COUNTABLE SPACES AND COUNTABLE DYNAMICS

by

DOLORES COLUMBA PEREZ FLORES

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College of Engineering and Physical Sciences
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Abstract

In this work we present results on three different topics. In the first of them, we consider the following situation: given a pair of functions $f : X \to Y$ and $g : X \to Z$, under which conditions can we find compact Hausdorff topologies on X, Y and Z with respect to which f and g are simultaneously continuous? We give a partial solution to the problem, solution that involves the One-point Compactification of a discrete space topology. Secondly, we extend the body of existing results on countable compact dynamical systems, which arise naturally in many dynamical settings. Among other results, we prove that these systems are ubiquitous in interval maps. The third part of this thesis is devoted to the study of the ordering by embeddability as a closed subset of closed sets of the real line. We characterise the poset $2^{\mathbb{R}}/\sim$, where $\sim$ denotes the mentioned relation. The structure of countable compact Hausdorff spaces is the underlying notion that unifies this work.

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Introduction

The present work is primarily a study of discrete dynamical systems (X, f) where X is a countable compact Hausdorff topological space. Dynamical systems of this kind appear in all sorts of situations; e.g. they arise as ω-limit sets of tent maps (see [15]). In Chapter 3, among many other results, we prove that countable compact systems are ubiquitous in interval maps (Theorem 3.3.2). Despite being subsystems of the most commonly studied maps, the attention that they have received is limited. Previous studies on properties of countable compact dynamical systems have been done by Bobok [4], Huang and Ye [17], and Kato and Park [18]. The results stated here extend the body of knowledge on this peculiar and highly interesting family of dynamical systems. We also address the problem of simultaneous topologisation of sets in order to make a pair of functions continuous (Chapter 2). Given a pair of surjective functions $f : X \to Y$ and $g : X \to Z$, we succeed at establishing necessary and sufficient conditions for f and g to guarantee that the One-point Compactification of a discrete space topology defined on both Y and Z will make f and g simultaneously continuous (Theorem 2.3.9). At the present time, no previous work on this problem is known to us. The last part of this thesis is devoted to the ordering by embeddability as a closed subspace on the family of closed sets of $\mathbb{R}$. Given two closed subsets A, B of $\mathbb{R}$ we define the relation $\sim$ as: $A \sim B$ if and only if A can be embedded as a closed subset into B and vice versa.

In Chapter 4 we provide a complete list of all the elements of $2^{\mathbb{R}}/\sim$ (Theorem 4.3.1). We did not find any trace of previous work in this direction either, which means that our results are the first on the subject.

The chapters appear in the order we studied their content. Therefore, the oldest part of this project was on simultaneous topologisation, whereas the most recent was the study on the embeddability ordering. Chapter 3, without doubt, can be seen as the main chapter of this work. The common factor of these 3 different parts is the ubiquitous presence of countable compact Hausdorff spaces and their scattered structure. We will now describe briefly the contents of each chapter.

Chapter 1 establishes the terminology and the basic concepts that are necessary throughout the whole body of the thesis. Theorem 1.2.1 enumerates important characteristics of countable compact Hausdorff spaces. At the end of Section 1.2, we prove that any compact metrisable space is scattered if and only if it is countable (Theorem 1.2.3). In Subsection 1.3.1 the proof of Purisch's Theorem (Theorem 1.3.5) can be found. This result establishes the relation between the limit type of a point and the limit type of the elements of its fibre under a continuous closed function, and we decided to name it after S. Purisch because this result is inspired by his work.

In Chapter 2 we attempt to solve the following problem: given two functions $f : X \to Y$ and $g : X \to Z$, under which conditions can we find compact Hausdorff topologies on X, Y and Z with respect to which f and g are simultaneously continuous? Section 2.2 shows necessary conditions for simultaneous topologisation and also shows an example of a pair of maps for which the problem has no solution. Theorem 2.3.9 is a characterisation of those pairs of maps for which the One-Point Compactification of a discrete space topology is a solution. However, in Section 2.4 we learn that the characterisation given by Theorem 2.3.9 is far from being exhaustive, since there are pairs of functions for which simultaneous topologising induces a complex scattered structure (Theorem 2.4.2).

Countable dynamical systems are the object of interest of Chapter 3. The ubiquity of these systems in interval maps is proved in Theorem 3.3.2. In Section 3.4 we discuss the structure of countable dynamical systems. We start by characterising continuous functions on countable compact spaces in terms of their orbit structure (Theorem 3.4.3). In Subsection 3.4.1 we prove that the top limit points of a countable compact dynamical system form cycles among them (Theorem 3.4.6). Besides, Theorem 3.4.11 establishes that a repelling fixed point of a countable compact system where the function is finite-to-one cannot have a limit ordinal as limit type. Section 3.5 deals with transitivity. We learn from work of Akin and Carlson [1] that the only definition that make some sense for countable compact systems is the one in terms of dense orbit sequences. Theorem 3.5.6 shows that, for every countable ordinal α, there is a transitive countable compact system with derived degree $\alpha + 1$. The concepts of ω-limit set and shadowing are presented in Section 3.6. The main result of this part of the chapter is Theorem 3.6.14, which says that the ω-limit set of any point in a compact countable system with shadowing is a periodic orbit.

In the Chapter 4 we deal with the ordering by embeddability on the family of closed subspaces of $\mathbb{R}$. Theorem 4.3.1 gives a complete list of the elements of the partially ordered set $2^{\mathbb{R}}/\sim$, and we can observe that the building blocks of the equivalence classes are countable ordinals, the closed unit interval $[0, 1]$, the unbounded interval $[1, \infty)$ and the Cantor set $\mathcal{C}$. A description of the ordering induced by the relation $\sim$ on $2^{\mathbb{R}}/\sim$ is also provided.

Chapter 1

Countable Compact Hausdorff Spaces

In this chapter, the central notions which are necessary to develop our results are reviewed. Because we mainly focus on countable compact Hausdorff spaces in this thesis, we consider it convenient to provide a list of some of their most important characteristics (Theorem 1.2.1). Among such attributes, the scattered nature of these spaces plays an important role in their study. Bearing that in mind, some examples which illustrate the way continuous functions affect the limit type of points in scattered spaces are presented in Section 1.3, and later in the same section a result that establishes the relation between a point and its image under a closed continuous surjective function, Theorem 1.3.5, is stated and proved.

1.1 Preliminaries

In this section we will establish the terminology and notation that will be used in most of the work. Later, in each chapter, more terms and notation will appear, based on the needs of that specific part. For undefined notions and more basic results we refer to the books of Engelking [11], Willard [31] and Kunen [21].

Throughout this thesis, the word *space* means *topological space*. In special occasions, a space X will be denoted by (X, τ_X), to emphasise the topology that is being considered on X. All spaces considered in this work are Hausdorff spaces, i.e., spaces where any two distinct points can be separated by disjoint open sets. The word neighbourhood means open neighbourhood. Given a subset A of a space X, $\overline{A}$ denotes the closure of A, and A' denotes the set of limit points of A. A is said to be clopen if it is closed and open in X. A space is called zero-dimensional if it has a basis consisting of clopen sets. If X is a metric space, $x \in X$ and ϵ is a positive real number, then the open ball with center x and radius ϵ is denoted by $B_\epsilon(x)$.

Let $f : X \to Y$ and choose a point $y \in Y$, and a set $A \subseteq X$. The fibre of y under f, i.e., the set $\{x \in X : f(x) = y\}$, is denoted by $f^{-1}(y)$. Also, $f \restriction_A$ denotes the restriction of f to A with image set $f(A)$.

As usual the set of real numbers, the set of natural numbers, the set of integer numbers, and the set of rational numbers are denoted by $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, respectively. The first infinite ordinal is denoted by ω, and occasionally we will write $|A| \geq \omega$ to say that the set A is infinite. The symbol $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$.

1.1.1 Some Remarks on Compact Spaces

Let X be a Hausdorff space. Recall that X is said to be *compact* provided that every open cover of X has a finite subcover. Below, we give a brief list of well-known facts on compact spaces that will be used in this work.

Compactness is a closed-hereditary property. A Hausdorff space X is compact if and only if any collection of closed subsets of X with the finite intersection property has non-empty intersection. The continuous image of a compact space is also a compact space. If $f : X \to Y$ is a continuous bijection and X is compact, then f is a homeomorphism. Important examples of compact spaces are: any finite space, the closed unit interval

$[0, 1]$, any successor ordinal, and the Cantor set.

A Hausdorff space Y is called *locally compact* if each of its points has a compact neighbourhood. A non-compact locally compact space Y can be embedded as a dense set into a compact space through the following technique. Consider the set $\tilde{Y} = Y \cup \{y^*\}$, with $y^* \notin Y$. Topologise $\tilde{Y}$ in this way: a neighbourhood basis for the point y^* will be the family $\{A \cup \{y^*\} : A \subseteq Y, Y \setminus A \text{ compact}\}$, and for each $y \in Y$ a neighbourhood basis of y will be the family $\{U \subseteq Y : y \in U, U \text{ is open in } Y\}$. $\tilde{Y}$ is known as the *Alexandroff Compactification* or *One-point Compactification* of Y. A special case that will be frequently studied in the next chapters (particularly in Chapter 2) is when Y is a discrete space. In such case, we will make reference to $\tilde{Y}$ as the *One-point Compactification of a Discrete Space*.

1.1.2 Some Remarks on Ordinals

Throughout this work we will frequently regard ordinals as topological spaces. Whenever this happens, the topology considered on them will be the order topology. It is worth remembering that any ordinal with the order topology is Hausdorff, scattered (see Subsection 1.1.3) because every set has a least element, and zero-dimensional because all the sets of the form $(\alpha, \beta]$ are clopen. The set of limit points of an ordinal α is precisely the set of all limit ordinals less than α. Observe that ω is homeomorphic to any countably infinite discrete space and that $\omega + 1$ is homeomorphic to its one-point compactification. An ordinal α is compact if and only if it is a successor ordinal (a proof can be found in [12, Page 42]).

Recall that any non-zero ordinal number α can be uniquely written as

$$\alpha = \omega^{\alpha_1} n_1 + ... + \omega^{\alpha_k} n_k$$

for some ordinals $\alpha_1, ..., \alpha_k$ such that $\alpha_1 > ... > \alpha_k \geq 0$, and some natural numbers $k, n_1, ..., n_k$. This is known as the *Cantor normal form* of α.

The following result is a classification of ordinal topologies (the reader interested in a proof can consult [19, Proposition 4, Main Theorem]). As we will see, it underlies our understanding of the topological structure of countable compact Hausdorff spaces.

THEOREM 1.1.1. *Let α be a non-zero ordinal with Cantor normal form*

$$\omega^{\alpha_1}n_1 + ... + \omega^{\alpha_k}n_k.$$

α is homeomorphic to either

(1) n_1, if $k = 1$ and $\alpha_1 = 0$; or

(2) $\omega^{\alpha_1}n_1$, if $k = 1$ and $\alpha_1 \neq 0$; or

(3) $\omega^{\alpha_1}n_1 + 1$, if $k > 1$ and $\alpha_k = 0$; or

(4) $\omega^{\alpha_1}n_1 + \omega^{\alpha_k}$, if $k > 1$ and $\alpha_k \neq 0$.

Notice that the only compact cases are (1) and (3). Therefore, any infinite ordinal space that is compact must be homeomorphic to some ordinal of the form $\omega^\alpha n + 1$.

1.1.3 Scattered Spaces

A space X is said to be *scattered* if every non-empty subset $A \subseteq X$ contains an isolated point in A. In this subsection we will mention some facts on scattered spaces and some related concepts that are indispensable for the development of our results in future chapters.

DEFINITION 1.1.2. Let X be a topological space. Recall that X' denotes the set of limit points of X. The *Cantor-Bendixson derivatives* of the space X are defined recursively by

$$\begin{aligned} X^{(0)} &= X, \\ X^{(\alpha+1)} &= (X^{(\alpha)})', \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)}, \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Since $X^{(\beta)} \subseteq X^{(\gamma)}$ for $\beta \geq \gamma$, for every space X there exists some ordinal α such that $X^{(\beta)} = X^{(\alpha)}$ whenever $\beta \geq \alpha$.

Definition 1.1.3. Let X be a topological space. The *derived degree (or scattered height)* $d(X)$ of the space X is the least ordinal α such that $X^{(\beta)} = X^{(\alpha)}$ whenever $\beta \geq \alpha$.

A space X is scattered if and only if there exists an ordinal α such that $X^{(\gamma)} = \emptyset$ for every $\gamma \geq \alpha$. Thus, the derived degree of a scattered space X is the least ordinal α such that $X^{(\alpha)} = \emptyset$. Now suppose that X is a compact scattered space. Since each $X^{(\gamma)}$ is closed, hence compact, if $X^{(\gamma)}$ is infinite, then it has a limit point so $X^{(\gamma+1)}$ is non-empty. Hence there exists β such that $X^{(\beta)}$ is finite and $X^{(\beta+1)} = \emptyset$.

Definition 1.1.4. Let X be a scattered topological space and let α be an ordinal. A point $x \in X$ has *limit type* α (also called *rank* or *scattered height*) if and only if $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. The limit type of x will be denoted by $lt(x)$. The set of all points of X which have limit type α is denoted by $\mathcal{L}^{\alpha}(X)$.

Example 1.1.5. The ordinal $\beta = \omega^{\alpha} n + 1$ has derived degree $\alpha + 1$ and $\mathcal{L}^{\alpha}(\beta)$ is a set of cardinality n.

Let x be a point of a scattered space X with $lt(x) = \alpha$. Since x is an isolated point of $X^{(\alpha)}$, there is a neighbourhood U of x in X such that $lt(y) < \alpha$ for every $y \in U \setminus \{x\}$. This leads to the following definition.

Definition 1.1.6. Let X be a scattered space and let $x \in X$. A neighbourhood U of x *witnesses the limit type* of x if $lt(y) < lt(x)$ for any $y \in U \setminus \{x\}$.

When X is a scattered metric space the following concept arises naturally.

Definition 1.1.7. Let X be a scattered metric space with countable derived degree and let $x \in X$. Suppose that $lt(x) > 0$. A sequence $(y_n)_{n \in \mathbb{N}}$, where $y_n \neq y_m$ if $n \neq m$, *witnesses the limit type* of x if $y_n \to x$ and

1) $lt(y_n) \to lt(x)$, if $lt(x)$ is a limit ordinal; or

2) $lt(y_n) = \alpha$ for all $n \in \mathbb{N}$, if $lt(x) = \alpha + 1$.

1.2 Countable compact Hausdorff spaces

Countable compact Hausdorff spaces are the main objects of study of this work. In Theorem 1.2.1 we enumerate some of their most important properties. Despite being well-known facts (see [4], [13] and [18]), we decided to include a proof of our own due to both the importance of this result and our wish to convey a general idea of what working on these spaces is like.

Theorem 1.2.1. *Any countable compact Hausdorff topological space is*

1) scattered;

2) second countable;

3) metrisable;

4) zero-dimensional;

5) homeomorphic to a countable successor ordinal;

6) homeomorphic to a subspace of $\mathbb{Q}$.

Proof. Let X be a countable compact Hausdorff space. Recall that any compact Hausdorff topological space is normal, i.e., for every pair of disjoint closed sets C and D there are disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$.

1) If X is finite, X is discrete and thus scattered. Let us assume then that X is infinite.

Suppose that X is not scattered. Then, there exists a countably infinite subspace A of X with no isolated points in A. We can assume without loss of generality that A is compact (observe that $\overline{A}$ is a countably infinite compact Hausdorff subspace of X with no isolated points). This implies that A is a normal space. Choose two distinct points x_0 and x_1 of A. By normality, there are two disjoint open sets U_0 and U_1 in A

such that $x_0 \in U_0$, $x_1 \in U_1$ and

$$\overline{U_0} \cap \overline{U_1} = \emptyset.$$

Let $\overline{U_0}$ and $\overline{U_1}$ be denoted by $A_{(0)}$ and $A_{(1)}$ respectively. $A_{(0)}$ and $A_{(1)}$ are compact Hausdorff spaces. Since both x_0 and x_1 are limit points of A, $A_{(0)}$ and $A_{(1)}$ are infinite. We claim that the spaces $A_{(0)}$ and $A_{(1)}$ have no isolated points. Suppose that $\overline{U_0}$ has an isolated point w. Then there exists an open set W of A such that $W \cap \overline{U_0} = \{w\}$. This implies that $w \notin \overline{U_0} \setminus U_0$. Then $w \in U_0$. But this would mean that $\{w\} = W \cap U_0$ is open in A, since U_0 is open in A. This contradicts the fact that the space A has no isolated points. Hence $A_{(0)}$ and $A_{(1)}$ cannot have isolated points.

Select two distinct points y_0 and y_1 in $A_{(0)}$, and two distinct points z_0 and z_1 in $A_{(1)}$. By normality of $A_{(0)}$, there exist two disjoint open sets V_0 and V_1 in $A_{(0)}$ such that $y_0 \in V_0$, $y_1 \in V_1$ and

$$\overline{V_0} \cap \overline{V_1} = \emptyset.$$

Analogously, there are two disjoint open sets W_0 and W_1 in $A_{(1)}$ such that $z_0 \in W_0$, $z_1 \in W_1$ and

$$\overline{W_0} \cap \overline{W_1} = \emptyset.$$

Let $A_{(00)}, A_{(01)}, A_{(10)}$ and $A_{(11)}$ denote $\overline{V_0}, \overline{V_1}, \overline{W_0}$ and $\overline{W_1}$ respectively. Then, since none of the points y_0, y_1, z_0 and z_1 is isolated, $A_{(00)}, A_{(01)}, A_{(10)}$ and $A_{(11)}$ are countably infinite compact Hausdorff spaces with no isolated points. Continuing this process ad infinitum we obtain for every $n > 2$ a family of pairwise disjoint countably infinite compact Hausdorff subspaces of X

$$\{A_{(a_1,...,a_n)} : (a_1, ..., a_n) \in \{0,1\}^n\}$$

with the following property: for each $(a_1, ..., a_{n-1}) \in \{0,1\}^{n-1}$

$$A_{(a_1,...,a_{n-1},0)} \subseteq A_{(a_1,...,a_{n-1})}$$

and

$$A_{(a_1,\dots,a_{n-1},1)} \subseteq A_{(a_1,\dots,a_{n-1})}.$$

Let $(a_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$. The set

$$\mathcal{A}_{(a_i)_{i\in\mathbb{N}}} = \{A_{(a_1,\dots,a_i)} : i \in \mathbb{N}\}$$

is a nested family of non-empty closed sets of X. By compactness of X, the set $\bigcap \mathcal{A}_{(a_i)_{i\in\mathbb{N}}}$ is non-empty. Since

$$\bigcap \mathcal{A}_{(a_i)_{i\in\mathbb{N}}} \cap \bigcap \mathcal{A}_{(b_i)_{i\in\mathbb{N}}} = \emptyset$$

whenever $(a_i)_{i\in\mathbb{N}} \neq (b_i)_{i\in\mathbb{N}}$, we obtain that $|X| \geq \mathfrak{c}$, which is a contradiction. Hence, X must be scattered.

2) We will now prove that X is second countable, i.e., that it has a countable basis. If X is finite, the assertion is clearly true. Assume then that $|X| = \omega$. As X is countable, it will suffice to show that X is first countable, i.e., that each point of X has a countable neighbourhood basis.

Let $x \in X$. If x is isolated, then it is evident that x has a countable neighbourhood basis. Suppose then that x is not isolated. Let $(x_n)_{n\in\mathbb{N}}$ be an enumeration of $X \setminus \{x\}$. By normality of X, there is an open set U_1 of X such that $x \in U_1$ and $x_1 \notin \overline{U_1}$. Now consider the disjoint closed sets $\{x\}$ and $\{x_2\} \cup (X \setminus U_1)$. Once again, by normality of X, there exists an open set U_2 such that $x \in U_2$ and $\{x_2\} \cup (X \setminus U_1) \subseteq (X \setminus \overline{U_2})$; in other words, $x \in U_2$, $x_2 \notin \overline{U_2}$ and $\overline{U_2} \subseteq U_1$. Continuing this process ad infinitum, we obtain a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets of X such that, for every $n \in \mathbb{N}$, $x \in U_n$, $x_n \notin \overline{U_n}$ and

$$U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n.$$

Then,

$$\bigcap_{n\in\mathbb{N}} \overline{U_n} = \{x\}.$$

We claim that $\{U_n : n \in \mathbb{N}\}$ is a neighbourhood basis for x. Suppose that it is not a basis. That would imply that there exists an open neighbourhood V of x such that $U_n \nsubseteq V$ for every $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, $\overline{U_n} \setminus V \neq \emptyset$. Thus, $\mathcal{A} = \{\overline{U_n} \setminus V : n \in \mathbb{N}\}$ is a decreasing family of non-empty closed sets of X such that $\bigcap \mathcal{A} = \emptyset$. But this contradicts the compactness of X. Therefore, $\{U_n : n \in \mathbb{N}\}$ is a local basis of x. Thus, X is first countable. Hence, as X is countable, X is second countable.

3) By Urysohn's Metrisation Theorem (see [31, Theorem 23.1]), any regular second countable space is metrisable. Since X is compact and Hausdorff, X is regular. By 2), X is also second countable. Therefore, X is metrisable.

4) We will now prove that X is zero-dimensional. We will show that every point $x \in X$ has a neighbourhood basis of clopen sets using induction on the limit type of x.

Let $x \in X$. If $lt(x) = 0$, then $\big\{\{x\}\big\}$ is a neighbourhood basis for x whose only element is clopen.

Suppose that $lt(x) = \alpha$, with α a countable non-zero ordinal, and suppose that every point $y \in X$ with $lt(y) < \alpha$ has a neighbourhood basis of clopen sets. Let U be an open neighbourhood of x which witnesses the limit type of x, i.e., for any point $z \in U \setminus \{x\}$, $lt(z) < \alpha$. Since X is normal, there is an open neighbourhood V of x such that $x \in V \subset \overline{V} \subseteq U$. Assume that $\overline{V} \setminus V \neq \emptyset$. Then, $\overline{V} \setminus V$ is a non-empty compact subspace of X such that $lt(z) < \alpha$ for every $z \in \overline{V} \setminus V$. For each $z \in \overline{V} \setminus V$, choose a clopen neighbourhood U_z of z. The family $\mathcal{U} = \{U_z : z \in \overline{V} \setminus V\}$ covers $\overline{V} \setminus V$. Since $\overline{V} \setminus V$ is compact, there is a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ which covers $\overline{V} \setminus V$. The set $\bigcup \mathcal{V}$ is clopen. Moreover, the set $\bigcup \mathcal{V} \cup \overline{V}$ is a clopen neighbourhood of x. This proves that x has a neighbourhood basis of clopen sets.

5) In part 1) it was proved that a countable compact Hausdorff space is scattered. As we mentioned in the remarks after Definition 1.1.3, the derived degree of such a space must be a successor ordinal. We will now show that any countable compact Hausdorff space X is homeomorphic to $\omega^\alpha n+1$ with the order topology, where $\alpha+1$ is the derived degree of X and $n \in \omega$, and we will do it using induction on the derived degree of X.

If $d(X) = 1$, then $lt(x) = 0$ for all $x \in X$, i.e., every point of X is isolated. Since X is compact, X must be finite. So, X is homeomorphic to n, where $|X| = n$.

Assume that $d(X) = \beta$ and suppose that, for every $\gamma < \beta$, every space Y of derived degree γ is homeomorphic to $\omega^\gamma m + 1$ with the order topology, for some $m \in \omega$. Since X is compact, $\beta = \alpha+1$ for some countable ordinal α, and $\mathcal{L}^\alpha(X)$, the set of all points in X with limit type α, is finite. Let $\mathcal{L}^\alpha(X) = \{x_1, x_2, ..., x_n\}$ for some $n > 0$. By 4), X is zero-dimensional. Then, there exists a finite clopen cover $\mathcal{U} = \{U_1, U_2, ..., U_n\}$ of X such that, for each $i, j \in \{1, 2, ..., n\}$, $x_i \in U_i$ and $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Then, for each i, U_i is a clopen neighbourhood of x_i which witnesses its limit type. Let $i \in \{1, 2, ..., n\}$. Choose a sequence $(y_j)_{j\geq 0}$ in U_i which witnesses the limit type of x_i, i.e., a sequence such that $y_k \neq y_j$ whenever $k \neq j$, $y_j \to x_i$ and

- $lt(y_j) \to lt(x_i)$, if $lt(x_i) = \alpha$ is a limit ordinal; or
- $lt(y_j) = \rho$, if $\alpha = \rho + 1$.

Since X is normal, there is a family $\mathcal{V} = \{V_j : j \geq 0\}$ of clopen subsets of U_i such that

- V_j is a clopen neighbourhood of y_j;
- $V_k \cap V_j = \emptyset$, if $k \neq j$;
- $\bigcup \mathcal{V} = U_i \setminus \{x_i\}$.

For every $j \geq 0$, the set V_j is compact and $d(V_j) = \rho$, where $lt(y_j) \leq \rho < \alpha$. Then, by inductive hypothesis, V_j is homeomorphic to $\omega^\rho p_j + 1$ for some $p_j \in \omega$. Since U_i is the disjoint union of the elements of $\mathcal{V}$, U_i is homeomorphic to $\omega^\alpha + 1$ with the order topology. Therefore, X is homeomorphic to $\omega^\alpha n + 1$.

6) Since every countable metric space is embeddable in $\mathbb{Q}$ (see [11, Exercise 6.2.A] and the remarks after [13, Theorem 1]), by (5) we have that X is homeomorphic to a subspace of $\mathbb{Q}$. $\square$

Corollary 1.2.2. *Let X and Y be countable compact Hausdorff spaces. If $d(X) = d(Y) = \alpha+1$ and $\mathcal{L}^\alpha(X)$ is homeomorphic to $\mathcal{L}^\alpha(Y)$, then X is homeomorphic to Y.* $\square$

The next result follows partially from Theorem 1.2.1.

Theorem 1.2.3. *Any compact metrisable space is scattered if and only if is countable.*

Proof. Suppose that there exists a compact metrisable scattered space which is not countable. Then $\{d(Y) : Y \text{ is uncountable compact metrisable scattered}\}$ is a non-empty family of ordinals and as such it has a minimum element. Let α be such minimum. Let X be an uncountable compact metrisable scattered space with $d(X) = \alpha$, and let $x_1, x_2, ..., x_k$ be its top limit points, i.e., the points in X with highest limit type. By normality of X, we can choose open neighbourhoods $U_1, U_2, ..., U_k$ of $x_1, x_2, ..., x_k$ respectively, such that, for each $i, j \in \{1, 2, ..., k\}$, U_i witnesses the limit type of x_i and $\overline{U_i} \cap \overline{U_j} = \emptyset$ if $i \neq j$. Then,

$$X = \overline{U_1} \cup \overline{U_2} \cup ... \cup \overline{U_k} \cup \Big(X \setminus \bigcup_{1 \le i \le k} U_i\Big).$$

Since $X \setminus \bigcup\limits_{1 \le i \le k} U_i$ is compact, metrisable, scattered and its derived degree is less than α, it is countable. Therefore, we can pick $i \in \{1, 2, ..., k\}$ such that $\overline{U_i}$ is uncountable. Since $\overline{U_i}$ is metrisable, there is a countable family $(V_n)_{n\in\mathbb{N}}$ of open sets of $\overline{U_i}$ such that $\bigcap\limits_{n\in\mathbb{N}} V_n = \{x_i\}$. For each $n \in \mathbb{N}$, $\overline{U_i} \setminus V_n$ is compact metrisable scattered with derived degree less than α, so countable. But this contradicts the fact of $\overline{U_i}$ being uncountable.

Therefore, if X is compact metrisable and scattered, it must be countable. The other implication is given by part (1) of Theorem 1.2.1. $\square$

1.3 Continuous Functions and Limit Type of Points

Let X and Y be scattered spaces and let $f : X \to Y$ be a continuous function. Given a point $x \in X$, it is natural to wonder if there is a relationship between the limit type of $f(x)$ and the limit type of x. Below, we provide some examples that illustrate different situations that can arise.

EXAMPLE 1.3.1. Consider the spaces $X = \omega + 1$ and $Y = \{0, 1, 2, ...k\}$ with the order topology, and

$$f(x) = \begin{cases} k & \text{if} \quad x \geq k \\ x & \text{if} \quad x < k \end{cases}$$

The function f is clearly continuous. In addition, $1 = lt(\omega) > lt(f(\omega)) = 0$. So, the limit type of a point can decrease under a continuous function.

Obviously, the limit type of a point can be maintained (just think about the case $X = Y$ and the identity function). But, is it possible that the limit type of a point in X increases under a continuous function?

EXAMPLE 1.3.2. Consider the constant function $f : X \to Y$, with $X = \omega + 1$ and $Y = \omega^2 + 1$, defined by $f(x) = \omega^2$ for all $x \in X$. Observe f is continuous and that $lt(f(x)) = 2 > 0 = lt(x)$ for any isolated point $x \in X$. Then, the limit type of a point can increase under a continuous function.

Notice that, though continuous, the function of Example 1.3.2 is not surjective. Surjectivity among other properties will play a significant role in the task of finding some relationship between the limit type of a point and the limit type of its image, as we will see in the next section.

1.3.1 Purisch's Theorem

In [27, Proposition 1], Purisch claims the following.

Proposition 1.3.3. *Let $f : X \to Y$ be a closed onto map with discrete fibres such that X and Y are scattered. For each $y \in Y$, $lt(y) = sup\{lt(x) : x \in f^{-1}(y)\}$.*

Unfortunately this result is not true. Example 1.3.4 proves the existence of two scattered spaces X and Y and a closed surjective function $f : X \to Y$ with discrete fibres with a point $y \in Y$ for which $lt(y) > \sup\{lt(x) : x \in f^{-1}(y)\}$. In addition, in the proof of Theorem 1.3.5 we will point out what is wrong in Purisch's original proof.

Example 1.3.4. Consider the ordinal ω^ω with the order topology. For every $n \in \omega$ the space

$$A_n = \omega^\omega \times \{n\}$$

is a homeomorphic copy of ω^ω. Let

$$X = \bigcup_{n\in\omega} A_n \cup \{x^*\},$$

where $x^* \notin \bigcup_{n\in\omega} A_n$. For each $n \in \omega$, A_n will be considered clopen. A set $G \subseteq \bigcup_{n\in\omega} A_n$ will be open in X if and only if $G \cap A_n$ is open in A_n for every $n \in \omega$. A neighbourhood basis for x^* will be the family

$$\Big\{X \setminus \bigcup_{k\in\mathcal{I}} A_k : \mathcal{I} \subset \omega, \mathcal{I} \text{ finite}\Big\}.$$

We claim that, endowed with such a topology, X is a Hausdorff countable scattered space. Suppose that $x, y \in X$ are distinct points. Assume without loss of generality that there exists $n \in \omega$ such that $x \in A_n$. If $y \notin A_n$, then A_n and $X \setminus A_n$ are disjoint open sets that separate x and y. If $y \in A_n$, then there exist disjoint open sets U_x, U_y of A_n which separate x and y, because A_n is a Hausdorff space. Since U_x and U_y are also open sets in X, there exists a pair of disjoint open sets in X which separates x and y. Thus, X is Hausdorff. Now suppose that A is a non-empty subset of X. If $A = \{x^*\}$, clearly the subspace A has an isolated point. If there exists $n \in \omega$ such that $A \cap A_n \neq \emptyset$, then the subspace $A \cap A_n$ has an isolated point since it is a non-empty subspace of the scattered space A_n. This means that there exist a point $x \in A \cap A_n$ and an open set U of A_n such that $U \cap (A \cap A_n) = \{x\}$. Since U is open in X, x is an isolated point of A. Therefore, X is scattered.

Define the relation $\sim$ on X as follows: $x \sim y$ if and only if there exist $n, m, l \in \omega$ such that $x = (\omega^n, l)$ and $y = (\omega^m, l)$, or $x = y$. It is easy to verify that $\sim$ is an equivalence relation. Consider now the space $Y = X/\sim$ with the quotient topology. We claim that Y is a Hausdorff space. To see this note that ω^ω is homeomorphic to the topological sum of the ordinal spaces of the family $\{\omega^n + 1 : n \in \omega\}$ with their order topologies. Each $\omega^n + 1$ can be embedded as a subspace of $[0, 1]$ with the point $\omega^n \in \omega^n + 1$ associated with 0. Therefore the quotient space Y is homeomorphic to a subspace of the hedgehog of spininess ω (see [11, Example 4.1.5]) and is therefore a metric space. Thus Y is Hausdorff.

Obviously, the quotient function

$$
\begin{aligned}
P : X &\to Y \\
x &\mapsto [x]
\end{aligned}
$$

is continuous and onto. Furthermore, since the set

$$B_l = \{(\omega^n, l) : n \in \omega\}$$

is a discrete subspace of A_l for each $l \in \omega$, P has discrete fibres.

We claim that Y is a scattered space. In order to prove it we will show first that $P(A_n)$ is scattered for every $n \in \omega$. Let $n \in \omega$. Since A_n is clopen, $P(A_n)$ is a clopen subset of Y. Let p_n denote $P((\omega, n))$ and let $\mathcal{C}$ be a non-empty subset of $P(A_n)$. If $p_n \notin \mathcal{C}$ then $P^{-1}(\mathcal{C})$ is homeomorphic to $\mathcal{C}$. Since $P^{-1}(\mathcal{C})$ is a non-empty subspace of the scattered space A_n, it has an isolated point and so does $\mathcal{C}$. Now suppose that $p_n \in \mathcal{C}$ and that p_n is not an isolated point of $\mathcal{C}$. In this case there exists an open set $\mathcal{D}$ of $P(A_n)$ such that $\mathcal{D} \cap \mathcal{C}$ contains p_n and $(\mathcal{D} \cap \mathcal{C}) \setminus \{p_n\}$ is non-empty. In addition, $(\mathcal{D} \cap \mathcal{C}) \setminus \{p_n\}$ is homeomorphic to $P^{-1}((\mathcal{D} \cap \mathcal{C}) \setminus \{p_n\})$. $P^{-1}((\mathcal{D} \cap \mathcal{C}) \setminus \{p_n\})$ has an isolated point because is a non-empty subspace of A_n, so $(\mathcal{D} \cap \mathcal{C}) \setminus \{p_n\}$ has an isolated point. Since $((D) \cap \mathcal{C}) \setminus \{p_n\}$ is open in $\mathcal{C}$, $\mathcal{C}$ has an isolated point. Therefore, $P(A_n)$ is scattered for each $n \in \omega$. Since $Y = \{P(x^*)\} \cup \bigcup_{n \in \omega} P(A_n)$ and $P(A_j) \cap P(A_m) = \emptyset$ for every $j, m \in \omega$, Y is scattered.

Now we will prove that P is closed. Let A be a closed subset of X. Observe that

$$P(A) = P(A \cap \{x^*\}) \cup \underset{n \in \omega}{\cup} P(A \cap A_n).$$

Now, given $n \in \omega$

$$P^{-1}(P(A \cap A_n)) = \begin{cases} A \cap A_n & \text{if } A \cap A_n \cap B_n = \emptyset \\ (A \cap A_n) \cup B_n & \text{if } A \cap A_n \cap B_n \neq \emptyset \end{cases}$$

Since B_n is a closed subset of A_n for each n, it is a closed subset of X. Therefore $P^{-1}(P(A \cap A_n))$ is closed in X, which means that $P(A \cap A_n)$ is closed in Y because P is the quotient map and Y is endowed with the quotient topology. Thus, $P(A_n) \setminus P(A \cap A_n)$ is open in Y. Therefore $Y \setminus P(A)$ is open in Y, i.e. $P(A)$ is closed. Hence P is a closed map.

Finally, observe that $lt(p_n) = \omega$ for every $n \in \omega$ and that $P(x^*)$ is a limit point of the set $\{p_n : n \in \omega\}$. Additionally, observe that sup$\{lt(x) : x \in \bigcup_{n\in\omega} A_n\} = \omega$. Since such supremum is not attained, it must be the limit type of x^*. Then

$$lt(P(x^*)) = \omega + 1 > lt(x^*) = \omega.$$

Although Purisch's Proposition is not actually correct, we can obtain some interesting results if we modify the hypotheses. Since his proposition has inspired the next result, we decided to name it after him.

THEOREM 1.3.5 (Purisch's Theorem). *Let X and Y be scattered Hausdorff topological spaces and let $f : X \to Y$ be a continuous closed surjective function.*

1) If

$$\textit{sup}\{lt(x) : x \in f^{-1}(z)\} = \textit{max}\{lt(x) : x \in f^{-1}(z)\}$$

for every $z \in Y$ (in particular, if f has compact fibres), then

$$lt(y) \le \textit{max}\{lt(x) : x \in f^{-1}(y)\}$$

for all $y \in Y$.

2) If f has discrete fibres, then

$$lt(y) \ge \textit{sup}\{lt(x) : x \in f^{-1}(y)\}$$

for all $y \in Y$.

3) If f is finite-to-one, then

$$lt(y) = \textit{max}\{lt(x) : x \in f^{-1}(y)\}$$

for all $y \in Y$.

Proof. 1) Suppose that

$$\text{sup}\{lt(x) : x \in f^{-1}(z)\} = \text{max}\{lt(x) : x \in f^{-1}(z)\}$$

for every $z \in Y$.

Let y be an isolated point of Y. Obviously

$$lt(y) = 0 \leq \text{max}\{lt(x) : x \in f^{-1}(y)\}.$$

Now choose $y \in Y$ such that $lt(y) > 0$ and assume that

$$lt(w) \leq \text{max}\{lt(x) : x \in f^{-1}(w)\}$$

is true for every $w \in Y$ such that $lt(w) < lt(y) = \lambda$.

For each $x \in f^{-1}(y)$ choose an open neighbourhood U_x which witnesses the limit type of x. Since $\text{max}\{lt(w) : w \in U_x\} = lt(x)$ for every $x \in f^{-1}(y)$, the set $U = \bigcup\limits_{x \in f^{-1}(y)} U_x$ is a neighbourhood of $f^{-1}(y)$ such that

$$\text{sup}\{lt(x) : x \in U\} = \text{sup}\{lt(x) : x \in f^{-1}(y)\} = \text{max}\{lt(x) : x \in f^{-1}(y)\}.$$

Since f is closed, $W = f(X \setminus U)$ is a closed set such that $y \notin W$. Therefore, there exists an open neighbourhood N of y such that $y \in N \subseteq Y \setminus W$ and N witnesses the limit type of y. Then $f^{-1}(N) \subseteq U$.

Case 1. Suppose that $\lambda = \beta + 1$. Since $y \in Y^{(\beta+1)}$, y is a limit point of $\mathcal{L}^{\beta}(Y)$.

Then, there exists $w \in N$ such that $lt(w) = \beta$. By induction hypothesis

$$\max\{lt(x) : x \in f^{-1}(w)\} \geq \beta.$$

Therefore, there exists a point $x_1 \in f^{-1}(w)$ such that $lt(x_1) \geq \beta$. Since $f^{-1}(N) \subseteq U$, $f^{-1}(w) \subseteq U$, i.e. $x_1 \in U$. Therefore, $x_1 \in U \setminus f^{-1}(y)$. Since $x_1 \in U$, there is a point $x_0 \in f^{-1}(y)$ such that $x_1 \in U_{x_0}$. Since U_{x_0} witnesses the limit type of x_0, we have that $lt(x_0) > lt(x_1) \geq \beta$, i.e. $lt(x_0) \geq \beta + 1$. Therefore

$$\max\{lt(x) : x \in f^{-1}(y)\} \geq \beta + 1 = \lambda = lt(y).$$

NOTE. This is the place where Purisch's Proposition has a mistake. In the first part of his proof he tried to show (without assuming that $\sup\{lt(x) : x \in f^{-1}(z)\} = \max\{lt(x) : x \in f^{-1}(z)\}$ for every $z \in Y$) that $\sup\{lt(x) : x \in f^{-1}(y)\} \geq lt(y) = \lambda$. He chooses neighbourhoods U and N of $f^{-1}(y)$ and y respectively such that N witnesses the limit type of y and $f^{-1}(N) \subseteq U$. Then he finds $y_\alpha \in N$ such that $lt(y_\alpha) = \alpha$ for every $\alpha < \lambda$. Using the inductive hypothesis $\sup\{lt(x) : x \in f^{-1}(y_\alpha)\} = lt(y_\alpha) = \alpha$ for every $\alpha < \lambda$, he claims that $\sup\{lt(x) : x \in f^{-1}(y)\} \geq \lambda$. But that claim cannot be proved if λ is a succesor ordinal (recall the point $P(x^*)$ of Example 1.3.4). This is why we analysed the case of succesor ordinals apart from the case of limit ordinals.

CASE 2. Suppose that λ is a limit ordinal. Since $y \in Y^{(\lambda)}$, y is a limit point of $\mathcal{L}^\alpha(Y)$ for every $\alpha < \lambda$. Then, for each $\alpha < \lambda$ there exists $y_\alpha \in N$ such that $lt(y_\alpha) = \alpha$. By induction hypothesis

$$\max\{lt(x) : x \in f^{-1}(y_\alpha)\} \geq \alpha.$$

Therefore, there exists a point $x_\alpha \in f^{-1}(y_\alpha)$ such that $lt(x_\alpha) \geq \alpha$. Since $f^{-1}(N) \subseteq U$, $f^{-1}(y_\alpha) \subseteq U$, i.e. $x_\alpha \in U$. Thus $x_\alpha \in U \setminus f^{-1}(y)$. Then

$$\lambda \leq \sup\{lt(x) : x \in U \setminus f^{-1}(y)\}.$$

Since

$$\mathsf{sup}\{lt(x) : x \in U \setminus f^{-1}(y)\} \leq \mathsf{sup}\{lt(x) : x \in U\} = \mathsf{max}\{lt(x) : x \in f^{-1}(y)\},$$

we have that

$$\lambda \leq \mathsf{max}\{lt(x) : x \in f^{-1}(y)\}.$$

2) Assume that f has discrete fibres.

Let $y \in Y$ such that $lt(y) = 0$. If there were a point $x \in f^{-1}(y)$ such that $lt(x) > 0$, since f is continuous, there were a neighbourhood U of x such that $f(U) \subseteq \{y\}$. Then $U \subset f^{-1}(y)$. But this would imply that $f^{-1}(y)$ is not discrete, which contradicts our assumption. Hence

$$\mathsf{sup}\{lt(x) : x \in f^{-1}(y)\} = 0 \leq lt(y).$$

Now choose $y \in Y$ such that $lt(y) > 0$ and assume that

$$lt(w) \geq \mathsf{sup}\{lt(x) : x \in f^{-1}(w)\}$$

for every $w \in Y$ such that $lt(w) < lt(y) = \lambda$. Let N be a neighbourhood of y which witnesses its limit type. Since f has discrete fibres, for each point $x \in f^{-1}(y)$ we can find a neighbourhood U_x of x such that $U_x \cap f^{-1}(y) = \{x\}$ and $U_x \subseteq f^{-1}(N)$. Then, for each $z \in U_x \setminus \{x\}$, we have that $f(z) \neq y$ and hence $lt(f(z)) < \lambda$. By induction hypothesis

$$\lambda > lt(f(z)) \geq \mathsf{sup}\{lt(x) : x \in f^{-1}(f(z))\} \geq lt(z).$$

Then, $lt(x) \leq \lambda$ for every $x \in f^{-1}(y)$. This means that

$$\mathsf{sup}\{lt(x) : x \in f^{-1}(y)\} \leq \lambda.$$

3) follows from 1) and 2). $\square$

None of the conditions required for the function f in Theorem 1.3.5 can be dropped, in this sense the result cannot be improved, as we will see in the next examples.

EXAMPLE 1.3.6. *Suppose that $f : \omega \to \omega + 1$ is a bijection. The function f has discrete fibres and is even continuous (because ω is discrete). However, if $A \subseteq \omega$ is infinite and the fibre of the point ω is not a subset of A, then $f(A)$ is not a closed set. Therefore f is not a closed map. On the other hand*

$$1 = lt(\omega) > \textit{sup}\{lt(x) : x \in f^{-1}(\omega)\} = 0.$$

EXAMPLE 1.3.7. *Let $X = \{0, ..., k\}$ and $Y = \omega + 1$. Define the map $f : X \to Y$ as $f(x) = \omega$ for any $x \in X$. It is evident that f is closed and has discrete fibres, but is not onto. Also*

$$1 = lt(\omega) > \textit{sup}\{lt(x) : x \in f^{-1}(\omega)\} = 0.$$

Finally, notice that the function described in the Example 1.3.1 is closed and surjective, but fails having only discrete fibres (since $f^{-1}(k)$ is infinite). Moreover

$$0 = lt(k) < \text{sup}\{lt(x) : x \in f^{-1}(k)\} = 1.$$

COROLLARY 1.3.8. *Let X be a scattered Hausdorff space and let $f : X \to X$ be a continuous closed surjective function with discrete fibres. For every $x \in X$*

$$lt(x) \leq lt(f(x)).$$

COROLLARY 1.3.9. *Suppose that $f : X \to Y$ is a continuous surjective function and that X and Y are compact Hausdorff scattered spaces. Then*

$$lt(y) \leq \textit{max}\{lt(x) : x \in f^{-1}(y)\}$$

for any $y \in Y$.

Proof. Since f is continuous and X and Y are compact Hausdorff spaces, f is a closed map. The result follows from the first part of the proof of Theorem 1.3.5. $\square$

Corollary 1.3.9 tells that, if we want to build an example of a countable compact Hausdorff space X and a continuous onto function $f : X \to X$, then we have to make sure that for any point $x \in X$ there is a point in its fibre with limit type greater than or equal to $lt(x)$. This fact will be considered in the construction of examples in Chapter 3.

Chapter 2

Topologising Sets Simultaneously

Given a set X, a function $f : X \to X$ and a topological property $\mathcal{P}$, it is natural to ask if we can endow X with a topology which satisfies $\mathcal{P}$ and with respect to which f is continuous. The origins of such a question can be traced back to 1952, when Ellis [10] raised the question of whether or not there exists a non-trivial topology on X which makes f continuous. Powderly and Tong [26] gave an affirmative answer to Ellis's question, though their topologies are not Hausdorff in general. In [9], De Groot and De Vries prove that, if X is infinite, there is a non-discrete metric on X under which f is continuous. They also mention that, even if $|X| = \omega$ or $|X| = \mathfrak{c}$, it is not always possible to make X into a compact metric space. In 2006, an elegant characterisation of the functions for which there is a compact Hausdorff topology on their domains which renders them continuous was given by Good et al. in [14].

Taking inspiration from this classic problem, we attempt to answer the following variation of the question: given the sets X, Y and Z, and the functions $f : X \to Y$ and

$g : X \to Z$, under which conditions can we provide X, Y and Z with compact Hausdorff topologies with respect to which f and g are simultaneously continuous? Presently, no previous work on this question is known to us. In Sections 2.1 and 2.2 we exhibit some preliminary results and we examine some necessary conditions for simultaneous topologisation. In Section 2.3 we offer a partial solution to the problem when we give a characterisation of those pairs of functions for which there exist compact Hausdorff topologies on X, Y and Z under which f and g are continuous, X is compact and Hausdorff, and Y and Z are one-point compactifications of discrete spaces. However, in Section 2.4 we show that this solution is far from being exhaustive, because a pair of functions may induce a scattered structure on X as complex as desired, leaving the door open for future research.

2.1 Establishing the Problem

We start by considering the following problem.

Problem 2.1.1. *Given a function $f : X \to Y$, is it possible to find topologies on X and Y which make*

- *f continuous, and*
- *X and Y compact Hausdorff spaces?*

It turns out that the answer to this question is always affirmative, as the next theorem shows.

Theorem 2.1.2. *Let $f : X \to Y$ be a function. There exist compact Hausdorff topologies on X and Y with respect to which f is a continuous function.*

Proof. First suppose that $|f(X)| > 1$. We will define a topology on $f(X)$. Pick a point y_0 of $f(X)$. A neighbourhood basis for this point will be the family of sets of the form $f(X) \setminus V$ where V is a finite subset of $f(X)$ which does not contain y_0. The points of $f(X) \setminus \{y_0\}$ will be isolated. With this topology, $f(X)$ is a compact Hausdorff space.

If $Y \setminus f(X)$ is non-empty, it can be endowed with a compact Hausdorff topology in an analogous way. Considering $f(X)$ and $Y \setminus f(X)$ as clopen sets, Y is turned into a compact Hausdorff space.

The next step is to define a topology on X. Take a point y of $f(X) \setminus \{y_0\}$. For continuity of f, the fibre $f^{-1}(y)$ must be a clopen set of X. Select a point x_y of $f^{-1}(y)$. A neighbourhood basis for this point will be the family of sets of the form $f^{-1}(y) \setminus V$ where V is a finite subset of $f^{-1}(y)$ which does not contain x_y. If x_y is not the only element of $f^{-1}(y)$, the points of $f^{-1}(y) \setminus \{x_y\}$ will be isolated. In this way $f^{-1}(y)$ is turned into a compact Hausdorff space. The fibre $f^{-1}(y_0)$ will be topologised in a slightly different way. Pick a point x_0 of $f^{-1}(y_0)$. A neighbourhood basis for x_0 will be the family of sets of the form

$$\big(f^{-1}(y_0) \setminus V\big) \cup \bigcup_{y \in f(X) \setminus W} f^{-1}(y),$$

where V is a finite subset of $f^{-1}(y_0)$ which does not contain x_0 and W is a finite subset of $f(X)$ which contains y_0. If x_0 is not the only point of $f^{-1}(y_0)$, the points of $f^{-1}(y_0) \setminus \{x_0\}$ will be isolated. We claim that X with this topology is also a compact Hausdorff space. If w and z are two different points of X, $f(w) \neq f(z)$ and $f(w) \neq y_0$, then the open sets $f^{-1}(f(w))$ and $X \setminus f^{-1}(f(w))$ are disjoint and separate w and z. On the other hand, if $f(w) = f(z)$ then w and z are both points of a fibre $f^{-1}(t)$, where $t \in f(X)$. We can assume without loss of generality that w is isolated, thus $\{w\}$ and $X \setminus \{w\}$ are disjoint open sets which separate w and z. Therefore X is Hausdorff. In addition, given any open cover $\mathcal{V}$ of X, there exists $V \in \mathcal{V}$ such that $x_0 \in V$. This implies the existence of finite sets $U_1 \subseteq f^{-1}(y_0)$ and $U_2 \subseteq f(X) \setminus \{y_0\}$ such that $X \setminus V \subseteq \bigcup_{x \in U_1} \{x\} \cup \bigcup_{y \in U_2} f^{-1}(y)$. Since $\bigcup_{x \in U_1} \{x\} \cup \bigcup_{y \in U_2} f^{-1}(y)$ can be covered by finitely many elements of $\mathcal{V}$, $\mathcal{V}$ has a finite subcover. Therefore X is compact.

Now we will prove the continuity of f with respect to the topologies previously defined. Let A be a basic open set of $f(X)$. It can be a singleton containing an isolated point or a set of the form $f(X) \setminus V$ where V is a finite subset of $f(X)$ which does not contain y_0. In the first case, $f^{-1}(A)$ would be a clopen fibre in X. In the second case, $f^{-1}(A)$ would be a set of the form

$$f^{-1}(y_0) \cup \bigcup_{y \in f(X) \setminus W} f^{-1}(y),$$

where W is a finite subset of $f(X)$ which contains y_0, which is open in X. Therefore, f is a continuous function.

When $|f(X)| = 1$ we can simply endow both X and Y with compact Hausdorff topologies in the same way we topologised $f(X)$ when $|f(X)| > 1$. The continuity of f follows from the fact of being a constant function. $\square$

Observation 2.1.3. Note that the topology on $f(X)$ and the topology on $Y \setminus f(X)$ are defined separately in the proof of Theorem 2.1.2. This makes it possible to work with surjective functions without loss of generality.

Observation 2.1.4. If X and $f(X)$ are countable sets, by Theorem 1.2.1 the topologies defined in the proof of Theorem 2.1.2 are in fact metrisable.

Corollary 2.1.5. *Let $f : X \to Y$ be a function. If X and Y are countable sets, then there exist compact metric topologies on X and Y with respect to which f is continuous.* $\square$

The simplicity of the solution to Problem 2.1.1 encourages questions in the same vein though of more complicated nature. One of them is the following problem, which we will attempt to solve throughout this chapter.

Problem 2.1.6. *Given two functions $f : X \to Y$ and $g : X \to Z$, do there exist topologies on X, Y and Z which make*

- *f and g continuous, and*
- *X, Y and Z compact Hausdorff spaces?*

2.2 Necessary Conditions for Simultaneous Topologisation

At first glance, there is no obvious way to approach Problem 2.1.6. Therefore, we will start by analysing some necessary conditions for the existence of the desired topologies.

Notation. *Throughout this chapter, given the functions $f : X \to Y$ and $g : X \to Z$, D_y will denote the set $g(f^{-1}(y))$ and C_z will denote the set $f(g^{-1}(z))$, for every $y \in Y$ and $z \in Z$.*

Proposition 2.2.1. *Let X, Y and Z be compact Hausdorff spaces, and let $f : X \to Y$ and $g : X \to Z$ be continuous functions.*

1. *For every $y \in Y$ and $z \in Z$, the sets $D_y = g(f^{-1}(y))$ and $C_z = f(g^{-1}(z))$ are compact.*
2. *If $\{y_n\}_{n\geq 1} \subseteq Y$ is a sequence of points such that $\{D_{y_n} : n \geq 1\}$ has the finite intersection property, then $\bigcap\limits_{n\geq 1} D_{y_n} \neq \emptyset$.*

Proof. 1) This is because D_y and C_z are continuous images of $f^{-1}(y)$ and $g^{-1}(z)$.

2) $\{D_{y_n} : n \geq 1\}$ is a family of closed sets of a compact Hausdorff space. If $\{D_{y_n} : n \geq 1\}$ has the finite intersection property, then its intersection is non-empty.

□

Consider two functions $f : X \to Y$ and $g : X \to Z$. By Proposition 2.2.1, if there exists a sequence of points $\{y_n\}_{n\geq 1} \subseteq Y$ such that $\{D_{y_n} : n \geq 1\}$ has the finite intersection property but $\bigcap\limits_{n\geq 1} D_{y_n} = \emptyset$, then it would be impossible to find compact Hausdorff topologies on X, Y and Z which make f and g continuous.

Is there such a pair of maps? Yes, as the next example shows.

Example 2.2.2. *Let $\mathcal{P}$ denote the set of prime numbers. For every $n \in \mathcal{P}$, let*

$$B_n = \{n^m : m \geq 1\}.$$

Consider the sets

$$X = \bigcup_{n \in \mathcal{P}} B_n,$$
$$Y = \mathcal{P}, \text{ and}$$
$$Z = \{\tfrac{1}{n} : n \geq 2\}$$

Let $f : X \to Y$ be the function which assigns to each $x \in X$ the value n if $x \in B_n$, and let $g : X \to Z$ be the function which maps x to $\frac{1}{n+m-1}$ if $x = n^m$, with $n \in \mathcal{P}$. Then, for every $n \in Y$, $D_n = g(B_n) = \{\frac{1}{j} : j \geq n\}$. It is clear that the family of sets $\{D_n : n \in Y\}$ has the finite intersection property, but $\bigcap_{n \in Y} D_n = \emptyset$.

Example 2.2.2 tell us that we cannot give a general answer to Problem 2.2.1. Henceforth we will direct our efforts to understand in which cases those topologies do exist.

Problem 2.2.3. *Given two functions $f : X \to Y$ and $g : X \to Z$, under which conditions is it possible to find topologies on X, Y and Z which make*

- *f and g continuous functions, and*
- *X, Y and Z compact Hausdorff spaces?*

2.3 Simultaneous Topologisation when Y and Z are One-point Compactifications

In this section, as a strategy to give a solution to Problem 2.2.3 and in order to ease the process, we will

- consider additional hypotheses which will help us to embed X in $Y \times Z$, providing a clearer picture of the question;
- focus on a particular case: when we want to turn Y and Z into one-point compactifications of discrete spaces.

As mentioned in Observation 2.1.3, it can be assumed without loss of generality that $f : X \to Y$ and $g : X \to Z$ are both surjective functions. Besides surjectivity, there is another condition which might help to identify X with a subset of $Y \times Z$, facilitating the visualisation of the sets $f^{-1}(y)$ and $g^{-1}(z)$ for every $y \in Y$ and $z \in Z$.

Notation. *Given $f : X \to Y$ and $g : X \to Z$, the function which assigns to each $x \in X$ the point $(f(x), g(x))$ in $Y \times Z$ will be denoted by $f \triangle g$.*

The following result is well-known and can be found in references like [11].

Proposition 2.3.1. *Let X, Y and Z be compact Hausdorff spaces and let $f : X \to Y$ and $g : X \to Z$ be continuous functions such that*

$$|f^{-1}(y) \cap g^{-1}(z)| \leq 1 \text{ for all } y \in Y \text{ and all } z \in Z.$$

Then the function $f \triangle g : X \to Y \times Z$, which assigns to each $x \in X$ the point $(f(x), g(x))$, is a homeomorphic embedding.

Proof. The continuity of $f \triangle g$ is given by continuity of f and g. If x_1 and x_2 are two points of X such that $(f \triangle g)(x_1) = (f \triangle g)(x_2) = (u, v)$, then x_1 and x_2 belong to $f^{-1}(u) \cap g^{-1}(v)$, which means that $x_1 = x_2$. So, $f \triangle g$ is one-to-one. To complete the proof we will demonstrate that $f \triangle g$ is a closed function (if $f \triangle g$ is closed, then the inverse of $h : X \to (f \triangle g)(X)$, where $h(x) = (f \triangle g)(x)$ for every $x \in X$, is continuous). Let A be a closed subset of X. By hypothesis X is compact and Hausdorff, condition that A inherits. The continuous image of A under $f\triangle g$ is a compact subset of the compact Hausdorff space $Y \times Z$. Hence $f\triangle g$ is closed. $\square$

OBSERVATION 2.3.2. Suppose that $f : X \to Y$ and $g : X \to Z$ are surjective functions such that

$$|f^{-1}(y) \cap g^{-1}(z)| \leq 1 \text{ for every } y \in Y \text{ and for every } z \in Z.$$

We claim that there exist compact Hausdorff topologies on X, Y and Z which make f and g continuous if and only if there exist compact Hausdorff topologies on Y and Z with respect to which $(f\triangle g)(X)$ is a compact subset of the product space $Y \times Z$. Indeed, if there exist compact Hausdorff topologies on X, Y and Z with respect to which f and g are continuous, by Proposition 2.3.1 the subspace $(f\triangle g)(X)$ is homeomorphic to X, hence compact. Now suppose that there exist compact Hausdorff topologies on Y and Z with respect to which $(f\triangle g)(X)$ is a compact subset of $Y \times Z$. Considering $A \subset X$ as open in X if and only if $(f\triangle g)(A)$ is open in $(f\triangle g)(X)$, the bijection $f\triangle g : X \to (f\triangle g)(X)$ will be a homeomorphism, X will be a compact Hausdorff space, and $f = \pi_Y \circ (f\triangle g)$ and $g = \pi_Z \circ (f\triangle g)$ will be continuous functions, where $\pi_Y : Y \times Z \to Y$ and $\pi_Z : Y \times Z \to Z$ are the projection maps.

OBSERVATION 2.3.3. Observe that the additional assumption that the sets of the form $f^{-1}(y) \cap g^{-1}(z)$ are empty or singletons provide us with a nice image of X as a subset of $Y \times Z$. In addition, for every $y \in Y$ the fibre $f^{-1}(y)$ can be identified with the set

$$(f\triangle g)(f^{-1}(y)) = \{(y, g(x)) : f(x) = y,\ x \in X\} \subseteq Y \times Z.$$

Similarly, for every $z \in Z$ the fibre $g^{-1}(z)$ can be identified with

$$(f\triangle g)(g^{-1}(z)) = \{(f(x), z) : g(x) = z,\ x \in X\} \subseteq Y \times Z.$$

Observation 2.3.3 indicates the convenience of working with the hypothesis that

$|f^{-1}(y) \cap g^{-1}(z)| \leq 1$ for all $y \in Y$ and $z \in Z$. Facilitating the visualisation of X and the fibres of f and g as subsets of $Y \times Z$ helps to identify those points which must be limit points when we attempt to topologise Y and Z, as we will see in Theorem 2.3.8. It turns out that we can actually assume this hypothesis without losing generality, as Theorem 2.3.5 will show. First, let us work towards a proof of this result.

Given the functions $f : X \to Y$ and $g : X \to Z$, consider the following relation on X: given $x, x' \in X$ we will say that $x \sim x'$ if and only if there exist $y \in Y$ and $z \in Z$ such that $x, x' \in f^{-1}(y) \cap g^{-1}(z)$. It is evident that $\sim$ is an equivalence relation, and the equivalence classes of $\sim$ are precisely the sets of the form $f^{-1}(y) \cap g^{-1}(z)$ for some $y \in Y$ and $z \in Z$. As it is customary to do, we will denote the equivalence class of a point x by $[x]$.

For every $[x] \in X/\sim$, let

$$\hat{f}([x]) = f(x) \text{ and } \hat{g}([x]) = g(x).$$

Observe that $\hat{f}([x])$ and $\hat{g}([x])$ are well-defined. On the other hand, if $q : X \to X/\sim$ is the natural quotient mapping, i.e., if q maps every element of X to its equivalence class, then

$$\hat{f} \circ q = f \quad \text{and} \quad \hat{g} \circ q = g.$$

Now suppose that X is a topological space. Recall that the *quotient topology* on $X/\sim$ is defined as follows: a set $A \subset X/\sim$ is open in $X/\sim$ if and only if $q^{-1}(A)$ is open in X. Endowing $X/\sim$ with this topology makes the quotient mapping q continuous. Furthermore, a function h from the quotient space $X/\sim$ to a topological space W is continuous if and only if the composition $h \circ q$ is continuous.

LEMMA 2.3.4. *Let $f : X \to Y$ and $g : X \to Z$ be continuous functions. Consider the equivalence relation $\sim$ on X defined by $x \sim x'$ if and only if $x, x' \in f^{-1}(y) \cap g^{-1}(z)$*

for some $y \in Y$ and $z \in Z$. If X, Y and Z are compact Hausdorff spaces, so is the quotient space $X/\sim$.

Proof. Given $y \in Y$ and $z \in Z$ such that $f^{-1}(y) \cap g^{-1}(z) \neq \emptyset$, let $p_{y,z}$ denote $f^{-1}(y) \cap g^{-1}(z)$. Let $p_{y,z}$ and $p_{y',z'}$ be two distinct points of $X/\sim$ (since $\emptyset \notin X/\sim$, whenever $p_{t,w} \in X/\sim$, where $t \in Y$ and $w \in Z$, we have that $p_{t,w} \neq \emptyset$). Without loss of generality we can assume that $y \neq y'$. Since Y is a Hausdorff space, there exist open sets $U, V \subset Y$ such that $y \in U$, $y' \in V$ and $U \cap V = \emptyset$. Then the sets

$$U' = \{p_{t,w} \in X/\sim : t \in U, w \in Z\}$$

$$V' = \{p_{t,w} \in X/\sim : t \in V, w \in Z\}$$

are disjoint subsets of $X/\sim$. In addition, $q^{-1}(U') = f^{-1}(U)$ and $q^{-1}(V') = f^{-1}(V)$. Due to the continuity of f, U' and V' are open sets in $X/\sim$ which separate $p_{y,z}$ and $p_{y',z'}$. Therefore, $X/\sim$ is a Hausdorff space.

The compactness of $X/\sim$ follows from the fact of being the continuous image of a compact space, because $q(X) = X/\sim$. □

Now we are in position to prove the following result.

THEOREM 2.3.5. *Let $f : X \to Y$ and $g : X \to Z$. Consider the equivalence relation $\sim$ on X defined by $x \sim x'$ if and only $x, x' \in f^{-1}(y) \cap g^{-1}(z)$ for some $y \in Y$ and $z \in Z$. There exist compact Hausdorff topologies on X, Y and Z with respect to which f and g are continuous functions if and only if there exist compact Hausdorff topologies on $X/\sim$, Y and Z with respect to which the functions $\hat{f} : X/\sim \to Y$ and $\hat{g} : X/\sim \to Z$, defined by $\hat{f}([x]) = f(x)$ and $\hat{g}([x]) = g(x)$ for each $[x] \in X/\sim$, are continuous.*

Proof. Suppose that (X, τ_X), (Y, τ_Y) and (Z, τ_Z) are compact Hausdorff spaces and that $f : (X, \tau_X) \to (Y, \tau_Y)$ and $g : (X, \tau_X) \to (Z, \tau_Z)$ are continuous functions. By Lemma 2.3.4, the quotient topology τ_q on $X/\sim$ is compact and Hausdorff. Since $\hat{f} \circ q = f$ and $\hat{g} \circ q = g$, the functions $\hat{f} : (X/\sim, \tau_q) \to (Y, \tau_Y)$ and $\hat{g} : (X/\sim, \tau_q) \to (Z, \tau_Z)$

are continuous.

Now assume that $(X/\sim, \tau)$, (Y, τ_Y) and (Z, τ_Z) are compact Hausdorff spaces and that $\hat{f} : (X/\sim, \tau) \to (Y, \tau_Y)$ and $\hat{g} : (X/\sim, \tau) \to (Z, \tau_Z)$ are continuous functions. We will define a compact Hausdorff topology τ_X on X which makes $q : (X, \tau_X) \to (X/\sim, \tau)$, $f : (X, \tau_X) \to (Y, \tau_Y)$ and $g : (X, \tau_X) \to (Z, \tau_Z)$ continuous functions.

For every $P \in X/\sim$, define a compact Hausdorff topology on P (for example, given a point p^* in P, we could consider the discrete topology on $P \setminus \{p^*\}$, and let P be the one-point compactification of $P \setminus \{p^*\}$) and select a point $x_P \in P$. Each of the sets $P \setminus \{x_P\}$ will belong to τ_X. A neighbourhood basis for the point x_P in (X, τ_X) will be the family of sets of the form

$$U \cup \bigcup_{Q \in V \setminus \{P\}} Q$$

where U is an open subset of P which contains x_P and V is an open set of $X/\sim$ which contains P. The space (X, τ_X) is said to be a *resolution* (see [30]). From the Fundamental Theorem of Resolutions ([30], Theorem 3.1.33) we can deduce that (X, τ_X) is compact and Hausdorff. However, in order to be as self-contained as possible, we include our own proof.

We claim that (X, τ_X) is a Hausdorff space. To see this, let x and y be two distinct points of X. If there exists $P \in X/\sim$ such that $x, y \in P$, then there exist disjoint open sets U_1 and U_2 of P such that $x \in U_1$ and $y \in U_2$ because P is a Hausdorff space. Thus $U_1 \setminus \{x_P\}$ and $U_2 \setminus \{x_P\}$ (or, in case $y = x_P$, $U_1 \setminus \{x_P\}$ and $U_2 \cup \bigcup_{Q \neq P} Q$) are disjoint open sets in X which separate x and y. If $[x] \neq [y]$, then there exist disjoint open sets V_1 and V_2 of $X/\sim$ such that $[x] \in V_1$ and $[y] \in V_2$ because $X/\sim$ is a Hausdorff space. This implies that $\bigcup_{Q \in V_1} Q$ and $\bigcup_{Q \in V_2} Q$ are disjoint open sets in X which separate x and y. Therefore, (X, τ_X) is Hausdorff.

Now we will prove that (X, τ_X) is compact. Let $\mathcal{U}$ be an open cover of X. Suppose without loss of generality that the elements of $\mathcal{U}$ are basic open sets. For every $P \in X/\sim$, there exists $U_P \in \mathcal{U}$ such that $x_P \in U_P$. Then, the family $\{q(U_P) : P \in X/\sim\}$ covers $X/\sim$ (notice that if U_P is a basic open neighbourhood of x_P, then U_P is of the form $U \cup \bigcup\limits_{Q \in V \setminus \{P\}} Q$, where U is an open subset of P which contains x_P and V is an open set of $X/\sim$ which contains P, therefore $q(U_P) = V$, which is open in $X/\sim$). Since $X/\sim$ is compact, there exist finitely many $P_1, P_2, ..., P_k \in X/\sim$ such that $q(U_{P_1}), q(U_{P_2}), ..., q(U_{P_k})$ cover $X/\sim$. Then $U_{P_1}, U_{P_2}, ..., U_{P_k}$ cover $X \setminus \bigcup\limits_{i \leq k} P_i$. Since each P_i is compact, $\bigcup\limits_{i \leq k} P_i$ can be covered by finitely many elements of $\mathcal{U}$. Therefore (X, τ_X) is compact.

Finally, notice that the function $q : (X, \tau_X) \to (X/\sim, \tau)$ is continuous because $q^{-1}(V) = \bigcup\limits_{Q \in V} Q$ is open in X for every open set V of $X/\sim$. Hence, the functions

$$\hat{f} \circ q : (X, \tau) \to (Y, \tau_Y)$$

and

$$\hat{g} \circ q : (X, \tau_X) \to (Z, \tau_Z)$$

are continuous, i.e., f and g are continuous. $\square$

Observation 2.3.6. *Theorem 2.3.5 enables us to work with the quotient space $X/\sim$ and once the problem is solved for it then the problem is automatically solved for X as well.*

Now, as it was said at the beginning of the section, we will focus on the following question.

Problem 2.3.7. *Given two surjective functions $f : X \to Y$ and $g : X \to Z$, under which conditions is it possible to find compact Hausdorff topologies on X, Y and Z which make*

- *f and g continuous functions, and*
- *Y and Z one-point compactifications of discrete spaces?*

Suppose that $f : X \to Y$ and $g : X \to Z$ are continuous and surjective, that X is a compact Hausdorff space, that $|f^{-1}(y) \cap g^{-1}(z)| \leq 1$ for all $y \in Y$ and $z \in Z$, and that Y and Z are one-point compactifications of discrete spaces. Due to Proposition 2.3.1 we will make no distinction between X and its image under $f \triangle g$, i.e. a point x in X and the point $(f(x), g(x))$ in $Y \times Z$ will be treated as the same object.

Let y^* and z^* be the points at infinity of Y and Z respectively. There are four types of points in $Y \times Z$ (see Figure 2.1).

1. The point (y^*, z^*), which has as a neighbourhood basis the family of sets of the form $U \times V$, where U is a cofinite set of Y which contains y^*, and V is a cofinite set of Z which contains z^*.

2. Points of the form (y^*, z), with z an isolated point of Z. Every set $U \times \{z\}$, where U is a cofinite set of Y which contains y^*, is a basic neighbourhood of (y^*, z).

3. Points of the form (y, z^*), with y an isolated point of Y. Every set $\{y\} \times V$, where V is a cofinite set of Z which contains z^*, is a basic neighbourhood of (y, z^*).

4. Points of the form (y, z), with y and z isolated points of Y and Z respectively. $\{\{(y, z)\}\}$ is a local base for (y, z) in $Y \times Z$.

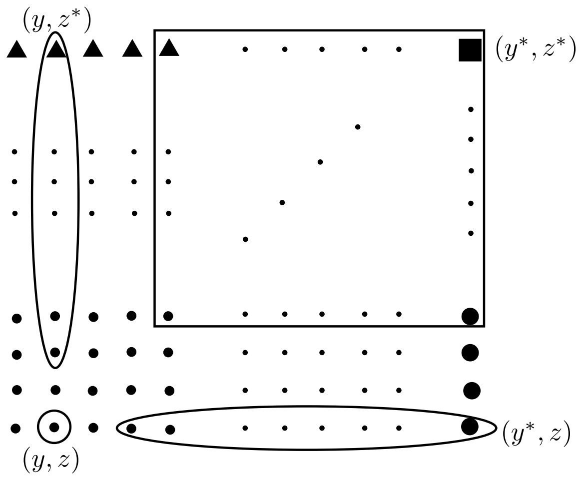

Figure 2.1: Types of points in $Y \times Z$ and their neighbourhoods

NOTATION. *Define* $\tilde{Y} = \{y \in Y : |f^{-1}(y)| \geq \omega\}$ *and* $\tilde{Z} = \{z \in Z : |g^{-1}(z)| \geq \omega\}$.

Observe that for every $y \in Y$ the subspace $\{(y, z) : z \in Z\}$ of $Y \times Z$ is homeomorphic to Z. Since the fibre $f^{-1}(y)$ can be identified with the set

$$M = \{(f(x), g(x)) \in Y \times Z : f(x) = y, x \in X\},$$

M is homeomorphic to a subspace of Z. From now on we will make no distinction between M and $f^{-1}(y)$. If $y \in \tilde{Y}$, then $f^{-1}(y)$ must have a limit point in $Y \times Z$, and this limit point can only be (y, z^*). As $f^{-1}(y)$ is compact, it is a closed set in $Y \times Z$, thus $(y, z^*) \in f^{-1}(y)$. In other words, for every $y \in \tilde{Y}$ there exists a point $x \in f^{-1}(y)$ such that $g(x) = z^*$, i.e., $f^{-1}(y) \cap g^{-1}(z^*) \neq \emptyset$. Analogously, for every point $z \in \tilde{Z}$ we have that $g^{-1}(z) \cap f^{-1}(y^*) \neq \emptyset$.

THEOREM 2.3.8. *Let X be a compact Hausdorff space, let Y and Z be one-point compactifications of discrete spaces, and let $f : X \to Y$ and $g : X \to Z$ be continuous and surjective functions such that*

$$|f^{-1}(y) \cap g^{-1}(z)| \leq 1 \text{ for all } y \in Y \text{ and } z \in Z.$$

If z^* is the point at infinity of Z and

$$\tilde{Y} = \{y \in Y : |f^{-1}(y)| \geq \omega\} \neq \emptyset,$$

then, for every $y \in \tilde{Y}$,

$$f^{-1}(y) \cap g^{-1}(z^*) \neq \emptyset.$$

In other words

$$z^* \in \bigcap_{y \in \tilde{Y}} D_y = \bigcap_{y \in \tilde{Y}} g(f^{-1}(y)).$$

Analogously, if y^* is the point at infinity of Y and

$$\tilde{Z} = \{z \in Z : |g^{-1}(z)| \geq \omega\} \neq \emptyset,$$

then, for every $z \in \tilde{Z}$,

$$g^{-1}(z) \cap f^{-1}(y^*) \neq \emptyset,$$

i.e.,

$$y^* \in \bigcap_{z \in \tilde{Z}} C_z = \bigcap_{z \in \tilde{Z}} f(g^{-1}(z)).$$

$\square$

Theorem 2.3.8 indicates where to find the possible candidates for limit points of Y and Z. They must be in $\bigcap_{z \in \tilde{Z}} C_z$ and $\bigcap_{y \in \tilde{Y}} D_y$. Of course any point of X which is mapped to y^* under f or to z^* under g is a potential limit point of X.

Theorem 2.3.9. *Let $f : X \to Y$ and $g : X \to Z$ be surjective functions such that*

$$|f^{-1}(y) \cap g^{-1}(z)| \leq 1 \text{ for all } y \in Y \text{ and } z \in Z.$$

There are compact Hausdorff topologies on X, Y and Z with respect to which Y and

Z are one-point compactifications of discrete spaces, and f and g are continuous, if and only if one of the following hold.

(a) The sets

$$\tilde{Y} = \{y \in Y : |f^{-1}(y)| \geq \omega\}$$
$$\tilde{Z} = \{z \in Z : |g^{-1}(z)| \geq \omega\}$$

and the sets

$$\bigcap_{y \in \tilde{Y}} D_y = \bigcap_{y \in \tilde{Y}} g(f^{-1}(y))$$
$$\bigcap_{z \in \tilde{Z}} C_z = \bigcap_{z \in \tilde{Z}} f(g^{-1}(z))$$

are non-empty, and either

(i) $g^{-1}(\bigcap_{y\in\tilde{Y}} D_y) \bigcap f^{-1}(\bigcap_{z\in\tilde{Z}} C_z)$ is non-empty; or

(ii) $g^{-1}(\bigcap_{y\in\tilde{Y}} D_y) \bigcap f^{-1}(\bigcap_{z\in\tilde{Z}} C_z)$ is empty, $\tilde{Y}$ and $\tilde{Z}$ are both finite, and

$$X \setminus \big(\bigcup_{y\in\tilde{Y}} f^{-1}(y) \cup \bigcup_{z\in\tilde{Z}} g^{-1}(z)\big)$$

is finite.

(b) $\tilde{Y}$ is empty, $\tilde{Z}$ is non-empty and finite, and $\bigcap_{z\in\tilde{Z}} C_z$ is non-empty.

(c) $\tilde{Z}$ is empty, $\tilde{Y}$ is non-empty and finite, and $\bigcap_{y\in\tilde{Y}} D_y$ is non-empty.

(d) Both $\tilde{Y}$ and $\tilde{Z}$ are empty.

Proof. Suppose that f and g are surjective and continuous functions with the property that $|f^{-1}(y) \cap g^{-1}(z)| \leq 1$ for all $y \in Y$ and $z \in Z$. Assume also that X, Y and Z

are compact Hausdorff spaces, and that Y and Z are one-point compactifications of discrete spaces.

Let y^* and z^* be the points at infinity of Y and Z respectively.

Case 1. Assume that $\tilde{Y} \neq \emptyset$ and $\tilde{Z} \neq \emptyset$. By Theorem 2.3.8, $y^* \in \bigcap_{z\in\tilde{Z}} C_z$ and $z^* \in \bigcap_{y\in\tilde{y}} D_y$, i.e., $\bigcap_{z\in\tilde{Z}} C_z \neq \emptyset$ and $\bigcap_{y\in\tilde{y}} D_y \neq \emptyset$. We will prove that, if $g^{-1}(\bigcap_{y\in\tilde{Y}} D_y)\bigcap f^{-1}(\bigcap_{z\in\tilde{Z}} C_z)$ is empty, then $\tilde{Y}$, $\tilde{Z}$ and

$$X \setminus \big(\bigcup_{y\in\tilde{Y}} f^{-1}(y) \cup \bigcup_{z\in\tilde{Z}} g^{-1}(z)\big)$$

are finite.

Suppose that $g^{-1}(\bigcap_{y\in\tilde{Y}} D_y)\bigcap f^{-1}(\bigcap_{z\in\tilde{Z}} C_z) = \emptyset$. Then there is no $x \in X$ such that $f(x) = y^*$ and $g(x) = z^*$, i.e., $f^{-1}(y^*)\cap g^{-1}(z^*) = \emptyset$. Since the fibre $f^{-1}(y)$ of a point y of $\tilde{Y}$ must intersect $g^{-1}(z^*)$, we have that $g^{-1}(z^*)$ is infinite whenever $\tilde{Y}$ is infinite. But, if $g^{-1}(z^*)$ were infinite, then $f^{-1}(y^*) \cap g^{-1}(z^*) \neq \emptyset$. Thus $\tilde{Y}$ and, analogously, $\tilde{Z}$ are finite.

In addition to this, we claim that the set

$$A = X \setminus \big(\bigcup_{y\in\tilde{Y}} f^{-1}(y) \cup \bigcup_{z\in\tilde{Z}} g^{-1}(z)\big)$$

is finite. First, we will prove that A does not have limit points. Let $x \in X$. If $f(x) \neq y^*$ and $g(x) \neq z^*$, then x is isolated, and it cannot be a limit point of A. If $f(x) = y^*$, then $g(x) \neq z^*$, so $g(x)$ is an isolated point of Z, which means $g^{-1}(g(x))$ is an open set of X which contains x. If $g(x) \in \tilde{Z}$ then $g^{-1}(g(x)) \cap A = \emptyset$. If $g(x) \notin \tilde{Z}$, then $g^{-1}(g(x))$ is a finite open set of X, which implies that x is isolated. Analogously, if $g(x) = z^*$, x cannot be a limit point of A . Thus, no point of X is limit point of A. Since X is compact, A must be a finite set.

Case 2. If $\tilde{Y} = \emptyset$ and $\tilde{Z} \neq \emptyset$, by Theorem 2.3.8, $f^{-1}(y^*) \cap g^{-1}(z) \neq \emptyset$ for every $z \in \tilde{Z}$. As $\tilde{Y} = \emptyset$, $f^{-1}(y^*)$ is finite. So $\tilde{Z}$ is finite.

Case 3. If $\tilde{Z} = \emptyset$ and $\tilde{Y} \neq \emptyset$, just as in Case 2, it can be proved that $\tilde{Y}$ is finite.

Now suppose that the functions $f : X \to Y$ and $g : X \to Z$ are surjective and that $|f^{-1}(y) \cap g^{-1}(z)| \leq 1$ for all $y \in Y$ and $z \in Z$. For each of the conditions a) - d) we aim to define compact Hausdorff topologies on Y and Z with respect to which Y and Z be one-point compactifications of discrete spaces and $(f \triangle g)(X)$ be a compact subset of the product space $Y \times Z$. If we succeed, by Observation 2.3.2, X will be endowed with a compact Hausdorff topology with respect to which f and g will be continuous functions.

Case (a). Assume that the sets $\tilde{Y}$, $\tilde{Z}$, $\bigcap_{y \in \tilde{Y}} D_y$, $\bigcap_{z \in \tilde{Z}} C_z$ are non-empty.

Subcase (a.i). Suppose that $g^{-1}(\bigcap_{y \in \tilde{Y}} D_y) \bigcap f^{-1}(\bigcap_{z \in \tilde{Z}} C_z)$ is non-empty. Choose a point $x^* \in g^{-1}(\bigcap_{y \in \tilde{Y}} D_y) \bigcap f^{-1}(\bigcap_{z \in \tilde{Z}} C_z)$. Then, $y^* = f(x^*)$ belongs to $\bigcap_{z \in \tilde{Z}} C_z$, and $z^* = g(x^*)$ to $\bigcap_{y \in \tilde{Y}} D_y$. Define a topology on Y as follows. Consider all the points of $Y \setminus \{y^*\}$ as isolated. A family of neighbourhoods of y^* will be the co-finite sets of Y in which y^* is contained. Equipped with this topology, Y is the one-point compactification of $Y \setminus \{y^*\}$. An analogous topology can be defined on Z taking z^* as the limit point. Once that Y and Z have been provided with a topology, consider in $Y \times Z$ the product topology. $Y \times Z$ and its subspace $(f \triangle g)(X)$ are Hausdorff spaces. The next step is to prove the compactness of $(f \triangle g)(X)$. Let $\mathcal{U}$ be an open cover of $(f \triangle g)(X)$. The point $(y^*, z^*) \in (f \triangle g)(X)$ must be covered by an element of $\mathcal{U}$, say U. For such U there exist finite sets $Y_1 \subset Y$ and $Z_1 \subset Z$ such that $(y^*, z^*) \in (Y \setminus Y_1) \times (Z \setminus Z_1)$ and $(Y \setminus Y_1) \times (Z \setminus Z_1) \subseteq U$. This means that U covers $\big((f \triangle g)(X)\big) \cap \big((Y \setminus Y_1) \times (Z \setminus Z_1)\big)$. It remains to see if the rest of $(f \triangle g)(X)$ can be covered by a finite subfamily of $\mathcal{U}$. Let y be a point of Y_1. If $f^{-1}(y) = \{(f(x), g(x)) : f(x) = y,\ x \in X\} \subseteq \mathrm{Im}(f \triangle g)$ is finite, then it can be covered by finitely many elements of $\mathcal{U}$. Assume now that

$f^{-1}(y)$ is infinite, i.e. $y \in \tilde{Y}$. The point $(y, z^*) \in f^{-1}(y) \subseteq \mathrm{Im}(f \triangle g)(X)$ because $z^* \in \bigcap\limits_{y \in \tilde{Y}} D_y$, and it must be covered by $\mathcal{U}$. If $V \in \mathcal{U}$ covers (y, z^*), then it covers all $f^{-1}(y)$ except, perhaps, finitely many points. Hence, the fibre $f^{-1}(y)$ can be covered by a finite subfamily of $\mathcal{U}$. Analogously, it can be proved that $g^{-1}(z)$ can be covered by finitely many elements of $\mathcal{U}$ for every $z \in Z_1$. Therefore, there is a finite subfamily of $\mathcal{U}$ which covers $(f \triangle g)(X)$. Then $(f \triangle g)(X)$ is compact.

Subcase (a.ii). Assume that $g^{-1}(\bigcap\limits_{y \in \tilde{Y}} D_y) \cap f^{-1}(\bigcap\limits_{z \in \tilde{Z}} C_z) = \emptyset$, that $\tilde{Y}$ and $\tilde{Z}$ are both finite, and that

$$X \setminus \big(\bigcup_{y \in \tilde{Y}} f^{-1}(y) \cup \bigcup_{z \in \tilde{Z}} g^{-1}(z)\big)$$

is finite.

Select a point $y^* \in \bigcap\limits_{z \in \tilde{Z}} C_z$ and a point $z^* \in \bigcap\limits_{y \in \tilde{Y}} D_y$. Make Y and Z into one-point compactifications of discrete spaces with y^* and z^* as limit points respectively. The product space $Y \times Z$ and its subspace $(f \triangle g)(X)$ are Hausdorff spaces. We will prove that $(f \triangle g)(X)$ is compact. Take an open cover $\mathcal{U}$ of $(f \triangle g)(X)$. We claim that, for each $y \in \tilde{Y}$, $f^{-1}(y)$ can be covered by a finite subfamily of $\mathcal{U}$. Indeed, for every $y \in \tilde{Y}$ the point (y, z^*) belongs to $f^{-1}(y)$ because $z^* \in \bigcap\limits_{y \in \tilde{Y}} D_y$. Since any open set which covers (y, z^*) covers almost all $f^{-1}(y)$ except, perhaps, finitely many points, then $f^{-1}(y)$ can be covered by a finite subfamily of $\mathcal{U}$. Something similar happens with the infinite fibres of g. As $\tilde{Y}$ and $\tilde{Z}$ are finite sets, the points of $(f \triangle g)(X)$ which belong to the infinite fibres of f or g can be covered by finitely many elements of $\mathcal{U}$. Since

$$X \setminus \big(\bigcup_{y \in \tilde{Y}} f^{-1}(y) \cup \bigcup_{z \in \tilde{Z}} g^{-1}(z)\big)$$

is finite, i.e., the subset of $(f \triangle g)(X)$ which is not contained in the union of the infinite fibres of f and g is finite, $(f \triangle g)(X)$ can be covered by a finite subfamily of $\mathcal{U}$. Hence, $(f \triangle g)(X)$ is compact.

$\textsc{Case}$ (B). Assume that $\tilde{Y}$ is empty, that $\tilde{Z}$ is finite and non-empty, and that $\bigcap_{z\in\tilde{Z}} C_z$ is non-empty.

Select a point $y^* \in \bigcap_{z\in\tilde{Z}} C_z$. Choose a point $x \in f^{-1}(y^*)$ and denote $g(x)$ by z^*. Make Y and Z into one-point compactifications of discrete spaces with y^* and z^* as limit points respectively. Then the product space $Y \times Z$ and its subspace $(f\triangle g)(X)$ are Hausdorff spaces. We will prove that $(f\triangle g)(X)$ is compact. Let $\mathcal{U}$ be an open cover of $(f\triangle g)(X)$. If U is an element of $\mathcal{U}$ which contains (y^*, z^*), then there exist finite sets $Y_1 \subset Y$ and $Z_1 \subset Z$ such that $(y^*, z^*) \in (Y\setminus Y_1)\times(Z\setminus Z_1)$ and $(Y\setminus Y_1)\times(Z\setminus Z_1) \subseteq U$. This means that U covers $((f\triangle g)(X)) \cap ((Y \setminus Y_1) \times (Z \setminus Z_1))$. In addition, for any $z \in Z_1$ the fibre $g^{-1}(z)$ can be covered by a finite subfamily of $\mathcal{U}$ (if $g^{-1}(z)$ is infinite, then it contains the point (y^*, z), so once we cover this point we cover almost all $g^{-1}(z)$ except, perhaps, a finite subset). On the other hand, for any $y \in Y_1$ the fibre $f^{-1}(y)$ is finite because $\tilde{Y} = \emptyset$. Therefore, $(f\triangle g)(X)$ can be covered by finitely many members of $\mathcal{U}$. Thus, $(f\triangle g)(X)$ is compact.

$\textsc{Case}$ (C). Analogously to Case (B), when $\tilde{Z}$ is empty, $\tilde{Y}$ is finite and nonempty, and $\bigcap_{y\in\tilde{Y}} D_y$ is non-empty, we can endow Y and Z with topologies with respect to which Y and Z are one-point compactifications of discrete spaces and $(f\triangle g)(X)$ is a compact subset of the product space $Y \times Z$.

$\textsc{Case}$ (D). Finally, consider the case when both $\tilde{Y}$ and $\tilde{Z}$ are empty. Select a point $x \in X$ and make Y and Z into one-point compactifications of discrete spaces with $y^* = f(x)$ and $z^* = g(x)$ as limit points respectively. Then, the product space $Y \times Z$ and its subspace $(f\triangle g)(X)$ are Hausdorff spaces. If $\mathcal{U}$ is an open cover of $(f\triangle g)(X)$, there must be an element B of $\mathcal{U}$ which contains (y^*, z^*). B should also contain a neighbourhood of (y^*, z^*) of the form $(Y\setminus Y_1)\times(Z\setminus Z_1)$, where Y_1 and Z_1 are finite subsets of Y and Z respectively. Then B covers $((f\triangle g)(X))\cap((Y\setminus Y_1)\times(Z\setminus Z_1))$. On the other hand, for each $y \in Y_1$ and $z \in Z_1$ the fibres $f^{-1}(y)$ and $g^{-1}(z)$ are finite, so they can be covered by finitely many elements of $\mathcal{U}$. Therefore, $(f\triangle g)(X)$ is compact.

$\square$

We can reformulate Theorem 2.3.9 for countable spaces as follows.

Corollary 2.3.10. *Assume that X, Y and Z are countable sets. Let $f : X \to Y$ and $g : X \to Z$ be surjective functions such that*

$$|f^{-1}(y) \cap g^{-1}(z)| \leq 1 \text{ for all } y \in Y \text{ and } z \in Z.$$

There exist compact metrisable topologies on X, Y and Z with respect to which Y and Z are homeomorphic to $\omega + 1$ with the order topology, and f and g are continuous, if and only if one of the following hold.

(a) The sets

$$\tilde{Y} = \{y \in Y : |f^{-1}(y)| = \omega\}$$
$$\tilde{Z} = \{z \in Z : |g^{-1}(z)| = \omega\}$$

and the sets

$$\bigcap_{y \in \tilde{Y}} D_y = \bigcap_{y \in \tilde{Y}} g(f^{-1}(y))$$
$$\bigcap_{z \in \tilde{Z}} C_z = \bigcap_{z \in \tilde{Z}} f(g^{-1}(z))$$

are non-empty, and either

(i) $g^{-1}(\bigcap_{y \in \tilde{Y}} D_y) \bigcap f^{-1}(\bigcap_{z \in \tilde{Z}} C_z)$ is non-empty; or

(ii) $g^{-1}(\bigcap_{y\in\tilde{Y}} D_y)\bigcap f^{-1}(\bigcap_{z\in\tilde{Z}} C_z)$ *is empty,* $\tilde{Y}$ *and* $\tilde{Z}$ *are both finite, and*

$$X\setminus(\bigcup_{y\in\tilde{Y}} f^{-1}(y)\cup\bigcup_{z\in\tilde{Z}} g^{-1}(z))$$

is finite.

(b) $\tilde{Y}$ *is empty,* $\tilde{Z}$ *is non-empty and finite, and* $\bigcap_{z\in\tilde{Z}} C_z$ *is non-empty.*

(c) $\tilde{Z}$ *is empty,* $\tilde{Y}$ *is non-empty and finite, and* $\bigcap_{y\in\tilde{Y}} D_y$ *is non-empty.*

(d) Both $\tilde{Y}$ *and* $\tilde{Z}$ *are empty.*

Theorem 2.3.9 gives a complete answer to Problem 2.3.7, but it is natural to wonder if this result solves completely Problem 2.2.3 for those pairs of functions with the property that

$$|f^{-1}(y)\cap g^{-1}(z)|\leq 1 \text{ for all } y\in Y \text{ and } z\in Z;$$

in other words, we have the following question.

Problem 2.3.11. *Is there a pair of surjective functions* $f:X\to Y$ *and* $g:X\to Z$*, with*

$$|f^{-1}(y)\cap g^{-1}(z)|\leq 1 \textit{ for all } y\in Y \textit{ and } z\in Z,$$

such that

1) there exist compact Hausdorff topologies on X*,* Y *and* Z *such that* f *and* g *are continuous;*

2) if τ_X*,* τ_Y *and* τ_Z *are compact Hausdorff topologies on* X*,* Y *and* Z *respectively, which make* f *and* g *continuous, then* (Y,τ_Y) *is not the one point compactification of a discrete space?*

Unfortunately, but not unexpectedly, the answer to the previous question is yes. We will provide an example of a pair of functions with the characteristics stated in Problem 2.3.11. But first notice the following facts. It follows from the definition of limit type of a point (Definition 1.1.4) that if X is the one-point compactification of a discrete space, then X has only one point of limit type 1 (the limit point) and the remaining points have limit type 0. Hence, clearly, if a space has a point of limit type greater than or equal to 2, then it cannot be the one-point compactification of a discrete space.

EXAMPLE 2.3.12. Let Y and Z be countably infinite sets. We will show the existence of a countably infinite set $X \subseteq Y \times Z$ such that

1) the functions $f = \pi_Y \restriction_X$ and $g = \pi_Z \restriction_X$ are surjective, i.e., $\pi_Y(X) = Y$ and $\pi_Z(X) = Z$ (where $\pi_Y : Y \times Z \to Y$ and $\pi_Z : Y \times Z \to Z$ are the projection maps);

2) there exist compact Hausdorff topologies on Y and Z with respect to which X is a closed (and hence compact) subset of $Y \times Z$ (recall Observation 2.3.2);

3) if we endow Y and Z with compact Hausdorff topologies with respect to which X is a compact subspace of $Y \times Z$, then Y has a point of limit type 2.

Suppose that

$$Y = \{y^*\} \bigcup \Big(\bigcup_{i \in \omega} F_i \Big),$$

where each F_i is countably infinite, $F_i \cap F_j = \emptyset$ if $i \neq j$, and $y^* \notin \bigcup_{i \in \omega} F_i$. In addition, assume that

$$Z = \{z^*\} \bigcup \{z_i : i \in \omega\},$$

where each $z_i \neq z^*$, and $z_i \neq z_j$ if $i \neq j$. Consider the following subsets of $Y \times Z$.

$$X_i = F_i \times \{z_i\};$$
$$X = \{(y^*, z^*)\} \cup \Big(\bigcup_{i\in\omega} X_i\Big).$$

It is evident that $\pi_Y(X) = Y$ and $\pi_Z(X) = Z$.

Now we will show that there exist compact Hausdorff topologies on Y and Z with respect to which X is a closed subset of $Y \times Z$. First, endow Z with a topology as follows. The points of $Z \setminus \{z^*\}$ will be isolated. A neighbourhood basis for the point z^* will be the family $\{Z \setminus V : V \subset Z, z^* \notin V, |V| < \omega\}$. Clearly, Z with this topology is homeomorphic to $\omega + 1$ with the order topology. Next, define a topology on Y. For each $i \in \omega$ pick a point $y_i \in F_i$. Just as we did with Z, we can topologise F_i in such a way that F_i be homeomorphic to $\omega + 1$ with y_i as limit point. A local basis for the point y^* will be the family of sets of the form $Y \setminus \Big(\bigcup_{i\in V} F_i\Big)$, where V is a finite subset of ω. With this topology, Y is homeomorphic to $\omega^2 + 1$. X is a closed set subset of the product space $Y \times Z$, and hence compact.

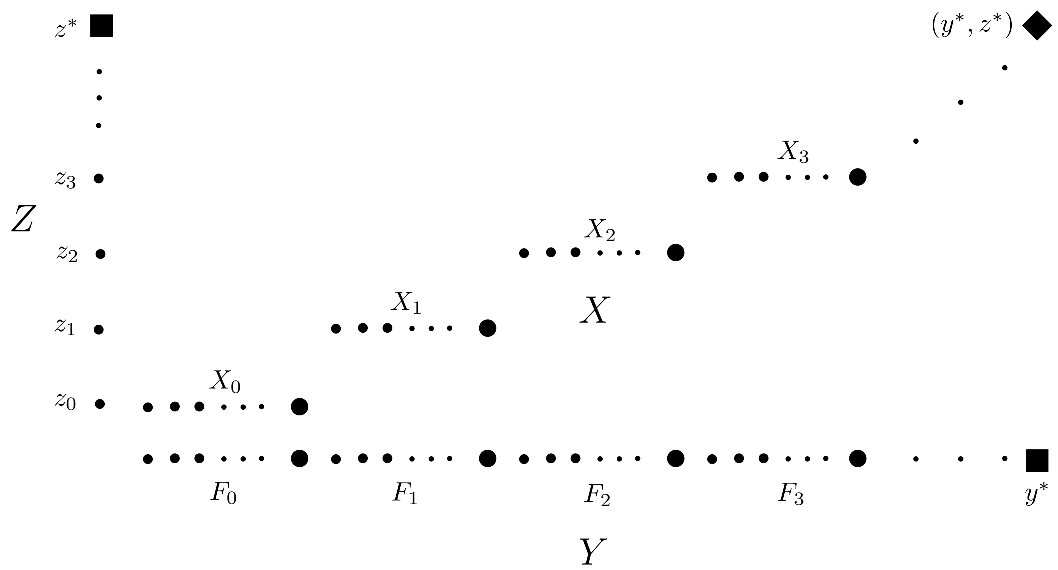

Figure 2.2: Simultaneous topologisation inducing derived degree 2

Finally, suppose we have endowed Y and Z with compact Hausdorff topologies (not necessarily the ones defined in the previous paragraph) with respect to which X is a compact subset of the product space $Y \times Z$. We claim that Y has a point of limit type 2. Indeed, $f = \pi_Y \restriction_X$ and $g = \pi_Z \restriction_X$ are continuous functions. For each $i \in \omega$,

$$g^{-1}(z_i) = \pi_Z^{-1}(z_i) \cap X = X_i,$$

which means that X_i is compact. This implies that $\pi_Y(X_i) = F_i$ is an infinite compact subspace of Y, because it is continuous image of a compact space. Then F_i must have a limit point, say $y_i \in F_i$. As $F_i \cap F_j = \emptyset$ if $i \neq j$, we have that $\{y_i : i \in \omega\}$ is infinite. By compactness of Y, $\{y_i : i \in \omega\}$ must have a limit point, say u. It is obvious that $lt(u) \geq 2$. Therefore, Y contains a point of limit type 2.

2.4 Complexity of the Scattered Structure Induced by a Pair of Functions

Theorem 2.4.1. *Let $f : X \to Y$ be a surjective function with infinitely many infinite fibres. If τ_X and τ_Y are compact Hausdorff topologies on X and Y respectively, with respect to which f is continuous, then (X, τ_X) has a point of limit type at least 2.*

Proof. Let $f : X \to Y$ be a surjective function with the property that there is an infinite set $B \subseteq Y$ such that $|f^{-1}(y)| \geq \omega$ for every $y \in B$. Suppose that X and Y have been endowed with compact Hausdorff topologies with respect to which f is continuous. Let $y \in B$. By the continuity of f and the compactness of X, $f^{-1}(y)$ is closed and compact. Since $f^{-1}(y)$ is infinite, it must contain a limit point, say x_y. Again, by the compactness of X, the infinite set $\{x_y : y \in B\}$ must have a limit point in X, say x^*. Evidently, $lt(x^*) \geq 2$. □

The existence of a pair of functions like the ones described in Example 2.3.12 suggests the idea that a couple of functions $f : X \to Y$ and $g : X \to Z$ may induce a scattered structure as complicated as desired when we attempt to topologise X, Y and Z.

THEOREM 2.4.2. *Let α be an ordinal. There exist sets X_α, Y_α and Z_α and surjective functions $f_\alpha : X_\alpha \to Y_\alpha$ and $g_\alpha : X_\alpha \to Z_\alpha$ such that*

1) $|f_\alpha^{-1}(y) \cap g_\alpha^{-1}(z)| \leq 1$ for every $y \in Y_\alpha$ and $z \in Z_\alpha$;

2) there exist compact Hausdorff topologies on X_α, Y_α and Z_α with respect to which f_α and g_α are continuous functions;

3) if τ_{X_α}, τ_{Y_α} and $\tau_{Z\alpha}$ are compact Hausdorff topologies on X_α, Y_α and Z_α respectively, which make f_α and g_α continuous, then $(X_\alpha, \tau_{X_\alpha})$ has a point of limit type α.

Proof. Transfinite induction will be used to prove this result.

For $\alpha = 0$ consider the sets $X_0 = \{x\}$, $Y_0 = \{y\}$ and $Z_0 = \{z\}$ and the functions $f_0 : X_0 \to Y_0$ and $g_0 : X_0 \to Z_0$ defined by $f_0(x) = y$ and $g_0(x) = z$. It is evident that f_0 and g_0 have the required characteristics.

Suppose that the assertion is true for the ordinal α. We will prove that the statement is also true for $\alpha + 1$.

Let $f_\alpha : X_\alpha \to Y_\alpha$ and $g_\alpha : X_\alpha \to Z_\alpha$ be a pair of surjective functions such that

1) $|f_\alpha^{-1}(y) \cap g_\alpha^{-1}(z)| \leq 1$ for every $y \in Y_\alpha$ and $z \in Z_\alpha$;

2) there exist compact Hausdorff topologies on X_α, Y_α and Z_α with respect to which f_α and g_α are continuous functions;

3) if τ_{X_α}, τ_{Y_α} and τ_{Z_α} are compact Hausdorff topologies on X_α, Y_α and Z_α respectively, which make f_α and g_α continuous, then $(X_\alpha, \tau_{X_\alpha})$ has a point of limit type α.

For every $i \in \omega$, let

$$X_{\alpha_i} = X_\alpha \times \{i\}$$
$$Y_{\alpha_i} = Y_\alpha \times \{i\}$$
$$Z_{\alpha_i} = Z_\alpha \times \{i\}$$

As mentioned in Proposition 2.3.1, since $|f_\alpha^{-1}(y) \cap g_\alpha^{-1}(z)| \leq 1$ for every $y \in Y_\alpha$ and $z \in Z_\alpha$, there is a bijection between X_α and the set $\{(f_\alpha(x), g_\alpha(x)) : x \in X_\alpha\} \subseteq Y_\alpha \times Z_\alpha$. In the same way, X_{α_i} will be identified with

$$\Big\{\big((f_\alpha(x), i), (g_\alpha(x), i)\big) : x \in X_\alpha\Big\} \subseteq Y_{\alpha_i} \times Z_{\alpha_i}.$$

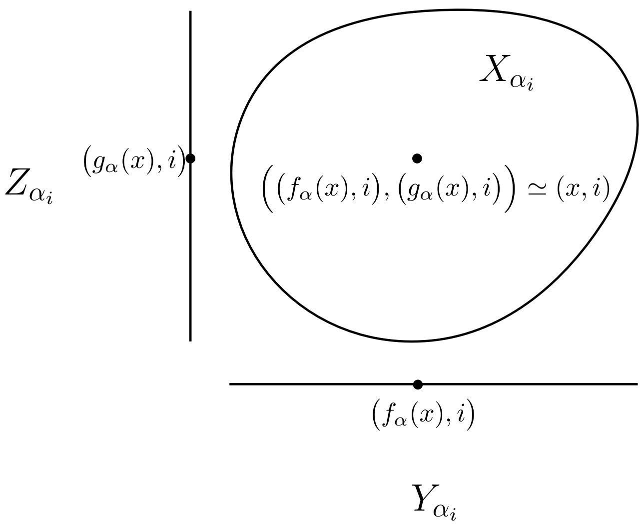

Figure 2.3: Embedding of X_{α_i} in $Y_{\alpha_i} \times Z_{\alpha_i}$

In addition, for every $i \in \omega$, choose distinct points a_{α_i} and b_{α_i} such that

- $a_{\alpha_i} \neq a_{\alpha_j}$ and $a_{\alpha_i} \neq b_{\alpha_j}$, if $i \neq j$;
- $\{a_{\alpha_i}, b_{\alpha_i}\} \cap \Big(\bigcup_{i\in\omega} X_{\alpha_i} \cup \bigcup_{i\in\omega} Y_{\alpha_i} \cup \bigcup_{i\in\omega} Z_{\alpha_i}\Big) = \emptyset$.

Let

$$F_{\alpha_i} = Y_{\alpha_i} \times \{b_{\alpha_i}\} = \{((y,i), b_{\alpha_i}) : y \in Y_\alpha, i \in \omega\}$$
$$G_{\alpha_i} = \{a_{\alpha_i}\} \times Z_{\alpha_i} = \{(a_{\alpha_i}, (z,i)) : z \in Z_\alpha, i \in \omega\}.$$

Choose distinct points $x_{\alpha+1}, y_{\alpha+1}$ and $z_{\alpha+1}$ such that

$$\{x_{\alpha+1}, y_{\alpha+1}, z_{\alpha+1}\} \cap \Big(\bigcup_{i\in\omega} X_{\alpha_i} \cup Y_{\alpha_i} \cup Z_{\alpha_i} \cup F_{\alpha_i} \cup G_{\alpha_i} \cup \{a_{\alpha_i}\} \cup \{b_{\alpha_i}\}\Big) = \emptyset.$$

Now, consider the following sets

$$X_{\alpha+1} = \{x_{\alpha+1}\} \cup \bigcup_{i\in\omega} X_{\alpha_i} \cup \bigcup_{i\in\omega} F_{\alpha_i} \cup \bigcup_{i\in\omega} G_{\alpha_i}$$
$$Y_{\alpha+1} = \{y_{\alpha+1}\} \cup \bigcup_{i\in\omega} Y_{\alpha_i} \cup \bigcup_{i\in\omega} \{a_{\alpha_i}\}$$
$$Z_{\alpha+1} = \{z_{\alpha+1}\} \cup \bigcup_{i\in\omega} Z_{\alpha_i} \cup \bigcup_{i\in\omega} \{b_{\alpha_i}\}$$

The sets previously defined can be observed more clearly in Figure 2.4.

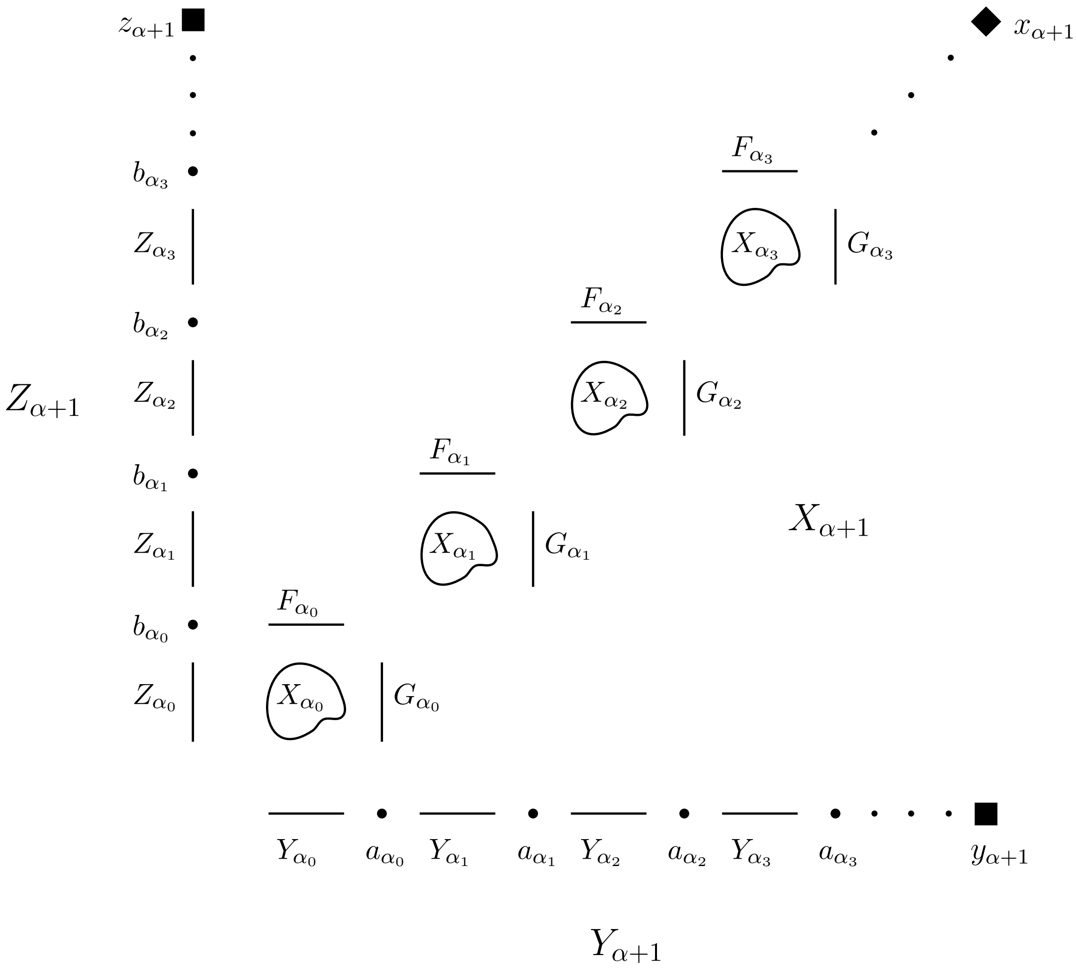

Figure 2.4: $X_{\alpha+1}$, $Y_{\alpha+1}$ and $Z_{\alpha+1}$

Define the functions $f_{\alpha+1} : X_{\alpha+1} \to Y_{\alpha+1}$ and $g_{\alpha+1} : X_{\alpha+1} \to Z_{\alpha+1}$ as follows: given $u \in X_{\alpha+1}$, let

$$f_{\alpha+1}(u) = \begin{cases} y_{\alpha+1} & \text{if } u = x_{\alpha+1} \\ (f_\alpha(x), i) & \text{if } u = (x, i),\ x \in X_\alpha,\ i \in \omega \\ (y, i) & \text{if } u = ((y, i), b_{\alpha_i}),\ y \in Y_\alpha,\ i \in \omega \\ a_{\alpha_i} & \text{if } u = (a_{\alpha_i}, (z, i)),\ z \in Z_\alpha,\ i \in \omega \end{cases}$$

and

$$g_{\alpha+1}(u) = \begin{cases} z_{\alpha+1} & \text{if } u = x_{\alpha+1} \\ (g_\alpha(x), i) & \text{if } u = (x,i),\ x \in X_\alpha,\ i \in \omega \\ (z,i) & \text{if } u = (a_{\alpha_i}, (z,i)),\ z \in Z_\alpha,\ i \in \omega \\ b_{\alpha_i} & \text{if } u = ((y,i), b_{\alpha_i}),\ y \in Y_\alpha,\ i \in \omega \end{cases}$$

The functions $f_{\alpha+1}$ and $g_{\alpha+1}$ can be visualised in Figure 2.4 as the projection maps. In particular, notice that both $f_{\alpha+1}$ and $g_{\alpha+1}$ are surjective.

Note also that, since X_{α_i}, Y_{α_i} and Z_{α_i} can be treated the same as X_α, Y_α and Z_α respectively, $f_{\alpha+1} \restriction_{X_{\alpha_i}} = f_\alpha$ and that $g_{\alpha+1} \restriction_{X_{\alpha_i}} = g_\alpha$ for every $i \in \omega$.

Now, it will be shown that $f_{\alpha+1}$ and $g_{\alpha+1}$ fulfil the required conditions.

Considering that there is a bijection between $X_{\alpha+1}$ and a subset of $Y_{\alpha+1} \times Z_{\alpha+1}$, and considering that $f_{\alpha+1}$ and $g_{\alpha+1}$ are precisely the composition of such bijection with each one of the projection maps, we have that for every $y \in Y_{\alpha+1}$ and every $z \in Z_{\alpha+1}$

$$|f_{\alpha+1}^{-1}(y) \cap g_{\alpha+1}^{-1}(z)| \leq 1.$$

The next step is to prove the existence of compact Hausdorff topologies on $X_{\alpha+1}$, $Y_{\alpha+1}$ and $Z_{\alpha+1}$ which make $f_{\alpha+1}$ and $g_{\alpha+1}$ continuous functions.

Provide X_α, Y_α and Z_α with compact Hausdorff topologies which make f_α and g_α continuous functions (this can be done by inductive hypothesis). Now, for every $i \in \omega$, X_{α_i} will be a clopen set of $X_{\alpha+1}$, Y_{α_i} will be a clopen set of $Y_{\alpha+1}$ and Z_{α_i} will be a clopen set of $Z_{\alpha+1}$. The points of the set $\{a_{\alpha_i} : i \in \omega\}$ will be isolated in $Y_{\alpha+1}$ and, analogously, the points of the set $\{b_{\alpha_i} : i \in \omega\}$ will be isolated in $Z_{\alpha+1}$. For each $i \in \omega$, the set F_{α_i} will be provided with the following topology: a set U will be open in F_{α_i} if and only if the set

$$\pi_{Y_{\alpha+1}}(U) = \{(y,i) \in Y_{\alpha_i} : \text{ there is } u \in U \text{ such that } u = ((y,i), b_{\alpha_i})\}$$

is open in Y_{α_i}. Analogously, for every $i \in \omega$, the set G_{α_i} will be provided with this topology: a set V will be open in G_{α_i} if and only if the set

$$\pi_{Z_{\alpha+1}}(V) = \{(z,i) \in Z_{\alpha_i} : \text{ there is } u \in V \text{ such that } u = (a_{\alpha_i}, (z,i))\}$$

is open in Z_{α_i}. Besides, for each $i \in \omega$, the sets F_{α_i} and G_{α_i} will be clopen sets of $X_{\alpha+1}$. A neighbourhood basis for the point $y_{\alpha+1}$ will be the sets of the form

$$\{y_{\alpha+1}\} \cup \bigcup_{i\in\omega\setminus W_1} Y_{\alpha_i} \cup \bigcup_{i\in\omega\setminus W_2} \{a_{\alpha_i}\}$$

with W_1, W_2 finite subsets of ω. Similarly, a neighbourhood basis for the point $z_{\alpha+1}$ will be the sets of the form

$$\{z_{\alpha+1}\} \cup \bigcup_{i\in\omega\setminus M_1} Z_{\alpha_i} \cup \bigcup_{i\in\omega\setminus M_2} \{b_{\alpha_i}\}$$

with M_1, M_2 finite subsets of ω. Finally, a neighbourhood basis for the point $x_{\alpha+1}$ will be the sets of the form

$$\{x_{\alpha+1}\} \cup \bigcup_{i\in\omega\setminus O_1} X_{\alpha_i} \cup \bigcup_{i\in\omega\setminus O_2} F_{\alpha_i} \cup \bigcup_{i\in\omega\setminus O_3} G_{\alpha_i}$$

with O_1, O_2 and O_3 finite subsets of ω. Equipped with these topologies, $X_{\alpha+1}$, $Y_{\alpha+1}$ and $Z_{\alpha+1}$ are compact Hausdorff spaces.

On the other hand, notice that the topology defined on $X_{\alpha+1}$ coincides with the topology of $X_{\alpha+1}$ as subspace of $Y_{\alpha+1} \times Z_{\alpha+1}$. Since $f_{\alpha+1}$ and $g_{\alpha+1}$ can be identified with the projection maps, the continuity of both functions follows.

The final step is to prove that $(X_{\alpha+1}, \tau_{X_{\alpha+1}})$ has a point of limit type $\alpha+1$ whenever $\tau_{X_{\alpha+1}}$, $\tau_{Y_{\alpha+1}}$ and $\tau_{Z_{\alpha+1}}$ are compact Hausdorff topologies on $X_{\alpha+1}$, $Y_{\alpha+1}$ and $Z_{\alpha+1}$ respectively, which make $f_{\alpha+1}$ and $g_{\alpha+1}$ continuous functions.

Suppose that $X_{\alpha+1}$, $Y_{\alpha+1}$ and $Z_{\alpha+1}$ have been endowed with compact Hausdorff topologies with respect to which $f_{\alpha+1}$ and $g_{\alpha+1}$ are continuous functions. For every $i \in \omega$, consider the set $\{a_{\alpha_i}\}$. Since it is a closed set of $Y_{\alpha+1}$ and $f_{\alpha+1}$ is continuous, $f_{\alpha+1}^{-1}(a_{\alpha_i}) = G_{\alpha_i}$ is a closed subset of $X_{\alpha+1}$, and hence compact. Then $g_{\alpha_i}(G_{\alpha_i}) = Z_{\alpha_i}$ must be a compact set of $Z_{\alpha+1}$. Analogously, it can be shown that $f_{\alpha_i}(F_{\alpha_i}) = Y_{\alpha_i}$ is a compact set of $Z_{\alpha+1}$. Now, since $f_{\alpha+1}^{-1}(Y_{\alpha_i}) = X_{\alpha_i} \cup F_{\alpha_i}$, the set $X_{\alpha_i} \cup F_{\alpha_i}$ is closed in $X_{\alpha+1}$. Similarly, $X_{\alpha_i} \cup G_{\alpha_i}$ is closed in $X_{\alpha+1}$. This implies that X_{α_i} is closed in $X_{\alpha+1}$, and thus compact. By inductive hypothesis, X_{α_i} must have a point of limit type α. Since this happens for every $i \in \omega$, $X_{\alpha+1}$ must have a point of limit type $\alpha+1$.

Finally, let α be a limit ordinal. Assume that the assertion is true for every $\beta < \alpha$. Select a cofinal subset $\mathcal{B} \subseteq \alpha$. For every $\beta \in \mathcal{B}$ there are sets X_β, Y_β and Z_β, and surjective functions $f_\beta : X_\beta \to Y_\beta$ and $g_\beta : X_\beta \to Z_\beta$ which fulfil the conditions 1), 2) and 3); in particular, every time we topologise our sets, each X_β has a point of limit type β. Using the same method that was used for successor ordinals, it is possible to construct sets X_α, Y_α and Z_α in terms of $\{X_\beta\}_{\beta\in\mathcal{B}}$, $\{Y_\beta\}_{\beta\in\mathcal{B}}$, and $\{Z_\beta\}_{\beta\in\mathcal{B}}$, and to construct surjective functions $f_\alpha : X_\alpha \to Y_\alpha$ and $g_\alpha : X_\alpha \to Z_\alpha$ in terms of $\{f_\beta\}_{\beta\in\mathcal{B}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$ with all the required characteristics; in particular, every time we topologise our sets, X_α has a point x_α which is limit of points $\{x_\beta\}_{\beta\in\mathcal{B}}$, where $x_\beta \in X_\beta$ and $lt(x_\beta) = \beta$ for every $\beta \in \mathcal{B}$, and therefore $lt(x_\alpha) = \alpha$ (Note that if a point w_α is a limit of points $\{w_\beta\}_{\beta\in\mathcal{B}}$, where $lt(w_\beta) = \beta$, then $lt(w_\alpha) \geq \alpha$. Furthermore, if w_α has a neighbourhood containing no points of limit type α, $lt(w_\alpha) = \alpha$).

$\square$

Chapter 3

Countable Dynamics

A discrete dynamical system is a pair (X, f) where X is a topological space and $f : X \to X$ is a continuous function. (X, f) is called a countable compact dynamical system if X is compact and countable. The purpose of this chapter is to extend the body of existing results on this particular kind of system.

3.1 A Brief Overview of Existing Results on Countable Dynamics

A dynamical system (X, f) such that X is a countable compact Hausdorff space is called a *countable compact system (or simply countable system)*. Despite being systems that appear in many situations, e.g. as ω-limit sets of unimodal maps (see [15]) or, more generally, as invariant subsets of interval maps (see Theorem 3.3.2), the literature on countable compact systems is limited. In this section we provide an account of existing results on them. In order to be concise we do not define the concepts involved in these results, but these notions can be found in references like [2] or [3].

The topological entropy of a countable dynamical system is zero (a proof can be consulted in [8]). A study on distributional chaos in countable compact systems can be found in Bobok's paper [4], where he proves among other results that the distributional chaos for countable systems (X, f), where X has at most one point of limit type 2, is zero. Another chaos-related notion is the property of being scrambled. In [17], Huang and Ye construct a beautiful and complex completely scrambled homeomorphism on a countable compact space of derived degree ω. They also prove that a countable compact space X admits a completely scrambled homeomorphism if and only if X has a unique top limit point. On the other hand, Kato and Park in their paper [18] give a characterisation of countable compact spaces which admit expansive homeomorphisms. They prove that a countable compact space admits an expansive homeomorphism if and only if its derived degree is not a limit ordinal. They also prove that if X is a countable compact space with a unique top limit point then there is no expansive homeomorphism on X with the shadowing property.

It is worth mentioning that common in all these references is the use of the scattered structure of the space X for the development of all kinds of results, something that we will do as well. In particular we want to emphasise the difficulty of dealing with points whose limit type is a limit ordinal. We experienced this situation in the proof of Theorem 3.4.11.

3.2 Preliminaries

In this section we will review fundamental notions which will be necessary for the study of countable dynamical systems.

Definition 3.2.1. A *(discrete) dynamical system* is a pair (X, f) where X is a topological space and $f : X \to X$ is a continuous function.

The identity function Id_X will also be denoted by f^0. For every $n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$. If f is invertible, then $f^{-n} = (f^{-1})^n$ for each $n \in \mathbb{N}$.

Once a dynamical system has been defined, a notion that arises naturally is the concept of orbit.

DEFINITION 3.2.2. Let (X, f) be a dynamical system.

1) The *forward orbit* of a point $x \in X$ is the sequence

$$\mathcal{O}_+(x) = (f^n(x))_{n \geq 0}.$$

If f is invertible, the *backward orbit* of x is the sequence

$$\mathcal{O}_-(x) = (f^{-n}(x))_{n \geq 0}.$$

2) An *orbit sequence* is either a sequence $(x_k)_{k \in \mathbb{Z}}$ or a sequence $(x_k)_{k \geq n}$ for some $n \in \mathbb{Z}$ such that $f(x_k) = x_{k+1}$ for all k.

3) Let $x \in X$. The set

$$\mathcal{O}(x) = \{y \in X : \text{ there exists } k \geq 0 \text{ such that } f^k(y) = x \text{ or } f^k(x) = y\}$$

is called the *full orbit through* x.

4) The relation $\sim$ on X, defined by $x \sim y$ if and only if there exist $m, n \in \mathbb{N}$ such that $f^m(x) = f^n(x)$, is an equivalence relation. The equivalence classes of $\sim$ are called *full orbits* of f.

5) Let $\mathcal{A}$ be a full orbit of f and let $n \in \mathbb{N}$. $\mathcal{A}$ is said to be an n*-cycle* if there exist distinct points $x_0, x_1, ..., x_{n-1}$ in $\mathcal{A}$ such that $f(x_i) = x_{i+1}$ for all i, where i is taken modulo n.

6) A full orbit $\mathcal{A}$ of f is called a $\mathbb{Z}$*-orbit* if there exists an orbit sequence $(x_k)_{k \in \mathbb{Z}}$ such that $x_k \in \mathcal{A}$ for every k and $x_k \neq x_j$ whenever $k \neq j$. If the $\mathbb{Z}$-orbit consists only of such an orbit sequence it will be called a *bijective* $\mathbb{Z}$*-orbit*.

The term *orbit* can make reference to any of the concepts above. However, whenever one of them is mentioned, it should be clear from the context which one is being used.

DEFINITION 3.2.3. Let (X, f) be a dynamical system. A point $x \in X$ will be called *pre-periodic* if $\{f^i(x) : i \geq 0\}$ is a finite set.

A point $x \in X$ is said to be a *periodic point* if there exist $n \in \mathbb{N}$ such that $f^n(x) = x$. Furthermore, if $f^i(x) \neq x$ for any $i < n$ it will be said that x has *period* n. The forward orbit of x will be called a *periodic orbit*.

In order to determine when two dynamical systems are the same from the dynamical point of view it is necessary to establish a structure-preserving map between them. Such map will be called a *topological conjugacy*.

DEFINITION 3.2.4. Let (X, f) and (Y, g) be dynamical systems. The functions f and g are said to be *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that

$$h \circ f = g \circ h,$$

i.e. the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes. A homeomorphism h which satisfies such condition is called a *topological conjugacy*.

The following defines an important family of subsets of a dynamical system.

DEFINITION 3.2.5. Let (X, f) be a dynamical system. A set $A \subseteq X$ is said to be *f-invariant* or just *invariant* if $f(A) \subseteq A$.

3.3 Ubiquity of countable dynamical systems in interval maps

Countable dynamical systems arise naturally in many situations. For example, in [15] Good et al. prove that if α is not a limit ordinal then there is a unimodal map with a critical point c such that $\omega(c) = \omega^\alpha n + 1$. If α is a limit ordinal then there is no such example (a related result can be found in Section 3.4.2). In this section we will show that countable compact dynamical systems are present in every interval map.

LEMMA 3.3.1. *Let $f : [0,1] \to [0,1]$ be a continuous function. If f has two distinct fixed points with no fixed point between them, then f has an infinite countable compact invariant set.*

Proof. Let a, b be fixed points of f, $a < b$. Suppose that f has no fixed point in (a, b). By continuity of f,

- either $f(x) > x$ for all $x \in (a, b)$, or $f(x) < x$ for each $x \in (a, b)$;
- $[a, b] \subseteq f([a, b])$.

Assume without loss of generality that $f(x) > x$ for all $x \in (a, b)$. Consider the set $f^{-1}(b)$.

CASE 1. Suppose that $f^{-1}(b) \cap [a, b] = \{b\}$. Since $f(x) > x$ for all $x \in (a, b)$, $[a, b] = f([a, b])$. Choose a point $c \in (a, b)$. Define $x_n = f^n(c)$ for each $n \geq 0$. For $n < 0$, define x_n inductively as follows

$$x_n = \min\{x \in [a, b] : f(x) = x_{n+1}\}.$$

Note that, since $f(x) > x$ for all $x \in (a, b)$, $x_{n+1} > x_n$ for every $n \in \mathbb{Z}$. Therefore, the set $A = \{a, b\} \cup \{x_n : n \in \mathbb{Z}\}$ is infinite, countable and invariant. Furthermore, by continuity of f, any limit point of A must be a fixed point. In addition, notice that the sequence $\{x_{-n}\}_{n\in\mathbb{N}}$ is strictly decreasing and the sequence $\{x_n\}_{n\in\mathbb{N}}$ is strictly increasing. This implies that the only limit points of A are a and b. Then A is closed and hence compact.

CASE 2. Suppose that $|f^{-1}(b) \cap [a, b]| \geq 2$, i.e. suppose there is a point in $[a, b]$ different from b whose image under f is b. For $n > 0$, define x_n inductively as follows

- $x_1 = \min\{x \in [a,b] : f(x) = b\}$;

- $x_n = \min\{x \in [a,b] : f(x) = x_{n-1}\}$, for every $n > 1$.

By similar arguments to those presented in Case 1, $A = \{a,b\} \cup \{x_n : n \in \mathbb{N}\}$ is an infinite, countable, invariant and compact set.

$\square$

Now we are in position to prove the next result.

THEOREM 3.3.2. *Any continuous surjective function $f : [0,1] \to [0,1]$ has an infinite countable compact invariant set.*

Proof. Since f is continuous, it must have a fixed point. There are three possibilities,

(1) for every two distinct fixed points there exists a fixed point between them;

(2) f has two fixed points with no fixed point between them;

(3) f has a unique fixed point.

CASE 1. Suppose that f has more than one fixed point. Moreover, suppose that for every pair of fixed points x and y with $x < y$ there is a fixed point w such that $x < w < y$. In this case, the family of all fixed points of f is infinite. Hence, it is enough to select an infinite convergent sequence of fixed points and its limit point which, by continuity of f, must be fixed as well. Such set will have all the required characteristics.

CASE 2. If f has two fixed points with no fixed point between them, the result is given by Lemma 3.3.1.

CASE 3. Suppose that f has a unique fixed point x. Then, f has at least two periodic points of period two. This means that f^2 has at least two fixed points. By Cases 1 and 2, there is an infinite countable compact f^2-invariant set $A \subseteq [0,1]$. Then $A \cup f(A)$ is an infinite countable compact f-invariant subset of $[0,1]$. $\square$

The following theorem shows that a countable compact dynamical system can be embedded in a system on the interval $[0, 1]$.

THEOREM 3.3.3. *Any countable compact dynamical system is topologically conjugate to a subsystem of a map on the interval* $[0, 1]$.

Proof. Let X be a countable compact Hausdorff space and let $f : X \to X$ be a continuous function. Let $e : X \to [0, 1]$ be an embedding of X into $[0, 1]$. Since X is compact, $e(X)$ is a closed subspace of $[0, 1]$. Besides, the function $e \circ f \circ e^{-1} : e(X) \to e(X)$ is continuous. The Tietze-Urysohn Extension Theorem asserts that every continuous function from a closed subspace A of a normal space Y to $[0, 1]$ or $\mathbb{R}$ is continuously extendable over Y. Then, $e \circ f \circ e^{-1}$ can be continuously extended over $[0, 1]$, say to a function $\tilde{f}$. Therefore, (X, f) is topologically conjugate to $(e(X), \tilde{f} \restriction e(X))$. □

Moreover, we can embed a homeomorphic countable system (i.e. a countable system (X, f) where f is a homeomorphism) in a higher dimension.

First we need the next theorem, which appears in [29] as Theorem 1.8.4.

THEOREM 3.3.4. *Let* $A, B \subseteq \mathbb{R}$ *be compact and let* $f : A \to B$ *be a homeomorphism. Then* f *can be extended to a homeomorphism* $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$.

THEOREM 3.3.5. *Any homeomorphism on a countable compact space is topologically conjugate to a subsystem of a homeomorphism on* $\mathbb{R}^2$.

Proof. Let X be a countable compact Hausdorff space and let $h : X \to X$ be a homeomorphism. Let $e : X \to \mathbb{R}$ be an embedding of X into $\mathbb{R}$. Then, $e(X)$ is a compact subspace of $\mathbb{R}$. The function $e \circ h \circ e^{-1} : e(X) \to e(X)$ is a homeomorphism. By Theorem 3.3.4, $e \circ h \circ e^{-1}$ can be extended over $\mathbb{R}^2$, say to a homeomorphism $\tilde{h}$. Therefore, (X, f) is topologically conjugate to $(e(X), \tilde{h} \restriction e(X))$. □

3.4 Structure of Countable Dynamical Systems

The following theorem characterises the orbit structure of continuous functions on compact spaces.

Theorem 3.4.1 (Good et al., 2006 [14]). *Let $f : X \to X$. There is a compact Hausdorff topology on X with respect to which f is continuous if and only if*

$$f(\bigcap_{m\in\mathbb{N}} f^m(X)) = \bigcap_{m\in\mathbb{N}} f^m(X) \neq \emptyset$$

and either

- *f has, in total, at least continuum many $\mathbb{Z}$-orbits or cycles; or*
- *f has both a $\mathbb{Z}$-orbit and a cycle; or*
- *f has no $\mathbb{Z}$-orbit and either*

  (a) f has n_i-cycle, for each $i \leq k$, with the property that whenever f has an n-cycle, then n is divisible by n_i, for some $i \leq k$; or

  (b) the restriction of f to $\bigcap_{m\in\mathbb{N}} f^m(X)$ is not one-to-one.

Characterising continuity on compact metric spaces in the same way is a hard problem. For countable compact metric spaces it turns out we have a simple and elegant characterisation in Theorem 3.4.3.

We will need the notion of finitely based permutation of periodics points, which is given in Definition 3.4.2.

Note. Given a dynamical system (X, f) and number $n \in \mathbb{N}$, we will say that n *is a period of* (X, f) if there exists a periodic point $x \in X$ which has period n.

Definition 3.4.2. A function $f : X \to X$ is called a *finitely based permutation of periodic points* if

1) every point of X is periodic; and

2) there are numbers $n_1, n_2, ..., n_k$ such that if n is any period of (X, f), n is divisible by n_i for some i.

THEOREM 3.4.3. *Let X be a countable set and let $f : X \to X$ be a surjection. There is a compact metrisable topology on X with respect to which f is continuous if and only if either*

1) f has a periodic point and an infinite full orbit; or

2) f is a finitely based permutation of periodic points.

Proof. Suppose that X is a countable set and that $f : X \to X$ is a surjection. By Theorem 1.2.1, any compact Hausdorff topology on X is metrisable. Additionally, observe that

- since f is a surjective function, $f^m(X) = X$ for all $m \geq 1$, so $f(\bigcap\limits_{m \in \mathbb{N}} f^m(X)) = \bigcap\limits_{m \in \mathbb{N}} f^m(X) \neq \emptyset$; and
- since X is countable, f cannot have continuum many $\mathbb{Z}$-orbits or cycles.

Then, by Theorem 3.4.1, there is a compact metrisable topology on X which makes the surjective function $f : X \to X$ continuous if and only if either

A) f has both a $\mathbb{Z}$-orbit and a cycle; or

B) f has no $\mathbb{Z}$-orbit and either

(a) f has n_i-cycle, for each $i \leq k$, with the property that whenever f has an n-cycle, then n is divisible by n_i, for some $i \leq k$; or

(b) f is not one-to-one.

If f has both a $\mathbb{Z}$-orbit and a cycle, then f has an infinite full orbit and a periodic point. Now, consider the case when f has no $\mathbb{Z}$-orbits. In such case, the only full orbits f can have are n-cycles, because f is surjective. If f is not one-to-one, then f has an infinite n-cycle, so f has an infinite full orbit and a periodic point. If f is one-to-one and fulfils condition $B)$ part $a)$, then all the points of X are periodic and f is a finitely based permutation of periodic points. Therefore, conditions $A) - B)$ imply conditions $1) - 2)$.

To see that conditions $1) - 2)$ imply conditions $A) - B)$ notice the following. If f has a periodic point and an infinite orbit, then f might have either a $\mathbb{Z}$-orbit and an n-cycle, or an infinite n-cycle and no $\mathbb{Z}$-orbit. In the second case f would not be one-to-one. On the other hand, if f is a finitely based permutation of periodic points, then f has no $\mathbb{Z}$-orbits. Thus, $1) - 2)$ imply conditions $A) - B)$.

Therefore, there exists a compact metrisable topology on X which makes the surjective function $f : X \to X$ continuous if and only if either

1) f has a periodic point and an infinite full orbit; or

2) f is a finitely based permutation of periodic points.

$\square$

3.4.1 Limit type structure of orbits

We know that countable compact Hausdorff spaces are scattered (see Theorem 1.2.1) and therefore each point has a well-defined limit type. In countable dynamics, limit type plays an important role.

It turns out that we can say a reasonable amount about the behaviour of limit types under f (see Purisch's Theorem 1.3.5). However, as Example 3.4.4 and Theorem 3.4.5 show, it is possible for limit types of periodic points to behave erratically.

NOTE. From now on, unless otherwise stated, the term n-cycle will make reference to a set $\{x_0, ..., x_{n-1}\}$ such that $f(x_i) = x_{i+1}$ where i is taken modulo n instead of making reference to a full orbit containing such a set.

Let us begin with the case when the orbit is a n-cycle.

Example 3.4.4. We will define a continuous surjective function $f : X \to X$ on a countable compact Hausdorff space X with a periodic point u of period 3 such that

$lt(u) = 0$, $lt(f(u)) = 1$ and $lt(f^2(u)) = 2$. Consider the following homeomorphic copies of the ordinals $\{0\}$, $\omega + 1$ and $\omega^2 + 1$ with the order topology:

$$X_0 = \{(0, x_0)\};$$

$$X_1 = (\omega + 1) \times \{x_1\};$$

$$X_2 = (\omega^2 + 1) \times \{x_2\};$$

$$X_{1,j} = (\omega + 1) \times \{x_{1,j}\},\ j \in \mathbb{N};$$

$$X_{2,j} = (\omega^2 + 1) \times \{x_{2,j}\},\ j \in \mathbb{N};$$

where $x_0, x_1, x_2, x_{1,j}$ and $x_{2,k}$, with $j, k \in \mathbb{N}$, are all distinct points which are not elements of $\omega^2 + 1$. Define

$$X = \Big(\bigcup_{i=0}^{2} X_i\Big) \bigcup \Big(\bigcup_{\substack{j\in\mathbb{N} \\ i\in\{1,2\}}} X_{i,j}\Big) \bigcup \Big(\bigcup_{i=1}^{2} \{y_i\}\Big)$$

where y_1, y_2 are distinct points that are not elements of $\Big(\bigcup_{i=0}^{2} X_i\Big) \bigcup \Big(\bigcup_{\substack{j\in\mathbb{N} \\ i\in\{1,2\}}} X_{i,j}\Big)$. Each X_i and $X_{i,j}$ will be considered clopen. On the other hand, a neighbourhood basis for the point y_i, with $i \in \{1, 2\}$, will be the family

$$\Big\{\{y_i\} \bigcup \bigcup_{j\in\mathbb{N}\setminus F} X_{i,j} : F \subset \mathbb{N}, |F| < \omega\Big\}.$$

With this topology, X is a compact countable Hausdorff space.

Consider now the following mappings.

1) If $i \in \{0, 1\}$, define $f_i : X_i \to X_{i+1}$ as $f_i((x, y)) = (\omega^{i+1}, x_{i+1})$ for every $(x, y) \in X_i$;

2) If $i \in \{1, 2\}$,

   a) define $f_{i,1} : X_{i,1} \to X_i$ as $f_{i,1}((x, x_{i,1})) = (x, x_i)$ for every $x \in \omega^i + 1$;

b) if $j \in \mathbb{N}$, $j > 1$, define the function $f_{i,j} : X_{i,j} \to X_{i,j-1}$ as $f_{i,j}((x, x_{i,j})) = (x, x_{i,j-1})$ for every $x \in \omega^i + 1$.

2) Define $f_2 : X_2 \to X_0$ as $f_2((x, y)) = (0, x_0)$ for every $(x, y) \in X_2$.

3) Finally, define $f : X \to X$ as

$$f(x) = \begin{cases} f_i(x) & x \in X_i,\ i \in \{0, 1, 2\} \\ f_{i,j}(x) & x \in X_{i,j},\ i \in \{1, 2\},\ j \in \mathbb{N} \\ y_i & x = y_i,\ i \in \{1, 2\} \end{cases}$$

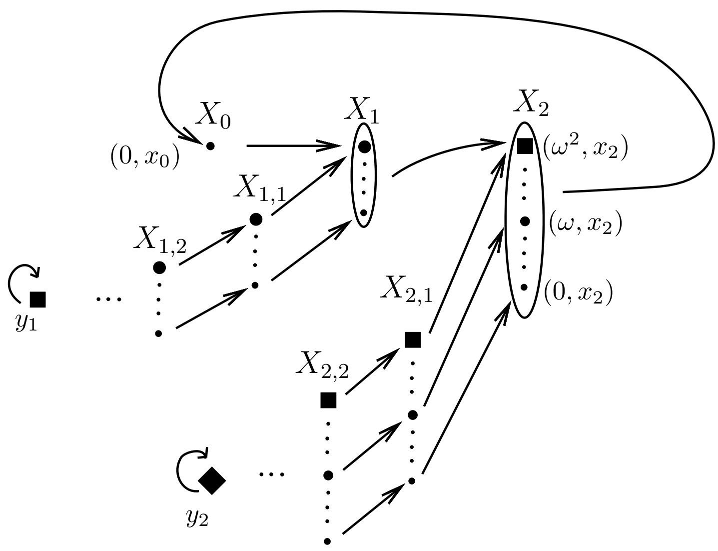

Figure 3.1: Example 3.4.4

The function f is continuous and surjective. Moreover, $u = (0, x_0)$ is a periodic point of period 3 such that $lt(u) = 0$, $lt(f(u)) = 1$ and $lt(f^2(u)) = 2$.

Example 3.4.4 suggests that we can get a similar construction for any finite set of countable limit types.

THEOREM 3.4.5. *Let $n \in \mathbb{N}$ and let $\alpha_0, ..., \alpha_{n-1}$ be countable ordinals. There exist a countable compact Hausdorff space X and a continuous surjective function $f : X \to X$ such that X has a periodic point x of period n such that $lt(f^i(x)) = \alpha_i$ for each i with $0 \leq i \leq n-1$.*

Proof. Given $i \in \{0, ..., n-1\}$, let $X_i = (\omega^{\alpha_i} + 1) \times \{x_i\}$ be a homeomorphic copy of $\omega^{\alpha_i} + 1$ with the order topology. Define $\mathcal{A} = \{i : \alpha_i \neq \alpha_{i+1}, i \text{ mod } n\}$. For every $i \in \mathcal{A}$ and every $j \in \mathbb{N}$, let $X_{i,j} = (\omega^{\alpha_{i+1}} + 1) \times \{x_{i,j}\}$ be a homeomorphic copy of $\omega^{\alpha_{i+1}} + 1$.

Let

$$X = \Big(\bigcup_{i=0}^{n-1} X_i\Big) \cup \Big(\bigcup_{\substack{j \in \mathbb{N} \\ i \in \mathcal{A}}} X_{i,j}\Big) \cup \Big(\bigcup_{i \in \mathcal{A}} \{y_i\}\Big),$$

where the points y_i, with $i \in \mathcal{A}$, are all distinct points which are not elements of $\Big(\bigcup_{i=0}^{n-1} X_i\Big) \cup \Big(\bigcup_{\substack{j \in \mathbb{N} \\ i \in \mathcal{A}}} X_{i,j}\Big)$. Define a topology on X as follows. Each X_i and $X_{i,j}$ will be clopen in X. The family $\Big\{\{y_i\} \cup \bigcup_{j \in \mathbb{N} \setminus F} X_{i,j} : F \subset \mathbb{N}, |F| < \omega\Big\}$ will be a neighbourhood basis for the point y_i, for every $i \in \mathcal{A}$. Clearly, X with this topology is a countable compact Hausdorff space.

Now, let us define some functions.

(1) If $i \notin \mathcal{A}$, i.e. if $\alpha_i = \alpha_{i+1}$, define $f_i : X_i \to X_{i+1}$ as $f_i((x, x_i)) = (x, x_{i+1})$ for any $x \in \omega^{\alpha_i} + 1$.

(2) If $i \in \mathcal{A}$,

- define $f_i : X_i \to X_{i+1}$ as $f_i((x, y)) = (\omega^{\alpha_{i+1}}, x_{i+1})$ for any $(x, y) \in X_i$;
- define $f_{i,1} : X_{i,1} \to X_{i+1}$ as $f_{i,1}((x, x_{i,1})) = (x, x_{i+1})$ for every $x \in \omega^{\alpha_{i+1}} + 1$;
- for every $j > 1$, define $f_{i,j} : X_{i,j} \to X_{i,j-1}$ as $f_{i,j}((x, x_{i,j})) = (x, x_{i,j-1})$ for any $x \in \omega^{\alpha_{i+1}} + 1$.

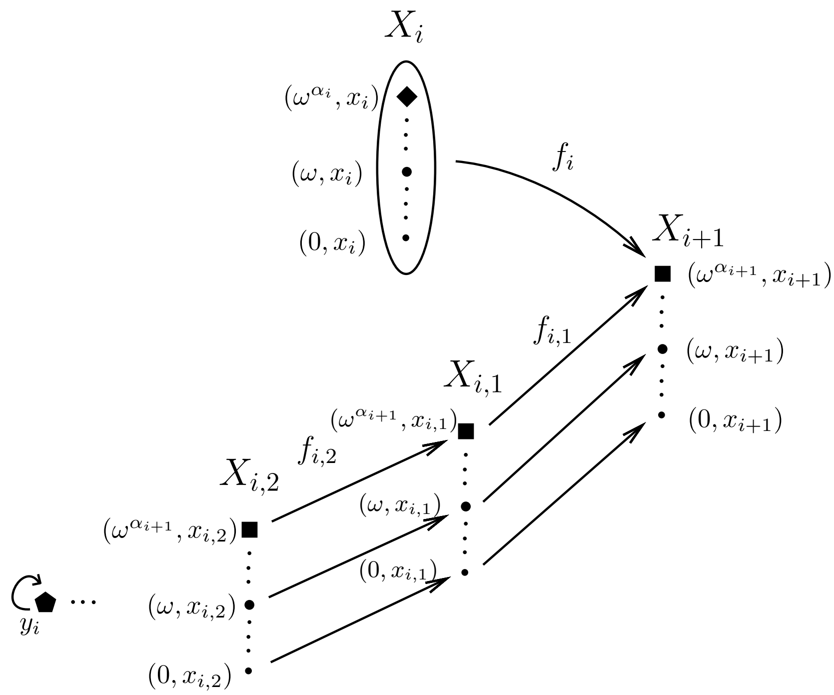

Figure 3.2: Case $i \in \mathcal{A}$

(3) Finally, define $f : X \to X$ as follows

$$f(x) = \begin{cases} f_i(x) & x \in X_i,\ 0 \leq i \leq n-1 \\ f_{i,j}(x) & x \in X_{i,j},\ i \in \mathcal{A},\ j \in \mathbb{N} \\ y_i & x = y_i,\ i \in \mathcal{A} \end{cases}$$

Observe that if $i \in \mathcal{A}$ and U is a neighbourhood of $(\omega^{\alpha_{i+1}}, x_{i+1})$, then, by definition, $f^{-1}(U) = f_{i,1}^{-1}(U) \cup X_i = U' \cup X_i$, where $U' \subseteq X_{i,1}$ is homeomorphic to U (notice that the function $f_{i,1} : X_{i,1} \to X_{i+1}$ is a homeomorphism). This implies continuity of f at (ω^{α_i}, x_i). Continuity at other points of X and the surjectivity of f are evident. The set $\{(\omega^{\alpha_i}, x_i) : i \bmod n\}$ is the cycle we needed. $\square$

In both Example 3.4.4 and Theorem 3.4.5 we can see that the top limit points, i.e. the points with the highest limit type, form cycles among them. This observation lead us to state the following result.

THEOREM 3.4.6. *Let X be countable compact Hausdorff space with derived degree $\alpha+1$, and let $f : X \to X$ be a continuous and surjective function. Then*

1) every top limit point of X, i.e., every $x \in \mathcal{L}^\alpha(X)$, is periodic; and

2) for any $x \in \mathcal{L}^\alpha(X)$, $f(x) \in \mathcal{L}^\alpha(X)$.

Proof. Let X be a countable compact Hausdorff space and let $f : X \to X$ be a continuous surjective function. Suppose that $d(X) = \alpha + 1$. Then, the set $\mathcal{L}^\alpha(X)$ is finite.

Let $x \in \mathcal{L}^\alpha(X)$. Observe that, since f is surjective, for every $n \in \mathbb{N}$ the set $f^{-n}(x)$ is non-empty. By Corollary 1.3.9,

$$\alpha = lt(x) \leq \max\{lt(z) : z \in f^{-1}(x)\}.$$

Since the highest limit type a point in X can have is α, we can build a sequence $(y_i)_{i\geq 0}$ in X such that $y_0 = x$, $y_i = f(y_{i+1})$, and $lt(y_i) = \alpha$ for every $i \geq 0$. The set $\{y_i : i \geq 0\}$ is finite because it is contained in $\mathcal{L}^\alpha(X)$. Then, there exists $k \geq 0$ such that the terms $y_0, y_1, ..., y_k$ are all distinct (indeed there must be $k \geq 0$ such that y_{k+1} is the first term of the sequence which is repeated). Thus, $y_{k+1} = y_j$ for some j with $0 \leq j \leq k-1$ (observe that $y_{k+1} \neq y_k$ because $f(y_{k+1}) = y_k$ and $f(y_k) = y_{k-1}$, and $y_{k-1} \neq y_k$ since $y_0, ..., y_k$ are all distinct). However, if $j > 0$ then $f(y_j) = y_k$ and $f(y_j) = y_{j-1}$, which is not possible because $y_{j-1} \neq y_k$. Thus, $f(y_0) = y_k$, i.e., $f(x) = y_k$. Therefore, x is periodic and its forward orbit $\mathcal{O}_+(x)$ consists entirely of points of limit type α. □

In plain words, under the conditions established by Theorem 3.4.6, the top limit points of a countable compact Hausdorff space form cycles among them.

Now we will see the case of a $\mathbb{Z}$-orbit. How might an orbit of this kind look? A simple example of a system with a $\mathbb{Z}$-orbit is the following.

Example 3.4.7. Consider the set

$$X = \{x_k : k \in \mathbb{Z}\} \cup \{x^*\}.$$

Topologise X in the following way: consider the discrete topology on $X \setminus \{x^*\}$ and let X be the one-point compactification of $X \setminus \{x^*\}$. Define $f : X \to X$ as

$$f(x) = \begin{cases} x_{k+1} & x = x_k,\ k \in \mathbb{Z} \\ x^* & x = x^* \end{cases}$$

The function f is continuous and onto. In addition, $\{x_k : k \in \mathbb{Z}\}$ is a $\mathbb{Z}$-orbit. Notice that all its points have the same limit type.

Among the several questions regarding limit types of the points of a $\mathbb{Z}$-orbit we are interested in this one: is it possible for a $\mathbb{Z}$-orbit to have no maximal limit type? The answer to this question is affirmative, as we will see in Example 3.4.9. But first we will prove an auxiliary proposition.

Proposition 3.4.8. *Let $i \in \omega \setminus \{0\}$. There exists a continuous function $f_i : \omega^{i+1} + 1 \to \omega^i + 1$ such that f_i is a quotient mapping and the image of the point ω^{i+1} under f_i is the point ω^i.*

Proof. We will prove the statement by induction. If $i = 1$, consider the spaces $\omega^2 + 1$ and $\omega + 1$ with the order topology, and the function $f_1 : \omega^2 + 1 \to \omega + 1$ defined as

$$f_1(x) = \begin{cases} 0 & x \in [0, \omega] \\ i & x \in (\omega \cdot i, \omega \cdot (i+1)],\ i \geq 1 \\ \omega & x = \omega^2 \end{cases}$$

It is evident that f_1 is a quotient mapping.

Now, let $i \in \omega$ such that $i > 1$. Assume the existence of a continuous quotient mapping $f_k : \omega^{k+1} + 1 \to \omega^k + 1$ for every $k < i$.

Let the interval $[0, \omega^i] \subset \omega^{i+1} + 1$ be denoted by $X_{i+1,1}$, and, for $n > 1$, let the interval $(\omega^i \cdot (n-1), \omega^i \cdot n]$ be denoted by $X_{i+1,n}$. Similarly, let $X_{i,1}$ denote $[0, \omega^{i-1}]$, and let $X_{i,n}$ denote $(\omega^{i-1} \cdot (n-1), \omega^{i-1} \cdot n]$ for each $n > 1$. Then

- $\omega^{i+1} + 1 = \{\omega^{i+1}\} \cup (\bigcup_{n \geq 1} X_{i+1,n})$;
- $\omega^i + 1 = \{\omega^i\} \cup (\bigcup_{n \geq 1} X_{i,n})$;
- for every $n \geq 1$, $X_{i+1,n}$ and $X_{i,n}$ are homeomorphic copies of $\omega^i + 1$ and $\omega^{i-1} + 1$, respectively (it is important to notice this because we will use the inductive hypothesis in these spaces);
- for every $n \geq 1$, $X_{i+1,n}$ and $X_{i,n}$ are clopen subspaces of $\omega^{i+1} + 1$ and $\omega^i + 1$, respectively.

Therefore, by inductive hypothesis, there exists a quotient map $f_{i,n} : X_{i+1,n} \to X_{i,n}$ for every $n \geq 1$. Hence, if we define the function $f_i : \omega^{i+1} + 1 \to \omega^i + 1$ as

$$f_i(x) = \begin{cases} f_{i,n}(x) & x \in X_{i+1,n},\ n \geq 1 \\ \omega^i & x = \omega^{i+1} \end{cases}$$

it turns out that f_i is a quotient mapping. We claim that f_i is also continuous. Let $\mathcal{V}_n$ be a basis of the space $X_{i,n}$ for every $n \in \mathbb{N}$. Since each $X_{i,n}$ is a clopen subspace of $\omega^i + 1$, the family

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n \cup \Big\{ \bigcup_{m \in W} X_{i,m} \cup \{\omega^i\} : W \text{ is a cofinite subset of } \mathbb{N} \Big\}$$

is a basis of $\omega^i + 1$. Let $B \in \mathcal{B}$. If $B \in \mathcal{V}_n$ for some $n \in \mathbb{N}$, then $f_i^{-1}(B) = f_{i,n}^{-1}(B)$ which is an open subset of $X_{i+1,n}$ because $f_{i,n}$ is continuous. Since $X_{i+1,n}$ is open in

$\omega^{i+1}+1$, $f_i^{-1}(B)$ is open in $\omega^{i+1}+1$. If $B = \bigcup_{m\in W} X_{i,m} \cup \{\omega^i\}$ for some cofinite subset W of $\mathbb{N}$, then $f_i^{-1}(B) = \bigcup_{m\in W} X_{i+1,m} \cup \{\omega^{i+1}\}$ which is open in $\omega^{i+1}+1$. Therefore f_i is continuous. $\square$

Proposition 3.4.8 will help us to build the following example.

EXAMPLE 3.4.9. *We will construct a countable compact Hausdorff space X and a continuous surjective function $f : X \to X$ such that (X, f) has a $\mathbb{Z}$-orbit with no maximal limit type.*

Consider the following homeomorphic copy of $\omega + 1$ with the order topology

$$X_0 = (\omega + 1) \times \{x_0\}.$$

In addition, for every $i \in \omega \setminus \{0\}$, consider the homeomorphic copy of $\omega^i + 1$ with the order topology

$$X_i = (\omega^i + 1) \times \{x_i\},\ i \in \omega \setminus \{0\}.$$

Let

$$X = \{y^*\} \cup (\bigcup_{i\in\omega} X_i),$$

where $y^* \notin \underset{i\in\omega}{X_i}$. Define a topology on X as follows. Each X_i will be clopen in X. A neighbourhood basis for y^* will be the family

$$\Big\{ \bigcup_{\substack{i\in\omega\setminus F \\ i\geq 1}} X_i \cup \{y^*\} : F \subset \omega,\ F \text{ finite} \Big\}.$$

With this topology, X is a countable compact Hausdorff space. In addition, since X_i has a point of limit type i (the point (ω^i, x_i)) for every $i \in \omega$, $lt(y^*) = \omega$, so X has derived degree $\omega + 1$.

By Proposition 3.4.8, for each $i \geq 1$, there exists a continuous quotient map $f_i : X_{i+1} \to X_i$ such that

$$f_i\left((\omega^{i+1}, x_{i+1})\right) = (\omega^i, x_i).$$

Define the function $f : X \to X$ as follows.

$$f(x) = \begin{cases} (\omega, x_0) & x = (\omega, x_0) \\ (i+1, x_0) & x = (i, x_0),\ i \in \omega \\ (0, x_0) & x \in X_1 \\ f_{i-1}(x) & x \in X_i,\ i > 1 \\ y^* & x = y^* \end{cases}$$

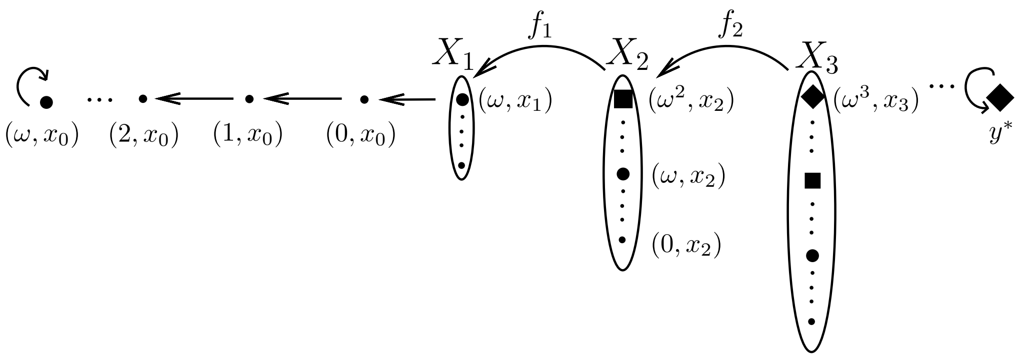

Figure 3.3: $\mathbb{Z}$-orbit with no maximal limit type

We claim that f is continuous. The family

$$\mathcal{U}_0 = \left\{X_0 \setminus F : F \subseteq X_0, F \text{ finite}, (\omega, x_0) \notin F, (0, x_0) \in F\right\} \cup \left\{\{(i, x_0)\} : i \neq \omega\right\}$$

is a basis of X_0. Let $\mathcal{U}_n$ be a basis for X_n for every $n \geq 1$. Since each X_n is clopen in X, the family

$$\mathcal{B} = \bigcup_{n \geq 0} \mathcal{U}_n \cup \left\{\bigcup_{i \in M} X_i \cup \{y^*\} : M \subseteq \omega,\ M \text{ cofinite}\right\}$$

is a basis of X. Let $B \in \mathcal{B}$. If $B \in \mathcal{U}_0$, then $f^{-1}(B)$ can be either a singleton containing an isolated point, or a set of the form $X_0 \setminus F$, where $F \subset X_0$, F is finite and $(\omega, x_0) \notin F$, or X_1 (when $B = \{(0, x_0)\}$). Then $f^{-1}(B)$ is open in X. If $B \in \mathcal{U}_n$ for some $n \geq 1$, then $f^{-1}(B) = f_n^{-1}(B)$, which is an open subset of X_{n+1} because f_n is continuous, so $f^{-1}(B)$ is open in X because X_{n+1} is open in X. If $B = \bigcup_{i \in M} X_i \cup \{y^*\}$ for some $M \subseteq \omega$, M cofinite, then $f^{-1}(B)$ is also a set of the form $\bigcup_{i \in M'} X_i \cup \{y^*\}$ for some $M' \subseteq \omega$, M' cofinite, which is open in X. Therefore f is continuous.

Clearly f is surjective. Besides, the set

$$\{(i, x_0) : i \in \omega\} \cup \{(\omega^i, x_i) : i \geq 1\}$$

is a $\mathbb{Z}$-orbit with no maximal limit type.

3.4.2 Repelling Fixed Points in Countable Systems

Consider a dynamical system (X, f), where (X, d) is a metric space. A point $x \in X$ is said to be *repelling* if there exists $\delta > 0$ such that, for every $y \in B_\delta(x) \setminus \{x\}$ there is $n \in \mathbb{N}$ such that $d(f^n(y), x) \geq \delta$. Repelling and fixed points are of interest in dynamics. It turns out that in countable compact systems such points have a particular structure. In this section we will prove that a repelling fixed point of a countable compact system where the function is finite-to-one cannot have a limit ordinal as limit type.

Observation 3.4.10. *Let X be a metrisable space. Note that if the metric d generates the topology on X, then saying that $x \in (X, d)$ is a repelling point of (X, f) is equivalent to saying that there is an open neighbourhood U of x such that for every $y \in U \setminus \{x\}$ there is $n \in \mathbb{N}$ for which $f^n(y) \notin U$.*

Recall the following definition (see Definition 1.1.7). Let X be a countable scattered metric space. Suppose that $lt(x_0) > 0$. A sequence $(y_n)_{n \in \mathbb{N}}$, where $y_n \neq y_m$ if $n \neq m$, witnesses the limit type of x_0 if $y_n \to x_0$ and

1) $lt(y_n) \to lt(x_0)$, if $lt(x_0)$ is a limit ordinal; or

2) $lt(y_n) = \alpha$ for each $n \in \mathbb{N}$, if $lt(x_0) = \alpha + 1$.

THEOREM 3.4.11. *Let X be a compact countable Hausdorff topological space, with compatible metric d, and let $f : X \to X$ be a continuous finite-to-one surjective function. Suppose that x_0 is a repelling fixed point of f. Then $lt(x_0)$ is not a limit ordinal. Moreover, if $lt(x_0) = \alpha + 1$ for some ordinal α, there exists $\delta > 0$ such that, for any $\eta \leq \delta$,*

1) if $y, f(y) \in B_\eta(x_0)$ and $lt(y) = \alpha$, then $lt(f(y)) = \alpha$;

2) $\{z \in B_\eta(x_0) : lt(z) = \alpha,\ f(z) \notin B_\eta(x_0)\}$ is finite.

Proof. Since x_0 is a repelling fixed point of f, there exists $\delta > 0$ such that for every $\eta \leq \delta$, if $y \in B_\eta(x_0) \setminus \{x_0\}$, then there is $n \in \mathbb{N}$ such that $d(f^n(y), x_0) \geq \eta$. Without loss of generality, we can assume that $\overline{B_\delta(x_0)}$ witnesses the limit type of x_0.

Since f has finite fibres, by Theorem 1.3.5, for every $x \in X$ it is true that

$$lt(f(x)) = \max\{lt(w) : f(w) = f(x)\} \geq lt(x).$$

Furthermore, for any $n \in \mathbb{N}$, $lt(f^n(x)) \geq lt(x)$.

Suppose that $lt(x_0) > 0$. Choose a sequence $(y_n)_{n\in\mathbb{N}}$ in $B_\delta(x_0)$, $y_n \neq y_m$ if $n \neq m$, witnessing the limit type of x_0.

For each y_n there exists $k_n \in \mathbb{N}$ such that

$$d(f^{k_n}(y_n), x_0) < \delta \text{ and } d(f^{k_n+1}(y_n), x_0) \geq \delta.$$

Let $\mathcal{A} = \{f^{k_n}(y_n) : n \in \mathbb{N}\}$. If $\mathcal{A}$ were infinite, then, considering a subsequence if necessary, we can assume that there would be $z \in X$ such that $f^{k_n}(y_n) \to z$. Since

$d(f^{k_n}(y_n), x_0) < \delta$ for all $n \in \mathbb{N}$, $d(z, x_0) \leq \delta$. On the other hand, since f is continuous, $f^{k_n+1}(y_n) \to f(z)$, which will imply that $d(f(z), x_0) \geq \delta$. Therefore, $z \neq x_0$. Besides, since $lt(f^{k_n}(y_n)) \geq lt(y_n)$ for every n, $lt(z) \geq lt(x_0)$. But this contradicts the fact that $\overline{B_\delta(x_0)}$ witnesses the limit type of x_0. Hence, $\mathcal{A}$ must be finite and, therefore, $lt(x_0)$ cannot be a limit ordinal.

Assume now that $lt(x_0) = \alpha + 1$, for some ordinal α.

Let $\eta \leq \delta$. If $y, f(y) \in B_\eta(x_0)$ and $lt(y) = \alpha$, then, as it was said before, $lt(f(y)) \geq lt(y) = \alpha$. Since $\overline{B_\delta(x_0)}$ witnesses the limit type of x_0, $lt(f(y)) < lt(x_0) = \alpha + 1$. Therefore, $lt(f(y)) = \alpha$. Now, consider the following set

$$\mathcal{B} = \{z \in B_\eta(x_0) : lt(z) = \alpha,\ \ f(z) \notin B_\eta(x_0)\}.$$

Suppose that $\mathcal{B}$ were infinite. Then, there would be a sequence $(y_n)_{n\in\mathbb{N}}$ of distinct points of $\mathcal{B}$, with limit type α, such that $d(f(y_n), x_0) \geq \eta$. Considering a subsequence if necessary, we can assume that there would be a $z \in X$ such that $y_n \to z$. Since $d(y_n, x_0) < \eta$, we have that $d(z, x_0) \leq \eta$. On the other hand, since f is continuous, $f(y_n) \to f(z)$, so $d(f(z), x_0) \geq \eta$. Therefore $z \neq x_0$. In addtion, since $lt(y_n) = \alpha$ for every n, we have that $lt(z) \geq \alpha+1$, But this contradicts the fact that $\overline{B_\delta(x_0)}$ witnesses the limit type of x_0. Therefore, $\mathcal{B}$ is finite. $\square$

3.5 Transitivity in countable dynamical systems

In Dynamics literature there are several different definitions of the property called *transitivity*. In [1], Akin and Carlson provide a nice and complete survey on the existing notions of transitivity and the relationship among them. Definition 3.5.1 lists the most common of these conceptions.

Definition 3.5.1. Let (X, f) be a dynamical system. To describe *topological transitivity*, we consider the following properties of (X, f).

(TT) For every pair U, V of open non-empty subsets of X, there exists $k \in \mathbb{Z}$ such that $U \cap f^{-k}(V) \neq \emptyset$.

(TT_+) For every pair U, V of open non-empty subsets of X, there exists $k \in \mathbb{N}$ such that $U \cap f^{-k}(V) \neq \emptyset$.

(DO) There exists an orbit sequence $\{x_k\}_{k\in\mathbb{Z}}$ or $\{x_k\}_{k\geq 1}$ dense in X.

The next theorem is an abridged version of [1, Theorem 5.1]

Theorem 3.5.2 (Akin and Carlson, 2012). *Let (X, f) be a dynamical system. Assume that X contains at least one isolated point.*

- *If (X, f) satisfies TT_+ then X is finite and consists of a single periodic orbit.*
- *TT is equivalent to DO for dynamical systems with isolated points.*

Theorem 3.5.2 tells us that in the particular case of countable dynamical systems the property (TT_+) is not ideal, because it only makes sense in finite spaces. Therefore, when referring to transitivity on countable dynamical spaces, we will mean condition DO.

The following theorem, which is an adaptation of [1, Theorem 5.11], illustrates how a dynamical system with isolated points and the property DO might look like.

Theorem 3.5.3 (Akin and Carlson, 2012). *Let (X, f) be a dynamical system such that the set of isolated points of X, Iso(X), is non-empty. Assume that (X, f) has the property DO, i.e., that (X, f) has a dense orbit sequence. Exactly one of the following cases occurs.*

(1) There exists a unique $x \in$ Iso(X) such that $f^{-1}(x) = \emptyset$. Then, the forward orbit of x, $\mathcal{O}_+(x)$, is dense in X. Exactly one of the following occurs.

(a) Iso$(X) = \mathcal{O}_+(x)$ *consists of infinitely many distinct points in a single forward orbit.*

(b) Iso$(X) = \mathcal{O}_+(x) = X$ *is a finite, pre-periodic forward orbit of period* ℓ.

(c) Iso$(X) = \{f^k(x) : 0 \leq k \leq n-1\}$ *is a finite sequence of distinct points, for some* $n \geq 0$. *For* $k \geq n$, $f^k(x)$ *is not isolated and* X *is infinite. The finite set Iso*(X) *is not dense in* X.

(2) For every point $z \in$ *Iso*(X) *the set* $f^{-1}(z) \neq \emptyset$, *and there exists* $x \in$ *Iso*(X) *such that* $f^{-1}(x)$ *contains two points. In that case the point* x *is unique and we have Iso*$(X) = \mathcal{O}(x)$, *where* $\mathcal{O}(x)$ *denotes the full orbit through* x, *is an infinite, pre-periodic orbit of period* ℓ, *and* $f^{-1}(x) = \{y, f^{\ell-1}(x)\}$ *for some* y. *For all* $k \in \mathbb{N}$, $f^{-k}(y)$ *is a single isolated point. The set Iso*(X) *is dense in* X.

(3) For every $x \in$ *Iso*(X), $f^{-1}(x)$ *is a singleton, and exactly one of the following occurs.*

(a) $X = f(X) =$ *Iso*(X) *is a single periodic orbit. This is the only case satisfying* TT, TT_+ *and* DO.

(b) For each $x \in$ *Iso*(X), $\mathcal{O}(x)$ *is a* $\mathbb{Z}$*-orbit such that* $\mathcal{O}(x) =$ *Iso*(X) *is dense in* X.

(c) There is a unique $y \in$ *Iso*(X) *such that* $f(y) \notin$ *Iso*(X). *Iso*$(X) = \{f^{-k}(y) : k \in \mathbb{N}\}$ *forms an infinite sequence ending in* y *and for* $k > 0$, $f^k(y)$ *is not isolated. Iso*(X) *may or may not be dense in* X.

EXAMPLE 3.5.4. We will construct a dynamical system (X, f) such that

- X is a countable compact Hausdorff space;
- f is a homeomorphism;
- the derived degree of X is 2, i.e., the top limit points of X have limit type 1;
- (X, f) has a dense bijective $\mathbb{Z}$-orbit.

Let $n,m \in \mathbb{N}$. For every $i \in \{0,...,n-1\}$, let $X_i = \{x_i\} \cup (\omega \times \{x_i\})$ be a homeomorphic copy of $\omega+1$ with the order topology (with x_i as limit point). Analogously, define for every $i \in \{0,...,m-1\}$ the space $Y_i = \{y_i\} \cup (\omega \times \{y_i\})$. Let

$$X = \left(\bigcup_{i=0}^{n-1} X_i\right) \cup \left(\bigcup_{i=0}^{m-1} Y_i\right).$$

Consider each X_i and Y_i as clopen in X. With this topology, X is a countable compact Hausdorff space.

Define $f : X \to X$ as follows

$$f(x) = \begin{cases} x_{i+1} & x = x_i,\ i \text{ mod } n \\ y_{i+1} & x = y_i,\ i \text{ mod } m \\ (k, x_{i+1}) & x = (k, x_i),\ i < n-1,\ k \in \omega \\ (k, y_{i+1}) & x = (k, y_i),\ i < m-1,\ k \in \omega \\ (k+1, x_0) & x = (k, x_{n-1}),\ k \in \omega \\ (k-1, y_0) & x = (k, y_{m-1}),\ k \in \omega,\ k > 0 \\ (0, x_0) & x = (0, y_{m-1}). \end{cases}$$

Evidently, the function f is a bijection. Let $i \in \{1,...,n-1\}$. Observe that, for every cofinite subset A of X_i containing x_i, the set $f^{-1}(A)$ is a cofinite subset of X_{i-1} containing x_{i-1}, because $f^{-1}(A) = \{(u, x_{i-1}) : (u, x_i) \in A\} \cup \{x_{i-1}\}$, so $f^{-1}(A)$ is open in X. We have a similar situation for Y_i for each $i \in \{1,...,m-1\}$. If A is a cofinite subset of Y_0 containing y_0, then $f^{-1}(A) = \{(u+1, y_{m-1}) : (u, y_0) \in A\} \cup \{y_{m-1}\}$, which is open in X. In the case of X_0 it might be possible that $f^{-1}(A)$, with A a cofinite subset of X_0 containing x_0, could be a cofinite subset of X_{n-1} containing x_{n-1} union the singleton $\{(0, y_{m-1})\}$, even so $f^{-1}(A)$ is open in X because $\{(0, y_{m-1})\}$ is open. These remarks imply the continuity of f at $\{x_0, ..., x_{n-1}, y_0, ..., y_{m-1}\}$. Notice that the set of isolated points of X is $I = X \setminus \{x_0, ..., x_{n-1}, y_0, ..., y_{m-1}\}$. For every

$x \in I$ we have that $f^{-1}(x)$ is a singleton containing an isolated point. Therefore f is continuous. Since X is compact, f is a homeomorphism. The set of isolated points of X form a dense $\mathbb{Z}$-orbit.

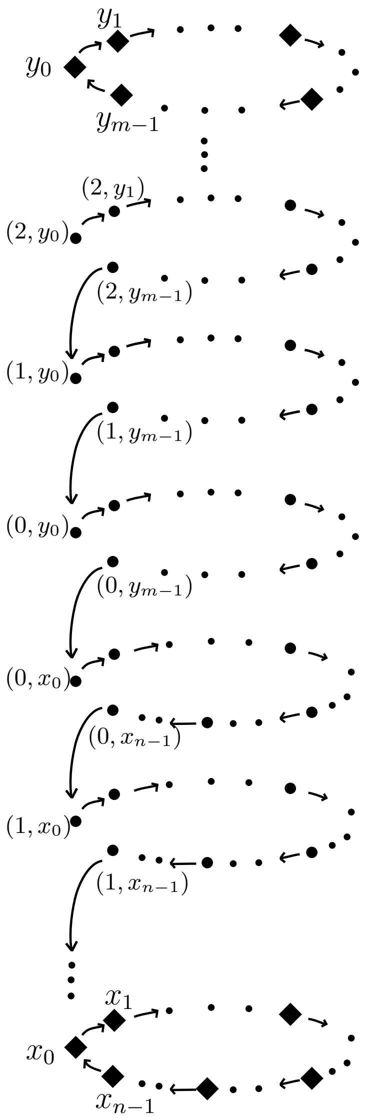

Figure 3.4: Countable system of derived degree 2 with dense $\mathbb{Z}$-orbit

A natural question arises from Example 3.5.4: can a bijective $\mathbb{Z}$-orbit of isolated points converge to a point of limit type higher than 1?

The following example motivates Theorem 3.5.6.

EXAMPLE 3.5.5. There exists a dynamical system (X, f) with the following properties:

- X is a countable compact Hausdorff topological space;
- f is a homeomorphism;
- X has a single top limit point of limit type 2, and hence, the derived degree of X is 3;
- f has a dense $\mathbb{Z}$-orbit (then such orbit should contain all the isolated points of X), i.e. f is topologically transitive.

Consider the set
$$A = \{x_i : i \in \mathbb{Z}\}.$$

For each $i < 0$ we define the set B_i as
$$B_i = \{x_{i,j} : j \geq -i\}.$$

Define $B_0 = \{x_{0,j} : j \geq 0\}$. If i is positive, then
$$B_i = \{x_{i,j} : j \geq i - 1\}.$$

Now consider
$$X = A \cup \Big(\bigcup_{i \in \mathbb{Z}} B_i\Big) \cup \{x^*\}.$$

A topology will be defined on X in the following way. All the elements of $\bigcup_{i \in \mathbb{Z}} B_i$ will be isolated points. For each $i \in \mathbb{Z}$, $B_i \cup \{x_i\}$ will be clopen in X and the family

$$\{(B_i \cup \{x_i\}) \setminus F : F \subseteq B_i, F \text{ finite}\}$$

will be a neighbourhood basis for the point x_i. For the point x^*, a neighbourhood basis will be the family

$$\Big\{ \bigcup_{i \in \mathbb{Z} \setminus F} B_i \cup \{x_i\} \cup \{x^*\} : F \subseteq \mathbb{Z},\ F \text{ finite} \Big\}.$$

With this topology X is a countable compact Hausdorff space, the point x^* has limit type 2, and the elements of A have limit type 1.

Let $f : X \to X$ be defined as follows

$$f(x) = \begin{cases} x^* & x = x^* \\ x_{i+1} & x = x_i \\ x_{i+1,j} & x = x_{i,j},\ i \leq 0,\ j \geq -i \\ x_{i+1,j} & x = x_{i,j},\ i > 0,\ j > i \\ x_{-(i-1),j-1} & x = x_{i,j},\ i > 0,\ j = i \\ x_{i+1,j+1} & x = x_{i,j},\ i > 0,\ j = i-1 \end{cases}$$

The function f is bijective. Besides, for each $i \in \mathbb{Z}$

$$f^{-1}(B_{i+1}) = (B_i \setminus F) \cup G$$

with F a finite subset of B_i and G a finite set of isolated points of $X \setminus B_i$. This implies the continuity of f. Since X is compact, f is a homeomorphism. In addition, all the isolated points of X (notice that the set of isolated points of X is the set $\bigcup\limits_{i \in \mathbb{Z}} B_i$) form a single orbit of f. Hence (X, f) is topologically transitive.

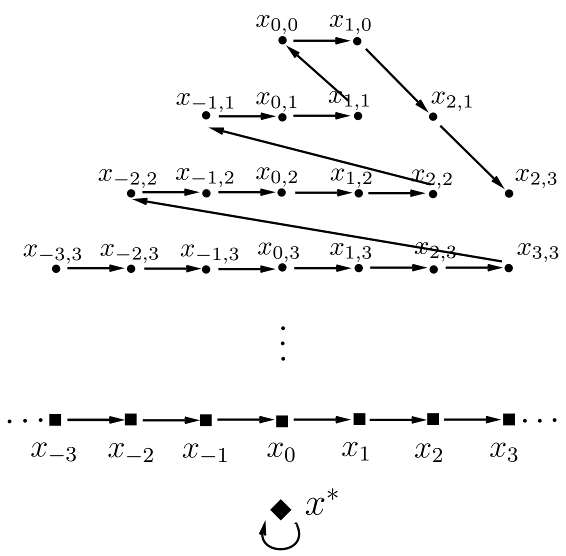

Figure 3.5: Countable system of derived degree 3 with dense $\mathbb{Z}$-orbit

THEOREM 3.5.6. *Let α be a countable ordinal, $\alpha > 0$. There exists a dynamical system (X, f) with the following characteristics.*

1) X is a countable compact Hausdorff space.

2) f is a homeomorphism.

3) The highest limit type of a point in X is α, and hence, the derived degree of X is $\alpha + 1$.

4) f has a dense $\mathbb{Z}$-orbit.

Proof. Examples 3.5.4 and 3.5.5 show that the statement is true for $\alpha = 1$ and $\alpha = 2$ respectively.

Let $\alpha > 2$. For each $j \in \mathbb{Z}$, consider a homeomorphic copy of ω^α with the order topology,

$$X_j = \omega^\alpha \times \{j\}.$$

Let $X = \bigcup_{j\in\mathbb{Z}} X_j \cup \bigcup_{n\in\mathbb{N}} \{y_n\}$. Define a topology on X in the following way. Each X_j will be considered clopen in X. Given a point $x \in X_j$ with $j \in \mathbb{Z}$, a local basis for x

in X_j will be a local basis for x in X as well. On the other hand, each point y_n will be isolated in X. In this way, it is clear that X is locally compact and Hausdorff. Let $\tilde{X} = X \cup \{x^*\}$ be the one-point compactification of X. Then, $lt(x^*) = \alpha$.

Consider the set $\mathcal{A} = \tilde{X}' \setminus \{x^*\}$. Define the function $h_1 : \mathcal{A} \to \mathcal{A}$ as

$$h_1(x, j) = (x, j + 1),$$

where x is a limit point of ω^α and $j \in \mathbb{Z}$.

Now, let $\{x_{i,j}\}_{i \geq 0}$ be an enumeration of the isolated points of X_j, for every $j \in \mathbb{Z}$. Define the function $h_2 : X \setminus \mathcal{A} \to X \setminus \mathcal{A}$ as follows: for every $j \in \mathbb{Z}$ and every $i \geq 0$, let

$$h_2(x_{i,j}) = \begin{cases} x_{0,-j} & \text{if } i = 0,\ j \geq 1 \\ x_{i-1,j} & \text{if } 0 < i \leq j,\ j \geq 1 \\ x_{i,j+1} & \text{if } i > j,\ j \geq 1 \\ x_{i+1,j} & \text{if } 0 \leq i \leq -j+1,\ j \leq 0 \\ x_{i,j+1} & \text{if } i > -j+1,\ j \leq 0; \end{cases}$$

and let

$$h_2(y_i) = \begin{cases} x_{0,0} & i = 1 \\ y_{i-1} & i > 1 \end{cases}$$

for every $i \in \mathbb{N}$ (see Figure 3.6).

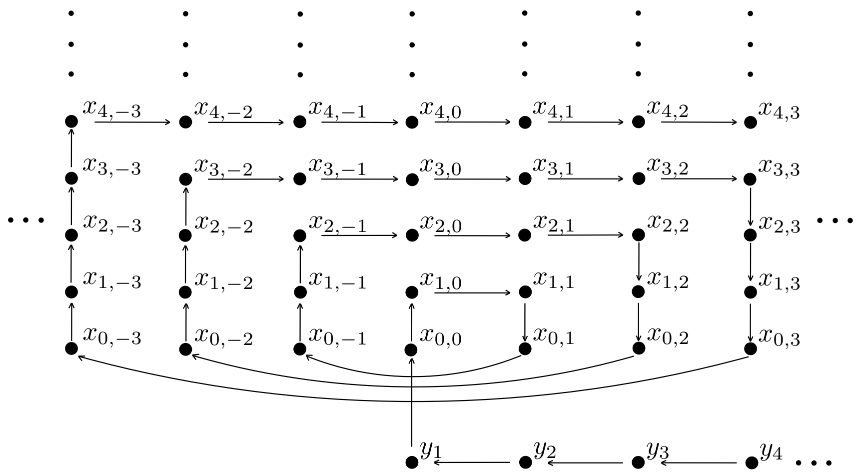

Figure 3.6: Function h_2

Finally, define the function $f : \tilde{X} \to \tilde{X}$ as

$$f(x) = \begin{cases} h_1(x) & x \in \mathcal{A} \\ h_2(x) & x \in X \setminus \mathcal{A} \\ x^* & x = x^* \end{cases}$$

The function f is bijective and has a dense orbit. On the other hand, for every $j \in \mathbb{Z}$,

$$f^{-1}(X_j) = (X_{j-1} \setminus F_j) \cup G_j,$$

where F_j, G_j are finite sets of isolated points of X such that $F_j \subset X_{j-1}$ and $G_j \subset X \setminus X_{j-1}$. Thus, f is continuous. Since $\tilde{X}$ is compact, f is closed. Hence, f is a homeomorphism.

$\square$

In plain words, Theorem 3.5.6 tell us that the scattered structure of a transitive countable compact dynamical system can be as complex as desired.

3.6 ω-limit Sets and Shadowing in Countable Dynamical Systems

An important notion in the field of Dynamics is the concept of ω-limit set. Its importance lies in the fact that it captures the behaviour of forward orbits. ω-limit sets are invariant subsets, so they are themselves dynamical systems. On the other hand, the notion of shadowing arises naturally in the numerical calculation of orbits as a way to track pseudo-orbits within a tolerance by real orbits. In this section we prove an interesting result which involves these concepts in the setting of countable compact dynamical systems (Theorem 3.6.14). Additionally, we provide an example of a countable system with the shadowing property and an example of a countable system with no shadowing condition.

We start by presenting the main three concepts of this section.

Definition 3.6.1. Let X be a topological space and $f : X \to X$ a continuous function. For every point $x \in X$, the *ω-limit set of x* is the set

$$\omega(x) = \omega(x, f) = \bigcap_{m\geq 0} \overline{\{f^n(x) : n \geq m\}}.$$

Observe the following facts.

1) Given a point $x \in X$, $y \in \omega(x)$ if and only if y is a limit point of the sequence $(f^n(x))_{n>0}$, i.e. $f^{n_k} \to y$ for some sequence of integers $n_k \to \infty$.

2) If x is pre-periodic, then $\omega(x)$ is finite and the set of accumulation points of the set $\{f^n(x) : n \geq 0\}$ is empty.

3) If x is not pre-periodic, $\{f^n(x) : n \geq 0\}$ is infinite and $\omega(x)$ is the set of accumulation points of $\{f^n(x) : n \geq 0\}$.

Definition 3.6.2. Let X be a metric space and let $f : X \to X$ be a continuous function. Given $\delta > 0$, a sequence $\{x_n\}_{n\geq 0}$ (or a finite sequence $\{x_n\}_{n=0}^{k}$) of points of

X is called a δ-*pseudo-orbit* if

$$d(f(x_n), x_{n+1}) \leq \delta$$

for every $n \geq 0$.

DEFINITION 3.6.3. Let X be a metric space. A continuous function $f : X \to X$ is said to have the *shadowing property* if for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit in X can be ϵ-shadowed by an actual orbit, i.e. for every δ-pseudo-orbit $\{x_n\}_{n\geq 0}$ there exists a point $x \in X$ such that

$$d(x_n, f^n(x)) \leq \epsilon$$

for every $n \geq 0$.

It turns out that the concept of shadowing can be seen from the purely topological point of view, as we can see in the next definition.

DEFINITION 3.6.4. Let X be a space, let $f : X \to X$, and let $\mathcal{U}$ be a finite open cover of X.

- A sequence $(x_n)_{n\geq 0}$ is said to be a $\mathcal{U}$-pseudo-orbit if for every $n \geq 0$ there exists $U_n \in \mathcal{U}$ such that $f(x_n), x_{n+1} \in U_n$.
- The point $z \in X$ $\mathcal{U}$-shadows a sequence $(x_n)_{n\geq 0}$ if for every $n \geq 0$ there exists $U_n \in \mathcal{U}$ such that $f^n(z), x_n \in U_n$.

OBSERVATION 3.6.5. Let (X, f) be a dynamical system and let $\mathcal{U}$ be a finite open cover of X. Given a finite refinement $\mathcal{V}$ of $\mathcal{U}$ (for all $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$) and a point $x \in X$,

- if x $\mathcal{V}$-shadows a sequence $(u_i)_{i\in\mathbb{N}}$, then x $\mathcal{U}$-shadows $(u_i)_{i\in\mathbb{N}}$;
- if a sequence $(u_i)_{i\in\mathbb{N}}$ is a $\mathcal{V}$-pseudo-orbit, then $(u_i)_{i\in\mathbb{N}}$ is a $\mathcal{U}$-pseudo-orbit.

The next theorem establishes the equivalence of Definition 3.6.3 and Definition 3.6.4 (the reader interested in a proof of this result can consult [16, Lemma 5]).

THEOREM 3.6.6. *Let X be a compact metric space and $f : X \to X$ a continuous function. (X, f) has the shadowing property if and only if for every finite open cover $\mathcal{U}$, there exists a finite open cover $\mathcal{V}$ such that every $\mathcal{V}$-pseudo-orbit can be $\mathcal{U}$-shadowed by some point of X.*

OBSERVATION 3.6.7. Notice that, given a finite open cover $\mathcal{U}$ of X, there exists a finite open cover $\mathcal{V}$ such that every $\mathcal{V}$-pseudo-orbit can be $\mathcal{U}$-shadowed by some point of X if and only if there exists a finite refinement $\mathcal{V}$ of $\mathcal{U}$ such that every $\mathcal{V}$-pseudo-orbit can be $\mathcal{U}$-shadowed by some point of X.

Before continuing we will establish the following notation.

NOTATION. *Let S be a set.*

1. *Given two finite sequences $(a_1, a_2, ..., a_k)$ and $(b_1, b_2, ..., b_j)$ in S, their concatenation is defined by*

$$A^\frown B = (a_1, a_2, ..., a_k, b_1, b_2, ..., b_j).$$

2. *Given a sequence $(A_i)_{i\in\mathbb{N}}$ of finite sequences in S, the sequence obtained concatenating the sequences A_i will be denoted by*

$$A_1{}^\frown A_2{}^\frown A_3{}^\frown ...{}^\frown A_i{}^\frown ...$$

EXAMPLE 3.6.8. We will exhibit an example of a countable dynamical system (X, f) with no shadowing.

Let $X = \{x^*\} \cup \{x_i : i \in \mathbb{Z}\}$. Define the discrete topology on $\{x_i : i \in \mathbb{Z}\}$ and let X be the one-point compactification of $\{x_i : i \in \mathbb{Z}\}$. Define the function $f : X \to X$

as follows

$$f(x) = \begin{cases} x_{i+1} & x = x_i, \ i \in \mathbb{Z} \\ x^* & x = x^* \end{cases}$$

Evidently, f is a homeomorphism.

Let $m \in \mathbb{N}$ and let $A = \{x^*\} \cup \{x_i : i < -m \text{ or } i > m\}$. Consider the finite open cover

$$\mathcal{U} = \{A, \{x_{-m}\}, \{x_{-m+1}\}, ..., \{x_0\}, ..., \{x_{m-1}\}, \{x_m\}\}$$

of X. Any finite refinement $\mathcal{W}$ of $\mathcal{U}$ has itself a refinement $\mathcal{V}$ of the form

$$\mathcal{V} = \{B, \{x_{-k}\}, \{x_{-k+1}\}, ..., \{x_0\}, ..., \{x_{k-1}\}, \{x_k\}\},$$

where $B = \{x^*\} \cup \{x_i : i < -k \text{ or } i > k\}$ for some $k > m$ (notice that, if $\mathcal{W}$ is a finite refinement of $\mathcal{U}$, then there exists $W \in \mathcal{W}$ such that $x^* \in W$ and $W \subseteq A$, so there is a finite set $M \subseteq \mathbb{Z}$ with $\{-m, ..., 0, ..m\} \subseteq M$ such that $W = X \setminus \{x_i : i \in M\}$).

Let S be the finite sequence $(x_0, x_1, ..., x_k, x_{-k-1}, x_{-k}, x_{-k+1}, ..., x_{-1})$. Then, the sequence

$$S' = (S)^\infty = S^\frown S^\frown S^\frown ...$$

is a $\mathcal{V}$-pseudo-orbit. We claim that S' cannot be $\mathcal{U}$-shadowed by any point in X. Indeed, the only point in X which might $\mathcal{U}$-shadow S' is x_0. But $f^{2k+2}(x_0) = x_{2k+2} \in A$, and x_0, which is the $(2k+2)$th term of S' (considering x_1 as the first term of S'), is not contained in A. Since the elements of $\mathcal{U}$ are pairwise disjoint, we have that x_0 does not $\mathcal{U}$-shadow S'. Therefore, S' is a $\mathcal{V}$-pseudo-orbit which cannot be $\mathcal{U}$-shadowed by any point in X. This proves that (X, f) does not have the shadowing property.

Example 3.6.9. We present an example of a countable dynamical system (X, f) with the shadowing property.

Let $X = \{x^*, y^*\} \cup \{x_i : i \in \mathbb{Z}\}$. Define on X the following topology. For each

$i \in \mathbb{Z}$, x_i will be an isolated point. A neighbourhood basis for the point x^* will be the family

$$\big\{\{x*\} \cup \{x_i : i \leq m\} : m \in \mathbb{Z}, m < 0\big\}.$$

Similarly, a neighbourhood basis for the point y^* will be the family

$$\big\{\{y*\} \cup \{x_i : i \geq m\} : m \in \mathbb{Z}, m > 0\big\}.$$

Endowed with this topology, X is a countable compact Hausdorff space. Define the function $f : X \to X$ as follows

$$f(x) = \begin{cases} x_{i+1} & x = x_i,\ i \in \mathbb{Z} \\ x^* & x = x^* \\ y^* & x = y^* \end{cases}$$

It is clear that f is a homeomorphism.

We will now show that X has the shadowing property. Let $\mathcal{U}$ be a finite open cover of X. $\mathcal{U}$ has a finite refinement $\mathcal{V}$ of the form

$$\{U, V\} \cup \{\{x_i\} : -m \leq i \leq m\},$$

where $U = \{x^*\} \cup \{x_i : i < -m\}$ and $V = \{y^*\} \cup \{x_i : i > m\}$, for some $m \in \mathbb{N}$. We claim that any $\mathcal{V}$-pseudo-orbit can be $\mathcal{V}$-shadowed by a point in X.

Let $(u_i)_{i \geq 0}$ be a $\mathcal{V}$-pseudo-orbit. There are three possibilities:

(1) $u_i \in U$ for every $i \geq 0$, or

(2) $u_i \in V$ for every $i \geq 0$, or

(3) there exists $n \in \mathbb{N}$ such that $u_i \in V$ for $i > n$ and $u_i \in X \setminus V$ for $i \leq n$ (so a finite non-empty set of terms of $(u_i)_{i \geq 0}$ is contained in $X \setminus V$) .

In case (1) the point x^* $\mathcal{V}$-shadows $(u_i)_{i\geq 0}$. Analogously, in case (2) the point y^* $\mathcal{V}$-shadows $(u_i)_{i\geq 0}$. If there exist $n \in \mathbb{N}$ such that $u_i \in V$ for $i > n$ (we can assume without loss of generality that $n = \min\{j \in \mathbb{N} : u_i \in V \text{ for } i > j\}$), then $u_n = x_k$ for some k with $-m \leq k \leq m$ (if u_n were an element of U, then $f(u_n) = x^*$ or $f(u_n) = x_l$ with $l < m$, in both cases $f(u_n) \notin V$, which is not possible since $(u_i)_{i\geq 0}$ is a $\mathcal{V}$-pseudo-orbit). In this case, the only element of $\mathcal{V}$ that covers u_n is $\{x_k\}$, thus if y is a point of X that $\mathcal{V}$-shadows $(u_i)_{i\geq 0}$, then $f^n(y) \in \{x_k\}$, i.e. $f^n(y) = x_k$. Hence the point x_{k-n} $\mathcal{V}$-shadows $(u_i)_{i\geq 0}$. Therefore, any $\mathcal{V}$-pseudo-orbit can be $\mathcal{V}$-shadowed, and hence $\mathcal{U}$-shadowed (see Observation 3.6.5), by a point in X. This proves that X has the shadowing property.

Now we will work towards a proof of Theorem 3.6.14. We begin by presenting the concepts of abstract ω-limit set and internally chain transitive set, which will be necessary to state Bowen-Sharkovsky Theorem (Theorem 3.6.12). A proof for this result can be found in [5, Lemma 2.5, Theorem 2.6]. Bowen-Sharkovsky Theorem will help us to prove Lemma 3.6.13.

Definition 3.6.10. Let X be a compact metric space and let $f : X \to X$ be a continuous function. A closed subset A of X is said to be *internally chain transitive* if for every $\epsilon > 0$ and for every $x, y \in A$ there is an ϵ-pseudo-orbit $(x_i)_{i=0}^n \subseteq A$ such that $x = x_0$ and $y = x_n$.

Definition 3.6.11. Let X be a compact Hausdorff space and let $f : X \to X$ be a continuous function. (X, f) is said to be an *abstract ω-limit set* if there exist a compact Hausdorff space Y and a continuous function $g : Y \to Y$ such that (X, f) is topologically conjugate to $(\omega(y, g), g\restriction_{\omega(y,g)})$ for some $y \in Y$.

Theorem 3.6.12 (Bowen and Sharkovsky). *A metrisable dynamical system is an abstract ω-limit set if and only if is internally chain transitive.*

Lemma 3.6.13. *Let X be a compact metric space, let $f : X \to X$ be a continuous surjective function, and let $x \in X$. If $\omega(x)$ is finite, then $\omega(x)$ is a single periodic orbit.*

Proof. By Theorem 3.6.12, the set $\omega(x)$ is internally chain transitive. Let $\omega(x) = \{x_1, x_2, ..., x_k\}$ for some $k \in \mathbb{N}$. Let ϵ be a positive number such that the elements of the family of balls $\{\mathcal{B}_\epsilon(x_i) : 1 \leq i \leq k\}$ are pairwise disjoint. Let $i, j \in \{1, 2, ..., k\}$. Since $\omega(x)$ is internally chain transitive, there exist $m \in \mathbb{N}$ and an ϵ-pseudo-orbit $(y_n)_{1\leq n\leq m}$ in $\omega(x)$ such that $x_i = y_1$ and $x_j = y_m$. Then, for every $n \in \{1, ..., m-1\}$, $d(y_{n+1}, f(y_n)) \leq \epsilon$. This implies that $y_{n+1} = f(y_n)$ for each n, i.e., $(y_n)_{1\leq n\leq m}$ is an actual orbit. Therefore, $x_j = f^m(x_i)$. Since this is possible for all $i, j \in \{1, 2, ..., k\}$, $\omega(x)$ is a single periodic orbit. □

Now we are in position to prove the main theorem of this section.

THEOREM 3.6.14. *Let X be a countable compact metric space and let $f : X \to X$ be a continuous surjective function with the shadowing condition. For every $x \in X$, $\omega(x)$ is a periodic orbit.*

Proof. Suppose that there exists $x \in X$ such that $\omega(x)$ is not a periodic orbit. If $\omega(x)$ is not a periodic orbit, then $\omega(x)$ is a compact infinite subspace of X by Lemma 3.6.13. By Theorem 3.4.6, the top limit points of $\omega(x)$, i.e., the points with the highest limit type in X, form cycles among them. Let $\{y_1, y_2, ..., y_k\}$ be a cycle of top limit points of $\omega(x)$, i.e., $f(y_i) = y_{i+1}$ where i is taken modulo k. Choose $z \in \omega(x)$ such that $z \neq y_i$ for every $i \leq k$. Let δ be a positive number such that the elements of the family

$$\{B_\delta(y_i) : i \leq k\} \cup \{B_\delta(z)\}$$

are pairwise disjoint, and let ϵ be a positive number such that $\epsilon < \delta/3$. Since (X, f) has the shadowing property, there exists a positive number $\delta_\epsilon < \epsilon$ such that every δ_ϵ-pseudo-orbit is ϵ-shadowed by a point of X. Now, since $y_1, z \in \omega(x)$, there exist

$m_1, m_2, n_1, n_2 \in \mathbb{N}$ such that $m_2 > m_1$, $n_2 > n_1$ and

$$f^{m_1}(x) \in B_{\delta_\epsilon}(y_1)$$
$$f^{m_2}(x) \in B_{\delta_\epsilon/2}(z)$$
$$f^{n_1}(x) \in B_{\delta_\epsilon/2}(z)$$
$$f^{n_2}(x) \in B_{\delta_\epsilon}(y_1)$$

Let M be a natural number such that $Mk > m_2 - m_1$. Consider the following finite sequences

$$\begin{aligned} A &= (f^{m_1}(x), f^{m_1+1}(x), ..., f^{m_2-1}(x), f^{n_1}(x), f^{n_1+1}(x), ..., f^{n_2-1}(x)) \\ B^0 &= (y_1, y_2, ..., y_k) \\ B^1 &= \underbrace{B^{0\frown} B^{0\frown} B^0 ... B^{0\frown} B^0}_{\text{M+1 times}} \end{aligned}$$

Now, for every $\{s_i\}_{i\in\mathbb{N}} \in 2^{\mathbb{N}}$, consider the following sequence of points of X

$$C_{\{s_i\}_{i\in\mathbb{N}}} = B^{s_1 \frown} A^\frown B^{s_2 \frown} A ... A^\frown B^{s_i \frown} A...$$

Notice that there are $\mathfrak{c}$-many sequences of this form.
We claim that, for each $\{s_i\}_{i\in\mathbb{N}} \in 2^{\mathbb{N}}$, the sequence $C_{\{s_i\}_{i\in\mathbb{N}}}$ is a δ_ϵ-pseudo-orbit. To see this, it is enough to observe that

$$\begin{aligned} d(f(y_k), f^{m_1}(x)) &= d(y_1, f^{m_1}(x)) \\ &< \delta_\epsilon; \end{aligned}$$

$$
\begin{aligned}
d\big(f\big(f^{m_2-1}(x)\big), f^{n_1}(x)\big) &= d(f^{m_2}(x), f^{n_1}(x)) \\
&\leq d(f^{m_2}(x), z) + d(z, f^{n_1}(x)) \\
&< \delta_\epsilon;
\end{aligned}
$$

$$
\begin{aligned}
d(f(f^{n_2-1}(x)), y_1) &= d(f^{n_2}(x), y_1) \\
&< \delta_\epsilon.
\end{aligned}
$$

Let $\{s_i\}_{i\in\mathbb{N}}$ and $\{t_i\}_{i\in\mathbb{N}}$ be two distinct elements of $2^{\mathbb{N}}$. We will prove that no point in X can ϵ-shadow both $C_{\{s_i\}_{i\in\mathbb{N}}}$ and $C_{\{t_i\}_{i\in\mathbb{N}}}$ at the same time.

Let $m = \min\{i \in \mathbb{N} : s_i \neq t_i\}$. Consider the finite sequence

$$
D = \begin{cases} B^{s_1\frown}A^\frown B^{s_2\frown}A...A^\frown B^{s_{m-1}\frown}A^\frown B^0, & \text{if } m > 1 \\ B^0, & \text{if } m = 1 \end{cases}
$$

Assume, without loss of generality that $s_m = 0$ and that $t_m = 1$. Then

$$
C_{\{s_i\}_{i\in\mathbb{N}}} = D^\frown A^\frown B^{s_{m+1}\frown}A...
$$

and

$$
C_{\{t_i\}_{i\in\mathbb{N}}} = D^\frown \underbrace{B^{0\frown}B^0...B^{0\frown}B^0}_{\text{M times}}{}^\frown A^\frown B^{t_{m+1}\frown}A...
$$

Let n the number of terms of D. Then, the $(n+1)$th term of $C_{\{s_i\}_{i\in\mathbb{N}}}$ is $f^{m_1}(x)$ and the $(n+1)$th term of $C_{\{t_i\}_{i\in\mathbb{N}}}$ is y_1. Besides, observe that the number of terms in the finite sequence

$$
(f^{m_1}(x), f^{m_1+1}(x), ..., f^{m_2-1}(x))
$$

is $m_2 - m_1$. Since $Mk > m_2 - m_1$, the $(n + m_2 - m_1 + 1)$th term of $C_{\{s_i\}_{i\in\mathbb{N}}}$ is $f^{n_1}(x)$ and the $(n + m_2 - m_1 + 1)$th term of $C_{\{t_i\}_{i\in\mathbb{N}}}$ is y_j for some $j \leq k$. If there were a point $u \in X$ such that $d(u, f^{n_1}(x)) < \epsilon$ and $d(u, y_j) < \epsilon$, then

$$\begin{aligned} d(z, y_j) &\leq d(z, f^{n_1}(x)) + d(f^{n_1}(x), y_j) \\ &\leq d(z, f^{n_1}(x)) + d(u, f^{n_1}(x)) + d(u, y_j) \\ &< \delta_\epsilon/2 + \epsilon + \epsilon \\ &< 3\epsilon \\ &< \delta, \end{aligned}$$

but this is not possible because $B_\delta(y_j) \cap B_\delta(z) = \emptyset$. Therefore, no point of X can ϵ-shadow both $C_{\{s_i\}_{i\in\mathbb{N}}}$ and $C_{\{t_i\}_{i\in\mathbb{N}}}$ at the same time. This implies that

$$|\{C_{\{s_i\}_{i\in\mathbb{N}}} : \{s_i\}_{i\in\mathbb{N}} \in 2^{\mathbb{N}}\}| \leq |X|$$

since each δ_ϵ-pseudo-orbit is ϵ-shadowed by a point of X. But this is a contradiction because X is countable and $|\{C_{\{s_i\}_{i\in\mathbb{N}}} : \{s_i\}_{i\in\mathbb{N}} \in 2^{\mathbb{N}}\}| = \mathfrak{c}$. Therefore, for every $x \in X$, $\omega(x)$ is finite, and hence, a single periodic orbit. $\square$

As a last result in this chapter we aim to prove that, in countable compact dynamical systems, shadowing and internal chain transitivity are not compatible unless the system is a single periodic orbit (Corollary 3.6.16).

LEMMA 3.6.15. *Let X be a countable compact metric space and let $f : X \to X$ be a continuous surjective function. If X is internally chain transitive, then either*

- *X is a single periodic orbit, or*
- *X is infinite and contains no periodic orbit of isolated points.*

Proof. Let X be a countable compact metric space and let $f : X \to X$ be a continuous surjective function. Assume that X is internally chain transitive.

If X is finite, then every point of X is periodic, because f is surjective. Suppose that X has at least two different cycles $(x_i)_{0\leq i\leq k}$ and $(y_i)_{0\leq i\leq m}$, with $k, m \in \mathbb{N}$. Choose a positive number ϵ such that the elements of the family

$$\{B_\epsilon(x) : x \in X\}$$

are pairwise disjoint. Then, any ϵ-pseudo-orbit is an actual orbit. This means that there is no ϵ-pseudo-orbit $(u_i)_{0\leq i\leq j}$ in X such that $u_0 = x_0$ and $u_j = y_0$. But this contradicts the assumption of X being internally chain transitive. Therefore, X must be a single periodic orbit.

Now suppose that X is infinite and contains a periodic isolated point x_0 such that the periodic orbit $\mathcal{O}_+(x_0)$ consists of isolated points. Choose a positive number δ such that $B_\delta(x) = \{x\}$ for every $x \in \mathcal{O}_+(x_0)$. Then, any δ-pseudo-orbit $(u_i)_{0\leq i\leq j}$ in X such that $u_0 = x_0$ will be the actual orbit $(f^i(x_0))_{0\leq i\leq j}$. This means that, if $y \notin \mathcal{O}_+(x)$, there is no δ-pseudo-orbit $(u_i)_{0\leq i\leq j}$ such that $u_0 = x_0$ and $u_j = y$, which is not possible because X is internally chain transitive. Hence, if X is infinite, then X contains no periodic orbit of isolated points. □

COROLLARY 3.6.16. *Let X be a countable compact metric space and let $f : X \to X$ be a continuous surjective function. X cannot have the shadowing property and be internally chain transitive unless X is a single periodic orbit.*

Proof. Let $f : X \to X$ be a continuous surjective function defined on a countable compact metric space X such that X is internally chain transitive. By Lemma 3.6.15, X is either a single periodic orbit or an infinite space with no periodic orbits of isolated points. If X is a single periodic orbit it is clear that X has the shadowing property.

Assume that X is infinite. By Theorem 3.6.12, X is an abstract ω-limit set. Just as we did in the proof of Theorem 3.6.14, we can find a positive number ϵ such that, for every δ with $0 < \delta < \epsilon$,

- there is a family $\mathcal{A}$ of δ-pseudo-orbits in X such that $|\mathcal{A}| = \mathfrak{c}$, and
- every point $x \in X$ can ϵ-shadow at most one element of $\mathcal{A}$.

If X has shadowing, for such ϵ there exists a positive number δ_ϵ, with $\delta_\epsilon < \epsilon$, such that every δ_ϵ-pseudo-orbit in X can be ϵ-shadowed by a point in X. But this contradicts the fact of X being countable. Therefore, if X is internally chain transitive and infinite, it does not have the shadowing property. $\square$

Chapter 4

Embeddability Order in the Family of Closed Sets of $\mathbb{R}$

The ordering by homeomorphic embeddability of topological spaces has raised interest since the early 20th century, when Kuratowski and Sierpiński established that it was possible to construct families of subspaces of the continuum whose embeddability ordering modelled $\mathfrak{c}^+$ [23] or the antichain on $2^{\mathfrak{c}}$ points [22]. Since then, some progress has been made in establishing the possible ordertypes of families of subspaces of a given topological space X under the embeddability ordering, for some of the most familiar instances of X, like $\mathbb{R}$ (see the work of Matthews and McMaster in [24], and McCluskey et al. in [25] and [20]) and $\mathbb{Q}$ (see the article of Gillam [13]).

Taking inspiration from this line of research, in this chapter we analyse the ordering by homeomorphic embeddability *as a closed subspace* on the family of closed sets $2^{\mathbb{R}}$ of $\mathbb{R}$. No previous work on this kind of ordering is known to us. Given the relation $\sim$ defined as $A \sim B$ if and only if A can be embedded into B as a closed subset and vice versa, we are interested in the set $2^{\mathbb{R}}/{\sim}$ and in the order induced on it by $\sim$. We succeed at characterising $2^{\mathbb{R}}/{\sim}$ (Theorem 4.3.1) and we provide a description of the ordering of its elements.

4.1 A Brief Overview

Given a family of topological spaces $\mathcal{F}$ and a partially ordered set P, P is said to be *realisable* within $\mathcal{F}$ if there is an injection $\phi : P \to \mathcal{F}$ such that $p \leq q$ if and only if $\phi(p)$ can be embedded homeomorphically into $\phi(q)$. In the particular case of the family of subspaces of the real line, discussions on realisability can be traced back to 1926, when Kuratowski and Sierpiński revealed, while working on extensibility of continuous maps over G_δ subsets, that is possible to realise within $\mathcal{P}(\mathbb{R})$ the following partially ordered sets: the antichain on $2^{\mathfrak{c}}$ points [22], and the ordinal $\mathfrak{c}^+$ [23]. More recently, Matthews and McMaster [24] proved that any partially ordered set of cardinality at most $\mathfrak{c}$ can be realised within $\mathcal{P}(\mathbb{R})$. Later, in [25] McCluskey et al. proved that the powerset $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$ ordered by set-inclusion can be realised within $\mathcal{P}(\mathbb{R})$. In addition, in [20], McCluskey and Knight proved that there is no ZFC analogous result to the one of Matthews and McMaster for cardinality $2^{\mathfrak{c}}$.

Some work in a more general direction has been done by Comfort and Gillam in [7], where the authors prove that, for every $\kappa \geq \omega$, there is a zero-dimensional Hausdorff space S such that every poset of cardinality κ can be realised within $\mathcal{P}(S)$. On the other hand, in [13] Gillam provides a characterisation of $\mathcal{P}(\mathbb{Q})$ with the embeddability ordering in terms of scattered subspaces of $\mathbb{Q}$ with finite derived degree. In this article the author establishes several results relating the derived degree of countable scattered spaces to their embeddability properties. This represents a connection between the embeddability ordering topic and the material presented in previous parts of this thesis.

4.2 Preliminaries

In this chapter we will consider the real line $\mathbb{R}$ with the usual topology. The symbol $2^{\mathbb{R}}$ will denote the family of all closed subsets of $\mathbb{R}$. By interval we mean a non-degenerate interval of $\mathbb{R}$.

Given $C \in 2^{\mathbb{R}}$, there exists a least ordinal α such that $C^{(\alpha)} = C^{(\beta)}$ for any $\beta \geq \alpha$, where $C^{(\beta)}$ denotes the βth Cantor-Bendixson derivative (see Definition 1.1.2). $Perf(C)$ will denote $C^{(\alpha)}$. Clearly, $Perf(C)$ is a closed set. C is said to be a perfect set if $C = Perf(C)$, i.e. C is closed and has no isolated points.

Recall that a space X is called zero-dimensional if it has a basis consisting of clopen sets. Besides, X is said to be totally disconnected if the only connected subsets of X are the empty set and the one-point sets.

$\mathcal{C}$ will denote the Cantor set. A space is called a Cantor space if it is homeomorphic to $\mathcal{C}$. In addition, for every $i \in \mathbb{N}$, $\mathcal{C}_i$ will denote a Cantor space embedded in the interval $[2i, 2i+1]$. It is worth remembering that $\mathcal{C}$ is a metrisable, zero-dimensional, totally disconnected, perfect and compact space.

In [6] Brouwer proves that any two non-empty compact Hausdorff spaces without isolated points and having countable bases consisting of clopen sets are homeomorphic to each other. On the other hand, it is well-known that a Hausdorff locally compact space is zero-dimensional if and only if is totally disconnected (see [31, Theorem 29.7]). Both results combined give us the following characterisation of all the spaces which are homeomorphic to $\mathcal{C}$.

THEOREM 4.2.1. *A topological Hausdorff space is a Cantor space if and only if it is non-empty, perfect, compact, totally disconnected, and metrisable.* □

COROLLARY 4.2.2. *Let $C \in 2^{\mathbb{R}}$. If $Perf(C)$ is non-empty, compact, and does not contain an interval, then $Perf(C)$ is a Cantor set.* □

Theorem 4.2.1 allows us to prove the next lemma, which will be a necessary tool in the proof of Theorem 4.3.1, the main result of this chapter.

LEMMA 4.2.3. *Any totally disconnected compact subset of $\mathbb{R}$ can be embedded into a Cantor space.*

Proof. Let C be a totally disconnected compact subspace of $\mathbb{R}$. Suppose that

$C\setminus\ Perf(C)\neq\emptyset$. There exists a family of open sets

$$\{U_x : x\in C,\ x \text{ isolated in } C\}$$

such that $U_x\cap C=\{x\}$ and, if $x,y\in C$ are distinct and isolated, $U_x\cap U_y=\emptyset$. Choose a Cantor subset of U_x containing x, say C_x. The set $D=C\cup\bigcup_{x\in C\setminus C'}\mathcal{C}_x$ is compact, perfect (since every isolated point of C is contained in a Cantor subset of D) and totally disconnected. Therefore, by Theorem 4.2.1, it is a Cantor space. □

4.3 Embeddability order in the family of closed sets of $\mathbb{R}$

Consider the relation $\sim$ defined on $2^{\mathbb{R}}$ as follows: given $A,B\in 2^{\mathbb{R}}$, $A\sim B$ if and only if A can be embedded homeomorphically into B as a closed subspace and vice versa. The relation $\sim$ is an equivalence relation. Our purpose is to provide a list of all the elements of $2^{\mathbb{R}}/\sim$.

NOTE. Throughout this section, by *embedding* we mean *embedding as a closed subset*.

THEOREM 4.3.1. *Let C be a closed subset of $\mathbb{R}$. C belongs to the equivalence class of one of the following sets with respect to relation $\sim$.*

1) $\emptyset$;

2) $\omega^\alpha n+1$, *for some countable ordinal* α *and some* $n\in\omega$;

3) $\mathcal{C}$;

4) $[0,1]$;

5) $\omega^\alpha n+\omega^\beta$, *for some countable ordinals* α,β *with* $\alpha>\beta>0$, *and some* $n\geq 1$;

6) $\omega^\alpha n$, *for some countable ordinal* α *and some* $n\geq 1$;

7) $(-\infty, -1]$*;*

8) $\mathcal{C} \cup \omega^\alpha$*, with* α *a countable ordinal;*

9) $[0, 1] \cup \omega^\alpha$*, with* α *a countable ordinal;*

10) $(-\infty, -1] \cup \omega^\alpha$*, with* α *a countable ordinal;*

11) $(-\infty, -1] \cup [1, \infty)$*;*

12) $\bigcup_{i \geq 1} \mathcal{C}_i$*;*

13) $[0, 1] \cup \bigcup_{i \geq 1} \mathcal{C}_i$*;*

14) $\bigcup_{i \geq 1} [2i, 2i + 1]$*;*

15) $(-\infty, -1] \cup \bigcup_{i \geq 1} \mathcal{C}_i$*;*

16) $(-\infty, -1] \cup \bigcup_{i \geq 1} [2i, 2i + 1]$*;*

17) $\mathbb{R}$*;*

Proof. Let C be a closed non-empty subset of $\mathbb{R}$.

Case 1. Suppose that C is bounded. If C contains an interval, clearly $\Big[C\Big] = \Big[[0, 1]\Big]$. If C is totally disconnected and $Perf(C)$ is non-empty, by Corollary 4.2.2, $Perf(C)$ is a Cantor space. Therefore $\mathcal{C}$ can be embedded into C. In addition, by Lemma 4.2.3, C can be embedded into $\mathcal{C}$. So $\Big[C\Big] = \Big[\mathcal{C}\Big]$. In the case that C is scattered, there exists a countable compact ordinal $\omega^\alpha n + 1$ such that $\Big[C\Big] = \Big[\omega^\alpha n + 1\Big]$ (see Theorem 1.1.1, Theorem 1.2.1 and Theorem 1.2.3).

Case 2. Assume that C is unbounded. Evidently, $\Big[\mathbb{R}\Big] = \{\mathbb{R}\}$.

If C has two unbounded connected components, then C is a set of the form $(-\infty, a] \cup D \cup [b, \infty)$ for some $a, b \in \mathbb{R}$ and $D \subset \mathbb{R}$ such that $a < b$ and D is properly contained in (a, b). Clearly, $(-\infty, -1] \cup [1, \infty)$ can be embedded into C as a closed subspace.

Choose a point $x \in (a, b) \setminus D$. Since C is closed, there exists an open interval (c, d) which contains x and that $(c, d) \subset (a, b) \setminus D$. Then, $C \cap (-\infty, c]$ and $C \cap [d, \infty)$ can be embedded homeomorphically, respectively, into $(-\infty, -1]$ and $[1, \infty)$ as closed subspaces. Therefore $\Big[C\Big] = \Big[(-\infty, -1] \cup [1, \infty)\Big]$.

Suppose now that C has just one unbounded connected component, say, without loss of generality, $(-\infty, a]$. In this case we have to analyse different situations.

Case a) If $C \setminus (-\infty, a]$ is bounded, then C can be embedded into $(-\infty, -1]$ as a closed subspace. So, $\Big[C\Big] = \Big[(-\infty, -1]\Big]$.

Case b) Suppose that $C \setminus (-\infty, a]$ is not bounded and it contains infinitely many disjoint closed intervals whose union is not bounded. Since $(-\infty, a]$ is the only unbounded connected component of C, we have that $\mathbb{R} \setminus C$ is a set which is not bounded from above. Pick $b_1 \in \mathbb{R}$ such that $b_1 > a$ and $b_1 \notin C$. We can get a sequence of points in $\mathbb{R} \setminus C$ such that $b_{n+1} \geq b_n + 1$ for every $n \geq 1$. Since C is closed, for every b_n there exists $\epsilon_n > 0$ such that $(b_n - \epsilon_n, b_n + \epsilon_n) \subseteq \mathbb{R} \setminus C$. We can embed $C \cap (-\infty, b_1 - \epsilon_1]$ into $(-\infty, -1]$, and, for each $n \geq 1$, $C \cap [b_n + \epsilon_n, b_{n+1} - \epsilon_{n+1}]$ can be embedded into $[2n, 2n+1]$. So, C can be embedded into $(-\infty, -1] \cup \bigcup\limits_{n \geq 1} [2n, 2n+1]$. On the other hand, since the union of intervals of $C \setminus (-\infty, a]$ is not bounded, there exists a subsequence $\{b_{k_n}\}_{n \in \mathbb{N}}$ of $\{b_n\}_{n \in \mathbb{N}}$ such that $C \cap [b_{k_n} + \epsilon_{k_n}, b_{k_n+1} - \epsilon_{k_n+1}]$ contains a closed interval, say C_{k_n}. Then $[2n, 2n+1]$ can be embedded into C_{k_n} for every n. Therefore, $\Big[C\Big] = \Big[(-\infty, -1] \cup \bigcup\limits_{i \geq 1} [2i, 2i+1]\Big]$.

Assume that $C \setminus (-\infty, a]$ contains closed intervals whose union is bounded. Let us call A such union. Then $(-\infty, a] \cup (C \cap (a, \sup A])$ can be embedded into $(-\infty, 1]$. Hence, we only need to study the cases when $C \setminus (-\infty, a]$ contains no intervals at all.

Case c) Now suppose that $C \setminus (-\infty, a]$ is not bounded, that $C \setminus (-\infty, a]$ does not contain intervals, and that it contains infinitely many disjoint Cantor sets whose union is not bounded. Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}\setminus C$ such that $b_{n+1} \geq b_n + 1$ for every $n \geq 1$. Since C is closed, for every b_n there exists $\epsilon_n > 0$ such that $(b_n - \epsilon_n, b_n + \epsilon_n) \subseteq \mathbb{R}\setminus C$. For each $n \in \mathbb{N}$, $[b_n + \epsilon_n, b_{n+1} - \epsilon_{n+1}] \cap C$ is a closed subset of $\mathbb{R}$ which does not contain an interval. Since the union of Cantor subsets of $C \setminus (-\infty, a]$ is not bounded, for infinitely many $n \in \mathbb{N}$, $Perf([b_n + \epsilon_n, b_{n+1} - \epsilon_{n+1}] \cap C)$ is not empty. Then, by Corollary 4.2.2, for infinitely many $n \in \mathbb{N}$, $Perf([b_n + \epsilon_n, b_{n+1} - \epsilon_{n+1}] \cap C)$ is a Cantor space. Therefore $(-\infty, -1] \cup \bigcup_{i\geq 1} \mathcal{C}_i$ can be embedded into C. In addition, by Lemma 4.2.3, $[b_n + \epsilon_n, b_{n+1} - \epsilon_{n+1}] \cap C$ can be embedded into $\mathcal{C}_n$ for every n. So C can be embedded into $(-\infty, -1] \cup \bigcup_{i\geq 1} \mathcal{C}_i$. This means that $\left[C\right] = \left[(-\infty, -1] \cup \bigcup_{i\geq 1} \mathcal{C}_i\right]$.

Case d) Assume that $C \setminus (-\infty, a]$ is not bounded and that there is a $b \geq a$ such that any interval or Cantor subset of C is contained in $(-\infty, b]$. Then $(-\infty, b] \cap C$ can be embedded into $(\infty, a]$. So we only need to study the case when $C \setminus (-\infty, a]$ is not bounded and it is scattered. In this case, $C \setminus (-\infty, a]$ is homeomorphic to a countable ordinal α. Let $\omega^{\beta_1}c_1 + \omega^{\beta_2}c_2 + ... + \omega^{\beta_k}c_k$ be the Cantor normal form of α (see Section 1.1.2). Since α is not compact, we can suppose that $\beta_k \geq 1$. Notice that $\alpha = \omega^{\beta_1}c_1 + \omega^{\beta_2}c_2 + ... + \omega^{\beta_k}(c_k - 1) + 1 + \omega^{\beta_k}$, because ω^{β_k} is homeomorphic to $1 + \omega^{\beta_k}$. The ordinal $\omega^{\beta_1}c_1 + \omega^{\beta_2}c_2 + ... + \omega^{\beta_k}(c_k - 1) + 1$ is compact since it is a succesor ordinal, therefore, the part of $C\setminus(-\infty, a]$ which is homeomorphic to it, say D, is bounded and can be embedded into $(-\infty, -1]$. This implies that $C \setminus ((-\infty, a] \cup D)$ is unbounded and homeomorphic to ω^{β_k}. So $\left[C\right] = \left[(-\infty, -1] \cup \omega^{\beta_k}\right]$.

Now suppose that C has a connected component but all connected components of C are bounded. Without loss of generality, we can assume that C is bounded from below, but not from above. Since all connected components of C are bounded, we can get a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $\mathbb{R} \setminus C$ such that $b_{n+1} \geq b_n + 1$ for every $n \geq 1$, just like we did in the previous cases, and since C is closed, for every b_n there exists $\epsilon_n > 0$ such that $(b_n - \epsilon_n, b_n + \epsilon_n) \subseteq \mathbb{R} \setminus C$. Furthermore we can choose $\{b_n\}_{n\in\mathbb{N}}$ such that

$[b_n+\epsilon_n, b_{n+1}-\epsilon_{n+1}] \cap C \neq \emptyset$ for every n and $C = \bigcup_{n\in\mathbb{N}} [b_n+\epsilon_n, b_{n+1}-\epsilon_{n+1}] \cap C$.

Case a) Suppose that there exists a subsequence $\{b_{k_n}\}_{n\in\mathbb{N}}$ of $\{b_n\}_{n\in\mathbb{N}}$ such that $C \cap [b_{k_n}+\epsilon_{k_n}, b_{k_n+1}-\epsilon_{k_n+1}]$ contains a closed interval, say C_{k_n}. Then $[2n, 2n+1]$ can be embedded into C_{k_n} for every n, which means that $\bigcup_{n\geq 1}[2n, 2n+1]$ can be embedded into C. On the other hand, for each n, $C \cap [b_n+\epsilon_n, b_{n+1}-\epsilon_{n+1}]$ can be embedded into $[2n, 2n+1]$. So, $\Big[C\Big] = \Big[\bigcup_{i\geq 1}[2i, 2i+1]\Big]$.

Case b) Now consider the case in which, only for finitely many (nonzero) $n \in \mathbb{N}$, $C \cap [b_n+\epsilon_n, b_{n+1}-\epsilon_{n+1}]$ contains an interval. Let M be the largest positive integer with this property. Then, $C \cap (-\infty, b_{M+1}-\epsilon_{M+1}]$ is compact and can be embedded into $[0,1]$ and vice versa. We have two cases.

Case b1) Assume that we can find a subsequence $\{b_{k_n}\}_{n\in\mathbb{N}}$ of $\{b_n\}_{n\in\mathbb{N}}$ such that $k_n \geq M$ for every n, and that $Perf(C\cap[b_{k_n}+\epsilon_{k_n}, b_{k_n+1}-\epsilon_{k_n+1}])$ is nonempty. As we said in previous cases, $Perf(C\cap[b_{k_n}+\epsilon_{k_n}, b_{k_n+1}-\epsilon_{k_n+1}])$ is a Cantor set. Therefore $[0,1]\cup\bigcup_{i\geq 1}\mathcal{C}_i$ can be embedded into C. In addition, by Lemma 4.2.3, $C\cap[b_n+\epsilon_n, b_{n+1}-\epsilon_{n+1}]$ can be embedded into $\mathcal{C}_n$ for every $n \geq M+1$. Then, $C\cap[b_{M+1}+\epsilon_{M+1}, \infty)$ can be embedded into $\bigcup_{n\in\mathbb{N}}\mathcal{C}_n$ and vice versa. This implies that $\Big[C\Big] = \Big[[0,1]\cup\bigcup_{i\geq 1}\mathcal{C}_i\Big]$.

Case b2) Suppose that there exists a largest positive number L such that $C \cap [b_L + \epsilon_L, b_{L+1}-\epsilon_{L+1}]$ contains a Cantor set. Let N be the maximum of $\{M, L\}$. $C \cap (-\infty, b_{N+1}-\epsilon_{N+1}]$ is compact, so we can embed $C \cap (-\infty, b_{N+1}-\epsilon_{N+1}]$ into $[0,1]$. On the other hand, $C \cap [b_{N+1}+\epsilon_{N+1}, \infty)$ is scattered and unbounded, therefore it is homeomorphic to a countable ordinal α. Let $\omega^{\beta_1}c_1 + \omega^{\beta_2}c_2 + ... + \omega^{\beta_k}c_k$ be the Cantor normal form of α. Since α is not compact, we can suppose that $\beta_k \geq 1$. Notice that $\alpha = \omega^{\beta_1}c_1+\omega^{\beta_2}c_2+...+\omega^{\beta_k}(c_k-1)+1+\omega^{\beta_k}$. The ordinal $\omega^{\beta_1}c_1+\omega^{\beta_2}c_2+...+\omega^{\beta_k}(c_k-1)+1$ is compact, then the part of $C\cap[b_{N+1}+\epsilon_{N+1}, \infty)$ which is homeomorphic to it, say D, is bounded and can be embedded into $[0,1]$ as well. Hence $C\setminus((-\infty, b_{N+1}-\epsilon_{N+1}]\cup D)$

is unbounded and homeomorphic to ω^{β_k}. So $\Big[C\Big] = \Big[[0,1] \cup \omega^{\beta_k}\Big]$.

Case c) Assume that C contains no intervals at all.

Case c1) If we can find a subsequence $\{b_{k_n}\}_{n\in\mathbb{N}}$ of $\{b_n\}_{n\in\mathbb{N}}$ such that $Perf(C \cap [b_{k_n} + \epsilon_{k_n}, b_{k_n+1} - \epsilon_{k_n+1}])$ is nonempty, just as we did in previous cases, $\mathcal{C}_n$ can be embedded into $C \cap [b_{k_n} + \epsilon_{k_n}, b_{k_n+1} - \epsilon_{k_n+1}]$. Then $\bigcup\limits_{i\geq 1} \mathcal{C}_i$ can be embedded into C. On the other hand, by Lemma 4.2.3, $C \cap [b_n + \epsilon_n, b_{n+1} - \epsilon_{n+1}]$ can be embedded into $\mathcal{C}_n$ for every $n \in \mathbb{N}$, since each $C \cap [b_n + \epsilon_n, b_{n+1} - \epsilon_{n+1}]$ is totally disconnected and compact. Hence, $\Big[C\Big] = \Big[\bigcup\limits_{i\geq 1} \mathcal{C}_i\Big]$.

Case c2) If there exists a largest positive number L such that $C \cap [b_L + \epsilon_L, b_{L+1} - \epsilon_{L+1}]$ contains a Cantor set, then $C \cap (-\infty, b_{L+1} - \epsilon_{L+1}]$ is compact and, since it is totally disconnected, by Lemma 4.2.3, can be embedded into $\mathcal{C}_1$. On the other hand, $C \cap [b_{L+1} + \epsilon_{L+1}, \infty)$ is scattered and unbounded, therefore it is homeomorphic to a countable ordinal α. Let $\omega^{\beta_1} c_1 + \omega^{\beta_2} c_2 + \ldots + \omega^{\beta_k} c_k$ be the Cantor normal form of α. Since α is not compact, we can suppose that $\beta_k \geq 1$. Notice that $\alpha = \omega^{\beta_1} c_1 + \omega^{\beta_2} c_2 + \ldots + \omega^{\beta_k}(c_k - 1) + 1 + \omega^{\beta_k}$. The ordinal $\omega^{\beta_1} c_1 + \omega^{\beta_2} c_2 + \ldots + \omega^{\beta_k}(c_k - 1) + 1$ is compact and totally disconnected, then, by Lemma 4.2.3, the part of $C \cap [b_{L+1} + \epsilon_{L+1}, \infty)$ which is homeomorphic to it, say D, can be embedded into $\mathcal{C}_1$ as well. Hence $C \setminus ((-\infty, b_{L+1} - \epsilon_{L+1}] \cup D)$ is unbounded and homeomorphic to ω^{β_k}. So $\Big[C\Big] = \Big[\mathcal{C}_1 \cup \omega^{\beta_k}\Big]$.

Case c3) Suppose that C is scattered. Then C is homeomorphic to a countable ordinal α. Let $\omega^{\beta_1} c_1 + \omega^{\beta_2} c_2 + \ldots + \omega^{\beta_k} c_k$ be the Cantor normal form of α. Since α is not compact, we can suppose that $\beta_k \geq 1$. Notice that $\alpha = \omega^{\beta_1} c_1 + \omega^{\beta_2} c_2 + \ldots + \omega^{\beta_k}(c_k - 1) + 1 + \omega^{\beta_k}$. The ordinal $\omega^{\beta_1} c_1 + \omega^{\beta_2} c_2 + \ldots + \omega^{\beta_k}(c_k - 1) + 1$ is compact, and the part of $C \cap [b_{L+1} + \epsilon_{L+1}, \infty)$ which is homeomorphic to it, say D, is homeomorphic copy of $\omega^{\beta_1} c_1 + 1$ (see Theorem 1.1.1). In addition $C \setminus D$ is unbounded and homeomorphic to

ω^{β_k}. So $\Big[C\Big] = \Big[\omega^{\beta_1}c_1 + \omega^{\beta_k}\Big]$. $\square$

Consider the relation $\lhd$ on $2^{\mathbb{R}}/\!\sim$ as follows: given $A, B \in 2^{\mathbb{R}}$, $[A] \lhd [B]$ if and only if A is embeddable into B. The partial order $\lhd$ will be called the *embeddability order* on $2^{\mathbb{R}}/\!\sim$. Our next goal is to provide a description of $(2^{\mathbb{R}}/\!\sim, \lhd)$.

The following list of conditions (A-F and C1-C11) describes the order $\lhd$ on $2^{\mathbb{R}}/\!\sim$ (for a summary, go to Figure 4.1).

(A) $\Big[\emptyset\Big] \lhd \Big[E\Big] \lhd \Big[\mathbb{R}\Big]$ for every $E \in 2^{\mathbb{R}}$;

(B) for every countable ordinal α,

$$\begin{aligned}
\Big[[0,1] \cup \bigcup_{i\geq 1}\mathcal{C}_i\Big] &\lhd \Big[\bigcup_{i\geq 1}[2i, 2i+1]\Big] \\
&\lhd \Big[(-\infty, -1]\Big] \\
&\lhd \Big[(-\infty, -1] \cup \omega^{\alpha}\Big] \\
&\lhd \Big[(-\infty, -1] \cup \bigcup_{i\geq 1}\mathcal{C}_i\Big] \\
&\lhd \Big[(-\infty, -1] \cup \bigcup_{i\geq 1}[2i, 2i+1]\Big] \\
&\lhd \Big[(-\infty, -1] \cup [1, \infty)\Big];
\end{aligned}$$

(C) for any countable ordinals α_1, α_2 and every $n \in \omega$,

$$\begin{aligned}
\Big[\omega^{\alpha_1}n + 1\Big] &\lhd \Big[\mathcal{C}\Big] \\
&\lhd \Big[[0,1]\Big] \\
&\lhd \Big[[0,1] \cup \omega^{\alpha_2}\Big] \\
&\lhd \Big[[0,1] \cup \bigcup_{i\geq 1}\mathcal{C}_i\Big];
\end{aligned}$$

(D) for any countable ordinals α_1, α_2 and every $n \in \omega$,

$$\begin{aligned}\left[\omega^{\alpha_1} n + 1\right] &\lhd \left[\mathcal{C}\right] \\ &\lhd \left[\mathcal{C} \cup \omega^{\alpha_2}\right] \\ &\lhd \left[\bigcup_{i \geq 1} \mathcal{C}_i\right] \\ &\lhd \left[[0,1] \cup \bigcup_{i \geq 1} \mathcal{C}_i\right];\end{aligned}$$

(E) for every pair of countable ordinals α, β, with $0 < \beta < \alpha$, and every $n \geq 1$,

$$\left[\omega^{\alpha} n + \omega^{\beta}\right] \lhd \left[\bigcup_{i \geq 1} \mathcal{C}_i\right]$$

(F) for any countable ordinal α, with $0 < \alpha$, and every $n \geq 1$,

$$\left[\omega^{\alpha} n\right] \lhd \left[\bigcup_{i \geq 1} \mathcal{C}_i\right]$$

Observe that the cases (2), (5), (6), (8), (9) and (10) of Theorem 4.3.1 are not single sets, but families of sets. Therefore, for these particular cases, we have the following conditions.

Note. Recall that a countable ordinal ω^{α} can be embedded as a closed subset into ω^{β} if $\alpha < \beta$, into $\bigcup_{i \geq 1} \mathcal{C}_i$, into $\bigcup_{i \geq 1} [2i, 2i+1]$, and into $(-\infty, -1]$.

(C1) For any countable ordinals α, β and any $n, m \in \omega$,

$$\left[\omega^{\alpha} n + 1\right] \lhd \left[\omega^{\beta} m + 1\right]$$

if either $\alpha < \beta$, or $\alpha = \beta$ and $n \leq m$.

(C2) For any countable ordinals α, β, γ, with $0 < \gamma < \beta$, and any $n, m \in \omega$,

$$\left[\omega^\alpha n + 1\right] \lhd \left[\omega^\beta m + \omega^\gamma\right]$$

if either $\alpha < \beta$, or $\alpha = \beta$ and $n \leq m$.

(C3) For any countable ordinals $\alpha, \beta, \gamma, \delta$, with $0 < \beta < \alpha$ and $0 < \delta < \gamma$, and any $n, m \geq 1$,

$$\left[\omega^\alpha n + \omega^\beta\right] \lhd \left[\omega^\gamma m + \omega^\delta\right]$$

if either $\alpha < \gamma$ and $\beta \leq \delta$; or $\alpha = \gamma$, $n \leq m$, and $\beta \leq \delta$.

(C4) For any countable ordinals α, β, γ, with $0 < \beta < \alpha$ and $0 < \gamma$, and any $n, m \geq 1$,

$$\left[\omega^\alpha n + \omega^\beta\right] \lhd \left[\omega^\gamma m\right]$$

if either $\alpha < \gamma$; or $\alpha = \gamma$ and $n < m$.

(C5) For any countable ordinals α, β, γ, with $0 < \beta < \alpha$ and $0 < \gamma$, and any $n, m \geq 1$,

$$\left[\omega^\gamma m\right] \lhd \left[\omega^\alpha n + \omega^\beta\right]$$

if $\gamma < \beta$.

(C6) For any countable ordinals α, β, with $0 < \alpha$ and $0 < \beta$, and any $n, m \geq 1$,

$$\left[\omega^\alpha n\right] \lhd \left[\omega^\beta m\right]$$

if either $\alpha < \beta$; or $\alpha = \beta$ and $n \leq m$.

(C7) For any non-zero countable ordinals $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$, with $\alpha_1 < \alpha_2$, and any $n \geq 1$,

$$\begin{aligned}
\left[\omega^{\alpha_1} n + \omega^{\alpha_2}\right] &\lhd \left[\mathcal{C} \cup \omega^{\alpha_3}\right] \\
&\lhd \left[\mathcal{C} \cup \omega^{\alpha_4}\right] \\
&\lhd \left[[0,1] \cup \omega^{\alpha_5}\right] \\
&\lhd \left[[0,1] \cup \omega^{\alpha_6}\right] \\
&\lhd \left[(-\infty,-1] \cup \omega^{\alpha_7}\right] \\
&\lhd \left[(-\infty,-1] \cup \omega^{\alpha_8}\right]
\end{aligned}$$

if $\alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \leq \alpha_7 \leq \alpha_8$.

(C8) For any countable ordinals α, β, with $0 < \beta$, and any $n, m \in \omega$, with $1 \leq m$,

$$\left[\omega^{\alpha} n + 1\right] \lhd \left[\omega^{\beta} m\right]$$

if either $\alpha < \beta$; or $\alpha = \beta$ and $n < m$.

(C9) For any two non-zero countable ordinals α, β, and any $n \geq 1$,

$$\left[\omega^{\alpha} n\right] \lhd \left[\mathcal{C} \cup \omega^{\beta}\right]$$

if $\alpha \leq \beta$.

(C10) For any non-zero countable ordinals α, β, γ, with $\beta < \alpha$, and any $n \geq 1$,

$$\left[\omega^{\alpha} n + \omega^{\beta}\right] \lhd \left[[0,1] \cup \omega^{\gamma}\right]$$

if $\beta \leq \gamma$.

(C11) For any non-zero countable ordinals α, β, and any $n \geq 1$,

$$\left[\omega^{\alpha} n\right] \triangleleft \left[[0,1] \cup \omega^{\beta}\right]$$

if $\alpha \leq \beta$.

Figure 4.1 illustrates the embeddability order on $2^{\mathbb{R}}/\sim$. In it, $A \triangleleft B$ is represented by an arrow or a sequence of arrows going from A to B. In the case of curved arrows, some condition from the list (C1-C11) is required for the existence of an embedding.

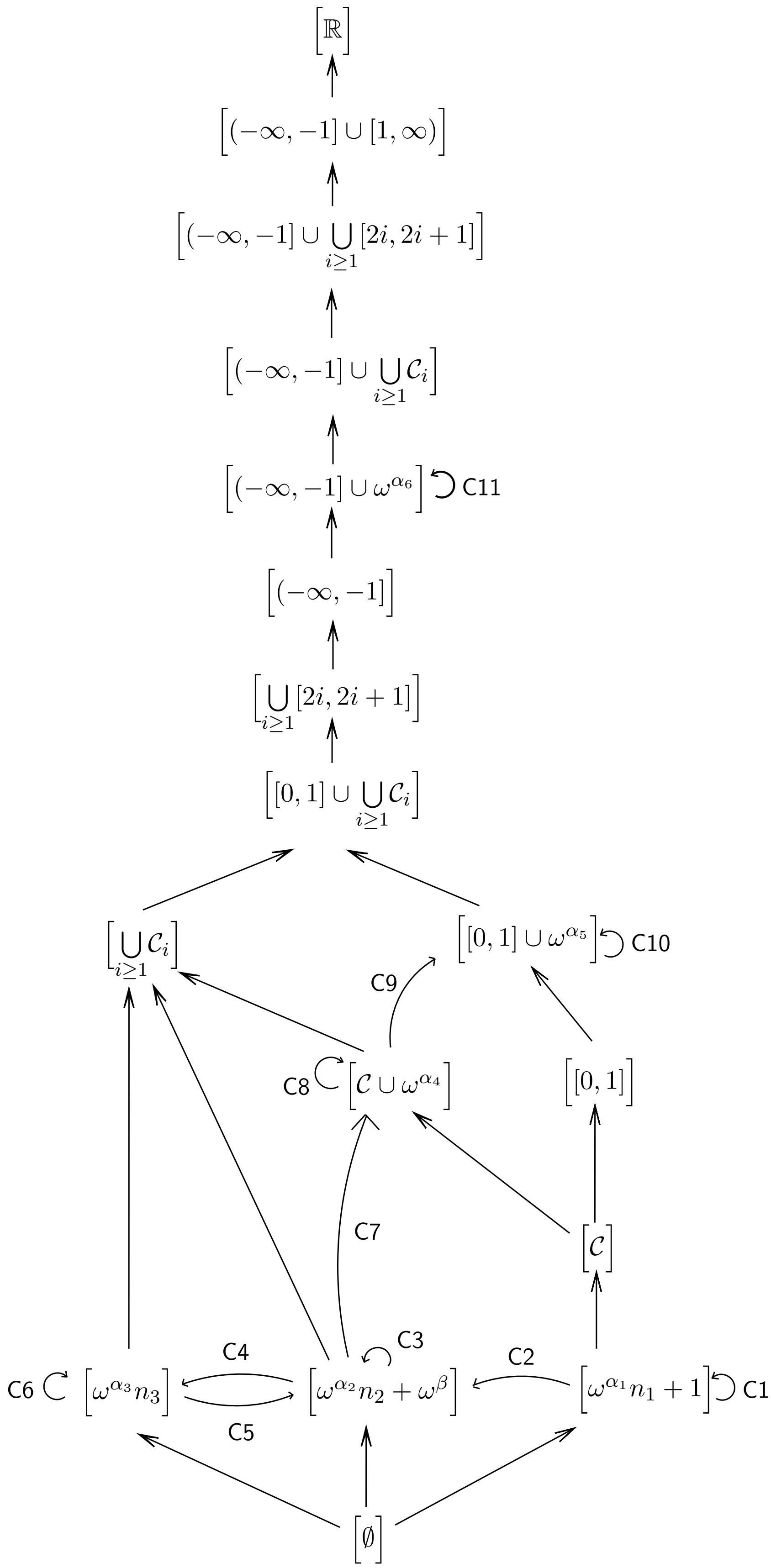

Figure 4.1: Embeddability order on $2^{\mathbb{R}}/\sim$

Theorem 4.3.1 also provides us with an immediate characterisation of $\mathcal{K}(\mathbb{R})/\sim$, where $\mathcal{K}(\mathbb{R})$ denotes the family of compact subspaces of $\mathbb{R}$.

COROLLARY 4.3.2. *Consider the relation $\sim_K$ defined on the family of compact subspaces of $\mathbb{R}$ as follows: $A \sim B$ if and only if A can be embedded homeomorphically into B as a compact subset and vice versa. Let C be a compact subset of $\mathbb{R}$. C belongs to the equivalence class of one of the following sets with respect to relation $\sim_K$.*

1) $\emptyset$*;*

2) $\omega^\alpha n + 1$*, for some countable ordinal* α *and some* $n \in \omega$*;*

3) $\mathcal{C}$*;*

4) $[0, 1]$. □

Concluding Remarks

In this thesis we studied countable dynamical systems from the perspective of three different topics. The scattered structure of countable dynamical systems was the fundamental notion that unified these subjects. Presently, we give a summary of the main results of each chapter and we raise some questions that can lead to future research (at present, we are unable to make any conjecture).

In Chapter 2 we characterised the pairs of maps $f : X \to Y$ and $g : X \to Z$ which can be made simultaneously continuous when defining the One-Point Compactification of a discrete space topology on Y and Z (Theorem 2.3.9). It was also proved that this result does not solve the problem of simultaneous topologising because a pair of functions can induce a scattered structure on the sets as complicated as desired (Theorem 2.4.2). The following are some of the questions that arise naturally:

(1) What can be said on simultaneous topologisation when considering a pair of functions $f : X \to Z$ and $g : Y \to Z$?;

(2) What can be said on simultaneous topologisation when considering a pair of functions $f : X \to X$ and $g : X \to X$? (In this case, Suabedissen [28] has made some remarks about the difficulty of finding a general solution, but probably it can be solved for particular instances of X, for example when X is countable);

(3) Given functions $f : X \to Y$ and $g : X \to Z$, is it possible to find compact metrisable topologies on X, Y and Z that make f and g simultaneously continuous?

In Chapter 3 we proved the ubiquity of countable dynamical systems in interval maps (Theorem 3.3.2). We also characterised the continuous functions on countable compact spaces (Theorem 3.4.3). Additionally, we learned that their top limit points form cycles among them (Theorem 3.4.6) and that a repelling fixed point of a countable system with finite fibres cannot have a limit ordinal as its limit type (Theorem 3.4.11). Regarding transitivity, Theorem 3.5.6 proves that there are countable systems with a scattered structure as complex as desired. On the other hand, Theorem 3.6.14 tell us that the ω-limit sets of a compact countable system with shadowing are periodic orbits.

(1) Can we obtain a characterisation of the countable dynamical systems with the shadowing property?;

(2) There are many dynamical notions we have not look at, such as recurrence, non-wondering sets, mixing, etc. What can we say about them in the context of countable dynamical systems?

In Chapter 4 we focus on the ordering by embeddability as a closed subset on the family of closed subspaces of the real line. We succeeded at characterising the poset $(2^{\mathbb{R}}/{\sim}, \lhd)$. We find interesting the following question.

(1) Is it possible to characterise the ordertypes of $(2^{\mathbb{R}}/{\sim}, \lhd)$?

List of References

[1] E. Akin, J. D. Carlson. *Conceptions of topological transitivity*, Topology and its Applications, 159: 2815-2830 (2012).

[2] E. Akin. *The General Topology of Dynamical Systems*, Graduate Studies in Mathematics, American Mathematical Society, 1993.

[3] L.S. Block, W.A. Coppel *Dynamics in One Dimension* Springer Verlag, 1992.

[4] J. Bobok. *Chaos in countable dynamical system*, Topology and its Applications, 126: 207-216 (2002).

[5] W. Brian. *Abstract ω-limit sets*, Preprint, 2010.

[6] L.E.J. Brouwer. *On the structure of perfect sets of points*, Proc. Akad. Amsterdam, 12: 785-794 (1910).

[7] W.W. Comfort, W.D. Gillam. *The embeddability ordering of topological spaces*, Topology and its Applications, 153: 2192-2198 (2006).

[8] M. Denker, C. Grillenberger, K. Sigmund. *Ergodic Theory on Compact Spaces*, in Lecture Notes in Math., Vol. 527, Springer-Verlag, Berlin, 1977.

[9] J. de Groot, H. de Vries. *Metrization of a set which is mapped into itself*, Quart. J. Math. Oxford (2), 144-148, 9 (1958).

[10] D. Ellis. *Orbital topologies*, Quart. J. Math. Oxford (2), 117-119, 4 (1953).

[11] R. Engelking. *General Topology*, Heldermann Verlag, 1989.

[12] J. Flum. *Model Theory of Regular and Compact Spaces* in Logic Colloquium '87, Vol. 129, edited by H. D. Ebbinghaus et al., North Holland, 1989.

[13] W. D. Gillam. *Embeddability properties of countable metric spaces*, Topology and its Applications, 148: 63-82 (2005).

[14] C. Good, S. Greenwood, R. W. Knight, D. W. McIntyre, S. Watson. *Characterizing continuous functions on compact spaces*, Advances in Mathematics, Vol. 206, Iss. 2, 695-728 (2006).

[15] C. Good, R. Knight, B. Raines. *Countable Inverse Limits of Poscritical ω-limit Sets of Unimodal Maps*, Preprint.

[16] C. Good, J. Meddaugh. *Shifts of Finite Type as Fundamental Objects in the Theory of Shadowing*, Preprint.

[17] W. Huang, X. Ye. *Homeomorphisms with the whole compacta being scrambled sets*, Ergod. Th. and Dynam. Sys., 21: 77-91 (2001).

[18] H. Kato, J. J. Park. *Expansive homeomorphisms of countable compacta*, Topology and its Applications, 95: 207-216 (1999).

[19] V. Kieftenbeld, B. Löwe. *A classification of ordinal topologies*, Preprint, 2006.

[20] R.W. Knight, A.E. McCluskey. *$\mathcal{P}(\mathbb{R})$, ordered by homeomorphic embeddability, does not represent all posets of cardinality $2^{\mathfrak{c}}$*, Topology and its Applications, 156: 1943-1945 (2009).

[21] K. Kunen. *Set Theory. An Introduction to Independence Proofs*, North Holland, 1983.

[22] C. Kuratowski. *Sur la puissance de l'ensemble des "nombres de dimension" au sens de M. Frechet*, Fund. Math. 8: 201-208 (1926).

[23] C. Kuratowski, W. Sierpiński. *Sur un problème de M. Fréchet concernant les dimensions des ensembles linéaires*, Fund. Math. 8: 193-200 (1926).

[24] P. T. Matthews and T.B.M. McMaster. *Families of spaces having prescribed embeddability ordertype*, Rend. Ist. Mat. Univ. Trieste 25: 345-352 (1993).

[25] A.E. McCluskey, T.B.M. McMaster, W.S. Watson. *Representing set-inclusion by embeddability (among the subspaces of the real line)*, Topology and its Applications, 96: 89-92 (1999).

[26] M. Powderly, H. Tong. *On orbital topologies*, Quart. J. Math. Oxford (2), 1-2, 7 (1956).

[27] S. Purisch. *The orderability and closed images of scattered spaces*, Transactions of the American Mathematical Society, Vol. 320, No. 2, 713-725 (1990).

[28] R. Suabedissen. *Topologizing Group Actions*, Doctoral thesis, University of Oxford, 2006.

[29] J. Van Mill. *The Infinite-Dimensional Topology of Function Spaces*, North Holland, 2001.

[30] W. S. Watson. *The construction of topological spaces:planks and resolutions*, in: M. Hušek and J. van Mill, eds., Recent Progress in General Topology, North-Holland (1992).

[31] S. Willard. *General Topology*, Dover, 2004.