i \in I\, \forall j \in I\, (j \geq i \to x_i \sim_i x_j)$. Recall that (x_i) is the truncation of a coded sequence $(x_i)_{i<k}$, where $k > I$. By overspill, there is some $m > I$ such that whenever $i \leq j \leq m$, we have $x_i \sim_i x_j$. In particular, $x_i \sim_i x_m$ for all $i \in I$. Therefore x_m is a strong limit. □

In order to obtain a model of a positive set theory, we require an I-compact model. The property of I-compactness readily splits into those of completeness and crowdedness.

Lemma 6.2.7. If I is strong, every I-Cauchy sequence has a strongly I-Cauchy I-subsequence, and hence $\mathcal{V}$ is I-complete.

Proof. First define a function F as follows. For $i, j \in I$ let $F(i, j)$ be the least $k \in M$ such that $x_k \not\sim_i x_j$, or $F(i, j) = \text{len}(x) > I$ if there is no such k.

By strength of I there is $c > I$ such that for all $i, j \in I$

$$F(i, j) \in I \text{ if and only if } F(i, j) < c.$$

Now we can define the subsequence. Define $f(0) < f(1) < \cdots < f(i) < \cdots$ by $f(0) = 0$ (so that $x_{f(0)} \sim_0 x_i$ for all i) and given $f(0) < f(1) < \cdots < f(i)$ all defined, define $f(i + 1)$ to be the least $j > f(i)$ such that $F(i + 1, j) \geq c$, if there is such j, $f(i + 1) = \text{len}(x)$ otherwise.

If $i \in I$, $f(i) \in I$ and (x_i) is I-Cauchy there is $j \in I$ such that $x_j \sim_{i+1} x_k$ for all $j \leq k \in I$ and clearly we may take $j > f(i)$. Thus by overspill, for this j, $F(i + 1, j) \geq c$. Therefore the least such j is indeed an element of I. The definition just given is first order. So it defines an M-finite function f which is increasing with the property that if $i \in I$ and $f(i) \in I$ then $f(i + 1) \in I$. Also, by strength, there is d such that $f(i) \in I$ iff $f(i) < d$ for all $i \in I$. So if for some $i \in I$ we have $f(i) > I$ then there is a greatest $i \in I$ with $f(i) < d$ i.e. $f(i) \in I$. This contradicts what has just been said, so f restricted to I is a coded function $I \to I$.

The subsequence $(x_{f(i)})_{i \in I}$ is evidently a strongly I-Cauchy subsequence of (x_i), as required. □

The following is a well understood combinatorial result, which shows that a tree property implies the crowdedness of a metric space. Briefly, it involves constructing a tree of the appropriate size for which any I-branch must be a Cauchy sequence. Our proof follows closely that given by Forti and Honsell [37].

Lemma 6.2.8. If I has the tree property and is closed under exponentiation then $\mathcal{V}$ is I-crowded.

Proof. Build a tree as follows. Let $T_0 = \{(0, x_0)\}$. Assume T_i has been defined. Then $(j, x_j) \in T_{i+1}$ if x_j is not included in any previous layer T_k, and j is the least m such that $x_m \tilde{x}_j$ (that is, x_j is the first unused representative of its equivalence class).

If $(j, x_j) \in T_m$ for some $m < i+1$ and $(k, x_k) \in T_{i+1}$, put $(j, x_j) <_T (k, x_k)$ if and only if $j < k$ and $x_j \sim_m x_k$. That is, at each stage $i+1$ we divide the i-equivalence classes up into their constituent $i+1$-equivalence classes, with representatives of least index for each class. Then the directed edges of the tree reflect inclusion of the represented classes.

Clearly T is a tree. Since (x_i) is an I-sequence it has length $M > I$, and so $\mathrm{card}(T) = M > I$. Further, each layer T_i has width at most $\exp(i)$, which is in I. Therefore by the tree property T has a branch of length greater than I. Clearly this branch represents a convergent I-sequence. □

Corollary 6.2.9. If I is strong then $\mathcal{V}/{\sim_I}$ is I-compact.

6.3 Positive set theory in $\mathcal{V}$

Here we will prove that a quotient of $\mathcal{V}$ satisfies the positive set theory GPK. As mentioned by Forti and Hinnion [30], GPK follows from the following axioms.

Definition 6.3.1 (Forti, Hinnion [30]). The theory BP is axiomatised by the existence of the following sets.

BP.**1** The identity set $I = \{(x, y) \colon x = y\}$;

BP.**2** The membership set $E = \{(x, y) \colon x \in y\}$;

BP.**3** For all a, $Q(a) = \{(x, y) \colon \forall z \in x\, ((x, y), z) \in a\}$;

BP.**4** For all a, the inverse $a^{-1} = \{(x, y) \colon (y, x) \in a\}$;

BP.**5** For all x, y, the doubleton $\{x, y\}$;

BP.**6** For all a, b, the pullback $\{((x, y), z) \colon (x, z) \in a \wedge (y, z) \in b\}$;

BP.**7** For all a, b, the image of b under a, i.e. $\{z \colon \exists y \in b\, (y, z) \in a\}$.

As explained by Forti and Hinnion [30], these axioms allow one to construct various familiar objects: a universal set, projections for sets of pairs, domains, ranges, unions, intersections, cartesian products, power sets, and so on. Further, we have the following theorem.

Theorem 6.3.2 (Forti & Hinnion [30])**.** Let $M = (A, E) \models \mathrm{BP}$. Then $M \models \mathrm{Comp(GPF)}$.

We will use this theorem to prove that a quotient of our model satisfies Comp(GPK).

Definition 6.3.3. The new membership E on $\mathcal{V}$ is defined by

$$\begin{aligned} x \; E \; y &\Leftrightarrow \exists x' \sim_I x \; \exists y' \sim_I y \; x' \in y' \\ &\Leftrightarrow x \in \overline{y}, \end{aligned}$$

where $\overline{y}$ denotes the closure of y.

This induces another membership on the quotient space $V/{\sim_I}$, which we also denote by E:

$$x/{\sim_I} \; E \; y/{\sim_I} \Leftrightarrow x \; E \; y.$$

Theorem 6.3.4. The structure $(\mathcal{V}/{\sim_I}, E)$ is a model of GPK.

Proof. Fix some set V which contains a representative for each I-equivalence class. This is possible by a theorem of Malitz [66], who showed in his thesis that for every ordinal α

the collection $\mathcal{M}_\alpha$ of canonical α-representatives is a set. In our case, we may take $\mathcal{M}_k$ for some $k > I$; then $\mathcal{M}_k$ contains a representative for each I-equivalence class, plus some other sets.

It is easily seen that in $\mathcal{U}$ the axioms BP.1-2 and BP.4-6 are satisfied. For instance, the identity set I is given by

$$\{(x, x)\colon x \in V\}/{\sim_I}$$

The axiom BP.3 is more difficult, and so we prove it here. Fix a set $s \in \mathcal{V}$; we find a set c whose equivalence class corresponds to $Q(s)$ from axiom BP.3. For each i, let

$$b_i = \{u = (x, y)\colon \forall z \in x\,((u, z) \in_i s)\}.$$

We prove that (b_i) converges to some set c, and that $\mathcal{U} \vDash c = Q(s)$.

Notice first that $b_0 \subseteq b_1 \subseteq \ldots$. Let (b_{m_i}) be a strongly convergent subsequence by compactness of $\mathcal{V}$, with limit c. If $u \in c$, then $u \in_i b_{m_i}$ for all i, hence $u \in_I b_{m_i}$ for all i, as (b_{m_i}) is a nonincreasing sequence. Further, by monotonicity the original sequence b converges to c.

Since $b_i \to c$, we have

$$\forall u = (x, y) \in c\; \forall z \in_I x\,((u, z) \in_I s).$$

Assume $u \in c$. Then $\forall i\; \exists j\; \forall k > j\,(u \in_i b_k)$. Therefore, for any i there is $j \geq i$ such that $u \in_i b_j$. Therefore u is a pair, say (x, y), and $u \in_i b_k$ so $u \sim_i v$ for some $v = (x_v, y_v)$ with

$$\forall z \in x_v\,((v, z) \in_i s).$$

If $z_i \in_I x_u$, then as $u \sim_i v$, $x_i \sim_{i-1} x_v$, so $z_u \sim_{i-2} z_v$ for some $z_v \in x_v$ As $z_v \in x_v$, $(v, z_v) \in_i s$. Therefore $(u, z_u) = \{\{u\}, \{u, z_u\}\} \sim_i (v, z_v) \in_i s$, so $(u, z_u) \in_i s$. This holds for all i, so that $(u, z_u) \in_I s$.

Conversely, suppose $u = (x, y)$ is such that $\forall z \in_I x\,((u, z) \in_I s)$. If $z \in x$, $(u, z) \in_I s$.

Therefore $u \in b_i$ for all i; hence $u \in_I c$. □

6.4 Reverse results

Given that strength is sufficient for our model $\mathcal{V}$ to satisfy the desired topological and logical properties, we now consider the question of necessity. In terms of topological properties, this problem again splits conveniently into those of crowdedness and completeness.

6.4.1 Crowdedness

It is relatively simple to show that crowdedness is equivalent to strength. This is unsurprising, as the former can be formulated as a purely combinatorial property, equivalent to closure under exponentiation and the tree property, by a well-understood tree argument. In order to obtain strength, then, we must first prove semiregularity follows from crowdedness; it is easier here to prove that regularity holds.

Lemma 6.4.1. If $\mathcal{V}$ is I-crowded, then I is regular.

Proof. Suppose $I \subseteq \bigcup_{i \leq e} A_i$, where the A_i form a coded family of disjoint subsets of M. By overspill there is some k such that $i \in \bigcup_{i \leq e} A_i$ for all $i \leq k$. Define a sequence x by taking x_i to be the unique $j \leq e$ such that $i \in A_j$. As $\mathcal{V}/I$ is crowded there is a Cauchy subsequence x_{m_i}. In particular, as the sequence x is bounded by $e \in I$, for $e \leq i, j \in I$ we must have $x_{m_i} = x_{m_j} = y$, say. Therefore $A_y \cap I$ is cofinal in I. □

Theorem 6.4.2. The space $\mathcal{V}$ is I-crowded if and only if I is strong.

Proof. If I is strong, then I satisfies the tree property and is closed under exponentiation. Therefore $\mathcal{V}$ is crowded, by Lemma 6.2.8. To see the other direction, assume $\mathcal{V}$ is I-crowded. Since I is closed under exponentiation by Lemma 6.4.1, I is strong if and only if I satisfies the tree property.

Suppose, for a contradiction, that I does not satisfy the tree property. Take an M-tree T such that $\forall i \in I\ |T_i| \in I$, but for which there is no M-coded branch B of T satisfying

$$\forall i \in I\ \exists b \in B \cap I\ \mathrm{rank}_T(b) = i.$$

Define a function f by $f(0) = 1$ and taking $f(i+1) = \max\{|T_{i+1}|, f(i)+1\}$. By semiregularity, $f(i) \in I$ for all $i \in I$.

Define a sequence x as follows. Let $x_0 = 0$. Assuming that $(x_k \colon k < n)$ are defined, corresponding to the layers $T_0, T_1, \ldots, T_i$ of T, take $x_n, \ldots, x_{n+|T_{i+1}|}$ to be representatives for the $\sim_{f(i+1)}$-equivalence classes of $\mathcal{V}$. Since I is regular, these representatives can be chosen in such a way that they are distinct and in I[1].

Now suppose $(x_{m_i})_{i \in I}$ is a Cauchy subsequence of (x_i). Let $b_0 = x_{m_0}$. Suppose $b_i = x_{m_k}$ has already been defined. Let X be the set of $<_T$-least y such that $b_i <_T y$. Let b_{i+1} be the $<$-least element y of X such that $y <_T x_{m_{k+1}}$. This guarantees that $y \in I$[2], but also that the next element of our sequence, b_{i+2}, will also be definable in this way. Clearly $B = \{b_i \colon i < I\}$ is a branch of T having an element of rank i for all $i \in I$. This contradicts the choice of T. □

Corollary 6.4.3. The space $\mathcal{V}/{\sim_I}$ is compact if and only if I is strong.

This is not what one might expect given the analogous result in standard set theory. For example, Forti and Honsell [37] use κ-completeness and κ-boundedness[3] to prove κ-compactness. Notice that κ-boundedness (analogous to the usual property of total boundedness) is true in any cut closed under exponentiation. However this does not enable us to prove crowdedness as usual, since we require subsequences to be coded in M.

[1] This also depends on our interpretation of finite set theory in M, however any sensible interpretation—such as Ackermann's—will suffice.

[2] Recall that Kirby's definition of an M-tree includes the requirement $a <_T b \to a < b$.

[3] The space is said to be κ-bounded if and only if, for every $\alpha < \kappa$, the number of α-equivalence classes is strictly less than κ.

6.4.2 Completeness

The problem of completeness is far less straightforward. Lemma 6.2.7 does not reverse, as the following example demonstrates.

Example 6.4.4. We give an example of a cut I in a model M such that I is not strong, yet for which $\mathcal{V}/I$ is complete. In particular every Cauchy sequence has a subsequence which is eventually strongly Cauchy, but I fails even to be semiregular.

Let M be ω_1-saturated. Let $\mathbb{N} < a \in M$, and take I to be the cut

$$\{m \in M \colon \text{for some } n \in \mathbb{N},\ m < a + n\}.$$

First we remark that I is not even semiregular: for some $b > \mathbb{N}$, define $f\colon [0, b] \to M$ by $f(n) = a + n$. Then $f''[0, a] \cap I$ is cofinal in I.

Let $(x_i)_i$ be an I-Cauchy sequence. Take $n_a \in [a, I)$ such that $n_a \le k < I$ implies $x_{n_a} \sim_a x_k$. By an external induction define, for $i \in \mathbb{N}$, n_{a+i+1} to be the least $j > n_{a+i}$ such that whenever $j \le k \in I$, we have $x_j \sim_{i+1} x_k$. Since $n_{a+i} \in I$ and $(x_k)_k$ is I-Cauchy, n_{a+i+1} is also in I. This defines a sequence $(n_{a+i})_{i \in \mathbb{N}}$ such that each n_{a+i} is in I. Find $(n_{a+i})_{i<K}$ extending $(n_{a+i})_i$ by saturation, and set

$$m_i = \begin{cases} i & \text{if } i \le a; \\ n_i & \text{otherwise.} \end{cases}$$

Then $m_i \in I$ for all $i \in I$, and $x_{m_i} \sim_i x_k$ whenever $i \ge a$; that is, $(x_{m_i})_i$ is eventually strongly Cauchy. This implies there is a limit, as in Lemma 6.2.6.

Although the cut I in Example 6.4.4 is very weak in the sense that it is not semiregular, the completeness of $\mathcal{V}/I$ is heavily influenced by another strong cut. Specifically, the following lemma shows that $\mathrm{cf}(I) = \mathbb{N}$ is strong in this case.

Lemma 6.4.5. Let M be $\aleph_1$-saturated. Then $\mathbb{N}$ is strong in M.

Proof. Let $f\colon [0,b] \to M$ be any coded function with $b > \mathbb{N}$. Let

$$p(x) = \{f(i) < x\colon i, f(i) < \mathbb{N}\} \cup \{x < f(i)\colon i < \mathbb{N} < f(i)\}.$$

Then $\mathrm{card}(p) = \aleph_0$ and clearly p is finitely satisfiable. Therefore $f(i) < x \Leftrightarrow f(i) < \mathbb{N}$, for all $i \in \mathbb{N}$. $\square$

In the standard realm, we have the following (see Forti and Honsell [37, Rem. 2.6, 2.7; Thm 2.7]).

Theorem 6.4.6. Suppose $\mathcal{U}$ is a model of ZFA and α is an ordinal. Then the quotient $\mathcal{U}/\alpha$ is α-complete if and only if either

- α is a limit ordinal with cofinality ω, or
- $\alpha = \kappa$ for some weakly compact, strongly inaccessible κ.

An analogue of this result in the nonstandard realm would certainly accommodate Example 6.4.4, since there $\mathbb{N}$ is strong by Lemma 6.4.5, and clearly $\mathbb{N}$ is cofinal in I (hence $\mathbb{N}$ *codes* I, which may be a more suitable analogue).

In order to better gauge the strength of I-completeness, we will have to introduce witnessing properties.

6.4.3 Witnessing principles

Definition 6.4.7. Suppose $\phi(x, y, z)$ is a formula (possibly with parameters from M), and

$$\forall x \in I\ \exists y \in I\ \forall z \in I\ \phi(x, y, z).$$

A *witness* for ϕ is a function f such that

$$\forall x \in I\ \forall z \in I\ \phi(x, f(x), z).$$

Witnesses can affect many notions involving sequences which are of particular interest to us.

Example 6.4.8. • Let $(x_i)_i$ be a Cauchy I-sequence, and $\phi(i,n,z)$ the formula

$$\forall m,k\,(z=\langle m,k\rangle \wedge m>k>n \to x_m \sim_i x_k).$$

Then $\forall i \in I\ \exists n \in I\ \forall z \in I\, \phi(i,n,z)$.

- Let $(x_i)_i$ be a convergent I-sequence, and $\phi(i,n,k)$ the formula

$$k \geq n \to x_k \sim_i y,$$

where y is a limit of $(x_i)_i$. Then

$$\forall i \in I\ \exists n \in I\ \forall k \in I\, \phi(i,n,k).$$

In each case a witness for the property ϕ is a function which allows us to discuss a non-definable notion (for example, $\exists n \in I\,(x_n \sim_i y)$) with a definable one (eg. $x_{f(i)} \sim_i y$). In general witnesses at least reduce the non-definable complexity of a formula by allowing us to replace an external quantifier with an internal one.

Definition 6.4.9. Let Σ be a collection of formulas. The *Witness Principle* WP(Σ) posits that whenever

$$\forall i \in I\ \exists n \in I\ \forall z \in I\, \phi(i,n,z)$$

for some $\phi \in \Sigma$, there is a witness f for ϕ.

Definition 6.4.10. Assume I is a cut closed under addition and multiplication. CDF is the assertion that whenever $(A_{i,n})_{i,n\in I}$ is a coded double family of sets such that

1. $A_{i,n}$ is monotonic nondecreasing in i;

2. $A_{i,n}$ is monotonic nonincreasing in n;

3. $\forall x \in A_{i,n}\,(x \geq n)$ for all n; and

4. $\forall i \in I\ \exists n \in I\ A_{i,n} \cap I = \varnothing$

there is a function f such that for all $i \in I$, $A_{i,f(i)} \cap I = \varnothing$.

We can state an equivalent condition in terms of functions; this allows us to also drop the condition that $A_{i,n} \geq n$.

Lemma 6.4.11. The following are equivalent when I is closed under addition and multiplication.

1. CDF

2. whenever $f(i,n)$ is a coded function which is nonincreasing in i, nondecreasing in n, and satisfies $\forall i, n \in I\ f(i,n) \geq n$ and $\forall i \in I\ \exists n \in I\,(f(i,n) > I)$, there is a coded function $g\colon I \to I$ such that

$$\forall i \in I\,(f(i, g(i)) > I).$$

3. whenever $f(i,n)$ is a coded function which is nonincreasing in i, nondecreasing in n, and satisfies $\forall i \in I\ \exists n \in I\,(f(i,n) > I)$, there is a coded function $g\colon I \to I$ such that

$$\forall i \in I\,(f(i, g(i)) > I).$$

4. whenever $A_{i,n}$ is a coded double family of sets satisfying all but the third hypothesis in the definition of CDF, there is a coded function $f\colon I \to I$ such that $A_{i,f(i)} \cap I = \varnothing$ for all $i \in I$.

Proof. To see that (1) and (2) are equivalent, first assume CDF and let $A_{i,n} = [f(i,n), N]$ where N is some number greater than I. Then $A_{i,n}$ satisfies the hypotheses of CDF, so there is a function g satisfying $\forall i \in I\,(A_{i,n} \cap I = \varnothing)$. But then $f(i,n) > I$ by choice of $A_{i,n}$. The reverse is proved similarly, by taking $f(i,n) = \min(A_{i,n} \cup \{N\})$ for some $N > I$. This argument will also prove the equivalence of items (3) and (4). Since (4) clearly implies (1), we are left to prove that (2) implies (3).

Given f satisfying the hypotheses of (3), define g by

$$g(i,n) = f(i,n) + n.$$

Clearly g is monotonic nonincreasing in i and increasing in n. Further, $g(i,n) \geq n$ and given any i, if $f(i,n) > I$ then $g(i,n) > I$. Find a function $h\colon I \to I$ such that

$$g(i,h(i)) > I$$

for all $i \in I$. Then $f(i,h(i)) = g(i,h(i)) - h(i)$ is in I precisely when $g(i,h(i)) \in I$. Hence h is a suitable function for f as well. $\square$

This result shows that, in the hypotheses of CDF, we may also assume the sets $A_{i,n}$ to be upwards-closed; in particular, only the minima of these sets are of interest. Considering sets which are not upwards closed introduces various unnecessary complications, and so we prefer to work with the equivalent function properties where possible.

Remark 6.4.12. Although our arguments are formulated in terms of cuts in a model of arithmetic, they hold in a model of second-order arithmetic with only minor adjustments. For instance, where we define a set A to be the interval $[f(i,n), N]$ where $N > I$, we could just as easily consider the set of elements in our model of second-order arithmetic which are at least $f(i,n)$ to obtain the result. In the reverse argument, where we consider functions f defined as the minimum of a set A, and otherwise defined as some $N > I$, we may alternatively consider partial functions, which are undefined when the appropriate set A is empty in the model, or simply upwards-closed sets.

Our reference to some $N > I$ is in fact just a means of discussing these notions in the context of a cut in a model of first-order arithmetic.

Theorem 6.4.13. Let Σ be the class of formulas $\phi(i,n,z)$ (possibly with parameters from M) such that

$$\phi(i,n,z) \to \phi(i,n+1,z)$$
$$\phi(i+1,n,z) \to \phi(i,n,z)$$

for all $i,n \in M$. Then $I \vDash \mathrm{CDF}$ if and only if $I \vDash \mathrm{WP}(\Sigma)$.

Proof. Suppose $I \vDash \mathrm{WP}(\Sigma)$. Given $(A_{i,n})$ satisfying the hypotheses of CDF, let $\phi(i,n,z)$ be the property $z \notin A_{i,n}$. Then clearly $\phi \in \Sigma$, hence there is a witness f such that

$$\forall i \in I\ \forall z \in I\ \phi(i,f(i),z);$$

that is, $A_{i,f(i)} \cap I = \varnothing$.

Conversely, suppose $\phi(i,n,z) \in \Sigma$, and let $A_{i,n} = \{z \colon \neg\phi(i,n,z)\}$. Then $A_{i,n}$ satisfies the CDF hypotheses and so there is a witness f for the emptiness of $A_{i,n}$—that is, f is a witness for ϕ in I. $\square$

Corollary 6.4.14. If $I \vDash \mathrm{CDF}$ then

- every convergent I-sequence has a witness;
- every Cauchy I-sequence has a witness;
- every Cauchy I-sequence has a strongly Cauchy (hence convergent) subsequence.

Proof. By Example 6.4.8 and theorem 6.4.13, if $I \vDash \mathrm{CDF}$ then both convergence and the Cauchy property are always witnessed. If $(x_i)_{i<I}$ is a Cauchy I-sequence, first find a witness f, i.e.

$$\forall i \in I\ \forall k,m \in I\ (m \geq k \geq f(i) \to x_m \sim_i x_k).$$

Define g by

$$g(i) = \max_{j \le i} f(j) + i.$$

For any $i \in I$, $\max_{j \le i} f(j) \in I$, hence—as I is closed under addition—$g(i) \in I$ for all $i \in I$. Further, g is strictly increasing, hence $(x_{g(i)})$ is a subsequence. Fix i. Then $g(i) \ge f(i)$, so for all $k \in I$ with $k \ge g(i)$, $x_k \sim_i x_{g(i)}$. Since g is increasing, in particular for $j \ge i$ we have $x_{g(j)} \sim_i x_{g(i)}$. Hence $(x_{g(i)})_i$ is a strongly Cauchy I-subsequence. $\square$

Corollary 6.4.15. If $I \vDash \mathrm{CDF}$ then $\mathcal{V}/I$ is I-complete.

This is all well and good, but we haven't yet learnt anything about the strength of completeness with respect to principles such as strength, regularity and semiregularity.

6.5 The relative strength of CDF

Recall Example 6.4.4, where we exhibited a cut $I = a + \mathbb{N}$ which failed to be semiregular, yet for which $\mathcal{V}$ was I-complete. This was because I was coded by the strong cut $\mathbb{N}$.

Proposition 6.5.1. Suppose J cd I. Then $I \vDash \mathrm{CDF} \Leftrightarrow J \vDash \mathrm{CDF}$.

Proof. Suppose there is a function $f\colon J \to I$ coded in M which is nondecreasing. Assume $\mathrm{dom}(f) = [0, a]$, and define g by

$$g(i) = \max\{j \le a\colon f(j) \le i\}.$$

Notice that $g(f(j)) \ge j$ for all $j \in J$ and $f(g(j)) \le i$ for all $i \in I$. This in fact proves I cd J; the remainder of this proof shows how to use the two codings to transfer CDF from one cut to the other.

Assume $I \vDash \mathrm{CDF}$ and that $h\colon J \to J$ is a coded function such that $h(j, m)$ is nonincreasing in j and nondecreasing in n. Assume also that for all $j \in J$ there exists $m \in J$

such that $h(j, m) > J$. Since g is nondecreasing, the function

$$(i, n) \mapsto h(g(i), g(n))$$

(for $i, n \in I$) is also nonincreasing in the first argument and nondecreasing in the second. Further, if $i \in I$ then $g(i) \in J$ hence there is $m \in J$ such that $h(g(i), m) > J$. Hence, as $g \circ f(m) \geq m$, $h(g(i), gf(m)) > J$. That is,

$$\forall i \in I \; \exists n \in I \; \big(h(g(i), g(n)) > J\big).$$

Let

$$k(i, n) = f \circ h(g(i), g(n)).$$

Then $\forall i \in I \; \exists n \in I \; (k(i, n) > I)$. Using CDF in I, there is a function l such that $k(i, l(i)) > I$ for all $i \in I$. But then

$$f(h(g(i), g \circ l(i))) > I,$$

so $h(g(i), g \circ l(i)) > J$. Fix $j \in J$. Then $j \leq g \circ f(j)$, so $h(j, m) \geq h(g \circ f(j), m)$ for all $m \in J$. In particular,

$$h(j, g \circ l \circ f(j)) \geq h(g \circ f(j), g \circ l \circ f(j)) > J.$$

Hence $g \circ l \circ f$ is a witness for h exceeding J; therefore $J \vDash \text{CDF}$. $\square$

The previous proposition will allow us to demonstrate that CDF is much weaker than strength.

Proposition 6.5.2. If I is strong, then $I \vDash \text{CDF}$.

Proof. Suppose $f(i, n)$ is a coded function satisfying the hypotheses of CDF, and that I

is strong. Take c such that

$$f(i,n) < c \Leftrightarrow f(i,n) < I$$

for all $i, n \in I$. Define

$$\begin{aligned} g(i) &= \min\{n \colon f(i,n) \geq c\} \\ &= \min\{n \colon f(i,n) > I\}. \end{aligned}$$

Then $f(i, g(i)) > I$ for all $i \in I$, and since f satisfies the hypotheses of CDF, $g``I \subseteq I$. $\square$

The previous two results immediately give the following.

Corollary 6.5.3. If I is coded by a strong cut, then $I \models \mathrm{CDF}$.

This result greatly generalises the observations of Example 6.4.4, and also demonstrates the weakness of CDF (and hence that of completeness in $\mathcal{V}/{\sim_I}$) relative to strength. Further, we suspect that CDF has exactly the same strength as CDF, although this may require extra assumptions on I.

Conjecture 6.5.4. Possibly under some weak additional assumptions (perhaps that I is closed under exponentiation), $I \models \mathrm{CDF}$ if and only $\mathcal{V}$ is I-complete.

Currently it is unknown whether the standard cut $\mathbb{N}$ satisfies CDF in general. This is of significance because in the standard realm, completeness is equivalent to either countable cofinality, or weak compactness and strong inaccessibility—hence we might expect that $\mathcal{V}/{\sim_I}$ is complete precisely when I is coded by $\mathbb{N}$ or I is strong. However, we expect the following to be true instead.

Conjecture 6.5.5. The standard cut $\mathbb{N}$ is not always a model of CDF.

In fact we suspect the much stronger statement:

Conjecture 6.5.6. If $I \models \mathrm{CDF} + \mathrm{SReg}$ then I is strong in M.

$$
\begin{tikzcd}
 & \mathrm{Str} \equiv \mathcal{V}\text{ crowded} \arrow[dl] \arrow[d, bend right] \arrow[dr] & \\
\mathrm{Reg} \arrow[d] & \mathrm{SReg} \wedge \mathrm{CDF} \arrow[u, bend right, "6.5.6"'] \arrow[dl] \arrow[dr] & \mathrm{Str}\text{-coded} \arrow[d, "6.5.3"] \\
\mathrm{SReg} & & \mathrm{CDF} \arrow[d, bend right, "6.4.14"'] \\
 & & \mathcal{V}\text{ complete} \arrow[u, bend right, "6.5.4"']
\end{tikzcd}
$$

Figure 6.1: Putting these principles in their place.

Notice that Conjecture 6.5.5 would follow from Conjecture 6.5.6, since $\mathbb{N}$ is always regular but there exist M in which $\mathbb{N}$ is not strong.

If Conjecture 6.5.5 were to fail, then we suspect that, at least, one of the following weaker statements would hold. Notice that the conjunction of these statements is precisely Conjecture 6.5.6.

Conjecture 6.5.7.

- If $I \models \mathrm{CDF} + \mathrm{SReg}$ then I is regular.
- If $I \models \mathrm{CDF} + \mathrm{Reg}$ then I is strong.

Our study of CDF may be useful in the reverse mathematics of analysis; in particular in the study of sequence properties such as completeness. Notice that all our arguments in this section would apply in second-order structures, since we assumed our cuts were closed under addition and multiplication. It is a simple task to translate the concepts to this context, and each result holds with only minor translations. See Figure 6.1 to see how the implications between the various principles we have introduced.

CHAPTER 7

CONCLUSIONS

7.1 In game theory

In Chapter 2 we introduced the two-sided theory Amphi-ZF. Despite a high level similarity with ZF, it has provided a useful set-theoretic foundation for our arguments, aiding all following game-theoretic discussion. In particular it has been useful for discussing issues of regularity; it is likely that Rieger–Bernays permutations will be of further use in this area, perhaps answering some questions of Chapter 5. One such application may be in determining the relative strength of the regularity axioms concerning minimal ATOs posed in Chapter 5.

Another potential application is determining whether an AZF-like theory with a free construction axiom (restricted, of course, to an appropriate class of functions) exists which will also admit the structure of a definable architecture. Equivalently, such a model would have no loops $x \in_{\mathrm{P}} x$, but would allow other kinds of set-formation. It may be possible to apply Rieger–Bernays permutations here (loops can easily be removed using this technique; preserving some form of restricted antifoundation is more difficult).

In chapter 3 we discussed the addition of a second, weaker order to posets as a generalisation of combinatorial games. The theory presented there should convince that such structures are worthy of study, although it is incomplete without some consideration of cases (in particular loopy games) where the relation $\triangleleft_{\mathrm{I}}$ is not antireflexive, and perhaps weakening other axioms, such as transitivity. Certainly this would be less interesting order-theoretically, but would be more accommodating to game theory. In particular we might hope such weakening would allow us to draw meaningful conclusions involving the greatest fixpoint order pairs of Honsell and Lenisa [47, 46].

A study of amphimorphisms in these general constructions may also be of interest, since they have a natural game-theoretic motivation—arguably more so than promorphisms. The case for amphi*functors* certainly needs to be explored in more depth. However the theory will never be complete with only amphimorphisms, since for example the value map fails to reflect the weak order.

7.2 In nonstandard topological set theory

In Chapter 6 we demonstrated that Malitz' own construction works in a nonstandard set theory, when ordinals and cardinals are replaced by distinguished cuts. Although it was previously known that ACA_0 implies the consistency of GPK, it was interesting to see the original construction—rather than those directly involving an antifoundation axiom—producing a compact model.

The more interesting part of that chapter was concerned with reversing these results, however. In particular the witnessing principles should be investigated further, both in nonstandard and standard contexts. Particularly interesting is the fact that we were often able to replace strength—equivalent to ACA_0—with a much weaker scheme, but one which is independent of the usual reverse-mathematical hierarchy

$$\Pi^1_1 - \mathrm{CA}_0 \Rightarrow \mathrm{ATR}_0 \Rightarrow \mathrm{ACA}_0 \Rightarrow \mathrm{WKL}_0 \Rightarrow \mathrm{RCA}_0$$

when we refuse to assume semiregularity (analogous to RCA_0) as a base theory. The exact strength of completeness needs to be determined, and we strongly suspect it is equivalent to CDF and the witnessing property.

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