ASPECTS OF ANISOTROPIC HARMONIC ANALYSIS BEYOND CALDERÓN-ZYGMUND THEORY

by

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Abstract

We consider three major parts of Fourier analysis and their role in Fefferman-Stein inequalities. The three areas can be considered as three separate topics in their own right, or as three steps to proving certain $L^p - L^q$ inequalities via the Fefferman-Stein inequalities of the form

$$\int_{\mathbb{R}^n} |Tf|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 \mathcal{M}w.$$

The first area discussed is that of maximal functions, specifically obtaining $L^p - L^q$ inequalities on large classes of maximal functions. We then use a simple duality argument to transfer these to operators where we have a Fefferman-Stein inequality via

$$\|T\|_{p\to q} \lesssim \|\mathcal{M}\|^{1/2}_{(q/2)'\to(p/2)'}.$$

The second area aims to control operators defined via multipliers by the previous section's geometrically defined maximal functions. In particular, we build up to a schema that can be used to prove Fefferman-Stein inequalities via the so called g-functions, originating in work of E. M. Stein [38] but having historic roots that can be easily seen by viewing g-functions as speciality square functions.

In the final section we consider some classes of operators with oscillatory kernels and obtain estimates on their multipliers, and by application of the previous two sections obtain some $L^p - L^q$ inequalities.

Dedication

To my father, a truly inspirational man of endless altruism and thunderous intellect.

Acknowledgements

First and foremost I would like to thank my supervisor, Jonathan Bennett; his calm approach to any problem has been a guiding light throughout the darkest hours. I would also like to thank the University of Birmingham Mountaineering Club, for their welcoming nature and continued friendship. Finally, I would like to thank Sean Plott, who has had a profound effect in shaping my love of mathematics, life and myself.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Calderón-Zygmund theory

The matters treated in this thesis are mostly anisotropic in nature, but it would be pertinent to first give an overview of the three main areas covered in their isotropic setting. In sketching the broad outlines we will separate out our discussion into three main pillars of harmonic analysis: maximal functions, weighted Littlewood-Paley theory and oscillatory integrals. However it should be stressed that these are not separate areas at all, but instead are all interconnected and all related, in one form or another, to the study of singular integrals. There are many differing introductions to harmonic analysis that depend on an even larger variety of perspectives, but our chosen point of departure is the real variable techniques introduced by A. P. Calderón and A. Zygmund in the 1950's. The goal of these techniques was to study the higher dimension analogues of the Hilbert transform, known as singular integrals, but our discourse will not heavily focus on these objects. Instead, we will consider questions outside of singular integrals and show that the tools and methods developed continue to work efficiently far beyond the framework they were intended for.

The overarching structure of this thesis has roots in a paper by Bennett, Carbery, Soria and Vargas [4] where they studied a conjecture of Stein on the circle. Further advancement was made by Bennett and Harrison [5] on the line. Later, Bennett [3] took a multiplier perspective of these same issues on the line, and Beltran and Bennett [2] extended this result to $\boldsymbol{R}^d$.

In Chapter 2 we will focus on maximal functions, the simplest of which is the Hardy-Littlewood maximal function, defined on an admissible function, f, at a point x as

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} f(x-y)dy.$$

This maximal function naturally arises when we consider the family of averages such as

$$A_r f(x) = \frac{1}{2r} \int_{-r}^{r} f(x-y)dy$$

for $r > 0$. The most interesting properties of these averages are their behaviour as $r \to 0$, which are extracted via consideration of their corresponding maximal function, M. These were originally studied as a means to understand the convergence of Fourier series, but have far reaching applications beyond this. Chapter 2 will build up to the study of our maximal function, given by

$$\mathcal{M}_{A,\alpha,\beta} f(x) = \sup_{(y,t)\in\Gamma_{A,\alpha}(x)} (t^{\nu})^{2\beta} |\vartheta_{A(t)} * f(y)|$$

where

$$\Gamma_{A,\alpha}(x) = \{(y,t) \in \mathbb{R}^2 \times \mathbb{R}_+ : 0 < t^{\alpha} \leqslant 1,\ \rho_A(x-y) \leqslant t^{1-\alpha}\},$$

ϑ is a positive, radial, decreasing (radially) Schwarz function with total mass 1, ρ_A is some anisotropic norm with respect to the dilation matrices A and the notation $\vartheta_{A(t)}$ refers to dilations of the function by these matrices. All of these terms will be more accurately defined in Section 1.3. $\Gamma_{A,\alpha}(x)$ are regions in the upper half space, and will be discussed at the end of Section 2.1.

This chapter will culminate in the $L^p - L^q$ bounds for $\mathcal{M}_{A,\alpha,\beta}$, which are given by the following theorem

Theorem 1.1.1 *Let* $1 < p \leqslant q \leqslant \infty$ *and* $\alpha, \beta \in \mathbb{R}$.

- *If* $\alpha < 0$ *and* $\beta \leqslant \frac{\alpha}{2q} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$;
- *or* $\alpha = 0$ *and* $\beta = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$;
- *or* $\alpha > 0$ *and* $\beta \geqslant \frac{\alpha}{2q} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$;

then for weights, w, we have

$$\|\mathcal{M}_{A,\alpha,\beta} w\|_q \leqslant C\|w\|_p,$$

for some constant $C > 0$.

Chapter 3 will focus on the area of Fourier multipliers, and we will obtain our results as an application of Littlewood-Paley theory. This area of study concerns itself with the extension of the Pythagorean theorem: if x in a Hilbert space is a sum of orthogonal basis vectors, then the sum of the squares of these basis vectors is equal to the square of the sum. This theorem clearly relies heavily on orthogonality; however, for more general Banach spaces, such as L^p ($p \neq 2$) spaces, we don't obviously have a notion of orthogonality. This is where Littlewood-Paley theory comes in and gives us ways of decomposing our functions, f, into special basis functions that essentially determine the size of f. We will use Littlewood-Paley theory in the spirit of Stein [38] to prove the multiplier theorem *

Theorem 1.1.2 *Let* $\gamma \in \mathbb{N}_0^2$. *If m is a Fourier multiplier such that*

$$|D^\gamma m(\xi)| \lesssim \rho_A(\xi)^{-\beta\nu + \|\gamma\|_A(\alpha-1)} \tag{1.1}$$

*Again, see Section 1.3 for undefined terms *for m with support in* $\{\xi \in \mathbb{R}^2 : |\xi|^\alpha \geqslant 1\}$ *and all* $|\gamma| \leqslant 3$, *then*

$$\int_{\mathbb{R}^2} |T_m f(x)|^2 w(x) dx \lesssim \int_{\mathbb{R}^2} |f(x)|^2 M_A^4 \mathcal{M}_{A,\alpha,\beta} M_A^3 w(x) dx, \tag{1.2}$$

where M_A *refers to an anisotropic adaption of the Hardy-Littlewood maximal function and is defined as*

$$M_A f(x) = \sup_{t>0} \chi_{A(t)} * |f|(y)$$

and M_A^n *is the n-fold composition of* M_A.

To finish our discourse Chapter 4 will focus on estimating the multipliers associated with certain Hirschmann kernels. This final chapter will make use of the deep results of the previous two chapters to bound large families of highly oscillatory and sometimes very singular kernels.

1.2 Fefferman-Stein inequalities

There has been interest in recent decades in finding Fefferman-Stein inequalities of the form

$$\int_{\mathbb{R}} |Tf|^p w \lesssim \int_{\mathbb{R}} |f|^p \mathcal{M}w, \tag{1.3}$$

where T is a suitable operator, $p \in [1, \infty)$, $\mathcal{M}$ is a maximal operator, f is an admissible input function and w is a non-negative locally integrable function, herein referred to as weights.*

By a simple duality argument, inequalities like (1.3) are of interest as they allow us to transfer bounds on $\mathcal{M}$ to bounds on T as follows. Let t' denote the Hölder conjugate of t and let $q, r \geqslant p$, then

$$\begin{aligned} \|Tf\|_q &= \sup_{\|w\|_{(q/p)'}=1} \left(\int_{\mathbb{R}} |Tf|^p w \right)^{1/p} \\ &\lesssim \sup_{\|w\|_{(q/p)'}=1} \left(\int_{\mathbb{R}} |f|^p \mathcal{M}w \right)^{1/p} \\ &\leqslant \sup_{\|w\|_{(q/p)'}=1} \|\mathcal{M}w\|_{(r/p)'}^{1/p} \|f\|_r, \end{aligned}$$

thus

$$\|T\|_{r\to q} \lesssim \|\mathcal{M}\|_{(q/p)'\to(r/p)'}^{1/p}. \tag{1.4}$$

So for such operators T, there is interest in identifying a corresponding geometrically defined maximal operator $\mathcal{M}$ that is optimal in the sense that all $L^q \to L^r$ mapping properties of T can be deduced from bounds on $\mathcal{M}$ and (1.4).

We will appear to digress momentarily from Fefferman-Stein inequalities in order to give a brief outline of some of the theory that is related to these inequalities and the power of using this

*See Section 1.3 for an explanation of notation approach, opposed to the method of Muckenhoupt or A_p weight type inequalities. Classically, Fefferman and Stein proved the following theorem.

Theorem 1.2.1 ([17]) *Let M denote the Hardy-Littlewood maximal function and let f be any admissible input function. If w is a weight, then for any $1 < p < \infty$*

$$\int_{\mathbb{R}} |Mf|^p w \lesssim \int_{\mathbb{R}} |f|^p Mw.$$

It is perhaps convenient at this moment to discuss the approach of Muckenhoupt, known as A_p theory. This approach can be defined from the perspective of this inequality quite directly; indeed, one can define the class of A_p as the class of weights such that

$$\int_{\mathbb{R}} |Mf|^p w \lesssim \int_{\mathbb{R}} |f|^p w.$$

This is a single weight inequality and a prototype inequality for this theory.
While the two weight inequality in Theorem 1.2.1 doesn't lend itself to the approach developed in light of (1.4), as the controlling maximal function is the same as the operator we are controlling, many inequalities that do benefit from the approach were built upon it and all share this standard structure.
The Calderón-Zygmund singular integral operators have been the focus of large amounts of study in harmonic analysis for decades. We will briefly give an overview of the role these Fefferman-Stein inequalities played in the area of singular integral operators after the definition of them.

Definition 1.2.2 *We call an operator T a Calderón-Zygmund operator if the following hold*

1. $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$;

2. *there exists a measurable function $K : \mathbb{R} \to \mathbb{R}$ such that for every $f \in L_0^\infty(\mathbb{R}^2)$ we have*

$$Tf(x) = \int_{\mathbb{R}} K(x-y)f(y)dy$$

for a.e. $x \notin \operatorname{supp}(f)$;

3. *the kernel K satisfies*

$$|K(x)| \lesssim \frac{1}{|x|}$$

for every $x \in \mathbb{R}$;

4. *the kernels K and K^* (defined by $K^*(x) = K(-x)$) satisfy the following pointwise Hörmander condition: There exist a positive constants $M > 1$ and $\gamma > 0$ such that whenever $\rho_A(y) < \frac{1}{M}\rho_A(x)$ we have*

$$|K(x) - K(x-y)| \lesssim \frac{|y|^\gamma}{|x|^{1+\gamma}}.$$

After his paper with Stein, Fefferman went on to write a paper with Córdoba where they proved the following theorem.

Theorem 1.2.3 ([14]) *If T is a Calderón-Zygmund singular integral operator** on the line then for any $p, s > 1$ we have*

$$\int_{\mathbb{R}} |Tf|^p w \lesssim \int_{\mathbb{R}} |f|^p (Mw^s)^{1/s},$$

where the implicit constant depends on at most p and s.

*Defined by the case $A = I$ in Definition 3.2.1

For a fixed $s > 1$, we conclude that T is L^p bounded for $p > s$, which is extracted via (1.4) and the known non-weighted L^p boundedness of M [16]. However, T is bounded on L^p for all $1 < p < \infty$; this gap was first reduced by Wilson.

Theorem 1.2.4 ([43]) *If T is a Calderón-Zygmund singular integral operator on the line*

$$\int_{\mathbb{R}} |Tf|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^3 w, \tag{1.5}$$

where the maximal operator, M^k, is the k-fold composition of M with itself.

In the same paper, Wilson also proved inequalities for p other than 2, one of which is the following

Theorem 1.2.5 ([43]) *For $1 < p < 2$, we have*

$$\int_{\mathbb{R}} |Tf|^p w \lesssim \int_{\mathbb{R}} |f|^p M^2 w.$$

Soon after, Pérez unified and extended these results by proving the following theorem.

Theorem 1.2.6 ([32]) *For $1 < p < \infty$ we have*

$$\int_{\mathbb{R}} |Tf|^p w \lesssim \int_{\mathbb{R}} |f|^p M^{[p]+1} w, \tag{1.6}$$

where $[p]$ is the integer part of p

With this we gain the full range of indices for which T is L^p bounded.
See [4] for further discussion of the flexibility of this method of capturing the behaviour of an operator by a maximal function.

1.3 Preliminaries

Every area of learning has its fair share of vices, and harmonic analysis is no different; indeed, the most famous is the disregard of constants, or the "constantly changing constant". This is due to the perspective harmonic analysis takes - we want to know the nature of how two quantities change with respect to each other. We will often forgo the use of a constant C, or c, to refer to a constant independent of the relevant variables to that equation by using the notation $A(t) \lesssim B(t)$ to mean that there exists $c > 0$ that does not depend upon t such that $A(t) \leqslant cB(t)$; likewise for $A \gtrsim B$. Some other abuses, akin to reusing C or c as different constants, in the area are referring to the Fourier transform of a function $f \in \mathcal{S}$, the class of Schwarz functions, via the definition

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx,$$

and the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{ix\xi} d\xi.$$

The scrutinous reader will take exception to the lack of appropriate scaling included in these definitions and they would be correct in pointing out that we actually incur a constant; that is if we apply the Fourier transform and then the inversion formula, we do not return to our original function but a constant multiple of it. However, with our view of constants in mind, as long as we are only taking finitely many iterations of the Fourier transform or its inverse, we have chosen to stick to the convention outlined above.

1.3.1 Definitions

We begin our study of anisotropic norms by first introducing our norms in the manner of Calderón and Torchinsky [7, 8]. However, in order to introduce our norms, we first must introduce our families of affine transformations, indexed by t, that we will base our norms on.

Definition 1.3.1 *For each $\sigma \geqslant 1$, and define for $t > 0$*

$$A(t) = \begin{pmatrix} t & 0 \\ 0 & t^\sigma \end{pmatrix}.$$

We claim that the dilations defined by

$$A'(t) = \begin{pmatrix} t^{a_1} & 0 \\ 0 & t^{a_2} \end{pmatrix},$$

where $a_2 \geqslant a_1 > 0$, are equivalent. In one direction this is simple, take $a_1 = 1$ and $a_2 = \sigma$. The reverse direction is almost as simple, take $\sigma = \frac{a_2}{a_1}$ and a dilation by $\tilde{A}(t)$ is equal, when taking a supremum in the sets $0 < t \leqslant 1$ or $t \geqslant 1$, to a dilation by $A(t^{a_1})$.

Note that this principle can be extended to any number of dimensions; indeed, if we have an ordered index set $a_1 \leqslant a_2 \leqslant ... \leqslant a_d$ we can merely take a scaling t^{a_1} instead of t and reduce the index set to $1 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant ... \leqslant \sigma_{d-1}$.

We will next outline some properties of the matrices $A(t)$.

Lemma 1.3.2 ([7]) *The affine transformations $A(t)$, indexed by t, form a continuous abelian group.*

Proof:

- Closure, let $t, s \in \mathbb{R}$

$$
\begin{aligned}
A(t)A(s) &= \begin{pmatrix} t & 0 \\ 0 & t^{\sigma} \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & s^{\sigma} \end{pmatrix} \\
&= \begin{pmatrix} ts & 0 \\ 0 & (ts)^{\sigma} \end{pmatrix} \\
&= A(ts).
\end{aligned}
$$

- Associativity, let $t, s, r \in \mathbb{R}$

$$
\begin{aligned}
(A(t)A(s))\, A(r) &= A(ts)A(r) \\
&= A(tsr) \\
&= A(t)A(sr) \\
&= A(t)\,(A(s)A(r))\,.
\end{aligned}
$$

- Identity, this is immediate as $A(1) = I$.

- Inverse element, given $t \in \mathbb{R}$ we have

$$
\begin{aligned}
A(t)A(t^{-1}) &= A(1) \\
&= I.
\end{aligned}
$$

- Commutativity, let $t, s \in \mathbb{R}$

$$\begin{aligned} A(t)A(s) &= A(ts) \\ &= A(st) \\ &= A(t)A(s). \end{aligned}$$

$\square$

Remark 1.3.3 *In fact, this group is an embedded Lie group by Cartan's closed subgroup theorem, as it is a closed subgroup of the general affine transformations, but this is outside the purview of this thesis.*

Remark 1.3.4 *We have used the notation D^γ, where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_d)$ is a multi-index to mean*

$$D^\gamma f(x) = \frac{\partial^{|\gamma|} f}{x_1^{\gamma_1} ... x_d^{\gamma_d}}.$$

We will now introduce the family of norms we will concern ourselves with for the major part of this thesis in the following definition.

Definition 1.3.5 *For each $\sigma \geqslant 1$, let $\rho_A : \mathbb{R}^2 \to [0, \infty)$ be defined by*

$$\begin{aligned} \rho_A(x) = 0 &\implies x = 0, \\ \rho_A(x) = t &\iff |A(t^{-1})x| = 1, \end{aligned}$$

for all $t > 0$.

Note that it is immediate that $\rho_A(x) = 1 \iff |x| = 1$ from the definition. Now we claim that the object we have defined is a norm associated with the matrix A, we will clarify and prove this in the following theorems starting with a proof that ρ_A is well-defined.

Proposition 1.3.6 *ρ_A is well-defined; more precisely, for a fixed x, $|A(t^{-1})x|$ is strictly decreasing as a function of t.*

Proof: We need to show that, if $0 > s > t$, then

$$|A(s^{-1})x| < |A(t^{-1})x|.$$

Consider

$$\begin{aligned}|A(s^{-1})x|^2 &= |s^{-1}x_1|^2 + |s^{-\sigma}x_2|^2\\ &= s^{-2}|x_1|^2 + s^{-2\sigma}|x_2|^2.\end{aligned}$$

As $s > t$, $\sigma > 1$, we have

$$s^{-2} < t^{-2},$$
$$s^{-2\sigma} < t^{-2\sigma}$$

therefore

$$\begin{aligned}|A(s^{-1})x|^2 &< t^{-2}|x_1|^2 + t^{-2\sigma}|x_2|^2\\ &= |A(t^{-1})x|^2.\end{aligned}$$

□

Proposition 1.3.7 *ρ_A is an A-norm; that is, for all $x, y \in \mathbb{R}^2$,*

1. *if $\rho_A(x) = 0$, then $x = 0$,*
2. *for all $t > 0$, $\rho_A(A(t)x) = t\rho_A(x)$,*
3. *$\rho_A(x + y) \leqslant \rho_A(x) + \rho_A(y)$.*

Proof: As the case when $\sigma = 1$ reduces immediately to the Euclidean norm, we consider only when $\sigma > 1$. Our definition of ρ_A immediately gives property 1, so consider property 2. To this end, let $q = \rho_A(A(t)x)$, then by definition

$$|A(q^{-1})A(t)x| = 1,$$

which, by Lemma 1.3.2, is equivalent to

$$|A(q^{-1}t)x| = 1.$$

Using the definition again, we have

$$\rho_A(x) = t^{-1}q,$$

so, finally,

$$t\rho_A(x) = q = \rho_A(A(t)x).$$

Now, to prove property 3, first we observe thyat if $\rho_A(x) = 0$ or $\rho_A(y) = 0$ the property is immediate, so we may assume both $\rho_A(x) > 0$ and $\rho_A(y) > 0$. Next we need to observe some trivial properties about the relationship between ρ_A and A. First, by definition,

$$\rho_A(x) = \rho_A(x) \iff |A(\rho_A(x)^{-1})x| = 1.$$

Now, let $t := \rho_A(x)$, $s := \rho_A(y)$, $\tilde{x} = A(t^{-1})x$ and $\tilde{y} = A(s^{-1})x$. So

$$\begin{aligned}|\tilde{x}| &= |A(t^{-1})x| \\ &= |A(\rho_A(x)^{-1})x| \\ &= 1.\end{aligned}$$

and likewise for $|\tilde{y}|$; also,

$$x + y = A(t)\tilde{x} + A(s)\tilde{y}.$$

Consider

$$\begin{aligned}\left|A\left(\frac{1}{t+s}\right)(x+y)\right| &= \left|A\left(\frac{t}{t+s}\right)\tilde{x} + A\left(\frac{s}{t+s}\right)\tilde{y}\right| \\ &\leqslant \left|A\left(\frac{t}{t+s}\right)\tilde{x}\right| + \left|A\left(\frac{s}{t+s}\right)\tilde{y}\right|.\end{aligned}$$

As $\sigma - 1 > 0$ and both $\frac{t}{t+s} < 1$ and $\frac{s}{t+s} < 1$, we have that

$$\begin{aligned}\left|A\left(\frac{1}{t+s}\right)(x+y)\right| &\leqslant \left(\frac{t}{t+s}\right)|\tilde{x}| + \left(\frac{s}{t+s}\right)|\tilde{y}| \\ &= 1,\end{aligned}$$

and therefore

$$\begin{aligned}\rho_A(x+y) &\leqslant t + s \\ &= \rho_A(x) + \rho_A(y).\end{aligned}$$

□

It is necessary next to introduce a few more objects that will be crucial to our analysis.

Definition 1.3.8 *Let ν denote our homogeneous dimension, specifically $\nu = 1 + \sigma$.*

We refer to this as our homogeneous dimension as in a lot of situations it completely replaces our usual dimension, the most obvious example of this is the following proposition.

Proposition 1.3.9 *Let $k_0 \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$, then $\rho_A(x)^{-\alpha}$ is in $L^1(\mathbb{R}^2 \setminus B_A(0, 2^{k_0}))$ if and only if $\alpha > \nu$.*

Proof: Consider

$$\int_{\mathbb{R}^2 \setminus B_A(0,2^{k_0})} \rho_A(x)^{-\alpha} dx = \sum_{k=k_0}^{\infty} \int_{2^k \leqslant \rho_A(x) \leqslant 2^{k+1}} \rho_A(x)^{-\alpha} dx.$$

Let $z = A(2^k)x$, then $x = A(2^k)z$ and the Jacobian of this transformation is given by $J = 2^{k\nu}$. Therefore we have that

$$\int_{\mathbb{R}^2 \setminus B_A(0,2^{k_0})} \rho_A(x)^{-\alpha} dx = \sum_{k=k_0}^{\infty} 2^{k\nu} \int_{2^k \leqslant \rho_A(A(2^k)z) \leqslant 2^{k+1}} \rho_A(A(2^k)z)^{-\alpha} dz \tag{1.7}$$

$$= \sum_{k=k_0}^{\infty} 2^{k(\nu-\alpha)} \int_{1 \leqslant \rho_A(z) \leqslant 2} \rho_A(z)^{-\alpha} dz. \tag{1.8}$$

The integral in (1.8) can be bounded above and below by a constant dependent only α and ν. The convergence of the sum is therefore dependent only on the term $2^{k(\nu-\alpha)}$, and so converges if and only if $\alpha > \nu$, the proposition follows. $\square$

Definition 1.3.10

- *Let $B_A(x, r)$ denote the ρ-ball with centre x and parameter r, that is*

$$B_A(x, r) = \{y \in \mathbb{R}^2 : \rho_A(x - y) \leqslant r\} = \{y \in \mathbb{R}^2 : |A(r^{-1})(x - y)| \leqslant 1\}.$$

- *Let $r_\rho(B)$ denote the ρ-radius of a ρ-ball B; that is, $r_\rho\left(B_A(x, r)\right) = r$.*
- *Throughout, we will use $\vartheta \in \mathcal{S}$ to denote a positive, radial, decreasing (in the obvious way) function with total mass 1 and we will denote parabolic dilations of any function f*

with respect to A as

$$f_{A(t)}(x) = t^{-\nu} f(A(t^{-1})x).$$

We will refer to the functions $\vartheta_{A(t)}$ as parabolic approximations of the identity.

It turns out that the family of norms ρ_A have an associated family of norms, the importance of which will become clear in Chapter 3, defined as follows

Definition 1.3.11 *Let $\|\cdot\|_A : \mathbb{R}^2 \to \mathbb{R}$ be a norm associated to A; that is, given $x = (x_1, x_2) \in \mathbb{R}^2$ define*

$$\|x\|_A = |x_1| + \sigma|x_2|.$$

Remark 1.3.12 *We can relate our ρ_A norms to the Euclidean norms as follows. Let $z = (z_1, z_2) \in \mathbb{R}^2$ and define $t := \rho_A(z)$.*

- *If $\rho_A(z) \leqslant 1$, then by the definition of ρ_A we have that*

$$\begin{aligned} 1 &= |A(t^{-1})z| \\ &= \left(t^{-2}|z_1|^2 + t^{-2\sigma}|z_2|^2\right)^{\frac{1}{2}} \\ &\geqslant \left(t^{-2}|z_1|^2 + t^{-2}|z_2|^2\right)^{\frac{1}{2}} \\ &= t^{-1}|z| \end{aligned}$$

  and therefore

$$\rho_A(z) = t \geqslant |z|.$$

In much the same way, we get

$$\begin{aligned} 1 &= |A(t^{-1})z| \\ &\leqslant t^{-\sigma}|z| \end{aligned}$$

thus

$$\rho_A(z)^\sigma \leqslant |z| \leqslant \rho_A(z).$$

- *If* $\rho_A(z) \geqslant 1$*, then we get the reverse*

$$\rho_A(z) \leqslant |z| \leqslant \rho_A(z)^\sigma.$$

Remark 1.3.13 *Let* $c > 0$ *and consider*

$$|A(ct)x| = |(ctx_1, c^\sigma t^\sigma x_2)|$$

then

$$\begin{aligned} |A(ct)x| &\leqslant \max(\{c, c^\sigma\})|(tx_1, t^\sigma x_2)| \\ &= \max(\{c, c^\sigma\})|A(t)x| \end{aligned}$$

and

$$\begin{aligned} |A(ct)x| &\geqslant \min(\{c, c^\sigma\})|(tx_1, t^\sigma x_2)| \\ &= \min(\{c, c^\sigma\})|A(t)x|. \end{aligned}$$

Note that the min or max of c, c^{σ} *depends only on if* $c \leqslant 1$ *or* $c > 1$*, as* $\sigma \geqslant 1$. *This remark is a formalisation of the observation that one can fit a circle in an ellipse and vice versa.*

Definition 1.3.14 *Let* $x, y \in \mathbb{R}^2$ *and* $B, C \subseteq \mathbb{R}^2$ *be sets, then we define* $\rho_A(x, y) = \rho_A(x - y)$*,*

$$\rho_A(B, C) = \inf_{\substack{b \in B \\ c \in C}} \rho_A(a - b),$$

and

$$\rho_A(B, x) = \inf_{b \in B} \rho_A(b - x).$$

CHAPTER 2

MAXIMAL FUNCTIONS

2.1 Two maximal functions

We will introduce two parabolic maximal functions in order to first highlight the difference between the classical maximal functions and the maximal functions that are the focus of this chapter.

Definition 2.1.1 *Let $a, b \in \mathbb{R}$. Denote the parabolic maximal operators associated with the dilation A as*

$$M_{A,a,b} f(x) = \sup_{(y,t)\in\Lambda_a(x)} t^{-b} \chi_{A(t)} * |f|(y),$$

where

$$\Lambda_{A,a}(x) = \{(y,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \,:\, \rho_A(x-y) \leqslant at\},$$

and χ is the indicator function of the unit ball (or the unit cube, which gives a pointwise equivalent definition).

Note that $M_{A,a,b}$ are parabolic fractional maximal functions, $M_{A,a,0}$ are parabolic nontangential maximal functions of aperture a, and $M_{A,0,0}$, which we denote just M_A, are parabolic versions *of the classic Hardy-Littlewood maximal function.*

Remark 2.1.2 *The maximal functions $M_{A,a,b}$ defined above are parabolic versions of the non-tangential fractional maximal operators, the isotropic case usually defined as*

$$M_{I,a,b}f(x) = \sup_{(y,t)\in\Lambda_{I,a}(x)} t^{-b} \fint_{B(y,r)} |f(z)|dz,$$

where

$$\Lambda_{I,a}(x) = \{(y, t \in \mathbb{R}^2 \times \mathbb{R}_+ \,:\, |x-y| \leqslant at\}.$$

The word nontangential here refers to the region $\Lambda_{I,a}(x)$, this region is a cone in $\mathbb{R}^2 \times \mathbb{R}_+$ with vertex at $(x, 0)$, the boundary of $\mathbb{R}^2$; we refer to a as the aperture of the cone. The word fractional here refers to the role of b, and the wording comes from the maximal function's close relationship and resemblance to fractional integration, or more accurately in the multidimensional case Riesz potentials, see [31] for more details.

For comparison, we now introduce our main object of study, $\mathcal{M}_{A,\alpha,\beta}$, which is a parabolic version of the maximal function introduced by Bennett and Beltran[2], see also [3]. We use very similar notation to the above maximal function to emphasise the close relationship between them, but will emphasise the differences shortly.

Definition 2.1.3 *Let $\alpha, \beta \in \mathbb{R}$ and define*

$$\mathcal{M}_{A,\alpha,\beta}f(x) = \sup_{(y,t)\in\Gamma_{A,\alpha}(x)} (t^{\nu})^{2\beta}|\vartheta_{A(t)} * f(y)|$$

where

$$\Gamma_{A,\alpha}(x) = \{(y, t) \in \mathbb{R}^2 \times \mathbb{R}_+ \,:\, 0 < t^{\alpha} \leqslant 1,\ \rho_A(x-y) \leqslant t^{1-\alpha}\}.$$

Indeed, these two maximal functions coincide precisely when $a = 1$, $\alpha = 0$ and $\beta = -b/2$. From this relationship it is easy to see that the roles of b and β differ only aesthetically, however the roles a and α play are quite different - they both pertain to the behaviour of the approach regions, but the nature of that change differs greatly.

The regions $\Lambda_{A,a}$ and $\Gamma_{A,\alpha}$ depend in the same way upon A, the eccentricity of the cross-sectional areas for each fixed t changes. The region $\Lambda_{A,a}$ is a cone for all values of a and only changes the aperture of the approach region. However, the region $\Gamma_{A,\alpha}$ can significantly change shape dependent upon α, in the same way as the Euclidean case. For $0 < \alpha < 1$, we have a slightly bulging cone shape, cut off at $t = 1$. For $\alpha > 1$, we get an inverted cone shape, which allows tangential approach. For $\alpha < 0$, the region does not include $t < 1$, which changes the nature of the region entirely, and would be more accurately described as an "escape" region. See [5, 2] for further discussion of these regions.

Remark 2.1.4 *Although $\mathcal{M}_{A,\alpha,\beta}$ depends on the choice of ϑ, all estimates involving this maximal function will be uniform for all parabolic approximations of the identity.*

2.2 A schema for the method

It is the goal of this chapter to prove $L^p - L^q$ inequalities for our maximal functions $\mathcal{M}_{A,\alpha,\beta}$, in light of this we will begin by outlining a schema for doing so using the parabolic Hardy-Littlewood maximal functions first. The main workload of such ventures is generally hidden within interpolation between endpoint spaces. The nature of L^∞ lends itself very well to being an endpoint space for maximal functions, and so the ideal other endpoint would be L^1. However, a quick calculation of $M_A\chi$, where χ is the indicator function of the unit ball, convinces us that f being integrable is not enough to ensure M_Af is intregrable. So instead we must settle for a weak-L^1 bound, Theorem 2.2.2. This is slightly different to how we will handle the endpoint estimate for the maximal functions defined in Definition 2.1.3, as we will need to take a brief foray into Hardy space estimates instead. To begin our estimates on M_A, we must first prove the following lemma.

Theorem 2.2.1 (Vitali's covering lemma, [12]) *Let $\{B_j : j \in J\}$ be a collection of ρ-balls in $\mathbb{R}^2$ such that*

$$\sup_{j\in J} r_\rho(B_j) < \infty.$$

Then there exists $J' \in J$, a countable subset of J, such that $\{B_j : j \in J'\}$ are disjoint and

$$\bigcup_{j\in J} B_j \subseteq \bigcup_{j\in J'} 5B_j.$$

Proof: As

$$\sup_{j\in J} r_\rho(B_j) < \infty,$$

we have that there exists $R > 0$ such that $\sup_{j\in J} r_\rho(B_j) < R$ for all $j \in J$.

Partition J into a countable collection of subsets, $\{J_i\}_{i\in N_0}$, such that J_i has all ρ-balls with parameter in $(\frac{R}{2^{i+1}}, \frac{R}{2^i}]$. Let $H_0 := J_0$ and E_0 be a maximal disjoint countable subcollection of H_0.

Let H_i and E_i be defined inductively by

$$H_i := \{B \in J_i : B \cap B' = \emptyset, \forall B' \in E_0 \cup ... \cup E_{i-1}\}$$

where E_i is a maximal disjoint countable subcollection of H_i. Then the desired set J' is defined as

$$J' = \bigcup_{i=0}^{\infty} E_i.$$

$\square$

With this lemma in hand, it is a relatively simple matter to gain a weak-L^1 estimate on M.

Theorem 2.2.2 (Weak PHL, [7] Theorem 1.7) *There exists $C_A > 0$ such that for all $\lambda > 0$,*

$$\left|\{x \in \mathbb{R}^2 : M_A f(x) > \lambda\}\right| \leqslant \frac{C_A}{\lambda}\|f\|_1.$$

Proof: Fix $\lambda > 0$. If there are no $x \in \mathbb{R}^2$ such that

$$M_A f(x) > \lambda,$$

then we are done. So fix $x \in \mathbb{R}^2$ such that

$$M_A f(x) > \lambda.$$

By the definition of M_A for each x there exists a finite $t_x > 0$ such that

$$\begin{aligned}
\lambda &< \chi_{A(t_x)} * |f|(x) \\
&= \int_{\mathbb{R}^2} t_x^{-\nu} \chi(A(t_x^{-1})(x-y))|f(y)|dy \\
&= t_x^{-\nu} \int_{|A(t_x^{-1})(x-y)|\leqslant 1} |f(y)|dy \\
&= \frac{1}{|B_A(x,t_x)|} \int_{B_A(x,t_x)} |f(y)|dy
\end{aligned}$$

then we have that

$$|B_A(x,t_x)| < \frac{1}{\lambda} \int_{B_A(x,t_x)} |f(y)|dy. \tag{2.1}$$

So, for each x such that $M_A f(x) > \lambda$ we obtain a t_x and corresponding $B_A(x,t_x)$ with the property (2.1) and assigning an index from an index set J to each x we have

$$\{x \in \mathbb{R}^2 : M_A f(x) > \lambda\} \subseteq \bigcup_{j\in J} B_j.$$

By the Vitali covering lemma, we have a subset of J, J', of disjoint balls such that

$$\{x \in \mathbb{R}^2 : M_A f(x) > \lambda\} \subseteq \bigcup_{j\in J} B_j \subseteq \bigcup_{j\in J'} 5B_j$$

and so by (2.1)

$$\begin{aligned}
|\{x \in \mathbb{R}^2 : M_A f(x) > \lambda\}| &\leqslant 5^\nu \sum_{j\in J'} |B_j| \\
&\leqslant \frac{5^\nu}{\lambda} \int_{\mathbb{R}^2} |f(y)|dy.
\end{aligned}$$

□

2.3 Parabolic Hardy spaces

Now, to give the same treatment to our maximal functions $\mathcal{M}_{A,\alpha,\beta}$, we must introduce parabolic Hardy spaces, so following Calderón and Torchinsky [7, 8]. In this section we give a very brief introduction of the required concepts.

Definition 2.3.1 *Let $1 \leqslant p \leqslant \infty$. We say a function f is in H^p_A if the parabolic maximal function of f is in $L^p(\mathbb{R}^2)$, that is*

$$\|f\|_{H^p_A} = \|M_A f\|_p < 1.$$

Now, one of the major advantages of using Hardy spaces to gain estimates on maximal functions is that functions in Hardy spaces can be decomposed into atoms. This process is outlined in the below definition and theorem due to Calderón.

Definition 2.3.2 *We shall call a function, a, a H^1_A-atom if there is a ρ_A-ball B such that*

1. $\mathrm{supp}(a) \subseteq B$;
2. $\|a\|_\infty \leqslant |B|^{-1}$;
3. $$\int_B a(x)dx = 0.$$

Theorem 2.3.3 (Atomic decomposition of H^1_A, [6]) *Given $f \in H^1_A$, there exists a sequence of H^1_A-atoms, a_j, and constants λ_j such that*

$$\|f - \sum_{j=1}^{N} \lambda_j a_j\|_{H^1_A} \to 0 \text{ as } N \to \infty$$

and there exists $c > 0$ *such that*

$$c^{-1}\|f\|_{H^1_A} \leqslant \sum_{j=1}^{\infty} |\lambda_j| \leqslant c\|f\|_{H^1_A}.$$

2.4 Estimates on parabolic subdyadic maximal functions

Much like in the case of the parabolic Hardy-Littlewood maximal function, the main difficulty is getting one of the endpoint estimates, the endpoint we require in this case is contained in the following theorem.

Theorem 2.4.1 *Let w be a weight, then*

$$\|\mathcal{M}_{A,\alpha,\alpha/2}w\|_1 \lesssim \|w\|_{H^1_A(\mathbb{R}^2)}.$$

Remark 2.4.2 *We have intentionally changed to using*

$$\tilde{A}(t) = \begin{pmatrix} t^{a_1} & 0 \\ 0 & t^{a_2} \end{pmatrix},$$

where $0 < a_1 \leqslant a_2$, as we feel that the argument is more illuminating with this convention as there is a non-trivial dependence on a_1, that is entirely hidden when $a_1 = 1$. The homogeneous dimension is therefore defined as $\nu = a_1 + a_2$ for the remainder of this chapter. Note that Definition 1.3.1 outlines that this is entirely equivalent when the supremum is taken in the set $t > 0$.

Proof: Let P be a bump function, strictly positive on $B_A(0,1) = B(0,1)$ and let $P_{A(t)}$ be $A(t)$-dilations of P

$$P_{A(t)}(x) = t^{-\nu} P\left(A(t^{-1})x\right).$$

Note that for any choice of ϑ we can bound $\mathcal{M}_{A,\alpha,\beta}$ from above pointwise by our maximal function with the choice $\vartheta = P$, modulo a constant. So we can use this dilated bump function to get upper estimates on $\mathcal{M}_{A,\alpha,\alpha/2}$.

Let $w \in H^1_A$, then by Theorem 2.3.3, there exists a sequence of H^1_A-atoms, a_j, and constants λ_j, such that

$$w = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty. \tag{2.2}$$

We first aim to prove

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} w(x) \leqslant \sum_{j=1}^{\infty} |\lambda_j| \mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a_j(x), \tag{2.3}$$

for almost all x. To this end, we will show that for a fixed t,

$$P_{A(t)} * w(x) = \sum_{j=1} \lambda_j P_{A(t)} * a_j(x), \tag{2.4}$$

for almost all x. Assuming (2.4) for now, we have

$$\begin{aligned}
\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} w(x) &= \sup_{(y,t)\in\Gamma_{A,\alpha}(x)} t^{\nu\alpha} |P_{A(t)} * w(y)| \\
&\leqslant \sup_{(y,t)\in\Gamma_{A,\alpha}(x)} t^{\nu\alpha} \sum_{j=1}^{\infty} |\lambda_j| |P_{A(t)} * a_j(y)| \\
&\leqslant \sum_{j=1}^{\infty} |\lambda_j| \sup_{(y,t)\in\Gamma_{A,\alpha}(x)} t^{\nu\alpha} |P_{A(t)} * a_j(y)| \\
&= \sum_{j=1}^{\infty} |\lambda_j| \mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a_j(x),
\end{aligned}$$

where we have used (2.2) and the triangle inequality on (2.4). So, to show (2.3), it is sufficient to prove (2.4). To this end, define T by

$$Tf = P_{A(t)} * f.$$

First, for $f \in L^1$, by Fubini's theorem

$$\begin{aligned}
\|Tf\|_1 &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} P_{A(t)}(x-y)f(y)dydx \\
&= \int_{\mathbb{R}^2} f(y)\int_{\mathbb{R}^2} P_{A(t)}(x-y)dxdy \\
&= \|P_{A(t)}\|_1\|f\|_1.
\end{aligned}$$

As $P_{A(t)}$ is normalised in L^1 and P has total mass 1, this implies

$$\|Tf\|_1 = \|f\|_1 \tag{2.5}$$

for all $f \in L^1$. Next, let a be an arbitrary H^1_A-atom, as $a \in L^1$ by (2.5) we have

$$\|T(a)\|_1 = \|a\|_1$$

and

$$\begin{aligned}
\|a\|_1 &= \int_{B_A(0,r)} |a(x)|dx \\
&\leqslant |B_A(0,r)|^{-1}|B_A(0,r)| \\
&= 1.
\end{aligned}$$

So we can conclude

$$\|T(a)\|_1 \leqslant 1, \tag{2.6}$$

uniformly in all H^1_A-atoms.

So, consider the set of points where (2.4) is not true, that is

$$|\{|T(w) - \sum_{j=1}^{\infty} \lambda_j T(a_j)| > \delta\}| \leqslant \frac{1}{\delta}\|T(w) - \sum_{j=1}^{\infty} \lambda_j T(a_j)\|_1,$$

by Chebyshev's inequality. Next, we split up the sum into the first N terms and the terms after N and use the reverse triangle inequality

$$|\{|T(w) - \sum_{j=1}^{\infty} \lambda_j T(a_j)| > \delta\}| \leqslant \frac{1}{\delta}\|T(w) - \sum_{j=1}^{N} \lambda_j T(a_j)\|_1 + \frac{1}{\delta}\|\sum_{j=N+1}^{\infty} \lambda_j T(a_j)\|_1.$$

Now, we have two terms on the right hand side, the first of which we can use linearity of T, (2.5) and the reverse L^1 bound on M_A to obtain

$$\begin{aligned}\|T(w) - \sum_{j=1}^{N} \lambda_j T(a_j)\|_1 &= \|T(w - \sum_{j=1}^{N} \lambda_j a_j)\|_1 \\ &= \|w - \sum_{j=1}^{N} \lambda_j a_j\|_1 \\ &\leqslant \|w - \sum_{j=1}^{N} \lambda_j a_j\|_{H^1_A}.\end{aligned}$$

For the second term we will use (2.6) and (2.2) as follows,

$$\begin{aligned}\|\sum_{j=N+1}^{\infty} \lambda_j T(a_j)\|_1 &\leqslant \sum_{j=N+1}^{\infty} |\lambda_j|\|T(a_j)\|_1 \\ &\leqslant \sum_{j=N+1}^{\infty} |\lambda_j|.\end{aligned}$$

Thus, we have

$$|\{|T(w) - \sum_{j=1}^{\infty} \lambda_j T(a_j)| > \delta\}| \leqslant \frac{1}{\delta}\|w - \sum_{j=1}^{N} \lambda_j a_j\|_{H^1_A} + \frac{1}{\delta}\sum_{j=N+1}^{\infty} |\lambda_j|.$$

By Theorem 2.3.2, $\sum_{j=1}^{N} \lambda_j a_j$ converges to w in H^1_A and (2.2), both terms on the right hand side converge to zero as $N \to \infty$, we conclude that

$$|\{|T(w) - \sum_{j=1}^{\infty} \lambda_j T(a_j)| > \delta\}| = 0,$$

for all $\delta > 0$, which implies (2.4).

Now, we can take an L^1 norm of both sides of (2.3) to get

$$\begin{aligned}\|\mathcal{M}_{A,\alpha,\alpha/2} w\|_1 &\leqslant \|\sum_{j=1}^{\infty} |\lambda_j| \mathcal{M}_{A,\alpha,\alpha/2} a_j\|_1 \\ &\leqslant \sum_{j=1}^{\infty} |\lambda_j| \|\mathcal{M}_{A,\alpha,\alpha/2} a_j\|_1,\end{aligned}$$

by the monotone convergence theorem. So if

$$\|\mathcal{M}_{A,\alpha,\alpha/2} a\|_1 \lesssim 1,$$

uniformly for all H^1_A-atoms a, then

$$\begin{aligned}\|\mathcal{M}_{A,\alpha,\alpha/2} w\|_1 &\lesssim \sum_{j=1}^{\infty} |\lambda_j| \\ &\lesssim \|w\|_{H^1_A},\end{aligned}$$

by Theorem 2.3.3. So, it is sufficient to show that

$$\|\mathcal{M}_{A,\alpha,\alpha/2}a\|_1 \lesssim 1,$$

uniformly for all H^1_A atoms a.

Note that as $\mathcal{M}_{A,\alpha,\alpha/2}$ is defined via convolution, it is translation invariant, so it is sufficient to consider only H^1_A-atoms, centred at the origin. Fix a H^1_A-atom a, let $B_A(0,r)$ be the support of a. For a fixed t and y, consider $P_{A(t)} * a(y)$. If $t \geqslant r$,

$$\begin{aligned} P_{A(t)} * a(y) &= \int_{\mathbb{R}^2} P_{A(t)}(y-z)a(z)dz \\ &= \int_{B_A(0,r)} \left(P_{A(t)}(y-z) - P_{A(t)}(y)\right) a(z)dz \end{aligned}$$

by the mean value property of a, that is property 3 of Definition 2.3.2. Now, by the mean value theorem, for each $z \in \mathbb{R}^2$, there exists some $\lambda \in (0,1)$ such that

$$P_{A(t)}(y-z) - P_{A(t)}(y) = \langle -z, \nabla P_{A(t)}(y-\lambda z)\rangle,$$

We can use the Cauchy-Schwarz inequality to get

$$\begin{aligned} |\langle -z, \nabla P_{A(t)}(y-\lambda z)\rangle| &= |\langle -A(r^{-1})z, A(r)\nabla P_{A(t)}(y-\lambda z)\rangle| \\ &\leqslant |A(r^{-1})z||A(r)\nabla P_{A(t)}(y-\lambda z)| \end{aligned}$$

and so

$$
\begin{aligned}
|P_{A(t)} * a(y)| &\leqslant \int_{B_A(0,r)} |A(r^{-1})z||A(r)\nabla P_{A(t)}(y-\lambda z)||a(z)|dz \\
&\leqslant \|A(r)\nabla P_{A(t)}\|_\infty \|a\|_\infty \int_{B_A(0,r)} |A(r^{-1})z|dz \\
&\leqslant t^{-\nu}\|A\left(\frac{r}{t}\right)\nabla P\|_\infty |B_A(0,r)|^{-1}\int_{B_A(0,r)} |A(r^{-1})z|dz \\
&\leqslant t^{-\nu}\left(\frac{r}{t}\right)^{a_1}\|\nabla P\|_\infty \fint_{B_A(0,r)} |A(r^{-1})z|dz.
\end{aligned}
$$

The estimate on $\|A\left(\frac{r}{t}\right)\nabla P\|_\infty$ is due to $t \geqslant r$ and $0 < a_1 \leqslant a_2$ giving that $\left(\frac{r}{t}\right)^{a_2-a_1} \leqslant 1$. Now, we have that $\|\nabla P\|_\infty \lesssim 1$ is independent of r and t and we have

$$
\begin{aligned}
\fint_{B_A(0,r)} |A(r^{-1})z|dz &= \fint_{|A(r^{-1})z|<1} |A(r^{-1})z|dz \\
&\leqslant 1.
\end{aligned}
$$

Thus we have that

$$
|P_{A(t)} * a(y)| \lesssim \frac{r^{a_1}}{t^{\nu+a_1}}
$$

for $t \geqslant r$.

If $t \leqslant r$,

$$
\begin{aligned}
\left|P_{A(t)} * a(y)\right| &= \left|\int_{\mathbb{R}^2} P_{A(t)}(y-z)a(z)dz\right| \\
&\leqslant |B_A(0,r)|^{-1}\|P_{A(t)}\|_1 \\
&\lesssim \frac{1}{r^\nu}.
\end{aligned}
$$

Consider the case when the supports of $P_{A(t)}(y-\cdot)$ and a do not overlap, then

$$P_{A(t)} * a(y) = \int_{\mathbb{R}^2} P_{A(t)}(y-z)a(z)dz = 0.$$

This definitely occurs when the rectangles with the supports of P_t and a inscribed do not overlap, which corresponds to the case when

$$\sup_{i=1,2}(t^{a_i}+r^{a_i})^{-1}|y_i| > 1 \implies \sup_{i=1,2} K_{a_i}(t+r)^{-a_i}|y_i| > 1,$$

for some constants K_{a_i}. This corresponds to taking a ℓ_∞ norm instead of a ℓ_2 norm in the definition of ρ_A. As these norms are equivalent we have that

$$|P_{A(t)} * a(y)| \lesssim \begin{cases} \frac{r^{a_1}}{t^{\nu+a_1}} & \text{if } \rho_A(y) \lesssim t+r, t \geqslant r \\ \frac{1}{r^\nu} & \text{if } \rho_A(y) \lesssim t+r, t \leqslant r \\ 0 & \text{if } \rho_A(y) \gtrsim t+r. \end{cases}$$

Now, we turn to estimating the maximal function via this estimate. We need to split our analysis up into five cases, where the set $\Gamma_{A,\alpha}(x)$ has significantly different behaviours. There are two cases singled out, $\alpha = 0$ and $\alpha = 1$. The case $\alpha = 0$ is degenerate, in that the maximal function considered reduces to the parabolic Hardy-Littlewood maximal function. The case $\alpha = 1$ is very similar to the case $0 < \alpha < 1$, but still significantly different enough to warrant a slightly different approach.

Case $\alpha < 0$

As we are taking a supremum over $\Gamma_{A,\alpha}(x)$, we reduce to when $t \geqslant 1$. As our supremum is of $t^{\nu\alpha}|P_{A(t)} * a(y)|$, which as we can see above will have only negative powers of t (either $t^{\nu\alpha}$ or $t^{\nu\alpha-(a_1+\nu)}$ as $\alpha < 0$), we need only find the smallest t (as again, $t \geqslant 1$). First, consider $r < 1$; if

$\rho_A(x) \lesssim 1$ then $\Gamma_{A,\alpha}(x)$ and $\operatorname{supp}(P_{A(t)} * a(x))$ have non-empty intersection for $t = 1$, so we have

$$\begin{aligned} \mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) &\lesssim r^{a_1} \\ &< 1. \end{aligned}$$

If $\rho_A(x) \gtrsim 1$ then $\Gamma_{A,\alpha}(x)$ and $\operatorname{supp}(P_{A(t)} * a(x))$ have empty intersection when $t = 1$ and so the smallest value of t for them to have non-empty intersection is when $r < 1 < t \sim \rho_A(x)^{\frac{1}{1-\alpha}}$. Then $t^{\nu\alpha}|P_{A(t)} * a(x)| \lesssim r^{a_1} t^{\nu\alpha-(\nu+a_1)}$ and we have

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim r^{a_1} \rho_A(x)^{-\frac{\alpha\nu-(\nu+a_1)}{\alpha-1}}.$$

So, collecting these we have

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim \begin{cases} 1 & \text{if } \rho_A(x) \lesssim 1 \\ \rho_A(x)^{-\frac{\alpha\nu-(\nu+a_1)}{\alpha-1}} & \text{if } \rho_A(x) \gtrsim 1. \end{cases}$$

Therefore, for fixed $\alpha < 0$ and $r < 1$ as $\nu\frac{\alpha-(1-\frac{a_1}{\nu})}{\alpha-1} > \nu$, thus $\mathcal{M}_{A,\alpha,\alpha/2}a$ is integrable by Proposition 1.3.9; that is $\|\mathcal{M}_{A,\alpha,\alpha/2}a\|_1 \lesssim 1$.

Now, consider when $r \geqslant 1$; if $\rho_A(x) \lesssim 1$, then $\Gamma_{A,\alpha}(x)$ and $\operatorname{supp}(P_{A(t)} * a(x))$ have non-empty intersection for $t = 1$, so we have

$$\begin{aligned} \mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) &\lesssim r^{-\nu} \\ &< 1. \end{aligned}$$

If $\rho_A(x) \gtrsim 1$ then $\Gamma_{A,\alpha}(x)$ and $\operatorname{supp}(P_{A(t)} * a(x))$ have empty intersection when $t = 1$ and so the smallest value of t for them to have non-empty intersection is when $t \sim \rho_A(x)^{\frac{1}{1-\alpha}}$. If

$1 \lesssim \rho_A(x)^{\frac{1}{1-\alpha}} \lesssim r$, then

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim r^{-\nu} \rho_A(x)^{-\nu\frac{\alpha}{\alpha-1}},$$

which for each $r > 1$, integrating over $\rho_A(x)^{\frac{1}{1-\alpha}} \lesssim r$ integrates to a constant independent of r.

Case $\alpha = 0$

We are taking a supremum over

$$\Gamma_{A,0}(x) = \{(y,t) \in \mathbb{R}^2 \times \mathbb{R}^+ : t > 0, \rho_A(x-y) \leqslant t\}.$$

Therefore the maximal operator reduces to an uncentred parabolic maximal operator given by

$$\begin{aligned}\mathcal{M}_{A,\alpha,\alpha/2} a(x) &= \sup_{(y,t)\in\Gamma_{A,0}(x)} |P_{A(t)} * a(y)| \\ &\lesssim M_A a(x).\end{aligned}$$

Case $0 < \alpha < 1$

We are taking a supremum over $\Gamma_{A,\alpha}(x)$, so we reduce to when $0 < t \leqslant 1$. First, if $r > 1$, then $t \leqslant 1 < r$, and so

$$t^{\nu\alpha} |P_{A(t)} * a(y)| \lesssim \begin{cases} \frac{t^{\nu\alpha}}{r^{\nu}} & \text{if } \rho_A(y) \leqslant t + r \\ 0 & \text{if } \rho_A(y) > t + r \end{cases}$$

and as $\alpha > 0$ and monomials of degree > 0 are monotonically increasing, we look for the largest $t \leqslant 1$ in the intersection of $\Gamma_{A,\alpha}(x)$ and $\rho_A(y) \lesssim t + r$. If $\rho_A(x) \lesssim r$, then $t = 1$ is in the intersection, therefore

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim \frac{1}{r^{\nu}}.$$

However, if $\rho_A(x) \gtrsim r$, then the intersection of $\Gamma_{A,\alpha}(x)$ and $\rho_A(y) \lesssim t + r$ is empty, and so

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim \begin{cases} r^{-\nu} & \text{if } \rho_A(x) \lesssim r \\ 0 & \text{if } \rho_A(x) \gtrsim r. \end{cases}$$

So, for a fixed $r > 1$, this integrates to a constant independent of r.

Now, if $r \leqslant 1$, we have three cases. If $\rho_A(x) \lesssim r$, then the supremum is when $t = r$, so

$$\begin{aligned} \mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) &\lesssim \frac{r^{\nu\alpha}}{r^{\nu}} \\ &\lesssim 1. \end{aligned}$$

If $r \lesssim \rho_A(x) \lesssim 1$, then the supremum is when $t \sim \rho_A(x)^{\frac{1}{1-\alpha}}$ and we have $r < 1$, so

$$\begin{aligned} \mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) &\lesssim r^{a_1} \rho_A(x)^{\frac{\nu\alpha-\nu-a_1}{\alpha-1}} \\ &\lesssim \rho_A(x)^{-\nu\frac{\alpha-1-\frac{a_1}{\nu}}{\alpha-1}}. \end{aligned}$$

Finally, if $\rho_A(x) \gtrsim 1$, then the intersection is empty, so we have

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim \begin{cases} 1 & \text{if } \rho_A(x) \lesssim r \\ \rho_A(x)^{-\nu\frac{\alpha-1-\frac{a_1}{\nu}}{\alpha-1}} & \text{if } r \lesssim \rho_A(x) \lesssim 1 \\ 0 & \text{if } \rho_A(x) \gtrsim 1. \end{cases}$$

So, for a fixed $r \leqslant 1$, this integrates to a constant independent of r since $\nu\frac{\alpha-1-\frac{a_1}{\nu}}{\alpha-1} > \nu$, therefore $\|\mathcal{M}_{A,\alpha,\alpha/2} a\|_1 \lesssim 1$.

Case $\alpha = 1$

We are taking a supremum over $\Gamma_{A,1}(x)$, we reduce to when $0 < t \leqslant 1$. First, if $r > 1$, then

$t \leqslant 1 < r$, and so

$$t^{\nu}|P_{A(t)} * a(y)| \lesssim \begin{cases} \frac{t^{\nu}}{r^{\nu}} & \text{if } \rho_A(y) \lesssim t + r \\ 0 & \text{if } \rho_A(y) \gtrsim t + r \end{cases}$$

and so we look for the biggest $t \leqslant 1$ in the intersection of $\Gamma_{A,1}(x)$ and $\rho_A(y) \lesssim t + r$. Now, $(y, t) \in \Gamma_{A,1}(x)$ implies $\rho_A(x - y) \leqslant 1$, and thus if $\rho_A(x) \gtrsim 1$, then the intersection is empty. If $\rho_A(x) \lesssim 1$ then $t = 1$ maximises $t^{\nu}|P_{A(t)} * a(y)|$ in the intersection and therefore,

$$\mathcal{M}_{A,1,\frac{1}{2}} a(x) \lesssim \begin{cases} 1 & \text{if } \rho_A(x) \lesssim 1 \\ 0 & \text{if } \rho_A(x) \gtrsim 1. \end{cases}$$

Integrability uniformly in r is immediate as the pointwise estimate on $\mathcal{M}_{A,1,\frac{1}{2}} a$ is independent of r and compactly supported.

Now we consider the case $r \leqslant 1$. If $\rho_A(x) \leqslant 1$ (or equivalently $|x| \leqslant 1$), then $t = r$ maximises $t^{\nu}|P_A(t) * a(y)|$ in the intersection of $\rho_A(y) \lesssim t + r$ and $\Gamma_{A,1}(x)$, and so

$$\mathcal{M}_{A,1,\frac{1}{2}} a(x) \lesssim 1.$$

If $1 < \rho_A(x) \lesssim 1+r$, then $t \sim \rho_A(x)$ maximises $t^{\nu}|P_{A(t)} * a(y)|$ in the intersection of $\rho_A(y) \lesssim t+r$ and $\Gamma_{A,1}(x)$, and so

$$\begin{aligned} \mathcal{M}_{A,1,\frac{1}{2}} a(x) &\lesssim r\rho_A(x)^{-a_1} \\ &\lesssim 1. \end{aligned}$$

Finally, if $\rho_A(x) \gtrsim 1+r$ we again have no intersection of the sets $\rho_A(y) \lesssim t+r$ and $\rho_A(x-y) \leqslant 1$ when $t \leqslant 1$, so all together that is

$$\mathcal{M}_{A,1,\frac{1}{2}} a(x) \lesssim \begin{cases} 1 & \text{if } \rho_A(x) \lesssim 1+r \\ 0 & \text{if } \rho_A(x) \gtrsim 1+r, \end{cases}$$

Note that we again have the same situation as the case $r > 1$ since $r \leqslant 1$ gives us that $1+r \leqslant 2$, therefore $\|\mathcal{M}_{A,1,\frac{1}{2}} a\|_1 \lesssim 1$.

The final case is when $\alpha > 1$, as we are taking a supremum over $\Gamma_{A,\alpha}(x)$, we reduce to when $0 < t \leqslant 1$. First, if $r > 1$, then $t \leqslant 1 < r$, and so

$$t^{\nu\alpha} |P_{A(t)} * a(y)| \lesssim \begin{cases} \frac{t^{\nu\alpha}}{r^\nu} & \text{if } \rho_A(y) \lesssim t+r \\ 0 & \text{if } \rho_A(y) \gtrsim t+r \end{cases}$$

and as $\alpha > 1$ and monomials of degree > 0 are monotonically increasing, we look for the largest $t \leqslant 1$ in the intersection of $\Gamma_{A,\alpha}(x)$ and $\rho_A(y) \lesssim t+r$. If $\rho_A(x) \lesssim r$, then $t = 1$ is in the intersection, therefore

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim \frac{1}{r^\nu}.$$

If $\rho_A(x) \gtrsim r$, then $t \sim \rho_A(x)^{\frac{1}{1-\alpha}}$ is the smallest t in the intersection and so

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim \begin{cases} \frac{1}{r^\nu} & \text{if } \rho_A(x) \lesssim r \\ \rho_A(x)^{-\nu\frac{\alpha}{\alpha-1}} & \text{if } \rho_A(x) \gtrsim r. \end{cases}$$

Note that the part of the estimate for $\rho_A(x) \lesssim r$ is integrable to constant independent of r. So, for fixed $\alpha > 1$ and $r > 1$ as $\nu\frac{\alpha}{\alpha-1} > \nu$, $\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a$ is integrable; that is $\|\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a\|_1 \lesssim 1$.

Now we consider when $r \leqslant 1$, if in addition we have $\rho_A(x) \leqslant 1$ (or equivalently $|x| \leqslant 1$), then

$t = 1$ maximises $t^{\nu\alpha}|P_{A(t)} * a(y)|$ in the intersection of $\rho_A(y) \lesssim t + r$ and $\Gamma_{A,\alpha}(x)$, and so

$$\begin{aligned}\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) &\lesssim r^{a_1} \\ &\lesssim 1.\end{aligned}$$

If $\rho_A(x) \leqslant 1$ (or equivalently $|x| \leqslant 1$), then $t \sim \rho_A(x)^{\frac{1}{1-\alpha}}$ maximises $t^{\nu\alpha}|P_{A(t)} * a(y)|$ in the intersection of $\rho_A(y) \lesssim t + r$ and $\Gamma_{A,\alpha}(x)$, and so for $\rho_A(x)^{\frac{1}{1-\alpha}} \sim t \geqslant r$, we have

$$\begin{aligned}\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) &\lesssim r^{a_1} \rho_A(x)^{\frac{\alpha\nu-\nu-a_1}{1-\alpha}} \\ &\lesssim \rho_A(x)^{\frac{a_1}{1-\alpha}} \rho_A(x)^{\frac{\alpha\nu-\nu-a_1}{1-\alpha}} \\ &\lesssim \rho_A(x)^{-\nu},\end{aligned}$$

and for $\rho_A(x)^{\frac{1}{1-\alpha}} \sim t \leqslant r$ we have

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim r^{-\nu} \rho_A(x)^{\frac{\alpha\nu}{1-\alpha}}.$$

Thus, combining these we have

$$\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} a(x) \lesssim \begin{cases} 1 & \text{if } \rho_A(x) \leqslant 1 \\ r^{-\nu} \rho_A(x)^{-\nu\frac{\alpha}{1-\alpha}} & \text{if } \rho_A(x) > 1. \end{cases}$$

So, for fixed $\alpha > 1$, as $r \leqslant 1$ and $\frac{\alpha}{\alpha-1} > 1$, $\mathcal{M}_{\alpha,\frac{\alpha}{2}} a$ is integrable by Proposition 1.3.9; that is $\|\mathcal{M}_{\alpha,\frac{\alpha}{2}} a\|_1 \lesssim 1$. $\square$

So now that we have this endpoint estimate, it is a case of applying interpolation to obtain a large range of other more general estimates via interpolation. The theorem that follows is the conclusion of this chapter and enclosed are all the $L^p - L^q$ estimates obtained.

Theorem 2.4.3 *Let* $1 \leqslant p \leqslant q \leqslant \infty$ *and* $\alpha, \beta \in \mathbb{R}$.

- *If* $\alpha < 0$ *and* $\beta \leqslant \frac{\alpha}{2q} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$;
- *or* $\alpha = 0$ *and* $\beta = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$;
- *or* $\alpha > 0$ *and* $\beta \geqslant \frac{\alpha}{2q} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$;

then

$$\|\mathcal{M}_{A,\alpha,\beta} w\|_q \lesssim \|w\|_p.$$

Proof: If $\alpha < 0$, then $(y,t) \in \Gamma_{A,\alpha}(x)$ implies that $t \geqslant 1$, thus if $\beta' \leqslant \beta$, $t^{2\beta'\nu} \leqslant t^{2\beta\nu}$ furthermore

$$\mathcal{M}_{A,\alpha,\beta'} w(x) \leqslant \mathcal{M}_{A,\alpha,\beta} w(x),$$

this allows us to reduce the case $\alpha < 0$ down to just the sharp line

$$\beta = \frac{\alpha}{2q} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right).$$

The same argument reduces the case $\alpha > 0$ to the same sharp line. As we did for the $H^1_A - L^1$ estimate, we will perform our analysis on the specific bump function P.

$$\begin{aligned} |\mathcal{M}_{A,\alpha,0} w(x)| &= \sup_{(y,t)\in\Gamma_{A,\alpha}(x)} |P_{A(t)} * w(y)| \\ &\leqslant \|P\|_1 \|w\|_\infty \end{aligned}$$

as $P_{A(t)}$ is normalised in L^1. Since this estimate is uniform in x, we immediately get

$$\|\mathcal{M}_{A,\alpha,0} w\|_\infty \lesssim \|w\|_\infty. \tag{2.7}$$

Additionally, we have

$$\begin{aligned}|\mathcal{M}_{A,\alpha,\frac{1}{2}} w(x)| &= \sup_{(y,t)\in\Gamma_{A,\alpha}(x)} t^{\nu} |P_{A(t)} * w(y)| \\ &\leqslant \|P\|_\infty \|w\|_1,\end{aligned}$$

again uniformly in x, so we have

$$\|\mathcal{M}_{A,\alpha,\frac{1}{2}} w\|_\infty \lesssim \|w\|_1. \tag{2.8}$$

Now, we use analytic interpolation on (2.7) and (2.8). Let $t \in (0,1)$ be our interpolation variable, then we obtain

$$\|\mathcal{M}_{A,\alpha,\beta_t} w\|_{q_t} \lesssim \|w\|_{p_t},$$

where $\beta_t = \frac{1}{2}(1-t)$, $p_t = \frac{1}{1-t}$ and $\frac{1}{q_t} = 0$. Rearranging these and eliminating t gives us

$$\|\mathcal{M}_{A,\alpha,\frac{1}{2s}} w\|_\infty \lesssim \|w\|_s, \tag{2.9}$$

for $s \in (1,\infty)$. Note that $\frac{1}{s} = 0, 1$ are the trivial endpoint estimates above, where we interpret $\frac{1}{s} = 0$ as $s = \infty$.

Additionally, by Theorem 2.4.1 we have

$$\|\mathcal{M}_{A,\alpha,\frac{\alpha}{2}} w\|_1 \lesssim \|w\|_{H^1_A}. \tag{2.10}$$

We shall interpolate between (2.9) and (2.10) using a form of analytic interpolation designed for spaces of homogeneous type, which H^1_A are known to be, see [35] and [13] Theorem D. Again let $t \in (0,1)$ be our interpolation variable, then we obtain

$$\|\mathcal{M}_{A,\alpha,\beta_t} w\|_{q_t} \lesssim \|w\|_{p_t},$$

where $\beta_t = \frac{\alpha}{2}(1-t) + \frac{t}{2s}$, $\frac{1}{q_t} = 1 - t$ and $\frac{1}{p_t} = 1 - t + \frac{t}{s}$. Rearranging these again and eliminating t gives us

$$\|\mathcal{M}_{A,\alpha,\beta} w\|_{q} \lesssim \|w\|_{p},$$

where $\beta = \frac{\alpha}{2q} + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, as required. $\square$

CHAPTER 3

MULTIPLIERS

3.1 Fourier multiplier theory

For appropriate $m : \mathbb{R} \to \mathbb{C}$, define T_m by

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$$

for $\xi \in \mathbb{R}$, then m is called a (Fourier) multiplier, and T_m is a (Fourier) multiplier operator. To each Fourier multiplier we have a corresponding convolution kernel, K, where

$$T_m f = K * f$$

and by the convolution theorem we have

$$\widehat{K} = m.$$

For example the Hilbert transform, H, has multiplier given by

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi).$$

The corresponding convolution kernel for the Hilbert transform is well known to be $\frac{1}{x}$. H is the prototype that the theory of Calderón-Zygmund singular integral operators was based, thus by [32] we have

$$\int_{\mathbb{R}} |Hf|^p w \lesssim \int_{\mathbb{R}} |f|^p M^{[p]+1} w \tag{3.1}$$

for $1 < p < \infty$, and so we can obtain all the known L^p bounds for H from bounds on M and the now familiar transfer of bounds (1.4).

However, there are Fourier multipliers that are not bounded on L^p and so clearly cannot be bounded in the above sense by powers of M. For these we must develop different maximal operators that have more general L^p-L^q bounds. One example is the fractional integral operator I_α of order $0 < \alpha < 1$, given by

$$I_\alpha f(x) = \int_{\mathbb{R}} \frac{f(x-y)}{|y|^{1-\alpha}} dy$$

with multiplier given by

$$\widehat{I_\alpha f}(\xi) = |\xi|^{-\alpha} \hat{f}(\xi),$$

modulo a constant. The L^p-L^q bounds for these operators have been known for a long time, in fact Hardy, Hardy and Littlewood showed in [19] that I_α is bounded from L^p to L^q when $1 < p < \frac{1}{\alpha}$ and $q = \frac{p}{1-p\alpha}$.

The history of these operators in a weighted context follows a similar path to that of the Calderón-Zygmund operators; we have the following theorem, due to Adams.

Theorem 3.1.1 ([1]) *For $p, r > 1$*

$$\int_{\mathbb{R}} |I_\alpha f|^p w \lesssim \int_{\mathbb{R}} |f|^p (M_{\alpha p r} w^r)^{1/r}$$

where M_β is the fractional maximal operator defined by

$$M_\beta f(x) = \sup_{x \in Q} \frac{|Q|^\beta}{|Q|} \int_Q |f|.$$

The above fractional maximal operator was introduced by Muckenhoupt and Wheedon, who in the same paper showed that

Theorem 3.1.2 ([31]) *Let $1 \leqslant p < \infty$ and $0 < \alpha < 1$, then*

$$\|I_\alpha f\|_p \lesssim \|M_\alpha f\|_p,$$

where the implicit constant depends on α.

From this we can see that bounds on I_α follow from bounds on M_α, in fact they are equivalent operators in this sense, see [16] for more details. However, in parallel with the Calderön-Zygmund theory the above inequality of Adams does not give us all the known L^p-L^q bounds for I_α. However, Pérez also considered these operators, where he produced a very similar result to that of his treatment of Calderón-Zygmund operators.

Theorem 3.1.3 ([33]) *For $0 < \alpha < 1$ and $1 < p < \infty$ we have*

$$\int_{\mathbb{R}} |I_\alpha f|^p w \lesssim \int_{\mathbb{R}} |f|^p M_{\alpha p}(M^{[p]} w),$$

where this time the implicit constant is independent of α.

To discuss more general multiplier theorems, we must return to the non-weighted setting momentarily. A classical multiplier theorem is the following.

Theorem 3.1.4 (See [16] for details) *If m is a function of bounded variation on $\mathbb{R}$, then m is an L^p multiplier for $1 < p < \infty$.*

The proof of this relies on use of the Hilbert transform, so it is unsurprising that we get similar bounds. Note that while this is a sufficient condition for m to be a multiplier, it is not a necessary one. We can see this directly from the example of the fractional integral operator, I_α, which has multiplier $|\xi|^{-\alpha}$. These multipliers are too singular (or rough) to have bounded variation on the whole of $\mathbb{R}$, yet we know they obey L^p-L^q bounds.

There are many theorems reducing this gap in our classification, one classical example is the Marcinkiewicz multiplier theorem, which asks slightly less than Theorem 3.1.4 by asking only that m has bounded variation on each dyadic interval uniformly.

Theorem 3.1.5 (See [16] for details) *If m has uniformly bounded variation on each dyadic interval in $\mathbb{R}$, then m is an L^p multiplier for $1 < p < \infty$.*

Another approach to multipliers makes use of the Sobolev space $L^2_a(\mathbb{R})$, defined for $a > 0$, which is the set of functions g such that

$$(1 + |\xi|^2)^{a/2}\widehat{g}(\xi) \in L^2,$$

where the norm is defined by

$$\|g\|_{L^2_a} = \left(\int_{\mathbb{R}} |(1 + |\xi|^2)^{a/2}\widehat{g}(\xi)|^2 d\xi \right)^{1/2}.$$

With this we can state another classical result in multiplier theory.

Theorem 3.1.6 (See [16] for details) *If $a > 1/2$ and $m \in L^2_a$, then m is a multiplier on L^p for $1 \leqslant p \leqslant \infty$*

In contrast with the previous approach, this method does not rely on the Hilbert transform and this gives us the end-point results $p = 1, \infty$.

Again, the hypotheses of this result can be weakened by considering a dyadic decomposition. To state this theorem, we need a dyadic partition of unity by smooth functions.
Let $\psi \in C^\infty(\mathbb{R})$ be supported on $\frac{1}{2} \leqslant |\xi| \leqslant 2$ such that

$$\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 = 1,$$

for $|\xi| \neq 0$. Then the Hörmander multiplier theorem is as follows.

Theorem 3.1.7 ([21]) *If $a > 1/2$ and m is such that*

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\psi\|_{L^2_a} < \infty,$$

then m is an L^p multiplier for $1 < p < \infty$.

Kurtz then went on to show a generalised single-weighted version of the Hörmander multiplier theorem [23]. Kurtz and Wheedon extended this to a generalised single-weight version of the Marcinkiewicz multiplier theorem [24]. These give very general weighted L^p bounds; however, as there is a restriction made on the weights allowed it does not allow us to use these weighted inequalities to obtain non-weighted L^p bounds via (1.4).
A Fefferman-Stein-type inequality of the Marcinkiewicz multiplier theorem can be obtained relatively easily, first consider m to be bounded and of bounded variation on $\mathbb{R}$. Then

$$T_m = \lim_{t\to-\infty} m(t)I + \frac{1}{2}\int_{\mathbb{R}} (I + iE_{-t}HE_t)|m'(t)|dt,$$

where I is the identity operator and the modulation operator E_t is given by $E_tf(x) = e^{-ixt}f(x)$. As E_t is bounded on $L^p(w)$ for $1 \leqslant p \leqslant \infty$ and by (3.1) we have

$$\int_{\mathbb{R}} |T_mf|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^3 w,$$

see [16] for details.

Using this, and classical Littlewood-Paley theory for dyadic decompositions of the line and (1.5), we obtain the following theorem.

Theorem 3.1.8 *If m has uniformly bounded variation on dyadic intervals, then*

$$\int_{\mathbb{R}} |T_m f|^2 w \lesssim \int_{\mathbb{R}} |f|^2 M^7 w.$$

Note that in contrast to putting a constraint on the weight w, obtaining Fefferman-Stein-type inequalities allows us to use (1.4) to immediately recover the classical Marcinkiewicz multiplier theorem. This idea of using Littlewood-Paley theory to reduce to dyadic intervals, thus reducing the problem to simpler behaviour that is easier to bound, is the impetus behind this chapter. We will discuss this method more in depth later, for further discussion see [3, 2, 38].

While these classical theorems deal with more singular multipliers, such as the fractional integral operator, there are much more singular multipliers that do not have bounded variation on even dyadic intervals. The following theorem can be found in the encyclopedic exposé of the topic by Miyachi.

Theorem 3.1.9 ([30]) *If $m \in C^1(\mathbb{R})$ has support in $|\xi| \geqslant 1$ such that for $a, b \geqslant 0$, we have $a(1/p - 1/2) = b$ and*

$$|m(\xi)| \lesssim |\xi|^{-b},$$
$$|m'(\xi)| \lesssim |\xi|^{-b+a-1}$$

or if $m \in C^1(\mathbb{R} \setminus \{0\})$ has support in $|\xi| \leqslant 1$ such that for $c, d \geqslant 0$, we have $c(1/p - 1/2) = d$

and

$$|m(\xi)| \lesssim |\xi|^{d},$$
$$|m'(\xi)| \lesssim |\xi|^{d-c-1},$$

then m is an L^p multiplier for $1 < p < 2$.

We note here that if we take $a = b = c = d = 0$, the above theorem implies both the Marcinkiewicz multiplier theorem and the Hörmander multiplier theorem, see [16] for details.

Remark 3.1.10 *A critique should be made of some of the above theorems, and furthermore of many classical multiplier theorems. That is, such theorems suffer from having hypotheses that are not translation invariant for the multiplier, yet it is well known that multiplier operators are translation invariant, furthermore the conclusions of the theorems are both translation and modulation invariant in the kernels due to the convolution structure of the operators and the following observation.*

Let m be a multiplier and T_m be the associated multiplier operator for which we have a Fefferman-Stein-type inequality with maximal operator M_m, that is for all admissible input functions f we have

$$\int_{\mathbb{R}} |T_m f|^p w \leqslant C \int_{\mathbb{R}} |f|^p M_m w,$$

and consider a translation of our multiplier by a, $m(\cdot - a)$. Then the convolution kernel associated with m, say K, would be modulated by $e^{ia\cdot}$, and so

$$\begin{aligned} T_{m(\cdot-a)}f(x) &= \int_{\mathbb{R}} K(y)e^{-iay}f(x-y)dy \\ &= e^{-iax}\int_{\mathbb{R}} K(y)e^{ia(x-y)}f(x-y)dy \end{aligned}$$

and so

$$T_{m(\cdot-a)}f(x) = e^{-iax}T_m(e^{ia\cdot}f)(x),$$

taking the modulus of each side

$$|T_{m(\cdot-a)}f(x)| = |T_m(e^{ia\cdot}f)(x)|$$

and since we have the Fefferman-Stein-type inequality for any admissible function f and weighted L^p spaces are invariant under modulation, we obtain

$$\int_{\mathbb{R}} |T_{m(\cdot-a)}f|^p w \lesssim \int_{\mathbb{R}} |f|^2 M_m w.$$

3.2 Anisotropic square function estimates

Definition 3.2.1 *We call an operator T a Calderón-Zygmund operator associated with the dilation A if the following hold*

1. $T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$*;*

2. *there exists a measurable function $K : \mathbb{R}^2 \to \mathbb{R}$ such that for every $f \in L_0^\infty(\mathbb{R}^2)$ we have*

$$Tf(x) = \int_{\mathbb{R}^2} K(x-y)f(y)dy$$

   for a.e. $x \notin \operatorname{supp}(f)$;

3. *the kernel K satisfies*

$$|K(x)| \lesssim \frac{1}{\rho_A(x)^\nu}$$

   for every $x \in \mathbb{R}^2$;

4. *the kernels K and K^* (defined by $K^*(x) = K(-x)$) satisfy the following pointwise Hörmander condition: There exist a positive constants $M > 1$ and $\gamma > 0$ such that whenever $\rho_A(y) < \frac{1}{M}\rho_A(x)$ we have*

$$|K(x) - K(x-y)| \lesssim \frac{\rho_A(y)^\gamma}{\rho_A(x)^{\nu+\gamma}}.$$

Note that in the isotropic definition of CZO, the constant $M > 1$ plays the role of keeping $x - y$ close to x in some sense. In our anisotropic definition, our understanding of "closeness" is necessarily dependent upon our space, thus our M will depend on A.
In M. Christ's book [11] page 94 Theorem 9 references the below theorem by R.R. Coifman and G. Weiss.

Theorem 3.2.2 ([12]) *Let $1 < q < \infty$ and w be a weight. If T is a Calderón-Zygmund operator associated with the dilation A then T is bounded on L^q.*

Additionally, in a paper by G. Pradolini and O. Salinas appears the following theorem.

Theorem 3.2.3 ([34]) *Let $1 < q < \infty$ and w be a weight. If T is a Calderón-Zygmund operator associated with the dilation A and $T : L^q(\mathbb{R}^2) \to L^q(\mathbb{R}^2)$ is linear and continuous for all $q \in (1, \infty)$ then*

$$\int_{\mathbb{R}^2} |Tf(x)|^p w(x) dx \lesssim \int_{\mathbb{R}^2} |f(x)| M_A^{[p]+1} w(x) dx$$

where $[p]$ is the largest integer smaller than p.

See Hu et al. [22] for further discussion.

We now introduce an anisotropic version of the continuous square functions.

Definition 3.2.4

$$s_A(f)(x) = \left(\int_0^\infty |f * \phi_{A(t)}(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $\widehat{\phi}$ has compact support away from the origin and for $\xi \neq 0$,

$$\int_0^\infty \widehat{\phi}(A(t)\xi) \frac{dt}{t} = 1. \tag{3.2}$$

This section concerns itself with the two-weighted problem of this anisotropic square function. The coming propositions show how to reduce the problem to an application of Theorem 3.2.3 in the case $p = 2$ and follows mostly follows the isotropic case. In these propositions we have not concerned ourselves with minimising the amount of applications of M_A to the weight; however, it would be surprising if Proposition 3.2.5 was not sharp in this sense and equally as surprising if Proposition 3.2.7 was sharp in this sense. See [10] for further discussion in the isotropic case.

Proposition 3.2.5

$$\|f\|_{L^2(w)} \lesssim \|s_A(f)\|_{L^2(M_A^3 w)}.$$

Proof: For $j = 0, 1, 2, 3, 4, 5$, let

$$f_j(x) = \sum_{k \in 6\mathbb{Z}+\{j\}} \int_{2^k}^{2^{k+1}} \phi_{A(t)} * f(x) \frac{dt}{t}.$$

Define $\widehat{h}(\xi) = \int_1^2 \widehat{\phi}(A(t)\xi)\frac{dt}{t}$. As $\mathrm{supp}(\widehat{\phi}) \subseteq \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leqslant \rho_A(\xi) \leqslant 3\}$ the function $\widehat{h}(\xi)$ has support when

$$\frac{3}{4} \leqslant \rho_A(A(t)\xi) \leqslant 3,$$

which using the A-homogeneity of ρ_A and the fact that in the definition of $\widehat{h}(\xi)$ the integral is over the set $t \in (1, 2)$, which gives the support of $\widehat{h}(\xi)$ as

$$\frac{3}{8} \leqslant \rho_A(\xi) \leqslant 3.$$

Now, define $\widehat{\chi}$ as the smooth function equal to 1 on the support of $\widehat{h}(\xi)$ and supported in the set $\{\xi \in \mathbb{R}^2 : \frac{1}{8} \leqslant \rho_A(\xi) \leqslant 4\}$. Additionally, let $\varepsilon = \{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a Rademacher distribution. Next define T_j^ε by

$$\widehat{T_j^\varepsilon f}(\xi) = \sum_{k \in 6\mathbb{Z}+\{j\}} \varepsilon_k \widehat{\chi}(A(2^{-k})\xi)\widehat{f}(\xi).$$

Now, consider

$$\widehat{T_j^\varepsilon T_j^\varepsilon f_j}(\xi) = \sum_{k \in 6\mathbb{Z}+\{j\}} \varepsilon_k \widehat{\chi}(A(2^{-k})\xi) \left(\sum_{s \in 6\mathbb{Z}+\{j\}} \varepsilon_s \widehat{\chi}(A(2^{-s})\xi)\widehat{f}_j(\xi) \right).$$

Note that by the choice of the support of $\widehat{\chi}$ that each of the summands in the definition of $\widehat{T_j^\varepsilon}$ have non-overlapping support, thus the only terms in the double sum above are when $s = k$. Therefore,

$$\begin{aligned}\widehat{T_j^\varepsilon T_j^\varepsilon f_j}(\xi) &= \sum_{k \in 6\mathbb{Z}+\{j\}} \varepsilon_k \widehat{\chi}(A(2^{-k})\xi)\varepsilon_k \widehat{\chi}(A(2^{-k})\xi)\widehat{f}_j(\xi) \\ &= \sum_{k \in 6\mathbb{Z}+\{j\}} \widehat{\chi}(A(2^{-k})\xi)\widehat{\chi}(A(2^{-k})\xi)\widehat{f}_j(\xi).\end{aligned}$$

Now, by how we defined $\widehat{\chi}$ it is clear that $\widehat{h}(\xi)\widehat{\chi}(\xi) = \widehat{h}(\xi)$, this allows us to conclude that

$$T_j^\varepsilon T_j^\varepsilon f_j(x) = f_j(x).$$

Next, consider (3.2) multiplied on both sides by $\widehat{f}(\xi)$, for $\xi \neq 0$

$$\widehat{f}(\xi) = \int_0^\infty \widehat{\phi}(A(t)\xi)\widehat{f}(\xi)\frac{dt}{t}.$$

Taking the inverse Fourier transform of both sides and splitting the t integral into dyadic intervals gives us

$$f(x) = \sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} \phi_{A(t)} * f(x)\frac{dt}{t},$$

which allows us to write $f(x) = \sum_{j=0}^{5} f_j(x)$, in particular

$$f(x) = \sum_{j=0}^{5} T_j^\varepsilon T_j^\varepsilon f_j(x).$$

Therefore we can write

$$\int_{\mathbb{R}^2} |f(x)|^2 w(x)dx = \int_{\mathbb{R}^2} \left| \sum_{j=0}^{5} T_j^\varepsilon T_j^\varepsilon f_j(x) \right|^2 w(x)dx$$
$$\lesssim \sum_{j=0}^{5} \int_{\mathbb{R}^2} \left| T_j^\varepsilon T_j^\varepsilon f_j(x) \right|^2 w(x)dx.$$

Now, we make a claim about the operator T_j^ε.

Claim 3.2.6 *For each $j = 0, ..., 5$, T_j^ε is a Calderón-Zygmund operator associated with the dilation A uniformly in ε.*

Assuming this claim for now, along with Theorem 3.2.2 and Theorem 3.2.3, we have that

$$\int_{\mathbb{R}^2} \left| T_j^\varepsilon T_j^\varepsilon f_j(x) \right|^2 w(x)dx \lesssim \int_{\mathbb{R}^2} \left| T_j^\varepsilon f_j(x) \right|^2 M_A^3 w(x)dx.$$

This gives us

$$\int_{\mathbb{R}^2} |f(x)|^2 w(x)dx \lesssim \sum_{j=0}^{5} \int_{\mathbb{R}^2} \left| T_j^\varepsilon f_j(x) \right|^2 M_A^3 w(x)dx. \tag{3.3}$$

Now, writing

$$y_k(x) = \int_{2^k}^{2^{k+1}} \phi_{A(t)} * f(x) \frac{dt}{t},$$

we can therefore write, due to the support of $\hat{\chi}$,

$$T_j^\varepsilon f_j(x) = \sum_{k \in 6\mathbb{Z} + \{j\}} \varepsilon_k y_k(x).$$

Thus, taking expectations in ε for both sides of (3.3) gives us

$$\int_{\mathbb{R}^2} |f(x)|^2 w(x)dx \lesssim \sum_{j=0}^{5} \int_{\mathbb{R}^2} \mathbb{E}\left(\left|\sum_{k\in 6\mathbb{Z}+\{j\}} \varepsilon_k y_k(x)\right|^2\right) M_A^3 w(x)dx.$$

Now, consider

$$\mathbb{E}\left(\left|\sum_{k\in 6\mathbb{Z}+\{j\}} \varepsilon_k y_k(x)\right|^2\right) = \mathbb{E}\left(\sum_{k\in 6\mathbb{Z}+\{j\}} \sum_{\ell\in 6\mathbb{Z}+\{j\}} \varepsilon_k \varepsilon_\ell y_k(x) y_\ell(x)\right).$$

Using the independence of ε and the fact that $\mathbb{E}(\varepsilon) = 0$

$$\mathbb{E}\left(\left|\sum_{k\in 6\mathbb{Z}+\{j\}} \varepsilon_k y_k(x)\right|^2\right) = \sum_{k\in 6\mathbb{Z}+\{j\}} \left|y_k(x)\right|^2.$$

Thus, with an application of Cauchy-Schwarz and the observation that

$$\int_{2^k}^{2^{k+1}} \frac{dt}{t} = \ln 2,$$

we have that

$$\begin{aligned}
\left|y_k(x)\right|^2 &= \left|\int_{2^k}^{2^{k+1}} \phi_{A(t)} * f(x) \frac{dt}{t}\right|^2 \\
&\leqslant \left(\int_{2^k}^{2^{k+1}} \left|\phi_{A(t)} * f(x)\right|^2 \frac{dt}{t}\right)\left(\int_{2^k}^{2^{k+1}} |1|^2 \frac{dt}{t}\right) \\
&\lesssim \int_{2^k}^{2^{k+1}} \left|\phi_{A(t)} * f(x)\right|^2 \frac{dt}{t}.
\end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned}\int_{\mathbb{R}^2} |f(x)|^2 w(x)dx &\lesssim \sum_{j=0}^{5} \int_{\mathbb{R}^2} \sum_{k\in 6\mathbb{Z}+\{j\}} \int_{2^k}^{2^{k+1}} \left|\phi_{A(t)} * f(x)\right|^2 \frac{dt}{t} M_A^3 w(x)dx \\ &= \int_{\mathbb{R}^2} \int_0^\infty \left|\phi_{A(t)} * f(x)\right|^2 \frac{dt}{t} M_A^3 w(x)dx \\ &= \int_{\mathbb{R}^2} s_A(f)(x)^2 M_A^3 w(x)dx.\end{aligned}$$

So, to complete the proof of Proposition 3.2.5 it is sufficient to prove Claim 3.2.6. □

Proposition 3.2.7

$$\|s_A(f)\|_{L^2(w)} \lesssim \|f\|_{L^2(M_A^3 w)}.$$

Proof: To prove this we will introduce a discrete version of the square function, this will allow us to expand out the square directly as we did above. Let $\kappa \in [1,2]$ and define

$$S_\kappa(f)^2(x) = \sum_{k\in\mathbb{Z}} |\phi_{A(\kappa 2^{-k})}(x)|^2.$$

With this, we can write our square function as

$$\begin{aligned}s_A(f)(x) &= \int_0^\infty \left|\phi_{A(t)} * f(x)\right|^2 \frac{dt}{t} \\ &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left|\phi_{A(t)} * f(x)\right|^2 \frac{dt}{t}.\end{aligned}$$

Let $t = \kappa 2^{-k}$, then $\frac{dt}{d\kappa} = 2^{-k}$ and so

$$\begin{aligned}
s_A(f)(x) &= \sum_{k\in\mathbb{Z}} \int_1^2 \left|\phi_{A(\kappa 2^{-k})} * f(x)\right|^2 \frac{d\kappa}{\kappa} \\
&\leqslant \int_1^2 \sum_{k\in\mathbb{Z}} \left|\phi_{A(\kappa 2^{-k})} * f(x)\right|^2 \frac{d\kappa}{\kappa} \\
&= \int_1^2 S_\kappa(f)^2(x) \frac{d\kappa}{\kappa}.
\end{aligned}$$

Let ε be a Rademacher distribution as above, expanding out the squares, we have

$$\begin{aligned}
S_\kappa(f)^2(x) &= \sum_{k\in\mathbb{Z}} \left|\phi_{A(\kappa 2^{-k})} * f(x)\right|^2 \\
&= \mathbb{E}\left(\left|\sum_{k\in\mathbb{Z}} \varepsilon_k \phi_{A(\kappa 2^{-k})} * f(x)\right|^2\right).
\end{aligned}$$

For the sake of simplicity of proof later, we introduce the same T_j^ε, for $j = 0, ..., 5$ as before, but this time it will have an extra dependence, κ. Define $T_{j,\kappa}^\varepsilon$ by

$$\widehat{T_{j,\kappa}^\varepsilon f}(\xi) = \sum_{k\in 6\mathbb{Z}+\{j\}} \varepsilon_k \widehat{\chi}(A(\kappa 2^{-k})\xi)\widehat{f}(\xi),$$

where $\widehat{\chi}$ is again the function equal to 1 on $\{\xi \in \mathbb{R}^2 : \frac{3}{8} \leqslant \rho_A(\xi) \leqslant 3\}$ and supported in the set $\{\xi \in \mathbb{R}^2 : \frac{1}{8} \leqslant \rho_A(\xi) \leqslant 4\}$. Note that by how we chose the support of $\widehat{\chi}$,

$$\begin{aligned}
S_\kappa(f)^2(x) &\leqslant \mathbb{E}\left(\left|\sum_{j=0}^5 T_j^\varepsilon f(x)\right|^2\right) \\
&\lesssim \mathbb{E}\left(\sum_{j=0}^5 \left|T_j^\varepsilon f(x)\right|^2\right).
\end{aligned}$$

Thus,

$$
\begin{aligned}
\|s_A(f)\|_{L^2(w)}^2 &= \int_{\mathbb{R}^2} s_A(f)(x)^2 w(x)dx \\
&\leqslant \int_{\mathbb{R}^2}\int_1^2 S_\kappa(f)^2(x)\frac{d\kappa}{\kappa}w(x)dx \\
&\lesssim \int_{\mathbb{R}^2}\int_1^2 \mathbb{E}\left(\sum_{j=0}^{5}\left|T_{j,\kappa}^{\varepsilon}f(x)\right|^2\right)\frac{d\kappa}{\kappa}w(x)dx \\
&\leqslant \int_1^2 \mathbb{E}\left(\sum_{j=0}^{5}\int_{\mathbb{R}^2}\left|T_{j,\kappa}^{\varepsilon}f(x)\right|^2 w(x)dx\right)\frac{d\kappa}{\kappa}.
\end{aligned}
$$

Claim 3.2.8 *For each $j = 0, ..., 5$, $T_{j,\kappa}^{\varepsilon}$ is a Calderón-Zygmund operator associated with the dilation A uniformly in ε and κ.*

Assuming this claim again for now, along with Theorem 3.2.2 and Theorem 3.2.3, we have that

$$
\int_{\mathbb{R}^2}\left|T_{j,\kappa}^{\varepsilon}f(x)\right|^2 w(x)dx \lesssim \int_{\mathbb{R}^2}|f(x)|^2 M_A^3 w(x)dx,
$$

and therefore

$$
\|s_A(f)\|_{L^2(w)}^2 \lesssim \int_1^2 \mathbb{E}\left(\sum_{j=0}^{5}\int_{\mathbb{R}^2}|f(x)|^2 M_A^3 w(x)dx\right)\kappa d\kappa.
$$

Finally, note that the RHS is independent of ε and j, and since $\kappa \leqslant 2$, we have

$$
\begin{aligned}
\|s_A(f)\|_{L^2(w)}^2 &\lesssim \int_{\mathbb{R}^2}|f(x)|^2 M_A^3 w(x)dx \\
&= \|f\|_{L^2(M_A^3 w)}^2.
\end{aligned}
$$

So, to complete the proof of Proposition 3.2.7 it is sufficient to prove Claim 3.2.8. $\square$

Note that Claim 3.2.6 is an immediate consequence of Claim 3.2.8 proof by taking $\kappa = 1$.

Proof: [Claim 3.2.8] By Definition 3.2.1 there are 4 properties to check; property 2 is trivially satisfied by the definition of $\widehat{T_j^{\varepsilon,\kappa}}$ where

$$K(x) = \sum_{k\in 6\mathbb{Z}+\{j\}} \varepsilon_k \chi_{A(\kappa 2^{-k})}(x).$$

Property 1 boils down to Plancherel's theorem as follows

$$\begin{aligned}
\|T_j^\varepsilon f\|_2^2 &= \|\widehat{T_j^\varepsilon f}\|_2^2 \\
&= \int_{\mathbb{R}^2} \left| \sum_{k\in 6\mathbb{Z}+\{j\}} \varepsilon_k \widehat{\chi}(A(\kappa 2^{-k})\xi)\widehat{f}(\xi) \right|^2 d\xi \\
&\leqslant \int_{\mathbb{R}^2} \sum_{k\in 6\mathbb{Z}+\{j\}} \left|\widehat{\chi}(A(\kappa 2^{-k})\xi)\right|^2 \left|\widehat{f}(\xi)\right|^2 d\xi.
\end{aligned}$$

As $\mathrm{supp}(\widehat{\chi}(A(\kappa 2^{-k})\cdot))$ is disjoint from $\mathrm{supp}(\widehat{\chi}(A(\kappa 2^{-s})\cdot))$ for all $s \neq k \in 6\mathbb{Z}+\{j\}$, we have that

$$\sum_{k\in 6\mathbb{Z}+\{j\}} \left|\widehat{\chi}(A(\kappa 2^{-k})\xi)\right|^2 \leqslant C.$$

Therefore, we conclude that

$$\begin{aligned}
\|T_j^\varepsilon f\|_2^2 &\lesssim \int_{\mathbb{R}^2} \left|\widehat{f}(\xi)\right|^2 d\xi \\
&= \|\widehat{f}\|_2^2 \\
&= \|f\|_2^2.
\end{aligned}$$

For properties 3 and 4 the arguments are longer, but no more complex. First let's start with property 3. Note that although the following proof is for K, it is identical for K^* due to our bounds on χ being even. So, fix i and x such that $2^{-i} \leqslant \rho_A(x) \leqslant 2^{-i+1}$. Then, as $1 \leqslant \kappa \leqslant 2$,

$$\begin{aligned}|K(x)| &= \left| \sum_{k \in 6\mathbb{Z}+\{j\}} \varepsilon_k \kappa^{-\nu} 2^{k\nu} \chi(A(\kappa^{-1}2^k)x) \right| \\ &\lesssim \sum_{k \in 6\mathbb{Z}+\{j\}} 2^{k\nu} \left| \chi(A(\kappa^{-1}2^k)x) \right|.\end{aligned}$$

As $\widehat{\chi}$ is a bump function, $\chi \in \mathcal{S}$ and therefore for each $N \in \mathbb{N}_0$, we can find $\lambda_N > 0$ such that

$$|\chi(y)| \leqslant \frac{\lambda_N}{(1+\rho_A(y))^N}$$

for all $y \in \mathbb{R}^2$. We note that our λ_N implicitly depends on the the homogeneous dimension ν, but we will explicitly choose N dependent on ν at a later stage in the proof. Now, if we split the sum using the triangle inequality as

$$|K(x)| \lesssim \sum_{\substack{k \in 6\mathbb{Z}+\{j\} \\ k \leqslant i}} 2^{k\nu} \left| \chi(A(\kappa^{-1}2^k)x) \right| + \sum_{\substack{k \in 6\mathbb{Z}+\{j\} \\ k > i}} 2^{k\nu} \left| \chi(A(\kappa^{-1}2^k)x) \right|$$

then use the estimate on χ separately in each sum

$$\begin{aligned}|K(x)| &\lesssim \sum_{\substack{k \in 6\mathbb{Z}+\{j\} \\ k \leqslant i}} 2^{k\nu} \frac{\lambda_0}{(1+\rho_A(A(\kappa^{-1}2^k)x))^0} + \sum_{\substack{k \in 6\mathbb{Z}+\{j\} \\ k > i}} 2^{k\nu} \frac{\lambda_N}{(1+\rho_A(A(\kappa^{-1}2^k)x))^N} \\ &= 2^{i\nu} \left(\sum_{\substack{k \in 6\mathbb{Z}+\{j\} \\ k \leqslant i}} \lambda_0 2^{(k-i)\nu} + \sum_{\substack{k \in 6\mathbb{Z}+\{j\} \\ k > i}} \frac{\lambda_N 2^{(k-i)\nu}}{(1+\kappa^{-1}2^k \rho_A(x))^N} \right) \\ &\leqslant 2^{i\nu} \left(2^\nu \lambda_0 + 2^{2-N} \lambda_N \sum_{\substack{k \in 6\mathbb{Z}+\{j\} \\ k > i}} \frac{2^{(k-i)\nu}}{(2^{k-i})^N} \right) \\ &= 2^{i\nu} \left(2^\nu \lambda_0 + 2^{2-N} \lambda_N \sum_{\substack{k \in 6\mathbb{Z}+\{j\} \\ k > i}} 2^{(k-i)(\nu-N)} \right),\end{aligned}$$

and so, choosing $N > \nu$, we have that

$$\begin{aligned}|K(x)| &\lesssim 2^{i\nu} \\ &\lesssim \rho_A(x)^{-\nu}.\end{aligned}$$

Again for property 4, fix i and x such that $2^{-i} \leqslant \rho_A(x) \leqslant 2^{-i+1}$, and fix y such that $\rho_A(y) \leqslant \frac{1}{M}\rho_A(x)$, for some $M > 1$ that we will choose. Now, consider

$$\begin{aligned}|K(x-y) - K(x)| &\leqslant \left| \sum_{k \in 6\mathbb{Z}+\{j\}} \varepsilon_k \kappa^{-\nu} 2^{k\nu} \chi(A(\kappa^{-1}2^k)(x-y)) - \chi(A(\kappa^{-1}2^k)x)) \right| \\ &\lesssim \sum_{k \in 6\mathbb{Z}+\{j\}} 2^{k\nu} \left| \chi(A(\kappa^{-1}2^k)(x-y)) - \chi(A(\kappa^{-1}2^k)x) \right|.\end{aligned}$$

Define for $s \in [0,1]$,

$$g(s) = \chi(A(\kappa^{-1}2^k)(x - sy))$$

and so by the mean value theorem applied to g we have, for some $c \in (0,1)$,

$$g'(c) = g(1) - g(0)$$

and so, by direct calculation of $g'(c)$, we have

$$\begin{aligned}\left|\chi(A(\kappa^{-1}2^k)(x-y)) - \chi(A(\kappa^{-1}2^k)x)\right| &\leqslant \left|\langle -A(\kappa^{-1}2^k)y, \nabla\chi(A(\kappa^{-1}2^k)(x-cy))\rangle\right| \\ &\leqslant |A(\kappa^{-1}2^k)y||\nabla\chi(A(\kappa^{-1}2^k)(x-cy)|\end{aligned}$$

by the Cauchy-Schwarz inequality. Next, define

$$I_1 = \{k \in 6\mathbb{Z} + \{j\} : k \leqslant i\},$$
$$I_2 = \{k \in 6\mathbb{Z} + \{j\} : \rho_A(A(\kappa^{-1}2^k)y) \leqslant 1, k \geqslant i\} \text{ and}$$
$$I_3 = \{k \in 6\mathbb{Z} + \{j\} : \rho_A(A(\kappa^{-1}2^k)y) \geqslant 1, k \geqslant i\}.$$

Here, we note that there is overlap in some of these sets, but as the summands are all positive, we have

$$\begin{aligned} |K(x-y) - K(x)| \lesssim & \sum_{k\in I_1} 2^{k\nu}|A(\kappa^{-1}2^k)y||\nabla\chi(A(\kappa^{-1}2^k)(x-cy)| \\ & + \sum_{k\in I_2} 2^{k\nu}|A(\kappa^{-1}2^k)y||\nabla\chi(A(\kappa^{-1}2^k)(x-cy)| \\ & + \sum_{k\in I_3} 2^{k\nu}|A(\kappa^{-1}2^k)y||\nabla\chi(A(\kappa^{-1}2^k)(x-cy)|. \end{aligned} \tag{3.4}$$

It's also crucial to note that a priori the sizes of I_2 and I_3 are dependent on y currently, but we will fix this issue shortly by making the sets larger. In fact, we will sum over the set defined by

$$I_4 = \{k \in 6\mathbb{Z} + \{j\} : k \geqslant i\}.$$

We will also need the fact that $\chi \in \mathcal{S}$, thus we can get bounds on $|\nabla\chi|$; that is, for each $N \in \mathbb{N}_0$, we can find $\lambda_N > 0$ such that

$$|\nabla\chi(z)| \leqslant \frac{\lambda_N}{(1+\rho_A(z))^N}$$

for all $z \in \mathbb{R}^2$.

We will consider each sum in turn, first for $k \in I_1$, $k \leqslant i$ and $\rho_A(y) \leqslant \frac{1}{M}\rho_A(x)$, and so if $M \geqslant 4$

we have

$$\rho_A(A(\kappa^{-1}2^k)y) = \kappa^{-1}2^k\rho_A(y) \leqslant \frac{\kappa^{-1}}{M}2^k\rho_A(x) \leqslant \frac{2\kappa^{-1}}{M}2^{k-i} \leqslant 1.$$

So by Remark 1.3.12 we have that $|A(\kappa^{-1}2^k)y| \leqslant \kappa^{-1}2^k\rho_A(y)$, thus we can estimate, with $N = 0$,

$$\begin{aligned}
\sum_{k\in I_1} 2^{k\nu}|A(\kappa^{-1}2^k)y||\nabla\chi(A(\kappa^{-1}2^k)(x-cy)| &\leqslant 2^{i(\nu+1)}\sum_{k\in I_1} 2^{k\nu}\kappa^{-1}2^k\rho_A(y)\lambda_0 2^{-i(\nu+1)} \\
&= 2^{i(\nu+1)}\rho_A(y)\sum_{k\in I_1} 2^{(k-i)(\nu+1)}\kappa^{-1}\lambda_0 \\
&\lesssim \frac{\rho_A(y)}{\rho_A(x)^{\nu+1}},
\end{aligned}$$

where the last line follows from the fact that for $k \in I_1$ we have $k \leqslant i$ and from the fact that $\rho_A(x) \sim 2^{-i}$.

Next, we consider $k \in I_2$. Immediately we have that $\rho_A(A(\kappa^{-1}2^k)y) \leqslant 1$, thus $|A(\kappa^{-1}2^k)y| \leqslant \kappa^{-1}2^k\rho_A(y)$ again, so for each $N_2 \in \mathbb{N}_0$, we have

$$\sum_{k\in I_2} 2^{k\nu}|A(\kappa^{-1}2^k)y||\nabla\chi(A(\kappa^{-1}2^k)(x-cy)| \leqslant 2^{i(\nu+1)}\rho_A(y)\sum_{k\in I_2}\frac{2^{(k-i)(\nu+1)}\kappa^{-1}\lambda_{N_2}}{(1+\rho_A(A(\kappa^{-1}2^k)(x-cy))^{N_2}}.$$

Now, since $\rho_A(y) \leqslant \frac{1}{M}\rho_A(x)$, we have that

$$\begin{aligned}
\rho_A(x-cy) &\geqslant \rho_A(x) - \rho_A(cy) \\
&\geqslant \rho_A(x) - \rho_A(y) \\
&\geqslant \rho_A(x) - \frac{1}{M}\rho_A(y) \\
&= \frac{M-1}{M}\rho_A(x) \\
&\geqslant \frac{1}{2}\rho_A(x)
\end{aligned}$$

where we used the reverse triangle inequality and the monotonicity of ρ_A since $c \in (0,1)$. We have also added a restriction of $M \geqslant 2$ for convenience. So, using this and the homogeneity of ρ_A, we have

$$\begin{aligned}\sum_{k\in I_2} 2^{k\nu}|A(\kappa^{-1}2^k)y||\nabla\chi(A(\kappa^{-1}2^k)(x-cy)| &\leqslant 2^{i(\nu+1)}\rho_A(y)\sum_{k\in I_2}\frac{2^{(k-i)(\nu+1)}\kappa^{-1}\lambda_{N_2}}{2^{kN_2}2^{-N_2}\rho_A(x)^{N_2}}\\ &\lesssim \frac{\rho_A(y)}{\rho_A(x)^{\nu+1}}\sum_{k\in I_2}\frac{2^{(k-i)(\nu+1)}}{2^{(k-i)N_2}}\\ &\leqslant \frac{\rho_A(y)}{\rho_A(x)^{\nu+1}}\sum_{k\in I_4}\frac{2^{(k-i)(\nu+1)}}{2^{(k-i)N_2}}\\ &\lesssim \frac{\rho_A(y)}{\rho_A(x)^{\nu+1}}\end{aligned}$$

for any $N_2 > \nu+1$ as $k \geqslant i$.

Finally, consider $k \in I_3$. By Remark 1.3.12, $\rho_A(A(\kappa^{-1}2^k)y) \geqslant 1$ implies $|A(\kappa^{-1}2^k)y| \leqslant \kappa^{-\sigma}2^{k\sigma}\rho_A(y)^\sigma$, thus for each $N_3 \in \mathbb{N}_0$, we have

$$\begin{aligned}\sum_{k\in I_3} 2^{k\nu}|A(2^k)y||\nabla\chi(A(2^k)(x-cy)| &\leqslant 2^{i(\nu+\sigma)}\sum_{k\in I_3}\frac{2^{k\nu}2^{k\sigma}\rho_A(y)^\sigma\kappa^{-\sigma}\lambda_{N_3}2^{-i(\nu+\sigma)}}{(1+\rho_A(A(2^k)(x-cy))^{N_3}}\\ &\lesssim \frac{\rho_A(y)^\sigma}{\rho_A(x)^{\nu+\sigma}}\sum_{k\in I_3}\frac{2^{(k-i)(\nu+\sigma)}}{2^{(k-i)N_3}}\\ &\lesssim \frac{\rho_A(y)^\sigma}{\rho_A(x)^{\nu+\sigma}},\end{aligned}$$

for any $N_3 > \nu+\sigma$, as $k \geqslant i$. Now, since $\rho_A(y) \leqslant \frac{1}{M}\rho_A(x)$, this gives us

$$\frac{\rho_A(y)}{\rho_A(x)} \leqslant \frac{1}{M} \leqslant 1$$

since $M > 1$, and since $\sigma > 1$,

$$\sum_{k \in I_3} 2^{k\nu} |A(2^k)y| |\nabla \chi(A(2^k)(x - cy)| \lesssim \frac{\rho_A(y)^\sigma}{\rho_A(x)^{\nu+\sigma}}$$
$$= \frac{\rho_A(y)}{\rho_A(x)^{\nu+1}} \left(\frac{\rho_A(y)}{\rho_A(x)}\right)^{\sigma-1} \leqslant \frac{\rho_A(y)}{\rho_A(x)^{\nu+1}}.$$

So we can substitute all of this back into (3.4) and we get

$$|K(x - y) - K(x)| \lesssim \frac{\rho_A(y)}{\rho_A(x)^{\nu+1}},$$

as required, thus we conclude the proof of the claim. $\square$

3.3 Subdyadic Littlewood-Paley theory

3.3.1 Isotropic subdyadic Littlewood-Paley theory

In Chapter 4 we will be producing oscillatory estimates on large classes of kernels and will need both the below theorems, due to Beltran and Bennett, and Bennett, respectively. We include them in this chapter as they are the isotropic and one dimensional versions of the main theorem for this chapter. We will discuss the nature of the first of these two theorems and its proof in more detail when we introduce the parabolic version. The second theorem is proved in [3] in a different way that contains elements that do not easily extend to higher dimensions, specifically using estimates on the Hilbert transform.

Theorem 3.3.1 ([2]) *Let $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{N}^d$ and let f be an admissible input function and w be a weight. If*

$$|D^\gamma m(\xi)| \lesssim |\xi|^{-\beta d+|\gamma|(\alpha-1)}$$

for m with support in $\{\xi \in \mathbb{R}^2 : |\xi|^\alpha \geqslant 1\}$ and $|\gamma| \leqslant \lfloor \frac{d}{2} \rfloor + 1$ then

$$\int_{\mathbb{R}^d} |T_m f|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^2 \mathcal{M}_{\alpha,\beta} M^4 w,$$

where T_m is defined by $\widehat{T_m f} = m\widehat{f}$ and

$$\mathcal{M}_{\alpha,\beta} w(x) = \sup_{(r,y)\in\Gamma_\alpha(x)} \frac{r^{2\beta d}}{r} \int_{|y-z|\leqslant r} w(z)dz, \tag{3.5}$$

where

$$\Gamma_\alpha(x) = \{(r, y) : 0 < r^\alpha \leqslant 1 \text{ and } |y - x| \leqslant r^{1-\alpha}\}.$$

Remark 3.3.2 *We have used the notation D^γ, where $\gamma = (\gamma_1, \gamma_2, ..., \gamma_d)$ is a multi-index to mean*

$$D^\gamma f(x) = \frac{\partial^{|\gamma|} f}{x_1^{\gamma_1} ... x_d^{\gamma_d}}.$$

Theorem 3.3.3 ([3]) *Let α, $\beta \in \mathbb{R}$ and μ, $C > 0$. If $m : \mathbb{R} \to \mathbb{C}$ is such that*

$$\operatorname{supp}(m) \subseteq \{\xi \in \mathbb{R} : |\xi|^\alpha \geqslant \mu^\alpha\}, \tag{3.6}$$

$$\sup_{\xi} |\xi|^\beta |m(\xi)| \leqslant C \tag{3.7}$$

and

$$\sup_{R^\alpha \geqslant \mu^\alpha} \sup_{\substack{I \subseteq [R, 2R] \\ \operatorname{len}(I) = (R/\mu)^{-\alpha} R}} R^\beta \int_{\pm I} |m'(\xi)| d\xi \leqslant C, \tag{3.8}$$

then there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}} |T_m f|^2 w \leqslant cC^2 \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta,\mu} M^4 w,$$

where T_m is defined by $\widehat{T_m f} = m \widehat{f}$,

$$\mathcal{M}_{\alpha,\beta,\mu} w(x) = \sup_{(y,r) \in \Gamma_{\alpha,\mu}(x)} \frac{r^{2\beta}}{r} \int_{y-r}^{y+r} w \tag{3.9}$$

and

$$\Gamma_{\alpha,\mu}(x) = \{(y, r) : 0 < r^\alpha \leqslant \mu^{-\alpha}, |x - y| \leqslant \mu^{-\alpha} r^{1-\alpha}\}.$$

Remark 3.3.4 *Theorem 3.3.3 is a scale invariant version of Theorem 3.3.1 in one dimension.*

We have been precise with the dependence of the conclusion on the constant in hypotheses (3.7) *and* (3.8) *so that we can keep track of the dependence of the constant on the scaling* μ.

3.3.2 g-functions and associated anisotropic subdyadic multipliers

The main result of this thesis is a parabolic version of Beltran and Bennett's Fefferman-Stein inequality [2] for a specific class of multipliers adapted to the dilations A.

Theorem 3.3.5 *Let $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{N}^d$ and let f be an admissible input function and w be a weight. If m is a Fourier multiplier such that*

$$|D^{\gamma} m(\xi)| \lesssim \rho_A(\xi)^{-\beta\nu + \|\gamma\|_A(\alpha-1)} \tag{3.10}$$

for m with support in $\{\xi \in \mathbb{R}^2 : |\xi|^{\alpha} \geqslant 1\}$ and $|\gamma| \leqslant 3$, then

$$\int_{\mathbb{R}^2} |T_m f(x)|^2 w(x) dx \lesssim \int_{\mathbb{R}^2} |f(x)|^2 M_A^4 \mathcal{M}_{A,\alpha,\beta} M_A^3 w(x) dx \tag{3.11}$$

where

$$\mathcal{M}_{A,\alpha,\beta} f(x) = \sup_{(y,t) \in \Gamma_{A,\alpha}(x)} (t^{\nu})^{2\beta} |\vartheta_{A(t)} * f(y)|$$

and

$$\Gamma_{A,\alpha}(x) = \{(y,t) \in \mathbb{R}^2 \times \mathbb{R}_+ : 0 < t^{\alpha} \leqslant 1,\ \rho_A(x-y) \leqslant t^{1-\alpha}\}.$$

Remark 3.3.6 *We have stated Theorem 3.3.5 with a Mikhlin-type condition on our multipliers, but it is possible to reduce the requirement to just $\lambda \leqslant 2$ and even further reduce to a broader class of multipliers with a Hörmander-type condition, see [2] for details.*

We will prove Theorem 3.3.5 by splitting our argument up into distinct steps. The main idea of this proof has roots in work of Stein, see [38], and it consists of finding square functions, g_1 and

g_2, adapted to our operators T_m such that we have the inequality

$$g_1(T_m f)(x) \lesssim g_2(f)(x).$$

The main aspects of this theorem are illuminated by understanding the proof of this pointwise inequality. Especially so for the structure of the recoupling decomposition, which is adapted to scales that are much finer than dyadic - referred to as *subdyadic*, see [3, 2]. At this subdyadic level, the multipliers considered are effectively reduced to bump functions - the archetype for this study in the one dimensional case are the Hirschmann multipliers [20]

$$m(\xi) = \frac{e^{i|x|^\alpha}}{|x|^{\beta d}}.$$

The multidimensional version in the isotropic case was studied by Wainger[42], and Fefferman and Stein [18]. Later, Miyachi studied a wider class of multipliers [30] defined by the Miyachi-condition

$$|D^\gamma m(\xi)| \lesssim |\xi|^{-\beta d + |\gamma|(\alpha-1)}$$

where the support of m is contained in $|\xi|^\alpha \geqslant 1$. This class of multipliers also encapsulated other multiplier classes, such as the class of multipliers famously considered by Hörmander in [21]. However, in the anisotropic case, it turns out the obvious adaption for the anisotropic version of the candidate multipliers given by

$$m(\xi) = \frac{e^{i\rho_A(x)^\alpha}}{\rho_A(x)^{\beta\nu}}$$

does not fit into the anisotropic Miyachi class (3.10). However, Theorem 3.3.5 still has many model multipliers; indeed, we can create them by summing up bump functions that have support on the subdyadic balls.

Returning to the overview of the proof, after we have the pointwise inequality in hand the problem is reduced to proving Fefferman-Stein-type inequalities for the two square functions g_1 and g_2, and putting these all together as follows

$$\|T_m f\|_{L^2(w_1)} \lesssim \|g_1(T_m f)\|_{L^2(w_2)} \lesssim \|g_2(f)\|_{L^2(w_2)} \lesssim \|f\|_{L^2(w_3)}$$

where w_1, w_2 and w_3 are weights. Our g-functions alluded to above are given in full in the following definitions.

Definition 3.3.7 *Let ϕ be as in Definition 3.2.4 and define*

$$g_{A,\alpha,\beta}(f)(x) = \left(\int_{0<t^\alpha\leqslant 1} \int_{\rho_A(x-y)\leqslant t^{(1-\alpha)}} |f * \phi_{A(t)}(y)|^2 \frac{dy}{(t^\nu)^{2\beta+(1-\alpha)}} \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$g_{A,\alpha,\beta,\Phi}(f)(x) = \left(\int_{t^\alpha\leqslant 1} |f * \phi_{A(t)}|^2 * \Phi_{A(t^{1-\alpha})}(x) \frac{dt}{t^{2\beta\nu+1}} \right)^{\frac{1}{2}}$$

where $\Phi \in \mathcal{S}$, $\operatorname{supp}(\widehat{\Phi}) \subseteq \{\xi \in \mathbb{R}^2 : |\xi| \leqslant 1\}$ *and $\Phi(x) \geqslant c$ for $|x| \leqslant 1$.*

Note that these two g-functions are intimately related to each other, and to our maximal functions $\mathcal{M}_{A,\alpha,\beta}$. The link between $\mathcal{M}_{A,\alpha,\beta}$ is rather immediate, as our approach regions, $\Gamma_{A,\alpha}(x)$, are the set that the integral in the definition of $g_{A,\alpha,\beta}$ is over. To see the relationship between these two g-functions, consider that $g_{A,\alpha,\beta,\Phi}$ dominates, modulo a constant, $g_{A,\alpha,\beta}$ pointwise; indeed, as $\Phi(x) \geqslant c$ for $|x| \leqslant 1$ we have $\Phi(A(t^{-(1-\alpha)})(x-y)) \geqslant c$ for

$$|A(t^{-(1-\alpha)})(x-y)| \leqslant 1 \iff \rho_A(x-y) \leqslant t^{(1-\alpha)},$$

which, for $0 < t^{\alpha} \leqslant 1$, is our set $\Gamma_{A,\alpha}(x)$, thus $\Phi(A(t^{-(1-\alpha)})(x-y)) \geqslant c$ on $\Gamma_{\alpha,A}(x)$. It is no accident that we have this pointwise majorant, and it will play a role in our analysis.

Primordial versions of these g-functions were introduced by Littlewood and Paley during their efforts to better understand the dyadic decomposition of Fourier series, see [25, 26, 27]. Later, the mantle of this study was taken up by Marcinkiewicz and Zygmund and great advances in understanding these g-functions was developed, including the introduction of the g^* function by Zygmund, see [46, 45]. However, the true power of these g^* functions were not realised until Stein's introduction of the g^*_λ function in [37], our version of which is given below.

Definition 3.3.8 *Let ϕ be as in Definition 3.2.4 and define*

$$g^*_{A,\alpha,\beta,\lambda}(f)(x) = \left(\int_{t^{\alpha} \leqslant 1} |f * \phi_{A(t)}|^2 * R^{\lambda}_{A(t^{\alpha-1})}(x) \frac{dt}{(t^{\nu})^{2\beta+1}} \right)^{\frac{1}{2}},$$

where $R^{\lambda}(x) = (1+|x|)^{-2\lambda}$ for $\lambda > 1$.

Note that as $\Phi \in \mathcal{S}$ in the definition of $g_{A,\alpha,\beta,\Phi}$, we can bound it by a constant multiple of $(1+|x|)^{-2\lambda}$ for any λ, thus $g_{A,\alpha,\beta,\Phi}(f)(x) \lesssim g^*_{A,\alpha,\beta,\lambda}(f)(x)$ for any admissible f. While this may seem like yet another pointwise majorant, it has a much more interesting property, given as

$$g_{A,\alpha,\beta,\Phi}(T_m(f))(x) \lesssim g^*_{A,\alpha,0,\frac{3}{2}}(f)(x).$$

This brings us full circle to the start of this discourse - finding square functions g_1, g_2 that are adapted to our multipliers, see [41] for more details.

Proof: [Theorem 3.3.5] First we apply Proposition 3.2.5 then Proposition 3.3.17 to obtain

$$\begin{aligned} \|T_m f\|_{L^2(w)} &\lesssim \|s_A(T_m f)\|_{L^2(M_A^3 w)} \\ &\lesssim \|g_{A,\alpha,\beta}(T_m f)\|_{L^2(M_{A,\alpha,\beta} M_A^3 w)}. \end{aligned}$$

Next, we use the observation that $g_{A,\alpha,\beta,\Phi}$ dominates, modulo a constant, $g_{A,\alpha,\beta}$ pointwise and Theorem 3.3.15 to obtain

$$g_{A,\alpha,\beta}(T_m f)(x) \lesssim g^*_{A,\alpha,0,\frac{3}{2}}(f)(x),$$

and thus

$$\|T_m f\|_{L^2(w)} \lesssim \|g^*_{A,\alpha,0,\frac{3}{2}}(f)\|_{L^2(M_{A,\alpha,\beta}M_A^3 w)}.$$

Finally, we use Proposition 3.3.18 and Proposition 3.2.7, with $\lambda = \frac{3}{2}$ to obtain

$$\begin{aligned}\|T_m f\|_{L^2(w)} &\lesssim \|s_A(f)\|_{L^2(M_A M_{A,\alpha,\beta}M_A^3 w)} \\ &\lesssim \|f\|_{L^2(M_A^4 M_{A,\alpha,\beta}M_A^3 w)}.\end{aligned}$$

□

Finally, we provide the $L^p - L^q$ bounds on our multipliers.

Corollary 3.3.9 *Let m be such that*

$$|D^\gamma m(\xi)| \lesssim \rho_A(\xi)^{-\beta\nu+\|\gamma\|_A(\alpha-1)} \tag{3.12}$$

for $|\gamma| \leqslant 3$ *with support in* $\{\xi \in \mathbb{R}^2 : |\xi|^\alpha \geqslant 1\}$, *and* $1 < p \leqslant q \leqslant \infty$ *and* $\alpha, \beta \in \mathbb{R}$.

- *If* $\alpha < 0$ *and* $\beta \leqslant \alpha\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{1}{p} - \frac{1}{q}$;
- *or* $\alpha = 0$ *and* $\beta = \frac{1}{p} - \frac{1}{q}$;
- *or* $\alpha > 0$ *and* $\beta \geqslant \alpha\left(\frac{1}{2} - \frac{1}{p}\right) + \frac{1}{p} - \frac{1}{q}$;

then T_m *is a Fourier multiplier from* L^p *to* L^q.

Proof: This is an immediate consequence of Theorem 3.3.5, Theorem 2.4.3 and (1.4). $\square$

3.3.3 Pointwise estimate

The aim of this section is the proof of our pointwise inequality

$$g_{A,\alpha,\beta,\Phi}(T_m(f))(x) \lesssim g^*_{A,\alpha,0,\frac{3}{2}}(f)(x).$$

To prove this pointwise estimate, we wish to reduce to a portions of the landscape where our multiplier's behaviour is much simpler, for this we must define what we mean by α-subdyadic, or more generally subdyadic. The general idea is to decompose dyadic rings into balls of size roughly their distance from the origin to the power $1-\alpha$. On these balls, the local behaviour of our multipliers is much simpler and makes gaining the pointwise estimate on each ball much easier.

Decomposition

Let $\mathcal{S}^\alpha = \{\xi \in \mathbb{R}^2 : |\xi|^\alpha \geqslant 1\}$. Let $\{A_j\}_{j\in\mathbb{Z}}$ be the set of annuli given by $A_j = \{\xi \in \mathbb{R}^2 : 2^{j-1} \leqslant \rho_A(\xi) \leqslant 2^j\}$. Let $\mathcal{B}_{\rho_A,j}$ be a family of ρ_A-balls, $B_{\rho_A,j}$, with $r_\rho(B_{\rho_A,j}) \sim 2^{j(1-\alpha)}$ such that each $B_{\rho_A,j}$ is entirely contained in $A_{j-1} \cup A_j \cup A_{j+1}$, $\mathcal{B}_{\rho_A,j}$ covers A_j and there is bounded overlap of the $B_{\rho_A,j}$. Finally, let

$$\mathcal{B}_{\rho_A} = \bigcup_{j\in\mathbb{Z}} \mathcal{B}_{\rho_A,j}.$$

For a fixed $B \in \mathcal{B}_{\rho_A}$, let $\psi_B \in \mathcal{S}$ such that $\widehat{\psi}_B$ has support in the concentric double of B, denoted $2B$,

$$\sum_{B\in\mathcal{B}_{\rho_A}} \widehat{\psi}_B(\xi) = 1,$$

for all $\xi \in \mathbb{R}^2 \setminus \{0\}$ and

$$|D^\gamma \widehat{\psi}_B(\xi)| \lesssim r_\rho(B)^{-\|\gamma\|_A}.$$

Then for f such that $\widehat{f}$ has support in $\mathcal{S}^\alpha$, we have

$$f = \sum_{B \in \mathcal{B}_{\rho_A}} f * \psi_B \tag{3.13}$$

Recoupling decomposition

For the recoupling estimate, we will use a specific example of the above decomposition based on a lattice structure.

Let $\Delta \in \mathcal{S}$ have Fourier support in A_0 such that

$$\sum_{j \in \mathbb{Z}} \widehat{\Delta}_j(\xi) = 1$$

for $\xi \in \mathcal{S}^\alpha$, where $\widehat{\Delta}_j(\xi) = \widehat{\Delta}(A(2^{-j})\xi)$, for each $j \in \mathbb{Z}$. Note that this is a partition of unity for the punctured real plane and that $\operatorname{supp}(\widehat{\Delta}_j) \subseteq A_j$. Next, let $\eta \in \mathcal{S}$ have Fourier support in $\{\xi \in \mathbb{R}^2 : |\xi| \leqslant 2\}$ such that

$$\sum_{k \in \mathbb{Z}^2} \widehat{\eta}(\xi + k) = 1$$

for $\xi \in \mathbb{R}^2$. Additionally, define $\widehat{\eta}_j(\xi) = \widehat{\eta}(A(2^{-j(\alpha-1)})\xi)$. For each $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^2$ define $\widehat{\zeta}_{j,k}(\xi) := \widehat{\Delta}_j(\xi)\widehat{\eta}_{j,k}(\xi)$, where $\widehat{\eta}_{j,k}(\xi) = \widehat{\eta}(A(2^{-j(\alpha-1)})\xi + k)$ and note that

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \widehat{\zeta}_{j,k}(\xi) = 1.$$

Finally, choose a family of ρ_A-balls $\mathcal{B}_{\rho_A}$ and functions $\{\psi_B\}_{B\in\mathcal{B}_{\rho_A}} \subseteq \mathcal{S}$ such that for each $B \in \mathcal{B}_{\rho_A}$ there is exactly one $(j,k) \in \mathbb{Z}\times\mathbb{Z}^2$ where $\psi_B = \zeta_{j,k}$ and $r_\rho(\mathrm{supp}(\widehat{\zeta}_{j,k})) \sim r_\rho(B)$. Note that due to the support of Δ_j, $\rho_A(B,0) \sim 2^j$, and by the support of η, η_j has support in a ρ_A-ball of ρ_A-radius given by $r_\rho(B) \sim 2^{-j(\alpha-1)}$. Now, consider

$$\begin{aligned}|D^\gamma \widehat{\psi}_B(\xi)| &= |D^\gamma \widehat{\zeta}_{j,k}(\xi)| \\ &\lesssim 2^{-j\|\gamma\|_A} + 2^{-j\|\gamma\|_A(\alpha-1)},\end{aligned}$$

and as we are considering $\xi \in \mathcal{S}^\alpha$, the support of η implies we only consider j such that $2^{j\alpha} \geqslant 1$, thus $2^{-j} \leqslant 2^{-j} \cdot 2^{j\alpha}$. We can then deduce

$$\begin{aligned}|D^\gamma \widehat{\psi}_B(\xi)| &\lesssim 2^{-j\|\gamma\|_A(\alpha-1)} \\ &\lesssim r_\rho(B)^{-\|\gamma\|_A},\end{aligned}$$

as $r_\rho(B) \sim 2^{-j(\alpha-1)}$.

Decoupling

Proposition 3.3.10 *For f such that $\widehat{f}$ has support in $\mathcal{S}^\alpha$*

$$g_{A,\alpha,\beta,\Phi}(f)(x)^2 \lesssim \sum_{B\in\mathcal{B}_{\rho_A}} g_{A,\alpha,\beta,\Phi}(f * \psi_B)(x)^2$$

Proof: By (3.13) we have

$$g_{A,\alpha,\beta,\Phi}(f)(x)^2 = \int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2}\left|\sum_{B\in\mathcal{B}_{\rho_A}} f * \psi_B * \phi_{A(t)}(y)\right|^2 \Phi_{A(t^{\alpha-1})}(x-y)dy\frac{dt}{t^{2\beta\nu+1}}.$$

Considering just the inner integral for a fixed $t^\alpha \leqslant 1$, multiplying out the square and applying Pascal's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \sum_{B,B' \in \mathcal{B}_{\rho_A}} (f * \psi_{B'} * \phi_{A(t)})(y)\overline{f * \psi_B * \phi_{A(t)}(y)}\Phi_{A(t^{\alpha-1})}(x-y)dy \\
&\quad= \int_{\mathbb{R}^2} \sum_{B,B' \in \mathcal{B}_{\rho_A}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \widehat{f}(\xi)\overline{\widehat{f}(\eta)}\widehat{\psi}_B(\xi)\overline{\widehat{\psi}_{B'}(\eta)}\widehat{\phi}(A(t)\xi)\overline{\widehat{\phi}(A(t)\eta)}e^{iy\cdot(\xi-\eta)}d\xi d\eta \Phi_{A(t^{\alpha-1})}(x-y)dy \\
&\quad= \sum_{B,B' \in \mathcal{B}_{\rho_A}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \widehat{f}(\xi)\overline{\widehat{f}(\eta)}\widehat{\psi}_B(\xi)\overline{\widehat{\psi}_{B'}(\eta)}\widehat{\phi}(A(t)\xi)\overline{\widehat{\phi}(A(t)\eta)}e^{ix\cdot(\xi-\eta)}\widehat{\Phi}(A(t^{\alpha-1})(\xi-\eta))d\xi d\eta
\end{aligned}$$

where the last step is simply an application of the Fourier inversion formula to Φ. The support of $\widehat{\phi}$ and $\widehat{\psi}_B$ ensure the integrand and therefore the summand above vanishes unless B and B' are both ρ_A-distance $\frac{1}{t}$ from the origin, thus $r_\rho(B) \sim r_\rho(B') \sim t^{(\alpha-1)}$ as $B, B' \in \mathcal{B}_{\rho_A}$. Furthermore, the support of $\widehat{\Phi}$ tells us that the integrand vanishes unless $|A(t^{\alpha-1})(\xi-\eta)| \leqslant 1$; that is, that the integrand vanishes unless $\rho_A(B,B') \lesssim t^{(\alpha-1)}$. For each $B \in \mathcal{B}_{\rho_A}$, let $n(B)$ be the set of $B' \in \mathcal{B}_{\rho_A}$ such that $\rho_A(B,B') \lesssim t^{(\alpha-1)}$. As the decomposition $\mathcal{B}_{\rho_A}$ has bounded overlap, we have that for each B there are finitely many B' in the summation, i.e. $|n(B)| \lesssim 1$. Thus,

$$\begin{aligned}
&g_{A,\alpha,\beta,\Phi}(f)(x)^2 \\
&\qquad= \int_{t^\alpha \leqslant 1}\int_{\mathbb{R}^2} \sum_{\substack{B,B' \in \mathcal{B}_{\rho_A} \\ \rho_A(B,B') \lesssim t^{(\alpha-1)} \\ |B| \sim |B'| \sim t^{(\alpha-1)\nu}}} (f * \psi_B * \phi_{A(t)})(y)\overline{(f * \psi_{B'} * \phi_{A(t)})(y)}\Phi_{A(t^{\alpha-1})}(x-y)dy\frac{dt}{t^{2\beta\nu+1}}.
\end{aligned}$$

So, rearranging then using the Cauchy-Schwarz inequality for the sum in B', we have

$$\begin{aligned}
&\sum_{\substack{B,B'\in\mathcal{B}_{\rho_A}\\ \rho_A(B,B')\lesssim t^{(\alpha-1)}\\ |B|\sim|B'|\sim t^{(\alpha-1)\nu}}} (f*\psi_B*\phi_{A(t)})(y)\overline{(f*\psi_{B'}*\phi_{A(t)})(y)}\\
&\qquad = \sum_{\substack{B\in\mathcal{B}_{\rho_A}\\ |B|\sim t^{(\alpha-1)\nu}}}\ \sum_{\substack{B'\in\mathcal{B}_{\rho_A}\\ \rho_A(B,B')\lesssim t^{(\alpha-1)}\\ |B'|\sim t^{(\alpha-1)\nu}}} (f*\psi_B*\phi_{A(t)})(y)\overline{(f*\psi_{B'}*\phi_{A(t)})(y)}\\
&\qquad = \sum_{\substack{B\in\mathcal{B}_{\rho_A}\\ |B|\sim t^{(\alpha-1)\nu}}} (f*\psi_B*\phi_{A(t)})(y) \sum_{\substack{B'\in n(B)\\ |B'|\sim t^{(\alpha-1)\nu}}} \overline{(f*\psi_{B'}*\phi_{A(t)})(y)}\cdot 1\\
&\qquad \leqslant \sum_{\substack{B\in\mathcal{B}_{\rho_A}\\ |B|\sim t^{(\alpha-1)\nu}}} (f*\psi_B*\phi_{A(t)})(y)\left(\sum_{\substack{B'\in n(B)\\ |B'|\sim t^{(\alpha-1)\nu}}} |f*\psi_{B'}*\phi_{A(t)})(y)|^2\right)^{\frac{1}{2}}\left(\sum_{\substack{B'\in n(B)\\ |B'|\sim t^{(\alpha-1)\nu}}} 1\right)^{\frac{1}{2}}\\
&\qquad \leqslant \sum_{\substack{B\in\mathcal{B}_{\rho_A}\\ |B|\sim t^{(\alpha-1)\nu}}} (f*\psi_B*\phi_{A(t)})(y)\left(|n(B)| \sum_{\substack{B'\in n(B)\\ |B'|\sim t^{(\alpha-1)\nu}}} |f*\psi_{B'}*\phi_{A(t)}(y)|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Next, using the Cauchy-Schwarz inequality for the sum in B, we have

$$\begin{aligned}
&\sum_{\substack{B,B'\in\mathcal{B}_{\rho_A}\\ \rho_A(B,B')\lesssim t^{(\alpha-1)}\\ |B|\sim|B'|\sim t^{(\alpha-1)\nu}}} (f*\psi_B*\phi_{A(t)})(y)\overline{(f*\psi_{B'}*\phi_{A(t)})(y)}\\
&\qquad \leqslant \left(\sum_{\substack{B\in\mathcal{B}_{\rho_A}\\ |B|\sim t^{(\alpha-1)\nu}}} |f*\psi_B*\phi_{A(t)}(y)|^2\right)^{\frac{1}{2}}\left(\sum_{\substack{B\in\mathcal{B}_{\rho_A}\\ |B|\sim t^{(\alpha-1)\nu}}} |n(B)| \sum_{\substack{B'\in n(B)\\ |B'|\sim t^{(\alpha-1)\nu}}} |f*\psi_{B'}*\phi_{A(t)}(y)|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Observe that in the last term on the right hand side of the above, the summation over each $B'\in n(B)$ for each $B\in\mathcal{B}_{\rho_A}$ is equivalent to just summing over all $B'\in\mathcal{B}_{\rho_A}$ and multiplying by $|n(B)|$ each time. Additionally, as $|n(B)|\lesssim 1$ for every $B\in\mathcal{B}_{\rho_A}$ we pick up a finite constant that is at most the maximum of $|n(B)|$ over all $B \in \mathcal{B}_{\rho_A}$, so we have

$$\sum_{\substack{B,B'\in\mathcal{B}_{\rho_A}\\ \rho_A(B,B')\lesssim t^{(\alpha-1)}\\ |B|\sim|B'|\sim t^{(\alpha-1)\nu}}} (f * \psi_B * \phi_{A(t)})(y)\overline{(f * \psi_{B'} * \phi_{A(t)})(y)}$$

$$\lesssim \left(\sum_{\substack{B\in\mathcal{B}_{\rho_A}\\ |B|\sim t^{(\alpha-1)\nu}}} |f * \psi_B * \phi_{A(t)}(y)|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{B'\in\mathcal{B}_{\rho_A}\\ |B'|\sim t^{(\alpha-1)\nu}}} |f * \psi_{B'} * \phi_{A(t)}(y)|^2\right)^{\frac{1}{2}}$$

$$= \sum_{\substack{B\in\mathcal{B}_{\rho_A}\\ |B|\sim t^{(\alpha-1)\nu}}} |f * \psi_B * \phi_{A(t)}(y)|^2.$$

Thus, we can conclude that

$$g_{A,\alpha,\beta,\Phi}(f)(x)^2 \lesssim \int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2}\sum_{B\in\mathcal{B}_{\rho_A}} \left|f * \psi_B * \phi_{A(t)}(y)\right|^2 \Phi_{A(t^{\alpha-1})}(x-y)dy\frac{dt}{t^{2\beta\nu+1}},$$

and finally, by Lebesgue's monotone convergence theorem, as the summands are positive, we have

$$g_{A,\alpha,\beta,\Phi}(f)(x)^2 \lesssim \sum_{B\in\mathcal{B}_{\rho_A}}\int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2} \left|f * \psi_B * \phi_{A(t)}(y)\right|^2 \Phi_{A(t^{\alpha-1})}(x-y)dy\frac{dt}{t^{2\beta\nu+1}}$$

$$= \sum_{B\in\mathcal{B}_{\rho_A}} g_{A,\alpha,\beta,\Phi}(f * \psi_B)(x)^2.$$

$\square$

Recoupling

In order to prove the recoupling estimate, we will require the following lemma.

Lemma 3.3.11 *Let $R_1, R_2 > 0$ and define*

$$\delta = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}.$$

Define

$$\widehat{\Phi}_{\delta^{-1}}(\xi) = \widehat{\Phi}(\delta^{-1}\xi)$$

and define for each $k \in \mathbb{Z}^2$

$$\widehat{\Phi}_{k,\delta^{-1}}(\xi) = \widehat{\Phi}(\delta^{-1}\xi + k)$$

and $f_k(x) = f * \Phi_{k,\delta^{-1}}(x)$. If

$$|\Phi(x)| \lesssim \frac{C_N}{(1+|x|)^N}$$

for every $N \in \mathbb{N}$, then

$$\sum_{k\in\mathbb{Z}^2} |f_k(x)|^2 \lesssim |f|^2 * |\Phi_{\delta^{-1}}|(x).$$

Proof: We have, for each $k \in \mathbb{Z}^2$,

$$\begin{aligned} f_k(x) &= f * \Phi_{k,\delta^{-1}}(x) \\ &= \int_{\mathbb{R}^2} f(y) e^{2\pi i \delta k \cdot (x-y)} \Phi_{\delta^{-1}}(x-y) dy \\ &= e^{2\pi i \delta k \cdot x} \int_{\mathbb{R}^2} f(y) e^{-2\pi i \delta k \cdot y} \Phi_{\delta^{-1}}(x-y) dy \\ &= e^{2\pi i \delta k \cdot x} (f(\cdot)\Phi_{\delta^{-1}}(x-\cdot))\widehat{\ }(\delta k) \\ &= e^{2\pi i \delta k \cdot x} \widehat{h}_{x,\delta}(k), \end{aligned}$$

where $h_x(y) = f(y)\Phi_{\delta^{-1}}(x-y)$ and $\widehat{h}_{x,\delta}(k) = \widehat{h}_x(\delta k)$. So

$$\begin{aligned} \sum_{k\in\mathbb{Z}^2} |f_k(x)|^2 &= \sum_{k\in\mathbb{Z}^2} |e^{2\pi i \delta k \cdot x} \widehat{h}_{x,\delta}(k)|^2 \\ &= \sum_{k\in\mathbb{Z}^2} |\widehat{h}_{x,\delta}(k)|^2. \end{aligned}$$

Now, by Parseval's identity,

$$\sum_{k\in\mathbb{Z}^2} |f_k(x)|^2 = \int_{[0,1]^2} \left| \sum_{k\in\mathbb{Z}^2} \widehat{h}_{x,\delta}(k) e^{iy\cdot k} \right|^2 dy,$$

using the change of variables $y = \delta z$ we have

$$\sum_{k\in\mathbb{Z}^2} |f_k(x)|^2 = R_1 R_2 \int_{[0,\frac{1}{R_1}]\times[0,\frac{1}{R_2}]} \left| \sum_{k\in\mathbb{Z}^2} \widehat{h}_x(\delta k) e^{iz\cdot\delta k} \right|^2 dz.$$

Now, using the Poisson summation formula, along with the scaling and translation properties of the Fourier transform, we can write

$$\sum_{k\in\mathbb{Z}^2} \widehat{h}_x(\delta k) e^{iz\cdot\delta k} = \frac{1}{R_1}\frac{1}{R_2} \sum_{k\in\mathbb{Z}^2} h_x(z + \delta^{-1} k),$$

so substituting this back in we get

$$\sum_{k\in\mathbb{Z}^2}|f_k(x)|^2=\frac{1}{R_1}\frac{1}{R_2}\int_{[0,\frac{1}{R_1}]\times[0,\frac{1}{R_2}]}\left|\sum_{k\in\mathbb{Z}^2}h_x(z+\delta^{-1}k)\right|^2dz.$$

Using the definition of h_x then the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2}|f_k(x)|^2&=\frac{1}{R_1}\frac{1}{R_2}\int_{[0,\frac{1}{R_1}]\times[0,\frac{1}{R_2}]}\left|\sum_{k\in\mathbb{Z}^2}f(z+\delta^{-1}k)\Phi_{\delta^{-1}}(x-z-\delta^{-1}k)\right|^2dz\\&\leqslant\frac{1}{R_1}\frac{1}{R_2}\int_{[0,\frac{1}{R_1}]\times[0,\frac{1}{R_2}]}\sum_{k\in\mathbb{Z}^2}\left|f(z+\delta^{-1}k)\right|^2\left|\Phi_{\delta^{-1}}(x-z-\delta^{-1}k)\right|\sum_{\ell\in\mathbb{Z}^2}\left|\Phi_{\delta^{-1}}(x-z-\delta^{-1}\ell)\right|dz.\end{aligned}$$

Next, using the substitution $z+\delta^{-1}k=w$, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^2}|f_k(x)|^2\\&\leqslant\sum_{k\in\mathbb{Z}^2}\int_{[\frac{k_1}{R_1},\frac{k_1+1}{R_1}]\times[\frac{k_2}{R_2},\frac{k_2+1}{R_2}]}|f(w)|^2|\Phi_{\delta^{-1}}(x-w)|\left(\frac{1}{R_1}\frac{1}{R_2}\sum_{\ell\in\mathbb{Z}^2}\left|\Phi_{\delta^{-1}}(x-w+\delta^{-1}(k-\ell))\right|\right)dw,\end{aligned}$$

and observe that

$$\begin{aligned}\frac{1}{R_1}\frac{1}{R_2}\sum_{\ell\in\mathbb{Z}^2}\left|\Phi_{\delta^{-1}}(x-w+\delta^{-1}(k-\ell))\right|&=\sum_{\ell\in\mathbb{Z}^2}|\Phi(\delta(x-w)+k-\ell)|\\&\lesssim\sum_{\ell\in\mathbb{Z}^2}\left(\frac{C_N}{1+|\delta(x-w)+k-\ell|}\right)^N\end{aligned}$$

for every $N\in\mathbb{N}$. Choosing N large enough, we can bound this term by some fixed constant, thus

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2}|f_k(x)|^2&\lesssim\int_{\mathbb{R}^2}|f(w)|^2|\Phi_{\delta^{-1}}(x-w)|dw\\&=|f|^2*|\Phi_{\delta^{-1}}|(x).\end{aligned}$$

$\square$

Proposition 3.3.12 *For f such that $\widehat{f}$ has support in $\mathcal{S}^\alpha$ and the specific decomposition $\mathcal{B}_{\rho_A}$ and $\psi_B = \zeta_{j,k} = \Delta_j * \eta_{j,k}$ described in Section 3.3.3, we have*

$$\sum_{B \in \mathcal{B}_{\rho_A}} g^*_{A,\alpha,\beta,\lambda}(f * \psi_B)(x)^2 \lesssim g^*_{A,\alpha,\beta,\lambda}(f)(x)^2. \tag{3.14}$$

Proof: First, consider the support of $\widehat{\phi}_{A(t)}$, this implies that $\phi_{A(t)} * \Delta_j * \eta_{j,k}(y) \neq 0$ only if $2^j \sim t^{-1}$, thus

$$\begin{aligned}
\sum_{B \in \mathcal{B}_{\rho_A}} g^*_{A,\alpha,\beta,\lambda}(f * \psi_B)(x)^2 &= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} |f * \phi_{A(t)} * \Delta_j * \eta_{j,k}(y)|^2 R^\lambda_{A(t^{\alpha-1})}(x-y) \frac{dy}{(t^\nu)^{2\beta}} \frac{dt}{t} \\
&= \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \sum_{2^j \sim t^{-1}} \sum_{k \in \mathbb{Z}^2} |f * \phi_{A(t)} * \Delta_j * \eta_{j,k}(y)|^2 R^\lambda_{A(t^{\alpha-1})}(x-y) \frac{dy}{(t^\nu)^{2\beta}} \frac{dt}{t}.
\end{aligned}$$

Now, we can use Lemma 3.3.11 where $\delta^{-1} = A(2^{-j(\alpha-1)})$ to get

$$\sum_{k \in \mathbb{Z}^2} |f * \phi_{A(t)} * \Delta_j * \eta_{j,k}(y)|^2 \lesssim |f * \phi_{A(t)} * \Delta_j|^2 * |\eta_j|(y)$$

uniformly in t, j and y, thus

$$\sum_{B \in \mathcal{B}_{\rho_A}} g^*_{A,\alpha,\beta,\lambda}(f * \psi_B)(x)^2 \lesssim \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \sum_{2^j \sim t^{-1}} |f * \Delta_j * \phi_{A(t)}|^2 * |\eta_j|(y) R^\lambda_{A(t^{\alpha-1})}(x-y) \frac{dy}{(t^\nu)^{2\beta}} \frac{dt}{t}.$$

Now, consider

$$
\begin{aligned}
&|f * \phi_{A(t)} * \Delta_j|^2 * |\eta_j|(y) \\
&\qquad = \int_{\mathbb{R}^2} |f * \phi_{A(t)} * \Delta_j(y-z)|^2 |\eta_j(z)| dz \\
&\qquad = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} f * \phi_{A(t)}(w) \Delta_j(y-z-w) dw \right|^2 |\eta_j(z)| dz \\
&\qquad \leqslant \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |f * \phi_{A(t)}(w)| |\Delta_j(y-z-w)| dw \right)^2 |\eta_j(z)| dz \\
&\qquad = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |f * \phi_{A(t)}(w)| |\Delta_j(y-(z+w))|^{\frac{1}{2}} |\Delta_j(y-z-w)|^{\frac{1}{2}} dw \right)^2 |\eta_j(z)| dz
\end{aligned}
$$

and using the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&|f * \phi_{A(t)} * \Delta_j|^2 * |\eta_j|(y) \\
&\qquad \leqslant \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |f * \phi_{A(t)}(w)|^2 |\Delta_j(y-(w+z))| dw \right) \left(\int_{\mathbb{R}^2} |\Delta_j(y-z-w)| dw \right) |\eta_j(z)| dz \\
&\qquad = \|\Delta_j\|_1 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f * \phi_{A(t)}(w)|^2 |\Delta_j(y-w-z)| |\eta_j(z)| dw dz
\end{aligned}
$$

using Fubini's theorem,

$$
\begin{aligned}
|f * \phi_{A(t)} * \Delta_j|^2 * |\eta_j|(y) &\lesssim \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f * \phi_{A(t)}(w)|^2 |\Delta_j(y-w-z)| |\eta_j(z)| dz dw \\
&= \int_{\mathbb{R}^2} |f * \phi_{A(t)}(w)|^2 |\Delta_j| * |\eta_j|(y-w) dw \\
&= |f * \phi_{A(t)}|^2 * |\Delta_j| * |\eta_j|(y).
\end{aligned}
$$

Hence,

$$\begin{aligned}
\sum_{B\in\mathcal{B}_{\rho_A}} g^*_{A,\alpha,\beta,\lambda}(f*\psi_B)(x)^2 &\lesssim \int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2}\sum_{2^j\sim t^{-1}} |f*\phi_{A(t)}|^2 * |\Delta_j| * |\eta_j|(y) R^\lambda_{A(t^{\alpha-1})}(x-y)dy\frac{dt}{t^{2\beta\nu+1}}\\
&= \int_{t^\alpha\leqslant 1}\sum_{2^j\sim t^{-1}} |f*\phi_{A(t)}|^2 * |\Delta_j| * |\eta_j| * R^\lambda_{A(t^{\alpha-1})}(x)\frac{dt}{t^{2\beta\nu+1}}\\
&= \int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2} |f*\phi_{A(t)}(y)|^2 \sum_{2^j\sim t^{-1}} |\Delta_j| * |\eta_j| * R^\lambda_{A(t^{\alpha-1})}(x-y)dy\frac{dt}{t^{2\beta\nu+1}}
\end{aligned}$$

thus, since $\Delta, \eta \in \mathcal{S}$ and by the fact that we're summing over j such that $2^j \sim t^{-1}$, we can use Lemmma A.1.1 to conclude that

$$\begin{aligned}
\sum_{B\in\mathcal{B}_{\rho_A}} g^*_{A,\alpha,\beta,\lambda}(f*\psi_B)(x)^2 &\lesssim \int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2} |f*\phi_{A(t)}(y)|^2 R^\lambda_{A(t^{\alpha-1})}(x-y)dy\frac{dt}{t^{2\beta\nu+1}}\\
&= g^*_{A,\alpha,\beta,\lambda}(f)(x).
\end{aligned}$$

$\square$

Pointwise estimate at subdyadic level

Now that we can efficiently decompose and recompose our landscape, all that remains to prove Theorem 3.3.15 is to prove the same estimate uniformly on each subdyadic ball.

Proposition 3.3.13 *Let $B \in \mathcal{B}_{\rho_A}$ and $\widehat{v}_B$ be a bump function supported on $3B$ and equal to 1 on $2B$. If*

$$|D^\gamma m(\xi)| \lesssim \rho_A(\xi)^{-\beta\nu+\|\gamma\|_A(\alpha-1)}$$

and

$$|D^\gamma \widehat{v}_B(\xi)| \lesssim r_\rho(B)^{-\|\gamma\|_A}$$

for $\xi \in \{\xi \in \mathbb{R}^2 : |\xi|^\alpha \geqslant 1\}$ *and* $|\gamma| \leqslant 3$, *then*

$$|T_m(v_B)(x)| \lesssim \rho_A(B,0)^{-\beta\nu} H_B(x), \tag{3.15}$$

where

$$H_B(x) = \frac{|B|}{(1 + |A(r_\rho(B))x|)^3}.$$

Proof: First, by the Fourier inversion formula

$$T_m(v_B)(x) = \int_{\mathbb{R}^2} m(\xi)\widehat{v}_B(\xi)e^{ix\cdot\xi}d\xi,$$

due to the support of $\widehat{\Phi}_B$ and then our hypothesis on m, we have

$$\begin{aligned}|T_m(v_B)(x)| &\leqslant \int_{3B} |m(\xi)||\widehat{v}_B(\xi)|d\xi \\ &\lesssim \int_{3B} \rho_A(\xi)^{-\beta\nu}|\widehat{v}_B(\xi)|d\xi.\end{aligned}$$

Since $|\widehat{v}_B(\xi)| \lesssim 1$, we have

$$|T_m(v_B)(x)| \lesssim \rho_A(B,0)^{-\beta} \cdot |B|. \tag{3.16}$$

Now, going back to our Fourier inversion formula, and using elementary properties of the Fourier transform, we have another estimate, that is

$$(ix)^\gamma T_m(v_B)(x) = \int_{\mathbb{R}^2} D^\gamma(m(\xi)\widehat{v}_B(\xi))e^{ix\cdot\xi}d\xi.$$

Using the support of $\widehat{v}_B$, this gives us the estimate

$$|x^\gamma||T_m(\Phi_B)(x)| \leqslant \int_{3B} |D^\gamma(m(\xi)\widehat{v}_B(\xi))|d\xi.$$

Now, setting $\gamma = (3,0)$ and using the chain rule

$$\begin{aligned}|x_1|^3|T_m(v_B)(x)| &\lesssim \int_{3B}\left(\sum_{j=0}^{3}\left|\frac{\partial^j}{\partial\xi_1^j}m(\xi)\right|\left|\frac{\partial^{(3-j)}}{\partial\xi_1^{(3-j)}}\widehat{v}_B(\xi)\right|\right)d\xi \\ &\lesssim \int_{3B}\left(\sum_{j=0}^{3}\rho_A(\xi)^{-\beta\nu+j(\alpha-1)}r_\rho(B)^{-(3-j))}\right)d\xi,\end{aligned}$$

where the second line is due to our hypotheses and the fact that

$$\left|\frac{\partial^j}{\partial\xi_1^j}\widehat{v}_B(\xi)\right| \lesssim r_\rho(B)^{-j}$$

for $j = 0,1,2,3.$ Thus, we have that

$$r_\rho(B)^3|x_1|^3|T_m(v_B)(x)| \lesssim \rho_A(B,0)^{-\beta\nu}|B|\left(\sum_{j=0}^{3}\rho_A(B,0)^{j(\alpha-1)}r_\rho(B)^j\right).$$

Likewise, if we set $\gamma = (0,3)$ and go through exactly the same steps, we obtain

$$r_\rho(B)^{3\sigma}|x_2|^3|T_m(v_B)(x)| \lesssim \rho_A(B,0)^{-\beta\nu}|B|\left(\sum_{j=0}^{3}\rho_A(B,0)^{\sigma j(\alpha-1)}r_\rho(B)^{\sigma j}\right).$$

Adding these two estimates together and using the fact that

$$\rho_A(B,0)^{(1-\alpha)} \sim r_\rho(B),$$

we deduce that

$$(r_\rho(B)^3|x_1|^3 + r_\rho(B)^{3\sigma}|x_2|^3)|T_m(v_B)(x)| \lesssim \rho_A(B,0)^{-\beta\nu}|B|.$$

So by equivalence of finite norms, we have

$$|A(r_\rho(B))x|^3|T_m(v_B)(x)| \lesssim \rho_A(B,0)^{-\beta\nu}|B|.$$

Adding this estimate to the estimate (3.16), we obtain

$$|T_m(v_B)(x)| \lesssim \rho_A(B,0)^{-\beta\nu}\frac{|B|}{(1+|A(r_\rho(B))x|)^3}.$$

$\square$

Proposition 3.3.14 *For each $B \in \mathcal{B}_{\rho_A}$,*

$$g_{A,\alpha,\beta,\Phi}(T_m(f * \psi_B))(x) \lesssim g^*_{A,\alpha,0,\frac{3}{2}}(f * \psi_B)(x). \tag{3.17}$$

Proof: Let v_B be smooth and such that $\operatorname{supp}(\widehat{v}_B) \subset 3B$, $\widehat{v}_B = 1$ on $\operatorname{supp}\widehat{\psi}_B$ and

$$|D^\gamma \widehat{v}_B(\xi)| \lesssim r_\rho(B)^{-\|\gamma\|_A}$$

for $\xi \in \mathbb{S}^\alpha$. Then we have

$$\begin{aligned} g_{A,\alpha,\beta,\Phi}(T_m(f * \psi_B))(x)^2 &= \int_{t^\alpha \leqslant 1}\int_{\mathbb{R}^2} |\check{m} * f * \psi_B * \phi_{A(t)}(y)|^2 \Phi_{A(t^{1-\alpha})}(x-y)\frac{dy}{t^{2\beta}}\frac{dt}{t} \\ &= \int_{t^\alpha \leqslant 1}\int_{\mathbb{R}^2} |\check{m} * v_B * f * \psi_B * \phi_{A(t)}(y)|^2 \Phi_{A(t^{1-\alpha})}(x-y)\frac{dy}{t^{2\beta}}\frac{dt}{t} \\ &= \int_{t^\alpha \leqslant 1}\int_{\mathbb{R}^2} |T_m(v_B) * f * \psi_B * \phi_{A(t)}(y)|^2 \Phi_{A(t^{1-\alpha})}(x-y)\frac{dy}{t^{2\beta}}\frac{dt}{t}. \end{aligned}$$

Expanding out the first convolution, we get

$$
\begin{aligned}
&g_{A,\alpha,\beta,\Phi}(T_m(f * \psi_B))(x)^2 \\
&\quad = \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} T_m(v_B)(y-z) f * \psi_B * \phi_{A(t)}(z) dz \right|^2 \Phi_{A(t^{1-\alpha})}(x-y) \frac{dy}{t^{2\beta}} \frac{dt}{t} \\
&\quad \leqslant \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |T_m(v_B)(y-z)| |f * \psi_B * \phi_{A(t)}(z)| dz \right)^2 \Phi_{A(t^{1-\alpha})}(x-y) \frac{dy}{t^{2\beta}} \frac{dt}{t} \\
&\quad = \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \left(|T_m(v_B)| * |f * \psi_B * \phi_{A(t)}|(y) \right)^2 \Phi_{A(t^{1-\alpha})}(x-y) \frac{dy}{t^{2\beta}} \frac{dt}{t}.
\end{aligned}
$$

Now, by Proposition 3.3.13, we have that

$$
g_{A,\alpha,\beta,\Phi}(T_m(f * \psi_B))(x)^2 \lesssim \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \left(\rho_A(B,0)^{-\beta\nu} H_B * |f * \psi_B * \phi_{A(t)}|(y) \right)^2 \Phi_{A(t^{1-\alpha})}(x-y) \frac{dy}{t^{2\beta}} \frac{dt}{t}.
$$

Now, by the support of $\phi_{A(t)}$, we need only consider B such that $\rho_A(B,0) \sim \frac{1}{t}$, thus

$$
g_{A,\alpha,\beta,\Phi}(T_m(f * \psi_B))(x)^2 \lesssim \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} t^{2\beta\nu} \left(H_B * |f * \psi_B * \phi_{A(t)}|(y) \right)^2 \Phi_{A(t^{1-\alpha})}(x-y) \frac{dy}{t^{2\beta}} \frac{dt}{t}.
$$

Using Fubini's theorem and the Cauchy-Schwarz inequality as we did in the proof of Proposition 3.3.12, we have

$$
g_{A,\alpha,\beta,\Phi}(T_m(f * \psi_B))(x)^2 \lesssim \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} |f * \psi_B * \phi_{A(t)}(y)|^2 H_B * \Phi_{A(t^{1-\alpha})}(x-y) dy \frac{dt}{t},
$$

where we have used the fact that $\|H_B\| \lesssim 1$. Note that $H_B(x) \lesssim R^{\frac{3}{2}}_{A(t^{1-\alpha})}(x)$ and $\Phi_{A(t^{1-\alpha})}(x) \lesssim R^{\lambda}_{A(t^{1-\alpha})}(x)$ for all $\lambda > 1$, in particular for $\lambda = \frac{3}{2}$, so by Lemma A.1.1 with $\lambda = \frac{3}{2}$ and $r = t^{1-\alpha}$ we have

$$g_{A,\alpha,\beta,\Phi}(T_m(f * \psi_B))(x)^2 \lesssim \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} |f * \psi_B * \phi_{A(t^{1-\alpha})}(y)|^2 R^{\frac{3}{2}}_{A(t^{\alpha-1})}(x-y)\,dy\frac{dt}{t}$$
$$= g^*_{A,\alpha,0,\frac{3}{2}}(f * \psi_B)(x)^2.$$

□

So, now we have the three vital ingredients, we can state and prove our theorem.

Theorem 3.3.15

$$g_{A,\alpha,\beta,\Phi}(T_m(f))(x) \lesssim g^*_{A,\alpha,0,\frac{3}{2}}(f)(x).$$

Proof: We use Proposition 3.3.10, then Proposition 3.17 and finally Proposition 3.3.12 as follows

$$\begin{aligned} g_{A,\alpha,\beta,\Phi}(T_m f)(x) &\lesssim \sum_{B \in \mathcal{B}_{\rho_A}} g_{A,\alpha,\beta,\Phi}(T_m f * \psi_B)(x) \\ &\lesssim \sum_{B \in \mathcal{B}_{\rho_A}} g^*_{A,\alpha,0,\frac{3}{2}}(f * \psi_B)(x) \\ &\lesssim g^*_{A,\alpha,0,\frac{3}{2}}(f)(x). \end{aligned}$$

□

3.3.4 Square functions and g-functions

It was noted by Wilson in [44] that large classes of square functions are essentially equivalent, so the final two propositions of this chapter should not come as a surprise, but the proofs of them may seem somewhat arbitrary, especially with respect to the parameter β as its seemingly added artificially. For this reason we have postponed these until after the pointwise estimate was completed, as in doing so the role of β will hopefully become clear. First we begin with the following lemma found in [2], Lemma 10.

Lemma 3.3.16 *Let $R > 0$. Then,*

$$\int_{\mathbb{R}^2} h_1(x)h_2(x)dx \lesssim R^{\nu} \int_{\mathbb{R}^2} \int_{y\in B_A(x,\frac{1}{R})} h_1(y)dy \sup_{z\in B_A(x,\frac{1}{R})} h_2(z)dx.$$

Proof: We follow the proof as found in [2] by first considering the one dimensional case. That is, if $r > 0$ then

$$\int_{\mathbb{R}} h_1(x)h_2(x)dx \leqslant 2r \int_{\mathbb{R}} \int_{|y-x|\leqslant\frac{1}{r}} h_1(y)dy \sup_{|z-x|\leqslant\frac{1}{r}} h_2(z)dx.$$

We start by decomposing the integral as

$$\int_{\mathbb{R}} h_1(x)h_2(x)dx = \sum_{k\in\mathbb{Z}} \int_{\frac{-1}{r}}^{\frac{1}{r}} h_1\left(x+u+\frac{2k}{r}\right) h_2\left(x+u+\frac{2k}{r}\right) dx$$

for every u. Let $y = x + u + \frac{2k}{r}$, then

$$\frac{-1}{r} \leqslant x \leqslant \frac{1}{r} \iff |x| \leqslant \frac{1}{r} \iff \left|y-u-\frac{2k}{r}\right| \leqslant \frac{1}{r}.$$

So we have

$$\int_{\mathbb{R}} h_1(x)h_2(x)dx = \sum_{k\in\mathbb{Z}} \int_{\left|y-u-\frac{2k}{r}\right|\leqslant\frac{1}{r}} h_1(y)h_2(y)dy,$$

where we only consider $|u| \leqslant \frac{1}{r}$. Taking the supremum of h_2 over the domain of integration, we have

$$\int_{\mathbb{R}} h_1(x)h_2(x)dx \leqslant \sum_{k\in\mathbb{Z}} \int_{\left|y-u-\frac{2k}{r}\right|\leqslant\frac{1}{r}} h_1(y)dy \sup_{\left|y-u-\frac{2k}{r}\right|\leqslant\frac{1}{r}} h_2(z).$$

Averaging over all values of u, we have

$$\begin{aligned}\int_{\mathbb{R}} h_1(x)h_2(x)dx &\leqslant \sum_{k\in\mathbb{Z}} 2r \int_{\frac{-1}{r}}^{\frac{1}{r}} \left(\int_{\left|y-u-\frac{2k}{r}\right|\leqslant\frac{1}{r}} h_1(y)dy \sup_{\left|z-u-\frac{2k}{r}\right|\leqslant\frac{1}{r}} h_2(z) \right) du \\ &= 2r \sum_{k\in\mathbb{Z}} \int_{\frac{-1+2k}{r}}^{\frac{1+2k}{r}} \left(\int_{|y-x|\leqslant\frac{1}{r}} h_1(y)dy \sup_{|z-x|\leqslant\frac{1}{r}} h_2(z) \right) dx \\ &= 2r \int_{\mathbb{R}} \left(\int_{|y-x|\leqslant\frac{1}{r}} h_1(y)dy \sup_{|z-x|\leqslant\frac{1}{r}} h_2(z) \right) dx,\end{aligned}$$

where in the penultimate line we have used the substitution $x = u + \frac{2k}{R}$.
The lemma follows by applying the one dimensional case in the x_1 direction with $r = 2R$ then the x_2 direction with $r = 2R^\sigma$ and observing that $\{(s_1, s_2) \in \mathbb{R}^2 : x = (x_1, x_2), |s_1 - x_1| \leqslant \frac{1}{2R}, |s_2 - x_2| \leqslant \frac{1}{(2R)^\sigma}\} \subseteq B_A(x, \frac{1}{R})$. □

Proposition 3.3.17 *Let $\alpha, \beta \in \mathbb{R}$. For functions f such that* $\mathrm{supp}(\widehat{f}) \subseteq \{\xi \in \mathbb{R}^2 : |\xi|^\alpha \geqslant 1\}$,

$$\|s_A(f)\|_{L^2(w)} \lesssim \|g_{\alpha,\beta,A}(f)\|_{L^2(\mathcal{M}_{A,\alpha,\beta}w)}.$$

Proof: Firstly, by Fubini's theorem

$$\begin{aligned}\|s_A(f)\|^2_{L^2(w)} &= \int_{\mathbb{R}^2} \int_0^\infty |f * \phi_{A(t)}(x)|^2 \frac{dt}{t} w(x)dx \\ &= \int_0^\infty \int_{\mathbb{R}^2} |f * \phi_{A(t)}(x)|^2 w(x)dx \frac{dt}{t}.\end{aligned}$$

Now, as

$$\mathrm{supp}(\widehat{\phi}) \subseteq \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leqslant \rho_A(\xi) \leqslant 3\}$$

and as

$$\widehat{f * \phi_{A(t)}}(\xi) = \widehat{f}(\xi)\widehat{\phi}(A(t)\xi),$$

we have support only when

$$\frac{3}{4} \leqslant \rho_A(A(t)\xi) \leqslant 3,$$

or equivalently

$$\frac{3}{4t} \leqslant \rho_A(\xi) \leqslant \frac{3}{t}.$$

Since $\widehat{f}$ has support only when $|\xi|^\alpha \geqslant 1 \iff \rho_A(\xi)^\alpha \geqslant 1$, we only consider values of t such that $0 < t^\alpha \leqslant 1$; since for $t^\alpha > 1$, $\widehat{f}(\xi)\widehat{\phi}(A(t)\xi)$ has support in a subset of the support when $t = 1$. Thus,

$$\|s_A(f)\|_{L^2(w)}^2 = \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} |f * \phi_{A(t)}(x)|^2 w(x) dx \frac{dt}{t}.$$

Next we define $\varphi \in \mathcal{S}$ such that $\operatorname{supp}(\widehat{\varphi}) \subseteq \{\xi \in \mathbb{R}^2 : \frac{1}{4} \leqslant |\xi| \leqslant 4\}$ and $\widehat{\varphi} = 1$ on $\operatorname{supp}(\widehat{\phi})$. Then $f * \phi_{A(t)}(x) = f * \phi_{A(t)} * \varphi_{A(t)}(x)$. So

$$\begin{aligned}
\|s_A(f)\|_{L^2(w)}^2 &= \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} |f * \phi_{A(t)} * \varphi_{A(t)}(x)|^2 w(x) dx \frac{dt}{t} \\
&= \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} f * \phi_{A(t)}(y) \varphi_{A(t)}(x-y) dy \right|^2 w(x) dx \frac{dt}{t} \\
&\leqslant \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |f * \phi_{A(t)}(y)| |\varphi_{A(t)}(x-y)| dy \right)^2 w(x) dx \frac{dt}{t} \\
&= \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |f * \phi_{A(t)}(y)| |\varphi_{A(t)}(x-y)|^{\frac{1}{2}} |\varphi_{A(t)}(x-y)|^{\frac{1}{2}} dy \right)^2 w(x) dx \frac{dt}{t}
\end{aligned}$$

and by the Cauchy-Schwarz inequality then Fubini's theorem, we have

$$\begin{aligned}
&\|s_A(f)\|_{L^2(w)}^2 \\
&\quad\leqslant \int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|f*\phi_{A(t)}(y)|^2|\varphi_{A(t)}(x-y)|dy\right)\left(\int_{\mathbb{R}^2}|\varphi_{A(t)}(x-y)|dy\right)w(x)dx\frac{dt}{t} \\
&\quad= \|\varphi_{A(t)}\|_1\int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2}|f*\phi_{A(t)}(y)|^2\left(\int_{\mathbb{R}^2}|\varphi_{A(t)}(x-y)|w(x)dx\right)dy\frac{dt}{t} \\
&\quad= \|\varphi\|_1\int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2}|f*\phi_{A(t)}(y)|^2|\varphi_{A(t)}|*w(y)dy\frac{dt}{t}.
\end{aligned}$$

Applying Lemma 3.3.16 with $h_1(x) = |f*\phi_{A(t)}(x)|^2$, $h_2(x) = |\varphi_{A(t)}|*w(x)$ and $R = \left(\frac{1}{t}\right)^{(1-\alpha)}$, we have

$$\begin{aligned}
&\|s_A(f)\|_{L^2(w)}^2 \\
&\quad\lesssim \int_{0<t^\alpha\leqslant 1}\left(\frac{1}{t}\right)^{(1-\alpha)\nu}\int_{\mathbb{R}^2}\int_{\rho_A(y-x)\leqslant t^{(1-\alpha)}}|f*\phi_{A(t)}(y)|^2dy\sup_{\rho_A(z-x)\leqslant t^{(1-\alpha)}}|\varphi_{A(t)}|*w(z)dx\frac{dt}{t}.
\end{aligned}$$

As $\varphi\in\mathcal{S}$ we can dominate it pointwise, modulo a constant, by some positive, radial function in $\mathcal{S}$ with total mass 1, namely ϑ. Thus,

$$\begin{aligned}
&\|s_A(f)\|_{L^2(w)}^2 \\
&\quad\lesssim \int_{0<t^\alpha\leqslant 1}\int_{\mathbb{R}^2}\int_{\rho_A(y-x)\leqslant t^{(1-\alpha)}}|f*\phi_{A(t)}(y)|^2\frac{dy}{(t^\nu)^{(1-\alpha)+2\beta}}(t^\nu)^{2\beta}\sup_{\rho_A(z-x)\leqslant t^{(1-\alpha)}}|\vartheta_{A(t)}*w(z)|dx\frac{dt}{t}.
\end{aligned}$$

Finally, taking the supremum in

$$(t^\nu)^{2\beta}\sup_{\rho_A(z-x)\leqslant t^{(1-\alpha)}}|\vartheta_{A(t)}*w(z)|$$

over t such that $0<t^\alpha\leqslant 1$, we have

$$\|s_A(f)\|_{L^2(w)}^2 \lesssim \int_{0<t^\alpha\leqslant 1}\int_{\mathbb{R}^2}\int_{\rho_A(y-x)\leqslant t^{(1-\alpha)}}|f*\phi_{A(t)}(y)|^2\frac{dy}{(t^\nu)^{(1-\alpha)+2\beta}}\mathcal{M}_{A,\alpha,\beta}w(x)dx\frac{dt}{t},$$

which, with a final application of Fubini's theorem, allows us to conclude

$$\|s_A(f)\|^2_{L^2(w)} \lesssim \int_{\mathbb{R}^2}\int_{0<t^\alpha\leqslant 1}\int_{\rho_A(y-x)\leqslant t^{(1-\alpha)}} |f * \phi_{A(t)}(y)|^2 \frac{dy}{(t^\nu)^{(1-\alpha)+2\beta}}\frac{dt}{t}\mathcal{M}_{A,\alpha,\beta}w(x)dx$$
$$= \|g_{A,\alpha,\beta}(f)\|^2_{L^2(\mathcal{M}_{A,\alpha,\beta}w)}.$$

$\square$

Proposition 3.3.18 *Let* $\lambda > 1$ *and* $\alpha \in \mathbb{R}$

$$\|g^*_{A,\alpha,0,\lambda}(f)\|^2_{L^2(w)} \lesssim \|s_A(f)\|^2_{L^2(M_A w)}.$$

While this proposition is written for the case $\beta = 0$, it is possible to run a very similar proof for other values of β, but it holds no content for our overall goal of proving Theorem 3.3.5 and would produce a different maximal average on the weight.

Proof: Using Fubini's theorem,

$$\begin{aligned}\|g^*_{A,\alpha,0,\lambda}(f)\|^2_{L^2(w)} &= \int_{\mathbb{R}^2}\int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2} |f * \phi_{A(t)}(y)|^2 R^\lambda_{A(t^{\alpha-1})}(x-y)dy\frac{dt}{t}w(x)dx \\ &= \int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2} |f * \phi_{A(t)}(y)|^2 \int_{\mathbb{R}^2} R^\lambda_{A(t^{\alpha-1})}(x-y)w(x)dxdy\frac{dt}{t} \\ &\leqslant \int_{t^\alpha\leqslant 1}\int_{\mathbb{R}^2} |f * \phi_{A(t)}(y)|^2 R^\lambda_{A(t^{\alpha-1})} * w(y)dy\frac{dt}{t}\end{aligned}$$

since $R^\lambda_{A(t^{\alpha-1})}(-x) = R^\lambda_{A(t^{\alpha-1})}(x)$.
Using the substitution $z = t^{1-\alpha}$,

$$\sup_{t^\alpha\leqslant 1} R^\lambda_{A(t^{\alpha-1})} * w = \sup_{z^{\frac{\alpha}{(1-\alpha)}}\leqslant 1} R^\lambda_{A(z)} \leqslant \sup_{z>0} R^\lambda_{A(z)} * w \lesssim M_A w,$$

where the final step is just dominating $\boldsymbol{R}^{\lambda}$ by ϑ pointwise.

Thus, we have

$$\|g_{A,\alpha,0,\Phi}(f)\|_{L^2(w)}^2 \lesssim \int_{t^\alpha \leqslant 1} \int_{\mathbb{R}^2} |f * \phi_{A(t)}(y)|^2 M_A w(y) dy \frac{dt}{t},$$

so finally by Fubini's theorem,

$$\begin{aligned}\|g_{A,\alpha,0,\Phi}(f)\|_{L^2(w)}^2 &\lesssim \int_{\mathbb{R}^2} \int_{t^\alpha \leqslant 1} |f * \phi_{A(t)}(y)|^2 \frac{dt}{t} M_A w(y) dy \\ &= \|s_A(f)\|_{L^2(M_A w)}^2.\end{aligned}$$

$\square$

CHAPTER 4

OSCILLATORY KERNELS

4.1 Background

4.1.1 Overview of the method

In this section we will be extending our previous methods that handled certain transforms defined via multipliers to oscillatory integrals. This method was employed by Bennett (see Section 2.2 of [3]) in the one dimensional case and Beltran and Bennett (see Section 1.1 of [2]) to kernels considered by Sjölin [36], defined for $a > 0$, $a \neq 0$ and $b < 1 - \frac{a}{2}$ on $R^d \setminus \{0\}$ by

$$\frac{e^{i|x|^a}}{|x|^{db}}.$$

We will first consider the complement to the kernels ($a < 0$ and $b > 1$) and then employ the same tactic to consider a class of kernels containing those considered by Bennett and Harrison in [5]. The heavy lifting will be done in most part by the previous section, a clever use of integration by parts and a powerful theorem known as van der Corput's lemma, or the much more restrictive multidimensional version. To do this we will decompose our kernels into parts that don't each have much oscillation, but the parts themselves will differ in size. For the part of the kernel that does not display much oscillation, we will bound using elementary methods without using any cancellation at all. The rest of the kernel will have some amount of oscillation, or "roughness", and we will split the kernel up into dyadic blocks corresponding to the amount of oscillation. Next, we will consider each of these blocks by estimating the size of the corresponding multiplier, and it's derivatives. Only boundedly many of these dyadic blocks will contribute to the multiplier as a whole, allowing us to sum up the parts and obtain an overall estimate on the multiplier. These estimates will coincide with the estimates on the multipliers in the previous sections, allowing us to apply the theorems there to obtain Fefferman-Stein inequalities on these transforms defined by oscillatory kernels.

4.1.2 Two important lemmas

Lemma 4.1.1 *Let $a, b \in \mathbb{R}$, $M \geqslant 2$ be an integer and $\lambda_1, \lambda_2 > 1$. Let h and ψ be real functions, such that ψ is smooth and has compact support in (a, b), and for each $2 \leqslant \gamma \leqslant M$ and for all $x \in [a, b]$, $|h'(x)| \geqslant c_0\lambda_1$, $|h^{(\gamma)}(x)| \leqslant c_\gamma \lambda_2$. Then*

$$\left|\int_a^b e^{ih(x)}\psi(x)dx\right| \lesssim \sum_{r=0}^{N} \lambda_1^{-(r+N)}\lambda_2^r,$$

for all natural numbers $N \leqslant M-1$, where the implicit constants depend only on ψ, c_0 and c_γ.

Remark 4.1.2 *Note that in the case $\lambda_1 = \lambda_2$ this reduces to the well known integration by parts argument that can be found in Stein[39].*

Proof: We start by defining a differential operator D by

$$Df(x) = (ih'(x))^{-1}\frac{df(x)}{dx}$$

and then let D^* denote it's adjoint,

$$D^* f(x) = \frac{-d}{dx}\left(\frac{f(x)}{ih'(x)}\right).$$

Then,

$$D(e^{ih(x)}) = \frac{1}{ih'(x)}\frac{d}{dx}\left(e^{ih(x)}\right)$$
$$= \frac{ih'(x)}{ih'(x)}e^{ih(x)}$$
$$= e^{ih(x)}.$$

And so by repeated application of this and integration by parts, we have, for each $N \in \mathbb{N}$,*

$$\int_a^b e^{ih(x)}\psi(x)dx = \int_a^b D^N(e^{ih(x)})\psi(x)dx$$
$$= \int_a^b e^{ih(x)}(D^*)^N(\psi(x))dx.$$

Thus, by the definition of D^*, we have that

$$\left|\int_a^b e^{ih(x)}\psi(x)dx\right| = \left|\int_a^b e^{ih(x)}(D^*)^N\psi(x)dx\right|$$
$$\leqslant \int_a^b \left|(D^*)^N\psi(x)\right| dx.$$

The lemma is thus immediate from a simple calculation of $\left|(D^*)^N\psi(x)\right|$, we provide the first few such calculations in Appendix B.1 for the scrutinous reader. □

This lemma has a very simple extension to multiple dimensions:

Lemma 4.1.3 *Let Ω be an open set in $\mathbb{R}^n$, $\zeta \in C_c^\infty(\Omega)$ and $M \in \mathbb{N}$. If ϕ is such that, for some $i \in \{1, ..., n\}$,*

$$\frac{\partial\phi}{\partial x_i}(x) \geqslant c_0\lambda_1,$$

*We have used D^N and $(D^*)^N$ to denote the N-fold composition of the differential operators with themselves, a la maximal function self composition *and for all positive integers $\gamma \leqslant M$,*

$$\frac{\partial^{(\gamma)}\phi}{\partial x_i^{(\gamma)}}(x) \leqslant c_\gamma \lambda_2,$$

then

$$\left|\int_\Omega e^{i\lambda\phi(x)}\zeta(x)dx\right| \lesssim \sum_{r=0}^{N} \lambda_1^{-(r+N)}\lambda_2^r$$

for all natural numbers $N \leqslant M - 1$.

Proof: Let Ω_1 be the interval in the x_i direction containing Ω, Ω_2 be the region containing Ω excluding the x_i direction, $\zeta_1 \in C_c^\infty(\Omega_1)$ and $\zeta_2 \in C_c^\infty(\Omega_2)$. By chopping up the integral over Ω into an integral over Ω_1 and an integral over Ω_2, we have

$$\int_\Omega e^{i\phi(x)}\zeta(x)dx = \int_{\Omega_2}\left(\int_{\Omega_1} e^{i\phi(x_1,\dots,x_n)}\zeta_1(x_i)dx_i\right)\zeta_2(x_1,...,x_{i-1},x_{i+1},x_n)dx_1,...,dx_{i-1},dx_{i+1},...,dx_n.$$

Then by Lemma 4.1.1 we have, for all $N \leqslant M - 1$,

$$\begin{aligned}\int_\Omega e^{i\phi(x)}\zeta(x)dx &\lesssim \int_{\Omega_2}\sum_{r=0}^{N}\lambda_1^{-(r+N)}\lambda_2^r\zeta_2(x_1,...,x_{i-1},x_{i+1},x_n)dx_1,...,dx_{i-1},dx_{i+1},...,dx_n\\ &\lesssim \sum_{r=0}^{N}\lambda_1^{-(r+N)}\lambda_2^r.\end{aligned}$$

$\square$

The second lemma we will be using is a simple corollary of a theorem due to J.G. van der Corput, see [39] for details.

Lemma 4.1.4 (van der Corput) *Let $\gamma \in \mathbb{N}$ and h be a function with continuous γth derivative. Let ψ be a smooth function with compact support in the interval (a, b), and let $\lambda > 0$. If $\gamma \geqslant 2$,* *or $\gamma = 1$ and h' monotonic, s.t. $|h^{(\gamma)}(x)| \gtrsim \lambda$ for $x \in (a, b)$, then*

$$\left| \int_a^b e^{ih(x)} \psi(x) dx \right| \lesssim \lambda^{-1/\gamma},$$

where the implicit constant is independent of λ.

Proof: We will obtain the desired result by first showing that

$$\left| \int_a^b e^{ih(x)} dx \right| \lesssim \lambda^{-1/\gamma}$$

holds independent of (a, b).

First, we will address the case $\gamma = 1$ and $h'(x)$ monotonic. Let D and D^* be the differential operators defined in the proof of Lemma 4.1.1, then

$$\begin{aligned} \int_a^b e^{ih(x)} dx &= \int_a^b D(e^{ih(x)}) dx \\ &= \int_a^b e^{ih(x)} D^*(1) dx + \left[(ih'(x))^{-1} e^{ih(x)} \right]_a^b. \end{aligned}$$

Then, by the triangle inequality, it is sufficient to consider each term on the right hand side separately. So, as $|h'(x)| \gtrsim \lambda$, we have

$$\left| \left[(ih'(x))^{-1} e^{ih(x)} \right]_a^b \right| \lesssim \lambda^{-1}.$$

By definition of D^* we have

$$\begin{aligned}
\left|\int_a^b e^{ih(x)} D^*(1) dx\right| &= \left|\int_a^b e^{ih(x)} \frac{d}{dx}\left(\frac{1}{h'(x)}\right) dx\right| \\
&\leqslant \int_a^b \left|\frac{d}{dx}\left(\frac{1}{h'(x)}\right)\right| dx \\
&= \left|\int_a^b \frac{d}{dx}\left(\frac{1}{h'(x)}\right) dx\right| \\
&= \left|\left[\frac{1}{h'(x)}\right]_a^b\right| \\
&\lesssim \lambda^{-1}
\end{aligned}$$

where the equality on line 3 holds by monotonicity of $h'(x)$ and the final line uses the bound $|h'(x)| \gtrsim \lambda$.

Now we will prove the lemma for $\gamma \geqslant 2$ by induction. First, suppose that the result holds for an integer $k < \gamma$ and assume that

$$h^{(k+1)}(x) \gtrsim \lambda,$$

replacing h with $-h$ if necessary. Let $c = \min_{[a,b]} |h^{(k)}(x)|$, then as $h^{(k+1)}(x) > 0$, if c is not a or b, then $h^{(k)}(c) = 0$. Let δ be such that outside of $[c-\delta, c+\delta]$ we have that $|h^{(k)}(x)| \gtrsim \lambda\delta$. Write (a, b) as $(a, c-\delta) \cup [c-\delta, c+\delta] \cup (c+\delta, b)$. By the inductive hypothesis we have

$$\left|\int_a^{c-\delta} e^{ih(x)} dx\right| \lesssim (\lambda\delta)^{-1/k},$$
$$\left|\int_{c+\delta}^b e^{ih(x)} dx\right| \lesssim (\lambda\delta)^{-1/k}.$$

The last part of the interval is estimated trivially by

$$\begin{aligned}\left|\int_{c-\delta}^{c+\delta} e^{ih(x)}dx\right| &\leqslant \int_{c-\delta}^{c+\delta} dx \\ &\lesssim \delta.\end{aligned}$$

If $c = a$ (or b, the cases are almost identical), then either $|h^{(k)}(a)| \gtrsim \lambda\delta$ and thus $|h^{(k)}(x)| \gtrsim \lambda\delta$ for all $x \in [a, b]$, so by using our inductive hypothesis we obtain

$$\left|\int_{a}^{b} e^{ih(x)}dx\right| \lesssim (\lambda\delta)^{-1/k}$$

or again we let δ be such that outside of $(a, a + \delta]$ we have $|h^{(k)}(x)| \gtrsim \lambda\delta$ and write $(a, b) = (a, a + \delta] \cup (a + \delta, b)$. By the inductive hypothesis we have that

$$\left|\int_{a+\delta}^{b} e^{ih(x)}dx\right| \lesssim (\lambda\delta)^{-1/k}.$$

Again, the other part of the interval is estimated trivially as

$$\begin{aligned}\left|\int_{a}^{a+\delta} e^{ih(x)}dx\right| &\leqslant \int_{a}^{a+\delta} dx \\ &\lesssim \delta.\end{aligned}$$

In all cases, by choosing $\delta = \lambda^{-1/(k+1)}$ we conclude the proof by noting that with this $(\lambda\delta)^{-1/k} = \lambda^{-1/(k+1)}$.

Now, we have the result

$$\left|\int_{a}^{b} e^{ih(x)}dx\right| \lesssim \lambda^{-1/\gamma} \tag{4.1}$$

so let

$$F(x) = \int_a^x e^{ih(y)} dy,$$

then $F'(x) = e^{ih(x)}$ and by 4.1 we have

$$|F(x)| \lesssim \lambda^{-1/\gamma}.$$

So, we have that

$$\begin{aligned}
\int_a^b e^{ih(x)} \psi(x) dx &= \int_a^b F'(x)\psi(x) dx \\
&= \Big[F(x)\psi(x) \Big]_a^b - \int_a^b F(x)\psi'(x) dx \\
&= - \int_a^b F(x)\psi'(x) dx,
\end{aligned}$$

where we have used integration by parts and then the fact that ψ has compact support in (a, b). Thus, we have

$$\begin{aligned}
\left| \int_a^b e^{ih(x)} \psi(x) dx \right| &= \left| \int_a^b F(x)\psi'(x) dx \right| \\
&\lesssim \lambda^{-1/\gamma} \left| \int_a^b |\psi'(x)| dx \right| \\
&\lesssim \lambda^{-1/\gamma} (b - a) \|\psi'\|_\infty \\
&\lesssim \lambda^{-1/\gamma},
\end{aligned}$$

since ψ is smooth, concluding the proof of Lemma 4.1.4. □

We will also need the multidimensional version of van der Corput's lemma that Sjölin [36] and Cao et al. [9] use in their papers. This lemma is essentially due to Littman[28], refined by Domar

[15].

Lemma 4.1.5 *Let Ω be an open set in $\mathbb{R}^n$ and $\zeta \in C_c^\infty(\Omega)$. If $\phi \in C^\infty(\Omega)$ is such that for each $i, j \in \{1, ..., n\}$*

$$\left|\det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x)\right)\right| \geqslant C_0 > 0$$

for all $x \in \Omega$, then

$$\left|\int_\Omega e^{i(\lambda\phi(x) - x\cdot\xi)}\zeta(x)dx\right| \leqslant C(1 + |\lambda|)^{-n/2},$$

where C depends on n, Ω, the uniform bounds on the absolute value of ζ and ϕ, the partial derivatives of ϕ and the inverse of the Hessian of ϕ over Ω.

4.2 Hirschmann kernels

In this section we will deal with hypersingular kernels $K_{a,b}$ that are tempered distributions and agree with the functions

$$\frac{e^{i|x|^{-a}}}{|x|^{db}} \qquad \text{for } x \in \mathbb{R}^d \setminus \{0\},$$

where $a > 0$ and $b > 1$.

Theorem 4.2.1 *Let T be an operator given by $Tf = K_{a,b} * f$, then*

$$\int_{\mathbb{R}^d} |Tf|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^2 \mathcal{M}_{\alpha,\beta} M^4 w,$$

where $\mathcal{M}_{\alpha,\beta}$ is the maximal operator given by (3.5) *with parameters $\alpha = \frac{a}{a+1}$ and $\beta = \frac{a/2-b+1}{a+1}$.**

*In this section our notation $A \lesssim B$ will have an implicit constant with dependence on at most a and b, unless otherwise specified.

Note that T is bounded on non-weighted $L^2(\mathbb{R}^d)$ if and only if $\beta \geqslant 0$, see [29]. However, with Theorem 4.2.1 we may obtain more non-weighted $L^p - L^q$ bounds as discussed before, see (1.4).

Corollary 4.2.2 *Let T be an operator given by $Tf = K_{a,b} * f$, then*

$$\|Tf\|_q \lesssim \|f\|_p.$$

whenever $b - 1 \geqslant \frac{-a}{q} - \left(\frac{1}{p} - \frac{1}{q}\right)$.

If we consider the case $p, q = 2$, then we obtain the requirement $b - 1 + \frac{a}{2} \geqslant 0$, which implies $\beta \geqslant 0$, implying in turn that $\mathcal{M}_{\alpha,\beta}$ is the optimal maximal operator for these kernels in the case $p, q = 2$, in the purview of the Fefferman-Stein inequality.

To prove Theorem 4.2.1 we will decompose the kernel $K_{a,b}$ into parts and use the linearity and continuity of the Fourier transform to reconstruct the multiplier for part of the kernel on the multiplier side, then we shall use Theorem 3.3.1 on the part of the multiplier that satisfies the relevant hypotheses. We will begin by separating the trivial part of our kernel and setting up our dyadic decomposition of the difficult part of the kernel.

Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ with compact support in $\{x \in \mathbb{R}^d : 1/2 < |x| < 2\}$ such that $\sum_{k\in\mathbb{Z}} \zeta(2^k x) = 1$ for $x \neq 0$. Define $\zeta_k(x) = \zeta(2^k x)$ and $K_{a,b,k} = \zeta_k K_{a,b}$ for each $k \in \mathbb{N}$ and define $K_{a,b,\infty} = (1 - \sum_{k=1}^\infty \zeta_k) K_{a,b}$. Then we have that

$$K_{a,b} = K_{a,b,\infty} + \sum_{k=1}^{\infty} K_{a,b,k}. \tag{4.2}$$

We note that the support for the first term on the RHS of (4.2) is away from the origin and the support of the sum of the rest of the terms is a small neighbourhood around, but not including, the origin.

Now, for the trivial part of the kernel, $K_{a,b,\infty}$, we can obtain an upper estimate:

$$
\begin{aligned}
|K_{a,b,\infty}(x)| &= \left| \frac{e^{i|x|^{-a}}}{|x|^{db}} \left(1 - \sum_{k=1}^{\infty} \zeta_k(x) \right) \right| \\
&\leqslant \frac{1 - \sum_{k=1}^{\infty} \zeta_k(x)}{|x|^{db}}
\end{aligned}
$$

for $x \neq 0$. Let

$$
\Phi(x) = \frac{1 - \sum_{k=1}^{\infty} \zeta_k(x)}{|x|^{db}},
$$

when $x \neq 0$ and $\Phi(0) = 0$. Let $B_n = \{x \in \mathbb{R}^d : |x| \leqslant 2^n\}$; and observe that, for each $x \in \mathbb{R}^d$,

$$
\Phi(x) \leqslant \sum_{n=1}^{\infty} 2^{-db(n-2)} \chi_{B_n}(x),
$$

and so

$$
\begin{aligned}
|K_{a,b,\infty}| * w(x) &= \int_{\mathbb{R}^d} |K_{a,b,\infty}(y)| w(x-y) dy \\
&\leqslant \int_{\mathbb{R}^d} \Phi(y) w(x-y) dy \\
&\leqslant \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} 2^{-b(n-2)} \chi_{B_n}(y) w(x-y) dy \\
&= \sum_{n=1}^{\infty} 2^{-db(n-2)} \int_{\mathbb{R}^d} \chi_{B_n}(y) w(x-y) dy \\
&= \sum_{n=1}^{\infty} 2^{-db(n-2)} \int_{|y| \leqslant 2^n} w(x-y) dy \\
&= 2^{2d} \sum_{n=1}^{\infty} 2^{-d(b-1)(n-2)} \frac{1}{2^{dn}} \int_{|x-y| \leqslant 2^n} w(z) dz \\
&\lesssim \sum_{n=1}^{\infty} 2^{-d(b-1)(n-2)} \sup_{r>1} \frac{1}{r^d} \int_{|x-z| \leqslant r} w(z) dz,
\end{aligned}
$$

where we have used the substitution $z = x - y$. Now, since $b > 1$ we have that

$$\sum_{n=1}^{\infty} 2^{-d(b-1)(n-2)} < +\infty$$

so we can conclude

$$|K_{a,b,\infty}| * w(x) \leqslant M^{(1)} w(x), \tag{4.3}$$

where

$$M^{(1)} w(x) = \sup_{r \geqslant 1} \frac{1}{r^d} \int_{|x-z| \leqslant r} w(z) dz. \tag{4.4}$$

So, we can estimate this part of our kernel

$$\begin{aligned}
\int_{\mathbb{R}^d} |K_{a,b,\infty} * f(x)|^2 w(x) dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} K_{a,b,\infty}(x-y) f(y) dy \right|^2 w(x) dx \\
&\leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_{a,b,\infty}(x-y)| |f(y)| dy \right)^2 w(x) dx \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_{a,b,\infty}(x-y)|^{1/2} |K_{a,b,\infty}(x-y)|^{1/2} |f(y)| dy \right)^2 w(x) dx
\end{aligned}$$

by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |K_{a,b,\infty} * f(x)|^2 w(x) dx \\
&\quad \leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_{a,b,\infty}(x-y) dy \right) \left(\int_{\mathbb{R}^d} |f(y)|^2 |K_{a,b,\infty}(x-y)| dy \right) w(x) dx \\
&\quad = \|K_{a,b,\infty}\|_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}} |f(y)|^2 |K_{a,b,\infty}(x-y)| w(x) dy dx \\
&\quad = \|K_{a,b,\infty}\|_1 \int_{\mathbb{R}^d} |f(y)|^2 \left(\int_{\mathbb{R}^d} |K_{a,b,\infty}(x-y)| w(x) dx \right) dy \\
&\quad = \|K_{a,b,\infty}\|_1 \int_{\mathbb{R}^d} |f(y)|^2 |K_{a,b,\infty}| * w(y) dy,
\end{aligned}$$

where the last step follows from the fact that our kernel is even, and we comment here that it makes sense to talk about the L^1 norm of $K_{a,b,\infty}$ as $\Phi(x)$ is clearly an L^1 function that dominates $K_{a,b,\infty}$. Then, by (4.3)

$$\int_{\mathbb{R}^d} |K_{a,b,\infty} * f|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^{(1)} w.$$

Now, we claim that if $|x-y|<1$, then

$$M^{(1)}w(x) \lesssim M^{(1)}w(y),$$

indeed;

$$\begin{aligned}
M^{(1)}w(x) &= \sup_{r>1} \frac{1}{r^d} \int_{|x-z|\leqslant r} w(z)dz \\
&\leqslant 2^d \sup_{r>1} \frac{1}{(2r)^d} \int_{|x-z|\leqslant 2r} w(z)dz \\
&\leqslant 3^d \sup_{r>1} \frac{1}{(3r)^d} \int_{|y-z|\leqslant 3r} w(z)dz \\
&= 3^d \sup_{r'>3} \frac{1}{(r')^d} \int_{|y-z|\leqslant r'} w(z)dz \\
&\leqslant 3^d \sup_{r'>1} \frac{1}{(r')^d} \int_{|y-z|\leqslant r'} w(z)dz \\
&= 3^d M^{(1)}w(y),
\end{aligned}$$

where we used the substitution $r' = 3r$. Now, using this claim, we have that

$$\begin{aligned}
AM^{(1)}w(x) &= \frac{1}{2} \int_{|x-y|<1} M^{(1)}w(y)dy \\
&\gtrsim \int_{|x-y|<1} M^{(1)}w(x)dy \\
&= CM^{(1)}w(x),
\end{aligned}$$

where the operator A is given by

$$Aw(x) = \frac{1}{2}\int_{|x-y|<1} w(y)dy.$$

Additionally, just by inspection of the maximal operator

$$\mathcal{M}_{\alpha,\beta}w(x) = \sup_{(r,y)\in\Gamma_\alpha(x)} \frac{r^{2\beta}}{r}\int_{|y-z|\leqslant r} w(z)dz,$$

where

$$\Gamma_\alpha(x) = \{(r,y) : 0 < r^\alpha \leqslant 1 \text{ and } |y-x| \leqslant r^{1-\alpha}\},$$

by taking $y = x$ and $r = 1$ in the supremum, we can see that

$$Aw(x) \leqslant \mathcal{M}_{\alpha,\beta}w(x).$$

Also, with the addition of the simple observation that

$$\begin{aligned} M^{(1)}w(x) &= \sup_{r>1}\frac{1}{r^d}\int_{|x-y|\leqslant r} w(y)dy \\ &\leqslant \sup_{r>0}\frac{1}{r^d}\int_{|x-y|\leqslant r} w(y)dy \\ &= Mw(x), \end{aligned}$$

where M is the classical Hardy-Littlewood maximal operator, and so finally, we have the pointwise bound

$$M^{(1)}w \lesssim AM^{(1)}w \leqslant \mathcal{M}_{\alpha,\beta}M^{(1)}w \leqslant \mathcal{M}_{\alpha,\beta}Mw \leqslant M^2\mathcal{M}_{\alpha,\beta}M^4w, \tag{4.5}$$

to give

$$\int_{\mathbb{R}^d} |K_{a,b,\infty} * f|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^2 \mathcal{M}_{\alpha,\beta} M^4 w.$$

It remains to prove Theorem 4.2.1 for the rest of the kernel. First define $m_k(\xi) = \widehat{K}_{a,b,k}(\xi)$ (defined by (4.2)) for each $k \in \mathbb{N}$ and

$$m(\xi) = \widehat{\left(\sum_{k=1}^{\infty} K_{a,b,k}\right)}(\xi).$$

We note here that the above definition of m excludes the part of the kernel away from the origin as we are only summing over $k \in \mathbb{N}$.

As the Fourier transform is an isomorphism of the Schwarz class, $\mathcal{S}(\mathbb{R})$ to itself, and so induces an isomorphism of the space of tempered distributions, $\mathcal{S}'(\mathbb{R})$, to itself, see [40]; and since the Fourier transform is continuous and linear on $\mathcal{S}'(\mathbb{R})$, we have

$$m(\xi) = \sum_{k=1}^{\infty} m_k(\xi).$$

For each $k \in \mathbb{N}$ we have

$$m_k(\xi) = \int_{\mathbb{R}^d} e^{i(|x|^{-a} - x\cdot\xi)} \frac{\zeta(2^k x)}{|x|^{db}} dx.$$

If we use the substitution $z = 2^k x$, then $x = 2^{-k} z$and so $J(x) = 2^{-dk}$ is the Jacobian determinant. We therefore have

$$\begin{aligned} m_k(\xi) &= 2^{kd(b-1)} \int_{\mathbb{R}} e^{i(2^{ka}|z|^{-a} - (2^{-k}z)\cdot\xi)} \frac{\zeta(z)}{|z|^{db}} dz \\ &= 2^{kd(b-1)} \int_{1/2 \leqslant |z| \leqslant 2} e^{ih_k(z)} \frac{\zeta(z)}{|z|^b} dz, \end{aligned}$$

where $h_k(z) = 2^{ka}|z|^{-a} - (2^{-k}z) \cdot \xi$, therefore $\nabla h_k(z) = -2^{ka}az|z|^{-(a+2)} - 2^{-k}\xi$.

We shall now use either Lemma 4.1.3 or Lemma 4.1.5, depending on k, to give bounds on each m_k. We let $c_1, c_2 \in \mathbb{R}^+$ be such that $c_1 < c_2$, later we will choose values of these that depend only on a.

Case 1: k is such that $k \in I_1 = \{k \in \mathbb{N} : 2^k \leqslant c_1|\xi|^{\frac{1}{a+1}}\}$.

Then $2^{ka} \leqslant c_1^{(a+1)}2^{-k}|\xi|$. So

$$\begin{aligned}|\nabla h_k(z)| &\geqslant 2^{-k}|\xi| - a2^{ka}|z|^{-(a+1)}\\ &\geqslant 2^{-k}|\xi| - ac_1^{(a+1)}2^{-k}|\xi||z|^{-(a+1)}\\ &\geqslant 2^{-k}|\xi|(1 - ac_1^{(a+1)}2^{(a+1)}),\end{aligned}$$

as $|z| > 1/2$, and so if we take $c_1 = \frac{1}{2}(2a)^{\frac{-1}{a+1}}$ we obtain

$$|\nabla h_k(z)| \gtrsim 2^{-k}|\xi|.$$

Now, this means there exists an i such that

$$\left|\frac{\partial h_k}{\partial x_i}(z)\right| \geqslant c|2^{-k}\xi|,$$

for some $c > 0$. We also have, for $i' = \{1, ..., d\}$, $j \geqslant 2$,

$$\frac{\partial^{(\gamma)} h_k}{\partial z_{i'}^{(\gamma)}}(z) = 2^{ka}\frac{\partial^{(j)}}{\partial x_{i'}^{(j)}}\left(|z|^{-a}\right)(z),$$

and so for $i' = i$,

$$\begin{aligned}\left|\frac{\partial^{(\gamma)} h_k}{\partial x_i^{(\gamma)}}(z)\right| &\lesssim 2^{ka}\\ &\lesssim 2^{-k}|\xi|.\end{aligned}$$

Thus there is a constant dependent only upon a, d and γ such that

$$\left|\frac{\partial^{(\gamma)} h_k}{\partial x_i^{(\gamma)}}(z)\right| \leqslant C_\gamma 2^{ka} \tag{4.6}$$

$$\leqslant C'_\gamma 2^{-k}|\xi|. \tag{4.7}$$

Then, by Lemma 4.1.3 we have that, for each $N \in \mathbb{N}$,

$$\begin{aligned}
|m_k(\xi)| &\lesssim 2^{kd(b-1)}(2^{-k}|\xi|)^{-N} \\
&= 2^{kd(b-1)}2^{kN}|\xi|^{-N} \\
&= 2^{kd(b-1)}2^{kN}|\xi|^{\frac{-N}{(a+1)}}|\xi|^{\frac{-aN}{2(a+1)}}|\xi|^{\frac{-aN}{2(a+1)}} \\
&\lesssim 2^{kd(b-1-\frac{aN}{2d})}|\xi|^{\frac{-aN}{2(a+1)}},
\end{aligned}$$

where the last line follows from the inequality $2^k \leqslant c_1|\xi|^{\frac{1}{a+1}}$.

Case 2: k is such that $k \in I_2 = \{k \in \mathbb{N} : 2^k \geqslant c_2|\xi|^{\frac{1}{a+1}}\}$.

Then $2^{ka}c_2^{-(a+1)} \geqslant 2^{-k}|\xi|$, and so

$$\begin{aligned}
|\nabla h_k(z)| &\geqslant a|z|^{-(a+1)}2^{ka} - 2^{-k}|\xi| \\
&\geqslant a|z|^{-(a+1)}2^{ka} - 2^{ka}c_2^{-(a+1)} \\
&\geqslant 2^{ka}(a2^{-(a+1)} - c_2^{-(a+1)}),
\end{aligned}$$

as $|z| \geqslant 2$, and so if we take $c_2 = 2(2a)^{\frac{-1}{a+1}}$ we obtain

$$|\nabla h_k(z)| \gtrsim 2^{ka}.$$

So, this means there exists an i such that

$$\left|\frac{\partial h_k}{\partial x_i}(z)\right| \geqslant c2^{ka},$$

for some $c > 0$. Again, (4.6) holds and so by applying Lemma 4.1.3 we have that, for each $N \in \mathbb{N}$,

$$\begin{aligned}|m_k(\xi)| &\lesssim 2^{kd(b-1)}(2^{ka})^{-N} \\ &\lesssim 2^{kd(b-1-\frac{aN}{d})}.\end{aligned}$$

Case 3: k is such that $k \in I_3 = \{k \in \mathbb{N} : c_1|\xi|^{\frac{1}{a+1}} < 2^k < c_2|\xi|^{\frac{1}{a+1}}\}$.

It is perhaps trivial to see that the function $|z|^{-a}$, for $a > 0$, has zero Hessian determinant only at $z = 0$, as this is the only point that could be critical. So, by Lemma 4.1.5, we have

$$\begin{aligned}|m_k(\xi)| &\lesssim 2^{kd(b-1)}(2^{ka})^{-d/2} \\ &= 2^{kd(b-1-a/2)} \\ &\lesssim |\xi|^{\frac{d(b-1-a/2)}{a+1}} \\ &= |\xi|^{-d\beta}.\end{aligned}$$

Now we will use these estimates on $|m_k(\xi)|$ to obtain an estimate on $|m(\xi)|$. First we sum over all k in *Case 1*, when N is large, we obtain

$$\sum_{k\in I_1} |m_k(\xi)| \lesssim |\xi|^{\frac{-aN}{2(a+1)}}.$$

Next we sum over all k in *Case 2*,

$$\sum_{k\in I_2}|m_k(\xi)| \lesssim \sum_{k\in I_2} 2^{kd(b-1-aN)}$$
$$= \sum_{k\in I_2} 2^{kd(b-1-aN/2)}2^{-kdaN/2}$$
$$\lesssim \sum_{k\in I_2} 2^{kd(b-1-aN/2)}|\xi|^{\frac{-daN}{2(a+1)}}$$

and again when N is large, we obtain

$$\sum_{k\in I_2}|m_k(\xi)| \lesssim |\xi|^{\frac{-daN}{2(a+1)}}.$$

Thus, choosing N large enough, we can obtain estimates for *Case 1* and *Case 2* such that

$$\sum_{k\in I_1\cup I_2}|m_k(\xi)| \lesssim |\xi|^{-d\beta}.$$

Finally, as there are only a bounded number of k in *Case 3*, summing over such k we have that

$$\sum_{k\in I_3}|m_k(\xi)| \lesssim |\xi|^{-d\beta},$$

thus we can conclude that

$$|m(\xi)| \lesssim |\xi|^{-d\beta}, \tag{4.8}$$

for $\xi \neq 0$.

Now we will attempt to get similar estimates on the derivatives, let $\gamma = \{\gamma_1, ..., \gamma_d\} \in \mathbb{N}_0^d$ then

$$D^\gamma m_k(\xi) = \int_{\mathbb{R}^d}(ix)^\gamma e^{i|x|^{-a}-ix\cdot\xi}\frac{\zeta(2^k x)}{|x|^{db}}dx,$$

and again using the substitution $z = 2^k x$, we get

$$
\begin{aligned}
D^\gamma m_k(\xi) &= i^{|\gamma|} 2^{k(d(b-1)-|\gamma|)} \int_{\mathbb{R}^d} e^{i(2^{ka}|z|^{-a}-(2^{-k}z)\cdot\xi)} \frac{\zeta(z) z^\gamma}{|z|^{db}} dz \\
&= i^{|\gamma|} 2^{k(d(b-1)-\gamma)} \int_{\frac{1}{2}\leqslant|z|\leqslant 2} e^{ih_k(z)} \frac{\zeta(z) z^\gamma}{|z|^b} dz.
\end{aligned}
$$

We observe here that the above integral is almost identical to the integral for $m_k(\xi)$, bar the z^γ term, which is easily controlled on the support of ζ. So by following the argument as before with very minor alterations we obtain the estimates

$$
|D^\gamma m_k(\xi)| \lesssim 2^{k(d(b-1)-\frac{aN}{2d})-|\gamma|)} |\xi|^{\frac{-aN}{2(a+1)}}
$$

for all $N \in \mathbb{N}$ and k in *Case 1*,

$$
|D^\gamma m_k(\xi)| \lesssim 2^{k(d(b-1-\frac{aN}{d})-|\gamma|)}
$$

for all $N \in \mathbb{N}$ and k in *Case 2* and finally

$$
\begin{aligned}
|D^\gamma m_k(\xi)| &\lesssim 2^{k(d(b-1)-|\gamma|)} (2^{ka})^{-d/2} \\
&= 2^{k(d(b-1-\frac{a}{2})-|\gamma|)} \\
&\lesssim |\xi|^{\frac{d(b-1-a/2)-|\gamma|}{a+1}} \\
&= |\xi|^{-d\beta+|\gamma|(\alpha-1)},
\end{aligned}
$$

for all k in *Case 3*. Again, by following an identical argument as before we obtain for large N

$$
\begin{aligned}
\sum_{k\in I_1} |D^\gamma m_k(\xi)| &\lesssim |\xi|^{\frac{-aN}{2(a+1)}}, \\
\sum_{k\in I_2} |D^\gamma m_k(\xi)| &\lesssim |\xi|^{\frac{-daN}{2(a+1)}}.
\end{aligned}
$$

Thus, choosing N large enough, we can obtain estimates for *Case 1* and *Case 2* such that

$$\sum_{k\in I_1\cup I_2} |D^\gamma m_k(\xi)| \lesssim |\xi|^{-d\beta+\gamma(\alpha-1)}.$$

Again, as there are only a bounded number of k in *Case 3*, summing over these k we obtain

$$\sum_{k\in I_3} |D^\gamma m_k(\xi)| \lesssim |\xi|^{-d\beta+\gamma(\alpha-1)}$$

thus we conclude that

$$|D^\gamma m(\xi)| \lesssim |\xi|^{-d\beta+\gamma(\alpha-1)}, \tag{4.9}$$

for $\xi \neq 0$.

Now we return to the proof of Theorem 4.2.1. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be such that $\eta(\xi) = 1$ on $\{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}$ and $\eta(\xi) = 0$ on $\{\xi \in \mathbb{R}^d : |\xi| \geqslant 2\}$, and define $m_0(\xi) = m(\xi)\eta(\xi)$.

First, consider the multiplier $m(\xi)(1 - \eta(\xi))$. By (4.8), (4.9) with $\gamma = 1$, and the support of η, we have that $m(\xi)(1 - \eta(\xi))$ satisfies the hypotheses of Theorem 3.3.1 so we can conclude Theorem 4.2.1 for this part of the multiplier.

For the rest of the multiplier, m_0, we will use more elementary methods to obtain our desired estimates.

Claim 4.2.3 *We claim that for* $|\xi| \leqslant 2$

$$|D^\gamma m(\xi)| \lesssim 1.$$

Proof: Let $c_0 > 0$ be small enough that when $|\xi| \leqslant c_0$, the above *Case 1* and *Case 3* do not occur.

Case A: ξ is such that $|\xi| < c_0$, then for some $i \in \{1, ..., d\}$

$$\left|\frac{\partial h}{\partial z_i}(z)\right| \gtrsim 2^{ka}.$$

Thus, following the arguments given before, we have that for each $\gamma \in \mathbb{N}_0^d$, for all $N \in \mathbb{N}$

$$|D^\gamma m_k(\xi)| \lesssim 2^{kd(b-1-\frac{aN}{d})-|\gamma|}$$

and so by taking N large enough we have

$$|D^\gamma m(\xi)| \lesssim 1.$$

Case B: ξ is such that $c_0 \leqslant |\xi| \leqslant 2$, then we consider

$$|D^\gamma m(\xi)| \lesssim |\xi|^{-d\beta+\gamma(\alpha-1)}.$$

If $\gamma < \frac{\beta}{\alpha-1}$ then

$$\begin{aligned}|D^\gamma m(\xi)| &\lesssim |\xi|^{-d\beta+\gamma(\alpha-1)}\\ &\leqslant c_0^{-\beta+\gamma(\alpha-1)}\\ &\lesssim 1.\end{aligned}$$

If $\gamma > \frac{\beta}{\alpha-1}$ then

$$\begin{aligned}|D^\gamma m(\xi)| &\lesssim |\xi|^{-d\beta+\gamma(\alpha-1)}\\ &\leqslant 2^{-d\beta+\gamma(\alpha-1)}\\ &\lesssim 1.\end{aligned}$$

$\square$

Next it is a simple observation that bounds on $|D^\gamma m_0(\xi)|$ follows from bounds on $|D^\gamma m(\xi)|$ for $|\xi| \leqslant 2$ as follows

$$
\begin{aligned}
|D^\gamma m_0(\xi)| &= \left| \sum_{\lambda \leqslant \gamma} D^\lambda m(\xi) D^{(\gamma-\lambda)} \eta(\xi) \right| \\
&\leqslant \sum_{\lambda \leqslant \gamma} |D^\lambda m(\xi)| |D^{(\gamma-\lambda)} \eta(\xi)| \\
&\lesssim \sum_{i=0}^{\gamma} 1 \\
&\lesssim 1,
\end{aligned}
$$

where we have used Claim 4.2.3 and the fact that $\eta \in C_c^\infty$ on the third line and the notation of multi-indices throughout.

Now that we have this estimate, we define K_{m_0} by

$$
\widehat{K_{m_0}}(\xi) = m_0(\xi).
$$

As η has compact support in $|\xi| < 2$, m_0 has compact support in $|\xi| < 2$; thus we can use the Fourier inversion formula to obtain

$$
K_{m_0}(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} m_0(\xi) d\xi,
$$

and furthermore, by standard properties of the Fourier transform, we obtain

$$
(ix)^{2d} K_{m_0}(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} D^{(2d)} m_0(\xi) d\xi.
$$

Thus, consider

$$\begin{aligned}
|K_{m_0}(x)| &= \left|\int_{\mathbb{R}^d} e^{ix\cdot\xi} m_0(\xi) d\xi\right| \\
&\leqslant \int_{|\xi|<2} |m_0(\xi)| d\xi \\
&\lesssim \int_{|\xi|<2} d\xi \\
&\lesssim 1,
\end{aligned}$$

and likewise

$$\begin{aligned}
|(ix)^{2d} K_{m_0}(x)| &= \left|\int_{\mathbb{R}^d} e^{ix\cdot\xi} D^{(2d)} m_0(\xi) d\xi\right| \\
&\leqslant \int_{|\xi|<2} |D^{(2d)} m_0(\xi)| d\xi \\
&\lesssim \int_{|\xi|<2} d\xi \\
&\lesssim 1.
\end{aligned}$$

So we can write

$$|x|^{2d} |K_{m_0}(x)| \lesssim 1,$$

and combining with the previous estimate we have

$$(1 + |x|^{2d}) |K_{m_0}(x)| \lesssim 1,$$

finally, rearranging we get

$$|K_{m_0}(x)| \lesssim \frac{1}{1 + |x|^{2d}}.$$

Now, define $\tilde{K}_{m_0}(x) = K_{m_0}(-x)$ and consider

$$\begin{aligned}|\tilde{K}_{m_0}(x)| &= |K_{m_0}(-x)|\\ &\lesssim \frac{1}{1+|x|^{2d}}\\ &\lesssim \sum_{n=1}^{\infty} 2^{-2d(n-1)}\chi_{B_n}(x).\end{aligned}$$

Thus, we have

$$\begin{aligned}|\tilde{K}_{m_0}| * w(x) &= \int_{\mathbb{R}^d} |\tilde{K}_{m_0}(y)|w(x-y)dy\\ &\lesssim \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} 2^{-2d(n-1)}\chi_{B_n}(y)w(x-y)dy\\ &= \sum_{n=1}^{\infty} 2^{-2d(n-1)} \int_{\mathbb{R}^d} \chi_{B_n}(y)w(x-y)dy\\ &= \sum_{n=1}^{\infty} 2^{-2d(n-1)} \int_{|y|<2^n} w(x-y)dy\\ &= \sum_{n=1}^{\infty} 2^{-d(n-2)} \frac{1}{2^{dn}} \int_{|x-z|<2^n} w(z)dz\\ &\leqslant \sum_{n=1}^{\infty} 2^{-d(n-2)} \sup_{r>1} \frac{1}{r^d} \int_{|x-z|<r} w(z)dz,\end{aligned}$$

where again we have used the substitution $z = x - y$. Therefore, since

$$\sum_{n=1}^{\infty} 2^{-d(n-2)} < +\infty$$

we have that

$$|\tilde{K}_{m_0}| * w(x) \lesssim M^{(1)}w(x) \tag{4.10}$$

where $M^{(1)}$ is defined by (4.4).

So we can estimate the final part of our kernel via the same method as before, that is

$$
\begin{aligned}
\int_{\mathbb{R}^d} |K_{m_0} * f(x)|^2 w(x)dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} K_{m_0}(x-y)f(y)dy \right|^2 w(x)dx \\
&\leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_{m_0}(x-y)||f(y)|dy \right)^2 w(x)dx \\
&= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_{m_0}(x-y)|^{1/2}|K_{m_0}(x-y)|^{1/2}|f(y)|dy \right)^2 w(x)dx
\end{aligned}
$$

again, by the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |K_{m_0} * f(x)|^2 w(x)dx \\
&\quad \leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |K_{m_0}(x-y)|dy \right) \left(\int_{\mathbb{R}^d} |f(y)|^2 |K_{m_0}(x-y)|dy \right) w(x)dx \\
&\quad = \|K_{m_0}\|_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)|^2 |K_{m_0}(x-y)| w(x)dydx \\
&\quad = \|K_{m_0}\|_1 \int_{\mathbb{R}^d} |f(y)|^2 \left(\int_{\mathbb{R}^d} |K_{m_0}(x-y)| w(x)dx \right) dy \\
&\quad = \|K_{m_0}\|_1 \int_{\mathbb{R}^d} |f(y)|^2 |\tilde{K}_{m_0}| * w(y)dy.
\end{aligned}
$$

Now, as K_{m_0} is dominated by $\frac{1}{1+|x|^{2d}}$, it is clearly an L^1 function. Additionally, via (4.10), we have

$$
\int_{\mathbb{R}^d} |f(y)|^2 |\tilde{K}_{m_0}| * w(y)dy \lesssim \int_{\mathbb{R}^d} |f(y)|^2 M^{(1)} w(y)dy
$$

and so

$$
\int_{\mathbb{R}^d} |K_{m_0} * f|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^{(1)} w.
$$

Therefore, via (4.5), we have that

$$\int_{\mathbb{R}^d} |K_{m_0} * f|^2 w \lesssim \int_{\mathbb{R}^d} |f|^2 M^2 \mathcal{M}_{\alpha,\beta} M^4 w,$$

as required, concluding our proof of Theorem 4.2.1.

4.3 Beyond the Hirschmann kernels

In this section we will deal with kernels that do not have a singularity, but allow a more general phase function. These are given pointwise as

$$K_{\lambda,\phi}(x) = e^{i\lambda\phi(x)}\psi(x),$$

where $\lambda > 0$, $\psi \in C_0^\infty(\mathbb{R})$ is a positive, smooth cutoff function and $\phi \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a phase function similar to x^ℓ for some $\ell > 1$, specifically $\phi \in C^{(4)}$ and satisfies the conditions

$$A_j |x|^{\ell - j} \leqslant |\phi^{(j)}(x)| \tag{4.11}$$

with $A_j > 0$ for $j = 1, 2$ and

$$|\phi^{(j)}(x)| \leqslant B_j |x|^{\ell - j}, \tag{4.12}$$

with $B_j > 0$, and $j = 1, 2, 3, 4$, on the support of ψ.

Theorem 4.3.1 *Let T be an operator given by $Tf = K_{\lambda,\phi} * f$, where $\lambda > 0$ and $\phi \in C^{(4)}$ such that* (4.11) *and* (4.12) *hold, then*

$$\int_{\mathbb{R}} |Tf|^2 w \lesssim \mu^{2\beta - 2} \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta,\mu} M^4 w,$$

where $\mathcal{M}_{\alpha,\beta,\mu}$ is the maximal operator given by (3.9) *with parameters $\alpha = \frac{\ell}{\ell - 1}$, $\beta = \frac{\ell - 2}{2(\ell - 1)}$ and*

$\mu = c\lambda^{1/\ell}$, *where* $c = A_1 2^{-\ell}$.*

Remark 4.3.2 *We note here the similarity of the phase function to monomials, and therefore the similarity of Theorem 4.3.1 to Theorem 2.1 of [5]. However, Theorem 4.3.1 deals with a greater range of phase functions, specifically monomials with real powers.*

Proof: [Theorem 4.3.1] Let $\tilde{\zeta} \in C_c^\infty(\mathbb{R})$ such that $\tilde{\zeta}$ has compact support in the set $\{x \in \mathbb{R} : \frac{1}{2} < |x| < 2\}$ and

$$\sum_{k\in\mathbb{Z}} \tilde{\zeta}(2^{-k}x) = 1$$

for $x \neq 0$.

Define

$$\zeta_k(x) = \psi(\lambda^{-1/\ell}2^k x)\tilde{\zeta}(x),$$

for each $k \in \mathbb{Z}$ and

$$\begin{aligned} K_{\lambda,\phi,k}(x) &= e^{i\lambda\phi(x)}\zeta_k(\lambda^{1/\ell}2^{-k}x) \\ &= e^{i\lambda\phi(x)}\psi(x)\tilde{\zeta}(\lambda^{1/\ell}2^{-k}x) \end{aligned}$$

for each $k \in \mathbb{N}$. Next, define

$$K_{\lambda,\phi,0}(x) = e^{i\lambda\phi(x)} \sum_{k=-\infty}^{0} \zeta_k(\lambda^{1/\ell}2^{-k}x).$$

*In this section our notation $A \lesssim B$ will have an implicit constant with dependence on quite a few introduced constants, including but not limited to ℓ, A_j for $j = 1, 2$, B_j for $j = 1, 2, 3, 4$ and the L^∞ norm of ψ. However, the implicit constant will never depend on λ, nor on the dyadic decomposition.

Note that the support of $K_{\lambda,\phi,0}$ is a subset of $|x| < 2\lambda^{-1/\ell}$, and we have that

$$K_{\lambda,\phi} = K_{\lambda,\phi,0} + \sum_{k=1}^{\infty} K_{\lambda,\phi,k}.$$

Now, we will begin the proof of Theorem 4.3.1 by estimating the part of the kernel that has little oscillation, $K_{\lambda,\phi,0}$. To this end, define

$$\tilde{K}_{\lambda,\phi,0}(x) = K_{\lambda,\phi,0}(-x)$$

and consider

$$\begin{aligned}
|\tilde{K}_{\lambda,\phi,0}(x)| &= \left| e^{i\lambda\phi(-x)}\psi(-x)\left(\sum_{k=-\infty}^{0} \tilde{\zeta}(-\lambda^{1/\ell}2^{-k}x)\right)\right| \\
&\leqslant \psi(-x)\left(\sum_{k=-\infty}^{0} \tilde{\zeta}(-\lambda^{1/\ell}2^{-k}x)\right) \\
&\leqslant \|\psi\|_\infty \frac{1}{1+(c\lambda^{1/\ell}x)^2} \\
&\leqslant \|\psi\|_\infty \sum_{n=1}^{\infty} 2^{-2(n-1)}\chi_{[-2^n,2^n]}(c\lambda^{1/\ell}x) \\
&= \|\psi\|_\infty \sum_{n=1}^{\infty} 2^{-2(n-1)}\chi_{[-c^{-1}\lambda^{-1/\ell}2^n,c^{-1}\lambda^{-1/\ell}2^n]}(x).
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
|\tilde{K}_{\lambda,\phi,0}| * w(x) &\leqslant \|\psi\|_\infty \int_{\mathbb{R}} \sum_{n=1}^{\infty} 2^{-2(n-1)} \chi_{[-c^{-1}\lambda^{-1/\ell}2^n, c^{-1}\lambda^{-1/\ell}2^n]}(y) w(x-y)dy \\
&= \|\psi\|_\infty \sum_{n=1}^{\infty} 2^{-2(n-1)} \int_{-c^{-1}\lambda^{-1/\ell}2^n}^{c^{-1}\lambda^{-1/\ell}2^n} w(x-y)dy \\
&= 2c^{-1}\lambda^{-1/\ell}\|\psi\|_\infty \sum_{n=1}^{\infty} 2^{-(n-2)} \frac{1}{2^n c^{-1}\lambda^{-1/\ell}} \int_{x-c^{-1}\lambda^{-1/\ell}2^n}^{x+c^{-1}\lambda^{-1/\ell}2^n} w(z)dz \\
&\leqslant 2c^{-1}\lambda^{-1/\ell}\|\psi\|_\infty \sum_{n=1}^{\infty} 2^{-(n-2)} \sup_{r>c^{-1}\lambda^{-1/\ell}} \frac{1}{2r} \int_{x-r}^{x+r} w(z)dz,
\end{aligned}
$$

where we have used the substitution $z = x - y$ on the third line and as

$$\sum_{n=1}^{\infty} 2^{-(n-2)} < +\infty,$$

we have

$$|\tilde{K}_{\lambda,\phi,0}| * w(x) \lesssim c^{-1}\lambda^{-1/\ell} \sup_{r>c^{-1}\lambda^{-1/\ell}} \frac{1}{2r} \int_{x-r}^{x+r} w(z)dz.$$

Finally, if we define

$$M^{(\lambda)} f(x) = \sup_{r>c^{-1}\lambda^{-1/\ell}} \frac{1}{2r} \int_{x-r}^{x+r} f(y)dy,$$

then we can conclude that

$$|\tilde{K}_{\lambda,\phi,0}| * w(x) \lesssim c^{-1}\lambda^{-1/\ell} M^{(\lambda)} w(x). \tag{4.13}$$

Now, consider

$$\begin{aligned}\int_{\mathbb{R}}|K_{\lambda,\phi,0} * f(x)|^2 w(x)dx &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{\lambda,\phi,0}(x-y)f(y)dy\right|^2 w(x)dx\\ &\leqslant \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|K_{\lambda,\phi,0}(x-y)||f(y)|dy\right)^2 w(x)dx\\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|K_{\lambda,\phi,0}(x-y)|^{1/2}|K_{\lambda,\phi,0}(x-y)|^{1/2}|f(y)|dy\right)^2 w(x)dx\end{aligned}$$

by the Cauchy-Schwarz inequality we have

$$\begin{aligned}&\int_{\mathbb{R}}|K_{\lambda,\phi,0} * f(x)|^2 w(x)dx\\ &\quad\leqslant \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|K_{\lambda,\phi,0}(x-y)|dy\right)\left(\int_{\mathbb{R}}|f(y)|^2|K_{\lambda,\phi,0}(x-y)|dy\right)w(x)dx\\ &\quad= \|K_{\lambda,\phi,0}\|_1\int_{\mathbb{R}}\int_{\mathbb{R}}|f(y)|^2|K_{\lambda,\phi,0}(x-y)|w(x)dydx\\ &\quad= \|K_{\lambda,\phi,0}\|_1\int_{\mathbb{R}}|f(y)|^2\left(\int_{\mathbb{R}}|K_{\lambda,\phi,0}(x-y)|w(x)dx\right)dy\\ &\quad= \|K_{\lambda,\phi,0}\|_1\int_{\mathbb{R}}|f(y)|^2|\tilde{K}_{\lambda,\phi,0}| * w(y)dy.\end{aligned} \tag{4.14}$$

Note that it makes sense to talk about the L^1 norm of K_{λ,ϕ,k_0}, as it is smooth and has finite support.

Now, to calculate $\|K_{\lambda,\phi,0}\|_1$, we have

$$\begin{aligned}\|K_{\lambda,\phi,0}\|_1 &= \int_{\mathbb{R}}|K_{\lambda,\phi,0}(x)|dx\\ &= \int_{\mathbb{R}}\left|e^{i\lambda\phi(x)}\psi(x)\left(\sum_{k=-\infty}^{0}\zeta(\lambda^{1/\ell}2^{-k}x)\right)\right|dx\\ &\leqslant \|\psi\|_\infty\int_{\mathbb{R}}\chi_{[-2\lambda^{-1/\ell},2\lambda^{-1/\ell}]}(x)dx\\ &\lesssim \lambda^{-1/\ell}.\end{aligned} \tag{4.15}$$

Then, we use (4.13) to obtain

$$\int_{\mathbb{R}} |K_{\lambda,\phi,0} * f(x)|^2 w(x)dx \lesssim \lambda^{-2/\ell} \int_{\mathbb{R}} |f(x)|^2 M^{(\lambda)} w(x)dx.$$

Now, we claim that if $|x - y| < c^{-1}\lambda^{-1/\ell}$ then

$$M^{(\lambda)} w(x) \lesssim M^{(\lambda)} w(y), \tag{4.16}$$

where the implicit constant is just an absolute constant.

Indeed, we have that

$$\begin{aligned}
M^{(\lambda)} w(x) &= \sup_{r > c^{-1}\lambda^{-1/\ell}} \frac{1}{2r} \int_{x-r}^{x+r} w \\
&\leqslant 2 \sup_{r > c^{-1}\lambda^{-1/\ell}} \frac{1}{4r} \int_{x-2r}^{x+2r} w \\
&\leqslant 3 \sup_{r > c^{-1}\lambda^{-1/\ell}} \frac{1}{6r} \int_{y-3r}^{y+3r} w \\
&= 3 \sup_{r' > 3c^{-1}\lambda^{-1/\ell}} \frac{1}{2r'} \int_{y-r'}^{y+r'} w \\
&\leqslant 3 \sup_{r' > c^{-1}\lambda^{-1/\ell}} \frac{1}{2r'} \int_{y-r'}^{y-r'} w \\
&= 3M^{(\lambda)} w(y),
\end{aligned}$$

where we have used the substitution $r' = 3r$. Next, if we define

$$A^{(\lambda)} f(x) = \frac{1}{2c^{-1}\lambda^{-1/\ell}} \int_{|x-y| < c^{-1}\lambda^{-1/\ell}} f(y)dy,$$

then

$$
\begin{aligned}
A^{(\lambda)}M^{(\lambda)}w(x) &= \frac{1}{2c^{-1}\lambda^{-1/\ell}}\int_{|x-y|<c^{-1}\lambda^{-1/\ell}} M^{(\lambda)}w(y)dy \\
&\geqslant \frac{1}{6c^{-1}\lambda^{-1/\ell}}\int_{|x-y|<c^{-1}\lambda^{-1/\ell}} M^{(\lambda)}w(x)dy \\
&= \frac{1}{3}M^{(\lambda)}w(x),
\end{aligned}
$$

where we used (4.16) on the second line. On the other hand, if we consider our maximal operator $\mathcal{M}_{\alpha,\beta,\mu}$ given by

$$
\mathcal{M}_{\alpha,\beta,\mu}w(x) = \sup_{(y,r)\in\Gamma_{\alpha,\mu}(x)} \frac{r^{2\beta}}{r}\int_{y-r}^{y+r} w,
$$

where

$$
\Gamma_{\alpha,\mu}(x) = \{(y,r) : 0 < r^{\alpha} \leqslant \mu^{-\alpha}, |x-y| \leqslant \mu^{-\alpha}r^{1-\alpha}\}
$$

and use the substitutions $\mu = c\lambda^{1/\ell}$, $\alpha = \frac{\ell}{\ell-1}$ and $\beta = \frac{\ell-2}{2(\ell-1)}$, then we can define another maximal function as

$$
\mathfrak{M}_{\ell,\lambda}w(x) = \sup_{(y,r)\in\tilde{\Gamma}_{\ell,\lambda}(x)} \frac{1}{r^{1/(\ell-1)}}\int_{y-r}^{y+r} w,
$$

where

$$
\tilde{\Gamma}_{\ell,\lambda}(x) = \{(y,r) : 0 < r \leqslant c^{-1}\lambda^{-1/\ell}, |x-y| \leqslant (c^{\ell}r\lambda)^{-1/(\ell-1)}\}.
$$

So now, if we fix $(y,r) = (x, c^{-1}\lambda^{-1/\ell})$, then we can see that

$$
Af(x) \leqslant 2c^{\frac{\ell-2}{\ell-1}}\lambda^{\frac{\ell-2}{\ell(\ell-1)}}\mathfrak{M}_{\ell,\lambda}f(x),
$$

and substituting back using $\mu = c\lambda^{1/\ell}$, $\alpha = \frac{\ell}{\ell-1}$ and $\beta = \frac{\ell-2}{2(\ell-1)}$, we conclude

$$Af(x) \leqslant 2\mu^{2\beta}\mathcal{M}_{\alpha,\beta,\mu}f(x).$$

Now, with the additional observations that

$$\begin{aligned} M^{(\lambda)}w(x) &= \sup_{r>\lambda^{-1/\ell}} \frac{1}{2r}\int_{x-r}^{x+r} w \\ &\leqslant \sup_{r>0} \frac{1}{2r}\int_{x-r}^{x+r} w \\ &= Mw(x), \end{aligned}$$

where M denotes the standard Hardy-Littlewood maximal operator, and

$$\mathcal{M}_{\alpha,\beta,\mu}Mw(x) \leqslant M^6\mathcal{M}_{\alpha,\beta,\mu}M^4w(x),$$

we can conclude our treatment of this part of the kernel; that is, we have shown that

$$\begin{aligned} M^{(\lambda)}w(x) &\leqslant 3A^{(\lambda)}M^{(\lambda)}w(x) \\ &\leqslant 6\lambda^{\frac{\ell-2}{\ell(\ell-1)}}\mathfrak{M}_{\ell,\lambda}M^{(\lambda)}w(x) \\ &\leqslant 6\mu^{2\beta}\mathcal{M}_{\alpha,\beta,\mu}Mw(x) \\ &\leqslant 6\mu^{2\beta}M^6\mathcal{M}_{\alpha,\beta,\mu}M^4w(x). \end{aligned} \tag{4.17}$$

Thus, by combining this with (4.14) and (4.15), we can conclude that

$$\int_{\mathbb{R}} |K_{\lambda,\phi,0} * f|^2 w \lesssim \mu^{2\beta-2}\int_{\mathbb{R}} |f|^2 M^6\mathcal{M}_{\alpha,\beta,\mu}M^4 w.$$

Now, for the rest of our kernel, we define $m_k(\xi) = \widehat{K_{\lambda,\phi,k}}(\xi)$ and

$$m(\xi) = \left(\sum_{k=0}^{\infty} K_{\lambda,\phi,k}\right)^{\widehat{}}(\xi),$$

thus again we have

$$m(\xi) = \sum_{k=0}^{\infty} m_k(\xi).$$

Note here that m is not the Fourier transform of our entire kernel, $K_{\lambda,\phi}$, but instead the part supported away from the origin. To control this part of the kernel, we will need the following 3 lemmas.

Lemma 4.3.3 *For* $|\xi| \geqslant \frac{1}{2}c\lambda^{1/\ell}$,

$$|m(\xi)| \lesssim \lambda^{-1/\ell}\left(\lambda^{-1/\ell}|\xi|\right)^{-\frac{\ell-2}{2(\ell-1)}},$$

where the implicit constant does not depend upon λ.

Proof: Consider

$$m_k(\xi) = \int_{\mathbb{R}} e^{i(\lambda\phi(x)-x\xi)}\zeta(\lambda^{1/\ell}2^{-k}x)dx$$

and let $z = \lambda^{1/\ell}2^{-k}x$, so $x = \lambda^{-1/\ell}2^k z$, thus

$$m_k(\xi) = \lambda^{-1/\ell}2^k \int_{\mathbb{R}} e^{i(\lambda\phi(\lambda^{-1/\ell}2^k z)-\lambda^{-1/\ell}2^k z\xi)}\zeta(z)dz.$$

Let

$$h_k(z) = \lambda\phi(\lambda^{-1/\ell}2^k z) - \lambda^{-1/\ell}2^k z\xi,$$

so

$$m_k(\xi) = \lambda^{-1/\ell} 2^k \int_{\frac{1}{2}<|z|<2} e^{ih(z)} \zeta(z) dz.$$

Case 1: $k \in U_1 = \{k \geqslant 1 : 2^k \leqslant (c_1 \lambda^{-1/\ell} |\xi|)^{1/\ell - 1}\}$.

It follows that $2^{k\ell} \leqslant c_1 2^k \lambda^{-1/\ell} |\xi|$ and we have that

$$h_k'(z) = \lambda \lambda^{-1/\ell} 2^k \phi'(\lambda^{-1/\ell} 2^k z) - \lambda^{-1/\ell} 2^k \xi.$$

So we have

$$\begin{aligned}
|h_k'(z)| &\geqslant \lambda^{-1/\ell} 2^k |\xi| - \lambda \lambda^{-1/\ell} 2^k |\phi'(\lambda^{-1/\ell} 2^k z)| \\
&\geqslant \lambda^{-1/\ell} 2^k |\xi| - \lambda \lambda^{-1/\ell} 2^k B_1 (\lambda^{-1/\ell} 2^k |z|)^{\ell - 1} \\
&\geqslant \lambda^{-1/\ell} 2^k |\xi| - B_1 2^{\ell-1} 2^{k\ell} \\
&\geqslant \lambda^{-1/\ell} 2^k |\xi| (1 - c_1 B_1 2^{\ell - 1}),
\end{aligned}$$

thus, by choosing $c_1 = (B_1 2^\ell)^{-1}$, we obtain

$$|h_k'(z)| \geqslant \frac{1}{2} \lambda^{-1/\ell} 2^k |\xi|. \tag{4.18}$$

Next, consider

$$h_k''(z) = \lambda \lambda^{-2/\ell} 2^{2k} \phi''(\lambda^{-1/\ell} 2^k z),$$

and so,

$$
\begin{aligned}
|h_k''(z)| &= \lambda\lambda^{-2/\ell}2^{2k}|\phi''(\lambda^{-1/\ell}2^k z)| \\
&\leqslant \lambda\lambda^{-2/\ell}2^{2k}B_2(\lambda^{-1/\ell}2^k|z|)^{\ell-2} \\
&\leqslant 2^{k\ell}B_2 2^{|\ell-2|} \\
&\lesssim 2^{k\ell}, \qquad (4.19)
\end{aligned}
$$

where we have used the fact that $\frac{1}{2} \leqslant |z| \leqslant 2$ in the 3rd line.

Likewise,

$$
h_k'''(z) = \lambda\lambda^{-3/\ell}2^{3k}\phi'''(\lambda^{-1/\ell}2^k z),
$$

and

$$
\begin{aligned}
|h_k'''(z)| &= \lambda\lambda^{-3/\ell}2^{3k}|\phi'''(\lambda^{-1/\ell}2^k z)| \\
&\leqslant \lambda\lambda^{-3/\ell}2^{3k}B_2(\lambda^{-1/\ell}2^k|z|)^{\ell-3} \\
&\leqslant 2^{k\ell}B_3 2^{|\ell-3|} \\
&\lesssim 2^{k\ell}, \qquad (4.20)
\end{aligned}
$$

where we have used the fact that $\frac{1}{2} \leqslant |z| \leqslant 2$ in the 3rd line.

So by Lemma 4.1.1, with $M = 2$, we have

$$
\begin{aligned}
\left|\int_{\frac{1}{2}<|z|<2} e^{ih_k(z)}\zeta(z)dz\right| &\lesssim \int_{\frac{1}{2}<|z|<2} \sum_{r=0}^{1} 2^{k\ell r}(\lambda^{-1/\ell}2^k|\xi|)^{-1-r}dz \\
&\lesssim \sum_{r=0}^{1} 2^{k\ell r}(\lambda^{-1/\ell}2^k|\xi|)^{-1-r}.
\end{aligned}
$$

Therefore we can now estimate our multiplier as

$$\begin{aligned}|m_k(\xi)| &\lesssim 2^k\lambda^{-1/\ell}\left(\sum_{r=0}^{1} 2^{k\ell r}(\lambda^{-1/\ell}2^k|\xi|)^{-1-r}\right)\\ &\lesssim 2^k\lambda^{-1/\ell}\left(\sum_{r=0}^{1}(\lambda^{-1/\ell}2^k|\xi|)^{-1}\right),\end{aligned}$$

as $2^k \leqslant (c_1\lambda^{-1/\ell}|\xi|)^{1/\ell-1}$ if and only if $2^{k\ell} \leqslant c_1 2^k\lambda^{-1/\ell}|\xi|$ and so

$$\begin{aligned}|m_k(\xi)| &\lesssim \lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-1}\\ &= \lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell}{2(\ell-1)}}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}}\\ &\lesssim \lambda^{-1/\ell}2^{-\frac{k\ell}{2}}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}},\end{aligned}$$

again using the fact that $2^{k\ell} \leqslant c_1 2^k\lambda^{-1/\ell}|\xi|$. Thus, summing over all $k \in U_1$, we have

$$\sum_{k\in U_1}|m_k(\xi)| \lesssim \lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}}$$

where the implicit constant depends only on c_1, ψ and B_1 for $i = 0, 1$ and the fact that

$$\sum_{k\in\mathbb{N}} 2^{-\frac{k\ell}{2}} < +\infty.$$

Case 2: $k \in U_2 = \{k \geqslant 1 : 2^k \geqslant (c_2\lambda^{-1/\ell}|\xi|)^{1/(\ell-1)}\}$.

Again, it follows that $c_2^{-1}2^{k\ell} \geqslant \lambda^{-1/\ell}2^k|\xi|$ and we have that

$$h_k'(z) = \lambda\lambda^{-1/\ell}2^k\phi'(\lambda^{-1/\ell}2^k z) - \lambda^{-1/\ell}2^k\xi.$$

Thus

$$\begin{aligned}
|h_k'(z)| &\geqslant \lambda\lambda^{-1/\ell}2^k|\phi'(\lambda^{-1/\ell}2^k z)| - \lambda^{-1/\ell}2^k|\xi| \\
&\geqslant \lambda\lambda^{-1/\ell}2^k A_1|\lambda^{-1/\ell}2^k z|^{\ell-1} - \lambda^{-1/\ell}2^k|\xi| \\
&\geqslant 2^{k\ell}A_1 2^{-(\ell-1)} - c_2^{-1}2^{k\ell} \\
&= 2^{k\ell}(A_1 2^{-(\ell-1)} - c_2^{-1}),
\end{aligned}$$

and choosing $c_2 = 2^\ell A_1^{-1}$, we have

$$|h_k'(z)| \geqslant c_2^{-1}2^{k\ell}. \tag{4.21}$$

Again, we have the estimate on $|h_k''(z)|$ given by (4.19), so again we can apply Lemma 4.1.1 with $M = 2$, so we can conclude

$$\begin{aligned}
|m_k(\xi)| &\lesssim \lambda^{-1/\ell}2^k\left(\sum_{r=0}^{1} 2^{k\ell r}(2^{k\ell})^{-1-r}\right) \\
&= \lambda^{-1/\ell}2^k\left(\sum_{r=0}^{1} 2^{-k\ell}\right) \\
&\lesssim \lambda^{-1/\ell}2^k 2^{-k\ell} \\
&= \lambda^{-1/\ell}2^{k(1-\frac{\ell}{2})}2^{-k\ell/2} \\
&\lesssim \lambda^{-1/\ell}2^{k(1-\frac{\ell}{2})}(2^k\lambda^{-1/\ell}|\xi|)^{-1/2} \\
&= 2^{-\frac{k}{2}(\ell-1)}\lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}}(\lambda^{-1/\ell}|\xi|)^{-\frac{1}{2(\ell-1)}}.
\end{aligned}$$

As $|\xi| \geqslant \frac{1}{2}c\lambda^{1/\ell}$ we have

$$(\lambda^{-1/\ell}|\xi|)^{-\frac{1}{2(\ell-1)}} \leqslant \left(\frac{c}{2}\right)^{-\frac{1}{2(\ell-1)}}$$

and since $\ell > 1$ we have

$$\sum_{k=1}^{\infty} 2^{-\frac{k}{2}(\ell-1)} < +\infty.$$

So summing over $k \in U_2$ we have

$$\sum_{k\in U_2} |m_k(\xi)| \lesssim \lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}}.$$

Case 3: $k \in U_3 = \{k \geqslant 1 : (c_1\lambda^{-1/\ell}|\xi|)^{1/\ell-1} < 2^k < (c_2\lambda^{-1/\ell}|\xi|)^{1/\ell-1}\}$

Now we consider

$$h_k''(z) = \lambda\lambda^{-2/\ell}2^{2k}\phi''(\lambda^{-1/\ell}2^k z),$$

and observe that

$$\begin{aligned} |h_k''(z)| &\geqslant \lambda\lambda^{-2/\ell}2^{2k}\phi''(\lambda^{-1/\ell}2^k z) \\ &\geqslant \lambda\lambda^{-2/\ell}2^{2k}A_2|\lambda^{-1/\ell}2^k z|^{\ell-2} \\ &\geqslant |z|^{\ell-2}2^{k\ell} \\ &\gtrsim 2^{k\ell}, \end{aligned}$$

where in the last step we used the fact that $\frac{1}{2} < |z| < 2$. So, by Lemma 4.1.4 with parameter 2 we can deduce that

$$\begin{aligned} |m_k(\xi)| &\lesssim \lambda^{-1/\ell}2^k(2^{k\ell})^{-1/2} \\ &= \lambda^{-1/\ell}2^{-\frac{k}{2}(\ell-2)} \\ &\lesssim \lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}}. \end{aligned}$$

Note here that as $\ell - 2$ may be negative that we have used either the upper or lower bounds in the definition of U_3 to obtain the last line, depending on the value of ℓ.

Finally, as there only a bounded number of $k \in U_3$, independent of λ, we can conclude that

$$\sum_{k \in U_3} |m_k(\xi)| \lesssim \lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}}.$$

By combining all three cases, we can therefore conclude that

$$\sum_{k \in \mathbb{N}} |m_k(\xi)| \lesssim \lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}},$$

and so we have

$$|m(\xi)| \lesssim \lambda^{-1/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-2}{2(\ell-1)}},$$

where the implicit constant depends on at most ψ, A_j and B_j for $j = 1, 2$, and ℓ; concluding the proof of Lemma 4.3.3. □

Lemma 4.3.4 *For* $|\xi| \geqslant \frac{1}{2}c\lambda^{1/\ell}$,

$$|m'(\xi)| \lesssim \lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}},^{*}$$

where the implicit constant does not depend upon λ.

Proof: This proof will follow the proof of Lemma 4.3.3 very closely. Consider

$$m_k'(\xi) \leqslant \int_{\mathbb{R}} (ix)e^{i(\lambda\phi(x)-x\xi)}\zeta(\lambda^{1/\ell}2^{-k}x)dx$$

*The exponent that appears here may seem strange, but using our substitutions for α and β it is equal to $-\beta+\alpha-1$, which is exactly what we would expect.

and let $z = \lambda^{1/\ell} 2^k x$, so $x = \lambda^{-1/\ell} 2^k z$, thus

$$m_k'(\xi) = i\lambda^{-2/\ell} 2^{2k} \int_{\mathbb{R}} e^{i(\lambda\phi(\lambda^{-1/\ell}2^k z) - \lambda^{-1/\ell}2^k z\xi)} z\zeta(z)dz,$$

and again setting

$$h_k(z) = \lambda\phi(\lambda^{-1/\ell}2^k z) - \lambda^{-1/\ell}2^k z\xi,$$

we have

$$m_k'(\xi) = i\lambda^{-2/\ell} 2^{2k} \int_{\frac{1}{2}<|z|<2} e^{ih(z)} z\zeta(z)dz.$$

As $h_k(z)$ is identical to that in Lemma 4.3.3, we can use the exact same estimates on $h_k(z)$, provided we have k from the same sets.

Case 1: $k \in U_1$.

So by (4.18), (4.19) and (4.20) we have that

$$\begin{aligned}
|h_k'(z)| &\gtrsim 2^k \lambda^{-1/\ell}|\xi|, \\
|h_k''(z)| &\lesssim 2^{k\ell}, \\
|h_k'''(z)| &\lesssim 2^{k\ell}.
\end{aligned}$$

So we can immediately use Lemma 4.1.1 with $M = 3$ and get

$$\begin{aligned}
\left| \int_{\frac{1}{2}<|z|<2} e^{ih(z)} z\zeta(z)dz \right| &\lesssim \int_{\frac{1}{2}<|z|<2} \sum_{r=0}^{2} 2^{k\ell r}(2^k\lambda^{-1/\ell}|\xi|)^{-r-2} dz \\
&\lesssim \sum_{r=0}^{2} 2^{k\ell r}(2^k\lambda^{-1/\ell}|\xi|)^{-r-2},
\end{aligned}$$

thus we can estimate $|m_k'(\xi)|$ as

$$
\begin{aligned}
|m_k'(\xi)| &\lesssim 2^{2k}\lambda^{-2/\ell}\sum_{r=0}^{2} 2^{k\ell r}(2^k\lambda^{-1/\ell}|\xi|)^{-r-2}\\
&\lesssim 2^{2k}\lambda^{-2/\ell}(2^k\lambda^{-1/\ell}|\xi|)^{-2}\\
&= \lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-2}\\
&= \lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{3\ell}{2(\ell-1)}}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}}\\
&\lesssim 2^{-3k\ell/2}\lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}},
\end{aligned}
$$

and therefore conclude that

$$
\sum_{k\in U_1}|m_k'(\xi)| \lesssim \lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}}.
$$

Case 2: $k \in U_2$. Again, we have (4.21), (4.19) and (4.20); that is,

$$
\begin{aligned}
|h_k'(z)| &\geqslant 2^{k\ell},\\
|h_k''(z)| &\lesssim 2^{k\ell},\\
|h_k'''(z)| &\lesssim 2^{k\ell},
\end{aligned}
$$

and using the same argument as in *Case 1*, we have that

$$
\begin{aligned}
|m_k'(\xi)| &\lesssim 2^{2k}\lambda^{-2/\ell}\sum_{r=0}^{2} 2^{k\ell r}(2^{k\ell})^{-r-2}\\
&= 2^{k(2-\ell)}\lambda^{-2/\ell}2^{-k\ell}\\
&\lesssim 2^{k(2-\ell)}\lambda^{-2/\ell}(2^k\lambda^{-1/\ell}|\xi|)^{-1}\\
&= 2^{-k(\ell-1)}\lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell+2}{2(\ell-1)}},
\end{aligned}
$$

again observing that $|\xi| \geqslant \frac{1}{2}c\lambda^{1/\ell}$ we have

$$(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell+2}{2(\ell-1)}} \leqslant \left(\frac{c}{2}\right)^{-\frac{\ell+2}{2(\ell-1)}},$$

and can therefore conclude that

$$\sum_{k\in U_2} |m_k'(\xi)| \lesssim \lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}}.$$

Case 3: $k \in U_3$. Again we have the estimate on $|h_k''(z)|$ from *Case 3* of Lemma 4.3.3, that is

$$|h_k''(z)| \gtrsim 2^{k\ell}.$$

Thus, by Lemma 4.1.4 with parameter 2 we have that

$$\begin{aligned}|m_k'(\xi)| &\lesssim \lambda^{-2/\ell}2^{2k}(2^{k\ell})^{-1/2}\\ &= \lambda^{-2/\ell}2^{-\frac{k}{2}(\ell-4)}\\ &\lesssim \lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}}.\end{aligned}$$

Finally, again there are only a bounded number of $k \in U_3$, so we conclude that

$$\sum_{k\in U_3} |m_k'(\xi)| \lesssim \lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}},$$

and so, combining all three cases we have

$$|m'(\xi)| \lesssim \lambda^{-2/\ell}(\lambda^{-1/\ell}|\xi|)^{-\frac{\ell-4}{2(\ell-1)}},$$

where the implicit constant depends on at most an absolute constant, E_i, C_i, for $i = 0, 1$, A_1, A_2, B_j for $j = 1, 2, 3$, and ℓ; concluding the proof of Lemma 4.3.4. $\square$

Lemma 4.3.5 *Let $\gamma \in \{0, 1, 2\}$. Then for $|\xi| \leqslant c\lambda^{1/\ell}$*

$$|m^{(\gamma)}(\xi)| \lesssim \lambda^{-(\gamma+1)/\ell},$$

where the implicit constant does not depend upon λ.

Proof: *Case 1: $\gamma = 0$.*

Consider

$$m_k(\xi) = \int_{\mathbb{R}} e^{i(\lambda\phi(x) - x\xi)} \zeta(\lambda^{1/\ell} 2^{-k} x) dx$$

and let $z = \lambda^{1/\ell} 2^{-k} x$, so $x = \lambda^{-1/\ell} 2^k z$, thus

$$m_k(\xi) = \lambda^{-1/\ell} 2^k \int_{\mathbb{R}} e^{i(\lambda\phi(\lambda^{-1/\ell} 2^k z) - \lambda^{-1/\ell} 2^k z\xi)} \zeta(z) dz.$$

Let

$$h_k(z) = \lambda\phi(\lambda^{-1/\ell} 2^k z) - \lambda^{-1/\ell} 2^k z\xi,$$

so

$$m_k(\xi) = \lambda^{-1/\ell} 2^k \int_{\frac{1}{2} < |z| < 2} e^{ih(z)} \zeta(z) dz,$$

and

$$h_k'(z) = \lambda\lambda^{-1/\ell} 2^k \phi'(\lambda^{-1/\ell} 2^k z) - \lambda^{-1/\ell} 2^k \xi.$$

As $|\xi| \leqslant c\lambda^{1/\ell}$, we have

$$c^{-1}\lambda^{-1/\ell}|\xi| \leqslant 1$$

and so

$$\begin{aligned}(c^{-1}\lambda^{-1/\ell}|\xi|)^{\frac{1}{\ell-1}} &\leqslant 1\\ &\leqslant 2^k\end{aligned}$$

for all $k \in \mathbb{N}$; that is, $c2^{k\ell} \geqslant \lambda^{-1/\ell}2^k|\xi|$ for all $k \in \mathbb{N}$. Thus, by the hypotheses on ϕ, we have

$$\begin{aligned}|h_k'(z)| &\geqslant \lambda\lambda^{-1/\ell}2^k|\phi'(\lambda^{-1/\ell}2^kz)| - \lambda^{-1/\ell}2^k|\xi|\\ &\geqslant \lambda\lambda^{-1/\ell}2^kA_1|\lambda^{-1/\ell}2^kz|^{\ell-1} - \lambda^{-1/\ell}2^k|\xi|\\ &\geqslant 2^{k\ell}A_12^{-(\ell-1)} - c2^{k\ell}\\ &\geqslant 2^{k\ell}(A_12^{-(\ell-1)} - c)\end{aligned}$$

and since $c = A_12^{-\ell}$ we obtain

$$\begin{aligned}|h_k'(z)| &\geqslant c2^{k\ell}\\ &\gtrsim 2^{k\ell}.\end{aligned}$$

So this case is identical to *Case 2* in Lemma 4.3.3, with the exception of c instead of c_2^{-1}, thus by following the exact same argument as we do there we obtain

$$\begin{aligned}|m_k(\xi)| &\lesssim \lambda^{-1/\ell}2^k2^{-k\ell}\\ &= \lambda^{-1/\ell}2^{-k(\ell-1)}.\end{aligned}$$

Thus, by summing over all $k \in \mathbb{N}$, we have

$$\sum_{k\in\mathbb{N}} |m_k(\xi)| \lesssim \lambda^{-1/\ell},$$

where the implicit constant depends only on an absolute constant, A_1, B_j, E_i, C_i for $i = 0, 1, \ell$ and the fact that since $\ell > 1$,

$$\sum_{k\in\mathbb{N}} 2^{-k(\ell-1)} < +\infty.$$

Case 2: $\gamma = 1$

In this case

$$m_k'(\xi) = \int_{\mathbb{R}} (ix) e^{i(\lambda\phi(x) - x\xi)} \zeta(\lambda^{1/\ell} 2^{-k} x) dx.$$

Again, let $z = \lambda^{1/\ell} 2^k x$, so $x = \lambda^{-1/\ell} 2^k z$, thus

$$m_k'(\xi) = i\lambda^{-2/\ell} 2^{2k} \int_{\mathbb{R}} e^{i(\lambda\phi(\lambda^{-1/\ell} 2^k z) - \lambda^{-1/\ell} 2^k z\xi)} z\zeta(z) dz,$$

and again setting

$$h_k(z) = \lambda\phi(\lambda^{-1/\ell} 2^k z) - \lambda^{-1/\ell} 2^k z\xi,$$

we have

$$m_k'(\xi) = i\lambda^{-2/\ell} 2^{2k} \int_{\frac{1}{2}<|z|<2} e^{ih_k(z)} z\zeta(z) dz.$$

Since $h_k(z)$ is identical to *Case 1* we can use the same estimate; that is,

$$|h'_k(z)| \gtrsim 2^{k\ell}.$$

Now, this case is identical to *Case 2* of Lemma 4.3.4, with the exception of c instead of c_2^{-1}, thus by again following the exact same argument as we do there we obtain

$$\begin{aligned} |m'_k(\xi)| &\lesssim 2^{2k}\lambda^{-2/\ell}\sum_{r=0}^{2} 2^{k\ell r}(2^{k\ell})^{-r-2} \\ &= \lambda^{-2/\ell}2^{-2k(\ell-1)}. \end{aligned}$$

Again, by summing over all $k \in \mathbb{N}$, we have

$$\sum_{k\in\mathbb{N}} |m_k(\xi)| \lesssim \lambda^{-2/\ell},$$

where the implicit constant depends only on an absolute constant, A_1, B_j, for $j = 2, 3$, E_i, C_i for $i = 0, 1, 2$, ℓ and the fact that since $\ell > 1$,

$$\sum_{k\in\mathbb{N}} 2^{-2k(\ell-1)} < +\infty.$$

Case 3: $\gamma = 2$. In this case

$$m''_k(\xi) = \int_{\mathbb{R}} (ix)^2 e^{i(\lambda\phi(x)-x\xi)}\zeta(\lambda^{1/\ell}2^{-k}x)dx.$$

Again, let $z = \lambda^{1/\ell}2^k x$, so $x = \lambda^{-1/\ell}2^k z$, thus

$$m''_k(\xi) = -\lambda^{-3/\ell}2^{3k}\int_{\mathbb{R}} e^{i(\lambda\phi(\lambda^{-1/\ell}2^k z)-\lambda^{-1/\ell}2^k z^2\xi)} z\zeta(z)dz,$$

and again setting

$$h_k(z) = \lambda\phi(\lambda^{-1/\ell}2^k z) - \lambda^{-1/\ell}2^k z\xi,$$

we have

$$m_k'(\xi) = -\lambda^{-3/\ell}2^{3k}\int_{\frac{1}{2}<|z|<2} e^{ih_k(z)}z^2\zeta(z)dz.$$

Since $h_k(z)$ is identical to *Case 1* we can use the same estimate; that is,

$$|h_k'(z)| \gtrsim 2^{k\ell}.$$

Additionally, since $h_k(z)$ is identical to *Case 1* of Lemma 4.3.3, we have (4.19) and (4.20); that is,

$$|h_k''(z)| \lesssim 2^{k\ell}$$

and

$$|h_k'''(z)| \lesssim 2^{k\ell}.$$

Finally, consider

$$h_k^{(4)}(z) = \lambda\lambda^{-4/\ell}2^{4k}\phi^{(4)}(\lambda^{-1/\ell}2^k z),$$

then

$$\begin{aligned}
|h_k^{(4)}(z)| &\leqslant \lambda\lambda^{-4/\ell}2^{4k}B_4|\lambda^{-1/\ell}2^k z|^{\ell-4} \\
&= B_4 2^{k\ell} \\
&\leqslant B_4 2^{k\ell}2^{|\ell-4|} \\
&\lesssim 2^{k\ell}, \qquad (4.22)
\end{aligned}$$

where on the penultimate line we have used the fact that $\frac{1}{2} \leqslant |z| \leqslant 2$.

So we can again use Lemma 4.1.1 with $M = 4$, we have

$$\begin{aligned}
\left|\int_{\frac{1}{2}<|z|<2} e^{ih(z)}z^2\zeta(z)dz\right| &\lesssim \sum_{r=0}^{3} 2^{k\ell r}(2^{k\ell})^{-r-3} \\
&\lesssim \sum_{r=0}^{3} 2^{-3k\ell}.
\end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned}
|m_k''(z)| &\lesssim \lambda^{-3/\ell}2^{3k}2^{-3k\ell} \\
&= \lambda^{-3/\ell}2^{-3k(\ell-1)}.
\end{aligned}$$

Again, by summing over all $k \in \mathbb{N}$, we have

$$\sum_{k\in\mathbb{N}} |m_k''(\xi)| \lesssim \lambda^{-3/\ell},$$

where the implicit constant depends only on A_1, B_j, for $j = 2, 3, 4$, ψ, ℓ and the fact that since $\ell > 1$,

$$\sum_{k\in\mathbb{N}} 2^{-3k(\ell-1)} < +\infty;$$

concluding the proof of Lemma 4.3.5. $\square$

We now return to the proof of Theorem 4.3.1. Let $\eta \in C_c^\infty(\mathbb{R})$ such that η has support in $(-c, c)$ and $1-\eta$ has support in $\mathbb{R} \setminus [-\frac{c}{2}, \frac{c}{2}]$. Then $1-\eta(\lambda^{-1/\ell}\cdot)$ has support in $\mathbb{R} \setminus [-\frac{1}{2}c\lambda^{1/\ell}, \frac{1}{2}c\lambda^{1/\ell}]$ and the multiplier $(1-\eta(\lambda^{-1/\ell}\cdot))m$ satisfies hypothesis (3.6) with $\mu = \frac{1}{2}c\lambda^{-1/\ell}$ by virtue of the support of $1-\eta(\lambda^{-1/\ell}\cdot)$. Additionally, by Lemma 4.3.3, for each $\xi \in \operatorname{supp}(1-\eta(\lambda^{-1/\ell}\cdot))$

$$\begin{aligned}
|\xi|^\beta |m(\xi)| &\lesssim |\xi|^\beta \lambda^{-\frac{1}{2(\ell-1)}} |\xi|^{-\frac{\ell-2}{2(\ell-1)}} \\
&\lesssim |\xi|^\beta \mu^{\beta-1} |\xi|^{-\beta} \\
&= \mu^{\beta-1}.
\end{aligned}$$

So the multiplier $(1-\eta(\lambda^{-1/\ell}\cdot))m$ satisfies hypothesis (3.7) with $C = s\mu^{\beta-1}$, where $s > 0$ is a constant independent of μ. Finally, by Lemma 4.3.4, for each $\xi \in \operatorname{supp}(1-\eta(\lambda^{-1/\ell}\cdot))$

$$\begin{aligned}
\sup_{\substack{I \subseteq [R,2R] \\ \operatorname{len}(I) = (R/\mu)^{-\alpha} R}} R^\beta \int_{\pm I} |m'(\xi)| d\xi &\lesssim \sup_{\substack{I \subseteq [R,2R] \\ \operatorname{len}(I) = (R/\mu)^{-\alpha} R}} R^\beta \int_{\pm I} \lambda^{-1/2(\ell-1)} \lambda^{1/(\ell-1)} |\xi|^{-\frac{\ell-4}{2(\ell-1)}} d\xi \\
&\lesssim \sup_{\substack{I \subseteq [R,2R] \\ \operatorname{len}(I) = (R/\mu)^{-\alpha} R}} R^\beta \int_{\pm I} \mu^{-(\beta-1)} |\xi|^{-\beta} |\xi|^\alpha |\xi| d\xi \\
&\lesssim \sup_{\substack{I \subseteq [R,2R] \\ \operatorname{len}(I) = (R/\mu)^{-\alpha} R}} R^\beta \operatorname{len}(I) \mu^{\beta-1} \mu^{2\beta-2} R^{-\beta} R^\alpha R d\xi \\
&= \sup_{\substack{I \subseteq [R,2R] \\ \operatorname{len}(I) = (R/\mu)^{-\alpha} R}} R^\beta (R/\mu)^{-\alpha} R \mu^{\beta-1} \mu^{2\beta-2} R^{-\beta} R^\alpha R d\xi \\
&= \mu^{\beta-1}.
\end{aligned}$$

So the multiplier $(1-\eta(\lambda^{-1/\ell}\cdot))m$ satisfies hypothesis (3.8) with $C = s'\mu^{\beta-1}$, where $s' > 0$ is a constant independent of μ. Thus, by Theorem 3.3.3 we have,

$$\int_{\mathbb{R}} |T_{(1-\eta)m} f|^2 w \lesssim \mu^{2\beta-2} \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta,\mu} M^4 w,$$

where $T_{(1-\eta)m}$ is defined by $\widehat{T_{(1-\eta)m}f} = (1-\eta(\lambda^{-1/\ell}\cdot))m\widehat{f}$, and the implicit constant is independent of μ.

So, to conclude Theorem 4.3.1 it is sufficient to show that

$$\int_{\mathbb{R}} |T_{\eta m}f|^2 w \lesssim \mu^{2\beta-2} \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta,\mu} M^4 w,$$

where $T_{\eta m}$ is defined by $\widehat{T_{\eta m}f} = \eta(\lambda^{-1/\ell}\cdot)m\widehat{f}$.

Now, by Lemma 4.3.5 we have that

$$|m^{(\gamma)}(\xi)| \lesssim \lambda^{-(\gamma+1)/\ell},$$

for $\gamma \in \{0, 1, 2\}$ and $|\xi| \leqslant c\lambda^{1/\ell}$.

Define $K_{\eta m}$ by $\widehat{K_{\eta m}}(\xi) = \eta(\lambda^{-1/\ell}\xi)m$. As $\eta(\lambda^{-1/\ell}\cdot)m$ has compact support, since $\eta(\lambda^{-1/\ell}\cdot)$ has compact support, we can use the Fourier inversion formula to obtain

$$K_{\eta m}(x) = \int_{\mathbb{R}} \eta(\lambda^{-1/\ell}\xi)m(\xi)e^{ix\xi}d\xi,$$

and furthermore, by standard properties of the Fourier transform, we obtain

$$(ix)^2 K_{\eta m}(x) = \int_{\mathbb{R}} \frac{d^2}{d\xi^2}(\eta(\lambda^{-1/\ell}\xi)m(\xi))e^{ix\xi}d\xi.$$

So, consider

$$
\begin{aligned}
|K_{\eta m}(x)| &= \left| \int_{\mathbb{R}} \eta(\lambda^{-1/\ell}\xi) m(\xi) e^{ix\xi} d\xi \right| \\
&\leqslant \int_{|\xi|<c\lambda^{1/\ell}} |\eta(\lambda^{-1/\ell}\xi)||m(\xi)| d\xi \\
&\lesssim \int_{|\xi|<c\lambda^{1/\ell}} \lambda^{-1/\ell} d\xi \\
&\lesssim 1,
\end{aligned}
$$

where we have used Lemma 4.3.5 on the third line, the fact that η is bounded and the implicit constants on both the third and last lines are independent of λ.

Next, consider

$$
\begin{aligned}
|(ix)^2 K_{\eta m}(x)| &= \left| \int_{\mathbb{R}} \frac{d^2}{d\xi^2} (\eta(\lambda^{-1/\ell}\xi) m(\xi)) e^{ix\xi} d\xi. \right| \\
&\leqslant \int_{|\xi|<c\lambda^{1/\ell}} |\eta(\lambda^{-1/\ell}\xi)||m''(\xi)| + \lambda^{-1/\ell}|\eta'(\lambda^{-1/\ell}\xi)||m'(\xi)| + \lambda^{-2/\ell}|\eta''(\lambda^{-1/\ell}\xi)||m(\xi)| d\xi \\
&\lesssim \int_{|\xi|<c\lambda^{1/\ell}} \lambda^{-3/\ell} d\xi \\
&\lesssim \lambda^{-2/\ell},
\end{aligned}
$$

where we have again used Lemma 4.3.5 on the third line, the fact that η, η' and η'' are all bounded and the implicit constants on both the third and last lines are independent of λ.

So, we can write

$$
|c\lambda^{1/\ell} x|^2 |K_{\eta m}(x)| \lesssim 1
$$

and so, combining with the above estimate on $|K_{\eta m}(x)|$, we have

$$
(1 + |c\lambda^{1/\ell} x|^2) |K_{\eta m}(x)| \lesssim 1,
$$

which implies that

$$|K_{\eta m}(x)| \lesssim \frac{1}{(1 + |c\lambda^{1/\ell} x|^2)}. \tag{4.23}$$

Now, define $\tilde{K}_{\eta m}(x) = K_{\eta m}(-x)$ and consider

$$\begin{aligned}
|\tilde{K}_{\eta m}(x)| &\leqslant |K_{\eta m}(-x)| \\
&\lesssim \frac{1}{(1 + |c\lambda^{1/\ell} x|^2)} \\
&\leqslant \sum_{n=1}^{\infty} 2^{-2(n-1)} \chi_{[-2^n, 2^n]}(c\lambda^{1/\ell} x) \\
&= \sum_{n=1}^{\infty} 2^{-2(n-1)} \chi_{[-c^{-1}\lambda^{-1/\ell} 2^n, c^{-1}\lambda^{-1/\ell} 2^n]}(x).
\end{aligned}$$

From this, we can estimate $|\tilde{K}_{\eta m}| * w(x)$ as

$$\begin{aligned}
|\tilde{K}_{\eta m}| * w(x) &= \int_{\mathbb{R}} |\tilde{K}_{\eta m}(y)| w(x-y) dy \\
&\lesssim \int_{\mathbb{R}} \sum_{n=1}^{\infty} 2^{-2(n-1)} \chi_{[-c^{-1}\lambda^{-1/\ell} 2^n, c^{-1}\lambda^{-1/\ell} 2^n]}(y) w(x-y) dy \\
&= \sum_{n=1}^{\infty} 2^{-2(n-1)} \int_{-2^n c^{-1}\lambda^{-1/\ell}}^{2^n c^{-1}\lambda^{-1/\ell}} w(x-y) dy \\
&= \sum_{n=1}^{\infty} 2^{-2(n-1)} 2^n c^{-1}\lambda^{-1/\ell} \frac{1}{2^n c^{-1}\lambda^{-1/\ell}} \int_{x-2^n c^{-1}\lambda^{-1/\ell}}^{x+2^n c^{-1}\lambda^{-1/\ell}} w(z) dz \\
&\leqslant \sum_{n=1}^{\infty} 2^{-(n-2)} c^{-1}\lambda^{-1/\ell} \sup_{r > c^{-1}\lambda^{-1/\ell}} \frac{1}{2r} \int_{x-r}^{x+r} w(z) dz \\
&\leqslant 2c^{-1}\lambda^{-1/\ell} M^{(\lambda)} w(x).
\end{aligned}$$

where we have used the substitution $z = x - y$ on the fourth line.

Next consider $\|K_{\eta m}\|_1$, by the previous estimate (4.23) we have

$$\|K_{\eta m}\|_1 \lesssim \int_{\mathbb{R}} \frac{1}{(1 + |c\lambda^{1/\ell} x|^2)} dx.$$

So, via the substitution $u = \lambda^{1/\ell} x$, we have

$$\begin{aligned}
\|K_{\eta m}\|_1 &\lesssim \lambda^{-1/\ell} \int_{\mathbb{R}} \frac{1}{1+u^2} du \\
&= \lambda^{-1/\ell} 2 \lim_{R\to\infty} \int_0^R \frac{1}{1+u^2} du \\
&= \lambda^{-1/\ell} 2 \lim_{R\to\infty} \arctan(R) \\
&= \lambda^{-1/\ell} \pi.
\end{aligned}$$

Thus, in the exact same way we handled the part of the kernel $K_{\lambda,\phi,0}$ via the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}} |K_{\eta m} * f(x)|^2 w(x) dx &\leqslant \|K_{\eta m}\|_1 \int_{\mathbb{R}} |f(x)|^2 |\tilde{K}_{\eta m}| * w(x) dx \\
&\lesssim \lambda^{-2/\ell} \int_{\mathbb{R}} |f(x)|^2 M^{(\lambda)} w(x) dx,
\end{aligned}$$

and using (4.17), we have

$$\int_{\mathbb{R}} |K_{\eta m} * f|^2 w \lesssim \mu^{2\beta-2} \int_{\mathbb{R}} |f|^2 M^6 \mathcal{M}_{\alpha,\beta,\mu} M^4 w,$$

concluding the proof of Theorem 4.3.1. □

Remark 4.3.6 *As Theorem 4.3.1 is stated, it's not quite obvious that our argument works for* $\ell = 2, 3$. *However, the obvious adaptation of the given argument for Lemma 4.3.4 and for* Case 2 *and* Case 3 *of Lemma 4.3.5 would yield Theorem 4.3.1 for such* ℓ. *It is also of some* *interest to note that if we change our requirement to* $\ell > 2$*, we may drop the requirement* $j = 4$ *in the hypotheses, again allowing* $\ell = 3$ *to be considered, by slightly adapting* Case 3 *of Lemma 4.3.5. Furthermore, if our requirement was actually* $\ell > 3$*, we may drop both the requirements* $\gamma = 3, 4$*, again adapting our argument slightly, implying some relationship between* ℓ *and the smoothness required by our phase.*

Appendix A
Multipliers

A.1 R-function lemma

The following lemma is elementary in nature and doesn't really fit within our discourse, but still a necessary step used in one of our proofs.

Lemma A.1.1 *Let $R^\lambda(x) := (1+|x|)^{-2\lambda}$ and $r > 0$ then,*

$$R^\lambda_{A(r)} * R^\lambda_{A(r)}(x) \lesssim R^\lambda_{A(r)}(x)$$

for all $\lambda > 1$.

Proof: Let $z = A(r^{-1})y$, then

$$\begin{aligned} R^\lambda_{A(r)} * R^\lambda_{A(r)}(x) &= \int_{\mathbb{R}^2} r^{-2\nu}(1+|A(r^{-1})(x-y)|)^{-2\lambda}(1+|A(r^{-1})y|)^{-2\lambda}dy \\ &= r^{-\nu}\int_{\mathbb{R}^2}(1+|A(r^{-1})x-z|)^{-2\lambda}(1+|z|)^{-2\lambda}dz \end{aligned}$$

Let $I = R^\lambda * R^\lambda$, that is

$$I(x) = \int_{\mathbb{R}^2}(1+|x-z|)^{-2\lambda}(1+|z|)^{-2\lambda}dz.$$

Then we have

$$R^\lambda_{A(r)} * R^\lambda_{A(r)}(x) = I_{A(r)}(x)$$

so by scaling it is sufficient to prove

$$I(x) \lesssim R^\lambda(x).$$

To prove this, we'll consider $I(x)$ separately for when $|x| \leqslant 2$ and when $|x| > 2$. First we shall deal with $|x| \leqslant 2$. As $|x - z| \geqslant 0$ trivially, we have

$$\begin{aligned} I(x) &= \int_{\mathbb{R}^2} (1 + |x - z|)^{-2\lambda} (1 + |z|)^{-2\lambda} dz \\ &\leqslant 1^{-2\lambda} \int_{\mathbb{R}^2} (1 + |z|)^{-2\lambda} dz \\ &\lesssim 1. \end{aligned}$$

Next, we shall deal with $|x| > 2$ by separating it into dyadic rings. Fix $k \in \mathbb{N}$, and fix x such that $2^k \leqslant |x| \leqslant 2^{k+1}$ then define $I_1(x)$ and $I_2(x)$ as

$$\begin{aligned} I(x) &= \int_{|x-z| \geqslant 2^{k-1}} (1 + |x - z|)^{-2\lambda} (1 + |z|)^{-2\lambda} dz + \int_{|x-z| \leqslant 2^{k-1}} (1 + |x - z|)^{-2\lambda} (1 + |z|)^{-2\lambda} dz \\ &= I_1(x) + I_2(x). \end{aligned}$$

For $I_1(x)$, we have

$$\begin{aligned} I_1(x) &= \int_{|x-z| \geqslant 2^{k-1}} (1 + |x - z|)^{-2\lambda} (1 + |z|)^{-2\lambda} dz \\ &\leqslant \int_{\mathbb{R}^2} (1 + 2^{k-1})^{-2\lambda} (1 + |z|)^{-2\lambda} dz \\ &\lesssim 2^{-k\lambda 2} \int_{\mathbb{R}^2} (1 + |z|)^{-2\lambda} dz \\ &\leqslant |x|^{-2\lambda}. \end{aligned}$$

For $I_2(x)$, we use the observation that if $z \in \mathbb{R}^2$ such that $|x - z| \leqslant 2^{k-1}$, since we have that $2^k \leqslant |x|$, then we have $|z| \geqslant 2^{k-1}$, and so

$$\begin{aligned} I_2(x) &= \int_{|x-z| \leqslant 2^{k-1}} (1 + |x - z|)^{-2\lambda} (1 + |z|)^{-2\lambda} dz \\ &\lesssim 2^{-k\lambda 2} \int_{|x-z| \leqslant 2^{k-1}} (1 + |x - z|)^{-2\lambda} dz \\ &\lesssim |x|^{-2\lambda}. \end{aligned}$$

So, by combining all of the above, we have

$$I(x) \lesssim \begin{cases} 1 & |x| < 2 \\ |x|^{-2\lambda} & |x| \geqslant 2 \end{cases}$$

and this can clearly be bounded above by $R^\lambda(x)$ modulo a constant. $\square$

Appendix B
Oscillatory kernels

B.1 Integration by parts lemma

Here we provide the first few applications of D^* to our function ψ from Lemma 4.1.1.

- $N = 1$
  Firstly, fix $x \in \mathbb{R}$ and consider

$$\begin{aligned} D^*\psi(x) &= -\frac{d}{dx}\left(\frac{\psi(x)}{ih'(x)}\right) \\ &= -\frac{\psi'(x)}{ih'(x)} + \frac{h''(x)\psi(x)}{ih'(x)^2}, \end{aligned}$$

  then by use of the triangle inequality, the estimate $|\psi(x)| \leqslant c$ for some constant $c > 0$ and the use of the hypotheses on the derivatives of h, we have

$$|D^*\psi(x)| \lesssim \lambda_1^{-1} + \lambda_2\lambda_1^{-2}.$$

- $N = 2$
  Again, fix $x \in \mathbb{R}$ and consider

$$\begin{aligned} (D^*)^2\psi(x) &= D^*(D^*\psi(x)) \\ &= -\frac{d}{dx}\left(-\frac{\psi'(x)}{(ih'(x))^2} + \frac{h''(x)\psi(x)}{(ih'(x))^3}\right) \\ &= -\frac{\psi''(x)}{h'(x)^2} + \frac{h''(x)\psi'(x)}{h'(x)^3} + \frac{h'''(x)\psi(x)}{h'(x)^3} + \frac{h''(x)\psi'(x)}{h'(x)^3} - \frac{h''(x)^2\psi(x)}{h'(x)^4}, \end{aligned}$$

  and we get

$$\left|(D^*)^2\psi(x)\right| \lesssim \lambda_1^{-2} + \lambda_1^{-3}\lambda_2 + \lambda_1^{-4}\lambda_2^2.$$

- $N = 3$

Fix $x \in \mathbb{R}$ and consider

$$
\begin{aligned}
(D^*)^3\psi(x) &= D^*(D^*(D^*\psi(x))) \\
&= -\frac{d}{dx}\left(-\frac{\psi''(x)}{ih'(x)^3} + \frac{h''(x)\psi'(x)}{ih'(x)^4} + \frac{h'''(x)\psi(x)}{ih'(x)^4} + \frac{h''(x)\psi'(x)}{ih'(x)^4} - \frac{h''(x)^2\psi(x)}{ih'(x)^5}\right) \\
&= \frac{\psi'''(x)}{ih'(x)^3} - \frac{h''(x)^2\psi'(x)}{ih'(x)^4} - \frac{h'''(x)\psi'(x)}{ih'(x)^4} - \frac{h'''(x)\psi'(x)}{ih'(x)^4} + \frac{h''(x)^2\psi'(x)}{ih'(x)^5} \\
&\quad - \frac{h^{(4)}(x)\psi}{ih'(x)^4} - \frac{h'''(x)\psi'(x)}{ih'(x)^4} + \frac{h''(x)h'''(x)\psi(x)}{ih'(x)^5} - \frac{h'''(x)\psi(x)}{ih'(x)^4} - \frac{h''(x)\psi''(x)}{ih'(x)^4} \\
&\quad + \frac{h''(x)^2\psi'(x)}{ih'(x)^5} + \frac{2h'''(x)h''(x)\psi(x)}{ih'(x)^5} + \frac{h''(x)^2\psi'(x)}{ih'(x)^5} - \frac{h''(x)^3\psi(x)}{ih'(x)^6}.
\end{aligned}
$$

and we get

$$\left|(D^*)^3\psi(x)\right| \lesssim \lambda_1^{-3} + \lambda_1^{-4}\lambda_2 + \lambda_1^{-5}\lambda_2^2 + \lambda_1^{-6}\lambda_2^3.$$

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