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Tomáš Jakl

d-Frames as algebraic duals of bitopological spaces

Department of Applied Mathematics (Charles University)

and

School of Computer Science (University of Birmingham)

Supervisors of the doctoral thesis:	Prof. Achim Jung Prof. RNDr. Aleš Pultr, DrSc.
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Author: Tomáš Jakl

Department: Department of Applied Mathematics (Charles University) and School of Computer Science (University of Birmingham)

Supervisor: Prof. Achim Jung, School of Computer Science
Prof. RNDr. Aleš Pultr, DrSc., Department of Applied Mathematics

Abstract: Achim Jung and Drew Moshier developed a Stone-type duality theory for bitopological spaces, amongst others, as a practical tool for solving a particular problem in the theory of stably compact spaces. By doing so they discovered that the duality of bitopological spaces and their algebraic counterparts, called d-frames, covers several of the known dualities.

In this thesis we aim to take Jung's and Moshier's work as a starting point and fill in some of the missing aspects of the theory. In particular, we investigate basic categorical properties of d-frames, we give a Vietoris construction for d-frames which generalises the corresponding known Vietoris constructions for other categories, and we investigate the connection between bispaces and a paraconsistent logic and then develop a suitable (geometric) logic for d-frames.

Keywords: d-frames, bitopological spaces, free constructions, Stone duality, Vietoris construction

Věnováno paní učitelce Čechové.

Dedicated to Mrs teacher Čechová.

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Contents

1

Introduction

Achim Jung and Drew Moshier developed in [JM06] a Stone-type duality theory for bitopological spaces mostly as a practical tool for solving a particular problem in the theory of stably compact spaces. By doing so they discovered that the duality of bitopological spaces and their algebraic counterparts, called *d-frames*, covers several of the known dualities such as Stone duality, Priestley duality, and the duality of topological spaces and frames. Furthermore, they also recognised that bitopological spaces, or *bispaces* for short, permit a logical reading strikingly similar to the paraconsistent logic proposed by Nuel Belnap, designed as a logic suitable for computer reasoning.

In this thesis we aim to take Jung's and Moshier's work as a starting point and fill in some of the missing aspects of the theory. In particular,

1. we investigate basic categorical properties of d-frames. In particular, we show that they are closed under forming colimits and develop a technique of free constructions from generators and relations.

2. We give a Vietoris construction for d-frames and show that it generalises the corresponding known Vietoris constructions for the categories (some of them mentioned above) that can be naturally viewed as subcategories of our category.

3. We investigate the connection between bispaces and a paraconsistent logic and then develop a suitable (geometric) logic for d-frames based on those ideas.

The point of bitopological techniques is that very often topological spaces or other structures come naturally, but often not explicitly, equipped with two topologies. Specifying the two topologies and expressing some of the properties in bitopological terms often sheds some light on the problem that we are trying to solve. To demonstrate this

4. we give new bitopological proofs of some old results in the theory of stably compact spaces and also, as an application of the d-frame Vietoris construction, obtain new results concerning powerspace constructions for stably compact spaces.

Before we dive into the actual theory, let us recall some motivations behind the study of duality theories and their connections to logics.

1.1 On algebraic dualities

Marshall Stone established, in his influential series of papers [Sto36; Sto37a], that Boolean algebras are in a (dual) correspondence with what we now call Stone spaces. At the same time, propositional logic is known to be sound and complete with respect to Boolean algebras and so, as a result, Boolean algebras provide a bridge between propositional logic and its topological semantics:

$$\text{Stone spaces} \quad \longleftrightarrow \quad \text{Boolean algebras} \quad \longleftrightarrow \quad \text{propositional logic}$$

Under this interpretation, models of the logic correspond to the points of a space and formulas to the (clopen) sets of models where the formula holds. As we describe shortly, Stone's work was generalised in two important ways.

(1) By extending the classes of algebras and spaces: One direction of generalisations was taken by Stone [Sto37b], Priestley [Pri70] and Esakia [Esa85]. In Priestley's case, one replaced Boolean algebras by distributive lattices and obtained a topological semantics for positive logic:

$$\text{Priestley spaces} \quad \longleftrightarrow \quad \text{distributive lattices} \quad \longleftrightarrow \quad \text{positive logic}$$

Because Priestley spaces are zero-dimensional, they do not include many important spaces needed for computations, such as the unit interval of reals. This led to another important generalisation carried out, in stages, by Smyth [Smy92b], Jung-Sünderhauf [JS96] and Jung-Kegelmann-Moshier [Keg02; JKM99]

$$\text{stably compact spaces} \quad \longleftrightarrow \quad \text{strong proximity lattices} \quad \longleftrightarrow \quad \text{MLS}$$

where MLS stands for the multilingual sequence calculus.

Alternatively, one can give up finitarity of the algebras and extend the duality to all topological spaces. Probably the most prominent example of such generalisation is the dual adjunction between spaces and algebraic structures called *frames*, introduced by Ehresmann and Bénabou [Bén59; Ehr58] and developed by Isbell, Johnstone, Banaschewski and many others [Isb72; Joh82]. In the dualities mentioned above, the link between spaces and algebras is made by a restriction to an appropriate collection of open sets; frames constitute an immediate approach to space of places axiomatizing the lattice behaviour of *all* open sets. Such *point-free* spaces are more general and hence we have, instead of a duality, an adjunction.

$$\text{topological spaces} \quad \overset{adj.}{\longleftrightarrow} \quad \text{frames} \quad \longleftrightarrow \quad \text{geometric logic}$$

Geometric logic, in the diagram, is a logic with strong connections to theoretical computer science. This comes from Smyth's insight [Smy83] that open sets can be viewed as *observable properties* or, from the point of view of computability theory, semidecidable properties. These ideas motivated Abramsky [Abr87a; Abr87b] to build his work around the duality of spaces and frames where he systematically relates a topological semantics with a program logic.

(2) By strengthening the logic. The second approach to generalisations of Stone's work is by extending Stone duality so that it provides a topological semantics for a more expressive logic. Jónnson and Tarski [JT51], gave an example of such extension which in turn gives a topological semantics to modal logic (see also [BRV01]):

$$\text{descriptive general frames}^{1} \quad \longleftrightarrow \quad \text{modal Boolean algebras} \quad \longleftrightarrow \quad \text{modal logic}$$

This correspondence can be well explained in categorical terms; descriptive general frames are equivalently represented as coalgebras of *Vietoris functor* $\mathbb{V}$ and modal Boolean algebras as algebras of a functor $\mathbb{M}$ which, morally, is an algebraic counterpart to $\mathbb{V}$. This means that the duality between Stone spaces and Boolean algebras lifts to a duality of coalgebras and algebras over those categories [Abr05b; KKV04]:

$$\mathsf{Coalg}(\mathbb{V}) \quad \longleftrightarrow \quad \mathsf{Alg}(\mathbb{M}) \quad \longleftrightarrow \quad \text{modal logic}$$

In fact, the appropriate reincarnations of $\mathbb{V}$ and $\mathbb{M}$ exist also for the other dualities discussed in (1) and match modal extensions of the corresponding logics [VV14; BBH12].

Apart from applications in modal logics, Vietoris constructions found many applications also in the semantics of programming languages for modelling non-determinism [Abr87b; Plo76]. This comes from the fact that the Vietoris functor $\mathbb{V}$ provides a topological variant of the powerset functor [Vie22] and so a (continuous) step function $X \to \mathbb{V}(X)$ represents a non-deterministic choice of the next state.

1.2 On d-frames

It was recognised by Jung and Moshier that all the above mentioned dualities have a very natural bitopological description. They all embed into a larger duality between bitopological spaces and d-frames in such a way that this embedding reveals

[1] Note that descriptive general frames are a special kind of relational structures as opposed to frames from point-free topology which are algebraic structures. This confusing use of terms is only ad hoc; in this text we do not study the descriptive general frames.

a precise relationship between those dualities [JM06]. Furthermore, as was the case for frames, d-frames could be thought of as an algebraic realisation of the notion of bispace. This means that, when reasoning in d-frames, we can fully rely on our geometric intuition.

All the dualities mentioned in the previous section have something in common. The algebras, in the middle, provide a bridge between a logic and its topological semantics. This aspect of d-frames, although hinted at in [JM06], has not been formally explored until now. It turns out that bispaces model – via d-frames – most of the logic of bilattices[2]. Because, as we believe, bilattice logic does not fulfil the program outlined by Belnap, we propose a different logic based on geometric logic of frames and obtain a logic sound and complete with respect to d-frames. Diagrammatically, we have:

$$\text{bitopological spaces} \quad \overset{adj.}{\longleftrightarrow} \quad \text{d-frames} \quad \longleftrightarrow \quad \text{geometric d-frame logic}$$

As usual, the points of bispaces represent models of d-frame logic and the two topologies correspond to the observably true and observably false properties, respectively.

At the same time, we also explore modal extensions of d-frames, similar to those explained in (2) of the previous section. For that we define a Vietoris construction for bispaces and d-frames and show that the duality between bispaces and d-frames lifts to the duality between the corresponding coalgebras and algebras. A neat feature of our construction is that all the standard categorical properties of Johnstone's Vietoris construction for frames [Joh85] are also satisfied. Moreover, we also show that our d-frame Vietoris construction is a generalisation of all the other defined Vietoris constructions on the categories mentioned above.

In order to define a Vietoris construction for d-frames and prove basic categorical properties of it, we need to develop a lot of categorical machinery first. Namely, we propose a technique for constructing d-frames from generators and relations, in the manner of universal algebra. It turns out that our technique is fairly versatile and many of the free frame constructions can be easily adapted to the context of d-frames. Furthermore, our free construction turns out to be crucial in the proof of completeness of geometric d-frame logic.

It has to be mentioned that, even though our main motivations come from logics and semantics of programming languages, the main emphasis of this thesis is on the categorical development of the theory of d-frames.

Constructivity disclaimer. Point-free topology is often praised for its constructivity. In particular, it is often the case that a statement about topological spaces which relies on a choice principle, such as the Axiom of Choice, has a choice-free proof

[2] Bilattices are certain algebraic structures considered to model a Belnap-Dunn paraconsistent logic.

of the corresponding frame-theoretic reformulation. The same is expected to be the case for d-frames. It was, however, beyond capabilities of the author to verify to which extent the constructions and arguments in this text are constructive. We believe that most of the purely point-free arguments are entirely choice-free and even do not rely on the law of excluded middle.

Some exceptions to this rule are the construction of coproducts of d-frames and Lemma 4.2.5, which is essential for the d-frame Vietoris construction. The proofs of those two statements we present here rely on the law of excluded middle. The author believes that the problem might disappear by selecting the right notion of finiteness. Moreover, although we use ordinals classically, we only need their universal property and so they could probably be replaced by a constructive notion of ordinals such as Taylor's ordinals [Tay96; Tay99].

Related publications. Some of the results in this thesis also appear elsewhere. Namely, Chapter 3 is an extension of the results in the papers [JJP17] and [JJ17b], Chapter 4 contains results of [JJ17a] and, lastly, Chapter 6 is based on the insights presented in [JJP16].

1.3 Who is this text for?

Because of the span of the theory of d-frames and its connections to other disciplines, we believe that this text has potentially three further target audiences, apart from the point-free topologists.

1. *For duality theorists and logicians:* In Chapter 3 we show how the basic dualities embed into the duality of bispaces and d-frames and then in Chapter 4 we present a common generalisation of Vietoris constructions on those dualities. We also obtain new Jónnson-Tarski-like models of positive modal logic (Corollary 4.5.9).

2. *For paraconsistent logicians:* In Chapter 6 we show that d-frames are of interest for their connection with bilattices. Not only they interpret most of bilattice logic, they also provide more (possibly non-symmetric) models and new, very clean, bitopological semantics for bilattices.

3. *For domain theorist:* Chapter 5 includes a novel presentation of basics of the theory of stably compact spaces from the bitopological perspective. It also includes a new fact about Vietoris for stably compact frames.

Note that there are very few dependencies between the individual chapters. In fact, Chapter 2 establishes the basics of the theory of d-frames and then all the other chapters are, for the most part, independent of each other.

2

Bispaces and d-frames

In this chapter we give a survey of basic properties of bitopological spaces and their algebraic duals – d-frames. We prove that known dualities such as Stone duality or Priestley duality embed into the duality of compact regular bispaces and d-frames.

Only basic knowledge of category theory and frame/locale theory is assumed. Also, it is for the benefit of the reader to be aware of the dualities of Stone and Priestley, but, their in-depth knowledge is not necessary and parts of this chapter which rely on them can be skipped without any harm.

All results from this section except for Theorem 2.6.11 have been already known and can be found, mostly, in one of [JM06; Kli12; Nac65; Kel63; Gou13; Sal83].

2.1 Bitopological spaces

Bitopological spaces, or bispaces for short, were introduced by Kelly in 1963 [Kel63] and soon became a subject of study on their own. Our main motivation for studying bispaces comes from the fact that many known mathematical structures are naturally bitopological; although this often might not be mentioned explicitly. Obvious examples include partially ordered spaces such as real line, unit interval or Priestley spaces. The same way category theory is a neat organising tool for many mathematical structures, bitopological view also often offers a good language and organising principle for our proofs and results.

Apart from practical benefits, studying bispaces has also good philosophical reasons. They provide models of four-valued logics. This will be further explored in Chapter 6.

2.1.1 Definition. (X, τ_+, τ_-) is a bitopological space if (X, τ_+) and (X, τ_-) are topological spaces.

To demonstrate how common bispaces are we give some of the well known examples. First set of examples constitutes of situations when the two topologies can be obtained as a refinement of the existing topology.

2.1.2 Example (A).

1. The usual topology of the real line $\mathbb{R}$ has the topology of opens upsets τ_+ and the topology of open downsets τ_- as its refinements. In fact, the join $\tau_+ \vee \tau_-$[1] gives us the standard topology of $\mathbb{R}$ back. Similarly, we can decompose the topology of the unit interval $[0,1]$ into two.

2. *Priestley space* space is a structure $(X, \tau, \preceq)$ where (X, τ) is a compact space and $(X, \preceq)$ is a partially ordered set such that, whenever $x \not\preceq y$, then there exists a clopen upset $U \in \tau$ such that $x \in U \not\ni y$.

   The bispace (X, τ_+, τ_-) associated with $(X, \tau, \preceq)$ is obtained by setting

   $$\tau_+ = \{U \in \tau \mid {\uparrow} U = U\} \quad \text{and} \quad \tau_- = \{U \in \tau \mid {\downarrow} U = U\}.$$

3. The last construction generalises to *partially ordered topological spaces* (also called pospaces), i.e. the structures $(X, \tau, \preceq)$ where (X, τ) is a topological space and $(X, \preceq)$ is a partially ordered set with $\preceq$ closed in $X \times X$. In fact also $\mathbb{R}$ and $[0,1]$ are partially ordered spaces. Note that it is not true in general that the join topology of the two topologies gives τ back, unless more is assumed about $(X, \tau, \preceq)$.

Another, quite common situation, is when we have a (mono)topological space and its topology has a natural "mate" topology associated to it. Then, we can think of the second topology as if it was a topology of "complex open sets" – by analogy to complex numbers.

2.1.3 Example (B).

1. *Stably compact spaces* are topological spaces (X, τ) which are sober, compact, locally compact and *coherent*, i.e. that any intersection of two compact saturated subsets is compact again.

   The second topology τ^{d} is generated from the base $\{X \setminus K \mid K \text{ is compact saturated }\}$. Then, the bispace associated with (X, τ) is $(X, \tau, \tau^{\mathrm{d}})$.

2. Stone's *spectral spaces* are those stably compact spaces which have a basis of compact open subsets.

It is a well-known fact that stably compact spaces correspond exactly to compact partially ordered spaces and that spectral spaces correspond to Priestley spaces. We will see later that the Priestley/Stone duality for distributive lattices becomes much more transparent under bitopological lenses.

[1] Join topology of two topologies is computed as the smallest topology containing both.

The last list of examples constitutes of such mathematical structures that are not equipped with any topology explicitly but which can be assumed to have one and, moreover, again they are naturally bitopological.

2.1.4 Example (C)**.**

1. Any dcpo $(D, \sqsubseteq)$ can be equipped with two topologies, the Scott topology σ_D and weak lower topology ω_D, and the join of those two yields the famous Lawson topology.

2. Quasi-metric spaces, the generalisation of metric spaces by dropping the symmetry axiom (i.e. $d(x, y) = d(y, x)$), are also naturally bitopological. The two topologies are generated from the sets of open balls:

$$B_+(x, r) = \{y \in X \mid d(x, y) < r\} \quad \text{and} \quad B_-(x, r) = \{y \in X \mid d(y, x) < r\}$$

   In fact, a lot from the analogy between what metric spaces are for monotopological spaces transfers to quasi-metric spaces and bispaces. See for example, [Wil31; MR93; Gou13].

2.1.1 Separation axioms and other topological properties

The theory of bispaces often mirrors many topological concepts. In the following we list some of the most important definitions similar to those we know from the monotopological setting.

Recall that every topology defines a *specialisation order*, i.e. for a bispace (X, τ_+, τ_-) and for any $x, y \in X$ we have

$$x \leq_+ y \overset{\text{def}}{\equiv} \forall U_+ \in \tau_+ .\, x \in U_+ \implies y \in U_+ \quad \text{and}$$
$$x \leq_- y \overset{\text{def}}{\equiv} \forall U_- \in \tau_- .\, x \in U_- \implies y \in U_- .$$

Then, all τ_+-open sets are $\leq_+$-upwards closed and, similarly, τ_--open sets are $\leq_-$-upwards closed. Specialisation order is often overlooked in monotopological setting as it is trivial in more geometrical spaces, e.g. because (X, τ_+) is T_1 iff $(\leq_+) = (=)$. However, it has been well studied in the context of point-free topology or the theory of computation where the non-T_1 spaces are quite common. Furthermore, because those two specialisation orders do not have to interact in any way, we define the *associated pre-order* as the intersection

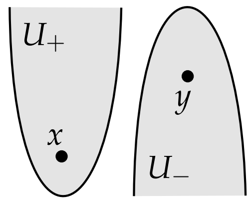
$$\leq \overset{\text{def}}{\equiv} \leq_+ \cap \geq_- .$$

Apart from its occurrence in the definitions, the associated pre-order is necessary for the development of our bitopological intuition. Under this order we can imagine τ_+-open sets as upsets and τ_--open sets as downsets.

Let us take a look at the bitopological versions of the separation axioms. We make use of the associated pre-order to define the bitopological variant of the T_2 axiom.

2.1.5 Definition. A bispace (X, τ_+, τ_-) is *order-separated* if

1. the associated pre-order $\leq$ is a partial order[2], and
2. $x \not\leq y$ implies that there is a pair of disjoint open sets $U_+ \in \tau_+$ and $U_- \in \tau_-$ such that $x \in U_+$ and $y \in U_-$.



Clearly, the bispace of real numbers and the bispaces arising from Priestley spaces are order-separated. In fact all examples we mentioned in the previous section except for partially ordered spaces, when equipped with two topologies, are order-separated. And, in order for this to be the case also for partially ordered spaces it is enough to assume compactness:

2.1.6 Lemma. *Let $(X, \tau, \preceq)$ be a partially ordered space and let (X, τ_+, τ_-) be the bispace of upper and lower opens (in the $\preceq$-order). Then,*

1. $\preceq \, = \, \leq_+ \, = \, \geq_-$ *and*
2. *if (X, τ) is compact, then (X, τ_+, τ_-) is order-separated.*

Proof. (1) Clearly $\preceq \, \subseteq \, \leq_+$ as $\tau_+ \subseteq \tau$. For the other direction, let $x \not\preceq y$. Then, $x \in (X \setminus {\downarrow} y) \not\ni y$ and we finish by realising that, for every $z \in X$, ${\downarrow} z$ (or ${\uparrow} z$) is closed.

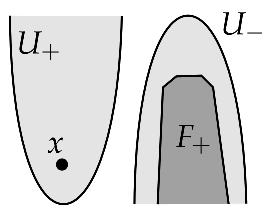
(2) from (1) we know that $\leq \, = \, \leq_+ \cap \geq_- \, = \, \preceq$. Let $x \not\preceq y$. By definition, $\preceq$ is closed in $(X, \tau) \times (X, \tau)$ and so there exist open sets $V_+, V_- \in \tau$ such that $(x, y) \in V_+ \times V_-$ and $(V_+ \times V_-) \cap (\preceq) = \emptyset$. Moreover, V_+ and V_- are disjoint as $=$ is a subset of $\preceq$.

Set $U_+ = X \setminus {\downarrow}(X \setminus V_+)$ and $U_- = X \setminus {\uparrow}(X \setminus V_-)$. Clearly $U_+ \subseteq V_+$ and $U_- \subseteq V_-$ and so $U_+ \cap U_- = \emptyset$. Finally U_+ and U_- are open because, for any closed subset $F \subseteq X$, ${\downarrow} F$ and ${\uparrow} F$ are closed (see Lemma 9.1.13 in [Gou13], for example). □

The next stronger notion after Hausdorffness is regularity.

2.1.7 Definition. A bispace (X, τ_+, τ_-) is *d-regular* if

1. Whenever $x \in V_+$ for some $V_+ \in \tau_+$, then there is an open set $U_+ \in \tau_+$ such that $x \in U_+ \subseteq \overline{U_+}^{\tau_-} \subseteq V_+$ where $\overline{U_+}^{\tau_-}$ is the τ_--closure of U_+.
2. and symmetrically for $y \in V_-$ for some $V_- \in \tau_-$.

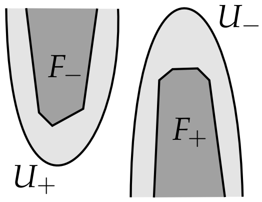


[2] This is equivalent to saying that, if $x \neq y$, then there exists a $U \in \tau_+ \cup \tau_-$ such that $x \in U \not\ni y$ or $x \notin U \ni y$. Notice the similarity with the monotopological: (X, τ_+) is T_0 iff $\leq_+$ is a partial order.

Similarly to the classical topology $x \in U_+ \subseteq \overline{U_+}^{\tau_-} \subseteq V_+$, for some τ_+-open U_+, iff there is a pair of disjoint opens U_+ and U_- which separates x and $F_+ \stackrel{\text{def}}{\equiv} X \setminus V_+$. However, we prefer to use the former definition because it exactly matches the corresponding point-free definition.

Next, complete regularity can be also defined for bispaces and because we do not need it for our investigations we proceed directly to normality.

2.1.8 Definition. A bispace (X, τ_+, τ_-) is *d-normal* if for any two disjoint sets F_+ and F_-, where F_+ is τ_+-closed and F_- is τ_--closed, there exist a pair of disjoint opens $U_+ \in \tau_+$ and $U_- \in \tau_-$ such that $F_- \subseteq U_+$ and $F_+ \subseteq U_-$.



2.1.9 Example. (A) Both the bispace of real numbers $\mathbb{R}$ and the bispace [0,1] are trivially order-separated, d-regular and d-normal. This is not true in general for bispaces arising from partially ordered spaces. However, it becomes true by assuming compactness. We will give a proof of this in later.

(B) Checking that the bispaces arising from spectral spaces and stably compact spaces are also order-separated, d-regular and d-normal is also true even though the proof is a bit more involved (see Chapter 5).

(C) Under reasonable conditions all the examples in 2.1.4 also yield bispaces which satisfy the separating conditions we just discussed.

These are all separation axioms that we are going to need. Next, we take a look at other topological notions such as a disconectedness property and compactness.

2.1.10 Definition. A bispace (X, τ_+, τ_-) is *d-zero-dimensional* if

1. Whenever $x \in V_+$ for some $V_+ \in \tau_+$, then there is a τ_+-open τ_--closed U_+ such that $x \in U_+ \subseteq V_+$.
2. and symmetrically for $y \in V_-$ for some $V_- \in \tau_-$.

As in monotopological spaces, zero-dimensionality is stronger property than regularity. Compare bitopological zero-dimensionality and regularity with the classical ones. The interplay of the two topologies is what makes many proofs in the theory of bitopological spaces more involved and often also quite beautiful.

2.1.11 Definition. A bispace (X, τ_+, τ_-) is *d-compact* if whenever $\bigcup_{i \in I} U_+^i \cup \bigcup_{j \in J} U_-^j = X$, for some $\{U_+^i\}_i \subseteq \tau_+$ and $\{U_-^j\}_j \subseteq \tau_-$, then there exist finite $F \subseteq_{\text{fin}} I$ and $G \subseteq_{\text{fin}} J$ such that $\bigcup_{i \in F} U_+^i \cup \bigcup_{j \in G} U_-^j = X$.

2.1.12 Example. As in the monotopological case, the unit interval [0,1] is d-compact. Also, Priestley spaces give d-compact and also d-zero-dimensional bispaces. Analogously, bispaces of stably compact spaces are always d-compact and those arising from spectral spaces are d-zero-dimensional. The last two cases will be explained in Chapter 5.

Following the analogues from the classical topology, all the definitions from this section interact the expected way. Let us denote order-separatedness by T_2 and the statement that "the associated pre-order $\leq_+ \cap \geq_-$ is a partial order" by T_0. Then we get the following diagram of implications:

$$\text{d-compact} + \mathrm{T}_2 \implies \text{d-normal} + \mathrm{T}_2 \implies \text{d-regular} + \mathrm{T}_0 \implies \mathrm{T}_2 \implies \mathrm{T}_0$$

$$\text{d-zero-dim.} + \mathrm{T}_0 \implies \text{d-regular} + \mathrm{T}_0$$

Figure 2.1: Implications of basic bitopological properties

The first part of the following proposition proves the second right-most implication in the diagram. All the other missing pieces will be addressed later by point-free techniques.

2.1.13 Proposition. *Let (X, τ_+, τ_-) be a bispace. Then,*

1. *if X is d-regular and T_0, then X is order-separated, and*
2. *if X is d-compact and order-separated, then X is d-regular.*

Proof. (1) Only the second condition of order-separated bispaces needs to be verified. Let $x \not\leq y$. By definition, this means that $x \not\leq_+ y$ or $x \not\geq_- y$. Assume the first. Then, there exists a $V_+ \in \tau_+$ such that $x \in V_+ \not\ni y$. By regularity there exists a $U_+ \in \tau_+$ such that $x \in U_+ \subseteq \overline{U_+}^{\tau_-} \subseteq V_+$. Set $U_- = X \setminus \overline{U_+}^{\tau_-}$. Clearly, $U_- \in \tau_-$ and $y \in U_-$. We have found disjoint U_+ and U_- such that $x \in U_+$ and $y \in U_-$. The case for $x \not\geq_- y$ is analogous.

(2) Let $x \in U_+$ for some $U_+ \in \tau_+$. This means that, for all $y \in X \setminus U_+$, $x \not\leq y$ as $\leq \subseteq \leq_+$ and every τ_+-open set is upwards closed w.r.t. $\leq_+$-order. From order-separatedness, there exists a pair of disjoint sets $V_+^y \in \tau_+$ and $V_-^y \in \tau_-$ such that $x \in V_+^y$ and $y \in V_-^y$. Then, $U_+ \cup (\bigcup_{y \in X \setminus U_+} V_-^y) = X$ and so, by compactness, there is a finite $F \subseteq_{\text{fin}} X \setminus U_+$ such that $U_+ \cup (\bigcup_{y \in F} V_-^y) = X$. Set $V_+ = \bigcap_{y \in F} V_+^y$ and $V_- = \bigcup_{y \in F} V_-^y$. Then, clearly, $x \in V_+ \subseteq (X \setminus V_-) \subseteq U_+$ and so $x \in V_+ \subseteq \overline{V_+}^{\tau_-} \subseteq U_+$. The case for $x \in U_-$ for some $U_- \in \tau_-$ is analogous. □

2.1.14 Remark. This is not an exhaustive list of separation axioms for bitopological spaces. For more axioms and their comparison consult [Kop95; Sal83; Sae71]. Note that in the older literature all the "d-" properties are usually prefixed with "pairwise", e.g. d-regular is called pairwise regular.

Convention. As we can see, because of the two sided nature of bispaces, proving properties about the τ_+-side usually mirrors to the τ_--side, and vice versa. For that reason we often omit the proof of either of the side without even mentioning it. Moreover, we introduce the symbol "$\pm$" to mean both "+" and "-". For example, saying that "$U_\pm$ have the property Q" means that "U_+ and U_- have the property Q", or "there exist $x_\pm \in U_\pm$" translates as "there exist $x_+ \in U_+$ and $x_- \in U_-$".

2.1.2 Bicontinuous maps

2.1.15 Definition. A map $f\colon (X, \tau_+, \tau_-) \to (Y, \sigma_+, \sigma_-)$ is *bicontinuous* if both maps $f\colon (X, \tau_+) \to (Y, \sigma_+)$ and $f\colon (X, \tau_-) \to (Y, \sigma_-)$ are continuous in the usual sense.

We denote the category of bitopological spaces and bicontinuous maps by **biTop**. In the classical topology, continuous maps are monotone with respect to the specialisation order and the analogue of this for bicontinuous maps is:

2.1.16 Observation. *Bicontinuous functions are monotone with respect to the associated order $\leq\, = \,\leq_+ \cap \geq_-$.*

The $\leq$ order is a natural part of the structure of bispaces. Not only it is essential for the formulations of weaker separation axioms (T_0 and order-separatedness, in our case), it is also the largest order such that τ_+-opens are upwards closed and τ_--opens are downwards closed.

2.1.17 Example.
1. Negation seen as a function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is not continuous when $\mathbb{R}$ is equipped with the usual Euclid topology. On the other hand, negation is bicontinuous when seen as a function $\mathbb{R} \times \mathbb{R}^{op} \to \mathbb{R}$ where $(X, \tau_+, \tau_-)^{op}$ is the bispace (X, τ_-, τ_+) and $\mathbb{R}$ is the bispace of real numbers, i.e. the bispace of upper and lower-opens.
2. Lower upper semi-continuous functions correspond exactly to the bicontinuous functions into the bispace $\mathbb{R}$.

2.2 Comparison with compact partially ordered spaces

In this section we prove that the categories of compact partially ordered spaces and d-compact order-separated bispaces are isomorphic. Lemma 2.1.6 already guarantees that the assignment

$$\mathbf{bi}\colon (X, \tau, \preceq) \quad \longmapsto \quad (X, \tau_+, \tau_-)$$

where $\tau_+ = \{U \in \tau \mid U = \uparrow U\}$ and $\tau_- = \{U \in \tau \mid U = \downarrow U\}$, always maps a compact partially ordered space to an order-separated bispace. Moreover, since $(X, \tau,)$ is compact, (X, τ_+, τ_-) is d-compact.

There is also a mapping back (the $^{-1}$ is only suggestive for now)

$$\mathbf{bi}^{-1}\colon (X, \tau_+, \tau_-) \quad \longmapsto \quad (X,\ \tau_+ \vee \tau_-,\ (\leq_+ \cap \geq_-))$$

If (X, τ_+, τ_-) is d-compact and order-separated, we know, by Alexander Subbase Lemma, that $(X, \tau_+ \vee \tau_-)$ is compact. To show that $\leq\ =\ \leq_+ \cap \geq_-$ is closed in $X \times X$ take any $x \not\leq y$. From order-separatedness of (X, τ_+, τ_-) we know that there exist disjoint $U_+ \in \tau_+$ and $U_- \in \tau_-$ such that $x \in U_+$ and $y \in U_-$. Moreover, $(U_+ \times U_-) \cap (\leq) = \emptyset$ as otherwise, if $a \leq b$ for some $(a, b) \in U_+ \times U_-$, since $\leq\ \subseteq\ \leq_+$ and U_+ is $\leq_+$-upwards closed, it would follow that $b \in U_+$.

In fact, going from compact partially ordered spaces to bispaces and back gives the same space. Before we show that, we prove the following technical lemma:

2.2.1 Lemma. *Let $(X, \tau, \preceq)$ be a compact partially ordered space and let $\mathcal{S}_+, \mathcal{S}_- \subseteq \tau$ be such that if $x \not\preceq y$, for some $x, y \in X$, then there exist $U_+ \in \mathcal{S}_+$ and $U_- \in \mathcal{S}_-$ such that $x \in U_+$, $y \in U_-$ and $U_+ \cap U_- = \emptyset$. Then, $\mathcal{S}_+ \cup \mathcal{S}_-$ is a subbase of τ.*

Proof. Let $U \in \tau$ and let $x \in U$. For every $y \notin U$, either $x \not\leq y$ or $y \not\leq x$. In the first case, there exist disjoint $U_+^y \in \mathcal{S}_+$ and $U_-^y \in \mathcal{S}_-$ such that $x \in U_+^y$ and $y \in U_-^y$. If, on the other hand, $y \not\leq x$, then we would have disjoint $U_+^y \in \mathcal{S}_+$ and $U_-^y \in \mathcal{S}_-$ but the position of x and y swaps, i.e. $y \in U_+^y$ and $x \in U_-^y$. By M_+ denote the set of all the y's such that $y \not\leq x$ and by M_- denote the set of y's where $x \not\leq y$.

Then, $\bigcup_{y \in M_+} U_+^y \cup \bigcup_{y \in M_-} U_-^y$ covers $X \setminus U$ and because (X, τ) is compact, $X \setminus U$ is also compact. Therefore, there exist finite $F_+ \subseteq_{\mathrm{fin}} M_+$ and $F_- \subseteq_{\mathrm{fin}} M_-$ such that $X \setminus U \subseteq \bigcup_{y \in F_+} U_+^y \cup \bigcup_{y \in F_-} U_-^y$. Clearly then, $x \in \bigcap_{y \in F_+} U_-^y \cap \bigcap_{y \in F_-} U_+^y \subseteq U$. Therefore, topology τ is generated from the subbase $\mathcal{S}_+ \cup \mathcal{S}_-$. □

2.2.2 Proposition (Proposition 2.11 in [JM06]). *Let $(X, \tau, \preceq)$ be a compact partially ordered space. Then,*

$$\mathbf{bi}^{-1}(\mathbf{bi}(X, \tau, \preceq)) = (X, \tau, \preceq).$$

Proof. Set $\mathcal{S}_+$ and $\mathcal{S}_-$ to be the sets of upper or lower-opens, respectively, that is $\mathcal{S}_+ = \tau_+$ and $\mathcal{S}_- = \tau_-$. Since $\mathbf{bi}(X, \tau, \preceq)$ is order-separated, $\mathcal{S}_+$ and $\mathcal{S}_-$ satisfy the condition of Lemma 2.2.1 and so the topology of $\mathbf{bi}^{-1}(\mathbf{bi}(X, \tau, \preceq))$ is equal to τ. Moreover, also the orders agree as, by (1) of Lemma 2.1.6, $\preceq\ =\ \leq_+\ =\ \geq_-$ and so $\preceq\ =\ \leq_+ \cap \geq_-$. □

2.2.3 Lemma. *Let (X, τ_+, τ_-) be order-separated. Then, $\leq_+ = \geq_-$.*

Proof. Let $x \not\leq_+ y$. Then, $x \not\leq y$ and so they can be separated by $U_+ \in \tau_+$ and $U_- \in \tau_-$. Therefore, $x \notin U_- \ni y$ and so $x \not\geq_- y$. The reverse direction is analogous. □

The proof of the following Proposition is inspired by Theorem 9.1.32 in [Gou13], although the true bitopological nature of the argument was not highlighted there.

2.2.4 Proposition. *Let (X, τ_+, τ_-) be a d-compact order-separated bispace. Then,*

$$\mathbf{bi}(\mathbf{bi}^{-1}(X, \tau_+, \tau_-)) = (X, \tau_+, \tau_-).$$

Proof. Let us denote $\mathbf{bi}(\mathbf{bi}^{-1}(X, \tau_+, \tau_-))$ by (X, τ_+^2, τ_-^2) with the corresponding specialisation orders $\leq_+^2$ and $\leq_-^2$. We know, by Lemma 2.2.3, that the associated order $\leq$ is equal to $\leq_+$ (resp. $\geq_-$) and so $\mathbf{bi}^{-1}(X, \tau_+, \tau_-) = (X,\ \tau_+ \vee \tau_-,\ \leq)$. Moreover, by Lemma 2.1.6, $\leq\, = \,\leq_+^2\, = \,\geq_-^2$. This will help us to prove that $\tau_+ = \tau_+^2$ and $\tau_- = \tau_-^2$.

Clearly, $\tau_+ \subseteq \tau_+^2$ and $\tau_- \subseteq \tau_-^2$. To prove the other direction, let $V_+ \in \tau_+^2$. Then, for all $y \in X \setminus V_+$, $x \not\leq y$ as V_+ is an upper set in $\leq$-order. Because (X, τ_+, τ_-) is order-separated, there exist $U_+^y \in \tau_+$ and $U_-^y \in \tau_-$ such that $x \in U_+^y$ and $y \in U_-^y$. Since (X, τ_+^2, τ_-^2) is d-compact and $\tau_- \subseteq \tau_-^2$, the covering of X by V_+ together with all U_-^y's has a finite subcover. Let F be a finite subset of $X \setminus V_+$ such that V_+ and $\bigcup_{y \in F} U_-^y$ cover X. It is easy to see that $x \in \bigcap_{y \in F} U_+^y \subseteq (X \setminus \bigcup_{y \in F} U_-^y) \subseteq V_+$. Hence, τ_+ generates τ_+^2. The inclusion $\tau_- \supseteq \tau_-^2$ is proved similarly. □

It is easy to check that if $f\colon (X, \tau_+^X, \tau_-^X) \to (Y, \tau_+^Y, \tau_-^Y)$ is bicontinuous, then the map $f\colon (X, \tau_+^X \vee \tau_-^X) \to (Y, \tau_+^Y \vee \tau_-^Y)$ is continuous. Moreover, by Observation 2.1.16,

$$f\colon \mathbf{bi}^{-1}(X, \tau_+^X, \tau_-^X) \to \mathbf{bi}^{-1}(Y, \tau_+^Y, \tau_-^Y)$$

is also monotone. The reverse direction is trivially true as τ_+ and τ_- are coarser than $\tau_+ \vee \tau_-$. We obtain:

2.2.5 Theorem.
The category of compact partially ordered spaces and monotone continuous maps is isomorphic to the category of d-compact order-separated bispaces and bicontinuous maps.

This isomorphisms of categories restrict to the isomorphism of the category of zero-dimensional compact partially ordered spaces and the category of d-zero-dimensional d-compact order-separated bispaces. Priestley spaces are exactly the zero-dimensional compact partially ordered spaces and because order-separatedness can be replaced by T_0 under d-zero-dimensionality, we have:

2.2.6 Corollary. *The category of Priestley spaces is isomorphic to the category of d-compact d-zero-dimensional T_0 bispaces.*

2.3 d-Frames

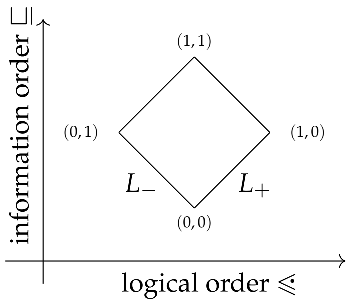
Classical point-free topology studies *frames*, i.e. complete lattices $L = (L, \bigvee, \wedge, 0, 1)$ satisfying the following distributivity law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\}$$

for all $A \subseteq L$ and $b \in L$. The frame associated with a space (X, τ) is the lattice of its open sets ordered by set-inclusion, i.e. it is the frame $\Omega(X, \tau) = (\tau, \bigcup, \cap, \emptyset, X)$. Frames are often thought of as the algebraic duals of (mono)topological spaces. In this section we define d-frames to play the role of the point-free algebraic duals of bispaces.

It is no surprise that, because bispaces consist of two topologies, we will have two frames L_+ and L_- as the core of the structure of d-frames. Before we give a full definition of d-frames, let us take a look at some consequences of this. It is a general fact that a product of two lattices[3] (or frames, in our case) $L_+ \times L_-$ introduces two orders which are somehow orthogonal to each other. Namely, for any $\alpha = (\alpha_+, \alpha_-), \beta = (\beta_+, \beta_-) \in L_+ \times L_-$ define

- *Information order:* $\alpha \sqsubseteq \beta$ if $\alpha_+ \leq \beta_+$ and $\alpha_- \leq \beta_-$, and
- *Logical order:* $\alpha \leqslant \beta$ if $\alpha_+ \leq \beta_+$ and $\alpha_- \geq \beta_-$.



2.3.1 Observation. *Let L_+, L_- be two lattices. Then, $(L_+ \times L_-, \sqcap, \sqcup, \bot, \top)$ is a lattice in $\sqsubseteq$-order and $(L_+ \times L_-, \curlywedge, \curlyvee, \mathrm{ff}, \mathrm{tt})$ is a lattice in $\leqslant$-order where, for any $\alpha, \beta \in L_+ \times L_-$,*

$$\alpha \sqcap \beta = (\alpha_+ \wedge \beta_+, \alpha_- \wedge \beta_-), \qquad \alpha \sqcup \beta = (\alpha_+ \vee \beta_+, \alpha_- \vee \beta_-),$$
$$\alpha \curlywedge \beta = (\alpha_+ \wedge \beta_+, \alpha_- \vee \beta_-), \qquad \alpha \curlyvee \beta = (\alpha_+ \vee \beta_+, \alpha_- \wedge \beta_-)$$

and

$$\bot = (0,0), \quad \top = (1,1), \quad \mathrm{ff} = (0,1), \quad \mathrm{tt} = (1,0).$$

Now we take a look at the remaining parts of the structure of d-frames. As will soon become clear, most of topological properties of bispaces can in fact be expressed in terms of two relations between the two frame components. Take a bispace

[3] In this text we always assume that lattices are distributive and bounded.

(X, τ_+, τ_-) and a pair of opens $(U_+, U_-) \in \tau_+ \times \tau_-$. We will say that (U_+, U_-) is *total* if $U_+ \cup U_- = X$ and that it is *consistent* if $U_+ \cap U_- = \emptyset$[4]. Consistency and totality can, in fact, be accurately axiomatised in a purely point-free fashion and in terms of the logical an information orders.

Notice that total pairs are upwards closed in the information order. Concretely, if (U_+, U_-) is total and $(V_+, V_-) \in \tau_+ \times \tau_-$ is such that $(U_+, U_-) \sqsubseteq (V_+, V_-)$, then (V_+, V_-) is also total. Being consistent, on the other hand, is downwards closed in the $\sqsubseteq$-order. Also, totality is closed under $\curlywedge$ and $\curlyvee$, e.g. let (U_+, U_-) and (V_+, V_-) be total, then $(U_+, U_-) \curlywedge (V_+, V_-) = (U_+ \cap V_+, U_- \cup V_-)$ is also total; indeed

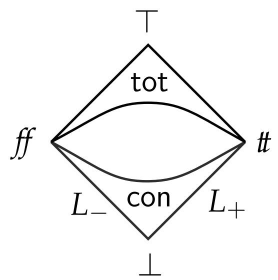
$$\begin{aligned}(U_+ \cap V_+) \cup (U_- \cup V_-) &= (U_+ \cup (U_- \cup V_-)) \cap (V_+ \cup (U_- \cup V_-)) \\ &\supseteq (U_+ \cup U_-) \cap (V_+ \cup V_-) = X \cap X = X.\end{aligned}$$

The same is true about consistent pairs of opens. We are now ready to define d-frames.

2.3.2 Definition. A *d-frame* is a quadruple $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ where L_+, L_- are frames, $\mathsf{con} \subseteq L_+ \times L_-$ is the *consistency* relation and $\mathsf{tot} \subseteq L_+ \times L_-$ is the *totality* relation such that

- (in the information order:)

$$\begin{aligned}&(\mathsf{tot}\text{-}\uparrow) && \alpha \sqsubseteq \beta \text{ and } \alpha \in \mathsf{tot} \implies \beta \in \mathsf{tot},\\ &(\mathsf{con}\text{-}\downarrow) && \alpha \sqsubseteq \beta \text{ and } \beta \in \mathsf{con} \implies \alpha \in \mathsf{con},\\ &(\mathsf{con}\text{-}{\textstyle\bigsqcup}^{\uparrow}) && \sqsubseteq\text{-directed } A \subseteq^{\uparrow} \mathsf{con} \implies {\textstyle\bigsqcup}^{\uparrow} A \in \mathsf{con}\end{aligned}$$



- (in the logical order:)

$$\begin{aligned}&(\mathsf{tot}\text{-}\curlyvee, \curlywedge) && \alpha, \beta \in \mathsf{tot} \implies \alpha \curlyvee \beta, \alpha \curlywedge \beta \in \mathsf{tot},\\ & && tt, ff \in \mathsf{tot},\\ &(\mathsf{con}\text{-}\curlyvee, \curlywedge) && \alpha, \beta \in \mathsf{con} \implies \alpha \curlyvee \beta, \alpha \curlywedge \beta \in \mathsf{con},\\ & && tt, ff \in \mathsf{con},\end{aligned}$$

- (interplay between con and tot:)

$$\begin{aligned}&(\mathsf{con}\text{-}\mathsf{tot}) && \alpha \in \mathsf{con} \text{ and } \beta \in \mathsf{tot} \text{ such that}\\ & && (\alpha_+ = \beta_+ \text{ or } \alpha_- = \beta_-) \implies \alpha \sqsubseteq \beta\end{aligned}$$

In order theoretic terms, the information-order axioms say that con is Scott-closed and tot is upwards closed in $(L_+ \times L_-, \sqsubseteq)$, and the logical-order axioms say that con and tot are sublattices of $(L_+ \times L_-, \curlywedge, \curlyvee, ff, tt)$.

2.3.3 Example. Let (X, τ_+, τ_-) be a bispace. Define $\Omega_d(X, \tau_+, \tau_-)$ to be the structure $(\tau_+, \tau_-, \mathsf{con}_X, \mathsf{tot}_X)$ where

$$(U_+, U_-) \in \mathsf{con}_X \overset{\text{def}}{\equiv} U_+ \cap U_- = \emptyset \quad \text{and} \quad (U_+, U_-) \in \mathsf{tot}_X \overset{\text{def}}{\equiv} U_+ \cup U_- = X.$$

[4]The reason behind this naming convention will become clear in Chapter 6.

Then, in the product $\tau_+ \times \tau_-$, $\mathit{tt} = (X, \emptyset)$, $\mathit{ff} = (\emptyset, X)$, $\bot = (\emptyset, \emptyset)$ and $\top = (X, X)$. From the discussion above, we already know that the axioms (tot-$\uparrow$), (con-$\downarrow$), (tot-⩛, ⩚) and (con-⩛, ⩚) hold for $\Omega_d(X, \tau_+, \tau_-)$. It is also immediate that (con-$\bigsqcup^{\uparrow}$) is satisfied because a $\sqsubseteq$-directed collection of disjoint sets has disjoint unions.

Finally, to justify (con-tot), let $(U_+, U_-) \in \mathsf{con}_X$ and $(V_+, V_-) \in \mathsf{tot}_X$ be such that $U_+ = V_+$. We see that $x \in U_-$ implies $x \notin U_+ = V_+$ and by totality $x \in V_-$; therefore, $U_- \subseteq V_-$. Not only the axiom (con-tot) is there to express an interplay between con and tot, it is absolutely crucial for any development of the theory of d-frames. Namely, it makes sure that $(\lhd) \subseteq (\leq)$ where the relation $\lhd$ is the well-inside relation which we define below. On the other hand, (con-tot) it makes calculations involving quotients and free constructions of d-frames more involved, as we will see in Chapter 3.

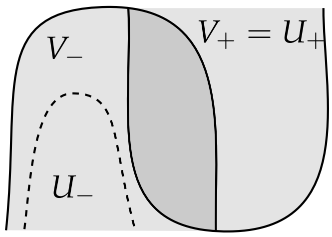


Figure 2.2: The axiom (con-tot) geometrically

2.3.4 Example. To give at least one explicit example consider the one element bispace $\mathbb{1} = (\{\star\}, \{\emptyset, \{\star\}\}, \{\emptyset, \{\star\}\})$. Both frame components of $\Omega_d(\mathbb{1})$ are isomorphic to the two element frame $\mathbf{2} = \{0 < 1\}$. The con and tot relations are defined simply as $(a, b) \in \mathsf{con}$ iff $a = 0$ or $b = 0$, and $(a, b) \in \mathsf{tot}$ iff $a = 1$ and $b = 1$. Notice that this is the only way the consistency and totality relations can be defined on the product $\mathbf{2} \times \mathbf{2}$ to satisfy all the d-frame axioms.

2.3.5 Example. In fact the procedure form the previous example can be used to endow a product of any two frames $L_+ \times L_-$ with the trivial consistency and totality relations $\mathsf{con}_{\mathrm{triv}}$ and $\mathsf{tot}_{\mathrm{triv}}$, where $(a, b) \in \mathsf{con}_{\mathrm{triv}}$ iff $a = 0$ or $b = 0$, and $(a, b) \in \mathsf{tot}_{\mathrm{triv}}$ iff $a = 1$ or $b = 1$.

2.3.1 Separation axioms and other topological properties

An essential notion of the theory of frames is the well-inside relation. Because d-frame consist of two frames, we have two well-inside relations; one for each frame component. Let $a, x \in L_+$. We say that a is *well-inside* x (and write $a \lhd_+ x$) if there exists a $c \in L_-$ such that $(a, c) \in \mathsf{con}$ and $(x, c) \in \mathsf{tot}$. Define $a \lhd_- x$ for L_- symmetrically. Later in the text we drop the subscript $\pm$ whenever it does not lead to a confusion.

By (con-tot), we can see that $a \lhd_\pm x$ implies $a \leq_\pm x$ where $\leq_\pm$ is the order of $L_\pm$. For a bispace (X, τ_+, τ_-), $U \lhd_+ V$ is true precisely whenever $\overline{U}^{\tau_-} \subseteq V$.

Following the same strategy as in classical point-free topology, we can directly translate most of the bitopological notions we introduced in Section 2.1.1 to d-frames:

2.3.6 Definition. We say that a d-frame $\mathcal{L}$ is

- *d-regular* if, $\forall x \in L_\pm$, $x = \bigvee\{a \in L_\pm \mid a \lhd_\pm x\}$,
- *d-normal* if, $\forall (x,y) \in \mathsf{tot}$, $\exists (u,v) \in \mathsf{con}$ such that $(x,v) \in \mathsf{tot}$ and $(u,y) \in \mathsf{tot}$,
- *d-zero-dimensional* if, $\forall x \in L_\pm$, $x = \bigvee\{a \in L_\pm \mid a \lhd_\pm a \leq x\}$, and
- *d-compact* if whenever $\bigsqcup A \in \mathsf{tot}$, then $\exists A' \subseteq_{\mathrm{fin}} A$ s.t. $\bigsqcup A' \in \mathsf{tot}$.

Equivalently, compactness can be restated as

- $\mathcal{L}$ is d-compact iff whenever $\bigsqcup^{\uparrow} A \in \mathsf{tot}$ then $A \cap \mathsf{tot} \neq \emptyset$.

It is also immediate to convince ourselves that these definitions mirror exactly the definitions introduced in Section 2.1.1.

2.3.7 Observation. *Let (X, τ_+, τ_-) be a bispace. Then, (X, τ_+, τ_-) is d-regular, d-normal, d-zero-dimensional or d-compact if and only if $\Omega_d(X, \tau_+, \tau_-)$ is.*

As in classical point-free topology where T_2 can not be directly translated, order-separatedness does not have a counterpart for d-frames either. To overcome this, frame theory introduces a separation property called *subfit* which then, plays the role of T_1 axiom for frames; in fact it is weaker than that. The d-frame variant is as follows.

2.3.8 Definition. We say that a d-frame $\mathcal{L}$ is *d-subfit*, if whenever $x \not\leq y$ in L_+, then there exists a $z \in L_-$ such that $(x,z) \in \mathsf{tot}$ and $(y,z) \notin \mathsf{tot}$; and the same, symmetrically, for $x \not\leq y$ in L_-.

Again, as in classical point-free topology, we have the following relations between the definitions:

2.3.9 Proposition. *Let $\mathcal{L}$ be a d-frame. Then,*

1. *if $\mathcal{L}$ is d-regular, then it is d-subfit,*
2. *if $\mathcal{L}$ is d-normal and d-subfit, then it is d-regular,*
3. *if $\mathcal{L}$ is d-compact and d-regular, then it is d-normal.*

Proof. (1) If $x \not\leq y$ in L_+, then there exists a $a \lhd_+ x$ and $a \not\leq y$. Then, by definition, $(x,c) \in \mathsf{tot}$ for some $c \in L_-$ such that $(a,c) \in \mathsf{con}$. On the other hand, $(y,c) \notin \mathsf{tot}$ as this would imply $a \leq y$ by (con-tot).

(2) Let $x \in L_+$ and let us denote $y = \bigvee\{a \mid a \lhd_+ x\}$. Clearly, $y \leq x$. To prove the other direction, let $(x,z) \in \mathsf{tot}$. Since $\mathcal{L}$ d-normal, there is some $(u,v) \in \mathsf{tot}$ such

that $(x,v) \in \mathsf{tot}$ and $(u,z) \in \mathsf{tot}$. By definition, $u \lhd_+ x$ and $u \leq y$. Since $(u,z) \in \mathsf{tot}$ and since tot is upwards closed, also $(y,z) \in \mathsf{tot}$. Therefore, by subfitness, $x \leq y$.

(3) Let $(x,y) \in \mathsf{tot}$. By regularity $x = \bigvee\{a \mid a \lhd_+ x\}$. The set $\{a \mid a \lhd_+ x\}$ is, in fact, directed. Indeed, let $a \lhd_+ x$ and $a' \lhd_+ x$ and let c and c' be the witnesses, i.e. $(a,c) \in \mathsf{con}$, $(x,c) \in \mathsf{tot}$ and $(a',c') \in \mathsf{con}$, $(x,c') \in \mathsf{tot}$. From (con-$\dot\vee$) and (tot-$\dot\wedge$) we have that $(a \vee a', c \wedge c') \in \mathsf{con}$ and $(x, c \wedge c') \in \mathsf{tot}$. Therefore $(a \vee a') \lhd_+ x$.

Finally, because $\mathcal{L}$ is d-compact, $(a,y) \in \mathsf{tot}$ for some $a \lhd_+ x$ (as witnessed by c). We are done because $(x,c) \in \mathsf{tot}$, $(a,y) \in \mathsf{tot}$ and $(a,c) \in \mathsf{con}$. □

Also, d-zero-dimensionality trivially implies d-regularity. To sum up, we have the diagram of implications as shown in Figure 2.3. In comparison with Figure 2.1, T_0 is not present because it makes no sense for point-free spaces. Also, because there is no point-free analogue of order-separatedness, in the left-most implication in Figure 2.3 we needed to assume regularity.

$$\text{d-compact} + \text{d-regular} \Longrightarrow \text{d-normal} + \text{d-subfit} \Longrightarrow \text{d-regular} \Longrightarrow \text{d-subfit}$$

$$\text{d-zero-dim.} \Longrightarrow \text{d-regular}$$

Figure 2.3: Implications of basic d-frame properties

The missing implications to prove in Figure 2.1 are that d-compact + T_2 implies d-normal and that d-normal + T_2 implies d-regular. However, these immediately follow from Proposition 2.1.13, Proposition 2.3.9 and the following Lemma

2.3.10 Lemma. *If (X, τ_+, τ_-) is order-separated, then $\Omega_d(X, \tau_+, \tau_-)$ is d-subfit.*

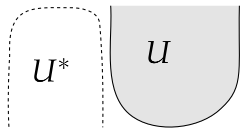
Proof. Let $U, V \in \tau_+$ such that $U \nsubseteq V$. Pick an $x \in U \setminus V$. The set $\uparrow x = \{y \mid x \leq_+ y\}$ is τ_--closed (this is easy to see, we will give a proof of this in Lemma 4.1.29). Since U and V are τ_+-opens, they are upwards closed in $\leq_+$-order and so, $(U, X \setminus \uparrow x) \in \mathsf{tot}_X$ and $(V, X \setminus \uparrow x) \notin \mathsf{tot}_X$. □

2.3.2 Pseudocomplements

From the structure of d-frames one can define an operator $(-)^*$ which assigns to each element $x \in L_\pm$, the largest element in the other frame (i.e. $x^* \in L_\mp$) which is consistent with x. Then, x^* is called the *pseudocomplement* of x.

To show that x^* is well-defined, for any $x \in L_+$, notice that the set $\{b \mid (a,b) \in \mathsf{con}\}$ is directed (by (con-$\dot\vee$, $\dot\wedge$)) and so, by (con-$\bigsqcup^\uparrow$), $(a, \bigvee^\uparrow\{b \mid (a,b) \in \mathsf{con}\}) \in \mathsf{con}$. Set x^* to be the join $\bigvee^\uparrow\{b \mid (a,b) \in \mathsf{con}\}$. To define $(-)^*$ for elements of L_- proceed symmetrically.

2.3.11 Example. For d-frames arising from bispaces, U^* (for some $U \in \tau_+$) is computed as the τ_--interior of $X \setminus U$. In other words, it is the largest τ_--open disjoint with U.



The formula which defines $\lhd_\pm$ becomes much simpler with pseudocomplements:

$$a \lhd_+ x \quad \text{iff} \quad (x, a^*) \in \mathsf{tot} \quad \text{and} \quad b \lhd_- y \quad \text{iff} \quad (b^*, y) \in \mathsf{tot}.$$

Next, we summarise basic properties of $\lhd_\pm$, or simply just $\lhd$.

2.3.12 Lemma (inspired by 5.2.1 in [PP12])**.** *For all $a, b, x, y \in L_\pm$:*

1. $a \lhd b \implies a \leq b$
2. $0 \lhd a \lhd 1$
3. $x \leq a \lhd b \leq y \implies x \lhd y$
4. $a \lhd b$ *and* $x \lhd y \implies (a \wedge x) \lhd (b \wedge y)$ *and* $(a \vee x) \lhd (b \vee y)$.

Proof. (1) follows from (con-tot). (2) is true because we always have that $(a, 1)$ resp. $(1, a) \in \mathsf{tot}$ and $(a, 0)$ resp. $(0, a) \in \mathsf{con}$. (3) follows because of (con-↓) and (tot-↑). For (4), by (con-$\curlyvee$, $\curlywedge$) and (tot-$\curlyvee$, $\curlywedge$), $(b, a^*), (y, x^*) \in \mathsf{tot}$ implies that $(b \vee y, a^* \wedge x^*) \in \mathsf{tot}$ and $(a, a^*), (x, x^*) \in \mathsf{con}$ implies that $(a \vee x, a^* \wedge x^*) \in \mathsf{con}$. Therefore, $(a \vee x) \lhd (b \vee y)$. $(a \wedge x) \lhd (b \wedge y)$ is proved similarly. □

2.3.13 Lemma. *Let $\mathcal{L}$ be a d-frame and $a, c \in L_\pm$, $\{a_i\}_i \subseteq L_\pm$ and $c \in L_\mp$.*

1. $a \leq a^{**}$
2. $(\bigvee_i a_i)^* = \bigwedge_i a_i^*$

(Interactions of $(-)^*$ with the rest of the structure:)

3. $a \lhd c \implies c^* \lhd a^*$
4. $(a, a^*) \in \mathsf{tot}$ *(resp.* $(a^*, a) \in \mathsf{tot}$*)* $\implies a = a^{**}$
5. $(a, b) \in \mathsf{con}$ *(resp.* $(b, a) \in \mathsf{con}$*) iff* $a \leq b^*$ *iff* $b \leq a^*$.

Proof. For $(a, b) \in L_+ \times L_-$, (5) follows from the definition of pseudocomplements, i.e.

$$a \leq b^* \quad \text{iff} \quad (a, b) \in \mathsf{con} \quad \text{iff} \quad b \leq a^*.$$

Therefore, pseudocomplement maps can be seen as two adjoint monotone maps $f \overset{\text{def}}{\equiv} (-)^* \colon L_- \to L_+^{\text{op}}$ and $g \overset{\text{def}}{\equiv} (-)^* \colon L_+^{\text{op}} \to L_-$. Because g is the right adjoint, it preserves meets in L_+^{op}, i.e. it transforms L_+-joins into L_--meets, and, similarly, f

also transforms L_--joins into L_+-meets. This gives (2) and (1) follows from $f \cdot g \leq \mathrm{id}$ and $\mathrm{id} \leq g \cdot f$.

(3) Since $a \lhd_+ c$ is equivalent to $(c, a^*) \in \mathsf{tot}$ it follows, by (1) and (tot-↑), that $(c^{**}, a^*) \in \mathsf{tot}$ and so $c^* \lhd_- a^*$. (4) is a consequence of (con-tot) as $(a, a^*) \in \mathsf{tot}$ implies $a^{**} \leq a$ and (1) implies the other inequality. □

Convention. In the rest of this text we will be using the properties of pseudocomplements that we derived in this subsection automatically and without mentioning.

2.3.3 The dual adjunction $\Omega_d \dashv \Sigma_d$

In order to define the category of d-frames we need to define what morphisms of this category are. Let $\mathcal{L}$ and $\mathcal{M}$ be d-frames, we say that a pair of frame homomorphisms[5] $h = (h_+, h_-) \colon \mathcal{L} \to \mathcal{M}$, where $h_+ \colon L_+ \to M_+$ and $h_- \colon L_- \to M_-$, is a *d-frame homomorphism* if, for all $\alpha \in L_+ \times L_-$,

$$\alpha \in \mathsf{con}_{\mathcal{L}} \implies h(\alpha) = (h_+(\alpha_+), h_-(\alpha_-)) \in \mathsf{con}_{\mathcal{M}}$$
$$\alpha \in \mathsf{tot}_{\mathcal{L}} \implies h(\alpha) \in \mathsf{tot}_{\mathcal{M}}$$

These conditions can be also written more economically as

$$h[\mathsf{con}_{\mathcal{L}}] \subseteq \mathsf{con}_{\mathcal{M}} \quad \text{and} \quad h[\mathsf{tot}_{\mathcal{L}}] \subseteq \mathsf{tot}_{\mathcal{M}}.$$

2.3.14 Observation. *For any bicontinuous map $f \colon (X, \tau_+^X, \tau_-^X) \to (Y, \tau_+^Y, \tau_-^Y)$. The pair of frame homomorphism $\Omega_d(f) = (\Omega(f_+), \Omega(f_-))$ where*

$$\Omega(f_+) \colon \quad \tau_+^Y \longrightarrow \tau_+^X \qquad\qquad \Omega(f_-) \colon \quad \tau_-^Y \longrightarrow \tau_-^X$$
$$U_+ \longmapsto f^{-1}[U_+] \qquad\qquad U_- \longmapsto f^{-1}[U_-]$$

is a d-frame homomorphism $\Omega_d(f) \colon \Omega_d(Y) \to \Omega_d(X)$ where $\Omega_d(X)$ and $\Omega_d(Y)$ are defined as in Example 2.3.3.

In fact, if we denote the category of d-frames and d-frame homomorphisms by **d-Frm**, then we get that $\Omega_d \colon \mathbf{biTop} \to \mathbf{d\text{-}Frm}$ is a well-defined contravariant functor. We will show that Ω_d has a right adjoint Σ_d.

To define the spectrum $\Sigma_d(\mathcal{L})$ for a d-frame $\mathcal{L}$, we need to establish what the points of this bispace should be. In the category of bispaces, the set of points of a bispace X is in a bijection with the set of bicontinuous maps $\mathbb{1} \to X$, where $\mathbb{1}$ is the one-point bispace. If we instantiate X with $\Sigma_d(\mathcal{L})$ we obtain that the set of points of $\mathcal{L}$ should be the set $\mathbf{biTop}(\mathbb{1}, \Sigma_d(\mathcal{L}))$ and for Σ_d to be the right adjoint to Ω_d, that

[5] Frame homomorphism is any map between two frames that distributes over all joins and all finite meets, e.g. it needs to preserve 0 and 1.

means that this should be in a bijection with **d-Frm**$(\mathcal{L}, \mathbf{2}{\times}\mathbf{2})$, as $\Omega_d(\mathbb{1}) \cong \mathbf{2}{\times}\mathbf{2}$ (see Exercise 2.3.4).

We define $\Sigma_d(\mathcal{L})$ to be the bispace $(\mathbf{d\text{-}Frm}(\mathcal{L}, \mathbf{2}{\times}\mathbf{2}), \Sigma_+[L_+], \Sigma_-[L_-])$ where, for every $a \in L_+$ and $b \in L_-$,

$$\Sigma_+(a) = \{p\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \mid p_+(a) = 1\} \quad \text{and} \quad \Sigma_-(b) = \{p\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \mid p_-(b) = 1\}.$$

It is not difficult to check that

$$\Sigma_\pm(a \wedge b) = \Sigma_\pm(a) \cap \Sigma_\pm(b) \quad \text{and} \quad \Sigma_\pm(\textstyle\bigvee_i a_i) = \bigcup_i \Sigma_\pm(a_i). \tag{2.3.1}$$

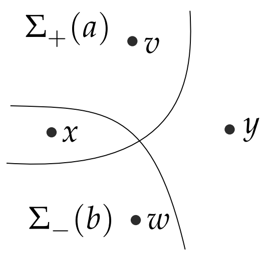
Therefore, $\Sigma_\pm[L_\pm] = \{\Sigma_\pm(x) \mid x \in L_\pm\}$ do define topologies. For simplicity, we will often refer to the set of points and to the bispace itself by the same name, i.e. by $\Sigma_d(\mathcal{L})$. We will often refer to elements of $\Sigma_d(\mathcal{L})$ as *d-points* (or simply *points*) of $\mathcal{L}$.

To extend Σ_d to a functor we need to define how it acts on morphisms. For every d-frame homomorphism $h\colon \mathcal{L} \to \mathcal{M}$, set

$$\begin{array}{rccc} \Sigma_d(h)\colon & \Sigma_d(\mathcal{M}) & \longrightarrow & \Sigma_d(\mathcal{L}) \\ & p\colon \mathcal{M} \to \mathbf{2}{\times}\mathbf{2} & \longmapsto & p \circ h\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \end{array}$$

Proving that $\Sigma_d(h)$ is bicontinuous is done by the standard argument; by checking that $h_\pm^{-1}[\Sigma_\pm(a)] = \Sigma_\pm(h_\pm(a))$. We obtain a functor $\Sigma_d\colon$ **d-Frm** $\to$ **biTop**.

2.3.15 Example.
- Since there is only one d-frame homomorphism $\mathbf{2}{\times}\mathbf{2} \to \mathbf{2}{\times}\mathbf{2}$, the bispace $\Sigma_d(\mathbf{2}{\times}\mathbf{2})$ has only one point and so it is bihomeomorphic to the space $\mathbb{1} = (\{\star\}, \tau, \tau)$ where $\tau = \{\emptyset, \{\star\}\}$.
- Consider the d-frame $\mathbf{3}{\times}\mathbf{3}$ defined as $(\mathbf{3}_+, \mathbf{3}_-, \mathrm{con}_{\mathrm{triv}}, \mathrm{tot}_{\mathrm{triv}})$ where the consistency and totality relations are defined as in Example 2.3.5 and $\mathbf{3}_+ = \{0 < a < 1\}$ and $\mathbf{3}_- = \{0 < b < 1\}$. A consequence of the triviality of the relations is that all pairs of frame homomorphisms $p_+\colon \mathbf{3}_+ \to \mathbf{2}$ and $p_-\colon \mathbf{3}_- \to \mathbf{2}$ define a point $(p_+, p_-)\colon \mathbf{3}{\times}\mathbf{3} \to \mathbf{2}{\times}\mathbf{2}$. We see that $\Sigma_d(\mathbf{3}{\times}\mathbf{3})$ has four points as shown in the figure below.



Furthermore, adding (a, b) to con removes the point x in the spectrum and adding (a, b) to tot removes y. This is a consequence of the general fact that making relations con and tot bigger corresponds to quotienting the d-frame (see Chapter 3) which, in the spectrum, is the same as taking a subspace.

2.3.3.1 An alternative description of $\Sigma_d(\mathcal{L})$

The fact that frame homomorphisms $p\colon L \to \mathbf{2}$ are in a bijective correspondence with completely prime filters $P \subseteq L$ offers an alternative presentation of $\Sigma_d(\mathcal{L})$. For d-points of $\mathcal{L}$ we only consider those pairs of frame homomorphisms which together constitute a d-frame homomorphism $\mathcal{L} \to \mathbf{2}{\times}\mathbf{2}$. To restrict the pairs of completely prime filters (P_+, P_-) accordingly, we require, for any $\alpha \in L_+{\times}L_-$

$$\text{(dp-con)} \qquad \alpha \in \mathsf{con}_{\mathcal{L}} \implies \alpha_+ \notin P_+ \text{ or } \alpha_- \notin P_-$$

$$\text{(dp-tot)} \qquad \alpha \in \mathsf{tot}_{\mathcal{L}} \implies \alpha_+ \in P_+ \text{ or } \alpha_- \in P_-$$

2.3.16 Observation. *A pair of frame homomorphisms $(p_+, p_-)\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2}$ is a d-frame homomorphism iff (P_+, P_-) satisfies (dp-*con*) and (dp-*tot*) where $P_\pm$ equals $\{x \mid p_\pm(x) = 1\}$.*

Then, the formula for the open sets of $\Sigma_d(\mathcal{L})$ can be translated to

$$\Sigma_+(a) = \{(P_+, P_-) \mid a \in P_+\} \quad \text{and} \quad \Sigma_-(b) = \{(P_+, P_-) \mid b \in P_-\}$$

and the action on morphisms is then computed as

$$\Sigma_d(h)\colon \quad (P_+, P_-) \quad \longmapsto \quad (h_+^{-1}[P_+],\ h_-^{-1}[P_-])$$

2.3.3.2 The adjunction

Following the classical point-free topology, we will show that Ω_d is the left adjoint of Σ_d. First, for any bispace X, define the unit of adjunction

$$\eta^X\colon\ X \longrightarrow \Sigma_d(\Omega_d(X))$$

to be the map $x \mapsto (\mathcal{U}_+(x), \mathcal{U}_-(x))$ where $\mathcal{U}_\pm(x) = \{U \in \tau_\pm \mid x \in U\}$. Standard argument shows that $(\mathcal{U}_+(x), \mathcal{U}_-(x))$ is a pair of completely prime filters. Observe also that this pair satisfies (dp-con) and (dp-tot). Moreover, we have:

2.3.17 Lemma.

1. *η^X is bicontinuous, for every bispace X.*
2. *η is a natural transformation $Id \Longrightarrow \Sigma_d\Omega_d$.*

Proof. (1) Let $U \in \tau_\pm$. Then, $(\eta^X_\pm)^{-1}[\Sigma_+(U)] = \{x \mid \mathcal{U}_\pm(x) \in \Sigma_\pm(U)\} = \{x \mid x \in U\} = U$ which is $\tau_\pm$-open. (2) Let $f\colon X \to Y$ be bicontinuous. By definitions, $\Sigma_d\Omega_d(f)(\eta^X(x))$ is equal to

$$\Sigma_d\Omega_d(f)(\mathcal{U}_+(x), \mathcal{U}_-(x)) = ((\Omega_d(f))_+^{-1}[\mathcal{U}_+(x)],\ (\Omega_d(f))_-^{-1}[\mathcal{U}_-(x)]).$$

Since $(\Omega_d(f))_\pm^{-1}(U)$ is equal to $\{V \in \tau_\pm^Y \mid f^{-1}[V] = U\}$, then

$$(\Omega_d(f))_\pm^{-1}[\mathcal{U}_\pm(x)] = \{V \mid f^{-1}[V] \in \mathcal{U}_\pm(x)\} = \mathcal{U}_\pm(f(x)).$$

Consequently, $\Sigma_d\Omega_d(f)(\eta^X(x)) = (\mathcal{U}_+(f(x)), \mathcal{U}_-(f(x)))$. □

Next, for any d-frame $\mathcal{L}$, define the (co)unit of adjunction

$$\varepsilon^{\mathcal{L}}\colon\ \mathcal{L} \longrightarrow \Omega_d(\Sigma_d(\mathcal{L}))$$

to be the mappings $\alpha \in L_+ \times L_- \mapsto (\Sigma_+(\alpha_+), \Sigma_-(\alpha_-))$. From (2.3.1) it is immediate that $\varepsilon_\pm$ are onto frame homomorphisms. It is easy to check that ε is a d-frame homomorphism.

2.3.18 Lemma. *ε is a natural transformation $Id \Longrightarrow \Omega_d\Sigma_d$.*

Proof. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a d-frame homomorphism. Then, for an $\alpha \in L_+ \times L_-$,

$$\Omega_d\Sigma_d(h)(\varepsilon^{\mathcal{L}}(\alpha)) = (\Sigma_d(h)^{-1}[\Sigma_+(\alpha_+)],\ \Sigma_d(h)^{-1}[\Sigma_-(\alpha_-)])$$

and $\Sigma_d(h)^{-1}[\Sigma_\pm(\alpha_\pm)] = \{(P_+, P_-) \mid \Sigma_d(h)(P_\pm) \in \Sigma_\pm(\alpha_p m)\} = \{(P_+, P_-) \mid \alpha_\pm \in h_\pm^{-1}[P_\pm]\}$. Therefore, $\Omega_d\Sigma_d(h)(\varepsilon^{\mathcal{L}}(\alpha)) = \varepsilon^{\mathcal{L}}(h(\alpha))$. □

2.3.19 Proposition ($\Omega_d \dashv \Sigma_d$)**.** *Ω_d and Σ_d constitute a dual adjunction with Ω_d to the left, Σ_d to the right and units η and ε.*

Proof. What is left to prove is that the triangle identities for the adjunction hold. First, we check that

$$\Sigma_d(\varepsilon^{\mathcal{L}}) \cdot \eta^{\Sigma_d\mathcal{L}}\colon \quad \Sigma_d(\mathcal{L}) \to \Sigma_d(\Omega_d(\Sigma_d(\mathcal{L}))) \to \Sigma_d(\mathcal{L})$$

composes to the identity. Let $p \in \Sigma_d(\mathcal{L})$. Let us compute an explicit formula for $\eta^{\Sigma_d\mathcal{L}}(p)\colon \Omega_d\Sigma_d(\mathcal{L}) \to \mathbf{2}\times\mathbf{2}$:

$$\eta^{\Sigma_d\mathcal{L}}(p) = (\mathcal{U}_+(p),\ \mathcal{U}_-(p)) = (\{\Sigma_+(a) \mid p_+(a) = 1\},\ \{\Sigma_-(b) \mid p_-(b) = 1\})$$

or equivalently, when represented as a d-frame homomorphism, it is the function $(\Sigma_+(a), \Sigma_-(b)) \mapsto (p_+(a), p_-(b))$. Then, $\Sigma_d(\varepsilon^{\mathcal{L}})(\eta^{\Sigma_d\mathcal{L}}(p)) = \eta^{\Sigma_d\mathcal{L}}(p) \circ \varepsilon^{\mathcal{L}}$ is the function

$$\alpha \mapsto (\Sigma_+(\alpha_+), \Sigma_-(\alpha_-)) \mapsto (p_+(\alpha_+), p_-(\alpha_-)) = p(\alpha).$$

Next, we compute that the following composition gives an identity

$$\Omega_d(\eta^X) \cdot \varepsilon^{\Omega_d X}\colon \quad \Omega_d(X) \to \Omega_d(\Sigma_d(\Omega_d(X))) \to \Omega_d(X)$$

Again, we give an explicit formula for $\Omega_d(\eta^X)$. Let $U \in \tau_\pm$, then $\Omega_d(\eta^X)_\pm(\Sigma_\pm(U))$ is equal to

$$\eta^{-1}[\Sigma_\pm(U)] = \{x \mid \eta(x) \in \Sigma_\pm(U)\} = \{x \mid U \in \mathcal{U}_\pm(x)\} = \{x \mid x \in U\} = U.$$

Finally, $\Omega_d(\eta^X)(\varepsilon^{\Omega_d X}(U_+, U_-)) = \Omega_d(\eta^X)(\Sigma_+(U_+), \Sigma_-(U_-)) = (U_+, U_-)$. □

2.3.3.3 Spatiality and sobriety

It is a remarkable feature of Stone-type adjunctions that they can be restricted to the largest (non-trivial) subcategories where the adjunction is an equivalence. In case of our adjunction between bispaces and d-frames, we have a subcategory of d-sober bispaces and spatial d-frames which are dually equivalent.

Being d-sober then means that all information about the bispace is already present in its d-frame of open sets. On the other hand, spatial d-frames have "enough points" and they can be obtained form an actual bispace. Moreover, they have a nice algebraic characterisation (Fact 2.3.21).

2.3.20 Fact (Theorem 4.1 in [JM06])**.** *Let X be a bispace. Then, the following are equivalent:*

1. *X is* d-sober, *i.e. $X \cong \Sigma_d(\mathcal{L})$ for some d-frame $\mathcal{L}$.*
2. *η^X is a bijection.*

2.3.21 Fact (Theorem 5.1 in [JM06])**.** *Let $\mathcal{L}$ be a d-frame. Then, the following are equivalent:*

1. *$\mathcal{L}$ is* spatial, *i.e. $\mathcal{L} \cong \Omega_d(X)$ for some bispace X.*
2. *$\varepsilon^{\mathcal{L}}$ is injective and reflects* con *and* tot.
3. *$\mathcal{L}$ satisfies the following conditions*

$$\begin{aligned}
&(s_\pm) && x \not\leq y \text{ in } L_\pm \implies \\
& && \exists p\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \text{ s.t. } p_\pm(x) = 1 \text{ and } p_\pm(y) = 0 \\
&(s\text{-con}) && \alpha \notin \mathsf{con} \implies \exists p\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \text{ s.t. } p(\alpha) = \top \\
&(s\text{-tot}) && \alpha \notin \mathsf{tot} \implies \exists p\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \text{ s.t. } p(\alpha) = \bot
\end{aligned}$$

2.3.22 Example. Let X be the bispace obtained from $\Sigma_d(\mathbf{3}{\times}\mathbf{3})$ (defined in Example 2.3.15) by removing the point v. Then, $\Omega_d(X) \cong \mathbf{3}{\times}\mathbf{3}$ and so X is not d-sober. This is in contrast with the classical topology where only infinite spaces are non-sober.

On the other hand, the bispace of reals equipped with the upper and lower topologies $(\mathbb{R}, \tau_l, \tau_u)$ is d-sober, even though neither $(\mathbb{R}, \tau_u)$ nor $(\mathbb{R}, \tau_l)$ is a sober space.

2.4 The duality of the compact regular

In this section we show that the duality of d-sober bispaces and spatial d-frames restrict further to the duality of d-compact order-separated bispaces and d-compact d-regular d-frames. This duality is the core of our investigations. Not only many other known dualities embed into it, as we show in Sections 2.5, 2.6 and 2.7, later, in Chapter 5, we also show that a well studied and important duality studied in the context of domain theory is equivalent to this duality.

As we are used to by now, many phenomena from frame theory carry to the context of d-frames. In the classical duality theory are all T_2 spaces sober and here our bitopological variant of T_2 implies d-sobriety.

2.4.1 Fact (Theorem 4.13 in [JM06]). *Order-separated bispaces are d-sober.*

Recall that d-compact order-separated bispaces are d-regular and that every bispace is d-compact and d-regular if and only if its Ω_d-image is. Then, Fact 2.4.1 gives us that d-compact order-separated bispaces are dually equivalent to the category of *spatial* d-compact d-regular d-frames.

In what follows we show that d-compact d-regular d-frames are actually already spatial. Before we do that we prove a few auxiliary lemmas.

2.4.2 Lemma. *For any d-compact d-frame $\mathcal{L}$ and any $y \in L_-$. The set $\mathcal{S}_+ = \{x \mid (x,y) \in \mathrm{tot}\}$ is a* Scott-open filter*, i.e. it is a filter such that anytime $\bigvee^{\uparrow}_{i\in I} b_i \in \mathcal{S}_+$, then $b_i \in \mathcal{S}_+$ for some $i \in I$.*

Analogously, for any $x \in L_+$ and $\mathcal{S}_- = \{y \mid (x,y) \in \mathrm{tot}\}$.

Proof. By (tot-$\curlyvee$) and (tot-$\uparrow$) we see that $\mathcal{S}_+$ is a filter. Next, if $\bigvee^{\uparrow}_{i\in I} b_i \in \mathcal{S}_+$, that means that $(x, \bigvee^{\uparrow}_{i\in I} b_i) \in \mathrm{tot}$. Since $\mathcal{L}$ is d-compact, there is an i such that $(x, b_i) \in \mathrm{tot}$. □

2.4.3 Lemma. *Let $\mathcal{L}$ be a d-compact d-regular d-frame. Let $P_+ \subseteq L_+$ be a completely prime filter. Then, (P_+, P_-) is a d-point where $P_- = \{y \mid \exists x \in L_+ \setminus P_+.\ (x,y) \in \mathrm{tot}\}$.*

Proof. First, notice that because P_+ is completely prime, the element $p = \bigvee(L_+ \setminus P_+)$ is not in P_+. Therefore, by (tot-$\uparrow$), $P_- = \{y \mid (p,y) \in \mathrm{tot}\}$. We check that P_- is a completely prime filter. Because $\mathcal{L}$ is d-compact, it is immediate that P_- is a Scott-open filter.

To see why it is a prime filter, let $y \vee z \in P_-$. By definition, $(p, y \vee z) \in \mathrm{tot}$. By compactness and regularity of $\mathcal{L}$, there must be some $a \lhd y$ and $b \lhd z$ such that $(p, a \vee b) \in \mathrm{tot}$. Then, by (con-tot) and Lemma 2.3.13, $a^* \wedge b^* \leq p$. As P_+ is a filter, it cannot be that both a^* and $b^* \in P_+$ as otherwise $p \in P_+$. Without loss of generality assume that $a^* \notin P_+$. Then, $a^* \leq p$ and because $(a^*, y) \in \mathrm{tot}$, $(p,y) \in \mathrm{tot}$ and so $y \in P_-$.

Next, we check that the pair (P_+, P_-) is a d-point. (dp-tot) by definition, if $\alpha \in \mathrm{tot}$ and $\alpha_+ \notin P_+$, then $\alpha_- \in P_-$. To check (dp-con), let $\alpha \in \mathrm{con}$ such that $\alpha_+ \in P_+$. If it were that $\alpha_- \in P_-$, that would mean that $(p, \alpha_-) \in \mathrm{tot}$ and so, by (con-tot), $\alpha_+ \leq p$ would imply that $p \in P_+$, a contradiction. □

Again, following the classical example we have:

2.4.4 Proposition. *d-Compact d-regular d-frames are spatial.*

Proof. We check the conditions of Fact 2.3.21. For (s-tot), let $\alpha \notin \mathrm{tot}$. This means that $\alpha_+ \notin \mathcal{S}_+$ where $\mathcal{S}_+ = \{x \mid (x, \alpha_-) \in \mathrm{tot}\}$ is a Scott-open filter (Lemma 2.4.2). By Zorn's Lemma, $\mathcal{S}_+$ can be extended to a completely prime filter P_+ such that $\alpha_+ \notin P_+ \supseteq \mathcal{S}_+$ (see, for example, Proposition 6.2 in [PP12]). By Lemma 2.4.3, we have a d-point (P_+, P_-). Now, assume that $\alpha_- \in P_-$. Then, by the construction of P_-, there would be an $x \in L_+ \setminus P_+$ such that $(x, \alpha_-) \in \mathrm{tot}$. Which would mean, by (con-tot), that $\alpha_+ \leq x$ and so $x \in P_+$, a contradiction.

For (s$_\pm$) let $a \not\leq b$ in L_+. By regularity, there exists a $c \lhd a$ and $c \not\leq b$. By (con-tot), also $(b, c^*) \notin \mathrm{tot}$. Then, by (s-tot) we know that there is a d-point (P_+, P_-) such that $b \notin P_+$ and $c^* \notin P_-$. However, $(a, c^*) \in \mathrm{tot}$ and (P_+, P_-) is a d-point and so, $a \in P_+$.

Finally, to check (s-con), let $\alpha \notin \mathrm{con}$. By regularity α_- is the join of the directed set $\{w \mid w \lhd \alpha_-\}$ and so, by (con-$\bigsqcup^\uparrow$), there must be a $w \lhd \alpha_-$ such that $(\alpha_+, w) \notin \mathrm{con}$. Then, $\alpha_+ \not\leq w^*$ and, by (s$_\pm$), there exists a d-point (P_+, P_-) such that $\alpha_+ \in P_+ \not\ni w^*$. As $(w^*, \alpha_-) \in \mathrm{tot}$ it must be that $\alpha_- \in P_-$. $\square$

2.4.5 Theorem.
The categories of d-compact order-separated bispaces and d-compact d-regular d-frames are dually equivalent.

Convention. Because a d-compact bispace is d-regular and T_0 iff it is order-separated, to mirror the situation in d-frames, we will refer to the d-compact order-separated bispaces as d-compact d-regular bispaces, i.e. we will implicitly assume T_0 without mentioning it.

With this convention, the theorem above proves the dual equivalence of the category d-compact d-regular bispaces **biKReg** and the category of d-compact d-regular d-frames **d-KReg**.

2.5 Embedding a frame duality

Recall that the category of compact regular (T_0) spaces **KRegSp** is dually equivalent to the category of compact regular frames **KRegFrm**. The functors witnessing this duality are $\Omega\colon \mathbf{KRegSp} \rightleftarrows \mathbf{KRegFrm} :\Sigma$, where $\Omega(X, \tau) = \tau$ and $\Sigma(L)$ is the space of all completely prime filters.

This duality can be fully embedded into the duality of d-compact d-regular bispaces and d-frames. Namely, there are two functors

$$I\colon \mathbf{KRegSp} \to \mathbf{biKReg} \quad \text{and} \quad J\colon \mathbf{KRegFrm} \to \mathbf{d\text{-}KReg}$$

defined as $I(X,\tau) = (X,\tau,\tau)$ and $J(L) = L^{\bowtie}$ where the consistency and totality relations for the d-frame $L^{\bowtie} = (L, L, \mathsf{con}_L, \mathsf{tot}_L)$ are defined as follows

$$(a,b) \in \mathsf{con}_L \overset{\text{def}}{\equiv} a \wedge b = 0 \quad \text{and} \quad (a,b) \in \mathsf{tot}_L \overset{\text{def}}{\equiv} a \vee b = 1$$

On morphisms they are define as expected, namely $I(f) = f$ and $J(h) = (h,h)$. It is immediate that I and J are well-defined faithful functors. It is also clear that I is full and we can show that the same is true for J.

2.5.1 Lemma. *J is a full embedding.*

Proof. The last thing to check is that if a pair of frame homomorphisms $h_\pm \colon L \to M$ constitutes a d-frame homomorphism $(h_+, h_-)\colon L^{\bowtie} \to M^{\bowtie}$, then $h_+ = h_-$. Let $b \in L$ and let a be well-inside b in L, i.e. there is a $c \in L$ such that $a \wedge c = 0$ and $b \vee c = 1$. Then, because (h_+, h_-) is a d-frame homomorphism

$$h_+(a) \wedge h_-(c) = 0 \quad \text{and} \quad h_+(b) \vee h_-(c) = 1$$

and, because h_- is a frame homomorphism

$$h_-(a) \wedge h_-(c) = 0 \quad \text{and} \quad h_-(b) \vee h_-(c) = 1.$$

Because, M is a distributive lattice, $x \wedge h_-(c) = 0$ with $x \vee h_-(c) = 1$ has a unique solution, therefore $h_+(a) = h_-(a)$. And, because $h_\pm$ are frame homomorphisms and b is a join of all elements that are well-inside it, $h_+(b) = h_-(b)$. □

As a consequence, since $\mathbf{2}^{\bowtie} = \mathbf{2} \times \mathbf{2}$, the d-points of $L^{\bowtie}$ are in a bijection with the points of L. Hence:

2.5.2 Proposition. *The duality of compact regular spaces and frames* embeds *into the duality of d-compact d-regular bispaces and d-frames:*

$$\begin{array}{ccc}
\mathbf{KRegSp}^{op} & \overset{\cong}{\longleftrightarrow} & \mathbf{KRegFrm} \\
{\scriptstyle I^{op}} \big\downarrow & & \big\downarrow {\scriptstyle J} \\
\mathbf{biKReg}^{op} & \overset{\cong}{\longleftrightarrow} & \mathbf{d\text{-}KReg}
\end{array}$$

(This means that $\Sigma_d \circ J \cong I \circ \Sigma$, or equivalently $\Omega_d \circ I \cong J \circ \Omega$.)

2.6 Embedding the Priestley duality

Recall that Priestley duality is the dual equivalence of the category of Priestley spaces (Example 2.1.2) and continuous monotone maps **Pries** and the category of distributive lattices and lattice homomorphisms **DLat**. The equivalence is witnessed by the pair of functors $\mathsf{Clp}_{\preceq}\colon \mathbf{Pries} \rightleftarrows \mathbf{DLat} : \mathsf{spec}_{\preceq}$ where $\mathsf{Clp}_{\preceq}(X, \tau, \preceq)$ is the lattice of clopen upsets and $\mathsf{spec}_{\preceq}(D)$ is the space of prime filters $\mathsf{PFilt}(D)$ ordered by set-inclusion, with the topology generated by the sets

$$\Phi_{+}^{D}(a) = \{P \mid a \in P\} \quad \text{and} \quad \Phi_{-}^{D}(a) = \{P \mid a \notin P\}, \tag{2.6.1}$$

for all $a \in D$.

Recall also that **Pries** is isomorphic to the category of *Priestley bispaces* **biPries**, i.e. d-compact d-zero-dimensional T_0 bispaces (Corollary 2.2.6). We have the following diagram of categories

$$\begin{array}{ccc} \mathbf{Pries}^{\mathrm{op}} & \xleftrightarrow{\cong} & \mathbf{DLat} \\ \updownarrow{\scriptstyle iso} & & \updownarrow \\ \mathbf{biPries}^{\mathrm{op}} & \xleftrightarrow{\cong} & \mathbf{d\text{-}Pries} \end{array}$$

where **d-Pries** is the category of *Priestley d-frames*, i.e. d-compact d-zero-dimensional d-frames (**d-Pries** is dual to **biPries** by Theorem 2.4.5). From the composition of the equivalences we see that **d-Pries** is equivalent to **DLat**. In the following we give a constructive proof of this fact and, as a consequence, we will also prove that $\mathsf{spec}_{\preceq}$ and $\mathsf{Clp}_{\preceq}$ factor through **d-Pries**.

In order to define a functor $\mathbf{d\text{-}Pries} \to \mathbf{DLat}$ we examine what does the functor $\mathsf{Clp}_{\preceq}$ do bitopologically. Clopen upsets of a Priestley space are exactly the τ_+-open τ_--closed sets of the corresponding bispace (via $\mathbf{bi}\colon \mathbf{Pries} \to \mathbf{biPries}$ from Corollary 2.2.6). Moreover, τ_+-open τ_--closed subsets of a bispace X are in a bijection with the pairs of opens $(U_+, U_-) \in \tau_+ \times \tau_-$ which are in tot_X and con_X.

This is a general construction which works for any d-frame. Define the functor

$$\mathsf{Clp}_d\colon \mathbf{d\text{-}Frm} \to \mathbf{DLat}$$

as $\mathcal{L} \mapsto (\mathsf{con} \cap \mathsf{tot}, \curlywedge, \curlyvee, ff, tt)$ on objects and $h \mapsto h$ on morphisms. Observe that it is well defined and, moreover, $\mathsf{Clp}_{\preceq} \cong \mathsf{Clp}_d \circ \Omega_d \circ \mathbf{bi}$.

Conversely, it is not difficult to check the elements of the form (2.6.1) constitute the sets of generators of τ_+ and τ_-, respectively, for the bispace $\mathbf{bi}(\mathsf{spec}_{\preceq}(D))$. Motivated by this observation, we define the functor

$$\mathcal{IF}\colon \mathbf{DLat} \to \mathbf{d\text{-}Frm}$$

on objects $D \mapsto (\mathsf{Idl}(D), \mathsf{Filt}(D); \mathsf{con}_D, \mathsf{tot}_D)$ where $\mathsf{Idl}(D)$ and $\mathsf{Filt}(D)$ are the frames of ideals and filters[6], respectively, and

$$(I, F) \in \mathsf{con}_D \overset{\text{def}}{\equiv} \forall i \in I\, \forall f \in F.\ i \leq f \qquad (I, F) \in \mathsf{tot}_D \overset{\text{def}}{\equiv} I \cap F \neq \emptyset.$$

[6] Both frames of ideals and filters are ordered by set-inclusion with the smallest elements $\{0\}$ and $\{1\}$, respectively.

On morphisms, for a lattice homomorphism $h\colon D \to E$, define $\mathcal{IF}(h)\colon \mathcal{IF}(D) \to \mathcal{IF}(E)$ as

$$\mathcal{IF}(h)\colon (I,F) \longmapsto ({\downarrow}h[I], {\uparrow}h[F]).$$

2.6.1 Lemma. *$\mathcal{IF}$ is well-defined.*

Proof. We need to check that the structure $(\mathsf{Idl}(D), \mathsf{Filt}(D); \mathsf{con}_D, \mathsf{tot}_D)$ is a d-frame. (con-$\downarrow$) and (tot-$\uparrow$) are immediate and $t\!t, f\!f \in \mathsf{con} \cap \mathsf{tot}$ since $t\!t = (D, \{1\})$ and $f\!f = (\{0\}, D)$. For the rest:

(con-$\bigsqcup^{\uparrow}$) Let $\{(I_j, F_j) : j \in J\}$ be a directed subset of con. Let $i \in \bigvee^{\uparrow}{}_j I_j$ and $f \in \bigvee^{\uparrow}{}_j F_j$. By definition, $i \in I_k$ for some $k \in J$ and $f \in F_l$ for some $l \in J$. Let j be an upper bound for k and l in J.Then, both $i \in I_j$ and $f \in F_j$ and, since $(I_j, F_j) \in \mathsf{con}$, we know that $i \leq f$.

(con-$\dot{\vee}$, $\dot{\wedge}$) Let $(I_1, F_1), (I_2, F_2) \in \mathsf{con}_D$. Then, for any $i \in I_1 \wedge I_2 = I_1 \cap I_2$ and $f_1 \wedge f_2 \in F_1 \vee F_2 = \{f_1 \wedge f_2 \mid f_1 \in F_1,\ f_2 \in F_2\}$, we have that $i \leq f_1$ and $i \leq f_2$ from our assumptions. Therefore, also $i \leq f_1 \wedge f_2$ as we wanted. To prove the second part, take any $i_1 \vee \imath_2 \in I_1 \vee I_2 = \{i_1 \vee i_2 \mid i_1 \in I_1,\ i_2 \in I_2\}$ and $f \in F_1 \wedge F_2 = F_1 \cap F_2$. Then again, $i_1 \leq f$ and $i_2 \leq f$ and so $i_1 \vee i_2 \leq f$.

(tot-$\dot{\vee}$, $\dot{\wedge}$) Let $(I_1, F_1), (I_2, F_2) \in \mathsf{tot}_D$. There must exist an $a_i \in I_i \cap F_i$ for $i = 1, 2$. Therefore, we have $a_1 \vee a_2 \in (I_1 \vee I_2) \cap (F_1 \wedge F_2)$ and $a_1 \wedge a_2 \in (I_1 \wedge I_2) \cap (F_1 \vee F_2)$.

(con-tot) Let $(I, F_1) \in \mathsf{con}_D$ and $(I, F_2) \in \mathsf{tot}_D$. From the second assumption we know that there exists an $x \in I \cap F_2$ and from the first assumption we know that $x \leq f$ for all $f \in F_1$. Therefore, $F_1 \subseteq {\uparrow}x \subseteq F_2$. Proving that $(I_1, F) \in \mathsf{con}_D$ and $(I_2, F) \in \mathsf{tot}_D$ implies $I_1 \subseteq I_2$ follows the exactly same reasoning.

For functoriality, let $h\colon D \to E$ be a homomorphism. Clearly, both components are frame homomorphisms. If $(I, F) \in \mathsf{con}_D$, then for every $a \in h[I]$ and $b \in h[F]$, $a \leq b$ as h is monotone. Therefore, the same is true also for ${\downarrow}h[I]$ and ${\uparrow}h[F]$. Finally, if $(I, F) \in \mathsf{tot}_D$, then for an $x \in I \cap F$, also $h(x) \in \mathsf{Idl}(h)(I) \cap \mathsf{Filt}(h)(F)$. □

The next lemma convinces us that $\mathcal{IF}$ does have the potential to be the embedding of **DLat** $\hookrightarrow$ **d-Frm**:

2.6.2 Lemma. *For a distributive lattice D, $\mathsf{con}_D \cap \mathsf{tot}_D = \{({\downarrow}a, {\uparrow}a) : a \in D\}$. Consequently, $\mathsf{Clp}_d(\mathcal{IF}(D)) \cong D$.*

Proof. Let $(I, F) \in \mathsf{con}_D \cap \mathsf{tot}_D$. Then, there exists an $x \in I \cap F$ since $(I, F) \in \mathsf{tot}_D$ and, for all $i \in I$ and $f \in F$, $i \leq x \leq f$ since $(I, F) \in \mathsf{con}_D$. Therefore, $I = {\downarrow}x$ and $F = {\uparrow}x$. □

2.6.3 The unit of coreflection. Let $\mathcal{L}$ be a d-frame. Since the lattice $\mathsf{Clp}_d(\mathcal{L})$ is isomorphic to the sublattice of L_+ of complemented elements $C_+ = \{x \mid (x, x^*) \in \mathsf{tot}\}$ (Lemma 2.3.13), we have two monotone maps:

$$v_+ \colon \mathsf{Idl}(C_+) \longrightarrow L_+ \qquad\qquad e_+ \colon L_+ \longrightarrow \mathsf{Idl}(C_+) \tag{2.6.2}$$
$$I \longmapsto \bigvee I \qquad\qquad x \longmapsto {\downarrow} x \cap C_+$$

A simple calculation shows that $v_+(e_+(x)) \leq x$ and $I \subseteq e_+(v_+(I))$ for all $x \in L_+$ and $I \in \mathsf{Idl}(C_+)$. Therefore, v_+ is a left adjoint and it preserves all joins. It also preserves finite meets because

$$\begin{aligned} v_+(I) \wedge v_+(J) &= (\bigvee I) \wedge (\bigvee J) = \bigvee \{i \wedge j \mid i \in I, j \in J\} \\ &\leq \bigvee \{a \mid a \in I \cap J\} = v_+(I \cap J) = v_+(I \wedge J). \end{aligned}$$

(Where the inequality holds because if $i \in I$ and $j \in J$, then $i \wedge i \in I \cap J$.) As a result, we obtain that

2.6.4 Lemma. *$v_+ \colon \mathsf{Idl}(C_+) \to L_+$, $I \mapsto \bigvee I$, is a frame homomorphism.*

Next, $\mathsf{Filt}(\mathsf{Clp}_d(\mathcal{L})) \cong \mathsf{Filt}(C_+) \cong \mathsf{Idl}(C_+^{\mathrm{op}})$ and C_+^{op} is isomorphic to the lattice of complemented $C_- = \{y \in L_- \mid (y^*, y) \in \mathsf{tot}\}$ (Lemma 2.3.13). The isomorphism map $\mathsf{Filt}(C_+) \cong \mathsf{Idl}(C_-)$ is computed as

$$(-)^{\circledast} \colon\ F \longmapsto F^{\circledast} = \{x^* \mid x \in F\}.$$

Then, by the same procedure as above we have a frame homomorphism $\mathsf{Idl}(C_-) \to L_-$ which, when precomposed with $(-)^{\circledast}$, gives:

2.6.5 Lemma. *$v_- \colon \mathsf{Filt}(C_+) \to L_-$, $F \mapsto \bigvee(F^{\circledast})$, is a frame homomorphism.*

In addition we also have:

2.6.6 Lemma. *$v = (v_+, v_-) \colon \mathcal{IF}(\mathsf{Clp}_d(\mathcal{L})) \to \mathcal{L}$ is a d-frame homomorphism.*

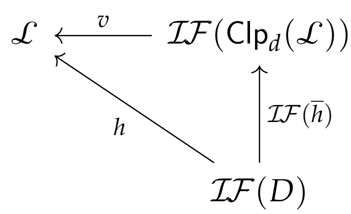
Proof. What is left to check is that v preserves con and tot. Let $(I, F) \in \mathsf{con}_{\mathsf{Clp}_d(\mathcal{L})}$. By definition, for every $i \in I$ and $f \in F$, $i \leq f$ and so $(i, f^*) \in \mathsf{con}_{\mathcal{L}}$. Then, $v({\downarrow} i, {\uparrow} f) = (i, f^*) \in \mathsf{con}_{\mathcal{L}}$ and, because con is closed under directed joins,

$$v(I, F) = v(\textstyle\bigvee^{\uparrow}\{{\downarrow} i \mid i \in I\}, \bigvee^{\uparrow}\{{\uparrow} f \mid f \in F\}) = \bigsqcup^{\uparrow}\{v({\downarrow} i, {\uparrow} f) \mid i \in I, f \in F\} \in \mathsf{con}_{\mathcal{L}}.$$

Next, let $(I, F) \in \mathsf{tot}_{\mathsf{Clp}_d(\mathcal{L})}$. By definition, there exists an $x \in I \cap F$. Since $({\downarrow} x, {\uparrow} x) \sqsubseteq (I, F)$ and $(x, x^*) \in \mathsf{tot}$, we have $v(I, F) \sqsupseteq v({\downarrow} x, {\uparrow} x) = (x, x^*) \in \mathsf{tot}_{\mathcal{L}}$. □

Moreover, v has the following universal property:

2.6.7 Proposition. *Let $\mathcal{L}$ be a d-frame and let D be a distributive lattice. Then, for any d-frame homomorphism $h\colon \mathcal{IF}(D) \to \mathcal{L}$ there exists a* unique *lattice homomorphism $\overline{h}\colon D \to \mathsf{Clp}_d(\mathcal{L})$ such that the following diagram commutes*



Proof. With $\mathsf{Clp}_d(\mathcal{L}) \cong C_+$ set $\overline{h}\colon\ d \mapsto h_+(\downarrow d)$. This is a well-defined lattice homomorphism as $(\downarrow d, \uparrow d) \in \mathsf{con}_D \cap \mathsf{tot}_D$ (Lemma 2.6.2) also $(h_+(\downarrow d), h_-(\uparrow d)) \in \mathsf{con}_{\mathcal{L}} \cap \mathsf{tot}_{\mathcal{L}}$. Let $(I, F) \in \mathcal{IF}(D)$. With $\mathcal{IF}(\overline{h})$ of the type $\mathsf{Idl}(D)\times\mathsf{Filt}(D) \to \mathsf{Idl}(C_+)\times\mathsf{Filt}(C_+)$ and, for any $i \in I$, $f \in F$, compute

$$v(\mathcal{IF}(\overline{h})(\downarrow i, \uparrow f)) = v(\downarrow\overline{h}[\downarrow i], \uparrow\overline{h}[\uparrow f]) = v(\downarrow\overline{h}(i), \uparrow\overline{h}(f)) = v(\downarrow h_+(\downarrow i), \uparrow h_+(\downarrow f))$$

Because $h_+(\downarrow f)^* = h_-(\uparrow f)$ and, therefore, $(\uparrow h_+(\downarrow f))^{\circledast} = \downarrow h_-(\uparrow f)$, the last term in the calculation above is equal to $(h_+(\downarrow i), h_-(\uparrow f))$. With this we see that $\overline{h}$ makes the diagram above commute as I and F are the joins of such $\downarrow i$'s and $\uparrow f$'s, respectively.

To show uniqueness, let $g\colon D \to \mathsf{Clp}_d(\mathcal{L})$ also makes the diagram commute. Then, for any $d \in D$, $g(d) = \bigvee \downarrow g(d) = \bigvee \downarrow g[\downarrow x] = v_+(\mathcal{IF}(g)_+(\downarrow d) = h_+(\downarrow d) = \overline{h}(d)$. □

As a result, $\mathcal{IF}$ is the left adjoint of Clp_d with $v\colon \mathcal{IF} \circ \mathsf{Clp}_d \Longrightarrow \mathrm{Id}$ the counit and the unit $\lambda\colon \mathrm{Id} \Longrightarrow \mathsf{Clp}_d \circ \mathcal{IF}$ is computed as

$$\lambda_D = \overline{1_{\mathcal{IF}(D)}}\colon\ a \in D \quad\longmapsto\quad (1_{\mathsf{Idl}(D)}(\downarrow a), 1_{\mathsf{Filt}(D)}(\uparrow a)) = (\downarrow a, \uparrow a)$$

However, by Lemma 2.6.2, we know that this is an isomorphism. Therefore, **DLat** is equivalent to a coreflective subcategory of d-frames.

In the next two lemmas we show that this coreflective subcategory is the category of Priestley d-frames.

2.6.8 Lemma. *$\mathcal{IF}(D)$ is a Priestley d-frame, for any distributive lattice D.*

Proof. Checking zero-dimensionality is immediate as, by Lemma 2.6.2, all $\downarrow a$ (resp. $\uparrow a$) are complemented and $I = \bigvee_{a\in I} \downarrow a$ (resp. $F = \bigvee_{a\in F} \uparrow a$) for every every ideal I (resp. filter F). To check compactness, let $\bigsqcup^{\uparrow}_j (I_j, F_j) \in \mathsf{tot}$. That means that, for some $x \in D$, $x \in \bigvee^{\uparrow}_j I_j = \bigcup_j I_j$ and $x \in \bigvee^{\uparrow}_j F_j = \bigcup_j F_j$. From directedness, there exists a $j \in J$ such that $x \in I_j \cap F_j$. □

2.6.9 Corollary. *$\mathcal{IF}\colon$ **DLat** $\to$ **d-Frm** is full and faithful.*

Proof. Faithfulness follows from the fact that the functors Idl and Filt are faithful. Next, let D and E be distributive lattices and $h\colon \mathcal{IF}(D) \to \mathcal{IF}(E)$ a d-frame homomorphism. Since $\mathsf{con}_D \cap \mathsf{tot}_D = \{(\downarrow x, \uparrow x) \mid x \in D\}$ and $h[\mathsf{con}_D \cap \mathsf{tot}_D] \subseteq \mathsf{con}_E \cap \mathsf{tot}_E$, we see that h preserves principal ideals and filters. In other words, $h_+ = \mathsf{Idl}(g_1)$ and $h_- = \mathsf{Filt}(g_2)$ for some lattice homomorphisms $g_1, g_2\colon D \to E$. Moreover, because d-frame homomorphisms between d-regular d-frames are uniquely determined by one of their components (as we will show in Lemma 5.3.10) and the plus coordinate of $\mathcal{IF}(g_1)$ is equal to h_+, it must be that $g_1 = g_2$. □

2.6.10 Proposition. *$\mathcal{L}$ is a Priestley d-frame if and only if v is an isomorphism.*

Proof. The implication from right follows from Lemma 2.6.8. For the opposite direction, let $\mathcal{L}$ be a Priestley d-frame and recall the definition of v_+ and e_+ from (2.6.2). Since $\mathcal{L}$ is d-zero-dimensional we see that $v_+(e_+(x)) = x$ for all $x \in L_+$. On the other hand let $I \in \mathsf{Idl}(C_+)$ and let $c \in e_+(v_+(I))$ (equivalently $c \le \bigvee I$). Because $c \in C_+$, $(c, c^*) \in \mathsf{tot}_{\mathcal{L}}$ and because $\mathcal{L}$ is d-compact, $c \le d$ for some $d \in I$. Therefore, $c \in I$ and so $e_+(v_+(I)) \subseteq I$. We have proved that v_+ is a frame isomorphism and, correspondingly, the same is true for v_-.

What is left to prove is that v reflects $\mathsf{con}_{\mathcal{L}}$ and $\mathsf{tot}_{\mathcal{L}}$. Let $(x, y) \in L_+ \times L_-$. From the previous we know that $v(I, F) = (x, y)$ where $I = \downarrow x \cap C_+$ and $F^{\circledast} = \downarrow y \cap C_-$. Assume $(x, y) \in \mathsf{con}_{\mathcal{L}}$. Then, for every $c_+ \in I$ and $c_- \in F^{\circledast}$, $(c_+, c_-) \in \mathsf{con}_{\mathcal{L}}$ and so $c_+ \le c_-^*$. Therefore, $(I, F) \in \mathsf{con}_{\mathsf{Clp}_d(\mathcal{L})}$. Finally, assume $(x, y) \in \mathsf{tot}_{\mathcal{L}}$. Since $(x, y) = \sqcup(I, F^{\circledast})$, by compactness, there are $c_+ \in I$ and $c_- \in F^{\circledast}$ such that $(c_+, c_-) \in \mathsf{tot}$. By (con-tot), $c_-^* \le c_+$ and so $c_-^* \in I \cap F$. Therefore, $(I, F) \in \mathsf{tot}_{\mathsf{Clp}_d(\mathcal{L})}$. □

We summarise our findings in the main theorem of this section.

2.6.11 Theorem.
The category of d-compact d-zero-dimensional d-frames is equivalent to the category of distributive lattices and, moreover, the category of d-frames coreflects onto it.

Since Priestley d-frames are always spatial (Proposition 2.4.4), we also have that the category of bispaces coreflects onto the category of Priestley bispaces. The reflection map is the composite, for any bispace X,

$$X \xrightarrow{\eta^X} \Sigma_d(\Omega_d(X)) \xrightarrow{\Sigma_d(v)} \Sigma_d(\mathcal{IF}(\mathsf{Clp}_d(\Omega_d(X)))).$$

2.6.12 Remark. Banaschewski, in [Ban79], proved that frames coreflect onto coherent frames and that (mono)topological spaces reflect onto spectral spaces. Then, in [Pic94], Picado showed that the category of Priestley spaces is isomorphic to the category of Priestley bispaces and, therefore, dually equivalent to the category of Priestley biframes (defined correspondingly). Coreflectivity of biframes onto a category of biframes equivalent to the category of distributive lattices is also showed therein. Theorem 2.6.11 is a version of both of those results in the context of d-frames.

Note that our construction of $\mathcal{IF}(D)$ is simpler than the corresponding construction in [Ban79]. This is because we do not have to construct an ambient frame which contains both $\mathsf{Idl}(D)$ and $\mathsf{Filt}(D)$ and is generated by them.

2.6.1 The bispace side

In this subsection we finally take a look at a variant of Proposition 2.5.2 but for Priestley duality. We have the required functors ready

$$I\colon \mathbf{Pries} \xrightarrow{\mathbf{bi}} \mathbf{biPries} \xrightarrow{\subseteq} \mathbf{biKReg} \quad \text{and} \quad J\colon \mathbf{DLat} \xrightarrow{\mathcal{IF}} \mathbf{d\text{-}Pries} \xrightarrow{\subseteq} \mathbf{d\text{-}KReg}$$

(by Corollary 2.2.6 and Theorem 2.6.11) What is left to show is that they commute with Ω_d and Σ_d. We will define a functor $\mathsf{spec}_{\mathrm{bi}}\colon \mathbf{DLat} \to \mathbf{biPries}$ such that (1) $\mathsf{spec}_{\preceq} \cong \mathbf{bi}^{-1} \circ \mathsf{spec}_{\mathrm{bi}}$, and (2) $\mathsf{spec}_{\mathrm{bi}} \cong \Sigma_d \circ \mathcal{IF}$. As a result we obtain:

2.6.13 Proposition. *The duality of Priestley spaces and distributive lattices embeds into the duality of d-compact d-regular bispaces and d-frames:*

$$\begin{array}{ccc} \mathbf{Pries}^{op} & \xleftrightarrow{\ \cong\ } & \mathbf{DLat} \\ {\scriptstyle I^{op}}\big\downarrow & & \big\downarrow{\scriptstyle J} \\ \mathbf{biKReg}^{op} & \xleftrightarrow{\ \cong\ } & \mathbf{d\text{-}KReg} \end{array}$$

2.6.14 Spectra bitopologically. Define a functor

$$\mathsf{spec}_{\mathrm{bi}}\colon \mathbf{DLat} \to \mathbf{biPries}$$

on objects as $D \mapsto (\mathsf{PFilt}(D); \tau_+^D, \tau_-^D)$ where the plus topology is generated by $\Phi_+^D(a)$'s and the minus topology by $\Phi_-^D(a)$'s (see (2.6.1)). On morphisms, for a lattice homomorphism $h\colon D \to E$, set $\mathsf{spec}_{\mathrm{bi}}(h)$ to be the bicontinuous map $P \mapsto h^{-1}[P]$.

2.6.15 Observation.

1. *Our condition (1) from above (i.e.* $\mathsf{spec}_{\preceq} \cong \mathbf{bi}^{-1} \circ \mathsf{spec}_{bi}$*) follows immediately from the definitions.*

2. *The sets* $\{\Phi_+^D(a) : a \in D\}$ *and* $\{\Phi_-^D(a) : a \in D\}$ *are closed under finite intersections and, therefore, form bases of* τ_+^D *and* τ_-^D*, respectively.*

2.6.16 Lemma. $\mathsf{Idl}(D) \cong \tau_+^D$ *via the frame isomorphism* $U_+ \colon I \mapsto \{P \mid I \cap P \neq \emptyset\}$.

Proof. Observe that $U_+(\downarrow a) = \Phi_+^D(a)$ for all $a \in D$. Since $\Phi_+^D(a)$'s form a basis we get that U_+ is an onto frame homomorphism from

$$\bigcup_{a \in M} \Phi_+^D(a) = \{P \mid P \cap (\bigvee_{a \in M} \downarrow a) \neq \emptyset\}$$

for every set $M \subseteq D$. To check injectivity, let $I, J \in \mathsf{Idl}(D)$ such that $I \not\subseteq J$. Then, there exists an $x \in I \setminus J$ and, by (AC), we can extend $\uparrow x$ to a prime filter P such that $P \cap J = \emptyset$. Therefore $U_+(I) \not\subseteq U_-(J)$. □

2.6.17 Lemma. $\mathsf{Filt}(D) \cong \tau_-^D$ *via the frame isomorphism* $U_- \colon F \mapsto \{P \mid F \not\subseteq P\}$.

Proof. The argument is almost the same as above: $U_-(\uparrow a) = \Phi_-^D(a)$ for all $a \in D$ and, since $a \notin P$ iff $\uparrow a \not\subseteq P$, we also have that $\bigcup_{a \in M} \Phi_-^D(a) = \{P \mid (\bigvee_{a \in M} \uparrow a) \not\subseteq P\}$ which proves that U_- is an onto frame homomorphism. Finally, U_- is injective because if $F \not\subseteq G$ in $\mathsf{Filt}(G)$, then the (AC) gives us a prime filter $P \supseteq G$ disjoint with $\downarrow x$ for some $x \in F \setminus G$. □

2.6.18 Proposition. $U_D = (U_+, U_-) \colon \mathcal{IF}(D) \to \Omega_d(\mathsf{spec}_{bi}(D))$ *is a d-frame isomorphism.*

Proof. All we need to check is that U_D preserves and reflects con and tot. The latter is equivalent to, for all $(I, F) \in \mathcal{IF}(D)$,

$$U_+(I) \cup U_-(F) = X \quad \text{iff} \quad I \cap F \neq \emptyset.$$

For "$\Leftarrow$", let $x \in I \cap F$. Then $x \in P$ implies that $P \in U_+(I)$ and $x \notin P$ implies that $F \not\subseteq P$. For "$\Rightarrow$", if $I \cap F = \emptyset$, then, by (AC), there exists a prime filter $P \supseteq F$ such that $I \cap P = \emptyset$. Therefore, $P \notin U_+(I) \cup U_-(F)$.

To show that U_D preserves and reflects con, we need to show that

$$U_+(I) \cap U_-(F) = \emptyset \quad \text{iff} \quad \forall i \in I, \forall f \in F.\, i \leq f.$$

For "$\Leftarrow$" notice that, for a prime filter P, $P \in U_+(I)$ iff $\exists i \in P \cap I$ but, from the assumption, $F \subseteq \uparrow i \subseteq P$ and therefore $P \notin U_-(F)$. Conversely, if $F \not\subseteq P$, no such $i \in P \cap I$ can exist. For "$\Rightarrow$", if there exists $i \in I$ and $f \in F$ such that $i \not\leq f$ then, by (AC), there exists a prime filter P such that $i \in P \not\ni f$. Then, $P \in U_+(I) \cap U_-(F)$. □

2.6.19 Lemma. *U_D is natural in D.*

Proof. Let $h\colon D \to E$ be a lattice homomorphism. Then, $U_E \circ \mathcal{IF}(h)$ is the following map

$$(I, F) \longmapsto (\{R \in \mathsf{spec}_{\preceq}(E) \mid R \cap h[I] \neq \emptyset\}, \{R \in \mathsf{spec}_{\preceq}(E) \mid h[F] \nsubseteq R\}),$$

and $\Omega_d(\mathsf{spec}_{\preceq}(h)) \circ U_D$ is the map which sends (I, F) to

$$(\mathsf{spec}_{\preceq}(h)^{-1}[\{P \in \mathsf{spec}_{\preceq}(D) \mid P \cap I \neq \emptyset\}], \mathsf{spec}_{\preceq}(h)^{-1}[\{P \in \mathsf{spec}_{\preceq}(D) \mid F \nsubseteq P\}]).$$

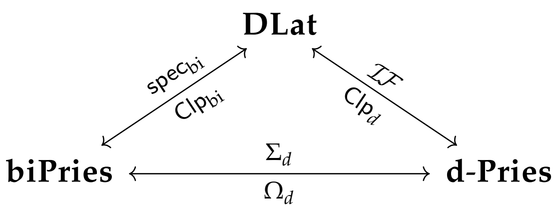
Both plus and minus coordinates can be further simplified to

$$\{R \in \mathsf{spec}_{\preceq}(E) \mid h^{-1}[R] \cap I \neq \emptyset\} \quad \text{and} \quad \{R \in \mathsf{spec}_{\preceq}(E) \mid F \nsubseteq h^{-1}[R]\}.$$

Hence $U_E \circ \mathcal{IF}(h)$ and $\Omega_d(\mathsf{spec}_{\preceq}(h)) \circ U_D$ give the same map. □

As a corollary we obtain that $\mathcal{IF} \cong \Omega_d \circ \mathsf{spec}_{\mathrm{bi}}$ and that is equivalent to our condition (2), i.e. that $\Sigma_d \circ \mathcal{IF} \cong \mathsf{spec}_{\mathrm{bi}}$. This finishes the proof of Proposition 2.6.13.

2.6.20 Remark. We saw in this section that the duality for distributive lattices is much more clear when expressed purely bitopologically. In fact, we believe that the most natural picture of the duality is the commutative triangle:

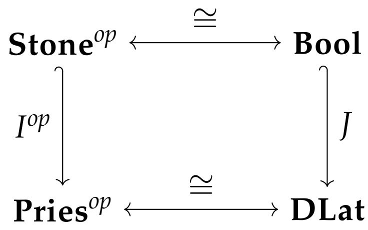


(Where $\mathsf{Clp}_{\mathrm{bi}} \stackrel{\text{def}}{\equiv} \mathsf{Clp}_d \circ \Omega_d$.) The isomorphism of categories $\mathbf{bi}\colon \mathbf{Pries} \cong \mathbf{biPries}$ is extraneous (as it requires the Axiom of Choice from its use of Alexander Subbase Lemma in Theorem 2.2.5) and it is rather just a non-trivial fact about partially ordered spaces and bispaces.

2.7 Embedding Stone duality

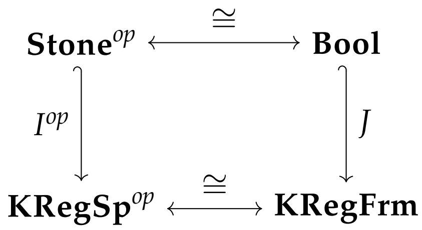
Lastly, recall that the category of *Stone spaces* **Stone**, i.e. compact zero-dimensional T_0 spaces, is dually equivalent to the category of *Boolean algebras* **Bool**. The functors witnessing this duality are $\mathsf{Clp}\colon \mathbf{Stone} \rightleftarrows \mathbf{Bool} : \mathsf{Ult}$ where $\mathsf{Clp}(X, \tau)$ is the Boolean algebra of clopen subsets of X and $\mathsf{Ult}(B)$ is the Stone space of ultrafilters.

2.7.1 Fact. *The duality of Stone spaces and Boolean algebras embeds into Priestley duality:*



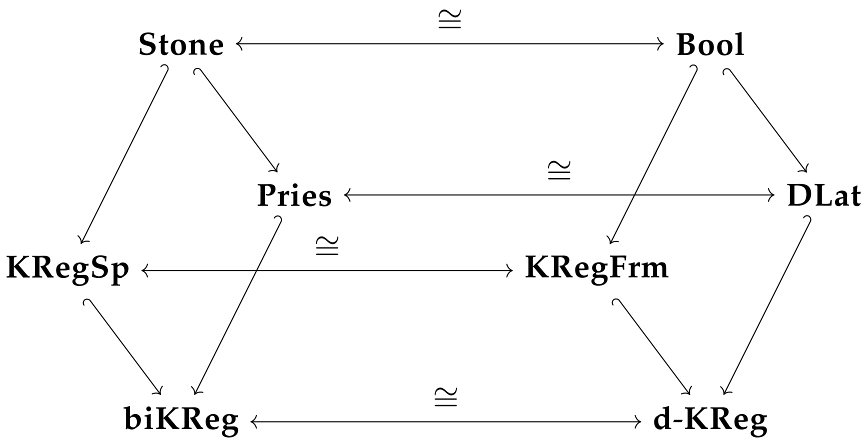
with I the functor $(X, \tau) \mapsto (X, \tau, =)$ and J is the inclusion.

2.7.2 Fact ([Ban79; Joh82]). *The duality of Stone spaces and Boolean algebras embeds into Priestley duality:*



with I the inclusion and J the functor $B \mapsto \mathsf{Idl}(B)$.

All the mentioned embeddings from the previous three sections combine into the following diagram:



To see why it commutes it is enough to prove that the right-hand side square of embeddings commutes:

2.7.3 Lemma. *Let B be a Boolean algebra. Then, $\mathcal{IF}(B) \cong \mathsf{Idl}(B)^{\bowtie}$.*

Proof. Since B is self-isomorphic via the homomorphisms $\neg\colon B \to B^{\mathrm{op}}$, also $\mathsf{Idl}(B) \cong \mathsf{Filt}(B)$ via $F \mapsto \neg F = \{\neg f \mid f \in F\}$. This establishes the pair of frame isomorphisms $\mathsf{Idl}(B) \times \mathsf{Filt}(B) \to \mathsf{Idl}(B) \times \mathsf{Idl}(B)$. Let us check that it preserves and reflects the consistency relation, let $(I, F) \in \mathcal{IF}(B)$,

$$\begin{aligned}(I,F) \in \mathsf{con}_B \quad &\text{iff} \quad \forall i \in I,\ f \in F.\, i \leq f \\ &\text{iff} \quad \forall i \in I,\ j \in \neg F.\, i \wedge j = 0 \quad \text{iff} \quad I \wedge \neg F = \{0\}.\end{aligned}$$

To check the same for the totality relation

$$(I,F) \in \mathsf{tot}_B \quad \text{iff} \quad \exists x \in I \cap F \quad \text{iff} \quad \exists i \in I,\ j \in \neg F.\, i \vee j = 1 \quad \text{iff} \quad I \vee F = B. \qquad \square$$

We conclude this section with a classification of the subcategory of d-frames which corresponds to the Boolean algebras with the embedding **Bool** $\hookrightarrow$ **d-Frm**.

2.7.4 Proposition. *The category of Boolean algebras is equivalent to the category of d-frames $(L, L, \mathsf{con}_L, \mathsf{tot}_L)$ such that L is compact and zero-dimensional, i.e. a Stone frame.*

Proof. By Lemma 2.7.3, $\mathcal{IF}(B) \cong (\mathsf{Idl}(B), \mathsf{Idl}(B), \mathsf{con}_{\mathsf{Idl}(B)}, \mathsf{tot}_{\mathsf{Idl}(B)})$ and $\mathsf{Idl}(B)$ are Stone frames (see [Joh82]). Moreover, the embedding $\mathsf{Idl}(-)^{\bowtie}$: **Bool** $\hookrightarrow$ **d-KReg** is full and faithful since both $\mathsf{Idl}(-)$: **Bool** $\hookrightarrow$ **KRegFrm** and $(-)^{\bowtie}$: **KRegFrm** $\hookrightarrow$ **d-KReg** are as well. $\square$

3

The category **d-Frm**

In the recent literature there has been little, if any, account of an investigation of d-frames from the categorical perspective. We aim to partially fill this hole with this chapter by showing that the category of d-frames is complete and cocomplete and that it has the extremal epi–mono factorisation system. Moreover, we also describe a construction of free objects from their presentations by generators and relations.

In order to show all of that, it proved to be really useful to show that the category of d-frames is a reflective subcategory of a category which is much easier to work with. Then, this categorical gadget makes many of the constructions in d-frames straightforward as they are often just a reflection of a construction in the less restrictive setting.

3.1 The simple case, category pd-Frm

A category in which **d-Frm** is reflective is the category of so called proto-d-frames. In this section we prove basic properties of the category of those. Then, after we show the reflection, we explain how thanks to this **d-Frm** inherits some of the properties of the category of proto-d-frames.

3.1.1 Definition. The structure $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ is a *proto-d-frame* if it satisfies axioms (tot-$\uparrow$), (con-$\downarrow$), (tot-$\curlyvee, \curlywedge$) and (con-$\curlyvee, \curlywedge$) of d-frames.

By **pd-Frm** denote the category of proto-d-frames and d-frame homomorphisms.

In other words, proto-d-frames are like d-frames but we do not require the two troublesome axioms (con-$\bigsqcup^{\uparrow}$) and (con-tot) to hold (see Definition 2.3.2).

3.1.1 Limits

The product $\prod_i \mathcal{L}^i$ of a family of proto-d-frames $\{\mathcal{L}^i = (L^i_+, L^i_-, \mathsf{con}^i, \mathsf{tot}^i)\}_{i\in I}$ is the proto-d-frame $(\prod_i L^i_+, \prod_i L^i_-, \mathsf{con}, \mathsf{tot})$ where $\prod_i L^i_\pm$ are products of frames and

$$\mathsf{con} = \{ ((\alpha^i_+)_i, (\alpha^i_-)_i) \in \prod_i L^i_+ \times \prod_i L^i_- \mid \forall i.\ \alpha^i \in \mathsf{con}^i \}$$
$$\mathsf{tot} = \{ ((\alpha^i_+)_i, (\alpha^i_-)_i) \in \prod_i L^i_+ \times \prod_i L^i_- \mid \forall i.\ \alpha^i \in \mathsf{tot}^i \}$$

3.1.2 Lemma. *$\prod_i \mathcal{L}^i$ is the product in* **pd-Frm**.

Proof. It is immediate that $\prod_i \mathcal{L}^i$ satisfies all axioms of proto-d-frames. From frame theory we know that the projections $\pi^i_\pm \colon \prod_i L^i_\pm \to L^i_\pm$ are frame homomorphisms and from the definition we see that all compounds $\pi^i = (\pi^i_+, \pi^i_-) \colon \prod_i \mathcal{L}^i \to \mathcal{L}^i$ are d-frame homomorphisms.

The universal property of products follows from the universal property of frame products. Indeed, let $\mathcal{M}$ be a proto-d-frame and $\{\delta^i \colon \mathcal{M} \to \mathcal{L}^i\}_i$ be a family of d-frame homomorphisms. Then, we have two frame homomorphisms $\overline{\delta}_\pm \colon M_\pm \to \prod_i L^i_\pm,\ x \mapsto (\delta^i_\pm(x))_i$, and, by definition, $\overline{\delta} = (\overline{\delta}_+, \overline{\delta}_-)$ is a d-frame homomorphism. □

Next, let $h, g \colon \mathcal{L} \to \mathcal{M}$ be two morphisms in **pd-Frm**. Define $\mathcal{N}$ to be the proto-d-frame $(N_+, N_-, \mathsf{con}_\mathcal{N}, \mathsf{tot}_\mathcal{N})$ where $N_\pm = \{x \in L_\pm \mid h_\pm(x) = g_\pm(x)\}$ and

$$\mathsf{con}_\mathcal{N} = \mathsf{con}_\mathcal{L} \cap (N_+ \times N_-) \quad \text{and} \quad \mathsf{tot}_\mathcal{N} = \mathsf{tot}_\mathcal{L} \cap (N_+ \times N_-).$$

Clearly, $\mathcal{N}$ is a proto-d-frame. Since the embeddings $e_\pm \colon N_\pm \subseteq L_\pm$ are the equalisers of $h_\pm$ and $g_\pm$ in the category of frames, the compound $e = (e_+, e_-) \colon \mathcal{N} \to \mathcal{L}$ is a d-frame homomorphism and has the required universal property in **pd-Frm**. We obtain:

3.1.3 Lemma. $\mathcal{N} \xrightarrow{e} \mathcal{L} \overset{f}{\underset{g}{\rightrightarrows}} \mathcal{M}$ *is the equaliser in* **pd-Frm**.

Combining Lemmas 3.1.2 and 3.1.3 we conclude:

3.1.4 Proposition. *The category* **pd-Frm** *is complete.*

3.1.2 Quotients

Let us recall how quotients are formed in frame theory. Let R be a binary relation on a frame L. An element x is said to be *R-saturated* if

$$\forall a, b, c. \quad (a, b) \in R \quad \text{and} \quad b \wedge c \le x \implies a \wedge c \le x$$

By L/R denote the set $\{x \in L \mid x \text{ is } R\text{-saturated}\}$ which is a frame in order induced by L. The quotient map is as follows

$$\mu^R\colon L \to L/R, \quad x \mapsto \bigwedge\{s \mid x \leq s \text{ and } s \text{ is } R\text{-saturated}\}.$$

Quotients have the expected universal property: For any frame homomorphisms $h\colon L \to M$ such that

$$\forall (a,b) \in R \implies h(a) \leq h(b)$$

the restriction $h{\restriction}_{L/R}\colon L/R \to M$ is a frame homomorphism such that $h = h{\restriction}_{L/R} \cdot \mu^R$.

3.1.5 The relation $\leq^R$. For every $R \subseteq L{\times}L$ there exists the largest relation $\leq^R$ such that $L/R \cong L/{\leq^R}$. It is defined as $\leq^R \stackrel{\text{def}}{\equiv} \{(x,y) \mid \mu^R(x) \leq \mu^R(y)\}$[1]. Observe that $R \subseteq {\leq^R}$. Alternatively, $\leq^R$ can be defined from the graph of μ^R, i.e. the set $\mathbb{G}^R = \{(x, \mu^R(x)) : x \in L\}$, as follows

$$\leq^R \,=\, \leq ;(\mathbb{G}^R)^{-1}$$

where $(-)\,;\,(-)$ is the relation composition and $(-)^{-1}$ is the reversed relation:

$$P\,;Q = \{(x,y) \mid \exists z.\ x\ P\ z\ Q\ y\} \quad \text{and} \quad P^{-1} = \{(y,x) \mid x\ P\ y\}.$$

(*Proof.* If (x,y) is such that $\mu^R(x) \leq \mu^R(y)$ then $x \leq \mu^R(x) \leq \mu^R(y)$ gives that $(x,y) \in {\leq}\,;(\mathbb{G}^R)^{-1}$. On the other hand if $x \leq \mu^R(y)(\mathbb{G}^R)^{-1}y$, because $\mu^R(y)$ is R-saturated, $\mu^R(x) \leq \mu^R(y)$.)

We will make a use of $\leq^R$ later in Sections 3.2.2 and 3.2.4 where we will also need the following:

3.1.6 Lemma. *The assignment $R \mapsto \leq^R$ is monotone (in the subset order).*

Proof. Observe that $x \leq^R y$ is equivalent to: whenever $y \leq s$ and s is R-saturated, then $x \leq s$. Let $R \subseteq S$ be subsets of $L{\times}L$ and let $x \leq^R y$. Then, for a S-saturated s such that $y \leq s$, because every S-saturated element is also R-saturated, $x \leq s$. Hence, $x \leq^S y$. □

3.1.7 Remark. Unlike as is in the standard literature of frames we have defined quotients of frames non-symmetrically. For that reason we include a full proof that L/R has the required universal property in Appendix, Section A.5.

A disadvantage of this reformulation is that the relation for coequalisers (in Section 3.1.3) becomes a bit more complicated but, on the other hand, everything else we needed to do in this text is simpler. An example when this slight modification shines is when doing free constructions of frames. A quotient of an absolutely free frame by R is definitionally equal to the frame of $\mathcal{C}$-ideals (see, for example, [Joh82]) where the set of coverages $\mathcal{C}$ is generated from R.

[1] In fact $\leq^R$ is a quasi-congruence, i.e. it is a preorder which is also a subframe of $L{\times}L$.

3.1.8 Quotients of proto-d-frames are computed componentwise. Let $\mathcal{L}$ be a proto-d-frame and $R_\pm$ two binary relations: $R_+ \subseteq L_+ \times L_+$ and $R_- \subseteq L_- \times L_-$. Consider $\mu^R = (\mu^R_+, \mu^R_-) \colon L_+ \times L_- \to (L_+/R_+) \times (L_-/R_-)$ where $\mu^R_\pm$ are the quotient maps $L_\pm \to L_\pm/R_\pm$, and set

$$\mathcal{L}/R = (L_+/R_+,\ L_-/R_-,\ \mu^R[\mathsf{con}_{\mathcal{L}}],\ \mu^R[\mathsf{tot}_{\mathcal{L}}]).$$

3.1.9 Lemma. *$\mathcal{L}/R$ is a quotient of $\mathcal{L}$ in* **pd-Frm**. *More precisely, $\mathcal{L}/R$ is a proto-d-frame, and for every d-frame homomorphism $h \colon \mathcal{L} \to \mathcal{N}$ for which $(a,b) \in R_\pm$ implies $h_\pm(a) \leq h_\pm(b)$, there is precisely one d-frame homomorphism $\widetilde{h} \colon \mathcal{L}/R \to \mathcal{N}$ such that $\widetilde{h} \cdot q = h$.*

Proof. Checking the axioms (con-⩒, ⩓) and (tot-⩒, ⩓) is straightforward. Now let $\alpha \in \mathsf{con}$ and $\beta \sqsubseteq \mu^R(\alpha)$. Since μ^R is onto we have a γ such that $\beta = h(\gamma)$. Then $\alpha \sqcap \gamma \in \mathsf{con}$, and $h(\alpha \sqcap \gamma) = \beta$ which proves (con-$\downarrow$); similarly we see that (tot-$\uparrow$) holds and we obtain that $(L_+/R_+, L_-/R_-, \mu^R[\mathsf{con}], \mu^R[\mathsf{tot}])$ is a proto-d-frame.

To check universality set $\widetilde{h} = (h_+\restriction_{L_+/R_+}, h_-\restriction_{L_-/R_-})$. It is a pair of frame homomorphisms and the definition of $\mathcal{L}/R$ assures that $\widetilde{h}$ preserves con and tot. □

3.1.10 Technical facts about quotients. For a proto-d-frame $\mathcal{L}$ and binary relations $R_\pm \subseteq L_\pm \times L_\pm$, define

$$
\begin{array}{rll}
(\forall x, y \in L_\pm) & x \leq^R_\pm y & \text{if} \quad \mu^R_\pm(x) \leq \mu^R_\pm(y) \\
(\forall \alpha, \beta \in L_+ \times L_-) & \alpha \sqsubseteq^R \beta & \text{if} \quad \alpha_+ \leq^R_+ \beta_+ \quad \text{and} \quad \alpha_- \leq^R_- \beta_- \\
 & \alpha =^R \beta & \text{if} \quad \alpha \sqsubseteq^R \beta \quad \text{and} \quad \beta \sqsubseteq^R \alpha \\
(\forall P \subseteq L_+ \times L_-) & {\downarrow^R} P = \{\alpha \mid \exists \beta \in P.\, \alpha \sqsubseteq^R \beta\} & \\
 & {\uparrow^R} P = \{\alpha \mid \exists \beta \in P.\, \alpha \sqsupseteq^R \beta\} &
\end{array}
$$

3.1.11 Observation.

1. *If $(\alpha_+, \beta_+) \in R_+$ and $(\alpha_-, \beta_-) \in R_-$, then $\alpha \sqsubseteq^R \beta$.*
2. *If $R_\pm$ is a subset of $\leq_\pm$, then $L/R \cong L$ and so $\sqsubseteq^R$ is the same as $\sqsubseteq$. Consequently, ${\downarrow^R} P = {\downarrow} P$ and ${\uparrow^R} P = {\uparrow} P$*

   (where ${\downarrow} P$ and ${\uparrow} P$ are the downwards and upwards closures in the $\sqsubseteq$-order, respectively).

Let $P \subseteq L_+ \times L_-$ be a any relation. We say that P is ⩓-closed (resp. ⩒-closed), if for every $\alpha, \beta \in R$, $\alpha ⩓ \beta \in R$ (resp. $\alpha ⩒ \beta \in R$). Being ${\downarrow^R}$-closed (resp. ${\uparrow^R}$-closed) means that ${\downarrow^R} P = P$ (resp. ${\uparrow^R} P = P$). Similarly, define $\bigsqcup$-, $\bigvee\!\!\!\!\vee$- and $\bigwedge\!\!\!\!\wedge$-closedness where $\bigvee\!\!\!\!\vee$ and $\bigwedge\!\!\!\!\wedge$ are infinitary versions of ⩒ and ⩓.

3.1.12 Lemma. *Let $P \subseteq L_+ \times L_-$ be a $(\curlywedge, \curlyvee)$-closed relation. Then, $\downarrow^R P$ and $\uparrow^R P$ are also $(\curlywedge, \curlyvee)$-closed. In particular, $\downarrow P$ is also $(\curlywedge, \curlyvee)$-closed.*

Proof. Let $\alpha, \beta \in \downarrow^R P$. This means that there are $\alpha', \beta' \in P$ such that $\alpha \sqsubseteq^R \alpha'$ and $\beta \sqsubseteq^R \beta'$. Observe that $(\alpha \wedge \beta)_+ = \alpha_+ \wedge \beta_+ \leq^R_+ \alpha'_+ \wedge \beta'_+ = (\alpha' \wedge \beta')_+$ and similarly $(\alpha \vee \beta)_- \leq^R_- (\alpha' \vee \beta')_-$. Therefore, $\alpha \curlywedge \beta \sqsubseteq^R \alpha' \curlywedge \beta' \in P$ and $\alpha \curlywedge \beta \in \downarrow P$. Proving closedness $\downarrow^R P$ under $\curlyvee$ is similar. The same reasoning also shows that $\uparrow P$ is $(\curlywedge, \curlyvee)$-closed. □

$\mathcal{L}$ is not the only proto-d-frame which yields $\mathcal{L}/R$ when taking a quotient by R. The following lemma shows that the proto-d-frame $(L_+, L_-, \downarrow^R \mathsf{con}, \uparrow^R \mathsf{tot})$ has the largest con and tot to still have the same quotient by R as $\mathcal{L}$ has, i.e. that

$$\mathcal{L}/R = (L_+, L_-, \downarrow^R \mathsf{con}, \uparrow^R \mathsf{tot})/R.$$

3.1.13 Lemma. $\downarrow^R \mathsf{con} = (\mu^R)^{-1}[\mu^R[\mathsf{con}]]$ *and* $\uparrow^R \mathsf{tot} = (\mu^R)^{-1}[\mu^R[\mathsf{tot}]]$.

Proof. We only prove the con part, the tot part is proved similarly. Let $\alpha \sqsubseteq^R \beta \in \mathsf{con}$. Then, $\mu^R(\alpha) \sqsubseteq \mu^R(\beta)$. By definition, $\mu^R(\beta) \in \mu^R[\mathsf{con}]$ and, $\mu^R[\mathsf{con}]$ is downwards closed (Lemma 3.1.9), and so $\mu^R(\alpha) \in \mu^R[\mathsf{con}]$. Therefore, $\alpha \in (\mu^R)^{-1}[\mu^R[\mathsf{con}]]$. For the other direction, let $\mu^R(\alpha) \in \mu^R[\mathsf{con}]$ for some $\alpha \in L_+ \times L_-$. This means that $\alpha =^R \beta$ for some $\beta \in \mathsf{con}$, and so $\alpha \in \downarrow^R \mathsf{con}$. □

We can think of $(L_+, L_-, \downarrow^R \mathsf{con}, \uparrow^R \mathsf{tot})$ as of $\mathcal{L}$ with con and tot extended by the "R-equal" elements. This extension has the following important lifting property.

3.1.14 Lemma. *Let $h \colon \mathcal{L} \to \mathcal{M}$ be a morphism in **pd-Frm** which preserves $R = (R_+, R_-)$, i.e. $(x, y) \in R_\pm$ implies $h_\pm(x) \leq h_\pm(y)$. Then,*

$$h \colon (L_+, L_-, \downarrow^R \mathsf{con}_{\mathcal{L}}, \uparrow^R \mathsf{tot}_{\mathcal{L}}) \to \mathcal{M}$$

is a d-frame homomorphism.

Proof. Let $\alpha \sqsubseteq^R \beta \in \mathsf{con}_{\mathcal{L}}$. Then, $h(\alpha) \sqsubseteq h(\beta)$, because h preserves R, and $h(\beta) \in \mathsf{con}_{\mathcal{M}}$ because h preserves con. Therefore, $h(\alpha) \in \mathsf{con}_{\mathcal{M}}$. The tot case is similar. □

The relation $\sqsubseteq^R = (\leq^R_+, \leq^R_-)$ can be, again, expressed as a relation composition: $\leq^R_\pm = \leq_\pm ; (\mathbb{G}^R_\pm)^{-1}$ where $\mathbb{G}^R_\pm$ is the graph of the function $\mu^R_\pm$. With this representation we also have that

$$\begin{aligned} \downarrow^R \mathsf{con} &= \leq_+ ; (\mathbb{G}^R_+)^{-1} ; \mathsf{con} ; \mathbb{G}^R_- ; \geq_- \\ \uparrow^R \mathsf{tot} &= \mathbb{G}^R_+ ; \geq_+ ; \mathsf{tot} ; \leq_- ; (\mathbb{G}^R_-)^{-1}. \end{aligned}$$

Convention. Unless specified otherwise, whenever we say that a set is downwards closed or that we close a set downwards, we always mean in $\sqsubseteq$-order. Similarly, $A \subseteq^{\uparrow} L_+ \times L_-$ means that A is a directed subset of $L_+ \times L_-$ in the $\sqsubseteq$-order.

3.1.3 Colimits

Colimits of proto-d-frames are a bit more complicated but still quite simple when compared to colimits of d-frames. The coproduct $\bigoplus_i \mathcal{L}^i$ of a family of proto-d-frames $\{\mathcal{L}^i\}_i$ is the structure $(\bigoplus_i L^i_+, \bigoplus_i L^i_-, \mathsf{con}_\oplus, \mathsf{tot}_\oplus)$ where $\mathsf{con}_\oplus$ and $\mathsf{tot}_\oplus$ are generated from

$$\mathsf{con}_1 = \bigcup_i \iota^i[\mathsf{con}^i] \quad \text{and} \quad \mathsf{tot}_1 = \bigcup_i \iota^i[\mathsf{tot}^i]$$

with $\iota^i = (\iota^i_+, \iota^i_-)$ being the pairs of the frame inclusions $\iota^i_\pm \colon L^i_\pm \to \bigoplus_i L^i_\pm$. Concretely, con is the smallest relation containing con_1 which is downwards closed in $\sqsubseteq$-order and closed under $\curlyvee$ and $\curlywedge$. Similarly, tot is the smallest upwards $\sqsubseteq$-closed and $(\curlyvee, \curlywedge)$-closed superset of tot_1.

3.1.15 Observation. *For any relation P, set $P_{\curlywedge,\curlyvee}$ to be the algebraic closure of P under all* finite *$\curlywedge$ and $\curlyvee$. By Lemma 3.1.12, $\downarrow P_{\curlywedge,\curlyvee}$ and $\uparrow P_{\curlywedge,\curlyvee}$ are still $(\curlywedge, \curlyvee)$-closed. Then, in this notation:*

$$\mathsf{con}_\oplus = \downarrow(\mathsf{con}_1)_{\curlywedge,\curlyvee} \quad \textit{and} \quad \mathsf{tot}_\oplus = \uparrow(\mathsf{tot}_1)_{\curlywedge,\curlyvee}.$$

From the definition we see that $\bigoplus_i \mathcal{L}^i$ is a proto-d-frame and all $\iota^i \colon \mathcal{L}^i \to \bigoplus_i \mathcal{L}^i$ are d-frame homomorphisms. The universal property follows from the universality of frame coproducts.

3.1.16 Lemma. *$\bigoplus_i \mathcal{L}^i$ is the coproduct in* **pd-Frm**.

3.1.17 Coequalisers. Let $h, g \colon \mathcal{M} \to \mathcal{L}$ be two morphisms in **pd-Frm**. Define two relations $R_\pm \subseteq L_\pm \times L_\pm$ as

$$R_+ = \{(g_+(x), h_+(x)),\ (h_+(x), g_+(x)) \mid x \in M_+\} \text{ and}$$
$$R_- = \{(g_-(x), h_-(x)),\ (h_-(x), g_-(x)) \mid x \in M_-\}.$$

Consider any d-frame homomorphism $k \colon \mathcal{L} \to \mathcal{N}$ such that $k \cdot h = k \cdot g$, The commutativity condition is equivalent to $k_\pm(a) \le k_\pm(b)$, for all $(a, b) \in R_\pm$, and so, by Lemma 3.1.9, k can be uniquely lifted to $\widetilde{k} \colon \mathcal{L}/R \to \mathcal{N}$ such that $k = \widetilde{k} \cdot \mu^R$. In other words, we have:

3.1.18 Lemma. $\mathcal{M} \overset{f}{\underset{g}{\rightrightarrows}} \mathcal{L} \xrightarrow{\mu^R} \mathcal{L}/R$ *is the coequaliser in* **pd-Frm**.

Combining Lemmas 3.1.16 and 3.1.18 we conclude:

3.1.19 Proposition. *The category* **pd-Frm** *is cocomplete.*

3.2 Reflecting pd-Frm onto d-Frm

Observe that the same proof which showed that **pd-Frm** is complete can be also adapted for **d-Frm**. Proving cocompleteness is not so straightforward. For this reason, instead of giving a direct proof we show that the embedding

$$\textbf{d-Frm} \xrightarrow{\subseteq} \textbf{pd-Frm}$$

has a left adjoint or, in other words, that **d-Frm** is a reflective subcategory of **pd-Frm**. Then, **d-Frm** is, as a consequence, complete and cocomplete (we give a categorical argument in Section 3.3.1).

The definition of proto-d-frames when compared to d-frames is missing two axioms: (con-$\bigsqcup^{\uparrow}$) and (con-tot). Before we give the full definition of the reflector $\mathfrak{r}$: **pd-Frm** $\to$ **d-Frm** we consider two easier cases. We take a look at how to "correct" either of the missing axioms separately.

3.2.1 Case 1: correcting (con-$\bigsqcup^{\uparrow}$)

Let $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ be a proto-d-frame. If we only wanted to correct (con-$\bigsqcup^{\uparrow}$), then the reflection of $\mathcal{L}$ would be the structure $(L_+, L_-, \mathsf{con}', \mathsf{tot})$ where con' is the smallest relation containing con and closed under all the required operations, i.e.

$$\mathsf{con}' = \bigcap\{C \subseteq L_+ \times L_- \mid \mathsf{con} \subseteq C \text{ and } C \text{ is } (\downarrow, \bigsqcup^{\uparrow}, \curlywedge, \curlyvee)\text{-closed}\}. \tag{3.2.1}$$

For now we omit proving universality of this construction. Instead, we show that con' can be computed by an iterative procedure from below. Let $P \subseteq L_+ \times L_-$ be any relation and define

$$\mathfrak{D}(P) = \{\bigsqcup^{\uparrow} A \mid A \subseteq^{\uparrow} P\}.$$

Note that $\mathfrak{D}(P)$ is only a *"one-step"* closure under joins of directed subsets in $\sqsubseteq$-order. $\mathfrak{D}(P)$ might still contain directed subsets which do not have suprema in $\mathfrak{D}(P)$. To close P under all directed suprema, one has to iterate the process.

3.2.1 Lemma. *Let L_+, L_- be two frames and let $P \subseteq L_+ \times L_-$ be a relation.*

1. *If P is $(\curlywedge, \curlyvee)$-closed then the relation $\mathfrak{D}(P)$ is still $(\curlywedge, \curlyvee)$-closed.*

2. *If P is downwards closed then the relation $\mathfrak{D}(P)$ is still downwards closed.*

Proof. For (1), let $\alpha, \beta \in \mathfrak{D}(P)$. From the definition $\alpha = \bigsqcup_i^{\uparrow} \alpha^i$ and $\beta = \bigsqcup_j^{\uparrow} \beta^j$ for some α^i's and β^j's from P. Calculate:

$$\begin{aligned}\alpha \curlywedge \beta &= (\bigvee_i^{\uparrow} \alpha_+^i \wedge \bigvee_j^{\uparrow} \beta_+^j, \bigvee_i^{\uparrow} \alpha_-^i \vee \bigvee_j^{\uparrow} \beta_-^j) \\ &= (\bigvee_i^{\uparrow} \bigvee_j^{\uparrow} (\alpha_+^i \wedge \beta_+^j), \bigvee_i^{\uparrow} \bigvee_j^{\uparrow} (\alpha_-^i \vee \beta_-^j)) \\ &= (\bigvee_{i,j}^{\uparrow} (\alpha_+^i \wedge \beta_+^j), \bigvee_{i,j}^{\uparrow} (\alpha_-^i \vee \beta_-^j))\end{aligned}$$

Notice that the set $\{\alpha^i \curlywedge \beta^j : i \in I, j \in J\}$ is directed since $\{\alpha^i\}_i$ and $\{\beta^j\}_j$ are and, moreover, $\alpha^i \curlywedge \beta^j \in P$ for all i, j since P is closed under logical meets.

For (2), let $\beta \sqsubseteq \bigsqcup_i^{\uparrow} \alpha_i$ where α_i's are from P. Then, $\beta = \bigsqcup_i^{\uparrow} (\beta \sqcap \alpha_i) \in \mathfrak{D}(P)$ because the set $\{\beta \sqcap \alpha_i\}_i$ is a directed subset of P. □

3.2.2 Lemma. *Let $h\colon \mathcal{L} \to \mathcal{M}$ be a morphism in* **pd-Frm** *such that $\mathcal{M}$ satisfies (con-$\bigsqcup^{\uparrow}$). Then, $h\colon (L_+, L_-, \mathfrak{D}(\mathsf{con}), \mathsf{tot}) \to \mathcal{M}$ is a d-frame homomorphism.*

Proof. Let $A \subseteq^{\uparrow} \mathsf{con}$. Since h is a d-frame homomorphisms, $h[A] \subseteq \mathsf{con}_{\mathcal{M}}$ and, since $\mathcal{M}$ satisfies (con-$\bigsqcup^{\uparrow}$), $h(\bigsqcup^{\uparrow} A) = \bigsqcup^{\uparrow} h[A] \in \mathsf{con}_{\mathcal{M}}$. □

Lemma 3.2.1 shows that one application of $\mathfrak{D}$ does not destroy the property of being $(\downarrow, \curlyvee, \curlywedge)$-closed. We obtain a growing sequence of $(\downarrow, \curlyvee, \curlywedge)$-closed relations $\{\mathfrak{D}^{\gamma}(\mathsf{con}) : \gamma \in \mathsf{Ord}\}$ where, for an ordinal γ and a limit ordinal λ,

$$\mathfrak{D}^0(P) = P, \quad \mathfrak{D}^{\gamma+1}(P) = \mathfrak{D}(\mathfrak{D}^{\gamma}(P)) \quad \text{and} \quad \mathfrak{D}^{\lambda}(P) = \bigcup_{\gamma < \lambda} \mathfrak{D}^{\gamma}(P)\,.$$

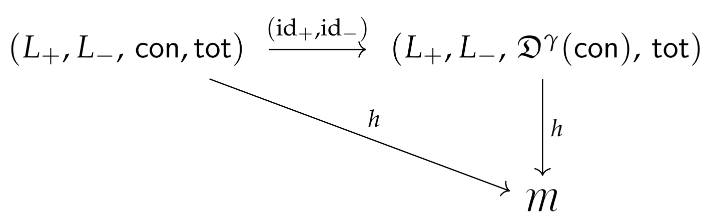
Moreover, because the size of $\mathfrak{D}^{\gamma}(\mathsf{con})$ is bounded by the size of $L_+ \times L_-$, $\mathfrak{D}(\mathfrak{D}^{\gamma}(\mathsf{con}))$ equals $\mathfrak{D}^{\gamma}(\mathsf{con})$, for some ordinal γ. Write $\mathfrak{D}^{\infty}(\mathsf{con})$ for such $\mathfrak{D}^{\gamma}(\mathsf{con})$. Obviously, $\mathfrak{D}^{\gamma}(\mathsf{con})$ is $\bigsqcup^{\uparrow}$-closed and, in fact, it is equal to con' (defined in (3.2.1)).

Finally, we define the reflector. For a proto-d-frame $\mathcal{L}$ set

$$\mathfrak{d}(\mathcal{L}) = (L_+, L_-, \mathfrak{D}^{\infty}(\mathsf{con}), \mathsf{tot})$$

and $\kappa^{\mathcal{L}} = (\mathrm{id}_+, \mathrm{id}_-)\colon \mathcal{L} \to \mathfrak{d}(\mathcal{L})$. Because $\mathfrak{D}(-)$ is increasing, i.e. $P \subseteq \mathfrak{D}(P)$, $\kappa^{\mathcal{L}}$ is a d-frame homomorphism.

By a transfinite application of Lemma 3.2.2, we get that for any d-frame homomorphism $h\colon (L_+, L_-, \mathsf{con}, \mathsf{tot}) \to \mathcal{M}$ with $\mathcal{M}$ satisfying (con-$\bigsqcup^{\uparrow}$)



commutes, for all $\gamma \in \mathsf{Ord}$. Moreover, $h\colon (L_+, L_-, \mathfrak{D}^{\gamma}(\mathsf{con}), \mathsf{tot}) \to \mathcal{M}$ is uniquely determined for all $\gamma \in \mathsf{Ord}$; in particular, for $\gamma = \infty$.

3.2.3 Proposition. *Proto-d-frames satisfying* (con-$\bigsqcup^{\uparrow}$) *form a reflective subcategory of* **pd-Frm**, *with the reflection given by* $\kappa^{\mathcal{L}} \colon \mathcal{L} \to \mathfrak{d}(\mathcal{L})$.

3.2.1.1 When is one step enough?

Before we continue with the (con-tot)-reflection, we would like to investigate some sufficient (though not necessarily minimal) conditions under which $\mathfrak{D}^{\infty}(\mathsf{con}) = \mathfrak{D}(\mathsf{con})$. This is going to have applications later in Section 3.4.3. Consider a fully general scenario: Let $L_{\pm}$ be two frames which are generated from their *bases* $B_{\pm} \subseteq L_{\pm}$, i.e. for every $x \in L_{\pm}$, $x = \bigvee({\downarrow}x \cap B_{\pm})$. Instead of con consider any $P \subseteq L_{+} \times L_{-}$ which is $(\downarrow, \curlywedge, \curlyvee)$-closed.

We start with an important definition. Two sets $A_{+} \subseteq L_{+}$ and $A_{-} \subseteq L_{-}$ are said to be P-*independent* if $\forall a_{+} \in A_{+}$ and $\forall a_{-} \in A_{-}$, $(a_{+}, a_{-}) \in P$. Equivalently, A_{+} and A_{-} are P-independent if $A_{+} \times A_{-} \subseteq P$.

3.2.4 Observation. *For every* $\alpha \in P$, *the sets* $\mathcal{B}_{+}(\alpha_{+})$ *and* $\mathcal{B}_{-}(\alpha_{-})$ *are* P-*independent where* $\mathcal{B}_{\pm}(\alpha_{\pm}) \overset{def}{\equiv} {\downarrow}\alpha_{\pm} \cap B_{\pm}$.

It turns out that $\mathfrak{D}(P)$ can be reformulated by using P-independent sets. Let $\alpha \in \mathfrak{D}(P)$. From the definition, there is some directed $A \subseteq^{\uparrow} P$ such that $\alpha = \bigsqcup^{\uparrow} A$. Because $\mathcal{B}_{\pm}(-)$ are monotone and A is directed, the sets $\{\mathcal{B}_{+}(\alpha_{+}) : \alpha \in A\}$ and $\{\mathcal{B}_{-}(\alpha_{-}) : \alpha \in A\}$ are both also directed (in the subset order) and so we have:

$$\forall A \subseteq^{\uparrow} R \implies \bigcup_{\alpha \in A} \mathcal{B}_{+}(\alpha_{+}) \text{ and } \bigcup_{\alpha \in A} \mathcal{B}_{-}(\alpha_{-}) \text{ are } P\text{-independent} \tag{3.2.2}$$

Moreover, because $L_{\pm}$ is generated by $B_{\pm}$ and every $x \in L_{\pm}$ is equal to $\bigvee \mathcal{B}_{\pm}(x)$, we obtain that $\alpha = (\bigvee^{\uparrow}{}_{\alpha \in A}\, \alpha_{+}, \bigvee^{\uparrow}{}_{\alpha \in A}\, \alpha_{-}) = (\bigvee \mathcal{A}_{+}, \bigvee \mathcal{A}_{-})$ where $\mathcal{A}_{\pm} = \bigcup_{\alpha \in A} \mathcal{B}_{\pm}(\alpha_{\pm})$.

It might seem that $\mathfrak{D}(-)$ is just a special case of a more general construction:

$$\mathfrak{D}^{\text{ind}}(P) = \{(\bigvee A_{+}, \bigvee A_{-}) \mid A_{\pm} \subseteq B_{\pm} \text{ s.t. } A_{+} \text{ and } A_{-} \text{ are } P\text{-independent}\}$$

What we have proved in the previous paragraphs is that $\mathfrak{D}(P) \subseteq \mathfrak{D}^{\text{ind}}(P)$. In fact, both closures are equivalent:

3.2.5 Lemma. $\mathfrak{D}(P) = \mathfrak{D}^{\text{ind}}(P)$

Proof. Only the right-to-left inclusion remains to be proved. Let $A_{+} \subseteq B_{+}$ and $A_{-} \subseteq B_{-}$ be P-independent. Observe that for two finite sets $F_{+} \subseteq_{\text{fin}} A_{+}$ and $F_{-} \subseteq_{\text{fin}} A_{-}$, $(\bigvee F_{+}, \bigvee F_{-}) \in P$. This is because R is $\vee$-closed and so $(\bigvee F_{+}, f_{-}) \in P$ for every $f_{-} \in F_{-}$ and, because R is $\wedge$-closed, $(\bigvee F_{+}, \bigvee F_{-}) \in P$. Clearly, the set $\mathcal{A} = \{(\bigvee F_{+}, \bigvee F_{-}) : F_{+} \subseteq_{\text{fin}} A_{+} \text{ and } F_{-} \subseteq_{\text{fin}} A_{-}\}$ is a directed subset of P and $(\bigvee A_{+}, \bigvee A_{-}) = \bigsqcup^{\uparrow} \mathcal{A} \in \mathfrak{D}(P)$. □

Because $\mathfrak{D}(P)$ is also downwards closed and closed under $\curlywedge$ and $\curlyvee$ (Lemma 3.2.1), $\mathfrak{D}(\mathfrak{D}(P)) = \mathfrak{D}^{\text{ind}}(\mathfrak{D}^{\text{ind}}(P))$ and it might seem that this is already equal to $\mathfrak{D}^{\text{ind}}(P)$. But this is not true in general. Take, for example, $\mathcal{A}_+ = \{a_+\}$ and $\mathcal{A}_- = \{a_-^1, a_-^2\}$ which are $\mathfrak{D}^{\text{ind}}(P)$-independent. Each of (a_+, a_-^1) and $(a_+, a_-^2) \in \mathfrak{D}^{\text{ind}}(P)$ is witnessed by a pair of P-independent sets A_+^1 and A_-^1, and A_+^2 and A_-^2, respectively, such that $a_+ = \bigvee A_+^1 = \bigvee A_+^2$ and $a_-^1 = \bigvee A_-^1$ and $a_-^2 = \bigvee A_-^2$. However, because there is no reason to believe that A_+^1 and A_+^2 are equal, there are no obvious candidates for P-independent sets which would have $(a_+, a_-^1 \vee a_-^2)$ as their supremum. To overcome this problem, we assume the following condition:

(P-ind) For all $\forall \alpha \in \mathfrak{D}^{\text{ind}}(P)$, $\mathcal{B}_+(\alpha_+)$ and $\mathcal{B}_-(\alpha_-)$ are P-independent.

This guarantees, for every $\alpha \in \mathfrak{D}^{\text{ind}}(P)$, a canonical choice of P-independent sets, namely $A_\pm = \mathcal{B}_\pm(\alpha_\pm)$.

3.2.6 Lemma. $\mathfrak{D}(\mathfrak{D}^{\text{ind}}(P)) \subseteq \mathfrak{D}^{\text{ind}}(P)$

Proof. Let $A \subseteq^{\uparrow} \mathfrak{D}^{\text{ind}}(P)$. By ($P$-ind), for every $\alpha \in A$, $\mathcal{B}_+(\alpha_+)$ and $\mathcal{B}_-(\alpha_-)$ are P-independent. As in (3.2.2), because A is directed, the sets $\mathcal{A}_+ \stackrel{\text{def}}{\equiv} \bigcup^{\uparrow}_{\alpha \in A} \mathcal{B}_+(\alpha_+)$ and $\mathcal{A}_- \stackrel{\text{def}}{\equiv} \bigcup^{\uparrow}_{\alpha \in A} \mathcal{B}_+(\alpha_+)$ are P-independent and $\bigsqcup^{\uparrow} A = (\bigvee \mathcal{A}_+, \bigvee \mathcal{A}_-)$. Hence, $\bigsqcup^{\uparrow} A \in \mathfrak{D}^{\text{ind}}(P)$. □

A combination of the preceding lemmas yields the desired result:

3.2.7 Proposition. *Let $P \subseteq L_+ \times L_-$ be downwards closed, closed under logical meets and joins. If (P-ind) holds for P, then $\mathfrak{D}(\mathfrak{D}(P)) = \mathfrak{D}(P)$.*

Proof. $\mathfrak{D}(\mathfrak{D}(P)) \stackrel{(3.2.5)}{=} \mathfrak{D}(\mathfrak{D}^{\text{ind}}(P)) \stackrel{(3.2.6)}{\subseteq} \mathfrak{D}^{\text{ind}}(P) \stackrel{(3.2.5)}{=} \mathfrak{D}(P) \subseteq \mathfrak{D}(\mathfrak{D}(P))$. □

3.2.8 Remark. Because $\mathfrak{D}(P) = \mathfrak{D}^{\text{ind}}(P)$ is downwards closed, for every $\alpha \in \mathfrak{D}(P)$ and every $(b_+, b_-) \in \mathcal{B}_+(\alpha_+) \times \mathcal{B}_-(\alpha_-)$, also $(b_+, b_-) \in \mathfrak{D}(P)$. Therefore, ($P$-ind) can be reformulated in the following more compact way:

(P-ind) $(B_+ \times B_-) \cap \mathfrak{D}(P) \subseteq P$

3.2.9 Remark. Lemma 3.2.5 has applications even beyond this section. As a byproduct it also proves that, for a d-frame $\mathcal{L}$ and $A_\pm \subseteq L_\pm$, if $A_+ \times A_- \subseteq \mathsf{con}$, the join $(\bigvee A_+, \bigvee A_-)$ is in con, and vice versa.

3.2.2 Case 2: correcting (con-tot)

The next step that gets us closer to defining our desired left adjoint to the inclusion **d-Frm** $\hookrightarrow$ **pd-Frm** is to show how to correct the axiom (con-tot). One can try to force the equations that the axiom requires to hold when quotienting $\mathcal{L}$ by the relations

$$R_+ = \{(x, y) \mid \exists z \in L_-. (x, z) \in \mathsf{con} \text{ and } (z, y) \in \mathsf{tot}\}$$
$$R_- = \{(x, y) \mid \exists z \in L_+. (z, x) \in \mathsf{con} \text{ and } (y, z) \in \mathsf{tot}\}$$

also equivalently written as (recall paragraph 3.1.5):

$$R_+ = \mathsf{con}\,;\mathsf{tot}^{-1} \quad \text{and} \quad R_- = \mathsf{con}^{-1}\,;\mathsf{tot}. \tag{3.2.3}$$

By Lemma 3.1.9, we know that $\mathcal{L}/R$ is a proto-d-frame. However, it might still not satisfy (con-tot) and so we have to iterate this procedure. Since, in every step the frame components of the resulting proto-d-frame get smaller, this process eventually stops and yields a proto-d-frame satisfying (con-tot). Because the carrier frames might be infinite it might require a transfinitely many steps before this process stops.

This iterative procedure would work but we would completely lose control over the resulting proto-d-frame. Instead, we describe a procedure which keeps the frame component intact and iteratively extends con, tot, R_+ and R_- and then, after it finishes, we factor only once by the computed relations and obtain a proto-d-frame satisfying (con-tot).

3.2.10 A sequence of quotient structures. Let us fix two frames L_+ and L_- and define $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-)$ to be a *quotient structure (on (L_+, L_-))* if $\mathsf{con}, \mathsf{tot} \subseteq L_+ \times L_-$ and $R_\pm \subseteq L_\pm \times L_\pm$ are binary relations. (We do not require any axioms of d-frames to hold for con and tot.)

We know that, if $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ is a proto-d-frame, then it has the same quotient by R as $(L_+, L_-, \downarrow^R \mathsf{con}, \uparrow^R \mathsf{tot})$ has (recall 3.1.10). In the next step, factoring $(L_+, L_-, \mathsf{con}, \mathsf{tot})/R$ by the missing inequalities (as in (3.2.3)), is the same as factoring $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ by R_+ extended by $(\downarrow^R \mathsf{con})\,;(\uparrow^R \mathsf{tot})^{-1}$ and similarly extended R_-. This motivates the following update operator on quotient structures:

$$\mathfrak{b}^\star(Q) \overset{\text{def}}{\equiv} (\downarrow^R \mathsf{con},\ \uparrow^R \mathsf{tot},\ (\mathsf{con}\,;\mathsf{tot}^{-1}) \cup R_+,\ (\mathsf{con}^{-1}\,;\mathsf{tot}) \cup R_-)$$

A growing (transfinite) sequence of quotient structures $\mathfrak{b}^\gamma(Q) \overset{\text{def}}{\equiv} (\mathsf{con}^\gamma, \mathsf{tot}^\gamma, R_+^\gamma, R_-^\gamma)$, for $\gamma \in \mathsf{Ord}$, is then defined as follows

$$\begin{aligned} \mathfrak{b}^0(Q) &\overset{\text{def}}{\equiv} (\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}, \uparrow\mathsf{tot}_{\mathbb{A},\mathbb{V}}, R_+, R_-)^{2} \\ \mathfrak{b}^{\gamma+1}(Q) &\overset{\text{def}}{\equiv} \mathfrak{b}^\star(\mathfrak{b}^\gamma(Q)) \\ \text{For a limit ordinal } \lambda\text{:} \quad \mathfrak{b}^\lambda(Q) &\overset{\text{def}}{\equiv} (\textstyle\bigcup_{\gamma<\lambda} \mathsf{con}^\gamma, \bigcup_{\gamma<\lambda} \mathsf{tot}^\gamma, \bigcup_{\gamma<\lambda} R_+^\gamma, \bigcup_{\gamma<\lambda} R_-^\gamma). \end{aligned}$$

Since $\mathfrak{b}^\star$ always increases the result and since the components of $\mathfrak{b}^\gamma(Q)$ are bounded by the size of $L_+ \times L_-$, there exists the smallest ordinal γ such that $\mathfrak{b}^\gamma(Q) = \mathfrak{b}^{\gamma+1}(Q)$. Write $\mathfrak{b}^\infty(Q)$ for this $\mathfrak{b}^\gamma(Q)$.

To compute the reflection of a proto-d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ let Q be the quadruple $(\mathsf{con}, \mathsf{tot}, \emptyset, \emptyset)$ and define

$$\mathfrak{b}(\mathcal{L}) = (L_+, L_-, \mathsf{con}', \mathsf{tot}')/R$$

[2]Here, we use the notation from Observation 3.1.15. The 0th step is included only to make sure that $(L_+, L_-, \mathsf{con}^0, \mathsf{tot}^0)$ is a proto-d-frame. Observe that if $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ already was a proto-d-frame, then $\mathfrak{b}^0(Q) = Q$.

where $(\mathsf{con}', \mathsf{tot}', R_+, R_-) = \mathfrak{b}^\infty(Q)$. Set $\kappa^{\mathcal{L}} = (\mu_+^R, \mu_-^R)\colon \mathcal{L} \to \mathfrak{b}(\mathcal{L})$.

Lemma 3.1.14 applied transfinitely gives that all $(L_+, L_-, \mathsf{con}^\gamma, \mathsf{tot}^\gamma)$'s in the sequence for $\mathfrak{b}(\mathcal{L})$ are proto-d-frames. Therefore, $\mathfrak{b}(\mathcal{L})$ is also a proto-d-frame (by Lemma 3.1.9).

3.2.11 Lemma. *Let $h\colon \mathcal{L} \to \mathcal{M}$ be a morphism in* **pd-Frm** *such that $\mathcal{M}$ satisfies* (con-tot)*. Then, h preserves $S_+ \overset{def}{\equiv} \mathsf{con} \,;\mathsf{tot}^{-1}$ and $S_- \overset{def}{\equiv} \mathsf{con}^{-1} ;\mathsf{tot}$, i.e. $h_\pm(x) \leq h_\pm(y)$ whenever $(x,y) \in S_\pm$.*

Proof. Whenever $(x,y) \in \mathsf{con}_{\mathcal{L}} \,;\mathsf{tot}_{\mathcal{L}}^{-1}$, then also $(h_+(x), h_+(y)) \in h[\mathsf{con}_{\mathcal{L}}] \,; h[\mathsf{tot}_{\mathcal{L}}^{-1}]$ and, because h is a d-frame homomorphism, $h[\mathsf{con}_{\mathcal{L}}] \,; h[\mathsf{tot}_{\mathcal{L}}^{-1}] = h[\mathsf{con}_{\mathcal{L}}] \,; h[\mathsf{tot}_{\mathcal{L}}]^{-1}$ is a subset of $\mathsf{con}_{\mathcal{M}} \,;\mathsf{tot}_{\mathcal{M}}$. Since $\mathcal{M}$ satisfies (con-tot), $h_+(x) \leq h_+(y)$. □

3.2.12 Proposition. *Proto-d-frames satisfying* (con-tot) *form a reflective subcategory of* **pd-Frm***, with the reflection given by the homomorphisms $\kappa^{\mathcal{L}}\colon \mathcal{L} \to \mathfrak{b}(\mathcal{L})$.*

Proof. First, we check that $\mathfrak{b}(\mathcal{L})$ satisfies (con-tot). Let $\alpha \in \mathsf{con}'$ and $\beta \in \mathsf{tot}'$ be such that $\mu_-^R(\alpha_-) = \mu_-^R(\beta_-)$. Then, $(\alpha_+, \beta_-) \in {\downarrow^R}\mathsf{con}'$ and so $(\alpha_+, \beta_+) \in ({\downarrow^R}\mathsf{con}') \,; ({\uparrow^R}\mathsf{tot}')^{-1}$. Because $\mathfrak{b}^\infty(Q) = q$ for $Q = (\mathsf{con}', \mathsf{tot}', R_+, R_-)$, ${\downarrow^R}\mathsf{con}' \subseteq \mathsf{con}'$, ${\uparrow^R}\mathsf{tot}' \subseteq \mathsf{tot}'$ and $\mathsf{con}' \,;\mathsf{tot}'^{-1} \subseteq R_+$. Therefore, $(a,b) \in R_+$ and $\mu_+^R(\alpha_+) \leq \mu_+^R(\beta_+)$.

Let $h\colon \mathcal{L} \to \mathcal{M}$ be a morphism in **pd-Frm** such that $\mathcal{M}$ satisfies (con-tot). If h preserves $\mathfrak{b}^\gamma(Q)$ for some ordinal γ, i.e. that $h[\mathsf{con}^\gamma] \subseteq \mathsf{con}_{\mathcal{M}}$, $h[\mathsf{tot}^\gamma] \subseteq \mathsf{tot}_{\mathcal{M}}$ and $(x,y) \in R_\pm^\gamma$ implies $h_\pm(x) \leq h_\pm(y)$, then it also preserves $\mathfrak{b}^{\gamma+1}(Q)$ by Lemmas 3.1.14 and 3.2.11.

By a transfinite induction, $h\colon (L_+, L_-, \mathsf{con}', \mathsf{tot}') \to \mathcal{M}$ is a homomorphism preserving R. Therefore, by Lemma 3.1.9, it can be uniquely extended to the homomorphism $\widetilde{h}\colon \mathfrak{b}(\mathcal{L}) \cong \mathcal{L}/R \to \mathcal{M}$ such that $h = \widetilde{h} \cdot \kappa^{\mathcal{L}}$. □

3.2.3 The reflection

Unfortunately, neither the (con-tot)-reflection (from Section 3.2.2) followed by the (con-$\bigsqcup^{\uparrow}$)-reflection (from Section 3.2.1) nor the vice versa application yields a reflection $\mathfrak{r}\colon$ **pd-Frm** $\to$ **d-Frm** (see Counterexample 3.6.4). Instead, to make this work, we have to combine both of them into one.

Fix two frames L_+ and L_-. Define, for a quotient structure $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-)$ on (L_+, L_-), an operator:

$$\mathfrak{r}^\star(Q) \overset{\text{def}}{\equiv} ({\downarrow^R}\mathfrak{D}(\mathsf{con}),\ {\uparrow^R}\mathsf{tot},\ (\mathsf{con} \,;\mathsf{tot}^{-1}) \cup R_+,\ (\mathsf{con}^{-1} ;\mathsf{tot}) \cup R_-)$$

Then, $\mathfrak{r}^\gamma(Q)$, for every $\gamma \in \mathsf{Ord}$, and $\mathfrak{r}^\infty(Q)$ are defined as in Section 3.2.2. With this is the reflection of a proto-d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ computed as

$$\mathfrak{r}(\mathcal{L}) = (L_+, L_-, \mathsf{con}', \mathsf{tot}')/R$$

where $(\mathsf{con}', \mathsf{tot}', R_+, R_-) = \mathfrak{r}^\infty(\mathsf{con}, \mathsf{tot}, \emptyset, \emptyset)$. Set $\kappa^{\mathcal{L}} = (\mu_+^R, \mu_-^R)\colon \mathcal{L} \to \mathfrak{r}(\mathcal{L})$.

3.2.13 Lemma. *Let $L_\pm$ be two frames and $R_\pm$ two relations on them. Then, for any directed subset $A \subseteq^\uparrow (L_+/R_+)\times(L_-/R_-)$, $(\mu^R)^{-1}[A]$ is directed in $L_+\times L_-$.*

Proof. Let $\alpha^1, \alpha^2 \in \mu^{-1}[A]$. Since A is directed, there is a $\beta \in A$ which is above $\mu(\alpha^1)$ and $\mu(\alpha^2)$. Let α be an element of $\mathcal{L}$ that is mapped to β. We get that $q(\alpha \sqcup \alpha^1 \sqcup \alpha^2) = \beta$ and hence the element $\alpha \sqcup \alpha^1 \sqcup \alpha^2$ belongs to $\mu^{-1}[A]$; it is clearly an upper bound for the α^1 and α^2. □

3.2.14 Lemma. *Let $L_\pm$ be two frames and $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-)$ such that $\mathfrak{r}^\infty(Q) = Q$. Then, $(L_+, L_-, \mathsf{con}, \mathsf{tot})/R$ is a d-frame. In particular, for any proto-d-frame $\mathcal{L}$, $\mathfrak{r}(\mathcal{L})$ is a d-frame.*

Proof. Because $\mathfrak{r}^0(Q) = Q$, $\mathcal{L} \stackrel{\text{def}}{\equiv} (L_+, L_-, \mathsf{con}, \mathsf{tot})$ is a proto-d-frame. Moreover, $\mathcal{L}/R$ satisfies (con-tot) because $\mathfrak{b}(Q) = Q$ (Proposition 3.2.12). To check (con-$\bigsqcup^\uparrow$), let $A \subseteq^\uparrow \mu^R[\mathsf{con}]$. By Lemma 3.2.13 we know that $(\mu^R)^{-1}[A]$ is also directed in $\mathcal{L}$. Since Q is a fixpoint of $\mathfrak{r}^\star$ we have that

$$\downarrow^R \mathsf{con} \subseteq \mathsf{con} \quad \text{and} \quad \mathfrak{D}(\mathsf{con}) \subseteq \mathsf{con} \tag{3.2.4}$$

The former and Lemma 3.1.13 imply that $(\mu^R)^{-1}[\mu^R[\mathsf{con}]] \subseteq \mathsf{con}$ and hence the directed set $(\mu^R)^{-1}[A]$ is contained in con. The second condition in (3.2.4) implies that the supremum of $(\mu^R)^{-1}[A]$ also belongs to con; clearly, it is mapped to the supremum of A by the d-frame homomorphism μ^R which is in $\mu^R[\mathsf{con}]$.

For the "in particular" part, a transfinite application of Lemma 3.1.12 and 3.2.1, proves that $(L_+, L_-, \mathsf{con}', \mathsf{tot}')$ is a proto-d-frame. Then, $Q' = (\mathsf{con}', \mathsf{tot}', R_+, R_-)$ is a fixpoint of $\mathfrak{r}^\infty$ and so the reasoning above proves that $\mathfrak{r}(\mathcal{L})$ is a d-frame. □

3.2.15 Theorem.
d-Frm *forms a reflective subcategory of* **pd-Frm**, *with the reflection given by the homomorphisms $\kappa^{\mathcal{L}} : \mathcal{L} \to \mathfrak{r}(\mathcal{L})$.*

Proof. Thanks to Lemma 3.2.14 we only need to prove the universal property of $\kappa^{\mathcal{L}}$. The proof follows the same lines as the proofs of Proposition 3.2.3 and 3.2.12. Denote $\mathfrak{r}^\gamma(\mathsf{con}, \mathsf{tot}, \emptyset, \emptyset)$ by $(\mathsf{con}^\gamma, \mathsf{tot}^\gamma, R_+^\gamma, R_-^\gamma)$, for all $\gamma \in \mathsf{Ord}$. By Lemma 3.2.2 and 3.1.14, a homomorphism $h\colon \mathcal{L} \to \mathcal{M}$ in **pd-Frm** such that $\mathcal{M}$ is a d-frame, makes the following diagram commute

$$\begin{array}{ccc} (L_+, L_-, \mathsf{con}, \mathsf{tot}) & \xrightarrow{(\mathrm{id}_+, \mathrm{id}_-)} & (L_+, L_-, \mathsf{con}^\gamma, \mathsf{tot}^\gamma) \\ & \searrow^{h} & \downarrow h \\ & & \mathcal{M} \end{array}$$

for all $\gamma \in \mathsf{Ord}$. Moreover such $h\colon (L_+, L_-, \mathsf{con}^\gamma, \mathsf{tot}^\gamma) \to \mathcal{M}$ is unique and, by Lemma 3.2.11, it preserves $R^\gamma_\pm$. This is true, in particular, for $\gamma = \infty$. Hence, h can be uniquely lifted to $\overline{h}$ such that

$$\mathcal{L} \xrightarrow{(\mathrm{id}_+,\mathrm{id}_-)} (L_+, L_-, \mathsf{con}^\infty, \mathsf{tot}^\infty) \xrightarrow{\mu^{R^\infty}} \mathfrak{r}(\mathcal{L}) = (L_+, L_-, \mathsf{con}^\infty, \mathsf{tot}^\infty)/R^\infty \xrightarrow{\overline{h}} \mathcal{M}$$

composes to $h\colon \mathcal{L} \to \mathcal{M}$. □

3.2.4 Reflection from above

Let $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-)$ be a quotient structure on (L_+, L_-). Notice that Q is a fixpoint of $\mathfrak{r}^\infty$ if and only if the following list of conditions is satisfied for Q

(R1) con and tot are $(\curlywedge, \curlyvee)$-closed

(R2) $\downarrow^R \mathsf{con} \subseteq \mathsf{con}$ and $\uparrow^R \mathsf{tot} \subseteq \mathsf{tot}$

(R3) $\mathsf{con}\,;\mathsf{tot}^{-1} \subseteq R_+$ and $\mathsf{con}^{-1}\,;\mathsf{tot} \subseteq R_-$

(R4) $\mathfrak{D}(\mathsf{con}) \subseteq \mathsf{con}$

Let us call the quotient structures which satisfy (R1)–(R4) *reasonable quotient structures* and denote by $\mathcal{RS}(L_+, L_-)$ (or just $\mathcal{RS}(L_\pm)$) the set of all such structures on $L_\pm$.

As we will see shortly we can define the functor $\mathfrak{r}(-)$ only from the structure of $\mathcal{RS}(L_\pm)$. This gives us an alternative description of the reflection **pd-Frm** $\to$ **d-Frm** by a construction "from above". This proved to be very useful as sometimes it fits certain problems better then the iterative description.

3.2.16 Example. For any d-frame $\mathcal{L}$, the quadruple $\Lambda(\mathcal{L}) \stackrel{\text{def}}{\equiv} (\mathsf{con}, \mathsf{tot}, \leq_+, \leq_-)$ is a reasonable quotient structure, i.e. $\Lambda(\mathcal{L}) \in \mathcal{RS}(L_\pm)$[3].

3.2.17 Proposition. *Let $(\mathsf{con}, \mathsf{tot}, R_+, R_-) \in \mathcal{RS}(L_\pm)$ and let $\mu_\pm$ be the quotient homomorphisms $L_\pm \to L_\pm/R_\pm$. Then,*

$$\mathcal{L}/R = (L_+/R_+,\ L_-/R_-,\ \mu[\mathsf{con}],\ \mu[\mathsf{tot}])$$

is a reasonable d-frame.

Proof. Follows from Lemma 3.2.14 as, by (R1) and (R2), is $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ a proto-d-frame and by (R2)–(R4) is Q a fixpoint of $\mathfrak{r}^\star$. □

[3] $\leq_\pm$ are the orders on the frames $L_\pm$. We hope that the reader will not confuse them with the specialisation orders on bispaces (defined in Section 2.1.1), despite their same notation.

3.2.18 Proposition. *$\mathcal{RS}(L_\pm)$ is closed under (coordinatewise) intersections.*

Consequently, $\mathcal{RS}(L_\pm)$ is a complete lattice when ordered by coordinatewise inclusions, i.e. by

$$Q \subseteq Q' \overset{def}{\equiv} \mathsf{con} \subseteq \mathsf{con}',\ \mathsf{tot} \subseteq \mathsf{tot}' \text{ and } R_\pm \subseteq R'_\pm.$$

Proof. Let $\{(\mathsf{con}^i, \mathsf{tot}^i, R^i_+, R^i_-)\}_{i\in I}$ be a collection of reasonable quotient structures. If J is void we have the trivial intersection (more precisely, void meet)

$$(L_+\times L_-,\ L_+\times L_-,\ L_+\times L_+,\ L_-\times L_-)$$

which is reasonable. Hence we can consider the meets $(\bigcap_I \mathsf{con}^i, \bigcap_I \mathsf{tot}^i, \bigcap_I R^i_+, \bigcap_I R^i_-)$ with non-void I. Obviously this system satisfies (R1) and (R4).

(R2): We have $\downarrow^{\bigcap_I R^i}(\bigcap_I \mathsf{con}^i) = (\leq_+^{\bigcap_I R^i}) ; (\bigcap_I \mathsf{con}^i) ; (\geq_-^{\bigcap_I R^i})$ and, since $\leq_\pm^{(-)}$ is monotone (Lemma 3.1.6), this is a subset of $\downarrow^{R^j}(\mathsf{con}^j) = (\leq_+^{R^j}) ; (\mathsf{con}^j) ; (\geq_-^{R^j}) \subseteq \mathsf{con}^j$ for each individual $j \in I$, and hence $(\bigcap_I R^i_+) ; (\bigcap_I \mathsf{con}^i) ; (\bigcap_I R^i_-) \subseteq (\bigcap_I \mathsf{con}^i)$.

(R4): As in (R2), we obtain that $(\bigcap_I \mathsf{con}^i) ; (\bigcap_I \mathsf{tot}^i)^{-1} \subseteq R^j_+$ for each individual $j \in I$ and so $(\bigcap_I \mathsf{con}^i) ; (\bigcap_I \mathsf{tot}^i)^{-1} \subseteq (\bigcap_I R^i_+)$. □

Now, let $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-)$ be a quotient structure. We will show that $\mathfrak{r}^\infty(Q)$ is, in fact, the smallest reasonable quotient structure containing Q. One direction is trivial: $\mathfrak{r}^\infty(Q)$ belongs to $\mathcal{RS}(L_\pm)$ and so the smallest one is smaller than $\mathfrak{r}^\infty(Q)$. For the other direction:

3.2.19 Lemma. *If Q is a quotient structure on (L_+, L_-) and $Q' \in \mathcal{RS}(L_\pm)$ such that $Q \subseteq Q'$. Then, $\mathfrak{r}^\star(Q) \subseteq Q'$.*

Proof. Since Q' is reasonable, it is a fixpoint of $\mathfrak{r}^\star$ and so $\mathsf{con} \subseteq \mathsf{con}'$ implies $\mathfrak{D}(\mathsf{con}) \subseteq \mathsf{con}'$ and, because $\downarrow^{(-)}$ is monotone, $\downarrow^R \mathfrak{D}(\mathsf{con}) \subseteq \downarrow^{R'} \mathfrak{D}(\mathsf{con}) \subseteq \downarrow^{R'} \mathfrak{D}(\mathsf{con}') \subseteq \mathsf{con}'$. Similarly, show that $\uparrow^R \mathsf{tot} \subseteq \mathsf{tot}'$. Also, $(\mathsf{con} ; \mathsf{tot}^{-1}) \cup R_+ \subseteq (\mathsf{con}' ; (\mathsf{tot}')^{-1}) \cup R'_+ = R'_+$ and $(\mathsf{con}^{-1} ; \mathsf{tot}) \cup R_- \subseteq R'_-$. □

We have obtained:

3.2.20 Proposition. *Let $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-)$ be a quotient structure on (L_+, L_-) such that $(L_+, L_-, \mathsf{con}, \mathsf{tot})$ is a proto-d-frame. Then,*

$$\mathfrak{r}^\infty(Q) = \bigcap\{Q' \in \mathcal{RS}(L_\pm) \mid Q \subseteq Q'\}.$$

Next, we prove an important technical proposition with applications later in the text:

3.2.21 Proposition. *Let $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-) \in \mathcal{RS}(M_\pm)$ and let $h_\pm \colon L_\pm \to M_\pm$ be a pair of frame homomorphisms. Then*

$$h^{-1}[[Q]] \overset{def}{\equiv} (h^{-1}[\mathsf{con}],\ h^{-1}[\mathsf{tot}],\ (h_+\times h_+)^{-1}[R_+],\ (h_-\times h_-)^{-1}[R_-])$$

is a reasonable quotient structure, i.e. $h^{-1}[[Q]] \in \mathcal{RS}(L_{\pm})$.

Proof. (R1) immediately follows from the definitions of ff, tt and the lattice structure of $(L_+ \times L_-, \leqslant)$, and from the fact that $h_\pm$ are frame, hence lattice homomorphisms.

(R2): Set $S_\pm = (h_\pm \times h_\pm)^{-1}[R_\pm]$. If $(a,b) \in S_+ ; h^{-1}[\mathsf{con}] ; S_-^{-1}$ then we have some a', b' such that $h_+(a)\ R_+\ h_+(a')\ \mathsf{con}\ h(b')\ R_-^{-1}\ h(b)$ and hence, by (R2) for Q, $(a,b) \in h^{-1}[\mathsf{con}]$. Similarly for tot.

(R3): Let $(a,c) \in h^{-1}[\mathsf{con}] ; (h^{-1}[\mathsf{tot}])^{-1}$. Hence, $(h_+(a), h_-(b)) \in \mathsf{con}$ and $(h_+(c), h_-(b)) \in \mathsf{tot}$ for some $b \in L_-$. By (R3) for Q, $(h_+(a), h_+(c)) \in R_+$, and hence $(a,c) \in S_+$.

(R4): Since $(h_+, h_-)\colon L_+ \times L_- \to M_+ \times M_-$ is a frame homomorphism in the order $\sqsubseteq$, it is obviously Scott continuous and hence $h^{-1}[\mathsf{con}]$ is closed since con is. In particular, $h^{-1}[\mathsf{con}]$ is closed under directed suprema. □

3.2.22 An alternative proof of Theorem 3.2.15. The proposition above allows us to give an alternative proof of the universality $\mathfrak{r}^\infty(-)$. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a d-frame homomorphism form a proto-d-frame $\mathcal{L}$ to a d-frame $\mathcal{M}$. By Proposition 3.2.21,

$$h^{-1}[[\Lambda(\mathcal{M})]] = (h^{-1}[\mathsf{con}_{\mathcal{M}}],\ h^{-1}[\mathsf{tot}_{\mathcal{M}}],\ E(h_+),\ E(h_-))$$

where $\Lambda(\mathcal{M})$ is defined as in Example 3.2.16, $E(h_\pm) \stackrel{\text{def}}{\equiv} (h_\pm \times h_\pm)^{-1}[\leq_\pm]$ (i.e. the relation $\{(x,y) \mid h_\pm(x) \leq h_\pm(y)\}$), is a reasonable quotient structure. From minimality of,

$$\mathsf{con}' \subseteq h^{-1}[\mathsf{con}],\ \mathsf{tot}' \subseteq h^{-1}[\mathsf{tot}],\ R_+ \subseteq E(h_+) \text{ and } R_- \subseteq E(h_-)$$

for $(\mathsf{con}', \mathsf{tot}', R_+, R_-) = \mathfrak{r}^\infty(\mathsf{con}_{\mathcal{L}}, \mathsf{tot}_{\mathcal{L}}, \emptyset, \emptyset)$. By the third and fourth inclusion and by Lemma 3.1.9, there is a d-frame homomorphism $\widetilde{h}_\pm\colon \mathcal{L}/R \to \mathcal{M}$ such that $h = \widetilde{h} \cdot \kappa$. The first and second inclusions yield $h[\mathsf{con}'] \subseteq \mathsf{con}_{\mathcal{M}}$ and $h[\mathsf{tot}'] \subseteq \mathsf{tot}_{\mathcal{M}}$, and since (recall Lemma 3.1.9 again) $\widetilde{h}_\pm$ are restrictions of $h_\pm$ we conclude that $\widetilde{h}[\mathsf{con}'] \subseteq \mathsf{con}_{\mathcal{M}}$ and $\widetilde{h}[\mathsf{tot}'] \subseteq \mathsf{tot}_{\mathcal{M}}$. Therefore, the same $\widetilde{h}$ is also a d-frame homomorphism $\mathfrak{r}(\mathcal{L}) \to \mathcal{M}$. Unicity is obvious since $\kappa_\pm$ are onto.

3.3 Consequences for d-Frm

3.3.1 Limits and colimits

Recall the standard extension of the reflector $\mathfrak{r}$ to a functor: for a morphism $h\colon \mathcal{L} \to \mathcal{M}$ in **pd-Frm** there is precisely one $\mathfrak{r}(h)$ in **d-Frm** such that $\mathfrak{r}(h) \cdot \kappa_{\mathcal{L}} = \kappa_{\mathcal{M}} \cdot h$. Hence we have commutative diagrams

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\kappa_{\mathcal{L}}} & \mathfrak{r}(\mathcal{L}) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathfrak{r}(h)} \\ \mathcal{M} & \xrightarrow{\kappa_{\mathcal{M}}} & \mathfrak{r}(\mathcal{M}) \end{array}$$

in other words, the system $(\kappa_{\mathcal{L}})_{\mathcal{L}}$ is a natural transformation. Note that for $\mathcal{L}$ in **d-Frm** we have $\mathfrak{r}(\mathcal{L}) = \mathcal{L}$ and $\kappa_{\mathcal{L}} = \mathrm{id}$.

In Section 3.1 we proved that **pd-Frm** is a complete and cocomplete category. From **d-Frm** being a reflective subcategory we see that **d-Frm** is also complete and cocomplete, and how the limits and colimits look like. This is a consequence of the following categorical fact.

3.3.1 Proposition. *Let $\mathcal{C}$ be a reflective subcategory of a complete and cocomplete category $\mathcal{D}$. Then, for every diagram $\mathbb{D}$ in $\mathcal{C}$,*

1. *the limit of $\mathbb{D}$ computed in $\mathcal{D}$ is isomorphic to its $\mathcal{C}$-reflection, and*
2. *the $\mathcal{C}$-reflection of the colimits of $\mathbb{D}$ computed in $\mathcal{D}$ is a colimit of $\mathbb{D}$ in $\mathcal{C}$.*

Consequently, $\mathcal{C}$ is also complete and cocomplete.

Proof (see [PP12], for example). Let $\kappa_D \colon D \to \mathfrak{r}(D)$ be the reflector and let $\mathbb{D}$ be a diagram in $\mathcal{C}$, seen as a functor $I \to \mathcal{C}$. A colimit of $\mathbb{D}$ in $\mathcal{D}$ is a cocone $(f_i \colon \mathbb{D}(i) \to A)_{i \in I}$ and its $\mathcal{C}$-reflection $(\kappa_A \cdot f_i \colon \mathbb{D}(i) \to \mathfrak{r}(A))_i$ is also a cocone. Universal property of A gives a $\phi \colon \mathfrak{r}(A) \to A$ such that $\phi \cdot \kappa_A = \mathrm{id}_A$. Hence, κ_A is a section and therefore an isomorphism. (This is a general fact: $\kappa_A \cdot \phi \cdot \kappa_A = \kappa_A$ and uniqueness of the solution $\overline{g} \cdot \kappa_A = g$ for $g = \kappa_A$, makes $\kappa_A \cdot \phi = \mathrm{id}_{\mathfrak{r}(A)}$.)

For (2) denote by $(f_i \colon B \to \mathbb{D}(i))_i$ the limit of $\mathbb{D}$ in $\mathcal{D}$. Then, for any cone $(g_i \colon C \to \mathbb{D}(i))_i$ there is a unique $g^\dagger \colon B \to C$ and if, moreover, $C \in \mathcal{C}$ then $\overline{g^\dagger} \cdot \kappa_B = g^\dagger$ for a unique $\overline{g^\dagger} \colon \mathfrak{r}(B) \to C$. □

This gives a purely categorical proof that limits in **pd-Frm** and **d-Frm** are computed exactly the same way. To sketch how colimits are computed we give explicit formulas for quotients and coproducts.

3.3.2 Quotients in d-Frm. Let $\mathcal{L}$ be a d-frame and let $R = (R_+, R_-)$ be a pair of relations, $R_\pm$ on $L_\pm$. Recall the proto-d-frame $\mathcal{L}/R$ and the quotient map $\mu \colon \mathcal{L} \to \mathcal{L}/R$ from 3.1.9. We set

$$\mu_{\mathfrak{r}} = \kappa_{\mathcal{L}/R} \cdot \mu \colon \mathcal{L} \to \mathfrak{r}(\mathcal{L}/R)$$

and get a morphism in **d-Frm**.

3.3.3 Proposition.

1. *$\mu_{\mathfrak{r}}$ is a quotient of $\mathcal{L}$ by the relation R in* **d-Frm**.
2. *$\mu_{\mathfrak{r}}$ is an extremal epimorphism in* **d-Frm**.

Proof. (1) Follows from Proposition 3.3.1 but we prove it explicitly. Let $h \colon \mathcal{L} \to \mathcal{M}$ be a morphism in **d-Frm** for which $(a, b) \in R_\pm$ implies $h_\pm(a) = h_\pm(b)$. By Lemma 3.1.9

there is $h' \colon \mathcal{L}/R \to M$ in **pd-Frm** such that $h' \cdot \mu = h$. Since $\mathcal{M}$ is in **d-Frm** there is $\widetilde{h}$ in **d-Frm** such that $\widetilde{h} \cdot \kappa_{\mathcal{L}} = h'$. Thus, $\widetilde{h} \cdot \mu_{\mathfrak{r}} = \widetilde{h} \cdot \kappa_{\mathcal{L}} \cdot \mu = h' \cdot \mu = h$. Unicity is obvious since $\mu_{\mathfrak{r}}$ is onto.

(2) Now let $\mu_{\mathfrak{r}} = m \cdot f$ for some $f \colon \mathcal{L} \to \mathcal{M}$ and $m \colon \mathcal{M} \to \mathfrak{r}(\mathcal{L}/R)$ a monomorphism. For $(a,b) \in R_{\pm}$ we have $m_{\pm}(f_{\pm}(a)) = m_{\pm}(f_{\pm}(b))$ and hence $f_{\pm}(a) = f_{\pm}(b)$. Thus, there is an $f' \colon \mathcal{L}/R \to \mathcal{M}$ such that $f' \cdot \mu = f$, and since $\mathcal{M}$ is in **d-Frm** there is $\widetilde{f} \colon \mathfrak{r}(\mathcal{L}/E) \to \mathcal{M}$ such that $\widetilde{f} \cdot \kappa_{\mathcal{L}} = f'$. Consequently, $m \cdot \widetilde{f} \cdot \mu_{\mathfrak{r}} = m \cdot \widetilde{f} \cdot \kappa_{\mathcal{L}} \cdot \mu = m \cdot f' \cdot \mu = m \cdot f = \mu_{\mathfrak{r}}$ and since $\mu_{\mathfrak{r}}$ is onto, $m \cdot \widetilde{f} = \mathrm{id}$. Finally, $m \cdot \widetilde{f} \cdot m = m$, and since m is a monomorphism we see that also $\widetilde{f} \cdot m = \mathrm{id}$. □

3.3.4 Coproducts in d-Frm. Let $\{\mathcal{L}^i\}_{i \in I}$ be a family of d-frames. By $\bigoplus_i \mathcal{L}^i$ denote the coproduct in **pd-Frm** together with the embeddings $\iota^i \colon \mathcal{L}^i \to \bigoplus_i \mathcal{L}^i$ (as in Section 3.1.3). The d-frame reflection gives the d-frame $\mathfrak{r}(\bigoplus_i \mathcal{L}^i)$ and the morphism $\kappa \colon \bigoplus_i \mathcal{L}^i \to \mathfrak{r}(\bigoplus_i \mathcal{L}^i)$. Then, by Proposition 3.3.1,

$$\mathcal{L}^i \xrightarrow{\ \iota^i\ } \bigoplus_i \mathcal{L}^i \xrightarrow{\ \kappa\ } \mathfrak{r}(\bigoplus_i \mathcal{L}^i)$$

are the embeddings into a coproduct in **d-Frm**. Moreover, in Section 3.5.4, we show that the frame components of κ are isomorphisms and also that only one application of $\mathfrak{D}(-)$ is enough to complete the consistency relation under directed suprema. In other words:

3.3.5 Proposition. *The coproduct $\{\mathcal{L}^i\}_{i \in I}$ of a family of d-frames in* **d-Frm** *is computed as*

$$\Big(\bigoplus_i L^i_+,\ \bigoplus_i L^i_-,\ \mathfrak{D}(\downarrow(\mathsf{con}_1)_{\Lambda,\forall}),\ \uparrow(\mathsf{tot}_1)_{\Lambda,\forall}\Big)$$

where $\mathsf{con}_1 = \bigcup_i \iota^i[\mathsf{con}^i]$ *and* $\mathsf{tot}_1 = \bigcup_i \iota^i[\mathsf{tot}^i]$.

3.3.2 (Co)reflections

Olaf Klinke showed in [Kli12], among other things, two important coreflections for the category of d-frames:

3.3.6 Proposition. *d-Normal d-frames form a coreflective subcategory of d-frames.*

Proof sketch (Theorem 2.4.4 in [Kli12]). Let $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ be a d-frame. Define $\lll$ to be the largest interpolative subrelation of $\lhd_+$.[4] Then, $(L_+, L_-, \mathsf{con}, (\lll)^{-1} ; \mathsf{tot})$ is the normal coreflection of $\mathcal{L}$. □

3.3.7 Proposition. *d-Compact d-regular d-frames form a coreflective subcategory of d-normal d-frames.*

[4] By the Axiom of Countable Dependent Choice, $\lll$ can be computed as the smallest fixpoint of $P \mapsto P ; P$. See [BP02] for further details.

Proof sketch (Proposition 4.2.5 [Kli12]). Set $\mathsf{Idl}^{\prec}(L)$ to be the set of such ideals $I \in \mathsf{Idl}(L)$ such that if $x \in I$ then there is some $y \in I$ such that $x \prec y$. Then, for a d-normal d-frame $\mathcal{L}$, define the d-compact d-regular coreflection to be the d-frame

$$(\mathsf{Idl}^{\lhd_+}(L_+), \mathsf{Idl}^{\lhd_-}(L_-), \mathsf{con}_\circ, \mathsf{tot}_\circ)$$

where $(I, J) \in \mathsf{con}_\circ$ iff $(I \times J) \subseteq \mathsf{con}_{\mathcal{L}}$ and $(I, J) \in \mathsf{tot}_\circ$ iff $(I \times J) \cap \mathsf{tot}_{\mathcal{L}} \neq \emptyset$. □

Moreover, we can also adapt the standard proof that regular frames are coreflective in **Frm** [Pul03] to the d-frame context:

3.3.8 Proposition. *d-Regular d-frames form a coreflective subcategory of d-frames.*

Proof. Let $\mathcal{L}$ be a d-frame. Define $\mathcal{L}'$ as $(L'_+, L'_-, \mathsf{con}', \mathsf{tot}')$ where

$$L'_\pm = \{a \mid a = \bigvee\{x \mid x \lhd_\pm a\}\}$$

and $\mathsf{con}' = \mathsf{con} \cap L'_+ \times L'_-$ and $\mathsf{tot}' = \mathsf{tot} \cap L'_+ \times L'_-$. We prove that $\mathcal{L}'$ is a d-frame which follows from the fact that $L'_\pm$ are subframes of $L_\pm$. Indeed, it is closed under finite meets as $1 \in L'_\pm$ and for $a, b \in L'_\pm$, $x \lhd a$ and $y \lhd b$, $(x \wedge y) \lhd (a \wedge b)$. Therefore, $a \wedge b = \bigvee\{x \wedge y \mid x \lhd a, y \lhd b\} \leq \bigvee\{z \mid z \lhd (a \wedge b)\} \leq a \wedge b$ and so $a \wedge b \in L'_\pm$. To check closeness under all joins let $M \subseteq L'_\pm$. Then, since for an $a \in M$ and $x \lhd a$, $x \lhd a \leq \bigvee M$ and $x \lhd \bigvee M$. Hence, $\bigvee M \in L'_\pm$ as

$$\bigvee M = \bigvee\{x \mid \exists a \in M. x \lhd a\} \leq \bigvee\{x \mid x \lhd \bigvee M\} \leq \bigvee M.$$

Next, define a decreasing sequence of d-frames $\{\mathcal{L}^{(\gamma)} = (L^\gamma_+, L^\gamma_-, \mathsf{con}^\gamma, \mathsf{tot}^\gamma)\}_{\gamma \in \mathsf{Ord}}$ by a transfinite induction

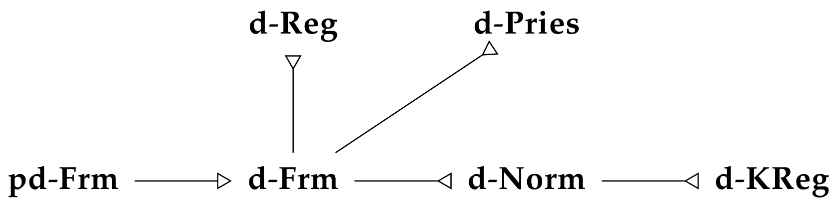
$$\mathcal{L}^{(0)} \stackrel{\text{def}}{\equiv} \mathcal{L}, \quad \mathcal{L}^{(\gamma+1)} = (\mathcal{L}^{(\gamma)})' \quad \text{and} \quad \mathcal{L}^{(\lambda)} = (\bigcap_{\gamma<\lambda} L^\gamma_+, \bigcap_{\gamma<\lambda} L^\gamma_-, \bigcap_{\gamma<\lambda} \mathsf{con}^\gamma, \bigcap_{\gamma<\lambda} \mathsf{tot}^\gamma)$$

(where λ is a limit ordinal.) By $\mathcal{L}^{(\infty)}$ denote the $\mathcal{L}^{(\gamma)}$ for which $\mathcal{L}^{(\gamma)} = \mathcal{L}^{(\gamma+1)}$. Clearly, $\mathcal{L}^{(\infty)}$ is d-regular. To check coreflectivity, let $h\colon \mathcal{M} \to \mathcal{L}$ be a homomorphism from a d-regular d-frame $\mathcal{M}$. Because, for an $a \in M_\pm$,

$$h_\pm(a) = \bigvee\{h_\pm(x) \mid x \lhd a\} \leq \bigvee\{y \mid y \lhd h(a)\} \leq h(a)$$

we see that $h_\pm(a) \in L'_\pm$. Therefore, by a transfinite induction, h lifts to $\overline{h}\colon \mathcal{M} \to \mathcal{L}^{(\infty)}$ such that $h = \iota \cdot \overline{h}$ where ι is the inclusion $\mathcal{L}^{(\infty)} \subseteq \mathcal{L}$. □

Let $\mathcal{C} \longrightarrow\!\!\!\!\triangleright\, \mathcal{D}$ and $\mathcal{C} \longrightarrow\!\!\!\!\triangleleft\, \mathcal{D}$ denote that the category $\mathcal{D}$ is a reflective and coreflective subcategory of $\mathcal{C}$, respectively. Then, we can summarise the relationship between categories of d-frames given by the previous three propositions, Priestley coreflection (Theorem 2.6.11) and the d-frame coreflection (Theorem 3.2.15) in the following diagram:



Furthermore, by the dual version of Proposition 3.3.1, we have that all the coreflective subcategories of d-frames in the diagram are complete and cocomplete and that their colimits coincide with the colimits in the category d-frames. Moreover, we have:

3.3.9 Lemma. *If $\mathcal{L}$ and $\mathcal{M}$ are two d-compact, d-regular, d-zero-dimensional or d-normal d-frames, then $\mathcal{L} \times \mathcal{M}$ also is.*

Proof. To check compactness, if $\bigsqcup_i \alpha^i \in \mathrm{tot}_{\mathcal{L}\times\mathcal{M}}$ then, also $\pi^{\mathcal{L}}(\bigsqcup_i \alpha^i) = \bigsqcup_i \pi^{\mathcal{L}}(\alpha^i) \in \mathrm{tot}_{\mathcal{L}}$ and $\bigsqcup_i \pi^{\mathcal{M}}(\alpha^i) \in \mathrm{tot}_{\mathcal{M}}$ where $\pi^{\mathcal{L}}$ and $\pi^{\mathcal{M}}$ are the projections. Because $\mathcal{L}$ and $\mathcal{M}$ are d-compact there is a finite $I_{\mathcal{L}}$ and $I_{\mathcal{M}}$ such that $\bigsqcup_{i\in I_{\mathcal{L}}\cup I_{\mathcal{M}}} \alpha^i \in \mathrm{tot}_{\mathcal{L}\times\mathcal{M}}$.

d-Regularity and d-zero-dimensionality follow from the fact that $(x, y) \lhd_{\pm} (a, b)$ iff $x \lhd_{\pm} a$ in $\mathcal{L}$ and $y \lhd_{\pm} b$ in $\mathcal{M}$. Checking d-normality is analogous. □

To emphasise all of this for the category of d-compact d-regular d-frames, because it plays an important role later in Chapter 6, we summarise the previous results in the following:

3.3.10 Corollary. *The category of d-compact d-regular d-frames is closed under colimits and finite products (in the category of d-frames).*

3.3.3 A factorisation system

It is well-known that the monomorphisms in **Frm** are precisely the injective frame homomorphisms, [PP12, Lemma III.1.1.1]. An analogous result holds for **pd-Frm** and **d-Frm**.

3.3.11 Proposition. *Let $\mathcal{L}, \mathcal{M}$ be proto-d-frames s (resp., d-frames). A d-frame homomorphism $h\colon \mathcal{L} \to \mathcal{M}$ is a monomorphism in* **pd-Frm** *(resp.,* **d-Frm***) iff both h_+ and h_- are injective frame homomorphisms.*

Proof. The right-to-left direction is immediate. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a morphism such that $h(\alpha) = h(\beta)$ for different $\alpha, \beta \in L_+ \times L_-$. Without loss of generality assume $\alpha_+ \neq \beta_+$. Then let **3** be the three element frame $\{0 < * < 1\}$[5] and let $\mathrm{con}_{\mathrm{triv}}$ and $\mathrm{tot}_{\mathrm{triv}}$ be the minimal consistency and totality relations on $\mathbf{3}\times L_-$. From Exercise 2.3.5, we know that $\mathcal{I} = (\mathbf{3}, L_-, \mathrm{con}_{\mathrm{triv}}, \mathrm{tot}_{\mathrm{triv}})$ is a d-frame.

Since consistency and totality are chosen minimally, we have morphisms f and f' of the type $\mathcal{I} \to \mathcal{L}$, where $f_+(*) = \alpha_+$ and $f'_+(*) = \beta_+$. For the other component

[5] To avoid the usage of the law of excluded middle replace **3** by the free frame of one generator $*$.

we may choose the identity on L_- in both cases. It now holds that $h \cdot f = h \cdot f'$ which shows that h is not a monomorphism. □

3.3.12 Lemma. *If $h\colon \mathcal{L} \to \mathcal{M}$ is a monomorphism in* **pd-Frm** *and $\mathcal{M}$ satisfies* (con-tot)*, then so does $\mathcal{L}$.*

Proof. If $\alpha \in \mathsf{con}_{\mathcal{L}}$, $\beta \in \mathsf{tot}_{\mathcal{L}}$ are elements such that $\alpha_+ = \beta_+$ then $h(\alpha) \in \mathsf{con}_{\mathcal{M}}$, $h(\beta) \in \mathsf{tot}_{\mathcal{M}}$, and $h_+(\alpha_+) = h_+(\beta_+)$. Since $\mathcal{M}$ is balanced, we have $h(\alpha) \sqsubseteq h(\beta)$ or equivalently, $h(\alpha) \sqcap h(\beta) = h(\alpha)$. Since h is an injective homomorphism, it follows that $\alpha \sqcap \beta = \alpha$ or $\alpha \sqsubseteq \beta$. □

The easy observation in the preceding lemma has a rather surprising consequence for the interplay between substructures and the reflection $\mathfrak{r}$.

3.3.13 Proposition. *Let $h\colon \mathcal{L} \to \mathcal{M}$ be a monomorphism from a proto-d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ to a d-frame $\mathcal{M}$. Then the reflection of $\mathcal{L}$ into* **d-Frm** *is given by a reasonable quotient structure $(\mathfrak{D}^\infty(\mathsf{con}), \mathsf{tot}, R_+, R_-)$ on (L_+, L_-) such that $R_\pm \subseteq \leq_\pm$.*

In other words, $\mathfrak{r}(\mathcal{L})$ is carried by the original frames, the original totality relation, and the DCPO-closure of the original consistency relation. The underlying homomorphism of $\kappa_{\mathcal{L}}$ is the identity.

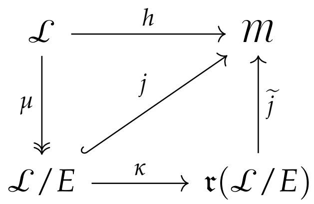
Proof. Set $R_+ = \mathfrak{D}^\infty(\mathsf{con})\,;\mathsf{tot}^{-1}$ and $R_- = \mathfrak{D}^\infty(\mathsf{con})^{-1}\,;\mathsf{tot}$. We will show that $(\mathfrak{D}^\infty(\mathsf{con}), \mathsf{tot}, R_+, R_-)$ is reasonable. Condition (R1) is satisfied by a transfinite application of Lemma 3.2.1. Also, by a transfinite application of Lemma 3.2.2, all $h\colon (L_\pm, \mathfrak{D}^\gamma(\mathsf{con}), \mathsf{tot}) \to \mathcal{M}$ are homomorphism and so, by Lemma 3.3.12,

$$\mathfrak{D}^\gamma(\mathsf{con})\,;\mathsf{tot}^{-1} \subseteq \leq_+ \quad \text{and} \quad \mathfrak{D}^\gamma(\mathsf{con})^{-1}\,;\mathsf{tot} \subseteq \leq_-.$$

In this case $\downarrow^R P$ and $\uparrow^R P$ are the same as $\downarrow P$ and $\uparrow P$, respectively. Hence, (R2) holds because it reduces to $\downarrow\mathfrak{D}^\infty(\mathsf{con}) \subseteq \mathfrak{D}^\infty(\mathsf{con})$ and $\uparrow\mathsf{tot} \subseteq \mathsf{tot}$. Finally, (R3) and (R4) are true by construction. □

3.3.14 Theorem.
The category **d-Frm** *admits the factorization system $(\mathbb{E}, \mathbb{M})$ with $\mathbb{E}$ consisting of all extremal epimorphisms and $\mathbb{M}$ consisting of all monomorphisms.*

Proof. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a morphism in **d-Frm** and consider the kernels $E_\pm$ of $h_\pm$. By Lemma 3.1.9 we may factor $\mathcal{L}$ by E to obtain a proto-d-frame $\mathcal{L}/E$ together with a decomposition of h into morphisms $\mu\colon \mathcal{L} \to \mathcal{L}/E$ and $j\colon \mathcal{L}/E \to \mathcal{M}$, where the latter is injective. We apply the reflection and obtain the d-frame $\mathfrak{r}(\mathcal{L}/E)$ together with the decomposition of j into $\kappa\colon \mathcal{L}/E \to \mathfrak{r}(\mathcal{L}/E)$ and $\widetilde{j}\colon \mathfrak{r}(\mathcal{L}/E) \to \mathcal{M}$. Consider a commutative diagram (in **pd-Frm**):



We know from Proposition 3.3.3 that $\kappa \cdot \mu$ is an extremal epimorphism in **d-Frm**, and from Proposition 3.3.13 that the underlying functions for j and $\widetilde{j}$ are the same; since they are injective, $\widetilde{j}$ is a monomorphism.

The unicity of the factorization (extremal epi, mono) is a standard categorical fact (see the proof of Theorem 14.17. in [Cas+91]). □

3.3.15 Remark.

1. By Proposition 3.3.13 we know a little bit more about the image factorization constructed above: The totality relation on $\mathfrak{r}(\mathcal{L}/E)$ is simply the image of $\mathsf{tot}_{\mathcal{L}}$ under μ and the consistency relation is the DCPO-closure of $\mu[\mathsf{con}_{\mathcal{L}}]$.

2. Extremal epimorphisms in the category **d-Frm** were recently characterised by Imanol Carollo and M. Andrew Moshier [CM17]. They proved that $h\colon \mathcal{L} \to \mathcal{M}$ is an extremal epimorphism iff both h_+ and h_- are surjective maps and, moreover, $\mathfrak{D}^\infty[h[\mathsf{con}_{\mathcal{L}}]] = \mathsf{con}_{\mathcal{M}}$ and $h[\mathsf{tot}_{\mathcal{L}}] = \mathsf{tot}_{\mathcal{M}}$.

3.4 Free constructions

3.4.1 Frames

The category of frames is known to be algebraic [Joh82] which, among other things, means that the forgetful functor $U\colon \mathbf{Frm} \to \mathbf{Set}$ has a left adjoint $\mathbf{Fr}\langle - \rangle$. To compute the *absolutely free frame* $\mathbf{Fr}\langle G \rangle$ on the set of generators G, first, compute the free meet-semilattice on G as the set of all finite subsets of G ordered by reverse inclusion $(\mathcal{F}(G), \supseteq)$ with the meet being the set union. Then, set

$$\mathbf{Fr}\langle G \rangle \stackrel{\text{def}}{\equiv} \mathsf{Down}(\mathcal{F}(G), \cup)$$

where $\mathsf{Down}(X, \wedge)$, for a semilattice $(X, \wedge)$, is the set of all downward closed subsets of X. We can think of elements of $\mathbf{Fr}\langle G \rangle$ as if they were frame terms of the form $\bigvee_i (\bigwedge_{j=1}^{n_i} g_{i,j})$ with $g_{i,j}$'s from G. This follows from the fact that the unit of adjunction $\mathbf{Fr} \dashv U$ is an embedding of the set of generators into the free frame:

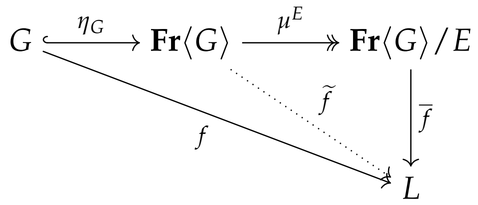
$$\eta_G\colon G \hookrightarrow \mathbf{Fr}\langle G \rangle$$

written explicitly as $g \mapsto {\downarrow}\{g\} = \{F \in \mathcal{F}(G) \mid g \in F\}$. Then, for example, the term $g_1 \vee (g_2 \wedge g_3)$ is represented by ${\downarrow}\{g_1\} \cup {\downarrow}\{g_2, g_3\}$.

A free construction of a frame is computed from its *presentation*, written as $\langle G \mid E \rangle$, where G is a set of generators and $E \subseteq \mathbf{Fr}\langle G \rangle \times \mathbf{Fr}\langle G \rangle$ is a set of equations. Each element of E is thought of as an inequality between freely generated frame terms:

$$\bigvee_i (\wedge_j \, g_{i,j}) \;\leq\; \bigvee_{i'} (\wedge_{j'} \, g'_{i',j'}). \tag{3.4.1}$$

The data $\langle G \mid E \rangle$ then represents the *freely generated frame* $\mathbf{Fr}\langle G \rangle / E$ which we also denote by $\mathbf{Fr}\langle G \mid E \rangle$. Free constructions enjoy an important universal property. Let L be a frame and $f \colon G \to L$ a map. By the adjunction $\mathbf{Fr} \dashv U$, there is a unique frame homomorphism $\widetilde{f}$ such that $f = \widetilde{f} \cdot \eta_G$. If f preserves E, i.e. if for every inequality $x \leq y$ in E, expressed as in (3.4.1), $\widetilde{f}(x) \leq \widetilde{f}(y)$ in L, then f can be extended to a homomorphism $\overline{f} \colon \mathbf{Fr}\langle G \mid E \rangle \to L$ such that the following diagram commutes



The map μ^E, which we for its importance denote as $[\![-]\!] \colon \mathbf{Fr}\langle G \rangle \to \mathbf{Fr}\langle G \mid E \rangle$, gives an *interpretation* of the terms from $\mathbf{Fr}\langle G \rangle$ in the freely generated frame $\mathbf{Fr}\langle G \mid E \rangle$. This also means that any element $x \in \mathbf{Fr}\langle G \mid E \rangle$ is expressible as a combination of generators interpreted in $\mathbf{Fr}\langle G \mid E \rangle$. Concretely, by B denote the $\wedge$-closure of $[\![G]\!]$ in $\mathbf{Fr}\langle G \mid E \rangle$, then $x = \bigvee(\downarrow x \cap B)$.

3.4.1 Example. Let $G = \{g\}$ be a one-element set. Then, because $\mathcal{F}(G) = \{\emptyset, \{g\}\}$, the carrier of $\mathbf{Fr}\langle G \rangle$ contains three elements: $\emptyset$, $\{\{g\}\}$ and $\{\{g\}, \emptyset\}$. Therefore, $\mathbf{Fr}\langle G \rangle$ is isomorphic to the three element lattice $\mathbf{3} = \{0 < \star < 1\}$ and $[\![g]\!] = \star$.

3.4.2 d-Frames

For free constructions of d-frames we need to represent all four pieces of the structure of d-frames. Define a *d-frame presentation* to be the structure $\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}} \rangle$ where $\langle G_\pm \mid E_\pm \rangle$ are frame presentations and $E_{\mathsf{con}}, E_{\mathsf{tot}} \subseteq \mathbf{Fr}\langle G_+ \rangle \times \mathbf{Fr}\langle G_- \rangle$ represent the sets of pairs from which we intend to generate the consistency and totality relations.

Before we explain how to obtain a d-frame from its presentation we introduce an auxiliary category. This category will help us to express the required universal property of free constructions (similarly to what we just explained above for frames).

3.4.2 Definition. Let **Pres** be the *category of presentations* with objects the tuples $(L_+, L_-, E_{\mathsf{con}}, E_{\mathsf{tot}}, E_+, E_-)$, or $(L_\pm, E)$ for short, where $L_\pm$ are frames and the rest are any relations $E_{\mathsf{con}}, E_{\mathsf{tot}} \subseteq L_+ \times L_-$ and $E_\pm \subseteq L_\pm \times L_\pm$.

A morphism of presentations $h \colon (L_\pm, E) \to (M_\pm, F)$ is a pair of frame homomorphisms $h_\pm \colon L_\pm \to M_\pm$ such that

$$h[E_{\mathsf{con}}] \subseteq F_{\mathsf{con}}, \quad h[E_{\mathsf{tot}}] \subseteq F_{\mathsf{tot}} \quad \text{and} \quad (h_\pm \times h_\pm)[E_\pm] \subseteq F_\pm.$$

Unlike in the category of d-frames, in **Pres** the two relations $E_\pm$ are not assumed to satisfy any axioms of d-frames. One of technical advantages of having this category is that, for a d-frame presentation $\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}} \rangle$ and a d-frame $\mathcal{L}$, the pairs of maps $(f_+, f_-) \colon (G_+, G_-) \to (L_+, L_-)$ which preserve $E_\pm$, E_{con} and E_{tot} exactly correspond to the morphisms

$$(\mathbf{Fr}\langle G_+ \rangle\,, \mathbf{Fr}\langle G_- \rangle\,, E_{\mathsf{con}}, E_{\mathsf{tot}}, E_+, E_-) \longrightarrow (L_\pm, \Lambda(\mathcal{L})) \quad \text{in } \mathbf{Pres}.$$

where $(L_\pm, \Lambda(\mathcal{L})) \stackrel{\text{def}}{\equiv} (L_+, L_-, \mathsf{con}_{\mathcal{L}}, \mathsf{tot}_{\mathcal{L}}, \leq_+, \leq_-)$,

3.4.3 Observation. *The mapping* $\mathfrak{i} \colon \mathbf{d\text{-}Frm} \to \mathbf{Pres}$, *defined on objects as*

$$\mathcal{L} \;\mapsto\; (L_\pm, \Lambda(\mathcal{L}))$$

and on morphisms as $h \mapsto h$, is a functor. Moreover, **d-Frm** *is isomorphic to the full subcategory of* **Pres** *consisting of $(L_\pm, \Lambda(\mathcal{L}))$'s where $\mathcal{L}$ is a d-frame.*

3.4.4 Functor dFr. On the other hand, we can use $\mathfrak{r}^\infty(-)$ from Section 3.2.3 to define a functor **Pres** $\to$ **d-Frm**. For any $(L_\pm, E)$ from **Pres**, set

$$(L_\pm, \mathfrak{r}^\infty(E)) \;\stackrel{\text{def}}{\equiv}\; (L_+, L_-, \mathsf{con}, \mathsf{tot}, R_+, R_-)$$

where $(\mathsf{con}, \mathsf{tot}, R_+, R_-) = \mathfrak{r}^\infty(E_{\mathsf{con}}, E_{\mathsf{tot}}, E_+, E_-)$. Observe that *the quotient* of the structure $(L_\pm, \mathfrak{r}^\infty(E))$ by its relations, i.e.

$$\mathbf{dFr}(L_\pm, E) \;\stackrel{\text{def}}{\equiv}\; (L_+, L_-, \mathsf{con}, \mathsf{tot})/R,$$

is always a d-frame (Propositions 3.2.17, 3.2.18 and 3.2.20).

3.4.5 Lemma. *Let $h \colon (L_\pm, E) \to (M_\pm, Q)$ be a morphism in* **Pres** *where $Q \in \mathcal{RS}(M_\pm)$. Then, the following is a commutative diagram in* **Pres**

$$\begin{array}{ccc} (L_\pm, E) & \xrightarrow{(id_+, id_-)} & (L_\pm, \mathfrak{r}^\infty(E)) \\ & \searrow^{h} & \downarrow h \\ & & (M_\pm, Q) \end{array}$$

Proof. All that we need to check is that $h \colon (L_\pm, \mathfrak{r}^\infty(E)) \to (M_\pm, Q)$ is a morphism in **Pres**. By Proposition 3.2.21, $h^{-1}[[Q]]$ is a reasonable quotient structure on (L_+, L_-). Because $h \colon (L_\pm, E) \to (M_\pm, Q)$ is a morphism in **Pres**, $E \subseteq h^{-1}[[Q]]$ and, because $\mathfrak{r}^\infty(E)$ is the smallest reasonable quotient structure containing E (Proposition 3.2.20), $\mathfrak{r}^\infty(E) \subseteq h^{-1}[[Q]]$. □

3.4.6 Proposition. **dFr** *defines the left adjoint to the embedding* $\mathfrak{i}\colon$ **d-Frm** $\to$ **Pres**.

Proof. Let $\mathcal{M}$ be a d-frame and $h\colon (L_\pm, E) \to (M_\pm, \Lambda(\mathcal{M}))$ a morphism in **Pres**. By Lemma 3.4.5, h lifts to $(L_\pm, \mathfrak{r}^\infty(E)) \to (M_\pm, \Lambda(\mathcal{M}))$. Set

$$\mathcal{L} \stackrel{\text{def}}{\equiv} (L_+, L_-, \mathsf{con}, \mathsf{tot})$$

for $(\mathsf{con}, \mathsf{tot}, R_\pm) \stackrel{\text{def}}{\equiv} \mathfrak{r}^\infty(E)$. On the whole, h is a d-frame homomorphism $\mathcal{L} \to \mathcal{M}$ which preserves $R_\pm$ and, by Lemma 3.1.9, it uniquely lifts to a d-frame homomorphism $\widetilde{h}\colon \mathcal{L}/R \to \mathcal{M}$ such that $h = \widetilde{h} \cdot \mu^R$ in **pd-Frm**. Finally, observe that the quotient

$$\rho \stackrel{\text{def}}{\equiv} (L_\pm, E) \xrightarrow{(\mathrm{id}_+, \mathrm{id}_-)} (L_\pm, \mathfrak{r}^\infty(E)) \xrightarrow{\mu^R} (L_\pm/R_\pm,\ \Lambda(\mathcal{L}/R))$$

is a morphism in **Pres**. Thanks to $h = \widetilde{h} \cdot \mu^R$ in **pd-Frm**,

$$(L_\pm, E) \xrightarrow{\rho} (L_\pm/R_\pm,\ \Lambda(\mathcal{L}/R)) \xrightarrow{\mathfrak{i}(\widetilde{h}) = \widetilde{h}} (M_\pm, \Lambda(\mathcal{M}))$$

is equal to $h\colon (L_\pm, E) \to (M_\pm, \Lambda(\mathcal{M}))$ in **Pres**. Therefore, the embedding **d-Frm** $\to$ **Pres** is a reflection with ρ being the unit of adjunction. □

3.4.7 Free d-frames from their presentations. Let $\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$ be a d-frame presentation. Define the *free d-frame* to be the d-frame

$$\mathbf{dFr}\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle \stackrel{\text{def}}{\equiv} \mathbf{dFr}(\mathbf{Fr}\langle G_\pm\rangle, E)$$

or, in other words,

$$(\mathbf{Fr}\langle G_+\rangle, \mathbf{Fr}\langle G_-\rangle, \mathsf{con}, \mathsf{tot})/R$$

where $(\mathsf{con}, \mathsf{tot}, R_+, R_-) = \mathfrak{r}^\infty(E)$.

Recall from Section 3.4.1 the interpretation of frame terms in the free frame, i.e. the homomorphisms $[\![-]\!]_\pm\colon \mathbf{Fr}\langle G_\pm\rangle \to \mathbf{Fr}\langle G_\pm \mid E_\pm\rangle$. Similarly we have two quotients $\mu^R_\pm\colon \mathbf{Fr}\langle G_\pm\rangle \to \mathbf{Fr}\langle G_\pm\rangle/R_\pm$ which, when restricted to the sets of generators, define an *interpretation* map

$$(\!|-|\!)\colon (G_+, G_-) \xrightarrow{(\eta_{G_+}, \eta_{G_-})} (\mathbf{Fr}\langle G_+\rangle, \mathbf{Fr}\langle G_-\rangle) \xrightarrow{\mu^R} \mathbf{dFr}\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$$

Correspondingly to frames, $(\!|-|\!)$ has the following universal property:

3.4.8 Theorem.
Let $\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$ *be a d-frame presentation and* $f = (f_+, f_-)\colon (G_+, G_-) \to \mathcal{L}$ *a pair of functions to a d-frame* $\mathcal{L}$*. If* f *preserves* $E_\pm$, E_{con} *and* E_{tot}*, then there is a* unique *d-frame homomorphism* $h\colon \mathbf{dFr}\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle \to \mathcal{L}$ *such that* $f = h \cdot (\!|-|\!)$

Proof. Let us denote the object in **Pres** corresponding to the d-frame presentation as $(\mathbf{Fr}\langle G_\pm\rangle, E)$ and set $(\mathsf{con}', \mathsf{tot}', R_\pm) \stackrel{\text{def}}{\equiv} \mathfrak{r}^\infty(E)$. We know that f corresponds to a morphism $\widetilde{f}\colon (\mathbf{Fr}\langle G_\pm\rangle, E) \to (L_\pm, \Lambda(\mathcal{L}))$ (in **Pres**). By Proposition 3.4.6, $\widetilde{f}$ uniquely lifts to the d-frame homomorphism h of the required type such that $\widetilde{f} = h \cdot \mu$ (here μ is the pair of natural quotients $\mathbf{Fr}\langle G_\pm\rangle \to \mathbf{Fr}\langle G_\pm\rangle/R$). Since $f_\pm = \widetilde{f}_\pm \cdot \eta_{G_\pm}$ and $(\!|-|\!) = \mu \cdot \eta_G$, we obtain the intended $f = h \cdot (\!|-|\!)$. □

Convention. When giving a d-frame by its presentation we prefer to use the "equational" form rather then writing out the relations $E_\pm$, E_{con} and E_{tot} explicitly. For example, the d-frame presented as

$$\mathbf{dFr}\Big\langle\, a_+;\, a_-, b_-, c_- \;\Big|\; a_- \leq b_-,\; (a_+, b_-) \in \mathsf{con},\; (a_+, c_-) \in \mathsf{tot}, \Big\rangle.$$

corresponds to the d-frame obtained from the presentation $\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$ where $G_+ = \{a_+\}$, $G_- = \{a_-, b_-, c_-\}$, $E_+ = \emptyset$, $E_- = \{(a_-, b_-)\}$, $E_{\mathsf{con}} = \{(a_+, b_-)\}$ and $E_{\mathsf{tot}} = \{(a_+, c_-)\}$.

3.4.3 A special case

It turned out that in all of our applications in this text, the free d-frames generated from their presentations $\langle G_\pm \mid E\rangle = \langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$, are always of a special form. This is because, in those cases, the reflection process stops after one step and then

$$\mathbf{dFr}\langle G_\pm \mid E\rangle = (\mathbf{Fr}\langle G_+ \mid E_+\rangle, \mathbf{Fr}\langle G_- \mid E_-\rangle, \mathfrak{D}({\downarrow}\mathsf{con}_{\mathbb{A},\mathbb{V}}), {\uparrow}\mathsf{tot}_{\mathbb{A},\mathbb{V}}) \tag{3.4.2}$$

where $\mathsf{con}_1 \stackrel{\text{def}}{\equiv} [\![E_{\mathsf{con}}]\!]$ and $\mathsf{tot}_1 \stackrel{\text{def}}{\equiv} [\![E_{\mathsf{tot}}]\!]$[6]. In this subsection we outline sufficient conditions under which this is the case. The starting point is the following general fact.

3.4.9 Proposition. *Let $(L_\pm, E)$ be an object of* **Pres**. *Then,* **dFr** *maps $(L_\pm, E)$ and*

$$(L_\pm/E_\pm,\, \mu[E_{\mathsf{con}}],\, \mu[E_{\mathsf{tot}}],\, \leq_\pm) \tag{3.4.3}$$

to isomorphic d-frames, where μ is the pair of frame quotients $L_\pm \to L_\pm/E_\pm$.

Proof. Denote (3.4.3) by $\mathbb{P}$. We prove the statement by showing that $\mathbf{dFr}(\mathbb{P})$ satisfies the same universal property as $\mathbf{dFr}(L_\pm, E)$ does in Theorem 3.4.8. Let $h\colon (L_\pm, E) \to (L_\pm, \Lambda(\mathcal{L}))$ be a morphism in **Pres**. Then, by the same proof as in Lemma 3.1.9, h uniquely lifts to $\widetilde{h}\colon \mathbb{P} \to (L_\pm, \Lambda(\mathcal{L}))$ such that $h = \widetilde{h} \cdot \mu$ and, by Proposition 3.4.6, $\widetilde{h}$ uniquely lifts to a d-frame homomorphism $\overline{h}\colon \mathbf{dFr}(\mathbb{P}) \to \mathcal{L}$ such that

[6] For brevity, we prefer to write $\mathsf{con}_{\mathbb{A},\mathbb{V}}$ and $\mathsf{tot}_{\mathbb{A},\mathbb{V}}$ instead of $(\mathsf{con}_1)_{\mathbb{A},\mathbb{V}}$ and $(\mathsf{tot}_1)_{\mathbb{A},\mathbb{V}}$ to mean the $(\mathbb{A},\mathbb{V})$-closures of $[\![E_{\mathsf{con}}]\!] = \{([\![\alpha_+]\!]_+, [\![\alpha_-]\!]_-) : \alpha \in E_{\mathsf{con}}\}$ and $[\![E_{\mathsf{tot}}]\!]$ in $\mathbf{Fr}\langle G_+ \mid E_+\rangle \times \mathbf{Fr}\langle G_- \mid E_-\rangle$, respectively.

$$(L_\pm, E) \xrightarrow{\ \mu\ } \mathbb{P} \xrightarrow{\ \rho\ } \mathfrak{i}(\mathbf{dFr}(\mathbb{P})) \xrightarrow{\ \mathfrak{i}(\overline{h})\ } (L_\pm, \Lambda(\mathcal{L}))$$

is equal to h where $\rho\colon \mathbb{P} \to \mathfrak{i}(\mathbf{dFr}(\mathbb{P}))$ is the unit of adjunction from Proposition 3.4.6. □

An immediate consequence of Proposition 3.4.9 is that, for any $\mathbb{P} = (L_\pm, E)$ such that $E_\pm = \leq_\pm$ (as in (3.4.3) above),

$$(L_\pm, \mathfrak{r}^0(E)) \;=\; (L_+, L_-, {\downarrow}(E_{\mathsf{con}})_{\mathbb{A},\mathbb{V}}, {\uparrow}(E_{\mathsf{tot}})_{\mathbb{A},\mathbb{V}}, \leq_+, \leq_-) \;=\; \mathfrak{i}(\mathcal{L})$$

where $\mathcal{L}$ is the proto-d-frame $(L_+, L_-, {\downarrow}(E_{\mathsf{con}})_{\mathbb{A},\mathbb{V}}, {\uparrow}(E_{\mathsf{tot}})_{\mathbb{A},\mathbb{V}})$[7]. In this case, $\mathbf{dFr}(\mathbb{P})$ is by definition equal to $\mathfrak{r}(\mathcal{L})$.

From this, the d-frame $\mathbf{dFr}\langle G_\pm \mid E\rangle$ generated from $\langle G_\pm \mid E\rangle$, is isomorphic to the d-frame reflection of the freely generated proto-d-frame $\langle G_\pm \mid E\rangle$. In other words, we have proved the following:

3.4.10 Corollary. *Let $\langle G_\pm \mid E\rangle$ be a d-frame presentation. Then, $\mathbf{dFr}\langle G_\pm \mid E\rangle$ is isomorphic to the d-frame reflection of the proto-d-frame*

$$\mathcal{L} = (\mathbf{Fr}\langle G_+ \mid E_+\rangle\,,\, \mathbf{Fr}\langle G_- \mid E_-\rangle\,,\, {\downarrow}[\![E_{\mathsf{con}}]\!]_{\mathbb{A},\mathbb{V}},\, {\uparrow}[\![E_{\mathsf{tot}}]\!]_{\mathbb{A},\mathbb{V}})$$

(i.e. the d-frame $\mathfrak{r}(\mathcal{L})$).

Next, we take a look at the conditions which would guarantee that the reflection of the $\mathcal{L}$ (given by the corollary above) stops after one step. To simplify the notation set $L_\pm = \mathbf{Fr}\langle G_\pm \mid E_\pm\rangle$, $\mathsf{con}_1 = [\![E_{\mathsf{con}}]\!]$ and $\mathsf{tot}_1 = [\![E_{\mathsf{tot}}]\!]$. We would like $\mathfrak{r}(\mathcal{L})$ to be equal to

$$(L_+,\ L_-,\ \mathfrak{D}({\downarrow}\mathsf{con}_{\mathbb{A},\mathbb{V}}),\ {\uparrow}\mathsf{tot}_{\mathbb{A},\mathbb{V}}). \tag{3.4.4}$$

In order for this to be the case it is necessary to meet the following two conditions:

1. *The relation $\mathfrak{D}({\downarrow}\mathsf{con}_{\mathbb{A},\mathbb{V}})$ is already $\bigsqcup^{\uparrow}$-closed.* We assure this by assuming (P-ind) from Section 3.2.1.1 for $P = {\downarrow}\mathsf{con}_{\mathbb{A},\mathbb{V}}$ and $B_\pm$ equal to the closure of $[\![G_\pm]\!] \subseteq L_\pm$ under all finite meets (in $L_\pm$). Then, by Proposition 3.2.7, the proto-d-frame in (3.4.4) satisfies (con-$\bigsqcup^{\uparrow}$).

2. *The reflection keeps the frame components unchanged.* In other words we require that the proto-d-frame in (3.4.4) already satisfies (con-tot). We address this requirement in the rest of this section.

3.4.11 Chasing down (con-tot). The axiom (con-tot) for the proto-d-frame in (3.4.4) written explicitly is as follows

[7] Observe that $(L_+, L_-, {\downarrow}(E_{\mathsf{con}})_{\mathbb{A},\mathbb{V}}, {\uparrow}(E_{\mathsf{tot}})_{\mathbb{A},\mathbb{V}})$ is a well-defined proto-d-frame by Lemma 3.1.12.

(con-tot) $\quad \alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\curlywedge,\curlyvee})$ and $\beta \in \uparrow\mathsf{tot}_{\curlywedge,\curlyvee}$ such that

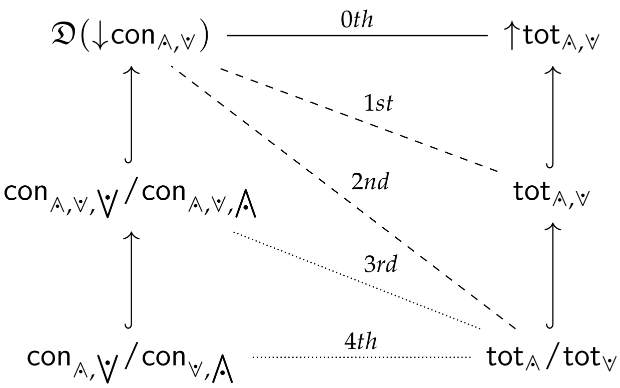
$$(\alpha_+ = \beta_+ \text{ or } \alpha_- = \beta_-) \implies \alpha \sqsubseteq \beta$$

We can split it into two parts:

(λ^0_+-con-tot) $\quad \alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\curlywedge,\curlyvee})$, $\beta \in \uparrow\mathsf{tot}_{\curlywedge,\curlyvee}$ and $\alpha_+ = \beta_+ \implies \alpha_- \leq \beta_-$

(λ^0_--con-tot) $\quad \alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\curlywedge,\curlyvee})$, $\beta \in \uparrow\mathsf{tot}_{\curlywedge,\curlyvee}$ and $\alpha_- = \beta_- \implies \alpha_+ \leq \beta_+$

Our aim now is to restrict α and β to smaller and smaller sets. First, we restate the axioms such that the β's come from $\mathsf{tot}_{\curlywedge,\curlyvee}$ and then from $\mathsf{tot}_{\curlywedge}$ (resp. $\mathsf{tot}_{\curlyvee}$). Then, we do the same with α until we obtain a version of the (con-tot) axiom stated purely in terms of formulas involving only elements from $\mathsf{con}_{\curlywedge,\dot{\bigvee}}$ (resp. $\mathsf{con}_{\curlyvee,\dot{\bigwedge}}$) and $\mathsf{tot}_{\curlywedge}$ (resp. $\mathsf{tot}_{\curlyvee}$). The individual stages are depicted in the diagram below (the λ superscripts in the axiom name correspond to the stages as shown in the diagram):



At every stage we introduce a pair of axioms (named ($\lambda^i_\pm$-con-tot), for $i = 1, \ldots, 4$) and show that they imply the previous axioms. Because the axioms (λ^i_+-con-tot) and (λ^i_--con-tot) are dual to each other, we will always only prove that, say, (λ^i_+-con-tot) implies (λ^{i-1}_+-con-tot) and leave out that (λ^i_--con-tot) implies (λ^{i-1}_--con-tot) as it is proved dually.

3.4.12 Remark. Above we have used a notation similar to the one introduced earlier. The relation $\mathsf{con}_{\curlyvee}$ is the algebraic closure of con_1 under all *finite* logical joins ($\curlyvee$) in $L_+ \times L_-$, and $\mathsf{con}_{\curlywedge}$, $\mathsf{tot}_{\curlyvee}$, $\mathsf{tot}_{\curlywedge}$, $\mathsf{tot}_{\curlywedge,\curlyvee}$ and $\mathsf{con}_{\curlywedge,\curlyvee}$ are defined correspondingly. Likewise, $\mathsf{con}_{\curlywedge,\dot{\bigvee}}$ is the closure of con_1 under finite meets followed by the closure under all joins, both in logical order[8], i.e.

$$\mathsf{con}_{\curlywedge,\dot{\bigvee}} = \{(\bigvee_i \alpha^i_+, \bigwedge_i \alpha^i_-) : \{\alpha^i\}_i \subseteq \mathsf{con}_{\curlywedge}\}.$$

The other versions, such as $\mathsf{con}_{\curlyvee,\dot{\bigwedge}}$, $\mathsf{con}_{\curlywedge,\curlyvee,\dot{\bigvee}}$ and $\mathsf{con}_{\curlywedge,\curlyvee,\dot{\bigwedge}}$, are defined correspondingly.

[8] This makes sense because, in any d-frame $(L_+, L_-, \mathsf{con}, \mathsf{tot})$, $\{(\bigvee_i \alpha^i_+, \bigwedge_i \alpha^i_-) : \{\alpha^i\}_i \subseteq \mathsf{con}\} \subseteq \mathsf{con}$. Indeed, from (con–↓), all $(\alpha^i_+, \bigwedge_i \alpha^i_-) \in \mathsf{con}$ and, by $\curlyvee$ and $\bigsqcup^\uparrow$-closedness, $(\bigvee_i \alpha^i_+, \bigwedge_i \alpha^i_-) \in \mathsf{con}$.

1st stage.

We intend to simplify the elements in the tot relation. Consider the following axioms:

$$(\lambda^1_+\text{-con-tot}) \qquad \alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}),\ \boxed{\beta \in \mathsf{tot}_{\mathbb{A},\mathbb{V}}},\ \beta_+ \leq \alpha_+ \implies \alpha_- \leq \beta_-$$

$$(\lambda^1_-\text{-con-tot}) \qquad \alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}),\ \boxed{\beta \in \mathsf{tot}_{\mathbb{A},\mathbb{V}}},\ \beta_- \leq \alpha_- \implies \alpha_+ \leq \beta_+$$

Now, let $\alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}})$ and let $\beta \in \mathtt{TOT}\langle\mathsf{tot}_1\rangle$ with $\alpha_+ = \beta_+$. That means that there is some $\beta' \in \mathsf{tot}_{\mathbb{A},\mathbb{V}}$ such that $\beta' \sqsubseteq \beta$. We have that $\beta'_+ \leq \alpha_+$ and so we can now apply (λ^1_+-con-tot) and get that $\alpha_- \leq \beta'_-$ and so $\alpha_- \leq \beta'_- \leq \beta_-$. To sum up, we have proved the first part of the following:

3.4.13 Lemma. *($\lambda^1_\pm$-con-tot) implies ($\lambda^0_\pm$-con-tot), and vice versa.*

For the converse assume $\beta_+ \leq \alpha_+$. Then the pair (α_+, β_-) belongs to $\uparrow\mathsf{tot}_{\mathbb{A},\mathbb{V}}$ and by ($\lambda^0_\pm$-con-tot) we can conclude $\alpha_- \leq \beta_-$.

2nd stage.

We can simplify the elements in tot even further. Take the axioms:

$$(\lambda^2_+\text{-con-tot}) \qquad \alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}),\ \boxed{\beta \in \mathsf{tot}_{\mathbb{A}}},\ \beta_+ \leq \alpha_+ \implies \alpha_- \leq \beta_-$$

$$(\lambda^2_-\text{-con-tot}) \qquad \alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}),\ \boxed{\beta \in \mathsf{tot}_{\mathbb{V}}},\ \beta_- \leq \alpha_- \implies \alpha_+ \leq \beta_+$$

Let $\alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}})$ and let $\beta \in \mathsf{tot}_{\mathbb{A},\mathbb{V}}$ with $\alpha_+ \leq \beta_+$. We can decompose β such that $\beta = \bigvee_{k=1}^n \beta^k$ where $\beta^k \in \mathsf{tot}_{\mathbb{A}}$, for every $k = 1, \ldots, n$. Then, for every k, we have that $\beta^k_+ \leq \beta_+ \leq \alpha$ and so $\alpha_- \leq \beta^k_-$. Because $\alpha_- \leq \beta^k_-$ for every k, also $\alpha_- \leq \beta_- = \bigwedge_{k=1}^n \beta^k_-$.

3.4.14 Lemma. *($\lambda^2_\pm$-con-tot) implies ($\lambda^1_\pm$-con-tot), and vice versa.*

Here the converse direction is trivial.

3rd stage.

Now we focus on the complexity of the elements α from con. To eliminate $\mathfrak{D}(-)$ consider the following auxiliary axioms:

$$(\alpha_+\text{-con-tot}) \qquad \boxed{\{(x^k, y)\}_k \subseteq \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}},\ \beta \in \mathsf{tot}_{\mathbb{A}},\ \beta_+ \leq \bigvee_k x^k \implies y \leq \beta_-$$

$$(\alpha_-\text{-con-tot}) \qquad \boxed{\{(x, y^k)\}_k \subseteq \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}},\ \beta \in \mathsf{tot}_{\mathbb{V}},\ \beta_- \leq \bigvee_k y^k \implies x \leq \beta_+$$

Let $\alpha \in \mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}})$. By Lemma 3.2.5, this means that there exist $A_\pm \subseteq B_\pm$ which are $(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}})$-independent and such that $\alpha = (\bigvee A_+, \bigvee A_-)$. Let us fix a $b_- \in A_-$. The $(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}})$-independence of A_+ and A_- means that $A_+ \times \{b\} \subseteq \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$. Because also $\beta_+ \leq \alpha_+ = \bigvee A_+$, we can apply ($\alpha_+$-con-tot) and obtain that $b_- \leq \beta_-$. Since $b_- \in A_-$ has been chosen arbitrarily, $\alpha_- = \bigvee A_- \leq \beta_-$. We have proved that ($\alpha_+$-con-tot) implies ($\lambda^2_+$-con-tot).

Finally, we can get rid of the downwards closure of $\mathsf{con}_{\curlywedge,\curlyvee}$. Consider the following axioms:

$$(\lambda^3_+\text{-con-tot}) \qquad \boxed{\alpha \in \mathsf{con}_{\curlywedge,\curlyvee,\bigcurlyvee}},\ \beta \in \mathsf{tot}_{\curlywedge},\ \beta_+ \leq \alpha_+ \implies \alpha_- \leq \beta_-$$

$$(\lambda^3_-\text{-con-tot}) \qquad \boxed{\alpha \in \mathsf{con}_{\curlywedge,\curlyvee,\bigcurlywedge}},\ \beta \in \mathsf{tot}_{\curlyvee},\ \beta_- \leq \alpha_- \implies \alpha_+ \leq \beta_+$$

Let $\{(x^k, y)\}_k \subseteq {\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}$ be such that $\beta_+ \leq \bigvee_k x^k$. For every k, there exists an $\alpha^k \in \mathsf{con}_{\curlywedge,\curlyvee}$ such that $(x^k, y) \sqsubseteq \alpha^k$. Clearly, $\beta_+ \leq \bigvee_k x^k \leq \bigvee_k \alpha^k_+$, and $\alpha = (\bigvee_k \alpha^k_+, \bigwedge_k \alpha^k_-) \in \mathsf{con}_{\curlywedge,\curlyvee,\bigcurlyvee}$. We can apply ($\lambda^3_+$-con-tot) and obtain that $y \leq \alpha_- \leq \beta_-$. Together with the previous result we have that:

3.4.15 Lemma. *(*$\lambda^3_\pm$-con-tot*) implies (*$\lambda^2_\pm$-con-tot*).*

4th stage.

The final simplification is similar to the 2nd stage but this time acts on the con side:

$$(\lambda^4_+\text{-con-tot}) \qquad \boxed{\alpha \in \mathsf{con}_{\curlywedge,\bigcurlyvee}},\ \beta \in \mathsf{tot}_{\curlywedge},\ \beta_+ \leq \alpha_+ \implies \alpha_- \leq \beta_-$$

$$(\lambda^4_-\text{-con-tot}) \qquad \boxed{\alpha \in \mathsf{con}_{\curlyvee,\bigcurlywedge}},\ \beta \in \mathsf{tot}_{\curlyvee},\ \beta_- \leq \alpha_- \implies \alpha_+ \leq \beta_+$$

Distributivity of $\curlywedge$ and $\curlyvee$ gives us that

$$\mathsf{con}_{\curlywedge,\curlyvee,\bigcurlyvee} = \mathsf{con}_{\curlywedge,\bigcurlyvee} \quad \text{and} \quad \mathsf{con}_{\curlywedge,\curlyvee,\bigcurlywedge} = \mathsf{con}_{\curlyvee,\bigcurlywedge}$$

from which we have:

3.4.16 Lemma. *(*$\lambda^4_\pm$-con-tot*) implies (*$\lambda^3_\pm$-con-tot*), and vice versa.*

Furthermore, ($\lambda^4_\pm$-con-tot) and ($\lambda^1_\pm$-con-tot) are equivalent because

$$\mathsf{con}_{\curlywedge,\bigcurlyvee} \subseteq \mathfrak{D}({\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}) \quad \text{and} \quad \mathsf{con}_{\curlyvee,\bigcurlywedge} \subseteq \mathfrak{D}({\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}). \tag{3.4.5}$$

To prove these inclusions, let $\alpha = (\bigvee_k \alpha^k_+, \bigwedge_k \alpha^k_-)$ where $\{\alpha^k\}_{k\in K} \subseteq \mathsf{con}_{\curlywedge}$. Then, for every $k \in K$, $(\alpha^k_+, \alpha_-) \sqsubseteq \alpha^k$ and so $(\alpha^k_+, \alpha_-) \in {\downarrow}\mathsf{con}_{\curlywedge} \subseteq {\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}$. Because ${\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}$ is $\curlyvee$-closed, $\{(\bigvee_{k\in F} \alpha^k_+, \alpha_-) : F \subseteq_{\mathrm{fin}} K\}$ is directed in ${\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}$ and so $\alpha \in \mathfrak{D}({\downarrow}\mathsf{con}_{\curlywedge,\curlyvee})$.

We can apply similar techniques to simplify (P-ind):

3.4.17 Lemma. *(P-ind) for* $P = {\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}$ *is equivalent to having the following two conditions*

$$({\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}\text{-}ind_+) \qquad (B_+ \times B_-) \cap {\downarrow}\mathsf{con}_{\curlywedge,\bigcurlyvee} \subseteq {\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}$$

$$({\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}\text{-}ind_-) \qquad (B_+ \times B_-) \cap {\downarrow}\mathsf{con}_{\curlyvee,\bigcurlywedge} \subseteq {\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}$$

where $B_\pm$ is the closure of $[\![G_\pm]\!]$ in $\mathbf{Fr}\langle G_\pm \mid E_\pm\rangle$ under all finite meets.

Proof. The left-to-right implication holds immediately from (3.4.5) and the fact that the relation $\mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}})$ is downwards closed. For the other implication, let $(b_+, b_-) \in (B_+ \times B_-) \cap \mathfrak{D}(\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}})$. By Lemma 3.2.5, there exist $A_\pm \subseteq B_\pm$ such that $b_\pm = \bigvee A_\pm$ and $A_+ \times A_- \subseteq \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$. Fix an $a_- \in A_-$. Because $A_+ \times \{a_-\} \subseteq \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$, for every $a_+^k \in A_+$, there exists an $\alpha^k \in \mathsf{con}_{\mathbb{A},\mathbb{V}}$ such that $(a_+^k, a_-) \sqsubseteq \alpha^k$. Then,

$$(b_+, a_-) = (\bigvee A_+, a_-) \sqsubseteq (\bigvee_k \alpha_+^k, \bigwedge_k \alpha_-^k) \in \mathsf{con}_{\mathbb{A},\mathbb{V},\dot{\bigvee}} = \mathsf{con}_{\mathbb{A},\dot{\bigvee}}$$

and because $(b_+, a_-) \in B_+ \times B_-$ we can use ($\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$-ind$_+$) and obtain that $(b_+, a_-) \in \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$.

Since $a_- \in A_-$ has been chosen arbitrarily, $\{b_+\} \times A_- \subseteq \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$. Similarly to the above, for every $a_-^k \in A_-$ there is an $\alpha^k \in \mathsf{con}_{\mathbb{A},\mathbb{V}}$ such that $(b_+, a_-^k) \sqsubseteq \alpha^k$, and $(b_+, b_-) \sqsubseteq (\bigwedge_k \alpha_+^k, \bigvee_k \alpha_-^k) \in \mathsf{con}_{\mathbb{V},\dot{\bigwedge}}$. Finally, by ($\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$-ind$_-$), $(b_+, b_-) \in \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$. □

We sum up all the previous results into this proposition:

3.4.18 Proposition. *If ($\lambda_\pm^4$-con-tot) and ($\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$-ind$_\pm$) hold for a d-frame presentation $\langle G_\pm \mid E\rangle$, then $\mathbf{dFr}\langle G_\pm \mid E\rangle$ is of the form (3.4.2) from page 66.*

3.4.19 Further simplification. In our applications, even stronger and simpler conditions hold for the presentations. Namely, consider the following *"micro version"* of (con-tot):

$$(\mu_+\text{-con-tot}) \qquad \alpha \in \mathsf{con}_{\mathbb{V}},\ \beta \in \mathsf{tot}_{\mathbb{A}},\ \beta_+ \le \alpha_+ \implies \alpha_- \le \beta_-$$

$$(\mu_-\text{-con-tot}) \qquad \alpha \in \mathsf{con}_{\mathbb{A}},\ \beta \in \mathsf{tot}_{\mathbb{V}},\ \beta_- \le \alpha_- \implies \alpha_+ \le \beta_+$$

and the following (more powerful) version of conditions ($\downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$-ind$_\pm$):

$$(\text{Indep}_+) \qquad (L_+ \times B_-) \cap \downarrow\mathsf{con}_{\mathbb{A},\dot{\bigvee}} \subseteq \downarrow\mathsf{con}_{\mathbb{V}}$$

$$(\text{Indep}_-) \qquad (B_+ \times L_-) \cap \downarrow\mathsf{con}_{\mathbb{V},\dot{\bigwedge}} \subseteq \downarrow\mathsf{con}_{\mathbb{A}}$$

3.4.20 Theorem.
If ($\mu_\pm$-con-tot) and (Indep$_\pm$) hold for $\langle G_\pm \mid E\rangle$, then $\mathbf{dFr}\langle G_\pm \mid E\rangle$ results in the following

$$(\mathbf{Fr}\langle G_+ \mid E_+\rangle\,,\ \mathbf{Fr}\langle G_- \mid E_-\rangle\,,\ \mathfrak{D}(\downarrow[\![E_{\mathsf{con}}]\!]_{\mathbb{A},\mathbb{V}}),\ \uparrow[\![E_{\mathsf{tot}}]\!]_{\mathbb{A},\mathbb{V}}).$$

Proof. We use Proposition 3.4.18. Clearly, (Indep$_\pm$) is a strengthening of ($\downarrow$con$_{\mathbb{A},\mathbb{V}}$-ind$_+$). To prove ($\lambda^4_\pm$-con-tot), let $\alpha \in \mathrm{con}_{\mathbb{A},\bigvee}$ and $\beta \in \mathrm{tot}_{\mathbb{A}}$ be such that $\beta_+ \leq \alpha_+$. Moreover, fix a $b_- \in B_-$ such that $b_- \leq \alpha_-$. Then, $(\alpha_+, b_-) \in {\downarrow}\mathrm{con}_{\mathbb{A},\bigvee}$. By, (Indep$_+$), $(\alpha_+, b_-) \in {\downarrow}\mathrm{con}_{\mathbb{V}}$ and so there must be some $\gamma \in \mathrm{con}_{\mathbb{V}}$ such that $(\alpha_+, b_-) \sqsubseteq \gamma$. Because $\beta_+ \leq \alpha_+ \leq \gamma_+$, by ($\mu_+$-con-tot), $b_- \leq \gamma_- \leq \beta_-$. Finally, because $b_- \in {\downarrow}\alpha_- \cap B_-$ has been chosen arbitrarily, then also $\alpha_- = \bigvee({\downarrow}\alpha_- \cap B_-) \leq \beta_-$. □

In this setting, we can make Theorem 3.4.8 more precise:

3.4.21 Proposition. *Let $\langle G_\pm \mid E\rangle$ be a d-frame presentation for which ($\mu_\pm$-con-tot) and (Indep$_\pm$) hold and let $f = (f_+, f_-)\colon (G_+, G_-) \to \mathcal{L}$ be a pair of functions to a d-frame $\mathcal{L}$ preserving $E_\pm$.*

1. *If, moreover, f preserves E_{con} and E_{tot} then the necessarily unique homomorphism $h\colon \mathbf{dFr}\langle G_\pm \mid E\rangle \to \mathcal{L}$ from Theorem 3.4.8 is equal to $\overline{f} = (\overline{f}_+, \overline{f}_-)$.*
2. *f preserves E_{con} and E_{tot} iff $\overline{f}$ preserves $[\![E_{\mathsf{con}}]\!]$ and $[\![E_{\mathsf{tot}}]\!]$.*

(The frame homomorphisms $\overline{f}_\pm\colon \mathbf{Fr}\langle G_\pm \mid E_\pm\rangle \to L_\pm$ are the lifts of $f_\pm$ from Section 3.4.1.)

Proof. By Theorem 3.4.20 we know that the frame components of $\mathbf{dFr}\langle G_\pm \mid E\rangle$ are the frames $\mathbf{Fr}\langle G_\pm \mid E_\pm\rangle$. This means that the inclusion of generators $(\!|-|\!)\colon (G_+, G_-) \to \mathbf{dFr}\langle G_\pm \mid E\rangle$ is equal to the restriction of $[\![-]\!] = ([\![-]\!]_+, [\![-]\!]_-)$ to (G_+, G_-) where $[\![-]\!]_\pm$ are the interpretation maps $\mathbf{Fr}\langle G_\pm\rangle \to \mathbf{Fr}\langle G_\pm \mid E_\pm\rangle$.

(1) If f preserves E_{con} and E_{tot}, by Theorem 3.4.8, there is an $h\colon \mathbf{dFr}\langle G_\pm \mid E\rangle \to \mathcal{L}$ such that $f = h \cdot (\!|-|\!) = h \cdot [\![-]\!]$. However, $\overline{f}_\pm$ are unique such $f_\pm = \overline{f}_\pm \cdot [\![-]\!]_\pm$. Hence, $h_\pm = \overline{f}_\pm$.

(2) Let $\widetilde{f}\colon (\mathbf{Fr}\langle G_+\rangle, \mathbf{Fr}\langle G_-\rangle) \to (L_+, L_-)$ be the pair of frame homomorphisms such that $\widetilde{f}_\pm{\restriction_{G_\pm}} = f_\pm$. Since $\widetilde{f} = \overline{f} \cdot [\![-]\!]$, if $\overline{f}$ preserves $[\![E_{\mathsf{con}}]\!]$ and $[\![E_{\mathsf{tot}}]\!]$ then $\widetilde{f}[E_{\mathsf{con}}] = \overline{f}[[\![E_{\mathsf{con}}]\!]] \subseteq \mathrm{con}_{\mathcal{L}}$ and similarly $\widetilde{f}[E_{\mathsf{tot}}] \subseteq \mathrm{tot}_{\mathcal{L}}$. The reverse implication follows from 1. □

We will find this proposition especially useful in Section 4.3 where we check that certain frame homomorphisms lift componentwise to d-frame homomorphisms and all that will be needed to verify is that the lifted pair of frame homomorphisms preserve $[\![E_{\mathsf{con}}]\!]$ and $[\![E_{\mathsf{tot}}]\!]$.

3.4.4 Single-sorted d-frame presentations

Because of the symmetric nature of d-frames many presentations are symmetric as well. We introduce an alternative and more compact representation of presentations which, effectively, is just a shortcut for the d-frame presentations introduced earlier.

3.4.22 Definition. The data $\langle G \mid S_{\sqsubseteq}, S_{\leqslant}, S_{\mathsf{con}}, S_{\mathsf{tot}}\rangle$ or just $\langle G \mid S\rangle$ constitute a *single-sorted d-frame presentation* if G is a set of generators, $S_{\sqsubseteq}, S_{\leqslant} \subseteq \mathbf{Fr}\langle G\rangle \times \mathbf{Fr}\langle G\rangle$ are relations representing inequalities of the form $\alpha \sqsubseteq \beta$ and $\alpha \leqslant \beta$, respectively, and $S_{\mathsf{con}}, S_{\mathsf{tot}} \subseteq \mathbf{Fr}\langle G\rangle$ are predicates representing constrains of the form $\alpha \in \mathsf{con}$ and $\alpha \in \mathsf{tot}$, respectively.

Every single-sorted d-frame presentation $\langle G \mid S_{\sqsubseteq}, S_{\leqslant}, S_{\mathsf{con}}, S_{\mathsf{tot}}\rangle$ yields an (ordinary) d-frame presentation $\langle G_{\pm} \mid E_{\pm}, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$ in the following way. The sets of generators are defined to be the sets

$$G_{+} \stackrel{\text{def}}{\equiv} \{\alpha_{+} \mid \alpha \in G\} \quad \text{and} \quad G_{-} \stackrel{\text{def}}{\equiv} \{\alpha_{-} \mid \alpha \in G\}$$

(Here the terms “α_{+}” and “α_{-}” are defined purely syntactically.) Then, the quotienting relations become

$$\begin{aligned} E_{+} &\stackrel{\text{def}}{\equiv} \{\alpha_{+} \leq \beta_{+} \mid \alpha \sqsubseteq \beta \text{ in } S_{\sqsubseteq}\} \cup \{\alpha_{+} \leq \beta_{+} \mid \alpha \leqslant \beta \text{ in } S_{\leqslant}\} \quad \text{and} \\ E_{-} &\stackrel{\text{def}}{\equiv} \{\alpha_{-} \leq \beta_{-} \mid \alpha \sqsubseteq \beta \text{ in } S_{\sqsubseteq}\} \cup \{\alpha_{-} \geq \beta_{-} \mid \alpha \leqslant \beta \text{ in } S_{\leqslant}\}, \end{aligned}$$

and, finally, the relations generating con and tot are

$$E_{\mathsf{con}} \stackrel{\text{def}}{\equiv} \{(\alpha_{+}, \alpha_{-}) \mid \alpha \in S_{\mathsf{con}}\} \quad \text{and} \quad E_{\mathsf{tot}} \stackrel{\text{def}}{\equiv} \{(\alpha_{+}, \alpha_{-}) \mid \alpha \in S_{\mathsf{tot}}\}.$$

We write $\mathbf{dFr}\langle G \mid S_{\sqsubseteq}, S_{\leqslant}, S_{\mathsf{con}}, S_{\mathsf{tot}}\rangle$ or just $\mathbf{dFr}\langle G \mid S\rangle$ for the free d-frame generated from $\langle G_{\pm} \mid E_{\pm}, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$.

3.4.23 Proposition. *The functors*

1. $\mathbf{d\text{-}Frm} \to \mathbf{Set} \times \mathbf{Set}$, $\mathcal{L} \mapsto (L_{+}, L_{-})$ *and*
2. $\mathbf{d\text{-}Frm} \to \mathbf{Set}$, $\mathcal{L} \mapsto L_{+} \times L_{-}$

have left adjoints.

Proof. Observe that, for any pair of sets (G_{+}, G_{-}), $\mathbf{dFr}\langle G_{\pm}\rangle$ (which is a presentation such that E_{con}, E_{tot} and $E_{\pm}$ are all empty) equals $\mathcal{L} \stackrel{\text{def}}{\equiv} (\mathbf{Fr}\langle G_{+}\rangle, \mathbf{Fr}\langle G_{-}\rangle, \mathsf{con}_{\mathsf{triv}}, \mathsf{tot}_{\mathsf{triv}})$. Indeed, $\mathcal{L}$ is a d-frame and we have a universal pair of maps $(\!|-|\!) \colon (G_{+}, G_{-}) \to \mathcal{L}$ in $\mathbf{Set} \times \mathbf{Set}$ as any pair of maps $(G_{+}, G_{-}) \to \mathcal{M}$ into a d-frame $\mathcal{M}$ lifts to a d-frame homomorphism $\mathcal{L} \to \mathcal{M}$. Moreover, this is exactly the universal property of a reflection needed for (1). The left adjoin in (2), is the functor $G \mapsto \mathbf{dFr}\langle G\rangle$, which generates a d-frame from its single-sorted presentation, with $g \mapsto (\!|g_{+}, g_{-}|\!)$ as the unit of adjunction. □

3.5 Examples of free constructions

In this section we show that a number of free frame constructions have their d-frame variants. Moreover, as we will also see, the conditions of Theorem 3.4.20 are satisfied for all of our examples. This then means that the frame components of our freely generated d-frames are the same as the original freely generated frames.

Given a d-frame presentation as $\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$ or $\langle G \mid S_\sqsubseteq, S_\leqslant, S_{\mathsf{con}}, S_{\mathsf{tot}}\rangle$ we will always follow the same procedure before we use Theorem 3.4.20:

1. Use Proposition 3.4.9 and transform the presentation into an object of the category **Pres** of the form $(\mathbf{Fr}\langle G_+ \mid E_+\rangle, \mathbf{Fr}\langle G_- \mid E_-\rangle, \mathsf{con}_1, \mathsf{tot}_1, \leq_+, \leq_-)$.
2. Examine $\mathbf{Fr}\langle G_\pm \mid E_\pm\rangle$.
3. Examine $\mathsf{con}_{⩓,⩔}$ and $\mathsf{tot}_{⩓,⩔}$, i.e. the $(⩓, ⩔)$-closures of con_1 and tot_1, respectively.
4. Prove ($\mu_\pm$-con-tot) and (Indep$_\pm$).

Consequently, Theorem 3.4.20 proves that the generated d-frame is of the form

$$(\mathbf{Fr}\langle G_+ \mid E_+\rangle, \mathbf{Fr}\langle G_- \mid E_-\rangle, \mathfrak{D}(\downarrow\mathsf{con}_{⩓,⩔}), \uparrow\mathsf{tot}_{⩓,⩔}).$$

Notice that examining the structure of $\mathbf{Fr}\langle G_\pm \mid E_\pm\rangle$ is not a waste of time because the frame components of the generated d-frame stay unchanged.

3.5.1 d-Frames 2×2 and 3×3

The first set of examples is actually even simpler and so we do not have to explicitly follow the procedure outlined above. In Proposition 3.4.23 we examined that $\mathbf{dFr}\langle G\rangle$, for a set G, is the d-frame

$$(\mathbf{Fr}\langle G\rangle, \mathbf{Fr}\langle G\rangle, \mathsf{con}_{\mathrm{triv}}, \mathsf{tot}_{\mathrm{triv}}).$$

Since, classically[9], $\mathbf{Fr}\langle\emptyset\rangle \cong \mathbf{2}$ and $\mathbf{Fr}\langle\{\star\}\rangle \cong \mathbf{3}$, the d-frame $\mathbf{dFr}\langle\emptyset\rangle$ is isomorphic to $\mathbf{2}\times\mathbf{2}$ and $\mathbf{dFr}\langle\{\star\}\rangle$ is isomorphic to $\mathbf{3}\times\mathbf{3}$ – two d-frames known from Example 2.3.15. Adding $\star$ to con or tot corresponds to the other two cases also discussed therein.

3.5.2 The functor $\mathcal{IF}$: DLat → d-Pries

Let D be a distributive lattice. Consider the d-frame $\mathcal{L}_D$ presented as follows

$$\mathbf{dFr}\Big\langle \langle d\rangle : d \in D \;\Big|\; \langle d\rangle ⩔ \langle e\rangle = \langle d \vee e\rangle,\ \langle 0\rangle = \mathit{ff},\quad \langle d\rangle ⩓ \langle e\rangle = \langle d \wedge e\rangle,\ \langle 1\rangle = \mathit{tt},\quad (\forall d \in D)\ \langle d\rangle \in \mathsf{con}, \langle d\rangle \in \mathsf{tot}\Big\rangle.$$

The first step in the proposed procedure at the beginning of Section 3.5 yields an object $\mathbb{P}_D$ of the form $(L_+, L_-, \mathsf{con}_1, \mathsf{tot}_1, \leq_+, \leq_-)$ such that $\mathcal{L}_D \cong \mathbf{dFr}(\mathbb{P}_D)$. This means that $L_\pm$ are the quotients of the free frames $\mathbf{Fr}\langle\langle d\rangle_\pm : d \in D\rangle$ by the two-sorted versions of the equations in the presentation of $\mathcal{L}_D$.

[9] By "classically" we mean that we use the Law of Excluded Middle.

3.5.1 Lemma. *The frame components of $\mathbb{P}_D$ are isomorphic to the frames of ideals and filter of D, respectively. The embedding $d \mapsto \langle d \rangle$ then corresponds to $d \mapsto (\downarrow d, \uparrow d)$.*

Proof. Follows from the fact that $\mathsf{Idl}(D)$ is isomorphic to $\mathbf{Fr}\langle G \mid R \rangle$ where $G = \{[d] : d \in D\}$ and R are the equations $[d] \vee [e] = [d \vee e]$, $[d] \wedge [e] = [d \wedge e]$, $[0] = \mathit{ff}$ and $[1] = \mathit{tt}$ (see Theorem 9.2.2 in [Vic89]). The presentation of L_- is just the upside down version of L_+; or, in other words, $L_- \cong \mathsf{Idl}(D^{\mathrm{op}})$ which is isomorphic to $\mathsf{Filt}(D)$. □

As a result $\mathbb{P}_D$ is actually of the form $(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{con}_1, \mathsf{tot}_1, \leq_\pm)$ where both con_1 and tot_1 are equal to the set $\{(\downarrow d, \uparrow d) : d \in D\}$.

3.5.2 Lemma.

1. $\mathsf{con}_{\mathbb{A},\mathbb{V}} = \mathsf{con}_1$ *and* $\mathsf{tot}_{\mathbb{A},\mathbb{V}} = \mathsf{tot}_1$,
2. $\mathbb{P}_D$ *satisfies the axioms* ($\mu_\pm$-con-tot) *and* (Indep$_\pm$).

Proof. The first part is immediate. For ($\mu_\pm$-con-tot) let $\alpha \in \mathsf{con}_1$ and $\beta \in \mathsf{tot}_1$. By definition, $\alpha = (\downarrow d, \uparrow d)$ and $\beta = (\downarrow e, \uparrow e)$ for some $d, e \in D$. Therefore, if $\beta_+ \leq \alpha_+$, that means that $e \leq d$, then $\alpha_- \leq \beta_-$.

Next, to check (Indep$_\pm$), let $\alpha \in L_+ \times B_-$ such that $\alpha \sqsubseteq \beta$ for some $\beta \in \mathsf{con}_{\mathbb{A},\dot{\bigvee}}$. By definition, α is of the form $(I, \uparrow b)$ and β is of the form $\dot{\bigvee}_i(\downarrow x_i, \uparrow x_i)$. Then, $\alpha_- \leq \beta_-$ means that $x_i \leq b$ (for all i). Therefore, $I \leq \bigvee_i(\downarrow x_i) \leq \downarrow b$ and so $\alpha = (I, \uparrow b) \sqsubseteq (\downarrow b, \uparrow b) \in \mathsf{con}_1 \subseteq \downarrow \mathsf{con}_{\mathbb{V}}$. □

3.5.3 Proposition. *$\mathcal{L}_D$ is isomorphic to $\mathcal{IF}(D)$, defined in Section 2.6.*

Proof. By Theorem 3.4.20 and the previous lemmas, $\mathcal{L}_D$ is isomorphic to the d-frame of the form $(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathfrak{D}(\downarrow \mathsf{con}_1), \uparrow \mathsf{tot}_1)$. Hence, the carrier frames of $\mathcal{L}_D$ and $\mathcal{IF}(D)$ are the same. To see that their totality relations agree as well notice that $(I, F) \in \uparrow \mathsf{tot}_1$ iff there exists a $d \in D$ such that $(\downarrow d, \uparrow d) \sqsubseteq (I, F)$ which is the same as $I \cap F \neq \emptyset$.

Similarly, $(I, F) \in \downarrow \mathsf{con}_1$ iff there is a $d \in D$ such that $(I, F) \sqsubseteq (\downarrow d, \uparrow d)$ which is equivalent to $\forall i \in I, f \in F\colon\ i \leq d \leq f$. Therefore, $\downarrow \mathsf{con}_1 \subseteq \mathsf{con}_D$ where con_D is the consistency relation of $\mathcal{IF}(D)$ (see Section 2.6). Hence, also $\mathfrak{D}(\downarrow \mathsf{con}_1) \subseteq \mathsf{con}_D$. For the other direction let $(I, F) \in \mathsf{con}_D$. Since, for all $i \in I$ and $f \in F$, $i \leq f$ and $(\downarrow i, \uparrow f) \sqsubseteq (\downarrow i, \uparrow i) \in \mathsf{con}_1$. We have that $(\downarrow i, \uparrow f) \in \downarrow \mathsf{con}_1$. Moreover, (I, F) is a directed union of such $(\downarrow i, \uparrow f)$'s and so $(I, F) \in \mathfrak{D}(\downarrow \mathsf{con}_1)$. □

3.5.3 The d-frame of reals

Now we will present the d-frame of reals $\mathcal{L}(\mathbb{R})$. The starting point of our definition is a presentation of the biframe of lower and upper topologies of $\mathbb{R}$ [GP07]. We rewrite García-Picado's presentation into the language of d-frames and obtain the d-frame $\mathcal{L}(\mathbb{R})$ presented as $\mathbf{dFr}\langle G_\pm \mid E\rangle$ where $G_+ = \{(-,q) : q \in \mathbb{Q}\}$, $G_- = \{(q,-) : q \in \mathbb{Q}\}$ and E is the following set of equations

(for the plus side:)

$$(-,q) \vee (-,q') = (-,\max(q,q')), \quad (-,q) = \bigvee_{q'<q} (-,q'),$$
$$(-,q) \wedge (-,q') = (-,\min(q,q')), \quad 1 = \bigvee_{q} (-,q),$$

(for the minus side:)

$$(q,-) \vee (q',-) = (\min(q,q'),-), \quad (q,-) = \bigvee_{q<q'} (q',-),$$
$$(q,-) \wedge (q',-) = (\max(q,q'),-), \quad 1 = \bigvee_{q} (q,-),$$

(for con and tot:)

$$((-,q),(q',-)) \in \mathsf{con} \quad \text{if} \quad q \leq q', \quad ((-,q),(q',-)) \in \mathsf{tot} \quad \text{if} \quad q' < q.$$

Each $(-,q)$ and $(q,-)$ is intended to abstractly represent the opens $(-\infty,q)$ and $(q,+\infty)$, respectively. As before, we have an object $\mathbb{P}_\mathbb{R}$ of **Pres** such that $\mathcal{L}(\mathbb{R}) \cong \mathbf{dFr}(\mathbb{P}_\mathbb{R})$. The frame components of $\mathbb{P}_\mathbb{R}$ are the frames of upper and lower opens $\mathcal{L}_l(\mathbb{R})$ and $\mathcal{L}_u(\mathbb{R})$ computed as the quotients of $\mathbf{Fr}\langle G_\pm\rangle$ by the equations for the plus or minus side, respectively.

3.5.4 Lemma. *$\mathcal{L}_l(\mathbb{R}) \cong \tau_l^\mathbb{Q}$ and $\mathcal{L}_u(\mathbb{R}) \cong \tau_u^\mathbb{Q}$ (as frames) where $(\mathbb{Q}, \tau_l^\mathbb{Q}, \tau_u^\mathbb{Q})$ is the bitopological space with the lower and upper topology on the set of rational numbers.*

Proof. The proof is very similar to the proof of Lemma 3.5.1. We show that the embedding $\iota\colon \mathbb{Q} \to \tau_l^\mathbb{Q}$, $q \mapsto \{x \in \mathbb{Q} \mid x < q\}$, has the universal property of $\mathcal{L}_l(\mathbb{R})$. Clearly, ι preserves the defining equations of $\mathcal{L}_l(\mathbb{R})$. For example, $\iota(q)$ is equal to the union $\bigcup_{q'<q} \iota(q')$. We need to verify that any $f\colon \mathbb{Q} \to M$ into a frame M which preserves E uniquely lifts to a frame homomorphism $\overline{f}\colon \mathsf{Down}(\mathbb{Q}) \to M$ such that $f = \overline{f} \cdot \iota$. Define $\overline{f}$ as the map $M \mapsto \bigvee f[M]$. We check that $\overline{f}$ is a frame homomorphism. Let $\{M_i\}_i \subseteq \mathsf{Down}(\mathbb{Q})$. Then, $\bigvee f[\bigcup_i M_i] = \bigvee(\bigcup_i f[M_i]) = \bigvee_i(\bigvee f[M_i]) = \bigvee_i \overline{f}[M_i]$. To check finite meets, $\bigvee f[M_1 \cap M_2] = \bigvee\{f(\min(x_1,x_2)) : x_1 \in M_1,\ x_2 \in M_2\}$ and, since f preserves the equations $\mathcal{L}_R(\mathbb{R})$, $\bigvee f[M_1 \cap M_2]$ is equal to $\bigvee\{f(x_1) \wedge f(x_2) : x_i \in M_i\} = \bigvee f[M_1] \wedge \bigvee f[M_2]$. For unicity, take an $h\colon \tau_l^\mathbb{Q} \to M$ such that $f = h \cdot \iota$. Then, $h(M) = h(\bigcup\{\iota(q) : q \in M\}) = \bigvee\{h(\iota(q)) : q \in M\} = \bigvee f[M]$ since $h(\iota(q)) = f(x) = \overline{f}(\iota(q))$. □

This means that $\mathbb{P}_\mathbb{R}$ is of the form $(\tau_l^\mathbb{Q}, \tau_u^\mathbb{Q}, \mathsf{con}_1, \mathsf{tot}_1, \leq_\pm)$ with the consistency and totality relations containing the pairs of $(-\infty,q)_\mathbb{Q} \stackrel{\text{def}}{\equiv} \{x \in \mathbb{Q} \mid x < q\}$ and $(q,+\infty)_\mathbb{Q} \stackrel{\text{def}}{\equiv} \{y \in \mathbb{Q} \mid q < y\}$ for which $q \leq q'$ and $q' < q$, respectively.

3.5.5 Lemma.

1. $\mathsf{con}_{\curlywedge,\curlyvee} = \mathsf{con}_1 \cup \{\mathtt{tt}, \mathtt{ff}\}$ *and* $\mathsf{tot}_{\curlywedge,\curlyvee} = \mathsf{tot}_1 \cup \{\mathtt{tt}, \mathtt{ff}\}$.
2. *The presentation of* $\mathcal{L}(\mathbb{R})$ *satisfies the axioms* ($\mu_\pm$-con-tot) *and* (Indep$_\pm$).

Proof. (1) tt appears in the relations as a consequence of closing them under all finite $\leqslant$-meets because $\mathit{tt} = \bigwedge \emptyset$ and, similarly, ff is the empty $\leqslant$-join.

(2) To check ($\mu_\pm$-con-tot), without loss of generality, let $((-\infty, q_+)_\mathbb{Q}, (q_-, +\infty)_\mathbb{Q}) \in \mathsf{con}_\curlyvee$ and $((-\infty, q'_+)_\mathbb{Q}, (q'_-, +\infty)_\mathbb{Q}) \in \mathsf{tot}_\curlywedge$ be such that $(-\infty, q'_+)_\mathbb{Q} \subseteq (-\infty, q_+)_\mathbb{Q}$. Therefore, $q'_- < q'_+ \leq q_+ \leq q_-$ and so $(q_-, +\infty)_\mathbb{Q} \subseteq (q'_-, +\infty)_\mathbb{Q}$. For (Indep$_\pm$), w.l.o.g., let $(U, (q, +\infty)_\mathbb{Q})$ be $\sqsubseteq$-smaller than $\curlyvee_{i=1}^n((-\infty, q^i_+)_\mathbb{Q}, (q^i_-, +\infty)_\mathbb{Q})$ where $q^i_+ \leq q^i_-$ for all i. Then, $q^i_- \leq q$ and therefore also $q^i_+ \leq q$, for all i. Hence, $(U, (q, +\infty)_\mathbb{Q}) \sqsubseteq ((-\infty, q)_\mathbb{Q}, (q, +\infty)_\mathbb{Q}) \in \mathsf{con}_1 \subseteq {\downarrow}\mathsf{con}_\curlyvee$. □

From this and Theorem 3.4.20 we know that $\mathcal{L}(\mathbb{R})$ equals $(\tau^\mathbb{Q}_l, \tau^\mathbb{Q}_u, \mathfrak{D}({\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}), {\uparrow}\mathsf{tot}_{\curlywedge,\curlyvee})$. We also have an exact description of the consistency and totality relations.

3.5.6 Lemma. *Let* $(U_l, U_u) \in \tau^\mathbb{Q}_l \times \tau^\mathbb{Q}_u$. *Then,*

1. $(U_l, U_u) \in \mathsf{con}_{\mathcal{L}(\mathbb{R})}$ *iff* $U_l \cap U_u = \emptyset$ *and*
2. $(U_l, U_u) \in \mathsf{tot}_{\mathcal{L}(\mathbb{R})}$ *iff* $U_l \cap U_u \neq \emptyset$ *or* $U_l = \mathbb{Q}$ *or* $U_u = \mathbb{Q}$.

Proof. Clearly con_1 is a subrelation of the consistency of $\Omega_d(\mathbb{Q})$ and so are its $\bigsqcup^\uparrow, \downarrow, \curlywedge$ and $\curlyvee$-closures, i.e. $\mathfrak{D}({\downarrow}\mathsf{con}_{\curlywedge,\curlyvee}) \subseteq \mathsf{con}_\mathbb{Q}$. This proves the left-to-right implication in (1). On the other hand, assume $U_l \cap U_u = \emptyset$. For all $q_+ \in U_l$ and $q_- \in U_u$, $q_+ < q_-$ and $((-\infty, q_+)_\mathbb{Q}, (q_-, +\infty)_\mathbb{Q}) \in \mathsf{con}_1$. Also, each of U_l and U_u are directed unions of such opens, that is $U_l = \bigcup^\uparrow\{(-\infty, q)_\mathbb{Q} : q \in U_l\}$ and $U_u = \bigcup^\uparrow\{(q, +\infty)_\mathbb{Q} : q \in U_u\}$. Therefore, $(U_l, U_u) \subseteq \mathfrak{D}(\mathsf{con}_1) \subseteq \mathfrak{D}({\downarrow}\mathsf{con}_1)$.

For (2), by definition, $(U_l, U_u) \in \mathsf{tot}_{\mathcal{L}(\mathbb{R})}$ iff either U_l or U_u is equal to $\mathbb{Q}$ or $((-\infty, q)_\mathbb{Q}, (q', +\infty)_\mathbb{Q}) \sqsubseteq (U_l, U_u)$ for some $q' < q$ from $\mathbb{Q}$. The latter case implies that $x \in U_l \cap U_u$ for some $x \in (q', q) \cap \mathbb{Q}$. Conversely, if $x \in U_l \cap U_u$ then, because U_l and U_u are open in Euclidean topology, $(-\infty, q)_\mathbb{Q} \subseteq U_l$ and $(q', +\infty)_\mathbb{Q} \subseteq U_p$ for some rational numbers $q' < x < q$. □

3.5.7 Proposition. $\Sigma_d(\mathcal{L}(\mathbb{R}))$ *is bihomeomorphic to the bispace of reals* $(\mathbb{R}, \tau^\mathbb{R}_+, \tau^\mathbb{R}_-)$ *with the topologies of the lower and upper opens.*

Proof. From [Ban97] we know that $\Sigma(\mathcal{L}_l(\mathbb{R}))$ is homeomorphic to the space $\mathbb{R} \cup \{+\infty\}$ with the topology of lower opens. Each $r \in \mathbb{R}$ corresponds to the completely prime filter $P^r_+ = \{(-\infty, q) \mid r < q\}$ and $+\infty$ corresponds to the completely prime filter $P^{+\infty}_+$ of all non-empty intervals (The interval $(-\infty, q)_\mathbb{Q}$ is interpreted as the interval of reals $(-\infty, q)$). Similarly, $\Sigma(\mathcal{L}_u(\mathbb{R}))$ is homeomorphic to the space $\mathbb{R} \cup$

$\{-\infty\}$ with the topology of upper opens with P^r_- and $P^{-\infty}_-$ defined accordingly. We will use the description of points from Section 2.3.3.1 to show that $(P^r_+, P^{r'}_-) \in \Sigma_d(\mathcal{L}(\mathbb{R}))$ if and only if $r = r'$ (therefore, implicitly, neither $r = +\infty$ nor $r' = -\infty$). This then proves that there is a bijection between the points of $\Sigma_d(\mathcal{L}(\mathbb{R}))$ and real numbers.

For "$\Rightarrow$", let $(P^r_+, P^{r'}_-)$ be a point of $\mathcal{L}(\mathbb{R})$. If $r = +\infty$ and $r' \in \mathbb{R}$, then take a rational number q from the interval $(r'-1, r')$. The pair $((-\infty, q), (q, +\infty))$ belongs to $\mathsf{con}_{\mathcal{L}(\mathbb{R})}$ but $(-\infty, q) \in P^{+\infty}_+$ and $(q, +\infty) \in P^{r'}_-$. Therefore, (dp-con) is violated and $(P^{+\infty}_+, P^{r'}_-) \notin \Sigma(\mathcal{L}_l(\mathbb{R}))$. The same is true if $r \in \mathbb{R}$ and $r' = -\infty$ or $r = +\infty$ and $r' = -\infty$. Next, assume that $r, r' \in \mathbb{R}$ but $r < r'$. Find two rational numbers $q' < q$ lying in between r and r'. Then, the pair $((-\infty, q), (q', +\infty))$ belongs to $\mathsf{tot}_{\mathcal{L}(\mathbb{R})}$ but $(-\infty, q) \notin P^r_+$ and $(q', +\infty) \notin P^{r'}_-$. Again, (dp-tot) is violated and $(P^r_+, P^{r'}_-) \notin \Sigma(\mathcal{L}_l(\mathbb{R}))$. Similarly, $r' < r$ violates (dp-con).

For "$\Leftarrow$" direction, let $r = r' \in \mathbb{R}$. We prove that (P^r_+, P^r_-) satisfies (dp-con) and (dp-tot). Let $(U_l, U_u) \in \mathsf{con}_{\mathcal{L}(\mathbb{R})}$. The cases when either $U_l = \emptyset$ or $U_u = \emptyset$ are immediate. Assume that neither of them is empty, which also means that neither of them is the whole space. Then, define l_+ to be the least upper bound of U_l and l_- to be the largest lower bound of U_u. Since U_l and U_u are disjoint, $l_+ \le l_-$. Next, either $r \le l_-$ or $l_+ \le r$ (or both). Assume the former. Then, since $q \in U_u$ iff $l_- < q$, none of $(q, +\infty)$'s is in P^r_-. Finally, because P^r_- is a completely prime filter and $U_u = \bigcup_{q \in U_u} (q, +\infty)$, also $U_u \notin P^r_-$. To check (dp-tot) assume that $(U_l, U_u) \in \mathsf{tot}_{\mathcal{L}(\mathbb{R})}$. Again, the cases when either $U_l = \mathbb{Q}$ or $U_u = \mathbb{Q}$ are trivial. Let $q \in U_l \cap U_u$ and w.l.o.g. assume that $q < r$ or $r < q$ (if q was equal to r, there is a $q' < r$ in an open neighbourhood of r still belonging to $U_l \cap U_u$). If $q < r$, then $(q, +\infty) \in P^r_-$ and also $U_u \in P^r_-$ as $(q, +\infty) \subseteq U_u$. □

The last proposition suggest that the d-frame $\mathcal{L}(\mathbb{R})$ can be the algebraic dual of the bispace of real numbers. Because, $(U_l, U_u) \in \mathsf{tot}_{\mathbb{R}}$ iff $U_l = \mathbb{R}$, $U_u = \mathbb{R}$ or $U_l \cap U_u \cap \mathbb{Q} \ne \emptyset$, we get, by Lemma 3.5.6, also the other part of the equivalence:

3.5.8 Proposition. *The d-frame $\mathcal{L}(\mathbb{R})$ is isomorphic to $\Omega_d(\mathbb{R}, \tau^{\mathbb{R}}_l, \tau^{\mathbb{R}}_u)$.*

3.5.9 Remark. 1. $\mathcal{L}(\mathbb{R})$ also has a single-sorted presentation. This is done simply by representing every pair $((-, q), (q, -))$ as a single generator $\langle q \rangle$ and rewriting all the equations correspondingly:

$$\mathbf{dFr}\Big\langle\ \langle q \rangle : q \in \mathbb{Q}\ \Big|\ \langle q \rangle \curlyvee \langle q' \rangle = \langle \max(q, q') \rangle,\ \ \langle q \rangle \curlywedge \langle q' \rangle = \langle \min(q, q') \rangle,$$

$$\langle q \rangle = \bigsqcup_{q' < q} (\langle q' \rangle \sqcap tt) \sqcup \bigsqcup_{q < q''} (\langle q'' \rangle \sqcap ff), \quad \top = \bigsqcup_{q} \langle q \rangle,$$

$$(\forall q, q' \in \mathbb{Q})\ \ \langle q \rangle \sqcap \langle q' \rangle \in \mathsf{con},\ \ \text{if } q \ne q' \text{:}\ \ \langle q \rangle \sqcup \langle q' \rangle \in \mathsf{tot} \Big\rangle.$$

2. $\mathcal{L}(\mathbb{R})$ is "almost" isomorphic to the d-frame $\Omega_d(\mathbb{Q}, \tau_+^{\mathbb{Q}}, \tau_-^{\mathbb{Q}})$. The only difference is that $\mathrm{tot}_{\mathcal{L}(\mathbb{R})}$ is missing the pairs of opens $((-\infty, r)_{\mathbb{Q}}, (r, +\infty)_{\mathbb{Q}})$ where $r \in \mathbb{R} \setminus \mathbb{Q}$.

3. For a real number r, the pair (P_+^r, P_-^r) represents the Dedekind cut for r with the intervals $(q, +\infty)_{\mathbb{Q}}$ in P_-^r being the rational lower bounds for r and the intervals $(-\infty, q)_{\mathbb{Q}}$ in P_+^r being the upper bounds.

4. Olaf Klinke defined the d-frame of the real interval [0,1] in his thesis [Kli12] by an open ideal completion of the dyadic rational numbers (see Lemma 2.3.5 and Definition 4.1.5 therein). This can be easily extended to obtain a d-frame of (all) reals. Then, because the dyadic numbers are dense in $\mathbb{Q}$, his d-frame of reals is isomorphic to our $\mathcal{L}(\mathbb{R})$.

3.5.4 Coproducts of d-frames explicitly

3.5.4.1 Coproducts of frames

Let $\{L^i\}_{i \in \mathcal{I}}$ be a family of frames. The coproduct of $\{L^i\}_i$ in the category of frames is the free frame $\bigoplus_i L^i \overset{\mathrm{def}}{\equiv} \mathbf{Fr}\langle G \mid E \rangle$ where G is equal to the disjoint union

$$\dot{\bigcup}_{j \in \mathcal{I}} \{a \oplus_j \overline{1} \mid a \in L^j\}$$

(defined syntactically) and E is the set of (in)equalities

$$\begin{aligned} (\bigwedge_{i=1}^{n} a^i) \oplus_j \overline{1} &= \bigwedge_{i=1}^{n} (a^i \oplus_j \overline{1}) \\ (\bigvee_k b^k) \oplus_j \overline{1} &\leq \bigvee_k (b^k \oplus_j \overline{1}) \end{aligned} \tag{3.5.1}$$

for some $j \in \mathcal{I}$, $a^1, a^2, \ldots, a^n \in L^j$ (allowing $n = 0$) and $\{b^k\}_{k \in K} \subseteq L^j$[10]. The frame embeddings $\iota^j \colon L^j \to \bigoplus_i L^i$, $a \mapsto a \oplus_j \overline{1}$, factor through the coproduct of $\{L^i\}_i$ in the category of semilattices, i.e. through $\prod_i' L^i$ which is the subset of $\prod_i L^i$ consisting of those elements with all but finitely many coordinates equal to 1. Concretely, ι^j is equal to the composition of the following meet-semilattice homomorphisms

$$\kappa^j \colon L^j \to \textstyle\prod_i' L^i \qquad\qquad \text{and} \qquad\qquad [\![-]\!] \colon \textstyle\prod_i' L^i \to \bigoplus_i L^i$$

$$a \mapsto a *_j \overline{1} \qquad\qquad\qquad\qquad\qquad u \mapsto \bigwedge_i (u_i \oplus_i \overline{1})$$

where $a *_j \overline{1}$ is the vector with 1 in all coordinates except for the jth one, which is equal to a. Note that, since all except for finitely many coordinates of u are finite, $[\![u]\!]$ is just a finite meet.

[10] Because coproducts of frames correspond to products of spaces, $a \oplus_j \overline{1}$ represents an open set from the subbasis of the product, that is $a \oplus_j \overline{1}$ represents the inverse image of the open set a by the jth projection.

3.5.10 Definition. Let $a \in L^j$ and $u \in \prod'_i L^i$. Define $a *_j u$ to be the element of $\prod'_i L^i$ such that $(a *_j u)_j = a$ and $(a *_j u)_i = u_i$ for $i \neq j$.

Similarly, set $a \oplus_j u$ to be element of $\bigoplus_i L^i$ defined as $[\![a *_j u]\!]$.

If we denote by $\overline{1}$ the top element of $\prod'_i L^i$, i.e. $(\overline{1})_i = 1$ for all $i \in \mathcal{I}$, then the notation we used above agrees with this newly introduced one.

3.5.11 The structure of $\bigoplus_i L^i$. Before we switch back to d-frames we give a more explicit description of the structure of coproducts since it will be essential later when we prove axioms ($\mu_\pm$-con-tot) and (Indep$_\pm$) for the coproduct of d-frames.

By definition, $\bigoplus_i L^i$ is equal to $\mathsf{Down}(\mathcal{F}(G))$ where $\mathcal{F}(G)$ is the absolutely free semilattice on the set G (Section 3.4.1). Note that $\prod'_i L^i$ is isomorphic to $\mathcal{F}(G)/E_1$, where E_1 represents the first equation in (3.5.1) The pair of meet-homomorphisms establishing this isomorphism is

$$\begin{array}{rcll} u \in \prod'_i L^i & \longmapsto & \{u_j \oplus_j \overline{1} \mid u_j \neq 1\} & \text{and} \\ [F] \in \mathcal{F}(G)/E_1 & \longmapsto & u(F) \quad \text{where} & u(F)_i = \bigwedge\{a : (a \oplus_i \overline{1}) \in F\}. \end{array}$$

3.5.12 Fact ([GPP14]). *Let S be a semilattice and let $E \subseteq S \times S$ be a sets of equations (i.e. a relation). Then,* $\mathsf{Down}(S/E) \cong \mathsf{Down}(S)/\widetilde{E}$ *where* $\widetilde{E} \overset{def}{\equiv} \{(\downarrow a, \downarrow b) : (a,b) \in E\}$ *is the set of equations E lifted to the frame* $\mathsf{Down}(S)$.

Combining this fact and the isomorphism $\prod'_i L^i \cong \mathcal{F}(G)/E_1$ we obtain that $\bigoplus_i L^i$ is formed of (E_2-)*saturated downsets* $D \in \mathsf{Down}(\prod'_i L^i)$, where E_2 is the second equation in (3.5.1), i.e.

$$\bigoplus_i L^i \cong \{D \in \mathsf{Down}(\textstyle\prod'_i L^i) \mid \{b^k *_j u : k \in K\} \subseteq D \implies (\bigvee_{k \in K} b^k) *_j u \in D\}.$$

It is immediate to check that the smallest element of $\bigoplus_i L^i$ is the downset

$$\mathbf{n} = \{u \in \textstyle\prod'_i L^i \mid u_i = 0 \text{ for some } i\}.$$

Also, with this new representation, we have a new description of the maps from above:

$$[\![-]\!]\colon \quad u \mapsto \downarrow u \cup \mathbf{n} \qquad\qquad \iota^j\colon \quad a \mapsto \downarrow(a *_j \overline{1}) \cup \mathbf{n}$$

This is valid because $\downarrow u \cup \mathbf{n}$ is the smallest saturated downset containing u. For this reason, we will often interpret the generators $a \oplus_j \overline{1}$ as the downsets $\iota^j(a)$.

Next we prove a few auxiliary lemmas.

3.5.13 Lemma ([PP12]). *Let $a, b \in L^j$, $\{a^k\}_k \subseteq L^j$ and $u, v \in B$. Then,*

1. *If $u \notin \mathbf{n}$, $[\![u]\!] \leq [\![v]\!]$ iff $u \leq v$.*
2. *$[\![-]\!]$ is injective on $\prod'_i L^i \setminus \mathbf{n}$.*
3. *$(a \oplus_j u) \wedge (b \oplus_j u) = (a \wedge b) \oplus_j u$*
4. *$\bigvee_k (a^k \oplus_j u) = (\bigvee_k a^k) \oplus_j u$*

Proof. (1) $[\![u]\!] \leq [\![v]\!]$ iff $\downarrow u \cup \mathbf{n} \subseteq \downarrow v \cup \mathbf{n}$ and, since $u \notin \mathbf{n}$, $u \leq v$. The other direction follows from monotonicity of $[\![-]\!]$. (2) follows from (1). For (3), recall that $[\![-]\!]$ is a meet-semilattice homomorphism, so $(a \oplus_j u) \wedge (b \oplus_j u) = [\![a *_j u]\!] \wedge [\![b *_j u]\!] = [\![(a *_j u) \wedge (b *_j u)]\!] = [\![(a \wedge b) *_j u]\!] = (a \wedge b) \oplus_j u$. Finally, we check (4):

$$\begin{aligned}\bigvee_k (a^k \oplus_j u) &= \bigvee_k ((a^k \oplus_j \overline{1}) \wedge (1 \oplus_j u)) = (\bigvee_k (a^k \oplus_j \overline{1})) \wedge (1 \oplus_j u) \\ &= ((\bigvee_k a^k) \oplus_j \overline{1}) \wedge (1 \oplus_j u) = (\bigvee_k a^k) \oplus_j u\end{aligned}$$

where the first and last equalities hold because $a^k \oplus_j u = (a \oplus_j \overline{1}) \wedge \bigwedge_{i \neq j} (u_i \oplus_i \overline{1})$. □

3.5.14 Lemma. *Let $\alpha^j = a^j \oplus_{i(j)} \overline{1}$, for $j = 1, \dots, n$. Then,*

$$\bigwedge_{j=1}^{n} \alpha^j = \bigwedge_{i \in I} (b^i \oplus_i \overline{1}) = \bigcap_{i \in I} (b^i \oplus_i \overline{1})$$

where $I = \{i(j) : j = 1, \dots, n\}$, $b^i = \bigwedge \{a^j \mid i(j) = i\}$).

Moreover, $\bigwedge_{j=1}^{n} \alpha^j = [\![u]\!] = \downarrow u \cup \mathbf{n}$ where $u \in \prod'_i L^i$ such that $(u)_i = b^i$ for every $i \in I$ and $(u)_i = 1$ otherwise.

Proof. By Lemma 3.5.13, $\bigwedge_{j=1}^{n} \alpha^j = \bigwedge_{i \in I} (b^i \oplus_i \overline{1})$ and, because meets of saturated downsets are computed as their intersections, $\bigwedge_{j=1}^{n} \alpha^j = \bigcap_{i \in I} (b^i \oplus_i \overline{1})$. The 'Moreover' part follows from this representation. □

3.5.15 Lemma. *Let $\alpha^j = a^j \oplus_{i(j)} \overline{1}$, for $j = 1, \dots, n$. Then,*

$$\bigvee_{j=1}^{n} \alpha^j = \bigvee_{i \in I} (b^i \oplus_i \overline{1}) = \bigcup_{i \in I} (b^i \oplus_i \overline{1})$$

where $I = \{i(j) : j = 1, \dots, n\}$ and $b^i = \bigvee \{a^j \mid i(j) = i\}$.

Proof. Let $\beta^i \stackrel{\text{def}}{\equiv} b^i \oplus_i \overline{1}$, for every $i \in I$. First, we will show that $\bigcup_{i \in I} \beta^i$ is saturated. Let $X = \{x^k *_l u\}_{k \in K} \subseteq \bigcup_{i \in I} \beta^i$. Without loss of generality, assume that $X \cap \mathbf{n} = \emptyset$, i.e. $u_i \neq 0$, for all $i \neq l$, and that $x^k \neq 0$, for all $k \in K$. If $X \subseteq \beta^i$, for some i, then also $\bigvee_{k\in K} x^k *_l u \in \beta^i$ because β^i is saturated. Otherwise, there must exist an $m \in K$ which is different from l such that $x^m *_l u \in \beta^m$. This means that $(x^m *_l u)_m = u_m \leq b^m$. From this we have that, for all $k \in K$, $(x^k *_l u)_m = u_m \leq b^m$ and, therefore, $x^k *_l u \in \beta^k$. Again, because β^k is saturated, $(\bigvee_{k\in K} x^k) *_l u \in \beta^k$.

Next, $\alpha^j \subseteq \beta^{i(j)}$, for all $j = 1, \dots, n$, and so $\bigcup_j \alpha^j \subseteq \bigcup_{i\in I} \beta^i$. Because $\bigvee_{j=1}^n \alpha^j$ is the smallest saturated containing $\bigcup_j \alpha^j$ and $\bigcup_{i\in I} \beta^i$ is also saturated, we get that $\bigvee_{j=1}^n \alpha^j \subseteq \bigcup_{i\in I} \beta^j$. Finally, every $\beta^i \subseteq \bigvee\{\alpha^j \mid i(j) = i\} \subseteq \bigvee_{j=1}^n \alpha^j$ and so $\bigcup_{i\in I} \beta^j \subseteq \bigvee_{j=1}^n \alpha^j$. □

3.5.4.2 Coproducts of d-frames

Let $\{\mathcal{L}^i = (L^i_+, L^i_-; \mathsf{con}^i, \mathsf{tot}^i)\}_{i\in\mathcal{I}}$ be a family of d-frames. In Section 3.1.3 we showed that the coproduct $\{\mathcal{L}^i\}_i$ in the category of proto-d-frames is computed as

$$\bigoplus_i \mathcal{L}^i \;=\; (\textstyle\bigoplus_i L^i_+, \bigoplus_i L^i_-, {\downarrow}\mathsf{con}_{\mathbb{A},\mathbb{V}}, {\uparrow}\mathsf{tot}_{\mathbb{A},\mathbb{V}})$$

where $\mathsf{con}_1 = \{(\alpha_+ \oplus_i \overline{1}, \alpha_- \oplus_i \overline{1}) : \alpha \in \mathsf{con}^i\}$ and $\mathsf{tot}_1 = \{(\alpha_+ \oplus_i \overline{1}, \alpha_- \oplus_i \overline{1}) : \alpha \in \mathsf{tot}^i\}$. $\bigoplus_i \mathcal{L}^i$ comes with the d-frame embeddings $\iota^j = (\iota^j_+, \iota^j_-) \colon \mathcal{L}^j \to \bigoplus_i \mathcal{L}^i$ with

$$\iota^j_\pm \;=\; L^j_\pm \xrightarrow{\kappa^j_\pm} \textstyle\prod'_i L^i_\pm \xrightarrow{[\![-]\!]_\pm} \bigoplus_i L^i_\pm$$

Recall from Section 3.3.1 that the coproduct of $\{\mathcal{L}^i\}_i$ in **d-Frm** is computed as the d-frame reflection of $\bigoplus_i \mathcal{L}^i$. Equivalently, this is defined as the freely generated d-frame $\mathbf{dFr}\langle G_\pm \mid E_\pm, E_{\mathsf{con}}, E_{\mathsf{tot}}\rangle$ where $\langle G_\pm \mid E_\pm\rangle$ are the presentation of the coproducts (for $\bigoplus_i L^i_\pm$), and E_{con} and E_{tot} are the relations:

$$(a \oplus_j \overline{1},\, b \oplus_j \overline{1}) \in E_{\mathsf{con}} \quad \text{whenever} \quad (a, b) \in \mathsf{con}^j$$
$$(a \oplus_j \overline{1},\, b \oplus_j \overline{1}) \in E_{\mathsf{tot}} \quad \text{whenever} \quad (a, b) \in \mathsf{tot}^j$$

Corollary 3.4.10 proves that this is isomorphic to $\mathfrak{r}(\bigoplus_i \mathcal{L}^i)$ defined above.

In the rest of this subsection we aim to prove Proposition 3.3.5. We do that by proving ($\mu_\pm$-con-tot) and (Indep$_\pm$) from Theorem 3.4.20 which then shows that the d-frame reflection of $\bigoplus_i \mathcal{L}^i$ is trivial; it is only the one-step DCPO-completion of the consistency relation, i.e. $\mathfrak{D}({\downarrow}\mathsf{con}_{\mathbb{A},\mathbb{V}})$.

To simplify our work to make sure that we can deal with indexes without worries, we prove the following lemma about normal forms of elements coming from $\mathsf{con}_{\mathbb{V}}$, $\mathsf{con}_{\mathbb{A}}$, $\mathsf{tot}_{\mathbb{V}}$ and $\mathsf{tot}_{\mathbb{A}}$:

3.5.16 Lemma. *Let $\alpha \in \mathsf{con}_{\mathbb{A}}/\mathsf{tot}_{\mathbb{A}}$. Then, it is of the form $(\bigwedge_i \alpha^i_+, \bigvee_i \alpha^i_-)$ such that*

1. *for every $i \in \mathcal{I}$: $\alpha^i = (a_+ \oplus_i \overline{1}, a_- \oplus_i \overline{1})$ for some $(a_+, a_-) \in \mathsf{con}^i$ (resp. tot^i), and*

2. *there exists a* finite *$I(\alpha) \subseteq_{fin} \mathcal{I}$ s.t. $i \in I(\alpha)$ iff $\alpha^i \neq \mathsf{tt}$*

Similarly, every $\alpha \in \mathsf{con}_{\mathbb{V}}$ (resp. $\mathsf{tot}_{\mathbb{A}}$) is of the form $(\bigvee_i \alpha^i_+, \bigwedge_i \alpha^i_-)$ where $\alpha^i \in \mathsf{con}_1$ (resp. tot_1) and $I(\alpha)$ denotes the finite set of indexes for which $\alpha^i \neq \mathsf{ff}$.

Notice that (1) and (2) make sense together. Anytime $\alpha^i = tt$ we have that $tt = ({\downarrow}\overline{1} \cup \mathbf{n}_+, \mathbf{n}_-) = (1 \oplus_i \overline{1}, 0 \oplus_i \overline{1})$ and $(1, 0) \in \mathsf{con}^i/\mathsf{tot}^i$. The case for $\alpha^i = ff$ is similar.

Proof. We prove that, for every $\alpha \in \mathsf{con}_{\mathbb{V}}$, $\alpha = \mathbb{V}_{i \in I}\, \alpha^i$ for some $I \subseteq_{\mathrm{fin}} \mathcal{I}$ and $\alpha^i = (b^i_+ \oplus_i \overline{1},\, b^i_- \oplus_i \overline{1}) \in \mathsf{con}_1$ (or tot_1) and from this the lemma follows.

Let $\bigvee_{j=1}^n \alpha^j \in \mathsf{con}_{\mathbb{V}}$ where $\alpha^j = (a^j_+ \oplus_{i(j)} \overline{1},\, a^j_- \oplus_{i(j)} \overline{1}) \in \mathsf{con}_1$, for all $j = 1, \ldots, n$. From Lemma 3.5.14 and Lemma 3.5.15, we have that

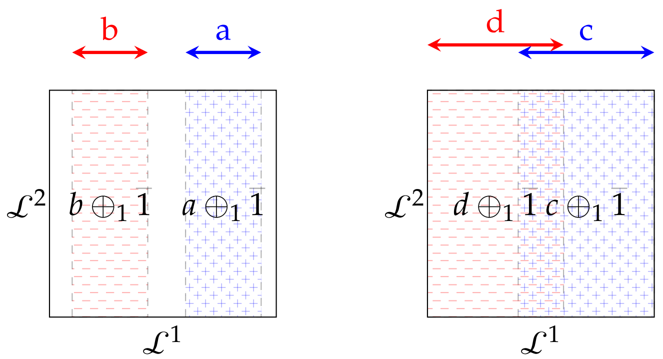
$$\mathop{\mathbb{V}}_{j=1}^{n} \alpha^j = \Big(\bigvee_{i \in I} (b^i_+ \oplus_i \overline{1}),\ \bigwedge_{i \in I} (b^i_- \oplus_i \overline{1})\Big)$$

where $I = \{i(j) : j = 1, \ldots, n\}$ and $b^i_+ = \bigvee\{a^j \mid i(j) = i\}$ and $b^i_- = \bigwedge\{a^j \mid i(j) = i\}$. For every $i \in I$, because (con-$\mathbb{V}$) holds for $\mathcal{L}^i$, $(b^i_+, b^i_-) \in \mathsf{con}^i$ and, therefore, also $(b^i_+ \oplus_i \overline{1},\, b^i_- \oplus_i \overline{1}) \in \mathsf{con}_1$. □

3.5.4.3 Strips, rectangles and crosses

Before we get to proving that $\bigoplus_i \mathcal{L}^i$ has a one-step reflection we look into the structure of $\mathsf{con}_{\mathbb{V}}$, $\mathsf{con}_{\mathbb{A}}$, $\mathsf{tot}_{\mathbb{V}}$ and $\mathsf{tot}_{\mathbb{A}}$. It turns out that there is a nice geometrical intuition that we can employ.

First, for an $a \in L^i_{\pm}$, we call $a \oplus_i \overline{1}$ *an i-strip*[11]. Then, anytime $(a, b) \in \mathsf{con}^i$, we can think of the corresponding pair $(a \oplus_i \overline{1},\, b \oplus_i \overline{1}) \in \mathsf{con}_1$ as of a pair of *"disjoint"* i-strips and, similarly, $(c, d) \in \mathsf{tot}^i$ gives a pair of strips that are *"covering the whole space"*, i.e. $(c \oplus_i \overline{1},\, d \oplus_i \overline{1}) \in \mathsf{tot}_1$. This terminology is motivated by the case when $\mathcal{I} = \{1, 2\}$. Both cases are displayed in the picture below for $\mathcal{L}^1 \oplus \mathcal{L}^2$:



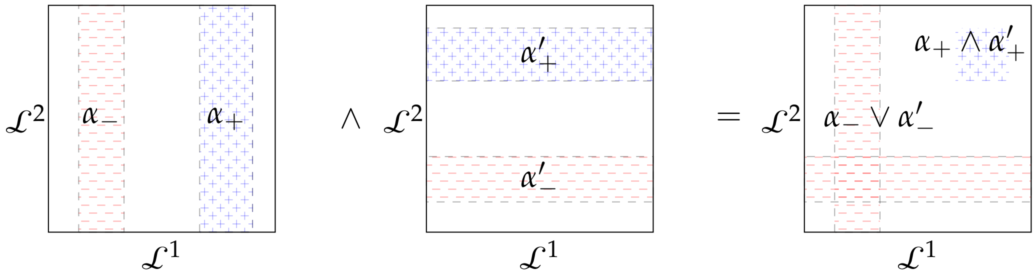
[11] We sometimes omit the index and call i-strips just strips whenever it does not lead to a confusion.

Therefore, all elements of con_1 and tot_1 are pairs of strips. As a direct consequence of Lemma 3.5.13 we see that the set of i-strips in the coproduct has exactly the same structure as the d-frame $\mathcal{L}^i$:

3.5.17 Lemma. *Let $S^i_\pm$ be the set of all i-strips in $\bigoplus_i L^i_\pm$. If all $L^i_\pm$'s are nontrivial*[12] *then*

$$(S^i_+, S^i_-; \mathsf{con}_1 \cap (S^i_+ \times S^i_-), \mathsf{tot}_1 \cap (S^i_+ \times S^i_-)) \cong \mathcal{L}^i.$$

Moreover, finite $\curlywedge$-combinations of pairs of strips is something that we can imagine as a pair consisting of a rectangle and a cross. For example, let $\alpha \in \mathsf{con}_1$ be a pair of 1-strips and $\alpha' \in \mathsf{con}_1$ a pair of 2-strips. Then, as the picture below suggests, the plus coordinate of $\alpha \curlywedge \alpha'$ in $\mathcal{L}^1 \oplus \mathcal{L}^2$ is a rectangle and the minus coordinate is a cross. Notice also that the cross and rectangle are disjoint.



The picture for two pairs of strips $\beta, \beta' \in \mathsf{tot}_1$ is similar but this time the cross and rectangle of $\beta \curlywedge \beta'$ cover the whole space.

This geometrical intuition builds up well for these formal definitions: $\gamma = \curlywedge_i \gamma^i$, where $\gamma^i = c^i \oplus_i \overline{1}$ ($\forall i \in \mathcal{I}$), is a *rectangle* if there exists a finite $I(\gamma) \subseteq_{\text{fin}} \mathcal{I}$ such that $c^i \neq 1$ iff $i \in I(\gamma)$. Similary, $\delta = \curlyvee_i \delta^i$, where $\delta^i = d^i \oplus_i \overline{1}$, is a *cross* if for some finite $I(\delta) \subseteq_{\text{fin}} \mathcal{I}$, $d^i \neq 0$ iff $i \in I(\delta)$.

Notice that, by Lemma 3.5.16, every element of $\mathsf{con}_\curlywedge$ (resp. $\mathsf{tot}_\curlywedge$) is of the form $(\bigwedge_i \alpha^i_+, \bigvee_i \alpha^i_-)$ with only finitely many nontrivial α^i's. In the present terminology, α is a pair rectangle–cross and this exactly matches the geometrical intuition we have just discussed.

3.5.18 Observation. *Rectangles are exactly the elements of $B_\pm \stackrel{def}{\equiv} [\![\prod'_i L^i_\pm]\!]_\pm$.*

Proof. Every $\gamma \in B_\pm$ is of the form $[\![u]\!]_\pm$ for some $u \in \prod'_i L^i_\pm$. Because u has only finitely many indexes different from 1, $[\![u]\!]_\pm = [\![(a^1 *_{i(1)} \overline{1}) \wedge \cdots \wedge (a^n *_{i(n)} \overline{1})]\!]_\pm = [\![a^1 *_{i(1)} \overline{1}]\!]_\pm \wedge \cdots \wedge [\![a^n *_{i(n)} \overline{1}]\!]_\pm = (a^1 \oplus_{i(1)} \overline{1}) \wedge \cdots \wedge (a^n \oplus_{i(n)} \overline{1})$. The reverse direction is similar. □

There is a nice interplay between rectangles and crosses:

[12] A frame is *trivial* if it is isomorphic to $\mathbf{1} = \{0 = 1\}$.

3.5.19 Lemma. *Let $\gamma = \bigwedge_i \gamma^i$ be a rectangle and let $\delta = \bigvee_i \delta^i$ be a cross such that $\gamma \leq \delta$. Then, there exists an $i \in I(\gamma)$ such that $\gamma^i \leq \delta^i$.*

Proof. Let $\gamma^i = c^i \oplus_i \bar{1}$ and $\delta^i = d^i \oplus_i \bar{1}$, for every $i \in \mathcal{I}$. By Observation 3.5.18, $\gamma = [\![u]\!]_\pm$ for some $u \in \prod_i' L^i_\pm$ such that, for every $i \in \mathcal{I}$, $(u)_i = c^i$. This means that $(u)_i \neq 1$ iff $i \in I(\gamma)$. Also, by Lemma 3.5.15, δ has a form of a finite union $\bigcup_{i \in I(\delta)} \delta^i$. If $c^i = 0$ for some $i \in I(\gamma)$, then $c^i \leq d^i$. Otherwise, $c^i \neq 0$ for all $i \in I(\gamma)$ and, since $\gamma \leq \delta$ iff $u \in \delta$, there must exist an $i \in I(\delta)$ such that $u \in \delta^i$ and then, by Lemma 3.5.13.1, $(u)_i = c^i \leq d^i$. Finally, because $i \in I(\delta)$, $d^i \neq 1$ and so also $c^i \neq 1$ and $i \in I(\gamma)$. □

3.5.4.4 Proof of ($\mu_\pm$-con-tot) and (Indep$_\pm$)

To simplify our proofs, we can assume that all $\mathcal{L}^i$'s are nontrivial thanks to the following lemma.

3.5.20 Lemma. *If $L^i_+ = \mathbf{1}$ or $L^i_- = \mathbf{1}$ for some $i \in \mathcal{I}$, then $\bigoplus_i \mathcal{L}^i$ is isomorphic to the trivial d-frame $(\mathbf{1}, \mathbf{1}, \mathsf{con}_{triv}, \mathsf{tot}_{triv})$.*

Proof. Observe that, by (con-tot) for $\mathcal{L}^i$, if $L^i_+ = \mathbf{1}$ then automatically also $L^i_- = \mathbf{1}$, and vice versa. Therefore, $\bigoplus_i L^i_\pm = \{\mathbf{n}_\pm\}$ and making $\bigoplus_i \mathcal{L}^i$ trivial. □

Finally, let us check that the conditions of the Theorem 3.4.20 (for $B_\pm = [\![\prod_i' L^i_\pm]\!]_\pm$) are satisfied:

3.5.21 Lemma. *($\mu_\pm$-con-tot) holds for $\bigoplus_i \mathcal{L}^i$:*

Proof. Let $\alpha = \dot\bigvee_i \alpha^i \in \mathsf{con}_{\dot\vee}$ and $\beta = \dot\bigwedge_i \beta^i \in \mathsf{tot}_{\dot\wedge}$ be in canonical forms, and assume that $\beta_+ \leq \alpha_+$. From canonicity of α and β, know that α_+ is a cross and β_+ is a rectangle. By Lemma 3.5.19, there is an $i \in I(\beta)$ such that $\beta^i_+ \leq \alpha^i_+$. From (con-tot) for $\mathcal{L}^i$, $\alpha^i_- \leq \beta^i_-$ and so $\alpha_- = \bigwedge_i \alpha^i_- \leq \alpha^i_- \leq \beta^i_- \leq \bigvee_i \beta^i_- = \beta_-$. □

3.5.22 Lemma. *(Indep$_\pm$) holds for $\bigoplus_i \mathcal{L}^i$.*

Proof. Let $(x, b_-) \in (L_+ \times B_-) \cap \downarrow\mathsf{con}_{\dot\wedge,\dot\bigvee}$. Denote its upper bound $(\bigvee_k \alpha^k_+, \bigwedge_k \alpha^k_-)$ where, for each k, $\alpha^k = (\bigwedge_i \alpha^{k,i}_+, \bigvee_i \alpha^{k,i}_-)$ is a pair rectangle–cross from $\mathsf{con}_{\dot\wedge}$. Because $b_- \in B_-$, it is a rectangle of the form $b_- = \bigwedge_i \gamma^i$ (Observation 3.5.18). Because, for every k, $b_- \leq \alpha^k_-$, by Lemma 3.5.19, there exists an $i(k) \in I(b_-)$ such that $\gamma^{i(k)} \leq \alpha^{k,i(k)}_-$. Fix an $i \in I(b_-)$ and set $K(i) = \{k \mid i(k) = i\}$. By Lemma 3.5.20, we can assume that all $L^i_\pm$'s are non-trivial and because $\{\alpha^{k,i} : k \in K(i)\}$ are all pairs of i-strips and γ^i is an i-strip, by Lemma 3.5.17, we can carry the reasoning in the rest of this paragraph in the d-frame $\mathcal{L}^i$. Since con^i is downwards closed and $\gamma^i \leq \alpha^{k,i}$

$(\forall k \in K(i))$, also $(\alpha_+^{k,i(k)}, \gamma^{i(k)}) \in \mathsf{con}_1$ and, therefore, by $\bigsqcup^\uparrow$ and $\curlyvee$-closeness of con^i, $(\bigvee_{k\in K(i)} \alpha_+^{k,i}, \gamma^i) \in \mathsf{con}_1$.

Finally, because $I(b_-)$ is finite,

$$\dot{\bigvee}_{i\in I(b_-)} (\bigvee_{k\in K(i)} \alpha_+^{k,i}, \gamma^i) = (\bigvee_{i\in I(b_-)} (\bigvee_{k\in K(i)} \alpha_+^{k,i}), \bigwedge_{i\in I(b_-)} \gamma^i) = (\bigvee_k \alpha_+^{k,i(k)}, b_-) \in \mathsf{con}_\curlyvee.$$

Because $\alpha_+^k = \bigwedge_i \alpha_+^{k,i} \leq \alpha_+^{k,i(k)}$ $(\forall k)$, $x \leq \bigvee_k \alpha_+^k \leq \bigvee_k \alpha_+^{k,i(k)}$ and so $(x, b_-) \in {\downarrow}\mathsf{con}_\curlyvee$. □

This concludes the proof of Proposition 3.3.5.

3.5.5 Vietoris constructions

Powerlocales are an important tool in frame/locale theory. For example, for a frame L, consider the *upper powerlocale* (also known as Smyth powerlocale) construction:

$$\mathbb{V}_\Box(L) \stackrel{\text{def}}{\equiv} \mathbf{Fr}\Big\langle \Box a : a \in L \;\Big|\; \Box(a \wedge b) = \Box a \wedge \Box b,\ \Box 1 = 1,\ \Box(\bigvee_i^\uparrow a^i) = \bigvee_i^\uparrow (\Box a^i) \Big\rangle.$$

Then, for a sober space (X,τ), the frame homomorphisms $\mathbb{V}_\Box(\Omega(X)) \to \mathbf{2}$ are in a bijective correspondence with compact saturated subsets of X [Esc04; Vic97][13]. Similarly, for a d-frame $\mathcal{L}$, define the *upper Vietoris d-frame* $\mathbb{W}_\Box(\mathcal{L})$ to be the d-frame

$$\mathbf{dFr}\Big\langle \Box\alpha : \alpha \in \mathcal{L} \;\Big|\; \begin{array}{l} \Box(\alpha \curlywedge \beta) = \Box\alpha \curlywedge \Box\beta,\ \Box tt = tt,\ \ \Box(\bigsqcup_i^\uparrow \alpha^i) = \bigsqcup_i^\uparrow (\Box\alpha^i), \\ (\forall \alpha \in \mathsf{con}_\mathcal{L})\ \ \Box\alpha \in \mathsf{con},\quad (\forall \alpha \in \mathsf{tot}_\mathcal{L})\ \ \Box\alpha \in \mathsf{tot} \end{array} \Big\rangle$$

Again, for a d-regular bispace (X, τ_+, τ_-), the $\bigsqcup^\uparrow$, $\curlywedge$, con and tot-preserving maps $\Omega_d(X) \to \mathbf{2}\times\mathbf{2}$ or, equivalently, the d-frame homomorphisms $\mathbb{W}_\Box(\Omega_d(X)) \to \mathbf{2}\times\mathbf{2}$ are in a bijective correspondence with compact upwards-closed subsets of X [JM08, Theorem 5.6].

Of a comparable importance is the *lower powerlocale* (also known as Hoare powerlocale) construction:

$$\mathbb{V}_\Diamond(L) \stackrel{\text{def}}{\equiv} \mathbf{Fr}\Big\langle \Diamond a : a \in L \;\Big|\; \Diamond(a \vee b) = \Diamond a \vee \Diamond b,\ \Diamond 0 = 0,\ \Diamond(\bigvee_i^\uparrow a^i) = \bigvee_i^\uparrow (\Diamond a^i) \Big\rangle.$$

The frame homomorphisms $\mathbb{V}_\Diamond(\Omega(X)) \to \mathbf{2}$ are in a bijection with closed subsets of X [Esc04, Proposition 5.4.2][14]. For a d-frame $\mathcal{L}$, the *lower Vietoris d-frame* $\mathbb{W}_\Diamond(\mathcal{L})$ is defined as

$$\mathbf{dFr}\Big\langle \Diamond\alpha : \alpha \in \mathcal{L} \;\Big|\; \begin{array}{l} \Diamond(\alpha \curlyvee \beta) = \Diamond\alpha \curlyvee \Diamond\beta,\ \Diamond ff = ff,\ \ \Diamond(\bigsqcup_i^\uparrow \alpha^i) = \bigsqcup_i^\uparrow (\Diamond\alpha^i), \\ (\forall \alpha \in \mathsf{con}_\mathcal{L})\ \ \Diamond\alpha \in \mathsf{con},\quad (\forall \alpha \in \mathsf{tot}_\mathcal{L})\ \ \Diamond\alpha \in \mathsf{tot} \end{array} \Big\rangle.$$

[13] This has further consequences. As $\bigvee^\uparrow$ and $\wedge$-preserving maps $\Omega(X) \to \mathbf{2}$ and frame homomorphisms $\mathbb{V}_\Box(\Omega(X)) \to \mathbf{2}$ are in a bijective correspondence, saturated subsets can be thought of as (continuous) universal quantifiers on X. For further details see Escardó's *Synthetic topology*, Theorem 5.3.1 in [Esc04].

[14] In Escardó's theory, closed subsets correspond to continuous existential quantifiers.

Lastly, the *Vietoris powerlocale* (also known as Plotkin or Johnstone powerlocale) is a combination of both the upper and lower constructions. Consequently, the set of generators and the set of axioms get combined, plus two extra axioms are added to express the relationship between the "boxed" and "diamonted" elements:

$$\mathbb{V}_{\mathrm{Fr}}(L) \overset{\text{def}}{\equiv} \mathbf{Fr}\Big\langle \Box a, \Diamond a : a \in L \;\Big|\; \text{(axioms of } \mathbb{V}_\Box(L)\text{)}, \quad \text{(axioms of } \mathbb{V}_\Diamond(L)\text{)}, \quad \Box a \wedge \Diamond b \leq \Diamond(a \wedge b), \quad \Box(a \vee b) \leq \Box a \vee \Diamond b \Big\rangle$$

Correspondingly, the *(full) Vietoris d-frame* is defined as follows

$$\mathbb{W}_d(\mathcal{L}) \overset{\text{def}}{\equiv} \mathbf{dFr}\Big\langle \Box\alpha, \Diamond\alpha : \alpha \in \mathcal{L} \;\Big|\; \text{(axioms of } \mathbb{W}_\Box(\mathcal{L})\text{)}, \quad \text{(axioms of } \mathbb{W}_\Diamond(\mathcal{L})\text{)}, \quad \Box\alpha \mathbin{\curlywedge} \Diamond\beta \preccurlyeq \Diamond(\alpha \mathbin{\curlywedge} \beta), \quad \Box(\alpha \mathbin{\curlyvee} \beta) \preccurlyeq \Box\alpha \mathbin{\curlyvee} \Diamond\beta, \Big\rangle.$$

Each of the different versions of the powerlocale constructions has a different role in the theory of computation and also in modal logic (see, for example, [Gou10]). We leave examining properties and applications of the d-frame versions of powerlocale constructions for Chapters 4, 5 and 6. For now, let us just take a look at how the generated d-frames look like.

3.5.23 Two-sorted reformulation. First, we examine how the single-sorted representation of $\mathbb{W}_\Box(\mathcal{L})$ gets translated into the two-sorted one. Every generator $\Box\alpha$ becomes a pair of generators which we suggestively denote as $\Box\alpha_+$ and $\Diamond\alpha_-$. Then, the axioms split into two symmetric sets of axioms as well. For example $\Box(\alpha \mathbin{\curlywedge} \beta) = \Box\alpha \mathbin{\curlywedge} \Box\beta$ becomes $\Box(\alpha_+ \wedge \beta_+) = \Box\alpha_+ \wedge \Box\beta_+$ and $\Diamond(\alpha_- \vee \beta_-) = \Diamond\alpha_- \vee \Diamond\beta_-$. We see that the quotienting relations on the plus side agree with the defining equations of $\mathbb{V}_\Box(L_+)$ and on the minus side, since they are dual, they agree with $\mathbb{V}_\Diamond(L_-)$. The consistency and totality relations are, after the translation, seen to be generated from the pairs $(\Box\alpha_+, \Diamond\alpha_-)$ where $\alpha \in \mathsf{con}_\mathcal{L}$ and $\alpha \in \mathsf{tot}_\mathcal{L}$, respectively.

Intuitively, this makes sense because $\alpha \in \mathsf{con}$ represents "$\alpha_+ \wedge \alpha_- = 0$" and then, from the axioms for the Vietoris powerlocale, "$\Box\alpha_+ \wedge \Diamond\alpha_- \leq \Diamond(\alpha_+ \wedge \alpha_-) = \Diamond 0 = 0$" suggesting that $(\Box\alpha_+, \Diamond\alpha_-)$ should be consistent. A similar intuitive reasoning justifies why the pairs $(\Box\alpha_+, \Diamond\alpha_-)$, for $\alpha \in \mathsf{tot}$, should be total.

In Chapter 4 we show that the presentation of $\mathbb{W}_\Box(\mathcal{L})$ satisfies the conditions of Theorem 3.4.20 from which we derive that $\mathbb{W}_\Box(\mathcal{L})$ is isomorphic to

$$(\mathbb{V}_\Box L_+,\ \mathbb{V}_\Diamond L_-,\ \mathfrak{D}(\downarrow\mathsf{con}_{\curlywedge,\curlyvee}),\ \uparrow\mathsf{tot}_{\curlywedge,\curlyvee})$$

where $\mathsf{con}_1 = \{(\Box\alpha_+, \Diamond\alpha_-) : \alpha \in \mathsf{con}\}$ and $\mathsf{tot}_1 = \{(\Box\alpha_+, \Diamond\alpha_-) : \alpha \in \mathsf{tot}\}$.

Similarly, the other two constructions satisfy the conditions of Theorem 3.4.20. Therefore, $\mathbb{W}_\Box(\mathcal{L})$ is isomorphic to $(\mathbb{V}_\Diamond L_+, \mathbb{V}_\Box L_-, \mathfrak{D}(\downarrow\mathsf{con}_{\curlywedge,\curlyvee}), \uparrow\mathsf{tot}_{\curlywedge,\curlyvee})$ where $\mathsf{con}_1 = \{(\Diamond\alpha_+, \Box\alpha_-) : \alpha \in \mathsf{con}\}$ and $\mathsf{tot}_1 = \{(\Diamond\alpha_+, \Box\alpha_-) : \alpha \in \mathsf{tot}\}$, and the d-frame Vietoris power-construction $\mathbb{W}_d(\mathcal{L})$ becomes

$$(\mathbb{V}_{\mathrm{Fr}} L_+,\ \mathbb{V}_{\mathrm{Fr}} L_-,\ \mathfrak{D}(\downarrow\mathsf{con}_{\curlywedge,\curlyvee}),\ \uparrow\mathsf{tot}_{\curlywedge,\curlyvee})$$

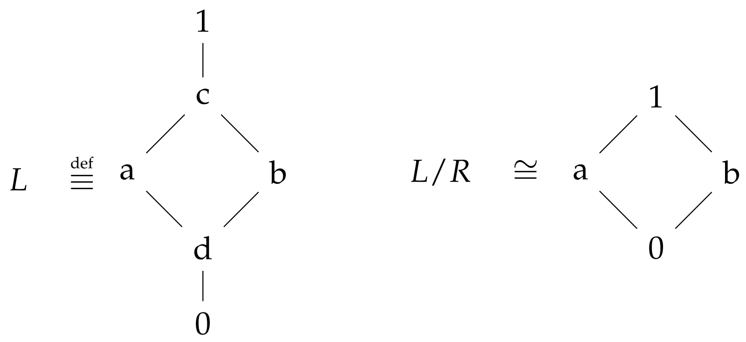
with

$$\mathsf{con}_1 = \{(\Box\alpha_+, \Diamond\alpha_-), (\Diamond\alpha_+, \Box\alpha_-) : \alpha \in \mathsf{con}\}, \text{ and}$$
$$\mathsf{tot}_1 = \{(\Box\alpha_+, \Diamond\alpha_-), (\Diamond\alpha_+, \Box\alpha_-) : \alpha \in \mathsf{tot}\}.$$

3.6 Final remarks

3.6.1 Further free constructions. The method of constructing a free d-frame from its presentation, that we described in this chapter, is quite flexible. In fact, whenever we translated a free frame construction to d-frames, all that we often needed to do was to specify the sets of generators for the consistency and totality relations. The author of this text believes that it should not be difficult to translate other frame constructions which can be given as a free construction, such as, Booleanization, frame of congruences (see page 85 in [Vic89]), or function space construction by Martin Hyland [Hyl81].

3.6.2 Quotienting twist d-frames. One may think that, if a d-frame is of the form $L^{\bowtie}$ for some frame L, then the quotient of $L^{\bowtie}$ by (R, R) should yield a d-frame isomorphic to $(L/R)^{\bowtie}$. This is, however, not the case. Consider the frame L and its quotient shown below



where $R = \{d \leq 0, 1 \leq c\}$. It is easy to check that quotients of d-frames with trivial consistency and totalities relations also have those relations trivial. Therefore, the quotient of $L^{\bowtie} = (L, L, \mathsf{con}_{\mathrm{triv}}, \mathsf{tot}_{\mathrm{triv}})$ by (R, R) is equal to $(L/R, L/R, \mathsf{con}_{\mathrm{triv}}, \mathsf{tot}_{\mathrm{triv}})$. On the other hand, the consistency and totality relations of $(L/R)^{\bowtie}$ are non-trivial.

3.6.3 Iterative reflection categorically. The construction of a free d-frame from its presentation, seen as a functor **dFr**: **Pres** $\to$ **d-Frm** (defined in paragraph 3.4.4), can be split into two separate constructions. First of all we have a functor $\mathfrak{f}$: **Pres** $\to$ **pd-Frm** which assigns to a d-frame presentation $(L_\pm, E)$ the proto-d-frame

$$(L_+, L_-, \downarrow(E_{\mathsf{con}})_{\mathbb{A},\mathbb{V}}, \uparrow(E_{\mathsf{tot}})_{\mathbb{A},\mathbb{V}}) / (E_+, E_-).$$

(And is defined on morphisms as expected.) It is immediate to check that the embedding $\mathfrak{i}\colon \mathbf{pd\text{-}Frm} \to \mathbf{Pres}$ is the right adjoint of $\mathfrak{f}$. Moreover, **Pres** reflects onto its full subcategory **cPres** consisting of *correct* presentations, i.e. presentations $(L_\pm, E)$ such that E is a reasonable quotient structure on (L_+, L_-) (in the sense of Section 3.2.4). The reflection is computed as the morphism of presentations $(\mathrm{id}_+, \mathrm{id}_-)\colon (L_\pm, E) \to (L_\pm, \mathfrak{r}^\infty(E))$. Denote the corresponding functor as $\widetilde{\mathfrak{r}}\colon \mathbf{Pres} \to \mathbf{cPres}$. We have the following diagram of categories

$$\begin{array}{ccc} \mathbf{d\text{-}Frm} & \underset{\mathfrak{f}}{\overset{\mathfrak{i}}{\rightleftarrows}} & \mathbf{cPres} \\ {\scriptstyle\mathfrak{r}}\uparrow\downarrow & & {\scriptstyle\widetilde{\mathfrak{r}}}\uparrow\downarrow \\ \mathbf{pd\text{-}Frm} & \underset{\mathfrak{f}}{\overset{\mathfrak{i}}{\rightleftarrows}} & \mathbf{Pres} \end{array}$$

Consequently, $\mathbf{dFr} = \mathfrak{f} \circ \widetilde{\mathfrak{r}}$ and $\mathfrak{r} = \mathfrak{f} \circ \widetilde{\mathfrak{r}} \circ \mathfrak{i}$ where $\mathfrak{i}$ is the embedding $\mathcal{L} \mapsto (L_\pm, \Lambda(\mathcal{L}))$ as defined in Section 3.4.2. Moreover, $\widetilde{\mathfrak{r}}$ can be seen as an iterative construction on d-frame presentations. Define $\widetilde{\mathfrak{r}}^\star\colon \mathbf{Pres} \to \mathbf{Pres}$ as follows

$$(L_\pm, E) \quad \longmapsto \quad (L_\pm,\ \mathfrak{r}^\star(\downarrow(E_{\mathsf{con}})_{\curlywedge,\curlyvee}, \uparrow(E_{\mathsf{tot}})_{\curlywedge,\curlyvee}, E_+, E_-))^{15}$$

where $\mathfrak{r}^\star$ is as in Section 3.2.3. Then, $\widetilde{\mathfrak{r}}(L_\pm, E)$ is the colimit of the transfinite sequence

$$(L_\pm, E) \xrightarrow{(\mathrm{id}_+, \mathrm{id}_-)} \widetilde{\mathfrak{r}}^\star(L_\pm, E) \xrightarrow{(\mathrm{id}_+, \mathrm{id}_-)} \widetilde{\mathfrak{r}}^\star(\widetilde{\mathfrak{r}}^\star(L_\pm, E)) \xrightarrow{(\mathrm{id}_+, \mathrm{id}_-)} \cdots$$

Moreover, if $(L_\pm, E) = \mathfrak{i}(\mathcal{L})$ for some proto-d-frame $\mathcal{L}$, then this sequence in **Pres**, when mapped back into **pd-Frm** by $\mathfrak{f}$, yields a sequence of quotients of proto-d-frames

$$\mathcal{L} \twoheadrightarrow \mathfrak{f}(\widetilde{\mathfrak{r}}^\star(\mathfrak{i}(\mathcal{L}))) \twoheadrightarrow \mathfrak{f}(\widetilde{\mathfrak{r}}^\star(\widetilde{\mathfrak{r}}^\star(\mathfrak{i}(\mathcal{L})))) \twoheadrightarrow \cdots$$

It can be checked that the colimit of this sequence is again isomorphic to $\mathfrak{r}(\mathcal{L})$.

3.6.4 Counterexample (by Achim Jung)**.** We show that there is an onto frame homomorphism $h\colon L \twoheadrightarrow M$ and a downset $C \subseteq L$, which is Scott-closed, such that $h[C]$ is not Scott-closed. Let L be the free frame $\mathbf{Fr}\langle A\rangle$ where A is the set $\{a_n : n \in \mathbb{N}\}$. Define $C \subseteq L$ as the downset closure of A embedded into L. Next we show that C is directed in L. Recall from Section 3.4.1 that L can be represented as the frame $\mathsf{Down}(\mathcal{F}(A), \cup)$. Then, under this representation, C is equal to

$$\{D \subseteq \mathcal{F}(A) \mid D = \downarrow D \text{ (in } (\mathcal{F}(A), \supseteq)) \text{ and } \exists n \in \mathbb{N}.\ A \subseteq \downarrow\{a_n\}\}.$$

To see why is C directed, consider a function $G\colon C \to \mathcal{P}(A)$, $D \mapsto \{a_n \mid D \subseteq \downarrow\{a_n\}\}$. Then, for every $D \in C$, $G(D)$ is either finite or the whole A and, moreover, $U \subseteq U'$ implies $G(U) \supseteq G(U')$. Hence, C is directed.

[15]This closure of E_{con} and E_{tot} under logical operations and downwards or upwards is, in fact, necessary only in the first application of $\widetilde{\mathfrak{r}}^\star$.

Next, consider the quotient of L by $R = \{a_n \leq a_{n+1} \mid n \in \mathbb{N}\}$. Then, L/R is is isomorphic to the chain $a_0 < a_1 < \cdots < \omega$, and the quotient map $q\colon L \twoheadrightarrow L/R$ maps C to the set which contains everything but ω, i.e. a not Scott-closed set.

4

Vietoris constructions for bispaces and d-frames

Leopold Vietoris, in his celebrated paper [Vie22], motivated his construction to be the powerset for metric spaces. The idea of having powerset-like constructions has been adapted and generalised to other contexts many times since. Common examples are power-constructions for topological spaces and domains [Smy83; Plo76] and frames [Joh85; Joh82; VV14]. Having a power-construction for a category, despite being an interesting question on its own, showed to be a practical tool too. For example, the Vietoris construction became a key construction in [Abr87b; Vic89; KKV04].

In this chapter we take a look at how to generalise the Vietoris construction to the categories of bispaces and d-frames.

4.1 Bispatial Vietoris constructions

To motivate the definition of our Vietoris construction for bispaces we first take a look at two examples of power-constructions from which we took inspiration. Since the bispaces we are the most interested in can be thought of as posets equipped with two (compatible) topologies, we first take a look at the category of posets **Pos** and the category of topological spaces **Top** and how power-constructions are defined for those.

4.1.1 Example 1: Partially ordered sets

Let $(Z, \leq)$ be a poset. A naive attempt to define a "powerposet" of Z can be to try to define a partial order on the set of all subsets $\mathcal{P}(Z)$ of Z. Here we have (at least) three reasonable options how to define an order on $\mathcal{P}(Z)$. For $M, N \in \mathcal{P}(Z)$, define

$$\begin{aligned} M \leq^{\mathrm{U}} N &\overset{\text{def}}{\equiv} \forall n \in N.\, \exists m \in M.\, m \leq n \\ M \leq^{\mathrm{L}} N &\overset{\text{def}}{\equiv} \forall m \in M.\, \exists n \in N.\, m \leq n \\ M \leq^{\mathrm{EM}} N &\overset{\text{def}}{\equiv} M \leq^{\mathrm{U}} N \text{ and } M \leq^{\mathrm{L}} N \end{aligned}$$

called *upper*, *lower* and *Egli-Milner lifting* of $\leq$, respectively. It is easy to verify that $(\mathcal{P}(Z), \leq^{\mathrm{U}})$, $(\mathcal{P}(Z), \leq^{\mathrm{L}})$ and $(\mathcal{P}(Z), \leq^{\mathrm{EM}})$ are all preordered sets; however, they might fail to satisfy antisymmetry and therefore being posets.

On the other hand, the assignment $\mathcal{P}'_{\mathrm{U}} \colon (Z, \leq) \mapsto (\mathcal{P}(Z), \leq^{\mathrm{U}})$ is functorial. For a monotone $f \colon (Y, \leq) \to (Z, \leq)$, setting $\mathcal{P}'_{\mathrm{U}}(f) \colon \mathcal{P}'_{\mathrm{U}}(Y) \to \mathcal{P}'_{\mathrm{U}}(Z)$ to be the map $M \mapsto f[M]$ defines a (covariant) endofunctor on the category of preordered sets and monotone maps **PreOrd**.

Moreover, $\mathcal{P}'_{\mathrm{U}}$ can be extended to a monad. Consider the following mappings

$$\begin{aligned} \eta_Z \colon Z &\to \mathcal{P}'_{\mathrm{U}}(Z) \\ z &\mapsto \{z\} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mu_Z \colon \mathcal{P}'_{\mathrm{U}}(\mathcal{P}'_{\mathrm{U}}(Z)) &\to \mathcal{P}'_{\mathrm{U}}(Z) \\ \mathcal{M} &\mapsto \bigcup \mathcal{M} \end{aligned}$$

Clearly, η_Z is monotone. To check that also μ_Z is consider $\mathcal{M}, \mathcal{N} \in \mathcal{P}'_{\mathrm{U}}(\mathcal{P}'_{\mathrm{U}}(Z))$ such that $\mathcal{M} \leq^{\mathrm{U}} \mathcal{N}$. This means that for every $N \in \mathcal{N}$ there is a $M \in \mathcal{M}$ such that $M \leq^{\mathrm{U}} N$. Therefore, if $n \in \bigcup \mathcal{N}$, then there is some $N \in \mathcal{N}$ such that $n \in N$ and, then, for some $M \in \mathcal{M}$ such that $M \leq^{\mathrm{U}} N$, there is some $m \in M$ such that $m \leq^{\mathrm{U}} n$. Next, the required categorical identities for the monad $(\mathcal{P}'_{\mathrm{U}}, \eta, \mu)$ hold because η and μ are computed exactly the same as the corresponding maps for the powerset monad $\mathcal{P}(-)$ on **Set**.

Similarly, we extend the other two constructions to endofunctors $\mathcal{P}'_{\mathrm{L}}$ and $\mathcal{P}'_{\mathrm{EM}}$ and, then, exactly the same reasoning applies. We obtain that:

4.1.1 Proposition. *$(\mathcal{P}'_{\mathrm{U}}, \eta, \mu)$, $(\mathcal{P}'_{\mathrm{L}}, \eta, \mu)$ and $(\mathcal{P}'_{\mathrm{EM}}, \eta, \mu)$ are monads on* **PreOrd**.

However, our task was to define a power-construction for the category of posets. We can reuse what we have just proved. Define the restrictions of the above endofunctors to the category of posets as the following compositions of functors

$$\mathbf{Pos} \xrightarrow{\ \subseteq\ } \mathbf{PreOrd} \xrightarrow{\mathcal{P}'_{\mathrm{U}} / \mathcal{P}'_{\mathrm{L}} / \mathcal{P}'_{\mathrm{EM}}} \mathbf{PreOrd} \xrightarrow{\ \mathfrak{r}_{\leq}\ } \mathbf{Pos}$$

Here, the first functor is the (full) inclusion and $\mathfrak{r}_{\leq}$ is the reflection of **PreOrd** into **Pos** (defined on objects as $Z \mapsto Z/{\sim}$ where $a \sim b$ iff $a \leq b$ and $a \geq b$). We denote such compositions as $\mathcal{P}_{\mathrm{U}}$, $\mathcal{P}_{\mathrm{L}}$ and $\mathcal{P}_{\mathrm{EM}}$, respectively.

4.1.2 $\mathscr{P}_U$, $\mathscr{P}_L$ and $\mathscr{P}_{EM}$ explicitly. Let $(Z, \leq)$ be a poset. By $M =^U N$ denote $M \leq^U N$ and $N \leq^U M$. Since $\mathscr{P}_U(Z)$ is defined as the quotient of $\mathscr{P}'_U(Z)$ by $=^U$, it is essential to understand how the equivalence classes of $=^U$ look like.

4.1.3 Lemma. *Let $M, N \in \mathscr{P}(Z)$. Then,*

1. $M \leq^U N$ *iff* $\uparrow M \supseteq N$,
2. $M =^U \uparrow M$, *and*
3. $M =^U N$ *iff* $\uparrow M = \uparrow N$.

Proof. (1) follows from the definition of $\leq^U$ and (2) is an immediate consequence of (1). (3) $M =^U N$ implies, by (1), that $\uparrow M \supseteq N$ and $M \subseteq \uparrow N$. Hence, $\uparrow M = \uparrow N$. On the other hand, if $\uparrow M = \uparrow N$, then, by (2), $M =^U \uparrow M = \uparrow N =^U N$. □

We see that each $=^U$-equivalence class has a canonical representative computed as the upwards closure. Consequently, instead of quotienting $\mathscr{P}'_U(Z)$, it is equivalent to define $\mathscr{P}_U(Z)$ as the poset of upsets $(\mathsf{Up}(Z), \leq^U)$. Moreover, $\rho_U\colon M \mapsto \uparrow M$ is the reflection map $\mathscr{P}'_U(Z) \to \mathscr{P}_U(Z)$ in **PreOrd**.

Next, define $=^L$ and $=^{EM}$ similarly to $=^U$. Again, the above reasoning can be directly translated to identifying the representatives of $=^L$ and $=^{EM}$-equivalence classes:

4.1.4 Lemma. *Let $M, N \in \mathscr{P}(Z)$. Then,*

1. $M \leq^L N$ *iff* $M \subseteq \downarrow N$,
2. $M =^L \downarrow M$,
3. $M =^L N$ *iff* $\downarrow M = \downarrow N$,
4. $M \leq^{EM} N$ *iff* $\uparrow M \supseteq N$ *and* $M \subseteq \downarrow N$,
5. $M =^{EM} \downarrow M \cap \uparrow M$, *and*
6. $M =^{EM} N$ *iff* $\downarrow M \cap \uparrow M = \downarrow N \cap \uparrow N$.

Proof. (1), (2) and (3) are just the upside-down versions of Lemma 4.1.3. (4) and (5) follow from the definition of $\leq^{EM}$. Then, right-to-left implication in (6) follows from (5) and for the reverse direction, observe that $\downarrow(\downarrow M \cap \uparrow M) = \downarrow M$ and $\uparrow(\downarrow M \cap \uparrow M) = \uparrow M$. Then, $\downarrow M \cap \uparrow M = \downarrow N \cap \uparrow N$ implies that $M =^U N$ and $M =^L N$. □

This time the unique representative of each equivalence $=^L$-equivalence class is computed as the downwards closure and for $=^{EM}$ it is the convex closure, i.e. $\downarrow M \cap \uparrow M$. Then, $\mathscr{P}_L(Z)$ is isomorphic to the poset of downsets $(\mathsf{Down}(Z), \leq^L)$ and $\mathscr{P}_{EM}(Z)$ is isomorphic to the poset of *convex subsets* $(\mathsf{Conv}(Z), \leq^{EM})$ of Z with the reflection morphisms: $\rho_L\colon M \mapsto \uparrow M$ and $\rho_{EM}\colon M \mapsto \downarrow M \cap \uparrow M$.

4.1.5 Several key observations. All these constructions have something in common. For each $\pi \in \{U, L, EM\}$, the following composition is equal to the identity on $\mathscr{P}_\pi(Z)$

$$\mathcal{P}_\pi(Z) \xrightarrow{\subseteq} \mathcal{P}'_\pi(Z) \xrightarrow{\rho_\pi} \mathcal{P}_\pi(Z)$$

Categorically speaking, $r \stackrel{\text{def}}{\equiv} \rho_\pi \colon \mathcal{P}'_\pi(Z) \to \mathcal{P}_\pi(Z)$ is a retraction and the inclusion map $s \stackrel{\text{def}}{\equiv} \mathcal{P}_\pi(Z) \stackrel{\subseteq}{\to} \mathcal{P}'_\pi(Z)$ is a section. Each retraction–section pair defines a split idempotent $e \stackrel{\text{def}}{\equiv} r\,;s \colon \mathcal{P}'_\pi(Z) \to \mathcal{P}'_\pi(Z)$[1] which interacts nicely with the monad structure $(\mathcal{P}'_\pi, \eta, \mu)$:

4.1.6 Lemma. *In the category of preordered sets, for any monotone $f \colon \mathcal{P}'_\pi(Z) \to Y$ and $g \colon \mathcal{P}'_\pi(\mathcal{P}'_\pi(Z)) \to Y$ such that Y is a poset, the following two diagrams commute*

$$\text{(N1)}\quad \begin{array}{ccc} \mathcal{P}'_\pi(Z) & \xrightarrow{e} & \mathcal{P}'_\pi(Z) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f} & \\ Y & & \end{array} \qquad\qquad \text{(N2)}\quad \begin{array}{ccc} \mathcal{P}'_\pi(\mathcal{P}'_\pi(Z)) & \xrightarrow{\mathcal{P}'_\pi(e)} & \mathcal{P}'_\pi(\mathcal{P}'_\pi(Z)) \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle g} & \\ Y & & \end{array}$$

Proof. (N1) For any $M \in \mathcal{P}(Z)$, $M =^\pi e(M)$ (Lemma 4.1.3 resp. 4.1.4). Because Y is a poset and f is monotone, $f(M) = f(e(M))$.

(N2) Let $\mathcal{M} \in \mathcal{P}(\mathcal{P}(Z))$. We will show that $\mathcal{P}'_\pi(e)(\mathcal{M}) = \{e(M) : M \in \mathcal{M}\}$ is $=^\pi$-equal to $\mathcal{M}$; which then gives that $g(\mathcal{M}) = g(\mathcal{P}'_\pi(e)(\mathcal{M}))$ because g is monotone and Y is a poset. For $\pi = \mathrm{U}$ (i.e. $e(M) = {\uparrow}M$), $\mathcal{P}'_\pi(e)(\mathcal{M})$ and $\mathcal{M}$ are $=^{\mathrm{U}}$-equal if

$$\begin{aligned} {\uparrow}_{\leq^{\mathrm{U}}} \{{\uparrow}M : M \in \mathcal{M}\} &= \{N \mid \exists M \in \mathcal{M}.\, {\uparrow}M \leq^{\mathrm{U}} N\} \quad \text{and} \\ {\uparrow}_{\leq^{\mathrm{U}}} \mathcal{M} &= \{N \mid \exists M \in \mathcal{M}.\, M \leq^{\mathrm{U}} N\} \end{aligned}$$

are equal and this follows from $M =^{\mathrm{U}} {\uparrow}M$. The case when $\pi = \mathrm{L}$ is similar. For $\pi = \mathrm{EM}$ we need

$$\{N \mid \exists M_i \in \mathcal{M}.\, \langle\!\langle M_1 \rangle\!\rangle \leq^{\mathrm{EM}} N \leq^{\mathrm{EM}} \langle\!\langle M_2 \rangle\!\rangle\} = \{N \mid \exists M_i \in \mathcal{M}.\, M_1 \leq^{\mathrm{EM}} N \leq^{\mathrm{EM}} M_2\}$$

where $\langle\!\langle M \rangle\!\rangle \stackrel{\text{def}}{\equiv} {\downarrow}M \cap {\uparrow}M$. But, this holds because $M =^{\mathrm{EM}} \langle\!\langle M \rangle\!\rangle$ (for all $M \in \mathcal{P}(Z)$). □

As we will see in the next section (Proposition 4.1.9) these conditions are powerful enough to make sure that $\mathcal{P}_\pi$ is a monad, for every $\pi \in \{\mathrm{U,L,EM}\}$. Moreover, Proposition 4.1.9 also gives an explicit formula for the action of the endofunctor on morphisms. For $f \colon Z \to Y$ a morphism in **Pos**, $\mathcal{P}_\pi(f)$ is defined as the composite $s\,; \mathcal{P}'_\pi(f)\,; r$ which translates to

$$\mathcal{P}_{\mathrm{U}}(f) \colon\; M \mapsto {\uparrow}f[M], \quad \mathcal{P}_{\mathrm{L}}(f) \colon\; M \mapsto {\downarrow}f[M] \quad \text{and} \quad \mathcal{P}_{\mathrm{EM}}(f) \colon\; M \mapsto {\downarrow}f[M] \cap {\uparrow}f[M].$$

4.1.2 Restricting a monad to a subcategory

Let (M, η, μ) be a monad on category $\mathcal{D}$ and let $(M, \eta, \widetilde{(-)})$ be its equivalent representation as a Kleisli triple (recall A.3.12). Moreover, let $\mathcal{C}$ be a full subcategory of $\mathcal{D}$ such that

[1] We use ; for function composition in the diagrammatic order.

(RS-1) for every $d \in \mathcal{D}$, there is an object $N(d) \in \mathcal{C}$ and a retraction–section pair of morphisms: $r\colon M(d) \to N(d)$ and $s\colon N(d) \to M(d)$ such that $s\,;r = \mathrm{id}_{N(d)}$.

(RS-2) for $e \overset{\text{def}}{\equiv} r\,;s\colon M(d) \to M(d)$ and any $h\colon M(d) \to M(d')$ and $g\colon d' \to M(d)$:

(a) $e\,;h\,;r = h\,;r$ (b) $\widetilde{g\,;e}\,;r = \widetilde{g}\,;r$

Condition (a) should formally be written as $e_d\,;h\,;r_{d'} = h\,;r_{d'}$ (and similarly (b)) but we choose to not write subscripts as they should be always clear from the context.

Next, for every $c \in \mathcal{C}$ and $f\colon c \to N(c')$, define

- η^N_c to be the composite $c \xrightarrow{\eta_c} M(c) \xrightarrow{r} N(c)$, and
- $\overline{f}$ to be $N(c) \xrightarrow{s} M(c) \xrightarrow{\widetilde{f;s}} M(c') \xrightarrow{r} N(c')$.

Because $\mathcal{C}$ is a full subcategory of $\mathcal{D}$, both η^N_c and $\overline{f}$ are in $\mathcal{C}$. Therefore, for N seen as a mapping on objects $\mathsf{obj}(\mathcal{C}) \to \mathsf{obj}(\mathcal{C})$ we get that it defines a monad:

4.1.7 Proposition. $(N, \eta^N, \overline{(-)})$ *is a Kleisli triple on* $\mathcal{C}$*.*

Proof. (M1) Let $c \in \mathcal{C}$. $\overline{\eta^N_c} = \overline{\eta_c\,;r} = s\,;\widetilde{\eta_c\,;r\,;s}\,;r \overset{(b)}{=} s\,;\widetilde{\eta_c}\,;r = s\,;r = \mathrm{id}$ because $\widetilde{\eta_c} = \mathrm{id}_{M(c)}$.

(M2) Let $f\colon c \to N(c')$. Then, $\eta^N_c\,;\overline{f} = \eta_c\,;r\,;s\,;\widetilde{f\,;s}\,;r \overset{(a)}{=} \eta_c\,;\widetilde{f\,;s}\,;r \overset{(M2)}{=} f\,;s\,;r = f$ where we applied (a) with $h = \widetilde{f\,;s}$ and in (M2) we used (M2) for the monad M.

(M3) Let $f\colon c \to N(c')$ and $g\colon c' \to N(c'')$. Then, $\overline{f}\,;\overline{g} = (s\,;\widetilde{f\,;s}\,;r)\,;(s\,;\widetilde{g\,;s}\,;r) \overset{(a)}{=} s\,;\widetilde{f\,;s}\,;\widetilde{g\,;s}\,;r \overset{(M3)}{=} s\,;\widetilde{f\,;s\,;\widetilde{g\,;s}}\,;r \overset{(b)}{=} s\,;\widetilde{f\,;s\,;\widetilde{g\,;s}\,;r\,;s}\,;r = s\,;\widetilde{f\,;\overline{g}\,;s}\,;r = \overline{f\,;\overline{g}}$. □

Next, we justify that the conditions from Lemma 4.1.6 are sufficient for N to be a monad (with N being in the place of $\mathcal{P}_{\mathrm{U}}$, $\mathcal{P}_{\mathrm{L}}$ or $\mathcal{P}_{\mathrm{EM}}$). Instead of (RS-2) consider the following

(RS-2′) for $e \overset{\text{def}}{\equiv} r\,;s$ and any $f\colon M(d) \to c$ and $g\colon M(M(d)) \to c$ where $c \in \mathcal{C}$:

(N1) $e\,;f = f$ (N2) $M(e)\,;g = g$

4.1.8 Lemma. *The condition (b) from (RS-2) is equivalent to* $M(e)\,;\mu\,;r = \mu\,;r$*.*

Proof. Use the $(M,\eta,\mu) \leftrightarrow (M,\eta,\widetilde{(-)})$ translation: For "⇒" set $g = \mathrm{id}_{M(d)}$. Then, $\widetilde{\mathrm{id}\,;e}\,;r = \widetilde{e}\,;r = M(e)\,;\mu\,;r$ and, by (b), $\widetilde{\mathrm{id}\,;e}\,;r = \widetilde{\mathrm{id}}\,;r = \mu\,;r$. For "⇐" $\widetilde{g\,;e}\,;r = M(g)\,;M(e)\,;\mu\,;r = M(g)\,;\mu\,;r = \widetilde{g}\,;r$. □

Because (N1) implies (a) from (RS-2) by setting $f = h\,;r$ and (N2) implies $M(e)\,;\mu\,;r = \mu\,;r$ by setting $g = \mu\,;r$, we see that (RS-2′) implies (RS-2). To sum up we have obtained:

4.1.9 Proposition. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{D}$ and let M be a monad on $\mathcal{D}$. If $N\colon \mathsf{obj}(\mathcal{D}) \to \mathsf{obj}(\mathcal{C})$ is a mapping on objects satisfying (RS-1) and (RS-2′), then N is a monad on $\mathcal{D}$ and its restriction to $\mathcal{C}$ is a monad on $\mathcal{C}$. Moreover, the action of N on morphisms is*

$$N\colon \quad f\colon c \to c' \quad \longmapsto \quad s\,;M(f)\,;r\colon N(c) \to N(c')$$

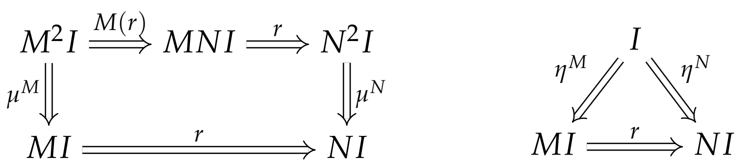
Proof. Only the "moreover" part needs an explanation. By definition $f\colon c \to c'$ is mapped to $\overline{f\,;\eta^N_{c'}} = s\,;\widetilde{f\,;\eta_{c'}\,;e}\,;r = s\,;\widetilde{f\,;\eta_{c'}}\,;r = s\,;M(f\,;\eta_{c'})\,;\mu\,;r = s\,;M(f)\,;r$. □

Notice that we do not require that N is a functor. It suffices that it is a mapping on objects. Functoriality is obtained by the translation from its Kleisli triple.

4.1.10 Remark. 1. Alternatively, (N2) can be replaced by $M(e)\,;r = r$ as then $g \overset{(N1)}{=} r\,;s\,;g = M(e)\,;r\,;s\,;g \overset{(N1)}{=} M(e)\,;g$.

2. Just by (a) from (RS-2) it follows that N is a functor and that r is a natural transformation $M \Longrightarrow N$. On the other hand, s is *not* a natural transformation $N \Longrightarrow M$ but, in fact, in all of our applications of Proposition 4.1.9 our categories are **Pos**-enriched and s is a lax-natural transformation (in sense of [Kel82]).

3. Let $I\colon \mathcal{C} \to \mathcal{D}$ be the embedding of the categories. Then, r seen as a natural transformation $MI \Longrightarrow IN$ is a morphism of monads (in sense of [LS02]). To see that, first, let us compute the multiplication of N: $\mu^N_c = \overline{\mathrm{id}_{N(c)}} = s\,;\widetilde{\mathrm{id}}\,;s\,;r = s\,;M(s)\,;\mu^M_c\,;r$. Therefore, the following two diagrams of natural transformations commute:



(The first diagram commutes by (RS-2) and the second commutes by the definition of η^N.)

4.1.3 Example 2: Spaces

Next, we take a look at how the power-constructions look like for spaces. Let us fix a topological space (X, τ) for the rest of this section. Again, there are three main possibilities how to define the topology on the subsets of X and they are related to the corresponding constructions for posets.

4.1.3.1 Upper Vietoris topology

Let $\mathcal{K}(X)$ be the set of compact subsets of X. Define, for every $U \in \tau$,

$$\boxtimes U \stackrel{\text{def}}{\equiv} \{K \in \mathcal{K}(X) \mid K \subseteq U\}.$$

The *upper Vietoris topology* $\mathbb{V}_{\boxtimes}\tau$ on $\mathcal{K}(X)$ is generated then from the basis $\{\boxtimes U : U \in \tau\}$. The assignment on objects

$$\mathbb{V}'_{\boxtimes}(X) \stackrel{\text{def}}{\equiv} (\mathcal{K}(X), \mathbb{V}_{\boxtimes}\tau)$$

extends to an endofunctor on the category of topological spaces: for a continuous $f\colon X \to Y$, define $\mathbb{V}'_{\boxtimes}(f)\colon \mathbb{V}'_{\boxtimes}(X) \to \mathbb{V}'_{\boxtimes}(Y)$ as $K \mapsto f[K]$.

4.1.11 Lemma.

1. *$\mathbb{V}'_{\boxtimes}$ is a well-defined functor.*
2. *$(\mathbb{V}'_{\boxtimes}, \eta, \mu)$ is a monad where the unit is the mapping $\eta_X\colon x \to \{x\}$ and multiplication is $\mu_X\colon \mathcal{M} \mapsto \bigcup \mathcal{M}$.*

Proof. (1) Since the image of a compact set is compact only continuity of $\mathbb{V}'_{\boxtimes}(f)$ needs to be checked:

$$\mathbb{V}'_{\boxtimes}(f)^{-1}[\boxtimes U] = \{K \mid f[K] \subseteq U\} = \{K \mid K \subseteq f^{-1}[U]\} = \boxtimes(f^{-1}[U]).$$

(2) Again, we only need to check continuity of η and μ; their naturality and commutativity of the monad diagrams follows from the fact that $\mathbb{V}'_{\boxtimes}$ is defined on morphisms identically to the powerset monad on sets. Observe that η is continuous because $\eta_X^{-1}[\boxtimes U] = \{x \mid \{x\} \subseteq U\} = U$.

Next, we check that μ is well defined. Let $\mathcal{M} \subseteq \mathcal{K}(X)$ be a compact subset of $\mathbb{V}'_{\boxtimes}(X)$. We need to show that $\bigcup \mathcal{M}$ is compact in X. Assume that $\bigcup \mathcal{M} \subseteq \bigcup_i^{\uparrow} U_i$. Then, for every $K \in \mathcal{M}$, since $K \subseteq \bigcup \mathcal{M} \subseteq \bigcup_i^{\uparrow} U_i$ and K is compact, there is an i such that $K \subseteq U_i$. Consequently, $\mathcal{M} \subseteq \bigcup_i^{\uparrow} \boxtimes U_i$ and because $\mathcal{M}$ is compact there is an i such that $\mathcal{M} \subseteq \boxtimes U_i$. Then, $\bigcup \mathcal{M} \subseteq U_i$.

Finally, we check that μ is continuous. Compute:

$$\mu_X^{-1}[\boxtimes U] = \{\mathcal{M} \mid \bigcup \mathcal{M} \subseteq U\} = \{\mathcal{M} \mid \mathcal{M} \subseteq \boxtimes U\} = \{\mathcal{M} \mid \mathcal{M} \in \boxtimes(\boxtimes U)\} = \boxtimes(\boxtimes U).$$

□

Notice that $\mathbb{V}'_{\boxtimes}(X)$ need not be T_0. Similarly to the upper power-construction for posets, we can fix this by defining $\mathbb{V}_{\boxtimes}(X)$ as the T_0 reflection $\mathbb{V}'_{\boxtimes}(X)$. That is, $\mathbb{V}_{\boxtimes}(X)$ is obtained from $\mathbb{V}'_{\boxtimes}(X)$ by quotienting it by $=^{\boxtimes}$ where $\leq^{\boxtimes}$ is the specialisation preorder of $\mathbb{V}_{\boxtimes}\tau$. As was the case for posets, $=^{\boxtimes}$ also has a more explicit description:

4.1.12 Lemma. *Let $K, H \in \mathcal{K}(X)$. Then,*

1. $K =^{\boxtimes} \uparrow K$, *and*
2. $K \leq^{\boxtimes} H$ *iff* $\uparrow K \supseteq H$,
3. $K =^{\boxtimes} H$ *iff* $\uparrow K = \uparrow H$.

(Where $\uparrow K$ is the upwards closure of K in the specialisation pre-order of (X, τ).)

Proof. (1) Because open sets are upwards closed in the specialisation order we have that $K \subseteq U$ iff $\uparrow K \subseteq U$ for every $U \in \tau$. (2) The right-to-left implication follows from (1). For "$\Rightarrow$", first notice that, for every subset $M \subseteq X$,

$$\uparrow M = \bigcap \{U \in \tau \mid M \subseteq U\}. \tag{4.1.1}$$

The "$\supseteq$" is because $M \subseteq U$ implies $\uparrow M \subseteq U$. For the other inclusion let $x \notin \uparrow M$. This means that for every $m \in M$ there is a $U^m \in \tau$ such that $x \notin U^m \ni m$. Then, $x \notin \bigcup_{m \in M} U^m \supseteq M$. Therefore if, for every $U \in \tau$, $K \subseteq U$ implies $H \subseteq U$, then, by (4.1.1), $\uparrow H \subseteq \uparrow K$. Finally, (3) follows from (1) and (2). □

We see that each $=^{\boxtimes}$-equivalence class has a canonical representative computed as the upwards-closure. Hence $\mathbb{V}_{\boxtimes}(X)$ is homeomorphic to the space $(\mathcal{K}^{\uparrow}(X), \mathbb{V}_{\boxtimes}\tau)$ of compact upsets of X and the quotient map $\rho_{\boxtimes} \colon \mathbb{V}'_{\boxtimes}(X) \to \mathbb{V}_{\boxtimes}(X)$ is $K \mapsto \uparrow K$.

4.1.3.2 Lower Vietoris topology

The second construction gives a topological space on the set $\mathcal{P}(X)$ of all subsets of the set X. For every $U \in \tau$, define

$$\diamond U \stackrel{\text{def}}{\equiv} \{M \in \mathcal{P}(X) \mid M \cap U \neq \emptyset\}$$

and set

$$\mathbb{V}'_{\diamond}(X) \stackrel{\text{def}}{\equiv} (\mathcal{P}(X), \mathbb{V}_{\diamond}\tau)$$

where $\mathbb{V}_{\diamond}\tau$ is the *lower Vietoris topology* generated from the subbasis $\{\diamond U : U \in \tau\}$.

4.1.13 Lemma.

1. $\mathbb{V}'_{\diamond}$ *is functorial with* $\mathbb{V}'_{\diamond}(f)$, *for a continuous map* $f \colon X \to Y$, *defined as* $M \mapsto f[M]$.
2. $(\mathbb{V}'_{\diamond}, \eta, \mu)$ *is a monad with the unit and multiplication defined as for* $\mathbb{V}'_{\boxtimes}$.

Proof. (1) Only continuity of $\mathbb{V}'_{\diamond}(f)$ needs to be checked:

$$\mathbb{V}'_{\boxtimes}(f)^{-1}[\diamond U] = \{M \mid f[M] \cap U \neq \emptyset\} = \{M \mid M \cap f^{-1}[U] \neq \emptyset\} = \diamond(f^{-1}[U]).$$

(2) As before, only continuity of η_X and μ_X needs to be checked. First, $\eta_X^{-1}[\oplus U] = \{x \mid \{x\} \cap U \neq \emptyset\} = U \in \tau$ and to check the former compute

$$\mu_X^{-1}[\oplus U] = \{m \mid \bigcup m \cap U \neq \emptyset\} = \{m \mid m \cap \oplus U \neq \emptyset\} = \{m \mid m \in \oplus(\oplus U)\}.$$

Therefore, $\mu_X^{-1}[\oplus U] = \oplus(\oplus U)$ which is open in $\mathbb{V}'_\boxtimes(\mathbb{V}'_\boxtimes(X))$. □

Furthermore, since $\mathbb{V}'_\oplus(X)$ might not be T_0 we need to examine the specialisation order $\leq^\oplus$ of $\mathbb{V}_\oplus\tau$ too:

4.1.14 Lemma. *Let $M, N \in \mathcal{P}(X)$. Then,*

1. $M =^\oplus \overline{M}$ *where $\overline{M}$ is the closure of M in X, and*
2. $M \leq^\oplus N$ *iff* $M \subseteq \overline{N}$,
3. $M =^\oplus N$ *iff* $\overline{M} = \overline{N}$.

Proof. (1) follows from the fact that $M \cap V \neq \emptyset$ iff $\overline{M} \cap V \neq \emptyset$. (2) The "$\Leftarrow$" implication follows from (1) as $M \subseteq \overline{N}$ implies $M \leq^\boxtimes \overline{N} =^\boxtimes N$. For the reverse implication, let $x \in M$. For any neighbourhood U of x, $M \in \oplus U$. Therefore, $N \in \oplus U$ or equivalently $N \cap U \neq \emptyset$ and, because U was an arbitrary neighbourhood of x, $x \in \overline{N}$. Finally, (3) is an immediate consequence of (2). □

We see that $\mathbb{V}_\oplus(X)$ defined as the T_0 reflection of $\mathbb{V}'_\oplus(X)$ is homeomorphic to the space of all closed subsets $(\mathsf{Clos}(X), \mathbb{V}_\oplus\tau)$ of X and the reflection map $\rho_\oplus \colon \mathbb{V}'_\oplus(X) \to \mathbb{V}_\oplus(X)$ is defined as $M \mapsto \overline{M}$.

4.1.3.3 (Full) Vietoris topology

In Lemmas 4.1.12 and 4.1.14 we saw that the upper and lower Vietoris constructions are tightly related to the upper and lower power-constructions on posets. It is then no surprise that a combination of both gives a construction related to the Egli-Milner lifting; set

$$\mathbb{V}'(X) \stackrel{\text{def}}{\equiv} (\mathcal{K}(X), \mathbb{V}\tau).$$

where $\mathbb{V}\tau$ is the *(full) Vietoris topology* generated from $\{\boxtimes U, \oplus U : U \in \tau\}$.

4.1.15 Lemma.

1. *$\mathbb{V}'$ is functorial with $\mathbb{V}'(f)$ defined as $M \mapsto f[M]$.*
2. *$(\mathbb{V}', \eta, \mu)$ is a monad with the unit and multiplication defined as for $\mathbb{V}'_\boxtimes$ and $\mathbb{V}'_\oplus$.*

Proof. Well-definedness of $\mathbb{V}'(f)$ and μ follows from well-definedness of $\mathbb{V}'_\boxtimes(f)$ and μ for $\mathbb{V}'_\boxtimes$. Continuity of $\mathbb{V}'(f)$, η and μ follows from continuity of the corresponding maps for $\mathbb{V}'_\boxtimes$ and $\mathbb{V}'_\oplus$. □

4.1.16 Lemma. *Let $K, H \in \mathcal{K}(X)$. Then,*

1. $K =^{\mathbb{V}} \uparrow K \cap \overline{K}$
2. $K \leq^{\mathbb{V}} H$ *iff* $\uparrow K \supseteq H$ *and* $K \subseteq \overline{H}$,
3. $K =^{\mathbb{V}} H$ *iff* $\uparrow K = \uparrow H$ *and* $\overline{K} = \overline{H}$ *iff* $\uparrow K \cap \overline{K} = \uparrow H \cap \overline{H}$.

Proof. (1) First, observe that $\uparrow K \cap \overline{K}$ is in $\mathcal{K}(X)$. This is because $\uparrow K \in \mathcal{K}(X)$ and a closed subset of a compact set is also compact. Next, $K \subseteq \uparrow K \cap \overline{K}$ proves the left-to-right implications of the following:

$$\uparrow K \cap \overline{K} \subseteq U \iff K \subseteq U \quad \text{and} \quad K \cap U \neq \emptyset \iff \uparrow K \cap \overline{K} \cap U \neq \emptyset$$

The first reverse implication is from $K =^{\boxtimes} \uparrow K$ and $\uparrow K \cap \overline{K} \subseteq \uparrow K$ and the second reverse implication follows from $K =^{\oplus} \overline{K}$ and because $\uparrow K \cap \overline{K} \cap U \neq \emptyset$ implies $\overline{K} \cap U \neq \emptyset$. Consequently, for any open set $\mathcal{U}$ from the base of $\mathbb{V}'(X)$, i.e. an open of the form $\boxtimes U \cap \bigcap_{i=1}^{n} \oplus V_i$, we have that $K \in \mathcal{U}$ iff $\uparrow K \cap \overline{K} \in \mathcal{U}$.

(2) The left-to-right implication follows from the fact that $\leq^{\mathbb{V}}$ is a subrelation of $\leq^{\boxtimes}$ and $\leq^{\oplus}$. For the reverse, if $K \in \boxtimes U \cap \bigcap_{i=1}^{n} \oplus V_i$ then, because $\uparrow K \supseteq H$, $H \in \boxtimes U$ and because $K \subseteq \overline{H}$, $\overline{H} \in \bigcap_{i=1}^{n} \oplus V_i$ which is equivalent to $H \in \bigcap_{i=1}^{n} \oplus V_i$.

(3) The first equivalence follows from (2) The left-to-right implication in the second equivalence is immediate and the reverse is a consequence of the calculations:

$$\uparrow K \subseteq \uparrow(\uparrow K \cap \overline{K}) \subseteq \uparrow(\uparrow K) = \uparrow K \quad \text{and} \quad \overline{K} \subseteq \overline{\uparrow K \cap \overline{K}} \subseteq \overline{\overline{K}} = \overline{K}. \qquad \square$$

Therefore, the T_0-reflection $\mathbb{V}(X)$ of $\mathbb{V}'(X)$ is homeomorphic to the space of *Plotkin lenses* $(\mathsf{Lens}(X), \mathbb{V}\tau)$ where $\mathsf{Lens}(X) = \{\uparrow K \cap C \mid K \in \mathcal{K}(X),\ C \in \mathsf{Clos}(X)\}$[2]. The reflection $\rho_{\mathbb{V}} \colon \mathbb{V}'(X) \to \mathbb{V}(X)$ is the map $K \mapsto \uparrow K \cap \overline{K}$.

Notice that (3) in Lemma 4.1.16 proves that $\rho_{\mathbb{V}}(\rho_{\mathbb{V}}(K)) = \rho_{\mathbb{V}}(K)$. This is because $K =^{\mathbb{V}} \rho_{\mathbb{V}}(K)$ and so $\uparrow K = \uparrow \rho_{\mathbb{V}}(K)$ and $\overline{K} = \overline{\rho_{\mathbb{V}}(K)}$. Consequently,

$$\mathsf{Lens}(X) = \{K \in \mathcal{K}(X) \mid \uparrow K \cap \overline{K} = K\}.$$

4.1.17 Remark. The fact that, for all $\mathcal{V} \in \{\mathbb{V}_{\boxtimes}, \mathbb{V}_{\oplus}, \mathbb{V}\}$, the T_0-reflection of $\mathcal{V}'(X)$ is homeomorphic to the space of compact upper sets, closed sets or Plotkin lenses, respectively, is not new. This has been known since Smyth's [Smy83, Theorem 2].

4.1.3.4 Categorical properties

Observe that, for all $\mathcal{V} \in \{\mathbb{V}_{\boxtimes}, \mathbb{V}_{\oplus}, \mathbb{V}\}$, as was the case for posets, $\mathcal{V}(X)$ is a subspace and reflection of $\mathcal{V}'(X)$ at the same time. Therefore, we have a pair of maps, retraction $r \stackrel{\text{def}}{\equiv} \rho_{\mathcal{V}} \colon \mathcal{V}'(X) \to \mathcal{V}(X)$ and section $s \stackrel{\text{def}}{\equiv} \mathcal{V}(X) \hookrightarrow \mathcal{V}'(X)$, defining a split idempotent $e \stackrel{\text{def}}{\equiv} r\,;s$.

[2]The name is motivated by the original construction on domains made by Gordon Plotkin [Plo76].

4.1.18 Lemma. *In the category of all topological spaces, for any continuous $f\colon \mathcal{V}'(X) \to Y$ and $g\colon \mathcal{V}'(\mathcal{V}'(X)) \to Y$ such that Y is T_0, the following two diagrams commute*

$$\text{(N1)}\quad \begin{array}{ccc} \mathcal{V}'(X) & \xrightarrow{\;e\;} & \mathcal{V}'(X) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f} & \\ Y & & \end{array} \qquad\qquad \text{(N2)}\quad \begin{array}{ccc} \mathcal{V}'(\mathcal{V}'(X)) & \xrightarrow{\;\mathcal{V}'(e)\;} & \mathcal{V}'(\mathcal{V}'(X)) \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle g} & \\ Y & & \end{array}$$

Proof. Recall that continuous functions are monotone w.r.t. the specialization order and also that being T_0 is equivalent to saying that the specialisation order is a poset. Then (N1) commutes because M and $e(M)$ are equal in the specialisation order and so $f(M) = f(e(M))$. To show commutativity of (N2) it is enough to show that, for any $\mathcal{M} \in \mathcal{V}'(\mathcal{V}'(X))$, $\mathcal{M}$ and $\mathcal{N} \stackrel{\text{def}}{\equiv} \{e(K) : K \in \mathcal{M}\}$ are equal in the specialisation order of $\mathcal{V}'(\mathcal{V}'(X))$.

Assume $\mathcal{V}' = \mathbb{V}'$ (i.e. $e(K) = \uparrow K \cap \overline{K}$) and let $\mathcal{U} \in \mathcal{V}'(X)$. Since $\mathcal{U}$ is a of the form $\bigcup_{i\in I}(\boxtimes U_i \cap \bigcap_{j=1}^{n_i} \diamondplus V_{ij})$, for some U_i's and V_{ij}'s from τ,

$$\mathcal{M} \in \boxtimes\mathcal{U} \quad\text{iff}\quad \mathcal{M} \subseteq \boxtimes\mathcal{U} \quad\text{iff}\quad \forall K \in \mathcal{M}\, \exists i \in I.\, K \in \boxtimes U_i \cap \bigcap_{j=1}^{n_i} \diamondplus V_{ij}.$$

Moreover, $K \in \boxtimes U_i \cap \bigcap_{j=1}^{n_i} \diamondplus V_{ij}$ iff $e(K) \in \boxtimes U_i \cap \bigcap_{j=1}^{n_i} \diamondplus V_{ij}$ and so, $\mathcal{M} \in \boxtimes\mathcal{U}$ iff $\mathcal{N} \in \boxtimes\mathcal{U}$. Similarly, $\mathcal{M} \in \diamondplus\mathcal{U}$ iff $\mathcal{N} \in \diamondplus\mathcal{U}$. This proves the commutativity of (N2) for $\mathcal{V} = \mathbb{V}$. The proof for the other two Vietoris constructions is essentially the same. □

This jointly proves, by Proposition 4.1.9, that $\mathbb{V}_\boxtimes$, $\mathbb{V}_\diamondplus$ and $\mathbb{V}$ are monads on the category of T_0 topological spaces and that their actions on morphisms are computed as

$$\mathbb{V}_\boxtimes(f)\colon K \mapsto \uparrow f[K], \quad \mathbb{V}_\diamondplus(f)\colon M \mapsto \overline{f[M]} \quad\text{and}\quad \mathbb{V}(f)\colon K \mapsto \uparrow f[K] \cap \overline{f[K]}.$$

4.1.19 Relationship with powerlocale constructions Johnstone showed in [Joh82] that, for the functor Ω restricted to **KRegSp** $\to$ **KRegFrm**, $\Omega \circ \mathbb{V} \cong \mathbb{V}_{\mathrm{Fr}} \circ \Omega$ and similar results hold for the other Vietoris constructions, that is $\Omega \circ \mathbb{V}_\boxtimes \cong \mathbb{V}_\Box \circ \Omega$ and $\Omega \circ \mathbb{V}_\diamondplus \cong \mathbb{V}_\Diamond \circ \Omega$ [Vic97; Bf96].

This analogy goes even beyond compact regular spaces/frames. Points of $\mathbb{V}_\Box(L)$ are in a bijection with the Scott-open filters, or by the Hofmann-Mislove Theorem, the compact upwards-closed subsets of $\Sigma(L)$. On the other hand, the points of $\mathbb{V}_\Diamond(L)$ correspond to elements of L, or also the closed subsets of $\Sigma(L)$ [Vic97]. Next, in Section 4.3.3 we will recall the fact that $\mathbb{V}_{\mathrm{Fr}}L$ can be viewed as a quotient of $\mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L$ [Vic09a]. As a result, any point $p\colon \mathbb{V}_{\mathrm{Fr}}L \to \mathbf{2}$ pre-composed with the quotient map $\mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L \to \mathbb{V}_{\mathrm{Fr}}L$ gives two points $\mathbb{V}_\Box L \to \mathbf{2}$ and $\mathbb{V}_\Diamond L \to \mathbf{2}$. This partially explains the definition of Plotkin lenses as the points of $\mathbb{V}(X)$.

For a different approach to powerspaces or powerlocales see [BK17], [VVV12] and [Vic09b].

4.1.4 Bispaces

To define a power-construction for the category of bispaces **biTop** we take inspiration from both the power-construction on posets and on spaces. For a bispace $X = (X, \tau_+, \tau_-)$ set $\mathcal{K}_+(X)$ and $\mathcal{K}_-(X)$ to be the sets of all τ_+ and τ_--compact subsets of X, respectively. Then, define *upper, lower and (full) Vietoris bispaces* to be the bispaces

$$\mathbb{W}'_{\boxtimes}(X) \stackrel{\text{def}}{\equiv} (\mathcal{K}_+(X), \mathbb{V}_{\boxtimes}\tau_+, \mathbb{V}_{\oplus}\tau_-), \qquad \mathbb{W}'_{\oplus}(X) \stackrel{\text{def}}{\equiv} (\mathcal{K}_-(X), \mathbb{V}_{\oplus}\tau_+, \mathbb{V}_{\boxtimes}\tau_-) \quad \text{and}$$
$$\mathbb{W}'(X) \stackrel{\text{def}}{\equiv} (\mathcal{K}_+(X) \cap \mathcal{K}_-(X), \mathbb{V}\tau_+, \mathbb{V}\tau_-),$$

respectively. For a bicontinuous $f\colon X \to Y$, define $\mathbb{W}'_{\boxtimes}(f)$, $\mathbb{W}'_{\oplus}(f)$ and $\mathbb{W}'(f)$ all as the map $K \mapsto f[K]$. All those constructions are endofunctors on **biTop** and, moreover, by a combination of Lemmas 4.1.11, 4.1.13 and 4.1.15, we also have that:

4.1.20 Lemma. *For every $\mathcal{W}' \in \{\mathbb{W}'_{\boxtimes}, \mathbb{W}'_{\oplus}, \mathbb{W}'\}$, $(\mathcal{W}', \eta_X\colon x \mapsto \{x\}, \mu_X\colon \mathcal{M} \mapsto \bigcup \mathcal{M})$ is a monad on the category of bispaces.*

Proof. We only show that $\eta_X^{\mathbb{W}'}\colon X \to \mathbb{W}'(X)$ is bicontinuous; the other cases are similar. It is enough to check that, for every $U \in \tau_\pm$, $\eta^{-1}[\oplus U]$ and $\eta^{-1}[\boxtimes U]$ are $\tau_\pm$-open and this is true because $\eta_X^{\mathbb{V}'}\colon (X, \tau_\pm) \to \mathbb{V}'(X, \tau_\pm)$ is continuous. □

Recall that the equations we used for defining the point-free Vietoris constructions (Section 3.5.5) hold for $\boxtimes(-)$ and $\oplus(-)$.

4.1.21 Lemma ([Joh82, Lemma 4.2])**.** *Let $\mathcal{W}' \in \{\mathbb{W}'_{\boxtimes}, \mathbb{W}'_{\oplus}, \mathbb{W}'\}$ and let M, N and M_i's be arbitrary subsets of a bispace X and $\{U_j\}_j \subseteq \tau_\pm$. Then, in $\mathcal{W}'(X)$:*

1. $\boxtimes X = \mathcal{K}_\pm(X)$ *or* $\mathcal{K}_+(X) \cap \mathcal{K}_-(X)$, *respectively,* $\quad \boxtimes(M \cap N) = \boxtimes M \cap \boxtimes N$,
2. $\oplus\emptyset = \emptyset$, $\quad M \subseteq N$ *implies* $\oplus M \subseteq \oplus N$,
3. ${\bigcup^{\uparrow}}_{j \in J} \boxtimes U_j = \boxtimes({\bigcup^{\uparrow}}_{j \in J} U_j)$, $\quad \bigcup_{i \in I} \oplus M_i = \oplus(\bigcup_{i \in I} M_i)$, *and*
4. $\boxtimes(M \cup N) \subseteq \boxtimes M \cup \oplus N$, $\quad \boxtimes M \cap \oplus N \subseteq \oplus(M \cap N)$.

As a consequence we see that the basis of $\mathbb{V}\tau_\pm$ consists of the elements of the form $\boxtimes U \cap \bigcap_{i=1}^n \oplus V_i$. Also, an immediate consequence of the last item is a description of some elements in the consistency and totality relations of $\mathbb{W}X$.

4.1.22 Lemma. *Let U_+ be a τ_+-open and U_- a τ_--open subsets of a bispace X, then:*

1. *If* $(U_+, U_-) \in \mathsf{con}_X$ *then* $(\boxtimes U_+, \oplus U_-) \in \mathsf{con}_{\mathbb{W}X}$ *and* $(\oplus U_+, \boxtimes U_-) \in \mathsf{con}_{\mathbb{W}X}$.
2. *If* $(U_+, U_-) \in \mathsf{tot}_X$ *then* $(\boxtimes U_+, \oplus U_-) \in \mathsf{tot}_{\mathbb{W}X}$ *and* $(\oplus U_+, \boxtimes U_-) \in \mathsf{tot}_{\mathbb{W}X}$.

And, similarly, for $\mathsf{con}_{\mathbb{W}_\boxtimes X}$, $\mathsf{con}_{\mathbb{W}_\oplus X}$, $\mathsf{tot}_{\mathbb{W}_\boxtimes X}$ *and* $\mathsf{tot}_{\mathbb{W}_\oplus X}$.

Proof. Let $(U_+, U_-) \in \mathsf{con}_X$. By (2) and (4) in Lemma 4.1.21, $\boxtimes U_+ \cap \oplus U_- \subseteq \oplus(U_+ \cap U_-) = \oplus\emptyset = \emptyset$ and so $(\boxtimes U_+, \oplus U_-) \in \mathsf{con}_{\mathbb{W}X}$. The other case is the same. For 2., we use (1) and (4) from Lemma 4.1.21. □

Recall, from Section 2.1.1, that each bitopological space has a preorder $\leq$ associated to it. It is defined as the intersection of the specialisation order of the τ_+ topology ($\leq_+$) and the reversed specialisation order of the τ_- topology ($\geq_-$), i.e. $\leq\, = \,\leq_+ \cap \geq_-$. We say that a bispace is T_0 if $\leq$ is a partial order. As was the case for the power-constructions of spaces, $\mathcal{W}'(X)$'s are hardly bitopologically T_0. As before, we fix this by a T_0 reflection:

4.1.23 Definition. Set $\mathbb{W}_\boxtimes(X)$, $\mathbb{W}_\oplus(X)$ and $\mathbb{W}(X)$ to be the T_0 reflections of $\mathbb{W}'_\boxtimes(X)$, $\mathbb{W}'_\oplus(X)$ and $\mathbb{W}'(X)$, respectively.

Next, we examine what the T_0-reflections look like.

4.1.24 Observation. *Let* K *and* H *be from* $\mathcal{K}_+(X)$, $\mathcal{K}_-(X)$ *or* $\mathcal{K}_+(X) \cap \mathcal{K}_-(X)$. *Then,*

1. $K \leq H$ *in* $\mathbb{W}'_\boxtimes(X)$ *iff* $\uparrow_+ K \cap \overline{K}^{\tau_-} \supseteq H$,
2. $K \leq H$ *in* $\mathbb{W}'_\oplus(X)$ *iff* $K \subseteq \uparrow_- H \cap \overline{H}^{\tau_+}$, *and*
3. $K \leq H$ *in* $\mathbb{W}'(X)$ *iff* $\uparrow_+ K \cap \overline{K}^{\tau_-} \supseteq H$ *and* $K \subseteq \uparrow_- H \cap \overline{H}^{\tau_+}$.

(Where $\uparrow_\pm M$ *is the upwards closure of* M *in the* $\tau_\pm$*-specialisation order.)*

Proof. An immediate consequence of Lemmas 4.1.12, 4.1.14 and 4.1.16: (1) $K \leq H$ iff $K \leq_+^\boxtimes H$ and $H \leq_-^\oplus K$ where $\leq_+^\boxtimes$ and $\leq_-^\oplus$ are the specialisation orders of $\mathbb{V}_\boxtimes \tau_+$ and $\mathbb{V}_\oplus \tau_-$, respectively. Because $K \leq_+^\boxtimes H$ iff $\uparrow_+ K \supseteq H$ and $H \leq_-^\oplus K$ iff $H \subseteq \overline{K}^{\tau_-}$, as in (3) of Lemma 4.1.16, $K \leq H$ is equivalent to $\uparrow_+ K \cap \overline{K}^{\tau_-} \supseteq H$. The proof of (2) is the same and (3) follows immediately from (2) of Lemma 4.1.16. □

4.1.25 Lemma. *Let* $K \in \mathcal{K}_\pm(X)$ *and* $M \subseteq X$ *be such that* $K \subseteq M \subseteq \uparrow_\pm K$. *Then,* $M \in \mathcal{K}_\pm(X)$.

Proof. Let $M \subseteq \bigcup_i U^i_\pm$ for some $\{U^i_\pm\}_i \subseteq \tau_\pm$. Since $K \subseteq M$, there is a finite F such that $K \subseteq \bigcup_{i \in F} U^i_\pm$. Because, for every $U \in \tau_\pm$, $K \subseteq U$ iff $\uparrow_\pm K \subseteq U$, also $M \subseteq \uparrow_\pm K \subseteq \bigcup_{i\in F} U^i_\pm$. □

Notice that, for a $K \in \mathcal{K}_+(X)$, both $K =_+^\boxtimes \uparrow_+ K \cap \overline{K}^{\tau_-}$ and $K =_-^\oplus \uparrow_+ K \cap \overline{K}^{\tau_-}$, where $\leq_+^\boxtimes$ and $\leq_-^\oplus$ are the specialisation orders of $\mathbb{V}_\boxtimes \tau_+$ and $\mathbb{V}_\oplus \tau_-$, respectively. This is because

$$K \subseteq \uparrow_+ K \cap \overline{K}^{\tau_-} \subseteq \uparrow_+ K, \qquad K \subseteq \uparrow_+ K \cap \overline{K}^{\tau_-} \subseteq \overline{K}^{\tau_-}$$

and so $\uparrow_+ K \cap \overline{K}^{\tau_-} \in \mathcal{K}_+(X)$ (Lemma 4.1.25), $K =^{\boxtimes}_{+} \uparrow_+ K$ and $K =^{\oplus}_{-} \overline{K}^{\tau_-}$. We see that each equivalence class has a distinguished element computed by $\rho\colon K \mapsto \uparrow_+ K \cap \overline{K}^{\tau_-}$ such that $K =^{\mathbb{W}'_{\boxtimes}(X)} H$ iff $\rho(K) = \rho(H)$. Consequently:

4.1.26 Lemma. *$\mathbb{W}_{\boxtimes}(X)$ is bihomeomorphic to $(\mathsf{Lens}_+(X), \mathbb{V}_{\boxtimes}\tau_+, \mathbb{V}_{\oplus}\tau_-)$ and $\mathbb{W}_{\oplus}(X)$ is bihomeomorphic to $(\mathsf{Lens}_-(X), \mathbb{V}_{\oplus}\tau_+, \mathbb{V}_{\boxtimes}\tau_-)$ where*

$$\mathsf{Lens}_+ = \{K \in \mathcal{K}_+(X) \mid K = \uparrow_+ K \cap \overline{K}^{\tau_-}\} \text{ and}$$
$$\mathsf{Lens}_- = \{K \in \mathcal{K}_-(X) \mid K = \uparrow_- K \cap \overline{K}^{\tau_+}\}.$$

A similar reasoning also works for $\mathbb{W}(X)$. Let $K \in \mathcal{K}_+(X) \cap \mathcal{K}_-(X)$ and set

$$M \stackrel{\text{def}}{\equiv} \uparrow_+ K \cap \uparrow_- K \cap \overline{K}^{\tau_+} \cap \overline{K}^{\tau_-}.$$

We know, by Lemma 4.1.16, that $K =^{\mathbb{V}}_{+} \uparrow_+ K \cap \overline{K}^{\tau_+}$ and $K =^{\mathbb{V}}_{-} \uparrow_- K \cap \overline{K}^{\tau_-}$ and, because

$$K \subseteq M \subseteq \uparrow_+ K \cap \overline{K}^{\tau_+} \quad \text{and} \quad K \subseteq M \subseteq \uparrow_- K \cap \overline{K}^{\tau_-},$$

we see that $K =^{\mathbb{V}}_{\pm} M$. Again, by Lemma 4.1.25, $M \in \mathcal{K}_+(X) \cap \mathcal{K}_-(X)$ and the mapping $\rho\colon K \mapsto \uparrow_+ K \cap \uparrow_- K \cap \overline{K}^{\tau_+} \cap \overline{K}^{\tau_-}$ picks out the distinguished elements from each equivalence class. We obtain:

4.1.27 Lemma. *$\mathbb{W}(X)$ is bihomeomorphic to $(\mathsf{Lens}_d(X), \mathbb{V}\tau_+, \mathbb{V}\tau_-)$ where*

$$\mathsf{Lens}_d = \{K \in \mathcal{K}_+(X) \cap \mathcal{K}_-(X) \mid K = \uparrow_+ K \cap \uparrow_- K \cap \overline{K}^{\tau_+} \cap \overline{K}^{\tau_-}\}.$$

4.1.28 Categorical properties. Exactly the same reasoning as in Section 4.1.3.4 applies for proving that $\mathbb{W}_{\boxtimes}$, $\mathbb{W}_{\oplus}$ and $\mathbb{W}$ are monads on the category of bispaces. Then, for a bicontinuous $f\colon X \to Y$, the corresponding action on morphisms is computed as

$$\mathbb{W}_{\boxtimes}(f)\colon K \mapsto \uparrow_+ f[K] \cap \overline{f[K]}^{\tau_-}, \quad \mathbb{W}_{\oplus}(f)\colon K \mapsto \uparrow_- f[K] \cap \overline{f[K]}^{\tau_+} \quad \text{and}$$
$$\mathbb{W}(f)\colon K \mapsto \uparrow_+ f[K] \cap \uparrow_- f[K] \cap \overline{f[K]}^{\tau_-} \cap \overline{f[K]}^{\tau_+}.$$

4.1.5 Order-separatedness and relation liftings

Recall that, by Lemma 2.2.3, for an order-separated bispace X, the specialisation orders of both topologies are aligned, i.e. $\leq_+ = \geq_-$, and so the associated order $\leq$ is equal to $\leq_+$ (or $\geq_-$). As a result, the definitions of the Vietoris functors for bispaces become significantly simpler for X. To start with, we have a lemma:

4.1.29 Lemma. *Let (X, τ_+, τ_-) be an order-separated bispace and let $K \subseteq X$. Then,*

1. *K is convex (w.r.t $\leq$) iff $K = \downarrow K \cap \uparrow K$ iff $K = \downarrow M \cap \uparrow N$ for some $M, N \subseteq X$.*
2. *If K is τ_+-compact and $x \notin \uparrow K$, then $\exists (U_+, U_-) \in \mathsf{con}_X$ s.t. $K \subseteq U_+$ and*

$x \in U_-$.

Consequently, if K is τ_+-compact, then $\uparrow K$ is τ_--closed and $\overline{K}^{\tau_-} = \uparrow K$.

3. *If K is τ_--compact and $x \notin \downarrow K$, then $\exists (U_+, U_-) \in \mathsf{con}_X$ s.t. $x \in U_+$ and $K \subseteq U_-$.*

   Consequently, f K is τ_--compact, then $\downarrow K$ is τ_+-closed and $\overline{K}^{\tau_+} = \downarrow K$.

Proof. (1) is true for any partially ordered set. For (2) assume that K is τ_+-compact and $x \notin \uparrow K$. From order-separateness, for every $k \in K$, as $k \not\leq x$, there exists a $(U_+^k, U_-^k) \in \mathsf{con}_X$ such that $k \in U_+$ and $x \in U_-$. Since K is τ_+-compact and $\{U_+^k\}_{k \in K}$ covers K, there must be a finite $F \subseteq_{\text{fin}} K$ such that $\{U_+^k\}_{k \in F}$ still covers K. Then, by (con-$\forall$), $U_+ \stackrel{\text{def}}{\equiv} \bigcup_{k \in F} U_+^k$ is disjoint from $U_- \stackrel{\text{def}}{\equiv} \bigcap_{k \in F} U_-^k$ and, clearly, $x \in U_-$. (3) is proved symmetrically. □

With this, we can simplify the formulas for the points of the Vietoris bispaces, for an order-separated X:

$$\begin{aligned}
\mathsf{Lens}_+(X) &= \{K \in \mathcal{K}_+(X) \mid K = \uparrow K\}, \\
\mathsf{Lens}_-(X) &= \{K \in \mathcal{K}_-(X) \mid K = \downarrow K\} \quad \text{and} \\
\mathsf{Lens}_d(X) &= \{K \in \mathcal{K}_+(X) \cap \mathcal{K}_-(X) \mid K = \uparrow K \cap \downarrow K\}
\end{aligned}$$

which can be also written as

$$\begin{aligned}
\mathsf{Lens}_+(X) &= \mathcal{K}_+(X) \cap \mathsf{Up}(X, \leq), \\
\mathsf{Lens}_-(X) &= \mathcal{K}_-(X) \cap \mathsf{Down}(X, \leq) \quad \text{and} \\
\mathsf{Lens}_d(X) &= \mathcal{K}_+(X) \cap \mathcal{K}_-(X) \cap \mathsf{Conv}(X, \leq).
\end{aligned}$$

Also, if $f \colon X \to Y$ is a bicontinuous function such that Y is order-separated then

$$\mathbb{W}_{\boxtimes}(f) \colon K \mapsto \uparrow f[K], \quad \mathbb{W}_{\oplus}(f) \colon K \mapsto \downarrow f[K] \quad \text{and} \quad \mathbb{W}(f) \colon K \mapsto \downarrow f[K] \cap \uparrow f[K].$$

The associated pre-orders on the Vietoris powerbispaces also simplify accordingly so that they exactly match those for the corresponding power-constructions on posets:

$(\mathrm{O}_{\boxtimes})$ $\quad K \leq H$ in $\mathbb{W}_{\boxtimes}(X)$ iff $\uparrow K \supseteq H$,

$(\mathrm{O}_{\oplus})$ $\quad K \leq H$ in $\mathbb{W}_{\oplus}(X)$ iff $K \subseteq \downarrow H$, and

$(\mathrm{O}_{\mathbb{V}})$ $\quad K \leq H$ in $\mathbb{W}(X)$ iff $\uparrow K \supseteq H$ and $K \subseteq \downarrow H$

4.1.30 Lemma. *Let X be an order-separated bispace, $K, H \in \mathcal{K}_-(X)$ and $K', H' \in \mathcal{K}_+(X)$. Then,*

1. $K \not\subseteq \downarrow H$ *iff* $\exists (U_+, U_-) \in \mathsf{con}_X$ *such that* $K \in \oplus U_+$ *and* $H \in \boxtimes U_-$.

2. $K' \not\subseteq \uparrow H'$ *iff* $\exists (U_+, U_-) \in \mathsf{con}_X$ *such that* $H' \in \boxtimes U_+$ *and* $K' \in \oplus U_-$.

Proof. For the implication from right to left, by Lemma 4.1.22, the sets $\diamondplus U_+$ and $\boxtimes U_-$ are disjoint and, because $K \in \diamondplus U_+$, it must be that $K \notin \boxtimes U_-$ and therefore $K \nsubseteq U_-$. Because $H \in \boxtimes U_-$, $H \subseteq U_-$ and also $\downarrow H \subseteq U_-$ as U_- is downwards closed. That means that $K \nsubseteq \downarrow H \subseteq U_-$.

For the other direction, let $x \in K \setminus \downarrow H$. By (3) of Lemma 4.1.29 there is some $(U_+, U_-) \in \mathsf{con}_X$ such that $x \in U_+$ and $H \subseteq U_-$. Therefore, $K \in \diamondplus U_+$ and $H \in \boxtimes U_-$, as required. (This proof also works for $H = \emptyset$ as then $U_+ = X$ and $U_- = \emptyset$.) □

A consequence of ($O_\boxtimes$), ($O_\diamondplus$) or ($O_\mathbb{V}$) is that the associated pre-order $\leq$ of $\mathcal{W}(X)$, for a $\mathcal{W} \in \{\mathbb{W}_\boxtimes, \mathbb{W}_\diamondplus, \mathbb{W}\}$, is a partial order (i.e. it satisfies antisymmetry). Furthermore, by the last lemma, we also have the separation property needed for order-separatedness. Indeed, assume $K \nsubseteq \downarrow H$, for example. Then, by Lemma 4.1.30, there exists a pair $(U_+, U_-) \in \mathsf{con}_X$ such that $K \in \diamondplus U_+$ and $H \in \boxtimes U_-$ and, moreover, $\diamondplus U_+$ and $\boxtimes U_-$ are disjoint (Lemma 4.1.22). In other words, $K \nleq H$ in $\mathcal{W}(X)$ implies that K and H can be separated by two disjoint opens. We obtain:

4.1.31 Proposition. *If X is order-separated, then also $\mathbb{W}_\boxtimes(X)$, $\mathbb{W}_\diamondplus(X)$ and $\mathbb{W}(X)$ are.*

4.1.6 Topological properties

4.1.32 Theorem.
Let $X = (X; \tau_+, \tau_-)$ be an bispace and let $\mathcal{W} \in \{\mathbb{W}_\boxtimes, \mathbb{W}_\diamondplus, \mathbb{W}\}$. Then:

1. *$\mathcal{W}X$ is T_0.*
2. *$\mathcal{W}X$ is order-separated if X is.*
3. *$\mathcal{W}X$ is d-regular if X is.*
4. *$\mathcal{W}X$ is d-zero-dimensional if X is.*
5. *$\mathcal{W}X$ is d-compact if X is.*

Proof. (1) follows from Observation 4.1.24 and Lemmas 4.1.26 and 4.1.27. We have already proved (2) in Proposition 4.1.31. Next, assume that X is d-regular. Let us focus on the case when $\mathcal{W} = \mathbb{W}$ first. We will prove that

$$\forall K \in \mathbb{W}(X),\ \forall \mathcal{U} \in \mathbb{V}\tau_+ \text{ s.t. } K \in \mathcal{U}, \quad \exists \mathcal{V} \in \mathbb{V}\tau_+ \text{ s.t. } K \in \mathcal{V} \subseteq \overline{\mathcal{V}}^{\mathbb{V}\tau_-} \subseteq \mathcal{U}. \qquad (\star)$$

First, we prove this for $\mathcal{U}$ coming from the subbasis of $\mathbb{V}\tau_+$. Let $\mathcal{U} = \boxtimes U$. Since X is d-regular and $K \subseteq U$, for every $k \in K$, there exists a $V_+^k \in \tau_+$ such that $k \in V_+^k \subseteq$

$\overline{V^k_+}^{\tau_-} \subseteq U$. Notice that the last inequality is equivalent to $(U, (V^k_+)^*) \in \mathsf{tot}_X$. Since K is τ_+-compact, there exists a finite $F \subseteq_{\text{fin}} K$ such that $K \subseteq V_+$ where $V_+ \stackrel{\text{def}}{\equiv} \bigcup_{k\in F} V^k_+$. Moreover, for $V_- \stackrel{\text{def}}{\equiv} \bigcap_{k\in F}(V^k_-)^*$, we have that $(U, V_-) \in \mathsf{tot}_X$, from (tot-$\curlyvee$), and so, by Lemma 4.1.22, also $(\boxtimes U, \oplus V_-) \in \mathsf{tot}_{\mathbb{W}X}$ and $K \in \boxtimes V_+ \subseteq \overline{\boxtimes V_+}^{\mathbb{V}\tau_-} \subseteq \boxtimes U$. Similarly, let $\mathcal{U} = \oplus U$. Then, $K \in \mathcal{U}$ implies that there exists an $x \in K \cap U$ and, from d-regularity, there is a $V_+ \in \tau_+$ such that $x \in V_+ \subseteq \overline{V_+}^{\tau_-} \subseteq U$. We have $K \in \oplus V_+ \subseteq \overline{\oplus V_+}^{\mathbb{V}\tau_-} \subseteq \oplus U$ where the last inequality holds because $(\oplus U, \boxtimes(W_+)^*) \in \mathsf{tot}_{\mathbb{W}X}$. Finally, we prove ($\star$) for $\mathcal{U} = \boxtimes U \cap \bigcap_{i=1}^n \oplus V_i$ by induction. We have the basic cases covered and now assume that $K \in \mathcal{U}_1 \cap \mathcal{U}_2$ such that there exists $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $K \in \mathcal{V}_i \subseteq \overline{\mathcal{V}_i}^{\mathbb{V}\tau_-} \subseteq \mathcal{U}_i$, for $i = 1,2$. Since $(\mathcal{U}_i, (\mathcal{V}_i)^*) \in \mathsf{tot}_{\mathbb{W}X}$, for $i = 1,2$, by (tot-$\curlywedge$), also $(\mathcal{U}_1 \cap \mathcal{U}_2, (\mathcal{V}_1)^* \cup (\mathcal{V}_2)^*) \in \mathsf{tot}_{\mathbb{W}X}$. Moreover, by (con-$\curlywedge$), $(\mathcal{V}_1 \cap \mathcal{V}_2, (\mathcal{V}_1)^* \cup (\mathcal{V}_2)^*) \in \mathsf{con}_{\mathbb{W}X}$ and so we get the last inequality in $K \in \mathcal{V}_1 \cap \mathcal{V}_2 \subseteq \overline{\mathcal{V}_1 \cap \mathcal{V}_2}^{\mathbb{V}\tau_-} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$. The cases when $\mathcal{W} = \mathbb{W}_\boxtimes$ or $\mathcal{W} = \mathbb{W}_\oplus$ are similar.

For d-zero-dimensionality, the proof is the same but V_+'s can always be chosen so that $\overline{V_+}^{\tau_-} = V_+$, or equivalently, $(V_+, (V_+)^*) \in \mathsf{tot}$, and so also $(\boxtimes V_+, \oplus(V_+)^*) \in \mathsf{tot}_{\mathbb{W}X}$ resp. $(\oplus V_+, \boxtimes(V_+)^*) \in \mathsf{tot}_{\mathbb{W}X}$.

Finally, assume that X is d-compact. Again, we only prove that $\mathbb{W}(X)$ is d-compact as the other cases are similar. By Alexander's subbasis lemma it is enough to check the case when $(\bigcup_{i\in I_+} \boxtimes U^i_+ \cup \bigcup_{j\in J_+} \oplus V^j_+, \bigcup_{i\in I_-} \boxtimes U^i_- \cup \bigcup_{j\in J_-} \oplus V^j_-) \in \mathsf{tot}_{\mathbb{W}X}$. Set

$$K \stackrel{\text{def}}{\equiv} X \setminus (\bigcup_{j\in J_+} V^j_+ \cup \bigcup_{j\in J_-} V^j_-) = (X \setminus \bigcup_{j\in J_+} V^j_+) \cap (X \setminus \bigcup_{j\in J_-} V^j_-).$$

Observe that K is d-compact because it is a complement of open sets (and X is d-compact); so $K \in \mathcal{K}_+(X) \cap \mathcal{K}_-(X)$. Set K' to be the $\leq$-equivalent element to K in $\mathsf{Lens}_d(X)$, i.e.

$$K' \stackrel{\text{def}}{\equiv} \uparrow_+ K \cap \uparrow_- K \cap \overline{K}^{\tau_+} \cap \overline{K}^{\tau_-}.$$

Clearly, K is not in $\bigcup_{j\in J_+} \oplus V^j_+ \cup \bigcup_{j\in J_-} \oplus V^j_-$ (when interpreted in $\mathbb{W}'(X)$) and so neither K' is (when interpreted in $\mathbb{W}(X)$). Hence, there exists either an $i \in I_+$ or $i \in I_-$ such that $K' \in \boxtimes U^i_+$ or $K' \in \boxtimes U^i_-$, respectively. Without loss of generality assume the former. Then, since $K \subseteq K' \subseteq U^i_+$, $(U^i_+ \cup \bigcup_{j\in J_+} V^j_+, \bigcup_{j\in J_-} V^j_-) \in \mathsf{tot}_X$ and because X is d-compact, there must exist a finite sets $F_\pm \subseteq_{\text{fin}} J_\pm$ such that $(U^i_+ \cup \bigcup_{j\in F_+} V^j_+, \bigcup_{j\in F_-} V^j_-) \in \mathsf{tot}_X$ and so $(\boxtimes(U^i_+ \cup \bigcup_{j\in F_+} V^j_+), \oplus(\bigcup_{j\in F_-} V^j_-)) \in \mathsf{tot}_{\mathbb{W}X}$ (Lemma 4.1.22). However, the last pair of opens is componentwise smaller than $(\boxtimes U^i_+ \cup \bigcup_{j\in F_+} \oplus V^j_+, \bigcup_{j\in F_-} \oplus V^j_-)$ (Lemma 4.1.21) by which

$$(\boxtimes U^i_+ \cup \bigcup_{j\in F_+} \oplus V^j_+, \bigcup_{j\in F_-} \oplus V^j_-) \in \mathsf{tot}_{\mathbb{W}X}. \qquad \square$$

Observe that also the reverse of (3) and (4) in Theorem 4.1.32 hold. Indeed, a bispace X is d-regular (resp. d-zero-dimensional) iff its T_0 reflection X' is and $\mathcal{W}(X') \cong \mathcal{W}(X)$. Moreover, if $\mathcal{W}(X)$ is d-regular (resp. d-zero-dimensional) then also X' is because X' is embedded into $\mathcal{W}(X')$ by the unit map $\eta_{X'} \colon X' \to \mathcal{W}(X')$.

In the following we observe that order-separatedness together with d-compactness allows for an even simpler description of the points of $\mathbb{W}(X)$'s than the one we had in the previous subsection:

4.1.33 Proposition. *Let X be a d-compact d-regular (T_0) bispace. Then,*

1. $\mathbb{W}_{\boxtimes}(X) \cong (\mathsf{Clos}_-(X), \mathbb{V}_{\boxtimes}\tau_+, \mathbb{V}_{\oplus}\tau_-)$,
2. $\mathbb{W}_{\oplus}(X) \cong (\mathsf{Clos}_+(X), \mathbb{V}_{\oplus}\tau_+, \mathbb{V}_{\boxtimes}\tau_-)$ *and*
3. $\mathbb{W}(X) \cong (\mathcal{K}_c(X), \mathbb{V}\tau_+, \mathbb{V}\tau_-)$

where $\mathsf{Clos}_\pm(X)$ *is the set of* $\tau_\pm$*-closed sets and* $\mathcal{K}_c(X)$ *is the set of all* d-compact convex *subsets of* X.

Proof. By Proposition 2.1.13 we know that X is order-separated and that $\mathsf{Lens}_\pm(X)$ and $\mathsf{Lens}_d(X)$ are the sets of $\tau_\pm$-compact upwards/downwards closed sets and the sets of τ_+ and τ_--compact convex subsets of X, respectively.

(1) Because X is d-compact, every τ_--closed is d-compact and upwards closed. On the other hand, by Lemma 4.1.29, each τ_+-compact upset is τ_--closed. (2) is dual. (3) Clearly, if K is d-compact convex, then it is also $\tau_\pm$-compact. For the other direction, if K is τ_+ and τ_--compact and convex, then, by Lemma 4.1.29, ${\downarrow}K$ is τ_+-closed and ${\uparrow}K$ is τ_--closed. Therefore, since X is d-compact, K must be d-compact because it is the complement of two open set $X \setminus {\uparrow}K$ and $X \setminus {\downarrow}K$. □

4.1.34 Example. d-Compactness is necessary for Proposition 4.1.33 to work. Let X be the bispace $[0,1] \setminus \{\frac{1}{2}\}$ with the upper and lower topologies. Then, for $K = \{\frac{1}{4}, \frac{3}{4}\}$, the $\le$-equivalent element in $\mathbb{W}(X)$ is $K' = [\frac{1}{4}, \frac{3}{4}] \setminus \{\frac{1}{2}\}$ which is τ_+ and τ_--compact and convex but not d-compact.

4.2 d-Frame Vietoris constructions

Recall Section 3.5.5 where we defined the upper, lower and (full) Vietoris d-frames for a d-frame $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$. We assumed (without a proof) that their presentations satisfy the axioms ($\mu_\pm$-con-tot) and (Indep$_\pm$) from Theorem 3.4.20, and on this basis showed that the resulting d-frames can be written as follows:

$$\mathbb{W}_\square(\mathcal{L}) = (\mathbb{V}_\square L_+,\ \mathbb{V}_\diamond L_-,\ \mathfrak{D}({\downarrow}\mathsf{con}^\square_{\mathbb{A},\mathbb{V}}),\ {\uparrow}\mathsf{tot}^\square_{\mathbb{A},\mathbb{V}})$$

$$\mathbb{W}_\diamond(\mathcal{L}) = (\mathbb{V}_\diamond L_+,\ \mathbb{V}_\square L_-,\ \mathfrak{D}({\downarrow}\mathsf{con}^\diamond_{\mathbb{A},\mathbb{V}}),\ {\uparrow}\mathsf{tot}^\diamond_{\mathbb{A},\mathbb{V}})$$

$$\mathbb{W}_d(\mathcal{L}) = (\mathbb{V}_{\mathrm{Fr}} L_+,\ \mathbb{V}_{\mathrm{Fr}} L_-,\ \mathfrak{D}({\downarrow}\mathsf{con}_{\mathbb{A},\mathbb{V}}),\ {\uparrow}\mathsf{tot}_{\mathbb{A},\mathbb{V}})$$

where

$$\mathsf{con}_1^{\Box} = \{(\Box\alpha_+, \Diamond\alpha_+) \mid \alpha \in \mathsf{con}\}, \quad \mathsf{tot}_1^{\Box} = \{(\Box\alpha_+, \Diamond\alpha_+) \mid \alpha \in \mathsf{tot}\},$$

$$\mathsf{con}_1^{\Diamond} = \{(\Diamond\alpha_+, \Box\alpha_+) \mid \alpha \in \mathsf{con}\}, \quad \mathsf{tot}_1^{\Diamond} = \{(\Diamond\alpha_+, \Box\alpha_+) \mid \alpha \in \mathsf{tot}\},$$

and

$$\mathsf{con}_1 = \{(\Box\alpha_+, \Diamond\alpha_+), (\Diamond\alpha_+, \Box\alpha_+) \mid \alpha \in \mathsf{con}\},$$
$$\mathsf{tot}_1 = \{(\Box\alpha_+, \Diamond\alpha_+), (\Diamond\alpha_+, \Box\alpha_+) \mid \alpha \in \mathsf{tot}\}.$$

In the following section we explore the structure of $\mathbb{V}_{\Box}(L_\pm)$, $\mathbb{V}_{\Diamond}(L_\pm)$ and $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$ to the level which is sufficient for proving the axioms ($\mu_\pm$-con-tot) and (Indep$_\pm$) for the presentations of $\mathbb{W}_{\Box}(\mathcal{L})$, $\mathbb{W}_{\Diamond}(\mathcal{L})$ and $\mathbb{W}_d(\mathcal{L})$, thus filling the gap left in Section 3.5.5.

Convention. In the rest of this chapter we often prove properties that all three power-constructions share by proving them only for $\mathbb{W}_d(\mathcal{L})$. This is justified since the corresponding arguments for the other two are actually simpler.

4.2.1 The structure of $\mathbb{V}_{\mathrm{Fr}}(L)$

Let us fix a frame L. Just by examining the equations of $\mathbb{V}_{\mathrm{Fr}}(L)$ we observe that the set

$$B \overset{\mathrm{def}}{\equiv} \{\Box a \wedge \bigwedge_{i=1}^{n} \Diamond b_i \mid a, b_1, \dots, b_n \in L\}$$

is closed under finite meets in $\mathbb{V}_{\mathrm{Fr}}(L)$. Also, we see that every element of $\mathbb{V}_{\mathrm{Fr}}(L)$ is of the form

$$\bigvee_k (\Box a_k \wedge \bigwedge_{i=1}^{n_k} \Diamond b_{k,i})$$

for some $a_k, b_{k,i} \in\subseteq L$. Said in other words, B generates $\mathbb{V}_{\mathrm{Fr}}(L)$.

However, not much more can be said at the moment; equational reasoning alone does not get us very far. We need a more concrete representation for $\mathbb{V}_{\mathrm{Fr}}(L)$. To begin with, let us state a general lemma about quotients of frames.

4.2.1 Lemma. *Let L be a frame and let $R_1, R_2 \subseteq L \times L$. Then,*

$$L/(R_1 \cup R_2) \;\cong\; (L/R_1)/(\mu\times\mu)[R_2]$$

where μ is the quotient map $L \to L/R_1$.

Proof. Observe that the composition of the quotient maps

$$L \longrightarrow L/R_1 \longrightarrow (L/R_1)/(\mu\times\mu)[R_2]$$

respects R_1 and R_1. Also, whenever there is a frame homomorphism $h\colon L \to M$ respecting R_1 and R_2, it uniquely lifts to $\overline{h}\colon L/R_1 \to M$ and, because $h = \overline{h} \cdot \mu$, $\overline{h}$ respects $(\mu{\times}\mu)[R_2]$. Hence $\overline{h}$ uniquely lifts to $\overline{\overline{h}}\colon (L/R_1)/(\mu{\times}\mu)[R_2] \to M$. □

Next, recall a useful fact about frames with finitary presentations:

4.2.2 Fact (Theorem 9.2.2 in [Vic89]). *Let $\langle G \mid E\rangle$ be a frame presentation with no infinite joins in its equations. Then,*

$$\boldsymbol{Fr}\langle G \mid E\rangle \;\cong\; \mathsf{Idl}(\boldsymbol{DL}\langle G \mid E\rangle)$$

where $\boldsymbol{DL}\langle G \mid E\rangle$ is the freely generated (bounded) distributive lattice from the presentation $\langle G \mid E\rangle$ and $\mathsf{Idl}(D)$ is the frame of ideals on D.

Lemma 4.2.1 together with Fact 4.2.2 suggest that we can obtain $\mathbb{V}_{\mathrm{Fr}}(L)$ in two stages. In the first stage we define a construction on distributive lattices which uses only the finitary equations from the presentation of $\mathbb{V}_{\mathrm{Fr}}(L)$; that is, for a lattice D, set

$$\begin{aligned}\mathbb{M}(D) \;\stackrel{\mathrm{def}}{\equiv}\; \mathbf{DL}\Big\langle \tilde{\Box}a, \tilde{\Diamond}a : a \in D \;\Big|\; & \tilde{\Box}(a \wedge b) = \tilde{\Box}a \wedge \tilde{\Box}b, \quad \tilde{\Box}1 = 1,\\ & \tilde{\Diamond}(a \vee b) = \tilde{\Diamond}a \vee \tilde{\Diamond}b, \quad \tilde{\Diamond}0 = 0,\\ & \tilde{\Box}a \wedge \tilde{\Diamond}b \le \tilde{\Diamond}(a \wedge b), \quad \tilde{\Box}(a \vee b) \le \tilde{\Box}a \vee \tilde{\Diamond}b \Big\rangle.\end{aligned}$$

Then, in the second stage, we factor $\mathsf{Idl}(\mathbb{M}(L))$ by the two missing infinitary equations of $\mathbb{V}_{\mathrm{Fr}}(L)$. Namely, we consider a relation $R_{\bigvee} \subseteq \mathsf{Idl}(\mathbb{M}(L)){\times}\mathsf{Idl}(\mathbb{M}(L))$ which represents the equations "$\Box(\bigvee^{\uparrow}_i a_i) \le \bigvee_i \Box a_i$" and "$\Diamond(\bigvee^{\uparrow}_i a_i) \le \bigvee_i \Diamond a_i$" but mapped into $\mathsf{Idl}(\mathbb{M}(L))$:

$$R_{\bigvee} \;\stackrel{\mathrm{def}}{\equiv}\; \{(\downarrow\tilde{\Box}(\bigvee^{\uparrow}_i a_i), \bigvee_i \downarrow\tilde{\Box}a_i),\; (\downarrow\tilde{\Diamond}(\bigvee^{\uparrow}_i a_i), \bigvee_i \downarrow\tilde{\Diamond}a_i) \mid \{a_i\}_i \subseteq^{\uparrow} L\}.$$

Then, Lemma 4.2.1 implies that

$$\mathbb{V}_{\mathrm{Fr}}(L) \;\cong\; \mathsf{Idl}(\mathbb{M}(L))/R_{\bigvee}. \tag{4.2.1}$$

Let us examine how this isomorphism works. By

$$S\colon \mathsf{Idl}(\mathbb{M}(L)) \to \mathsf{Idl}(\mathbb{M}(L))/R_{\bigvee}$$

denote the quotient map, also called *$R_{\bigvee}$-saturation*. $S(I)$, for an ideal $I \in \mathsf{Idl}(\mathbb{M}(L))$, is computed as the smallest $R_{\bigvee}$-saturated ideal containing I (Section 3.1.2). The isomorphism in (4.2.1) then interprets every $\Box a$ and $\Diamond b$ from $\mathbb{V}_{\mathrm{Fr}}(L)$ as $S(\downarrow\tilde{\Box}a)$ and $S(\downarrow\tilde{\Diamond}b)$ in $\mathsf{Idl}(\mathbb{M}L)$, respectively. Next, a general element of $\mathbb{V}_{\mathrm{Fr}}(L)$, i.e. an element of the form $\bigvee_{j\in J}(\Box a_k \wedge \bigwedge_{i=1}^n \Diamond b_{k,i})$, is interpreted as the $R_{\bigvee}$-saturation of the ideal

$$\bigvee_{j\in J}(\downarrow\tilde{\Box}a_k \wedge \bigwedge\nolimits_{i=1}^n \downarrow\tilde{\Diamond}b_{k,i}) \tag{4.2.2}$$

This is because the join of ideals $\{I_i\}_i$ in $\mathsf{Idl}(\mathbb{M}(L))/R_\bigvee$ is computed as the $R_\bigvee$-saturation of the join $\bigvee_i I_i$ in $\mathsf{Idl}(\mathbb{M}L)$, that is $S(\bigvee_i I_i)$.

In fact, certain principal ideals such as $\downarrow\tilde{\Box}a$ and $\downarrow\tilde{\Diamond}b$ are already $R_\bigvee$-saturated and so $S(\downarrow\tilde{\Box}a) = \downarrow\tilde{\Box}a$ and $S(\downarrow\tilde{\Diamond}b) = \downarrow\tilde{\Diamond}b$. We prove this in Lemma 4.2.5. Before we see why this is the case, let us take a look at a fact by Jan Cederquist and Thierry Coquand [CC00, Theorem 10] about the order in $\mathbb{M}(D)$:

4.2.3 Fact. *Let D be a distributive lattice and let $a = \bigwedge_{i=1}^{n} a_i$ and $c = \bigvee_{j=1}^{n'} c_j$. Then, in $\mathbb{M}(D)$:*

$$\bigwedge_{i=1}^{n} \tilde{\Box}a_i \wedge \bigwedge_{j=1}^{m} \tilde{\Diamond}b_j \leq \bigvee_{j=1}^{n'} \tilde{\Diamond}c_j \vee \bigvee_{k=1}^{m'} \tilde{\Box}d_k \quad \text{iff} \quad \exists i.\ a \wedge b_i \leq c \ \text{ or } \ \exists j.\ a \leq c \vee d_j$$

As we will see later a variant of this fact is also true for $\mathbb{V}_{\mathrm{Fr}}(L)$ (Proposition 4.2.6) which is an essential tool for proving that the presentation of $\mathbb{W}_d(\mathcal{L})$ satisfies the axioms ($\mu_\pm$-con-tot) and (Indep$_\pm$). Before we show that, let us first show that checking $R_\bigvee$-saturatedness reduces to checking it only for the elements of B:

4.2.4 Lemma. *Let $I \in \mathsf{Idl}(\mathbb{M}(L))$. Then, I is $R_\bigvee$-saturated (i.e. $I \in \mathsf{Idl}(\mathbb{M}(L))/R_\bigvee$) iff*

$$(R\Box) \qquad \{\tilde{\Box}a_i \wedge \textstyle\bigwedge_{j=1}^{n} \tilde{\Diamond}b_j\}_i \subseteq I \implies \tilde{\Box}(\bigvee^{\uparrow}_i a_i) \wedge \bigwedge_{j=1}^{n} \tilde{\Diamond}b_j \in I.$$

$$(R\Diamond) \qquad \{\tilde{\Diamond}a_i \wedge \tilde{\Box}b_0 \wedge \textstyle\bigwedge_{j=1}^{n} \tilde{\Diamond}b_j\}_i \subseteq I \implies \tilde{\Diamond}(\bigvee^{\uparrow}_i a_i) \wedge \tilde{\Box}b_0 \wedge \bigwedge_{j=1}^{n} \tilde{\Diamond}b_j \in I.$$

for all $\{a_i\}_{i\in I} \subseteq^{\uparrow} L$ and $b_0, b_1, \ldots, b_n \in L$.

Proof. Recall that I is $R_\bigvee$-saturated if, for all $(J_1, J_2) \in R$ and $K \in \mathsf{Idl}(\mathbb{M}(L))$, $J_2 \wedge K \subseteq I$ implies $J_1 \wedge K \subseteq I$. For the left-to-right implication it is enough to instantiate J_2 and K by the ideals $\bigvee_i \downarrow\tilde{\Box}a_i$ and $\downarrow(\bigwedge_{j=1}^{n} \tilde{\Diamond}b_j)$ (for $(R\Box)$) or $\bigvee_i \downarrow\tilde{\Diamond}a_i$ and $\downarrow(\tilde{\Box}b_0 \wedge \bigwedge_{j=1}^{n} \tilde{\Diamond}b_j)$ (for $(R\Diamond)$). In this case, $J_2 \wedge K \subseteq I$ iff the premise of $(R\Box)$ or $(R\Diamond)$ is satisfied.

For the reverse, let J_2 stays the same and let K be any ideal. Then, $J_2 \wedge K \subseteq I$ is equivalent to

$$\forall i \in I\, \forall \kappa \in K. \quad \tilde{\Box}a_i \wedge \kappa \in I$$

Let $\bigvee_{l=1}^{n(\kappa)} \kappa_l$ be the disjunctive normal form of κ. Then, by distributivity of $\mathbb{M}(L)$, the above line can be rewritten as:

$$\forall i \in I\, \forall \kappa \in K\, \forall l = 1, \ldots, n(\kappa). \quad \tilde{\Box}a_i \wedge \kappa_l \in I$$

Therefore, for a fixed $\kappa \in K$, since $\{\tilde{\Box}a_i \wedge \kappa_l\}_i \subseteq I$, $\tilde{\Box}(\bigvee^{\uparrow}_i a_i) \wedge \kappa_l \in I$ by $(R\tilde{\Box})$. By distributivity, also $\tilde{\Box}(\bigvee^{\uparrow}_i a_i) \wedge \kappa \in I$ and, because $\kappa \in K$ was chosen arbitrarily, $\downarrow\tilde{\Box}(\bigvee^{\uparrow}_i a_i) \wedge K = \{\tilde{\Box}(\bigvee^{\uparrow}_i a_i) \wedge \kappa : \kappa \in K\} \subseteq I$. The case for $(R\Diamond)$ is similar. □

4.2.5 Lemma. *Let $c_0, c_1, \dots, c_m \in L$. Then, the ideals $\downarrow(\tilde{\Diamond} c_0 \vee \bigvee_{l=1}^m \tilde{\Box} c_j)$ and $\downarrow(\tilde{\Box} c_0 \wedge \bigwedge_{j=1}^n \tilde{\Diamond} c_j)$ are $R_\bigvee$-saturated.*

Proof. We use Lemma 4.2.4 to show that $\downarrow(\tilde{\Diamond} c_0 \vee \bigvee_{l=1}^m \tilde{\Box} c_j)$ is $R_\bigvee$-saturated. Let $b_0, b_1, \dots, b_n \in \mathbb{M}(L)$ and let $\{a_i\}_{i\in I} \subseteq^\uparrow L$. To check $(R\Diamond)$, assume

$$\{\tilde{\Diamond} a_i \wedge \tilde{\Box} b_0 \wedge \bigwedge_{j=1}^n \tilde{\Diamond} b_j\}_{i\in I} \;\subseteq\; \downarrow\gamma \qquad \text{where} \quad \gamma \overset{\text{def}}{\equiv} \tilde{\Diamond} c_0 \vee \bigvee_{l=1}^m \tilde{\Box} c_l.$$

By Fact 4.2.3, there are three options: 1. there is an l such that $b_0 \le c_0 \vee c_l$, 2. there is a j such that $b_0 \wedge b_j \le c_0$, or 3. $b_0 \wedge a_i \le c_0$ for all $i \in I$. In the first two cases $\tilde{\Box} b_0 \wedge \bigwedge_{j=1}^n \tilde{\Diamond} b_j \le \gamma$ and in the third case $b_0 \wedge \bigvee^\uparrow{}_i a_i \le c_0$ and so $\tilde{\Diamond}(\bigvee^\uparrow{}_i a_i) \wedge \tilde{\Box} b_0 \wedge \bigwedge_{j=1}^n \tilde{\Diamond} b_j \le \tilde{\Diamond}(\bigvee^\uparrow{}_i a_i) \wedge \tilde{\Box} b_0 \le \tilde{\Diamond} c_0 \le \gamma$.

Next, we aim to show $(R\Box)$. Assume that

$$\{\tilde{\Box} a_i \wedge \bigwedge_{j=1}^n \tilde{\Diamond} b_j\}_{i\in I} \;\subseteq\; \downarrow\gamma.$$

By Fact 4.2.3, for all $i \in I$, either $a_i \wedge b_j \le c_0$ for some j or $a_i \le c_0 \vee c_l$ for some l. For every $i \in I$ set $M(i)$ to be the minimal subset of $M \overset{\text{def}}{\equiv} \{0, 1, \dots, m\}$ such that

1. $0 \in M(i)$ if $a_i \wedge b_j \le c_0$ for some j, and
2. $l \in M(i)$ if $a_i \le c_0 \vee c_l$.

Observe that $\bigcap_i M(i) \neq \emptyset$. Indeed, if $\bigcap_i M(i) = \emptyset$ then, for every $m \in M$, there exists a $i(m)$ such that $m \notin M(i(m))$. Because M is finite and $\{a_i\}_i$ directed there exists a k such that $a_{i(m)} \le a_k$, for all $m \in M$. However, $a_{i(m)} \le a_k$ implies $M(i(m)) \supseteq M(k)$ and $M(k) \neq \emptyset$, a contradiction.

Since $\bigcap_i M(i) \neq \emptyset$, let $l \in \bigcap_i M(i)$. If $l \neq 0$, that means that, for all $i \in I$, $a_i \le c_0 \vee c_l$. Then, also $\bigvee^\uparrow{}_i a_i \le c_0 \vee c_l$ and so $\tilde{\Box}(\bigvee^\uparrow{}_i a_i) \wedge \bigwedge_{j=1}^n \tilde{\Diamond} b_j \le \tilde{\Box}(\bigvee^\uparrow{}_i a_i) \le \tilde{\Diamond} c_0 \vee \tilde{\Box} c_l \le \gamma$. If, on the other hand, $l = 0$, then for every $i \in I$ there is a j such that $a_i \wedge b_j \le c_0$. Set $J(i) = \{j \mid a_i \wedge b_j \le c_0\}$. For the same reason as before, $\bigcap_i J(i)$ is not empty. Let $j \in \bigcap_i J(i)$. Then, $\bigvee^\uparrow{}_i a_i \wedge b_j \le c_0$ and so $\tilde{\Box}(\bigvee^\uparrow{}_i a_i) \wedge \bigwedge_{j=1}^n \tilde{\Diamond} b_j \le \tilde{\Box}(\bigvee^\uparrow{}_i a_i) \wedge \tilde{\Diamond} b_j \le \tilde{\Diamond} c_0 \le \gamma$.

We have proved that $\downarrow\gamma$, for every $\gamma \in \mathbb{M}(L)$ of the shape $\tilde{\Diamond} c_0 \vee \bigvee_{l=1}^m \tilde{\Box} c_j$, is $R_\bigvee$-saturated. A special case of this is when $\gamma = \tilde{\Box} e$ or $\gamma = \tilde{\Diamond} e$, for some $e \in L$. Consequently, because meets in $\mathsf{Idl}(\mathbb{M}(L))/R_\bigvee$ are computed as in $\mathsf{Idl}(\mathbb{M}(L))$, i.e. as intersections, $\downarrow(\tilde{\Box} c_0 \wedge \bigwedge_{j=1}^n \tilde{\Diamond} c_j) = \downarrow\tilde{\Box} c_0 \cap \bigcap_{j=1}^n \downarrow\tilde{\Diamond} c_j$ is $R_\bigvee$-saturated. □

An immediate consequence of Lemma 4.2.5 is that the quotient map$/R_\bigvee$-saturation

$$\mathsf{Idl}(\mathbb{M}(L)) \xrightarrow{\;S\;} \mathsf{Idl}(\mathbb{M}(L))/R_\bigvee \cong \mathbb{V}(L)$$

when restricted to the elements of the shape $\downarrow(\widetilde{\Box}a \wedge \bigwedge_{i=1}^{n} \widetilde{\Diamond}b_i)$ and $\downarrow(\widetilde{\Diamond}c \vee \bigvee_{j=1}^{m} \widetilde{\Box}d_j)$ is injective and also that the isomorphism in (4.2.1) is indeed computed as $\Box a, \Diamond b \mapsto \downarrow\widetilde{\Box}a, \downarrow\widetilde{\Diamond}b$ on the generators. A consequence of this is a $\mathbb{V}_{\mathrm{Fr}}(L)$ variant of Cederquist and Coquand's result:

4.2.6 Proposition. *Let L be a frame. Then, in $\mathbb{V}_{\mathrm{Fr}}(L)$:*

$$\Box a \wedge \bigwedge_{i=1}^{n} \Diamond b_i \leq \Diamond c \vee \bigvee_{j=1}^{m} \Box d_j \quad \text{iff} \quad a \wedge b_i \leq c \text{ for some } i \quad \text{or} \quad a \leq c \vee d_j \text{ for some } j.$$

Proof. Since $\Box a \wedge \bigwedge_{i=1}^{n} \Diamond b_i \leq \Diamond c \vee \bigvee_{j=1}^{m} \Box d_j$ in $\mathbb{V}_{\mathrm{Fr}}(L)$ iff $\downarrow(\widetilde{\Box}a \wedge \bigwedge_{i=1}^{n} \widetilde{\Diamond}b_i) \subseteq \downarrow(\widetilde{\Diamond}c \vee \bigvee_{j=1}^{m} \widetilde{\Box}d_j)$ in $\mathsf{Idl}(\mathbb{M}(L))/R_{\bigvee}$ which is equivalent to $\widetilde{\Box}a \wedge \bigwedge_{i=1}^{n} \widetilde{\Diamond}b_i \leq \widetilde{\Diamond}c \vee \bigvee_{j=1}^{m} \widetilde{\Box}d_j$ in $\mathbb{M}(L)$ and this, again, is equivalent, by Fact 4.2.3, to the right-hand-side. □

Equipped with Proposition 4.2.6 we can finally prove the axioms ($\mu_{\pm}$-con-tot) and (Indep$_{\pm}$) from Theorem 3.4.20 for $\mathbb{W}_d(\mathcal{L})$:

4.2.7 Lemma. *The presentation of $\mathbb{W}_d\mathcal{L}$ satisfies the axiom ($\mu_{\pm}$-con-tot).*

Proof. Let, $\alpha = \mathbb{V}_{j=0}^{m} \alpha^j$ where $\alpha^j \in \mathsf{con}_1$, for all j, and $\beta = \mathbb{A}_{i=0}^{n} \beta^i$ where $\beta^i \in \mathsf{tot}_1$, for all i, be such that $\beta_+ \leq \alpha_+$. Because $\Diamond x \vee \Box y = \Diamond x \vee \Box(x \vee y)$ and $\Box x \wedge \Diamond y = \Box x \wedge \Diamond(x \wedge y)$, we can assume that $\alpha^0 = (\Diamond c_+, \Box c_-)$ and $\alpha^j = (\Box d_+^j, \Diamond d_-^j)$ are such that $c_+ \leq d_+^j$ and $d_-^j \leq c_-$, for all $j = 1, \ldots, m$. Similarly, assume that $\beta^0 = (\Box a_+, \Diamond a_-)$ and $\beta^i = (\Diamond b_+^i, \Box b_-^i)$ are such that $b_+^i \leq a_+$ and $a_- \leq b_-^i$, for all $i = 1, \ldots, n$.

Then, $\beta_+ \leq \alpha_+$, by Proposition 4.2.6, implies that either $b_+^i \leq c_+$, for some i, or $a_+ \leq d_+^j$, for some j. Assume the first is the case. Since $(b_+^i, b_-^i) \in \mathsf{tot}_{\mathcal{L}}$ and $(c_+, c_-) \in \mathsf{con}_{\mathcal{L}}$, from (con-tot) for $\mathcal{L}$, $c_- \leq b_-^i$ and so $\alpha_-^0 = \Box c_- \leq \Box b_-^i = \beta_-^i$. Therefore, $\alpha_- = \bigvee_{j=0}^{m} \alpha_-^j \leq \alpha_-^0 \leq \beta_-^i \leq \bigwedge_{i=0}^{n} \beta_-^i = \beta_-$. The case when $a_+ \leq d_+^j$ is similar. □

4.2.8 Lemma. *The presentation of $\mathbb{W}_d\mathcal{L}$ satisfies the axiom (Indep$_{\pm}$).*

Proof. Recall that the elements of B_-, which is the $\wedge$-closure of the set $\{\Box a, \Diamond a : a \in L_-\}$ in $\mathbb{V}_{\mathrm{Fr}}(L_-)$, are of the form $\Box a \wedge \bigwedge_{i=1}^{n} \Diamond b_i$. Let $(\gamma_+, \beta_-) \in (\mathbb{V}_{\mathrm{Fr}} L_+ \times B_-) \cap \downarrow\mathsf{con}_{\mathbb{A},\mathbb{V}}$. This means that there is some $\{\alpha^k\}_{k \in K} \subseteq \mathsf{con}_{\mathbb{A}}$ such that $(\gamma_+, \beta_-) \sqsubseteq (\bigvee_{k \in K} \alpha_+^k, \bigwedge_{k \in K} \alpha_-^k)$. We want to show that $(\gamma_+, \beta_-) \in \downarrow\mathsf{con}_{\mathbb{V}}$.

From the definition we have that, for every $k \in K$,

$$\alpha^k = (\Box a_+^{k,0} \wedge \bigwedge_{j=1}^{m_k} \Diamond a_+^{k,j},\ \Diamond a_-^{k,0} \vee \bigvee_{j=1}^{m_k} \Box a_-^{k,j})$$

where $(a_+^{k,j}, a_-^{k,j}) \in \mathsf{con}_{\mathcal{L}}$ for all $j \in \{0, 1, \ldots, m_k\}$. Also, $\beta_- \in B_-$ means that $\beta_- = \Box b^0 \wedge \bigwedge_{i=1}^{n} \Diamond b^i$ for some $b^0, b^1, \ldots, b^n \in L_-$. As in the proof of Lemma 4.2.7, assume that $a_-^{k,0} \leq a_-^{k,j}$, for all $j = 1, \ldots, m_k$, and that $b^j \leq b^0$, for all $i = 1, \ldots, n$.

For every $k \in K$, $\beta_- \leq \bigwedge_k \alpha_-^k \leq \alpha_-^k$, by Proposition 4.2.6, there exist an $i(k) \in \{1, \ldots, n\}$ such that $b^{i(k)} \leq a_-^{k,0}$ or there exists a $j(k) \in \{1, \ldots, m_k\}$ such that $b^0 \leq a_-^{k,j(k)}$. Set $j(k) = 0$ in the first case and $i(k) = 0$ in the second case. We obtain that

$$\forall k \in K. \quad b^{i(k)} \leq a_-^{k,j(k)}$$

Combining this with $(a_+^{k,j(k)}, a_-^{k,j(k)}) \in \mathsf{con}_{\mathcal{L}}$ we obtain, by (con-$\downarrow$), that $(a_+^{k,j(k)}, b^{i(k)}) \in \mathsf{con}_{\mathcal{L}}$ for all $k \in K$. Fix an $i \in \{0, \ldots, n\}$ and set $K(i) = \{k \mid i(k) = i\}$. Then, by (con-$\curlyvee$) and (con-$\bigsqcup^{\uparrow}$), $(\bigvee_{k \in K(i)} a_+^{k,j(k)}, b^i) \in \mathsf{con}_{\mathcal{L}}$. Next, set

$$\delta^i \stackrel{\text{def}}{\equiv} \begin{cases} (\Diamond(\bigvee_{k \in K(i)} a_+^{k,j(k)}),\ \Box b^i) & \text{if } i = 0 \\[2ex] (\Box(\bigvee_{k \in K(i)} a_+^{k,j(k)}),\ \Diamond b^i) & \text{if } i \neq 0 \end{cases}$$

By the definition, we see that $\delta^i \in \mathsf{con}_1$, for all $i \in \{0, 1, \ldots, n\}$. Therefore,

$$\mathop{\curlyvee}_{i=0}^{n} \delta^i = (\Box(\bigvee_{k \in K(i)} a_+^{k,j(k)}) \vee \bigvee_{i=1}^{n} \Diamond(\bigvee_{k \in K(i)} a_+^{k,j(k)}),\ \beta_-) \in \mathsf{con}_{\curlyvee}.$$

Lastly, notice that, if $j(k) = 0$, $\alpha_+^k \leq \Box a_+^{k,j(k)}$ and $\alpha_+^k \leq \Diamond a_+^{k,j(k)}$ otherwise. Also, because for every $k \in K$: $k \in K(i(k))$, $\Diamond a_+^{k,j(k)}$ resp. $\Box a_+^{k,j(k)}$ is smaller than $\delta_+^{i(k)}$ and so $\gamma_+ \leq \alpha_+^k \leq \bigvee_{i=0}^n \delta_+^i$. Therefore, $(\gamma_+, \beta_-) \in {\downarrow}\mathsf{con}_{\curlyvee}$. □

4.2.9 Proposition. *The d-frames $\mathbb{W}_\Box(\mathcal{L})$, $\mathbb{W}_\Diamond(\mathcal{L})$ and $\mathbb{W}_d(\mathcal{L})$ are of the form described at the beginning of Section 4.2.*

4.2.2 Topological properties

Next, we prove a d-frame variant of the Theorem 4.1.32. Before we do so we take a look at two purely frame-theoretic results:

4.2.10 Lemma. *Let $h\colon \mathcal{L} \twoheadrightarrow \mathcal{M}$ be an onto d-frame homomorphism. If $\mathcal{L}$ is d-regular or d-zero-dimensional, then $\mathcal{M}$ is as well.*

In particular, if $\mathcal{L}$ is d-regular or d-zero-dimensional, then any of its quotients also is.

Proof. Let $y \in M_\pm$. Since, h is onto, there is an $x \in L_\pm$ such that $h_\pm(x) = y$. Then, if $\mathcal{L}$ is d-regular, since $z \lhd x$ implies $h_\pm(z) \lhd y$, $y = \bigvee\{h_\pm(z) \mid z \lhd x\} \leq \bigvee\{v \mid v \lhd y\} \leq y$. Proving d-zero-dimensionality is similar.

In Section 3.3.1 we have established that the quotient map $\mu_{\mathfrak{r}}\colon \mathcal{L} \to \mathfrak{r}(\mathcal{L}/R)$ is a composition of two onto maps, which proves the second part. □

In the following technical lemma we show that the saturation process, which we use to compute joins in $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$, has an explicit (non-iterative) description in the case when L_+ and L_- come from a d-compact d-regular d-frame. The statement is inspired by Johnstone's Theorem 4.4 in [Joh82].

4.2.11 Lemma. *Let $\mathcal{L}$ be a d-compact d-regular d-frame. For any $I \in \mathsf{Idl}(\mathbb{M}(L_\pm))$, the $R_\bigvee$-saturation $S_\pm(I)$ of I is the ideal*

$$\{\alpha \in \mathbb{M}(L_\pm) \mid \forall \beta \otimes \alpha.\ \beta \in I\}$$

where $\beta \otimes \alpha$, for α written in a disjunctive normal form as $\bigvee_{i=1}^{n}(\widetilde{\Box} a_{i,0} \wedge \bigwedge_{j=1}^{n_i} \widetilde{\Diamond} a_{i,j})$, if there exist $b_{i,j} \lhd a_{i,j}$, for all i, j, such that $\beta = \bigvee_{i=1}^{n}(\widetilde{\Box} b_{i,0} \wedge \bigwedge_{j=1}^{n_i} \widetilde{\Diamond} b_{i,j})$.

Proof. It is immediate that $K \stackrel{\text{def}}{\equiv} \{\alpha \in \mathbb{M}(L) \mid \forall \beta \otimes \alpha.\ \beta \in I\}$ is closed under finite joins. To show that it is also downwards closed take a $\gamma \leq \alpha$ for some $\alpha \in K$. Let $\delta \otimes \gamma$ and set $\beta = \bigvee_{i=1}^{n}(\widetilde{\Box} 0 \wedge \bigwedge_{j=1}^{n_i} \widetilde{\Diamond} 0)$, i.e. set all $b_{i,j}$'s to 0. Since $\beta \otimes \alpha$, $\delta \vee \beta \otimes \gamma \vee \alpha = \alpha$. Hence, $\delta \vee \beta \in I$ and because I is downwards closed also $\delta \in I$.

Next, we show that K is $R_\bigvee$-saturated and we do that by checking the conditions of Lemma 4.2.4. To show $(R\Box)$ let $\{\widetilde{\Box} a_i \wedge \bigwedge_{j=1}^{n} \widetilde{\Diamond} b_j\}_i \subseteq K$. This is equivalent to saying that

$$\forall i \in I\ \forall a_i' \lhd a_i,\ b_j' \lhd b_j.\quad \widetilde{\Box} a_i' \wedge \bigwedge_{j=1}^{n} \widetilde{\Diamond} b_j' \in I. \tag{$\star$}$$

Now, let $a' \lhd \bigvee^{\uparrow}_i a_i$. By d-compactness, there is some a_i such that $a' \lhd a_i$. Therefore, by $(\star)$, $\widetilde{\Box} a' \wedge \bigwedge_{j=1}^{n} \widetilde{\Diamond} b_j' \in I$ for all $b_j' \lhd b_j$. Since $a' \lhd \bigvee^{\uparrow}_i a_i$ was chosen arbitrarily, $\widetilde{\Box}(\bigvee^{\uparrow}_i a_i) \wedge \bigwedge_{j=1}^{n} \widetilde{\Diamond} b_j \in K$. The proof of $(R\Diamond)$ is exactly the same.

Since $\beta \otimes \alpha$ implies $\beta \leq \alpha$, I is a subset of K. On the other hand, clearly $K \subseteq S_\pm(I)$ and because we proved that K is $R_\bigvee$-saturated and $S_\pm(I)$ is the smallest such containing I it must be that $S_\pm(I) = K$. □

4.2.12 Theorem.
Let $\mathcal{L}$ be a d-frame and let $\mathcal{W} \in \{\mathbb{W}_\Box, \mathbb{W}_\Diamond, \mathbb{W}_d\}$. Then we have that,

1. *$\mathcal{W}(\mathcal{L})$ is d-regular iff $\mathcal{L}$ is,*
2. *$\mathcal{W}(\mathcal{L})$ is d-zero-dimensional iff $\mathcal{L}$ is; and*
3. *$\mathcal{W}(\mathcal{L})$ is d-compact if $\mathcal{L}$ is d-regular and d-compact.*

Proof. Before we prove (1) and (2) first observe that if $a \lhd_+ b$ then also $\Box a \lhd_+ \Box b$. Indeed, we have $(a, a^*) \in \mathsf{con}_{\mathcal{L}}$ and $(b, a^*) \in \mathsf{tot}_{\mathcal{L}}$. Therefore, $(\Box a, \Diamond(a^*)) \in \mathsf{con}_{\mathbb{W}_d(\mathcal{L})}$ and $(\Box b, \Diamond(a^*)) \in \mathsf{tot}_{\mathbb{W}_d(\mathcal{L})}$. Similarly, $a \lhd_+ b$ implies $\Diamond a \lhd_+ \Diamond b$. This means that whenever $a \in L_+$ is complemented then also $\Box a$ and $\Diamond a$ are because $a \lhd_+ a$ implies $\Box a \lhd_+ \Box a$ and $\Diamond a \lhd_+ \Diamond a$.

Define $\widetilde{(-)}\colon B_\pm \to \mathbb{M}(L_\pm)$ as the injective map $\Box a \wedge \bigwedge_{i=1}^n \Diamond a_i \mapsto \widetilde{\Box} a \wedge \bigwedge_{i=1}^n \widetilde{\Diamond} a_i$ (injectivity follows from the discussion below Lemma 4.2.5). Observe that $\widetilde{\alpha}' \circledcirc \widetilde{\alpha}$ implies $\alpha' \lhd_+ \alpha$, for all $\alpha \in B_+$. This is because for an $\alpha = \Box a \wedge \bigwedge_{i=1}^n \Diamond b_i$ and $\alpha' = \Box a' \wedge \bigwedge_{i=1}^n \Diamond b_i'$ where $a' \lhd_+ a$ and $b_i' \lhd_+ b_i$, for the element $\alpha^\star = \Diamond (a')^* \vee \bigvee_{i=1}^n \Box (b_i')^*$ is $(\alpha', \alpha^\star) \in \mathsf{con}_{\mathbb{W}_d(\mathcal{L})}$ and $(\alpha, \alpha^\star) \in \mathsf{tot}_{\mathbb{W}_d(\mathcal{L})}$ by (con-$\mathbb{A}$) and (tot-$\mathbb{A}$).

If $\mathcal{L}$ is d-regular, we know that $a = \bigvee^\uparrow \{x : x \lhd a\}$, and because $\Box$ distributes over directed suprema we have that $\Box a = \bigvee^\uparrow \{\Box x : x \lhd a\} \le \bigvee^\uparrow \{\Box x : \Box x \lhd \Box a\} \le \Box a$ and also that $\Diamond a = \bigvee^\uparrow \{\Diamond x : \Diamond x \lhd \Diamond a\}$. The same is true for any $\alpha \in B_\pm$, because finite meets distribute over joins, $\alpha = \bigvee^\uparrow \{\alpha' \mid \widetilde{\alpha}' \circledcirc \widetilde{\alpha}\} \le \{x \mid x \lhd \alpha\} \le \alpha$. Since $B_\pm$ generates $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$ we see that $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$ is also d-regular.

The d-zero-dimensional case is proved similarly and the right-to-left implications of (1) and (2) follows from Lemma 4.2.10 and the fact that we have an onto homomorphism $\mathbb{W}_d(\mathcal{L}) \to \mathcal{L}$ as we are going to show in Lemma 4.3.1.

To prove (3) let $\{I^j = (I^j_+, I^j_-)\}_{j \in J}$ be a *directed* subset of $\mathbb{V}_{\mathrm{Fr}}(L_+) \times \mathbb{V}_{\mathrm{Fr}}(L_-)$ such that $\bigsqcup^\uparrow_j I^j \in \mathsf{tot}_{\mathbb{W}_d(\mathcal{L})}$. Recall that we can view $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$ as $\mathsf{Idl}(\mathbb{M}(L_\pm))/R_\bigvee$ and $I^j_\pm$'s as $R_\bigvee$-saturated ideals. By definition, there is a $K = (K_+, K_-) \in \mathsf{tot}_{\mathbb{A},\mathbb{V}}$ such that $K \sqsubseteq \bigsqcup^\uparrow_j I^j$. Because tot_1 consists of pairs $({\downarrow}\widetilde{\Box} a_+, {\downarrow}\widetilde{\Diamond} a_-)$ and $({\downarrow}\widetilde{\Diamond} a_+, {\downarrow}\widetilde{\Box} a_-)$ for $(a_+, a_-) \in \mathsf{tot}_\mathcal{L}$, the pair (K_+, K_-) is equal to $(S_+({\downarrow}\alpha_+), S_-({\downarrow}\alpha_-))$ where

$$\alpha = \Big(\bigvee_{k=1}^{n} (\widetilde{\Box} a_+^{k,0} \wedge \bigwedge_{l=1}^{m_k} \widetilde{\Diamond} a_+^{k,l}),\ \bigwedge_{k=1}^{n} (\widetilde{\Diamond} a_-^{k,0} \vee \bigvee_{l=1}^{m_k} \widetilde{\Box} a_-^{k,l})\Big).$$

for some $(a_+^{k,l}, a_-^{k,l}) \in \mathsf{tot}_\mathcal{L}$. Next, we find a $\beta \in \mathbb{M}(L_+) \times \mathbb{M}(L_-)$ such that $\beta \circledcirc \alpha$ and $H \stackrel{\text{def}}{\equiv} (S_+({\downarrow}\beta_+), S_-({\downarrow}\beta_-)) \in \mathsf{tot}_{\mathbb{A},\mathbb{V}}$. Indeed, from d-regularity of $\mathcal{L}$, each $a_\pm^{k,l}$ is equal to $\bigvee^\uparrow \{b \mid b \lhd a_\pm^{k,l}\}$ and, because $\mathcal{L}$ is also d-compact, there are some $b_\pm^{k,l} \lhd a_\pm^{k,l}$ such that $(b_+^{k,l}, b_-^{k,l}) \in \mathsf{tot}_\mathcal{L}$. Finally, set

$$\beta \stackrel{\text{def}}{\equiv} \Big(\bigvee_{k=1}^{n} (\Box b_+^{k,0} \wedge \bigwedge_{l=1}^{m_k} \Diamond b_+^{k,l}),\ \bigwedge_{k=1}^{n} (\Diamond b_-^{k,0} \vee \bigvee_{l=1}^{m_k} \Box b_-^{k,l})\Big).$$

Because $K_\pm \subseteq \bigvee^\uparrow_j I^j_\pm = S_\pm(\bigcup_j I^j_\pm)$ and because $\alpha_\pm \in K_\pm$, by Lemma 4.2.11, $\beta_\pm \in \bigcup_j I^j_\pm$. Therefore, there are some $j_\pm \in J$ such that $\beta_\pm \in I_\pm^{j_\pm}$. Because J is directed, there is some j such that $I^{j_+}, I^{j_-} \subseteq I^j$. As $\beta_\pm \in I^j$ and I^j is $R_\bigvee$-saturated, $H_\pm = S_\pm({\downarrow}\beta_\pm) \subseteq I^j_\pm$. Therefore, $I^j \in {\uparrow}\mathsf{tot}_{\mathbb{A},\mathbb{V}} = \mathsf{tot}_{\mathbb{W}_d(\mathcal{L})}$. □

4.3 Categorical properties

In this section we study basic categorical properties of the Vietoris constructions for d-frames. We show that all the basic properties of the functors $\mathbb{V}_\Box$, $\mathbb{V}_\Diamond$ and $\mathbb{V}_{\mathrm{Fr}}$ are also valid for their corresponding d-frame variants, that is for $\mathbb{W}_\Box$, $\mathbb{W}_\Diamond$ and $\mathbb{W}_d$.

4.3.1 Comonad structure

Johnstone proved that $\mathbb{V}_{\mathrm{Fr}}$ is a comonad on the category of frames [Joh85, Proposition 1.1]. For a frame homomorphism $h\colon L \to L'$, the frame homomorphism $\mathbb{V}_{\mathrm{Fr}}(h)\colon \mathbb{V}_{\mathrm{Fr}}L \to \mathbb{V}_{\mathrm{Fr}}L'$ is uniquely determined by its action on generators

$$\Box a \mapsto \Box(h(a)) \quad \text{and} \quad \Diamond a \mapsto \Diamond(h(a)).$$

The comonad $(\mathbb{V}_{\mathrm{Fr}}, \eta^{\mathrm{Fr}}, \mu^{\mathrm{Fr}})$ has as its counit $\eta^{\mathrm{Fr}}_L\colon \mathbb{V}_{\mathrm{Fr}}L \to L$ the map $\Box a, \Diamond a \mapsto a$, and comultiplication $\mu^{\mathrm{Fr}}_L\colon \mathbb{V}_{\mathrm{Fr}}L \to \mathbb{V}_{\mathrm{Fr}}(\mathbb{V}_{\mathrm{Fr}}L)$ is the map determined by $\Box a \mapsto \Box(\Box a)$ and $\Diamond a \mapsto \Diamond(\Diamond a)$. Alternatively, one describes this comonad as a co-Kleisli triple $(\mathbb{V}_{\mathrm{Fr}}, \eta^{\mathrm{Fr}}_L, \overline{(-)})$. For frame homomorphisms $h\colon \mathbb{V}_{\mathrm{Fr}}L \to L'$ and $g\colon \mathbb{V}_{\mathrm{Fr}}L' \to L''$, it has to satisfy:

(C1) $\overline{\eta^{\mathrm{Fr}}_L} = \mathrm{id}_{\mathbb{V}_{\mathrm{Fr}}L}$

(C2) $\overline{h}\,;\eta^{\mathrm{Fr}}_{L'} = h$

(C3) $\overline{\overline{h}\,;g} = \overline{h}\,;\overline{g}$

where the *lifting* $\overline{h}\colon \mathbb{V}_{\mathrm{Fr}}L \to \mathbb{V}_{\mathrm{Fr}}L'$ of h is the frame homomorphism uniquely determined by the mapping on generators: $\Box a \mapsto \Box(h(\Box a))$ and $\Diamond a \mapsto \Diamond(h(\Diamond a))$ (use the dual of Theorem A.3.13).

Based on this we define a co-Kleisli triple $(\mathbb{W}_d, \eta^d_{\mathcal{L}}, \overline{(-)})$ for $\mathbb{W}_d$. For a d-frame $\mathcal{L}$, set $\eta^d\colon \mathbb{W}_d\mathcal{L} \to \mathcal{L}$ to be the pair of maps $(\eta^{\mathrm{Fr}}_{L_+}, \eta^{\mathrm{Fr}}_{L_-})\colon \mathbb{V}_{\mathrm{Fr}}L_+ \times \mathbb{V}_{\mathrm{Fr}}L_- \to L_+ \times L_-$ and, for a d-frame homomorphism $h\colon \mathbb{W}_d\mathcal{L} \to \mathcal{M}$ define the lifting $\overline{h}\colon \mathbb{W}_d\mathcal{L} \to \mathbb{W}_d\mathcal{M}$ also pairwise, that is as the pair $(\overline{h}_+, \overline{h}_-)$. Because $\eta^d_{\mathcal{L}}$ and the lifting $\overline{(-)}$ are defined in terms of the corresponding parts of the frame co-Kleisli triple, it is clear that the axioms (C1), (C2) and (C3) hold. The only thing we need to do, in order to show that this co-Kleisli triple is well defined, is to prove that $\eta^d_{\mathcal{L}}$ and $\overline{h}$ are d-frame homomorphisms.

4.3.1 Proposition. *$\eta^d_{\mathcal{L}}$ is a d-frame homomorphism. Moreover, it embeds $\mathcal{L}$ into $\mathbb{W}_d\mathcal{L}$.*

Proof. By Proposition 3.4.21 it is enough to check that $\eta^d_{\mathcal{L}} = (\eta^{\mathrm{Fr}}_{L_+}, \eta^{\mathrm{Fr}}_{L_-})$ preserves E_{con} and E_{tot}. Let, for example, $(\Box a, \Diamond b)$ be from E_{con}. By definition, this only happens if $(a, b) \in \mathsf{con}_{\mathcal{L}}$. Then, $\eta(\Box a, \Diamond b) = (a, b) \in \mathsf{con}_{\mathcal{L}}$. The other cases for E_{con} and E_{tot} are the same.

Next is the "moreover" part. It is clear that both frame homomorphism components of $\eta^d_{\mathcal{L}}$ are onto such that $\overline{f}[\mathsf{con}_{\mathbb{W}_d\mathcal{L}}] = \mathsf{con}_{\mathcal{L}}$ and $\overline{f}[\mathsf{tot}_{\mathbb{W}_d\mathcal{L}}] = \mathsf{tot}_{\mathcal{L}}$. This makes $\eta^d_{\mathcal{L}}$ an extremal epimorphism in the category of d-frames (see Remark 3.3.15). □

4.3.2 Proposition. *Let $h\colon \mathbb{W}_d\mathcal{L} \to \mathcal{M}$ be a d-frame homomorphism. Then, $\overline{h}$ is also a d-frame homomorphism.*

Proof. Again, we use Proposition 3.4.21 and only check that the componentwise lift $\overline{h} = (\overline{h}_+, \overline{h}_-)$ preserves E_{con} (resp. E_{tot}) of $\mathbb{W}_d\mathcal{L}$. Let $(a,b) \in \mathsf{con}_{\mathcal{L}}$ (resp. $\mathsf{tot}_{\mathcal{L}}$). Then, because h is a d-frame homomorphism, $(h_+(\Box a), h_-(\Diamond b)) \in \mathsf{con}_{\mathcal{M}}$ (resp. $\mathsf{tot}_{\mathcal{M}}$) and so $\overline{h}$ maps $(\Box a, \Diamond b)$ to $(\Box(h_+(\Box a)), \Diamond(h_-(\Diamond b))) \in \mathsf{con}_{\mathbb{W}_d\mathcal{M}}$ (resp. $\mathsf{tot}_{\mathbb{W}_d\mathcal{M}}$). □

We have proved that $(\mathbb{W}_d, \eta^d_{\mathcal{L}}, \overline{(-)})$ is a co-Kleisli triple and so, by Theorem A.3.13, is $\mathbb{W}_d$ a functor and we also get:

4.3.3 Theorem.
$\mathbb{W}_\Box$, $\mathbb{W}_\Diamond$ and $\mathbb{W}_d$ are comonads on the category of d-frames.

One can also compute how $\mathbb{W}_d$ acts on morphisms. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a d-frame homomorphism. Then, $\mathbb{W}_d(h) = \overline{h \circ \eta^d_{\mathcal{L}}}$ and so on generators it is defined as follows

$$\mathbb{W}_d(h)_\pm\colon \quad \Box a, \Diamond a \quad \longmapsto \quad \Box h_\pm(a),\ \Diamond h_\pm(a)$$

for all $a \in L_\pm$.

4.3.2 Intrinsic co-semilattice structure of $\mathbb{W}_d(\mathcal{L})$

Johnstone proved in [Joh85] that $\mathbb{V}_{\mathrm{Fr}}L$ carries a co-semilattice structure, for any frame L. First, he shows that the quotient of the frame $\mathbb{V}_{\mathrm{Fr}}L$ by the equation $\Diamond 0 = 1$ is isomorphic to $\mathbf{2}$; the quotient map

$$p\colon \mathbb{V}_{\mathrm{Fr}}L \longrightarrow \mathbb{V}_{\mathrm{Fr}}L/_{(\Diamond 0=1)} \cong \mathbf{2}$$

is defined as $\Box a \mapsto 0$ and $\Diamond a \mapsto 1$. Another ingredient Johnstone needs is the fact that $\mathbb{V}_{\mathrm{Fr}}(L\times M) \cong \mathbb{V}_{\mathrm{Fr}}L \oplus \mathbb{V}_{\mathrm{Fr}}M$ as witnessed by the frame homomorphism q and its inverse $q^{-1} \stackrel{\mathrm{def}}{\equiv} q_L^{-1} \oplus q_M^{-1}$:

$$\begin{array}{rrcl} q\colon & \mathbb{V}_{\mathrm{Fr}}(L\times M) & \longrightarrow & \mathbb{V}_{\mathrm{Fr}}L \oplus \mathbb{V}_{\mathrm{Fr}}M \\ & \Box(l,m) & \mapsto & \Box l \oplus \Box m \\ & \Diamond(l,m) & \mapsto & \Box l \oplus 1 \vee 1 \oplus \Box m \end{array}$$

$$\begin{array}{rrcl} q_L^{-1}\colon & \mathbb{V}_{\mathrm{Fr}}L & \longrightarrow & \mathbb{V}_{\mathrm{Fr}}(L\times M) \\ & \Box l & \mapsto & \Box(l,0) \\ & \Diamond l & \mapsto & \Diamond(l,1) \\ q_M^{-1}\colon & \mathbb{V}_{\mathrm{Fr}}M & \longrightarrow & \mathbb{V}_{\mathrm{Fr}}(L\times M) \\ & \Box m & \mapsto & \Box(0,m) \\ & \Diamond m & \mapsto & \Diamond(1,m) \end{array}$$

where $\alpha \oplus \beta$ is the shortcut for $[\![(\alpha,\beta)]\!] = (\alpha \oplus_1 (1,1)) \wedge (\beta \oplus_2 (1,1))$ from Section 3.5.4.1. With this, the frame co-semilattice structure on $\mathbb{V}_{\mathrm{Fr}}L^3$are the morphisms

$$p\colon \mathbb{V}_{\mathrm{Fr}}L \longrightarrow \mathbf{2} \quad \text{and} \quad q \cdot \mathbb{V}_{\mathrm{Fr}}(\Delta)\colon \mathbb{V}_{\mathrm{Fr}}L \longrightarrow \mathbb{V}_{\mathrm{Fr}}L \oplus \mathbb{V}_{\mathrm{Fr}}L, \tag{4.3.1}$$

where Δ is the diagonal map $L \to L\times L$, $a \mapsto (a,a)$. Furthermore, this extends to all coalgebras of the comonad:

4.3.4 Fact ([Joh85]). *The forgetful functor from the category of coalgebras of the comonad $(\mathbb{V}_{\mathrm{Fr}}, \eta^{\mathrm{Fr}}, \mu^{\mathrm{Fr}})$ to* **Frm** *factors through the category of frame co-semilattices.*

The functor assigns to every coalgebra $(L, \xi\colon L \to \mathbb{V}_{\mathrm{Fr}} L)$ a co-semilattice (L, x_ξ, s_ξ) such that

$$x_\xi \stackrel{def}{\equiv} L \xrightarrow{\alpha} \mathbb{V}_{\mathrm{Fr}} L \xrightarrow{p} \mathbf{2}$$

$$s_\xi \stackrel{def}{\equiv} L \xrightarrow{\alpha} \mathbb{V}_{\mathrm{Fr}} L \xrightarrow{\mathbb{V}_{\mathrm{Fr}}(\Delta)} \mathbb{V}_{\mathrm{Fr}}(L\times L) \xrightarrow{q} \mathbb{V}_{\mathrm{Fr}} L \oplus \mathbb{V}_{\mathrm{Fr}} L \xrightarrow{\eta^{\mathrm{Fr}}_L \oplus \eta^{\mathrm{Fr}}_L} L \oplus L$$

Moreover, a $\mathbb{V}_{\mathrm{Fr}}$-coalgebra structure that gives rise to this frame co-semilattice is uniquely determined.

4.3.5 Remark. The co-semilattice structure assigned to the free coalgebra $(\mathbb{V}_{\mathrm{Fr}} L, \mu^{\mathrm{Fr}}\colon \mathbb{V}_{\mathrm{Fr}} L \to \mathbb{V}_{\mathrm{Fr}}(\mathbb{V}_{\mathrm{Fr}} L))$, for a frame L, is equal to the co-semilattice structure we computed for $\mathbb{V}_{\mathrm{Fr}} L$ in (4.3.1). Moreover, if $L = \Omega(X)$ for a compact regular space X, since $\mathbb{V}_{\mathrm{Fr}} L \cong \Omega(\mathbb{V}X)$, we can interpret $\Sigma(x)\colon \Sigma(\mathbf{2}) \to \Sigma(L)$ as a map that picks the empty subset of X, and the binary map $\Sigma(s)\colon \Sigma(L \oplus L) \to \Sigma(L)$ as the operation that computes the union of two compact subsets of X [Joh82; Rob86].

Inspired by the previous lines, we show a similar result to Fact 4.3.4 but for the comonad $\mathbb{W}_d$. First we observe that the maps p and q from above have their d-frame counterparts:

4.3.6 Lemma. *Let $\mathcal{L}$ be a d-frame. The pair of frame homomorphisms (p_+, p_-) forms a d-frame homomorphism $\mathbb{W}_d\mathcal{L} \to \mathbf{2}\times\mathbf{2}$.*

Proof. It is enough to check that $p = (p_+, p_-)\colon (\mathbb{V}_{\mathrm{Fr}} L_+, \mathbb{V}_{\mathrm{Fr}} L_-) \to (\mathbf{2},\mathbf{2})$ preserves E_{con} and E_{tot} (Proposition 3.4.21). Let $\alpha \in \mathsf{con}_{\mathcal{L}}$ (resp. $\mathsf{tot}_{\mathcal{L}}$). Then, $p(\Box\alpha_+, \Diamond\alpha_-) = (0,1) \in \mathsf{con}_{\mathbf{2}\times\mathbf{2}}$ (resp. $\mathsf{tot}_{\mathbf{2}\times\mathbf{2}}$) and $p(\Diamond\alpha_+, \Box\alpha_-) = (1,0) \in \mathsf{con}_{\mathbf{2}\times\mathbf{2}}$ (resp. $\mathsf{tot}_{\mathbf{2}\times\mathbf{2}}$). □

4.3.7 Lemma. *$\mathbb{W}_d(\mathcal{L} \times \mathcal{M}) \cong \mathbb{W}_d\mathcal{L} \oplus \mathbb{W}_d\mathcal{M}$ as witnessed by the d-frame homomorphisms (q_+, q_-) and (q_+^{-1}, q_-^{-1}).*

Proof. Again, we only check that $q = (q_+, q_-)$ and $q^{-1} = (q_+^{-1}, q_-^{-1})$ are d-frame homomorphisms; then, $q \cdot q^{-1} = \mathrm{id}$ and $q^{-1} \cdot q = \mathrm{id}$ follows from the corresponding identities for $q_\pm$ and $q_\pm^{-1}$.

Let $\alpha \in \mathsf{con}_{\mathcal{L}\times\mathcal{M}}$. We can view $\alpha = (\alpha_+, \alpha_-)$ as a quadruple where $\alpha_\pm = (\alpha_{L_\pm}, \alpha_{M_\pm}) \in L_\pm \times M_\pm$. Then, since $(\alpha_{L_+}, \alpha_{L_-}) \in \mathsf{con}_{\mathcal{L}}$ and $(\alpha_{M_+}, \alpha_{M_-}) \in \mathsf{con}_{\mathcal{M}}$,

$$(\Box\alpha_{L_+} \oplus 1, \Diamond\alpha_{L_-} \oplus 1) \in \mathsf{con}_{\mathbb{W}_d\mathcal{L}\oplus\mathbb{W}_d\mathcal{M}} \quad\text{and}\quad (1 \oplus \Box\alpha_{M_+}, 1 \oplus \Diamond\alpha_{M_-}) \in \mathsf{con}_{\mathbb{W}_d\mathcal{L}\oplus\mathbb{W}_d\mathcal{M}}.$$

[3] Frame co-semilattices are the triples $(L, x\colon L \to \mathbf{2}, s\colon L \to L \oplus L)$ which satisfy the co-semilattice equations when written as categorical diagrams in the opposite category.

Therefore, $q(\Box\alpha_+, \Diamond\alpha_-) = (\Box\alpha_{L_+} \oplus \Box\alpha_{M_+}, \Diamond\alpha_{L_-} \oplus 1 \vee 1 \oplus \Diamond\alpha_{M_-}) \in \mathsf{con}_{\mathbb{W}_d\mathcal{L}\oplus\mathbb{W}_d\mathcal{M}}$ by (con-$\curlywedge$). The cases $q(\Diamond\alpha_+, \Box\alpha_-) \in \mathsf{con}$ and $q[\mathsf{tot}_1] \subseteq \mathsf{tot}_{\mathbb{W}_d\mathcal{L}\oplus\mathbb{W}_d\mathcal{M}}$ are proved similarly.

Next, we show that also $q^{-1} = (q_+^{-1}, q_-^{-1})$ is a d-frame homomorphisms by showing that $q_{\mathcal{L}} = (q_{L_+}^{-1}, q_{L_-}^{-1})\colon \mathbb{W}_d\mathcal{L} \to \mathbb{W}_d(\mathcal{L} \times \mathcal{M})$ and $q_{\mathcal{M}} = (q_{M_+}^{-1}, q_{M_-}^{-1})\colon \mathbb{W}_d\mathcal{M} \to \mathbb{W}_d(\mathcal{L} \times \mathcal{M})$ are d-frame homomorphisms. Let $\alpha \in \mathsf{con}_{\mathcal{L}}$. Then, $q_{\mathcal{L}}^{-1}(\Box\alpha_+, \Diamond\alpha_-) = (\Box(\alpha_+, 1), \Diamond(\alpha_-, 0))$ which is in $\mathsf{con}_{\mathbb{W}_d(\mathcal{L}\times\mathcal{M})}$ since $((\alpha_+, 1), (\alpha_-, 0)) \in \mathsf{con}_{\mathcal{L}\times\mathcal{M}}$. The remaining cases are proved similarly. □

4.3.8 Theorem.
The forgetful functor from the category of coalgebras of the comonad $(\mathbb{W}_d, \eta^d, \mu^d)$ to **d-Frm** *factors through the category of d-frame co-semilattices.*

Moreover, a $\mathbb{W}_d$-coalgebra structure that gives rise to a d-frame co-semilattice is uniquely determined.

Proof. Lemmas 4.3.6 and 4.3.7 establish that the construction from Fact 4.3.4 extends to d-frames. Namely, for every d-frame coalgebra $(\mathcal{L}, \xi\colon \mathcal{L} \to \mathbb{W}_d\mathcal{L})$ we have a co-semilattice $(\mathcal{L}, x_\xi, s_\xi)$ such that

$$x_\xi \overset{\text{def}}{\equiv} \mathcal{L} \xrightarrow{\ \alpha\ } \mathbb{W}_d\mathcal{L} \xrightarrow{(p_+,p_-)} \mathbf{2}\times\mathbf{2}$$
$$s_\xi \overset{\text{def}}{\equiv} \mathcal{L} \xrightarrow{\ \alpha\ } \mathbb{W}_d\mathcal{L} \xrightarrow{\mathbb{W}_d(\Delta)} \mathbb{V}_{\mathrm{Fr}}(\mathcal{L}\times\mathcal{L}) \xrightarrow{(q_+,q_-)} \mathbb{W}_d\mathcal{L} \oplus \mathbb{W}_d\mathcal{L} \xrightarrow{\eta^d_{\mathcal{L}}\oplus\eta^d_{\mathcal{L}}} \mathcal{L} \oplus \mathcal{L}$$

The fact that all the required equations hold and that this is unique is a consequence of Fact 4.3.4 and because all the constructions we made are componentwise the same as for frames. □

4.3.3 A relationship between $\mathbb{W}_\Box$, $\mathbb{W}_\Diamond$ and $\mathbb{W}_d$

Another known fact about Vietoris constructions for frames and DCPO's/domains is that, for a frame L, $\mathbb{V}_{\mathrm{Fr}}L$ can be seen as the quotient of $\mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L$ by the equations $R_{\mathbb{V}}$ [Vic09a], [Gie+80, Theorem IV-8.14]:

$$\begin{aligned} \Box a \oplus \Diamond b &\leq 1 \oplus \Diamond(a \wedge b) \\ \Box(a \vee b) \oplus 1 &\leq (\Box a \oplus 1) \vee (1 \oplus \Diamond b) \end{aligned} \tag{4.3.2}$$

This is because the frame homomorphism $i = i^\Box \oplus i^\Diamond\colon \mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L \to \mathbb{V}_{\mathrm{Fr}}L$, where $i^\Box\colon \mathbb{V}_\Box L \to \mathbb{V}_{\mathrm{Fr}}L$ and $i^\Diamond\colon \mathbb{V}_\Diamond L \to \mathbb{V}_{\mathrm{Fr}}L$ are the inclusions, preserves the equations in (4.3.2) and *uniquely* lifts to

$$k\colon (\mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L)/R_{\mathbb{V}} \to \mathbb{V}_{\mathrm{Fr}}L.$$

On the other hand, by Lemma 4.2.1 we know that $(\mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L)/R_\mathbb{V}$ can be given by a presentation where $G = \{\Box a \oplus 1,\ 1 \oplus \Diamond a : a \in L\}$ is the set of generators and the equations are obtained as a combination of equations for both Vietoris constructions, the coproducts, and $R_\mathbb{V}$. Then, the map $G \to \mathbb{V}_{\mathrm{Fr}} L$ defined as $\Box a \oplus 1 \mapsto \Box a$ and $1 \oplus \Diamond a \mapsto \Diamond a$ preserves all the equations of $(\mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L)/R_\mathbb{V}$ and it is the universal such. This proves that the frame homomorphism

$$l \colon \mathbb{V}_{\mathrm{Fr}} L \to (\mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L)/R_\mathbb{V},$$

defined as $\Box a \mapsto \Box a \oplus 1$ and $\Diamond a \mapsto 1 \oplus \Diamond a$, is inverse to k, proving that $\mathbb{V}_{\mathrm{Fr}} L \cong (\mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L)/R_\mathbb{V}$. To summarise, we have a diagram

$$\begin{array}{ccc} \mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L & \xrightarrow{\ i\ } & \mathbb{V}_{\mathrm{Fr}} L \\ {\scriptstyle q}\big\downarrow & \overset{l}{\swarrow}\ \underset{k}{\nearrow} & \\ \mathbb{V}_\Box L \oplus \mathbb{V}_\Diamond L / R_\mathbb{V} & & \end{array}$$

This result can be immediately adapted to the theory of d-frames:

4.3.9 Proposition. *Let $\mathcal{L}$ be a d-frame. Then, $\mathbb{W}_d \mathcal{L}$ is isomorphic to the d-frame quotient of $\mathbb{W}_\Box \mathcal{L} \oplus \mathbb{W}_\Diamond \mathcal{L}$ by $R = (R_+, R_-)$, defined as*

$$\begin{array}{rl} R_+ \colon & \\ \Box a \oplus \Diamond b & \le 1 \oplus \Diamond(a \wedge b) \\ \Box(a \vee b) \oplus 1 & \le (\Box a \oplus 1) \vee (1 \oplus \Diamond b) \end{array} \qquad \begin{array}{rl} R_- \colon & \\ \Diamond a \oplus \Box b & \le \Diamond(a \wedge b) \oplus 1 \\ 1 \oplus \Box(a \vee b) & \le (\Diamond a \oplus 1) \vee (1 \oplus \Box b) \end{array}$$

Proof. By $\mathcal{L}_\oplus$ denote the d-frame $\mathbb{W}_\Box \mathcal{L} \oplus \mathbb{W}_\Diamond \mathcal{L}$ and by $\mathrm{con}_\oplus$ and $\mathrm{tot}_\oplus$ denote its consistency and totality relations, respectively. Since R_+ and R_- introduce a pair of frame quotients

$$q_+ \colon \mathbb{V}_\Box L_+ \oplus \mathbb{V}_\Diamond L_+ \twoheadrightarrow \mathbb{V}_\Box L_+ \oplus \mathbb{V}_\Diamond L_+ / R_+$$
$$q_- \colon \mathbb{V}_\Diamond L_- \oplus \mathbb{V}_\Box L_- \twoheadrightarrow \mathbb{V}_\Diamond L_- \oplus \mathbb{V}_\Box L_- / R_-$$

we have the corresponding quotient homomorphism in the category of proto-d-frames $q = (q_+, q_-) \colon \mathcal{L}_\oplus \twoheadrightarrow q[\mathcal{L}_\oplus]$ where

$$q[\mathcal{L}_\oplus] = ((\mathbb{V}_\Box L_+ \oplus \mathbb{V}_\Diamond L_+)/R_+,\ (\mathbb{V}_\Diamond L_- \oplus \mathbb{V}_\Box L_-)/R_-,\ q[\mathrm{con}_\oplus], q[\mathrm{tot}_\oplus]).$$

In the following we show two things at once: (1) the d-frame reflection of $q[\mathcal{L}_\oplus]$, denoted as $\mathcal{L}_\oplus / R$, is obtained from $q[\mathcal{L}_\oplus]$ by the $\bigsqcup^\uparrow$-completion of the consistency relation (the rest of the structure stays unchanged), (2) $\mathcal{L}_\oplus / R$ is isomorphic to $\mathbb{W}_d \mathcal{L}$. To follow the frame construction, consider the pairs of frame isomorphisms

$$k = (k_+, k_-) \colon ((\mathbb{V}_\Box L_+ \oplus \mathbb{V}_\Diamond L_+)/R_+, (\mathbb{V}_\Diamond L_- \oplus \mathbb{V}_\Box L_-)/R_-) \to (\mathbb{V}_{\mathrm{Fr}} L_+, \mathbb{V}_{\mathrm{Fr}} L_-)$$

and

$$l = (l_+, l_-) \colon (\mathbb{V}_{\mathrm{Fr}} L_+, \mathbb{V}_{\mathrm{Fr}} L_-) \to ((\mathbb{V}_\Box L_+ \oplus \mathbb{V}_\Diamond L_+)/R_+, (\mathbb{V}_\Diamond L_- \oplus \mathbb{V}_\Box L_-)/R_-)$$

defined componentwise as above. (In fact, because k_- and l_- are witnessing the isomorphism $\mathbb{V}_{\mathrm{Fr}} L_- \cong (\mathbb{V}_\Diamond L_- \oplus \mathbb{V}_\Box L_-)/R_-$, they are defined symmetrically). First, we check that k (seen as a pair of frame homomorphisms $q[\mathcal{L}_\oplus] \to \mathbb{W}_d\mathcal{L}$) is a d-frame homomorphism, i.e. that it preserves $q[\mathsf{con}_\oplus]$ and $q[\mathsf{tot}_\oplus]$. Let $\alpha \in \mathsf{con}_\mathcal{L}$. Then, $i(\Box\alpha_+ \oplus 1, \Diamond\alpha_- \oplus 1) = (\Box\alpha_+, \Diamond\alpha_-) \in \mathsf{con}_{\mathbb{W}_d\mathcal{L}}$ and, similarly, $i(\Box\alpha_+ \oplus 1, \Diamond\alpha_- \oplus 1) \in \mathsf{con}_{\mathbb{W}_d\mathcal{L}}$. Checking that i also preserves the generators of $\mathsf{tot}_\oplus$ is the same. Because k is a monomorphism, Proposition 3.3.13 gives that the d-frame reflection of $q[\mathcal{L}_\oplus]$, i.e. the d-frame $\mathcal{L}_\oplus/R$, is computed as a $\bigsqcup^\uparrow$-completion of the consistency relation. Consequently, $k \colon \mathcal{L}_\oplus/R \to \mathbb{W}_d\mathcal{L}$ is a d-frame homomorphism.

Finally, we check that $l \colon \mathbb{W}_d\mathcal{L} \to \mathcal{L}_\oplus/R$ preserves the generators of $\mathsf{con}_{\mathbb{W}_d\mathcal{L}}$ and $\mathsf{tot}_{\mathbb{W}_d\mathcal{L}}$. Let $\alpha \in \mathsf{con}_\mathcal{L}$. Then, $(\Box\alpha_+, \Diamond\alpha_-)$ is mapped by l to $(\Box\alpha_+ \oplus 1, \Diamond\alpha_- \oplus 1)$, which is an element in the presentation of $\mathsf{con}_\oplus$. Similarly, one proves that $l(\Diamond\alpha_+, \Box\alpha_-)$ is in con and that l also preserves the totality relation. □

This explains why the proof that the presentation of $\mathbb{W}_d\mathcal{L}$ satisfies ($\mu_\pm$-con-tot) and (Indep$_\pm$) (in Propositions 4.2.7 and 4.2.8) looks so much like the corresponding proof for the presentation of a coproduct of two d-frames (Section 3.5.4.4).

4.3.4 A comparison with constructions in DLat

Recall, from Section 4.2.1, the Vietoris-like construction $\mathbb{M}$ for distributive lattices. It is immediate to see that $\mathbb{M}$ is functorial; with the action on a morphisms defined as

$$\begin{array}{rccc} \mathbb{M}(h)\colon & \mathbb{M}(D) & \longrightarrow & \mathbb{M}(E) \\ & \tilde{\Box}a, \tilde{\Diamond}b & \longmapsto & \tilde{\Box}h(a), \tilde{\Diamond}h(b) \end{array}$$

where $h \colon D \to E$ is a lattice homomorphism.

Johnstone proved in [Joh82] that the functor $\mathsf{Idl}(-) \colon \mathbf{DLat} \to \mathbf{Frm}$, which assigns to a distributive lattice D its frame of ideals $\mathsf{Idl}(D)$, relates $\mathbb{V}_{\mathrm{Fr}}$ and $\mathbb{M}$. Concretely, the compositions of functors $\mathbb{V}_{\mathrm{Fr}} \circ \mathsf{Idl}$ and $\mathsf{Idl} \circ \mathbb{M}$ are naturally isomorphic.

4.3.10 Fact. $\mathbb{V}_{\mathrm{Fr}} \circ \mathsf{Idl} \cong \mathsf{Idl} \circ \mathbb{M}$ *with the components of the natural isomorphisms uniquely determined by*

$$\begin{array}{rccc} l_+ \colon \mathbb{V}_{\mathrm{Fr}}(\mathsf{Idl}(D)) & \xrightarrow{\cong} & \mathsf{Idl}(\mathbb{M}D) \\ \Box I & \longmapsto & \bigvee\{\downarrow\tilde{\Box}d : d \in I\} \\ \Diamond I & \longmapsto & \bigvee\{\downarrow\tilde{\Diamond}d : d \in I\} \end{array} \qquad \begin{array}{rccc} k_+ \colon \mathsf{Idl}(\mathbb{M}D) & \xrightarrow{\cong} & \mathbb{V}_{\mathrm{Fr}}(\mathsf{Idl}(D)) \\ \downarrow\tilde{\Box}a & \longmapsto & \Box(\downarrow a) \\ \downarrow\tilde{\Diamond}a & \longmapsto & \Diamond(\downarrow a) \end{array}$$

4.3.4.1 Comparison with $\mathbb{W}_d$

Moving on to d-frames, recall the functor $\mathcal{IF}\colon \mathbf{DLat} \to \mathbf{d\text{-}Frm}$ from Section 2.6 defined as

$$D \longmapsto (\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{con}_D, \mathsf{tot}_D)$$

where $(I, F) \in \mathsf{con}_D$ iff $\forall i \in I\, \forall f \in F.\, i \leq f$, and $(I, F) \in \mathsf{tot}_D$ iff $I \cap F \neq \emptyset$.

We will see that $\mathcal{IF}$ relates $\mathbb{W}_d$ and $\mathbb{M}$. Notice that, for a lattice D, the plus-frame in $\mathbb{W}_d(\mathcal{IF}(D))$ is isomorphic to the plus-frame of $\mathcal{IF}(\mathbb{M}D)$. Moreover, consider the isomorphism

$$i\colon (\mathbb{M}D)^{\mathrm{op}} \xrightarrow{\ \cong\ } \mathbb{M}(D^{\mathrm{op}})$$

defined on generators as $\tilde{\Box}a, \tilde{\Diamond}b \mapsto \tilde{\Diamond}a, \tilde{\Box}b$. Then, we have

$$\mathbb{V}_{\mathrm{Fr}}(\mathsf{Filt}(D)) \cong \mathbb{V}_{\mathrm{Fr}}(\mathsf{Idl}(D^{\mathrm{op}})) \cong \mathsf{Idl}(\mathbb{M}(D^{\mathrm{op}})) \cong \mathsf{Idl}((\mathbb{M}D)^{\mathrm{op}}) \cong \mathsf{Filt}(\mathbb{M}D)$$

which is computed as

$$\begin{array}{rcl} l_-\colon \mathbb{V}_{\mathrm{Fr}}(\mathsf{Filt}(D)) & \xrightarrow{\ \cong\ } & \mathsf{Filt}(\mathbb{M}D) \\ \Box F & \longmapsto & \bigvee\{\uparrow\tilde{\Diamond}d : d \in F\} \\ \Diamond F & \longmapsto & \bigvee\{\uparrow\tilde{\Box}d : d \in F\} \end{array} \qquad \begin{array}{rcl} k_-\colon \mathsf{Filt}(\mathbb{M}D) & \xrightarrow{\ \cong\ } & \mathbb{V}_{\mathrm{Fr}}(\mathsf{Filt}(D)) \\ \uparrow\tilde{\Box}a & \longmapsto & \Diamond(\uparrow a) \\ \uparrow\tilde{\Diamond}a & \longmapsto & \Box(\uparrow a) \end{array}$$

This shows that also the minus-frames of $\mathbb{W}_d(\mathcal{IF}(D))$ and $\mathcal{IF}(\mathbb{M}D)$ are isomorphic.

4.3.11 The d-frame isomorphism. Next, we show that the pairs of frame homomorphisms $k = (k_+, k_-)$ and $l = (l_+, l_-)$ actually establish a (natural) isomorphism of the d-frames $\mathbb{W}_d(\mathcal{IF}(D))$ and $\mathcal{IF}(\mathbb{M}(D))$. To see that, let $(I, F) \in \mathsf{con}_D$. Then, $(\Box I, \Diamond F)$, which is in $\mathsf{con}_{\mathbb{W}_d(\mathcal{IF}(D))}$, gets mapped by l to $(\bigvee\{\downarrow\tilde{\Box}a : a \in I\}, \bigvee\{\uparrow\tilde{\Box}b : b \in F\})$. Next, let $A \subseteq_{\mathrm{fin}} I$ and $B \subseteq_{\mathrm{fin}} F$. We have that

$$\bigvee_{a\in A} \tilde{\Box}a \leq \tilde{\Box}(\bigvee A) \leq \tilde{\Box}b \quad \text{for all } b \in B$$

because $\bigvee A \in I$ and $a \leq b$ for every $a \in A$ and $b \in B$. Hence, $\bigvee_{a\in A} \tilde{\Box}a \leq \bigwedge_{b\in B} \tilde{\Box}b$ proving that $l(\Box I, \Diamond F) \in \mathsf{con}_{\mathbb{M}D}$. A similar argument also proves that $l(\Diamond I, \Box F)$ is in $\mathsf{con}_{\mathbb{M}D}$. Let, on the other hand, (I, F) be from tot_D. The pair $(\Box I, \Diamond F)$ from $\mathsf{tot}_{\mathbb{W}_d(\mathcal{IF}(D))}$ gets mapped by l into $\mathsf{tot}_{\mathbb{M}D}$ because, for an $x \in I \cap F$, $\tilde{\Box}x \in l_+(\Box I) \cap l_-(\Diamond F)$. We have proved that the images by l of the generators of the consistency and totality relations of $\mathbb{W}_d(\mathcal{IF}(D))$ are also consistent and total in $\mathcal{IF}(\mathbb{M}D)$, respectively. Therefore,

$$l[\mathsf{con}_{\mathbb{W}_d(\mathcal{IF}(D))}] \subseteq \mathsf{con}_{\mathbb{M}D} \quad \text{and} \quad l[\mathsf{tot}_{\mathbb{W}_d(\mathcal{IF}(D))}] \subseteq \mathsf{tot}_{\mathbb{M}D}$$

In Section 3.5.2 we showed that $\mathsf{con}_{\mathbb{M}D}$ and $\mathsf{tot}_{\mathbb{M}D}$ are both generated from the elements of the form $(\downarrow x, \uparrow x)$ for $x \in \mathbb{M}D$. Hence, to show that k is a d-frame homomorphism, it is enough to check that $(\downarrow x, \uparrow x)$'s are mapped to $\mathsf{con} \cap \mathsf{tot}$ of $\mathbb{W}_d(\mathcal{IF}(D))$. Let $\bigvee_{i=1}^n (\tilde{\Box} a_i \wedge \bigwedge_{j=1}^{m_i} \tilde{\Diamond} b_{i,j})$ be a disjunctive normal form of x. We see that

$$k_+(\downarrow x) = k_+(\bigvee_{i=1}^n (\downarrow\tilde{\Box} a_i \wedge \bigwedge\nolimits_{j=1}^{m_i} \downarrow\tilde{\Diamond} b_{i,j})) = \bigvee_{i=1}^n (\Box{\downarrow} a_i \wedge \bigwedge\nolimits_{j=1}^{m_i} \Diamond{\downarrow} b_{i,j})$$

$$k_-(\uparrow x) = k_-(\bigwedge_{i=1}^n (\uparrow\tilde{\Box} a_i \vee \bigvee\nolimits_{j=1}^{m_i} \uparrow\tilde{\Diamond} b_{i,j})) = \bigwedge_{i=1}^n (\Diamond{\uparrow} a_i \vee \bigvee\nolimits_{j=1}^{m_i} \Box{\uparrow} b_{i,j})$$

Then, because $(\Box{\downarrow} a_i, \Diamond{\uparrow} a_i)$ and $(\Diamond{\downarrow} b_{i,j}, \Box{\uparrow} b_{i,j})$ are in $\mathsf{con}_{\mathbb{W}_d(\mathcal{IF}(D))}$ and $\mathsf{tot}_{\mathbb{W}_d(\mathcal{IF}(D))}$, also their $(\curlywedge, \curlyvee)$-combinations are; proving $k(\downarrow x, \uparrow x) \in \mathsf{con}_{\mathbb{W}_d(\mathcal{IF}(D))} \cap \mathsf{tot}_{\mathbb{W}_d(\mathcal{IF}(D))}$. Consequently:

$$k[\mathsf{con}_{\mathbb{M}D}] \subseteq \mathsf{con}_{\mathbb{W}_d(\mathcal{IF}(D))} \quad \text{and} \quad k[\mathsf{tot}_{\mathbb{M}D}] \subseteq \mathsf{tot}_{\mathbb{W}_d(\mathcal{IF}(D))}.$$

Because i and k are componentwise natural in D we obtain:

4.3.12 Theorem.
The functor $\mathbb{W}_d \circ \mathcal{IF}$ is naturally isomorphic to the functor $\mathcal{IF} \circ \mathbb{M}$.

It should be no surprise that, for the following two lattice constructions

$$\mathbb{M}_\Box(D) \stackrel{\text{def}}{\equiv} \mathbf{DL}\Big\langle \tilde{\Box} a : a \in D \;\Big|\; \tilde{\Box}(a \wedge b) = \tilde{\Box} a \wedge \tilde{\Box} b, \quad \tilde{\Box} 1 = 1 \Big\rangle$$

$$\mathbb{M}_\Diamond(D) \stackrel{\text{def}}{\equiv} \mathbf{DL}\Big\langle \tilde{\Diamond} a : a \in D \;\Big|\; \tilde{\Diamond}(a \vee b) = \tilde{\Diamond} a \vee \tilde{\Diamond} b, \quad \tilde{\Diamond} 0 = 0 \Big\rangle$$

we have that

$$\mathbb{V}_\Box \circ \mathsf{Idl} \cong \mathsf{Idl} \circ \mathbb{M}_\Box \quad \text{and} \quad \mathbb{V}_\Diamond \circ \mathsf{Idl} \cong \mathsf{Idl} \circ \mathbb{M}_\Diamond$$

and, since $\mathbb{M}_\Box(D)^{\mathrm{op}} \cong \mathbb{M}_\Diamond(D^{\mathrm{op}})$, also

$$\mathbb{V}_\Diamond \circ \mathsf{Filt} \cong \mathsf{Filt} \circ \mathbb{M}_\Box \quad \text{and} \quad \mathbb{V}_\Box \circ \mathsf{Filt} \cong \mathsf{Filt} \circ \mathbb{M}_\Diamond.$$

By the same reasoning as above, we obtain:

4.3.13 Proposition. *$\mathbb{W}_\Box \circ \mathcal{IF} \cong \mathcal{IF} \circ \mathbb{M}_\Box$ and $\mathbb{W}_\Diamond \circ \mathcal{IF} \cong \mathcal{IF} \circ \mathbb{M}_\Diamond$.*

As a consequence of these natural isomorphisms, we obtain, by a general diagram chasing argument:

4.3.14 Corollary. *For $\mathbb{W}_\Box, \mathbb{W}_\Diamond$ and $\mathbb{W}_d$ restricted to the category of Priestley d-frames:*

1. $\mathsf{Alg}(\mathbb{W}_\Box) \cong \mathsf{Alg}(\mathbb{M}_\Box)$,

2. $\mathsf{Alg}(\mathbb{W}_\diamond) \cong \mathsf{Alg}(\mathbb{M}_\diamond)$ *and*

3. $\mathsf{Alg}(\mathbb{W}_d) \cong \mathsf{Alg}(\mathbb{M})$.

4.3.5 Initial algebra construction

Consider the following countable sequence in the category of d-frames

$$\mathbf{2}{\times}\mathbf{2} \xrightarrow{!} \mathbb{W}_d(\mathbf{2}{\times}\mathbf{2}) \xrightarrow{\mathbb{W}_d(!)} \mathbb{W}_d^2(\mathbf{2}{\times}\mathbf{2}) \xrightarrow{\mathbb{W}_d^2(!)} \mathbb{W}_d^3(\mathbf{2}{\times}\mathbf{2}) \xrightarrow{\mathbb{W}_d^3(!)} \cdots \tag{4.3.3}$$

where $!\colon \mathbf{2}{\times}\mathbf{2} \to \mathbb{W}_d(\mathbf{2}{\times}\mathbf{2})$ is the unique morphism from the initial object $\mathbf{2}{\times}\mathbf{2}$ of **d-Frm**. Observe that $\mathbf{2}{\times}\mathbf{2}$ is a Priestley d-frame because $\mathbf{2}{\times}\mathbf{2} \cong \mathcal{IF}(\mathbf{2})$, where $\mathcal{IF}\colon \mathbf{DLat} \to \mathbf{d\text{-}Frm}$ is the functor from Section 2.6 mapping distributive lattices into the equivalent category of Priestley d-frames. Consequently, repeated application of Theorem 4.2.12 proves that $\mathbb{W}_d^n(\mathbf{2}{\times}\mathbf{2})$ is a Priestley d-frame, for every $n \in \mathbb{N}$.

Define $\mathbb{W}_d^\omega(\mathbf{2}{\times}\mathbf{2})$ to be the colimit of the sequence in (4.3.3). Since **d-Pries** closed under colimits in **d-Frm** (Section 3.3.2), $\mathbb{W}_d^\omega(\mathbf{2}{\times}\mathbf{2})$ is a Priestley d-frame. Further, since $\mathbb{W}_d^n(\mathbf{2}{\times}\mathbf{2}) \cong \mathbb{W}_d^n(\mathcal{IF}(\mathbf{2})) \cong \mathcal{IF}(\mathbb{M}^n(\mathbf{2}))$ by Theorem 4.3.12, we have the following sequence in **DLat**

$$\mathbf{2} \xrightarrow{!} \mathbb{M}(\mathbf{2}) \xrightarrow{\mathbb{M}(!)} \mathbb{M}^2(\mathbf{2}) \xrightarrow{\mathbb{M}^2(!)} \mathbb{M}^3(\mathbf{2}) \xrightarrow{\mathbb{M}^3(!)} \cdots$$

Where, this time, $!\colon \mathbf{2} \to \mathbb{M}(\mathbf{2})$ is the morphism from the initial object $\mathbf{2}$ in **DLat**. Set $\mathbb{M}^\infty(\mathbf{2})$ to be the colimit of this sequence in **DLat**. Since $\mathcal{IF}$ is a left adjoint (Theorem 2.6.11) it preserves colimits and so it must be that $\mathbb{W}_d^\infty(\mathbf{2}{\times}\mathbf{2}) \cong \mathcal{IF}(\mathbb{M}^\infty(\mathbf{2}))$.

Lastly, it is a well-known fact ([Joh82]) that $\mathbb{M}(\mathbb{M}^\infty(\mathbf{2})) \cong \mathbb{M}^\infty(\mathbf{2})$[4]or, in other words, that $\mathbb{M}$ has an initial algebra such that its initial sequence stops after $|\mathbb{N}|$-many steps. Consequently,

$$\mathbb{W}_d(\mathbb{W}_d^\infty(\mathbf{2}{\times}\mathbf{2})) \cong \mathbb{W}_d(\mathcal{IF}(\mathbb{M}^\infty(\mathbf{2}))) \cong \mathcal{IF}(\mathbb{M}(\mathbb{M}^\infty(\mathbf{2}))) \cong \mathcal{IF}(\mathbb{M}^\infty(\mathbf{2})) \cong \mathbb{W}_d^\infty(\mathbf{2}{\times}\mathbf{2}).$$

Observe that there was nothing specific about $\mathbb{W}_d$ and the same argument would also work for $\mathbb{W}_\square$ and $\mathbb{W}_\diamond$. We have proved:

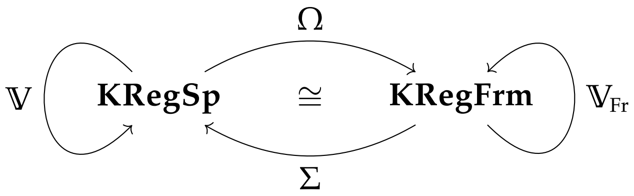
4.3.15 Theorem.
Every $\mathcal{W} \in \{\mathbb{W}_\square, \mathbb{W}_\diamond, \mathbb{W}_d\}$ has an initial algebra and its initial sequence stops after $|\mathbb{N}|$-many steps.

---

[4] The map $\mathbb{M}^\infty(\mathbf{2}) \to \mathbb{M}(\mathbb{M}^\infty(\mathbf{2}))$ is the unique morphism for the cocone $\{\mathbb{M}^n(\mathbf{2}) \xrightarrow{\mathbb{M}^n(!)} \mathbb{M}(\mathbb{M}^\infty(\mathbf{2}))\}_n$.

4.4 Connecting the constructions

The functors $\mathbb{V}\colon \mathbf{Top} \to \mathbf{Top}$ and $\mathbb{V}_{\mathrm{Fr}}\colon \mathbf{Frm} \to \mathbf{Frm}$ are thought of as the same constructions, but acting on different categories. To make this precise, consider their restrictions to dually equivalent categories, that is to the category of compact regular (T_0) spaces and frames, respectively. The situation is as follows:



To establish that $\mathbb{V}$ and $\mathbb{V}_{\mathrm{Fr}}$ are related constructions constructions (on the compact regular part) one proves that there is a natural isomorphism of functors $\Omega \circ \mathbb{V} \cong \mathbb{V}_{\mathrm{Fr}} \circ \Omega$ or, equivalently,

$$\mathbb{V} \circ \Sigma \cong \Sigma \circ \mathbb{V}_{\mathrm{Fr}}. \tag{4.4.1}$$

To compute this, Johnstone (in [Joh82]) fully exploited the fact that the points of $\mathbb{V}(X)$ for a compact regular space X have a direct frame-theoretic description. Namely, each compact subset $K \in \mathcal{K}(X)$ is closed and so the mapping $K \mapsto X \setminus K$ establishes a bijection between the points of $\mathbb{V}(X)$ and open sets of X. Therefore, we can rephrase the definition of $\mathbb{V}(X)$ with the points being open subsets instead of compact subsets of X; basic opens generating the topology $\mathbb{V}\tau$ are then interpreted as:

$$\boxtimes U = \{V \in \tau \mid U \cup V = X\} \quad \text{and} \quad \diamondplus U = \{V \in \tau \mid U \not\subseteq V\}.$$

This way we have expressed $\mathbb{V}$ entirely in terms of open sets. Consequently, for a compact regular frame L, $\mathbb{V}(\Sigma(L))$ can be computed directly from the structure of L. Define an auxiliary contravariant construction $\widetilde{\mathbb{V}}\colon \mathbf{KRegFrm} \to \mathbf{KRegSp}$, $L \mapsto (L, \widetilde{\mathbb{V}}L)$, where the topology $\widetilde{\mathbb{V}}L$ is generated from the basic open sets $\boxtimes x$ and $\diamondplus x$, for every $x \in L$, where

$$\boxtimes x = \{a \in L \mid x \vee a = 1\} \quad \text{and} \quad \diamondplus x = \{a \in L \mid x \not\leq a\}.$$

We see that $\widetilde{\mathbb{V}}(L)$ is homeomorphic to $\mathbb{V}(\Sigma(L))$.

With this, the last step in showing (4.4.1) is to prove that $\widetilde{\mathbb{V}}(L) \cong \Sigma(\mathbb{V}_{\mathrm{Fr}}(L))$. The bijection between the elements of L and homomorphisms $\mathbb{V}_{\mathrm{Fr}}(L) \to \mathbf{2}$ is established by maps $a \mapsto P^a$ and $P \mapsto a^P$, with P^a defined on generators as

$$P^a(\Box x) = 1 \ \text{ iff } \ x \vee a = 1 \quad \text{and} \quad P^a(\Diamond x) = 1 \ \text{ iff } \ x \not\leq a \tag{4.4.2}$$

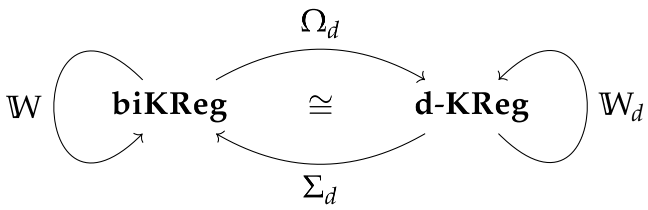
and a^P defined as

$$a^P \stackrel{\text{def}}{\equiv} \bigvee\{x \in L \mid P(\Diamond x) = 0\}.$$

To explain the intuition behind the last formula consider a compact regular space X such that $L \cong \Omega(X)$. The open set $X \setminus K$, corresponding to a compact subset K, of X is computed as the union of all $V \in \tau$ such that $K \cap V = \emptyset$. Alternatively, since $\mathbb{V}(X)$ is sober, K uniquely determines a frame homomorphism $P^K \colon \Omega(\mathbb{V}(X)) \to \mathbf{2}$. Because P^K is the characteristic function of the neighbourhood of K in $\mathbb{V}(X)$, $V \cap K = \emptyset$ iff $P(\oplus V) = 0$. Therefore,

$$X \setminus K = \bigcup \{V \in \tau \mid P(\oplus V) = 0\}.$$

As we will see, we can pretty much follow the same methodology as Johnstone outlined to show that $\mathbb{W}$ and $\mathbb{W}_d$ are also intimately bound together. Namely, for their restrictions in the diagram



we show, in the following subsections, that

$$\mathbb{W} \circ \Sigma_d \cong \Sigma_d \circ \mathbb{W}_d.$$

4.4.1 Remark. $\widetilde{\mathbb{V}}$ can be extended to a functor. For a frame homomorphism $h \colon L \to M$ between two compact regular frames, set $\widetilde{\mathbb{V}}(h) \colon \widetilde{\mathbb{V}}(M) \to \widetilde{\mathbb{V}}(L)$ to be the right adjoint $h_\bullet \colon M \to L$ of h; it is the (localic) map

$$\widetilde{\mathbb{V}}(h) \colon \quad a \in M \quad \longmapsto \quad \bigvee \{x \in L \mid h(x) \leq a\}$$

The intuition for this formula comes again from the spatial interpretation. For a continuous function $f \colon X \to Y$ and a compact $K \subseteq X$, the open set $Y \setminus f[K]$ is equal to

$$\bigcup \{W \in \tau^Y \mid W \cap f[K] = \emptyset\} = \bigcup \{W \mid f[W] \cap K = \emptyset\} = \bigcup \{W \mid f[W] \subseteq X \setminus K\}.$$

4.4.1 Frame-theoretic points of $\mathbb{W}(X)$

Let $X = (X, \tau_+, \tau_-)$ be a d-compact d-regular bispace. We would like to redefine $\mathbb{W}(X)$ such that it is expressed only in terms of open sets and their relationships, i.e. in the language of $\Omega_d(X)$. The crucial step is to be able to express the points this way. Recall that by Proposition 4.1.33 we can characterise the points of $\mathbb{W}(X)$ as the d-compact convex subsets of X.

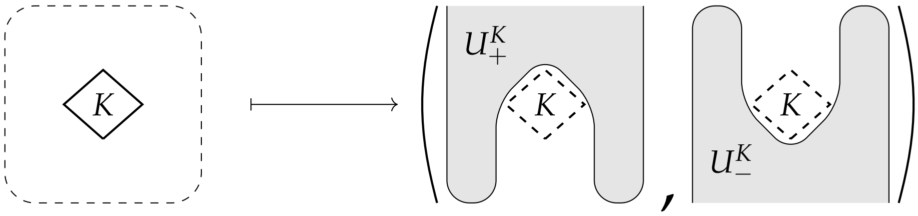


Figure 4.1: Computing $U_\pm^K$'s

As a starting point we reuse the intuition behind the reformulation of the mono-topological Vietoris construction where we used that every compact subset of a compact regular space is identified with an open in its complement. Now, for a d-compact convex subset $K \subseteq X$ we have two such candidates: the largest τ_+-open U_+^K and τ_--open U_-^K which are disjoint with K (see Figure 4.1). Lemma 4.1.29 suggests how to compute them. Because $\downarrow K$ and $\uparrow K$ are τ_+-closed and τ_--closed, respectively,

$$U_+^K = X \setminus \downarrow K \quad \text{and} \quad U_-^K = X \setminus \uparrow K.$$

On the other hand, every pair of opens $(U_+, U_-) \in \tau_+ \times \tau_-$ determines a d-compact convex subset of X in the complement of U_+ and U_-, i.e. set

$$K^U \stackrel{\text{def}}{\equiv} X \setminus (U_+ \cup U_-).$$

However, this mapping is not injective. We might have $(U_+, U_-), (V_+, V_-) \in \tau_+ \times \tau_-$ such that $K^U = K^V$. Moreover, the d-compact convex computed from $(U_+ \cup V_+, U_- \cup V_-)$ is the same as K^U and K^V. In other words, we need to restrict to those $(U_+, U_-) \in \tau_+ \times \tau_-$ which are *"the largest such"*. Ultimately, this will give us that $U_+^{K^U} = U_+$ and $U_-^{K^U} = U_-$.

Assume that (U_+, U_-) does not have the property of being "the largest such", for example $U_+ \subsetneqq U_+^K$ for $K = X \setminus (U_+ \cup U_-)$. By Lemma 4.1.29, for an $x \in U_+^K \setminus U_+$, there exists $(W_+, W_-) \in \mathrm{con}_X$ such that $x \in W_+$ and $K \subseteq W_-$. But, then

$$U_+ \cup U_- \cup W_- = X \quad \text{and} \quad U_+ \cup W_- \neq X \tag{4.4.3}$$

where the inequality holds because neither U_+ nor W_- contain x. Moreover, K is still the complement of the larger pair $(U_+ \cup W_+, U_-)$. This is illustrated in Figure 4.2.

In fact, preventing (4.4.3) from happening gives us exactly "the largest such" pairs with bonus that it is stated in the language of d-frames. For a d-compact d-regular d-frame $\mathcal{L}$, denote by $\mathbb{K}_c(\mathcal{L})$ the set of all pairs $\alpha \in L_+ \times L_-$ such that

(K+) $\quad \forall u_+ \in L_+$: if $(\alpha_+ \vee u_+, \alpha_-) \in \mathrm{tot}$ then $(u_+, \alpha_-) \in \mathrm{tot}$

(K−) $\quad \forall u_- \in L_-$: if $(\alpha_+, \alpha_- \vee u_-) \in \mathrm{tot}$ then $(\alpha_+, u_-) \in \mathrm{tot}$

In the following statements we show that this exactly identifies the points of $\mathbb{W}(X)$.

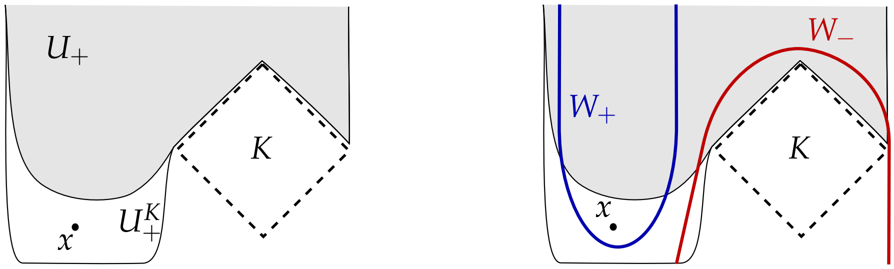


Figure 4.2: (K−) violated

4.4.2 Lemma. *Let X be an order-separated bispace and $U = (U_+, U_-) \in \mathbb{K}_c(\Omega_d(X))$. Then, ${\downarrow} K^U = X \setminus U_+$ and ${\uparrow} K^U = X \setminus U_-$, where $K^U = X \setminus (U_+ \cup U_-)$.*

Proof. Let $x \leq k$ for some $k \in K^U$. Since $k \notin U_+$ and $X \setminus U_+$ is downwards closed, $x \in X \setminus U_+$. For "$\supseteq$", let $x \in X \setminus U_+$. Assume $x \notin {\downarrow} K^U$. By Lemma 4.1.29, there exists a $(V_+, V_-) \in \mathrm{con}_X$ s.t. $x \in V_+$ and $K \subseteq V_-$. Then, $(U_+, V_- \cup U_-) \in \mathrm{tot}_X$ and by (K−) also $(U_+, V_-) \in \mathrm{tot}_X$. But, this yields a contradiction because $x \notin U_+$ and $x \notin V_-$. □

4.4.3 Proposition. *Let X be a d-compact d-regular bispace. Then, the mapping $(U_+, U_-) \mapsto X \setminus (U_+ \cup U_-)$ is a bijection between $\mathbb{K}_c(\Omega_d(X))$ and $\mathcal{K}_c(X)$, i.e. the set of points of $\mathbb{W}X$.*

Proof. Let $U = (U_+, U_-)$ satisfies (K+) and (K−). It is clear that $K^U = X \setminus (U_+ \cup U_-)$ is d-compact and convex. For the way back, let K be a d-compact convex subset of X. Set $U^K = (U_+^K, U_-^K)$ and observe that, for every $V_- \in \tau_+$,

$$U_+^K \cup V_- = X \quad \text{iff} \quad K \subseteq V_-. \tag{$\star$}$$

We will show that U^K satisfies (K−); let $U_+^K \cup U_-^K \cup V_- = X$. Assume, for a contradiction, that $U_+^K \cup V_- \neq X$. From ($\star$) we know that $K \nsubseteq V_-$, therefore there exists an $x \in K \setminus V_-$. Then, $U_+^K \cup U_-^K \cup V_- \neq X$ because none of the sets contains x.

Now, we show that $K = K^{U^K}$: $x \in K^{U^K}$ iff $x \notin U_+^K \cup U_-^K$ iff $x \in {\downarrow} K$ and $x \in {\uparrow} K$ iff $x \in K$. To show $U = U^{K^U}$, observe that $x \in (U^{K^U})_+$ iff $x \in X \setminus {\downarrow} K^U$ iff, by Lemma 4.4.2, $x \in X \setminus (X \setminus U_+)$ iff $x \in U_+$. □

4.4.4 Auxiliary functor $\widetilde{\mathbb{W}}$. For a d-compact d-regular d-frame $\mathcal{L}$, we can now define $\widetilde{\mathbb{W}}(\mathcal{L})$ to be the bispace $(\mathbb{K}_c(\mathcal{L}), \widetilde{\mathbb{V}}(L_+), \widetilde{\mathbb{V}}(L))$ where $\widetilde{\mathbb{V}}(L_\pm)$ are generated by the sets $\boxtimes x_\pm$ and $\diamondplus x_\pm$, for every $x_\pm \in L_+$, where

$$\boxtimes x_+ = \{\alpha \in \mathbb{K}_c(\mathcal{L}) \mid (x_+, \alpha_-) \in \mathrm{tot}\} \quad \text{and} \quad \diamondplus x_+ = \{\alpha \in \mathbb{K}_c(\mathcal{L}) \mid x_+ \nleq \alpha_+\},$$

and, similarly,

$$\boxtimes x_- = \{\alpha \in \mathbb{K}_c(\mathcal{L}) \mid (\alpha_+, x_-) \in \mathrm{tot}\} \quad \text{and} \quad \diamondplus x_- = \{\alpha \in \mathbb{K}_c(\mathcal{L}) \mid x_- \nleq \alpha_-\}.$$

4.4.5 Observation. $\widetilde{\mathbb{W}}(\mathcal{L}) \cong \mathbb{W}(\Sigma_d(\mathcal{L}))$

Proof. From spatiality is $\mathcal{L} \cong \Omega_d(X)$ for some d-compact d-regular bispace X. In Proposition 4.4.3 we established that there is a bijection between $\mathbb{K}_c(\Omega_d(X))$ and $\mathcal{K}_c(X)$, i.e. between the points of $\widetilde{\mathbb{W}}(\Omega_d(X))$ and $\mathbb{W}(X)$. Next we show that this is both-ways continuous. For a $K \in \mathcal{K}_c(X)$ we have

$$K \in \boxtimes U_+ \quad \text{iff} \quad K \subseteq U_+ \quad \text{iff} \quad \uparrow K \subseteq U_+ \quad \text{iff} \quad (U_+, X \setminus \uparrow K) \in \mathrm{tot}_X$$

and, similarly,

$$K \in \oplus U_+ \quad \text{iff} \quad K \cap U_+ \neq \emptyset \quad \text{iff} \quad \downarrow K \cap U_+ \neq \emptyset \quad \text{iff} \quad U_+ \not\subseteq X \setminus \downarrow K. \qquad \square$$

Next, we extend $\widetilde{\mathbb{W}}$ to a functor **d-KReg** $\to$ **biKReg**$^{\mathrm{op}}$. As in Remark 4.4.1, for a d-frame homomorphism $h\colon \mathcal{L} \to \mathcal{M}$, define

$$\widetilde{\mathbb{W}}(h)\colon \quad \alpha \in \mathbb{K}_c(\mathcal{M}) \quad \longmapsto \quad h_\bullet(\alpha) = \bigsqcup\{\, \beta \in L_+ \times L_- \mid h(\beta) \sqsubseteq \alpha\}.$$

4.4.6 Lemma. *$\widetilde{\mathbb{W}}$ is a well-defined functor.*

Proof. All we need to check is well-definedness and continuity of such $\widetilde{\mathbb{W}}(h)$'s. Functoriality follows from the fact that the translation $h \mapsto h_\bullet$ from frame homomorphisms to localic maps is functorial.

Let $\beta = h_\bullet(\alpha)$ for some $\alpha \in \mathbb{K}_c(\mathcal{M})$ and let $u \in L_+$ be such that $(\beta_+ \vee u, \beta_-) \in \mathrm{tot}_{\mathcal{L}}$. Because h is a frame homomorphism and $h \dashv h_\bullet$, $(h_+(\beta_+) \vee h_+(u), h_-(\beta_-)) = (hh_\bullet(\alpha)_+ \vee h_+(u),\ hh_\bullet(\alpha)_-) \in \mathrm{tot}_{\mathcal{M}}$ and, therefore, $(\alpha_+ \vee h_+(u), \alpha_-) \in \mathrm{tot}_{\mathcal{M}}$. By (A+) for α, $(h_+(u), \alpha_-) \in \mathrm{tot}_{\mathcal{M}}$. Next, because $u = \bigvee^{\uparrow}\{x \mid x \lhd u\}$ and $\mathcal{M}$ is d-compact, there is an $x \lhd u$ such that $(h_+(x), \alpha_-) \in \mathrm{tot}_{\mathcal{M}}$. Also, since $(h_+(x), h(x^*)) \in \mathrm{con}_{\mathcal{M}}$, $h(x^*) \leq \alpha_-$ and, by adjointness, $x^* \leq h_\bullet(\alpha)_- = \beta_-$. Therefore, $(u, \beta_+) \in \mathrm{tot}_{\mathcal{L}}$ as $(u, x^*) \in \mathrm{tot}_{\mathcal{L}}$.

To check continuity let $z \in L_+$. We will show that the sets

$$\widetilde{\mathbb{W}}(h)^{-1}[\boxtimes z] = \{\alpha \in \mathbb{K}_c(\mathcal{M}) \mid h_\bullet(\alpha) \in \boxtimes z\} \text{ and}$$
$$\widetilde{\mathbb{W}}(h)^{-1}[\oplus z] = \{\alpha \in \mathbb{K}_c(\mathcal{M}) \mid h_\bullet(\alpha) \in \oplus z\}$$

are equal to $\boxtimes h(z)$ and $\oplus h(z)$, respectively. First, we check the former:

$$\widetilde{\mathbb{W}}(h)^{-1}[\boxtimes z] = \{\alpha \mid (z,\ h_\bullet(\alpha)_-) \in \mathrm{tot}_{\mathcal{L}}\} = \{\alpha \mid \exists x.\ h_-(x) \leq \alpha_- \text{ and } (z, x) \in \mathrm{tot}_{\mathcal{L}}\}$$

where the last equality holds because $\mathcal{L}$ is d-compact and the set $\{x \mid h_-(x) \leq \alpha_-\}$ is directed. Moreover, if x is such that $h_-(x) \leq \alpha_-$ and $(z, x) \in \mathrm{tot}_{\mathcal{L}}$, then also $(h_+(z), \alpha_-) \in \mathrm{tot}_{\mathcal{M}}$ and so $\alpha \in \boxtimes h_+(z)$. Conversely, from regularity and compactness, if $(h_+(z), \alpha_-) \in \mathrm{tot}_{\mathcal{M}}$, then there is some $l \lhd z$ such that $(h_+(l), \alpha_-) \in \mathrm{tot}_{\mathcal{M}}$. With $x \stackrel{\mathrm{def}}{\equiv} l^*$, clearly, $(z, x) \in \mathrm{tot}_{\mathcal{L}}$ and also $h_-(x) \leq \alpha_-$. We have proved that $\widetilde{\mathbb{W}}(h)^{-1}[\boxtimes z] = \{\alpha \mid (h_+(z), \alpha_-) \in \mathrm{tot}_{\mathcal{M}}\} = \boxtimes h(z)$.

Finally, $\widetilde{\mathbb{W}}(h)^{-1}[\oplus z] = \{\alpha \mid z \not\leq h_\bullet(\alpha)_+\} = \{\alpha \mid h_+(z) \not\leq \alpha_+\} = \oplus h(z)$. $\square$

4.4.7 Proposition. *The functors* $\widetilde{\mathbb{W}}$ *and* $\mathbb{W} \circ \Sigma_d$ *are naturally isomorphic.*

Proof. What is left to check is that the bihomeomorphism in Observation 4.4.5 is natural in $\mathcal{L}$. Let $h\colon \mathcal{L} \to \mathcal{M}$ be a d-frame homomorphism. By $f\colon Y \to X$ denote the bicontinuous map such that $\Omega_d(f) = h$. Then, the naturality square that we need to check for commutativity is the following:

$$\begin{array}{ccc} \widetilde{\mathbb{W}}(\Omega_d(Y)) & \xrightarrow{K^{(-)}} & \mathbb{W}(Y) \\ {\scriptstyle \widetilde{\mathbb{W}}(\Omega_d(f))}\downarrow & & \downarrow{\scriptstyle \mathbb{W}(f)} \\ \widetilde{\mathbb{W}}(\Omega_d(X)) & \xrightarrow{K^{(-)}} & \mathbb{W}(X) \end{array}$$

Let $(V_+, V_-) \in \mathbb{K}_c(\Omega_d(Y))$. Set K to be the image of (V_+, V_-) under $K^{(-)}$, that is the set $Y \setminus (V_+ \cup V_-)$ such that $V_+ = Y \setminus {\downarrow}K$ and $V_- = Y \setminus {\uparrow}K$. By definition

$$\widetilde{\mathbb{W}}(\Omega_d(f))_+(V_+) = \bigcup\{U_+ \in \tau_+^X \mid f^{-1}[U_+] \subseteq V_+\}.$$

However, $f^{-1}[U_+] \subseteq V_+ = Y \setminus {\downarrow}K$ iff $f^{-1}[U_+] \cap {\downarrow}K = \emptyset$ and, because $f^{-1}[U_+]$ is upwards closed, it is equivalent to $f^{-1}[U_+] \cap K = \emptyset$ and $U_+ \cap f[K] = \emptyset$. Similarly, $U_+ \cap f[K] = \emptyset$ iff $U_+ \cap {\downarrow}f[K] = \emptyset$ iff $U_+ \subseteq X \setminus {\downarrow}f[K]$. Finally, since ${\downarrow}f[K]$ is τ_+-closed (Lemma 4.1.29), $X \setminus {\downarrow}f[K]$ is τ_+-open and so $\widetilde{\mathbb{W}}(\Omega_d(f))_+(V_+) = X \setminus {\downarrow}f[K]$. Correspondingly, $\widetilde{\mathbb{W}}(\Omega_d(f))_-(V_-) = X \setminus {\uparrow}f[K]$. Finally, $K^{(-)}$ maps

$$(X \setminus {\downarrow}f[K],\ X \setminus {\uparrow}f[K])$$

to $X \setminus ((X \setminus {\downarrow}f[K]) \cup (X \setminus {\uparrow}f[K])) = X \setminus (X \setminus ({\downarrow}f[K] \cap {\uparrow}f[K])) = {\downarrow}f[K] \cap {\uparrow}f[K]$. □

4.4.2 Natural equivalence between $\widetilde{\mathbb{W}}$ and $\Sigma_d \circ \mathbb{W}_d$

Next, we would like to establish a bijection between the points of the bispaces $\widetilde{\mathbb{W}}(\mathcal{L})$ and $\Sigma_d(\mathbb{W}_d(\mathcal{L}))$, for a fixed d-compact d-regular d-frame $\mathcal{L}$.

In Lemma 4.4.9 we will show how to construct an element of $\mathbb{K}_c(\mathcal{L})$ from a d-frame homomorphism $P\colon \mathbb{W}_d(\mathcal{L}) \to \mathbf{2}{\times}\mathbf{2}$ but, before we do that, we prove the following auxiliary lemma about $\eta\colon \mathbb{W}_d(\mathcal{L}) \to \mathcal{L}$ from Proposition 4.3.1:

4.4.8 Lemma.

1. *Let* $\kappa \in \mathbb{W}_d(\mathcal{L})$*, then* $(\Diamond\eta_+(\kappa_+), \Diamond\eta_-(\kappa_-)) \sqsubseteq \kappa$.
2. $\eta[\mathbb{K}_c(\mathbb{W}_d(\mathcal{L}))] \subseteq \mathbb{K}_c(\mathcal{L})$.

Proof. (1) Recall from Section 4.2.1 that $\kappa_\pm$ is a join of the form $\bigvee_k(\Box a_k \wedge \bigwedge_{i=1}^{n_k} \Diamond b_{k,i})$. Since $\eta_\pm$ is a frame homomorphism and $\Diamond$ distributes over all joins,

$$\Diamond\eta_\pm(\kappa_\pm) = \Diamond(\bigvee_k (a_k \wedge \bigwedge_{i=1}^{n_k} b_{k,i})) = \bigvee_k \Diamond(a_k \wedge \bigwedge_{i=1}^{n_k} b_{k,i}).$$

Moreover, for every k, $\Diamond(a_k \wedge \bigwedge_{i=1}^{n_k} b_{k,i}) \leq \Box a_k \wedge \Diamond(\bigwedge_{i=1}^{n_k} b_{k,i}) \leq \Box a_k \wedge \bigwedge_{i=1}^{n_k} \Diamond b_{k,i}$. Hence, $\Diamond\eta_\pm(\kappa_\pm) \leq \kappa$.

(2) Let $\kappa \in \mathbb{K}_c(\mathbb{W}_d(\mathcal{L}))$, we show that $\eta(\kappa)$ satisfies (K+) and (K−). Let $u \in L_+$ be such that $(\eta_+(\kappa_+) \vee u, \eta_-(\kappa_-)) \in \mathrm{tot}_{\mathcal{L}}$. Then, $(\Box(\eta_+(\kappa_+) \vee u), \Diamond\eta_-(\kappa_-)) \in \mathrm{tot}_{\mathbb{W}_d(\mathcal{L})}$ and thanks to $\Box(\eta_+(\kappa_+) \vee u) \leq \Diamond\eta_+(\kappa_+) \vee \Box u$ and (1), also $(\kappa_+ \vee \Box u, \kappa_-) \in \mathrm{tot}_{\mathbb{W}_d(\mathcal{L})}$. Finally, by (K+) for κ, $(\Box u, \kappa_-) \in \mathrm{tot}_{\mathbb{W}_d(\mathcal{L})}$ and, because η is a homomorphism, $(u, \eta_-(\kappa_-)) \in \mathrm{tot}_{\mathcal{L}}$. □

4.4.9 Lemma. *Let $\mathcal{L}$ be a d-compact d-regular d-frame, $P\colon \mathbb{W}_d\mathcal{L} \to \mathbf{2}\times\mathbf{2}$ be a d-frame homomorphism and define α^P as $\bigsqcup\{\gamma \in L_+ \times L_- \mid P(\Diamond\gamma_+, \Diamond\gamma_-) = \bot\}$. Then,*

1. *$\alpha^P \in \mathbb{K}_c(\mathcal{L})$.*
2. *$P_\pm(\Diamond\alpha^P_\pm) = 0$, and*

Proof. It is immediate to check that $\mathbb{K}_c(\mathbf{2}\times\mathbf{2}) = \{\bot, \top\}$. We will show that $\alpha^P = \eta(P_\bullet(\bot))$ from which (1) follows by Lemmas 4.4.8 and 4.4.6. Because $P_\bullet(\bot) = \bigsqcup\{\delta \mid P(\delta) = \bot\}$ we immediately get that $\alpha^P \sqsubseteq \eta(P_\bullet(\bot))$. For the other direction, let $\delta \in \mathbb{W}_d(\mathcal{L})$ be such that $P(\delta) = \bot$. Because $P(\Diamond\eta_+(\delta_+), \Diamond\eta_-(\delta_-)) \sqsubseteq P(\delta) = \bot$, $\eta(\delta) \sqsubseteq \alpha^P$. Consequently,

$$\eta(P_\bullet(\bot)) = \eta(\bigsqcup\{\delta \mid P(\delta) = \bot\}) = \bigsqcup\{\eta(\delta) \mid P(\delta) = \bot\} \sqsubseteq \alpha^P.$$

To show (2) we use Lemma 4.4.8: $(\Diamond\alpha^P_+, \Diamond\alpha^P_-) = (\Diamond\eta(P_\bullet(\bot))_+, \Diamond\eta(P_\bullet(\bot))_-) \sqsubseteq P_\bullet(\bot)$. Then, from adjointness, $P(\Diamond\alpha^P_+, \Diamond\alpha^P_-) \sqsubseteq \bot$. □

In fact, Lemma 4.4.9 can be proved directly without any reference to $\mathbb{K}_c(\mathbb{W}_d(\mathcal{L}))$ but then, for the first item, one needs to use a trick similar to the one we used in the proof of Lemma 4.4.6.

In the following, we use a similar formula as we had in (4.4.2) to give us a mapping from $\widetilde{\mathbb{W}}(\mathcal{L})$ to $\Sigma_d(\mathbb{W}_d(\mathcal{L}))$.

4.4.10 Lemma. *Let $\mathcal{L}$ is a d-compact d-regular d-frame and let $\alpha \in \mathbb{K}_c(\mathcal{L})$. Then, $P^\alpha\colon \mathbb{W}_d(\mathcal{L}) \to \mathbf{2}\times\mathbf{2}$ is a d-frame homomorphism, where P^α is defined on generators as follows:*

$$\begin{aligned} \forall x \in L_+ :\quad & P^\alpha_+(\Box x) = 1 \text{ iff } (x, \alpha_-) \in \mathrm{tot}, && \text{and} && P^\alpha_+(\Diamond x) = 1 \text{ iff } x \not\leq \alpha_+, \\ \forall x \in L_- :\quad & P^\alpha_-(\Box x) = 1 \text{ iff } (\alpha_+, x) \in \mathrm{tot}, && \text{and} && P^\alpha_-(\Diamond x) = 1 \text{ iff } x \not\leq \alpha_-. \end{aligned}$$

Proof. We need to prove that P^α defined this way is well-defined, i.e. that it preserves the defining equations of $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$, con_1 and tot_1 and, therefore, can be uniquely extended to a d-frame homomorphism $\mathbb{W}_d(\mathcal{L}) \to \mathbf{2}\times\mathbf{2}$ by Theorem 3.4.8. First, we check the equations of $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$:

1. $P_+^\alpha(\Box x \wedge \Box y) = P_+^\alpha(\Box(x \wedge y))$ is equal to 1 iff $P_+^\alpha(\Box x) = 1$ and $P_+^\alpha(\Box y) = 1$ because $(x \wedge y, \alpha_-) \in \mathsf{tot}$ iff $(x, \alpha_-) \in \mathsf{tot}$ and $(y, \alpha_-) \in \mathsf{tot}$. The latter follows from the tot being upwards and $\curlywedge$-closed.

2. $P_+^\alpha(\Box 1_{L_+}) = 1_{\mathbf{2}} = P_+^\alpha(1_{\mathbb{V}_{\mathrm{Fr}} L_+})$ because $(1, \alpha_-)$ is always in tot and, correspondingly, $P_+^\alpha(\Diamond 0) = 0 = P_+^\alpha(0)$.

3. For a non-empty directed subset $\{x_i\}_i$ of L_+,

$$\begin{aligned} \bigvee\nolimits_i^{\uparrow} P_+^\alpha(\Box x_i) = 1 \quad &\text{iff} \quad \exists i.\, (x_i, \alpha_-) \in \mathsf{tot} \\ &\overset{(\star)}{\text{iff}} \quad (\bigvee\nolimits_i^{\uparrow} x_i, \alpha_-) \in \mathsf{tot} \quad \text{iff} \quad P_+^\alpha(\Box(\bigvee\nolimits_i^{\uparrow} x_i)) = 1. \end{aligned}$$

   Where the right-left implication of $(\star)$ follows from compactness of $\mathcal{L}$.

4. For any non-empty subset $\{x_i\}_i$ of L_+, because $\exists i.\, x_i \not\leq \alpha_+$ is equivalent to the negation of $\forall i.\, x_i \leq \alpha_+$,

$$\bigvee\nolimits_i P_+^\alpha(\Diamond x_i) = 1 \quad \text{iff} \quad \exists i.\, x_i \not\leq \alpha_+ \quad \text{iff} \quad \bigvee\nolimits_i x_i \not\leq \alpha_+ \quad \text{iff} \quad P_+^\alpha(\Diamond(\bigvee\nolimits_i x_i)) = 1.$$

5. To see why $P_+^\alpha(\Box x \wedge \Diamond y) \leq P_+^\alpha(\Diamond(x \wedge y))$, assume $P_+^\alpha(\Diamond(x \wedge y)) = 0$ (i.e. $x \wedge y \leq \alpha_+$) and $P_+^\alpha(\Box x) = 1$ (i.e. $(x, \alpha_-) \in \mathsf{tot}$). Then, for every $z \lhd y$, since $(y, z^*) \in \mathsf{tot}$ also $(x \wedge y, \alpha_- \vee z^*) \in \mathsf{tot}$. Therefore, also $(\alpha_+, \alpha_- \vee z^*) \in \mathsf{tot}$ and, by (K−), also $(\alpha_+, z^*) \in \mathsf{tot}$. But, this together with (con-tot) implies that $z \leq \alpha_+$ and so $y = \bigvee\{z \mid z \lhd y\} \leq \alpha_+$ and $P_+^\alpha(\Diamond y) = 0$.

6. For $P_+^\alpha(\Box(x \vee y)) \leq P_+^\alpha(\Box x \vee \Diamond y)$, let $(x \vee y, \alpha_-) \in \mathsf{tot}$. If $y \leq \alpha_+$ (i.e. $P_+^\alpha(\Diamond y) = 0$) then $(x \vee \alpha_+, \alpha_-) \in \mathsf{tot}$. From (K+) then follows that $(x, \alpha_-) \in \mathsf{tot}$ and also $P_+^\alpha(\Box x) = 1$.

Finally, we check that P^α preserves con_1 and tot_1. Let $(\Box x, \Diamond y) \in \mathsf{con}_{\mathbb{W}_d \mathcal{L}}$ for some $(x, y) \in \mathsf{con}$. If $(x, \alpha_-) \in \mathsf{tot}$ (i.e. $P_+^\alpha(\Box x) = 1$) then from (con-tot) for $\mathcal{L}$ we know that $y \leq \alpha_-$, therefore $P_-^\alpha(\Diamond y) = 0$ and $P^\alpha(\Box x, \Diamond y) = (1, 0) \in \mathsf{con}_{\mathbf{2}\times\mathbf{2}}$. Similarly, if $(x, y) \in \mathsf{tot}$ and so $(\Box x, \Diamond y) \in \mathsf{tot}_{\mathbb{W}_d \mathcal{L}}$, then $y \leq \alpha_-$ implies that $(x, \alpha_-) \in \mathsf{tot}$ as tot is upwards closed and so $P^\alpha(\Box x, \Diamond y) = (1, 0) \in \mathsf{tot}_{\mathbf{2}\times\mathbf{2}}$. □

Notice that each of the assumptions of Lemma 4.4.10 was needed precisely once in its proof. Compactness was used only in (3), regularity in (5) and the fact that $\alpha \in \mathbb{K}_c(\mathcal{L})$ was needed in (6).

4.4.11 Lemma. *Let $\mathcal{L}$ be a d-compact d-regular d-frame and $P\colon \mathbb{W}_d \mathcal{L} \to \mathbf{2}\times\mathbf{2}$ a d-frame homomorphism. Then,*

1. $P_+(\Diamond x) = 1$ *iff* $x \not\leq \alpha_+^P$,

2. $P_+(\Box x) = 1$ *iff* $(x, \alpha_-^P) \in \mathsf{tot}$

and similarly for P_-.

Proof. (1) If $P_+(\Diamond x) = 1$, then it has to be that $x \not\leq \alpha_+^P$ because otherwise $\Diamond x \leq \Diamond\alpha_+^P$ which gives a contradiction with $P_+(\Diamond\alpha_+^P) = 0$ (Lemma 4.4.9).

(2) If $P_+(\Box x) = 1$ then $P_+(\Box y) = 1$ already for some $y \lhd_+ x$ because $x = \bigvee^{\uparrow}\{y \in L_+ \mid y \lhd x\}$ and $\Box$ distributes over directed joins. Because $(y, y^*) \in \mathrm{con}$, we know that $P_-(\Diamond(y^*)) = 0$ and so, by 1., $y^* \leq \alpha_-^P$. Since $(x, y^*) \in \mathrm{tot}$ and tot is upwards closed, $(x, \alpha_-^P) \in \mathrm{tot}$. On the other hand, if $(x, \alpha_-^P) \in \mathrm{tot}$ then $(P_+(\Box x), P_-(\Diamond\alpha_-)) \in \mathrm{tot}_{\mathbf{2}\times\mathbf{2}}$ because P is tot-preserving and, since $P_-(\Diamond\alpha_-^P) = 0$, necessarily $P_+(\Box x) = 1$. □

4.4.12 Proposition. *Let $\mathcal{L}$ be a d-compact d-regular d-frame. Then, the map $P \mapsto \alpha^P$ is a bijection between the set of points of $\Sigma\mathbb{W}_d\mathcal{L}$ and $\mathbb{K}_c(\mathcal{L})$.*

Proof. We need to show that $\alpha = \alpha^{P^\alpha}$ and that $P = P^{\alpha^P}$. For the first one, let $\alpha \in \mathbb{K}_c(\mathcal{L})$ and compute: $\alpha_\pm^{P^\alpha} = \bigvee\{x \in L_\pm \mid P_\pm^\alpha(\Diamond x) = 0\} = \bigvee\{x \in L_\pm \mid x \leq \alpha_\pm\} = \alpha_\pm$. For the latter, take a d-frame homomorphism $P\colon \mathbb{W}_d L \to \mathbf{2}{\times}\mathbf{2}$. By Lemma 4.4.11, P^{α^P} and P agree on generators and, therefore, must be equal. □

4.4.13 Proposition. *The functors $\widetilde{\mathbb{W}}$ and $\Sigma_d \circ \mathbb{W}_d$ are naturally isomorphic.*

Proof. We need to show that the bijection from Proposition 4.4.12 is a bihomeomorphism and that it is natural in $\mathcal{L}$. Bicontinuity follows from Lemma 4.4.11 as, for any $x \in L_+$ and homomorphism $P\colon \mathbb{W}_d(\mathcal{L}) \to \mathbf{2}{\times}\mathbf{2}$,

$$\begin{array}{llllllll} P \in \Sigma_+(\Box x) & \text{iff} & P(\Box x) = 1 & \text{iff} & (x, \alpha_-^P) \in \mathrm{tot}_{\mathcal{L}} & \text{iff} & \alpha^P \in \boxtimes x, & \text{and} \\ P \in \Sigma_+(\Diamond x) & \text{iff} & P(\Diamond x) = 1 & \text{iff} & x \not\leq \alpha_-^P \in \mathrm{tot}_{\mathcal{L}} & \text{iff} & \alpha^P \in \diamondplus x. & \end{array}$$

To check naturality, let $h\colon \mathcal{L} \to \mathcal{M}$ be a d-frame homomorphism. We have the diagram

$$\begin{array}{ccc} \Sigma_d(\mathbb{W}_d\mathcal{M}) & \xrightarrow{\alpha^{(-)}} & \widetilde{\mathbb{W}}(\mathcal{M}) \\ {\scriptstyle \Sigma_d(\mathbb{W}_d h)}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\mathbb{W}}(h)} \\ \Sigma_d(\mathbb{W}_d\mathcal{L}) & \xrightarrow{\alpha^{(-)}} & \widetilde{\mathbb{W}}(\mathcal{L}) \end{array}$$

For a homomorphism $P\colon \mathbb{W}_d(\mathcal{M}) \to \mathbf{2}{\times}\mathbf{2}$, compute

$$\alpha^{(-)} \cdot \Sigma_d(\mathbb{W}_d h)(P) = \alpha^{(-)}(P \circ \mathbb{W}_d h) = \alpha^{P \circ \mathbb{W}_d h} = \bigsqcup\{\beta \mid (P \circ \mathbb{W}_d h)(\Diamond\beta_+, \Diamond\beta_-) = \bot\}$$

and, because $(P \circ \mathbb{W}_d h)(\Diamond\beta_+, \Diamond\beta_-) = P(\Diamond h_+(\beta_+), \Diamond h_-(\beta_-))$, we see that $\alpha^{(-)} \cdot \Sigma_d(\mathbb{W}_d h)(P)$ is equal to $\widetilde{\mathbb{W}}(\alpha^P) = h_\bullet(\alpha^P) = \bigsqcup\{\beta \mid h(\beta) \sqsubseteq \alpha^P\}$ (recall that $h(\beta) \sqsubseteq \alpha^P$ iff $P_\pm(\Diamond\beta_\pm) = 0$). □

The composition of the isomorphisms in Propositions 4.4.7 and 4.4.13 proves the anticipated result:

4.4.14 Theorem.
The functors $\mathbb{W} \circ \Sigma_d$ and $\Sigma_d \circ \mathbb{W}_d$ are naturally isomorphic.

4.4.15 Remark. The bihomeomorphisms from the natural isomorphism in Theorem 4.4.14 can be written explicitly in a single formula. The bihomeomorphism between $\widetilde{\mathbb{W}}(\mathcal{L})$ and $\mathbb{W}(\Sigma_d(\mathcal{L}))$ is witnessed by the map $(U_+, U_-) \mapsto (\Sigma_d\mathcal{L}) \setminus (U_+ \cup U_-)$. Because $\Omega_d(\Sigma_d(\mathcal{L})) \cong \mathcal{L}$, every $\tau_\pm$-open in $\Sigma_d\mathcal{L}$ (e.g. $U_\pm$) is of the form $\Sigma_\pm(a) = \{p\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \mid p_\pm(a) = 1\}$. Therefore, the formula above reduces to

$$(a_+, a_-) \mapsto \{p\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \mid p \notin \Sigma_+(a_+) \text{ and } p \notin \Sigma_-(a_-)\} = \{p \mid p(a_+, a_-) = \bot\}.$$

Next, we pre-compose this with the bihomeomorphism $\Sigma_d\mathbb{W}_d(\mathcal{L}) \xrightarrow{\cong} \widetilde{\mathbb{W}}(\mathcal{L})$ (Proposition 4.4.12) and obtain

$$P \mapsto \{p\colon \mathcal{L} \to \mathbf{2}{\times}\mathbf{2} \mid p(\alpha^P) = \bot\}$$

as the bihomeomorphism $\Sigma_d\mathbb{W}_d(\mathcal{L}) \xrightarrow{\cong} \mathbb{W}\Sigma_d(\mathcal{L})$.

4.4.16 An adaptation to $\mathbb{W}_\square$ and $\mathbb{W}_\diamond$. Since the points of $\mathbb{W}_\boxtimes(X)$ for a d-compact d-regular bispace X are in a bijection with τ_--closed subsets of X (Proposition 4.1.33), $\mathbb{W}_\boxtimes(\Sigma_d(\mathcal{L}))$, for a d-compact d-regular d-frame $\mathcal{L}$, has an immediate point-free description. Namely, it is bihomeomorphic to the bispace $(L_-, \widetilde{\mathbb{V}}_\square(L_+), \widetilde{\mathbb{V}}_\diamond(L_-))$ where $\widetilde{\mathbb{V}}_\square(L_+)$ is generated by $\boxtimes x$'s and $\widetilde{\mathbb{V}}_\diamond(L_-)$ by $\diamondplus y$'s, for $x \in L_+$ and $y \in L_-$, such that

$$\boxtimes x = \{a \in L_- \mid (x, a) \in \mathsf{tot}\} \quad \text{and} \quad \diamondplus y = \{a \in L_- \mid y \not\leq a\}.$$

Furthermore, $\Sigma_d(\mathbb{W}_\square(\mathcal{L}))$ is also bihomeomorphic to this bispace via the bihomeomorphism

$$P\colon \mathbb{W}_\square(\mathcal{L}) \to \mathbf{2}{\times}\mathbf{2} \quad\longmapsto\quad a^P \overset{\text{def}}{\equiv} \bigvee\{b \mid P_-(\diamond b) = 0\}.$$

Consequently, $\mathbb{W}_\boxtimes(\Sigma_d(\mathcal{L})) \cong \Sigma_d(\mathbb{W}_\square(\mathcal{L}))$ and, correspondingly, also $\mathbb{W}_\diamondplus(\Sigma_d(\mathcal{L})) \cong \Sigma_d(\mathbb{W}_\diamond(\mathcal{L}))$.

Moreover, by a general diagram chasing argument (Proposition A.3.16):

4.4.17 Corollary.

1. $\mathsf{Coalg}(\mathbb{W}_\boxtimes)^{op} \cong \mathsf{Alg}(\mathbb{W}_\square)$,
2. $\mathsf{Coalg}(\mathbb{W}_\diamondplus)^{op} \cong \mathsf{Alg}(\mathbb{W}_\diamond)$ *and*
3. $\mathsf{Coalg}(\mathbb{W})^{op} \cong \mathsf{Alg}(\mathbb{W}_d)$.

(The base category for coalgebras and algebras is the category of d-compact d-regular bispaces and d-frames, respectively.)

4.4.18 Theorem.
For every $\mathcal{W} \in \{\mathbb{W}_{\boxtimes}, \mathbb{W}_{\oplus}, \mathbb{W}\}$, $\mathcal{W}$ restricted to the category of d-compact d-regular (T_0) bispaces has a final coalgebra.

4.5 $\mathbb{W}$, $\mathbb{W}_d$ and other Vietoris-like functors

In Sections 2.5, 2.6 and 2.7 we showed that three famous dualities embed into the duality of d-compact d-regular bispaces and d-frames. Concretely, we have commutative diagrams of categories

$$\begin{array}{ccc} \mathcal{X}^{\mathrm{op}} & \xleftrightarrow{\;\cong\;} & \mathcal{A} \\ {\scriptstyle I^{\mathrm{op}}}\big\downarrow & & \big\downarrow{\scriptstyle J} \\ \mathbf{biKReg}^{\mathrm{op}} & \xleftrightarrow{\;\cong\;} & \mathbf{d\text{-}KReg} \end{array} \tag{4.5.1}$$

where $\mathcal{X}^{\mathrm{op}} \cong \mathcal{A}$ is either Stone duality, duality of compact regular spaces and frames, or Priestley duality.

In all of those instances, we also have a pair of Vietoris(-like) functors $V \colon \mathcal{X} \to \mathcal{X}$ and $M \colon \mathcal{A} \to \mathcal{A}$ such that $M \circ Q \cong Q \circ V$ (or, equivalently, $S \circ M \cong V \circ S$) where $Q \colon \mathcal{X} \rightleftarrows \mathcal{A} : S$ are the (contravariant) functors witnessing the duality $\mathcal{X}^{\mathrm{op}} \cong \mathcal{A}$. Concrete instantiations of the functors V, M, Q and S with respect to $\mathcal{X}$ and $\mathcal{A}$ are summarised in the following table:

$\mathcal{X}$	$\mathcal{A}$	V	M	Q	S	$M \circ Q \cong Q \circ V$
Stone	**Bool**	$\mathbb{V}$	$\mathbb{M}$	Clp	Ult	Fact 4.3.10 + $\mathbb{V}_{\mathrm{Fr}} \circ \Omega \cong \Omega \circ \mathbb{V}$
KRegSp	**KRegFrm**	$\mathbb{V}$	$\mathbb{V}_{\mathrm{Fr}}$	Ω	Σ	Proposition 4.6 in [Joh82]
Pries	**DLat**	$\mathbb{V}_{\mathrm{P}}$	$\mathbb{M}$	$\mathrm{Clp}_{\preceq}$	$\mathrm{spec}_{\preceq}$	[Pri70]
biKReg	**d-KReg**	$\mathbb{W}$	$\mathbb{W}_d$	Ω_d	Σ_d	Theorem 4.4.14

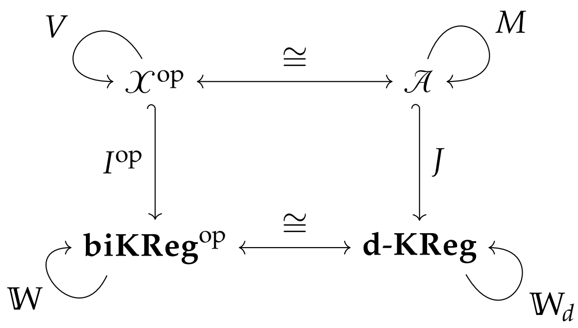
Table 4.1

The only previously not mentioned functor in Table 4.1 is the Vietoris functor for Priestley spaces $\mathbb{V}_{\mathrm{P}} \colon$ **Pries** $\to$ **Pries** [Pal04; Pal03]

$$\mathbb{V}_{\mathrm{P}} \colon \quad (X, \tau, \preceq) \quad \longmapsto \quad (\mathcal{K}_c(X), \mathbb{V}(\tau), \preceq^{\mathrm{EM}})$$

Frederik Lauridsen proved in his thesis [Lau15, Proposition 10 and 11] that $\mathbf{bi} \circ \mathbb{V}_{\mathrm{P}} \cong \mathbb{W} \circ \mathbf{bi}$. Moreover, in Section 4.3.4 we also showed that $\mathbb{W}_d \circ \mathcal{IF} \cong \mathcal{IF} \circ \mathbb{M}$.

In fact, similar equations hold for the other constructions as well. That is, for the diagram (4.5.1) extended to



we have that $I \circ V \cong \mathbb{W} \circ I$ and $J \circ M \cong \mathbb{W}_d \circ J$ for all instantiations mentioned in Table 4.1. We will show that this is indeed the case for the remaining two cases, for the Stone and frame dualities, in Propositions 4.5.1 and 4.5.3. Moreover, we only prove the latter equation as the former one follows from it by a simple diagram chasing:

$$IV \cong ISMQ \cong \Sigma_d JMQ \cong \Sigma_d \mathbb{W}_d JQ \cong \Sigma_d \mathbb{W}_d \Omega_d I \cong \mathbb{W} I$$

This justifies that we can think of $\mathbb{W}$ and $\mathbb{W}_d$ as *generalisations* of all the other Vietoris constructions mentioned in Table 4.1.

Now, we show that the embedding $J\colon \mathbf{KRegFrm} \hookrightarrow \mathbf{d\text{-}KReg}$, dubbed $(-)^{\bowtie}$ from Section 2.5, transforms $\mathbb{V}_{\mathrm{Fr}}$ into $\mathbb{W}_d$:

4.5.1 Proposition. *Let L be a compact regular frame. Then, $\mathbb{W}_d(L^{\bowtie}) = (\mathbb{V}_{\mathrm{Fr}} L)^{\bowtie}$.*

Proof. We see from the definition that the frame components of both d-frames are identical. We will show that also their consistency and totality relations are the same. Let $a \wedge b = 0$ or, in other words, $(a, b) \in \mathsf{con}_L$. Then, the generator $(\Box a, \Diamond b)$ of $\mathsf{con}_{\mathbb{W}_d(L^{\bowtie})}$ is also an element of $\mathsf{con}_{\mathbb{V}_{\mathrm{Fr}} L}$ as $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b) = \Diamond 0 = 0$. Similarly, whenever $a \vee b = 1$, the generator $(\Box a, \Diamond b)$ of $\mathsf{tot}_{\mathbb{W}_d(L^{\bowtie})}$ is also in $\mathsf{tot}_{\mathbb{V}_{\mathrm{Fr}} L}$ as $\Box a \vee \Diamond b \geq \Box(a \vee b) = \Box 1 = 1$.

For the other inclusions, first, let $(U, V) \in \mathsf{con}_{\mathbb{V}_{\mathrm{Fr}} L}$. In Remark 3.2.9 we understood that this is equivalent to the statement:

$$\forall \alpha \in {\downarrow} U \cap B \text{ and } \forall \beta \in {\downarrow} V \cap B.\ (\alpha, \beta) \in \mathsf{con}_{\mathbb{V}_{\mathrm{Fr}} L}$$

where $B \subseteq \mathbb{V}_{\mathrm{Fr}} L$ is the basis of $\mathbb{V}_{\mathrm{Fr}} L$ from Section 4.2.1 (and so $U = \bigvee({\downarrow} U \cap B)$ and $V = \bigvee({\downarrow} V \cap B)$). Namely, such α is of the form $\Box a \wedge \bigwedge_{i=1}^{n} \Diamond b_i$ and β is of the form $\Box c \wedge \bigwedge_{j=1}^{m} \Diamond d_j$ for some $a, b_1, \dots, b_n, c, d_1, \dots, d_m$ from L. And, $(\alpha, \beta) \in \mathsf{con}_{\mathbb{V}_{\mathrm{Fr}} L}$ translates as $\alpha \wedge \beta \leq 0 = \Diamond 0$. By Proposition 4.2.6, this must be because either $a \wedge c \wedge b_i \leq 0$, for some i, or $a \wedge c \wedge d_j \leq 0$, for some j. W.l.o.g. assume the former. Then, $(\Diamond(a \wedge b), \Box c) \in \mathsf{con}_{\mathbb{W}_d(L^{\bowtie})}$ and the same is true for (α, β) since $\alpha \leq \Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ and $\beta \leq \Box c$.

Finally, let $(U, V) \in \mathrm{tot}_{\mathbb{V}_{\mathrm{Fr}} L}$. Since $\mathbb{V}_{\mathrm{Fr}} L$ is compact ([Joh82]), there is a finite $F \subseteq_{\mathrm{fin}} B$ and $G \subseteq_{\mathrm{fin}} B$ such that $(\bigvee F, \bigvee G) \sqsubseteq (U, V)$ and $(\bigvee F, \bigvee G) \in \mathrm{tot}_{\mathbb{V}_{\mathrm{Fr}} L}$ (we again use that $U = \bigvee(\downarrow U \cap B)$ and $V = \bigvee(\downarrow V \cap B)$). Next, we express $\bigvee F$ and $\bigvee G$ as a combination of generators of $\mathbb{V}_{\mathrm{Fr}} L$ in their conjunctive-normal form, i.e.

$$\bigvee F = \bigwedge_{i=1}^{n} \alpha_i \quad \text{for some} \quad \alpha_i = \Diamond a_i \vee \bigvee_{j=1}^{m_i} \Box b_{i,j} \quad \text{and}$$

$$\bigvee G = \bigwedge_{p=1}^{n'} \beta_p \quad \text{for some} \quad \beta_p = \Diamond c_p \vee \bigvee_{q=1}^{m'_p} \Box d_{p,q}.$$

Since $\tilde{\Box} 1 = 1_{\mathbb{V}_{\mathrm{Fr}} L} \le (\bigwedge_i \alpha_i) \vee (\bigwedge_p \beta_p) = \bigwedge_{i,p} (\alpha_i \vee \beta_p) \le \alpha_i \vee \beta_p$ $(\forall i, p)$, by Proposition 4.2.6, we have, for every i and p, either a j such that $1 \le a_i \vee c_p \vee b_{i,j}$ or a q such that $1 \le a_i \vee c_p \vee d_{p,q}$. W.l.o.g. assume the former. Then, because $(a_i \vee b_{i,j}, c_p) \in \mathrm{tot}_{L^{\bowtie}}$, $(\Box(a_i \vee b_{i,j}), \Diamond c_p) \in \mathrm{tot}_{\mathbb{W}_d(L^{\bowtie})}$ and, consequently, also $(\alpha_i, \beta_p) \in \mathrm{tot}_{\mathbb{W}_d(L^{\bowtie})}$ as $\Box(a_i \vee b_{i,j}) \le \Box(a_i \vee b_{i,j}) \le \alpha_i$ and $\Diamond c_p \le \beta_p$.

We have proved that, for all i and p, (α_i, β_p) is total in $\mathbb{W}_d(L^{\bowtie})$. From this it follows that also $(\bigvee F, \bigvee G)$ is since, by (tot-$\curlywedge$), $(\bigwedge_i \alpha_i, \beta_p)$ is total, for all p's, and then, by (tot-$\curlyvee$), also $(\bigwedge_i \alpha_i, \bigwedge_p \beta_p) \in \mathrm{tot}_{\mathbb{W}_d(L^{\bowtie})}$. Therefore, $(U, V) \in \mathrm{tot}_{\mathbb{W}_d(L^{\bowtie})}$ as the totality relation of $\mathbb{W}_d(L^{\bowtie})$ is upwards closed. □

4.5.2 Remark. This time, there is no hope for the same to be true for the other two Vietoris constructions. Namely, we will not get that $(\mathbb{V}_\Box L)^{\bowtie} \cong \mathbb{W}_\Box(L^{\bowtie})$ or $(\mathbb{V}_\Diamond L)^{\bowtie} \cong \mathbb{W}_\Diamond(L^{\bowtie})$ as already their frame components differ. However, we will see in Chapter 5 that $(\mathbb{V}_\Box L)^{\mathrm{bi}} \cong \mathbb{W}_\Box(L^\wedge)$ and $(\mathbb{V}_\Diamond L)^{\mathrm{bi}} \cong \mathbb{W}_\Diamond(L^\wedge)$ where $(-)^{\mathrm{bi}}$ and $(-)^\wedge$ are also defined therein.

Next, we prove that the embedding J: **Bool** $\hookrightarrow$ **d-KReg** also transforms one Vietoris construction into the other. Moreover, since J is equivalently expressible as $\mathcal{IF}$ and $\mathsf{Idl}(-)^{\bowtie}$ (Section 2.7) we show that in both cases we get the same result.

4.5.3 Proposition. *Let B be a Boolean algebra. Then,*

1. $\mathbb{M}(B)$ *is a Boolean algebra which can be equivalently presented as*

$$\mathbf{BA}\left\langle \tilde{\Box} a : a \in B \;\middle|\; \tilde{\Box}(a \wedge b) = \tilde{\Box} a \wedge \tilde{\Box} b,\ \tilde{\Box} 1 = 1 \right\rangle$$

2. $\mathcal{IF}(\mathbb{M}B) \cong \mathbb{W}_d(\mathcal{IF}(B))$ *and* $\mathsf{Idl}(\mathbb{M}B)^{\bowtie} \cong \mathbb{W}_d(\mathsf{Idl}(B)^{\bowtie})$.

Proof. (1) $\tilde{\Box} a \wedge \tilde{\Diamond}(\neg a) \le \tilde{\Diamond}(a \wedge \neg a) = \tilde{\Diamond} 0 = 0$ and $\tilde{\Box} a \vee \tilde{\Diamond}(\neg a)) \ge \tilde{\Box}(a \vee \neg a) = \tilde{\Box} 1 = 1$. We proved that each generator $\tilde{\Box} a$ of $\mathbb{M}(B)$ has a complement and a similar argument would show the same for $\tilde{\Diamond} a$'s. Since each element of $\mathbb{M}(B)$ is a $(\wedge, \vee)$-combination of $\tilde{\Box} a$'s and $\tilde{\Diamond} a$'s, $\mathbb{M}(B)$ is a Boolean algebra. Moreover, $\tilde{\Diamond} a = \neg \tilde{\Box}(\neg a)$ giving that generators $\tilde{\Diamond} a$ are expressible from $\tilde{\Box} a$'s. For details see [Abr05b].

(2) The first natural equivalence is the same as for distributive lattices and the second follows from Fact 4.3.10 and Proposition 4.5.1: $\mathsf{Idl}(\mathbb{M}B)^{\bowtie} \cong (\mathbb{V}_{\mathrm{Fr}}\,\mathsf{Idl}(B))^{\bowtie} \cong \mathbb{W}_d(\mathsf{Idl}(B)^{\bowtie})$. □

For the sake of completeness we also give a direct proof of Lauridsen's result but in a bit more general setting:

4.5.4 Proposition. *For every compact partially ordered space $X = (X, \tau, \preceq)$,*

$$\mathbf{bi}(\mathbb{V}_{\mathrm{P}}(X)) = \mathbb{W}(\mathbf{bi}(X))$$

where $\mathbb{V}_{\mathrm{P}}$ is an endofunctor on the category of compact partially ordered spaces defined exactly the same way as Vietoris endofunctor for Priestley spaces from Table 4.1.

Proof. We show $\mathbb{V}_{\mathrm{P}}(X) = \mathbf{bi}^{-1}(\mathbb{W}(\mathbf{bi}(X)))$, which is equivalent by Theorem 2.2.5. Recall that the associated order $\leq$ of a bispace $\mathbf{bi}(X)$ is the same as $\preceq$. In Section 4.1.5 we showed that the associated order of $\mathbb{W}(\mathbf{bi}(X))$ is the Egli-Milner lifting of $\leq$, therefore, it is equivalent to $\preceq^{\mathrm{EM}}$.

Moreover, also $\mathbb{V}_{\mathrm{Fr}}\tau = \mathbb{V}_{\mathrm{Fr}}\,\tau_+ \vee \mathbb{V}_{\mathrm{Fr}}\,\tau_-$. To right-to-left inclusion is trivial and the reverse inclusion is true because $\mathbb{W}(\mathbf{bi}(X))$ is order-separated and, as we showed in Section 4.1.5, the assignment $\mathcal{S}_+ = \mathbb{V}_{\mathrm{Fr}}\tau_+$ and $\mathcal{S}_- = \mathbb{V}_{\mathrm{Fr}}\tau_-$ satisfies the conditions of Lemma 2.2.1. □

4.5.1 Applications to modal and coalgebraic logics

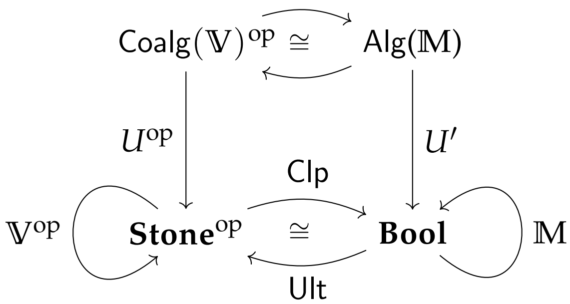
Boolean algebras are often seen as algebraic models or an algebraization of propositional logic. Stone spaces then, via Stone duality, provide a topological semantics for propositional logic. This is important because Stone spaces are defined purely in topological terms as opposed to Boolean algebras which can be seen as a translation of the syntactic description of propositional logic to universal algebra. In other words, this demonstrates how fundamental propositional logic is because it has a natural description in a seemingly unrelated discipline of mathematics.

Jónnson and Tarski extended Stone duality to the duality of so called *descriptive general frames* and *Boolean algebras with (modal) operators* [JT51; BRV01]. Because Boolean algebras with operators are sound and complete with respect to finitary modal logics, this way Jónsson-Tarski duality demonstrates that the same is true for the category of descriptive general frames. Further, Ghilardi observed in [Ghi95] that the category of Boolean algebras with operators is *isomorphic* to the category of $\mathbb{M}$-algebras on **Bool** and, a bit later, it was showed by Kupke, Kurz and Venema that the category of descriptive general frames is *isomorphic* to the category of $\mathbb{V}_{\mathrm{Fr}}$-coalgebras on **Stone** [KKV04][5]. In other words, $\mathbb{V}_{\mathrm{Fr}}$-coalgebras and $\mathbb{M}$-algebras provide an adequate semantics for finitary modal logic. Again, this suggests that modal

[5] In the paper, Kupke et al. attribute this result to Abramsky in [Abr05a], although, [KKV04] is clearly the first paper which spells this categorical fact in full details.

logic is somehow fundamental since it admits a very natural and non-syntactic description by some independently studied topological objects.

The link between the model theory of modal logic and $\mathbb{V}_{\mathrm{Fr}}$-coalgebras goes even further. For example, the canonical model of modal logic is equivalently represented as the final coalgebra of $\mathbb{V}_{\mathrm{Fr}} + C$ on **Stone** (here C represents the set of constants) [BK07]. In categorical terms, the connection between propositional logic and (finitary) modal logic is expressed as the following diagram of categories (where U and U' are the obvious forgetful functors):



For a more complete account of the history of the subject we refer the reader to [VV14; BK07], [Vos10] and [Rob86].

4.5.1.1 Positive modal logic

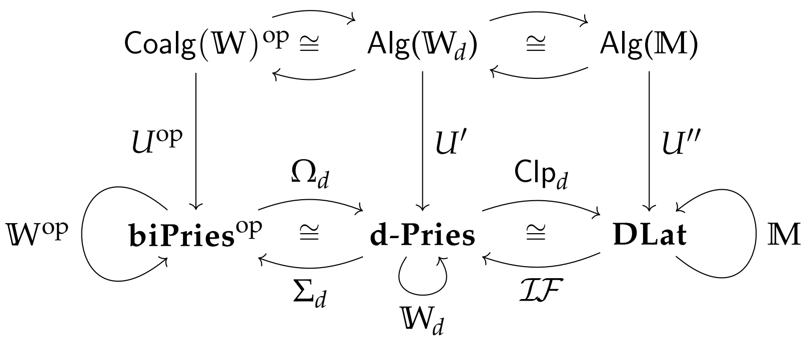
Distributive lattices are an algebraization of the negation free fragment of propositional logic. Therefore, all of the topological duals we talked about earlier, that is Priestley spaces, spectral spaces or Priestley bispaces, provide topological semantics for positive propositional logic. A simple adaptation of Ghilardi's observation [Ghi95] gives that $\mathbb{M}$-algebras on **DLat** provide an algebraic semantics for positive modal logic [Dun95; Jan02]. In the following we list the appropriate constructions on the categories of spaces which give topological semantics for positive modal logics:

1. A combination of results by Celani and Jansana [Cel99] and Palmigiano [Pal03; Pal04] shows that $\mathsf{Coalg}(\mathbb{V}_{\mathrm{P}})^{\mathrm{op}} \cong \mathsf{Alg}(\mathbb{M})$ making positive modal logic sound and complete with respect to $\mathbb{V}_{\mathrm{P}}$-coalgebras on the category of Priestley spaces.

2. Johnstone showed in [Joh82; Joh85] that $\mathbb{V}_{\mathrm{Fr}}$ is an endofunctor on the category of *spectral frames*, which are exactly the frame theoretic duals of spectral spaces (see Chapter 5). Moreover, the fact that $\Sigma \circ \mathbb{V}_{\mathrm{Fr}} \cong \mathbb{V} \circ \Sigma$ for the spectral spaces–spectral frames part of the duality is also presented therein. Consequently, for a distributive lattice D, $\mathsf{spec}_{\mathrm{s}}(\mathbb{M}D) \cong \Sigma(\mathsf{Idl}(\mathbb{M}D)) \cong \Sigma(\mathbb{V}_{\mathrm{Fr}}(\mathsf{Idl}(D))) \cong \mathbb{V}(\Sigma(\mathsf{Idl}(D))) \cong \mathbb{V}(\mathsf{spec}_{\mathrm{s}} D)$ where $\mathsf{spec}_{\mathrm{s}}$ is the functor assigning to a distributive lattice its spectral space. This proves that positive modal logic is sound and complete with respect to $\mathbb{V}$-coalgebras on the category of spectral spaces.

3. Lastly, Frederik Lauridsen proved in his thesis that $\mathbb{V}_P$-coalgebras on Priestley spaces and $\mathbb{W}$-coalgebras on **biPries** are isomorphic categories [Lau15] which shows that positive modal logic is sound and complete with respect to $\mathbb{W}$-coalgebras on Priestley bispaces.

All three items in the list are consequences of the general theory that we have developed in this text. In particular, a combination of Corollaries 4.3.14 and 4.4.17 yields $\mathsf{Coalg}(\mathbb{W})^{\mathrm{op}} \cong \mathsf{Alg}(\mathbb{W}_d) \cong \mathsf{Alg}(\mathbb{M})$ for $\mathbb{W}_d$ restricted to **d-Pries** and $\mathbb{W}$ restricted to **biPries**. The picture of categories is as follows (with U, U' and U'' being the obvious forgetful functors):



This proves that not only $\mathbb{W}_d$-algebras on **d-Pries** but also $\mathbb{W}$-coalgebras on Priestley bispaces are adequate models of positive modal logic and, by Proposition 4.5.4, the same follows for $\mathbb{V}_P$-coalgebras on Priestley spaces. Likewise, Proposition 5.2.14 together with Theorem 5.4.1 (which we prove in Chapter 5) imply that also $\mathbb{V}$-coalgebras on spectral spaces provide an adequate semantics for positive modal logic.

In the following proposition we compute the components of the natural isomorphism $\mathbb{W} \circ \mathsf{spec}_{\mathrm{bi}} \cong \mathsf{spec}_{\mathrm{bi}} \circ \mathbb{M}$ explicitly.

4.5.5 Proposition. *The map* $\mathsf{spec}_{bi}(\mathbb{M}D) \to \mathbb{W}(\mathsf{spec}_{bi}(D))$ *defined as*

$$P\colon \mathbb{M}D \to \mathbf{2} \quad \longmapsto \quad \{h\colon D \to \mathbf{2} \mid h[I_P] = \{0\} \ \text{and}\ \ h[F_P] = \{1\}\}$$

where $I_P = \{d \in D \mid P(\widetilde{\Diamond} d) = 0\}$ *and* $F_P = \{d \in D \mid P(\widetilde{\Box} d) = 1\}$, *determines the bihomeomorphism* $\mathbb{W} \circ \mathsf{spec}_{bi} \cong \mathsf{spec}_{bi} \circ \mathbb{M}$. *Moreover, its inverse is computed by the formula*

$$K \quad \longmapsto \quad P_K\colon \mathbb{M}D \to \mathbf{2} \quad \text{defined as} \quad \begin{array}{lll} P_K(\widetilde{\Box} a) = 1 & \text{iff} & a \in F_K \\ P_K(\widetilde{\Diamond} a) = 0 & \text{iff} & a \in I_K \end{array}$$

where $I_K = \{d \in D \mid \forall h \in K.\, h(a) = 0\}$ *and* $F_K = \{d \in D \mid \forall h \in K.\, h(a) = 1\}$.

Proof. Let us first establish how the bihomeomorphism $\mathsf{spec}_{\mathrm{bi}}(E) \to \Sigma_d(\mathcal{IF}(E))$, for a lattice E, looks like (recall Proposition 2.6.18). It sends a lattice homomorphism

$p\colon E \to \mathbf{2}$ to a d-frame homomorphism $\hat{p}\colon \mathcal{IF}(E) \to \mathbf{2}\times\mathbf{2}$, uniquely determined by

$$\hat{p}_+(I) = 1 \text{ iff } p \in U_+(I) \quad \text{and} \quad \hat{p}_-(F) = 1 \text{ iff } p \in U_-(F)$$

where $U_+(I) = \{p\colon E \to \mathbf{2} \mid 1 \in p[I]\}$ and $U_-(F) = \{p\colon E \to \mathbf{2} \mid 0 \in p[F]\}$.

Let $P\colon \mathbb{M}D \to \mathbf{2}$ be a lattice homomorphism and consider the composition of bihomeomorphisms:

$$\mathsf{spec}_{\mathsf{bi}}(\mathbb{M}D) \overset{①}{\to} \Sigma_d(\mathcal{IF}(\mathbb{M}D)) \overset{②}{\to} \Sigma_d(\mathbb{W}_d(\mathcal{IF}(D))) \overset{③}{\to} \mathbb{W}(\Sigma_d(\mathcal{IF}(D))) \overset{④}{\to} \mathbb{W}(\mathsf{spec}_{\mathsf{bi}}(D))$$

(The middle two maps are from Theorems 4.3.12 and 4.4.14.) We compute how P gets send to $\mathbb{W}(\mathsf{spec}_{\mathsf{bi}}(D))$ in a step-by-step application of those maps:

①: P is send to $\hat{P}\colon \mathcal{IF}(\mathbb{M}D) \to \mathbf{2}\times\mathbf{2}$.

②: $\hat{P}$ gets precomposed with the isomorphism $l\colon \mathbb{W}_d(\mathcal{IF}(D)) \to \mathcal{IF}(\mathbb{M}D)$ from Section 4.3.4. In the following, we need to know how $\hat{P}\cdot l$ behaves in the following two cases, for an $a \in D$:

$$\hat{P}_+(l_+(\Diamond{\downarrow}a)) = \hat{P}_+({\downarrow}\tilde{\Diamond}a) = 0 \text{ iff } P(\tilde{\Diamond}a) = 0$$
$$\hat{P}_-(l_-(\Diamond{\uparrow}a)) = \hat{P}_+({\uparrow}\tilde{\Box}a) = 0 \text{ iff } P(\tilde{\Box}a) = 1$$

③: $\hat{P}\cdot l$ is send to $K_0 \overset{\text{def}}{\equiv} \{q\colon \mathcal{IF}(D) \to \mathbf{2}\times\mathbf{2} \mid q(\alpha^{\hat{P}\cdot l}) = \bot\}$. Moreover,

$$\alpha^{\hat{P}\cdot l} = \bigsqcup\{(I,F) \mid \hat{P}(l(\Diamond I, \Diamond F)) = \bot\} = \bigsqcup\{({\downarrow}a, {\uparrow}b) \mid \hat{P}(l(\Diamond{\downarrow}a, \Diamond{\uparrow}b)) = \bot\}$$

which means that $\alpha^{\hat{P}\cdot l} = (I_P, F_P)$.

④: K_0 is sent to the convex closure of $\{h\colon D \to \mathbf{2} \mid h[I_P] = \{0\} \text{ and } h[F_P] = \{1\}\}$. However, this set is easily seen to be convex.

For the inverse map, let K be a convex d-compact subset of $\mathsf{spec}_{\mathsf{bi}}(D)$. It is immediate to verify that a $P\colon \mathbb{M}D \to \mathbf{2}$ is mapped to K if and only if $P = P_K$. □

4.5.6 Remark. With the formula for the natural isomorphism from Proposition 4.5.5 one can prove $\mathbb{W} \circ \mathsf{spec}_{\mathsf{bi}} \cong \mathsf{spec}_{\mathsf{bi}} \circ \mathbb{M}$ directly and without the detour to d-frames. In order to do that, it is useful to notice that the set $\mathbb{K}_c(\mathcal{IF}(D))$ can be equivalently expressed in simpler terms. Namely, for $(I,F) \in \mathsf{Idl}(D)\times\mathsf{Filt}(D)$, (I,F) is in $\mathbb{K}_c(\mathcal{IF}(D))$ if and only if it satisfies:

$$(\text{A}+) \quad \forall i \in I\, \forall y \notin F.\, i \vee y \notin F \qquad (\text{A}-) \quad \forall x \notin I\, \forall f \in F.\, x \wedge f \notin I$$

4.5.7 Remark. By a similar argument we get that the adequate algebraic semantics for the $\Box$ and $\Diamond$ fragments of positive modal logic are the $\mathbb{M}_\Box$ and $\mathbb{M}_\Diamond$-algebras and consequently also $\mathbb{W}_\boxtimes$ and $\mathbb{W}_\oplus$-coalgebras on Priestley bispaces, respectively.

This was explored in the context of spectral spaces by Bonsangue, Kurz and Rewitzky in [BKR07].

4.5.1.2 Jónnson-Tarski-like models of positive modal logic

Coalgebras of $\mathbb{V}_P$ on the category of Priestley spaces are known to be in a correspondence with so called $\mathbf{K}^+$-spaces. These are the tuples $\langle X, \leq, R, \mathcal{A} \rangle$ satisfying a list of conditions such that $\langle X, \leq, \mathcal{A} \rangle$ uniquely determines a Priestley space. Moreover, $\leq$ and R are required to be compatible, to make sure that the mapping $x \mapsto \{y \mid (x,y) \in R\}$ is monotone and that the image of every point is convex and compact (see [Pal04] for details).

When starting from $\mathbb{W}$-coalgebras on **biPries** as models of positive modal logic, an alternative description can be given. An advantage of this, when compared to $\mathbf{K}^+$-spaces (or $\mathbb{V}_P$-coalgebras), is that the result is much more combinatorial and closer to the original Jónsson-Tarski duality.

4.5.8 Proposition. *Bicontinuous maps $X \to \mathbb{W}(X)$ in* **biPries** *are in a bijective correspondence with the triples $\langle X, R, \mathcal{A}_+ \rangle$, where $R \subseteq X \times X$ is a relation and $\mathcal{A}_+$ is a set of subsets of X, such that*

- *(JT-1) $\mathcal{A}_+$ is closed under finite unions and intersections,*
- *(JT-2) $\mathcal{A}_+$ is closed under $\Box(-)$ and $\Diamond(-)$.*
- *(JT-3) $x \neq y$ in X iff $x \in A \not\ni y$ for some $A \in \mathcal{A}_+ \cup \mathcal{A}_-$,*
- *(JT-4) if $\forall A \in \mathcal{A}_+ \cup \mathcal{A}_-$, $y \in A$ implies $x \in \Diamond A$, then $(x,y) \in R$,*
- *(JT-5) for any $\mathcal{M} \subseteq \mathcal{A}_+ \cup \mathcal{A}_-$ with finite intersection property, $\bigcap \mathcal{M} \neq \emptyset$,*

where $\mathcal{A}_- = \{X \setminus A \mid A \in \mathcal{A}_+\}$ and, for a subset $M \subseteq X$,

$$\Box M = \{x \in X \mid \forall y.\, (x,y) \in R \text{ implies } y \in M\},$$
$$\Diamond M = \{x \in X \mid \exists y \text{ s.t. } (x,y) \in R \text{ and } y \in M\}.$$

Proof (inspired by [KKV04]). First, we notice that, for a fixed set X, Priestley bispaces (X, τ_+, τ_-) and the pairs $\langle X, \mathcal{A}_+ \rangle$, which satisfy (JT-1), (JT-3) and (JT-5), are in a bijective correspondence. In one direction, define $\mathcal{A}_+$ to be the set of τ_+-open τ_--closed subsets of X and, conversely, define $X_{\mathcal{A}}$ to be the bispace $(X, \tau_+^{\mathcal{A}}, \tau_-^{\mathcal{A}})$, where $\tau_\pm^{\mathcal{A}}$ is the smallest topology containing $\mathcal{A}_\pm$. It is immediate that $\mathcal{A}_+$ obtained from a bispace satisfies (JT-1) (by (con-$\curlyvee$, $\curlywedge$) and (tot-$\curlyvee$, $\curlywedge$)) and that $X_{\mathcal{A}}$ is always d-zero-dimensional. Moreover, d-compactness and (JT-5) are equivalent notions under those translations. A combination of T_0 and being d-zero-dimensional implies that $\mathcal{A}_+$ obtained from a bispace satisfies (JT-3). Conversely, if $x \neq y$ in a bispace $X_{\mathcal{A}}$, then there is some $A \in \mathcal{A}_+$ such that $x \in A \not\ni y$ or $x \notin A \ni y$ by (JT-3). Therefore, the specialisation order $\leq_+$ of $\tau_+^{\mathcal{A}}$ is antisymmetric. Moreover, $\leq_+ = \geq_-$ also makes $\leq_+ \cap \geq_-$ antisymmetric. Finally, it is easy to see that the translations $(X, \tau_+, \tau_-) \leftrightarrow \langle X, \mathcal{A}_+ \rangle$ are inverse to each other.

Let us fix a Priestley bispace (X, τ_+, τ_-) and the corresponding $\langle X, \mathcal{A}_+\rangle$. Observe that a bicontinuous map

$$\xi\colon (X, \tau_+^{\mathcal{A}}, \tau_-^{\mathcal{A}}) \to \mathbb{W}(X, \tau_+^{\mathcal{A}}, \tau_-^{\mathcal{A}})$$

determines a relation $R \subseteq X \times X$ by $(x,y) \in R$ iff $y \in \xi(x)$, which satisfies (JT-4). Indeed, if $(x,y) \notin R$ then, because $\xi(x)$ is d-compact convex, either $y \notin {\downarrow}\xi(x)$ or $y \notin {\uparrow}\xi(x)$. W.l.o.g. assume the former. By Lemma 4.1.29, there is an $U \in \tau_+$ such that $y \in U$ and $U \cap \xi(x) = \emptyset$. Because X is d-zero-dimensional, $y \in A \subseteq U$ for some τ_+-open τ_--closed A and, also, $x \notin \Diamond A = \{x \mid \xi(x) \cap A \neq \emptyset\}$.

Conversely, assume that $R \subseteq X \times X$ satisfies (JT-4). We show that the set $\xi(x) \stackrel{\text{def}}{\equiv} \{y \mid (x,y) \in R\}$ is a d-compact convex subset of $X_{\mathcal{A}}$, for every $x \in X$. If $y \notin \xi(x)$, for some y, then by (JT-4) there is some $A_y \in \mathcal{A}_+ \cup \mathcal{A}_-$ such that $y \in A_y$ and $x \notin \Diamond A_y$, where the latter is equivalent to $\xi(x) \cap A_y = \emptyset$. Hence $\xi(x)$ is in the complement of $U_+ \cup U_-$ for some $U_\pm \in \tau_\pm^{\mathcal{A}}$. Because $X_{\mathcal{A}}$ is a Priestley bispace, $\xi(x)$ is d-compact and convex.

Before we show that ξ is bicontinuous iff (JT-2) holds for $\langle X, R, \mathcal{A}_+\rangle$, let us prove that (JT-2) is equivalent to its minus-version: $\mathcal{A}_-$ is closed under $\Box(-)$ and $\Diamond(-)$. This is because, for an $A \in \mathcal{A}_+$,

$$\Box(X \setminus A) = \{x \mid \xi(x) \subseteq (X \setminus A)\} = \{x \mid \xi(x) \cap A = \emptyset\} = X \setminus \Diamond A, \text{ and}$$
$$\Diamond(X \setminus A) = \{x \mid \xi(x) \cap (X \setminus A) \neq \emptyset\} = \{x \mid \xi(x) \nsubseteq A\} = X \setminus \Box A.$$

Further, observe that $\mathbb{V}\tau_+^{\mathcal{A}}$ is generated from the elements $\boxtimes A$ and $\diamondplus A$, for $A \in \mathcal{A}_+$. Therefore, it is enough to check continuity for such opens. By a simple computation we have that

$$\xi^{-1}[\boxtimes A] = \{x \mid \xi(x) \subseteq A\} = \Box A \qquad \xi^{-1}[\diamondplus(X \setminus A)] = \Diamond(X \setminus A) = X \setminus \Box A$$
$$\xi^{-1}[\diamondplus A] = \{x \mid \xi(x) \cap A \neq \emptyset\} = \Diamond A \qquad \xi^{-1}[\boxtimes(X \setminus A)] = \Box(X \setminus A) = X \setminus \Diamond A$$

Therefore, if (JT-2) holds, ξ is bicontinuous. Conversely, if ξ is bicontinuous, $\Box A$ has to be τ_+-open and $X \setminus \Box A$ and τ_--open. Then, $\Box A$ is τ_+-open and τ_--closed, i.e. $\Box A \in \mathcal{A}_+$. Similarly, we see that $\Diamond A \in \mathcal{A}_+$. □

4.5.9 Corollary. *Positive modal logic is sound and complete with respect to the class of triples $\langle X, R, \mathcal{A}_+\rangle$ satisfying the conditions (JT-1), ..., (JT-5) of Proposition 4.5.8.*

4.5.10 Remark. A remarkable feature of this representation is that it brings us closer to Canonical (Kripke) Models as defined by Dunn in his original paper [Dun95]. When proving completeness he needed to consider sets of *theories* and *counter-theories*. In our setting, those are some collections of elements in $\mathcal{A}_+$ and in $\mathcal{A}_-$ or, more specifically, ideals of $\mathbb{M}D$ and filters of $\mathbb{M}D$ (for some lattice D). Also, a theory and a counter-theory is *disjoint* in Dunn's terms iff the corresponding ideal and filter are consistent in $\mathcal{IF}(\mathbb{M}D)$. There is also an apparent similarity between *maximally disjoint pairs* representing points of Dunn's Canonical Model and points of our $\mathsf{spec}_{\mathrm{bi}}(\mathbb{M}D) \cong \Sigma_d(\mathcal{IF}(\mathbb{M}D))$.

4.5.1.3 Coalgebraic logic

In the recent years some people have noticed that the fact that certain modal logics have semantics given by a class of coalgebras is not just an interesting fact but there is actually more to it. Namely, it is now widely accepted that the same way equational logic is closely connected to universal algebra, modal logics are closely connected to coalgebras [Kur01; Cir+09].

Because of the generality of our approach, it is now possible to set up modal logics such that the base category for coalgebras would be **biKReg**, a much broader category than Priestley (bi)spaces or even compact regular spaces. Moreover, since **d-KReg** is complete and cocomplete (Section 3.3.2) also **biKReg** is. Thanks to this, all *Vietoris polynomial functors*, i.e. functors recursively given by the grammar

$$T ::= \mathrm{Id} \mid X \mid T_1 + T_2 \mid \prod_{a \in A} T_a \mid \mathbb{W}T, \qquad \text{(where } A \in \mathbf{Set} \text{ and } X \in \mathbf{biKReg}\text{)}$$

when restricted to the category of d-compact d-regular bispaces are endofunctors. (Recall that products and finite coproducts of d-compact d-regular bispaces agree with products and finite coproducts in **biTop** thanks to Corollary 3.3.10.)

Furthermore, every such Vietoris polynomial endofunctor T has a "mate" endofunctor on **d-KReg** associated to it. Namely, there is a *Vietoris copolynomial endofunctor* on **d-KReg** recursively given by the following grammar

$$L ::= \mathrm{Id} \mid \Omega_d(X) \mid L_1 \times L_2 \mid \bigoplus_{a \in A} L_a \mid \mathbb{W}_d L, \qquad \text{(where } A \in \mathbf{Set} \text{ and } X \in \mathbf{biKReg}\text{)}$$

such that

$$\mathsf{Coalg}(T)^{\mathrm{op}} \cong \mathsf{Alg}(L). \tag{4.5.2}$$

As a result, we have set up all the basic machinery of the framework of coalgebraic logics [BK05; Kli07; KKV04; Jac01]. In that setting the category $\mathsf{Coalg}(T)$ plays the role of semantics and one can think of T-coalgebras as if those were "T-shaped" Kripke frames or, alternatively, step based models of computation (for example, automata). The category $\mathsf{Alg}(L)$ are algebraic presentations of a geometric (intuitionistic) modal logic (because of the equivalence with stably compact frames, see Chapter 5) or a geometric paraconsistent modal logic (see Chapter 6). The equivalence of categories in (4.5.2) then guarantees that such a logic will be sound, complete, and expressive with respect to T-coalgebras [BK06; KKP04].

5

Stably compact spaces

In many ways, stably compact spaces are the T_0 analogues of compact Hausdorff spaces. They arise naturally in the theory of computation where most spaces are not even T_1. The key feature of stably compact spaces is that compact sets enjoy essentially the same properties as compact sets of compact Hausdorff spaces.

The purpose of this chapter is to highlight some of the applications of the theory of bitopological spaces and d-frames to the theory of stably compact spaces. We believe that our bitopological reorganisation of the material leads to shorter and arguably more transparent proofs of some of the known results. In Section 5.4 we reuse results from Chapter 4 and as a by-product get new results concerning powerspace constructions of stably compact frames (Theorem 5.4.2).

Unless stated otherwise, the author does not claim novelty of the results presented in this chapter. Standard literature of stably compact spaces includes [Jun04], [JS96], [Law10], [Gou10], [Gie+80], and many more. For other sources treating stably compact spaces bitopologically see [JM06] and [Law10].

Convention. In this chapter, we assume that every topological space (X, τ) automatically comes equipped with a preorder $\leq_\tau$ (or simply just $\leq$) defined as the specialisation preorder of τ. Note that this preorder is a partial order iff X is T_0.

Further, upsets in the specialisation orders are traditionally called *saturated sets*. We stick to this terminology even though it conflicts with what we call the elements of a quotient of a frame (from Section 3.1.2), but these never occur in this chapter.

5.1 Frame duality for stably compact spaces

In this section we recapitulate the well known duality between stably compact spaces and stably compact frames. First, we recall a few basic definitions and facts.

5.1.1 Definition.

1. A space X is *sober* if $X \cong \Sigma(L)$ for some frame L.
2. A space X is *locally compact* if for every $U \in \tau$ and $x \in U$ there exists a compact saturated $K \subseteq X$ such that $x \in K^\circ \subseteq K \subseteq U$ where K° is the interior of K.
3. A frame L is *locally compact* (or *continuous*) if $x = \bigvee\{a \in L \mid a \ll x\}$ for every $x \in L$, where $\ll$ is the *way below* relation on L defined as

$$a \ll x \overset{\text{def}}{\equiv} \forall D \text{ directed s.t. } x \leq \bigvee\nolimits^{\uparrow} D \implies a \leq d \text{ for some } d \in D$$

5.1.2 Observation. *If L is locally compact then $\ll$ interpolates and has the following property:*

$$a \ll \bigvee\nolimits^{\uparrow} D \implies a \ll d \text{ for some } d \in D$$

Proof. To show that $a \ll b$ interpolates we use local compactness twice. Since $b = \bigvee^{\uparrow}\{x \mid x \ll b\} = \bigvee^{\uparrow}\{y \mid y \ll x \ll b\}$, $a \leq y \ll x \ll b$ for some x and y.

Next, let $a \ll \bigvee^{\uparrow} D$. Interpolate $a \ll x \ll \bigvee^{\uparrow} D$ and, by definition, $a \ll x \leq d$ for some $d \in D$. □

A result that connects the topological and frame-theoretic definitions stated above is the famous Hofmann-Lawson Theorem:

5.1.3 Fact ([HL78]). *The restriction of adjoint functors $\Sigma \dashv \Omega\colon$ **Top** $\rightleftarrows$ **Frm** to the categories of sober locally compact spaces and locally compact frames, respectively, constitutes a dual equivalence of categories.*

For a more recent proof see Theorem 6.4.3 in [PP12]. In the core of the argument behind the Hofmann-Lawson Theorem is the Hofmann-Mislove Theorem:

5.1.4 Fact ([HM81; KP94]). *Let (X, τ) be a sober space. There is a bijection between Scott-open filters[1] of the frame τ and compact saturated sets of X, as defined by the pair of antitone maps*

$$\mathcal{S} \subseteq \tau \ \mapsto\ \bigcap \mathcal{S} \quad \textit{and} \quad K \subseteq X \ \mapsto\ \{U \in \tau \mid K \subseteq U\}$$

Not only are Scott-open filters essential for the duality of locally compact spaces and frames, their importance spans throughout this whole chapter. For this reason

[1] For the definition of Scott-open filters see Lemma 2.4.2.

we explore some of the basic properties of the poset $L^\wedge$ which consists of all Scott-open filters of a frame L ordered by set inclusion:

5.1.5 Lemma. *Let L be a frame.*

1. *$L^\wedge$ is closed under directed joins and finite meets in* $\mathsf{Filt}(L)$ *or, in other words, $L^\wedge$ is a sub-preframe of* $\mathsf{Filt}(L)$.
2. *If L is locally compact: $a \ll b$ iff there exists a Scott-open filter $\mathcal{S} \subseteq L$ such that $b \in \mathcal{S} \subseteq \uparrow a$.*
3. *If L is locally compact: $L^\wedge$ is closed under binary joins in* $\mathsf{Filt}(L)$ *iff L is* coherent, *that is, whenever $a \ll b$ and $a \ll c$ in L, then also $a \ll b \wedge c$.*
4. *If L is compact: $\{1\}$ is the smallest element of $L^\wedge$.*

Consequently, $L^\wedge$ is a subframe of $\mathsf{Filt}(L)$ *if L is compact, locally compact and coherent.*

Proof. (1) Let $\mathcal{S}_i$ be a directed subset of $L^\wedge$. Then, $\bigvee^\uparrow D \in \bigcup_i^\uparrow \mathcal{S}_i$ if $\bigvee^\uparrow D \in \mathcal{S}_i$ for some i. Since $\mathcal{S}_i$ is Scott-open, $d \in \mathcal{S}_i \subseteq \bigcup_i^\uparrow \mathcal{S}_i$ for some d; which proves that $\bigcup_i^\uparrow \mathcal{S}_i \in L^\wedge$. Let $\mathcal{S}, \mathcal{T} \in L^\wedge$. If $\bigvee^\uparrow D \in \mathcal{S} \cap \mathcal{T}$, then there are $d_1, d_2 \in D$ such that $d_1 \in \mathcal{S}$ and $d_2 \in \mathcal{T}$. From directedness find a $d \in D$ such that $d_1, d_2 \leq d$. Then, $d \in \mathcal{S} \cap \mathcal{T}$. Also, the empty meet of filters, i.e. L, is Scott-open.

(2) We borrow a proof from [PP12, Lemma 6.3.2]. In Observation 5.1.2 we proved that $\ll$ interpolates. Therefore, we have a sequence

$$a \ll \ldots \ll x_n \ll x_{n-1} \ll \ldots \ll x_2 \ll x_1 \ll x_0 = b.$$

Define $\mathcal{S} = \uparrow\{x_i : i \in \mathbb{N}\}$. Clearly, $b \in \mathcal{S} \subseteq \uparrow a$. $\mathcal{S}$ is a filter as $x_i \leq z$ and $x_j \leq z'$ implies $x_{\min(i,j)} \leq z \wedge z'$, and it is Scott-open because, if $x_i \leq \bigvee^\uparrow D$, then $x_{i+1} \leq d$ for some $d \in D$. The reverse direction is immediate from local compactness.

(3) Assume that $L^\wedge$ is closed under binary joins in $\mathsf{Filt}(L)$. If $a \ll b$ and $a \ll c$, then by (2) there are $\mathcal{S}, \mathcal{T} \in L^\wedge$ such that $b \in \mathcal{S} \subseteq \uparrow a$ and $c \in \mathcal{T} \subseteq \uparrow a$. By the assumption $\mathcal{S} \vee \mathcal{T} = \{s \wedge t \mid s \in \mathcal{S},\ t \in \mathcal{T}\}$ is Scott-open and contains $b \wedge c$. Therefore, by (2) again, $a \ll b \wedge c$.

For the other direction, first, observe that, for a filter $F \subseteq L$,

$$\twoheaduparrow F = \{x \mid f \ll x \text{ for some } f \in F\}$$

is a Scott-open filter. Clearly, $\twoheaduparrow F$ is upwards closed and also Scott-open as $\bigvee^\uparrow D \in \twoheaduparrow F$ implies that $f \ll d$ for some $d \in D$ (Observation 5.1.2). By coherence it is also a filter since, if $f \ll a$ and $g \ll b$ for some $f, g \in F$, then also $f \wedge g \ll a \wedge b$ and $f \wedge g \in F$.

Next, let $\mathcal{S}, \mathcal{T} \in L^\wedge$. We show that their join in $\mathsf{Filt}(L)$ is Scott-open by showing that $\mathcal{S} \vee \mathcal{T} = \twoheaduparrow(\mathcal{S} \vee \mathcal{T})$. Indeed, let $s \in \mathcal{S}$ and $t \in \mathcal{T}$. By local compactness of L, there are $x \ll s$ and $y \ll t$ such that $x \in \mathcal{S}$ and $y \in \mathcal{T}$. Then, by coherence, $x \wedge y \ll s \wedge t$ and so $s \wedge t \in \twoheaduparrow(\mathcal{S} \vee \mathcal{T})$. The other inclusion is obvious.

(4) Because $1 \ll 1$, the smallest filter $\{1\}$ is Scott-open. □

5.1.6 Lemma. *Let $h\colon L \to M$ be a frame homomorphism between two locally compact frames. If h preserves the way-below relation, i.e. $h(x) \ll h(y)$ whenever $x \ll y$ in L, then the map*

$$h^{\wedge}\colon L^{\wedge} \to M^{\wedge}, \quad \mathcal{S} \mapsto {\uparrow} h[\mathcal{S}]$$

is a preframe homomorphism.

If, moreover, L and M are coherent and compact, then $h^{\wedge}$ is a frame homomorphism.

Proof. First, we compute that $h^{\wedge}(\mathcal{S})$ is a Scott-open filter. Clearly it is a filter. To verify Scott-openness let $\bigvee^{\uparrow} D \in {\uparrow} h[\mathcal{S}]$. This means that $h(x) \leq \bigvee^{\uparrow} D$ for some $x \in \mathcal{S}$. Because L is locally compact and $\mathcal{S}$ Scott-open, there is a $y \in \mathcal{S}$ such that $y \ll x$. Since h preserves the way-below relation, $h(y) \ll h(x) \leq \bigvee^{\uparrow} D$. Therefore, $h(y) \leq d$ for some $d \in D$.

Next, we verify that $h^{\wedge}$ is a preframe homomorphism. Observe that $h^{\wedge}$ preserves finite meets as ${\uparrow} h[\mathcal{S} \cap \mathcal{T}] = {\uparrow} h[\mathcal{S}] \cap {\uparrow} h[\mathcal{T}]$ and ${\uparrow} h[L] = M$. Also, directed joins are preserved: $\bigcup_i^{\uparrow} {\uparrow} h[\mathcal{S}_i] = {\uparrow} h[\bigcup_i^{\uparrow} \mathcal{S}_i]$.

For the "moreover" part, clearly, $h^{\wedge}$ preserves the smallest element as ${\uparrow} h[\{1_L\}] = {\uparrow} 1 = \{1_M\}$. Lastly, we check that it preserves binary joins: ${\uparrow} h[\mathcal{S} \vee \mathcal{T}] = {\uparrow}\{h(s) \wedge h(t) \mid s \in \mathcal{S},\ t \in \mathcal{T}\} = {\uparrow}\{x \wedge y \mid x \in h[\mathcal{S}],\ y \in h[\mathcal{T}]\} = \{a \wedge b \mid x \in h[\mathcal{S}],\ y \in h[\mathcal{T}],\ x \leq a,\ y \leq b\} = {\uparrow} h[\mathcal{S}] \vee {\uparrow} h[\mathcal{T}]$. □

In the next statement we explore the property which exactly mirrors coherence of frames in the theory of (sober locally compact) topological spaces.

5.1.7 Proposition.

1. *Let U and V be open sets of a sober locally compact space. Then, $U \ll V$ iff there exists a compact saturated set K such that $U \subseteq K \subseteq V$.*

2. *For a sober locally compact space X: $\Omega(X)$ is coherent iff X is (topologically)* coherent, *that is, any intersection of two compact saturated sets is compact too.*

Proof. (1) For the right-to-left implication, by Fact 5.1.3, $\mathcal{S} \stackrel{\text{def}}{\equiv} \{W \in \tau \mid K \subseteq W\}$ is a Scott-open filter such that $V \in \mathcal{S} \subseteq {\uparrow} U$. Therefore, by (2) of Lemma 5.1.5, $U \ll V$. Conversely, if $U \ll V$, then there is a Scott-open filter $\mathcal{S} \subseteq \tau$ such that $V \in \mathcal{S} \subseteq {\uparrow} U$. Then, for its associated saturated set $K \stackrel{\text{def}}{\equiv} \bigcap \mathcal{S}$, $U \subseteq K \subseteq V$.

(2) For "⇒" direction, let $U \ll V$ and $U \ll W$. From (2) there are two compact saturated K and Q such that $U \subseteq K \subseteq V$ and $U \subseteq Q \subseteq W$. Because $K \cap Q$ is compact, $U \subseteq K \cap Q \subseteq V \cap W$ implies $U \ll V \cap W$.

For the reverse direction, let K and H be two compact saturated subsets of X. Set $\mathcal{S} = \{U \in \tau \mid K \subseteq U\}$ and $\mathcal{T} = \{V \in \tau \mid H \subseteq V\}$ be the Scott-open filters associated to K and H, respectively. Observe that $K \cap H = (\bigcap \mathcal{S}) \cap (\bigcap \mathcal{T}) = \bigcap(\mathcal{S} \vee \mathcal{T})$ and by (3) of Lemma 5.1.5, $\mathcal{S} \vee \mathcal{T}$ is a Scott-open filter. Therefore, by Hofmann-Mislove Theorem, $\bigcap(\mathcal{S} \vee \mathcal{T})$ is compact. □

This proposition establishes that the dual equivalence between sober locally compact spaces and locally compact frames restricts to the duality of those which are coherent (in their respective categories). Moreover, this duality also restricts to the subcategories which consist of important classes of morphisms:

5.1.8 Lemma. *For a continuous map $f\colon (X, \tau) \to (Y, \sigma)$ between sober locally compact spaces, the following are equivalent:*

1. *f is a* perfect *map, that is, $f^{-1}[K]$ is compact for every compact saturated $K \subseteq Y$, and*

2. *$\Omega(f)$ is a* perfect *frame homomorphism, that is, $\Omega(f)(U) \ll \Omega(f)(V)$ whenever $U \ll V$ in $\Omega(Y)$.*

Proof. (1) ⇒ (2) follows from Proposition 5.1.7: $U \ll V$ in $\Omega(Y)$ whenever there is a compact saturated K such that $U \subseteq K \subseteq V$. By the assumption $f^{-1}[K]$ is compact and so $f^{-1}[U] \ll f^{-1}[V]$ because $f^{-1}[U] \subseteq f^{-1}[K] \subseteq f^{-1}[V]$.

Conversely, let $K \subseteq Y$ be compact saturated and let $\mathcal{S} = \{U \mid K \subseteq U\}$ be the associated Scott-open filter by Hofmann-Mislove Theorem. By Lemma 5.1.6, $\uparrow\Omega(f)[\mathcal{S}]$ is also a Scott-open filter and so $\bigcap(\uparrow\Omega(f)[\mathcal{S}])$ is compact. Moreover, this compact set is equal to $f^{-1}[K]$ as $\bigcap(\uparrow\Omega(f)[\mathcal{S}]) = \bigcap(\Omega(f)[\mathcal{S}]) = \bigcap\{f^{-1}[U] \mid U \in \mathcal{S}\} = f^{-1}[\bigcap \mathcal{S}]$. □

5.1.9 Definition.

1. A space is *stably compact* if it is sober, compact, locally compact and coherent.

2. A frame is *stably compact* if it is compact, locally compact and coherent.

Denote by **SCTop** the category of stably compact spaces and perfect maps, and **SCFrm** the category of stably compact frames and perfect frame homomorphisms.

Because a topological space is compact if and only if its Ω-image is compact, it is a corollary of Proposition 5.1.7 and Lemma 5.1.8 that the equivalence of categories given by Hofmann-Lawson Theorem restricts further to those two categories of stably compact spaces and frames:

5.1.10 Corollary. *The restriction of adjoint functors $\Sigma \dashv \Omega\colon$ **Top** $\rightleftarrows$ **Frm** to the categories* **SCTop** *and* **SCFrm**, *respectively, constitutes a dual equivalence of categories.*

This result is by no means new, although a direct proof of coherence is not so easy to find. It first appeared in [Hof84], it is also indirectly proved in [Gie+03] and Smyth included a proof sketch in [Smy92b].

5.1.11 Remark. Stably compact spaces can be alternatively defined as T_0, compact, locally compact, coherent and *well-filtered* spaces where the last property means that, for an open set U and a filtered set of compact saturated sets $\mathcal{F}$ (i.e. directed in the reverse $\subseteq$-order), if $\bigcap \mathcal{F} \subseteq U$, then $F \subseteq U$ for some $F \in \mathcal{F}$.

This is equivalent to the previous definition because a locally compact space is T_0 and well-filtered if and only if it is sober [Gie+03, Theorem II-1.21].

5.1.12 Definition.

1. A frame L is *spectral*[2] if it is compact, its compact elements are closed under finite meets (i.e. $k \ll k$ and $h \ll h$ implies $k \wedge h \ll k \wedge h$) and, for each $x \in L$, $x = \bigvee^{\uparrow}\{k \mid k \ll k \leq x\}$.
2. A space X is *spectral* if it is sober and $\Omega(X)$ is a spectral frame.
3. By **SpecTop** denote the categories of spectral spaces and perfect maps and by and **SpecFrm** denote the category of spectral frames and perfect frame homomorphisms.

This is a slightly different definition from the one originally given by Stone in [Sto37b]. He defined spectral spaces as topological spaces which are (when written in modern parlance),

1. T_0, compact, well-filtered, such that
2. their compact open sets are closed under finite intersection and
3. their topology is generated by compact open sets.

Our definition is equivalent to Stone's because (3) implies local compactness and then T_0 together with well-filteredness is equivalent to sobriety (see Remark 5.1.11).

5.1.13 Proposition. *The categories* **SpecTop** *and* **SpecFrm** *are dually equivalent.*

Proof. We use Corollary 5.1.10. It is enough to show that every spectral frame is stably compact. It is immediate that being spectral implies being compact and locally compact. For coherence, let $x \ll a$ and $x \ll b$. Since a is a directed join of compact elements below a, $x \leq k \ll k \leq a$ and, similarly, $x \leq h \ll h \leq b$ for some compact k and h. Then, because closed elements are closed under meets: $x \leq k \wedge h \ll k \wedge h \leq a \wedge b$ and so $x \ll a \wedge b$. □

[2]The adjective *"spectral"* for frames is non-standard and it does not exist in the literature. Instead, it common to use "coherent" (as in [Joh82] and [Vic89]) but that would cause confusion with coherence from Lemma 5.1.5. Also, coherent spaces in [JS96] is what we call stably compact spaces here.

5.2 Stably compact spaces and bispaces

An important construction in the theory of T_0 spaces is the de Groot dual of a space. Namely, for a space $X = (X, \tau)$, define the dual of X to be the space $X^{\mathrm{d}} \overset{\mathrm{def}}{\equiv} (X, \tau^{\mathrm{d}})$ where τ^{d} is generated from the basis $\{X \setminus K \mid K \text{ is compact saturated}\}$[3].

The reason why X^{d} is called the dual space of X is because its specialisation order is reverse to the one of X:

5.2.1 Observation. *For a locally compact space X, $x \leq_\tau y$ iff $x \geq_{\tau^{\mathrm{d}}} y$.*

Proof. If $x \not\leq_\tau y$, then there is an open U such that $x \in U \not\ni y$. By local compactness, let K be compact saturated such that $x \in K^\circ \subseteq K \subseteq U$. Then, $x \notin X \setminus K \ni y$.

Conversely, let $y \in X \setminus K \not\ni x$ for some compact saturated K. From K being saturated, $K = \bigcap\{U \mid K \subseteq U\}$ and so $y \notin U$ for some $U \supseteq K$. Because $x \in K$, also $x \in U$ whereas $y \notin U$. □

If we put our bitopological lenses on, then there is an obvious question we should ask: Can wan classify the class of bitopological spaces that arise as $X^{\mathrm{bi}} \overset{\mathrm{def}}{\equiv} (X, \tau, \tau^{\mathrm{d}})$ for some stably compact space X? First, we examine what properties do the bispaces arising this way have:

5.2.2 Lemma. *If X is T_0 and locally compact, then $X^{\mathrm{bi}} = (X, \tau, \tau^{\mathrm{d}})$ is order-separated.*

Proof (inspired by [Gou10]). By the previous observation we know that $\leq_\tau \; = \; \geq_{\tau^{\mathrm{d}}}$ and so the associated order $\leq \; \overset{\mathrm{def}}{\equiv} \; \leq_\tau \cap \geq_{\tau^{\mathrm{d}}}$ of X^{bi} is equal to $\leq_\tau$; hence, it is a partial order. Next, if $x \not\leq y$, because $x \not\leq_\tau y$, there is some $U \in \tau$ such that $x \in U \not\ni y$ and, from local compactness, $x \in K^\circ \subseteq K \subseteq U$. Therefore, $K^\circ \cap (X \setminus K) = \emptyset$, $x \in K^\circ$ and $y \in X \setminus K$. □

5.2.3 Lemma. *If X is compact, coherent and well-filtered, then X^{bi} is d-compact.*

Proof (inspired by [XL17]). We use Alexander Subbasis Lemma. Let $\bigcup_i U_i \cup \bigcup_j (X \setminus K_j) = X$ for some $\{U_i\}_{i \in I} \in \tau$ and $\{X \setminus K_j\}_{j \in J} \subseteq \tau^{\mathrm{d}}$. If J is empty, then a finite sub-cover of $\{U_i\}_i$ exists from compactness of X. Otherwise, since $\bigcup_j (X \setminus K_j) = X \setminus \bigcap_j K_j$, we have that $\bigcap_j K_j \subseteq \bigcup_i U_i$. Because X is coherent, every finite intersection of K_j's is compact saturated and so $\bigcap_j K_j$ is equal to the filtered family of all finite combinations of intersections of elements in $\{K_j\}_i$. By, well-filteredness, $K_{j_1} \cap \cdots \cap K_{j_n} \subseteq \bigcup_i U_i$ for some $j_1, \ldots, j_n \in J$. Because $K_{j_1} \cap \cdots \cap K_{j_n}$ is compact, $K_{j_1} \cap \cdots \cap K_{j_n} \subseteq U_{i_1} \cup \cdots \cup U_{i_m}$ for some $i_1, \ldots, i_m \in I$. In other words, $U_{i_1} \cup \cdots \cup U_{i_m} \cup (X \setminus K_{j_1}) \cup \cdots \cup (X \setminus K_{j_n}) = X$. □

[3] This set is a basis because it is closed under finite intersections: $(X \setminus K) \cap (X \setminus H) = X \setminus (K \cup H)$ and $K \cup H$ is compact saturated whenever K and H are.

To summarise, we have proved the following (recall Convention from page 28 and Remark 5.1.11):

5.2.4 Proposition. *Let X be a stably compact space. Then, $X^{\mathrm{bi}} = (X, \tau, \tau^{\mathrm{d}})$ is a d-compact d-regular (T_0) bispace.*

On the other hand, also the reverse direction holds. Every d-compact d-regular bispace determines a stably compact space as follows:

5.2.5 Proposition. *Let $X = (X, \tau_+, \tau_-)$ be a d-compact d-regular bispace. Then, (X, τ_+) and (X, τ_-) are stably compact spaces.*

Proof. We prove that (X, τ_+) is a stably compact space according to the alternative definition from Remark 5.1.11:

1. *T_0 separation and compactness* follow from the fact that the bispace is order-separated and d-compact.

2. *Local compactness:* Let $x \in U$ for some $U \in \tau_+$. Then, from d-regularity, $x \in V \subseteq \overline{V}^{\tau_-} \subseteq U$ for some $V \in \tau_-$. Because X is d-compact, $\overline{V}^{\tau_-}$ is d-compact and, in particular, also τ_+-compact.

3. *Coherence:* Let K and H be τ_+-compact. By Lemma 4.1.29, they are also τ_--closed. Then, $K \cap H$ is also τ_--closed and, because X is d-compact, $K \cap H$ is also τ_+-compact.

4. *Well-filteredness:* If $\bigcap_i K_i \subseteq U$ for $U \in \tau_+$ and a filtered collection of $\{K_i\}_i$ of τ_+-compact saturated sets. We have that $U \cup \bigcup_i (X \setminus K_i) = X$. By Lemma 4.1.29 again, we see that each $X \setminus K_i$ is τ_--open and so, by d-compactness, there is an i such that $U \cup X \setminus K_i = X$. Consequently, $K_i \subseteq U$. □

The last two propositions establish mapping between the objects of categories of stably compact spaces and d-compact d-regular bispaces; namely we have

$$(X, \tau) \mapsto (X, \tau, \tau^{\mathrm{d}}) \qquad \text{and} \qquad (X, \tau_+, \tau_-) \mapsto (X, \tau_+)$$

In the following we show that those mappings are inverse to each other:

5.2.6 Lemma. *If $X^1 = (X, \tau_+, \tau_-^1)$ and $X^2 = (X, \tau_+, \tau_-^2)$ are two d-compact d-regular bispaces, then $\tau_-^1 = \tau_-^2$.*

Proof. Let $U \in \tau_-^1$. Because $X \setminus U$ is d-compact in X^1, it is also τ_+-compact in X^2. Moreover, $X \setminus U$ is upwards closed both in X^1 and X^2 as their associated orders agree. Therefore, by Lemma 4.1.29, $X \setminus U$ is τ_-^2-closed. Hence, $U \in \tau_-^2$. The converse inclusion is the same. □

This means that each stably compact space *uniquely* extends to a d-compact d-regular bispace and each d-compact d-regular bispace can be obtained this way. Moreover, our bitopological approach immediately implies two known facts about duals of stably compact spaces:

5.2.7 Corollary. *For a stably compact space X,*

1. (X, τ^{d}) *is a stably compact space, and*
2. $\tau = \tau^{\mathrm{dd}}$.
3. *If X is compact Hausdorff, then* $\tau = \tau^{\mathrm{d}}$.

Proof. (1) is a consequence of the fact that d-compactness and d-regularity are symmetric axioms. Hence, $(X, \tau^{\mathrm{d}}, \tau)$ is d-compact and d-regular by Proposition 5.2.4 and (X, τ^{d}) is stably compact by Proposition 5.2.5.

(2) From (1) and Proposition 5.2.4 we know that both $(X, \tau^{\mathrm{d}}, \tau)$ and $(X, \tau^{\mathrm{d}}, \tau^{\mathrm{dd}})$ are d-compact and d-regular and, therefore, τ is equal to τ^{dd} by Lemma 5.2.6.

(3) Since (X, τ, τ) and $(X, \tau, \tau^{\mathrm{d}})$ are both d-compact d-regular, by Lemma 5.2.6, τ is equal to τ^{d}. □

5.2.8 Example. Let D be a continuous coherent dcpo with bottom, as defined in [Jun04]. Then, (D, σ_D) is known to be stably compact (2.4 Examples in [Jun04]), where σ_D is the Scott-topology on D, and the bispace (D, σ_D, ω_D) is d-regular where ω_D is the weak-lower topology [Kli12, Proposition 3.5.4]. Since (D, σ_D) is compact iff (D, σ_D, ω_D) is d-compact [Law10], by Lemma 5.2.6 $(\sigma_D)^{\mathrm{d}} = \omega_D$. This is, in fact, true even if D is just a continuous poset [Law10].

In the following statement we observe that this bijection on objects can be extended to an isomorphism of categories, if one restricts morphisms between stably compact spaces to perfect maps:

5.2.9 Proposition. *Let X and Y be stably compact spaces. A map* $f\colon X^{\mathrm{bi}} \to Y^{\mathrm{bi}}$ *is bicontinuous if and only if* $f\colon X \to Y$ *is a perfect map.*

Proof. Since τ_+-compact saturated $K \subseteq Y$ is also τ_--closed, $f^{-1}[K]$ is τ_--closed in X and, therefore, d-compact.

Conversely, each τ_--open $U \subseteq Y$ has a τ_+-compact saturated complement and, because $f^{-1}[Y \setminus U]$ is τ_+-compact saturated, $X \setminus f^{-1}[Y \setminus U] = f^{-1}[U]$ is τ_--open in X by Lemma 4.1.29. □

5.2.10 Theorem.
The categories of stably compact spaces and perfect maps and d-compact d-regular bispaces and bicontinuous maps are isomorphic.

Although the name “d-compact d-regular bispace” might seem mouthful, the actual definition is much simpler than the monotopological definition of stably compact spaces. We already saw a benefit of this in the proof of Corollary 5.2.7 and will see again in Observation 5.2.13 and Theorem 5.4.1. Bitopological proofs of statements about stably compact spaces are tent to be simpler than their monotopological counterparts and this is even more pronounced in more sophisticated constructions such as Vietoris powerspaces, e.g. see Theorem 5.4.1.

The isomorphism of categories from Theorem 5.2.10 composes with the isomorphism we proved in Theorem 2.2.5, yielding the following:

5.2.11 Corollary. *The categories of stably compact spaces and perfect maps and compact partially ordered spaces and monotone continuous maps are isomorphic.*

5.2.12 Remark. We believe that both proofs of the two isomorphisms of categories in Theorems 2.2.5 and 5.2.10 combined are much simpler than any of the known proofs of Corollary 5.2.11 (see, for example, the proof of Proposition VI-6.23 in [Gie+03] or Proposition 9.1.20 and Theorem 9.1.32 in [Gou13]). This is probably because the arguments needed to prove Corollary 5.2.11 are fundamentally bitopological. A similar observation appears also in [Bez+10].

Another by-product of Proposition 5.2.9 is also the following:

5.2.13 Observation. *A continuous map between stably compact spaces $f\colon (X,\tau) \to (Y,\sigma)$ is a perfect map if and only if $f\colon (X,\tau^{\mathrm{d}}) \to (Y,\sigma^{\mathrm{d}})$ is a perfect map.*

Proof. f is perfect iff $f\colon X^{\mathrm{bi}} \to Y^{\mathrm{bi}}$ is bicontinuous and, because $\tau = \tau^{\mathrm{dd}}$ and $\sigma = \sigma^{\mathrm{dd}}$ (Corollary 5.2.7), $f\colon X^{\mathrm{bi}} \to Y^{\mathrm{bi}}$ is bicontinuous iff $f\colon (X,\tau^{\mathrm{d}}) \to (Y,\sigma^{\mathrm{d}})$ is perfect. □

In the following we show that the isomorphism of categories from Theorem 5.2.10 restricts to an isomorphisms of d-compact d-zero-dimensional bispaces and spectral spaces. Then, a combination of this result and Corollary 2.2.6 gives the famous result of Cornish [Cor75] that the categories of spectral spaces and Priestley spaces are isomorphic.

5.2.14 Proposition. *The category of d-compact d-zero-dimensional (T_0) bispaces is isomorphic to the category of spectral spaces.*

Proof. We use Theorem 5.2.10. First, assume that X is a spectral space and denote by $\mathcal{Q}$ the set of all its compact open subsets. Let $x \in U$ for some $U \in \tau$. Because X is spectral, $x \in Q \subseteq U$ for some $Q \in \mathcal{Q}$ and the pair $(Q, X \setminus Q)$ is both disjoint and covers the whole space. On the other hand, let K be some compact saturated. Because X is spectral, every open U is equal to $\bigcup^{\uparrow}\{Q \in \mathcal{Q} \mid Q \subseteq U\}$. Therefore, $K \subseteq U$ implies that $K \subseteq Q \subseteq U$ for some $Q \in \mathcal{Q}$. We have that

$$X \setminus K = X \setminus \bigcap\{U \mid K \subseteq U\} = X \setminus \bigcap\{Q \in \mathcal{Q} \mid K \subseteq Q\} = \bigcup\{X \setminus Q \mid Q \in \mathcal{Q},\, K \subseteq Q\}.$$

In other words, $X \setminus K$ is a union of τ^{d}-open τ-closed sets. We have proved that X^{bi} is d-zero-dimensional.

For the other direction, let (X, τ_+, τ_-) be a d-compact d-zero-dimensional bispace. We know that (X, τ_+) is stably compact. We need to verify that its compact open sets generate τ_+ and that they are closed under finite intersections. To check the former, let $x \in U$ for some $U_+ \in \tau_+$. From d-zero-dimensionality, there is a pair $(V_+, V_-) \in \tau_+ \times \tau_-$ which is both disjoint and covers X such that $x \in V_+ \subseteq U$. Because V_+ is τ_--closed it is also τ_+-compact. To check the latter, let $K, H \subseteq X$ be two τ_+-compact open sets. They are both upwards closed and so, by Lemma 4.1.29, τ_--closed. Therefore, $K \cap H$ is also τ_--closed and hence d-compact. □

The Hofmann-Mislove theorem says that, under local compactness and sobriety, the set $\mathcal{B} \overset{\text{def}}{\equiv} \{X \setminus K \mid K \text{ is compact saturated}\}$ ordered by set-inclusion and the set of Scott-open filters $\Omega(X)^{\wedge}$ ordered by set-inclusion are isomorphic posets. We conclude this section by giving a direct proof of a point-set variant of Lemma 5.1.5:

5.2.15 Proposition. *Let X be a sober topological space:*

1. *If X is coherent and compact, then τ^{d}-closed subsets of X are exactly the compact saturated subsets of X.*
2. *$\mathcal{B}$ (defined above) is a sub-preframe of* $\mathsf{Filt}(\Omega(X))$.
3. *If X is stably compact, then τ^{d} is a subframe of* $\mathsf{Filt}(\Omega(X))$.

Proof. (1) Clearly, $\mathcal{B}$ is closed under finite joins, if X is coherent. Since X is sober (and, therefore, well-filtered), $\mathcal{B}$ is closed under directed unions. Indeed, if $\bigcap_i K_i \subseteq \bigcup_j U_j$ for a filtered family of compact saturated sets $\{K_i\}_i$ and open sets $\{U_j\}_j$, then $K_i \subseteq \bigcup_j U_j$ for some i. Because K_i is compact, a finite subcover of U_j's covers K_i as well as $\bigcap_i K_i$. Moreover, if X is compact, then $\emptyset \in \mathcal{B}$. In summary, if X is sober, coherent and compact, then $\mathcal{B}$ is closed under all finite unions, intersections and directed unions, that is, it is a topology.

(2) First, we check that $\mathcal{B}$ is a sub-poset of $\mathsf{Filt}(\Omega(X))$. Consider the mapping

$$X \setminus K \quad \longmapsto \quad F_K \overset{\text{def}}{\equiv} \{U \mid K \subseteq U\}. \tag{5.2.1}$$

This is clearly monotone. The fact that it is injective follows from Hofmann-Mislove theorem: If $F_K = F_H$, for some compact saturated K and H, then $K = \bigcap F_K$ and $H = \bigcap F_H$ implies $K = H$. Further, the mapping in (5.2.1) preserves finite meets as $F_\emptyset = \Omega(X)$ and $F_K \cap F_H = \{U \mid K \subseteq U \text{ and } H \subseteq U\} = \{U \mid K \cup H \subseteq U\} = F_{K \cup H}$. It also preserves directed joins by well-filteredness since $\bigvee^{\uparrow}_i F_{K_i} = \{U_1 \cap \cdots \cap U_n \mid \forall j \in \{1, \ldots, n\}\ \exists i \text{ s.t. } K_i \subseteq U_j\} = \{U \mid \exists i. \text{ s.t. } K_i \subseteq U\} = \{U \mid \bigcap_i K_i \subseteq U\} = F_{\bigcap_i K_i}$.

(3) Because X is stably compact, $\tau^{\mathrm{d}} = \mathcal{B}$ by (1) and (2) is τ^{d} a sub-preframe of $\mathsf{Filt}(\Omega(X))$. By compactness the smallest filter $\{X\}$ is in the image of $F_{(-)}$ as $F_X = \{X\}$. Lastly, we show that τ^{d} is closed under meets in $\mathsf{Filt}(\Omega(X))$. We need

to show that $F_{K\cap H}$ is equal to $F_K \vee F_H = \{U \cap V \mid K \subseteq U,\ H \subseteq V\}$. Because $K \cap H = \bigcap F_{K\cap H}$ is equal to $\bigcap(F_K \vee F_H)$ and both $F_{K\cap H}$ and $F_K \vee F_H$ are Scott-open (Lemma 5.1.5), it follows that $F_{K\cap H} = F_K \vee F_H$ by Hofmann-Mislove theorem. □

5.3 Stably compact frames and d-frames

It is now clear, from what we have proved in the previous sections, that the categories of stably compact frames and d-compact d-regular d-frames are equivalent, by being dually equivalent to two isomorphic categories (Corollary 5.1.10 and Theorem 5.2.10). In this section we show that this can be proved directly and without the detour to (bi)spaces. One of the benefits of doing this is that, as a result, the new proof will not rely on the Axiom of Choice. Also, by doing so we develop techniques essential for the results in Section 5.4.

Let us first take a look at a construction which mimics $(-)^{\mathrm{bi}}$ for spaces. For a stably compact frame L define L^{bi} to be the structure $(L, L^{\wedge}, \mathrm{con}^{\mathrm{bi}}, \mathrm{tot}^{\mathrm{bi}})$ where, for an $x \in L$ and a Scott-open filter $\mathcal{S} \in L^{\wedge}$,

$$(x, \mathcal{S}) \in \mathrm{con}^{\mathrm{bi}} \overset{\mathrm{def}}{\equiv} \mathcal{S} \subseteq \uparrow x \qquad \text{and} \qquad (x, \mathcal{S}) \in \mathrm{tot}^{\mathrm{bi}} \overset{\mathrm{def}}{\equiv} x \in \mathcal{S}$$

5.3.1 Example. For a stably compact space X, the definition of $\mathrm{con}^{\mathrm{bi}}$ in $\Omega(X)^{\mathrm{bi}}$ exactly matches the consistency relation of $\Omega_d(X^{\mathrm{bi}})$ as $U \cap (X \setminus K) = \emptyset$ iff $U \subseteq V$ for every open $V \supseteq K$. The same is true for the totality relation as $U \cup (X \setminus K) = X$ iff $K \subseteq U$.

5.3.2 Lemma. *Let L be a stably compact frame. Then, L^{bi} is a d-frame.*

Proof. By Lemma 5.1.5 we know that $L^{\wedge}$ is a frame. We need to verify that the axioms of d-frames are satisfied.

Information order axioms: (tot-↑) If $(x, \mathcal{S}) \in \mathrm{tot}^{\mathrm{bi}}$ and $(x, \mathcal{S}) \sqsubseteq (y, \mathcal{T})$, then y is in $\mathcal{T}$ because $\mathcal{T}$ is upwards closed and $x \in \mathcal{S} \subseteq \mathcal{T}$. (con-↓) is similar: if $(y, \mathcal{T}) \in \mathrm{tot}^{\mathrm{bi}}$ and $(x, \mathcal{S}) \sqsubseteq (y, \mathcal{T})$, then $\mathcal{S} \subseteq \mathcal{T} \subseteq \uparrow y \subseteq \uparrow x$. For (con-$\bigsqcup^{\uparrow}$) let $\{(x_i, \mathcal{S}_i)\}_i$ be a directed subset of $\mathrm{con}^{\mathrm{bi}}$ and let $x \in \bigvee_i^{\uparrow} \mathcal{S}_i = \bigcup_i^{\uparrow} \mathcal{S}_i$. There is an i such that $x \in \mathcal{S}_i$. Moreover, for every $j \geq i$, $x \in \mathcal{S}_i \subseteq \mathcal{S}_j \subseteq \uparrow x_j$. Therefore, $x \in \bigcap_i \uparrow x_i = \uparrow(\bigvee_i^{\uparrow} x_i)$. This proves that $\bigvee_i^{\uparrow} \mathcal{S}_i \subseteq \uparrow(\bigvee_i^{\uparrow} x_i)$.

Logical order axioms: (con-⩒, ⩓) Let $(x, \mathcal{S}), (y, \mathcal{T}) \in \mathrm{con}^{\mathrm{bi}}$. Then, $(x \wedge y, \mathcal{S} \vee \mathcal{T}) \in \mathrm{con}^{\mathrm{bi}}$ because the join $S \vee \mathcal{T}$ is computed in $\mathrm{Filt}(L)$, i.e. it is the set $\{s \wedge t \mid s \in \mathcal{S}, t \in \mathcal{T}\}$. Also, $(x \vee y, \mathcal{S} \wedge \mathcal{T}) \in \mathrm{con}^{\mathrm{bi}}$ because $\mathcal{S} \wedge \mathcal{T}$ is the intersection $\mathcal{S} \cap \mathcal{T}$ which is upwards closed. Finally $\{1\} \subseteq \uparrow 1$ and $L \subseteq \uparrow 0$ proves that $tt, ff \in \mathrm{con}^{\mathrm{bi}}$. Next, for (tot-⩒, ⩓) assume $(x, \mathcal{S}), (y, \mathcal{T}) \in \mathrm{tot}^{\mathrm{bi}}$. Their logical join and meet are also in $\mathrm{tot}^{\mathrm{bi}}$ since $x \vee y \in \mathcal{S} \cap \mathcal{T}$, from being upwards closed, and $x \in \mathcal{S} \subseteq \mathcal{S} \vee \mathcal{T}$ and, similarly, $y \in \mathcal{S} \vee \mathcal{T}$ implies that $x \wedge y \in \mathcal{S} \vee \mathcal{T}$. Also, $1 \in \{1\}$ and $0 \in L$ proves $tt, ff \in \mathrm{tot}^{\mathrm{bi}}$.

Lastly, to show (con-tot), let $(x, \mathcal{S}) \in \mathrm{con}^{\mathrm{bi}}$ and $(y, \mathcal{T}) \in \mathrm{tot}^{\mathrm{bi}}$. If $x = y$, then $\mathcal{S} \subseteq \uparrow x \subseteq \mathcal{T}$ from $x \in \mathcal{T}$. If, on the other hand, $\mathcal{S} = \mathcal{T}$, then $y \in \mathcal{S} \subseteq \uparrow x$ and so $x \leq y$. □

5.3.3 Observation. *Let L be a locally compact frame and $\mathcal{S} \subseteq L$ a Scott-open filter. Then, $\mathcal{S} = \bigcup^{\uparrow}_{x \in \mathcal{S}} \twoheaduparrow x$ where $\twoheaduparrow x = \{y \mid x \ll y\}$.*

Proof. Let $z \in \mathcal{S}$. From local compactness, $z = \bigvee^{\uparrow}\{x \mid x \ll z\}$ and so $x \in \mathcal{S}$ for some $x \ll z$. Hence, $z \in \twoheaduparrow x$. On the other hand, for every $x \in \mathcal{S}$, $\twoheaduparrow x \subseteq \mathcal{S}$, therefore, $\mathcal{S} = \bigcup^{\uparrow}_{x \in \mathcal{S}}(\twoheaduparrow x)$. □

5.3.4 Proposition. *For a stably compact frame L, L^{bi} is a d-compact d-regular d-frame.*

Proof. First we check d-regularity. Because L is locally compact, we know that every x in L is equal to $\bigvee^{\uparrow}\{a \mid a \ll x\}$. Moreover, $a \ll x$ implies that $x \in \mathcal{S} \subseteq \uparrow a$ for some Scott-open filter $\mathcal{S}$ (Lemma 5.1.5) and so $a \lhd_{+} x$ because $(a, \mathcal{S}) \in \mathsf{con}^{\mathrm{bi}}$ and $(x, \mathcal{S}) \in \mathsf{tot}^{\mathrm{bi}}$. On the other hand, if $x \in \mathcal{S}$ for some Scott-open filter $\mathcal{S}$, then $(x, \mathcal{S}) \in \mathsf{tot}^{\mathrm{bi}}$, $(x, \twoheaduparrow x) \in \mathsf{con}^{\mathrm{bi}}$ and $\twoheaduparrow x \lhd_{-} \mathcal{S}$. Therefore, since $\mathcal{S} = \bigvee^{\uparrow}_{x \in \mathcal{S}} \twoheaduparrow x = \bigcup^{\uparrow}_{x \in \mathcal{S}} \twoheaduparrow x$ (Observation 5.3.3), every Scott-open filter $\mathcal{S}$ is a (directed) join of elements $\lhd$-below it.

Next, for d-compactness, let $\{(x_i, \mathcal{S}_i)\}_{i \in I}$ be a directed subset of $L \times L^{\wedge}$ with the join $\bigsqcup^{\uparrow}_i (x_i, \mathcal{S}_i)$ in $\mathsf{tot}^{\mathrm{bi}}$. This means that $\bigvee^{\uparrow}_i x_i \in \bigcup^{\uparrow}_i \mathcal{S}_i$ and, therefore, $\bigvee^{\uparrow}_i x_i \in \mathcal{S}_j$ for some $j \in I$. Since $\mathcal{S}_j$ is Scott-open, $x_k \in \mathcal{S}_j$ for some $k \in I$. This means that, for some $i \in I$ such that $i \geq j$ and $i \geq k$, $x_i \in \mathcal{S}_i$. □

5.3.5 Lemma. *For a d-compact d-regular d-frame $\mathcal{L} = (L_{+}, L_{-}, \mathsf{con}, \mathsf{tot})$ and $a, b \in L_{+}$ or $a, b \in L_{-}$,*

1. *$a \lhd b$ iff $a \ll b$, and*

2. *L_{+} and L_{-} are stably compact frames.*

Proof. (1) Whenever $b \leq \bigvee^{\uparrow} D$, then, from d-compactness, because $(b, a^*) \in \mathsf{tot}$, $(d, a^*) \in \mathsf{tot}$ for some $d \in D$. Hence, $a \leq d$. On the other hand, if $a \ll b$ then, because $\mathcal{L}$ is d-regular, $b = \bigvee^{\uparrow}\{c \mid c \lhd b\}$ and so $a \leq c \lhd b$ for some c.

(2) L_{+} is locally compact and also compact since $1 \lhd 1$. Moreover, it is coherent because $x \lhd a$ and $x \lhd b$ implies $x \lhd (a \wedge b)$ (Lemma 2.3.12). □

We have defined mappings between the category of stably compact frames and d-compact d-regular d-frames and back

$$L \mapsto L^{\mathrm{bi}} \quad \text{and} \quad \mathcal{L} \mapsto L_{+}.$$

The following lemma shows that those two mappings are, up to isomorphism, inverse to each other.

5.3.6 Lemma. *Let* $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ *be a d-compact d-regular d-frame. Then,*

1. *The mapping* $y \mapsto \mathcal{S}_y \overset{def}{\equiv} \{x \mid (x,y) \in \mathsf{tot}\}$ *establishes an isomorphisms between* L_- *and* $(L_+)^\wedge$
2. $(x,y) \in \mathsf{con}$ *iff* $\mathcal{S}_y \subseteq \uparrow x$,
3. $(x,y) \in \mathsf{tot}$ *iff* $x \in \mathcal{S}_y$.

In other words, $\mathcal{L}$ *is isomorphic to the d-frame* $(L_+)^{\mathrm{bi}}$ *and, symmetrically, also to* $((L_-)^{\mathrm{bi}})^{-1}$ *where*

$$\mathcal{L}^{-1} \overset{def}{\equiv} (L_-, L_+, \mathsf{con}^{-1}, \mathsf{tot}^{-1}).$$

Proof (inspired by Theorem 6.15 in [JM06]). (1) Define the maps:

$$k\colon L_- \longrightarrow (L_+)^\wedge \qquad l\colon (L_+)^\wedge \longrightarrow L_-$$
$$y \longmapsto \mathcal{S}_y \qquad\qquad \mathcal{S} \longmapsto \bigvee I_{\mathcal{S}}$$

where $I_{\mathcal{S}}$ is the ideal $\downarrow\{x^* \mid x \in \mathcal{S}\}$ or, equivalently, $\{y \in L_- \mid \exists x \in \mathcal{S}.\ (x,y) \in \mathsf{con}\}$. The mapping k is well-defined by Lemma 2.4.2 and, moreover, both k and l are monotone. Next, observe that $l \cdot k = \mathrm{id}$. By definition, $lk(y) = \bigvee\{z \mid \exists x \in k(y).\ (x,z) \in \mathsf{con}\} = \bigvee\{z \mid \exists x.\ (x,y) \in \mathsf{tot} \text{ and } (x,z) \in \mathsf{con}\} = \bigvee\{z \mid z \lhd_- y\} = y$ from d-regularity. Also, $k \cdot l = \mathrm{id}$ as

$$kl(\mathcal{S}) = \{w \mid (w, \bigvee I_{\mathcal{S}}) \in \mathsf{tot}\} = \{w \mid \exists x \in \mathcal{S}.\ (w, x^*) \in \mathsf{tot}\} = \mathcal{S}$$

where the middle equality follows from d-compactness and the last equality is a consequence of the fact that $w \in \mathcal{S}$ iff $x \in \mathcal{S}$ for some $x \lhd w$, by d-regularity.

Because (3) holds by definition, the last thing to check is (2): (con-tot) implies that $(x,y) \in \mathsf{con}$ and so $\mathcal{S}_y \subseteq \uparrow x$. Conversely, we have that $w \lhd y$ implies that $x \leq w^*$ and so $(x,w) \in \mathsf{con}$. Therefore, by d-regularity and (con-$\bigsqcup^\uparrow$), $(x,y) = (x, \bigvee^\uparrow\{w \mid w \lhd y\}) \in \mathsf{con}$.

The isomorphism between $\mathcal{L}$ and $(L_+)^{\mathrm{bi}}$ is given by the pair of frame isomorphisms $(\mathrm{id}_{L_+}, k)\colon \mathcal{L} \xrightarrow{\cong} (L_+)^{\mathrm{bi}}$. Similarly, (k', id_{L_-}) establishes an isomorphisms between $\mathcal{L}$ and $((L_-)^{\mathrm{bi}})^{-1} = ((L_-)^\wedge, L_-, (\mathsf{con}^{\mathrm{bi}})^{-1}, (\mathsf{tot}^{\mathrm{bi}})^{-1})$ where k' is the isomorphisms between L_+ and $(L_-)^\wedge$ computed symmetrically to k. □

5.3.7 Proposition. *A d-frame* $\mathcal{L} = (L_+, L_-, \mathsf{con}, \mathsf{tot})$ *is d-compact and d-regular if and only if* $\mathcal{L} \cong L^{\mathrm{bi}}$ *for some stably compact frame L.*

Moreover, if $\mathcal{L}$ *and* $\mathcal{M} = (M_+, M_-, \mathsf{con}', \mathsf{tot}')$ *are both d-compact and d-regular such that either* $M_+ \cong L_+$ *or* $M_- \cong M_-$*, then* $\mathcal{L} \cong \mathcal{M}$.

Proof. The right-to-left direction follows from Proposition 5.3.4. Conversely, by Lemmas 5.3.5 and 5.3.6, L_+ is stably compact and, therefore $\mathcal{L} \cong (L_+)^{\mathrm{bi}}$.

For the "Moreover" part, assume $L_+ \cong M_+$. Then, $\mathcal{L} \cong (L_+)^\wedge \cong (M_+)^\wedge \cong \mathcal{M}$. Also, $\mathcal{L}$ is d-compact and d-regular iff $\mathcal{L}^{-1}$ is, and so $\mathcal{L}$ is isomorphic to $\mathcal{M}$ whenever $L_- \cong M_-$ for the same reason: $\mathcal{L} \cong ((L_-)^{\mathrm{bi}})^{-1} \cong ((M_-)^{\mathrm{bi}})^{-1} \cong \mathcal{M}$. □

This proposition implies that, in case of d-compact d-regular d-frames, either of the frame components determines the rest of the structure. A direct consequence of this is a result similar to Corollary 5.2.7 for stably compact spaces.

5.3.8 Corollary. *Let L be a stably compact frame. Then,*

1. *$L^\wedge$ is a stably compact frame,*
2. *$L \cong (L^\wedge)^\wedge$, and*
3. *$(L^\wedge)^{\mathrm{bi}} \cong (L^{\mathrm{bi}})^{-1}$*
4. *for any Scott-open filters $\mathcal{S}, \mathcal{T} \in L^\wedge$, $\mathcal{S} \ll \mathcal{T}$ in $L^\wedge$ iff there exists an $x \in L$ such that $x \in \mathcal{T}$ and $\mathcal{S} \subseteq {\uparrow} x$.*

Proof. L^{bi} is d-compact d-regular by Proposition 5.3.4. Therefore, $L^\wedge$ is stably compact (Lemma 5.3.5) and $L \cong (L^\wedge)^\wedge$ (Lemma 5.3.6). The "Moreover" part of Proposition 5.3.7 implies that $(L^\wedge)^{\mathrm{bi}}$ is isomorphic $(L^{\mathrm{bi}})^{-1}$.

4. follows from the fact that $(L, L^\wedge, \mathsf{con}, \mathsf{tot})$ is d-compact d-regular and, therefore, by Lemma 5.3.5, $\mathcal{S} \ll \mathcal{T}$ iff $\mathcal{S} \lhd \mathcal{T}$. This is, by definition, whenever $x \in \mathcal{T}$ and $\mathcal{S} \subseteq {\uparrow} x$ for some $x \in L$. □

Furthermore, there is also one immediate consequence for the frames of filters and ideals of distributive lattices:

5.3.9 Corollary. *Let D be a distributive lattice. Then, $\mathsf{Idl}(D)^\wedge \cong \mathsf{Filt}(D)$ and $\mathsf{Filt}(D)^\wedge \cong \mathsf{Idl}(D)$.*

Proof. Since $\mathcal{IF}(D)$ is d-compact and d-regular (Lemma 2.6.8), $\mathcal{IF}(D) \cong \mathsf{Idl}(D)^{\mathrm{bi}}$ and $\mathcal{IF}(D)^{-1} \cong \mathsf{Filt}(D)^{\mathrm{bi}}$ by Lemma 5.3.6. □

Coming back to the relationship between stably compact frames and d-compact d-regular d-frames, we established in Propositions 5.3.4 and 5.3.7 that the mappings between objects of **SCFrm** and **d-KReg**, i.e. $L \mapsto L^{\mathrm{bi}}$ and $(L_+, L_-, \mathsf{con}, \mathsf{tot}) \mapsto L_+$, are inverse to each other (up to isomorphism). To establish an equivalence of categories we need to extend this correspondence to morphisms.

5.3.10 Lemma. *If $(h_+, h_-), (h_+, h'_-) \colon \mathcal{L} \to \mathcal{M}$ are d-frame homomorphisms between two d-regular d-frames, then $h_- = h'_-$.*

Proof. Let $x \in L_-$ and $a \lhd x$. This means that $(a^*, x) \in \mathsf{tot}_{\mathcal{L}}$ and $(a^*, a) \in \mathsf{con}_{\mathcal{L}}$, and so $(h_+(a^*), h'_-(x)) \in \mathsf{tot}_{\mathcal{M}}$ and $(h_+(a^*), h_-(a)) \in \mathsf{con}_{\mathcal{M}}$, from which axiom (con-tot) gives $h_-(a) \leq h'_-(x)$. Therefore, because $h_-(x) = \bigvee\{h'_-(a) \mid a \lhd x\}$, we get that $h_-(x) \leq h'_-(x)$. The reverse inequality is the same. □

The last lemma is a result shown by Klinke (Theorem 3.5.8 in [Kli12] or Theorem 2 in [KJM11]). We use this to show the wanted equivalence of morphisms:

5.3.11 Lemma. *Let L and M be stably compact frames. The mapping $h \mapsto (h, h^\wedge)$ establishes a bijection between perfect frame homomorphisms $L \to M$ and d-frame homomorphisms $L^{\mathrm{bi}} \to M^{\mathrm{bi}}$.*

Proof. In Lemma 5.1.6 we showed that, if h is a perfect homomorphism, then $h^\wedge$ is a frame homomorphism, therefore $h^{\mathrm{bi}} = (h, h^\wedge)$ is well-defined. Moreover, it is a d-frame homomorphism: If $(x, \mathcal{S}) \in \mathsf{con}^{\mathrm{bi}}$, i.e. $\mathcal{S} \subseteq {\uparrow}x$, then, clearly, ${\uparrow}h[\mathcal{S}] \subseteq {\uparrow}h[{\uparrow}x] = {\uparrow}h(x)$. Also, $(x, \mathcal{S}) \in \mathsf{tot}^{\mathrm{bi}}$, i.e. $x \in \mathcal{S}$, implies that $h(x) \in h[\mathcal{S}] \subseteq {\uparrow}h[\mathcal{S}]$.

Moreover, if $g = (g_+, g_-)\colon L^{\mathrm{bi}} \to M^{\mathrm{bi}}$ is a d-frame homomorphism, then g_+ preserves $\lhd_+$ because g preserves con and tot. Therefore, by Lemma 5.3.5, g_+ also preserves the way-below relation. The fact that $g_- = {g_+}^\wedge$ follows from Lemma 5.3.10. □

5.3.12 Theorem.
The category of stably compact frames and perfect maps is equivalent to the category of d-compact d-regular d-frames and d-frame homomorphisms.

Consequently, since the category of d-compact d-regular d-frames is complete and cocomplete (Section 3.3.2), so are **SCFrm** and (by duality) also **SCTop**. In the following we prove a frame-theoretic variant of Proposition 5.2.14.

5.3.13 Proposition. *The category of spectral frames and perfect maps is equivalent to the category of d-compact d-zero-dimensional d-frames.*

Proof. In the proof of Proposition 5.1.13 we showed that every spectral frame is stably compact. Therefore, it is enough to show that the equivalence from Theorem 5.3.12 restricts to **SpecFrm** and **d-Pries**.

Let L be a stably compact frame. Observe that $(x, \mathcal{S}) \in \mathsf{con}^{\mathrm{bi}} \cap \mathsf{tot}^{\mathrm{bi}}$ iff $\mathcal{S}$ is equal to ${\uparrow}x$ iff $x \ll x$. Therefore, if L^{bi} is d-zero-dimensional, then L is generated by its compact elements and also $k \ll k$ and $h \ll h$ imply $(k \wedge k) \ll (k \wedge h)$ because $(k, {\uparrow}k), (h, {\uparrow}h) \in \mathsf{con}^{\mathrm{bi}} \cap \mathsf{tot}^{\mathrm{bi}}$ implies that $(k \wedge h, ({\uparrow}k) \vee ({\uparrow}h)) \in \mathsf{con}^{\mathrm{bi}} \cap \mathsf{tot}^{\mathrm{bi}}$.

On the other hand, if L is spectral, L^{bi} is d-zero-dimensional. Indeed, a Scott-open filter $\mathcal{S} \in L^\wedge$ is equal to the union $\bigcup\{{\uparrow}k \mid k \in \mathcal{S} \text{ and } k \ll k\}$ because each $x \in \mathcal{S}$ is equal to $\bigvee^{\uparrow}\{k \mid k \ll k \leq x\}$. Further, each $x \in L$ is generated by k's from L which are $k \ll k$. Because $k \ll k$ implies $(k, {\uparrow}k) \in \mathsf{con}^{\mathrm{bi}} \cap \mathsf{tot}^{\mathrm{bi}}$, both x and $\mathcal{S}$ are generated by complemented elements of L^{bi}. □

5.3.14 Remark. Banaschewski and Brümer show in [BB88] a biframe variant of Theorem 5.3.12 and later a generalisation of this also appeared in [Mat01]. In both cases, the authors needed to construct an ambient frame L_0 in which L_+ and L_- live. They do this by making L_0 a subframe of of a frame of congruences which leads to further difficulties, when compared to our approach. More recently Escardó [Esc99] and Klinke [Kli13] described a more elementary construction of the ambient frame L_0.

A biframe version of Proposition 5.3.13 was first published in [Pic94]. Then, Achim Jung and Drew Moshier gave a constructive and d-frame proof of the same result in [JM06]. The main difference here is that we give a direct proof; as opposed to Jung's and Moshier's proof which goes via distributive lattices and partial frames.

5.3.15 Compact regular and spectral frames. It is immediate to verify that each compact regular frame L is stably compact and that all frame homomorphisms between compact regular frames are perfect; that is, **KRegFrm** is a full subcategory of **SCFrm**. Moreover, the plus-frame of the d-compact d-regular d-frame $L^{\bowtie} = (L, L, \mathsf{con}_L, \mathsf{tot}_L)$ from Section 2.5 agrees with the plus frame of $L^{\mathrm{bi}} = (L, L^{\wedge}, \mathsf{con}^{\mathrm{bi}}, \mathsf{tot}^{\mathrm{bi}})$. Hence, by Proposition 5.3.7, $L \cong L^{\wedge}$ and $L^{\bowtie} \cong L^{\mathrm{bi}}$.

Furthermore, because homomorphisms between d-regular d-frames are determined be either of their components (Lemma 5.3.10), the functor

$$(-)^{\mathrm{bi}} \colon \mathbf{SCFrm} \to \mathbf{d\text{-}KReg}$$

when restricted to the category of compact regular frames must be exactly the same as the functor $(-)^{\bowtie} \colon \mathbf{KRegFrm} \to \mathbf{d\text{-}KReg}$.

Similarly, in the proof of Corollary 5.3.9, we pointed out that $\mathsf{Idl}(D)^{\mathrm{bi}} \cong \mathcal{IF}(D)$. This means that, for a spectral frame L, $L^{\mathrm{bi}} \cong \mathcal{IF}(D)$ where D is the lattice such that $L \cong \mathsf{Idl}(D)$. Also, for the same reason as above, $(-)^{\mathrm{bi}} \circ \mathsf{Idl}(-)$ and $\mathcal{IF}$ agree on morphisms.

In other words, $(-)^{\mathrm{bi}}$ generalises the two previously defined constructions since

$$\begin{aligned} (-)^{\mathrm{bi}} &\cong (-)^{\bowtie} &&: \mathbf{KRegFrm} \to \mathbf{d\text{-}KReg}, \quad \text{and} \\ (-)^{\mathrm{bi}} \circ \mathsf{Idl}(-) &\cong \mathcal{IF} &&: \mathbf{DLat} \to \mathbf{d\text{-}KReg}. \end{aligned}$$

5.4 Vietoris constructions and their duals

We conclude this chapter by showing some applications of results from Chapter 4 to the theory of stably compact spaces and frames.

The Vietoris powerspace constructions from Section 4.1.3 play an important role in domain theory because they are used to give a semantics to non-deterministic computations. For a stably compact space X, elements of $\mathbb{V}_{\boxtimes}(X)$, $\mathbb{V}_{\oplus}(X)$ and $\mathbb{V}(X)$ are considered to be models of *demonic, angelic* or *erratic choice*, respectively (as explained in [Gou10]). Since powerspace constructions have been intensively studied in the past, we do not aim to provide any new results in the following theorem. The novelty lies in the proof which is bitopological, shorter and, arguably, much easier to follow:

5.4.1 Theorem.
For a stably compact space (X, τ),

1. $\mathbb{V}_{\boxtimes}(X)$, $\mathbb{V}_{\oplus}(X)$ *and* $\mathbb{V}(X)$ *are all stably compact spaces, and*
2. $\mathbb{V}_{\boxtimes}(X)^{\mathrm{d}} \cong \mathbb{V}_{\oplus}(X^{\mathrm{d}})$, $\mathbb{V}_{\oplus}(X)^{\mathrm{d}} \cong \mathbb{V}_{\boxtimes}(X^{\mathrm{d}})$ *and* $\mathbb{V}(X)^{\mathrm{d}} \cong \mathbb{V}(X^{\mathrm{d}})$.

Proof. Let, $X^{\mathrm{bi}} = (X, \tau, \tau^{\mathrm{d}})$ be the associated d-compact d-regular bispace to X. We know that $\mathbb{W}_{\boxtimes}(X^{\mathrm{bi}}) = (\mathsf{Clos}_{-}(X), \mathbb{V}_{\boxtimes}\tau, \mathbb{V}_{\oplus}\tau^{\mathrm{d}})$ is again d-compact and d-regular (Theorem 4.1.32 and Proposition 4.1.33) where $\mathsf{Clos}_{-}(X)$ is the set of τ^{d}-closed subsets of X, i.e. compact saturated subsets of X (Proposition 5.2.15). Therefore, $\mathbb{V}_{\boxtimes}(X)$ is equal to the stably compact space $(\mathsf{Clos}_{-}(X), \mathbb{V}_{\boxtimes}\tau)$ (Proposition 5.2.5) and its dual is $\mathbb{V}_{\oplus}(X^{\mathrm{d}}) = (\mathsf{Clos}_{-}(X), \mathbb{V}_{\oplus}\tau^{\mathrm{d}})$ by Proposition 5.2.4 and Lemma 5.2.6.

The other cases are proved accordingly. □

The main ingredient of the proof is the fact that Vietoris construction is closed on the category of d-compact d-regular bispaces. The proof of this fact is arguably more straightforward then the corresponding purely monotopological argument about Vietoris construction on stably compact spaces [Gou10]. This is again a consequence of the fact that the bitopological definitions are simpler, as discussed below Theorem 5.2.10

Similarly to Theorem 5.4.1 an analogous result holds for the frame-theoretic powerspace constructions and frame-theoretic duals of stably compact frames:

5.4.2 Theorem.
For a stably compact frame L,

1. $\mathbb{V}_{\Box}(L)$, $\mathbb{V}_{\Diamond}(L)$ *and* $\mathbb{V}_{\mathrm{Fr}}(L)$ *are all stably compact frames, and*
2. $\mathbb{V}_{\Box}(L)^{\wedge} \cong \mathbb{V}_{\Diamond}(L^{\wedge})$, $\mathbb{V}_{\Diamond}(L)^{\wedge} \cong \mathbb{V}_{\Box}(L^{\wedge})$ *and* $\mathbb{V}_{\mathrm{Fr}}(L)^{\wedge} \cong \mathbb{V}_{\mathrm{Fr}}(L^{\wedge})$.

Proof. From Proposition 5.3.4 we know that L^{bi} is d-compact and d-regular. Therefore, $\mathbb{W}_{\Box}(L^{\mathrm{bi}}) = (\mathbb{V}_{\Box}(L), \mathbb{V}_{\Diamond}(L^{\wedge}), \mathsf{con}_{\mathbb{W}_{\Box}}, \mathsf{tot}_{\mathbb{W}_{\Box}})$ is also d-compact d-regular (Theorem 4.2.12) and $\mathbb{V}_{\Box}(L)$ is stably compact (Lemma 5.3.5). Hence, by Lemma 5.3.6, $\mathbb{V}_{\Box}(L)^{\wedge}$ is isomorphic to $\mathbb{V}_{\Diamond}(L^{\wedge})$.

The other cases are proved accordingly. □

5.4.3 Remark. Because $\Omega(\Sigma(L)^{\mathrm{d}}) \cong L^{\wedge}$, the previous theorem is a direct consequence of Theorem 5.4.1. However, such a detour to spaces requires the Axiom of

Choice. A purely frame-theoretic proof of this fact that, for a stably compact frame L, $\mathbb{V}_{\Box}(L)$ and $\mathbb{V}_{\Diamond}(L)$ are also stably compact, was already published in [Vic04]. Although, both [Vic04] and also [Keg02] suggest a framework which can be used to tackle this, they both leave the case for $\mathbb{V}_{\mathrm{Fr}}(L)$ open. Therefore, to the knowledge of the author, our proof that $\mathbb{V}_{\mathrm{Fr}}(L)$ is stably compact is the first purely frame-theoretic proof of this result which does not require the Axiom of Choice. Moreover, a choice-free proof of the second part of Theorem 5.4.2 is also entirely new.

6

Belnap-Dunn logic of bispaces

In this chapter we show that bispaces and d-frames are naturally associated with a paraconsistent version of the logic of observable properties. Such logic allows reasoning even in the presence of conflicting or missing information. We also make a comparison with bilattices, which are structures also designed to be well suited for paraconsistent reasoning. It turns out that d-compact d-regular d-frames, while providing a much larger class of models, model most of the logic of bilattices. Contrary to Belnap's proposal, bilattice logic cannot capture predicates which are only obtainable by an approximating computation. For this reason, we design a logic of d-frames, which is aimed to solve this problem, and show its soundness and completeness.

6.1 The logic of bispaces and d-frames

6.1.1 Geometric logic of spaces

The topological view of a *property* or *predicate* is as the set of models or states which satisfy it. An important class of properties in theoretical computer science are those which are, so called, *observable* – that is, properties for which we can determine their validity in a state based only on a finite observation or, in other words, by inspecting only a finite amount of information about the state [Abr87a; Abr87b]. Borrowing terminology from computability theory, observable properties are exactly the semidecidable or recursively enumerable sets.

Since observable properties are closed under unions and finite intersections, the set of states (or models) equipped with the set of all observable properties forms a topological space. This combines well with the fact that frames of open sets are associated with theories of *geometric logic*; that is, infinitary joins and finite meets correspond to infinitary disjunctions and finitary conjunctions. As a result, we can think of topological spaces or frames as logical theories of observable properties. Furthermore, the same way open sets approximate the notion of an observable property, under Dana Scott's interpretation, continuous functions approximate the notion of a computable function.

It has to be said that none of these ideas and interpretations is new. In fact, the research program studying the connections between computation and topology goes back to Scott's [Sco70; Sco76] which was immediately picked up by many followers, such as [Vic89; Smy83; Smy92a; Abr87b; Esc04; AJ94], and resulted in a fruitful exchange of ideas between the theory of computation, topology and logic. For example, Martín Escardó formally proved that "A subset Q of a topological space X is compact if and only if its universal-quantification functional is continuous." (Lemma 1.4.1 in [Esc04]).

For ease of reference, we summarise the correspondences between some topological, algebraic and logical notions in the following table:

Space X	**Frame** L	**Logical interpretation**
Open set $U \subseteq X$	Element $a \in L$	Observable property
Point $x \in X$	Completely prime filter $P \subseteq L$	Model, i.e. maximally consistent geometric theory
Union of open sets	Join of elements in L	Infinitary disjunction
Finite intersection of open sets	Finite meet in L	Finite conjunction
Compact set $K \subseteq X$	Scott-open filter $\mathcal{S} \subseteq L$	(Computationally) universally quantifiable property

Table 6.1: Topological vs. logical interpretation

6.1.2 Logical aspects of bispaces and d-frames

Taking inspiration from (mono)topological spaces, we can also interpret the structure of a bitopological space (X, τ_+, τ_-) in logical terms. Each of the topologies corresponds to a logical theory of observable properties. As suggested by the notation, τ_+ represents the frame of all *positive observations* and τ_- all *negative observations*. Then, performing an observation results in a pair of open sets $(U_+, U_-) \in \tau_+ \times \tau_-$ where U_+ determines the states where the examined property *observably holds* and U_- determines the states where the predicate *observably fails*.

Notice that observably "failing" (i.e. $x \in U_-$) is something different from "not holding" (i.e. $x \notin U_+$). Take, for example, a property given informally as:

"the program stops and outputs an even number".

If a program never stops, its output is neither even nor odd and so it does not belong to U_+ and neither to U_-. Although observably failing and not-holding are different in this example, they are somehow related. Namely, if a program stops and outputs an odd number, the property cannot hold. This can be manifested as

"failing implies not holding" (con)

or equivalently "holding implies not failing". Mathematically speaking, this happens whenever $U_+ \cap U_- = \emptyset$ as it cannot happen that state would both observably satisfy and fail the property; in other words, (con) expresses the fact that the observation (U_+, U_-) is *consistent*.

In addition, the reverse implication to (con), that is,

"not holding implies failing", (tot)

also has a logical reading. It corresponds to $U_+ \cup U_- = X$ which translates as: for each state $x \in X$, either the property holds ($x \in U_+$) or it fails ($x \in U_-$). We call such observations *total*.

An example of a property which is both consistent and total would be "the program outputs number 56 or nothing after 100 steps". On the other hand, properties which are total but not consistent are common for concurrent programs. Consider running an instance of "the value of the variable x is 56" in two parallel threads. In this case, we might get both "true" and "false" as answers, depending on the scheduler.

To summarise, a general picture of an observation $(U_+, U_-) \in \tau_+ \times \tau_-$ which is not consistent nor total is shown in Figure 6.1. The states $x \in U_+ \setminus U_-$ in the picture are observably true but not observably false and the opposite is the case for the states in $U_- \setminus U_+$. Further, the points in $U_+ \cap U_-$ are non-classical as they are both true and false, i.e. they represent a *contradiction*, and points in $X \setminus (U_+ \cup U_-)$ are also non-classical as they are neither true nor false, i.e. they represent having *no information*. Those four options determine four logical values in the logic associated to bispaces: true, false, contradiction and no-information.

All these ideas inspired the terminology and naming conventions of d-frames. We can think of d-frames as of a logical manifestation of bitopological spaces. What is left to explain is the motivation behind the logical and information orders. If $U_+ \subseteq V_+$ and $U_- \subseteq V_-$, for some $(V_+, V_-) \in \tau_+ \times \tau_-$, that means that wherever (U_+, U_-) is true (resp. false) (V_+, V_-) also is and, moreover, some states which were previously not true, not false or even carried no information, might have become true or false. In brief, (V_+, V_-) caries more information than (U_+, U_-). If, on the

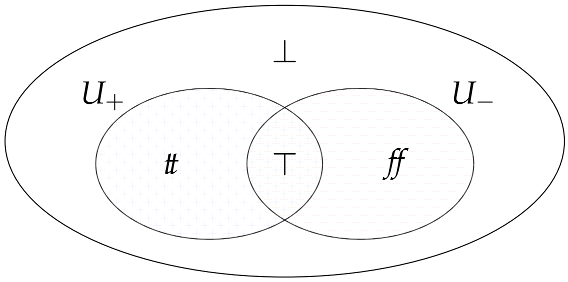


Figure 6.1: A general predicate (U_+, U_-)

other hand, $U_+ \subseteq V_+$ and $U_- \supseteq V_-$, then there is more true and less false evidence in (V_+, V_-) than is in (U_+, U_-), making the former "closer to" the everywhere true observation $(X, \emptyset)$ than (U_+, U_-) is.

6.2 Comparison with bilattices

6.2.1 Bilattices

Ginsberg, in his famous paper [Gin88], also considered splitting the information and logical orders to represent the knowledge of an AI system. This led him to consider an example which, by the nature, is very close to bispaces. Namely, Ginsberg took the (complete) Boolean algebra $\mathscr{P}(X)$ of all subsets[1] of a set of states X as the lattice of both positive and negative predicates. With this, every pair $(U_+, U_-) \in \mathscr{P}(X) \times \mathscr{P}(X)$ represents a predicate which assigns one of the four-values to the states of X the same way as in Figure 6.1.

An algebraic manifestation of these ideas are *(implicative) bilattices* which are structures

$$(B \times B, \sqcap, \sqcup, \mathbin{\wedge\!\!\!\!\cdot}, \mathbin{\vee\!\!\!\!\cdot}, \bot, \top, \mathit{ff}, \mathit{tt}, \neg, \supset)^{2}$$

where B is a Boolean algebra, the $\supset$ and $\neg$–free reduct is defined as in d-frames (see Observation 2.3.1) and, for $\alpha = (\alpha_+, \alpha_-), \beta = (\beta_+, \beta_-) \in B \times B$, *negation* and *weak implication* are defined as

$$\neg\alpha \overset{\text{def}}{\equiv} (\alpha_-, \alpha_+) \qquad \text{and} \qquad \alpha \supset \beta \overset{\text{def}}{\equiv} (\alpha_+ \to \beta_+, \alpha_+ \wedge \beta_-). \tag{6.2.1}$$

Ginsberg's example from above can be seen as a bilattice if we instantiate B by $\mathscr{P}(X)$.

[1] This is in contrast with bispaces which allowed only *observable* subsets/predicates.

[2] In fact, Ginsberg defined bilattices to be the structure $(A, \sqcap, \sqcup, \mathbin{\wedge\!\!\!\!\cdot}, \mathbin{\vee\!\!\!\!\cdot}, \bot, \top, \mathit{ff}, \mathit{tt}, \neg)$ and only later they were extended by an implication in [AA96] which Umberto Rivieccio showed to have an algebraic presentation as a product of two Boolean algebras in [Riv14; Riv10].

6.2.1 On Belnap-Dunn logic of bilattices. The minimal non-trivial bilattice (Figure 6.2) matches exactly what Belnap described as a practical logical lattice for a computer reasoning [Bel76; Bel77]. Belnap argued that it is in the very nature of computers to make decisions even in the presence of contradictions and, for that reason, a classical two-valued logic does not suffice.

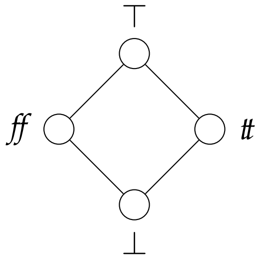


Figure 6.2: $\mathcal{FOUR}$

The work of Belnap and Ginsberg was initially picked up by Fitting, who successfully used bilattices to give a semantics to logical programming [Fit91], and by Arieli and Avron who formalised a bilattice-based logic [AA96]. This logic happened to be *paraconsistent*. This means that it does not allow one to derive all formulas from a contradiction (i.e. $\bot \not\vdash \varphi$ for some φ). The reason for its paraconsistency is that it does not permit *disjunctive syllogism* (i.e. $\varphi \vee \psi, \neg\varphi \not\vdash \psi$ for some φ and ψ).

6.2.2 Comparison

Although motivated differently, d-frames and bilattices have a lot in common. They both consist of pairs of elements which are closed under all *finite* logical and information meets and joins. On the other hand, they are not entirely the same because of the following differences:

- In d-frames, the two component lattices may be different, in bilattices they are identical;
- (consequently) it is not possible to define negation or weak implication on d-frames in the same way as it is done for bilattices;
- frames are *complete* Heyting algebras (but frame homomorphisms may not preserve Heyting implication);
- the two predicates con and tot are *relational*, not *algebraic* structure.

Moreover, Belnap, inspired by Scott's [Sco70], argued that (when paraphrased),

($\star$) *predicates ought to be constructed as directed joins of their (finite) approximations*

(§81.1 and §81.3.2 in [Bel76]).

A directed join in the information order is understood as a computation which generates its output gradually, in a limiting process. Requiring ($\star$) simply means that all predicates must be somehow computable, even though some predicates may only be represented by an infinite computation which produces them.

Ginsberg and Fitting, although not aiming to satisfy ($\star$), originally considered bilattices complete in both logical and information orders. Besides, completeness is usually not assumed for bilattices, not to mention the Scott property ($\star$). d-Frames, on the other hand, when restricted to the class of d-compact d-regular d-frames, satisfy ($\star$). Also, having the two component lattices of d-frames different can be seen as a bonus when compared to bilattices. This way, d-frames allow us to have different procedures for the confirmation and refutation of properties.

Moreover, as we will see in Corollary 6.2.5 and Section 6.2.5, d-compact d-regular d-frames can not only be freely extended with a negation and weak implication but they also "conservatively generalise" bilattices. This brings us to the brave conclusion that d-frames might actually be a better fit than bilattices to model Belnap's paraconsistent logic suitable for computers.

To summarise, the three respective categories of four-valued structures compare as shown in the following table:

bilattices	d-frames	d-compact d-regular d-frames
carrier symmetric $(B \times B)$	carrier non-symmetric $(L_+ \times L_-)$	carrier non-symmetric $(L_+ \times L_-)$
$\neg, \supset$	—	$\neg, \supset$
—	$\bigsqcup^{\uparrow}$, con, tot	$\bigsqcup^{\uparrow}$, con, tot

6.2.3 Negation as interior operations

We saw in (6.2.1) that, in order to define negation and implication for bilattices, we heavily use the fact that the carrier is the product of a Boolean algebra with itself. Therefore, we can freely send elements from one component of the product to the other.

Since the two component lattices of d-frames do not have to be the same, in order to define $\neg$ and $\supset$ for d-frames by a similar formula we need two *order-preserving* maps

$$\mathfrak{o}_+ : L_- \to L_+ \quad \text{and} \quad \mathfrak{o}_- : L_+ \to L_-.$$

Then, similarly to (6.2.1) we can define, for $\alpha, \beta \in \mathcal{L}$,

$$\neg\alpha \overset{\text{def}}{\equiv} (\mathfrak{o}_+(\alpha_-), \mathfrak{o}_-(\alpha_+)) \quad \text{and} \quad \alpha \supset \beta \overset{\text{def}}{\equiv} (\alpha_+ \to \beta_+, \mathfrak{o}_-(\alpha_+) \wedge \beta_-). \tag{6.2.2}$$

However, just in the structure of d-frames, there are no natural candidates for $\mathfrak{o}_+$ and $\mathfrak{o}_-$. This changes when we look at the semantic counterparts of d-frames, i.e. bitopological spaces, as there are very natural candidates for maps between both frames of open sets. For a bispace (X, τ_+, τ_-), assigning to every τ_+-open (or τ_--open) set its interior with respect to the other topology is a monotone map. These, then, are our maps

$$\mathfrak{o}_+ : \tau_- \to \tau_+ \quad \text{and} \quad \mathfrak{o}_- : \tau_+ \to \tau_-;$$

or more precisely: $\mathfrak{o}_+(V) = V^{\circ\tau_+}$ and $\mathfrak{o}_-(U) = U^{\circ\tau_-}$. Moreover, if X is d-regular, the interior operations $\mathfrak{o}_+$ and $\mathfrak{o}_-$ can be expressed explicitly in the structure of the d-frame $\Omega_d(X)$. Indeed, for a $U \in \tau_+$, $U^{\circ\tau_-}$ is equal to the union of all $V' \in \tau_-$ such that $V' \subseteq U$ and, by regularity, V' is an union of all $V \subseteq V'$ such that $V' \cup W = X$ and $V \cap W = \emptyset$ for some $W \in \tau_+$. Consequently,

$$U^{\circ\tau_-} = \bigcup\{V \in \tau_- \mid \exists W \in \tau_+.\ V \cap W = \emptyset \text{ and } W \cup U = X\}.$$

We can turn this into a definition of interior operations $\mathfrak{o}_+ \colon L_- \to L_+$ and $\mathfrak{o}_- \colon L_+ \to L_-$ for a general d-frame $\mathcal{L}$. For an $a \in L_+$, define

$$\mathfrak{o}_-(a) = \bigvee\{y \in L_- \mid \exists x \in L_+.\ (x, y) \in \mathsf{con} \text{ and } x \vee a = 1\},$$

and $\mathfrak{o}_+(b)$, for $b \in L_-$, symmetrically. The formulas can be slightly simplified with pseudocomplements as follows

$$\mathfrak{o}_-(a) = \bigvee\{y \in L_- \mid y^* \vee a = 1\} \quad \text{and} \quad \mathfrak{o}_+(b) = \bigvee\{x \in L_+ \mid x^* \vee b = 1\}. \tag{6.2.3}$$

Finally, combining (6.2.2) and (6.2.3) gives us an explicit formula for negation. For an $\alpha \in \mathcal{L}$,

$$\neg\alpha = (\bigvee\{x \mid \alpha_- \vee x^* = 1\}, \bigvee\{y \mid \alpha_+ \vee y^* = 1\} = \bigsqcup\{\gamma \mid \alpha \sqcup \gamma^* = \top\} \tag{6.2.4}$$

where γ^* is the componentwise pseudocomplementation (γ_-^*, γ_+^*).

6.2.2 Properties of negation. The following statements summarise the basic d-frame theoretic properties of the negation defined above. In the proofs we mostly just refer to the corresponding results in [JJP16] and only show how to translate the results from there to our setting.

6.2.3 Proposition. *Let $\mathcal{L}$ be a d-frame, then $\neg \colon \mathcal{L} \to \mathcal{L}$ defined as in (6.2.4) satisfies:*

(pm-1) $\neg(\alpha \sqcap \beta) = \neg\alpha \sqcap \neg\beta$

(pm-2) $\neg\mathtt{tt} = \mathtt{ff}, \quad \neg\mathtt{ff} = \mathtt{tt}$

(con-$\neg$) $\dfrac{\alpha \sqcap \beta \in \mathsf{con}}{\alpha \sqcap \neg\beta \in \mathsf{con}}$

Proof. In Proposition 6.1 of [JJP16] it is proved that $\mathfrak{o}_+$ and $\mathfrak{o}_-$ defined as in (6.2.3) preserve finite meets, send 1 to 1 and 0 to 0. This proves (pm-1) and (pm-2).

The same proposition also shows that $(a \wedge b,\ c) \in \mathsf{con}$ implies $(a,\ \mathfrak{o}_-(b) \wedge c) \in \mathsf{con}$ and that $(a,\ b \wedge c) \in \mathsf{con}$ implies $(a \wedge \mathfrak{o}_+(b),\ c) \in \mathsf{con}$. From this we can prove (con-$\neg$): $\alpha \sqcap \beta \in \mathsf{con}$ implies $(\alpha_+, \mathfrak{o}_-(\beta_+) \wedge \beta_- \wedge \alpha_-) \in \mathsf{con}$ which implies $\alpha \sqcap \neg\beta \in \mathsf{con}$. □

6.2.4 Proposition. *Let $\mathcal{L}$ be a spatial d-frame. Then, $\neg$ defined in (6.2.4) further satisfies*

(pm-3) $\neg\neg\alpha \sqsubseteq \alpha$

(tot-$\neg$) $\dfrac{\alpha \sqcup \neg\beta \in \mathsf{tot}}{\alpha \sqcup \beta \in \mathsf{tot}}$

If, moreover, $\mathcal{L}$ is also d-regular, then $\neg(U, V) = (V^{\circ\tau_+}, U^{\circ\tau_-})$ for every $(U, V) \in$

$\tau_+ \times \tau_-$ where $X = (X, \tau_+, \tau_-)$ is the bispace such that $\mathcal{L} \cong \Omega_d(X)$.

Proof. In Proposition 6.1 of [JJP16], it is proved that, for a spatial $\mathcal{L}$, $\mathfrak{o}_+ \circ \mathfrak{o}_- \leq \mathrm{id}$ and $\mathfrak{o}_- \circ \mathfrak{o}_+ \leq \mathrm{id}$ and also that $(a, \mathfrak{o}_-(b) \vee c) \in \mathsf{tot}$ implies $(a \vee b, c) \in \mathsf{tot}$ and that $(a \vee \mathfrak{o}_+(b), c) \in \mathsf{tot}$ implies $(a, b \vee c) \in \mathsf{tot}$. The first pair of inequalities gives (pm-3) and the last two implications give (tot-$\neg$) by an argument similar to the one in the proof of Proposition 6.2.3. The "moreover" part is exactly Proposition 6.6 in [JJP16]. □

The list of valid equalities or rules does not stop here. One can, for example, also prove that the negation of d-frames is monotone with respect to the information order and antitone with respect to the logical order, that $\neg\top = \top$, $\neg\bot = \bot$ or that $\alpha \in \mathsf{con}$ implies $\alpha \sqcap \neg\alpha = \bot$. The point of the equalities proved in Propositions 6.2.3 and 6.2.4 is that they are powerful enough to show that most of the logic of bilattices is preserved (Theorem 6.2.24).

As we discussed earlier, we are mostly interested in the category of d-compact d-regular d-frames to model a Belnap-Dunn logic. Such d-frames are always spatial (Proposition 2.4.4) and so we can assume that d-compact d-regular d-frames always come equipped with a negation:

6.2.5 Corollary. *For a d-compact d-regular d-frame $\mathcal{L}$, $\neg$ defined as in (6.2.4) satisfies (pm-1), (pm-2), (pm-3), (con-$\neg$) and (tot-$\neg$) from above and, moreover, it exactly corresponds to taking interiors in X where $\mathcal{L} \cong \Omega_d(X)$.*

Note that, in contrast with bilattices, the negation of d-frames is not required to satisfy $\neg\neg\alpha = \alpha$. From the point of view of bilattice semantics $\neg\alpha$ only flips the positive and negative evidence of α whereas in d-frames we can understand $\neg\alpha$ as the procedure which *translates* the positive evidence into the negative context and the negative evidence into the positive context. It then makes sense that such translation can be *lossy*, hence $\neg\neg\alpha \sqsubseteq \alpha$. Maybe, it would be better to call $\neg$ a *switch of context* rather than a negation.

Another difference between bilattices and d-frames is that a form of the Law of Excluded Middle $\alpha \curlyvee (\alpha \supset \bot) = tt$ is true for bilattices and not true for d-frames. This corresponds to the fact that the logic of observable properties is naturally intuitionistic, e.g. the set of states for which a program stops is observable but its complement is not. However, there is no philosophical reason to require that bilattices have to be of the form $B \times B$, for some Boolean algebra B; in fact, a generalisation where in the place of B is a Heyting algebra were already considered in [BJR11].

Because d-frame homomorphisms only preserve geometric connectives, one can show that $h(\neg\alpha) \sqsubseteq \neg h(\alpha)$ and $h(\alpha \supset \beta) \sqsubseteq h(\alpha) \supset h(\beta)$. This again is not a problem as, in applications of bilattices, both Fitting and Ginsberg already considered situations where $\neg \circ \neg \neq \mathrm{id}$ or $\neg v(\varphi) \neq v(\neg\varphi)$, for a valuation v, [Fit88; Gin90].

6.2.4 Modal extensions

In the following we show that many of the new d-frame constructions we have build in this text, such as coproducts or Vietoris functor, can be used to extend d-frames with a modal operator similarly to how this is done for bilattices.

Among existing modal extensions of bilattice logic the most elegant one is probably *minimal* modal logic studied in [JR13b; RJJ15]. Algebraic models of this logic are *modal bilattices*, that is, bilattices equipped with a modal operator $\Box\colon A \to A$ such that

(M1) $\Box tt = tt$

(M2) $\Box(\alpha \curlywedge \beta) = \Box\alpha \curlywedge \Box\beta$

(M3) $\Box(\bot \to \alpha) = \bot \to \Box\alpha$

where $\alpha \to \beta$ is the shorthand for $(\alpha \supset \beta) \curlywedge (\neg\beta \supset \neg\alpha)$. In addition, every such modal bilattice $(A, \Box)$ can be uniquely represented as a Boolean algebra B with two modal operators $\Box_1, \Box_2\colon B \to B$ which preserve 1 and distribute over binary meets. Then, to get a modal bilattice $(A, \Box)$ from this data, set $A = B \times B$ and

$$\Box\alpha \stackrel{\text{def}}{\equiv} (\Box_1\alpha_+ \wedge \Box_2(\sim\alpha_-),\ \Diamond_1\alpha_-), \tag{6.2.5}$$

for every $\alpha = (\alpha_+, \alpha_-)$ from A, where $\sim x$ is the negation in B and $\Diamond_1\alpha_-$ is an abbreviation for $\sim(\Box_1\sim\alpha_-)$.

6.2.6 Lemma. *In bilattices, the rule (M3) is equivalent to the rule*

(M3') $\Box(\alpha \supset \bot) = (\Diamond\alpha) \supset \bot$ *where* $\Diamond\alpha \stackrel{def}{\equiv} \neg\Box(\neg\alpha)$.

Proof. First, observe that $\Box(\bot \to \alpha) \stackrel{(M2)}{=} \Box(\bot \supset \alpha) \curlywedge \Box(\neg\alpha \supset \neg\bot)$ and, because $\bot \supset \alpha = (0 \to \alpha_+, \alpha_+ \wedge 0) = tt$ and $\neg\bot = \bot$, by (M1), $\Box(\bot \to \alpha) = \Box(\neg\alpha \supset \bot)$. Similarly, $\bot \to \Box\alpha = (\bot \supset \Box\alpha) \curlywedge (\neg\alpha \supset \neg\bot) = \neg\Box\alpha \supset \bot$.

We show that $\Box(\neg\alpha \supset \bot) = (\neg\Box\alpha) \supset \bot$ is equivalent to (M3'). Set $\alpha = \neg\beta$ for the right-to-left implication: $\Box(\neg\neg\beta \supset \bot) = (\neg\Box(\neg\beta) \supset \bot)$ from which, by $\neg\neg\beta = \beta$, $\Box(\beta \supset \bot) = (\Diamond\beta \supset \bot)$. Conversely, again set $\alpha = \neg\beta$: $\Box(\neg\beta \supset \bot) = (\Diamond(\neg\beta)) \supset \bot = (\neg\Box\neg(\neg\beta)) \supset \bot = (\neg\Box\beta) \supset \bot$. □

We can likewise extend d-frames with a modal operator preserving most of the properties from the bilattice setting. Observe that every 1 and meet-preserving map $B \to B$ uniquely extends to a Boolean algebra homomorphism $\mathbb{M}B \to B$ (this follows from the presentation of $\mathbb{M}$ described in Proposition 4.5.3). Then, Boolean algebras with two modal operators $(B, \Box_1, \Box_2)$ are in bijective correspondence with structures (B, e^1, e^2) where $e^i\colon \mathbb{M}B \to B$ (for $i = 1, 2$) are $\mathbb{M}$-algebras in **Bool**. The d-frame analogue of $\mathbb{M}$ is the Vietoris functor $\mathbb{W}_d$ (Theorem 4.3.12). Therefore, define a *modal d-frame* to be any triple $(\mathcal{L}, e^1, e^2)$ such that $\mathcal{L}$ is a d-compact d-regular d-frame and

$$e^1\colon \mathbb{W}_d\mathcal{L} \to \mathcal{L} \quad \text{and} \quad e^2\colon \mathbb{W}_d\mathcal{L} \to \mathcal{L}$$

are d-frame homomorphism. In fact, a modal d-frame $(\mathcal{L}, e^1, e^2)$ can be represented as an algebra $\mathbb{W}_d\mathcal{L} \oplus \mathbb{W}_d\mathcal{L} \to \mathcal{L}$ in **d-KReg** or, equivalently, as $\mathbb{W}_d(\mathcal{L} \times \mathcal{L}) \to \mathcal{L}$ because $\mathbb{W}_d(\mathcal{L} \times \mathcal{L}) \cong \mathbb{W}_d\mathcal{L} \oplus \mathbb{W}_d\mathcal{L}$ (Lemma 4.3.7).

In order to define a modal operator $\Box\colon \mathcal{L} \to \mathcal{L}$ similarly to (6.2.5) we use e^1 and e^2 to represent the $\Box_1$, $\Box_2$ and $\Diamond_1$, Heyting pseudocomplementation $(-) \to 0$ in the place of negation and we also need $\mathfrak{o}_+ \colon L_- \to L_+$ from page 173. For every $\alpha \in \mathcal{L}$, define

$$\Box\alpha \stackrel{\text{def}}{\equiv} (\widetilde{\Box}_1\alpha_+ \wedge \widetilde{\Box}_2(\mathfrak{o}_+(\alpha_- \to 0)), \widetilde{\Diamond}_1\alpha_-) \tag{6.2.6}$$

where $\widetilde{\Box}_i x$, for $x \in L_\pm$, is a shorthand for $e^i_\pm(\Box x)$ and $\widetilde{\Diamond}_i x$ for $e^i_\pm(\Diamond x)$. Next, we show how $\widetilde{\Box}_i$ and $\widetilde{\Diamond}_i$ interact with the rest of the structure:

6.2.7 Lemma. *For $x, y \in L_+$ and $i \in \{1, 2\}$,*

1. $\widetilde{\Box}_i 1 = 1$, $\widetilde{\Diamond}_i 0 = 0$, $\widetilde{\Box}_i(x \wedge y) = \widetilde{\Box}_i x \wedge \widetilde{\Box}_i y$ *and* $\widetilde{\Diamond}_i(x \vee y) = \widetilde{\Diamond}_i x \vee \widetilde{\Diamond}_i y$,
2. $\widetilde{\Box}_i(x \to 0) \leq (\widetilde{\Diamond}_i x) \to 0$ *and* $\widetilde{\Diamond}_i(x \to 0) \leq (\widetilde{\Box}_i x) \to 0$
3. $\widetilde{\Box}_i(z^*) \leq (\widetilde{\Diamond}_i z)^*$ *and* $\widetilde{\Diamond}_i(z^*) \leq (\widetilde{\Box}_i z)^*$, *and*
4. $\widetilde{\Box}_i(\mathfrak{o}_-(x)) \leq \mathfrak{o}_-(\widetilde{\Box}_i x)$ *and* $\widetilde{\Diamond}_i(\mathfrak{o}_-(x)) \leq \mathfrak{o}_-(\widetilde{\Diamond}_i x)$.

The dual inequalities hold for x and y coming from L_-.

Proof. (1) Because $\Box 1 = 1$ in $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$ and $e^i_\pm$ is a frame homomorphism, $\widetilde{\Box}_i 1 = e^i_\pm(\Box 1) = 1$. Similarly, $\Diamond_i 0 = 0$. Next, because $\Box$ distributes over finite meets in $\mathbb{V}_{\mathrm{Fr}}(L_\pm)$, $\widetilde{\Box}_i(x \wedge y) = e^i_\pm(\Box(x \wedge y)) = e^i_\pm(\Box x \wedge \Box y) = e^i_\pm(\Box x) \wedge e^i_\pm(\Box y) = \widetilde{\Diamond}_i x \vee \widetilde{\Diamond}_i y$. The proof of the last part is similar.

(2) For the same reason as in (1) $\widetilde{\Box}_i$ distributes over directed joins. Therefore, we have that $\widetilde{\Box}_i(x \to 0) = \widetilde{\Box}_i(\bigvee^\uparrow\{y \mid x \wedge y = 0\}) = \bigvee^\uparrow\{\widetilde{\Box}_i y \mid x \wedge y = 0\}$. On the other hand, $(\widetilde{\Diamond}_i x) \to 0 = \bigvee^\uparrow\{z \mid z \wedge \widetilde{\Diamond}_i x = 0\}$. Because $x \wedge y = 0$ implies $\widetilde{\Box}_i x \wedge \widetilde{\Diamond}_i x = 0$, $\widetilde{\Box}_i(x \to 0) \leq (\widetilde{\Diamond}_i x) \to 0$. The second part is analogous.

(3) Since $(\Box z, \Diamond(z^*)) \in \mathsf{con}_{\mathbb{W}_d\mathcal{L}}$, the pair $(\widetilde{\Box}_i z, \widetilde{\Diamond}_i(z^*)) = e^i(\Box z, \Diamond(z^*))$ is in $\mathsf{con}_{\mathcal{L}}$ because e^i is a d-frame homomorphism. Therefore, $\widetilde{\Diamond}_i(z^*) \leq (\widetilde{\Box}_i z)^*$. The proof of the first part is similar.

(4) Recall that $\mathfrak{o}_-(x)$ is the join $\bigvee\{y \in L_- \mid y^* \vee x = 1\}$ and this join is directed because $y_1^* \wedge y_2^* = (y_1 \vee y_2)^*$ (Lemma 2.3.13). Therefore, because $\widetilde{\Box}_i$ distributes over directed joins $\widetilde{\Box}_i(\mathfrak{o}_-(x)) = \bigvee\{\widetilde{\Box}_i y \in L_- \mid y^* \vee x = 1\}$. Since $y^* \vee x = 1$ implies $\widetilde{\Diamond}_i(y^*) \vee \widetilde{\Box}_i x = 1$ and (3) implies that also $(\widetilde{\Box}_i y)^* \vee \widetilde{\Box}_i x = 1$. Therefore, $\widetilde{\Box}_i(\mathfrak{o}_-(x))$ is smaller than $\mathfrak{o}_-(\widetilde{\Box}_i x) = \bigvee\{x \mid x^* \vee \widetilde{\Box}_i x = 1\}$. □

6.2.8 Proposition. *For a modal d-frame $(\mathcal{L}, e^1, e^2)$, the following inequalities hold:*

1. $\Box\mathtt{tt} = \mathtt{tt}$

2. $\square(\alpha \curlywedge \beta) = \square\alpha \curlywedge \square\beta$

3. $\square(\alpha \supset \bot) \leqslant (\Diamond\alpha) \supset \bot$

4. $(\neg\square\alpha) \supset \bot \leqslant (\Diamond\neg\alpha) \supset \bot$

where $\square\alpha$ *is as in* (6.2.6) *and* $\Diamond\alpha = (\widetilde{\Diamond}_1\alpha_+, \widetilde{\square}_1\alpha_- \wedge \widetilde{\square}_2(\mathfrak{o}_-(\alpha_+ \to 0)))$.

Proof. We use the properties proved in the previous lemma: (1) By definition $\square tt = (\widetilde{\square}_1 1 \wedge \widetilde{\square}_2(\mathfrak{o}_+(0 \to 0)), \widetilde{\Diamond}_1 0) = (1 \wedge 1, 0) = tt$ where $\widetilde{\square}_2(\mathfrak{o}_+(0 \to 0)) = \widetilde{\square}_2(\mathfrak{o}_+(1)) = \widetilde{\square}_2 1 = 1$ because $\mathfrak{o}_+$ preserves 1.

(2) $\square(\alpha \curlywedge \beta) = (\widetilde{\square}_1(\alpha_+ \wedge \beta_+) \wedge \widetilde{\square}_2(\mathfrak{o}_+((\alpha_- \vee \beta_-) \to 0)), \widetilde{\Diamond}_1(\alpha_- \vee \beta_-))$ and, since $\widetilde{\Diamond}_1$ distributes over joins, $(\alpha_- \vee \beta_-) \to 0 = (\alpha_- \to 0) \wedge (\beta_- \to 0)$ and both $\widetilde{\square}_2$ and $\mathfrak{o}_+$ distribute over finite meets, we have that $\square(\alpha \curlywedge \beta) = \square\alpha \curlywedge \square\beta$.

(3) Since $\gamma \supset \bot = (\gamma_+ \to 0, 0)$ and, whenever $\gamma_- = 0$, $\square\gamma = (\widetilde{\square}_1\gamma_+, 0)$, we have that $\square(\alpha \supset \bot) = (\widetilde{\square}_1(\alpha_+ \to 0), 0)$ is $\leqslant$-smaller than $(\Diamond\alpha) \supset \bot = ((\widetilde{\Diamond}_1\alpha_+) \to 0, 0)$.

(4) Similarly, $(\neg\square\alpha) \supset \bot = (\mathfrak{o}_+(\widetilde{\Diamond}_1\alpha_-) \to 0, 0)$ is $\leqslant$-smaller than $(\Diamond\neg\alpha) \supset \bot = ((\widetilde{\Diamond}_1\mathfrak{o}_+(\alpha_-)) \to 0, 0)$. □

This proposition shows that the axioms (M1) and (M2) hold for modal d-frames. Moreover, without any extra assumptions on $(\mathcal{L}, e^1, e^2)$, it does not seem possible to prove (M3) or even compare $\square(\bot \to \alpha)$ and $\bot \to \square\alpha$ in any way. However, (3) proves that one inequality of (M3') (from Lemma 6.2.6) holds in modal d-frames. In some sense this means that (M3') expresses a relationship between box and diamond although they are not interdefinable, as was the case for bilattices. This is similar to the situation in intuitionistic modal logic where there also have to be extra axioms postulated to relate box and diamond modalities.

6.2.9 Remark. (1) Proposition 6.2.8 suggests that an intuitionistic modal bilattice logic ought have both box and diamond. It would make sense to postulate axioms similar to the inequalities (1)–(4) and possibly even their dual versions: $\Diamond ff = ff$, $\Diamond(\alpha \curlyvee \beta) = \Diamond\alpha \curlyvee \Diamond\beta$, $\Diamond(\alpha \supset' \bot) \leqslant (\square\alpha) \supset' \bot$ and $(\neg\Diamond\alpha) \supset' \bot \leqslant (\square\neg\alpha) \supset' \bot$ where $\supset'$ is the dual implication $\alpha \supset' \beta \overset{\text{def}}{\equiv} (\beta_+ \wedge \mathfrak{o}_+(\alpha_-), \alpha_- \to \beta_-)$ which was already introduced in [MRJ17].

(2) Note also that the choice of the definition of $\square$ for modal d-frames is not arbitrary. In the place of $\sim\alpha_-$ in (6.2.5) we needed an antitone map $L_- \to L_+$ and among the apparent options $\mathfrak{o}_+(\alpha_- \to 0)$, α_-^* and $\mathfrak{o}_+(\alpha_-) \to 0$ only the first one satisfies (M2).

6.2.5 Bilattices bitopologised

In the following we show a simple fact, namely, that bilattices can be identified with a subcategory of d-frames and also the corresponding dual category of bispaces. Let $A = B \times B$ be a bilattice, for some Boolean algebra B. Define $\mathsf{Idl}_{\mathrm{BL}}(A)$ to be the d-frame $\mathsf{Idl}(B)^{\bowtie}$, i.e.

$$(\mathsf{Idl}(B),\ \mathsf{Idl}(B),\ \mathsf{con},\ \mathsf{tot})$$

where $(I, J) \in \mathsf{con}$ iff $I \cap J = 0_{\mathsf{Idl}(B)}$ and $(I, J) \in \mathsf{tot}$ iff $I \vee J = 1_{\mathsf{Idl}(B)}$.

6.2.10 Observation. *For every bilattice A, the d-frame $\mathsf{Idl}_{\mathrm{BL}}(A)$ is a* Stone d-frame, *that is, a Priestley d-frame with its frame components being zero-dimensional frames.*

Proof. Let B be the Boolean algebra such that $A = B \times B$. It is well known that $\mathsf{Idl}(B)$ is compact and zero-dimensional [Joh82]. Moreover, because $\mathsf{Idl}(B)^{\bowtie}$ is isomorphic to $\mathcal{IF}(B)$ (Lemma 2.7.3), $\mathsf{Idl}_{\mathrm{BL}}(A)$ is a Priestley d-frame. □

6.2.11 Remark. Note that having zero-dimensional frame components is a separate property from the d-frame d-zero-dimensionality. The former is a property of the frame components and it is independent from the rest of the d-frame.

Next, define *bilattice homomorphisms* $h_{\times} \colon B \times B \to B' \times B'$ to be mappings

$$\alpha \mapsto (h(\alpha_+), h(\alpha_-)),$$

for some Boolean homomorphism $h \colon B \to B'$. Observe that they distribute over $\neg$ and $\supset$ as well as the other bilattice operations and constants, e.g. $h_{\times}(\neg\alpha) = \neg h_{\times}(\alpha)$, $h_{\times}(\alpha \sqcap \beta) = h_{\times}(\alpha) \sqcap h_{\times}(\beta)$, $h_{\times}(\bot) = \bot$ and so on.

The action of $\mathsf{Idl}_{\mathrm{BL}}$ on morphisms is defined as $\mathsf{Idl}_{\mathrm{BL}}(h_{\times}) = (\mathsf{Idl}(h), \mathsf{Idl}(h))$.

6.2.12 Proposition. *The functor $\mathsf{Idl}_{\mathrm{BL}}(-)$ establishes an equivalence of categories between the category of bilattices and bilattice homomorphisms, and the category of Stone d-frames and d-frame homomorphisms.*

Proof. Because the functor Idl restricted to **Bool** $\to$ **Stone** establishes a duality of categories [Joh82], $\mathsf{Idl}_{\mathrm{BL}}$ is a faithful embedding. Moreover, it is full because morphisms between d-regular d-frames are determined by either of their components (Lemma 5.3.10).

On the other hand, if $\mathcal{L}$ is a Priestley d-frame with either of the frame component zero-dimensional, for example L_+, then, because L_+ is also compact, $L_+ \cong \mathsf{Idl}(B)$ for some Boolean algebra B. Furthermore, $\mathcal{L} \cong \mathsf{Idl}(B)^{\bowtie}$ by Proposition 5.3.7. □

From the duality between Priestley d-frames and Priestley bispaces it follows that:

6.2.13 Corollary. *The category of bilattices is dually equivalent to the category of Priestley bispaces with zero-dimensional topologies, i.e. bispaces (X, τ, τ) such that (X, τ) is a Stone space.*

The simplicity of this bitopological duality is in contrast with monotopological approaches. Traditionally, in order to model negation in the spectrum of a bilattice, an extra structure needs to be assumed, e.g. a continuous endomap [JR12; JR13a;

Mob+00]. On the other hand, in our bitopological duality, negation of (U_+, U_-) is computed by taking the interiors with respect to the other topology and because the topologies are the same we obtain just (U_-, U_+). See Lemma 6.2.17 for a point-free proof of this.

6.2.6 Bilattice logic and d-frames

Given a bilattice A, since the d-frame $\mathsf{Idl}_{\mathrm{BL}}(A)$ is d-compact and d-regular, it has negation and weak implication as defined in Section 6.2.3. This means that both A and $\mathsf{Idl}_{\mathrm{BL}}(A)$ admit an interpretation of the connectives of bilattice logic. In this subsection we show how A and $\mathsf{Idl}_{\mathrm{BL}}(A)$ compare from the perspective of bilattice logic.

6.2.14 Syntax and satisfaction relation of bilattice logic. Formulas of bilattice logic are defined inductively from the *language of bilattices*

$$\mathbf{BL} = \langle \mathcal{V}ar, \sqcap, \sqcup, \curlywedge, \curlyvee, \bot, \top, \mathit{ff}, \mathit{tt}, \neg, \supset \rangle.$$

We assume that the set of variables $\mathcal{V}ar$ is countable. Let M be a structure which can interpret the language of bilattices, i.e. M is either a bilattice or a d-compact d-regular d-frame. As always, any valuation of variables $\mathfrak{v}\colon \mathcal{V}ar \to M$ uniquely extends to $\mathfrak{v}\colon \mathcal{F}m_{\mathbf{BL}} \to M$ where $\mathcal{F}m_{\mathbf{BL}}$ is the term algebra of bilattice formulas. Validity of a bilattice formula φ under the valuation $\mathfrak{v}$ is, following the example of [Riv10], given as follows

$$M \vDash^{\mathrm{BL}}_{\mathfrak{v}} \varphi \quad \overset{\text{def}}{\equiv} \quad \mathfrak{v}(\varphi) = \mathfrak{v}(\varphi) \supset \mathfrak{v}(\varphi) \tag{6.2.7}$$

Again, define the abbreviations $M \vDash^{\mathrm{BL}} \varphi$ as $M \vDash^{\mathrm{BL}}_{\mathfrak{v}} \varphi$ for all valuations $\mathfrak{v}\colon \mathcal{V}ar \to M$.

6.2.15 Lemma. *For a bilattice or d-compact d-regular d-frame M, $M \vDash^{\mathrm{BL}}_{\mathfrak{v}} \varphi$ iff $\mathit{tt} \sqsubseteq \mathfrak{v}(\varphi)$.*

Proof. We only prove this for d-compact d-regular d-frames since the proof for a bilattices is the same. Assume that $\mathfrak{v}(\varphi) = (1, a)$. Then, by definition $\mathfrak{v}(\varphi) \supset \mathfrak{v}(\varphi) = (1 \to 1, \mathfrak{o}_-(1) \wedge a) = (1, 1 \wedge a) = (1, a) = \mathfrak{v}(\varphi)$. Conversely, $\mathfrak{v}(\varphi) = \mathfrak{v}(\varphi) \supset \mathfrak{v}(\varphi)$ implies that $\mathfrak{v}(\varphi)_+ = \mathfrak{v}(\varphi)_+ \to \mathfrak{v}(\varphi)_+$ and this is equivalent to $\mathfrak{v}(\varphi)_+ = 1$ in every Heyting algebra. □

This lemma will be useful later when we show that the set of implication-free formulas true in A is the same as the set of such formulas true in $\mathsf{Idl}_{\mathrm{BL}}(A)$. In order to show that, we first prove a similar statement but for positive two-valued logic:

6.2.16 Lemma. *Let* **L2** *be the language of positive two-valued logic, i.e.* $\langle \mathcal{V}ar, \wedge, \vee, 0, 1\rangle$. *Then, for a distributive lattice* D *and a formula* φ *in* **L2**,

$$D \models^{\mathrm{L2}} \varphi \quad \textit{iff} \quad \mathsf{Idl}(D) \models^{\mathrm{L2}} \varphi.$$

(Where, for a valuation $\mathfrak{v}\colon \mathcal{V}ar \to D$, *we define* $D \models^{\mathrm{L2}}_{\mathfrak{v}} \varphi$ *as* $\mathfrak{v}(\varphi) = 1$.*)*

Proof. Assume $\mathsf{Idl}(D) \models^{\mathrm{L2}} \varphi$ and let $\mathfrak{v}\colon \mathcal{V}ar \to D$ be a valuation. The composition of $\mathfrak{v}$ with the inclusion $i\colon D \to \mathsf{Idl}(D)$, $a \mapsto {\downarrow}a$, is a valuation $i \cdot \mathfrak{v}\colon \mathcal{V}ar \to \mathsf{Idl}(D)$ and so, by the assumption, $\mathsf{Idl}(D) \models^{\mathrm{L2}}_{i\cdot\mathfrak{v}} \varphi$. Therefore, $i(\mathfrak{v}(\varphi)) = 1_{\mathsf{Idl}(D)} = i(1)$ and, by injectivity of i, $\mathfrak{v}(\varphi) = 1$.

For the reverse direction we show something stronger. We show that, if $\mathfrak{v}'(\psi) \leq \mathfrak{v}'(\varphi)$ for every valuation $\mathfrak{v}'\colon \mathcal{V}ar \to D$, then also $\mathfrak{v}(\psi) \leq \mathfrak{v}(\varphi)$ for every valuation $\mathfrak{v}\colon \mathcal{V}ar \to \mathsf{Idl}(D)$. The original statements follows by assigning ψ to 1. Without loss of generality, assume that ψ and φ do not contain 0 nor 1 (otherwise, replace them with two unused variables w_0 and w_1 and define $\mathfrak{v}(w_0) = 0_{\mathsf{Idl}(D)}$ and $\mathfrak{v}(w_1) = 1_{\mathsf{Idl}(D)}$). Further, let $\bigvee_{j=1}^{n} \bigwedge_{k=1}^{m_j} v_{jk}$ be ψ in a disjunctive normal form where v_{jk}'s are its variables (note that they might repeat). Then, every element a of the ideal $\mathfrak{v}(\psi)$ is of the form $\bigvee_j^n \bigwedge_k^{m_j} a_{jk}$ for some a_{jk}'s such that $a_{jk} \in \mathfrak{v}(v_{jk})$.

Define $\mathfrak{v}'\colon \mathcal{V}ar \to D$ such that

$$\mathfrak{v}'\colon v \quad \longmapsto \quad \begin{cases} \bigvee\{a_{jk} \mid v \text{ is the same variable as } v_{jk}\} & \text{if } v \in \mathsf{Var}(\psi) \\ 0 & \text{if } v \notin \mathsf{Var}(\psi) \end{cases}$$

where $\mathsf{Var}(\psi)$ is the set of all variables occurring in ψ. We have that $a \leq \mathfrak{v}'(\psi)$ and so $i(a) \leq i(\mathfrak{v}'(\psi)) \leq i(\mathfrak{v}'(\varphi))$ by the assumption. Moreover, because $\mathfrak{v}'(v) \in \mathfrak{v}(v)$ for every $v \in \mathsf{Var}(\varphi)$, $i(\mathfrak{v}'(\varphi)) \leq \mathfrak{v}(\varphi)$. Hence $a \in \mathfrak{v}(\varphi)$ and, since a was chosen arbitrarily, also $\mathfrak{v}(\psi) \subseteq \mathfrak{v}(\varphi)$. □

6.2.17 Lemma. *For a bilattice* A *and any* $\alpha \in \mathsf{Idl}_{\mathrm{BL}}(A)$, $\neg\alpha = (\alpha_-, \alpha_+)$.

Proof. α is a pair of ideals $(I_+, I_-) \in \mathsf{Idl}(B) \times \mathsf{Idl}(B)$ where B is the Boolean algebra such that $A = B \times B$. Recall that $\neg\alpha = (\mathfrak{o}_+(I_-), \mathfrak{o}_-(I_+))$ where $\mathfrak{o}_+$ and $\mathfrak{o}_-$ are defined as in (6.2.3). This means that $\mathfrak{o}_-(I_+) = \bigvee\{J \mid I_+ \vee J^* = 1_{\mathsf{Idl}(B)}\}$ and

$$J^* = \bigvee\{K \mid J \wedge K = \{0\}\} = \bigvee\{{\downarrow}z \mid J \wedge {\downarrow}z = \{0\}\} = \{z \mid \forall j \in J.\, j \wedge z = 0\}.$$

In particular, $({\downarrow}x)^* = {\downarrow}({\sim}x)$ where $\sim$ is the negation in B. Observe that, whenever $I_+ \vee J^* = 1$, then $i \vee z = 1$ for some $i \in I$ and $z \in J^*$, and each $j \in J$ has the property that $i \vee ({\sim}j) = 1$. Because this means that $I \vee {\downarrow}({\sim}j) = 1$ and also $I_+ \vee ({\downarrow}j)^* = 1$, for every $j \in J$, $\mathfrak{o}_-(I_+)$ simplifies to $\bigvee\{{\downarrow}j \mid \exists i \in I.\, i \vee {\sim}j = 1\} = \{j \mid \exists i \in I.\, i \vee {\sim}j = 1\}$. Clearly $\mathfrak{o}_-(I_+)$ contains I. Moreover, because $i \vee {\sim}j = 1$ implies that $j \leq i$, we have that $\mathfrak{o}_-(I_+) = I_+$. The proof that $\mathfrak{o}_+(I_-) = I_-$ is similar. □

6.2.18 Proposition. *For a bilattice A and a bilattice formula φ which does not contain $\supset$,*

$$A \vDash^{\mathrm{BL}} \varphi \quad \textit{iff} \quad \mathsf{Idl}_{\mathrm{BL}}(A) \vDash^{\mathrm{BL}} \varphi.$$

Proof. Since the validity of φ is determined only by its value in the plus component (Lemma 6.2.15), we can inductively construct a formula $\varphi^\bullet$ such that $A \vDash^{\mathrm{BL}} \varphi$ if and only if $B \vDash^{\mathrm{L2}} \varphi^\bullet$, for every bilattices $A = B \times B$.

Let $\mathcal{F}m_d$ be the term algebra of **L2**-formulas but containing every variable twice; denoted as v_+ and v_-. Next, we inductively define a mapping $(-)^\circ$ which assigns an element of $\mathcal{F}m_d \times \mathcal{F}m_d$ to every bilattice formula which does not contain $\supset$:

$$\begin{aligned}
\bot^\circ = (0,0), \quad \top^\circ = (1,1), \quad ff^\circ = (0,1), \quad tt^\circ = (1,0),\\
(\neg\varphi)^\circ = (\varphi^\circ_-, \varphi^\circ_+), \quad v^\circ = (v_+, v_-), \qquad (\forall v \in \mathcal{V}ar)\\
(\varphi \sqcap \psi)^\circ = (\varphi^\circ_+ \wedge \psi^\circ_+,\ \varphi^\circ_- \wedge \psi^\circ_-), \quad (\varphi \sqcup \psi)^\circ = (\varphi^\circ_+ \vee \psi^\circ_+,\ \varphi^\circ_- \vee \psi^\circ_-),\\
(\varphi \mathbin{\curlywedge} \psi)^\circ = (\varphi^\circ_+ \wedge \psi^\circ_+,\ \varphi^\circ_- \vee \psi^\circ_-), \quad (\varphi \mathbin{\curlyvee} \psi)^\circ = (\varphi^\circ_+ \vee \psi^\circ_+,\ \varphi^\circ_- \wedge \psi^\circ_-),
\end{aligned}$$

where $\varphi^\circ = (\varphi^\circ_+, \varphi^\circ_-)$ and $\psi^\circ = (\psi^\circ_+, \psi^\circ_-)$. Finally, define $\varphi^\bullet$ as the projection to the first/plus coordinate of φ°. Observe that $A \vDash^{\mathrm{BL}} \varphi$ iff $B \vDash^{\mathrm{L2}} \varphi^\bullet$ and this is equivalent to $\mathsf{Idl}(B) \vDash^{\mathrm{L2}} \varphi^\bullet$ by Lemma 6.2.16. Finally, $\mathsf{Idl}(B) \vDash^{\mathrm{L2}} \varphi^\bullet$ iff $\mathsf{Idl}_{\mathrm{BL}}(A) \vDash^{\mathrm{L2}} \varphi$ by Lemma 6.2.17. □

6.2.19 Lemma. *Every infinite Boolean algebra B has an infinite strictly ascending sequence of elements $b_1 < b_2 < \dots$.*

Proof. The Stone space $\mathsf{spec}_\preceq(B)$ of an infinite Boolean algebra B is infinite. Take $x \neq y \in \mathsf{spec}_\preceq(B)$, there must exist a clopen C_1 such that $x \in C_1 \not\ni y$. Either C_1 or $\mathsf{spec}_\preceq(B) \setminus C_1$ must be infinite. Without loss of generality assume that C_1 is. Observe that C_1 is strictly smaller than $\mathsf{spec}_\preceq(B)$. Continue by induction on C_1 and obtain an infinite descending chain $C_0 = \mathsf{spec}_\preceq(B) \supset C_1 \supset C_2 \supset \dots$. Complements of C_i's form a strictly ascending chain. □

6.2.20 Proposition. *For a Boolean algebra B, if there exists a distributive lattice D such that B is isomorphic to $\mathsf{Idl}(D)$, then B is finite.*

Proof. Let $I_{(-)} \colon B \to \mathsf{Idl}(D)$ be the isomorphism map. Observe that all ideals of D must be principal. Indeed, for every ideal $I \in \mathsf{Idl}(D)$, there exist some $I^c \in \mathsf{Idl}(D)$ such that

$$I \vee I^c = D \quad \text{and} \quad I \wedge I^c = \{0\}.$$

By $I \vee I^c = D$, there exist $x \in I$ and $x' \in I^c$ such that $x \vee x' = 1$. Therefore, we have that $I \vee {\downarrow}x' = D$ and $I \wedge {\downarrow}x' = \{0\}$ and, since $\mathsf{Idl}(D)$ is a distributive lattice, $I^c = {\downarrow}x'$. A symmetrical argument gives that $I = {\downarrow}x$.

From this it is immediate that D is also a Boolean algebra. For every $x \in D$, the complement of $\downarrow x$ in $\mathsf{Idl}(D)$ is a principal ideal $\downarrow x'$. We can see that $x \vee x' = 1$ and $x \wedge x' = 0$.

Now, we will prove that D is Noetherian, that is, it does not have an infinite strictly ascending sequence of elements. Assume, for a contradiction, that there is a sequence $a_1 < a_2 < a_3 < \ldots$ in D. Denote $\bigvee_i \downarrow a_i = \{x \mid \exists i.\ x \leq a_i\}$ by I. Then, from previous, we know that $I = \downarrow a$ for some $a \in D$. Since $a \in I$, there exists an i such that $a \leq a_i$ and also $a_i < a_{i+1} \leq a$. A contradiction. Consequently, by Lemma 6.2.19, D must be finite and the same is the case for B as $B \cong \mathsf{Idl}(D)$. □

Consequently, because the frame of ideals of an infinite distributive lattice is infinite, we also have the following.

6.2.21 Corollary. *For an infinite Boolean algebra B, $\mathsf{Idl}(B)$ is not a Boolean algebra.*

It follows from this corollary that it cannot be the case that $A \vDash^{\mathrm{BL}} \varphi$ iff $\mathsf{Idl}_{\mathrm{BL}}(A) \vDash^{\mathrm{BL}} \varphi$ for every bilattice formula φ. Indeed, let $A = B \times B$ for some infinite Boolean algebra B. Then, for the bilattice form of the excluded middle $\varphi \stackrel{\text{def}}{\equiv} x \curlyvee (x \supset f\!f)$, $A \vDash^{\mathrm{BL}} \varphi$ but not $\mathsf{Idl}_{\mathrm{BL}}(A) \vDash^{\mathrm{BL}} \varphi$.

However, an equivalence can be retained if we restrict to the right kind of valuations. Observe that the valuations $\mathcal{V}ar \to A$ are in a bijection with the valuations $\mathcal{V}ar \to \mathsf{Cmp}_d(\mathsf{Idl}_{\mathrm{BL}}(A))$ where $\mathsf{Cmp}_d(\mathcal{L})$ is the set of all elements of $\mathcal{L}$ which are compact in the information order, i.e. $\mathsf{Cmp}_d(\mathsf{Idl}_{\mathrm{BL}}(A)) = \{(\downarrow x, \downarrow y) \mid (x,y) \in A\}$. Moreover, we have the following fact:

6.2.22 Lemma. *Let B be a Boolean algebra. For $a, b \in B$, $\downarrow a \to \downarrow b = \downarrow(a \to b)$ (where he first $\to$ is Heyting implication in the frame $\mathsf{Idl}(B)$).*

Proof. $\downarrow a \to \downarrow b = \bigvee\{J \mid \downarrow a \wedge J \leq \downarrow b\} = \bigvee\{\downarrow c \mid \downarrow a \wedge \downarrow c \leq \downarrow b\} = \{c \mid a \wedge c \leq b\} = \{c \mid c \leq a \to b\} = \downarrow(a \to b)$. □

From this lemma it follows that, for a valuation $\mathcal{V}ar\colon \mathfrak{v} \to A$ and a formula φ $\mathcal{V}ar(\varphi) \sqsupseteq t\!t$ iff $i(\mathcal{V}ar(\varphi)) \sqsupseteq t\!t$ where $i\colon A \to \mathsf{Idl}_{\mathrm{BL}}(A)$ is the inclusion $(x,y) \mapsto (\downarrow x, \downarrow y)$.

6.2.23 Corollary. *For a bilattice A and a bilattice formula φ,*

$$A \vDash^{\mathrm{BL}} \varphi \quad \textit{iff} \quad \mathsf{Idl}_{\mathrm{BL}}(A) \vDash^{\mathrm{BL}}_{\mathfrak{v}} \varphi \quad \textit{for all valuations } \mathfrak{v}\colon \mathcal{V}ar \to \mathsf{Cmp}_d(\mathsf{Idl}_{\mathrm{BL}}(A)),$$

Even when no restriction on valuations is assumed, most axioms of bilattice logic are preserved. Arieli and Avron, [AA96], gave a Hilbert-style axiomatisation of a four-valued logic which is sound and complete with respect to bilattices. Here we show that a large part of their logic is still valid in d-compact d-regular d-frames.

6.2.24 Theorem.
The following axioms of four-valued logic are valid in any d-compact d-regular d-frame:

(Weak implication)

$(\supset 1)$	$\varphi \supset (\psi \supset \varphi)$	
$(\supset 2)$	$(\varphi \supset (\psi \supset \gamma)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \gamma))$	
$(\neg\neg\, R)$	$\neg\neg\varphi \supset \varphi$	**$(\star A)$**

(Logical conjunction and disjunction)

$(⩚ \supset)$	$(\varphi ⩚ \psi) \supset \varphi$ *and* $(\varphi ⩚ \psi) \supset \psi$	
$(\supset ⩚)$	$\varphi \supset (\psi \supset (\varphi ⩚ \psi))$	
$(\supset \mathtt{tt})$	$\varphi \supset \mathtt{tt}$	
$(\supset ⩛)$	$\varphi \supset (\varphi ⩛ \psi)$ *and* $\psi \supset (\varphi ⩛ \psi)$	
$(⩛ \supset)$	$(\varphi \supset \gamma) \supset ((\psi \supset \gamma) \supset ((\varphi ⩛ \psi) \supset \gamma))$	
$(\supset \mathtt{ff})$	$\mathtt{ff} \supset \varphi$	

(Informational conjunction and disjunction)

$(\sqcap \supset)$	$(\varphi \sqcap \psi) \supset \varphi$ *and* $(\varphi \sqcap \psi) \supset \psi$	
$(\supset \sqcap)$	$\varphi \supset (\psi \supset (\varphi \sqcap \psi))$	
$(\supset \top)$	$\varphi \supset \top$	
$(\supset \sqcup)$	$\varphi \supset (\varphi \sqcup \psi)$ *and* $\psi \supset (\varphi \sqcup \psi)$	
$(\sqcup \supset)$	$(\varphi \supset \gamma) \supset ((\psi \supset \gamma) \supset ((\varphi \sqcup \psi) \supset \gamma))$	
$(\supset \bot)$	$\bot \supset \varphi$	

(Negation)

$(\neg ⩚\, L)$	$\neg(\varphi ⩚ \psi) \subset \neg\varphi ⩛ \neg\psi$	**$(\star B)$**
$(\neg ⩛)$	$\neg(\varphi ⩛ \psi) \equiv \neg\varphi ⩚ \neg\psi$	
$(\neg \sqcap)$	$\neg(\varphi \sqcap \psi) \equiv \neg\varphi \sqcap \neg\psi$	
$(\neg \sqcup\, L)$	$\neg(\varphi \sqcup \psi) \subset \neg\varphi \sqcup \neg\psi$	**$(\star B)$**
$(\neg \supset R)$	$\neg(\varphi \supset \psi) \supset \varphi ⩚ \neg\psi$	**$(\star A)$**

where $\varphi \equiv \psi$ is a shorthand for $(\varphi \supset \psi) ⩚ (\psi \supset \varphi)$. Furthermore, the rule of Modus Ponens is sound:

(MP)	$\varphi, (\varphi \supset \psi) \vdash \psi$

Proof. See Theorem 4.2 in [JJP16]. □

The axioms marked with **$(\star A)$** or **$(\star B)$** are the only axioms that differ from the original axioms of bilattices because they are only implications, whereas the original axioms are equivalences. Requiring equivalence instead of implication in the axioms marked by **(**$\star A$**)** is equivalent to requiring that $\mathfrak{o}_+ \circ \mathfrak{o}_- = \mathrm{id}$ and requiring equivalences for axioms marked by **(**$\star B$**)** is the same as requiring that $\mathfrak{o}_+$ preserves finite suprema. Also, the following axiom, called Peirce's law, is usually added [AA96]

$$(\supset 3) \qquad ((\varphi \supset \psi) \supset \varphi) \supset \varphi$$

Assuming this to hold is equivalent to assuming that L_+ is a Boolean frame [JJP16].

6.2.25 Remark. Notice that assuming $\mathfrak{o}_+ \circ \mathfrak{o}_- = \mathrm{id}$ implies that $\mathfrak{o}_-$ is an injective frame homomorphism and $\mathfrak{o}_+$ is its right adjoint. This follows from the fact that we always have that $\mathfrak{o}_- \circ \mathfrak{o}_+ \leq \mathrm{id}$ and that $\mathfrak{o}_-$ already distributes over meets and preserves 1. Conversely, let $m\colon L \to M$ be an injective frame homomorphism. It is not difficult to check that $(L, M, \mathsf{con}_m, \mathsf{tot}_m)$ defined as

$$(x,y) \in \mathsf{con}_m \overset{\text{def}}{\equiv} m(x) \wedge y = 0 \quad \text{and} \quad (x,y) \in \mathsf{tot}_m \overset{\text{def}}{\equiv} m(x) \vee y = 1$$

is a d-frame. This d-frame can be equipped with negation and implication

$$\neg\alpha = (m_\bullet(\alpha_-), m(\alpha_-)) \quad \text{and} \quad \alpha \supset \beta = (\alpha_+ \to \beta_+, m(\alpha_+) \wedge \beta_-)$$

which satisfies the conditions of Propositions 6.2.3 and 6.2.4 and so the same axioms as in Theorem 6.2.24 hold, plus ($\supset$ 3).

Whenever we have a continuous map between spaces $f\colon X \to Y$, we can factor it into an onto map $g\colon X \twoheadrightarrow f[X]$ followed by an injection $f[X] \hookrightarrow Y$. Then, $m = \Omega(g)$ is an injective frame homomorphism giving rise to a d-frame with the properties described above. Because continuous = computable, this says that computable processes come with paraconsistent language attached to them.

For example, if $\neg(\varphi \supset \psi) = \neg\varphi \supset' \neg\psi$ holds in a d-frame arising from an injective m if and only if m is an open homomorphism, as defined in [PP12] (where $\supset'$ was defined in Remark 6.2.9).

6.3 Belnap-Dunn geometric logic

In the previous section we briefly discussed to which extent d-frames model the logic of bilattices. However, as we argued in Section 6.2.2, bilattices are not suited for modelling infinite processes. For this reason we outline a new logic inspired by Belnap which is sound and complete with respect to d-frames. We mostly get our inspiration from Vicker's presentation of *geometric logic*, that is, the logic of frames [Vic07].

Note that the proof of completeness is not just a mere adaptation of the classical proof for propositional (intuitionistic) logic. It is more involved mostly because the logic of d-frames is infinitary. In fact, it uses the iterative machinery of quotients described in Chapter 3.

6.3.1 Language. Let $\mathcal{V}ar$ be a fixed set of *variables*. The logic of d-frames contains four *basic constant symbols* $\bot$, $\top$, tt and ff. The class $\mathcal{F}m$ of *d-frame formulas* is defined by a transfinite induction from variables and basic constant symbols as follows

$$\begin{aligned} \bot, \top, tt, ff \in \mathcal{F}m, \quad & \mathcal{V}ar \subseteq \mathcal{F}m, \\ S \text{ is a subset of } \mathcal{F}m \implies & \bigsqcup S \in \mathcal{F}m, \\ \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}m \implies & \alpha_1 \sqcap \alpha_2 \sqcap \ldots \sqcap \alpha_n \in \mathcal{F}m. \end{aligned}$$

It follows from this construction that the formulas form a proper class. However, as we will see in Section 6.3.1, this is not a problem because the collection of equivalence classes of those formulas is in bijection with a set. Next, define $\mathcal{J}ud_{\Rightarrow}$, $\mathcal{J}ud_{\mathsf{con}}$ and $\mathcal{J}ud_{\mathsf{tot}}$ to be the classes of *judgements* of the form

$$\alpha \Rightarrow \beta, \qquad \mathtt{con}(\alpha), \quad \text{and} \quad \mathtt{tot}(\alpha),$$

respectively, where α and β range over the elements of $\mathcal{F}m$. Often, we will denote the class of all judgements as just $\mathcal{J}ud$, i.e. $\mathcal{J}ud = \mathcal{J}ud_{\Rightarrow} \dot\cup \mathcal{J}ud_{\mathsf{con}} \dot\cup \mathcal{J}ud_{\mathsf{tot}}$.

6.3.2 Satisfaction relation. Let $\mathcal{L}$ be a d-frame and let $\mathfrak{v}\colon \mathcal{V}ar \to \mathcal{L}$ be an assignment of variables. In the following, we will always use the same symbol for the natural extension of $\mathfrak{v}$ to $\mathcal{F}m \to \mathcal{L}$ such that it preserves basic constants $\bot$, $\top$, tt and ff. Then, define the *satisfaction relation* as follows

$$\begin{aligned} \mathcal{L} \vDash_{\mathfrak{v}} \alpha \Rightarrow \beta \quad & \text{iff} \quad \mathfrak{v}(\alpha) \sqsubseteq \mathfrak{v}(\beta) \text{ in } \mathcal{L} \\ \mathcal{L} \vDash_{\mathfrak{v}} \mathtt{con}(\alpha) \quad & \text{iff} \quad \mathfrak{v}(\alpha) \in \mathsf{con}_{\mathcal{L}} \\ \mathcal{L} \vDash_{\mathfrak{v}} \mathtt{tot}(\alpha) \quad & \text{iff} \quad \mathfrak{v}(\alpha) \in \mathsf{tot}_{\mathcal{L}} \end{aligned}$$

From this relation we also define the usual abbreviations. For a judgement φ, define $\mathcal{L} \vDash \varphi$ to mean that $\mathcal{L} \vDash_{\mathfrak{v}} \varphi$, for all valuations $\mathfrak{v}\colon \mathcal{V}ar \to \mathcal{L}$. For a set of judgements Γ, $\mathcal{L} \vDash \Gamma$ means $\mathcal{L} \vDash \psi$, for all $\psi \in \Gamma$, and $\Gamma \vDash \varphi$ means that, whenever $\mathcal{L} \vDash \Gamma$ for some d-frame $\mathcal{L}$, then also $\mathcal{L} \vDash \varphi$.

6.3.3 Axioms of d-frame logic. Our next task is to define an entailment relation which we later prove to be sound and complete with respect to the satisfaction relation. Before we look at how it is defined, let us first list the *basic rules* of d-frame logic. The rules are quantified over all formulas $\alpha, \beta, \gamma \in \mathcal{F}m$, subsets of formulas $S \subseteq \mathcal{F}m$ and subsets of judgements $\Gamma \subseteq \mathcal{J}ud$.

(Frm-1) *Basic pre-order and entailment rules:*

$$\frac{}{\alpha \Rightarrow \alpha} \qquad \frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma} \qquad \frac{\Gamma}{\varphi}\ (\varphi \in \Gamma)$$

(Frm-2) *Information-wise implication and meets:*

$$\frac{}{\alpha \Rightarrow \top} \qquad \frac{\gamma \Rightarrow \alpha \quad \gamma \Rightarrow \beta}{\gamma \Rightarrow \alpha \sqcap \beta} \qquad \frac{}{\alpha \sqcap \beta \Rightarrow \alpha} \qquad \frac{}{\alpha \sqcap \beta \Rightarrow \beta}$$

(Frm-3) *Information-wise implication and joins, and frame distributivity:*

$$\frac{}{\bigsqcup \emptyset \Rightarrow \bot} \qquad \frac{}{\alpha \Rightarrow \bigsqcup S}\,(\alpha \in S) \qquad \frac{\alpha \Rightarrow \beta \text{ (all } \alpha \in S)}{\bigsqcup S \Rightarrow \beta}$$

$$\frac{}{\bot \Rightarrow \alpha} \qquad \frac{}{\alpha \sqcap \bigsqcup S \Rightarrow \bigsqcup \{\alpha \sqcap \beta \mid \beta \in S\}}$$

(Frm-4) *Relationship between the constants:*

$$\frac{}{\mathit{tt} \sqcap \mathit{ff} \Rightarrow \bot} \qquad \frac{}{\top \Rightarrow \mathit{tt} \sqcup \mathit{ff}}$$

(d-Frm-1) *Consistency relation:*

$$\frac{}{\mathtt{con}(\mathit{tt})} \qquad \frac{}{\mathtt{con}(\mathit{ff})} \qquad \frac{\mathtt{con}(\alpha) \quad \mathtt{con}(\beta)}{\mathtt{con}(\alpha \curlywedge \beta)} \qquad \frac{\mathtt{con}(\alpha) \quad \mathtt{con}(\beta)}{\mathtt{con}(\alpha \curlyvee \beta)}$$

$$\frac{\mathtt{con}(\beta) \quad \alpha \Rightarrow \beta}{\mathtt{con}(\alpha)} \qquad \frac{\mathtt{con}(\alpha) \text{ (all } \alpha \in S) \quad S \text{ is } \Rightarrow\text{-directed}}{\mathtt{con}(\bigsqcup S)}$$

(d-Frm-2) *Totality relation:*

$$\frac{}{\mathtt{tot}(\mathit{tt})} \qquad \frac{}{\mathtt{tot}(\mathit{ff})} \qquad \frac{\mathtt{tot}(\alpha) \quad \mathtt{tot}(\beta)}{\mathtt{tot}(\alpha \curlywedge \beta)} \qquad \frac{\mathtt{tot}(\alpha) \quad \mathtt{tot}(\beta)}{\mathtt{tot}(\alpha \curlyvee \beta)}$$

$$\frac{\mathtt{tot}(\alpha) \quad \alpha \Rightarrow \beta}{\mathtt{tot}(\beta)}$$

(d-Frm-3) *Interaction between* $\mathtt{con}$ *and* $\mathtt{tot}$*:*

$$\frac{\mathtt{con}(\alpha) \quad \mathtt{tot}(\beta) \quad \alpha \sqcap \mathit{tt} \Leftrightarrow \beta \sqcap \mathit{tt}}{\alpha \Rightarrow \beta} \qquad \frac{\mathtt{con}(\alpha) \quad \mathtt{tot}(\beta) \quad \alpha \sqcap \mathit{ff} \Leftrightarrow \beta \sqcap \mathit{ff}}{\alpha \Rightarrow \beta}$$

Where $\alpha \sqcup \beta$ is a shorthand for $\bigsqcup\{\alpha, \beta\}$. In the rules for the consistency and totality relations we needed to use logical meets and joins. Those are defined as shorthands:

$$\alpha \curlywedge \beta \stackrel{\text{def}}{\equiv} ((\alpha \sqcap \beta) \sqcap \mathit{tt}) \sqcup ((\alpha \sqcup \beta) \sqcap \mathit{ff})$$
$$\alpha \curlyvee \beta \stackrel{\text{def}}{\equiv} ((\alpha \sqcup \beta) \sqcap \mathit{tt}) \sqcup ((\alpha \sqcap \beta) \sqcap \mathit{ff})$$

The $\Leftrightarrow$ used in (d-Frm-3) translates as two $\Rightarrow$ conditions, one for each direction.

6.3.4 Entailment relation. A *substitution* σ is an assignment of variables $\sigma\colon \mathcal{V}ar \to \mathcal{F}m$ which, when applied to a formula φ, yields a formula $\sigma(\varphi)$ obtained from φ by replacing all of its variables simultaneously, as given by the mapping σ. For a set of judgements define $\sigma\Gamma$ to be the closure of Γ under all substitutions, i.e. $\sigma\Gamma$ is the minimal set of judgements such that, for all substitutions $\sigma\colon \mathcal{V}ar \to \mathcal{F}m$ and $\varphi \in \Gamma$, $\sigma(\varphi) \in \sigma\Gamma$.

Next, define the *entailment relation* $\vdash$ as a relation between sets of judgements and judgements. To define it formally, consider the function $\mathfrak{ent}\colon \mathcal{P}(\mathcal{J}ud) \to \mathcal{P}(\mathcal{J}ud)$ given by

$$\mathfrak{ent}(\Gamma) = \{\varphi \mid \text{a rule } \frac{\Gamma'}{\varphi} \text{ is in 6.3.3, for some } \Gamma' \text{ subset of } \sigma\Gamma\}.$$

For any set of formulas Γ, we have a growing sequence of sets $\{\Gamma^\delta \mid \delta \in \mathsf{Ord}\}$ where, for an ordinal δ and a limit ordinal λ,

$$\Gamma^0 = \Gamma, \quad \Gamma^{\delta+1} = \mathfrak{ent}(\Gamma^\delta) \quad \text{and} \quad \Gamma^\lambda = \bigcup_{\delta<\lambda} \Gamma^\delta.$$

Then, define

$$\Gamma \vdash \varphi \quad \stackrel{\text{def}}{\equiv} \quad \varphi \in \Gamma^\delta, \text{ for some ordinal } \delta.$$

Because the basic axioms from 6.3.3 are all true in d-frames, it is an immediate observation that we have *soundness* of the entailment relation with respect to the satisfaction relation:

6.3.5 Proposition. *For a judgement φ and set of judgements Γ, $\Gamma \vdash \varphi$ implies $\Gamma \vDash \varphi$.*

In the next two subsections we also prove completeness. Note that this has to be done by a d-frame specific method because there is no general theorem for infinitary logics[3]. This is because infinitary logics are usually not complete and they often do not even have a Lindenbaum-Tarski algebra.

6.3.6 Remark. In comparison with the geometric logic of frames defined in [Vic07], we extended the language with constants tt and ff and two more types of judgements, that is, $\mathfrak{con}(\alpha)$ and $\mathfrak{tot}(\alpha)$. We can read any judgement of the form $\alpha \Rightarrow \beta$ as: the positive evidence of α implies that of β and the same is the case for the negative evidence. This reading comes from the interpretation in the models because $\alpha \Rightarrow \beta$ translates as $\alpha \sqsubseteq \beta$ (i.e. α is information-wise smaller than β).

Also, except for the axioms for $\mathfrak{con}$ and $\mathfrak{tot}$, the logical order does not appear in the axiomatisation of our logic. This is because the logical order is definable from the information order; we can define $\alpha \leqslant \beta$ to be the shorthand for $\alpha \curlywedge \beta \Leftrightarrow \alpha$.

[3]Exception for Theorem 5.1.7 in [MR77] which works for some logics with a countable set of variables.

6.3.1 Completeness of (plain) geometric logic

In the following we compute the Lindenbaum-Tarski algebra for the d-frame logic. However, in the presence of constants tt and ff one cannot use the standard theory. For this reason we first extend the geometric/frame logic with constants and prove completeness for it.

Define (propositional) geometric logic as follows:

- Fix a set of *variables* $\mathcal{V}ar$ and a set of *constant symbols* $\mathcal{C}ons$, disjoint for each other. Then, the set of *atoms* $\mathcal{A}t$ consists of both variables and constant symbols, i.e. $\mathcal{A}t = \mathcal{V}ar \cup \mathcal{C}ons$.
- *Formulas* are defined by a transfinite induction as in 6.3.1 for d-frame logic, except that we include also all constant symbols, $\mathcal{C}ons \subseteq \mathcal{F}m$, and from the basic constants we only include $\bot$ and $\top$.
- The only judgements we consider are of the form $\alpha \Rightarrow \beta$.
- For a frame L and a valuation of atoms $\mathfrak{v}\colon \mathcal{A}t \to L$, define $L \vDash^{\mathrm{Fr}}_{\mathfrak{v}} \alpha \Rightarrow \beta$ to mean $\mathfrak{v}(\alpha) \sqsubseteq \mathfrak{v}(\beta)$.
- $\Gamma \vdash^{\mathrm{Fr}} \varphi$ iff we can deduce φ from Γ by only using the rules from (Frm-1), (Frm-2) and (Frm-3) from above.

The other abbreviations for $\vDash^{\mathrm{Fr}}$ need to take into an account that we also have constants in our language: For an assignment of constants $\mathfrak{c}\colon \mathcal{C}ons \to L$, define $(L, \mathfrak{c}) \vDash^{\mathrm{Fr}} \varphi$ to mean that $L \vDash^{\mathrm{Fr}}_{\mathfrak{v}} \varphi$, for all valuations $\mathfrak{v}$ such that the restriction $\mathfrak{v}\restriction_{\mathcal{C}ons}$ is equal to $\mathfrak{c}$. Further, $(L, \mathfrak{c}) \vDash^{\mathrm{Fr}} \Gamma$ means $(L, \mathfrak{c}) \vDash^{\mathrm{Fr}} \psi$, for all $\psi \in \Gamma$, and $\Gamma \vDash^{\mathrm{Fr}} \varphi$ means that, whenever $(L, \mathfrak{c}) \vDash^{\mathrm{Fr}} \Gamma$ for some frame L and assignment $\mathfrak{c}\colon \mathcal{C}ons \to L$, then also $(L, \mathfrak{c}) \vDash^{\mathrm{Fr}} \varphi$.

In the following proposition we show that there is a bijection between $\Leftrightarrow$-equivalence classes of formulas in geometric logic and elements of $\mathbf{Fr}\langle\mathcal{A}t\rangle$ (which is the free frame defined in Section 3.4.1). This will allow us to treat $\mathbf{Fr}\langle\mathcal{A}t\rangle$ as if it was the set of formulas $\mathcal{F}m$. We will obtain the bijection as the natural extension $\mathcal{F}m \to \mathbf{Fr}\langle\mathcal{A}t\rangle$ of the inclusion of atoms as generators $\mathcal{A}t \hookrightarrow \mathbf{Fr}\langle\mathcal{A}t\rangle$.

6.3.7 Lemma. ***Fr***$\langle\mathcal{A}t\rangle \vDash^{\mathrm{Fr}} \alpha \Rightarrow \beta$ *if and only if* $\vdash^{\mathrm{Fr}} \alpha \Rightarrow \beta$

Proof. The right-to-left direction is immediate. Conversely, each formula in the language of geometric logic is $\Leftrightarrow$-equivalent to a formula of the form

$$\bigsqcup_{F \in A} (\sqcap_{v \in F}\, v), \quad \text{for some } A \subseteq \mathcal{F}(\mathcal{A}t)^{4}. \tag{6.3.1}$$

This is proved by a transfinite induction; one inequality follows from the last rule of (Frm-3) and the other from the last two rules of (Frm-2) and the third rule of (Frm-3).

[4] Recall that $\mathcal{F}(\mathcal{A}t)$ is the set of finite subsets of $\mathcal{A}t$, as defined in Section 3.4.1.

Next, since $\mathbf{Fr}\langle\mathcal{A}t\rangle = \mathsf{Down}(\mathcal{F}(\mathcal{A}t), \cup)$, we aim to relate the formula in (6.3.1) to a downset of finite meets. Define

$$A^\dagger \stackrel{\text{def}}{\equiv} \{G \subseteq_{\text{fin}} \mathcal{A}t \mid F \subseteq G, \text{ for some } F \in A\},$$

Then, each formula of the form (6.3.1) is equivalent to $\bigsqcup_{F\in A^\dagger}(\sqcap_{v\in F} v)$ by the last two rules in (Frm-2) and the second and third rule of (Frm-3).

Let $\mathbf{Fr}\langle\mathcal{A}t\rangle \vDash^{\mathrm{Fr}} \alpha \Rightarrow \beta$ for some $\alpha = \bigsqcup_{F\in A}(\sqcap_{v\in F} v)$ and $\beta = \bigsqcup_{F\in B}(\sqcap_{v\in F} v)$. Since $\mathfrak{v}(\alpha) \sqsubseteq \mathfrak{v}(\beta)$ for all valuations $\mathfrak{v}\colon \mathcal{A}t \to \mathbf{Fr}\langle\mathfrak{v}\rangle$, it must be true, in particular, for the valuation $\mathfrak{v}$ which is the inclusion of the generators $\mathcal{A}t \hookrightarrow \mathbf{Fr}\langle\mathcal{A}t\rangle$. In this case $A^\dagger$ and $B^\dagger$ are elements of $\mathbf{Fr}\langle\mathcal{A}t\rangle = \mathsf{Down}(\mathcal{F}(\mathcal{A}t), \cup)$ and $\mathfrak{v}(\alpha) \Rightarrow \mathfrak{v}(\beta)$ translates as $A^\dagger \subseteq B^\dagger$. Therefore, $\sqcap_{v\in F} v \Rightarrow \beta$, for every $F \in A$, by the second rule of (Frm-3), and so $\bigsqcup_{F\in A}(\sqcap_{v\in F} v) \Rightarrow \beta$ by the third rule of (Frm-3). □

In Proposition 6.3.9 below we show completeness of geometric logic with respect to frames. Before we do that let us take a look at a special case.

Convention. By abuse of notation we define $\mathbf{Fr}\langle\mathcal{A}t \mid \Gamma\rangle$, for a set of frame judgements Γ, to be the freely generated frame $\mathbf{Fr}\langle\mathcal{A}t\rangle$ quotiented by equations $\alpha \sqsubseteq \beta$ for every $\alpha \Rightarrow \beta$ from Γ , i.e. we see Γ as a subset of $\mathbf{Fr}\langle\mathcal{A}t\rangle\times\mathbf{Fr}\langle\mathcal{A}t\rangle$ and $\mathbf{Fr}\langle\mathcal{A}t \mid \Gamma\rangle$ as the quotient $\mathbf{Fr}\langle\mathcal{A}t\rangle/\Gamma$ (recall Section 3.4.1).

6.3.8 Lemma. *For a set of frame judgements Γ and the valuation which is an inclusion of atoms as generators $\mathfrak{v}\colon \mathcal{A}t \hookrightarrow \mathbf{Fr}\langle\mathcal{A}t \mid \Gamma\rangle$,*

$$\mathbf{Fr}\langle\mathcal{A}t \mid \Gamma\rangle \vDash^{\mathrm{Fr}}_{\mathfrak{v}} \alpha \Rightarrow \beta \quad \textit{implies} \quad \Gamma \vdash^{\mathrm{Fr}} \alpha \Rightarrow \beta.$$

Proof. Consider the factorisation of $\mathfrak{v}$ into the composition of the inclusion $\mathcal{A}t \hookrightarrow \mathbf{Fr}\langle\mathcal{A}t\rangle$ and the quotient $\mathbf{Fr}\langle\mathcal{A}t\rangle \twoheadrightarrow \mathbf{Fr}\langle\mathcal{A}t \mid \Gamma\rangle$. Recall that $\mathbf{Fr}\langle\mathcal{A}t \mid \Gamma\rangle$ is the set of all Γ-saturated elements of $\mathbf{Fr}\langle\mathcal{A}t\rangle$ (Section A.5) and that the quotient map $\mu\colon \mathbf{Fr}\langle\mathcal{A}t\rangle \twoheadrightarrow \mathbf{Fr}\langle\mathcal{A}t \mid \Gamma\rangle$ can be computed as $\gamma \mapsto \bigsqcup_{\delta\in\mathsf{Ord}} \gamma^\delta$ where, for an ordinal δ and a limit ordinal λ,[5]

$$\begin{aligned}
\gamma^0 &= \gamma,\\
\gamma^{\delta+1} &= \gamma^\delta \sqcup \bigsqcup\{\alpha \sqcap \varepsilon \mid \beta \sqcap \varepsilon \sqsubseteq \gamma^\delta \text{ for some } \varepsilon \text{ and } \alpha \Rightarrow \beta \text{ in } \Gamma\} \quad \text{and}\\
\gamma^\lambda &= \bigsqcup_{\delta<\lambda} \gamma^\delta.
\end{aligned}$$

Moreover, for every $\gamma \in \mathbf{Fr}\langle\mathcal{A}t\rangle$, there is an ordinal λ such that $\gamma^\lambda = \gamma^{\lambda+1}$. We show $\Gamma \vdash^{\mathrm{Fr}} \gamma^\lambda \Rightarrow \gamma$ by a transfinite induction. First, $\Gamma \vdash^{\mathrm{Fr}} \gamma^0 \Rightarrow \gamma$ is in (Frm-1). Next, assume $\Gamma \vdash^{\mathrm{Fr}} \gamma^\delta \Rightarrow \gamma$. Whenever $\beta \sqcap \varepsilon \sqsubseteq \gamma^\delta$ for some $\varepsilon \in \mathbf{Fr}\langle\mathcal{A}t\rangle$ (and so $\vdash^{\mathrm{Fr}} \beta \sqcap \varepsilon \Rightarrow \gamma^\delta$ by Lemma 6.3.7) and $\alpha \Rightarrow \beta$ in Γ, then (Frm-2) gives that $\alpha \sqcap \varepsilon \Rightarrow \beta \sqcap \varepsilon$

[5]In this instance, we denote the order of $\mathbf{Fr}\langle\mathcal{A}t\rangle$ as $\sqsubseteq$ and, therefore, its frame operations as $\bigsqcup$ and $\sqcap$.

and, then, the second rule in (Frm-1) entails that $\alpha \sqcap \varepsilon \Rightarrow \gamma^{\delta}$ and also that $\alpha \sqcap \varepsilon \Rightarrow \gamma$. Therefore, by the third rule of (Frm-3), $\Gamma \vdash^{\mathrm{Fr}} \gamma^{\delta+1} \Rightarrow \gamma$. The transfinite step also follows from (Frm-3).

Because $\alpha \sqsubseteq \beta$ in $\mathbf{Fr}\langle \mathcal{A}t \mid \Gamma\rangle$ if and only if $\alpha^{\delta} \sqsubseteq \beta^{\delta'}$ in $\mathbf{Fr}\langle \mathcal{A}t\rangle$, for some ordinals δ and δ', by Lemma 6.3.7, $\vdash^{\mathrm{Fr}} \alpha^{\delta} \Rightarrow \beta^{\delta'}$, also $\alpha \Rightarrow \alpha^{\delta}$ by (Frm-3) and $\beta^{\delta'} \Rightarrow \beta$ by what we proved above. Therefore, by (Frm-1), $\Gamma \vdash^{\mathrm{Fr}} \alpha \Rightarrow \beta$. □

Finally, we use this lemma to show that $\Gamma \vDash^{\mathrm{Fr}} \alpha \Rightarrow \beta$ implies $\Gamma \vdash^{\mathrm{Fr}} \alpha \Rightarrow \beta$.

6.3.9 Proposition (Completeness). *Let Γ be a set of frame judgements,*

1. $(\mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle, \mathfrak{c}) \vDash^{\mathrm{Fr}} \Gamma$
2. $(\mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle, \mathfrak{c}) \vDash^{\mathrm{Fr}} \alpha \Rightarrow \beta$ *implies* $\Gamma \vdash^{\mathrm{Fr}} \alpha \Rightarrow \beta$

where $\sigma\Gamma$ is the closure of Γ under substitution (as in 6.3.4) and the picked assignment $\mathfrak{c}\colon \mathcal{C}ons \to \mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle$ is the inclusion of constants as generators.

Proof. (1) Let $\mathfrak{v}\colon \mathcal{A}t \to \mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle$ be an assignment such that $\mathfrak{v}\restriction_{\mathcal{C}ons} = \mathfrak{c}$. Since the frame $\mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle$ is a subset of $\mathbf{Fr}\langle \mathcal{A}t\rangle$ (i.e. it is a sublocale), we can define a substitution $\sigma\colon \mathcal{V}ar \to \mathbf{Fr}\langle \mathcal{A}t\rangle$ as the composition of $\mathfrak{v}\restriction_{\mathcal{V}ar}$ with the set inclusion $\mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle \subseteq \mathbf{Fr}\langle \mathcal{A}t\rangle$. Then, for an equation $\alpha \Rightarrow \beta$ in Γ, the equation $\sigma(\alpha) \Rightarrow \sigma(\beta)$ is in $\sigma\Gamma$ because $\sigma\Gamma$ is closed under substitutions. Therefore, for the quotient map $\mu\colon \mathbf{Fr}\langle \mathcal{A}t\rangle \twoheadrightarrow \mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle$, $\mu\sigma(\alpha) \sqsubseteq \mu\sigma(\beta)$ is true in $\mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle$ and so $\mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle \vDash^{\mathrm{Fr}}_{\mathfrak{v}} \alpha \Rightarrow \beta$ because $\mathfrak{v} = \mu \cdot \sigma$.

(2) If $(\mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle, \mathfrak{c}) \vDash^{\mathrm{Fr}} \alpha \Rightarrow \beta$ then, in particular, $\mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle \vDash^{\mathrm{Fr}}_{\mathfrak{v}} \alpha \Rightarrow \beta$ where $\mathfrak{v}$ is the inclusion of generators $\mathcal{A}t \hookrightarrow \mathbf{Fr}\langle \mathcal{A}t \mid \sigma\Gamma\rangle$. Lemma 6.3.8 implies that $\sigma\Gamma \vdash^{\mathrm{Fr}} \alpha \Rightarrow \beta$ and, because $\Gamma \vdash^{\mathrm{Fr}} \varphi$ for every $\varphi \in \sigma\Gamma$, also $\Gamma \vdash^{\mathrm{Fr}} \alpha \Rightarrow \beta$. □

6.3.2 Completeness of d-frame logic

In the following we show that completeness of geometric logic with constants can be directly used to compute the Lindenbaum-Tarski algebra for d-frame logic. The starting point is Proposition 6.3.9 used with $\mathcal{C}ons = \{tt, ff\}$ and $\Gamma = \{tt \sqcap ff \Rightarrow \bot,\ \top \Rightarrow tt \sqcup ff\}$. It follows that the $\Leftrightarrow$-equivalence classes of d-frame formulas $\mathcal{F}m$ can be identified with elements of the freely generated frame

$$\mathcal{T} \overset{\text{def}}{\equiv} \mathbf{Fr}\langle \mathcal{V}ar, tt, ff \mid tt \sqcap ff \sqsubseteq \bot,\ \top \sqsubseteq tt \sqcup ff\rangle$$

where $\sqsubseteq$ denotes the order of the free frame and $\bot$ and $\top$ are the smallest and largest elements, respectively[6]. As before, the mapping $\mathcal{F}m \to \mathcal{T}$ which establishes the

[6] As before, we interpret statements $\alpha \Rightarrow \beta$ in the logic as inequalities $\alpha \sqsubseteq \beta$ in $\mathcal{T}$. Observe that the rules (d-Frm-1), (d-Frm-2) and (d-Frm-3) do not add any new equations since, unless more is assumed, only tt and ff are “in” $\mathfrak{con}$ and $\mathfrak{tot}$.

bijection between $\Leftrightarrow$-equivalence classes of formulas and elements of $\mathcal{T}$ is obtained as the natural extension of the inclusion of generators $\mathcal{V}ar \hookrightarrow \mathcal{T}$.

$\mathcal{T}$ is a lattice with two (non-trivial) complemented elements tt and ff. A general result from lattice theory says that any lattice L with two complemented elements can be equivalently represented as a product $L_+ \times L_-$ of two lattices L_+ and L_- [JM06]. In the following we explicitly compute this decomposition for $\mathcal{T}$.

6.3.10 Lemma.

1. *$\mathcal{T} \cong \mathbf{Fr}\langle \mathcal{V}ar_+ \rangle \times \mathbf{Fr}\langle \mathcal{V}ar_- \rangle$ where $\mathcal{V}ar_\pm$ are defined syntactically as $\{v_+ \mid v \in \mathcal{V}ar\}$.*

2. *For a $\Gamma \subseteq \mathcal{J}ud_\Rightarrow$, the quotient frame $\mathcal{T}/\Gamma$ is isomorphic to the product*

$$\mathbf{Fr}\langle \mathcal{V}ar_+ \mid \Gamma_+ \rangle \times \mathbf{Fr}\langle \mathcal{V}ar_- \mid \Gamma_- \rangle$$

where the relations $\Gamma_\pm = \{\alpha_\pm \sqsubseteq \beta_\pm \mid \alpha \Rightarrow \beta \text{ in } \Gamma\}$ are the images of Γ under the isomorphism $\alpha \mapsto (\alpha_+, \alpha_-)$ from (1).

Proof. (1) We show that $\mathbf{Fr}\langle \mathcal{V}ar \rangle \times \mathbf{Fr}\langle \mathcal{V}ar \rangle$ has the same universal property as $\mathcal{T}$ has. First, define the embedding $i\colon \mathcal{V}ar \cup \{ff, tt\} \to \mathbf{Fr}\langle \mathcal{V}ar \rangle \times \mathbf{Fr}\langle \mathcal{V}ar \rangle$ as $v \mapsto (v_+, v_-)$, $tt \mapsto (1,0)$ and $ff \mapsto (0,1)$. Then, for a mapping $g\colon \mathcal{V}ar \cup \{ff, tt\} \to L$ into a frame $(L, \bigsqcup, \sqcap, \bot, \top)$ which preserves the defining equations of $\mathcal{T}$ define its lift $\overline{g}\colon \mathbf{Fr}\langle \mathcal{V}ar \rangle \times \mathbf{Fr}\langle \mathcal{V}ar \rangle \to L$ as

$$\overline{g}\colon (x,y) \mapsto (\widetilde{g}_+(x) \sqcap g(tt)) \sqcup (\widetilde{g}_-(y) \sqcap g(ff))$$

where $\widetilde{g}_\pm$ is the lift of $g{\restriction_{\mathcal{V}ar}}\colon \mathcal{V}ar \to L$ to $\widetilde{g}_\pm\colon \mathbf{Fr}\langle \mathcal{V}ar_\pm \rangle \to L$ (after the obvious renaming). Checking that $\overline{g} \cdot i = g$ and that $\overline{g}$ is unique is standard.

(2) follows from (1) by a general fact about quotients of a product of two frames (see Proposition A.5.4 in the appendix). □

This establishes that $\mathcal{T}$ can be represented as the product $\mathbf{Fr}\langle \mathcal{V}ar_+ \rangle \times \mathbf{Fr}\langle \mathcal{V}ar_- \rangle$. In fact, these frames are identical to the frame components of the freely generated d-frame given by a single-sorted presentation as $\mathbf{dFr}\langle \mathcal{V}ar \rangle$ (recall Section 3.4.4). In order to achieve completeness of the d-frame logic, we need to show that the d-frame

$$\mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma \rangle$$

behaves as the Lindenbaum-Tarski algebra for Γ. As in Convention on page 189 and interpret each judgement in $\sigma\Gamma$ as a d-frame equation in the obvious way, e.g. $\alpha \Rightarrow \beta$ translates as $\alpha \sqsubseteq \beta$, $\mathtt{con}(\alpha)$ as $\alpha \in \mathrm{con}$ and $\mathtt{tot}(\alpha)$ as $\alpha \in \mathrm{tot}$.

The first step in proving completeness is a d-frame variant of Lemma 6.3.8 for the $\Rightarrow$-fragment of our logic. Most of the proof is about establishing a translation between the single-sorted and two-sorted views on the free d-frame.

6.3.11 Lemma. *For a set of frame judgements Γ (i.e. $\Gamma \subseteq \mathcal{J}ud_{\Rightarrow}$) and the valuation which is an inclusion of variables as generators $\mathfrak{v}\colon \mathcal{V}ar \hookrightarrow \mathbf{dFr}\langle \mathcal{V}ar \mid \Gamma\rangle$,*

$$\mathbf{dFr}\langle \mathcal{V}ar \mid \Gamma\rangle \vDash_{\mathfrak{v}} \alpha \Rightarrow \beta \quad \textit{implies} \quad \Gamma \vdash \alpha \Rightarrow \beta.$$

Proof. First, let us examine how the d-frame $\mathbf{dFr}\langle \mathcal{V}ar \mid \Gamma\rangle$ looks like. Recall from Section 3.4.4, that each $\alpha \Rightarrow \beta$ in Γ translates as two inequalities $\alpha_+ \leq \beta_+$ and $\alpha_- \leq \beta_-$ for $\mathbf{Fr}\langle \mathcal{V}ar_+\rangle$ and $\mathbf{Fr}\langle \mathcal{V}ar_-\rangle$, respectively. Because there are no judgements involving $\mathfrak{con}$ or $\mathfrak{tot}$ in Γ, the presentation of $\mathbf{dFr}\langle \mathcal{V}ar \mid \Gamma\rangle$ automatically satisfies the conditions of Theorem 3.4.20 making the resulting d-frame isomorphic to

$$(\mathbf{Fr}\langle \mathcal{V}ar_+ \mid \Gamma_+\rangle\,,\ \mathbf{Fr}\langle \mathcal{V}ar_- \mid \Gamma_-\rangle\,,\ \mathsf{con}_{\mathsf{triv}},\ \mathsf{tot}_{\mathsf{triv}}).$$

Here $\mathsf{con}_{\mathsf{triv}}$ and $\mathsf{tot}_{\mathsf{triv}}$ are the trivial consistency and totality relations (as in Example 2.3.5). Moreover, by Lemma 6.3.10, $\mathbf{Fr}\langle \mathcal{V}ar_+ \mid \Gamma_+\rangle \times \mathbf{Fr}\langle \mathcal{V}ar_- \mid \Gamma_-\rangle \cong \mathcal{T}/\Gamma$ which is isomorphic to $\mathbf{Fr}\langle \mathcal{V}ar, \mathit{tt}, \mathit{ff} \mid \mathit{tt} \sqcap \mathit{ff} \sqsubseteq \bot,\ \top \sqsubseteq \mathit{tt} \sqcup \mathit{ff},\ \Gamma\rangle$ (notice the extra Γ at the end).

It is immediate to see that, for a judgement φ of the form $\alpha \Rightarrow \beta$, $\mathbf{dFr}\langle \mathcal{V}ar \mid \Gamma\rangle \vDash_{\mathfrak{v}} \varphi$ if and only if $\mathcal{T}/\Gamma \vDash^{\mathrm{Fr}}_{\mathfrak{v}} \varphi$. However, in the latter case, Lemma 6.3.11 implies that $\Gamma \vdash^{\mathrm{Fr}} \varphi$ and so also $\Gamma \vdash \varphi$. □

In the proof of completeness we will also need to relate the procedure which computes $\mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma\rangle$ with the entailment relation. For that we will need that quotient structures on $(\mathbf{Fr}\langle \mathcal{V}ar_+\rangle\,, \mathbf{Fr}\langle \mathcal{V}ar_-\rangle)$, as defined in paragraph 3.4.7, uniquely determine sets of d-frame judgements (up-to interprovability), and vice versa. In the following we write $\Gamma \vdash \Delta$ to denote that $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$:

6.3.12 Lemma. *There is a pair of maps $\Delta_{(-)}\colon Q \mapsto \Delta_Q$ and $Q_{(-)}\colon \Delta \mapsto Q_\Delta$ between quotient structures on $(\mathbf{Fr}\langle \mathcal{V}ar_+\rangle\,, \mathbf{Fr}\langle \mathcal{V}ar_-\rangle)$ and sets of judgements such that*

$$\mathfrak{r}^\star(Q) = \mathfrak{r}^\star(Q_{\Delta_Q}) \quad \textit{and also} \quad \Delta \vdash \Delta_{Q_\Delta} \quad \textit{and} \quad \Delta_{Q_\Delta} \vdash \Delta,$$

for every quotient structure Q and set of judgements Δ.
(where $\mathfrak{r}^\star$ was defined in Section 3.2.3.)

Proof. We use the isomorphism $\mathcal{T} \cong \mathbf{Fr}\langle \mathcal{V}ar_+\rangle \times \mathbf{Fr}\langle \mathcal{V}ar_-\rangle$ from Lemma 6.3.10. Denote by $\alpha \mapsto (\alpha_+, \alpha_-)$ and $(\alpha_+, \alpha_-) \mapsto \langle \alpha_+, \alpha_-\rangle$ the corresponding pair of inverse maps. For any set of judgements $\Delta \subseteq \mathcal{J}ud$, define the following sets of equations

$$E_{\mathsf{con}} = \{(\alpha_+, \alpha_-) \in \mathsf{con} \mid \mathfrak{con}(\alpha) \text{ is in } \Delta\}, \qquad E_+ = \{\alpha_+ \leq \beta_+ \mid \alpha \Rightarrow \beta \text{ is in } \Delta\},$$
$$E_{\mathsf{tot}} = \{(\alpha_+, \alpha_-) \in \mathsf{tot} \mid \mathfrak{tot}(\alpha) \text{ is in } \Delta\} \quad \text{and} \quad E_- = \{\alpha_- \leq \beta_- \mid \alpha \Rightarrow \beta \text{ is in } \Delta\}.$$

Then, set the quotient structure Q_Δ corresponding to Δ to be $(E_{\mathsf{con}}, E_{\mathsf{tot}}, E_+, E_-)$. Conversely, for a quotient structure $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-)$ on $(\mathbf{Fr}\langle \mathcal{V}ar_+\rangle\,, \mathbf{Fr}\langle \mathcal{V}ar_-\rangle)$

define Δ_Q to be the following set of judgements

$$\{\mathfrak{con}(\langle \alpha_+, \alpha_- \rangle) \mid (\alpha_+, \alpha_-) \in \mathsf{con}\} \cup \{\langle \alpha_+, 1\rangle \Rightarrow \langle \beta_+, 1\rangle \mid \alpha_+ \leq \beta_+ \text{ in } R_+\}$$
$$\cup \{\mathfrak{tot}(\langle \alpha_+, \alpha_- \rangle) \mid (\alpha_+, \alpha_-) \in \mathsf{tot}\} \cup \{\langle 1, \alpha_-\rangle \Rightarrow \langle 1, \beta_-\rangle \mid \alpha_- \leq \beta_- \text{ in } R_-\}.$$

Observe that Q_{Δ_Q} is equal to Q if R_+ and R_-, both, contain the inequalities $1 \leq 1$. Moreover, adding this pair of inequalities to Q does not make any difference for the image of $\mathfrak{r}^\star$. On the other hand, Δ_{Q_Δ} contains the same consistency and totality judgements as Δ does, and each judgement $\alpha \Rightarrow \beta$ in Δ yields a pair of judgements $\langle \alpha_+, 1\rangle \Rightarrow \langle \beta_+, 1\rangle$ and $\langle 1, \alpha_-\rangle \Rightarrow \langle 1, \beta_-\rangle$ in Δ_{Q_Δ}. Because $\langle \alpha_+, 1\rangle = \alpha \sqcup \mathit{ff}$ and $\langle \beta_+, 1\rangle = \beta \sqcup \mathit{ff}$ in $\mathcal{T}$, $\alpha \Rightarrow \beta \vdash \langle \alpha_+, 1\rangle \Rightarrow \langle \beta_+, 1\rangle$ by (Frm-3) and, similarly, $\alpha \Rightarrow \beta \vdash \langle 1, \alpha_-\rangle \Rightarrow \langle 1, \beta_-\rangle$. Conversely, $\{\langle 1, \alpha_-\rangle \Rightarrow \langle 1, \beta_-\rangle, \langle \alpha_+, 1\rangle \Rightarrow \langle \beta_+, 1\rangle\} \vdash \alpha \Rightarrow \beta$ by (Frm-2). □

Recall the iterative version $\mathfrak{r}^\infty$ of the operation on quotient structures from Section 3.2.3. We show that $\Delta_{(-)}$ from the previous lemma maps $Q \subseteq \mathfrak{r}^\star(Q)$ and $Q \subseteq \mathfrak{r}^\infty(Q)$ (which hold automatically) to $\Delta_Q \vdash \Delta_{\mathfrak{r}^\star(Q)}$ and $\Delta_Q \vdash \Delta_{\mathfrak{r}^\infty(Q)}$, respectively.

6.3.13 Proposition. *Let Q be a quotient structure on $(\mathbf{Fr}\langle \mathcal{V}ar_+\rangle, \mathbf{Fr}\langle \mathcal{V}ar_-\rangle)$. Then, for the translation from Lemma 6.3.12,*

1. $\Delta_Q \vdash \Delta_{\mathfrak{r}^\star(Q)}$, *and*
2. $\Delta_Q \vdash \Delta_{\mathfrak{r}^\infty(Q)}$.

Proof. Let $Q = (\mathsf{con}, \mathsf{tot}, R_+, R_-)$ be a quotient structure on $(\mathbf{Fr}\langle \mathcal{V}ar_+\rangle, \mathbf{Fr}\langle \mathcal{V}ar_-\rangle)$. (1) Define $(\mathsf{con}^\star, \mathsf{tot}^\star, R_+^\star, R_-^\star)$ to be the quotient structure $\mathfrak{r}^\star(Q)$, i.e. it is equal to

$$(\downarrow^R \mathfrak{D}(\mathsf{con}),\ \uparrow^R \mathsf{tot},\ (\mathsf{con}\,;\mathsf{tot}^{-1}) \cup R_+,\ (\mathsf{con}^{-1}\,;\mathsf{tot}) \cup R_-)$$

Let $\sqsubseteq^R$ be as in paragraph 3.1.10 and let $\sqsubseteq^\star$ be an abbreviation for $\sqsubseteq^{R^\star}$ where $R^\star = (R_+^\star, R_-^\star)$; i.e. $\sqsubseteq^R$ and $\sqsubseteq^\star$ are pre-orders on $\mathbf{Fr}\langle \mathcal{V}ar_+\rangle \times \mathbf{Fr}\langle \mathcal{V}ar_-\rangle$. Observe the following:

- *$\alpha \sqsubseteq^R \beta$ implies $\Delta_Q \vdash \alpha \Rightarrow \beta$*: Notice that $\alpha \sqsubseteq^R \beta$ implies that $\mu(\alpha) \sqsubseteq \mu(\beta)$ where μ is the quotient map $\mathbf{dFr}\langle \mathcal{V}ar\rangle \twoheadrightarrow \mathbf{dFr}\langle \mathcal{V}ar \mid R\rangle$. Moreover, by Lemma 6.3.12, $\mathbf{dFr}\langle \mathcal{V}ar \mid R\rangle$ is isomorphic to the d-frame $\mathbf{dFr}\langle \mathcal{V}ar \mid \Delta_R\rangle$ where Δ_R is the set of judgements $\Delta_{(\emptyset,\emptyset,R_+,R_-)}$, i.e. it is the union of $\{\langle a, 1\rangle \Rightarrow \langle b, 1\rangle \mid a \leq b \text{ in } R_+\}$ and $\{\langle 1, c\rangle \Rightarrow \langle 1, d\rangle \mid c \leq d \text{ in } R_-\}$. Therefore, by Lemma 6.3.11, $\Delta_R \vdash \alpha \Rightarrow \beta$ and, because $\Delta_R \subseteq \Delta_Q$, also $\Delta_Q \vdash \alpha \Rightarrow \beta$.

In the following we use this fact and show that $\alpha \in \mathsf{con}^\star$ implies $\Delta_Q \vdash \mathfrak{con}(\alpha)$, that $\alpha \in \mathsf{tot}^\star$ implies $\Delta_Q \vdash \mathfrak{tot}(\alpha)$ and that $\alpha \sqsubseteq^\star \beta$ implies $\Delta_Q \vdash \alpha \Rightarrow \beta$. We do this in a case by case fashion:

- $\alpha \sqsubseteq^{\star} \beta$: This implies that $\mu'(\alpha) \sqsubseteq \mu'(\beta)$ where μ' is the quotient map $\mathbf{dFr}\langle \mathcal{V}ar \rangle \twoheadrightarrow \mathbf{dFr}\langle \mathcal{V}ar \mid R^{\star} \rangle$. As before, by Lemma 6.3.12, $\mathbf{dFr}\langle \mathcal{V}ar \mid R^{\star} \rangle \cong \mathbf{dFr}\langle \mathcal{V}ar \mid \Delta^{\star}_{\Rightarrow} \rangle$ where $\Delta^{\star}_{\Rightarrow}$ is the set of judgements $\Delta_{(\emptyset, \emptyset, R^{\star}_{+}, R^{\star}_{-})}$, i.e. it is the set

$$\{\langle a, 1\rangle \Rightarrow \langle b, 1\rangle \mid a \leq b \text{ in } R^{\star}_{+}\} \cup \{\langle 1, c\rangle \Rightarrow \langle 1, d\rangle \mid c \leq d \text{ in } R^{\star}_{-}\}.$$

  Then, by Lemma 6.3.11, $\Delta^{\star}_{\Rightarrow} \vdash \alpha \Rightarrow \beta$. Therefore, if we prove that $\Delta_Q \vdash \varphi$, for all $\varphi \in \Delta^{\star}_{\Rightarrow}$, then we have that $\Delta_Q \vdash \alpha \Rightarrow \beta$. If $(a, b) \in R^{\star}_{+}$, then either $(a, b) \in R_{+}$ or $(a, b) \in \mathsf{con} \,;\mathsf{tot}^{-1}$. In the first case we, $\langle a, 1\rangle \sqsubseteq^{R} \langle b, 1\rangle$ and so, by the previous bullet point, $\Delta_Q \vdash \langle a, 1\rangle \Rightarrow \langle b, 1\rangle$. In the latter case, there must be some $c \in \mathbf{Fr}\langle \mathcal{V}ar_{-}\rangle$ such that $(a, c) \in \mathsf{con}$ and $(b, c) \in \mathsf{tot}$. Then, because $\mathfrak{con}(\langle a, c\rangle)$ and $\mathfrak{tot}(\langle b, c\rangle)$ are in Δ_Q, by the first rule of (d-Frm-3), $\Delta_Q \vdash \langle a, c\rangle \Rightarrow \langle b, c\rangle$. We also entail that $\langle a, c\rangle \Rightarrow \langle b, c\rangle \sqcup \mathit{ff}$ and then $\langle a, c\rangle \sqcup \mathit{ff} \Rightarrow \langle b, c\rangle \sqcup \mathit{ff}$ by (Frm-3). Finally, because $\langle a, c\rangle \sqcup \mathit{ff} = \langle a, 1\rangle$ and $\langle b, c\rangle \sqcup \mathit{ff} = \langle b, 1\rangle$ in $\mathcal{T}$ (resp. $\mathbf{dFr}\langle \mathcal{V}ar\rangle$), $\Delta_Q \vdash \langle a, 1\rangle \Rightarrow \langle b, 1\rangle$.

- $\alpha \in \mathfrak{D}(\mathsf{con})$: By definition, $\alpha = \bigsqcup S$ for some directed $S \subseteq \mathsf{con}$. We know that $\Delta_Q \vdash \mathfrak{con}(\gamma)$, for all $\gamma \in S$ and so, by the last rule of (d-Frm-1), $\Delta_Q \vdash \mathfrak{con}(\alpha)$.

- $\alpha \in {\downarrow^{R}} \mathfrak{D}(\mathsf{con})$: There is an $\alpha' \in \mathfrak{D}(\mathsf{con})$ such that $\alpha \sqsubseteq^{R} \alpha'$. Then, $\Delta_Q \vdash \alpha \Rightarrow \alpha'$ and also $\Delta_Q \vdash \mathfrak{con}(\alpha')$ by the previous case. Consequently, $\Delta_Q \vdash \mathfrak{con}(\alpha)$ by the penultimate rule of (d-Frm-1).

- $\alpha \in {\uparrow^{R}} \mathsf{tot}$: by an argument analogous to the previous case, $\Delta_Q \vdash \mathfrak{tot}(\alpha)$.

(2) Q is the first element of the growing sequence $\{\mathfrak{r}^{\delta}(Q) : \delta \in \mathsf{Ord}\}$ (defined as in Section 3.2.3) and $\mathfrak{r}^{\infty}(Q)$ is equal to some $\mathfrak{r}^{\delta}(Q)$ such that $\mathfrak{r}^{\delta}(Q) = \mathfrak{r}^{\star}(\mathfrak{r}^{\delta}(Q))$. The statement follows from a transfinite induction: In the previous we showed that $Q_{\mathfrak{r}^{\delta}(Q)} \vdash Q_{\mathfrak{r}^{\delta+1}(Q)}$, for every ordinal δ, and the limit steps follow from the fact that, for a limit ordinal λ, $\Delta_{\bigcup\{\mathfrak{r}^{\delta}(Q) \mid \delta < \lambda\}}$ is equal to $\bigcup\{\Delta_{\mathfrak{r}^{\delta}(Q)} \mid \delta < \lambda\}$. □

6.3.14 Theorem (Completeness)**.**
For a judgement φ and a set of judgements Γ, $\Gamma \vDash \varphi$ implies $\Gamma \vdash \varphi$.

Proof. Recall that $\mathcal{L}_{\Gamma} \stackrel{\text{def}}{\equiv} \mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma\rangle$ is computed as the quotient d-frame

$$(\mathbf{Fr}\langle \mathcal{V}ar_{+}\rangle, \mathbf{Fr}\langle \mathcal{V}ar_{-}\rangle, \mathsf{con}^{\infty}, \mathsf{tot}^{\infty}) / R^{\infty}$$

where $(\mathsf{con}^{\infty}, \mathsf{tot}^{\infty}, R^{\infty}_{+}, R^{\infty}_{-})$ is equal to the quotient structure $\mathfrak{r}^{\infty}(Q_{\sigma\Gamma})$. The construction of $\mathfrak{r}^{\infty}(Q_{\sigma\Gamma})$ has the property that, for the quotient map $\mu \colon \mathbf{dFr}\langle \mathcal{V}ar\rangle \twoheadrightarrow \mathcal{L}_{\Gamma}$, $\mu(\alpha) \sqsubseteq \mu(\beta)$ iff $\alpha \sqsubseteq^{R^{\infty}} \beta$, that $\mu(\alpha) \in \mathsf{con}$ iff $\alpha \in \mathsf{con}^{\infty}$ and that $\mu(\alpha) \in \mathsf{tot}$ iff $\alpha \in \mathsf{tot}^{\infty}$.

Now we show that, for any valuation $\mathfrak{v} \colon \mathcal{V}ar \to \mathcal{L}_{\Gamma}$, $\mathcal{L}_{\Gamma} \vDash_{\mathfrak{v}} \Gamma$. Let $\mathfrak{con}(\alpha)$ be a judgement from Γ. We have that $\mathfrak{con}(\sigma(\alpha))$ is in $\sigma\Gamma$, where the substitution σ is obtained as the composition of $\mathfrak{v}$ and the pairwise inclusion $\mu_\bullet\colon \mathcal{L}_\Gamma \hookrightarrow \mathbf{dFr}\langle \mathcal{V}ar\rangle$, and so $\sigma(\alpha) \in \mathrm{con}^\infty$. Hence, $\mathfrak{v}(\alpha) = \mu(\sigma(\alpha))$ is consistent in $\mathcal{L}_\Gamma$ because $\mu \cdot \mu_\bullet = \mathrm{id}$. Similarly, we have that whenever $\alpha \Rightarrow \beta$ (resp. $\mathtt{tot}(\alpha)$) is in Γ, then $\mathfrak{v}(\alpha) \sqsubseteq \mathfrak{v}(\beta)$ (resp. $\mathfrak{v}(\alpha) \in \mathrm{tot}$).

Next, because $\mathcal{L}_\Gamma \models \Gamma$, then $\mathcal{L}_\Gamma \models_{\mathfrak{v}} \varphi$ for all valuations $\mathfrak{v}$. Fix a valuation $\mathfrak{v}\colon \mathcal{V}ar \to \mathcal{L}_\Gamma$ defined as the composition of the inclusion of variables $\iota\colon \mathcal{V}ar \hookrightarrow \mathbf{dFr}\langle \mathcal{V}ar\rangle$ and the quotient μ (defined above). If $\mathcal{L}_\Gamma \models_{\mathfrak{v}} \mathtt{con}(\alpha)$, for some formula α, then it follows from the discussion in the first paragraph that $\iota(\alpha) \in \mathrm{con}^\infty$. Consequently, $\Gamma \vdash \mathtt{con}(\alpha)$ because $\Delta_{Q_{\sigma\Gamma}} \vdash \mathtt{con}(\alpha)$ (Proposition 6.3.13) and $\Gamma \vdash \Delta_{Q_\Gamma}$ (Lemma 6.3.12) and $\Delta_{Q_\Gamma} \vdash \Delta_{Q_{\sigma\Gamma}}$ (the latter judgements are only substitutions). The case for judgements of the form $\mathtt{tot}(\alpha)$ or $\alpha \Rightarrow \beta$ is similar. We have proved that $\mathcal{L}_\Gamma \models_{\mathfrak{v}} \varphi$ implies $\Gamma \vdash \varphi$. □

6.3.15 Example. Let Γ be a set of judgements of d-frame logic. Let $\mathcal{L}_\Gamma$ to be the d-frame representing the Lindenbaum-Tarski algebra for Γ, i.e. $\mathcal{L}_\Gamma = \mathbf{dFr}\langle \mathcal{V}ar \mid \sigma\Gamma\rangle$. Then, the points of the bispace $X_\Gamma \overset{\text{def}}{\equiv} \Sigma_d(\mathcal{L}_\Gamma)$ are the d-frame homomorphisms $p\colon \mathcal{L}_\Gamma \to \mathbf{2}{\times}\mathbf{2}$. Each such homomorphism p uniquely determines a pair of maximally consistent geometric theories (completely prime filters) $T_+ \subseteq \mathbf{Fr}\langle \mathcal{V}ar_+\rangle$ and $T_- \subseteq \mathbf{Fr}\langle \mathcal{V}ar_-\rangle$ such that

- if $\alpha_+ \in T_+$, $\alpha_- \in T_-$ and $\Gamma \vdash \alpha \Rightarrow \beta$, then $\beta_+ \in T_+$ and $\beta_+ \in T_-$;
- if $\Gamma \vdash \mathtt{con}(\alpha)$, then either $\alpha_+ \notin T_+$ or $\alpha_- \notin T_-$; and
- if $\Gamma \vdash \mathtt{tot}(\alpha)$, then either $\alpha_+ \in T_+$ or $\alpha_- \in T_-$.

This comes from the fact that we compute $T_\pm$ as the preimage of $1 \in \mathbf{2}$ by the map $p_\pm \cdot \mu_\pm$ where μ is the quotient $\mathbf{dFr}\langle \mathcal{V}ar\rangle \twoheadrightarrow \mathcal{L}_\Gamma$.

Consequently, every point $x \in X_\Gamma$ determines a *theory* T_+ and a *counter-theory* T_-[7] and we interpret the elements of T_+ as the properties which *observably hold* for x and the elements of T_- are the properties which *observably fail* for x. Therefore, we can think of points of x as *models* of Γ (the same way as in Table 6.1).

There is no guarantee that, for a set of judgements Γ, the space X_Γ has any points at all. Moreover, even if it has some points, it "might not have enough of them". This means that there might be α and β such that $\Gamma \nvdash \alpha = \beta$ but for all points $x = (T_+, T_-)$, $\alpha_\pm \in T_\pm$ iff $\beta_\pm \in T_\pm$. In other words, α and β are indistinguishable by any model $x \in X_\Gamma$. This happens whenever the d-frame $\mathcal{L}_\Gamma$ is not spatial.

6.3.16 Adaptation to d-KReg. In order for $\mathcal{L}_\Gamma$, from the example above, to be spatial, it is enough if it is d-compact and d-regular (Proposition 2.4.4). At the same time, every d-compact d-regular d-frame $\mathcal{L}$ is isomorphic to the d-frame presented

[7] We borrow the name "theory" and "counter-theory" from Dunn [Dun95].

as follows

$$\mathbf{dFr}\Big\langle \langle\alpha\rangle : \alpha \in \mathsf{con}_{\mathcal{L}} \;\Big|\; \langle\alpha\rangle \curlywedge \langle\beta\rangle = \langle\alpha \curlywedge \beta\rangle\,,\ \langle tt\rangle = tt,\ \langle ff\rangle = ff, \\ (\forall\alpha \prec \alpha', \beta \prec \beta')\ \ \langle\alpha \curlyvee \beta\rangle \lesseqgtr \langle\alpha'\rangle \curlyvee \langle\beta'\rangle\,, \\ \langle\alpha\rangle \sqsubseteq \bigsqcup\{\langle\gamma\rangle \sqcap \langle\delta\rangle \mid \gamma \prec \alpha,\ \alpha \prec \delta\}, \\ (\forall\alpha \in \mathsf{con}_{\mathcal{L}})\ \langle\alpha\rangle \in \mathsf{con},\quad (\forall\alpha \prec \beta)\ \ \langle\alpha\rangle \sqcup \langle\beta\rangle \in \mathsf{tot}\Big\rangle. \tag{6.3.2}$$

where the *strongly implies* relation $\alpha \prec \beta$ is defined as $(\beta_+, \alpha_-) \in \mathsf{tot}_{\mathcal{L}}$. This can be proved by an adaptation of a construction by Achim Jung and Drew Moshier [JM06, Proposition 8.3][8]. In fact, all that is needed from $\mathsf{con}_{\mathcal{L}}$ to show that the presentation in (6.3.2) yields a d-compact d-regular frame is that the structure $(\mathsf{con}_{\mathcal{L}}, \curlywedge, \curlyvee, tt, ff, \prec)$ is a *strong proximity lattice*. Those are distributive lattices $(X, \wedge, \vee, 1, 0)$ equipped with a relation $\prec \subseteq X \times X$ satisfying six simple axioms (see, for example, Definition 2.18 in [JM06]). Moreover, strong proximity lattices can be thought of as an algebraisation of *Multi Lingual Sequent Calculus* introduced in [JKM97] and [Keg95]. It should not be difficult to spell out the necessary conditions under which the d-frame $\mathcal{L}_\Gamma$ is d-compact and d-regular, for a set of judgements Γ.

For more information about (strong) proximity lattices refer to [Smy92b], [JS96] and [BH14].

[8] This is because the first three lines of (6.3.2) make sure that the frame components are the frames of *rounded ideals* and *rounded filters*. The proof of this is exactly as in [BM03]. Then, to check that also the consistency and totality relations agree, one has to adapt the proof of Proposition 3.5.3.

A

Appendix: Mathematical background

In this chapter we give an overview of basic terminology needed to understand this text. For further information about order theory see [DP02], for topology and frame theory see [PP12; PPT04; Joh82; Vic89] and for category theory see [Mac71; AHS90].

A.1 Set theory

For the purpose of this text, we do not rely on any particular foundations of mathematics. We only need to make a distinction between *classes* and *sets* as is common in set theory.

For a function $f\colon X \to Y$ between sets, we write $f[M]$ and $f^{-1}[N]$ for the sets $\{f(x) \mid x \in M\}$ and $\{x \mid f(x) \in N\}$, respectively. Denote the restriction of f to $M \subseteq X$ by $f{\restriction_M}\colon M \to Y$. Next, $A \subseteq_{\text{fin}} B$ denotes that A is a finite subset of B, $M \times N$ is the set $\{(x, y) \mid x \in M,\ y \in N\}$ and $\mathscr{P}(X) = \{A \mid A \subseteq X\}$ is the set of all subsets of X. The quotient of a set X by an equivalence relation $\sim$, denoted $X/{\sim}$, is defined as the set $\{[x]_\sim \mid x \in X\}$ where $[x]_\sim = \{y \in X \mid x \sim y\}$.

We also often need the class Ord of ordinal numbers to perform transfinite inductions. All that we require from our ordinals is that they are pre-ordered, that we can take a successor and a supremum of a set of ordinals. The exact definition of ordinals is not important for us. One can, for example, use von Neumann ordinals, which are the transitive well-ordered sets.

A.2 Order theory

Most of structures of our investigations are *pre-ordered sets*, that is, a set equipped with a binary relation $(Z, \leq)$ such that, for every $a, b, c \in Z$,

1. $a \leq a$ (reflexivity)

2. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

A pre-ordered set is a *partially ordered set*, or simply *poset*, if it further satisfies

3. $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry).

For a set $A \subseteq Z$, define the downwards closure of A, denoted $\downarrow A$, and the upwards closure of A, denoted $\uparrow A$, to be the sets $\{b \mid b \leq a \text{ for some } a \in A\}$ and $\{b \mid a \leq b \text{ for some } a \in A\}$, respectively. Also, $\downarrow a$ and $\uparrow a$ are shorthands for $\downarrow\{a\}$ and $\uparrow\{a\}$, respectively. Then, a set A is a *downset*, *upset* or *convex set* if $A = \downarrow A$, $A = \uparrow A$ and $A = \downarrow A \cap \uparrow A$, respectively.

Throughout the text we come across many different posets and so the symbol we use for the order relation depends on the context. Apart from $\leq$ and $\preceq$ we also use, for example, $\sqsubseteq$ and $\leqslant$.

A.2.1 Finitary lattices. A poset equipped with a binary operation called *infimum* (or *meet*) and a constant $(S, \wedge, 1)$ is a *meet semilattice* if $\wedge$ is an associative, commutative and idempotent binary operation such that

1. $a \wedge 1 = a$

2. $a \wedge b \leq a$

3. $a \leq b, a \leq c$ implies $a \leq b \wedge c$.

Dually, $(S, \vee, 0)$ is a *join semilattice* if the dual poset $(S, \leq^{\text{op}})$ together with $\vee$ and 0 is a meet semilattice, where $a \leq^{\text{op}} b$ iff $b \leq a$. We call the operation $\vee$ *supremum* or *join*.

Next, we say that $(D, \wedge, \vee, 0, 1)$ is a *(distributive) lattice* if $(D, \wedge, 1)$ and $(D, \vee, 0)$ are semilattices such that the following equality holds

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

An important consequence of the lattice laws is that, for a fixed a, b, c, any pair of equations of the form

$$a \wedge x = b \quad \text{and} \quad a \vee x = c$$

has at most one solution x. Finally, we say that $(B, \wedge, \vee, 0, 1, \sim)$ is a *Boolean algebra* if $(B, \wedge, \vee, 0, 1)$ is a distributive lattice such that

$$x \wedge {\sim}x = 0 \quad \text{and} \quad x \vee {\sim}x = 1, \quad \text{for all } x \in B.$$

A.2.2 Infinitary lattices and dcpos. A non-empty subset A of a poset $(P, \leq)$ is *directed*, $A \subseteq^{\uparrow} P$, if for every $a, b \in A$ there exists a $c \in A$ such that $a \leq c$ and $b \leq c$. Then, a structure $(P, \bigvee^{\uparrow})$ is a *directed-complete partial order*, or *dcpo* for short, if P is a poset and every directed subset A of P has a supremum (or join) $\bigvee^{\uparrow} A$, i.e.

1. $\forall a \in A, a \leq \bigvee^{\uparrow} A$, and
2. $\forall a \in A, a \leq b$ implies $\bigvee^{\uparrow} A \leq b$.

If a poset is a join semilattice and a dcpo at the same time, we say that it is a *complete lattice*. Complete lattices enjoy the following important properties:

- *Every subset has a supremum.* A supremum $\bigvee A$ of an arbitrary set A is computed as the directed join of the set $\{a_1 \vee a_2 \vee \cdots \vee a_n \mid a_1, a_2, \ldots, a_n \in A\}$.
- Dual poset is also a complete lattice, i.e. *every subset of a complete lattice has an infimum (or meet)*. An infimum $\bigwedge A$ of an arbitrary set A is computed as the supremum of the set $\{x \mid x \leq a \text{ for every } a \in A\}$.

An example of a poset which is a complete lattice is $(\mathscr{P}(X), \subseteq)$. The joins and meets are exactly the operations of taking unions and intersections.

A.2.3 Structure preserving maps. A *monotone map* $f \colon X \to Y$ between two preorders X and Y is a map such that

$$a \leq b \text{ in } X \implies f(a) \leq f(b) \text{ in } Y.$$

Similarly, a map $h \colon S \to T$ between two meet semilattices is a *(meet) semilattice homomorphism* if $h(a \wedge b) = h(a) \wedge h(b)$, for every $a, b \in S$. Define join semilattice homomorphisms dually. Next, a map $h \colon D \to E$ between two distributive lattices is a *lattice homomorphism* if, for every $a, b \in D$,

$$h(a \wedge b) = h(a) \wedge h(b), \quad h(a \vee b) = h(a) \vee h(b), \quad h(0) = 0 \quad \text{and} \quad h(1) = 1.$$

Similarly, a map $h \colon B \to C$ between two Boolean algebras is a *Boolean homomorphism* if it is a lattice homomorphism such that ${\sim}h(x) = h({\sim}x)$ for every $x \in B$.

These definitions generalise to the infinitary cases as well. We say that a map between posets $f \colon X \to Y$ preserves the supremum of $A \subseteq X$ (provided that it exists) if $f(\bigvee A) = \bigvee f[A]$. Define preservation of infima dually. Then, for example, a map between two dcpo is *Scott-continuous* if it preserves suprema of all directed sets.

A.2.4 Galois adjunction. Let $f \colon X \to Y$ and $g \colon Y \to X$ be maps between two posets. We say that f is the *left Galois adjoint* of g and that g is the *right Galois adjoint* of f if, for every $x \in X$ and $y \in Y$,

$$f(x) \leq y \quad \text{if and only if} \quad x \leq g(y)$$

or equivalently

$$f(g(y)) \leq y \quad \text{and} \quad x \leq g(f(x)).$$

Furthermore, in special cases, Galois adjoints have another characterisation. For a map $f\colon L \to M$ between two complete lattices, the following two are equivalent:

1. f preserves all suprema.
2. f has a right adjoint $f_\bullet$ (computed as $f_\bullet(x) = \bigvee\{y \in L \mid f(y) \leq x\}$).

A.3 Category theory

A *category* $\mathcal{C}$ consists of a class of objects $\mathsf{obj}(\mathcal{C})$, a set of morphisms $\mathsf{Hom}(A, B)$, for every pair of objects $A, B \in \mathsf{obj}(\mathcal{C})$, and a function for composition of morphisms

$$(\cdot)\colon \mathsf{Hom}(B, C) \times \mathsf{Hom}(A, B) \to \mathsf{Hom}(A, C)$$

for every $A, B, C \in \mathsf{obj}(\mathcal{C})$ such that

1. For every object $A \in \mathsf{obj}(\mathcal{C})$, there is a morphism $\mathrm{id}_A \in \mathsf{Hom}(A, A)$ such that $\mathrm{id}_A \cdot f = f$ and $g \cdot \mathrm{id}_A = g$ for every $f \in \mathsf{Hom}(B, A)$ and $g \in \mathsf{Hom}(A, B)$.
2. $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for every $h \in \mathsf{Hom}(A, B)$, $g \in \mathsf{Hom}(B, C)$ and $f \in \mathsf{Hom}(C, D)$.

Morphisms $f \in \mathsf{Hom}(A, B)$ are often denoted $f\colon A \to B$ or $A \xrightarrow{f} B$, and Hom-sets are alternatively denoted $\mathsf{Hom}_{\mathcal{C}}(A, B)$ or $\mathcal{C}(A, B)$.

A.3.1 Example. In the previous sections we came across the category **Set** of sets and functions, the category **PreOrd** of pre-ordered sets and monotone functions, the category **Pos** of partially ordered sets and monotone functions, the category **DLat** of distributive lattice and lattice homomorphisms, and the category **Bool** of Boolean algebras and Boolean homomorphisms.

Moreover, every poset P can be seen as category with the object of the category being the elements of P and $\mathsf{Hom}_P(a, b)$ is a singleton if $a \leq b$ in P, otherwise it is empty.

Every category $\mathcal{C}$ has an *opposite* category $\mathcal{C}^{\mathrm{op}}$ associated to it. It is defined as the category with morphisms going in the opposite direction; i.e. $\mathsf{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathsf{Hom}_{\mathcal{C}}(B, A)$ and $f \cdot^{\mathrm{op}} g = g \cdot f$.

A.3.2 Special morphisms. A morphism $f\colon A \to B$ is a *monomorphism* (or just *mono*) if $f \cdot g = f \cdot h$ implies $g = h$. Similarly, f is an *epimorphism* (or just *epi*) if f is a monomorphism in the opposite category, i.e. $g \cdot f = h \cdot f$ implies $g = h$. Further, f is an *isomorphism* (or just *iso*) if there exists and $f^{-1}\colon B \to A$ such that $f \cdot f^{-1} = \mathrm{id}_B$ and $f^{-1} \cdot f = \mathrm{id}_A$.

A.3.3 Remark. In the categories mentioned in Example A.3.1, monomorphisms, epimorphisms and isomorphisms correspond precisely to injective, surjective and bijective maps, respectively, but this is not the case for a general category. The category of frames (which we define below) is an example of a category where epimorphisms are not always surjective.

To capture surjectivity, a stronger notion has to be often assumed. One example is the following. An *extremal epimorphism* is a morphism e such that, whenever $e = m \cdot f$ for some monomorphisms m, then m is an isomorphisms.[1]

Next, anytime a pair of morphisms $A \xrightarrow{s} B \xrightarrow{r} A$ composes to the identity id_A we say that s is a *section* of r and that r is a *retraction* of s. Because r is always an epi and s is always a mono, we also say that r is a *split epimorphism* and s is a *split monomorphisms* (and that the composition $r \cdot s$ is called *splitting idempotent*). In fact, r is always an extremal epimorphism.

A.3.4 Functors and natural transformations. A *(covariant) functor* $F\colon \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping on objects $F\colon \mathsf{obj}(\mathcal{C}) \to \mathsf{obj}(\mathcal{D})$ together with mappings between sets of morphisms $F\colon \mathsf{Hom}(A,B) \to \mathsf{Hom}(F(A),F(B))$, for every $A, B \in \mathsf{obj}(\mathcal{C})$, which preserve the identity morphisms and composition, i.e.

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \quad \text{and} \quad F(f) \cdot F(g) = F(f \cdot g).$$

Alternatively, we define a *contravariant functor* similarly, except that the mappings between Hom-sets is contravariant, i.e. it is of the type $\mathsf{Hom}(A,B) \to \mathsf{Hom}(F(B),F(A))$.

We say that F is an *embedding* if it is injective on morphisms, *faithful* if all the mappings $F\colon \mathsf{Hom}(A,B) \to \mathsf{Hom}(F(A),F(B))$ are injective and *full* if those mappings are onto.

A.3.5 Example. (1) For every category $\mathcal{C}$, there is the *identity functor* $\mathrm{Id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is defined as the identity mapping on both objects and morphisms. Also, for any other category $\mathcal{D}$ and an object $C \in \mathsf{obj}(\mathcal{C})$, there is a *constant functor* $\Delta_C\colon \mathcal{D} \to \mathcal{C}$ defined as $\Delta_C(D) = C$ and $\Delta_C(f) = \mathrm{id}_C$ for every $D \in \mathsf{obj}(\mathcal{D})$ and any morphism f in $\mathcal{D}$.

(2) **Pos** being a subcategory of **PreOrd** induces a functor $I\colon \mathbf{Pos} \to \mathbf{PreOrd}$ which is a full embedding.

(3) Consider a *forgetful functor* $U_1\colon \mathbf{Pos} \to \mathbf{Set}$ which sends a poset $(Z, \leq)$ to Z and leaves the morphisms unchanged. Similarly define the following two forgetful

[1] Even though e is not required to be an epimorphism it often follows from mild categorical assumptions such as existence of equalisers.

functors $U_2\colon \mathbf{DLat} \to \mathbf{Pos}$, and $U_3\colon \mathbf{DLat} \to \mathbf{Set}$. We see that U_3 is, in fact, the composition of the two previous functors, i.e. $U_3 = U_1 \circ U_2$ (defined as a composition of mappings on objects and morphisms separately). Observe that U_1 and U_3 are full and faithful but not embeddings.

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\varepsilon\colon F \Longrightarrow G$ is defined as a collection of morphisms $\{\varepsilon_C : F(C) \to G(C)\}_{C \in \mathsf{obj}(\mathcal{C})}$ such that, for every $f\colon A \to B$ in $\mathcal{C}$, $G(f) \cdot \varepsilon_A = \varepsilon_B \cdot F(f)$; expressed as a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\varepsilon_A} & G(A) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\varepsilon_B} & G(B) \end{array}$$

(Natural transformations for contravariant functors are defined accordingly.)

Then, two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there is a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ where $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$ denotes that there is a natural isomorphism $\eta\colon F \circ G \Longrightarrow \mathrm{Id}_{\mathcal{D}}$, i.e. η is a natural transformation which consists of isomorphisms.

A.3.6 Limits and colimits. A *diagram* $\mathbb{D}$ in $\mathcal{C}$ is any functor $\mathbb{D}\colon I \to \mathcal{C}$ such that the category I has only set many objects. A *limit* of a digram $\mathbb{D}$ in $\mathcal{C}$ is a constant functor $\lim(\mathbb{D})\colon I \to \mathcal{C}$ and a natural transformation $\eta\colon \lim(\mathbb{D}) \Longrightarrow \mathbb{D}$ such that for any other constant functor $\Delta_C\colon I \to \mathcal{C}$ and a natural transformation $\delta\colon \Delta_C \Longrightarrow \mathbb{D}$ there is a *unique* natural transformation $\overline{\delta}\colon \Delta_C \Longrightarrow \lim(\mathbb{D})$ such that $\delta = \eta \cdot \overline{\delta}$ (composed componentwise). Correspondingly, a *colimit* of a diagram $\mathbb{D}$ in $\mathcal{C}$ is a limit in the opposite category, $\mathcal{C}^{\mathrm{op}}$.

A.3.7 Example. (1) An example of a limit is the *product* $\prod_i A_i$ of a collection of objects $\{A_i\}_{i \in I}$ in $\mathcal{C}$ (provided that it exists), i.e. $\mathbb{D}(i) = A_i$ and the category I has no morphisms except for the identities. The natural transformation $\lim(\mathbb{D}) \Longrightarrow \mathbb{D}$ consists exactly of the projection maps $\pi_i\colon \prod_i A_i \to A_i$.

(2) Another type of limit is an *equaliser*. For a pair of morphisms $f, g\colon A \to B$, an equaliser is a morphism $e\colon C \to A$ such that $f \cdot e = g \cdot e$ and, whenever there is an h such that $f \cdot h = g \cdot h$, then $h = e \cdot u$ for a unique u. In the category of sets, **Set**, is an equaliser of a pair of maps $f, g\colon A \to B$ the set $\{a \in A \mid f(a) = g(a)\}$ together with the inclusion map to A.

(3) *Coproducts* are defined as products but in the opposite category. They are usually denoted as $\coprod_i A_i$ or $\bigoplus_i A_i$ and the natural transformation $\mathbb{D} \Longrightarrow \mathsf{colim}(\mathbb{D})$ consists of the inclusion morphisms/maps $\iota_i\colon A_i \to \coprod_i A_i$.

(4) Finally, *coequaliser* is a colimit of a diagram $A \underset{g}{\overset{f}{\rightrightarrows}} B$. It consists of an object C and a morphism $c\colon B \to C$ such that $c \cdot f = c \cdot g$ with the expected universal

property. In **Set**, for example, an equaliser of f and g is the quotient $q\colon B \to B/{\sim}$ where $\sim$ is the smallest equivalence relation such that $f(a) \sim g(a)$, for all $a \in A$.

A.3.8 Fact. *If a category has (co)products and (co)equalisers then it has (co)limits of all diagrams.*

A.3.9 Adjoint functors. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be two functors. A functor F is a *left adjoint*[adjoint] of G and G is a *right adjoint* of F, written as $F \dashv G$, if there are two natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \Longrightarrow G \circ F$ and $\varepsilon\colon F \circ G \Longrightarrow \mathrm{Id}_{\mathcal{D}}$ such that the compositions

$$F \xRightarrow{F\eta} F \circ G \circ F \xRightarrow{\varepsilon_F} F \quad \text{and} \quad G \xRightarrow{\eta_G} G \circ F \circ G \xRightarrow{G\varepsilon} G$$

are equal to the identity natural transformations, i.e. those which consist of the identity morphisms $\mathrm{id}_{F(A)}\colon F(A) \to F(A)$ and $\mathrm{id}_{G(B)}\colon G(B) \to G(B)$, respectively.

Dual adjunction is defined as a pair of *contravariant* functors F and G and two natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \Longrightarrow G \circ F$ and $\varepsilon\colon \mathrm{Id}_{\mathcal{D}} \Longrightarrow F \circ G$ satisfying the corresponding pair of equalities.

A.3.10 Example. (1) Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be such that $F \circ G = \mathrm{Id}_{\mathcal{C}}$ and $G \circ F = \mathrm{Id}_{\mathcal{D}}$ (in other words the categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*) then the functors F and G are both left and right adjoints to each other.

(2) Let $U\colon \mathbf{DLat} \to \mathbf{Set}$ be the forgetful functor and let $F\colon \mathbf{Set} \to \mathbf{DLat}$ be defined on objects as the mapping $X \mapsto \mathbf{DL}\langle X\rangle$ where $\mathbf{DL}\langle X\rangle$ is the freely generated distributive lattice over X. Then, F is the left adjoint of U.

(3) Let $f\colon X \to Y$ be a monotone map between two posets and let $g\colon Y \to X$ be its right Galois adjoint. Then, for X and Y seen as categories, f and g are two functors which are adjoint to each other, i.e. $f \dashv g$.

(4) Let $\mathcal{D}$ be a *full subcategory* of $\mathcal{C}$, that is, the embedding functor $J\colon \mathcal{D} \hookrightarrow \mathcal{C}$ is full. $\mathcal{D}$ is said to be *reflective* (resp. *coreflective*) if J has a left (resp. right) adjoint. Reflectivity is equivalent to postulating that, for every object C in $\mathcal{C}$, there is an object $F(C)$ in $\mathcal{D}$ and a morphism $\rho_C\colon C \to J(F(C))$ in $\mathcal{C}$ such that, for any other morphism $f\colon A \to J(D)$, $f = J(\overline{f}) \cdot \rho_C$ for a *unique* $\overline{f}\colon F(C) \to D$. Coreflection can be equivalently restated as a mapping on objects $G\colon \mathsf{obj}(\mathcal{C}) \to \mathsf{obj}(\mathcal{D})$ together with a collection of maps $\lambda_C\colon J(G(C)) \to C$, for every $C \in \mathsf{obj}(\mathcal{C})$, satisfying the expected universal property.

A concrete example of a reflection is the reflection of **PreOrd** onto its full subcategory **Pos**. $F(Z, \leq)$ is defined as the poset $Z/{\sim}$ where $x \sim y$ if $x \leq y$ and $x \geq y$ and $\rho_Z\colon Z \to Z/{\sim}$ is the quotient map.

A.3.11 Fact. *Let $F \dashv G \colon \mathcal{C} \rightleftarrows \mathcal{D}$ be a pair of adjoint functors. Then F preserves all existing colimits and G preserves all existing limits, i.e. $F(\mathsf{colim}\mathbb{D}) \cong \mathsf{colim}(F \circ \mathbb{D})$ for any diagram $\mathbb{D}$ which has a colimit in $\mathcal{C}$ and, dually, $G(\lim \mathbb{D}) \cong \mathsf{colim}(G \circ \mathbb{D})$ for any diagram $\mathbb{D}$ which has a limit in $\mathcal{D}$.*

A.3.12 Monads and comonads. A *monad* on a category $\mathcal{C}$ is a structure (T, η, μ) where $T \colon \mathcal{C} \to \mathcal{C}$ is an (endo)functor and the rest are natural transformations of the type $\eta \colon \mathrm{Id}_{\mathcal{C}} \Longrightarrow T$ and $\mu \colon T \circ T \Longrightarrow T$, such that the following diagrams commute

$$\begin{array}{ccc} T^3 & \overset{T\mu}{\Longrightarrow} & T^2 \\ {\scriptstyle \mu_T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T^2 & \overset{\mu}{\Longrightarrow} & T \end{array} \qquad \begin{array}{ccc} T & \overset{\eta_T}{\Longrightarrow} & T^2 \\ & {\scriptstyle \mathrm{id}_T}\searrow & \Downarrow{\scriptstyle \mu} \\ & & T \end{array} \qquad \begin{array}{ccc} T^2 & \overset{T\eta}{\Longrightarrow} & T \\ {\scriptstyle \mu}\Downarrow & \nearrow{\scriptstyle \mathrm{id}_T} & \\ T & & \end{array}$$

Instead of providing the full structure of the monad and proving commutativity of those three diagrams, it is often easier to find an associated Kleisli triple for it. A huge advantage of this representation is that it is enough to define the action of T on objects and the action on morphisms is derived from it. A *Kleisli triple* over a category $\mathcal{C}$ is a structure $(T, \eta, \overline{(-)})$ where

- T is a mapping on objects $T \colon \mathsf{obj}(\mathcal{C}) \to \mathsf{obj}(\mathcal{C})$,
- $\eta_A \colon A \to T(A)$ is a morphism in $\mathcal{C}$, for every $A \in \mathsf{obj}(\mathcal{C})$, and
- $\overline{(-)}$ *"lifts"* every morphism $f \colon A \to T(B)$ to $\overline{f} \colon T(A) \to T(B)$.

Moreover, the following equations hold

(M1) $\overline{\eta_A} = \mathrm{id}_{T(A)}$,

(M2) $\overline{f} \cdot \eta_A = f$ for all $f \colon A \to T(B)$,

(M3) $\overline{\overline{g} \cdot f} = \overline{g} \cdot \overline{f}$ for all $f \colon A \to T(B)$ and $g \colon B \to T(C)$.

Every Kleisli triple $(T, \eta, \overline{(-)})$ defines a monad (T, η, μ) where the extension of T to morphisms is computed as $T(h \colon A \to B) = \overline{\eta_b \cdot f}$ and μ_A is defined as $\overline{\mathrm{id}_{T(A)}}$. Conversely, every monad (T, η, μ) gives a Kleisli triple the following way: T and η stay the same and $\overline{f}$, for an $f \colon A \to T(B)$, is the morphism $\mu_A \cdot T(f)$.

A.3.13 Fact (Proposition 1.6 in [Mog91] or Theorem 3.16 in [Man76])**.** *This correspondence between Kleisli triples and monads is a bijection.*

Comonads and *co-Kleisli triples* are defined dually.

A.3.14 Example. The powerset monad $(\mathcal{P}, \eta \colon x \mapsto \{x\}, \mu \colon m \mapsto \bigcup m)$ on **Set** represented as Kleisli triple $(\mathcal{P}, \eta, \overline{(-)})$ where the lift operation maps $f \colon X \to \mathcal{P}(Y)$ to $\overline{f} \colon (X) \to \mathcal{P}(Y), M \mapsto \bigcup f[M]$.

A.3.15 Algebras and coalgebras. Any endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ introduces categories of T-algebras and T-coalgebras and their homomorphisms, denoted as $\mathsf{Alg}(T)$ and $\mathsf{Coalg}(T)$, where

- a *T-algebra* (A, α) is any morphism $\alpha\colon T(A) \to A$ in $\mathcal{C}$ and a homomorphism of T-algebras $h\colon (A, \alpha) \to (B, \beta)$ is any morphisms $h\colon A \to B$ in $\mathcal{C}$ such that $h \cdot \alpha = \beta \cdot T(h)$; and dually

- a *T-coalgebra* (X, ξ) is any morphism $\xi\colon X \to T(X)$ and a homomorphism of T-coalgebras $f\colon (X, \xi) \to (Y, \rho)$ is a morphism $f\colon X \to Y$ such that $\rho \cdot f = T(f) \cdot \xi$.

We say that a T-algebra (I, ι) is an *initial algebra* if for every T-algebra (A, α) there is a *unique* homomorphism of T-algebras $(I, \iota) \to (A, \alpha)$. *Final coalgebras are defined dually.*

Finally, for a comonad $(T, \eta\colon T \Longrightarrow \mathrm{Id}_{\mathcal{C}}, \mu\colon T \Longrightarrow T^2)$ and a coalgebra (X, ξ), we say that (X, ξ) is a *comonad T-coalgebra* if $T(\xi) \cdot \xi = \mu_X \cdot \xi$ and $\eta_X \cdot \xi = \mathrm{id}_X$.

A.3.16 Proposition. *Let $Q\colon \mathcal{X} \rightleftarrows \mathcal{A} : S$ be adjoint functors, $Q \dashv S$, establishing a dual equivalence of categories. If $T\colon \mathcal{X} \to \mathcal{X}$ and $L\colon \mathcal{A} \to \mathcal{A}$ are two endofunctors such that $LQ \cong QT$, then* $\mathsf{Coalg}(T)$ *is dually equivalent to* $\mathsf{Alg}(L)$.

Proof. By $QS \cong \mathrm{Id}_{\mathcal{A}}$, the morphisms $L(A) \to A$ are in a (natural) bijection with morphisms $LQS(A) \to A$ which are, by $LQ \cong QT$, in bijection with morphisms $QTS(A) \to A$. From the dual adjunction $Q \dashv S$, those correspond to morphisms $S(A) \to TS(A)$ and therefore to T-coalgebras, because every X is isomorphic to $SQ(X)$. □

A.4 Point-free topology

A *topological space* is a structures structure (X, τ) where $\tau \subseteq \mathcal{P}(X)$ is a collection of *open sets* such that

1. $\emptyset, X \in \tau$,

2. $U, V \in \tau$ implies $U \cap V \in \tau$, and

3. $\mathcal{U} \subseteq \tau$ implies $\bigcup \mathcal{U} \in \tau$.

Complements of open sets are called *closed sets*[closed set] and the sets which are both open and closed are called *clopen*. For a subset $M \subseteq X$ we define its *interior* M° as the largest open set contained in M, i.e. $M^\circ = \bigcup\{U \in \tau \mid U \subseteq M\}$. Similarly, the *closure* $\overline{M}$ of M is the smallest closed set containing M, that is, $\overline{M} = \bigcap\{\text{closed } C \mid M \subseteq C\}$.

Morphisms in the category of topological spaces **Top** are *continuous maps*, that is, the maps $f\colon (X, \tau^X) \to (Y, \tau^Y)$ which satisfy

$$f^{-1}[U] \in \tau^X \quad \text{for every} \quad U \in \tau^Y.$$

A.4.1 Example. The classical example of a space is the unit interval of real numbers [0,1] with the smallest topology which contains all the intervals of opens (a, b) where $0 \leq a < b \leq 1$.

A non-Hausdorff example of a space is the space of all *Scott-open sets* on a dcpo $(D, \bigsqcup^\uparrow)$ where $U \subseteq D$ is Scott-open provided that

$$\bigsqcup\nolimits^\uparrow A \in U \quad \text{implies} \quad a \in U \text{ for some } a \in A.$$

A.4.2 Frames and frame homomorphisms. The basic objects of study in the point-free topology are *frames* (or their covariant variants, *locales*). When compared to topological spaces, we take (abstract) open sets as the basic notion and then the points of frames are only derived from the relationships between opens.

The objects of the category of frames **Frm** are complete lattices $L = (L, \bigvee, \wedge, 0, 1)$ which satisfy

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\} \tag{A.4.1}$$

for all $A \subseteq L$ and $b \in L$. Morphisms of **Frm** are *frame homomorphisms*, that is, maps $h\colon L \to M$ which preserve all joins and all finite meets. This means that they also preserve the empty join, $h(0) = 0$, and the empty meet, $h(1) = 1$.

A.4.3 Example. (1) A topological space (X, τ) gives rise to a frame $\Omega(X)$, defined as $(\tau, \bigcup, \cap, \emptyset, X)$, and a continuous map $f\colon X \to Y$ gives rise to a frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$, $U \mapsto f^{-1}[U]$. In other words, we have a *contravariant* functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$.

(2) $\mathsf{Down}\colon \mathbf{Pos} \to \mathbf{Frm}$ is a functor which assigns to a poset $(Z, \leq)$ the frame of downsets $\mathsf{Down}(Z, \leq)$ ordered by set inclusion. Joins and meets in $\mathsf{Down}(Z, \leq)$ are computed as unions and intersections. On morphisms, it is defined as $\mathsf{Down}(f)\colon U \mapsto {\downarrow} f[U]$ for every monotone $f\colon Z \to Y$.

(3) $\mathsf{Idl}\colon \mathbf{DLat} \to \mathbf{Frm}$ is a functor which assigns to a distributive lattice D its frame of ideals $\mathsf{Idl}(D)$ ordered by set inclusion, where an *ideal* is a subset $I \subseteq D$ such that

(I1) $0 \in I$ (I2) $a, b \in I$ implies $a \vee b \in I$ (I3) $a \leq b$ and $b \in I$ implies $a \in I$.

Similarly, define a functor $\mathsf{Filt}\colon \mathbf{DLat} \to \mathbf{Frm}$ assigning to D the frame of *filters* $\mathsf{Filt}(D)$ ordered by set inclusion, where a filter is an ideal in D^{op}.

The actions on morphisms are defined as $\mathsf{Idl}(h)\colon I \mapsto {\downarrow} h[I]$ and $\mathsf{Filt}(h)\colon F \mapsto {\uparrow} h[F]$, respectively. Note that directed joins and finite meets of ideals/filters are defined as the union and finite intersection, and binary joins are defined as follows

$$I_1 \vee I_2 = \{x \vee y \mid x \in I_1,\ y \in I_2\} \quad \text{and} \quad F_1 \vee F_2 = \{x \wedge y \mid x \in F_1,\ y \in F_2\}.$$

A.4.4 The adjunction $\Omega \dashv \Sigma$. Every frame L gives rise to a topological space $\Sigma(L)$ whose points are frame homomorphism $p\colon L \to \mathbf{2}$, where $\mathbf{2} = \{0 < 1\}$, and the topology of $\Sigma(L)$ consists of the open sets $\Sigma(a) = \{p\colon L \to \mathbf{2} \mid p(a) = 1\}$, for every $a \in L$.

This definition extends to morphisms. For a frame homomorphism $h\colon L \to M$ define $\Sigma(h)\colon \Sigma(M) \to \Sigma(L)$ as $p\colon M \to \mathbf{2} \quad\longmapsto\quad p \cdot h\colon L \to \mathbf{2}$. The contravariant functors Ω and Σ constitute a dual adjunction $\Omega \dashv \Sigma$ with the unit and counit morphisms defined as

$$\begin{aligned} \eta_X\colon X &\longrightarrow \Sigma(\Omega(X)) \\ x &\longmapsto \{U \in \tau \mid x \in U\} \end{aligned} \qquad\qquad \begin{aligned} \varepsilon_L\colon L &\longrightarrow \Omega(\Sigma(L)) \\ a &\longmapsto \{\Sigma(a) \mid a \in L\} \end{aligned}$$

A space X is called *sober* if $X \cong \Sigma(L)$ for some frame L and, similarly, a frame L is called *spatial* if $L \cong \Omega(X)$ for some space X.

Note that frame homomorphisms $p\colon L \to \mathbf{2}$ are in a bijective correspondence with *completely prime filters*, which are the filters $P \subseteq L$ satisfying

$$\bigvee A \in P \quad \text{implies} \quad a \in P \text{ for some } a \in A.$$

A.4.5 Heyting implication and pseudocomplements. Because of the distributivity law (A.4.1), for a fixed $a \in L$, the mapping $x \mapsto x \wedge a$ preserves all joins. Therefore, $(-) \wedge a$ has a right (Galois) adjoint $a \to (-)$ such that

$$x \wedge a \leq y \quad \text{iff} \quad x \leq a \to y \quad (\text{for every } x, y \in L).$$

In particular, for every $a \in L$, the *pseudocomplement* $a \to 0$ is the largest x such that $a \wedge x = 0$, i.e. $a \to 0 = \bigvee\{x \mid x \wedge a = 0\}$. It is more common to write a^* instead of $a \to 0$ but in the context of frames we solely use $a \to 0$ as we reserve a^* for d-frame pseudocomplements.

A.4.6 Example. For a topological space X and an open set $U \subseteq X$, $U \to \emptyset$ is precisely the interior of the complement of U, i.e. the open set $(X \setminus U)^\circ$.

A.4.7 Topological properties. For $a, b \in L$, define the *well-inside* relation $a \lhd b$ as $(a \to 0) \vee b = 1$. For topological space X, $U \lhd V$ in $\Omega(X)$ is equivalent to $\overline{U} \subseteq V$.

We can now rephrase basic topological properties in the language of frames:

- L is regular if $a = \bigvee\{x \mid x \lhd a\}$ for every $a \in L$,
- L is zero-dimensional if $a = \bigvee\{x \mid x \lhd x \leq a\}$ for every $a \in L$
- L is normal if, whenever $a \vee b = 1$, then there is some $u, v \in L$, such that $u \wedge v = 0$, $a \vee v = 1$ and $u \vee b = 1$, and
- L is compact if $\bigvee A = 1$ implies that $\bigvee F = 1$ for some finite $F \subseteq_{\text{fin}} A$.

Compactness can be also equivalently rephrased as: whenever $\bigvee^{\uparrow} A = 1$ then $a = 1$ for some $a \in A$.

A.4.8 Fact. *A topological space X is regular, zero-dimensional, normal or compact iff $\Omega(X)$ is as a frame.*

A.5 Quotients of frames

In the following we introduce a basic theory of quotients of frames. Our construction is heavily inspired by Theorem III.11.3.1 in [PP12]. The main difference is that our quotient relation does not represent a set of equations to quotient by but rather a set of *inequalities*. This turned out to be a great advantage because, for example, computing frame quotients can be expressed as an iterative procedure (Proposition A.5.3).

Let R be a binary relation on frame L. An element x is said to be *R-saturated* if

$$\forall a, b, c. \quad (a, b) \in R \quad \text{and} \quad b \wedge c \leq x \implies a \wedge c \leq x$$

By L/R denote the set $\{x \in L \mid x \text{ is } R\text{-saturated}\}$ with the order induced by L.

A.5.1 Lemma. *L/R is a frame.*

Proof. By Proposition III.2.2 in [PP12], it is enough to show that L/R is so called "sublocale" of L; that is, it is closed under arbitrary meets and also for any $x \in L$ and $s \in L/R$, $x \to s \in L/R$. The first part is immediate and for the latter, we show that for x and s selected as above, $x \to s$ is R-saturated. For $(a, b) \in R$,

$$b \wedge c \leq x \to s \quad \text{iff} \quad b \wedge c \wedge x \leq s \quad \text{implies} \quad a \wedge c \wedge x \leq s \quad \text{iff} \quad a \wedge c \leq x \to s. \qquad \square$$

Next, define the *quotient map* as follows

$$\mu^R \colon L \to L/R, \quad x \mapsto \bigwedge\{s \mid x \leq s \text{ and } s \text{ is } R\text{-saturated}\}.$$

A.5.2 Lemma. *For a frame homomorphisms $h \colon L \to M$ such that*

$$\forall (a, b) \in R \implies h(a) \leq h(b) \tag{A.5.1}$$

the restriction $h{\restriction}_{L/R} \colon L/R \to M$ is the unique *frame homomorphism such that*

$$h = h{\restriction}_{L/R} \cdot \mu^R.$$

Proof. First, we prove that μ^R is well defined, i.e. that it is a frame homomorphism. Let j be the embedding $L/R \to L$. Then, $\mu^R(a) \leq s$ iff $a \leq j(s) = s$ and so μ^R is the left adjoint of j. Therefore, μ^R preserves all joins. Because 1_L is always R-saturated, also $\mu^R(1_L) = 1_{L/R}$. Lastly, $\mu^R(a) \wedge \mu^R(b) = \mu^R(a \wedge b)$ by definition.

Next, we show that μ^R has the required universal property. Let $h\colon L \to M$ be a frame homomorphism which preserves R as in (A.5.1). To show $h = h{\restriction}_{L/R} \cdot \mu^R$ we observe that $h_\bullet(x)$ is R-saturated for every $x \in M$ (where $h_\bullet$ is the right adjoint of h). For $(a, b) \in R$,

$$b \wedge c \leq h_\bullet(x) \text{ iff } h(b) \wedge h(c) \leq x \text{ implies } h(a) \wedge h(c) \leq x \text{ iff } a \wedge c \leq h_\bullet(x).$$

Then, because $a \leq h_\bullet(h(a))$ and $h_\bullet(h(a))$ is R-saturated, $a \leq \mu^R(a) \leq h_\bullet(h(a))$. From adjointness also $h(a) = h(h_\bullet(h(a)))$, therefore $h(a) = h(\mu^R(a))$.

The last two things to check is that $h{\restriction}_{L/R}$ is a frame homomorphism and that it is unique. Uniqueness follows from the fact that μ^R is onto. Next, because meets of L/R are computed in L, $h{\restriction}_{L/R}$ preserves finite meets. Join of a set $\{s_i\}_i$ of R-saturated elements in L/R is computed as $\mu^R(\bigvee_i s_i)$. Hence, $h{\restriction}_{L/R}(\mu^R(\bigvee_i s_i))$ is equal to $h(\bigvee_i s_i) = \bigvee_i h(s_i) = \bigvee_i h{\restriction}_{L/R}(s_i)$. □

Moreover, we show that $\mu^R(x)$, for some $x \in L$, can be computed by an iterative procedure in L. First, define an auxiliary function

$$m(x) \stackrel{\text{def}}{\equiv} x \vee \bigvee\{a \wedge c \mid b \wedge c \leq x \text{ for some } c \in L \text{ and } (a, b) \in R\}.$$

Then, for an ordinal γ and a limit ordinal λ, set

$$m^0(x) = x, \quad m^{\gamma+1} = m(m^\gamma(x)) \quad \text{and} \quad m^\lambda(x) = \bigvee_{\gamma<\lambda} m^\gamma(x)$$

and set $m^\infty(x)$ to be $m^\gamma(x)$ for some ordinal γ such that $m^{\gamma+1}(x) = m^\gamma(x)$.

A.5.3 Proposition. $m^\infty(x) = \mu^R(x)$.

Proof. For "$\leq$", let $x \leq s$ for some R-saturated s. If $b \wedge c \leq x$ for some $c \in L$ and $(a, b) \in R$, then because s is R-saturated, $a \wedge c \leq s$. Therefore, $m(x) \leq s$ and by a transfinite induction also $m^\infty(x) \leq s$.

For "$\geq$", it is enough to show that $m^\infty(x)$ is R-saturated and this follows from the observation that $m(a) = a$ if and only if a is R-saturated. □

In the text we also need the following fact about quotients of products of two frames:

A.5.4 Proposition. *Let $L = L_+ \times L_-$ for two frames L_+ and L_- and R be a relation on L, i.e. $R \subseteq L \times L$. Then,*

$$L/R \cong (L_+/R_+) \times (L_-/R_-)$$

where $R_+ = \{(\alpha_+, \beta_+) \mid ((\alpha_+, \alpha_-), (\beta_+, \beta_-)) \in R\}$ and R_- is defined similarly.

Proof. Assume that $h\colon L \to M$ is a frame homomorphism preserving R. Define $h_+\colon L_+ \to M$ as $a \mapsto h(a,1)$ and $h_-\colon L_- \to M$ as $b \mapsto h(1,b)$. We show that h_+ and h_- preserve R_+ and R_-, respectively. For a $((\alpha_+,\alpha_-),(\beta_+,\beta_-)) \in R$, $h_+(\alpha_+) = h(\alpha_+,1) = h((\alpha_+,\alpha_-) \sqcup (0,1)) = h(\alpha_+,\alpha_-) \vee h(0,1) \leq h(\beta_+,\beta_-) \vee h(0,1) = h(\beta_+,1)$.

Therefore, by universality of L_+/R_+ and L_-/R_-, h_+ and h_- lift to

$$\overline{h}_+\colon L_+/R_+ \to M \quad \text{and} \quad \overline{h}_-\colon L_-/R_- \to M.$$

Define $\overline{h}\colon (L_+/R_+)\times(L_-/R_-) \to M$ as $(a,b) \mapsto \overline{h}_+(a) \wedge \overline{h}_-(b)$. Then, for the quotient maps $\mu_+\colon L_+ \twoheadrightarrow L_+/R_+$, $\mu_-\colon L_- \twoheadrightarrow L_-/R_-$ and $(a,b) \in L_+\times L_-$, is $\overline{h}(\mu_+(a),\mu_-(b))$ equal to $\overline{h}_+(\mu_+(a)) \wedge \overline{h}_-(\mu_-(b)) = h(a,1) \wedge h(1,b) = h((a,1) \sqcap (1,b)) = h(a,b)$. Therefore, $\overline{h} \cdot (\mu_+\times\mu_-)$ is equal to h. Unicity of $\overline{h}$ follows from $\mu_+\times\mu_-$ being onto. □

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