OPTIMAL CONSTANTS AND MAXIMISING FUNCTIONS FOR STRICHARTZ INEQUALITIES

by

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ABSTRACT

We prove sharp weighted bilinear inequalities which are global in time and for general dimensions for the free wave, Schrödinger and Klein–Gordon propagators. This extends work of Ozawa–Rogers for the Klein–Gordon propagator, work of Foschi–Klainerman and Bez–Rogers for the wave propagator, and work of Ozawa–Tsutsumi, Planchon–Vega and Carneiro for the Schrödinger propagator. In each case, we make a connection to estimates involving certain dispersive Sobolev norms.

As a consequence of these estimates we obtain, among other things, a new sharp form of a linear Strichartz estimate for the solution of the Klein–Gordon equation in five spatial dimensions for data belonging to H^1, and that maximisers do not exist for this estimate. We also obtain a new sharp form of a linear Sobolev–Strichartz estimate for the wave equation in four space dimensions for initial data in $\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}$, and characterisation of the maximisers.

Finally, we study the variational problems associated to the linear Sobolev–Strichartz estimates for the Schrödinger and wave equations. We establish that Gaussian functions are not maximisers for the $\dot{H}^m$ to L^p inequalities for the Schrödinger propagator, for any $m > 0$, and make a conjecture about the nature of the maximisers for the $\dot{H}^{\frac{d-1}{4}} \times \dot{H}^{\frac{d-5}{4}}$ to L^4 inequalities for the wave equation.

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Contents

Definitions and Preliminaries

We begin by stating some preliminary definitions, and quoting some standard results that we will be using in the forthcoming chapters. The results stated in this chapter are quite classical; for proofs, see (for example) [61] for results on L^p-spaces and the Hardy–Littlewood–Sobolev inequalities and [81] for the results on fractional integration and Sobolev inequalities. For $1 \leq p < \infty$ we define $L^p(X)$ to be the usual space of equivalence classes of complex-valued measurable functions f on a measure space X such that the norm

$$\|f\|_{L^p(X)} := \left(\int_X |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

is finite. We also make the standard modification for $p = \infty$:

$$\|f\|_{L^\infty(X)} := \inf \{ C : |f(x)| \leq C \text{ for a.e. } x \},$$

and as above say $f \in L^\infty(X)$ if and only if $\|f\|_\infty$ is finite. Throughout this thesis we will primarily be taking X to be either $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$ equipped with the usual Lebesgue measure, or the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ with surface measure, where $d \in \mathbb{N}$; we will sometimes shorten $\|f\|_{L^p(X)}$ to just $\|f\|_{L^p}$ or even $\|f\|_p$ if there is no chance of confusion. We also introduce the mixed norm space $L^q(L^r)$ on $\mathbb{R}^{d+1}$, defined by the norm

$$\|g\|_{L^q_t L^r_x} = \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}^d} |g(x,t)|^r \, \mathrm{d}x \right|^{\frac{q}{r}} \mathrm{d}t \right)^{\frac{1}{q}}.$$

Here $g = g(x,t)$ is a function on $\mathbb{R}^d \times \mathbb{R}$ and of course, if $q = r$ and $X = \mathbb{R}^{d+1}$ then this coincides with the L^q-norm. It is well-known that $L^p(X)$ with the norm $\|\cdot\|_p$ is a Banach space, and that if $p = 2$ then $L^p(X)$ is also a Hilbert space as we have the inner product

$$\langle f, g\rangle := \int_X f(x)\overline{g(x)}\,\mathrm{d}x, \quad f, g \in L^2(X).$$

We then have:

Theorem (Cauchy–Schwarz)**.** *Let $\langle\cdot,\cdot\rangle$ be a (complex) inner product on a vector space V and let $u, v \in V$ be two non-zero vectors. Then,*

$$|\langle u, v\rangle| \leq \langle u, u\rangle \langle v, v\rangle\,,$$

where equality holds if and only if $u = zv$ for some $z \in \mathbb{C}$.

In the context of the L^p-spaces, we also have:

Theorem (Hölder's inequality)**.** *Suppose that X is a measure space and that $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(X)$ and $g \in L^q(X)$ then the pointwise product $fg \in L^1(X)$ and we have*

$$\left|\int_X fg\right| \leq \|f\|_p \|g\|_q.$$

We will also use:

Theorem (Minkowski's inequality for integrals)**.** *Suppose that X and Y are any two spaces with σ-finite measures μ and ν respectively, and suppose that $f : X \times Y \to [0,\infty)$ is measurable with respect to the product measure. Let $1 \leq p < \infty$. Then*

$$\left(\int_X \left(\int_Y f(x,y), \mathrm{d}\nu(y)\right)^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X f(x,y)^p\,\mathrm{d}\mu(x)\right)^{\frac{1}{p}} \mathrm{d}\mu(y).$$

Next, we will need:

Theorem (Hardy–Littlewood–Sobolev inequality). *Let $p > 1$ and $0 < \lambda < d$ with $\frac{2}{p} + \frac{\lambda}{d} = 2$. Let $f, h \in L^p(\mathbb{R}^d)$. Then*

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) h(y) |x - y|^{-\lambda} \, \mathrm{d}x \mathrm{d}y \right| \leq C_{d,\lambda,p} \|f\|_p \|h\|_p$$

holds with constant

$$C_{d,\lambda,p} = \pi^{\frac{\lambda}{2}} \frac{\Gamma\left(\frac{d-1-\lambda}{2}\right)}{\Gamma\left(d - 1 - \frac{\lambda}{2}\right)} \left(\frac{\Gamma\left(d-1\right)}{\Gamma\left(\frac{d-1}{2}\right)} \right)^{1 - \frac{\lambda}{d-1}} \tag{1}$$

and this constant is sharp.

We remark that a full characterisation of the functions for which one has equality in this inequality with the sharp constant is also known, but we will not require this here. The Hardy–Littlewood–Sobolev inequality has an equivalent formulation on the sphere, which we also state.

Theorem (Hardy–Littlewood–Sobolev on the sphere). *Let $p > 1$ and $0 < \lambda < d$ with $\frac{2}{p} + \frac{\lambda}{d} = 2$. Let $f, h \in L^p(\mathbb{S}^d)$. Then the inequality*

$$\left| \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} f(x) h(y) |x - y|^{-\lambda} \, \mathrm{d}x \mathrm{d}y \right| \leq C_{d,\lambda,p} \|f\|_{L^p(\mathbb{S}^d)} \|h\|_{L^p(\mathbb{S}^d)},$$

holds with constant $C_{d,\lambda,p}$ given by (1). Further, this constant is sharp and equality holds if and only if there exists $c, c_1 \in \mathbb{C}$ and $\xi \in \mathbb{R}^d$ with $|\xi| < 1$ and

$$f(\omega) = c h(\omega) = \frac{c_1}{(1 + \xi \cdot \omega)^{\frac{2(d-1)-\lambda}{2}}},$$

unless either $f \equiv 0$ or $h \equiv 0$.

For $1 \le p \le 2$, if $f \in L^p\left(\mathbb{R}^d\right)$ then we can define its (spatial) Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}^d,$$

and if $g \in L^p\left(\mathbb{R}^{d+1}\right)$ we use the notation

$$\widetilde{g}(\tau, \xi) = \int_{\mathbb{R}^{d+1}} g(t, x) e^{-it\tau - ix\cdot\xi}\,\mathrm{d}\tau\mathrm{d}\xi,$$

and refer to the operator $\widehat{\cdot}$ as the Fourier transform in space and time. We then have:

Theorem (The Plancherel theorem). *If $f \in L^2\left(\mathbb{R}^d\right)$, then $\widehat{f} \in L^2\left(\mathbb{R}^d\right)$ and*

$$\|f\|^2_{L^2\left(\mathbb{R}^d\right)} = \frac{1}{(2\pi)^d} \|\widehat{f}\|^2_{L^2\left(\mathbb{R}^d\right)}.$$

We will use the notation $\mathcal{S}(\mathbb{R}^d)$ for the usual Schwartz space, that is the class of C^∞ functions on $\mathbb{R}^d$ whose derivatives to all orders rapidly decrease at infinity. For such functions we have the following formula for inverting the Fourier transform; we shall frequently use this implicitly.

Theorem (Fourier inversion formula). *For functions $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ we have*

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi}\,\mathrm{d}\xi, \ \ x \in \mathbb{R}^d.$$

On occasions we shall adopt the notation $^\vee$ for the inverse Fourier transform, where it is defined. We can use the inversion formula and the fact that $\left(\sqrt{-\Delta}(f)\right)^\wedge(\xi) = |\xi|\widehat{f}(\xi)$ to define functions of the Laplacian in d dimensions. Given a bounded, continuous function M from $[0, \infty)$ to $\mathbb{C}$ we can define the operator $M(\sqrt{-\Delta})$ from $\mathcal{S}(\mathbb{R}^d)$ to itself as follows:

$$(M(\sqrt{-\Delta})f)^\wedge(\xi) = M(|\xi|)\widehat{f}(\xi), \ \ \xi \in \mathbb{R}^d,$$

or explicitly,

$$M(\sqrt{-\Delta})f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} M(|\xi|)\widehat{f}(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}^d.$$

The examples we will primarily be working with are the Schrödinger evolution operator

$$e^{it\Delta} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi - it|\xi|^2}\,\mathrm{d}\xi,$$

the one-sided Klein–Gordon propagator

$$e^{it\sqrt{1-\Delta}} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi + it(1+|\xi|^2)^{\frac{1}{2}}}\,\mathrm{d}\xi,$$

and the one-sided wave propagator

$$e^{it\sqrt{-\Delta}} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi + it|\xi|}\,\mathrm{d}\xi,$$

each defined for $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. We will also be working with the inhomogeneous and homogeneous (L^2) Sobolev norms $\|\,.\,\|_{H^m(\mathbb{R}^d)}$ and $\|\,.\,\|_{\dot{H}^m(\mathbb{R}^d)}$respectively for $m \geq 0$, which we define to be

$$\|f\|_{H^m(\mathbb{R}^d)} = \|(1-\Delta)^{\frac{m}{2}} f\|_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^m |\widehat{f}(\xi)|^2\,\mathrm{d}\xi \right)^{\frac{1}{2}},$$

and

$$\|f\|_{\dot{H}^m(\mathbb{R}^d)} = \|(-\Delta)^{\frac{m}{2}} f\|_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\int_{\mathbb{R}^d} |\xi|^{2m} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi \right)^{\frac{1}{2}};$$

again we say that $f \in H^m(\mathbb{R}^d)$ or $f \in \dot{H}^m(\mathbb{R}^d)$ if the corresponding norm is finite. Closely related to these notions are the following:

Theorem (Sobolev inequalities on $\mathbb{R}^d$)**.** *Let $0 < \alpha < d$, $1 < p < q < \infty$, and suppose that*

$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. *Then there exist finite constants* $C_{p,q}$ *and* $C'_{p,q}$ *such that*

$$\|(-\Delta)^{-\frac{\alpha}{2}} f\|_{L^q(\mathbb{R}^d)} \leq C_{p,q} \|f\|_{L^p(\mathbb{R}^d)}. \tag{2}$$

and

$$\|(1-\Delta)^{-\frac{\alpha}{2}} f\|_{L^q(\mathbb{R}^d)} \leq C'_{p,q} \|f\|_{L^p(\mathbb{R}^d)} \tag{3}$$

The preceding theorem implies various embeddings of the Sobolev spaces into L^p-spaces for certain, easily computable values of the Lebesgue exponent. As such, we shall (adopting common convention) use the catch-all term 'Sobolev embedding' to mean an application of either inequality (2) or (3); precisely which we are using will be clear from the context in each case.

In addition to the above, some of our results (particularly those in Chapter 4) appeal to the theory of spherical harmonics; we use [2] as our reference for the standard theory. A spherical harmonic is defined to be the restriction of a homogeneous harmonic polynomial on $\mathbb{R}^d$ to the unit sphere $\mathbb{S}^{d-1}$. We will use the notation $\mathbb{Y}^d_k$ to denote the space of spherical harmonics of degree k in d dimensions, and we will denote an arbitrary element of this space by Y_k where the dimension is clear from the context. For convenience we introduce notation for the dimension of the space $\mathbb{Y}^d_k$, this is

$$N_{k,d} = \frac{(2k+d-2)(k+d-3)!}{k!(d-2)!}.$$

The spherical harmonics of different degrees are orthogonal with respect to the L^2 inner product on $\mathbb{S}^{d-1}$, that is

$$\int_{\mathbb{S}^{d-1}} Y_m(x) \overline{Y_n(x)} \, \mathrm{d}x = 0$$

if $m \neq n$. Closely related to the spherical harmonics are the Legendre polynomials $P_{k,d}$, which may be defined using the generating function

$$\frac{1}{(1+r^2-2rt)^{\frac{d-2}{2}}} = \sum_{k=0}^{\infty} \binom{k+d-3}{d-3} r^k P_{k,d}(t), \quad |r|<1, |t|\le 1,$$

these satisfy the following orthogonality relation:

$$\int_{-1}^{1} P_{k,d}(t) P_{l,d}(t) (1-t^2)^{\frac{d-3}{2}} \, \mathrm{d}t = 0, \quad k \neq l.$$

We use the notation Π_k for the projection operator from $L^2(\mathbb{S}^{d-1})$ into $\mathbb{Y}_k^d$, and we have

$$\Pi_k(f)(x) = \frac{N_{k,d}}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(\omega) P_{k,d}(\omega \cdot x) \, \mathrm{d}\omega \in \mathbb{Y}_k^d,$$

for $x \in \mathbb{S}^{d-1}$. Then:

Theorem (Spherical harmonic decomposition of $L^2(\mathbb{S}^{d-1})$)**.** *We have that $L^2(\mathbb{S}^{d-1})$ decomposes as the direct sum*

$$L^2(\mathbb{S}^{d-1}) = \bigoplus_{k\in\mathbb{N}} \mathbb{Y}_k^d,$$

and if $f \in L^2(\mathbb{S}^{d-1})$ then

$$f(x) = \sum_{k\in\mathbb{N}} \Pi_k f(x),$$

for any $x \in \mathbb{S}^{d-1}$.

Therefore, we shall sometimes refer to $\Pi_k f$ as the spherical harmonic component of f of degree k. Some of our calculations in the following chapters will be simplified by appealing to the Rodrigues formula for the Legendre polynomials, which states that

$$(1-t^2)^{\frac{d-3}{2}} P_{k,d}(t) = (-1)^k R_{k,d} \frac{\mathrm{d}^k}{\mathrm{d}t^k} (1-t^2)^{k+\frac{d-3}{2}},$$

for $t \in [-1,1]$, and where

$$R_{k,d} := \frac{\Gamma\left(\frac{d-1}{2}\right)}{2^k \Gamma\left(k + \frac{d-1}{2}\right)}.$$

We now state a key result which we will be using concerning integration of the spherical harmonics, the Funk–Hecke theorem. In order to state it we define the weighted L^1-space $L^1([-1,1],(1-t^2)^{\frac{d-3}{2}})$ of functions $F : [-1,1] \to \mathbb{R}$ such that

$$\int_{-1}^{1} |F(t)|(1-t^2)^{\frac{d-3}{2}} \, \mathrm{d}t$$

is finite.

Theorem (Funk–Hecke theorem)**.** *Suppose that* $F \in L^1([-1,1],(1-t^2)^{\frac{d-3}{2}})$*, and that* $Y_k \in \mathbb{Y}_k^d$*. Then,*

$$\int_{\mathbb{S}^{d-1}} Y_k(\eta) F(\omega \cdot \eta) \, \mathrm{d}\eta = \Lambda_k Y_k(\omega)$$

for $\omega \in \mathbb{S}^{d-1}$ *and* $k \in \mathbb{N}_0$*, where*

$$\Lambda_k := |\mathbb{S}^{d-2}| \int_{-1}^{1} F(t) P_{k,d}(t)(1-t^2)^{\frac{d-3}{2}} \, \mathrm{d}t.$$

We conclude this section by making precise the titular notion of the thesis, that is, the optimal constant and maximising function for an estimate. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are normed vector spaces and that T is a bounded operator from $\mathbf{X}$ to $\mathbf{Y}$, i.e. there exists a finite constant C such that the inequality

$$\|\mathrm{T}\, x\|_{\mathbf{Y}} \leq C \, \|x\|_{\mathbf{X}} \tag{4}$$

holds for every $x \in \mathbf{X}$. We define the optimal (or sharp) constant in the estimate (4) to be

$$\sup_{x \in \mathbf{X} \setminus \{0\}} \frac{\|\mathrm{T}\, x\|_{\mathbf{Y}}}{\|x\|_{\mathbf{X}}}.$$

If this supremum is attained for some $x_* \in \mathbf{X}$ then we say that x_* is a (global) maximiser for the inequality (4). If the sequence $(x_n)_{n\geq 1} \subseteq \mathbf{X}$ with $\|x_n\|_{\mathbf{X}} \leq 1$ is such that

$$\frac{\|\mathrm{T}\, x_n\|_{\mathbf{Y}}}{\|x_n\|_{\mathbf{X}}} \to \sup_{x \in \mathbf{X}\setminus\{0\}} \frac{\|\mathrm{T}\, x\|_{\mathbf{Y}}}{\|x\|_{\mathbf{X}}}$$

as $n \to \infty$, we say that $(x_n)_{n\geq 1}$ is a maximising sequence for the inequality (4). An important related concept is the notion of a *local* maximiser (or local minimiser) for the inequality (4). If we define a bounded nonlinear functional $\Phi = \Phi_{\mathrm{T}}$ on $\mathbf{X}$

$$\Phi(x) = \frac{\|\mathrm{T}\, x\|_{\mathbf{Y}}}{\|x\|_{\mathbf{X}}},$$

then we define a local extremiser for the inequality (4) to be a critical point of the nonlinear functional Φ, that is $x \in \mathbf{X}$ which satisfies

$$\Phi(x + \varepsilon x') = \Phi(x) + o(|\varepsilon|), \quad \varepsilon \to 0 \quad (\varepsilon \in \mathbb{C})$$

for any $x' \in \mathbf{X}$. Clearly any maximiser for (4) is also a local extremiser for (4); in practice this provides us with a convenient way to check that a given function is *not* a maximiser for inequalities of the form (4).

Chapter 1

Introduction

1.1 Strichartz estimates

The central objects of study in this thesis are the Strichartz and Sobolev–Strichartz inequalities and their bilinear generalisations, and in particular optimal constants and maximising functions for these inequalities. We will begin by introducing the particular equations and estimates we will be considering and briefly describing some of the history surrounding the estimates themselves.

1.1.1 Linear estimates

We first consider the linear Klein–Gordon equation

$$\begin{cases} \partial_{tt}u - \Delta_x u + u = 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a function on $\mathbb{R}^d \times \mathbb{R}$, for $d \geq 1$. For $d \geq 2$, Strichartz proved in [85] that the space-time estimate

$$\|u\|_{L^q(\mathbb{R}^{d+1})} \leq C \left(\|u_0\|^2_{H^m(\mathbb{R}^d)} + \|u_1\|^2_{H^{m-1}(\mathbb{R}^d)} \right)^{\frac{1}{2}} \tag{1.2}$$

holds for any solution u to (1.1), provided that the triple (q,d,m) satisfies

$$\frac{1}{2} \leq m < \frac{d}{2}, \qquad\qquad 2 + \frac{4}{d} \leq q \leq \frac{2d+2}{d-2m};$$

we define any such triple to be *KG-admissible*. When $d = 1$, and $m = \frac{1}{2}$, (1.2) also holds for $6 \leq q < \infty$; this was in fact proved slightly earlier by Segal in [76], following an argument by Carleson–Sjölin [21] and also Hörmander [43]. These ranges cannot be improved in the sense that if $m < \frac{d}{2}$ the stated range of q cannot be extended; this is checked by testing the inequality on particular functions f. We will consider this in more detail (at least in the endpoint case $m = \frac{1}{2}$ where the inequality (1.2) is directly analogous to such a problem) when we make the connection to Fourier restriction problems in Section 1.2. If u solves (1.1) then we can write $u = u_+ + u_-$ where

$$u(0) = f_+ + f_-, \quad \partial_t u(0) = i\sqrt{1-\Delta}(f_+ - f_-),$$

and therefore to prove (1.2) it suffices to consider the 'one-sided' estimate

$$\|e^{it\sqrt{1-\Delta}} u_0\|_{L^q(\mathbb{R}^{d+1})} \leq C\|u_0\|_{H^m(\mathbb{R}^d)}, \tag{1.3}$$

for triples (q,d,m) as above. We remark that in the endpoint case $m = \frac{1}{2}$ the estimates (1.3) are precisely (dual formulations of) Fourier restriction inequalities to the hyperboloid. In Chapter 2, we will obtain a new sharp form of inequality (1.3), and obtain the sharp constant in (1.2) as a consequence, in the case $(q,d,m)=(4,5,1)$. Moreover, we will show that maximisers for this estimate do not exist and that any maximising sequence concentrates at spatial infinity in a certain precise sense.

We consider next the linear Schrödinger equation

$$\begin{cases} -i\partial_t u + \Delta_x u & = 0 \\ u(0,x) & = u_0(x). \end{cases} \tag{1.4}$$

By taking the Fourier transform in x, it is easy to see that for any solution u of (1.4), one has that $u(t,x)=e^{it\Delta}u_0(x)$, at least for initial data for $u_0 \in \mathcal{S}(\mathbb{R}^d)$; we can extend it by density to any $f \in L^2(\mathbb{R}^d)$ in the standard way. The classical Strichartz estimate states that there exists a finite constant C such that

$$\left\| e^{it\Delta} u_0 \right\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \leq C \left\| u_0 \right\|_{L^2(\mathbb{R}^d)}, \tag{1.5}$$

where the value of the Lebesgue exponent on the left is fixed by homogeneity. Inequality (1.5) is due to Strichartz [85], following the proof of the closely-related Stein–Tomas Fourier extension inequality for the sphere (see Section 1.2, below). The estimate (1.5) has also been generalised to the mixed-norm setting, in the sense that

$$\left\| e^{it\Delta} u_0 \right\|_{L^r_t L^q_x(\mathbb{R}^{d+1})} \leq C \left\| u_0 \right\|_{L^2(\mathbb{R}^d)}, \tag{1.6}$$

for $2 \leq q, r \leq \infty$ with $(q,r,d) \neq (\infty, 2, 2)$, and where

$$\frac{d}{q} + \frac{2}{r} = \frac{d}{2}$$

is fixed by scale invariance. Away from the endpoints (that is, $(q,r) \neq (\frac{2d}{d-2}, 2)$), inequality

(1.6) was proved by Ginibre–Velo [39] and independently by Yajima [98]; the estimate for the endpoint values is more difficult and is due to Keel–Tao [48]. We also note that the estimate (1.6) is not true when $q = \infty$ and $r = d = 2$, this is proved by Montgomery-Smith in [64] (see [89] for further discussion). Using Sobolev embedding and (1.6), one can prove an alternative generalisation of (1.5):

$$\left\| e^{it\Delta} u_0 \right\|_{L^{\frac{2(d+2)}{d-2m}}(\mathbb{R}^{d+1})} \leq C \left\| u_0 \right\|_{\dot{H}^m(\mathbb{R}^d)}, \tag{1.7}$$

where now $0 \leq m < \frac{d}{2}$. In Chapter 5, we will study the maximisers for (1.7), and in particular we will show that the (conjectured, in general) class of maximisers in the case $m = 0$ is different to the class of maximisers when $m > 0$.

The final equation we will be considering in this thesis is the linear wave equation

$$\begin{cases} \partial_{tt} u - \Delta_x u = 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \end{cases} \tag{1.8}$$

For $d \geq 2$, if u is a solution to (1.8) then it is known that the estimate

$$\|u\|_{L^{\frac{2(d+1)}{d-2s}}(\mathbb{R}^{d+1})} \leq C_{s,d} \left(\|u_0\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|u_1\|^2_{\dot{H}^{s-1}(\mathbb{R}^d)} \right)^{\frac{1}{2}} \tag{1.9}$$

holds, provided that $\frac{1}{2} \leq s < \frac{d}{2}$. If u solves (1.8) it is well known that we have the decomposition $u = e^{it\sqrt{-\Delta}} f_+ + e^{-it\sqrt{-\Delta}} f_-$, where the functions f_+ and f_- are defined using the initial data by

$$u(0) = f_+ + f_-, \quad \partial_t u(0) = i\sqrt{-\Delta}(f_+ - f_-), \tag{1.10}$$

therefore to prove (1.9) it suffices to consider the 'one-sided' estimate

$$\|e^{it\sqrt{-\Delta}}f\|_{L^{\frac{2(d+1)}{d-2s}}} \leq C_{s,d}\|f\|_{\dot{H}^s(\mathbb{R}^d)} \tag{1.11}$$

for d and s as above. In the case $s = \frac{1}{2}$, the estimates (1.9) and (1.11) were proved by Strichartz in [85] as a consequence of a Fourier restriction inequality to the cone (cf. Section 1.2). For the general case $s > \frac{1}{2}$, by using Sobolev embedding in the space variable, inequality (1.11) follows from the mixed-norm estimate

$$\|e^{it\sqrt{-\Delta}}f\|_{L^r_t L^q_x} \leq C_{q,d}\|f\|_{\dot{H}^{\frac{1}{r}-\frac{1}{q}+\frac{1}{2}}(\mathbb{R}^d)}, \tag{1.12}$$

which holds for $r \geq 2$ and

$$\frac{2}{r} + \frac{d-1}{q} = \frac{d-1}{2}, \quad (q,r,d) \neq (\infty, 2, 3).$$

Away from the endpoint cases, that is, for $(q,r) \neq (\frac{2(d-1)}{d-3}, 2)$, inequality (1.12) was proved a by Ginibre–Velo in [40], and in the more difficult endpoint cases by Keel–Tao in [48]. The estimate in the exceptional case is once again known to be false, this is also proved in [64]. In Chapter 4, we will compute the optimal constant in inequalities (1.9) and (1.11) in the case $d = 4$ and $s = \frac{3}{4}$, and describe some progress on related problems.

Estimates such as those presented in this section, as well as related estimates for these and other equations, have become central to the the study of nonlinear dispersive equations - space-time estimates for the solution of a linear partial differential equation are frequently used to deduce desirable properties, for example local or global well-posedness, existence of low regularity solutions, and scattering, of certain nonlinear equations. This has been (and continues to be) a very active area of research in the field of nonlinear partial differential equations; it would be unwise to attempt an exhaustive survey of all such applications. We refer the reader to Section 2.3 of the book [88] for a general introduction to the linear Strichartz estimates, as well as a more comprehensive historical overview of the wider topic.

We conclude this section by noting that the results presented above have also been refined and further generalised to a number of different settings, see for example [18] and [75], as well as more recent work such as [74] (see Section 1.3). The generalisation we will be considering, however, is to *weighted bilinear estimates*, these have been studied for the propagators $e^{it\Delta}$, $e^{it\sqrt{-\Delta}}$ and $e^{it\sqrt{1-\Delta}}$, among others. Such estimates are interesting in their own right, and particularly in the case of the Schrödinger and wave equations have themselves found applications in the study of nonlinear problems. In the next section we introduce the bilinear estimates we will be considering and describe some of our results concerning sharp constants for these estimates.

1.1.2 Bilinear estimates

We begin with the wave equation. For $d \geq 2$, consider the estimate

$$\|(-\Delta)^{\frac{\beta_0}{2}} D_-^{\beta_-} D_+^{\beta_+}(uv)\|_{L^2(\mathbb{R}^{d+1})} \leq C\|(u_0,u_1)\|_{\dot{H}^{\alpha_1}\times\dot{H}^{\alpha_1-1}}\|(v_0,v_1)\|_{\dot{H}^{\alpha_2}\times\dot{H}^{\alpha_2-1}} \tag{1.13}$$

for u and v solutions to the linear wave equation (1.8) with initial data (u_0,u_1) and (v_0,v_1) respectively, and where the operators D_+ and D_- are defined using the space-time Fourier transform:

$$\widetilde{D_\pm f}(\tau,\xi) = \big||\tau| \pm |\xi|\big|\widetilde{f}(\tau,\xi),$$

for a suitable function f on $\mathbb{R} \times \mathbb{R}^d$. The estimate (1.13) is motivated as a replacement for the linear L^4 Sobolev–Strichartz inequality

$$\|u\|_{L^4(\mathbb{R}^{d+1})} \leq C\|(u_0, u_1)\|_{\dot{H}^{\frac{d-1}{4}}(\mathbb{R}^d) \times \dot{H}^{\frac{d-5}{4}}(\mathbb{R}^d)},$$

which requires more regularity on the initial data as the dimension increases. It is a simple exercise to show that (1.13) can hold only if $(\beta_-, \beta_+, \beta_0, \alpha_1, \alpha_2)$ satisfy the scaling relation

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{d-1}{2}. \tag{1.14}$$

Therefore, the estimate (1.13) can be interpreted as a statement about 'trading' regularity between the initial data and the product of the solutions to (1.8), the latter measured by powers of the Laplacian as well as of the so-called 'elliptic' and 'hyperbolic' derivatives corresponding to the operators D_+ and D_-, respectively.

The estimates (1.13) were first considered by Beals in [3] and Klainerman–Machedon in [50]; certain special cases were established and used in the study of some nonlinear wave equations. Further special cases were proved, and some applications presented, in [51], [52], [53], [54], [55] and [56] among others, and the full range of exponents $(\beta_0, \beta_-, \beta_+, \alpha_+, \alpha_-)$ for which (1.13) holds was found by Foschi–Klainerman in [37].

Further generalisations of (1.13), where the L^2 norm on the right hand side is replaced by a mixed space-time $L_t^q L_x^r$ norm, are also known. For such estimates when $q = r$, without the multiplier weights but under a certain frequency separation assumption, this problem was first studied by Bourgain in [16], subsequently generalised by Tao–Vargas [90] and Tao–Vargas–Vega in [91], and the full range of exponents for which one has such an estimate under these assumptions was found by Wolff in [96] and Tao in [86] (see also [58]). For the mixed norm generalisation of (1.13) with multiplier weights, the optimal range of exponents in this case for $d \geq 3$ was found by Lee–Vargas in [59] and Lee–Rogers–Vargas in [60] although a number of partial results were proved earlier, see the discussion and references in the latter paper. Further necessary conditions for the estimate to hold when $d = 2$ were also established in [60], but there remains a gap between the known necessary and sufficient conditions for an estimate to hold in this case.

The estimates (1.13) have found applications in the nonlinear theory. In addition to applications given in the papers [51]–[56] cited above, more recently in [1] d'Ancona–Foschi–Selberg were able to establish a local well-posedness result for the Dirac–Klein–Gordon system. A key part of the proof is showing that a crucial estimate needed for one of the nonlinear terms follows from an estimate of the form (1.13) proved in [37]. This argument has also proved useful in the study of other PDEs: see [68] for a recent application to the nonlinear Dirac equation.

Using the decomposition of the solution to (1.8) into $(+)$ and $(-)$ parts as in (1.10), the inequality (1.13) is deduced from the bilinear estimates for the operator $e^{it\sqrt{-\Delta}}$

$$\|(-\Delta)^{\frac{\beta_0}{2}} D_-^{\beta_-} D_+^{\beta_+} \left(e^{it\sqrt{-\Delta}} f e^{it\sqrt{-\Delta}} g\right) \|_{L^2(\mathbb{R}^{d+1})} \leq C \|f\|_{\dot{H}^{\alpha_1}(\mathbb{R}^d)} \|g\|_{\dot{H}^{\alpha_2}(\mathbb{R}^d)}, \tag{1.15}$$

and

$$\|(-\Delta)^{\frac{\beta_0}{2}} D_-^{\beta_-} D_+^{\beta_+} \left(e^{it\sqrt{-\Delta}} f \overline{e^{it\sqrt{-\Delta}} g}\right) \|_{L^2(\mathbb{R}^{d+1})} \leq C \|f\|_{\dot{H}^{\alpha_1}(\mathbb{R}^d)} \|g\|_{\dot{H}^{\alpha_2}(\mathbb{R}^d)}. \tag{1.16}$$

Some necessary and sufficient conditions for these estimates to hold were also established in [37]. We remark that both estimates are necessary to deal with the $(+,+)$ and $(+,-)$ interactions that arise when considering the product of two solutions to the wave equation and the decomposition (1.10) of each, in view of the basic equality $\overline{e^{it\sqrt{-\Delta}} g(x)} = e^{-it\sqrt{-\Delta}} \overline{g}(x)$. In addition to the works cited above, estimates related to

(1.15) appear in the book [17], and applications to well-posedness problems for some nonlinear wave equations are also discussed there.

Closely related to the estimates (1.15) and (1.16), at least in the symmetric cases $\beta_+ = \beta_-$ and $\alpha_1 = \alpha_2$, is the following inequality, which may (at least for $d > 3$) be viewed as a refinement of a bilinear Strichartz inequality corresponding to the cases $\beta_0 = \beta_+ = \beta_- = 0$ and $\alpha_1 = \alpha_2$ of (1.15) and (1.16):

$$\|e^{it\sqrt{-\Delta}} f e^{it\sqrt{-\Delta}} g\|_{L^2}^2 \leq \mathbf{BR}(d) \int_{\mathbb{R}^{2d}} |\widehat{f}(y_1)|^2 |\widehat{g}(y_2)|^2 |y_1|^{\frac{d-1}{2}} |y_2|^{\frac{d-1}{2}} (1 - y_1' \cdot y_2')^{\frac{d-3}{2}} \, \mathrm{d}y_1 \mathrm{d}y_2, \tag{1.17}$$

where

$$\mathbf{BR}(d) := \frac{|\mathbb{S}^{d-1}|}{2^{\frac{d-1}{2}} (2\pi)^{3d-1}}.$$

Inequality (1.17) is due to Bez–Rogers [13]; moreover when $d \geq 3$ they prove that the constant is sharp, and characterise the maximisers, generalising the case $d = 3$ proved earlier by Foschi in [34]. The estimate (1.17) has proved useful in obtaining sharp constants for some of the linear L^4 Sobolev–Strichartz estimates (1.9) and (1.11) discussed above: the case $d = 5$ was a key tool in the derivation of the sharp constant and characterisation of the maximisers in five space dimensions in [13], and we also apply it when $d = 4$ to obtain the sharp constant and characterisation of the maximisers in four space dimensions. In Chapter 4 we will obtain a family of sharp bilinear inequalities for the one-sided propagator $e^{it\sqrt{-\Delta}}$ which unify the estimate (1.17) with some estimates of the type (1.15) and (1.16), and as a consequence we obtain the sharp constant for some of the latter estimates.

Our final result in Chapter 4 concerns the validity of some estimates of the type (1.15) and (1.16) under the assumption of radially symmetric initial data. The fact that the admissible ranges of estimates, including Strichartz-type estimates, improve on radially symmetric inputs is well-known and has been studied in more general contexts such as those considered in [6], [25], [30], [41], [42], [80], [84]. For the estimates (1.13) and their generalisations, this was studied in the papers [36] and [50], where it is also shown that the estimates (1.13) fail (for general data) at certain endpoints. In [37] it is proved in particular that a necessary condition for (1.15) to hold in general is

$$\beta_- \geq \frac{3-d}{4}. \tag{1.18}$$

Using the scale invariance (1.14) of the estimates (1.15) and (1.16), when $\beta_0 = 0$, we must have

$$\alpha_1 + \alpha_2 = \beta_- + \beta_+ + \frac{d-1}{2}$$

for these estimates to hold. If we now assume that $\beta_+ = \beta_-$, combining these two conditions yields another necessary condition

$$\alpha_1 + \alpha_2 \geq 1,$$

which we can interpret as a requirement for a certain amount of regularity on the input functions, measured by certain homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$. As a corollary of our main result in Chapter 4 we prove that, in the cases $\beta_- = \beta_+$, $\beta_0 = 0$ and $\alpha_1 = \alpha_2$ it is possible to relax the condition (1.18) to

$$\beta_- \geq \frac{1-d}{4}$$

and obtain (1.15) and (1.16), if the initial data is assumed to be radial. This implies that some estimates of the type (1.13) can hold with less regularity (specifically, we can go down to $f, g \in \dot{H}^{\alpha_1}$ with $\alpha_1 = \alpha_2 > 0$) on initial data satisfying this assumption. We believe that the particular case of estimate (1.16) with $\alpha_1 + \alpha_2 < \frac{1}{2}$ of our work is new; in [37] it is proved that this estimate is false for general data. Our result complements a result from [36] where extensions of (1.15) for radial data are considered for a certain range of β_0, in the case $\beta_- = \beta_+ = 0$.

The bilinear estimates presented above also have analogues for the free Schrödinger equation. We begin with the paper [67], where Ozawa and Tsutsumi showed that any two solutions u and v of the free Schrödinger equation (1.4) with initial data u_0 and v_0, respectively, satisfy the global space-time bilinear estimate

$$\|(-\Delta)^{\frac{2-d}{4}}(u\overline{v})\|_{L^2}^2 \leq \frac{2^{-d}\pi^{\frac{2-d}{2}}}{\Gamma(\frac{d}{2})}\|u_0\|_{L^2}^2\|v_0\|_{L^2}^2, \tag{1.19}$$

and that the constant is optimal. The inequality (1.19) was motivated as a natural generalisation to d dimensions of the one-dimensional identity

$$\|(-\Delta)^{\frac{1}{4}}(u\overline{v})\|_{L^2(\mathbb{R}\times\mathbb{R})}^2 = \frac{\sqrt{\pi}}{2}\|u_0\|_{L^2(\mathbb{R})}^2\|v_0\|_{L^2(\mathbb{R})}^2,$$

which holds for any $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. This identity was established in [67], gives control on the so-called null gauge form $\partial(u\overline{v})$ for the Schrödinger equation in one spatial dimension, and was used as a tool in the proof of local well-posedness of some nonlinear Schrödinger equations with nonlinearities involving $\partial(|u|^2)u$. In the case where u_0 is equal to v_0, one may view the estimate (1.19) as a replacement, in the case where the initial data is in $L^2(\mathbb{R}^d)$, for the Sobolev–Strichartz estimate

$$\||u|^2\|_{L^2}^2 = \|u\|_{L^4}^4 \leq C_d\|u_0\|_{\dot{H}^{\frac{d-2}{4}}}^4$$

which requires rather more regularity on the initial data as the dimension gets large. This "trade-off" of derivatives on the initial data on the right-hand side for derivatives on the square of the solution on the left-hand side was first studied by Beals and Klainerman–Machedon in the context of the estimates (1.13); see the citations and discussion above.

A related bilinear estimate for (1.4), where instead one asks for control of the *unweighted* L^2 norm of the product of two solutions, is given by

$$\|uv\|_{L^2}^2 \leq \mathbf{C}(d) \int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2 |\widehat{v}_0(\eta)|^2 |\zeta - \eta|^{d-2} \,\mathrm{d}\zeta \mathrm{d}\eta \tag{1.20}$$

for $d \geq 2$, u and v as above, and any u_0 and v_0 for which the right hand side is finite. Here,

$$\mathbf{C}(d) := \frac{2^{2-4d} \pi^{\frac{2-5d}{2}}}{\Gamma(\frac{d}{2})}.$$

Inequality (1.20) in its full generality is due to Carneiro [22], moreover he proves in particular that the constant is sharp. As a consequence, a collection of estimates are obtained for the L^p-norm of the solution to (1.4) in terms of products of Sobolev norms on the initial data. Of course, when $d = 2$, estimates (1.19) and (1.20) are the same and, if in addition $u = v$, coincide with the classical L^2 to L^4 linear Strichartz inequality (1.5) for the equation (1.4).

Yet another closely related estimate is the following:

$$\|(-\Delta)^{\frac{3-d}{4}} (u\overline{v})\|_{L^2}^2 \leq \mathbf{PV}(d) \int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2 |\widehat{v}_0(\eta)|^2 |\zeta - \eta| \,\mathrm{d}\zeta \mathrm{d}\eta \tag{1.21}$$

for solutions u and v of (1.4), and where

$$\mathbf{PV}(d) := \frac{2^{-3d} \pi^{\frac{1-5d}{2}}}{\Gamma(\frac{d+1}{2})}$$

is the sharp constant. Inequality (3.4) is a particular consequence of some far-reaching identities proved by Planchon and Vega in [70] (see also [93]) using an innovative and radically different approach to those used to prove the estimates (1.19) and (1.20).

The preceding three estimates hint at a 'trade-off' of lowering the exponent of the kernel

$|\zeta - \eta|$ on the right hand side, with a lowering of the order of the derivatives of the product of the solutions on the left hand side, in the spirit of the estimate (1.13). Our main result in Chapter 3 is that this is the case - we obtain certain generalisations of results proved in [22] and [34] as corollaries. It turns out that our work also has an interesting connection to the Maxwell–Boltzmann equation from kinetic theory - we shall return to this point when we discuss the sharp constants and maximisers for (1.19), (1.20) and (1.21) in more detail in Section 1.3.

Finally, we consider the Klein–Gordon propagator $e^{it\sqrt{1-\Delta}}$. On $\mathbb{R}^{1+1}$, in [65] it was shown that the bilinear estimate

$$\begin{aligned}\left\| e^{it\sqrt{1-\Delta}} f_1 \; e^{it\sqrt{1-\Delta}} f_2 \right\|^2_{L^2(\mathbb{R}^2)} \\ \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f_1}(y_1)|^2 \, |\widehat{f_2}(y_2)|^2 \, \frac{(1+y_1^2)^{\frac{3}{4}}(1+y_2^2)^{\frac{3}{4}}}{|y_1 - y_2|} \, \mathrm{d}y_1 \mathrm{d}y_2 \qquad (1.22)\end{aligned}$$

holds whenever f_1 and f_2 have disjoint Fourier supports, and that the constant $\frac{1}{(2\pi)^2}$ is sharp. The main motivation behind the work in Chapter 2 was to identify a natural generalisation of this sharp bilinear estimate to arbitrary dimensions. In achieving this, we simultaneously extend work of Quilodrán in [73] and generalise work of Bez–Rogers in [13]. We will also obtain a new sharp form of the linear Strichartz estimate for the Klein–Gordon equation on $\mathbb{R}^{5+1}$ with H^1-initial data, and also outline a connection of our estimate to certain dispersive Sobolev norms, somewhat in the spirit of the estimates (1.13).

A number of the results presented in this section are not described as space-time estimates for a linear propagator as we do here, but rather are equivalently formulated as *Fourier restriction inequalities* to a certain hypersurface. This idea goes back at least as far as Segal, whose estimate for the Klein–Gordon equation in [76] was obtained as a consequence of a Fourier restriction inequality to the hyperboloid. The connection between the Strichartz estimates and the famous Fourier restriction problem provides some independent motivation to study the former; the next section is therefore devoted to a brief exposition of this connection and some background on the Fourier restriction problem, using the survey article [87] of Tao as a reference.

1.2 Connection to Fourier restriction problems

Suppose that $S \subseteq \mathbb{R}^{d+1}$ and equip it with a measure μ, and consider a function $f \in L^p(\mathbb{R}^d)$, for $1 \leq p \leq 2$. If $p = 1$ then the Fourier transform $\widehat{f}$ is continuous and bounded, and hence its restriction to S is defined. On the other hand, if $p = 2$ and S has measure zero then a restriction to S of $\widehat{f}$ does not make sense, since in this case $\widehat{f}$ could be an arbitrary L^2 function, which is only defined almost everywhere. In one form, the Fourier restriction problem is then to determine for which $p \in [1, 2)$ (and which S) one can hope for a meaningful restriction of $\widehat{f}$ to S. We say that S has the L^p to L^q Fourier restriction property if for some finite constant $C_{p,q} > 0$ the inequality

$$\|\widehat{f}\|_{L^q(S_0, \mathrm{d}\mu)} \leq C_{p,q} \, \|f\|_{L^p(\mathbb{R}^{d+1})} \tag{1.23}$$

holds for any open subset S_0 of S with compact closure in S. If the estimate (1.23) holds for some class of functions dense in L^p (say, smooth functions with compact support), then if $f \in L^p(\mathbb{R}^{d+1})$, by standard limiting arguments we can define the restriction of the Fourier transform of f as a function in $L^q(S, d\mu)$. We now let R formally denote the Fourier restriction operator to S, that is

$$\mathrm{R}\, f := \widehat{f}\Big|_S .$$

The estimate (1.23) may be stated equivalently (using duality) as

$$\|\mathrm{R}^* g\|_{L^{p'}(\mathbb{R}^{d+1})} \leq C_{p,q} \|g\|_{L^{q'}(S_0, \mathrm{d}\mu)}, \tag{1.24}$$

where R^* denotes the adjoint operator of R, which may be computed to be

$$\mathrm{R}^* g(x,t) = \int_{\mathbb{R}^{d+1}} e^{i(x,t)\cdot\xi} g(\xi)\, \mathrm{d}\mu(\xi), \quad x \in \mathbb{R}^d, t \in \mathbb{R},$$

and where g is a function on S. The operator R^* is sometimes referred to as the Fourier extension operator.

In the case where S is a hypersurface, it turns out that it is in fact its *curvature* which plays a crucial role in the estimates (1.23) and (1.24). For example, by testing on the function

$$f(x) = \exp(-\sum_{j=2}^{d} x_j^2)\frac{1}{1+|x_1|}, \quad x = (x_1, x_2, ...x_d) \in \mathbb{R}^d,$$

it follows that $f \in L^p(\mathbb{R}^d)$ for each $p > 1$ but $\widehat{f}$ is infinite at every point on a certain hyperplane. By modifying this example appropriately, it follows that the estimate (1.23) cannot be true outside of the trivial estimate given by the pair $(p,q) = (1,\infty)$, when S is a hyperplane. On the other hand, when S is the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and μ is surface measure, it is known that the estimates hold in the case $q = 2$; this is the Stein–Tomas theorem.

Theorem 1.2.1 (Stein–Tomas restriction inequality)**.** ([83], [92]) *Inequality* (1.23) *holds for $S = \mathbb{S}^d$ and $q = 2$ for all p such that*

$$1 \leq p \leq \frac{2(d+2)}{d+4}. \tag{1.25}$$

The stated range of exponents is optimal in this case, see below.

Let now $\rho : \mathbb{R}^d \to \mathbb{R}$ and define $S = S_\rho$ to be the hypersurface graphed by ρ, that is

$$S = \left\{ (x, \rho(x)) : x \in \mathbb{R}^d \right\}, \tag{1.26}$$

equipped with a measure μ. Assume in addition that S has everywhere nonvanishing Gaussian curvature; as an example we can take the paraboloid $\mathbb{P}^d$ ($\rho(x) = |x|^2$) and μ to be surface measure. It can be shown that, for such surfaces S the inequality (1.24) cannot hold unless

$$p' \geq \frac{2(d+2)}{d}, \quad p' \geq \frac{d+2}{d} q. \tag{1.27}$$

We first remark that the conditions are the same when $q = 2$ and coincide with the Stein–Tomas range (1.25), above. The proofs of the necessary conditions (1.27) may be found, for example, in [97]. The first condition arises from elementary considerations about the growth at infinity of measures on hypersurfaces S satisfying the curvature properties described above, and the second follows from the well-known *Knapp example*. In the latter case we choose $\widehat{f}$ to be the characteristic function of a 'cap' C_δ of radius $\delta \ll 1$ centered at a point x in S. Then, it turns out that since S has nonvanishing Gaussian curvature at x in particular, the size of C_δ may be computed to give the desired relation between p and q.

The Fourier restriction conjecture, posed by Stein in 1967 (see also [82, p.374], and the references there) states that the range given by (1.27) is actually sufficient for the estimates (1.23) to hold, for hypersurfaces S with everywhere nonvanishing Gaussian curvature. Although a number of partial results have been proved towards this goal (most recently in [19]), the problem has proved to be extremely difficult and remains open in general.

We can also consider the cone in $\mathbb{R}^{d+1}$, defined using (1.26) by $\rho(x) = |x|$. Although this does have vanishing curvature in the radial direction, it turns out that for an appropriate choice of measure one can obtain nontrivial restriction estimates (see for example [85]). In the case of the hyperboloid $\mathbb{H}^d$ ($\rho(x) = \sqrt{1+|x|^2}$), it turns out that the measure σ_1 formally defined on the ambient space $\mathbb{R}^{d+1}$ by $\sigma_1(t,x) = \frac{1}{\sqrt{1+|x|^2}}\delta(\tau - \sqrt{1+|x|^2})$ is the natural measure to consider, as it can be shown [85] that it is the unique measure on $\mathbb{H}^d$ which is invariant under Lorentz transformations, and this property has proved useful in the analysis of Fourier restriction inequalities in this setting in the past, see for instance [73].

To see the connection between the inequalities (1.23) and the inequalities for the propagators described in Section 1.1, we consider the particular case of the hyperboloid. Let $S = \mathbb{H}^d$ and set $\mu = \sigma_1$ as above. We note that a function f on $\mathbb{H}^d$ may be identified uniquely with a function on $\mathbb{R}^d$ using the natural correspondence

$$f(x) \leftrightarrow f(x, \phi_1(|x|)), \quad x \in \mathbb{R}^d,$$

and for convenience we use the notation $\phi_s(|x|) := \sqrt{s^2+|x|^2}$, for $s \geq 0$. Using this identification, we have that

$$\begin{aligned} \mathrm{R}^* f(x,t) &= \int_{\mathbb{R}^d} e^{ix\cdot\xi + it\phi_1(|\xi|)} f(\xi)\phi_1(|\xi|)^{-1}\,\mathrm{d}\xi \\ &= (2\pi)^d e^{it\phi_1(\sqrt{-\Delta})}\left(\left(\phi_1^{-1}f\right)^\vee\right)(x). \end{aligned} \tag{1.28}$$

Hence, the adjoint restriction operator R^* may be expressed in terms of the evolution operator $e^{it\phi_1(\sqrt{-\Delta})}$. Using (1.28), and our forthcoming Theorem 2.1.1 for the Klein–Gordon propagator $e^{it\sqrt{1-\Delta}}$, we will recover recent work on sharp inequalities for the adjoint Fourier restriction operator R^* due to Quilodrán [73].

At this point, we make some remarks concerning the relationship between the endpoints of the KG-admissible range and the wave and Schrödinger admissible Lebesgue exponents alluded to in the previous sections. Analogously to (1.28) one can identify the adjoint to the restriction operator to the paraboloid $\mathbb{P}^d$ with the propagator $e^{it\Delta}$, and the propagator $e^{it\sqrt{-\Delta}}$ can be related to the adjoint restriction operator to the cone. As in the general case, the upper bound for q is obtained as a consequence of the growth of a certain measure for large $|x|$ (see [85]). Since $\sqrt{1+|x|^2} \sim \phi_0(|x|)$ as $|x| \to \infty$, one would expect that a necessary condition that arises from considerations 'at infinity' to be the same as in the case of the cone, and indeed this is the case. On the other hand, for the Knapp example which yields the lower bound for q (see [85]), via a Taylor expansion we have that $\phi_s(|x|) \sim 1 + |x|^2 + o(|x|^2)$ as $|x| \to 0$, and so (since translating and rotating a hypersurface in $\mathbb{R}^{d+1}$ does not affect its curvature) we would expect a necessary condition whose proof relies upon curvature of the hyperboloid in a neighbourhood of a point to be no better than the corresponding condition for the paraboloid; once again, this is the case.

1.3 Optimal constants and maximising functions

One of the problems we will consider in this thesis can be summarised in the following three questions:

- What are the optimal constants in the estimates (1.3), (1.7) and (1.11) and their bilinear generalisations?
- Can we find sufficient conditions on the initial data for these constants to be attained?
- Can we, in addition, find *necessary* conditions for these constants to be attained? That is, can we characterise the whole class of maximisers for these estimates?

In this section we shall survey what is known in this area and briefly describe some of our results.

1.3.1 Linear estimates

We first consider the Sobolev–Strichartz estimate for the Schrödinger equation; we use the notation

$$\mathbf{S}(d,m) := \sup_{u_0 \in \dot{H}^m} \frac{\|e^{it\Delta}u_0\|_{L^{\frac{2(d+2)}{d-2m}}}}{\|u_0\|_{\dot{H}^m}}$$

for the sharp constant for the estimate (1.7). It is known that $\mathbf{S}(d,m)$ is attained for every $0 \le m < \frac{d}{2}$: The first progress on this problem was made in [57], where Kunze proved that $\mathbf{S}(d,m)$ is attained in the case $(d,m) = (1,0)$, and in [77] Shao proved that $\mathbf{S}(d,m)$ is attained for *every* admissible pair (d,m). These results follow from a profile decomposition for the Schrödinger equation and a compactness argument, and as such do not provide any information about the shape of the maximisers; on the other hand, the result of [77] in particular is applicable much more generally than the currently-known constructive results. As we shall see in the rest of this section, this is a common theme - broadly speaking the existence of maximisers for Strichartz inequalities is well-understood, but the identification of their shape has only been established in isolated cases.

The first progress towards the latter goal was made by Foschi who in [34] computed the values of $\mathbf{S}(1,0)$ and $\mathbf{S}(2,0)$, using a computation in Fourier space and a careful application of the Cauchy–Schwarz inequality. Moreover, he proved that u_0 is a maximiser if and only if

$$u_0(x) = e^{a|x|^2+b\cdot x+c} \quad a,c \in \mathbb{C},\ b \in \mathbb{C}^d\ \operatorname{Re} a < 0. \tag{1.29}$$

Shortly after, in [44] Hundertmark–Zharnitsky gave an alternative derivation of $\mathbf{S}(1,0)$ and $\mathbf{S}(2,0)$, and the characterisation of maximisers via a representation of the Strichartz norm in these cases as a projection operator. Yet another derivation of the sharp constant in these cases is due to Bennett–Bez–Carbery–Hundertmark in [7], this time by demonstrating that the sharp inequality follows as a consequence of the monotonicity of a certain nonlinear functional under heat-flow, and another proof of the characterisation of maximisers was given by Jiang–Shao in [47], specifically in the case $(d, m) = (2, 0)$, which means that a number of proofs of this sharp estimate are now known.

Some motivation to consider this problem comes from the following result in the nonlinear theory: in [29] Duyckaerts–Merle–Roudenko proved that for any solution of the mass-critical nonlinear Schrödinger equation, the maximal Strichartz norm is attained by some $u_0(\delta)$ with $\|u_0(\delta)\|_{L^2} = \delta$, where $\delta > 0$ is sufficiently small. Further, when $d = 1, 2$, they can compute this norm and show that any initial data $u_0(\delta)$ for which this value is attained is close to an appropriately normalised Gaussian. For this additional information, it was crucial to know the precise value of $\mathbf{S}(d, 0)$, as well as a full characterisation of maximisers for the inequality (1.7) in this case.

It is conjectured that the sharp constant $\mathbf{S}(d, m)$ is attained on the Gaussian functions (1.29) for all $d \geq 1$. when $m = 0$. One of the reasons for this conjecture is that functions (1.29) are local extremisers for the estimate (1.7) in this case; see [44] for a sketch proof. In Chapter 5 we will prove that this is not true for any $m > 0$, that is to say that the functions (1.29) are not local extremisers, and hence not maximisers, for the estimate (1.7), for such m.

We now turn to the wave equation (1.8); for $d \geq 2$ and $\frac{1}{2} \leq s < \frac{d}{2}$ we adopt the notation

$$\mathbf{W}(d, s) := \sup_{u_0 \in \dot{H}^s} \frac{\|e^{it\sqrt{-\Delta}} u_0\|_{L^{\frac{2(d+1)}{d-2s}}}}{\|u_0\|_{\dot{H}^s}}$$

for the best constant in the estimate (1.11). As in the case of the Schrödinger Strichartz inequality, it is known that there exists a maximiser for the inequalities (1.9) and (1.11), for all admissible pairs (d, s). This was first proved in the case $s \geq 1$ and $d \geq 3$ by Bulut in [20], then extended by Fanelli–Vega–Visciglia in [32] to the cases $d \geq 2$ and $s > \frac{1}{2}$, and finally completed by Ramos in [74] who dealt with the endpoint case $s = \frac{1}{2}$, also for $d \geq 2$.

The constructive results concerning maximisers for (1.11) are once again limited to isolated cases. In [34], Foschi calculated the precise value of $\mathbf{W}\left(2, \frac{1}{2}\right)$ and $\mathbf{W}\left(3, \frac{1}{2}\right)$ and characterised the maximisers for (1.11); as a corollary he was able to obtain the sharp constants for the estimates (1.9) by using the decomposition (1.10) and an orthogonality argument. The paper [34] is of particular significance to us as we will be adapting some of the key ideas used there to prove some of our results. In [13] Bez–Rogers calculated $\mathbf{W}(5,1)$ and showed that f is a maximiser for (1.11) if and only if

$$|\xi|\, \widehat{f}(\xi) = e^{a|\xi| + ib\cdot\xi + c}, \qquad a, c \in \mathbb{C},\, b \in \mathbb{R}^d,\, \operatorname{Re} a < 0. \tag{1.30}$$

As a consequence they obtain the sharp constant and characterisation of maximisers for the full Strichartz estimate (1.9) for initial data in the so-called energy space $\dot{H}^1 \times L^2$. In Chapter 4 we will calculate the value of $\mathbf{W}(4, \frac{3}{4})$ and prove that this is attained if and only if u_0 satisfies (1.30). In addition we can recover the derivation of $\mathbf{W}(5,1)$ from [13] via our argument, which gives an alternative proof of this estimate. In Chapter 5 we obtain that the functions (1.30) are not local extremisers for the L^6 estimate (unless $d = 2$), and are local extremisers for the L^4 estimate, in general dimensions.

For the Strichartz estimates for the linear Klein–Gordon equation, one can also ask for optimal constants and a full description of the maximisers; we define a three-parameter family

$$\mathbf{KG}(q, d, m) := \sup_{u_0 \in H^m} \frac{\|e^{it\sqrt{1-\Delta}} u_0\|_{L^q}}{\|u_0\|_{H^m}}$$

of sharp constants for the estimates (1.3), where (q, d, m) is KG-admissible. In this case the existence theory is not as well-developed; however the question has been addressed in a few cases, in the past these have been restricted to the case $m = \frac{1}{2}$. In [73], Quilodrán calculated the values of $\mathbf{KG}\left(4, 3, \frac{1}{2}\right)$, $\mathbf{KG}\left(4, 2, \frac{1}{2}\right)$ and $\mathbf{KG}\left(6, 2, \frac{1}{2}\right)$ and proved that in these cases maximisers do not exist for (1.3), by giving explicitly maximising sequences for the estimate (1.3) in each case. As a corollary of our main result in Chapter 2 we can compute $\mathbf{KG}(4, 5, 1)$ and show that in this case, maximisers for (1.3) do not exist, extending some of the results in [73]. We can then apply the orthogonality argument from [34] to obtain as a corollary a sharp form of the two-sided inequality (1.2) in this case. We can also recover two of the results from [73] stated above, particularly the cases where the Lebesgue exponent $q = 4$.

Remark. The results from [73] are not presented in terms of space-time bounds for a linear propagator as we do here, but rather are equivalently formulated as sharp forms of the Fourier restriction inequalities introduced in Section 1.2, in the case where S is a hyperboloid. Many of the results cited above also have equivalent formulations in this setting (see [22]), and further results have been obtained in [26] for the paraboloid, and in [31] for more general hypersurfaces. The problem of determining the optimal constant and identifying maximisers for the endpoint Stein–Tomas inequality (Theorem 1.2.1), was studied for $d = 2$ by Christ–Shao in [27] and [28], and was resolved recently in the remarkable article of Foschi [35] (subsequently generalised in [23]). In Chapter 4 we will apply some of the techniques used in [35] to obtain one of our corollaries.

We conclude this section by noting that the cases where constructive results are known concerning maximisers for the estimates in this section are the cases where the Lebesgue exponent is an even integer. Indeed, this is essential for the arguments leading to the sharp inequalities in [7], [13], [34], [44], and [73] as they rely on 'multiplying out' the

L^p-norm. As far as we are aware, a way to circumvent this difficulty is not currently known.

1.3.2 Bilinear estimates

We first consider the Schrödinger equation. For the the estimate (1.19), in [67] it is shown that the constant $\mathbf{OT}(d)$ is optimal since equality holds when u_0 and v_0 satisfy

$$u_0(x) = \lambda v_0(x) = e^{-a|x|^2+b\cdot x+c}, \tag{1.31}$$

with $\lambda = a = c = 1$ and $b = 0$. For the estimate (1.21) from [70], the emphasis is not placed on optimal constants or on identifying maximisers, but one can show that their argument gives (1.21) with the stated constant $\mathbf{PV}(d)$, that the constant is optimal and that it is attained if u_0 and v_0 satisfy (1.31), provided that $a, c \in \mathbb{C}$, $b \in \mathbb{C}^d, \operatorname{Re} a < 0$. In [22], Carneiro determined that (1.20) held with constant $\mathbf{C}(d)$ and proved that this was attained if and only if u_0 and v_0 satisfy (1.31), again for $a, c \in \mathbb{C}$, $b \in \mathbb{C}^d, \operatorname{Re} a < 0$. More recently, in [8] Bennett–Bez–Iliopolou were able to obtain a different proof of each of these estimates, with sharp constants, as a consequence of the monotonicity of a certain nonlinear functional under heat-flow; we adapt some of the ideas from this paper to prove the results in Chapters 3 and 4, extending some of the results in [8].

As a corollary of our main result in Chapter 3 we will deduce in particular that the constants $\mathbf{OT}(d)$ and $\mathbf{PV}(d)$ are attained if and only if u_0 and v_0 satisfy (1.31) with $a, c \in \mathbb{C}$, $b \in \mathbb{C}^d$ $\operatorname{Re} a < 0$, and provide an alternative proof of the characterisation of maximisers for (1.20). We do this by proving them all simultaneously as a one-parameter family of sharp bilinear inequalities, and our argument shows that any maximiser for this family must satisfy the so-called Maxwell–Boltzmann functional equation. One advantage of our approach is that we can reduce the equality condition in all of the estimates (1.19), (1.20) and (1.21) to this equation, where previously known equality conditions (which can be read off, respectively, from the proofs from [67], [22] and [70]) were rather different. We then provide a new, self-contained proof that this equation admits only Gaussian solutions which we believe is of independent interest.

For the corresponding estimates (1.13) and (1.17) for the wave equation, we first note that it is proved in [13] that the constant $\mathbf{BR}(d)$ is optimal, and a full characterisation of maximisers is given; in that case it is shown that one has equality in (4.5) if and only if $u_0 = \lambda v_0$ and

$$|\xi|\,\widehat{u_0}(\xi) = e^{a|\xi|+b\cdot\xi+c}, \qquad a, c \in \mathbb{C},\ b \in \mathbb{C}^d,\ \operatorname{Re} a < -|\operatorname{Re} b|. \tag{1.32}$$

Note that this is not the same as (1.30), as in principle the functions on the right hand side of (1.32) could have non-radial modulus. For the Klainerman–Machedon-type estimates (1.13), (1.15) and (1.16), however, as in [70] the emphasis has not been on optimal constants and identifying maximisers, but rather on determining the optimal range of exponents for which one has such an estimate with *some* finite constant. Although there are some cases where the arguments in, for example, [37] yield the optimal constant in (1.15) and (1.16) (typically a special case where some cancellation renders an upper bound used in the general case trivial), the optimality of such constants and identification of maximisers is not discussed. As a corollary of our main result in Chapter 4 we will obtain the sharp constant for some estimates of the type (1.15) and (1.16), and prove that these are attained if and only if $u_0 = \lambda v_0$ and u_0 satisfies (1.32).

Lastly, we return to the Klein–Gordon equation. As in the discussion on the linear theory the corresponding bilinear estimates associated to the propagator $e^{it\sqrt{1-\Delta}}$ are not as well-studied as for the wave and Schrödinger equations. In [65], it is proved that the bilinear estimate (1.22) is sharp, but that there are no nontrivial maximisers since an inequality is used which is only attained in trivial circumstances. On the other hand, the bilinear estimate we will prove in Chapter 2 *does* have maximisers, in fact an example of such will be given on the Fourier side by the function

$$\frac{e^{-\sqrt{1+|\xi|^2}}}{\sqrt{1+|\xi|^2}}.$$

Note the similarity with (1.32) for the corresponding estimate (1.17) for the wave equation; we note that this is to be expected given that our approach follows the general strategy used in [13] to prove (1.17). The fact then that the resulting linear estimates discussed in Section 1.3.1 do not have maximisers follows from the fact that upper bounds are used on certain integral kernels which are only attained in trivial circumstances, as in [65].

We conclude this section by noting that our results, and the results cited in this section, form part of a broad category of recent work concerning sharp constants in estimates for dispersive PDEs. Some related results which do not fit into the framework of the discussion in this section include those contained in [79] and [95], as well as the more recent papers [14], [15] and [66]; we refer the reader to the latter papers for discussion and further references.

1.4 Organisation of results

In Chapter 2 we will consider the Klein–Gordon equation and prove a sharp bilinear inequality for the Klein–Gordon propagator $e^{it\sqrt{1-\Delta}}$. From this, we will obtain a new sharp form of a classical Strichartz estimate for the solution to the Klein–Gordon equation with initial data in the space $H^1 \times L^2$. Further, we will show that maximisers for this inequality do not exist and establish that the H^1-norm of a maximising sequence for this estimate concentrates at spatial infinity. This builds upon earlier work of Bez–Rogers for the wave equation and generalises the estimate (1.22) of Ozawa–Rogers for the Klein–

Gordon equation. The work in this chapter has been accepted for publication and is contained in [45].

In Chapter 3 we provide a comprehensive analysis of sharp bilinear estimates of Ozawa–Tsutsumi type for solutions u of the free Schrödinger equation, which give sharp control on $|u|^2$ in classical Sobolev spaces. In particular, we provide a generalisation of their estimates in such a way that provides a unification of the Ozawa–Tsutsumi estimate (1.19) with the sharp bilinear estimates (1.20) and (1.21) proved respectively by Carneiro and Planchon–Vega, via entirely different methods, by seeing them all as special cases of a one-parameter family of sharp estimates. We show that the extremal functions are solutions of the Maxwell–Boltzmann functional equation and provide a new proof that this equation admits only gaussian solutions. We also make a connection to certain sharp estimates on u^2 involving certain dispersive Sobolev norms. The work in this chapter is joint with Jonathan Bennett, Neal Bez and Nikolaos Pattakos and is contained in [9].

In Chapter 4 we obtain a sharp bilinear estimate for the one-sided wave propagator $e^{it\sqrt{-\Delta}}$, which unifies some Klainerman–Machedon type space time estimates such as (1.15) with a sharp bilinear inequality (1.17) proved by Bez–Rogers in [13]. As a corollary, we obtain the sharp constant and characterisation of the maximisers for the L^4 linear Sobolev–Strichartz estimate for the wave equation in four space dimensions, the initial data in this case is in $\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}$. The work in this chapter is joint with Neal Bez and Tohru Ozawa and is contained in [10] and [11].

In Chapter 5 we collect a number of results concerning local extremisers for linear estimates for the Schrödinger and wave operators. Specifically, we will show that radial Gaussian functions are not maximisers for the pure and mixed-norm Sobolev–Strichartz estimates for the Schrödinger propagator, in the spirit of work by Christ and Quilodrán in [26]. We will also consider the Sobolev–Strichartz estimates for the wave equation, and obtain that the functions (1.30) are not local extremisers for the L^6 estimate, and are local extremisers for the L^4 estimate, in general dimensions. Some of the results in this chapter are joint work with Neal Bez and Nikolaos Pattakos and are contained in [12].

Finally, in Chapter 6 we conclude by summarising our work and identifying possible directions for further work in this area.

Chapter 2

A sharp bilinear estimate for the Klein–Gordon equation

2.1 Introduction

For the Klein–Gordon equation on $\mathbb{R}^{1+1}$, recently in [65] it was shown that the bilinear estimate

$$\begin{aligned}\left\| e^{it\sqrt{1-\Delta}} f_1 \; e^{it\sqrt{1-\Delta}} f_2 \right\|_{L^2(\mathbb{R}^2)}^2 \\ \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\widehat{f}_1(y_1)|^2 \, |\widehat{f}_2(y_2)|^2 \, \frac{(1+y_1^2)^{\frac{3}{4}}(1+y_2^2)^{\frac{3}{4}}}{|y_1 - y_2|} \, \mathrm{d}y_1 \mathrm{d}y_2 \qquad (2.1)\end{aligned}$$

holds whenever f_1 and f_2 have disjoint Fourier supports, and that the constant $\frac{1}{(2\pi)^2}$ is sharp. The main motivation behind the work in the present chapter was to identify a natural generalisation of this sharp bilinear estimate to arbitrary dimensions; in achieving this, we also extend work of Quilodrán in [73]. We will also obtain a new Strichartz estimate with sharp constant for the Klein–Gordon equation

$$\begin{cases} \partial_{tt}u - \Delta_x u + u = 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \end{cases} \tag{2.2}$$

on $\mathbb{R}^{d+1}$, for $d = 5$ and with initial data $(u_0, u_1) \in H^1 \times L^2$.

In order to state the main result, in what follows we define

$$\phi(r) = \sqrt{1 + r^2}, \quad r \geq 0,$$

we let

$$K(y_1, y_2) = \frac{(\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 - 1)^{\frac{d-2}{2}}}{(\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 + 1)^{\frac{1}{2}}}$$

for $(y_1, y_2) \in \mathbb{R}^{2d}$, and we introduce the constant

$$\mathbf{KG}(d) = \frac{2^{-\frac{d-1}{2}} \left|\mathbb{S}^{d-1}\right|}{(2\pi)^{3d-1}}$$

for $d \geq 1$, which will appear throughout the chapter.

Theorem 2.1.1. *If $d \geq 2$, then*

$$\begin{aligned} &\left\| e^{it\phi(\sqrt{-\Delta})} f_1 e^{it\phi(\sqrt{-\Delta})} f_2 \right\|_{L^2(\mathbb{R}^{d+1})}^2 \\ &\qquad \leq \mathbf{KG}(d) \int_{\mathbb{R}^{2d}} |\widehat{f}_1(y_1)|^2 \, |\widehat{f}_2(y_2)|^2 \, \phi(|y_1|)\phi(|y_2|) K(y_1, y_2) \, \mathrm{d}y_1 \, \mathrm{d}y_2, \end{aligned} \tag{2.3}$$

where the constant $\mathbf{KG}(d)$ is best possible since we have equality for functions of the form

$$\widehat{f}_1(\xi) = \widehat{f}_2(\xi) = \frac{e^{-a\phi(|\xi|)}}{\phi(|\xi|)}, \tag{2.4}$$

for $a > 0$. Further, if $d = 1$, and $\widehat{f_1}, \widehat{f_2}$ have disjoint support, then we have

$$\left\| e^{it\phi(\sqrt{-\Delta})} f_1 e^{it\phi(\sqrt{-\Delta})} f_2 \right\|_{L^2(\mathbb{R}^{1+1})}^2$$
$$= \frac{\mathbf{KG}(1)}{2} \int_{\mathbb{R}^{2d}} |\widehat{f_1}(y_1)|^2 \, |\widehat{f_2}(y_2)|^2 \, \phi(|y_1|)\phi(|y_2|) K(y_1, y_2) \, \mathrm{d}y_1 \, \mathrm{d}y_2. \tag{2.5}$$

Before we state the next result, we make a series of remarks concerning Theorem 2.1.1. It turns out that the functions described in (2.4) above play an important role in some of the applications of (2.3), as we will see below in Corollary 2.1.3, where we obtain a new sharp Strichartz estimate for Klein–Gordon propagator. We also note that (2.3) may be interpreted as a sharp smoothing estimate measured by dispersive Sobolev spaces; we present some of these results in Section 2.5.

In the case $d = 1$, we observe that

$$K(y_1, y_2) \leq \frac{(1 + y_1^2)^{\frac{1}{4}} (1 + y_2)^{\frac{1}{4}}}{|y_1 - y_2|} \tag{2.6}$$

for almost every $(y_1, y_2) \in \mathbb{R}^2$ with $y_1 \neq y_2$. One can see this by a (somewhat involved) direct argument which shows that the claimed inequality is equivalent to

$$0 \leq (y_1^2 + y_2^2)(y_1 - y_2)^4 + y_1^2 y_2^2 (y_1 - y_2)^4,$$

which is clearly true. Since $\frac{1}{2}\mathbf{KG}(1) = \frac{1}{(2\pi)^2}$, we see that (2.1) follows from (2.5)[1] and we claim that (2.3) in Theorem 2.1.1 provides a natural generalisation of this to higher dimensions.

Furthermore, one can deduce certain Strichartz estimates from (2.3) with sharp constants, some of which recover sharp Strichartz estimates due to Quilodrán in [73], and we also

[1] in fact, the argument in [65] leading to (2.1) goes via the identity (2.5), and they prove (2.6) differently using some trigonometric identities.

obtain a new sharp Strichartz estimate for the Klein–Gordon equation in five spatial dimensions (see the forthcoming Corollary 2.1.3). When $d = 2$, we have

$$K(y_1, y_2) = (\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 + 1)^{-\frac{1}{2}} \leq 2^{-\frac{1}{2}},$$

for $(y_1, y_2) \in \mathbb{R}^4$. Taking $f_1 = f_2$ in (2.3) it follows that

$$\left\| e^{it\sqrt{1-\Delta}} f \right\|_{L^4(\mathbb{R}^{2+1})} \leq \frac{1}{2^{5/4}\pi} \|f\|_{H^{1/2}(\mathbb{R}^2)}. \tag{2.7}$$

Estimate (2.7) is due to Quilodrán [73] and he showed that the constant is sharp but that maximisers do not exist.

Similarly, when $d = 3$, we get

$$K(y_1, y_2) = \frac{(\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 - 1)^{\frac{1}{2}}}{(\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 + 1)^{\frac{1}{2}}} \leq 1,$$

and (2.3) implies

$$\left\| e^{it\sqrt{1-\Delta}} f \right\|_{L^4(\mathbb{R}^{3+1})} \leq \frac{1}{(2\pi)^{7/4}} \|f\|_{H^{1/2}(\mathbb{R}^3)}. \tag{2.8}$$

Again, the constant is sharp and maximisers do not exist; this is also proved in [73][2]. We remark that we prove Theorem 2.1.1 using the approach of Foschi in [34], as did Quilodrán, and so it is not at all a surprise that (2.7) and (2.8) follow from Theorem 2.1.1.

In this chapter, we also obtain the following new sharp form of a classical Strichartz estimate for the full solution of the Klein–Gordon equation.

[2]in [73] the perspective is that of adjoint Fourier restriction inequalities for the hyperboloid, and we choose to present the estimates (2.7) and (2.8) in terms of the Klein–Gordon propagator.

Corollary 2.1.2. *Suppose that* $\partial_{tt}u - \Delta u + u = 0$ *on* $\mathbb{R}^{5+1}$*, then*

$$\|u\|_{L^4(\mathbb{R}^{5+1})} \leq \left(\frac{1}{8\pi}\right)^{\frac{1}{2}} \left(\|u(0)\|^2_{H^1(\mathbb{R}^5)} + \|\partial_t u(0)\|^2_{L^2(\mathbb{R}^5)}\right)^{\frac{1}{2}}. \tag{2.9}$$

The constant $\left(\frac{1}{8\pi}\right)^{\frac{1}{2}}$ *is sharp, but there are no nontrivial functions for which we have equality.*

A nonsharp form of (2.9) was proved by Strichartz in [85]. The sharp inequality (2.9) is deduced from the following sharp estimate for the one-sided propagator $e^{it\phi(\sqrt{-\Delta})}$.

Corollary 2.1.3. *We have that*

$$\left\|e^{it\phi(\sqrt{-\Delta})}f\right\|_{L^4(\mathbb{R}^{5+1})} \leq \left(\frac{1}{24\pi^2}\right)^{\frac{1}{4}} \left\|\phi(\sqrt{-\Delta})f\right\|_{L^2(\mathbb{R}^5)}. \tag{2.10}$$

The constant $\left(\frac{1}{24\pi^2}\right)^{\frac{1}{4}}$ *is sharp as we have the maximising sequence* $(g_a)_{a>0}$ *defined by*

$$g_a = \frac{f_a}{\left\|\phi(\sqrt{-\Delta})f_a\right\|_{L^2(\mathbb{R}^5)}},$$

where

$$\widehat{f_a}(\xi) = \frac{e^{-a\phi(|\xi|)}}{\phi(|\xi|)} \tag{2.11}$$

as $a \to 0+$*, but there are no functions for which we have equality.*

The estimate (2.10) is new; with nonsharp constant, (2.10) follows from [85].

As our final main result in this chapter, we establish that *any* maximising sequence for the estimate (2.10) must concentrate at spatial infinity in the following precise sense.

Proposition 2.1.4. *If* $(g_n)_{n\geq 1}$ *is any maximising sequence for* (2.10)*, then for each* $\varepsilon, R >$

0 *there exists* $N \in \mathbb{N}$ *so that if* $n \geq N$,

$$\left\| \phi(\widehat{\sqrt{-\Delta}) g_n} \right\|_{L^2(B(0,R))} < \varepsilon, \tag{2.12}$$

where $B(0, R)$ *denotes the ball of radius* R *centered at the origin in* $\mathbb{R}^5$.

The motivation for this result comes from the observation that the particular maximising sequence (g_a) considered in Corollary 2.1.3 satisfies these conditions. A result analogous to (2.12) was established in [73], where it was shown that any maximising sequence for either (2.7) or (2.8) must concentrate at spatial infinity. We also remark here that Proposition 2.1.4 may be interpreted as a statement about the concentration of the H^1-norm of a maximising sequence for the inequality (2.10).

2.2 Proof of Theorem 2.1.1

2.2.1 The case $d \geq 2$

We note firstly that the space-time Fourier transform of $v_j = e^{it\phi(\sqrt{-\Delta})} f_j$ will be the measure

$$\widetilde{v_j}(\xi, \tau) = 2\pi\delta(\tau - \phi(|\xi|))\widehat{f_j}(\xi)$$

for $j = 1, 2$, each supported on the hyperboloid in $\mathbb{R}^{d+1}$,

$$\left\{ (y, (1 + |y|^2)^{\frac{1}{2}}) : y \in \mathbb{R}^d \right\}.$$

Note that if $d \geq 2$, the function defined by K is well-defined and finite for any $y = (y_1, y_2) \in \mathbb{R}^{2d}$. For example, if $d = 2$, then

$$K(y_1, y_2) = \frac{1}{(1 + \phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2)^{\frac{1}{2}}},$$

and the denominator is strictly positive since

$$y_1 \cdot y_2 \leq |y_1|\,|y_2| < (|y_1|^2 + 1)^{\frac{1}{2}}(|y_2|^2 + 1)^{\frac{1}{2}} + 1 = \phi(|y_1|)\phi(|y_2|) + 1;$$

the claim for $d > 2$ follows from this as the power $\frac{d-2}{2}$ is positive in this case.

If we now write $u = e^{it\phi(\sqrt{-\Delta})} f_1 e^{it\phi(\sqrt{-\Delta})} f_2$, then the space-time Fourier transform of u will be the convolution of the measures $\widetilde{v_1}$ and $\widetilde{v_2}$, which may be written as

$$\widetilde{u}(\xi, \tau) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} \frac{\widehat{F}(y)}{(1 + |y_1|^2)^{\frac{1}{4}}(1 + |y_2|^2)^{\frac{1}{4}}} \delta\begin{pmatrix} \tau - \phi(|y_1|) - \phi(|y_2|) \\ \xi - y_1 - y_2 \end{pmatrix} \mathrm{d}y, \tag{2.13}$$

where $\xi \in \mathbb{R}^d$ and $\tau \in \mathbb{R}$ are fixed, we set

$$\widehat{F}(y) = \widehat{f_1}(y_1)\widehat{f_2}(y_2)(1 + |y_1|^2)^{\frac{1}{4}}(1 + |y_2|^2)^{\frac{1}{4}},$$

and we use the notation $\delta\binom{t}{x}$ for the product $\delta(t)\delta(x)$ on $\mathbb{R}^{d+1}$. It is proved in [73] that the function $\widetilde{u}$ is supported on the set

$$\mathcal{H} := \left\{ (\xi, \tau) \in \mathbb{R}^{d+1} : \tau \geq (4 + |\xi|^2)^{\frac{1}{2}} \right\},$$

for completeness we include the proof here. If $\xi = y_1 + y_2$ and $\tau = \phi(|y_1|) + \phi(|y_2|)$ we have that

$$\begin{aligned}\tau^2 &= 2 + |y_1|^2 + |y_2|^2 + 2\phi(|y_1|)\phi(|y_2|)\\ &\geq |y_1|^2 + |y_2|^2 + 2\,|y_1|\,|y_2| + 4\\ &\geq 4 + |\xi|^2\,,\end{aligned}$$

since

$$\phi(|y_1|)\phi(|y_2|) \geq 1 + |y_1|\,|y_2|\,, \tag{2.14}$$

for all $y_1, y_2 \in \mathbb{R}^d$. By the Cauchy–Schwarz inequality,

$$|\widetilde{u}(\xi,\tau)|^2 \leq \frac{I(\xi,\tau)}{(2\pi)^{2d-2}} \int_{\mathbb{R}^{2d}} |\widehat{F}(y)|^2\,\delta\begin{pmatrix}\tau - \phi(|y_1|) - \phi(|y_2|)\\ \xi - y_1 - y_2\end{pmatrix} \mathrm{d}y, \tag{2.15}$$

where

$$I(\xi,\tau) := \int_{\mathbb{R}^{2d}} \frac{1}{\phi(|y_1|)\phi(|y_2|)} \delta\begin{pmatrix}\tau - \phi(|y_1|) - \phi(|y_2|)\\ \xi - y_1 - y_2\end{pmatrix} \mathrm{d}y,$$

Lemma 2.2.1. *For all $(\xi,\tau) \in \mathcal{H}$ we have that*

$$I(\xi,\tau) = \frac{\left|\mathbb{S}^{d-1}\right|}{2^{d-2}} \left(\tau^2 - |\xi|^2 - 4\right)^{\frac{d-2}{2}} \left(\tau^2 - |\xi|^2\right)^{-\frac{1}{2}}.$$

Proof. We could prove this using the Lorentz invariance of an appropriate measure as in [13], [34] and [73] (as was done in [45]), but we will give a different proof here inspired by a result from [37] concerning integration over ellipsoids. Firstly we evaluate the integral in y_1 to get

$$I(\tau,\xi) = \int_{\mathbb{R}^d} \frac{1}{\phi(|x|)\phi(|\xi - x|)} \delta\left(\tau - \phi(|x|) - \phi(|\xi - x|)\right) \mathrm{d}x.$$

Next, using the homogeneity of the delta measure we can multiply its argument by

$\tau - \phi(|x|) + \phi(|\xi - x|)$ and obtain

$$
\begin{aligned}
I(\tau, \xi) &= \int_{\mathbb{R}^d} \frac{\tau - \phi(|x|) + \phi(|\xi - x|)}{\phi(|x|)(\tau - \phi(|x|))} \delta\left((\tau - \phi(|x|))^2 - \phi(|\xi - x|)^2\right) \,\mathrm{d}x \\
&= 2 \int_{\mathbb{R}^d} \frac{1}{\phi(|x|)} \delta\left(\tau^2 - |\xi|^2 - 2\tau\phi(|x|) + 2\xi \cdot x\right) \,\mathrm{d}x \\
&= 2 \int_0^{\sqrt{\tau^2 - 1}} \int_{\mathbb{S}^{d-1}} \frac{r^{d-1}}{\phi(r)} \delta\left(\tau^2 - |\xi|^2 - 2\tau\phi(r) + 2r\xi \cdot \omega\right) \,\mathrm{d}\omega \mathrm{d}r,
\end{aligned}
$$

by polar co-ordinates. Using the Funk–Hecke theorem, after some rearranging we obtain that

$$
|\xi| I(\tau, \xi) = |\mathbb{S}^{d-2}| \int_0^{\sqrt{\tau^2-1}} \int_{-1}^1 r^{d-3}(1 - t^2)^{\frac{d-3}{2}} \frac{r}{\phi(r)} \delta\left(\frac{\tau^2 - |\xi|^2 - 2\tau\phi(r)}{2r|\xi|} + t\right) \,\mathrm{d}t\mathrm{d}r.
$$

Applying the change of variables $\phi(r) = u$ so that $\frac{r}{\phi(r)}\mathrm{d}r = \mathrm{d}u$, we have that

$$
\begin{aligned}
\frac{|\xi| I(\tau, \xi)}{|\mathbb{S}^{d-2}|} &= \int_1^\tau \int_{-1}^1 \left[(u^2 - 1)(1 - t^2)\right]^{\frac{d-3}{2}} \delta\left(\frac{\tau^2 - |\xi|^2 - 2\tau u}{2|\xi|\sqrt{u^2 - 1}} + t\right) \,\mathrm{d}t\mathrm{d}u \\
&= \int_1^\tau \int_{-1}^1 \left[(u^2 - 1) - t^2(u^2 - 1)\right]^{\frac{d-3}{2}} \delta\left(\frac{\tau^2 - |\xi|^2 - 2\tau u}{2|\xi|\sqrt{u^2 - 1}} + t\right) \,\mathrm{d}t\mathrm{d}u \\
&= \int_1^\tau \int_{-1}^1 \left[(u^2 - 1) - \frac{(\tau^2 - |\xi|^2 - 2\tau u)^2}{4|\xi|^2}\right]^{\frac{d-3}{2}} \delta\left(\frac{\tau^2 - |\xi|^2 - 2\tau u}{2|\xi|\sqrt{u^2 - 1}} + t\right) \,\mathrm{d}t\mathrm{d}u \\
&= \int_S \left[u^2 - 1 - \frac{(\tau^2 - |\xi|^2 - 2\tau u)^2}{4|\xi|^2}\right]^{\frac{d-3}{2}} \,\mathrm{d}u,
\end{aligned}
$$

where

$$
S := \left\{u \in \mathbb{R} : -1 \leq \frac{\tau^2 - |\xi|^2 - 2\tau u}{2|\xi|\sqrt{u^2 - 1}} \leq 1\right\} \cap [1, \tau].
$$

Note that the square bracketed term is quadratic in u:

$$
\begin{aligned}
u^2 - 1 - \frac{(\tau^2 - |\xi|^2 - 2\tau u)^2}{4|\xi|^2} &= \frac{1}{4|\xi|^2}\left(4(|\xi|^2 - \tau^2)u^2 + 4\tau(\tau^2 - |\xi|^2)u - (\tau^2 - |\xi|^2)^2 - 4|\xi|^2\right) \\
&= \frac{\tau^2 - |\xi|^2}{|\xi|^2}\left(u - \frac{\tau}{2} + \frac{|\xi|}{2}\sqrt{\frac{\tau^2 - |\xi|^2 - 4}{\tau^2 - |\xi|^2}}\right)\left(\frac{\tau}{2} + \frac{|\xi|}{2}\sqrt{\frac{\tau^2 - |\xi|^2 - 4}{\tau^2 - |\xi|^2}} - u\right) \\
&= \frac{\tau^2 - |\xi|^2}{|\xi|^2}\left(\Phi_+ - u\right)\left(u - \Phi_-\right),
\end{aligned}
$$

for

$$
\Phi_\pm := \frac{\tau}{2} \pm \frac{|\xi|}{2}\sqrt{\frac{\tau^2 - |\xi|^2 - 4}{\tau^2 - |\xi|^2}}.
$$

Using this calculation and the fact that $t \in [-1, 1]$ if and only if $1 - t^2 \geq 0$, it follows that

$$
S = [\Phi_-, \Phi_+] \cap [1, \tau] = [\Phi_-, \Phi_+],
$$

where the latter equality is a consequence of the inclusion $[\Phi_-, \Phi_+] \subseteq [1, \tau]$. To see that this holds, we first note that the inequality $\Phi_- > 1$ is equivalent to

$$
\tau - 2 > |\xi|\left(1 - \frac{4}{\tau^2 - |\xi|^2}\right)^{\frac{1}{2}}. \tag{2.16}
$$

Our assumption on (τ, ξ) implies in particular that $\tau > 2$. Therefore, after squaring both sides of (2.16) and rearranging, we see that it suffices to show

$$
\tau^4 - 4\tau^3 + 4\tau^2 - 2\tau^2|\xi|^2 + 4\tau|\xi|^2 + |\xi|^4 > 0,
$$

but this follows at once from the observation that

$$
\tau^4 - 4\tau^3 + 4\tau^2 - 2\tau^2|\xi|^2 + 4\tau|\xi|^2 + |\xi|^4 > \left((\tau - 1)^2 - |\xi|^2 - 1\right)^2,
$$

and hence $\Phi_- > 1$. On the other hand, that $\Phi_+ < \tau$ is obvious since $|\xi| < \tau$ and $1 - \frac{4}{\tau^2 - |\xi|^2} \leq 1$ whenever $\tau^2 \geq 4 + |\xi|^2$. In all we obtain that

$$\begin{aligned}
I(\tau,\xi) &= |\mathbb{S}^{d-2}| \frac{(\tau^2 - |\xi|^2)^{\frac{d-3}{2}}}{|\xi|^{d-2}} \int_{\Phi_-}^{\Phi_+} (u - \Phi_-)^{\frac{d-3}{2}} (\Phi_+ - u)^{\frac{d-3}{2}} \, \mathrm{d}u \\
&= |\mathbb{S}^{d-2}| \frac{(\tau^2 - |\xi|^2)^{\frac{d-3}{2}}}{|\xi|^{d-2}} \int_{0}^{\Phi_+ - \Phi_-} u^{\frac{d-3}{2}} (\Phi_+ - \Phi_- - u)^{\frac{d-3}{2}} \, \mathrm{d}u \\
&= |\mathbb{S}^{d-2}| \frac{(\tau^2 - |\xi|^2)^{\frac{d-3}{2}}}{|\xi|^{d-2}} (\Phi_+ - \Phi_-)^{d-2} \mathrm{B}\left(\tfrac{d-1}{2}, \tfrac{d-1}{2}\right),
\end{aligned}$$

and the desired expression for $I(\tau,\xi)$ follows from the fact that

$$\Phi_+ - \Phi_- = |\xi| \left(\frac{\tau^2 - |\xi|^2 - 4}{\tau^2 - |\xi|^2} \right)^{\frac{1}{2}},$$

and the identities

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Gamma(x)\Gamma(x + \tfrac{1}{2}) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x)$$

for the beta and Gamma functions, as well as the formula

$$\frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^{d-2}|} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}$$

relating the measure of the unit spheres in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$. $\square$

By a simple calculation, one can show that if $y = (y_1, y_2)$ with $y_1 + y_2 = \xi$ and $\phi(|y_1|) + \phi(|y_2|) = \tau$, then

$$K(y) = \frac{1}{2^{\frac{d-3}{2}}} \frac{(\tau^2 - |\xi|^2 - 4)^{\frac{d-2}{2}}}{(\tau^2 - |\xi|^2)^{\frac{1}{2}}},$$

and so if we integrate the inequality (2.15) for $|\widetilde{u}|^2$ with respect to τ and ξ, apply

Plancherel's theorem and change the order of integration, we obtain

$$\begin{aligned}\|u\|^2_{L^2_{x,t}} &= \frac{1}{(2\pi)^{d+1}} \|\widetilde{u}\|^2_{L^2_{\xi,\tau}} \\ &\leq \frac{2^{-\frac{d-1}{2}} \left|\mathbb{S}^{d-1}\right|}{(2\pi)^{3d-1}} \int_{\mathbb{R}^{2d}} \left|\widehat{f}_1(y_1)\right|^2 \left|\widehat{f}_2(y_2)\right|^2 K(y)\phi(|y_1|)\phi(|y_2|)\,\mathrm{d}y.\end{aligned}$$

Moreover, if we consider the functions f_j defined by

$$\phi(|y_j|)\widehat{f}_j(y_j) = e^{-a\phi(|y_j|)},$$

for $a > 0$ (and $j = 1, 2$), we immediately obtain that

$$\widehat{F}(y) = \frac{e^{-a\tau}}{\sqrt{\phi(|y_1|)\phi(|y_2|)}}, \tag{2.17}$$

on the support of the delta measures. Since the only place an inequality was used was in the application of the Cauchy–Schwarz inequality, this implies that we have equality for such functions. Indeed, the equality (2.17) implies the existence of a scalar function $g = g(\xi, \tau)$ so that

$$\widehat{F}(y) = g(\xi, \tau)(\phi(|y_1|)\phi(|y_2|))^{-\frac{1}{2}}$$

almost everywhere on the support of the delta measures. Hence we have equality in (2.15) for these functions f_j, and thus also in (2.3) for the constant

$$\mathbf{KG}(d) = \frac{2^{-\frac{d-1}{2}} \left|\mathbb{S}^{d-1}\right|}{(2\pi)^{3d-1}},$$

implying that it is best possible.

2.2.2 The case $d = 1$

We note that formally, the calculation allowing us to derive (2.3) also makes sense for $d = 1$. However, substituting $d = 1$ into the expression for K gives

$$\begin{aligned} K(y) &= [(\phi(|y_1|)\phi(|y_2|) - y_1y_2 + 1)\,(\phi(|y_1|)\phi(|y_2|) - y_1y_2 - 1)]^{-\frac{1}{2}} \\ &= \left((\phi(|y_1|)\phi(|y_2|) - y_1y_2)^2 - 1\right)^{-\frac{1}{2}} \\ &= \left(y_1^2 + y_2^2 + 2y_1^2y_2^2 - 2y_1y_2\phi(|y_1|)\phi(|y_2|)\right)^{-\frac{1}{2}}, \end{aligned}$$

and since this weight is singular on the diagonal $\{(y_1, y_2) \in \mathbb{R}^2 : y_1 = y_2\}$, it is not difficult to construct a pair of integrable functions (f_1, f_2) for which the integral given by the right-hand side of (2.3) is unbounded. However the weight K is well-defined for $y_1 \neq y_2$ and if we assume that f_1 and f_2 have disjointly supported Fourier transforms, we have the identity (2.5). To prove (2.5) we follow a method used in [33] for restriction estimates on the sphere (see also [21], [43], [65] and [76]). Specifically, we write

$$\begin{aligned} &e^{it\phi(\sqrt{-\Delta})}f_1(x)\overline{e^{it\phi(\sqrt{-\Delta})}f_2(x)} \\ &\qquad\qquad = \int_{\mathbb{R}^2} e^{ix(y_1-y_2)}e^{it((1+y_1^2)^{\frac{1}{2}}-(1+y_2^2)^{\frac{1}{2}})}\widehat{f}_1(y_1)\overline{\widehat{f}_2(y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2. \end{aligned}$$

If we use the one-to-one change of variables $(y_1, y_2) \mapsto (u, v)$, where $u = y_1 - y_2$ and $v = \phi(|y_1|) - \phi(|y_2|)$, then the Jacobian will be

$$\left|\det\begin{pmatrix} 1 & -1 \\ \frac{y_1}{\sqrt{1+y_1^2}} & -\frac{y_2}{\sqrt{1+y_2^2}} \end{pmatrix}\right|^{-1} = \frac{\phi(|y_1|)\phi(|y_2|)}{|\phi(|y_1|)y_2 - \phi(|y_2|)y_1|}.$$

Hence, we have

$$e^{it\phi(\sqrt{-\Delta})}f_1(x)\overline{e^{it\phi(\sqrt{-\Delta})}f_2(x)} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2_+} e^{ixu}e^{itv}H(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

where H is defined by

$$H(u,v) = \frac{\phi(|y_1|)\phi(|y_2|)}{|\phi(|y_1|)y_2 - \phi(|y_2|)y_1|}\widehat{f_1}(y_1)\widehat{f_2}(y_2).$$

By Plancherel's theorem,

$$\left\| e^{it\phi(\sqrt{-\Delta})}f_1 e^{it\phi(\sqrt{-\Delta})}f_2 \right\|^2_{L^2_{x,t}} = \frac{1}{(2\pi)^2}\|H\|^2_{L^2_{u,v}} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} |H(u,v)|^2\,\mathrm{d}u\mathrm{d}v.$$

By reversing the change of variables done in the previous step, this becomes

$$\|u\|^2_{L^2} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} |\widehat{f_1}(y_1)|^2|\widehat{f_2}(y)|^2\frac{\phi(|y_1|)\phi(|y_2|)}{|y_1\phi(|y_2|) - y_2\phi(|y_1|)|}\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Further, by a direct calculation, it is easily verified that

$$\begin{aligned}
\Big((1+y_1^2)^{\frac{1}{2}}y_2 - (1+y_2^2)^{\frac{1}{2}}y_1\Big)^2 &= (1+y_1^2)y_2^2 + (1+y_2^2)y_1^2 - 2y_1y_2(1+y_1^2)^{\frac{1}{2}}(1+y_2^2)^{\frac{1}{2}} \\
&= y_1^2 + y_2^2 + 2y_1^2y_2^2 - 2y_1y_2(1+y_1^2)^{\frac{1}{2}}(1+y_2^2)^{\frac{1}{2}} \\
&= K(y)^{-2}.
\end{aligned}$$

We remark that from the above we can see that the only singularity of the weight K would be at a point in $\mathbb{R}^2$ where

$$y_1(1+y_2^2)^{\frac{1}{2}} = y_2(1+y_1^2)^{\frac{1}{2}},$$

which can only happen if $y_1 = y_2$.

2.3 Proof of Corollaries 2.1.2 and 2.1.3

We begin by establishing Corollary 2.1.3 and show how to deduce Corollary 2.1.2. If we set $d = 5$ and $f_1 = f_2 = f$ in (2.3), then the right-hand side reduces to

$$\begin{aligned}
&\int_{\mathbb{R}^{10}} |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 \phi(|y_1|)\phi(|y_2|) \frac{(\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 - 1)^{\frac{3}{2}}}{(\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 + 1)^{\frac{1}{2}}} \,\mathrm{d}y \\
&\qquad \leq \int_{\mathbb{R}^{10}} |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 \phi(|y_1|)\phi(|y_2|) \left(\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 - 1\right) \,\mathrm{d}y \\
&\qquad \leq I_1 - I_2,
\end{aligned}$$

where

$$I_1 = \int_{\mathbb{R}^{10}} \left(\phi(|y_1|)\phi(|y_2|)\right)^2 |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 \,\mathrm{d}y_1 \,\mathrm{d}y_2,$$

and

$$I_2 = \int_{\mathbb{R}^{10}} |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 \phi(|y_1|)\phi(|y_2|) y_1 \cdot y_2 \,\mathrm{d}y_1 \,\mathrm{d}y_2.$$

We can now use the observation from [22] that

$$\int_{\mathbb{R}^{2d}} f(x) f(y) x \cdot y \,\mathrm{d}x \,\mathrm{d}y \geq 0 \tag{2.18}$$

for any function f, with equality if f is radial, to obtain that $I_2 \geq 0$. Hence, by Plancherel's theorem we have that

$$\left\| e^{it\phi(\sqrt{-\Delta})} f \right\|_{L^4}^4 \leq (2\pi)^{10} \mathbf{KG}(5) \left\| \sqrt{1-\Delta} f \right\|_{L^2(\mathbb{R}^5)}^4 = \frac{1}{24\pi^2} \|f\|_{H^1}^4. \tag{2.19}$$

Note however that we have used that

$$K(y) \leq \phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 \tag{2.20}$$

for any $y = (y_1, y_2) \in \mathbb{R}^{5\times 2}$, and this inequality is of course pointwise strict, but as with the L^∞ analysis of the convolution of the measure σ in [73] (Corollary 4.3, Lemma 4.4 and Lemma 4.5) we claim that when normalised, the functions f_a form a maximising sequence for the inequality (2.19), as $a \to 0+$. As a consequence of this and inequality (2.20) we will obtain that the inequality (2.19) is sharp, and that there are no maximisers. We recall that the functions f_a are defined by

$$\widehat{f_a}(x) = \frac{e^{-a\phi(|x|)}}{\phi(|x|)},$$

for $a > 0$. By Theorem 2.1.1, these satisfy inequality (2.3) with equality, and by the observation after inequality (2.18) we also have that $I_2 = 0$ for such functions.

Lemma 2.3.1. *Suitably normalised, the functions f_a form a maximising sequence for the inequality* (2.19). *That is, we have that*

$$\lim_{a\to 0+} \frac{\left\| e^{it\phi(\sqrt{-\Delta})} f_a \right\|_{L^4}^4}{\left\| \sqrt{1-\Delta} f_a \right\|_{L^2(\mathbb{R}^5)}^4} = (2\pi)^{10} \mathbf{KG}(5).$$

Proof. To prove Lemma 2.3.1 we modify the approach in [73]. Firstly, we calculate

$$\begin{aligned}
(2\pi)^5 \left\| \phi(\sqrt{-\Delta}) f_a \right\|_{L^2}^2 &= \int_{\mathbb{R}^5} e^{-2a\phi(|x|)}\,\mathrm{d}x \\
&= \left|\mathbb{S}^4\right| \int_0^\infty e^{-2a\sqrt{1+r^2}} r^4\,\mathrm{d}r \\
&= \left|\mathbb{S}^4\right| \int_1^\infty e^{-2au} \left(u^2-1\right)^{\frac{3}{2}} u\,\mathrm{d}u \\
&= \frac{\left|\mathbb{S}^4\right|}{a^2} \int_a^\infty e^{-2x} \left(\left(\frac{x}{a}\right)^2 - 1\right)^{\frac{3}{2}} x\,\mathrm{d}x \\
&= \frac{\left|\mathbb{S}^4\right|}{a^5} \int_a^\infty e^{-2x} \left(x^2-a^2\right)^{\frac{3}{2}} x\,\mathrm{d}x,
\end{aligned}$$

so that

$$\lim_{a\to 0+} a^5(2\pi)^5 \left\|\sqrt{1-\Delta} f_a\right\|_{L^2}^2 = \frac{3}{4}\left|\mathbb{S}^4\right|. \tag{2.21}$$

We now wish to evaluate

$$\lim_{a\to 0+} a^{10} \left\|e^{it\phi(\sqrt{-\Delta})} f_a\right\|_{L^4(\mathbb{R}^{5+1})}^4.$$

Using Plancherel's theorem and then (2.13) we have

$$\begin{aligned}
\left\|e^{it\phi(\sqrt{-\Delta})} f_a\right\|_{L^4}^4 &= \left\|\left(e^{it\phi(\sqrt{-\Delta})} f_a\right)^2\right\|_{L^2}^2 \\
&= \frac{1}{(2\pi)^{18}} \left\|\widetilde{e^{it\phi(\sqrt{-\Delta})} f_a} * \widetilde{e^{it\phi(\sqrt{-\Delta})} f_a}\right\|_{L^2}^2 \\
&= \frac{1}{(2\pi)^{14}} \left\|\int_{\mathbb{R}^{10}} \frac{e^{-a(\phi(|y_1|)+\phi(|y_2|))}}{\phi(|y_1|)\phi(|y_2|)} \delta\begin{pmatrix}\tau-\phi(|y_1|)-\phi(|y_2|)\\ \xi-y_1-y_2\end{pmatrix} \mathrm{d}y_1\mathrm{d}y_2\right\|_{L^2_{\xi,\tau}}^2 \\
&= \frac{1}{(2\pi)^{14}} \left\|\int_{\mathbb{R}^{10}} \frac{e^{-a\tau}}{\phi(|y_1|)\phi(|y_2|)} \delta\begin{pmatrix}\tau-\phi(|y_1|)-\phi(|y_2|)\\ \xi-y_1-y_2\end{pmatrix} \mathrm{d}y\right\|_{L^2_{\xi,\tau}}^2.
\end{aligned}$$

By Lemma 2.2.1, we obtain

$$\begin{aligned}
\left\|e^{it\phi(\sqrt{-\Delta})} f_a\right\|_{L^4}^4 &= \frac{\left|\mathbb{S}^4\right|^2}{2^6(2\pi)^{14}} \int_{\mathbb{R}^{5+1}} e^{-2a\tau} \frac{(\tau^2-|\xi|^2-4)^3}{\tau^2-|\xi|^2} \chi_{\left\{\tau\geq\sqrt{4+|\xi|^2}\right\}} \mathrm{d}\xi\mathrm{d}\tau \\
&= \frac{\left|\mathbb{S}^4\right|^2}{2^6(2\pi)^{14}} \int_{\mathcal{H}} e^{-2a\tau} \left(\tau^2-|\xi|^2-4\right)^2 \left(1-\frac{4}{\tau^2-|\xi|^2}\right) \mathrm{d}\xi\,\mathrm{d}\tau \\
&= \frac{\left|\mathbb{S}^4\right|^2}{2^6(2\pi)^{14}} (\mathrm{I}_a - \mathrm{II}_a),
\end{aligned}$$

where

$$\mathrm{I}_a := \int_{\mathcal{H}} e^{-2a\tau} \left(\tau^2-|\xi|^2-4\right)^2 \mathrm{d}\xi\,\mathrm{d}\tau$$

and

$$\mathrm{II}_a := 4\int_{\mathcal{H}} e^{-2a\tau} \frac{\left(\tau^2-|\xi|^2-4\right)^2}{\tau^2-|\xi|^2} \mathrm{d}\xi\,\mathrm{d}\tau.$$

Claim 2.3.2. *We have that*

$$\lim_{a\to 0+} a^{10}\mathrm{I}_a = \left(\frac{1}{5}-\frac{2}{7}+\frac{1}{9}\right)\frac{|\mathbb{S}^4|\,9!}{2^{10}},$$

and

$$\lim_{a\to 0+} a^{10}\mathrm{II}_a = 0.$$

Assuming Claim 2.3.2 to be true for the moment, it then follows that

$$\lim_{a\to 0+} a^{10}\left\|e^{it\phi(\sqrt{-\Delta})}f_a\right\|^4_{L^4(\mathbb{R}^6)} = \frac{|\mathbb{S}^4|^3}{(2\pi)^{14}}\left(\frac{1}{5}+\frac{1}{9}-\frac{2}{7}\right)\frac{9!}{2^{16}}.$$

By (2.21),

$$\lim_{a\to 0+} a^{10}(2\pi)^{10}\left\|\sqrt{1-\Delta}f_a\right\|^4_{L^2(\mathbb{R}^5)} = \frac{9}{16}\left|\mathbb{S}^4\right|^2,$$

and so we obtain

$$\lim_{a\to 0+}\frac{\left\|e^{it\phi(\sqrt{-\Delta})}f_a\right\|^4_{L^4(\mathbb{R}^6)}}{(2\pi)^{10}\left\|\sqrt{1-\Delta}f_a\right\|^4_{L^2(\mathbb{R}^5)}} = \mathbf{KG}(5)$$

as claimed, and therefore the constant $(2\pi)^{10}\mathbf{KG}(5)$ is optimal for the inequality (2.10). It now remains to prove Claim 2.3.2.

Proof of Claim 2.3.2. We deal with the integral I_a first. Using polar co-ordinates and simple changes of variables, we have

$$
\begin{aligned}
a^{10}\mathrm{I}_a &= a^{10}\left|\mathbb{S}^4\right| \int_2^\infty e^{-2a\tau} \int_0^{\sqrt{\tau^2-4}} \left(\tau^2 - r^2 - 4\right)^2 r^4 \,\mathrm{d}r\mathrm{d}\tau \\
&= a^{10}\left|\mathbb{S}^4\right| \int_2^\infty e^{-2a\tau}(\tau^2-4)^{\frac{9}{2}} \int_0^1 \left(1 - r^2\right)^2 r^4 \,\mathrm{d}r\mathrm{d}\tau \\
&= a^{10}\left(\frac{1}{5}+\frac{1}{9}-\frac{2}{7}\right)\left|\mathbb{S}^4\right| \int_2^\infty e^{-2a\tau}(\tau^2-4)^{\frac{9}{2}} \,\mathrm{d}\tau \\
&= \left(\frac{1}{5}+\frac{1}{9}-\frac{2}{7}\right)\left|\mathbb{S}^4\right| \int_{2a}^\infty e^{-2x}(x^2-(2a)^2)^{\frac{9}{2}} \,\mathrm{d}x.
\end{aligned}
$$

In all, since

$$
\int_0^\infty x^\ell e^{-2x}\,\mathrm{d}x = \frac{\ell!}{2^{\ell+1}},
$$

we obtain that

$$
a^{10}\mathrm{I}_a \to \left(\frac{1}{5}-\frac{2}{7}+\frac{1}{9}\right)\frac{|\mathbb{S}^4|\,9!}{2^{10}}
$$

as $a \to 0+$. The term II_a is more easily dealt with since

$$
\begin{aligned}
0 \le a^{10}\mathrm{II}_a &\le Ca^{10}\int_{\mathcal{H}} e^{-2a\tau}(\tau^2 - |\xi|^2 - 4)\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\le Ca^{10}\int_2^\infty e^{-2a\tau}\tau^2 \int_0^{\sqrt{\tau^2-4}} r^4\,\mathrm{d}r\,\mathrm{d}\tau \\
&= Ca^{10}\int_2^\infty e^{-2a\tau}\tau^2(\tau^2-4)^{\frac{5}{2}}\,\mathrm{d}\tau \\
&= Ca^2\int_{2a}^\infty e^{-2x}x^2(x^2-(2a)^2)^{\frac{5}{2}}\,\mathrm{d}x \to 0
\end{aligned}
$$

as $a \to 0+$, as required. □

We conclude the section by showing how Corollary 2.1.2 is deduced from Corollary 2.1.3, to do this we follow the approach of Foschi in [34]. Suppose u solves (2.2), we recall the decomposition $u = u_+ + u_-$, where

$$u_+ = e^{it\phi(\sqrt{-\Delta})} f_+, \quad u_- = e^{-it\phi(\sqrt{-\Delta})} f_-,$$

for

$$u(0) = f_+ + f_-, \quad \partial_t u(0) = i\phi(\sqrt{-\Delta})(f_+ - f_-).$$

Then

$$\|u\|_{L^4}^4 = \|u_+ + u_-\|_{L^4}^4 = \left\|u_+^2 + u_-^2 + 2u_+u_-\right\|_{L^2}^2.$$

We claim that the supports of the space-time Fourier transforms of the functions u_+^2, u_-^2 and u_+u_- are pairwise disjoint. We have already seen that

$$\operatorname{supp} \widetilde{u_+^2} \subseteq \left\{(\xi,\tau) \in \mathbb{R}^{d+1} : \tau \geq \sqrt{4+|\xi|^2}\right\},$$

and by an identical argument we have that

$$\operatorname{supp} \widetilde{u_-^2} \subseteq \left\{(\xi,\tau) \in \mathbb{R}^{d+1} : \tau \leq -\sqrt{4+|\xi|^2}\right\}.$$

It remains to show that

$$\operatorname{supp} \widetilde{u_+u_-} \subseteq \left\{(\xi,\tau) \in \mathbb{R}^{d+1} : |\tau| \leq \sqrt{4+|\xi|^2}\right\}.$$

We note that this was shown in [73], we include it here for completeness. Note that analogously to (2.13) we will have, for $(\xi,\tau) \in \mathbb{R}^{d+1}$,

$$\widetilde{u_+u_-}(\xi,\tau) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{2d}} \frac{\widehat{F}(y)}{(1+|y_1|^2)^{\frac{1}{4}}(1+|y_2|^2)^{\frac{1}{4}}} \delta\begin{pmatrix} \tau - \phi(|y_1|) + \phi(|y_2|) \\ \xi - y_1 - y_2 \end{pmatrix} \mathrm{d}y.$$

Set $\xi = y_1 + y_2$ and $\tau = \phi(|y_1|) - \phi(|y_2|)$, then we have

$$|\xi|^2 = |y_1|^2 + |y_2|^2 + 2y_1 \cdot y_2,$$

and

$$\tau^2 = 2 + |y_1|^2 + |y_2|^2 - 2\phi(|y_1|)\phi(|y_2|),$$

so that

$$\tau^2 - |\xi|^2 = 2 - 2\phi(|y_1|)\phi(|y_2|) - 2y_1 \cdot y_2 \leq 4,$$

where the final inequality follows from (2.14). Hence,

$$\|u\|_{L^4}^4 = \left\|u_+^2\right\|_2^2 + \left\|u_-^2\right\|_2^2 + 4\left\|u_+ u_-\right\|_2^2.$$

Combining the preceding equality with the sharp polynomial inequality for non-negative real numbers X and Y

$$X^2 + Y^2 + 4XY \leq \frac{3}{2}(X + Y)^2,$$

with equality if and only if $X = Y$, we obtain the following:

$$\|u\|_{L^4}^4 \leq \frac{3}{2}\left(\|u_+\|_{L^4}^2 + \|u_-\|_{L^4}^2\right)^2.$$

Applying (2.10), we obtain that

$$\|u\|_{L^4}^4 \leq \frac{1}{16\pi^2}\left(\|f_+\|_{H^1}^2 + \|f_-\|_{H^1}^2\right)^2.$$

But then, by the definition of f_+ and f_- and the parallelogram law, the right-hand side equals

$$\frac{1}{16\pi^2}\left(\frac{1}{2}\|u(0)\|_{H^1}^2+\frac{1}{2}\left\|(\sqrt{1-\Delta})^{-1}\partial_t u(0)\right\|_{H^1}^2\right)^2=\frac{1}{64\pi^2}\left(\|u(0)\|_{H^1}^2+\|\partial_t u(0)\|_{L^2}^2\right)^2,$$

which completes the proof of Corollary 2.1.2. □

2.4 Proof of Proposition 2.1.4

In this section, it will be convenient to abuse notation slightly and think of ϕ as a function on $\mathbb{R}^5$ by identifying with $\phi(|.|)$. If we define

$$\|G\|_{(\tau,\xi)}^2=\int_{\mathbb{R}^{10}}|G(y_1,y_2)|^2\,\delta\begin{pmatrix}\tau-\phi\left(|y_1|\right)-\phi(|y_2|)\\ \xi-y_1-y_2\end{pmatrix}\mathrm{d}y_1\mathrm{d}y_2,$$

where G is a function on $\mathbb{R}^{2d}$, then by the proof of the bilinear inequality of Theorem 2.1.1 and by Lemma 2.2.1, we have that

$$\begin{aligned}\left\|e^{it\phi(\sqrt{-\Delta})}g_n\right\|_{L^4(\mathbb{R}^6)}^4 &\leq \frac{\mathbf{KG}(5)}{2}\int_{\mathcal{H}}\|\phi^{\frac{1}{2}}\widehat{g_n}\otimes\phi^{\frac{1}{2}}\widehat{g_n}\|_{(\tau,\xi)}^2\frac{(\tau^2-|\xi|^2-4)^{\frac{3}{2}}}{(\tau^2-|\xi|^2)^{\frac{1}{2}}}\,\mathrm{d}\xi\mathrm{d}\tau\\ &\leq \frac{\mathbf{KG}(5)}{2}\int_{\mathcal{H}}\|\phi^{\frac{1}{2}}\widehat{g_n}\otimes\phi^{\frac{1}{2}}\widehat{g_n}\|_{(\tau,\xi)}^2\left(\tau^2-|\xi|^2-4\right)\,\mathrm{d}\xi\,\mathrm{d}\tau\\ &\leq \frac{1}{24\pi^2}-\mathcal{I}_n-\mathcal{J}_n,\end{aligned}$$

where

$$\mathcal{I}_n=\mathbf{KG}(5)\int_{\mathbb{R}^{10}}|\widehat{g_n}(y_1)|^2|\widehat{g_n}(y_2)|^2\phi(|y_1|)\phi(|y_2|)y_1\cdot y_2\,\mathrm{d}y_1\mathrm{d}y_2\geq 0,$$

and

$$\mathcal{J}_n=\mathbf{KG}(5)\int_{\mathbb{R}^{10}}|\widehat{g_n}(y_1)|^2|\widehat{g_n}(y_2)|^2\phi(|y_1|)\phi(|y_2|)\,\mathrm{d}y_1=\mathbf{KG}(5)\left(\int_{\mathbb{R}^5}|\widehat{g_n}(y)|^2\phi(|y|)\,\mathrm{d}y\right)^2.$$

But then since $(g_n)_{n\geq 1}$ is a maximising sequence for inequality (2.10), it follows that $\mathcal{I}_n, \mathcal{J}_n \to 0$ as $n \to \infty$. In particular,

$$\int_{\mathbb{R}^5} |\widehat{g_n}(y_1)|^2 \, \phi(|y_1|) \, \mathrm{d}y_1 \to 0$$

as $n \to \infty$.

Now, to prove (2.12), using the fact that on the delta measures we have that

$$\phi(|y_1|)\phi(|y_2|) - y_1 \cdot y_2 + 1 = \frac{1}{2}\left(\tau^2 - |\xi|^2\right), \tag{2.22}$$

if $y_1, y_2 \in B(0,R)$ it is easy to see that for such τ, ξ,

$$\tau^2 - |\xi|^2 \leq 2(R^2+1).$$

Thus,

$$\begin{aligned}\int_{\mathcal{H}} \int_{B(0,R)} &|\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \phi(|y_1|)\phi(|y_2|)\left(\tau^2 - |\xi|^2\right) \\ &\qquad\qquad \times \delta\begin{pmatrix}\tau - \phi(|y_1|) - \phi(|y_2|) \\ \xi - y_1 - y_2\end{pmatrix} \mathrm{d}y \, \mathrm{d}\xi \mathrm{d}\tau \\ &\leq 2(R^2+1)\left(\int_{\mathbb{R}^5} |\widehat{g_n}(y_1)|^2 \phi(|y_1|) \, \mathrm{d}y_1\right)^2 \to 0\end{aligned}$$

as $n \to \infty$. Using (2.22) we obtain

$$\begin{aligned}
&\left(\int_{B(0,R)} |\widehat{g_n}(y_1)|^2 \phi(|y_1|)^2 \,\mathrm{d}y_1\right)^2 \\
&\leq \int_{B(0,R)} |\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \left(\phi(|y_1|)^2\phi(|y_2|)^2 + \phi(|y_1|)\phi(|y_2|)\right) \mathrm{d}y_1\mathrm{d}y_2 \\
&= \int_{B(0,R)} |\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \phi(|y_1|)\phi(|y_2|) y_1 \cdot y_2 \,\mathrm{d}y_1\mathrm{d}y_2 \\
&\qquad + \frac{1}{2}\int_{\mathcal{H}}\int_{B(0,R)} |\widehat{g_n}(y_1)|^2 |\widehat{g_n}(y_2)|^2 \phi(|y_1|)\phi(|y_2|) \left(\tau^2 - |\xi|^2\right) \\
&\qquad\qquad\qquad\qquad \times \delta\begin{pmatrix} \tau - \phi(|y_1|) - \phi(|y_2|) \\ \xi - y_1 - y_2 \end{pmatrix} \mathrm{d}y\mathrm{d}\xi\mathrm{d}\tau \\
&\leq \mathcal{I}_n + (R^2+1)\left(\int_{\mathbb{R}^5} |\widehat{g_n}(y_1)|^2 \phi(|y_1|)\,\mathrm{d}y_1\right)^2 \to 0
\end{aligned}$$

as $n \to \infty$. But then if ε, R are given we can choose N, as desired. □

2.5 Further remarks

1. If, after the application of the Cauchy–Schwarz inequality in the proof of (2.3), we instead take the function $I(\xi,\tau)$ to the left hand side of (2.15) we see that

$$\frac{1}{I(\xi,\tau)} |\widetilde{u}(\xi,\tau)|^2 \leq \frac{1}{(2\pi)^{2d-2}} \int_{\mathbb{R}^{2d}} |\widehat{F}(y)|^2 \,\delta\begin{pmatrix} \tau - \phi(|y_1|) - \phi(|y_2|) \\ \xi - y_1 - y_2 \end{pmatrix} \mathrm{d}y.$$

If we now integrate this inequality in τ and ξ as before, we can obtain the following alternative formulation of the inequality (2.3):

$$\left\| M(e^{it\phi(\sqrt{-\Delta})}f_1 e^{it\phi(\sqrt{-\Delta})}f_2) \right\|^2_{L^2(\mathbb{R}^{d+1})} \leq \frac{\left|\mathbb{S}^{d-1}\right|}{2^{d-2}(2\pi)^{d-1}} \|f_1\|^2_{H^{\frac{1}{2}}(\mathbb{R}^d)} \|f_2\|^2_{H^{\frac{1}{2}}(\mathbb{R}^d)}, \qquad (2.23)$$

where M is given by

$$\widetilde{Mu}(\xi,\tau) = \frac{(\tau^2 - |\xi|^2)^{\frac{1}{4}}}{(\tau^2 - |\xi|^2 - 4)^{\frac{d-2}{4}}} \widetilde{u}(\xi,\tau).$$

The constant is sharp and is attained on the functions given by (2.4); this follows from the argument used to deduce this property for (2.3). Further, using (2.23) and the pointwise inequality $I(\xi,\tau) \leq \frac{|\mathbb{S}^{d-1}|}{2^{d-2}}(\tau^2 - |\xi|^2 - 4)^{\frac{d-3}{2}}$ as in the proof of inequality (2.10) of Corollary 2.1.3, we can obtain the following:

$$\left\| (D_+D_-)^{\frac{3-d}{4}} \left(e^{it\sqrt{1-\Delta}} f_1 e^{it\sqrt{1-\Delta}} f_2 \right) \right\|^2_{L^2(\mathbb{R}^{d+1})} \leq \frac{\left|\mathbb{S}^{d-1}\right|}{2^{d-2}(2\pi)^{d-1}} \|f_1\|^2_{H^{\frac{1}{2}}(\mathbb{R}^d)} \|f_2\|^2_{H^{\frac{1}{2}}(\mathbb{R}^d)}, \tag{2.24}$$

where now the operators D_+ and D_- are given by

$$\widetilde{D_\pm u}(\xi,\tau) = 2\left(\frac{\tau}{2} \pm \phi\left(\left|\frac{\xi}{2}\right|\right)\right)\widetilde{u}(\xi,\tau).$$

The constant in (2.24) is sharp, but there are no maximising functions; the proof of this is a simple generalisation of the proof of the sharpness of the inequality (2.10) of Corollary 2.1.3. Estimate (2.24) is reminiscent of the Klainerman–Machedon type space-time estimates (1.13) for the wave equation, and the Ozawa–Tsutsumi-type estimates (1.19) for the Schrödinger equation. We will expand upon these points in Chapter 3 and Chapter 4.

2. We note that the proof of Corollary 2.1.3 actually yields

$$\left\| e^{it\phi(\sqrt{-\Delta})} f \right\|_{L^4(\mathbb{R}^6)} \leq \left(\frac{1}{24\pi^2}\right)^{\frac{1}{4}} \left(\|f\|^4_{H^1(\mathbb{R}^5)} - \|f\|^4_{H^{\frac{1}{2}}(\mathbb{R}^5)} \right)^{\frac{1}{4}}, \tag{2.25}$$

which is a refinement of the usual Strichartz estimate (2.10). In the proof of Proposition 2.1.4, we showed that

$$\|g_n\|_{H^{\frac{1}{2}}(\mathbb{R}^5)} \to 0$$

as $n \to \infty$, where $(g_n)_{n\geq 1}$ is any maximising sequence for (2.10). In particular, if $(f_a)_{a>0}$

is given by (2.11) then we have that

$$a^5 \|f_a\|_{H^{\frac{1}{2}}(\mathbb{R}^5)} \to 0$$

as $a \to 0+$. One can show that when suitably normalised (and modulo technicalities), the functions given by (2.11) form a maximising sequence for the refinement (2.25). Therefore, (2.25) is also sharp, and there are no maximisers.

Chapter 3

On sharp bilinear Strichartz estimates of Ozawa–Tsutsumi type

This chapter is devoted to the proofs of the sharp bilinear estimates for the free Schrödinger equation. The results in this chapter are joint with Jonathan Bennett, Neal Bez and Nikolaos Pattakos and are contained in [9]; further, we thank the anonymous referee for pointing out a simpler proof of the results in Section 3.3.

3.1 Introduction

For $d \geq 2$, consider the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0, \qquad u(0) = u_0 \tag{3.1}$$

on $\mathbb{R}^{1+d}$ with initial data $u_0 \in L^2(\mathbb{R}^d)$, where $d \geq 1$. In [67], Ozawa and Tsutsumi showed that any two solutions u and v of (3.1) with initial data u_0 and v_0, respectively, satisfy the global space-time bilinear estimate

$$\|(-\Delta)^{\frac{2-d}{4}}(u\overline{v})\|_{L^2}^2 \leq \mathbf{OT}(d)\|(u_0, v_0)\|_{L^2\times L^2}^2, \tag{3.2}$$

where we take

$$\|(u_0, v_0)\|_{L^2\times L^2} := \|u_0\|_{L^2}\|v_0\|_{L^2},$$

and we recall

$$\mathbf{OT}(d) = \frac{2^{-d}\pi^{\frac{2-d}{2}}}{\Gamma(\frac{d}{2})}.$$

They also showed that the constant $\mathbf{OT}(d)$ is optimal by observing that if $u_0(x) = v_0(x) = \exp(-|x|^2)$ then (3.2) is an equality; i.e. (u_0, v_0) is a maximiser for this estimate.

The case of one spatial dimension is rather special and in this case (3.2) is true as an identity

$$\|(-\Delta)^{\frac{1}{4}}(u\overline{v})\|_{L^2}^2 = \mathbf{OT}(1)\|(u_0, v_0)\|_{L^2\times L^2}^2$$

for any $(u_0, v_0) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. This identity was established in [67], gives control on the so-called null gauge form $\partial(u\overline{v})$ for the Schrödinger equation in one spatial dimension, and was used as a tool in the proof of local well-posedness of some nonlinear Schrödinger equations with nonlinearities involving $\partial(|u|^2)u$.

Based on the approach in [44], Carneiro proved in [22] that any two solutions u and v of (3.1) satisfy

$$\|uv\|_{L^2}^2 \leq \mathbf{C}(d)\int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2|\widehat{v}_0(\eta)|^2|\zeta-\eta|^{d-2}\,\mathrm{d}\zeta\mathrm{d}\eta, \tag{3.3}$$

where $d \geq 2$ and we recall that

$$\mathbf{C}(d) = \frac{2^{2-4d}\pi^{\frac{2-5d}{2}}}{\Gamma(\frac{d}{2})}.$$

It was shown in [22] that the constant in (3.3) is optimal and (u_0, v_0) is a maximiser if and only if $u_0(x) = v_0(x) = \exp(-|x|^2)$, up to certain transformations. A very closely related bilinear estimate

$$\|(-\Delta)^{\frac{3-d}{4}}(u\overline{v})\|_{L^2}^2 \leq \mathbf{PV}(d)\int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2|\widehat{v}_0(\eta)|^2|\zeta-\eta|\,\mathrm{d}\zeta\mathrm{d}\eta \tag{3.4}$$

for solutions u and v of (3.1) and $d \geq 2$ is a particular case of some far-reaching identities proved by Planchon and Vega in [70] using an innovative and radically different approach to those in [22], [34], [44] and [67]. We recall that the constant $\mathbf{PV}(d)$ is given by

$$\mathbf{PV}(d) = \frac{2^{-3d}\pi^{\frac{1-5d}{2}}}{\Gamma(\frac{d+1}{2})}$$

and can be shown to be optimal for the inequality (3.4). In this chapter, we show how to unify (3.2), (3.3) and (3.4) by seeing these sharp estimates as special cases of a one-parameter family of sharp estimates. Varying this parameter represents to a trade-off of lowering the exponent on the kernel $|\zeta-\eta|$ on the right-hand side, which may be viewed as lowering the "derivatives" on the right-hand side, with a lowering of the order of derivatives on $|u|^2$ on the left-hand side (very much in the spirit of [50]).

To state our first main result, we introduce the space

$$\Upsilon_\lambda := \{(f,g) : f,g : \mathbb{R}^d \to \mathbb{C} \text{ measurable and } I_\lambda(f,g) < \infty\},$$

where

$$I_\lambda(f,g) := \int_{\mathbb{R}^{2d}} |\widehat{f}(\zeta)|^2|\widehat{g}(\eta)|^2|\zeta-\eta|^{4\lambda+d-2}\,\mathrm{d}\zeta\mathrm{d}\eta\,.$$

Theorem 3.1.1. *Let $d \geq 2$ and $\sigma > \frac{1-d}{4}$. Then*

$$\|(-\Delta)^{\sigma}(u\overline{v})\|_{L^2}^2 \leq \mathbf{OT}(d,\sigma)\int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2|\widehat{v}_0(\eta)|^2|\zeta-\eta|^{4\sigma+d-2}\,\mathrm{d}\zeta\mathrm{d}\eta \tag{3.5}$$

for solutions u and v of (3.1) *with initial data $(u_0, v_0) \in \Upsilon_\sigma$, respectively. Here,*

$$\mathbf{OT}(d,\sigma) = 2^{-3d}\pi^{\frac{1-5d}{2}} \frac{\Gamma(2\sigma + \frac{d-1}{2})}{\Gamma(2\sigma + d - 1)}$$

is the optimal constant. Furthermore, if $\sigma \in (\frac{1-d}{4}, \frac{2+d}{4}]$ then the pair of initial data (u_0, v_0) is a maximiser for (3.5) *if and only if*

$$\widehat{u}_0(\eta) = \exp(a|\eta|^2 + b \cdot \eta + c), \qquad \widehat{v}_0(\eta) = \exp(a|\eta|^2 + b \cdot \eta + d)$$

for some $a, c, d \in \mathbb{C}$, $b \in \mathbb{C}^d$ and $\mathrm{Re}(a) < 0$.

We have

$$\mathbf{OT}(d,\sigma) = \begin{cases} (2\pi)^{-2d}\mathbf{OT}(d) & \text{if } \sigma = \frac{2-d}{4} \\ \mathbf{C}(d) & \text{if } \sigma = 0 \\ \mathbf{PV}(d) & \text{if } \sigma = \frac{3-d}{4}. \end{cases}$$

To verify this for $\sigma = 0$ one should use the duplication formula,

$$\Gamma(z)\Gamma(z + \tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

for the Gamma function. Hence, when $\sigma = \frac{2-d}{4}$, estimate (3.5) obviously coincides with (3.2) after an application of Plancherel's theorem on the right-hand side. When $\sigma = 0$, (3.5) coincides with (3.3) since once the operator $(-\Delta)^\sigma$ disappears, the complex conjugate on v has no effect (we will soon see that for $\sigma \neq 0$, the complex conjugate plays an important role). Thus (3.5) unifies the sharp estimates (3.2), (3.3) and (3.4) of Ozawa–Tsutsumi [67], Carneiro [22] and Planchon–Vega [70], respectively.

As a consequence of our argument, we will obtain that any maximiser gives rise to a solution of the so-called Maxwell–Boltzmann functional equation; this functional equation is so named since it also arises in the proof of Boltzmann's H-theorem in connection with the derivation of hydrodynamic equations from Boltzmann's equation. We provide a proof that this equation admits only gaussian solutions under assumptions natural to our context, which will imply in particular that any maximiser for our estimate is as stated. For interest, in Section 3.5 we also provide a new, self-contained proof that the Maxwell–Boltzmann functional equation admits only gaussian solutions under the assumption of local integrability on the input functions which we believe is interesting in its own right, following and extending results of Foschi in [34] where the case $d = 2$ is established. The main feature of this approach is that it only requires the local integrability of $\widehat{u}_0$ and $\widehat{v}_0$ which is more natural in our context.

A new proof of the Ozawa–Tsutsumi estimate (3.2) was given in [8]. An advantage of this new proof was that it exposed an underlying heat-flow monotonicity phenomenon. Here, we prove (3.5) following the argument in [8] and provide a full characterisation of maximisers for every $\sigma \in (\frac{1-d}{4}, \frac{2+d}{4}]$. When $\sigma = \frac{2-d}{4}$, it was observed in [67] that equality holds with $u_0(x) = v_0(x) = \exp(a|x|^2)$ for any $a < 0$. We should point out that when $\sigma = 0$, a full characterisation of maximisers was provided in [22] using substantially different arguments to our own. The lower bound $\sigma > \frac{1-d}{4}$ is necessary; in particular, the optimal constant blows up at this threshold. The restriction on the upper bound for σ for the characterisation arises since we require integrability of $\widehat{u}_0$ and $\widehat{v}_0$ for our argument to solve the aforementioned Maxwell–Boltzmann functional equation. Our range contains all cases of particular interest $\sigma \in \{0, \frac{2-d}{4}, \frac{3-d}{4}, \frac{4-d}{4}\}$, but it is quite possible that this restriction can be relaxed by a refined, or alternative, analysis of these functional equations.

For $\sigma \in (\frac{1-d}{4}, \frac{2-d}{4})$ (so that, in particular, the exponent $4\sigma + d - 2$ on the kernel in (3.5) is negative) and $p, q \in (2, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = \frac{4\sigma+3d-2}{2d}$, it follows from the (forward) Hardy–Littlewood–Sobolev inequality that

$$\mathcal{F}L^p(\mathbb{R}^d) \times \mathcal{F}L^q(\mathbb{R}^d) \subseteq \Upsilon_\sigma\,,$$

where $\mathcal{F}L^p$ denotes the Fourier–Lebesgue space of measurable functions whose Fourier transform belongs to L^p; such spaces also capture smoothness by the correspondence between decay of the Fourier transform and smoothness. This gives control (albeit, no longer necessarily with optimal constants) on $(-\Delta)^\sigma(u\overline{v})$ in $L^2(\mathbb{R}\times\mathbb{R}^d)$ for initial data $(u_0, v_0) \in \mathcal{F}L^p(\mathbb{R}^d) \times \mathcal{F}L^q(\mathbb{R}^d)$, with σ, p and q as above.

We also remark that for such σ, via the Parseval identity, the quantity $I_\sigma(u_0, v_0)$ is given by

$$I_\sigma(u_0, v_0) = C_{d,\sigma} \int_{\mathbb{R}^d} \frac{\widehat{\mathrm{d}\mu}(x)\widehat{\mathrm{d}\nu}(x)}{|x|^{4\sigma+2d-2}}\, \mathrm{d}x$$

and is the mutual $(4\sigma + d - 2)$-dimensional energy of the measures $\mathrm{d}\mu(\eta) = |\widehat{u}_0(\eta)|^2\mathrm{d}\eta$ and $\mathrm{d}\nu(\eta) = |\widehat{v}_0(\eta)|^2\mathrm{d}\eta$.

When $\sigma = \frac{2-d}{4}$, clearly we have $\Upsilon_0 = L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. The case $\sigma = \frac{4-d}{4}$ in Theorem 3.1.1 is also distinguished as we then have that $d - 2 - 4\sigma = 2$, and we can then apply an observation (which we learnt from [22]) to obtain the following corollary.

Corollary 3.1.2. *Let $d \geq 2$. Then*

$$\|(-\Delta)^{\frac{4-d}{4}}(|u|^2)\|_{L^2}^2 \leq \frac{2^{-d}\pi^{\frac{2-d}{2}}}{\Gamma(\frac{d+2}{2})}\|u_0\|_{\dot{H}^1}^2\|u_0\|_{L^2}^2 \tag{3.6}$$

for solutions u of (3.1) *with initial data $u_0 \in H^1$, and the constant is optimal. Furthermore, the initial data u_0 is an extremiser if and only if*

$$\widehat{u}_0(\eta) = \exp(a|\eta|^2 + ib\cdot\eta + c) \tag{3.7}$$

for some $a, c \in \mathbb{C}$, $b \in \mathbb{R}^d$ and $\mathrm{Re}(a) < 0$.

Note that the maximisers u_0 in Corollary 3.1.2 are such that $|\widehat{u}_0|$ is radially symmetric, which means the class of maximisers is smaller than in Theorem 3.1.1. In the case $d = 4$, Corollary 3.1.2 was proved by Carneiro [22] and our result generalises this to $d \geq 2$. We remark that the case $d = 2$ involves only classical derivatives, with the estimate (3.6) simplifying to

$$\|\nabla(|u|^2)\|_{L^2(\mathbb{R}^{1+2})} \leq \frac{1}{2}\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$$

for any $u_0 \in H^1(\mathbb{R}^2)$, where the constant is optimal and attained precisely when u_0 satisfies (3.7).

We also consider some related estimates to those in Theorem 3.1.1 and Corollary 3.1.2 in terms of certain dispersive Sobolev norms. We prove the following sharp estimates and in Section 3.4 describe connections with Theorem 3.1.1.

Theorem 3.1.3. *Let* $d \geq 2$ *and* $\beta > \frac{1-d}{2}$. *Then*

$$\left\| \, |\tfrac{\tau}{2} + |\tfrac{\xi}{2}|^2|^{\beta} \widetilde{uv}(\tau,\xi) \right\|_{L^2}^2 \leq \mathbf{C}(d,\beta) \int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2 |\widehat{v}_0(\eta)|^2 |\zeta - \eta|^{4\beta+d-2} \, \mathrm{d}\zeta \mathrm{d}\eta \tag{3.8}$$

for solutions u *and* v *of* (3.1) *with initial data* $(u_0, v_0) \in \Upsilon_\beta$, *respectively. Here,*

$$\mathbf{C}(d,\beta) = \frac{2^{3-3d-4\beta}\pi^{2-\frac{3}{2}d}}{\Gamma(\frac{d}{2})}$$

is the optimal constant and for $\beta \in (\frac{1-d}{2}, \frac{2+d}{4}]$ *the initial data* (u_0, v_0) *is a maximiser for* (3.8) *if and only if*

$$\widehat{u}_0(\eta) = \exp(a|\eta|^2 + b \cdot \eta + c), \qquad \widehat{v}_0(\eta) = \exp(a|\eta|^2 + b \cdot \eta + d)$$

for some $a, c, d \in \mathbb{C}$, $b \in \mathbb{C}^d$ *and* $\mathrm{Re}(a) < 0$.

Corollary 3.1.4. *Let* $d \geq 2$. *Then*

$$\left\| \, |\tfrac{\tau}{2} + |\tfrac{\xi}{2}|^2|^{\frac{4-d}{4}} \widetilde{u^2}(\tau,\xi) \right\|_{L^2}^2 \leq \frac{\pi^{2+\frac{d}{2}}}{\Gamma(\frac{d}{2})} \|u_0\|_{\dot{H}^1}^2 \|u_0\|_{L^2}^2 \tag{3.9}$$

for solutions u of (3.1) *with initial data $u_0 \in H^1$. The constant is optimal and the initial data u_0 is an extremiser if and only if*

$$\widehat{u}_0(\eta) = \exp(a|\eta|^2 + ib \cdot \eta + c)$$

for some $a, c \in \mathbb{C}$, $b \in \mathbb{R}^d$ and $\mathrm{Re}(a) < 0$.

The relationship between Theorems 3.1.1 and 3.1.3 can be seen most easily by considering the case of one spatial dimension. In fact, when $d = 1$ the natural analogues of the estimates (3.5) and (3.8) are identities, explaining why this case is not included in the statements of these theorems; we expound this point in Section 3.4.

It is possible to prove (3.5) by modifying to the approach of Ozawa–Tsutsumi in [67], and similarly, one can prove (3.8) by appropriately modifying the approach of Foschi in [34]; these approaches are rather different. Here, our proofs of (3.5) and (3.8) are based on the alternative perspective in [8], which has the main advantage of being simultaneously applicable to (3.5) and (3.8), thus permitting a streamlined presentation. A consequence of this is that the characterisation of extremisers in both Theorems 3.1.1 and 3.1.3 may be reduced immediately to finding the solution of the same functional equation. Furthermore, by using the approach based on [8] we are able to expose underlying heat-flow monotonicity phenomena in the general context of (3.5) and (3.8), extending some of the results in [8]. In particular, we shall prove the following.

Theorem 3.1.5. *Suppose $d \geq 2$. For any $\sigma > \frac{1-d}{4}$ and initial data $(u_0, v_0) \in \Upsilon_\sigma$, the quantity*

$$\rho \mapsto \mathbf{OT}(d,\sigma) I_\sigma(e^{\rho\Delta}u_0, e^{\rho\Delta}v_0) - \left\| (-\Delta)^\sigma (e^{\rho\Delta}u\, \overline{e^{\rho\Delta}v}) \right\|_{L^2_{t,x}}^2$$

is nonincreasing on $(0,\infty)$*. Similarly, for any* $\beta > \frac{1-d}{2}$ *and* $(u_0,v_0) \in \Upsilon_\beta$*, the quantity*

$$\rho \mapsto \mathbf{C}(d,\beta) I_\beta(e^{\rho\Delta}u_0, e^{\rho\Delta}v_0) - \Big\| \, |\tfrac{\tau}{2} + |\tfrac{\xi}{2}|^2|^\beta \widetilde{e^{\rho\Delta}u \, e^{\rho\Delta}v} \Big\|_{L^2_{\tau,\xi}}^2$$

is nonincreasing on $(0,\infty)$.

Organisation. In the next section we prove the sharp estimates appearing in Theorems 3.1.1 and 3.1.3 and Corollaries 3.1.2 and 3.1.4, along with the heat-flow monotonicity in Theorem 3.1.5. The statements concerning characterisations of extremisers in these results are proved in Section 3.3. In Section 3.4 we discuss the case $d = 1$ and the relationship between Theorems 3.1.1 and 3.1.3. This naturally leads to consideration of the importance of the complex conjugate appearing on one of the solutions in Theorem 3.1.1; we highlight this by showing that gaussians are not extremisers for the corresponding Ozawa–Tsutsumi estimate (1.19) when the conjugate is removed. Finally, an alternative derivation of the results concerning the characterisation of maximisers is provided in Section 3.5.

3.2 Proof of the sharp estimates (3.5)–(3.9)

Proof of (3.5). An application of Plancherel's theorem in space-time gives

$$\|(-\Delta)^\sigma (u\overline{v})\|_{L^2}^2 = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} |\xi|^{4\sigma} |\widetilde{(u\overline{v})}(\tau,\xi)|^2 \, \mathrm{d}\xi \mathrm{d}\tau$$

and since $\widetilde{u\overline{v}} = \frac{1}{(2\pi)^{d+1}} \, \widetilde{u} * \widetilde{\overline{v}}$ we obtain

$$\begin{aligned}
&\|(-\Delta)^\sigma (u\overline{v})\|_{L^2}^2 \\
&= \frac{1}{(2\pi)^{3d-1}} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |\zeta_1 + \zeta_2|^{4\sigma} \widehat{u}_0(\zeta_1) \widehat{\overline{v}}_0(\zeta_2) \overline{\widehat{u}_0(\eta_1) \widehat{\overline{v}}_0(\eta_2)} \, \times \\
&\qquad\qquad \delta(-|\zeta_1|^2 + |\zeta_2|^2 + |\eta_1|^2 - |\eta_2|^2) \delta(\zeta_1 + \zeta_2 - \eta_1 - \eta_2) \, \mathrm{d}\zeta \mathrm{d}\eta \, .
\end{aligned}$$

Relabelling the variables $(\zeta_1, \eta_1, \zeta_2, \eta_2) \to (\zeta_1, \eta_1, \eta_2, \zeta_2)$, we have

$$\|(-\Delta)^\sigma (u\overline{v})\|_{L^2}^2 = \frac{1}{(2\pi)^{3d-1}} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \widehat{U}_0(\zeta)\overline{\widehat{U}_0(\eta)}\, \mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta \tag{3.10}$$

where $U_0 = u_0 \otimes v_0(-\,\cdot)$ and the measure $\mathrm{d}\Sigma_\zeta(\eta)$ is given by

$$\mathrm{d}\Sigma_\zeta(\eta) = |\zeta_1 + \eta_2|^{4\sigma}\delta(|\eta_1|^2 + |\eta_2|^2 - |\zeta_1|^2 - |\zeta_2|^2)\delta(\eta_1 - \eta_2 - (\zeta_1 - \zeta_2))\mathrm{d}\eta\,. \tag{3.11}$$

Lemma 3.2.1. *For each $\zeta \in \mathbb{R}^{2d}$ we have*

$$\int_{\mathbb{R}^{2d}} \mathrm{d}\Sigma_\zeta = \pi^{\frac{d-1}{2}} \frac{\Gamma(2\sigma + \frac{d-1}{2})}{2\Gamma(2\sigma + d - 1)} |\zeta_1 + \zeta_2|^{4\sigma + d - 2}\,.$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \mathrm{d}\Sigma_\zeta(\eta) &= \frac{1}{2}\int_{\mathbb{R}^d} |\xi_2|^{4\sigma}\delta(|\xi_2|^2 - \xi_2\cdot(\zeta_1+\zeta_2))\mathrm{d}\xi_2 \\ &= \frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_0^\infty r^{4\sigma+d-2}\delta(r - \omega\cdot(\zeta_1+\zeta_2))\mathrm{d}r\mathrm{d}\omega\end{aligned}$$

via the change of variables $(\xi_1, \xi_2) = (\eta_1 + \zeta_2, \eta_2 + \zeta_1)$ and subsequently polar coordinates $\xi_2 = r\omega$. By applying a rotation, we may replace $\zeta_1 + \zeta_2$ with $|\zeta_1 + \zeta_2|\mathbf{e}_1$, and thus (via, for example, the Funk–Hecke theorem)

$$\int_{\mathbb{R}^{2d}} \mathrm{d}\Sigma_\zeta(\eta) = \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}|\zeta_1 + \zeta_2|^{4\sigma+d-2}\int_0^1 s^{4\sigma+d-2}(1-s^2)^{\frac{d-3}{2}}\mathrm{d}s\,.$$

To obtain the claimed expression for the constant we change variables once more

$$\int_0^1 s^{4\sigma+d-2}(1-s^2)^{\frac{d-3}{2}}\mathrm{d}s = \frac{1}{2}\int_0^1 t^{2\sigma+\frac{d-3}{2}}(1-t)^{\frac{d-3}{2}}\mathrm{d}t = \frac{1}{2}\mathrm{B}(\tfrac{d-1}{2}, 2\sigma + \tfrac{d-1}{2})\,,$$

where B is the beta function. An application of the identity $\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ completes the proof. $\square$

Lemma 3.2.1 and the symmetry relation $\mathrm{d}\Sigma_\eta(\zeta)\mathrm{d}\eta = \mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta$ imply that

$$\begin{aligned}\mathbf{OT}(d,\sigma)I_\sigma(u_0,v_0) &= \frac{1}{(2\pi)^{3d-1}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |\widehat{U}_0(\zeta)|^2 \,\mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta \\ &= \frac{1}{2(2\pi)^{3d-1}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \left(|\widehat{U}_0(\zeta)|^2 + |\widehat{U}_0(\eta)|^2\right) \mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta\,.\end{aligned}$$

Since the left-hand side of (3.10) is nonnegative, we may take the real part of both sides and apply the arithmetic–geometric mean inequality

$$2\mathrm{Re}\left(\widehat{U}_0(\zeta)\overline{\widehat{U}_0(\eta)}\right) \le |\widehat{U}_0(\zeta)|^2 + |\widehat{U}_0(\eta)|^2$$

to obtain

$$\|(-\Delta)^\sigma(u\overline{v})\|_{L^2}^2 \le \mathbf{OT}(d,\sigma)I_\sigma(u_0,v_0)$$

which establishes (3.5). $\square$

Proof of (3.8). Writing $\widetilde{uv} = \frac{1}{(2\pi)^{d+1}}\,\widetilde{u} * \widetilde{v}$ leads to

$$\begin{aligned}&\left\| \,|\tfrac{\tau}{2} + |\tfrac{\xi}{2}|^2|^\beta \widetilde{uv}(\tau,\xi)\right\|_{L^2}^2 \\ &= \frac{1}{2^{4\beta}(2\pi)^{2(d-1)}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |\zeta_1-\zeta_2|^{4\beta}\widehat{u}_0(\zeta_1)\widehat{v}_0(\zeta_2)\overline{\widehat{u}_0(\eta_1)\widehat{v}_0(\eta_2)} \,\times \\ &\qquad\qquad \delta(|\zeta_1|^2+|\zeta_2|^2-|\eta_1|^2-|\eta_2|^2)\delta(\zeta_1+\zeta_2-\eta_1-\eta_2)\,\mathrm{d}\zeta\mathrm{d}\eta \\ &= \frac{1}{2^{4\beta}(2\pi)^{2(d-1)}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \widehat{U}_0(\zeta)\overline{\widehat{U}_0(\eta)}\,\mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta\,,\end{aligned}$$

where $U_0 = u_0 \otimes v_0$, the measure $\mathrm{d}\Sigma_\zeta(\eta)$ is given by

$$\mathrm{d}\Sigma_\zeta(\eta) = |\zeta_1-\zeta_2|^{4\beta}\delta(|\zeta_1|^2+|\zeta_2|^2-|\eta_1|^2-|\eta_2|^2)\delta(\zeta_1+\zeta_2-\eta_1-\eta_2)\mathrm{d}\eta\,. \tag{3.12}$$

and where we have used the fact that whenever $\tau = -|\zeta_1|^2 - |\zeta_2|^2$ and $\xi = \zeta_1 + \zeta_2$, we have

$$|\tfrac{\tau}{2} + |\tfrac{\xi}{2}|^2| = \tfrac{1}{4}|\zeta_1 - \zeta_2|^2.$$

Remark. Notice that the function U_0 and the measure $\mathrm{d}\Sigma_\zeta$ in the current proof of (3.8) are slightly different to the U_0 and $\mathrm{d}\Sigma_\zeta$ used in the previous proof of (3.5). We have decided to use the same notation in order to highlight that the two proofs are structurally the same.

Lemma 3.2.2. *For each $\zeta \in \mathbb{R}^{2d}$ we have*

$$\int_{\mathbb{R}^{2d}} \mathrm{d}\Sigma_\zeta = \frac{\pi^{\frac{d}{2}}}{2^{d-1}\Gamma(\frac{d}{2})}|\zeta_1 - \zeta_2|^{4\beta+d-2}.$$

Proof. Using the change of variables $(\xi_1, \xi_2) = (\frac{1}{2}(\zeta_1 + \zeta_2) - \eta_1, \frac{1}{2}(\zeta_1 + \zeta_2) - \eta_2)$ and a subsequent polar coordinate change of variables in ξ_2, we have

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} \mathrm{d}\Sigma_\zeta(\eta) &= |\zeta_1 - \zeta_2|^{4\beta} \int_{\mathbb{R}^{2d}} \delta(\tfrac{1}{2}|\zeta_1 - \zeta_2|^2 - |\xi_1|^2 - |\xi_2|^2)\delta(\xi_1 + \xi_2)\,\mathrm{d}\xi \\
&= |\mathbb{S}^{d-1}||\zeta_1 - \zeta_2|^{4\beta} \int_0^\infty \delta(\tfrac{1}{2}|\zeta_1 - \zeta_2|^2 - 2r^2) r^{d-1}\,\mathrm{d}r \\
&= \frac{\pi^{\frac{d}{2}}}{2^{d-1}\Gamma(\frac{d}{2})}|\zeta_1 - \zeta_2|^{4\beta+d-2}.
\end{aligned}$$

In the last step, we used the well-known formula $|\mathbb{S}^{d-1}| = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ for the measure of the unit sphere in $\mathbb{R}^d$. $\square$

As in the proof of (3.5), we now use the symmetry relation $\mathrm{d}\Sigma_\eta(\zeta)\mathrm{d}\eta = \mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta$, Lemma

3.2.2 and the arithmetic–geometric mean inequality to obtain

$$\begin{aligned}\Big\| \, |\tfrac{\tau}{2}+|\tfrac{\xi}{2}|^2|^{\beta}\widetilde{uv}(\tau,\xi)\Big\|_{L^2}^2 &= \frac{1}{2^{4\beta}(2\pi)^{2(d-1)}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \widehat{U}_0(\zeta)\overline{\widehat{U}_0(\eta)}\,\mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta \\ &\le \frac{1}{2^{4\beta+1}(2\pi)^{2(d-1)}}\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \big(|\widehat{U}_0(\zeta)|^2+|\widehat{U}_0(\eta)|^2\big)\,\mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta \\ &= \mathbf{C}(d,\beta)I_\beta(u_0,v_0)\end{aligned}$$

as desired. □

Proof of (3.6) *and* (3.9). Expanding $|\zeta-\eta|^2$ and using Plancherel's theorem we obtain

$$I_{\frac{4-d}{4}}(u_0,u_0) = 2(2\pi)^{2d}\|u_0\|_{L^2}\|u_0\|_{\dot{H}^1} - 2\int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2|\widehat{u}_0(\eta)|^2\zeta\cdot\eta\,\mathrm{d}\zeta\mathrm{d}\eta$$

and therefore

$$I_{\frac{4-d}{4}}(u_0,u_0) \le 2(2\pi)^{2d}\|u_0\|_{L^2}\|u_0\|_{\dot{H}^1} \tag{3.13}$$

for any $u_0\in H^1$. The estimates (3.6) and (3.9) now follow at once from (3.5) and (3.8). □

Proof of Theorem 3.1.5. The above proof of (3.5) in fact shows that

$$\mathbf{OT}(d,\sigma)I_\sigma(u_0,v_0) - \|(-\Delta)^\sigma(u\overline{v})\|_{L^2}^2 = c\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \big|\widehat{U}_0(\zeta)-\widehat{U}_0(\eta)\big|^2\,\mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta$$

where $\frac{1}{c} = 2(2\pi)^{3d-1}$, $U_0 = u_0\otimes v_0(-\,\cdot)$ and the measure $\mathrm{d}\Sigma_\zeta(\eta)$ is given by (3.11). Replacing (u_0,v_0) with $(e^{\rho\Delta}u_0, e^{\rho\Delta}v_0)$ for fixed $\rho>0$, commuting the Schrödinger and heat flows, and using the support of $\mathrm{d}\Sigma_\zeta$, we obtain

$$\begin{aligned}\mathbf{OT}(d,\sigma)I_\sigma(e^{\rho\Delta}u_0, e^{\rho\Delta}v_0) - \left\|(-\Delta)^\sigma(e^{\rho\Delta}u\,\overline{e^{\rho\Delta}v})\right\|^2_{L^2_{t,x}} \\ = c\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} e^{-2\rho(|\zeta_1|^2+|\zeta_2|^2)}\left|\widehat{U}_0(\zeta)-\widehat{U}_0(\eta)\right|^2 \mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta\end{aligned}$$

which is manifestly nonincreasing for $\rho \in (0,\infty)$.

A similar argument based on the previous proof of (3.8) shows that

$$\begin{aligned}\mathbf{C}(d,\beta)I_\beta(e^{\rho\Delta}u_0, e^{\rho\Delta}v_0) - \left\|\,|\tfrac{\tau}{2}+|\tfrac{\xi}{2}|^2|^\beta \widetilde{e^{\rho\Delta}u\,e^{\rho\Delta}v}\right\|^2_{L^2_{\tau,\xi}} \\ = c\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} e^{-2\rho(|\zeta_1|^2+|\zeta_2|^2)}\left|\widehat{U}_0(\zeta)-\widehat{U}_0(\eta)\right|^2 \mathrm{d}\Sigma_\zeta(\eta)\mathrm{d}\zeta\end{aligned}$$

where, now, $\frac{1}{c} = 2^{4\beta+1}(2\pi)^{2(d-1)}$, $U_0 = u_0 \otimes v_0$ and the measure $\mathrm{d}\Sigma_\zeta(\eta)$ is given by (3.12). This completes our proof of Theorem 3.1.5. □

Remark. It is clear from the proof of Theorem 3.1.5 that the monotone quantities are in fact *completely monotone* since their ρ-derivatives have sign $(-1)^j$ for every $j \in \mathbb{N}$.

3.3 Characterisation of maximisers for (3.5)–(3.9)

It was shown in Section 3.2 that (3.5) and (3.8) follow from a single application of the arithmetic–geometric mean inequality

$$2\mathrm{Re}\left(\widehat{U}_0(\zeta)\overline{\widehat{U}_0(\eta)}\right) \leq |\widehat{U}_0(\zeta)|^2 + |\widehat{U}_0(\eta)|^2$$

for each $\zeta \in \mathbb{R}^{2d}$ and each η in the support of $\mathrm{d}\Sigma_\zeta$, which is obviously an equality if and only if $\widehat{U}_0(\zeta)$ and $\widehat{U}_0(\eta)$ coincide. For each estimate, U_0 and $\mathrm{d}\Sigma_\zeta$ are slightly different.

For (3.8), $U_0 = u_0 \otimes v_0$ and $\mathrm{d}\Sigma_\zeta$ is given by (3.12), which means (u_0, v_0) is a maximiser if and only if

$$\widehat{u}_0(\zeta_1)\widehat{v}_0(\zeta_2) = \widehat{u}_0(\eta_1)\widehat{v}_0(\eta_2)$$

for almost every $\zeta \in \mathbb{R}^{2d}$ and almost every $\eta \in \mathbb{R}^{2d}$ satisfying $|\eta_1|^2 + |\eta_2|^2 = |\zeta_1|^2 + |\zeta_2|^2$ and $\eta_1 + \eta_2 = \zeta_1 + \zeta_2$, or equivalently

$$\widehat{u}_0(\zeta_1)\widehat{v}_0(\zeta_2) = \Lambda(|\zeta_1|^2 + |\zeta_2|^2, \zeta_1 + \zeta_2) \tag{3.14}$$

for almost every $\zeta \in \mathbb{R}^{2d}$, and where Λ is a scalar function.

For (3.5), $U_0 = u_0 \otimes v_0(-\,\cdot)$ and $\mathrm{d}\Sigma_\zeta$ is given by (3.11). Since $\widehat{U_0}(\zeta) = \widehat{u}_0(\zeta_1)\widehat{v}_0(-\zeta_2)$ and for η in the support of $\mathrm{d}\Sigma_\zeta$ we have $|\eta_1|^2 + |\eta_2|^2 = |\zeta_1|^2 + |\zeta_2|^2$ and $\eta_1 - \eta_2 = \zeta_1 - \zeta_2$, it follows that (u_0, v_0) is a maximiser for (3.5) if and only if (3.14) holds.

If $\widehat{u}_0(\eta) = \exp(a|\eta|^2 + b\cdot\eta + c)$ and $\widehat{v}_0(\eta) = \exp(a|\eta|^2 + b\cdot\eta + d)$ for some $a, c, d \in \mathbb{C}$, $b \in \mathbb{C}^d$ and $\mathrm{Re}(a) < 0$, then it is trivial to see that (3.14) holds. In the next section we will prove that there are in fact no other solutions provided that $I_\beta(u_0, v_0)$ is finite, following the approach from [13].

3.3.1 Characterisation for (3.5) and (3.8)

First note that the right-hand side of (3.14) is symmetric in ζ_1 and ζ_2 and so it must be true that $\widehat{u}_0$ and $\widehat{v}_0$ are linearly dependent. Next, we show that whenever $\beta \in (-\infty, \frac{2+d}{4}]$ then $(u_0, u_0) \in \Upsilon_\beta$ guarantees that $\widehat{u}_0$ is locally integrable. First, for any ball B in $\mathbb{R}^d$ centred at the origin, we may use the Cauchy–Schwarz inequality to obtain

$$\left(\int_B |\widehat{u}_0|\right)^2 \leq I_\beta(u_0, u_0)^{\frac{1}{2}} \int_B \int_B |\zeta - \eta|^{-(4\beta+d-2)}\, \mathrm{d}\zeta \mathrm{d}\eta\,,$$

and this is finite as long as $\beta < \frac{1}{2}$. To extend this range, we may also use the reverse Hardy–Littlewood–Sobolev inequality (see [5]) to obtain

$$I_\beta(u_0, u_0) \geq C\|\widehat{u}_0\|_{L^p}^4$$

whenever $\beta > \frac{2-d}{4}$ and with $p = \frac{4d}{4\beta+3d-2}$. We may conclude that $\widehat{u}_0$ is locally integrable as long as $p \geq 1$ and this gives the constraint $\beta \leq \frac{2+d}{4}$. This implies that $f_\delta := e^{-\delta|\cdot|^2}\widehat{u_0}$ is in fact integrable, for any $\delta > 0$. Further, it is easy to see that both $\widehat{u_0}$ and f_δ also solve

$$f(\zeta)f(\eta) = \Lambda(|\zeta|^2 + |\eta|^2, \zeta + \eta) \qquad \text{for almost every } (\zeta, \eta) \in \mathbb{R}^d \times \mathbb{R}^d; \tag{3.15}$$

hence it suffices to find all integrable solutions $f : \mathbb{R}^d \to \mathbb{C}$ of this equation.

Remark. The functional equation is known as the Maxwell–Boltzmann functional equation. The system of equations

$$\zeta' + \eta' = \zeta + \eta \qquad \text{and} \qquad |\zeta'|^2 + |\eta'|^2 = |\zeta|^2 + |\eta|^2$$

express the conservation of momentum and kinetic energy, respectively, during a binary collision, where (ζ, η) are the velocities of a pair of particles before collision, and (ζ', η') are the velocities of the same pair after collision. In Section 3.5 we will provide an alternative proof that any solutions to the Maxwell–Boltzmann equation are necessarily of gaussian form, even if the initial data is assumed to be only locally integrable.

It is well-known that (3.15) has only gaussian solutions when the function f is assumed to be integrable. A proof of the following result can be found, for example, in lecture notes of Villani [94] (see also Lions [63] and Perthame [69]).

Theorem 3.3.1. *Suppose that f is an integrable solution to* (3.15)*. Then there exists*

$a, c \in \mathbb{C}, b \in \mathbb{C}^d$ *with* $\operatorname{Re} a < 0$ *such that*

$$f(x) = e^{a|x|^2 + b\cdot x + c}.$$

We may therefore conclude that f_δ has the desired form, and hence, by passing to the limit $\delta \to 0$, that u_0 does also, and this completes the proof of Theorems 3.1.1 and 3.1.3.

3.3.2 Characterisation for (3.6) and (3.9)

We saw in Section 3.2 that the estimates (3.6) and (3.9) follow from (3.5) and (3.8), respectively, followed by (3.13). When $u_0 = v_0$, maximisers of (3.5) and (3.8) are of the form $\widehat{u}_0(\eta) = \exp(a|\eta|^2 + b \cdot \eta + c)$, for some $a, c, d \in \mathbb{C}$, $b \in \mathbb{C}^d$ and $\operatorname{Re}(a) < 0$, and since

$$\int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2 |\widehat{u}_0(\eta)|^2 \zeta \cdot \eta \,\mathrm{d}\zeta \mathrm{d}\eta \tag{3.16}$$

vanishes when $|\widehat{u}_0|$ is radial, it is clear that we have equality in (3.6) and (3.9) whenever

$$\widehat{u}_0(\eta) = \exp(a|\eta|^2 + b \cdot \eta + c)$$

for some $a, c \in \mathbb{C}$, $b \in \mathbb{C}^d$, $\operatorname{Re}(a) < 0$ and $\operatorname{Re}(b) = 0$. In order to show that there are no further maximisers, it suffices to show that the quantity in (3.16) is nonzero whenever $\operatorname{Re}(b)$ is nonzero. For such $b \in \mathbb{C}^d$ we may perform a change of variables $(\zeta, \eta) \mapsto (R\zeta, R\eta)$ in (3.16), for a suitably chosen rotation R, so that it suffices to consider $b \in \mathbb{C}^d$ such that $\operatorname{Re}(b) = b_1 \mathbf{e}_1$, where b_1 is a strictly positive real number. Now

$$\int_{\mathbb{R}^{2d}} |\widehat{u}_0(\zeta)|^2 |\widehat{u}_0(\eta)|^2 \zeta \cdot \eta \,\mathrm{d}\zeta \mathrm{d}\eta \geq \left(\int_{\mathbb{R}^d} |\widehat{u}_0(\eta)|^2 \eta_1 \,\mathrm{d}\eta \right)^2$$

and for such u_0 we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\widehat{u}_0(\eta)|^2\eta_1 \,\mathrm{d}\eta &= \exp(2\mathrm{Re}(c)) \int_{\mathbb{R}^d} \exp(2\mathrm{Re}(a)|\eta|^2 + 2b_1\eta_1)\eta_1 \,\mathrm{d}\eta \\ &= C \int_{\mathbb{R}} \exp(2\mathrm{Re}(a)\eta_1^2 + 2b_1\eta_1)\eta_1 \,\mathrm{d}\eta_1 \\ &= C \int_0^\infty \exp(2\mathrm{Re}(a)\eta_1^2 + 2b_1\eta_1)\eta_1(1-\exp(-2b_1\eta_1)) \,\mathrm{d}\eta_1 \,,\end{aligned}$$

where C is some strictly positive constant depending on a and c. Since $b_1 > 0$ it follows that the quantity in (3.16) is nonzero, as desired. $\square$

3.4 Further results

3.4.1 One spatial dimension and the role of the conjugate

In the case of one spatial dimension, there are identities which are the analogues of the sharp estimates in Theorems 3.1.1 and 3.1.3. We present these identities briefly here, for completeness and to elucidate the role of the complex conjugation. The role of the complex conjugate on one of the solutions in Theorem 3.1.1 appears to be crucial at the level of optimal constants and extremisers; see the forthcoming Theorem 3.4.1.

For the analogue of (3.5), we have

$$\|(-\Delta)^\sigma(u\overline{v})\|_{L^2(\mathbb{R}^2)}^2 = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2} |\widehat{u}_0(\zeta)|^2|\widehat{v}_0(\eta)|^2|\zeta-\eta|^{4\sigma-1}\,\mathrm{d}\zeta\mathrm{d}\eta$$

by the well-known approach of writing

$$(u\overline{v})(t,x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \exp(ix(\zeta-\eta))\exp(-it(\zeta^2-\eta^2))\widehat{u}_0(\zeta)\overline{\widehat{v}_0(\eta)}\,\mathrm{d}\zeta\mathrm{d}\eta\,, \qquad (3.17)$$

changing variables $(\zeta,\eta)\mapsto(\zeta-\eta,\zeta^2-\eta^2)$, using Plancherel's Theorem, and then undoing the previous change of variables. The jacobian from the change of variables is $2|\zeta-\eta|$ and it is clear from (3.17) that this interacts precisely on taking $(-\partial_x)^\sigma$-derivatives of $(u\overline{v})(t,x)$.

On the other hand, for the analogue of (3.8), we have

$$\Big\| \, |\tfrac{\tau}{2}+|\tfrac{\xi}{2}|^2|^\beta \widetilde{uv}(\tau,\xi)\Big\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{2^{4\beta+2}}\int_{\mathbb{R}^2} |\widehat{u}_0(\zeta)\widehat{v}_0(\eta)+\widehat{u}_0(\eta)\widehat{v}_0(\zeta)|^2|\zeta-\eta|^{4\beta-1}\,\mathrm{d}\zeta\mathrm{d}\eta$$

and therefore, if $\widehat{u}_0$ and $\widehat{v}_0$ have separated supports,

$$\Big\| \, |\tfrac{\tau}{2}+|\tfrac{\xi}{2}|^2|^\beta \widetilde{uv}(\tau,\xi)\Big\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{2^{4\beta+1}}\int_{\mathbb{R}^2} |\widehat{u}_0(\zeta)|^2|\widehat{v}_0(\eta)|^2|\zeta-\eta|^{4\beta-1}\,\mathrm{d}\zeta\mathrm{d}\eta\,.$$

This follows in a similar way by writing

$$(uv)(t,x)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\exp(ix(\zeta+\eta))\exp(-it(\zeta^2+\eta^2))\widehat{u}_0(\zeta)\widehat{v}_0(\eta)\,\mathrm{d}\zeta\mathrm{d}\eta\,,$$

and then using complex conjugation, Plancherel's Theorem and the change of variable $(\zeta,\eta)\mapsto(\zeta+\eta,-\zeta^2-\eta^2)$ on the half-plane $\mathbb{H}=\{(\zeta,\eta)\in\mathbb{R}^2:\zeta<\eta\}$. The jacobian from the change of variables is again $2|\zeta-\eta|$, so it no longer interacts precisely with $(-\partial_x)^\sigma$-derivatives. The derivative with multiplier $|\frac{\tau}{2}+|\frac{\xi}{2}|^2|^\beta$ does interact precisely with uv since $-|\frac{\tau}{2}+(\frac{\xi}{2})^2|=\frac{1}{4}|\zeta-\eta|^2$, where $(\tau,\xi)=(-\zeta^2-\eta^2,\zeta+\eta)$.

Despite the above observations concerning the delicate role of the complex conjugate on one of the solutions, we know, for example, that for the Ozawa–Tsutsumi exponent, the estimate

$$\|(-\Delta)^{\frac{2-d}{4}}(uv)\|^2_{L^2}\le C_d\|u_0\|^2_{L^2}\|v_0\|^2_{L^2} \tag{3.18}$$

holds for some finite constant C_d, $d\ge 2$, independent of the initial data $(u_0,v_0)\in L^2(\mathbb{R}^d)\times$

$L^2(\mathbb{R}^d)$. This can easily be seen using Sobolev embedding, Hölder's inequality, and the mixed-norm linear Strichartz estimate

$$L^2(\mathbb{R}^d) \to L^4_t L^{\frac{2d}{d-1}}_x(\mathbb{R} \times \mathbb{R}^d)$$

for the solution of (3.1). Although we do not know the optimal constant in (3.18) we can at least determine that a pair of isotropic centred gaussians is not a maximiser, highlighting the importance of the complex conjugate in Theorem 3.1.1.

Theorem 3.4.1. *The gaussian pair of initial data*

$$(u_0, v_0) = (\exp(-|\cdot|^2, \exp(-|\cdot|^2)) \tag{3.19}$$

is not a critical point for the functional

$$(u_0, v_0) \mapsto \frac{\|(-\Delta)^{-\frac{d-2}{4}}(uv)\|_{L^2}}{\|u_0\|_{L^2}\|v_0\|_{L^2}}.$$

Proof. If Φ is the functional given by

$$\Phi(u_0, v_0) = \frac{\|(-\Delta)^{-\frac{d-2}{4}}(uv)\|_{L^2}}{\|u_0\|_{L^2}\|v_0\|_{L^2}}$$

then one can show that

$$\lim_{\varepsilon \to 0} \frac{\Phi(u_0 + \varepsilon U_0, v_0 + \varepsilon V_0) - \Phi(u_0, v_0)}{\varepsilon} = 0$$

for all $(U_0, V_0) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ if and only if

$$\mathrm{Re} \int_{\mathbb{R}^{d+1}} M\, \widetilde{uv}\, \overline{(\widetilde{uV} + \widetilde{Uv})} = \mathrm{Re}\big(\Phi(u_0, v_0)(\|v_0\|_2^2 \langle u_0, U_0\rangle + \|u_0\|_2^2 \langle v_0, V_0\rangle)\big)$$

for all $(U_0, V_0) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, and in our case $M(\tau, \xi) = |\xi|^{2-d}$. Therefore, by taking

$V_0 = 0$ and complex conjugation on both sides, it follows that if (u_0, v_0) is a critical point then necessarily

$$\int_{\mathbb{R}^{d+1}} M(\tau,\xi)\widetilde{\overline{uv}}(\tau,\xi)\widetilde{Uv}(\tau,\xi)\,\mathrm{d}\tau\mathrm{d}\xi = \lambda_{u_0,v_0}\langle u_0, U_0\rangle \tag{3.20}$$

for some constant λ_{u_0,v_0}, for any $U_0 \in L^2(\mathbb{R}^{d+1})$, where $u = e^{it\Delta}u_0$, $v = e^{it\Delta}v_0$, and $U = e^{it\Delta}U_0$. This implies that

$$\int_{\mathbb{R}^{d+1}} \widetilde{M\widetilde{\overline{uv}}}(t,x)v(t,x)e^{it\Delta}U_0(x)\,\mathrm{d}x\mathrm{d}t = \lambda_{u_0,v_0}\langle u_0, U_0\rangle\,,$$

and hence for (3.20) to hold we must have

$$\int_{\mathbb{R}} e^{it\Delta}\left(\widetilde{M\widetilde{\overline{uv}}}(t,\cdot)v(t,\cdot)\right)(y)\,\mathrm{d}t = \lambda u_0(y) \tag{3.21}$$

almost everywhere on $\mathbb{R}^d$. We want to check if (3.21) holds for the pair (u_0, v_0) given by (3.19). In what follows we denote by C_d a positive constant depending only on d which may change from line to line. First, we have

$$\widetilde{\overline{uv}}(\tau,\xi) = C_d\int_{\mathbb{R}^{2d}} \widehat{u_0}(\xi_1)\widehat{v_0}(\xi_2)\delta\begin{pmatrix}\tau - |\xi_1|^2 - |\xi_2|^2\\ \xi - \xi_1 - \xi_2\end{pmatrix}\mathrm{d}\xi_1\mathrm{d}\xi_2.$$

For (u_0, v_0) given by (3.19) we therefore have

$$\begin{aligned} M(\tau,\xi)\widetilde{\overline{uv}}(\tau,\xi) &= C_d|\xi|^{2-d}\int_{\mathbb{R}^{2d}} e^{-|\xi_1|^2}e^{-|\xi_2|^2}\delta\begin{pmatrix}\tau - |\xi_1|^2 - |\xi_2|^2\\ \xi - \xi_1 - \xi_2\end{pmatrix}\mathrm{d}\xi_1\mathrm{d}\xi_2\\ &= C_d\int_{\mathbb{R}^{2d}} |\xi_1+\xi_2|^{2-d}e^{-|\xi_1|^2}e^{-|\xi_2|^2}\delta\begin{pmatrix}\tau - |\xi_1|^2 - |\xi_2|^2\\ \xi - \xi_1 - \xi_2\end{pmatrix}\mathrm{d}\xi_1\mathrm{d}\xi_2.\end{aligned}$$

If we take the space-time Fourier transform of this expression, use the support condition of the delta measures and then integrate in (τ, ξ) using Fubini, we obtain that

$$
\begin{aligned}
\widetilde{M\overline{\widetilde{uv}}}(t,x) &= C_d \int_{\mathbb{R}^{d+1}_{\xi,\tau}} e^{-ix\cdot\xi - it\tau} \int_{\mathbb{R}^{2d}_{\xi_1,\xi_2}} |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2} \\
&\qquad\qquad\qquad \times \delta\begin{pmatrix} \tau - |\xi_1|^2 - |\xi_2|^2 \\ \xi - \xi_1 - \xi_2 \end{pmatrix} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \mathrm{d}\tau \\
&= C_d \int_{\mathbb{R}^{3d+1}} e^{-ix\cdot(\xi_1+\xi_2) - it(|\xi_1|^2+|\xi_2|^2)} |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2} \\
&\qquad\qquad\qquad \times \delta\begin{pmatrix} \tau - |\xi_1|^2 - |\xi_2|^2 \\ \xi - \xi_1 - \xi_2 \end{pmatrix} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \mathrm{d}\tau \\
&= C_d \int_{\mathbb{R}^{2d}} e^{-ix\cdot(\xi_1+\xi_2) - it(|\xi_1|^2+|\xi_2|^2)} |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2.
\end{aligned}
$$

Now, in our case

$$
v(t,x) = e^{it\Delta} v_0(x) = C_d \int_{\mathbb{R}^d} e^{ix\cdot\xi + it|\xi|^2} e^{-|\xi|^2} \, \mathrm{d}\xi,
$$

so

$$
\begin{aligned}
&\widetilde{M\overline{\widetilde{uv}}}(t,x) v(t,x) \\
&= C_d \int_{\mathbb{R}^{3d}} e^{-ix\cdot(\xi_1+\xi_2-\xi) - it(|\xi_1|^2+|\xi_2|^2-|\xi|^2)} |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2-|\xi|^2} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi.
\end{aligned}
$$

We need to calculate the Schrödinger extension of this function of x, for fixed t, and then integrate in t. Firstly,

$$
\begin{aligned}
\left(\widetilde{M\overline{\widetilde{uv}}}(t,\cdot) v(t,\cdot)\right)^{\wedge}(\eta) &= C_d \int_{\mathbb{R}^{4d}} e^{-ix\cdot(\xi_1+\xi_2-\xi+\eta) - it(|\xi_1|^2+|\xi_2|^2-|\xi|^2)} \\
&\qquad\qquad \times |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2-|\xi|^2} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \mathrm{d}x.
\end{aligned}
$$

Hence for $t \in \mathbb{R}$ fixed,

$$e^{it\Delta}\left(\widetilde{M\widetilde{\widetilde{uv}}}(t,\cdot)v(t,\cdot)\right)(y) = C_d \int_{\mathbb{R}^{5d}} e^{iy\cdot\eta} e^{-ix\cdot(\xi_1+\xi_2-\xi+\eta)-it(|\xi_1|^2+|\xi_2|^2-|\xi|^2-|\eta|^2)}$$
$$\times |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2-|\xi|^2}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \mathrm{d}x \mathrm{d}\eta$$
$$= \int_{\mathbb{R}^{5d}} e^{-iy\cdot\eta} e^{-ix\cdot(\xi_1+\xi_2-\xi-\eta)-it(|\xi_1|^2+|\xi_2|^2-|\xi|^2-|\eta|^2)}$$
$$\times |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2-|\xi|^2}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \mathrm{d}x \mathrm{d}\eta,$$

using the change of variable $\eta \mapsto -\eta$. Integrating this expression with respect to t, it follows that the left hand side of (3.21) evaluates to a constant multiple of

$$\int_{\mathbb{R}^{5d+1}} e^{-iy\cdot\eta} e^{-ix\cdot(\xi_1+\xi_2-\xi-\eta)-it(|\xi_1|^2+|\xi_2|^2-|\xi|^2-|\eta|^2)}$$
$$\times |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2-|\xi|^2}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \mathrm{d}x \mathrm{d}\eta \mathrm{d}t$$
$$= \int_{\mathbb{R}^{4d}} e^{-iy\cdot\eta} \delta\begin{pmatrix} |\xi_1|^2+|\xi_2|^2-|\xi|^2-|\eta|^2 \\ \xi_1+\xi_2-\xi-\eta \end{pmatrix} |\xi_1+\xi_2|^{2-d} e^{-|\xi_1|^2-|\xi_2|^2-|\xi|^2}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \mathrm{d}\eta$$
$$= \int_{\mathbb{R}^{4d}} e^{-iy\cdot\eta} \delta\begin{pmatrix} |\xi_1|^2+|\xi_2|^2-|\xi|^2-|\eta|^2 \\ \xi_1+\xi_2-\xi-\eta \end{pmatrix} |\xi+\eta|^{2-d} e^{-|\eta|^2-2|\xi|^2}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \mathrm{d}\eta,$$

and we observe that an explicit formula for the integral in ξ_1, ξ_2 is known; see [13].

Lemma 3.4.2. *We have that if $\tau > \frac{|\xi|^2}{2}$, then*

$$\int_{\mathbb{R}^{2d}} \delta\begin{pmatrix} \tau - |\xi_1|^2 - |\xi_2|^2 \\ v - \xi_1 - \xi_2 \end{pmatrix} \mathrm{d}\xi_1 \mathrm{d}\xi_2 = C_d (2\tau - |\xi|^2)^{\frac{d-2}{2}},$$

where $C_d > 0$ and is easily computable.

Hence, the left hand side of (3.21) simplifies to

$$C_d \int_{\mathbb{R}^{2d}} e^{-iy\cdot\eta} \left(2|\xi|^2 + 2|\eta|^2 - |\xi+\eta|^2\right)^{\frac{d-2}{2}} |\xi+\eta|^{2-d} e^{-|\eta|^2-2|\xi|^2}\, \mathrm{d}\xi \mathrm{d}\eta,$$

which is (up to constants) the Fourier transform of the function

$$e^{-|\eta|^2}\int_{\mathbb{R}^d} e^{-2|\xi|^2}|\xi-\eta|^{d-2}|\xi+\eta|^{2-d}\,\mathrm{d}\xi$$

on $\mathbb{R}^d$. Hence, by taking the (inverse) Fourier transform of both sides of (3.21), it follows that the equation to be satisfied is

$$e^{-|\eta|^2}\int_{\mathbb{R}^d} e^{-2|\xi|^2}|\xi-\eta|^{d-2}|\xi+\eta|^{2-d}\,\mathrm{d}\xi = \lambda e^{-|\eta|^2},$$

for some constant λ. However, unless $d = 2$ it is clear that this equality cannot hold, as the function

$$F(\eta) = \int_{\mathbb{R}^d} e^{-2|\xi|^2}|\xi-\eta|^{d-2}|\xi+\eta|^{2-d}\,\mathrm{d}\xi$$

is not constant in η if $d \geq 3$. By a rotation, this function depends only on $|\eta|$, and hence it suffices to show that the function on $[0,\infty)$ given by

$$G(r) = \int_{\mathbb{R}^d} e^{-2|\xi|^2}|\xi-r\mathbf{e}_1|^{d-2}|\xi+r\mathbf{e}_1|^{2-d}\,\mathrm{d}\xi,$$

is not constant, where $\mathbf{e}_1$ denotes the first basis vector in $\mathbb{R}^d$. If we let

$$H(r) = \frac{|\xi-r\mathbf{e}_1|}{|\xi+r\mathbf{e}_1|}^{d-2},$$

then one can calculate that up to a nonzero constant depending only on d,

$$\frac{\mathrm{d}}{\mathrm{d}r}H(r) = (\xi\cdot\mathbf{e}_1)\frac{|\xi-r\mathbf{e}_1|^{d-6}}{|\xi+r\mathbf{e}_1|^{d-2}}(|\xi|^2-r^2),$$

and that (again up to constants)

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}H(r)\Big|_{r=0} = (\xi \cdot \mathbf{e}_1)^2|\xi|^{-2}$$

But if we multiply the latter expression by $e^{-2|\xi|^2}$ and integrate in ξ, then we see that (after interchange of limit and integral) the coefficient of the third term in a Taylor expansion of the function G will be nonzero, which implies the function is not constant in r. $\square$

Remark. As a 'reality check', we also note that this calculation can be adapted to consider the Ozawa–Tsutsumi estimate (3.2) by replacing v with $\overline{v}$ everywhere and applying Lemma 2.1 of [9] instead of the corresponding result from [13], and deduce (as expected) that the functions given by (3.19) are indeed critical points of the corresponding nonlinear functional in this case.

3.5 Alternative solution to the Maxwell–Boltzmann equation

In this section we provide the alternative derivation of the characterisation of maximisers for the estimates (3.5)–(3.9). The purpose is to provide a self-contained proof that the Maxwell–Boltzmann equation (3.15) has only gaussian solutions even if the input functions are assumed to be not necessarily integrable, but only locally integrable; showing that such solutions necessarily have gaussian form is a non-trivial task. Foschi [34] solved (3.14) in the case $d = 2$ under the assumption that the initial data are locally integrable, and here we show how to solve (3.14) for all $d \geq 2$. The obvious extension of Foschi's argument for $d = 2$ appears only to go through to arbitrary *even* dimensions, with the Hairy Ball theorem (see [71]) providing the obstacle in odd dimensions; our argument works in all dimensions.

As we saw in Section 3.3, this is not necessary to deduce the characterisation of maximisers for the particular estimates we are considering as one can reduce to the integrable case via an additional argument, but we hope that our analysis may find applications to the study of the Maxwell–Boltzmann equation in different contexts.

Following the overall strategy used by Foschi [34], we proceed with Steps (I)–(III) as follows:

(I) Locally integrable solutions f of (3.15) must be continuous.

(II) Continuous and nonzero solutions of (3.15) never vanish.

(III) Continuous and never vanishing solutions of (3.15) must be gaussian.

The above strategy is familiar in the literature on solving functional equations and, as is often the case, the most difficult is the first.

For $x, y \in \mathbb{R}^d$ we introduce the notation $\mathbb{S}(x, y)$ for the sphere in $\mathbb{R}^d$ with centre $\frac{1}{2}(x + y)$ and radius $\frac{1}{2}|x - y|$. It is also helpful to introduce the notation $\Pi(x, y)$ for the bisector plane of the points $x, y \in \mathbb{R}^d$; that is

$$\Pi(x, y) = \{u \in \mathbb{R}^d : |u - x| = |u - y|\}.$$

Notice that x and y are antipodal points on $\mathbb{S}(x, y)$, and we have the following simple lemma.

Lemma 3.5.1. *Let $x, y \in \mathbb{R}^d$. If $P, Q \in \mathbb{S}(x, y)$ are such that $P + Q = x + y$ then*

$$|P|^2 + |Q|^2 = |x|^2 + |y|^2 .$$

If, in addition, $P, Q \in \Pi(x, y)$ then

$$|P - y| = |Q - y| = \frac{1}{\sqrt{2}}|x - y|\,.$$

Proof. We have

$$\begin{aligned}|P|^2 &= \left|P - \frac{x+y}{2}\right|^2 + \left|\frac{x+y}{2}\right|^2 + \left(P - \frac{x+y}{2}\right)\cdot(x+y)\\ &= \left|\frac{x-y}{2}\right|^2 + \left|\frac{x+y}{2}\right|^2 + \left(P - \frac{x+y}{2}\right)\cdot(x+y)\end{aligned}$$

and using a similar identity for Q, we obtain

$$|P|^2 + |Q|^2 = 2\left|\frac{x-y}{2}\right|^2 + 2\left|\frac{x+y}{2}\right|^2 + (P + Q - (x+y))\cdot(x+y).$$

Using the parallelogram law and the assumption that $P+Q = x+y$, we obtain $|P|^2+|Q|^2 = |x|^2 + |y|^2$ as desired.

If we also assume that $P, Q \in \Pi(x, y)$ then

$$\left(P - \frac{x+y}{2}\right)\cdot(x-y) = \left(Q - \frac{x+y}{2}\right)\cdot(x-y) = 0$$

and therefore, using Pythagoras' Theorem and $P, Q \in \mathbb{S}(x, y)$, we obtain $|P - y|^2 = |Q - y|^2 = \frac{1}{2}|x - y|^2$. $\square$

For Step (I), the fundamental result on which our argument is based is Proposition 7.5 from [34], whose statement we now recall.

Proposition 3.5.2. *Suppose $\Omega \subset \mathbb{R}^d \times \mathbb{R}^d$ is open and for each nonzero $x \in \mathbb{R}^d$ we have that*

$$\Omega_x = \{y \in \mathbb{R}^d : (x, y) \in \Omega\}$$

is dense in $\mathbb{R}^d$. *If we have smooth maps* $P : \Omega \to \mathbb{R}^d$ *and* $Q : \Omega \to \mathbb{R}^d$ *such that*

$$\det \frac{\partial P}{\partial y}(x, y) \neq 0, \qquad \det \frac{\partial Q}{\partial y}(x, y) \neq 0$$

for $(x, y) \in \Omega$, *then any locally integrable solution* $f : \mathbb{R}^d \to \mathbb{C}$ *of*

$$f(x)f(y) = f(P(x, y))f(Q(x, y))$$

almost everywhere on Ω, *must be continuous.*

When $d = 2$, Foschi [34] used the mappings $P : \mathbb{R}^2 \to \mathbb{R}^2$ and $Q : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$P(x, y) = \frac{x + y}{2} + H\left(\frac{x - y}{2}\right) = \left(\frac{I + H}{2}\right)x + \left(\frac{I - H}{2}\right)y \tag{3.22}$$

$$Q(x, y) = \frac{x + y}{2} - H\left(\frac{x - y}{2}\right) = \left(\frac{I - H}{2}\right)x + \left(\frac{I + H}{2}\right)y\,, \tag{3.23}$$

where $H = H_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is the map given by $H_0(x_1, x_2) = (-x_2, x_1)$. The nice properties of H are that it is smooth, isometric and $H(x)$ is orthogonal to x for every $x \in \mathbb{R}^2$. Lemma 3.5.1 implies that solutions of (3.15) satisfy $f(x)f(y) = f(P(x, y))f(Q(x, y))$ and such functions are automatically continuous using Proposition 3.5.2. This argument almost immediately extends to $\mathbb{R}^d$ when d is *even*, by taking P and Q exactly as in (3.22) and (3.23), where

$$H(x_1, x_2, \ldots, x_{d-1}, x_d) = (-x_2, x_1, -x_4, x_3, \ldots, -x_d, x_{d-1})\,,$$

so that H is the block diagonal matrix with $\frac{d}{2}$ copies of H_0 on the diagonal; clearly, such a map H is smooth, isometric and $H(x)$ is orthogonal to x for every $x \in \mathbb{R}^d$. However, it seems we cannot proceed like this when d is *odd* because of the Hairy Ball theorem from algebraic topology. In particular, it follows (see, for example, [71]) from the Hairy Ball theorem that any continuous map H from an even dimensional sphere to itself cannot have the property that $H(x)$ is orthogonal to x for every x (because there must exist some point on the sphere which is fixed, or some point on the sphere which is sent to its antipode). So, we cannot find an isometric map $H : \mathbb{R}^d \to \mathbb{R}^d$ which is continuous and is such that $H(x)$ is orthogonal to x for every x when d is odd.

Our argument below applies to all dimensions $d \geq 2$ independently of its parity. It is motivated to some extent by the proof of Lemma 7.20 from [34], which concerns the functional equation $f(x)f(y) = \Lambda(|x| + |y|, x + y)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ (in connection with sharp Strichartz estimates for the wave equation), and where the analogous geometric object to $\mathbb{S}(x, y)$ is the ellipsoid

$$\mathbb{E}(x, y) = \{u \in \mathbb{R}^3 : |u| + |x + y - u| = |x| + |y|\}.$$

Foschi selects $P(x, y)$ to be the unique point lying on $\mathbb{E}(x, y) \cap \langle y\rangle \setminus \{y\}$. Here, we are using the notation $\langle y\rangle$ for the span of $y \in \mathbb{R}^d$. We cannot proceed in this way because if $P(x, y)$ is the unique point lying on $\mathbb{S}(x, y) \cap \langle y\rangle \setminus \{y\}$ then $P(x, y) = \frac{x \cdot y}{|y|^2} y$. Hence, for fixed x, we have $P(x, \lambda y) = P(x, y)$ and this means the invertibility of $y \mapsto P(x, y)$ fails rather strongly.

Define smooth mappings $P : \widetilde{\Omega} \to \mathbb{R}^d$ and $Q : \widetilde{\Omega} \to \mathbb{R}^d$ by

$$P(x, y) = \left(1 - \frac{|x - y|}{|x + y|}\right) \frac{x + y}{2}$$

and

$$Q(x, y) = \left(1 + \frac{|x - y|}{|x + y|}\right) \frac{x + y}{2},$$

where

$$\widetilde{\Omega} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x \neq \pm y\}.$$

Geometrically, $P(x,y)$ and $Q(x,y)$ are the two intersection points of $\mathbb{S}(x,y)$ with the straight line passing through the origin and the point $\frac{1}{2}(x+y)$, with $P(x,y)$ the closest of these intersection points with the origin. It is also clear that $P(x,y)$ and $Q(x,y)$ are antipodal points on $\mathbb{S}(x,y)$ with

$$P(x,y)+Q(x,y)=x+y.$$

We also have that

$$|P(x,y)|^2+|Q(x,y)|^2=|x|^2+|y|^2$$

and this follows immediately from Lemma 3.5.1.

Define

$$\Omega=\{(x,y)\in\widetilde{\Omega}:(x-P(x,y))\cdot P(x,y)\neq 0 \text{ and } (x-Q(x,y))\cdot Q(x,y)\neq 0\}.$$

For each nonzero $x\in\mathbb{R}^d$, the section $\Omega_x=\{y\in\mathbb{R}^d:(x,y)\in\Omega\}$ is dense in $\mathbb{R}^d$. This is a straightforward consequence of the following.

Lemma 3.5.3. *For each nonzero $x\in\mathbb{R}^d$ we have*

$$\mathbb{R}^d\setminus\Omega_x=\langle x\rangle\cup\langle x\rangle^{\perp}.$$

Proof. Clearly

$$2(x-P(x,y))=x-y+\frac{|x-y|}{|x+y|}(x+y)$$

and therefore

$$4(x-P(x,y))\cdot P(x,y)=\left(1-\frac{|x-y|}{|x+y|}\right)\left((x-y)\cdot(x+y)+|x-y||x+y|\right).$$

Obviously, $|x-y| = |x+y|$ if and only if $y \in \langle x \rangle^{\perp}$. Also,

$$(x-y)\cdot(x+y) = -|x-y||x+y|$$

implies that $y = \lambda x$ for some $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$.

Similarly

$$4(x - Q(x,y)) \cdot Q(x,y) = \left(1 + \frac{|x-y|}{|x+y|}\right)((x-y)\cdot(x+y) - |x-y||x+y|)$$

and

$$(x-y)\cdot(x+y) = |x-y||x+y|$$

implies that $y = \lambda x$ for some $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$. The lemma now follows. $\square$

For fixed $x \in \mathbb{R}^d$, our goal now is to show that the mappings $y \mapsto P(x,y)$ and $y \mapsto Q(x,y)$ are locally invertible on Ω_x. For this, we argue by construction that a smooth inverse exists, and to ease notation, we may write $P_x(y) = P(x,y)$ and $Q_x(y) = Q(x,y)$.

If $P\cdot(x-P) \neq 0$ then we define the point $C(x,P) \in \langle P \rangle$ by

$$C(x,P) = \frac{|x|^2 - |P|^2}{2P\cdot(x-P)}\, P.$$

Lemma 3.5.4. *If $x, P \in \mathbb{R}^d$ are such that $P\cdot(x-P) \neq 0$ then*

$$\Pi(x,P) \cap \langle P \rangle = \{C(x,P)\}.$$

Proof. We have that $\lambda P \in \Pi(x,P)$ if and only if $|\lambda P - x|^2 = (\lambda-1)^2|P|^2$. This is clearly equivalent to

$$2\lambda P\cdot(x-P) = |x|^2 - |P|^2$$

and the claim now follows. $\square$

We shall show that the mapping $\mathfrak{I}_x$ given by

$$\mathfrak{I}_x(P) = -x + \frac{|x|^2 - |P|^2}{P \cdot (x - P)}\, P$$

is a smooth inverse for P_x. Observe that

$$C(x, P) = \frac{x + \mathfrak{I}_x(P)}{2} \tag{3.24}$$

so, geometrically, $C(x, P)$ is the centre of a sphere containing x and P, and $\mathfrak{I}_x(P)$ is the antipodal point to x on this sphere.

Observe that if $y \in \Omega_x$ then $\mathfrak{I}_x(P_x(y))$ is well-defined. Since $y \notin \langle x\rangle^\perp$ we have

$$\frac{x+y}{2} = \left(1 - \frac{|x-y|}{|x+y|}\right)^{-1} P_x(y)\,;$$

that is, $\frac{1}{2}(x+y) \in \langle P_x(y)\rangle$. Also

$$\left|\frac{x+y}{2} - x\right| = \frac{|x-y|}{2} = \left|\frac{x+y}{2} - P_x(y)\right|$$

so $\frac{1}{2}(x+y) \in \Pi(x, P_x(y))$. Lemma 3.5.4 implies that $C(x, P_x(y)) = \frac{1}{2}(x+y)$ and therefore, by (3.24) we get $y = \mathfrak{I}_x(P_x(y))$.

We remark that whenever $x \in \mathbb{R}^d$ is nonzero and $y \in \mathbb{R}^d$, we have $|P_x(y)| < |x|$ and $|Q_x(y)| > |x|$. These inequalities without strictness may be seen using the triangle inequality, and the strictness comes from the fact that $y \notin \langle x\rangle$ for $y \in \Omega_x$. So the image of Ω_x under P_x is contained in the open ball of radius $|x|$ centred at the origin, and image of Ω_x under Q_x is contained in the complement of the closed ball of radius $|x|$ centred at the origin. The same argument given above to establish that $y = \mathfrak{I}_x(P_x(y))$ also shows that $y = \mathfrak{I}_x(Q_x(y))$ for each $y \in \Omega_x$.

It now follows from Proposition 3.5.2 that all locally integrable solutions of (3.15) must be continuous and this completes Step (I).

For Step (II), we must show that if $f : \mathbb{R}^d \to \mathbb{C}$ is a continuous and nonzero solution of (3.15), then f never vanishes. To prove this, suppose f vanishes at some $x_0 \in \mathbb{R}^d$ and take an arbitrary point $y \in \mathbb{R}^d$. It obviously suffices to prove that $f(y) = 0$. To see this, choose any two points $\widetilde{P}, \widetilde{Q} \in \mathbb{S}(x_0, y) \cap \Pi(x_0, y)$ satisfying $\widetilde{P} + \widetilde{Q} = x + y$. Such $\widetilde{P}$ and $\widetilde{Q}$ are easily seen to exist; they are essentially unique when $d = 2$ and there is some choice for $d \geq 3$. Then Lemma 3.5.1 implies $|\widetilde{P}|^2 + |\widetilde{Q}|^2 = |x_0|^2 + |y|^2$ and

$$|\widetilde{P} - y| = |\widetilde{Q} - y| = \frac{1}{\sqrt{2}}|x_0 - y|\,.$$

Using (3.15) it follows that either $f(\widetilde{P}) = 0$ or $f(\widetilde{Q}) = 0$. So, we may conclude that there exists $x_1 \in \mathbb{R}^d$ such that $f(x_1) = 0$ and $|x_1 - y| = \frac{1}{\sqrt{2}}|x_0 - y|$. By repeating this procedure, we obtain a sequence $(x_n)_{n\geq 0}$ such that $|x_n - y| = \frac{1}{\sqrt{2}}|x_{n-1} - y|$ for each $n \geq 1$ and $f(x_n) = 0$ for each $n \geq 0$. Thus, x_n is a convergent sequence to y, and the continuity of f implies that $f(y) = 0$, as desired.

Remark. The above proof for Step (II) is a simple extension of the proof of Lemma 7.13 in [34] to higher dimensions. For this argument, we do not need to consider the invertibility properties of the mappings $\widetilde{P}$ and $\widetilde{Q}$ (thus side-stepping the obstacle from the Hairy Ball Theorem alluded to earlier) and the important consideration here is the distance of $\widetilde{P}$ and $\widetilde{Q}$ to y.

For the final Step (III), we must show that whenever $f : \mathbb{R}^d \to \mathbb{C}$ is continuous, never vanishes and satisfies (3.15), then

$$f(x) = \exp(a|x|^2 + b \cdot x + c)$$

for some $a, c \in \mathbb{C}$ and $b \in \mathbb{C}^d$. By replacing f with $f(0)^{-1}f$, we may now assume that $f(0) = 1$. Therefore, whenever $x \perp y$ we have

$$f(x)f(y) = \Lambda(|x|^2 + |y|^2, x + y) = \Lambda(|x + y|^2, x + y) = f(x + y)f(0) = f(x + y),$$

so that f satisfies an orthogonal Cauchy functional equation.

Let g be given by $g(x) = f(x)f(-x)$; then g is continuous, g is even, $g(0) = 1$ and whenever $x \perp y$ we have

$$g(x + y) = f(x + y)f(-x - y) = f(x)f(y)f(-x)f(-y) = g(x)g(y)\,.$$

Similarly, let h be given by $h(x) = \frac{f(x)}{f(-x)}$; since f never vanishes, h is well-defined and continuous. Also, $h(0) = 1$, $h(x)h(-x) = 1$ and whenever $x \perp y$ we have

$$h(x + y) = \frac{f(x + y)}{f(-x - y)} = \frac{f(x)f(y)}{f(-x)f(-y)} = h(x)h(y)\,.$$

Since $f^2 = gh$, we can apply the following classical result; we give its proof for completeness and since we do not know a direct reference.

Proposition 3.5.5. *Suppose $k : \mathbb{R}^d \to \mathbb{C}$ is continuous, k never vanishes, $k(0) = 1$ and*

$$k(x)k(y) = k(x + y)$$

whenever x is orthogonal to y. If $k(x) = k(-x)$ for all $x \in \mathbb{R}^d$ then there exists $a \in \mathbb{C}$

such that

$$k(x) = \exp(a|x|^2).$$

If, instead, $k(x)k(-x) = 1$ *for all* $x \in \mathbb{R}^d$ *then there exists* $b \in \mathbb{C}^d$ *such that*

$$k(x) = \exp(b \cdot x).$$

Proof. First suppose that $k(x) = k(-x)$ for all $x \in \mathbb{R}^d$. If $u, v \in \mathbb{R}^d$ are such that $|u| = |v|$ then $u + v$ is orthogonal to $u - v$ and hence

$$k(u) = k(\tfrac{1}{2}(u + v) + \tfrac{1}{2}(u - v)) = k(\tfrac{1}{2}(u + v))k(\tfrac{1}{2}(u - v)).$$

Since k is even we have

$$\begin{aligned} k(\tfrac{1}{2}(u + v))k(\tfrac{1}{2}(u - v)) &= k(\tfrac{1}{2}(u + v))k(\tfrac{1}{2}(v - u)) \\ &= k(u) = k(\tfrac{1}{2}(u + v) + \tfrac{1}{2}(v - u)) = k(v). \end{aligned}$$

Hence, $k(u) = k(v)$ which means k is radial and we may write $k(x) = \Phi(|x|^2)$ for some continuous function $\Phi : [0, \infty) \to \mathbb{C}$.

If $\lambda, \mu > 0$ and e_j is the jth standard basis vector, then

$$\Phi(\lambda + \mu) = k(\sqrt{\lambda}e_1 + \sqrt{\mu}e_2) = k(\sqrt{\lambda}e_1)k(\sqrt{\mu}e_2) = \Phi(\lambda)\Phi(\mu)$$

and since Φ is continuous it follows that $\Phi(t) = \exp(at)$ for some $a \in \mathbb{C}$. This gives $k(x) = \exp(a|x|^2)$, as claimed.

Now assume that instead k satisfies $k(x)k(-x) = 1$ for all $x \in \mathbb{R}^d$. We first claim that $k(x) > 0$ for all $x \in \mathbb{R}^d$. This follows because if $x \in \mathbb{R}^d$ and we take any y orthogonal to

x with $|y| = |x|$ then $x - y$ is orthogonal to $x + y$ and therefore

$$k(2x) = k((x - y) + (x + y)) = k(x - y)k(x + y) = k(x)k(y)k(x)k(y) = k(x)^2.$$

For $n \in \mathbb{N}$ choose any y_n orthogonal to x with $|y| = \sqrt{n}|x|$ so that

$$(nx + y) \cdot (x - y) = n|x|^2 - |y|^2 = 0$$

and hence

$$k((n+1)x) = k((nx+y)+(x-y)) = k(nx+y)k(x-y) = k(nx)k(y)k(x)k(-y) = k(nx)k(x).$$

By induction, it follows that

$$k(nx) = k(x)^n$$

for each $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Consequently, if $m \in \mathbb{N}$ then

$$k(x) = k(m\tfrac{x}{m}) = k(\tfrac{x}{m})^m,$$

so that

$$k(\tfrac{x}{m}) = k(x)^{1/m}.$$

Altogether

$$k(\tfrac{n}{m}x) = k(x)^{n/m}$$

for each $n, m \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$. By continuity, it follows that

$$k(\lambda x) = k(x)^\lambda \tag{3.25}$$

for all $\lambda > 0$ and $x \in \mathbb{R}^d$. Observe that if $k(x_0) = 0$ for some $x_0 \neq 0$ then

$$k(\lambda x_0) = k(x_0)^\lambda = 0$$

for all $\lambda > 0$. Taking $\lambda \to 0+$ and use the continuity of k to obtain that $k(0) = 0$, which is a contradiction. Hence, $k(x) \neq 0$ for all $x \in \mathbb{R}^d$.

For each $\lambda > 0$ and $x \in \mathbb{R}^d$ we have $k(\lambda x)k(-\lambda x) = 1$ and by another application of (3.25) we get

$$k(-\lambda x) = k(x)^{-\lambda}.$$

Here, we use that k never vanishes, and we have shown that (3.25) in fact holds for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$.

Finally, we fix $x, y \in \mathbb{R}^d$ and write $x = \lambda y + z$ where $\lambda \in \mathbb{R}$ and z is orthogonal to y. Then

$$k(x+y) = k((\lambda+1)y + z) = k((\lambda+1)y)k(z) = k(y)^{\lambda+1}k(z) = k(y)^\lambda k(z)\, k(y)$$

and

$$k(x)k(y) = k(\lambda y + z)k(y) = k(\lambda y)k(z)k(y) = k(y)^\lambda k(z)\, k(y)$$

so that $k(x+y) = k(x)k(y)$ for all $x, y \in \mathbb{R}^d$. Since k is continuous it follows that $k(x) = \exp(b \cdot x)$ for some $b \in \mathbb{C}^d$. $\square$

This completes our alternative proof that locally integrable solutions $f : \mathbb{R}^d \to \mathbb{C}$ of (3.15) must have the form

$$f(x) = \exp(a|x|^2 + b \cdot x + c)$$

for some $a, c \in \mathbb{C}$ and $b \in \mathbb{C}^d$, and the characterisation of extremisers in Theorems 3.1.1

and 3.1.3 follow as before. $\square$

Remark. We may regard (3.15) as a Cauchy functional equation restricted to the paraboloid $\{(x, |x|^2) : x \in \mathbb{R}^d\}$ and an impressive recent result of Charalambides [24], on the solution of Cauchy functional equations restricted to a rather general class of submanifolds, implies that (3.15) has solution $f(x) = \exp(a|x|^2 + b \cdot x + c)$ under the assumption that $f^{-1}(0)$ is null. Our argument above, of course, means that f is continuous and consequently never vanishes (for non-trivial f) so that this pre-image set is empty. From this point, we may slightly shorten our argument by invoking [24]; however, the main purpose of including this solution of (3.15) is that it is self-contained and it is hoped the geometric construction leading to the continuity of f may be useful in other related contexts.

Chapter 4

Some sharp bilinear estimates for the wave equation

This chapter is devoted to the proofs of the sharp linear and bilinear estimates for the wave equation. The work on the linear estimate in four space dimensions appearing here is joint with Neal Bez and is contained in [10]. The work on the bilinear estimates is joint with Neal Bez and Tohru Ozawa and is contained in [11].

4.1 Introduction

For $d \geq 2$, suppose that u, v satisfy $\partial_{tt} u = \Delta u$ and $\partial_{tt} v = \Delta v$ on $\mathbb{R}^{d+1}$ with $u(0) = u_0$, $\partial_t u(0) = u_1$, $v(0) = v_0$ and $\partial_t v(0) = v_1$. Consider the estimate

$$\|(-\Delta)^{\frac{\beta_0}{2}} D_-^{\beta_-} D_+^{\beta_+} (uv)\|_{L^2(\mathbb{R}^{d+1})} \leq C \|(u_0, u_1)\|_{\dot{H}^{\alpha_1} \times \dot{H}^{\alpha_1 - 1}} \|(v_0, v_1)\|_{\dot{H}^{\alpha_2} \times \dot{H}^{\alpha_2 - 1}} \tag{4.1}$$

where we recall that the operators D_+ and D_- are defined as

$$\widetilde{D_\pm f}(\tau, \xi) := \big||\tau| \pm |\xi|\big| \widetilde{f}(\tau, \xi)$$

using the space-time Fourier transform.

If u solves the linear wave equation we recall the standard decomposition $u = e^{it\sqrt{-\Delta}} f_+ + e^{it\sqrt{-\Delta}} f_-$, where

$$u(0) = f_+ + f_-, \quad \partial_t u(0) = i\sqrt{-\Delta}(f_+ - f_-), \tag{4.2}$$

Using this decomposition, the inequalities (4.1) are deduced from the following bilinear estimates for the propagator $e^{it\sqrt{-\Delta}}$:

$$\|(-\Delta)^{\frac{\beta_0}{2}} D_-^{\beta_-} D_+^{\beta_+} \left(e^{it\sqrt{-\Delta}} f e^{it\sqrt{-\Delta}} g \right) \|_{L^2(\mathbb{R}^{d+1})} \leq C \|f\|_{\dot{H}^{\alpha_1}(\mathbb{R}^d)} \|g\|_{\dot{H}^{\alpha_2}(\mathbb{R}^d)}, \tag{4.3}$$

and

$$\|(-\Delta)^{\frac{\beta_0}{2}} D_-^{\beta_-} D_+^{\beta_+} \left(e^{it\sqrt{-\Delta}} f \overline{e^{it\sqrt{-\Delta}} g} \right) \|_{L^2(\mathbb{R}^{d+1})} \leq C \|f\|_{\dot{H}^{\alpha_1}(\mathbb{R}^d)} \|g\|_{\dot{H}^{\alpha_2}(\mathbb{R}^d)}. \tag{4.4}$$

Some necessary conditions for estimates (4.3) and (4.4) to hold were also given in [37]. Closely related to the estimates (4.3) and (4.4), at least in the symmetric cases $\beta_- = \beta_+$ and $\alpha_1 = \alpha_2$, is the following inequality:

$$\begin{aligned} &\|e^{it\sqrt{-\Delta}} f e^{it\sqrt{-\Delta}} g\|_{L^2(\mathbb{R}^{d+1})}^2 \\ &\leq \frac{|\mathbb{S}^{d-1}|}{2^{\frac{d-1}{2}} (2\pi)^{3d-1}} \int_{\mathbb{R}^{2d}} |\widehat{f}(y_1)|^2 |\widehat{g}(y_2)|^2 |y_1|^{\frac{d-1}{2}} |y_2|^{\frac{d-1}{2}} (1 - y_1' \cdot y_2')^{\frac{d-3}{2}} \, \mathrm{d}y_1 \mathrm{d}y_2, \end{aligned} \tag{4.5}$$

and we recall that $x' := \frac{x}{|x|}$, for $x \in \mathbb{R}^d$. When $d \geq 3$ the constant is sharp, and a full characterisation of maximisers is known; this was first proved by Foschi [34] in the case $d = 3$ and $f = g$, and the general case is contained in [13].

Our main result is the following one-parameter family of sharp bilinear inequalities for the one-sided propagator $e^{it\sqrt{-\Delta}}$. Before proceeding, we introduce some notation. For $\beta \in \mathbb{R}$ we define the operator $M_\beta = D_-^\beta D_+^\beta$, or equivalently using the space-time Fourier transform:

$$\widetilde{M_\beta u}(\tau,\xi) := |\tau^2 - |\xi|^2|^\beta \widetilde{u}(\tau,\xi),$$

and we introduce the quantity

$$\mathcal{I}_\beta(f,g) = \int_{\mathbb{R}^{2d}} |\widehat{f}(y_1)|^2 |\widehat{g}(y_2)|^2 |y_1||y_2| \left(|y_1||y_2| - y_1\cdot y_2\right)^{\frac{d-3}{2}+2\beta} \,\mathrm{d}y_1\mathrm{d}y_2,$$

defined for suitable functions f and g; for instance, $f,g \in \dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)$ suffices.

Theorem 4.1.1. *Let $d \geq 2$ and $\beta > \frac{1-d}{4}$. Then*

$$\|M_\beta(e^{it\sqrt{-\Delta}} f \overline{e^{it\sqrt{-\Delta}} g})\|_{L^2}^2 \leq \mathbf{KM}(\beta,d)\mathcal{I}_\beta(f,g) \tag{4.6}$$

holds with constant

$$\mathbf{KM}(\beta,d) = \frac{|\mathbb{S}^{d-2}| 2^{2\beta+\frac{d-3}{2}}}{(2\pi)^{3d-1}} \mathrm{B}\left(2\beta + \frac{d-1}{2}, \frac{d-1}{2}\right).$$

If in addition $d \geq 3$ and $\beta < \frac{d+1}{4}$, or $d=2$ and $\beta \in \left(0,\frac{3}{4}\right)$, then the constant is sharp and equality holds if and only if

$$\widehat{f}(\xi) = \lambda \widehat{g}(\xi) = \frac{e^{a|\xi| + b\cdot\xi + c}}{|\xi|} \tag{4.7}$$

for some $\lambda, a, c \in \mathbb{C}$ with $\operatorname{Re} a < 0$, and $b \in \mathbb{C}^d$ with $|\operatorname{Re} b| < -\operatorname{Re} a$.

Remarks.

- The restriction $\beta > \frac{1-d}{4}$ is necessary in Theorem 4.1.1; it is evident that the optimal constant $\mathbf{KM}(\beta,d)$ blows up at this threshold. On the other hand, the upper bounds on β here are necessary only since the method we use for solving the forthcoming functional equation (4.26) requires the assumption of local integrability on the input functions, and it is possible that these restrictions could be relaxed further by an alternative analysis of this equation.

- The lower bound on β for $d = 2$ is more restrictive than the one predicted by the general case $d \geq 3$. In the case $d = 2$ it was observed in [13] that $\mathcal{I}_0(f,g)$ is unbounded for $f = \lambda g$ integrable, and hence no sharp estimate attained on such functions is possible. A similar argument shows that a necessary condition for a sharp estimate as in Theorem 4.1.1 to hold for general dimensions is $\beta > \frac{2-d}{2}$; of course, when $d \geq 3$ this condition is redundant.

When $\beta = 0$, inequality (4.6) is the same as (4.5); one can easily check that

$$\mathbf{KM}(0,d) = \frac{|\mathbb{S}^{d-1}|}{2^{\frac{d-1}{2}}(2\pi)^{3d-1}}.$$

The case $\beta = \frac{3-d}{4}$ in Theorem 4.1.1 is also distinguished since the power of the angular weight is zero and so by Plancherel's theorem,

$$\mathcal{I}_{\frac{3-d}{4}}(f,g) = (2\pi)^{2d}\|f\|^2_{\dot{H}^{\frac{1}{2}}}\|g\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Therefore, Theorem 4.1.1 recovers immediately inequality (4.4) for the exponents

$$(\beta_0, \beta_-, \beta_+, \alpha_1, \alpha_2) = \left(0, \frac{3-d}{4}, \frac{3-d}{4}, \frac{1}{2}, \frac{1}{2}\right)$$

with sharp constant; in this sense Theorem 4.1.1 unifies and generalises the estimates (4.4) and (4.5).

Note that when $\beta_0 = \beta_+ = \beta_- = 0$ and $\alpha_1 = \alpha_2$, inequalities (4.3) and (4.4) (in the diagonal case $f = g$) are the same and are the L^4 Sobolev–Strichartz inequalities for the one-sided wave propagator

$$\| |e^{it\sqrt{-\Delta}} f|^2 \|^{\frac{1}{2}}_{L^2(\mathbb{R}^{d+1})} = \|e^{it\sqrt{-\Delta}} f\|_{L^4(\mathbb{R}^{d+1})} \leq \mathbf{W}\left(d, \tfrac{d-1}{4}\right) \|f\|_{\dot{H}^{\frac{d-1}{4}}(\mathbb{R}^d)}, \tag{4.8}$$

where we recall that $\mathbf{W}\left(d, \frac{d-1}{4}\right)$ is defined to be the optimal constant for this estimate. As we comment in the introduction, the problem of computing this constant and giving a full characterisation of the maximisers for the estimate (4.8) remains unresolved outside of special cases. As a consequence of our next result we can obtain progress on this problem, by computing the value of $\mathbf{W}(d, \frac{d-1}{4})$ in the case $d = 4$.

Corollary 4.1.2. *Suppose that $d \geq 3$ and that $\beta \in \left[\frac{3-d}{4}, \frac{5-d}{4}\right]$. Then,*

$$\left\| M_\beta |e^{it\sqrt{-\Delta}} f|^2 \right\|^2_{L^2(\mathbb{R}^{d+1})} \leq \mathbf{C}(\beta, d) \|f\|^4_{\dot{H}^{\beta+\frac{d-1}{4}}(\mathbb{R}^d)} \tag{4.9}$$

where

$$\mathbf{C}(\beta, d) = \frac{2^{d-3+4\beta}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2\beta + \frac{d-1}{2}\right)}{\pi^{\frac{d}{2}}(2\beta + d - 2)\Gamma\left(2\beta + \frac{3d-5}{2}\right)}.$$

Further, the constant is sharp and equality holds in (4.9) *if and only if f satisfies* (4.7) *with $\lambda, a, c \in \mathbb{C}$ and $b \in \mathbb{C}^d$ where $0 = |\operatorname{Re} b| < -\operatorname{Re} a$.*

The cases where $\beta = 0$ in Corollary 4.1.2 occur when $d \in \{3, 4, 5\}$; the case $d = 3$ is due to Foschi [34] and the case $d = 5$ to Bez–Rogers [13]. The case $d = 4$ was new, it implies that

$$\mathbf{W}(4, \tfrac{3}{4}) = \left(\frac{4}{15\pi^2}\right)^{\frac{1}{4}},$$

and gives a full characterisation of maximisers for inequality (4.8) for $d = 4$. As a consequence, we can apply the orthogonality argument from [34] and obtain the following estimate for the full solution of the linear wave equation.

Corollary 4.1.3. *The solution of the wave equation $\partial_{tt} u = \Delta u$ on $\mathbb{R}^4 \times \mathbb{R}$ with initial* *data* $(u(0), \partial_t u(0))$ *satisfies*

$$\|u\|_{L^4(\mathbb{R}^5)} \leq \left(\frac{1}{10\pi^2}\right)^{\frac{1}{4}} \left(\|u(0)\|^2_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^4)} + \|\partial_t u(0)\|^2_{\dot{H}^{-\frac{1}{4}}(\mathbb{R}^4)}\right)^{\frac{1}{2}}$$

and the constant is sharp. Furthermore, the initial data given by

$$(u(0), \partial_t u(0)) = (0, (1+|x|^2)^{-\frac{5}{2}}),$$

is extremal and generates the set of all extremal initial data under the action of the group generated by the transformations:

- *space-time translations* $u(t,x) \to u(t+t_0, x+x_0)$ *with* $(t_0, x_0) \in \mathbb{R}^{d+1}$*;*
- *space-time dilations* $u(t,x) \to u(\mu t, \mu x)$ *with* $\mu > 0$*;*
- *change of scale* $u(t,x) \to \mu u(t,x)$ *with* $\mu > 0$*;*
- *phase shift* $u(t,x) \to e^{i\theta_+} e^{it\sqrt{-\Delta}} f_+ + e^{i\theta_-} e^{it\sqrt{-\Delta}} f_-$ *with* $\theta_+, \theta_- \in \mathbb{R}$*, and* f_+ *and* f_- *given by* (4.2)*.*

In addition, the proof of Corollary 4.1.2 provides a new derivation of the sharp constant $\mathbf{W}(5,1)$ and characterisation of the maximisers for (4.8) in this case from [13], which unifies (at the level of the proof) with the derivation of $\mathbf{W}(4, \frac{3}{4})$ described above.

In the case where f and g are radial, using polar co-ordinates one can calculate that

$$\mathcal{I}_\beta(f,g) = 2^{\frac{7(d-1)}{2}+2\beta} \pi^{\frac{4d-1}{2}} \frac{\Gamma(\frac{d}{2})\Gamma(d-2+2\beta)}{\Gamma(\frac{3d-5}{2}+2\beta)} \|f\|^2_{\dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)} \|g\|^2_{\dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)},$$

so Theorem 4.1.1 implies that, for radial functions f, g and any $\beta > \max\left\{\frac{1-d}{4}, \frac{2-d}{2}\right\}$ we have

$$\|M_\beta(e^{it\sqrt{-\Delta}} f \overline{e^{it\sqrt{-\Delta}} g})\|^2_{L^2} \leq \mathbf{C}(\beta, d) \|f\|^2_{\dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)} \|g\|^2_{\dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)}, \tag{4.10}$$

which is (4.4) when

$$(\beta_0, \beta_+, \beta_-, \alpha_1, \alpha_2) = \left(0, \beta, \beta, \beta + \frac{d-1}{4}, \beta + \frac{d-1}{4}\right).$$

Here, $\mathbf{C}(\beta, d)$ is the same as in Corollary 4.1.2 and is sharp. Further, if $\beta < \frac{d+1}{4}$ it is attained if and only if (f, g) satisfy (4.7) for some $\lambda, a, c \in \mathbb{C}$ with $\operatorname{Re} a < 0$, and where b is the zero vector in $\mathbb{C}^d$ (this is simply the class of maximisers for inequality (4.6) restricted to pairs of radial functions).

Given the above discussion, it seems of interest to determine if an estimate such as (4.10), with sharp constant, could be obtained for general (that is, not necessarily radial) pairs (f, g). Our next result is that this is not possible when $\beta < \frac{3-d}{4}$; in fact the estimate (4.10) fails to hold for *any* finite constant, even in the diagonal case $f = g$.

Proposition 4.1.4. *If $\beta < \frac{3-d}{4}$, then for any $A > 0$ there exists $f \in \dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)$ such that*

$$\frac{\left\| M_\beta |e^{it\sqrt{-\Delta}} f|^2 \right\|_{L^2(\mathbb{R}^{d+1})}}{\|f\|^2_{\dot{H}^{\beta+\frac{d-1}{4}}(\mathbb{R}^d)}} > A.$$

Remark. Proposition 4.1.4 of course implies that the lower bound $\beta \geq \frac{3-d}{4}$ is tight for a sharp estimate such as (4.9) (or (4.10)) to hold. On the other hand, the upper bound $\beta \leq \frac{5-d}{4}$ from Corollary 4.1.2 is a technical condition which arises as a consequence of our method and it should be possible to remove this condition, however it is not clear to us how to do this.

Proposition 4.1.4 may be interpreted as the statement that we cannot expect to recover control of the left hand side of (4.10) for general functions in $\dot{H}^s$ with $s < \frac{1}{2}$, unless we restrict to considering radial functions; this is in the spirit of earlier work in [36] and [50] where extensions of (4.1) and related estimates are proved for radial data. A similar restriction arises for the more general estimate (4.4) in [37], where the estimate is proved to be false when $\alpha_1 + \alpha_2 < \frac{1}{2}$ ([37], Example 5.5). Our next result provides a quantitative replacement for (4.10) in some of these cases, with sharp constant, by increasing the right hand side. In order to state it, we introduce a collection of maps $T_\beta : \dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d) \to L^1(\mathbb{S}^{d-1})$ defined by

$$T_\beta f(\omega) = \frac{1}{(2\pi)^d} \int_0^\infty |\widehat{f}(r\omega)|^2 r^{\frac{3d-3}{2}+2\beta} \, \mathrm{d}r,$$

for $\omega \in \mathbb{S}^{d-1}$.

Corollary 4.1.5. *Suppose that $d \geq 3$ and that $\beta \in \left(\frac{1-d}{4}, \frac{3-d}{4}\right]$, or $d = 2$ and $\beta \in \left(0, \frac{1}{4}\right]$. Then,*

$$\left\| M_\beta(e^{it\sqrt{-\Delta}} f \overline{e^{it\sqrt{-\Delta}} g}) \right\|_{L^2}^2 \leq \mathbf{C}(\beta, d) |\mathbb{S}^{d-1}|^{\frac{3-d-4\beta}{d-1}} \|T_\beta f\|_{L^p(\mathbb{S}^{d-1})} \|T_\beta g\|_{L^p(\mathbb{S}^{d-1})}. \tag{4.11}$$

for $p := \frac{2(d-1)}{3d-5+4\beta}$, and where $\mathbf{C}(\beta, d)$ is as in Corollary 4.1.2. The constant is sharp and equality holds in (4.11) *if and only if $f = \lambda g$ satisfy* (4.7) *with $\lambda, a, c \in \mathbb{C}$ and $b \in \mathbb{C}^d$ where $|\operatorname{Re} b| < -\operatorname{Re} a$.*

Notice that $p > 1$ if and only if $\beta < \frac{3-d}{4}$, and so by Plancherel's theorem and Hölder, for such β we have

$$\|f\|^2_{\dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)} \leq |\mathbb{S}^{d-1}|^{\frac{3-d-4\beta}{2(d-1)}} \|T_\beta f\|_{L^p(\mathbb{S}^{d-1})}, \tag{4.12}$$

with strict inequality unless f is radial. This implies that we can recover control of the left hand side of (4.10) in some cases for functions f and g not necessarily radial, but satisfying a more restrictive size condition.

Our final result in this chapter is a family of sharp inequalities analogous to those of Theorem 4.1.1, but motivated instead by the estimate (4.3) (the '$(+,+)$ case') rather than (4.4) (the '$(+,-)$ case'). It is essentially contained in [13]; we state it here for completeness and to provide a new proof which elucidates the role of the complex conjugation in inequality (4.6).

Proposition 4.1.6. *Let $d \geq 2$ and $\beta > \frac{2-d}{2}$. Then,*

$$\|M_\beta(e^{it\sqrt{-\Delta}} f e^{it\sqrt{-\Delta}} g)\|_{L^2}^2 \leq \mathbf{KM}'(\beta, d)\mathcal{I}_\beta(f,g) \tag{4.13}$$

holds with constant

$$\mathbf{KM}'(\beta, d) = \frac{|\mathbb{S}^{d-2}| 2^{2\beta + \frac{d-3}{2}}}{(2\pi)^{3d-1}} \mathrm{B}\left(\frac{d-1}{2}, \frac{d-1}{2}\right) = \frac{|\mathbb{S}^{d-1}|}{2^{\frac{d-1}{2} - 2\beta}(2\pi)^{3d-1}}.$$

If in addition $\beta < \frac{d+1}{4}$, then the constant is sharp and equality holds if and only if (f,g) satisfies (4.7) *for some $\lambda, a, c \in \mathbb{C}$ with $\operatorname{Re} a < 0$, and $b \in \mathbb{C}^d$ with $|\operatorname{Re} b| < -\operatorname{Re} a$.*

Note that the range of β in Proposition 4.1.6 is the same as in Theorem 4.1.1 when $d = 2$, but is strictly larger otherwise. This is due to the presence of additional symmetry in the left hand side of (4.13) which does not appear to occur in (4.6), permitting a simpler argument for which the restriction in Theorem 4.1.1 does not arise. This is reminiscent of the work in Chapter 3 where bilinear estimates (with and without complex conjugate) are studied for the Schrödinger evolution operator $e^{it\Delta}$. In that case, however, the presence of a complex conjugate causes a change in the shape of the multiplier at the level of sharp estimates; in particular the class of extremisers for the estimates analogous to (4.6) and (4.13) are different. It will become clear from our argument that the complex conjugate also plays an important role in the estimates we present for the wave equation; in particular Theorem 4.1.1 does not follow directly from the arguments in [13] and our focus is therefore on the more difficult $(+,-)$ case described by this result.

Organisation. In Section 4.2 we show how to deduce Corollaries 4.1.2, 4.1.3 and 4.1.5 from Theorem 4.1.1 and prove Proposition 4.1.4, and in Section 4.3 we prove Theorem 4.1.1 and Proposition 4.1.6 by first establishing the estimates (4.6) and (4.13) and then characterising the maximisers.

4.2 Proofs of corollaries

We begin this section by proving Corollary 4.1.5 and then show how to deduce Corollaries 4.1.2, 4.1.4 and 4.1.3. Before proceeding, we introduce notation

$$H_\lambda(f,g) := \int_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} f(\omega_1)\overline{g(\omega_2)}|\omega_1-\omega_2|^{-\lambda}\,\mathrm{d}\omega_1\mathrm{d}\omega_2$$

where $|\cdot|$ in this context means chordal distance on $\mathbb{S}^{d-1}$ (that is, euclidean distance on $\mathbb{R}^d$), and throughout this section we define $\lambda = 3-d-4\beta$. The proofs of Corollaries 4.1.2 and 4.1.5 are based on the observation that

$$\begin{aligned}\mathcal{I}_\beta(f,g) &= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty |\widehat{f}(r\omega)|^2|\widehat{g}(s\eta)|^2 r^{\frac{3d-3+4\beta}{2}} s^{\frac{3d-3+4\beta}{2}}(1-\omega\cdot\eta)^{-\frac{\lambda}{2}}\,\mathrm{d}r\mathrm{d}s\mathrm{d}\omega\mathrm{d}\eta \\ &= \frac{(2\pi)^{2d}}{2^{\frac{d-3}{2}+\beta}} H_\lambda(T_\beta f, T_\beta g).\end{aligned}$$

Proof of Corollary 4.1.5. We have that $\lambda > 0$. Using the Hardy–Littlewood–Sobolev inequality on the sphere, it follows that

$$|H_\lambda(T_\beta f, T_\beta g)| \le \pi^{\frac{\lambda}{2}} \frac{\Gamma\left(\frac{d-1-\lambda}{2}\right)}{\Gamma\left(d-1-\frac{\lambda}{2}\right)}\left(\frac{\Gamma(d-1)}{\Gamma\left(\frac{d-1}{2}\right)}\right)^{1-\frac{\lambda}{d-1}} \|T_\beta f\|_{L^p}\|T_\beta g\|_{L^p}, \tag{4.14}$$

for

$$p := \frac{2(d-1)}{2(d-1)-\lambda} = \frac{2(d-1)}{3d-5+4\beta},$$

and where the constant is sharp and equality holds if and only if there exists $C_0, C_1 \in \mathbb{C}$

and $\xi \in \mathbb{R}^d$ with $|\xi| < 1$ and

$$T_\beta f(\omega) = \frac{C_0}{(1+\xi\cdot\omega)^{\frac{2(d-1)-\lambda}{2}}}, \quad T_\beta g(\omega) = \frac{C_1}{(1+\xi\cdot\omega)^{\frac{2(d-1)-\lambda}{2}}}. \tag{4.15}$$

That inequality (4.11) is true with the stated constant now follows immediately from Theorem 4.1.1 using (4.14); the optimality of this constant and characterisation of extremisers is deduced from the additional observation that the functions f, g given by (4.7) also satisfy (4.15). To see this, assume $\operatorname{Re} a < -|\operatorname{Re} b|$, then for example for f we have

$$\begin{aligned}
T_\beta f(\omega) &= e^{2\operatorname{Re} c}\int_0^\infty e^{2r\operatorname{Re} a + 2r\operatorname{Re} b\cdot\omega} r^{\frac{3d-7}{2}+2\beta}\,\mathrm{d}r \\
&= e^{2\operatorname{Re} c}\int_0^\infty e^{r(2\operatorname{Re} a + 2\operatorname{Re} b\cdot\omega)} r^{\frac{3d-7}{2}+2\beta}\,\mathrm{d}r \\
&= \frac{e^{2\operatorname{Re} c}}{(-2\operatorname{Re} a - 2\operatorname{Re} b\cdot\omega)^{\frac{3d-5}{2}+2\beta}}\int_0^\infty e^{-s} s^{\frac{3d-7}{2}+2\beta}\,\mathrm{d}s \\
&= \frac{C}{(1+\gamma\cdot\omega)^{\frac{3d-5}{2}+2\beta}},
\end{aligned}$$

where C is a positive constant, and $\gamma := \frac{\operatorname{Re} b}{\operatorname{Re} a} \in \mathbb{R}^d$ satisfies $|\gamma| < 1$ by assumption, and this completes the proof of Corollary 4.1.5.

Proof of Corollary 4.1.2. We have that $\lambda < 0$. In this case, we use the following result to bound H_λ; its proof is based on a spectral argument using a spherical harmonic decomposition of g and the Funk–Hecke formula to obtain explicit expressions for the eigenvalues, inspired by recent work of Foschi in [35] on the problem of determining maximisers for an adjoint Fourier restriction inequality on the sphere.

Lemma 4.2.1. *Let $d \geq 2$, $-2 \leq \lambda < 0$, and let g be any L^1 function on $\mathbb{S}^{d-1}$. Then,*

$$H_\lambda(g,g) \leq 2^{d-2-\lambda}\mathrm{B}(\tfrac{d-1-\lambda}{2}, \tfrac{d-1}{2})\frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}\left|\int_{\mathbb{S}^{d-1}} g\right|^2, \tag{4.16}$$

with equality if g is constant. If, in addition $\lambda > -2$, then equality holds only if g is constant.

Proof. We deal with $\lambda > -2$ first; the information we need concerning the eigenvalues in this case is contained in the following lemma. Here we recall that $P_{k,d}$ denotes the Legendre polynomial of degree k in d dimensions.

Lemma 4.2.2. *Let $-2 < \lambda < 0$, and define*

$$\mathrm{I}_k(d,\lambda) = |\mathbb{S}^{d-2}| \int_{-1}^{1} (1-t)^{-\frac{\lambda}{2}} P_{k,d}(t)(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t.$$

Then

$$\mathrm{I}_0(d,\lambda) = |\mathbb{S}^{d-2}| 2^{d-2-\frac{\lambda}{2}} \mathrm{B}(\tfrac{d-1-\lambda}{2}, \tfrac{d-1}{2}) > 0$$

and $\mathrm{I}_k(d,\lambda) < 0$ for all $k \geq 1$.

Remark. Inequality (4.16) is false if $\lambda < -2$. This is because $(-1)^k \mathrm{I}_k(d,\lambda) > 0$ for $k \geq 0$ up to some threshold; for example $\mathrm{I}_2(d,\lambda) > 0$ for such λ. This is the reason why our approach does not allow us to compute $\mathbf{W}(d, \frac{d-1}{4})$ for $d \geq 6$ (note that $\beta = 0$ implies that $\lambda = 3-d$). A similar obstacle arises in [23] when generalising Foschi's argument to obtain the result in [35] in higher dimensions.

Assume Lemma 4.2.2 to be true for the moment, then to prove Lemma 4.2.1 for $\lambda > -2$, we first observe that it suffices by density and continuity of the functional H_λ on $L^1(\mathbb{S}^{d-1})$ to consider $g \in L^2(\mathbb{S}^{d-1})$. We may then write $g = \sum_{k\geq 0} Y_k$ as a sum of orthogonal spherical harmonics; upon which it follows that

$$H_\lambda(g,g) = 2^{-\frac{\lambda}{2}} \sum_{k\geq 0} \int_{\mathbb{S}^{d-1}} g(\eta_1) \int_{\mathbb{S}^{d-1}} \overline{Y_k(\eta_2)}(1-\eta_1\cdot\eta_2)^{-\frac{\lambda}{2}}\,\mathrm{d}\eta_2 \mathrm{d}\eta_1. \tag{4.17}$$

To deal with the inner integral in (4.17) we use the Funk–Hecke theorem (see the preliminaries for a statement of this result) to obtain that the inner integral in (4.17) evaluates to a (positive) constant multiple of $\mathrm{I}_k(d,\lambda)\overline{Y_k(\eta_1)}$. Precisely, using the orthogonality of the spherical harmonics of different degrees and Lemma 4.2.2,

$$H_\lambda(g,g) = 2^{-\frac{\lambda}{2}} \sum_{k\geq 0} \mathrm{I}_k(d,\lambda) \int_{\mathbb{S}^{d-1}} |Y_k(\eta)|^2 \, \mathrm{d}\eta \leq 2^{-\frac{\lambda}{2}} \mathrm{I}_0(d,\lambda) \int_{\mathbb{S}^{d-1}} |Y_0|^2 \, \mathrm{d}\eta = H_\lambda(\mu_g \mathbf{1}).$$

Equality is clearly satisfied for $g = Y_0$ or equivalently g which are constant. There are no further cases of equality since $\mathrm{I}_k(d,\lambda)$ is strictly negative for $k \geq 1$, by Lemma 4.2.2. Using the expression for $\mathrm{I}_0(d,\lambda)$ in Lemma 4.2.2 and the definition of μ_g, it is then easy to derive the claimed expression for $H_\lambda(\mu_g \mathbf{1})$, which completes the proof of Lemma 4.2.1 in the case $\lambda > -2$.

Proof of Lemma 4.2.2. By a simple change of variables, it is easily checked that $\mathrm{I}_0(d,\lambda)$ satisfies the claimed equality in terms of the beta function. To prove the strict negativity of $\mathrm{I}_k(d,\lambda)$ for $k \geq 1$, we recall the Rodrigues formula for $P_{k,d}$:

$$(1-t^2)^{\frac{d-3}{2}} P_{k,d}(t) = (-1)^k R_{k,d} \frac{\mathrm{d}^k}{\mathrm{d}t^k} (1-t^2)^{k+\frac{d-3}{2}}, \quad t \in [-1,1],$$

with

$$R_{k,d} = \frac{\Gamma(\frac{d-1}{2})}{2^k \Gamma(k+\frac{d-1}{2})} > 0,$$

to obtain that

$$\mathrm{I}_k(d,\lambda) = (-1)^k R_{k,d} \int_{-1}^{1} (1-t)^{-\frac{\lambda}{2}} \frac{\mathrm{d}^k}{\mathrm{d}t^k} (1-t^2)^{k+\frac{d-3}{2}} \, \mathrm{d}t.$$

Integrating by parts, the boundary terms disappear and we obtain

$$\mathrm{I}_k(d,\lambda) = (-1)^k R_{k,d} \left(-\frac{\lambda}{2}\right) \int_{-1}^{1} (1-t)^{-\frac{\lambda}{2}-1} \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} (1-t^2)^{k+\frac{d-3}{2}} \, \mathrm{d}t. \tag{4.18}$$

Since $-\frac{\lambda}{2} > 0$, the sign of the constant in front of the integral in (4.18) does not change at the first integration by parts. However, since $-\frac{\lambda}{2} - 1 < 0$, at every integration by parts step after the first, we will incur a sign change. Hence, integrating by parts a total of k times, we see that $\mathrm{I}_k(d, \lambda)$ evaluates to

$$-C_k(d,\lambda) \int_{-1}^{1} (1-t)^{-\frac{\lambda}{2}-k}(1-t^2)^{k+\frac{d-3}{2}} \, \mathrm{d}t$$

for some strictly positive constant $C_k(d, \lambda)$. Hence $\mathrm{I}_k(d, \lambda) < 0$ as claimed. $\square$

For $\lambda = -2$, the above argument breaks down as Lemma 4.2.2 is false in this case, however it is straightforward to see that

$$\mathrm{I}_k(d,-2) = |\mathbb{S}^{d-2}| \int_{-1}^{1} (1-t) P_{k,d}(t)(1-t^2)^{\frac{d-3}{2}} \, \mathrm{d}t$$

satisfies $\mathrm{I}_0(d,-2) > 0$, $\mathrm{I}_1(d,-2) < 0$ and $\mathrm{I}_k(d,-2)$ vanishes for all $k \geq 2$. Thus

$$H_{-2}(g,g) = \frac{\mathrm{I}_0(d,-2)}{2} \|Y_0\|^2_{L^2(\mathbb{S}^{d-1})} + \frac{\mathrm{I}_1(d,-2)}{2} \|Y_1\|^2_{L^2(\mathbb{S}^{d-1})} \leq H_{-2}(\mathbf{1},\mathbf{1})|\mu_g|^2,$$

where $g = \sum_{k\geq 0} Y_k$ is the expansion of g into spherical harmonics. A straightforward computation of the constants involved then completes the proof of Lemma 4.2.1. $\square$

Remark. Similar types of arguments to the one presented above have proved profitable in understanding sharp forms of other important estimates; see, for example, [4], [14] and [38]. The connection to the latter paper deserves a further remark; indeed, in [38], Frank and Lieb provide a reproof of the sharp Hardy–Littlewood–Sobolev inequality on the sphere, originally due to Lieb [62], which gives the sharp upper bound on $H_\lambda(g, g)$ for $0 < \lambda < d - 1$ in terms of the L^p norm of g, where $p = \frac{2(d-1)}{2(d-1)-\lambda}$.

Since $T_\beta f \geq 0$ and we have that

$$\|T_\beta f\|_{L^1(\mathbb{S}^{d-1})} = \|f\|^2_{\dot{H}^{\beta+\frac{d-1}{4}}(\mathbb{R}^d)}$$

by Plancherel's theorem, Lemma 4.2.1 implies that the right hand side of (4.6) is at most

$$(2\pi)^{2d}\mathbf{KM}(2\beta,d)2^{\frac{3d-7}{2}+\beta}\mathrm{B}(d-2+2\beta,\tfrac{d-1}{2})\frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}\|f\|^4_{\dot{H}^{\beta+\frac{d-1}{4}}(\mathbb{R}^d)},$$

with equality if $T_\beta f$ is constant on $\mathbb{S}^{d-1}$, which happens when $|\widehat{f}|$ is radial. In particular, equality holds in both (4.6) and (4.16) for f given by

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+ib\cdot\xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$ such that $\mathrm{Re}(a) < 0$, and $b \in \mathbb{R}^d$; note that, for such f, we have that $|\widehat{f}|$ is radial (and hence g is constant).

On the other hand, if f is a maximiser for (4.9), then we must have equality in inequality (4.6), and in inequality (4.16) for $g = T_\beta f$. From the equality in (4.6), using Theorem 4.1.1, we see that necessarily

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+b\cdot\xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$, $b \in \mathbb{C}^d$ and $\mathrm{Re}(a) < -|\mathrm{Re}(b)|$. For such f, we know from the proof of Corollary 4.1.5 that

$$T_\beta f(\omega) = \frac{e^{2\,\mathrm{Re}\,c}}{(-2\,\mathrm{Re}\,a - 2\,\mathrm{Re}\,b\cdot\omega)^{\frac{3d-5}{2}+2\beta}}\int_0^\infty e^{-s}s^{\frac{3d-7}{2}+2\beta}\,\mathrm{d}s. \tag{4.19}$$

If equality holds in (4.16) when $\lambda > -2$ then we must have that g is constant, and for the function $T_\beta f$ to be constant in ω it is clear from (4.19) that we must have $\mathrm{Re}(b) = 0$. This establishes the characterisation of the maximisers for inequality (4.9) in the case

$\beta < \frac{5-d}{4}$. Of course, this argument does not immediately give the same result for $\beta = \frac{5-d}{4}$ (equivalently, $\lambda = -2$): equality holds in (4.16) if g is constant, but unlike the case $\lambda > -2$, there are further cases of equality. To prove the characterisation in this case we can argue as follows: using (4.19) again, we have

$$T_\beta f(\omega) = \frac{e^{2\,\mathrm{Re}\,c}}{(-2\,\mathrm{Re}\,a - 2\,\mathrm{Re}\,b\cdot\omega)^d}\int_0^\infty e^{-r} r^{d-1}\,\mathrm{d}r,$$

which is of course bounded and hence in $L^p(\mathbb{S}^{d-1})$ for $p = 2$ in particular, provided that $-|\,\mathrm{Re}\,b| > \mathrm{Re}\,a$. Accordingly, we can consider its expansion into spherical harmonics $T_\beta f = \sum Y_k$, where $Y_k = \Pi_k(T_\beta f)$, and we recall that Π_k is the projection onto the space of spherical harmonics of degree k. Since $\mathrm{I}_1(d, -2) < 0$, for equality to hold in (4.16) we must have that $\|Y_1\|_{L^2} = 0$. On the other hand, applying the projection Π_1 to $T_\beta f$ we have that

$$Y_1(\eta) = C\int_{\mathbb{S}^{d-1}} \frac{P_{1,d}(\eta\cdot\omega)}{(-\mathrm{Re}(a) - \mathrm{Re}(b)\cdot\eta)^d}\,\mathrm{d}\omega \tag{4.20}$$

for some absolute constant $C > 0$. If we suppose, for a contradiction, that $\mathrm{Re}(b) \neq 0$, then an application of the Funk–Hecke formula implies that

$$Y_1(\eta) = CP_{1,d}(\eta\cdot\mathrm{Re}(b)')\int_{-1}^{1} \frac{t(1-t^2)^{\frac{d-3}{2}}}{(1+At)^d}\,\mathrm{d}t \tag{4.21}$$

for each $\eta \in \mathbb{S}^{d-1}$, where $A := \frac{|\mathrm{Re}(b)|}{\mathrm{Re}(a)} \in (-1, 0]$. The absolute constants $C > 0$ in (4.20) and (4.21) may not be the same. Since Y_1 vanishes almost everywhere on $\mathbb{S}^{d-1}$, it follows that the integral on the right-hand side of (4.21) vanishes. This forces $A = 0$, which gives the desired contradiction, and this completes the proof that all maximisers for inequality (4.9) are as described. To complete the proof of Corollary 4.1.2, we note that the claimed expression for the constant in (4.9) follows by tedious, although routine, calculations using standard formulae for the total measure of the sphere and identities for the beta function such as

$$|\mathbb{S}^{d-1}| = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})},$$

also

$$\frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^{d-2}|} = \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})},$$

and

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for the beta function. $\square$

Remark. At the endpoint $\beta = \frac{5-d}{4}$, we have that $\frac{d-3}{2} + 2\beta = 1$ and so inequality (4.9) may also be seen to follow more directly by arguing as in [13] and [22], using the observation that

$$\int_{\mathbb{R}^{2d}} f(x)f(y)x \cdot y \,\mathrm{d}x\mathrm{d}y \geq 0,$$

with equality if f is radial.

Proof of Proposition 4.1.4. Fix $f = f_{|b|}$ to be a function of the form (4.7) with $c = 0$, $a = -1$ and $b = |b|\mathbf{e_1}$ with $|b| < 1$, that is

$$\widehat{f}(\xi) = \frac{e^{-|\xi| + |b|\mathbf{e_1}\cdot\xi}}{|\xi|}.$$

For the rest of this section we shall let C denote an arbitrary positive constant which may depend on d and β but not on $|b|$, and we use $x \lesssim y$ (respectively, $x \gtrsim y$) to mean $x \leq Cy$ ($x \geq Cy$), where C may be different even in a single chain of inequalities. From the proof of inequality (4.11) we have that

$$T_\beta f(\omega) = \frac{C}{(1 - |b|\mathbf{e_1}\cdot\omega)^{\frac{3d-5}{2}+2\beta}} = \frac{C}{(1 - |b|\mathbf{e_1}\cdot\omega)^{\frac{d-1}{p}}},$$

where p is as in (4.11); note in particular that $\beta < \frac{3-d}{4}$ implies that $p > 1$. Since f is extremal for (4.11) it is enough to show that

$$\frac{\|T_\beta f\|_{L^p(\mathbb{S}^{d-1})}}{\|f\|^2_{\dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)}} = C\frac{\|T_\beta f\|_{L^p(\mathbb{S}^{d-1})}}{\|T_\beta f\|_{L^1(\mathbb{S}^{d-1})}} \to \infty$$

as $|b| \to 1$. Using the Funk–Hecke theorem, we have that

$$\|T_\beta f\|_{L^p(\mathbb{S}^{d-1})} = C\left(\int_{-1}^{1} \frac{(1-t^2)^{\frac{d-3}{2}}}{(1-|b|t)^{d-1}}\,\mathrm{d}t\right)^{\frac{1}{p}} =: I_1(|b|),$$

and

$$\|T_\beta f\|_{L^1(\mathbb{S}^{d-1})} = C\int_{-1}^{1} \frac{(1-t^2)^{\frac{d-3}{2}}}{(1-|b|t)^{\frac{d-1}{p}}}\,\mathrm{d}t =: I_2(|b|).$$

To complete the proof of Corollary 4.1.4 it therefore suffices to prove that for $|b|$ sufficiently close to 1,

$$I_1(|b|) \gtrsim (1-|b|)^{\frac{1-d}{2p}}, \tag{4.22}$$

and

$$I_2(|b|) \lesssim (1-|b|)^{\frac{1-d}{p}+\frac{d-1}{2}}. \tag{4.23}$$

We deal with (4.22) first: for any $|b| < 1$ we have that the integrand in I_1 is positive and so

$$I_1(|b|) \gtrsim (1-|b|^2)^{\frac{d-3}{2p}}\left(\int_{-|b|}^{|b|} \frac{1}{(1-|b|t)^{d-1}}\,\mathrm{d}t\right)^{\frac{1}{p}},$$

also

$$\int_{-|b|}^{|b|} \frac{1}{(1-|b|t)^{d-1}}\,\mathrm{d}t = \frac{C}{|b|}\left(\frac{(1+|b|^2)^{d-2}-(1-|b|^2)^{d-2}}{(1-|b|^4)^{d-2}}\right),$$

so

$$I_1(|b|) \gtrsim \left((1-|b|^2)^{\frac{d-3}{2}-d+2}\left(\frac{(1+|b|^2)^{d-2}-(1-|b|^2)^{d-2}}{(d-2)|b|(1+|b|^2)^{d-2}}\right)\right)^{\frac{1}{p}},$$

which implies (4.22) for $|b|$ sufficiently close to 1. To prove (4.23), by a simple change of variables and the fact that if $t > 0$ then $1 + |b|t > 1 - |b|t$, we have

$$I_2(|b|) \lesssim \int_0^1 \frac{(1-t^2)^{\frac{d-3}{2}}}{(1-|b|t)^{\frac{d-1}{p}}}\,\mathrm{d}t$$

and then by making the further change of variables $S = \frac{1-|b|t}{1-|b|}$, it follows that

$$\int_0^1 \frac{(1-t^2)^{\frac{d-3}{2}}}{(1-|b|t)^{\frac{d-1}{p}}}\,\mathrm{d}t \lesssim \frac{1}{|b|}(1-|b|)^{\frac{d-1}{2}+\frac{1-d}{p}} \int_1^\infty S^{\frac{1-d}{p}}(S-1)^{\frac{d-3}{2}}\,\mathrm{d}S;$$

the integral here is finite since for $\beta > \frac{1-d}{4}$ we have in particular that $p \leq 2$, from which (4.23) (and hence Corollary 4.1.4) follows. $\square$

Proof of Corollary 4.1.3. We proceed following closely the orthogonality argument in [34] (see also [13]). Write the solution of the wave equation u as $e^{it\sqrt{-\Delta}}f_+ + e^{-it\sqrt{-\Delta}}f_-$, where the functions f_+ and f_- are defined using the initial data by

$$u(0) = f_+ + f_-, \quad \partial_t u(0) = i\sqrt{-\Delta}(f_+ - f_-).$$

Using orthogonality and the Cauchy–Schwarz inequality on $L^2(\mathbb{R}^5)$, we get

$$\begin{aligned}\|u\|^4_{L^4(\mathbb{R}^5)} &= \|e^{it\sqrt{-\Delta}}f_+\|^4_{L^4(\mathbb{R}^5)} + \|e^{-it\sqrt{-\Delta}}f_-\|^4_{L^4(\mathbb{R}^5)} + 4\|e^{it\sqrt{-\Delta}}f_+e^{-it\sqrt{-\Delta}}f_-\|^2_{L^2(\mathbb{R}^5)} \\ &\leq \|e^{it\sqrt{-\Delta}}f_+\|^4_{L^4(\mathbb{R}^5)} + \|e^{-it\sqrt{-\Delta}}f_-\|^4_{L^4(\mathbb{R}^5)} + 4\|e^{it\sqrt{-\Delta}}f_+\|^2_{L^4(\mathbb{R}^5)}\|e^{-it\sqrt{-\Delta}}f_-\|^2_{L^4(\mathbb{R}^5)}.\end{aligned}$$

The basic inequality $2(X^2+Y^2+4XY) \leq 3(X+Y)^2$ and the case $(\beta, d) = (0,4)$ of Corollary 4.1.2, which clearly also holds for $e^{-it\sqrt{-\Delta}}$, now yield

$$\|u\|^4_{L^4(\mathbb{R}^5)} \leq \frac{3}{8}\mathbf{W}(4, \tfrac{3}{4})^4 \left(\|u(0)\|^2_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^4)} + \|\partial_t u(0)\|^2_{\dot{H}^{-\frac{1}{4}}(\mathbb{R}^4)}\right)^2$$

which gives the claimed inequality in Corollary 4.1.3.

The above argument was used by Foschi in [34] when $(d,s) = (3, \frac{1}{2})$ and in [13] when $(d,s) = (5,1)$. The characterisation of maximisers also follows in the analogous way, and so we refer the reader to [13] or [34] and omit the details. $\square$

We conclude this section by stating an analogue of Corollaries 4.1.2 and 4.1.5 for Proposition 4.1.6. Its proof is identical to that of these results, so we skip it.

Corollary 4.2.3. *Suppose that $d \geq 2$ and that $\beta \in \big(\frac{2-d}{2}, \frac{5-d}{4}\big]$. If $\beta \leq \frac{3-d}{4}$ then*

$$\Big\| M_\beta(e^{it\sqrt{-\Delta}}fe^{it\sqrt{-\Delta}}g)\Big\|_{L^2}^2 \leq \mathbf{C}'(\beta,d)|\mathbb{S}^{d-1}|^{\frac{3-d-4\beta}{d-1}}\|T_\beta f\|_{L^p(\mathbb{S}^{d-1})}\|T_\beta g\|_{L^p(\mathbb{S}^{d-1})}.$$

for $p = \frac{2(d-1)}{3d-5+4\beta}$, and if $\beta > \frac{3-d}{4}$ then

$$\Big\| M_\beta(e^{it\sqrt{-\Delta}}f)^2\Big\|_{L^2(\mathbb{R}^{d+1})}^2 \leq \mathbf{C}'(\beta,d)\|f\|^4_{\dot{H}^{\beta+\frac{d-1}{4}}(\mathbb{R}^d)}$$

where the constant $\mathbf{C}'(\beta,d)$ is easily computable. Further, equality holds in (4.2.3) *if and only if $f = \lambda g$ satisfy* (4.7) *with $\lambda, a, c \in \mathbb{C}$ and $b \in \mathbb{C}^d$ where $|\operatorname{Re} b| < -\operatorname{Re} a$. Equality holds in* (4.2.3) *if and only if f satisfies* (4.7) *under the same conditions, but with the additional restriction* $\operatorname{Re} b = 0$.

4.3 Proof of Theorem 4.1.1 and Proposition 4.1.6

4.3.1 Proof of the sharp inequalities (4.6) and (4.13)

We prove (4.6) first, broadly following the strategy used in the proof of the corresponding results in Chapter 3 (which in turn was based on the approach in [8]) to reduce to evaluating a certain integral over a submanifold of $\mathbb{R}^{2d}$. By Plancherel's theorem and using the relabelling $(x_1, x_2, \eta_1, \eta_2) \mapsto (x_1, \eta_2, \eta_1, x_2)$,

$$
\begin{aligned}
&2^{-2\beta}(2\pi)^{3d-1}\|M_\beta(e^{it\sqrt{-\Delta}}f\overline{e^{it\sqrt{-\Delta}}g})\|^2_{L^2(\mathbb{R}^{d+1})} = 2^{-2\beta}(2\pi)^{2d-2}\|M_\beta(\widetilde{e^{it\sqrt{-\Delta}}f\overline{e^{it\sqrt{-\Delta}}g}})\|^2_{L^2(\mathbb{R}^{d+1})} \\
&= \int_{\mathbb{R}^{4d}} (|\eta_1||\eta_2| + \eta_1\cdot\eta_2)^{2\beta}\widehat{f}(x_1)\overline{\widehat{g}(-x_2)}\,\overline{\widehat{f}(\eta_1)}\widehat{g}(-\eta_2)\delta\begin{pmatrix} -|x_1|+|x_2|+|\eta_1|-|\eta_2| \\ x_1+x_2-\eta_1-\eta_2 \end{pmatrix}\mathrm{d}x\mathrm{d}\eta \\
&= \int_{\mathbb{R}^{4d}} (|\eta_1||x_2| + \eta_1\cdot x_2)^{2\beta}\widehat{f}(x_1)\overline{\widehat{g}(-\eta_2)}\,\overline{\widehat{f}(\eta_1)}\widehat{g}(-x_2)\delta\begin{pmatrix} -|x_1|+|\eta_2|+|\eta_1|-|x_2| \\ x_1+\eta_2-\eta_1-x_2 \end{pmatrix}\mathrm{d}x\mathrm{d}\eta \\
&= \int_{\mathbb{R}^{4d}} (|\eta_1||x_2| + \eta_1\cdot x_2)^{2\beta}\frac{\widehat{F}(\eta)\overline{\widehat{F}(x)}}{(|\eta_1||\eta_2||x_1||x_2|)^{\frac{1}{2}}}\delta\begin{pmatrix} |\eta_2|+|\eta_1|-|x_2|-|x_1| \\ x_1+\eta_2-\eta_1-x_2 \end{pmatrix}\mathrm{d}x\mathrm{d}\eta
\end{aligned}
$$

for $\widehat{F}(y) := |y_1|^{\frac{1}{2}}|y_2|^{\frac{1}{2}}\widehat{f}(y_1)\widehat{g}(-y_2)$, for $y = (y_1, y_2) \in \mathbb{R}^{2d}$, and where we have used that if $\tau = |\eta_1| - |\eta_2|$ and $\xi = \eta_1 + \eta_2$ then

$$
\left|\tau^2 - |\xi|^2\right| = 2\left|-|\eta_1||\eta_2| - \eta_1\cdot\eta_2\right| = 2(|\eta_1||\eta_2| + \eta_1\cdot\eta_2).
$$

We now define a non-negative function on $\mathbb{R}^{4d}$,

$$
\Psi_{x,\eta} = \Psi(x,\eta) = \left(\frac{|x_1||x_2|}{|\eta_1||\eta_2|}\right)^{\frac{1}{2}}.
$$

Taking real parts and then applying the arithmetic-geometric mean inequality to $\widehat{F}(x)\Psi(x,\eta)^{\frac{1}{2}}$ and $\widehat{F}(\eta)\Psi(x,\eta)^{-\frac{1}{2}}$, it follows that

$$
\begin{aligned}
&2^{-2\beta}(2\pi)^{3d-1}\|M_\beta(e^{it\sqrt{-\Delta}}f\overline{e^{it\sqrt{-\Delta}}g})\|^2_{L^2} \\
&\leq \frac{1}{2}\int_{\mathbb{R}^{4d}} (|\eta_1||x_2| + \eta_1\cdot x_2)^{2\beta}\frac{|\widehat{F}(\eta)|^2\Psi^{-1}_{x,\eta} + |\widehat{F}(x)|^2\Psi_{x,\eta}}{(|\eta_1||\eta_2||x_1||x_2|)^{\frac{1}{2}}}\delta\begin{pmatrix} |\eta_2|+|\eta_1|-|x_2|-|x_1| \\ x_1+\eta_2-\eta_1-x_2 \end{pmatrix}\mathrm{d}x\mathrm{d}\eta \\
&= \int_{\mathbb{R}^{4d}} (|\eta_1||x_2| + \eta_1\cdot x_2)^{2\beta}\frac{|\widehat{F}(\eta)|^2}{|x_1||x_2|}\delta\begin{pmatrix} |\eta_2|+|\eta_1|-|x_2|-|x_1| \\ x_1+\eta_2-\eta_1-x_2 \end{pmatrix}\mathrm{d}x\mathrm{d}\eta,
\end{aligned}
$$

and equality holds if and only if

$$\widehat{F}(x)\Psi(x,\eta)^{\frac{1}{2}} = \widehat{F}(\eta)\Psi(x,\eta)^{-\frac{1}{2}},$$

or equivalently

$$|x_1||x_2|\widehat{f}(x_1)\widehat{g}(-x_2) = |\eta_1||\eta_2|\widehat{f}(\eta_1)\widehat{g}(-\eta_2) \tag{4.24}$$

almost everywhere on the support of the delta measures; for example this equation is satisfied by f, g given by (4.7). We now define

$$\mathrm{I}_\beta(\eta) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(|\eta_1||x_2| - \eta_1 \cdot x_2)^{2\beta}}{|x_1||x_2|}\delta\begin{pmatrix}|x_1| + |x_2| - |\eta_1| - |\eta_2| \\ x_1 + x_2 - \eta_1 - \eta_2\end{pmatrix}\mathrm{d}x_1\mathrm{d}x_2,$$

for $\eta = (\eta_1, \eta_2) \in \mathbb{R}^d \times \mathbb{R}^d$; by simple changes of variables, to complete the proof of the sharp inequality it suffices to prove the following.

Lemma 4.3.1. *For $\eta \in \mathbb{R}^{2d}$ and $\beta > \frac{1-d}{4}$ we have that*

$$\mathrm{I}_\beta(\eta) = C_{\beta,d}(|\eta_1||\eta_2| - \eta_1 \cdot \eta_2)^{\frac{d-3}{2}+2\beta}$$

for constant

$$C_{\beta,d} = |\mathbb{S}^{d-2}|2^{\frac{d-3}{2}}\mathrm{B}\left(2\beta + \frac{d-1}{2}, \frac{d-1}{2}\right).$$

Remark. In the case $\beta = 0$, Lemma 4.3.1 was proved for $d = 3$ by Foschi in [34] and for general dimensions by Bez–Rogers in [13]. In each case the proof proceeds by applying an appropriate Lorentz transformation to reduce to the case $\eta_1 = -\eta_2$; we shall see that in fact this argument allows us to obtain the result for more general values of β. We also remark that in the case $\beta = 0$, Lemma 4.3.1 may be seen to hold via a direct calculation using the homogeneity of the delta measure, and is contained in Lemma 4.1 of the earlier paper [37].

Proof of Lemma 4.3.1. In order to shorten the formulas, in what follows we define $\tau = |\eta_1| + |\eta_2|$ and $\xi = \eta_1 + \eta_2$. It is then easy to see that $\mathrm{I}_\beta(\eta)$ equals

$$\int_{\mathbb{R}^{2d+2}} \frac{\delta(\sigma_1 - |x_1|)}{|x_1|} \frac{\delta(\sigma_2 - |x_2|)}{|x_2|} \delta\begin{pmatrix} \sigma_1 + \sigma_2 - \tau \\ x_1 + x_2 - \xi \end{pmatrix} (|\eta_1||x_2| - \eta_1 \cdot x_2)^{2\beta} \, \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \mathrm{d}x_1 \mathrm{d}x_2$$

$$= \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} \frac{\delta(\sigma_1 - |x_1|)}{|x_1|} \frac{\delta(\sigma_2 - |x_2|)}{|x_2|} \delta\left(\begin{pmatrix} \sigma_1 \\ \\ x_1 \end{pmatrix} + \begin{pmatrix} \sigma_2 \\ x_2 \end{pmatrix} - \begin{pmatrix} \tau \\ \xi \end{pmatrix} \right)$$
$$\times (|\eta_1|\sigma_2 - \eta_1 \cdot x_2)^{2\beta} \, \mathrm{d}\begin{pmatrix} \sigma_1 \\ x_1 \end{pmatrix} \mathrm{d}\begin{pmatrix} \sigma_2 \\ x_2 \end{pmatrix}.$$

We now introduce the Lorentz transformation L, given by

$$L\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma(t - v \cdot x) \\ x + \left(\frac{\gamma - 1}{|v|^2} v \cdot x - \gamma t\right) v \end{pmatrix}, \quad x \in \mathbb{R}^d, t \in \mathbb{R}$$

with $v = -\frac{\xi}{\tau}$ and

$$\gamma := \frac{1}{(1 - |v|^2)^{\frac{1}{2}}} = \frac{\tau}{(\tau^2 - |\xi|^2)^{\frac{1}{2}}}. \tag{4.25}$$

It is not hard to check that

$$L\begin{pmatrix} (\tau^2 - |\xi|^2)^{\frac{1}{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \tau \\ \xi \end{pmatrix},$$

that $|\det L| = 1$ and that the measure $|x|^{-1}\delta(t - |x|)$ is invariant under the transformation

L. Applying the change of variables $\binom{\widetilde{\sigma_j}}{\widetilde{x_j}} = L^{-1}\binom{\sigma_j}{x_j}$ for $j = 1, 2$ it follows that

$$\begin{aligned}
\mathrm{I}_\beta(\eta) = \int_{\mathbb{R}^{2(d+1)}} & \frac{\delta(\widetilde{\sigma_1} - |\widetilde{x_1}|)}{|\widetilde{x_1}|} \frac{\delta(\widetilde{\sigma_2} - |\widetilde{x_2}|)}{|\widetilde{x_2}|} \delta\left(L\begin{pmatrix} \widetilde{\sigma_1} \\ \\ \widetilde{x_1} \end{pmatrix} + L\begin{pmatrix} \widetilde{\sigma_2} \\ \widetilde{x_2} \end{pmatrix} - L\begin{pmatrix} (\tau^2 - |\xi|^2)^{\frac{1}{2}} \\ 0 \end{pmatrix} \right) \\
& \times \left(\begin{pmatrix} |\eta_1| \\ -\eta_1 \end{pmatrix} \cdot L\begin{pmatrix} \widetilde{\sigma_2} \\ \widetilde{x_2} \end{pmatrix} \right)^{2\beta} \mathrm{d}\begin{pmatrix} \widetilde{\sigma_1} \\ \widetilde{x_1} \end{pmatrix} \mathrm{d}\begin{pmatrix} \widetilde{\sigma_2} \\ \widetilde{x_2} \end{pmatrix} \\
= \int_{\mathbb{R}^{2d+2}} & \frac{\delta(\widetilde{\sigma_1} - |\widetilde{x_1}|)}{|\widetilde{x_1}|} \frac{\delta(\widetilde{\sigma_2} - |\widetilde{x_2}|)}{|\widetilde{x_2}|} \delta\begin{pmatrix} \widetilde{\sigma_1} + \widetilde{\sigma_2} - (\tau^2 - |\xi|^2)^{\frac{1}{2}} \\ \widetilde{x_1} + \widetilde{x_2} \end{pmatrix} \\
& \times \left(\begin{pmatrix} |\eta_1| \\ -\eta_1 \end{pmatrix} \cdot L\begin{pmatrix} \widetilde{\sigma_2} \\ \widetilde{x_2} \end{pmatrix} \right)^{2\beta} \mathrm{d}\widetilde{\sigma_1}\mathrm{d}\widetilde{\sigma_2}\mathrm{d}\widetilde{x_1}\mathrm{d}\widetilde{x_2} \\
= \int_{\mathbb{R}^d} & \frac{1}{|x|^2} \delta\left(2|x| - (\tau^2 - |\xi|^2)^{\frac{1}{2}} \right) \left(\begin{pmatrix} |\eta_1| \\ -\eta_1 \end{pmatrix} \cdot L\begin{pmatrix} |x| \\ x \end{pmatrix} \right)^{2\beta} \mathrm{d}x,
\end{aligned}$$

where the last line follows by evaluating the integrals in $\widetilde{\sigma_1}, \widetilde{\sigma_2}$ and $\widetilde{x_1}$ and then relabeling $\widetilde{x_2} = x$. We are now required to compute the quantity

$$\begin{pmatrix} |\eta_1| \\ -\eta_1 \end{pmatrix} \cdot L\begin{pmatrix} |x| \\ x \end{pmatrix} = |\eta_1|\gamma(|x| - v \cdot x) - \eta_1 \cdot \left(x + \left(\frac{\gamma - 1}{|v|^2} v \cdot x - \gamma|x| \right) v \right)$$

for $v = -\frac{\xi}{\tau}$, γ given by (4.25) and x on the support of the remaining delta measure; this is contained in the following.

Lemma 4.3.2. *For each $\eta_1, \eta_2 \in \mathbb{R}^d$ there exists $\omega_* \in \mathbb{S}^{d-1}$ such that*

$$\begin{pmatrix} |\eta_1| \\ -\eta_1 \end{pmatrix} \cdot L\begin{pmatrix} |x| \\ x \end{pmatrix} = \frac{|\eta_1||\eta_2| - \eta_1 \cdot \eta_2}{2} \left(1 + \frac{x}{|x|} \cdot \omega_* \right)$$

for any $x \in \mathbb{R}^d$ with $2|x| = (\tau^2 - |\xi|^2)^{\frac{1}{2}}$.

Assuming Lemma 4.3.2 to be true for the moment, we have by polar co-ordinates, the definitions of τ and ξ and then a rotation,

$$\begin{aligned}\mathrm{I}_\beta(\eta) &= \frac{(|\eta_1||\eta_2| - \eta_1\cdot\eta_2)^{2\beta}}{2^{2\beta+1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty r^{d-3}\delta\left(r - \frac{(|\eta_1||\eta_2| - \eta_1\cdot\eta_2)^{\frac{1}{2}}}{\sqrt{2}}\right)(1+\omega\cdot\omega_*)^{2\beta}\,\mathrm{d}r\mathrm{d}\omega \\ &= \frac{(|\eta_1||\eta_2| - \eta_1\cdot\eta_2)^{2\beta+\frac{d-3}{2}}}{2^{2\beta+\frac{d-1}{2}}}\int_{\mathbb{S}^{d-1}}(1+\omega\cdot\mathbf{e}_1)^{2\beta}\,\mathrm{d}\omega;\end{aligned}$$

as usual $\mathbf{e}_1$ denotes the first basis vector in $\mathbb{R}^d$. But then using for example the Funk–Hecke formula (see e.g. [2]) we have that if $\beta > \frac{1-d}{4}$ then

$$\begin{aligned}\mathrm{I}_\beta(\eta) &= |\mathbb{S}^{d-2}|\frac{1}{2^{2\beta+\frac{d-1}{2}}}\int_{-1}^1(1+t)^{2\beta+\frac{d-3}{2}}(1-t)^{\frac{d-3}{2}}\,\mathrm{d}t\,(|\eta_1||\eta_2| - \eta_1\cdot\eta_2)^{2\beta+\frac{d-3}{2}} \\ &= |\mathbb{S}^{d-2}|2^{\frac{d-3}{2}}\mathrm{B}\left(2\beta+\frac{d-1}{2},\frac{d-1}{2}\right)(|\eta_1||\eta_2| - \eta_1\cdot\eta_2)^{\frac{d-3}{2}+2\beta},\end{aligned}$$

as claimed. It now remains to prove Lemma 4.3.2.

Proof of Lemma 4.3.2. After some straightforward calculations and simplifications we deduce that

$$\begin{aligned}L\begin{pmatrix}|x|\\ x\end{pmatrix} &= \frac{1}{(\tau^2-|\xi|^2)^{\frac{1}{2}}}\begin{pmatrix}\tau|x| + \xi\cdot x \\ x(\tau^2-|\xi|^2)^{\frac{1}{2}} + \xi\big(|x| + \xi\cdot x(\tau + (\tau^2-|\xi|^2)^{\frac{1}{2}})^{-1}\big)\end{pmatrix} \\ &= \frac{|x|}{(\tau^2-|\xi|^2)^{\frac{1}{2}}}\begin{pmatrix}\tau + \xi\cdot x' \\ x'(\tau^2-|\xi|^2)^{\frac{1}{2}} + \xi\big(1 + \xi\cdot x'(\tau + (\tau^2-|\xi|^2)^{\frac{1}{2}})^{-1}\big)\end{pmatrix},\end{aligned}$$

where $x' := \frac{x}{|x|}$. By our assumption $|x| = \frac{1}{2}(\tau^2-|\xi|^2)^{\frac{1}{2}}$,

$$\begin{pmatrix}|\eta_1|\\ -\eta_1\end{pmatrix}\cdot L\begin{pmatrix}|x|\\ x\end{pmatrix} = \frac{1}{2}\left(|\eta_1|(\tau+\xi\cdot x') - 2\eta_1\cdot x - \eta_1\cdot\xi(1+\frac{\xi\cdot x'}{\tau+2|x|})\right).$$

Moreover, using the definitions $\tau = |\eta_1| + |\eta_2|$ and $\xi = \eta_1 + \eta_2$,

$$|\eta_1|\tau - \eta_1\cdot\xi = |\eta_1||\eta_2| - \eta_1\cdot\eta_2,$$

and

$$\begin{aligned}|\eta_1|\xi\cdot x' - 2\eta_1\cdot x - \xi\cdot x'\frac{\xi\cdot\eta_1}{\tau+2|x|} &= x'\cdot\left(|\eta_1|\xi - 2|x|\eta_1 - \xi\frac{\xi\cdot\eta_1}{\tau+2|x|}\right)\\ &= \frac{1}{\tau+2|x|}x'\cdot\Big(\xi\big(|\eta_1|(\tau+2|x|)-\xi\cdot\eta_1\big) - 2|x|(\tau+2|x|)\eta_1\Big)\\ &= x'\cdot\left(\eta_2\left(\frac{|\eta_1||\eta_2|-\eta_1\cdot\eta_2+2|x||\eta_1|}{|\eta_1|+|\eta_2|+2|x|}\right) - \eta_1\left(\frac{|\eta_1||\eta_2|-\eta_1\cdot\eta_2+2|x||\eta_2|}{|\eta_1|+|\eta_2|+2|x|}\right)\right),\end{aligned}$$

so that

$$\begin{pmatrix}|\eta_1|\\-\eta_1\end{pmatrix}\cdot L\begin{pmatrix}|x|\\x\end{pmatrix} = \frac{1}{2}\left(|\eta_1||\eta_2|-\eta_1\cdot\eta_2+x'\cdot z\right),$$

where

$$z = z(\eta_1,\eta_2) := \eta_2\left(\frac{|\eta_1||\eta_2|-\eta_1\cdot\eta_2+2|x||\eta_1|}{|\eta_1|+|\eta_2|+2|x|}\right) - \eta_1\left(\frac{|\eta_1||\eta_2|-\eta_1\cdot\eta_2+2|x||\eta_2|}{|\eta_1|+|\eta_2|+2|x|}\right).$$

But then, when $2|x| = \sqrt{\tau^2-|\xi|^2} = \sqrt{2(|\eta_1||\eta_2|-\eta_1\cdot\eta_2)}$, the vector z in fact has norm equal to $|\eta_1||\eta_2|-\eta_1\cdot\eta_2$. To see this, write

$$\begin{aligned}&\eta_2(|\eta_1||\eta_2|-\eta_1\cdot\eta_2+2|x||\eta_1|) - \eta_1(|\eta_1||\eta_2|-\eta_1\cdot\eta_2+2|x||\eta_2|)\\ &\quad= 2|x|\left(\eta_2(|x|+|\eta_1|)-\eta_1(|x|+|\eta_2|)\right)\end{aligned}$$

so that

$$\begin{aligned}&\left|\eta_2(|\eta_1||\eta_2|-\eta_1\cdot\eta_2+2|x||\eta_1|) - \eta_1(|\eta_1||\eta_2|-\eta_1\cdot\eta_2+2|x||\eta_2|)\right|^2\\ &= 4|x|^2\left(|\eta_2|^2(|x|^2+|\eta_1|^2)+|\eta_1|^2(|x|^2+|\eta_2|^2)-2\eta_1\cdot\eta_2(|x|+|\eta_1|)(|x|+|\eta_2|)\right)\\ &= 4|x|^2\left(4|\eta_1||\eta_2||x|^2+4|x|^3(|\eta_1|+|\eta_2|)+|x|^2(|\eta_1|^2+|\eta_2|^2-2\eta_1\cdot\eta_2)\right)\\ &= 4|x|^4\left(2|x|+|\eta_1|+|\eta_2|\right)^2 = \left(|\eta_1||\eta_2|-\eta_1\cdot\eta_2\right)^2\left(2|x|+|\eta_1|+|\eta_2|\right)^2,\end{aligned}$$

and so the proof is completed by taking $\omega_* = \frac{z}{|z|}$.

Next, we indicate the necessary modifications to the above argument in order to prove (4.13). In this case we have

$$\begin{aligned}
&2^{-2\beta}(2\pi)^{3d-1}\|M_\beta(e^{it\sqrt{-\Delta}}fe^{it\sqrt{-\Delta}}g)\|^2_{L^2(\mathbb{R}^{d+1})}\\
&=\int_{\mathbb{R}^{4d}}(|\eta_1||\eta_2|-\eta_1\cdot\eta_2)^{2\beta}\widehat{f}(x_1)\widehat{g}(x_2)\overline{\widehat{f}(\eta_1)\widehat{g}(\eta_2)}\delta\begin{pmatrix}|x_1|+|x_2|-|\eta_1|-|\eta_2|\\ x_1+x_2-\eta_1-\eta_2\end{pmatrix}\mathrm{d}x\mathrm{d}\eta\\
&=\int_{\mathbb{R}^{4d}}(|\eta_1||\eta_2|-\eta_1\cdot\eta_2)^{2\beta}\frac{\widehat{F}(x)\overline{\widehat{F}(\eta)}}{(|x_1||x_2||\eta_1||\eta_2|)^{\frac{1}{2}}}\delta\begin{pmatrix}|x_1|+|x_2|-|\eta_1|-|\eta_2|\\ x_1+x_2-\eta_1-\eta_2\end{pmatrix}\mathrm{d}x\mathrm{d}\eta,
\end{aligned}$$

but in this case we do not need to relabel any indices and we take $\widehat{F}:=(|\cdot|^{\frac{1}{2}}\widehat{f})\otimes(|\cdot|^{\frac{1}{2}}\widehat{g})$. Proceeding exactly as in the proof of (4.6) we then obtain

$$\frac{\|M_\beta(e^{it\sqrt{-\Delta}}fe^{it\sqrt{-\Delta}}g)\|^2_{L^2}}{2^{2\beta}(2\pi)^{1-3d}}\le\int_{\mathbb{R}^{4d}}(|\eta_1||\eta_2|-\eta_1\cdot\eta_2)^{2\beta}\frac{|\widehat{F}(\eta)|^2}{|x_1||x_2|}\delta\begin{pmatrix}|x_1|+|x_2|-|\eta_1|-|\eta_2|\\ x_1+x_2-\eta_1-\eta_2\end{pmatrix}\mathrm{d}x\mathrm{d}\eta,$$

where we have used the fact that $|x_1||x_2|-x_1\cdot x_2=|\eta_1||\eta_2|-\eta_1\cdot\eta_2$ on the support of the delta measures in this case. The term $|\eta_1||\eta_2|-\eta_1\cdot\eta_2$ does not depend on x and hence inequality (4.13) (for any admissible β) follows from the $\beta=0$ case of Lemma 4.3.1. As in (4.6), equality holds in (4.13) if and only if

$$|x_1||x_2|\widehat{f}(x_1)\widehat{g}(x_2)=|\eta_1||\eta_2|\widehat{f}(\eta_1)\widehat{g}(\eta_2)$$

almost everywhere on the support of the delta measures; an example is given by $\widehat{f}=\widehat{g}=\frac{e^{-|\cdot|}}{|\cdot|}$.

Remark. It is also possible to prove (4.13) using the Cauchy–Schwarz inequality applied with respect to an appropriately chosen measure, following the proof of (4.5) from [13]; we choose to prove (4.13) using the method presented above to highlight an alternative proof which unifies with the proof of (4.6), and also to point out why the latter inequality is more difficult.

It follows from the above argument that equality holds in (4.6) and (4.13) for functions (f, g) given by (4.7). In order to complete the proofs of Theorem 4.1.1 and Proposition 4.1.6 it now suffices to show that there are no other cases of equality under the stated conditions on β; to do this we follow the approach used in [13] to characterise the extremisers in the case $\beta = 0$.

4.3.2 Cases of equality

Since the only place an inequality was used in the proof of (4.6) was in the application of the arithmetic-geometric mean inequality, it follows that equality holds in (4.6) if and only if (replacing x_2 and η_2 with $-x_2$ and $-\eta_2$ in (4.24))

$$|x_1||x_2|\widehat{f}(x_1)\widehat{g}(x_2) = |\eta_1||\eta_2|\widehat{f}(\eta_1)\widehat{g}(\eta_2)$$

for almost every $(\eta_1, \eta_2, x_1, x_2) \in \mathbb{R}^{4d}$ satisfying $\eta_1+\eta_2 = x_1+x_2$ and $|\eta_1|+|\eta_2| = |x_1|+|x_2|$. So, equality holds in both (4.6) and (4.13) if and only if

$$h_1(x_1)h_2(x_2) = \Lambda(|x_1| + |x_2|, x_1 + x_2) \tag{4.26}$$

for almost every $x_1, x_2 \in \mathbb{R}^d$ and some scalar function Λ, where $h_1 := |\cdot|\widehat{f}$ and $h_2 := |\cdot|\widehat{g}$, and by symmetry we can assume that $h_1 = h_2 = h$. The functional equation (4.26) was solved by Foschi in [34] in the case $d = 3$ under the assumption that h is locally integrable, and this argument was generalised to obtain the solution in all dimensions (under the same assumption) in [13]. Therefore, the proof of Theorem 4.1.1 will be completed if we can show that finiteness of the right hand side of (4.6) implies that h is locally integrable, for $\beta \in \left(\frac{2-d}{2}, \frac{d+1}{4}\right)$. Proceeding as in [13], if B is the Euclidean ball centered at the origin of radius N then by Cauchy–Schwarz,

$$\left(\int_B h\right)^4 \leq C_\beta \int_B \int_B (|x_1||x_2|)^{\frac{5-d}{2}-2\beta}(1-x_1'\cdot x_2')^{\frac{3-d}{2}-2\beta}\,\mathrm{d}x_1\mathrm{d}x_2$$
$$= C_\beta \left(\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}(1-\omega_1\cdot\omega_2)^{\frac{3-d}{2}-2\beta}\,\mathrm{d}\omega_1\mathrm{d}\omega_2\right)\left(\int_0^N r^{\frac{d+3}{2}-2\beta}\,\mathrm{d}r\right)^2, \qquad (4.27)$$

where C_β is a constant multiple of $\mathcal{I}_\beta(f,f)$, and where we recall that

$$\mathcal{I}_\beta(f,g) := \int_{\mathbb{R}^{2d}} |\widehat{f}(y_1)|^2|\widehat{g}(y_2)|^2|y_1||y_2|\left(|y_1||y_2|-y_1\cdot y_2\right)^{\frac{d-3}{2}+2\beta}\,\mathrm{d}y_1\mathrm{d}y_2.$$

The integral in r in (4.27) is finite when $\beta < \frac{d+5}{4}$, but the integral over the sphere equals

$$\int_{-1}^1 (1-t)^{\frac{d-3}{2}}(1+t)^{-2\beta}\,\mathrm{d}t,$$

which is only finite if $\beta < \frac{1}{2}$. As in [9], we extend this range slightly by combining the above argument with a reverse form of the Hardy–Littlewood–Sobolev inequality on the sphere (for a proof see [4]), which states that if $g \geq 0$ then

$$|H_\lambda(g)| \gtrsim \|g\|^2_{L^p(\mathbb{S}^{d-1})},$$

where now $\lambda < 0$ and $p := \frac{2(d-1)}{2(d-1)-\lambda} \in (0,1)$. Therefore, we can conclude that h is locally integrable whenever $\int_B h$ is bounded above by a constant multiple of $\|T_\beta f\|_{L^p(\mathbb{S}^{d-1})}$. Using the Cauchy–Schwarz inequality, Minkowski's inequality for integrals and then Hölder's inequality, we obtain

$$\begin{aligned}
\int_B h(x)\,\mathrm{d}x &= \int_B \left|\widehat{f}(x)\right| |x|\,\mathrm{d}x \\
&= \int_0^N \left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(r\omega)| r^d\,\mathrm{d}\omega\right)\,\mathrm{d}r \\
&\leq C_{N,d}\left(\int_0^N \left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(r\omega)| r^d\,\mathrm{d}\omega\right)^2\,\mathrm{d}r\right)^{\frac{1}{2}} \\
&\leq C_{N,d}\int_{\mathbb{S}^{d-1}} \left(\int_0^N |\widehat{f}(r\omega)|^2 r^{2d}\,\mathrm{d}r\right)^{\frac{1}{2}}\,\mathrm{d}\omega \\
&\leq C_{N,d,q}\left(\int_{\mathbb{S}^{d-1}} \left(\int_0^N |\widehat{f}(r\omega)|^2 r^{2d}\,\mathrm{d}r\right)^{\frac{q}{2}}\,\mathrm{d}\omega\right)^{\frac{1}{q}} \\
&\leq C'_{N,d,q}\left(\int_{\mathbb{S}^{d-1}} \left(\int_0^N |\widehat{f}(r\omega)|^2 r^{\frac{3d-3}{2}+2\beta}\,\mathrm{d}r\right)^{\frac{q}{2}}\,\mathrm{d}\omega\right)^{\frac{1}{q}}
\end{aligned}$$

for any $q \geq 1$, at least provided that $2d > \frac{3d-3}{2} + 2\beta \Leftrightarrow \beta < \frac{d+3}{4}$. Recalling that $\lambda = 3 - d - 4\beta$, the above calculation with $q := 2p = \frac{4(d-1)}{2(d-1)-\lambda}$ allows us to conclude as long as $q \in [1, 2)$, or equivalently

$$6d - 10 + 8\beta > 4(d-1) \geq 3d - 5 + 4\beta \Leftrightarrow \beta \in \left(\frac{3-d}{4}, \frac{d+1}{4}\right],$$

and so we deduce that h is locally integrable whenever $\beta \leq \frac{d+1}{4}$. $\square$

Chapter 5

Sobolev–Strichartz estimates - Euler–Lagrange equations

This chapter is devoted to the proofs of two results concerning local extremisers for Sobolev–Strichartz inequalities. The result for the Schrödinger equation is joint with Neal Bez and Nikolaos Pattakos, has been accepted for publication and is contained in [12].

5.1 Introduction

For $d \geq 1$, we recall the linear Sobolev–Strichartz inequality for the Schrödinger equation

$$\left\| e^{it\Delta} u_0 \right\|_{L^{\frac{2(d+2)}{d-2m}}(\mathbb{R}^{d+1})} \leq C \left\| u_0 \right\|_{\dot{H}^m(\mathbb{R}^d)}, \tag{5.1}$$

where $0 \leq m < \frac{d}{2}$. Our first result is the following.

Theorem 5.1.1. *Only if $m = 0$ are the initial data*

$$u_0(x) = e^{a|x|^2 + b\cdot x + c} \quad a, c \in \mathbb{C},\ b \in \mathbb{C}^d,\ \operatorname{Re} a < 0 \tag{5.2}$$

local extremisers for the inequality (5.1)*. In particular if* $0 < m < \frac{d}{2}$ *the optimal constant*

$$\mathbf{S}(d,m) = \sup_{u_0 \in \dot{H}^m} \frac{\|e^{it\Delta}u_0\|_{L^{\frac{2(d+2)}{d-2m}}}}{\|u_0\|_{\dot{H}^m}}$$

is not attained on the functions (5.2)*.*

In the case $m = 0$, Theorem 5.1.1 was proved by Hundertmark–Zharnitsky in [44], and if in addition $d \in \{1, 2\}$ the functions (5.2) are known to be global maximisers for (5.1); this was first proved by Foschi in [34]. It is also conjectured that these functions are maximisers when $m = 0$ for any d (see [34] and [44]), but Theorem 5.1.1 implies that this is not true as soon as we move away from this endpoint exponent. Although maximisers are known to exist for any admissible (d, m) (see [57] and [77]), it is not clear to us what form they should take for $m > 0$.

Our second result concerns the linear Sobolev–Strichartz inequality for the (half-)wave equation

$$\|e^{it\sqrt{-\Delta}}f\|_{L^{\frac{2(d+1)}{d-2s}}} \leq C_{s,d}\|f\|_{\dot{H}^s(\mathbb{R}^d)} \tag{5.3}$$

where $\frac{1}{2} \leq s < \frac{d}{2}$.

Theorem 5.1.2. *Suppose that*

$$|\xi|\,\widehat{f}(\xi) = e^{a|\xi| + ib\cdot\xi + c}, \qquad a, c \in \mathbb{C},\ b \in \mathbb{R}^d,\ \operatorname{Re} a < 0. \tag{5.4}$$

Then the functions f*:*

- *Are local extremisers for* (5.3) *for* $s = \frac{d-1}{4}$ *(so that* $p := \frac{2(d+1)}{d-2s} = 4$*) for all* $d \geq 3$*, and*
- *Are not local extremisers (and hence not global maximisers) for* (5.3) *for* $s = \frac{2d-1}{6}$ *(so that* $p = 6$*), unless* $d = 2$*.*

In particular, when $s = \frac{2d-1}{6}$*, the optimal constant*

$$\mathbf{W}(d,s) = \sup_{u_0 \in \dot{H}^s} \frac{\|e^{it\sqrt{-\Delta}}u_0\|_{L^{\frac{2(d+1)}{d-2s}}}}{\|u_0\|_{\dot{H}^s}}$$

is not attained on the functions (5.4)*, for any* $d > 2$*.*

Theorem 5.1.2 builds upon work of Foschi, who showed in [34] that the functions (5.4) are maximisers for (5.3) in the cases $(d,s) \in \{(2,\frac{1}{2}),(3,\frac{1}{2})\}$, and of Bez–Rogers, who in [13] showed the same thing in the case $(d,s) = (5,1)$. We also recall Corollary 4.1.2 from Chapter 4, which implies that the functions (5.4) are also maximisers for (5.3) in the case $(d,s) = (4,\frac{3}{4})$. The results from [13] and Corollary 4.1.2 are interesting in part because they highlight a fundamental difference between the Sobolev–Strichartz inequalities (1.7) and (1.11) for the Schrödinger and wave equations, respectively: for the latter the maximisers are known to be the same at *two different regularities*, and this does not occur for the former by Theorem 5.1.1. The first conclusion of Theorem 5.1.2 sheds some light on this phenomenon by providing further evidence that the functions (5.4) are in fact maximisers for the L^4 Sobolev–Strichartz inequalities for general dimensions; using an explicit calculation of the ratio of the left and right sides of inequality (5.3) for the functions (5.4), we make the following conjecture.

Conjecture 5.1. *For all* $d \geq 3$*, we have that*

$$\mathbf{W}\left(d, \frac{d-1}{4}\right) = \frac{(d-3)!}{2\pi^{\frac{d-1}{2}}\Gamma\left(\frac{3d-5}{2}\right)}$$

and this is attained if and only if f *satisfies* (5.4)*.*

This would imply a range of sharp Strichartz estimates for the full solution of the linear wave equation (1.8). As far as we know, this is the first conjecture made about the precise nature of the extremisers for the Sobolev–Strichartz inequalities for (1.8) away from the endpoint regularity $s = \frac{1}{2}$, in general dimensions.

The second conclusion of Theorem 5.1.2 fails for $d = 2$ because we then have $s = \frac{2d-1}{6} = \frac{1}{2}$, so we are at an endpoint, and in fact the functions (5.4) are known to be global maximisers for (5.3); this is the case $(d, s) = (2, \frac{1}{2})$ from [34] cited above. In this respect, the second conclusion of Theorem 5.1.2 implies that Foschi's result in this case is, at the level of L^6 Sobolev–Strichartz estimates, an isolated fact. We recall that the existence of maximisers is known for (5.3), for *all* admissible pairs (d, s) (see [20], [32], [74]), however analogously to Theorem 5.1.1 our methods do not appear to yield a natural replacement candidate for the maximisers in the cases where they do not coincide with (5.4).

Remark. We could state Theorem 5.1.2 more generally; for instance we believe that the negative result may be extended to include those values of s for which p is an even integer strictly greater than 4. We keep it in this form for brevity and because our goal is to highlight that, analogously to Theorem 5.1.1, the shape of the maximisers for (5.3) is more complicated in general than (5.4). We also comment that it should be possible to adapt the proof from [44] that the functions (5.2) are local extremisers for (5.1) in the endpoint case $m = 0$, to prove that the functions (5.4) are local extremisers for (5.3) when $s = \frac{1}{2}$, for all $d \geq 2$. Although a statement or proof of this result has not appeared in the literature we do not do this here as it is likely to be known, see Conjecture 1.11 of [34].

Using the fact that any power of the spatial Laplacian $\Delta = \Delta_x$ commutes with $e^{it\sqrt{-\Delta}}$ and $e^{it\Delta}$, it is convenient to restate (5.1) and (5.3) as follows:

$$\|(-\Delta_x)^{-\frac{m}{2}} e^{it\Delta} u_0\|_{L^{\frac{2(d+2)}{d-2m}}(\mathbb{R}^{d+1})} \leq C \|u_0\|_{L^2(\mathbb{R}^d)} \tag{5.5}$$

and

$$\|(-\Delta_x)^{-\frac{s}{2}} e^{it\sqrt{-\Delta}} f\|_{L^{\frac{2(d+1)}{d-2s}}(\mathbb{R}^{d+1})} \leq C \|f\|_{L^2(\mathbb{R}^d)} \,. \tag{5.6}$$

The optimal constants for these estimates will be (respectively) $\mathbf{S}(d,m)$ and $\mathbf{W}(d,s)$, $(-\Delta)^{\frac{m}{2}}u_0$ is a maximiser for (5.5) if and only if u_0 is a maximiser for (5.1), and $(-\Delta)^{\frac{s}{2}}f$ is a maximiser for (5.6) if and only if f is a maximiser for (5.3). The fundamental result upon which the proofs of Theorems 5.1.1 and 5.1.2 are based is then the following lemma, which is contained in a more general result due to Christ–Quilodrán in [26]. We remark that in [26] (and the literature on this topic more generally) the equations defining local extremisers for various inequalities are referred to as Euler–Lagrange equations. Although for brevity we do not do this here, it is this convention we follow in the title of this chapter.

Lemma 5.1.3. *Let* T *be a linear operator from* $L^2(\mathbb{R}^d)$ *to* $L^p(\mathbb{R}^{d+1})$ *for which there exists a finite constant* C *such that*

$$\|\mathrm{T}\, f\|_{L^p} \leq C\|f\|_{L^2} \tag{5.7}$$

for any $f \in L^2(\mathbb{R}^d)$. *Then* f *is a local extremiser of* (5.7) *if and only if*

$$\mathrm{T}^*(|\mathrm{T}\, f(x,t)|^{p-2}\,\mathrm{T}\, f(x,t)) = \lambda f \tag{5.8}$$

for some $\lambda \geq 0$*, where* T^* *denotes the adjoint of* T.

For the proofs in this chapter, we shall not need to keep track of the precise values of the constants that appear; as such C, C_d and $C_{p,d}$ shall denote positive constants depending on the dimension d and the Lebesgue exponent p only, and which may be different even in a single chain of equalities.

5.2 Proof of Theorem 5.1.1

In this section, we fix $0 < m < \frac{d}{2}$, set

$$p = \frac{2(d+2)}{d-2m},$$

and define the operator

$$\mathrm{T}\, f(x,t) = (-\Delta_x)^{-\frac{m}{2}} e^{it\Delta} f(x).$$

By a straightforward calculation one can show that the adjoint T^* satisfies

$$\mathrm{T}^*\, g(\xi) = C_d \left(|\,.\,|^{-m} \int_{\mathbb{R}^{d+1}_{x,t}} g(x,t) e^{-ix\cdot\,\cdot} e^{it|\,\cdot\,|^2} \,\mathrm{d}x\mathrm{d}t \right)^{\vee} (\xi),$$

for $\xi \in \mathbb{R}^d$ and a suitable $g : \mathbb{R}^{d+1} \to \mathbb{C}$, recalling that we use $^\vee$ to denote the inverse Fourier transform on $\mathbb{R}^d$. If we apply Lemma 5.1.3 to T and take the Fourier transform of both sides of the resulting equation (5.8), we see that f is a local extremiser for (5.5) if and only if there exists a positive constant λ such that

$$\int_{\mathbb{R}^{d+1}} |\,\mathrm{T}\, f(x,t)|^{p-2}\, \mathrm{T}\, f(x,t) e^{-ix\cdot\xi} e^{it|\xi|^2} \,\mathrm{d}x\mathrm{d}t = \lambda |\xi|^m \widehat{f}(\xi), \quad \xi \in \mathbb{R}^d.$$

To prove Theorem 5.1.1 it then suffices to check that this equation does not hold for any λ when $f = (-\Delta)^{\frac{m}{2}} e^{-|\,\cdot\,|^2}$, or equivalently, that

$$\int_{\mathbb{R}^{d+1}} |e^{it\Delta} f(x)|^{p-2} e^{it\Delta} f(x) e^{-ix\cdot\xi} e^{it|\xi|^2} \,\mathrm{d}x\mathrm{d}t = \lambda |\xi|^{2m} \widehat{f}(\xi) \tag{5.9}$$

is not true when $f = f_* := e^{-\frac{|\,\cdot\,|^2}{4}}$. We choose this particular Gaussian for convenience since with our normalisation for the Fourier transform, $\widehat{f}$ equals a constant multiple of $e^{-|\,\cdot\,|^2}$, and it suffices to consider only this Gaussian since all functions of the form (5.2) may be obtained from this one using a group of transformations which preserves the ratio of the left and right sides of (5.1), see for example [26] or [34].

Our first step in showing that (5.9) does not hold for $f = f_*$ is the following Lemma. Before stating it we should clarify that, for $t, \omega \in \mathbb{R}$, by $(1+it)^\omega$ we mean the real power of a complex number defined using the branch of the logarithm taken to be analytic in the cut plane $\mathbb{C} \setminus \{iy : y \in [1, \infty)\}$, and $(1 - it)^{\omega}$ defined similarly, but analytic in the cut plane $\mathbb{C} \setminus \{-iy : y \in [1, \infty)\}$.

Lemma 5.2.1. *We have*

$$e^{it\Delta} f_*(x,t) = C_d e^{-\frac{|x|^2}{4}(1+it)^{-1}} (1+it)^{-\frac{d}{2}} \tag{5.10}$$

for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}$.

Proof. It suffices to consider $d = 1$ as $e^{it\Delta} f$ may be written as a product of d identical integrals in 1 variable. In this case we have

$$e^{it\Delta} f_*(x) = C \int_{\mathbb{R}} e^{ix\xi - it\xi^2} e^{-\xi^2} \, \mathrm{d}\xi = C \int_{\mathbb{R}} e^{ix\xi} e^{-\xi^2(1+it)} \, \mathrm{d}\xi.$$

If we now change variables $z = (1+it)^{\frac{1}{2}} \xi$, we obtain

$$C \int_{\mathbb{R}(1+it)^{\frac{1}{2}}} e^{ixz(1+it)^{-\frac{1}{2}}} e^{-z^2} (1+it)^{-\frac{1}{2}} \, \mathrm{d}z = Ce^{-\frac{x^2}{4}(1+it)^{-1}} (1+it)^{-\frac{1}{2}}.$$

The last equality here follows from a simple contour integration argument using Cauchy's theorem and the rapid decay of the Gaussian function for large $|x|$, by noting that if $z \in \mathbb{C}$ the contour $\mathbb{R}z$ is the line through z and the origin in the complex plane (see for example the calculation in [82, p.334]). Returning to d dimensions, we obtain equality (5.10). $\square$

It is clear that for $f = f_*$ the right hand side of (5.9) equals $\lambda |\xi|^{2m} e^{-|\xi|^2}$, and by Lemma

5.2.1, we see that the left hand side is equal to

$$
\begin{aligned}
&C_{p,d}\int_{\mathbb{R}^{d+1}} \frac{e^{-(p-2)\frac{|x|^2}{4(1+t^2)}}}{(1+t^2)^{\frac{(p-2)d}{4}}} \frac{e^{-\frac{|x|^2}{4(1+it)}}}{(1+it)^{\frac{d}{2}}} e^{-ix\cdot\xi} e^{it|\xi|^2}\,\mathrm{d}x\,\mathrm{d}t \\
&= C_{p,d}\int_{\mathbb{R}_t}(1+t^2)^{-\frac{(p-2)d}{4}}(1+it)^{-\frac{d}{2}} e^{it|\xi|^2}\int_{\mathbb{R}^d_x} e^{-ix\cdot\xi} e^{-(p-2)\frac{|x|^2}{4(1+t^2)}} e^{-\frac{|x|^2}{4(1+it)}}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Note that the inner integral once again may be written as a product of d integrals of one variable. We can then proceed as in the proof of Lemma 5.2.1 to obtain

$$
\begin{aligned}
\int_{\mathbb{R}} e^{-ix\xi} e^{-(p-2)\frac{x^2}{4(1+t^2)}} e^{-\frac{x^2}{4}(1+it)^{-1}}\,\mathrm{d}x &= \int_{\mathbb{R}} e^{-ix\xi} e^{-(\frac{p-2}{1+t^2}+\frac{1}{1+it})\frac{x^2}{4}}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} e^{-ix\xi} e^{-\frac{x^2}{4}\frac{p-1-it}{1+t^2}}\,\mathrm{d}x \\
&= \int_{(\frac{p-1-it}{1+t^2})\mathbb{R}} e^{-iz\left(\frac{1+t^2}{p-1-it}\right)^{\frac{1}{2}}\xi} e^{-\frac{z^2}{4}} \left(\frac{1+t^2}{p-1-it}\right)^{\frac{1}{2}}\,\mathrm{d}z \\
&= C\left(\frac{1+t^2}{p-1-it}\right)^{\frac{1}{2}} e^{-\xi^2\left(\frac{1+t^2}{p-1-it}\right)}.
\end{aligned}
$$

Hence,

$$
\int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-(p-2)\frac{|x|^2}{4(1+t^2)}} e^{-\frac{|x|^2}{4}(1+it)^{-1}}\,\mathrm{d}x = C_d\left(\frac{1+t^2}{p-1-it}\right)^{\frac{d}{2}} e^{-|\xi|^2\left(\frac{1+t^2}{p-1-it}\right)}.
$$

Since

$$
it - \frac{1+t^2}{p-1-it} = \frac{1-(p-1)it}{p-1-it},
$$

we have that the left hand side of (5.9) equals

$$
\begin{aligned}
&C_{p,d}\int_{\mathbb{R}}(1+t^2)^{-\frac{d(p-2)}{4}}(1+it)^{-\frac{d}{2}} e^{it|\xi|^2}\left(\frac{1+t^2}{p-1-it}\right)^{\frac{d}{2}} e^{-|\xi|^2\left(\frac{1+t^2}{p-1-it}\right)}\,\mathrm{d}t \\
&= C_{p,d}\int_{\mathbb{R}}\left(1+t^2\right)^{-\frac{d(p-2)}{4}}\left(\frac{1-it}{p-1-it}\right)^{\frac{d}{2}} e^{-|\xi|^2\left(\frac{1-it(p-1)}{p-1-it}\right)}\,\mathrm{d}t,
\end{aligned}
$$

for $\xi \in \mathbb{R}^d$. Since for our choice of p we have

$$\frac{d}{4}(p-2) = \frac{d(1+m)}{d-2m},$$

we have that (5.9) is equivalent to

$$\lambda\,|\xi|^{2m}\,e^{-\frac{|\xi|^2}{2}} = C_{p,d}\int_{\mathbb{R}} \left(1+t^2\right)^{-d\left(\frac{1+m}{d-2m}\right)} \left(\frac{1-it}{p-1-it}\right)^{\frac{d}{2}} e^{-|\xi|^2\left(\frac{1-it(p-1)}{p-1-it}\right)}\,\mathrm{d}t. \tag{5.11}$$

The following lemma will complete the proof of Theorem 5.1.1.

Lemma 5.2.2. *Define*

$$I(a) = \int_{\mathbb{R}} \left(1+t^2\right)^{-d\left(\frac{1+m}{d-2m}\right)} \left(\frac{1-it}{p-1-it}\right)^{\frac{d}{2}} e^{-a\left(\frac{1-it(p-1)}{p-1-it}\right)}\,\mathrm{d}t$$

for $a \geq 0$. Then, $I(a)$ is not a constant multiple of the function $a^m e^{-a}$.

Proof. As in [26] we want to consider those values of a for which a power series expansion of $I(a)$ is valid. Firstly, we notice that $I(a)$ converges for every $a \geq 0$. Indeed, the integrand is dominated by

$$t^{-2d(\frac{1+m}{d-2m})}e^{-a\,\mathrm{Re}\left(\frac{1-it(p-1)}{p-1-it}\right)} = t^{-2d(\frac{1+m}{d-2m})}e^{-a(p-1)\left(\frac{1+t^2}{(p-1)^2+t^2}\right)}.$$

Since $m < \frac{d}{2}$, this is integrable (for all $a \geq 0$) when

$$\frac{2d(1+m)}{d-2m} > 1,$$

which is equivalent, again since $m < \frac{d}{2}$, to

$$-m < \frac{d}{2(d+1)}$$

and hence the integral converges for any $0 \leq m < \frac{d}{2}$, since d and admissible values for m are non-negative. Write

$$e^a \exp\left(-a\left(\frac{1-i(p-1)t}{(p-1)-it}\right)\right) = \sum_{k=0}^{\infty}(-1)^k\frac{a^k}{k!}(p-2)^k\left(\frac{1+it}{(p-1)-it}\right)^k;$$

we may then apply the dominated convergence theorem to interchange the sum and integral. This yields

$$J(a) := e^a I(a) = \sum_{k=0}^{\infty}\frac{a^k}{k!}(-1)^k(p-2)^k I_k,$$

where

$$I_k = \int_{\mathbb{R}}(1+it)^{k-\frac{d(1+m)}{d-2m}}H_k(t)\,\mathrm{d}t,$$

and

$$H_k(t) = (1-it)^{-d\left(\frac{1+m}{d-2m}\right)+\frac{d}{2}}(p-1-it)^{-k-\frac{d}{2}}. \tag{5.12}$$

We want to show that the expression defined by $J(a)$ is not equal to a^m for some $a \geq 0$. There are two cases to consider.

Case 1. *m is not an integer.* If we denote $\lfloor m \rfloor$ by k_1 and choose any integer $k \geq k_1$ we get (by differentiating the power series term by term, since it converges for any $a \geq 0$) that the function a^m is not k-times differentiable at the origin, whereas the function J is: indeed $J^{(k)}(0)$ will be $(-1)^k(p-2)^k I_k$, and I_k converges for all k since the integrand is $O(t^{-\frac{2d(1+m)}{d-2m}})$ as $|t| \to \infty$. Hence, the two sides of (5.11) cannot be equal, for such m.

Case 2. *m is an integer.* In this case the above argument breaks down since the function a^m *is* differentiable for any a, so we need the following result, which is Lemma 4.1 of [26].

Lemma 5.2.3. *Let $\gamma > -1$ and suppose that $H : \mathbb{C} \to \mathbb{C}$ is holomorphic in the upper half-plane, continuous in its closure, and that there exists $\delta > 0$ so that* $|(1+it)^{\gamma} H(t)| =$

$O(|t|^{-1-\delta})$ as $|t| \to \infty$. Then

$$\int_{\mathbb{R}} (1+it)^{\gamma}\, H(t)\, \mathrm{d}t = -2\sin(\gamma\pi) \int_0^{\infty} y^{\gamma} H(i+iy) dy,$$

In particular, if H is real-valued and non-negative when restricted to the imaginary axis, then $\int_{\mathbb{R}} (1+it)^{\gamma} H(t)\, \mathrm{d}t = 0$ if and only if γ is an integer. Further, we have

$$\int_{\mathbb{R}} (1+it)^{-1}\, H(t)\, \mathrm{d}t = 2\pi H(i).$$

For $m \in \mathbb{Z}$, notice that we may write equality (5.11) as the equality of two power series

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} (-1)^k (p-2)^k I_k = \sum_{k=0}^{\infty} b_k a^k, \tag{5.13}$$

for coefficients

$$b_k = \begin{cases} \lambda & \text{if } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose for now that the functions H_k satisfy the conditions of Lemma 5.1.3 above. There are two further cases to consider:

Case 2a. *The exponent $\frac{d(1+m)}{d-2m}$ is not an integer.* In this case if we choose any $k \geq k_2 = \max\left\{ \left\lceil \frac{d(1+m)}{d-2m} \right\rceil, m+1 \right\}$ then by Lemma 5.1.3 we get that $I_k \neq 0$, which contradicts equality (5.13) as $p > 2$.

Case 2b. *The exponent $\frac{d(1+m)}{d-2m}$ is an integer.* In this case we notice that we can calculate explicitly the right hand side of (5.11) using the residue theorem, for any $0 < m < \frac{d}{2}$. Suppose now that m is positive and $\frac{d(1+m)}{d-2m} = N \in \mathbb{Z}$, or

$$m = \frac{d(N-1)}{d+2N}.$$

Observe that N is strictly greater than one as $0 < m < \frac{d}{2}$, and that the integrand on the right hand side of (5.11) has a pole of order N at $t = i$. The only other singularities are at $t = -i(p-1)$ and $t = -i$, both in the lower half-plane since $p > 2$ and so we consider the integral

$$\mathcal{I}_R = \int_\Gamma \left(1+t^2\right)^{-d\left(\frac{1+m}{d-2m}\right)} \left(\frac{1-it}{p-1-it}\right)^{\frac{d}{2}} e^{-|\xi|^2\left(\frac{1-it(p-1)}{p-1-it}\right)}\,\mathrm{d}t.$$

Here $\Gamma = \Gamma^{(R)}$ is the upper semi-circle contour in the complex plane defined by $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$$\begin{aligned} \Gamma_1 &= [-R, R]\,, \text{ and} \\ \Gamma_2 &= \left\{ Re^{i\theta} : \theta \in [0,\pi] \right\}, \end{aligned}$$

with usual orientation. Note that as $R \to \infty$ the integral along Γ_2 will converge to zero. Indeed, it is enough to note that for R sufficiently large the integral is bounded by

$$CR^{-\frac{d(1+m)}{d-2m}+1} \to 0$$

as $R \to \infty$ (for some constant C), since $\frac{d(1+m)}{d-2m} > 1$. Thus, by the residue theorem we get that

$$\begin{aligned} \lim_{R\to\infty} \mathcal{I}_R &= \int_{\mathbb{R}} \left(1+t^2\right)^{-N} \left(\frac{1-it}{p-1-it}\right)^{\frac{d}{2}} e^{-|\xi|^2\left(\frac{1-it(p-1)}{p-1-it}\right)}\,\mathrm{d}t \\ &= 2\pi i \operatorname*{Res}_{t=i} \left(1+t^2\right)^{-N} \left(\frac{1-it}{p-1-it}\right)^{\frac{d}{2}} e^{-|\xi|^2\left(\frac{1-it(p-1)}{p-1-it}\right)} \\ &= 2\pi i \frac{d^{N-1}}{\mathrm{d}t^{N-1}} \left(\frac{1}{(1-it)^N} \left(\frac{1-it}{p-1-it}\right)^{\frac{d}{2}} e^{-|\xi|^2\left(\frac{1-it(p-1)}{p-1-it}\right)} \right)\Bigg|_{t=i}. \end{aligned}$$

Since at $t = i$ the exponent of the Gaussian will be

$$-\left(\frac{1+(p-1)}{(p-1)+1}\right)|\xi|^2 = -|\xi|^2,$$

by the product rule we will obtain a finite sum of the form

$$\sum_{j=0}^{N-1} c_{j,N}\,|\xi|^{2j}\,e^{-|\xi|^2}$$

for some coefficients $c_{j,N} \in \mathbb{C}$. To show that the two sides of (5.11) cannot be equal we calculate the coefficient $c_{N-1,N}$ defined above. Aside from a real constant, this will equal

$$i^{1-N}\left(\frac{1-it}{p-1-it}\right)^{\frac{d}{2}}\left(\frac{2i(p-1)^2-2i}{((p-1)-it)^2}\right)^{N-1}\Bigg|_{t=i} = i^{1-N}\left(\frac{2}{p}\right)^{\frac{d}{2}}\left(\frac{2i(p-1)^2-2i}{p^2}\right)^{N-1} \neq 0,$$

since $p > 2$. Hence, in order for the powers of $|\xi|^2$ to match up with the right hand side of (5.11), we necessarily require $N-1 = m$, or

$$\frac{m(d+2)}{d-2m} = m,$$

which implies that $m = 0$. Hence, the two sides of (5.11) cannot be equal for $m > 0$. We remark at this point that this is where we use that $m > 0$ (and hence $N > 1$), as in the case $m = 0$ and $N = 1$ this condition is of course not violated and Gaussian functions do satisfy the corresponding Euler–Lagrange equation, as proved in [44].

It remains to check that the H_k defined by (5.12) satisfy the conditions of Lemma 5.1.3. First note (fix a $k \in \mathbb{N}$) that $p = p(m) > 2$ for all m and hence $(p-1) > 0$, so H_k has no singularities in the lower half-plane, as for t in this region both bracketed terms will be non-zero. Note also that by the agreed definition for the real powers of $(1-it)$ and $(p-1-it) = (p-1)(1-i\frac{t}{p-1})$, the function H_k is holomorphic and continuous in the required regions since it is a product of compositions of such functions. Moreover, if we restrict it to the positive imaginary axis, we see that H_k is real and positive, again since $p > 1$. Lastly we note that as $|t| \to \infty$

$$(1+it)^{k-d\left(\frac{1+m}{d-2m}\right)} H_k(t) = O(t^{-2d\left(\frac{1+m}{d-2m}\right)}),$$

but we already know that

$$\frac{2d(1+m)}{d-2m} > 1$$

in particular, and so we see that H_k satisfies the conditions of Lemma 5.1.3 for all $k \geq 0$.

$\square$

5.3 Proof of Theorem 5.1.2

Throughout this section, we shall fix

$$p = \frac{2(d+1)}{d-2s}, \tag{5.14}$$

and define the linear operator T by $\mathrm{T} f = (-\Delta_x)^{-\frac{s}{2}} e^{it\sqrt{-\Delta}} f$. By a simple calculation, for our operator T we have that

$$\mathrm{T}^* g(\xi) = \frac{1}{(2\pi)^d} \left(|\,.\,|^{-s} \int_{\mathbb{R}^{d+1}_{x,t}} g(x,t) e^{-ix\cdot} e^{-it|\,.\,|} \,\mathrm{d}x\mathrm{d}t \right)^{\vee} (\xi),$$

for $\xi \in \mathbb{R}^d$, suitable $g : \mathbb{R}^{d+1} \to \mathbb{C}$. Applying Lemma 5.1.3 and taking the Fourier transform of both sides of (5.8), we see that f is a local extremiser of (5.3) if and only if

$$\int_{\mathbb{R}^{d+1}} e^{-ix\cdot\xi} e^{-it|\xi|} |\,\mathrm{T} f(x,t)|^{p-2}\, \mathrm{T} f(x,t) \,\mathrm{d}x\mathrm{d}t = \lambda |\xi|^s \widehat{f}(\xi), \quad \xi \in \mathbb{R}^d, \tag{5.15}$$

where $\lambda > 0$ is a constant which may be different to the one in (5.8). To prove Theorem 5.1.2 it now suffices to test the equation (5.8) on the function f_* defined by

$$\widehat{f_*}(\xi) = |\xi|^{s-1} e^{-|\xi|},$$

in the cases where $p = 4$ and $p = 6$. First, we note that if $f = f_*$, the right hand side of (5.15) evaluates to a positive constant multiple of $|\xi|^{2s-1} e^{-|\xi|}$, and also

$$\mathrm{T}\, f_*(x,t) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|} \frac{e^{-|\xi|}}{|\xi|}\, \mathrm{d}\xi.$$

5.3.1 The case $p = 4$

In this case,

$$\begin{aligned} |\,\mathrm{T} f_*(x,t)|^{p-2}\, \mathrm{T}\, f_*(x,t) &= \mathrm{T}\, f_*(x,t)^2 \overline{\mathrm{T}\, f_*(x,t)} \\ &= \int_{\mathbb{R}^{3d}} e^{ix\cdot(y_1+y_2-y_3)} e^{it(|y_1|+|y_2|-|y_3|)} \frac{e^{-(|y_1|+|y_2|+|y_3|)}}{|y_1||y_2||y_3|}\, \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3. \end{aligned}$$

For $\eta \in \mathbb{R}^d$, multiply this expression by $e^{-it|\eta|} e^{-ix\cdot\eta}$ and integrate in x and t using Fubini's theorem. It then follows that the left hand side of (5.15) equals

$$\mathrm{I}(\eta) := \int_{\mathbb{R}^{3d}} \frac{e^{-(|y_1|+|y_2|+|y_3|)}}{|y_1||y_2||y_3|} \delta\begin{pmatrix} |y_1| + |y_2| - |y_3| - |\eta| \\ y_1 + y_2 - y_3 - \eta \end{pmatrix} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3,$$

where we adopt the usual notation $\delta\binom{t}{x}$ for the product $\delta(t)\delta(x)$ on $\mathbb{R}^{d+1}$. Using that $|y_1| + |y_2| = |y_3| + |\eta|$ on the support of the delta measures, we have that

$$\mathrm{I}(\eta) = e^{-|\eta|} \int_{\mathbb{R}^{3d}} \frac{e^{-2|y_3|}}{|y_1||y_2||y_3|} \delta\begin{pmatrix} |y_1| + |y_2| - |y_3| - |\eta| \\ y_1 + y_2 - y_3 - \eta \end{pmatrix} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3.$$

We now observe that the integrals in y_1 and y_2 here may be dealt with using the following result from [13].

Lemma 5.3.1 ([13], Lemma 3.1)**.** *For each $d, k \geq 2$ and each $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$ satisfying $|\xi| < \tau$, we have*

$$\int_{\mathbb{R}^{kd}} \prod_{j=1}^{k} \left(|y_j|^{-1}\right) \delta\Big(\tau - \sum_{j=1}^{k} |y_j|\Big) \delta\Big(\xi - \sum_{j=1}^{k} y_j\Big) \, \mathrm{d}y = C_{k,d} \left(\tau^2 - |\xi|^2\right)^{\alpha(k,d)},$$

where $C_{k,d}$ is a positive constant which is easily computable, and

$$\alpha(k, d) := \frac{(d-1)(k-1)}{2} - 1.$$

Set $(\tau, \xi) = (|y_3| + |\eta|, y_3 + \eta)$ and $k = 2$, and apply Lemma 5.3.1 to obtain that

$$\begin{aligned} \mathrm{I}(\eta) &= C_d e^{-|\eta|} \int_{\mathbb{R}^d} \frac{e^{-2|y|}}{|y|} ((|y| + |\eta|)^2 - |y + \eta|^2)^{\frac{d-3}{2}} \, \mathrm{d}y \\ &= C_d e^{-|\eta|} \int_{\mathbb{R}^d} \frac{e^{-2|y|}}{|y|} (|y||\eta| - y \cdot \eta)^{\frac{d-3}{2}} \, \mathrm{d}y \\ &= C_d e^{-|\eta|} \int_{\mathbb{R}^d} \frac{e^{-2|y|}}{|y|} (|y||\eta| - y \cdot \eta)^{\frac{d-3}{2}} \, \mathrm{d}y \\ &= C_d e^{-|\eta|} |\eta|^{\frac{d-3}{2}} \int_{\mathbb{R}^d} e^{-2|y|} |y|^{\frac{d-5}{2}} (1 - y' \cdot \eta')^{\frac{d-3}{2}} \, \mathrm{d}y, \end{aligned}$$

where $x' := \frac{x}{|x|}$ for $x \in \mathbb{R}^d \setminus \{0\}$. Note that the constant C_d changes from line to line, but is always positive and depends only on d. The remaining integral does not depend on η', is positive and converges for all $d \geq 3$ (as can be seen using polar co-ordinates), and so we see that

$$\mathrm{I}(\eta) = C_d |\eta|^{\frac{d-3}{2}} e^{-|\eta|}.$$

It then remains to note that by (5.14) and our choice $p = 4$,

$$s = \frac{d-1}{4} \Leftrightarrow 2s - 1 = \frac{d-3}{2},$$

and so equation (5.15) is indeed satisfied for $f = f_*$ for all dimensions when $p = 4$.

5.3.2 The case $p = 6$

In this case, the methods described above still apply since we can multiply out the quantity $|\operatorname{T} f(x,t)|^{p-2} = |\operatorname{T} f(x,t)|^4$. For $f = f_*$ and T as in the previous section we have

$$\begin{aligned}|\operatorname{T} f_*(x,t)|^{p-2} \operatorname{T} f_*(x,t) &= (\operatorname{T} f_*(x,t))^3 (\overline{\operatorname{T} f_*(x,t)})^2 \\ &= \int_{\mathbb{R}^{5d}} e^{ix\cdot(\sum_{j=1}^3 y_j - \sum_{l=4}^5 y_l)} e^{it(\sum_{j=1}^3 |y_j| - \sum_{l=4}^5 |y_l|)} \prod_{j=1}^5 \frac{e^{-|y_j|}}{|y_j|} \,\mathrm{d}y.\end{aligned}$$

Proceeding as in the previous section, multiply this expression by $e^{-it|\eta|}e^{-ix\cdot\eta}$ for $\eta \in \mathbb{R}^d$ and integrate in x and t to obtain that the left hand side of (5.15) equals a constant multiple of

$$\operatorname{I}_3(\eta) := \int_{\mathbb{R}^{5d}} \delta\begin{pmatrix} \sum_{j=1}^3 |y_j| - \sum_{l=4}^5 |y_l| - |\eta| \\ \sum_{j=1}^3 y_j - \sum_{l=4}^5 y_l - \eta \end{pmatrix} \prod_{j=1}^5 \frac{e^{-|y_j|}}{|y_j|} \,\mathrm{d}y.$$

Using that $\sum_{j=1}^3 |y_j| = \sum_{l=4}^5 |y_l| + |\eta|$ on the support of the delta measures it follows that

$$\operatorname{I}_3(\eta) = e^{-|\eta|} \int_{\mathbb{R}^{5d}} \delta\begin{pmatrix} \sum_{j=1}^3 |y_j| - \sum_{l=4}^5 |y_l| - |\eta| \\ \sum_{j=1}^3 y_j - \sum_{l=4}^5 y_l - \eta \end{pmatrix} \left(\prod_{j=1}^5 \frac{1}{|y_j|}\right) e^{-2\sum_{j=4}^5 |y_j|} \,\mathrm{d}y.$$

Set $\tau = \sum_{l=4}^5 |y_l| + |\eta|$ and $\xi = \sum_{l=4}^5 y_l + \eta$, apply Lemma 5.3.1 to do the integrals in y_1, y_2, y_3 and obtain that up to a positive constant $\operatorname{I}_3(\eta)$ equals

$$e^{-|\eta|} \int_{\mathbb{R}^{2d}} \left(\frac{1}{|y_1||y_2|}\right) e^{-2(|y_1|+|y_2|)} \left((|y_1| + |y_2| + |\eta|)^2 - |y_1 + y_2 + \eta|^2\right)^{\alpha(3,d)} \,\mathrm{d}y.$$

For general p the right hand side of (5.15) is essentially unchanged from the case $p = 4$ and so in order for this equality to hold for $f = f_*$ we would need

$$\mathrm{I}_3(\eta) = \lambda|\eta|^{2s-1}e^{-|\eta|}, \quad \eta \in \mathbb{R}^d. \tag{5.16}$$

It is now easy to see that (5.16) holds when $d = 2$. Indeed, we have that $\alpha(3,2) = 0$, and also in this case $s = \frac{1}{2}$, so (5.16) holds as expected as both sides evaluate to a constant multiple of $e^{-|\eta|}$. However, for general dimensions we have $\alpha(3,d) = d-2$, $s = \frac{2d-1}{6}$, and

$$(|y_1| + |y_2| + |\eta|)^2 - |y_1 + y_2 + \eta|^2 = 2\left((|y_1||\eta| - y_1 \cdot \eta) + (|y_2||\eta| - y_2 \cdot \eta) + (|y_1||y_2| - y_1 \cdot y_2)\right).$$

If $d > 2$ then (5.16) does not hold: for example if we take $d = 3$ then $\alpha(3,3) = 1$ and so I_3 may be written as a sum of three integrals. In the first two, a factor of $|\eta|$ comes out (also by symmetry they are the same), but in the third we get

$$\int_{\mathbb{R}^{2\times 3}} e^{-2|y_1|-2|y_2|}(|y_1||y_2|)^{-1}(|y_1||y_2| - y_1 \cdot y_2)\,\mathrm{d}y_1\mathrm{d}y_2 = \int_{\mathbb{R}^{2\times 3}} e^{-2|y_1|-2|y_2|}\,\mathrm{d}y_1\mathrm{d}y_2,$$

which is nonzero. This implies that there exist positive constants A and B such that $\mathrm{I}_3(\eta) = Ae^{-|\eta|} + B|\eta|e^{-|\eta|}$ and so $e^{|\eta|}\mathrm{I}_3(\eta)$ is not homogeneous in η, which contradicts equality (5.16). For general dimensions we proceed similarly: by multiplying out

$$\left((|y_1||\eta| - y_1 \cdot \eta) + (|y_2||\eta| - y_2 \cdot \eta) + (|y_1||y_2| - y_1 \cdot y_2)\right)^{d-2}$$

and using homogeneity, we can write $\mathrm{I}_3(\eta)$ as a constant multiple of $P(|\eta|)e^{-|\eta|}$, where $P(x) = \sum \alpha_k x^k$ is a polynomial of degree $d-2$ with

$$\alpha_k = \sum_{\substack{0\leq\alpha\leq k\\ \alpha\in\mathbb{N}}} C_{\alpha,k}\int_{\mathbb{R}^{2d}} e^{-2|y_1|-2|y_2|}(|y_1| - \eta'\cdot y_1)^{\alpha}(|y_2| - \eta'\cdot y_2)^{k-\alpha}(|y_1||y_2| - y_1\cdot y_2)^{d-2-k}\,\mathrm{d}y_1\mathrm{d}y_2,$$

where $C_{\alpha,k}$ are strictly positive absolute constants which can be read off from the multinomial theorem. It is easy to see that $\alpha_k \geq 0$ for all k, and by direct computation it can be shown in addition that α_0 and α_1 are both strictly positive and so homogeneity considerations once again imply that the equality (5.16) cannot be satisfied. $\square$

Chapter 6

Conclusions and further work

In this thesis we have established some new weighted bilinear Strichartz-type estimates for the wave, Schrödinger and Klein–Gordon equation, generalising and unifying a number of prior results, as described. Moreover, we have considered the linear Sobolev–Strichartz estimates for the Schrödinger, wave and Klein–Gordon equations. For the Schrödinger equation, we used a variational argument to show that Gaussian functions are not maximisers for these estimates when the Sobolev exponent is strictly positive, in contrast to the case where it is zero. For the wave equation, we have established a new sharp L^4 Sobolev–Strichartz estimate in four space dimensions and characterised the maximisers, and made a conjecture about the nature of the maximisers for the L^4 estimates for general dimensions. For the Klein–Gordon equation, we have computed the sharp constant and shown that maximisers do not exist for a certain linear L^4 Strichartz estimate, and that the H^1 norm of any maximising sequence for this estimate must concentrate at spatial infinity in the precise sense we have described. We conclude by describing some possible directions for further work in this area.

Characterisation of maximisers. A natural question that arises from the sharp inequality of Theorem 2.1.1 for the Klein–Gordon propagator is one of characterising *all* maximisers for this inequality. For the wave equation, it is proved in [13] that any maximisers for this estimate must take the form

$$\widehat{f}_j(\xi) = |\xi|^{-1} e^{a|\xi|+b\cdot\xi+c_j}, \quad j = 1, 2$$

for $a, b, c_1, c_2 \in \mathbb{C}$ satisfying $\operatorname{Re} a < 0$, $|\operatorname{Re} b| < -|\operatorname{Re} a|$; this is proved by generalising the argument of Foschi from [34]. We believe that using a similar argument, it should be possible to prove that all maximisers for (2.3) should be of the form

$$\widehat{f}_j(\xi) = \phi(|\xi|)^{-1} e^{a\phi(|\xi|)+b\cdot\xi+c_j}.$$

In the general case, we notice that since the only place an inequality is used in the proof of Theorem 2.1.1 is in the Cauchy–Schwarz inequality; using the equality condition, one can show that maximising functions for inequality (2.3) must solve the functional equation

$$g(\eta_1)g(\eta_2) = G(\phi(|\eta_1|) + \phi(|\eta_2|), \eta_1 + \eta_2) \tag{6.1}$$

almost everywhere, where $g_j = \phi(|\cdot|)\widehat{f}_j$ for $j = 1, 2$ and G is a scalar function on $\mathbb{R}^{d+1}$, and we recall that $\phi(r) = \sqrt{1 + r^2}$ for $r \in \mathbb{R}$. To deduce the desired characterisation, it would be sufficient to show that g_1 and g_2 must be of a certain exponential type. The key ideas involved are as follows:

1. Deduce that solutions to (6.1) for which the right hand side of (2.3) is finite are locally integrable,
2. Locally integrable solutions to (6.1) are in fact continuous,
3. Continuous solutions to (6.1) are either trivial or nowhere vanishing, and
4. Continuous solutions which never vanish are of the desired (exponential) form.

We believe (1.) to be true, we can prove that (2.) implies (3.), and (4.) would follow from (3.) by applying a recent result of Charalambides (see [24]). The main difficulty is in deducing that (1.) implies (2.). This appears to be rather difficult due to the more complicated geometry which arises as a result of replacing the function $x \mapsto |x|$ with $x \mapsto \phi(|x|)$, when working in general dimensions; nonetheless, we believe that a geometric argument adapted from one considered by Foschi in [34] may yield a corresponding result. An advantage of a resolution to this problem is that it is likely to have implications for more general equations, where our function ϕ is replaced by a more general (convex) function.

More general propagators $e^{ith(\sqrt{-\Delta})}$. Another question raised by the proof of the bilinear inequalities in the preceding chapters is that of generalisations to other pseudo-differential operators. For example, consider the fourth-order homogeneous Schrödinger equation:

$$\begin{cases} -i\partial_t u + \mu\Delta_x u + \Delta_x^2 u & = 0 \\ u(x,0) & = f(x), \end{cases}$$

where μ is a real parameter. In this case, a Strichartz-type estimate in one spatial dimension is certainly known:

$$\left\| D_\mu^{\frac{1}{3}} e^{it(\Delta^2 - \mu\Delta)} f \right\|_{L^6(\mathbb{R}^{1+1})} \leq C \left\| f \right\|_{L^2(\mathbb{R})},$$

and where $D_\mu = (\mu + 6\Delta)^{\frac{1}{2}}$ (see [49]). For this estimate, Jiang, Pausader and Shao [46] established a dichotomy result concerning the existence of maximisers, using the method of profile decomposition considered in earlier work on the Airy equation (see [78]). In the context of Fourier restriction inequalities, Quilodrán proved the related (cf. Section 1.2) sharp inequality ([72], unpublished)

$$\|\widehat{f\sigma_a}\|_{L^4(\mathbb{R}^3)} \leq 2^{\frac{3}{4}}\pi \, \|f\|_{L^2(S_a)} \, ,$$

and that there are no maximising functions for this estimate. Here,

$$\sigma_a(\xi,\tau) := \delta\left(\tau - \frac{1}{2}\,|\xi|^2 - a\,|\xi|^4\right), \quad (\xi,\tau) \in \mathbb{R}^3$$

is a measure supported on the set $S_a = \left\{\left(y, \frac{1}{2}\,|y|^2 + a\,|y|^4\right) : y \in \mathbb{R}^2\right\} \subseteq \mathbb{R}^3$. As in our proof of Theorem 2.1.1, the proof reduces to the calculation of the convolution of the measure σ_a with itself. Although this convolution could not be calculated explicitly, it turns out that a sharp inequality could be obtained by using certain upper bounds. Hence, it seems interesting to ask if other results could be obtained in this manner, by considering other measures, or if this result could be generalised to higher dimensions or multiple convolutions, or even to the case $d = 1$, which would connect the two results above.

'Multilinear' version of Theorem 2.1.1. The discussion in the previous remark also may be considered more directly in view of our Theorem 2.1.1. In these circumstances it is reasonable to ask if, for $k \in \mathbb{N}$ with $k \geq 3$, the integral

$$\int_{\mathbb{R}^{kd}} \frac{1}{\prod_{j=1}^{k} \phi(|x_k|)} \delta\begin{pmatrix} \tau - \sum_{j=1}^{k} \phi(|x_j|) \\ \xi - \sum_{j=1}^{k} x_j \end{pmatrix} \mathrm{d}x$$

could be calculated, or bounded in such a way that we could deduce a sharp multilinear version of inequality (2.3). If this could be done, we could obtain further estimates for the propagator $e^{it\phi(\sqrt{-\Delta})}$, but where the L^4-norm on the right hand side is replaced by an L^p-norm for certain even integers p. This is motivated further by the fact that such estimates hold in the case of the wave propagator (see [13]), and also for the Schrödinger propagator (see [22]).

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