SET THEORETIC AND TOPOLOGICAL CHARACTERISATIONS OF ORDERED SETS

by

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Abstract

Van Dalen and Wattel have shown that a space is LOTS (linearly orderable topological space) if and only if it has a T_1-separating subbase consisting of two interlocking nests. Given a collection of subsets $\mathcal{L}$ of a set X, van Dalen and Wattel define an order $\lhd_{\mathcal{L}}$ by declaring $x \lhd_{\mathcal{L}} y$ if and only if there exists some $L \in \mathcal{L}$ such that $x \in L$ but $y \notin L$. We examine $\lhd_{\mathcal{L}}$ in the light of van Dalen and Wattel's theorem. We go on to give a topological characterisation of ordinal spaces, including ω_1, in these terms, by first observing that the T_1-separating union of more than two nests generates spaces that are not of high order-theoretic interest. In particular, we give an example of a countable space X, with three nests $\mathcal{L}, \mathcal{R}, \mathcal{P}$, each T_0-separating X, such that their union T_1-separates X, but does not T_2-separate X. We then characterise ordinals in purely topological terms, using neighbourhood assignments, with no mention of nest or of order. We finally introduce a conjecture on the characterisation of ordinals via selections, which may lead into a new external characterisation.

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Contents

Chapter 1

Introduction

1.1 Research Motivation and Structure of the Thesis

"...confusion connotes something which possesses no **order**,

the individual parts of which are so strangely admixed and interwined,

that it is impossible to detect where each element actually belongs..."

(Extract from The Musical Dialogue, by *Nikolaus Harnoncourt*, Amadeus Press, 1997.)

In this thesis we introduce set-theoretic and topological characterisations of *ordered sets*. In addition, we dedicate quite a few pages on revisiting the *orderability theorem* and we propose a different perspective to look at it. But, first of all, what is an orderability theorem? In particular in S. Purisch's account of results on orderability and suborderability (see [21]), one can read the formulation and development of several orderability theorems, starting from the beginning of the 20th century and reaching our days. By an orderability theorem, in topology, we mean the following. Let $(X, \mathcal{T})$ be a topological space. Under what conditions does there exist an order relation $<$ on X such that the topology $\mathcal{T}_<$ induced by the order $<$ is equal to $\mathcal{T}$? As we can see, this problem is very fundamental as it is of the same weight as the metrizability problem, for example (let X

be a topological space: is there a metric d, on X, such that the metric topology generated by this metric to be equal to the original topology of X?). We will come to this in more detail in Chapter 3. For the meanwhile, let us introduce the material of this Thesis chapter by chapter.

In **Chapter 2**, we give the set-theoretical and topological background that one needs to be aware of in order to follow the material in the next chapters. In Theorem 2.50 we give a proof of the Pressing Down Lemma; it seems that this well-known result is strongly related to our characterisation of ω_1, in Corollary 5.8. In addition, we clearly use this result in our Example 5.14.

Chapter 3 is divided into four sections. In the first section we define an ordering $\lhd_{\mathcal{L}}$ generated by a nest $\mathcal{L}$. We examine properties of the ordering $\lhd_{\mathcal{L}}$ from a set-theoretic perspective, and we see the close link between nests, linear orders and interlockingness. In the second section we revisit the main characterisation theorem for GO-spaces and LOTS of van Dalen and Wattel, and we give necessary and sufficient conditions for a space to be LOTS, using tools that we present in the first section. We finally illustrate our ideas with two well-known examples: the Sorgenfrey line and the Michael line. In the third section, we give necessary conditions for a connected space to be LOTS. In the fourth section, we view the interval topology in the light of nests (subsection 3.4.1), and we investigate some of its order-theoretic properties which imply LOTS (subsection 3.4.2).

In **Chapter 4** we argue that more than two nests “destroy” the structure that we get from van Dalen and Wattel’s characterisations of GO-spaces and LOTS, and we get weaker topological properties. In particular, we give an example of three nests, whose T_1-separating union generates a space which is not Hausdorff.

In **Chapter 5** we characterise ordinals topologically. In the first section, we look at the equivalent facts of a topological space X being scattered, right-separated and being scattered by a nest of open subsets of X, and we observe that a space being scattered by a nest is well-ordered. We add a few extra conditions that we discuss in Chapter 3, in sections 1 and 2, in order to examine when a space X is *homeomorphic* to an ordinal, and this leads to Corollary 5.8, which characterises uncountable ordinals, like ω_1, in these terms. After examining a few properties related to subspaces of ordinals, we illustrate our results with three particular examples. In the second section, we give a characterisation of ordinals, which is entirely topological, with no mention of order or of nest. In the third section, we review the most important solutions to the orderability problem via selections and we state a conjecture on the characterisation of ordinals via selections (something that we also mention in our Open Problems chapter).

In **Chapter 6** we state open problems in our field, that appeared while researching for this thesis, and we will hopefully attempt solving them in the near future.

1.2 A Short Historical Overview

*"**Order** is a concept as old as the idea of number*
and much of early mathematics was devoted to
constructing and studying various subsets of the
real line." (Steve Purisch [21])

The great German mathematician Georg Cantor (1845-1918) is credited to be one of the inventors of set theory. This fact makes him automatically one of the inventors of order-theory as well, as he is the one who first introduced the class of cardinals and the class of ordinal numbers, two classes of rich order-theoretic properties. We give the definitions and we state fundamental properties of ordinals and cardinals in Chapter 2.

Cantor was not only interested in defining classes of ordered sets, and studying their arithmetic; he also produced major results while examining order-isomorphisms, that is, bijective order-preserving mappings between sets whose inverses are also order-preserving. S. Purisch gives a complete list of these historic papers written by Cantor, in his article

“A History of Results on Orderability and Suborderability” [21].

Together with set theory, the field of topology met a rapid rising in the early 20^{th} century and new problems, combining both fields, appeared. A topologist’s temptation is always to examine what sort of topology can be introduced in a given set. So, a very early question was what is the relationship between the natural topology of a set and the topology which is induced by an ordering in this set; this question led to the formulation of the *orderability problem*.

According to Purisch, one of the earliest orderability theorems was introduced by O. Veblen and N.J. Lennes, who were both students of the American mathematician E.H. Moore (1862-1932), and who attended his geometry seminar. This theorem stated that *every metric continuum, with exactly two non-cut points, is homeomorphic to the unit interval.* For the statement of the theorem, Veblen combined the notions of ordered set and topology, for defining a simple arc. Lennes used up-to-date machinery to prove Veblen’s statement, a proof that was published in 1911.

In the meanwhile, some of the greatest mathematicians of the first half of the 20th century, like the French mathematicians R. Baire, M. Fréchet, the Dutch mathematician L.E.J. Brouwer, the Jewish-German mathematician F. Hausdorff, the Polish mathematicians S. Mazurkiewicz, W. Sierpínski, the Russian mathematicians P. Alexandroff and P. Urysohn and others, were devoted to constructing various subsets of the real line. In particular, Baire used ideas of the Yugoslavian mathematician D. Kurepa and of the Dutch mathematician A.F. Monna, on non-Archimedean spaces, in order to characterise the set of irrational numbers. The British mathematician, A.J. Ward, found a topological characterisation of the real line (1936), stating that *the real line is homeomorphic to a separable, connected and locally connected metric space* X*, such that* $X - \{p\}$ *consists of exactly two components, for every* $p \in X$.

A more general result (1920), by Mazurkiewicz and Sierpínski, stated that *compact,* *countable metric spaces are homeomorphic to well-ordered sets*; this is one of the first, if not the first, topological characterisation of abstract ordered sets.

Having in mind that a special version of the orderability problem was solved in the beginning of the 70s (J. van Dalen and E. Wattel), its formulation started from the beginning of the 40s. In particular, the Polish-American mathematician S. Eilenberg, gave in 1941 the following result: *a connected space, X, is weakly orderable, if and only if $X \times X$ minus the diagonal is not connected.* This condition is also necessary and sufficient for a connected, locally connected space to be orderable.

The British mathematician, E. Michael, extended this work, and showed, in 1951, that *a connected Hausdorff space X is a weakly orderable space, if and only if X admits a continuous selection.* In Chapter 5, Section 3, we talk about selections and orderability in particular.

It took two more decades, for a complete topological characterisation of GO-spaces and LOTS to appear. In 1972 J. de Groot and P.S. Schnare showed [2] that *a compact T_1 space X is LOTS, if and only if there exists an open subbase $\mathcal{S}$ of X which is the union of two nests, such that every cover of the space, by elements of $\mathcal{S}$, has a two element subcover.* **J. van Dalen and E. Wattel** used the characterisation of de Groot and Schnare as a basis for their construction, which led to a solution of the orderability problem via nests. We revisit van Dalen and Wattel's characterisation in Chapter 3, Section 2, and we introduce a simpler proof of their main characterisation theorem.

The study of ordered spaces did not finish with the solution to the orderability problem that was proposed by van Dalen and Wattel. On the contrary, many interesting and important results have appeared since then. We will now refer to those results which have motivated our own research in particular.

In 1986, G.M. Reed published an article with title "The Intersection Topology w.r.t. the Real Line and the Countable Ordinals" [22]. The author constructed there a class which was shown to be a surprisingly useful tool in the study of abstract spaces. We know that, if $\mathcal{T}_1, \mathcal{T}_2$ are topologies on a set X, then the *intersection topology*, with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$, is the topology $\mathcal{T}$ on X such that the set $\{U_1 \cap U_2 : U_1 \in \mathcal{T}_1 \text{ and } U_2 \in \mathcal{T}_2\}$ forms a base for $(X, \mathcal{T})$. Reed introduced the class $\mathcal{C}$, where $(X, \mathcal{T}) \in \mathcal{C}$ if and only if $X = \{x_\alpha : \alpha < \omega_1\} \subset \mathbb{R}$, where $\mathcal{T}_1 = \mathcal{T}_\mathbb{R}$ and $\mathcal{T}_2 = \mathcal{T}_{\omega_1}$ and $\mathcal{T}$ is the intersection of $\mathcal{T}_\mathbb{R}$ (the subspace real line topology on X) and $\mathcal{T}_{\omega_1}$ (the order topology on X, of type ω_1). In particular, Reed showed that if $(X, \mathcal{T}) \in \mathcal{C}$, then X has rich topological, but not very rich order-theoretic properties. In particular, X is a completely regular, submetrizable, pseudo-normal, collectionwise Hausdorff, countably metacompact, first countable, locally countable space, with a base of countable order, that is neither subparacompact, metalindelöf, cometrizable nor locally compact. That an $(X, \mathcal{T}) \in \mathcal{C}$ does not necessarily have rich order-theoretic properties comes from the fact that there exists, in ZFC, an $(X, \mathcal{T}) \in \mathcal{C}$ which is not normal. As we shall see (Chapter 2), monotonically normal spaces appear to have rich order-theoretic properties. In Section 4.3, Example 4.12, we use Reed's argument to support further our own argument that more than two nests, whose union is T_1-separating, do not give strong topological properties.

Eric K. van Douwen characterised in 1993 [26] the noncompact spaces, whose every noncompact image is orderable, as the noncompact continuous images of ω_1. Van Douwen refers to a closed non-compact set as *cub* (corresponding to closed unbounded sets in ordinals - we will refer to it as club, throughout the thesis), and he calls *bear* a space which is noncompact and has no disjoint cubs. Here we state his result that has motivated our research on ordinals (Chapter 5):

For a noncompact space X, the following are equivalent:

1. X is a continuous image of ω_1.

2. Every noncompact continuous image of X is orderable.

3. X is scattered first countable orderable bear.

4. X is locally countable orderable bear.

5. X has a compatible linear order, all initial closed segments of which are compact and countable.

Chapter 2

Preliminaries

The aim of this introductory chapter is to present the mathematical tools which will be used in our applications in the next chapters. We aim to do this by giving definitions and appropriate examples.

Our basic reference for set theory will be the books of Moschovakis [18] and Kunen [14]. For theory on topological spaces we will use the classic book of Engelking [3]. The Handbook of Set Theoretic Topology [16] is one of the best accounts in the field, and it will be a very important reference, too. An instant helper, a sort of dictionary for definitions on topology, will always be the Encyclopedia of General Topology [10].

The empty set will be denoted by $\emptyset$ throughout the text. The power set of a set A, i.e. the set of all subsets of A, will be denoted either by $\mathcal{P}(A)$ or by 2^A and any subset of the power set will be denoted by a calligraphic Latin letter, like for example $\mathcal{S}$ or $\mathcal{T}$. For the set of natural numbers, the set of integers, the set of rational numbers and the set of real numbers we will use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively, but we will also denote the set of natural numbers by ω, in some special cases. By $\mathfrak{c}$ we will denote the power of the continuum. The closure of a set A will be denoted by $\overline{A}$ unless otherwise stated. The Axiom of Choice will be abbreviated as A.C. and the Zermelo Fraenkel set theory, together with the A.C. will be abbreviated as ZFC.

Some authors refer to T_3 (respectively T_4) spaces as those spaces which are both T_1 and regular (respectively normal). Some other authors define regular (respectively normal) spaces as those which are both T_1 and T_3 (respectively T_1 and T_4). Since this is a matter of convention, we choose that whenever we state T_3 or T_4 spaces, in this thesis, we will require these spaces to be T_1 plus regular or normal, respectively. Furthermore, a topological space X is $T_{3\frac{1}{2}}$, if X is both T_1 and Tychonoff.

2.1 Topological Preliminaries

In this section we present the topological machinery that will be needed for the understanding of our research results.

Our source for basic notions on scattered spaces will be [23].

Definition 2.1 *A topological space X is* scattered*, if every non-empty subset $A \subset X$ has an isolated point, i.e. for every non-empty $A \subset X$, there exists $a \in A$ and U open in X, such that $U \cap A = \{a\}$.*

Therefore, a space X is *scattered*, if for every non-empty $A \subset X$, there exists U open in X, such that $|U \cap A| = 1$.

Definition 2.2 *Let $\mathcal{S}$ be a family of subsets of a set X. We say that X is* scattered *by $\mathcal{S}$, if and only if for every $A \subset X$, there exists $S \in \mathcal{S}$, such that $|A \cap S| = 1$.*

Remark 2.3 *An equivalent definition for scattered spaces says that a topological space X is* scattered*, if for every non-empty subset K of X, the set of isolated points of K is dense in K. In addition, a subset A of X is scattered, if it is scattered with respect to the subspace topology. Last, but not least, every discrete space is scattered, as every singleton set is open and, hence, isolated.*

Definition 2.4 *A set A is said to be* perfect*, if it is equal to its set of limit points.*

The Cantor-Bendixson Theorem (see for example [9]), characterises topological spaces with respect to their limit points.

Theorem 2.5 (*Cantor-Bendixson) Every topological space can be decomposed uniquely into the union of two disjoint sets, one of which is perfect and the other is scattered.*

Definition 2.6 *Let X be a nonempty topological space and let A be a subset of X. Let $A' = \{x : x \text{ is a limit point of } A\}$. We call A' the* Cantor-Bendixson derivative *of A.*

We can define inductively the *iterated Cantor-Bendixson derivatives* of X, as follows:

$$X^{(0)} = X,$$

$$X^{(\alpha+1)} = (X^{(\alpha)})',$$

$$X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)},$$

where λ is some limit ordinal (see section on ordinals).

Clearly, for some ordinal γ, $X^{(\gamma)} = X^{(\gamma+1)}$. If this set is nonempty, then it is called the *perfect kernel* and, if it is empty, then X is *scattered*. In the scattered case, a point x of X has a well-defined Cantor-Bendixson rank, often called the *limit type* (or *scattered height*) of x, denoted by $lt(x) = \alpha$, if and only if $x \in X^{(\alpha)} - X^{(\alpha+1)}$. The set of all points, of limit type α, is then called the α^{th} level of X, denoted by L_α. Clearly, L_α is the set of all isolated points of $X^{(\alpha)}$.

M.A.D. families will be needed in order to define the Mrówka Ψ-space. The latter will be used in our characterisations of ordinals, as a counterexample. A good source of information for almost disjoint and M.A.D. families is the Handbook of Set Theoretic Topology [16] and for the Mrówka Ψ-space we refer to [19] and [3].

Definition 2.7 *A family $\mathcal{F}$, of infinite subsets of a set X, is said to be an* almost disjoint family *(a.d.f.), if and only if for all distinct $F, G \in \mathcal{F}$, $F \cap G$ is finite.*

Definition 2.8 *A family $\mathfrak{F}$ of infinite subsets of ω, is a Maximal Almost Disjoint family (M.A.D. family), if and only if it is an a.d.f. and, if $H \subset \omega$ is infinite, $H \notin \mathfrak{F}$, then there exists $F \in \mathfrak{F}$, such that $F \cap H$ is infinite.*

M.A.D. families exist by the A.C., and there are uncountably many of them.

Example 2.9 *Let us consider the family $\mathfrak{g} = \{\mathbb{O}, \mathbb{E}\}$, where $\mathbb{O}$ denotes the set of all positive odd numbers and $\mathbb{E}$ the set of all positive even numbers. Then, the intersection of $\mathbb{O}$ and $\mathbb{E}$ is empty, so it is finite. Then, $\mathfrak{g}$ is a M.A.D. family, because if we take any infinite set of natural numbers, it will meet one of $\mathbb{O}, \mathbb{E}$ infinitely many times.*

Another example of a M.A.D. family is $\mathfrak{F} = \{\omega\}$.

Proposition 2.10 *There exists a M.A.D. family $\mathfrak{F}$, such that it has the same size as the set of real numbers, i.e. $|\mathfrak{F}| = |\mathbb{R}|$.*

Proof. Let us consider a set $\mathbb{Q}$ which is dense in the real line and is indexed by the set of natural numbers, i.e. $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$. For the real number $r \in \mathbb{R}$ we choose a subsequence from $\mathbb{Q}$, namely $\{q_{n_{r_j}} : j \in \mathbb{N}\}$, such that $q_{n_{r_j}}$ tends to r, as j tends to infinity. Let also $F_r = \{n : n = n_{r_j}, j \in \mathbb{N}\}$. Then, $F_r \subset \mathbb{N}$ is infinite. We note that $q_{n_{r_j}} \neq q_{n_{r_k}}$, if $j \neq k$. If $r \neq s$, then $F_r \cap F_s$ is finite. Also, $|\mathfrak{F}| = |\{F_r : r \in \mathbb{R}\}| = |\mathbb{R}|$.

Now, let $\mathbb{A} = \{\mathcal{A} : \mathcal{A}$ is an almost disjoint family of subsets of $\mathbb{N}$, such that $\mathfrak{F} \subset \mathcal{A}\}$. Let $\mathbb{B}$ be a chain in $\mathbb{A}$. Then, we claim that $\bigcup \mathbb{B} \in \mathbb{A}$. The latter is true, because clearly for every $\mathcal{B} \in \mathbb{B}$, $\mathfrak{F} \subset \mathcal{B}$, so $\mathfrak{F} \subset \bigcup \mathbb{B}$. But, it is also true that $\bigcup \mathbb{B}$ is almost disjoint. For let distinct $B, C \in \bigcup \mathbb{B}$, such that $B \in \mathcal{B}$ and $C \in \mathcal{C}$, some $\mathcal{B}, \mathcal{C} \in \mathbb{B}$. Now, $\mathbb{B}$ is a chain, so, without loss of generality, $\mathcal{B} \subset \mathcal{C}$. Hence, $B, C \in \mathcal{C}$, which is almost disjoint and so $B \cap C$ is finite. Hence, $\bigcup \mathbb{B}$ is almost disjoint. Thus, by Zorn's Lemma, $\mathbb{A}$ has a maximal element, which proves that $\mathbb{A}$ has a M.A.D. family, of size continuum. $\square$

Definition 2.11 *Let $\mathfrak{F}$ be an uncountable M.A.D. family on ω. Then, the set X, where $X = \omega \cup \mathfrak{F}$, is called the* Mrówka Ψ-space.

We note that if $a \in \mathfrak{F}$, then a is an infinite subset of ω.

For constructing a base for a topology, in the Mrówka Ψ-space, we consider an element $n \in \omega$, which will be by definition isolated, i.e. the set $\{n\}$ will be open. A basic open set about $a \in \mathfrak{F}$ will be of the form:

$$B_k a = \{a\} \cup \{n \in a : n \geq k\}$$

where $k \in \omega$.

We will revisit the Mrówka Ψ-space and its one-point compactification in Chapter 5, in Example 5.13, where we see that a space which is scattered by a nest is not necessarily homeomorphic to an ordinal.

We will now give the definition of *monotone normality*, which is a basic property of linearly orderable topological spaces. Let us recall the definition of *Tychonoff* space, first.

Definition 2.12 *A topological space X is called Tychonoff, if for every point $x \in X$ and for every closed set C, such that C does not contain x, the sets C and $\{x\}$ are* separated *by a function. That is, there exists a continuous function $f : X \to [0,1]$, such that $f(x) = 0$ and $f(y) = 1$, for every $y \in C$. Furthermore, a topological space X is $T_{3\frac{1}{2}}$, if it is both T_1 and Tychonoff.*

Definition 2.13 *A T_1-topological space X is called* monotonically normal*, if and only if for every pair of disjoint closed sets H and K, there exists an open set $D(H,K)$, such that the following two conditions are satisfied:*

1. *$H \subset D(H,K) \subset \overline{D(H,K)} \subset X - K$.*

2. *If $H \subset H'$ and $K' \subset K$, then $D(H,K) \subset D(H',K')$,*

where H', K' are disjoint closed sets in the topology of X.

An alternative definition for monotone normality follows.

Definition 2.14 *A T_1-topological space X is called* monotonically normal, *if for every open set U, in X, and $x \in U$ there exists an open set $\mu(x,U)$, such that $x \in \mu(x,U) \subset U$ and if $\mu(x,U) \cap \mu(y,V) \neq \emptyset$, then either $x \in V$ or $y \in U$, where V is some open set in X and $y \in V$.*

Definition 2.15 *A topological space is called* 0*-dimensional, if it has a base of clopen sets.*

The property of a space being $0-$dimensional is linked to the property of being $T_{3\frac{1}{2}}$, in the following way.

Theorem 2.16 *Let X be a $T_{3\frac{1}{2}}$ space, such that $1 < |X| < \mathfrak{c}$. Then, X is $0-$dimensional.*

Proof. Let U be an open set in X and let $x \in U$, such that $U \neq X$. Then, there exists a continuous function $f : X \to [0,1]$, such that $f(x) = 0$ and $f(X - U) = 1$. But, since the cardinality of X is less than the power of the continuum, there exists $a \in [0,1]$, such that a does not belong to the range of values of f. Thus, $x \in f^{-1}[0,a) = f^{-1}[0,a]$. But, $f^{-1}[0,a) = f^{-1}[0,a] \subset U$, too; so $f^{-1}[0,a]$ is clopen. $\square$

The following definition will be fundamental for our construction of an order relation, via arbitrary collections of sets; it is a set-theoretic version of the T_0, T_1 and T_2 separation axioms, as there is no mentioning of topology.

Definition 2.17 *Let X be a set. We say that a collection of subsets $\mathcal{S}$ of X:*

1. T_0-separates *X, if and only if for all $x,y \in X$, such that $x \neq y$, there exists $S \in \mathcal{S}$, such that $x \in S$ and $y \notin S$ or $y \in S$ and $x \notin S$,*

2. T_1-separates *X, if and only if for all $x,y \in X$, such that $x \neq y$, there exist $S, T \in \mathcal{S}$, such that $x \in S$ and $y \notin S$ and also $y \in T$ and $x \notin T$ and*

3. T_2-separates X, *if and only if for all* $x, y \in X$, *such that* $x \neq y$, *there exist* $S, T \in \mathcal{S}$, *such that* $S \cap T = \emptyset$ *and* $x \in S$, $y \in T$.

The set theoretic and topological versions of the T_0, T_1 and T_2 separation axioms are linked as follows.

Proposition 2.18 *Let* $(X, \mathcal{T})$ *be a topological space.*

1. *X is a T_0 topological space, if and only if there exists a subbase $\mathcal{S}$ for $\mathcal{T}$ which T_0-separates X.*

2. *X is a T_1 topological space, if and only if there exists a subbase $\mathcal{S}$ for $\mathcal{T}$ which T_1-separates X.*

3. *X is a T_2 topological space, if and only if there exists a subbase $\mathcal{S}$ for $\mathcal{T}$ which T_2-separates X.*

Proof. 1. Let us suppose that there exists a subbase $\mathcal{S}$, for $\mathcal{T}$, such that for every $x, y \in X$, with $x \neq y$, there exists $U \in \mathcal{S}$, such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. Since $\mathcal{S} \subset \mathcal{T}$, we have that $U \in \mathcal{T}$ and also $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. Thus, $(X, \mathcal{T})$ is a T_0 topological space.

Let us now consider X to be a T_0 topological space and let $\mathcal{S}$ be a subbase for $\mathcal{T}$. Let $x, y \in X$, such that $x \neq y$. Then, there exists $V \in \mathcal{T}$, such that without loss of generality $x \in V$ and $y \notin V$. But $V = \bigcup_{i \in I} V_i$, where $V_i = \bigcap_{k=1}^{n_i} V_k^i$, $V_k^i \in \mathcal{S}$. So, $x \in \bigcup_{i \in I} V_i$ implies that there exists $i \in I$, such that $x \in V_i$. But $y \notin V_i$. Thus, there exists j, where $1 \leqslant j \leqslant n$, such that $y \notin V_k^j$. But $x \in V_i$ implies that $x \in V_k^j$. Thus, there exists $V_k^j \in \mathcal{S}$, such that $x \in V_k^j$ and $y \notin V_k^j$.

The proofs for 2. and 3. are similar to the proof of 1. □

Corollary 2.19 1. *A collection of sets $\mathcal{S}$ is T_0-separating, if and only if the topology that is generated by $\mathcal{S}$ is a T_0 topology.*

2. A collection of sets $\mathcal{S}$ is T_1-separating, if and only if the topology that is generated by $\mathcal{S}$ is a T_1 topology.

3. A collection of sets $\mathcal{S}$ is T_2-separating, if and only if the topology that is generated by $\mathcal{S}$ is a T_2 topology.

2.2 Partial Orderings, Linear Orderings and Well Orderings

Here we highlight important order-theoretic notions which will interact with several topological ideas in the constructions that will be presented in the next chapters.

Definition 2.20 *A binary relation, $\leq$, in a set A, is called a* partial order *on A, if and only if for any $a, b, c \in A$, the following conditions are satisfied:*

1. *$a \leq a$ (reflexivity);*
2. *$a \leq b$ and $b \leq a$ implies that $a = b$ (antisymmetry) and*
3. *$a \leq b$ and $b \leq c$ implies that $a \leq c$ (transitivity).*

The pair $(A, \leq)$ is called a partially ordered set *(or simply* poset*).*

Definition 2.21 *If a set X is equipped with a relation which is reflexive and transitive, but not necessarily* antisymmetric*, then this relation on X will be called a* preorder *on X.*

Definition 2.22 *Let $\leq$ be a relation which defines a partial order on a set X. Then, the* reverse order*, denoted by $\geq$, is defined by $a \geq b$, if and only if $b \leq a$, for any $a, b \in X$.*

Lemma 2.23 *If $\leq$ is a partial order on a set X, then $\geq$ is also a partial order on X.*

If for any two distinct elements a, b, in a partially ordered set X satisfying $a \leq b$, we write $a < b$, we will formally read this as *a precedes b* or as *b dominates a* and, informally, simply state "*a* is *less than b*" or "*b* is *greater than a*".

Definition 2.24 *Let $\leq$ be a partial order, on a set X, and let $p, q \in X$. Then:*

1. *p, q are* comparable *with respect to $\leq$ if and only if p is greater than q, or q is greater than p, or $p = q$.*

2. *The order $\leq$ satisfies* Trichotomy *if for any $x, y \in X$, exactly one of the following holds: $x < y$, $x = y$ or $x > y$.*

3. *The partial order $\leq$ is a* linear order *(or total order), if $\leq$ satisfies* trichotomy.

4. *$C \subset X$ is a* chain, *if and only if for all $p, q \in C$, p and q are comparable, i.e. C is linearly ordered by $\leq$.*

5. *$A \subset X$ is an* anti-chain, *if and only if for all $p, q \in A$, such that $p \neq q$, then p and q are not comparable. In this case, we write $p \perp q$.*

Definition 2.25 *Let $(X, \leq)$ be a partially ordered set. An element $a_0 \leq X$ is a* least element *of X if and only if $a_0 \leq x$ for all $x \in X$. An element $b_0 \in X$ is a* greatest element *of X if and only if $x \leq b_0$ for all $x \in X$. In addition, an element $M \in X$ is* maximal *if and only if $M \leq x$ implies $x = M$ for all $x \in X$ and an element $m \in X$ is* minimal *if and only if $x \leq m$ implies that $x = m$ for all $x \in X$.*

Proposition 2.26 *Let $(X, \leq)$ be a linearly ordered set and let $A \subset X$. Then, A has at most one minimal and at most one maximal element.*

Proof. Let $(A, \leq)$ have two different minimal elements, say 0_A and $0'_A$. Then, $0_A \leq 0'_A$ and $0'_A \leq 0_A$, which leads into the contradiction that $0_A = 0'_A$. A similar argument can be applied by considering two maximal elements, which will again lead into a contradiction.$\square$

Definition 2.27 *Let X be a linearly ordered set. Let $A \subset X$. A is said to be* cofinal *in X, if and only if for every $x \in X$, there exists $a \in A$, such that $x \leq a$.*

Definition 2.28 *A linearly ordered set A is called* well-ordered*, if every nonempty subset of A has a minimal element.*

Let us now introduce some notation. By (a, b) we denote the set $\{x \in X : a < x < b\}$, by $(-\infty, a)$ we denote the set $\{x : x < a\}$ and by (a, ∞) we denote the set $\{x : x > a\}$. Also, by $[a, b]$ we denote the set $\{x \in X : a \leq x \leq b\}$, by $(-\infty, a]$ we denote the set $\{x : x \leq a\}$ and by $[a, \infty)$ we denote the set $\{x : x \geq a\}$. Last, we denote by $[a, b)$ the set $\{x \in X : a \leq x < b\}$ and by $(a, b]$ the set $\{x \in X : a < x \leq b\}$. No confusion should be made with the real line intervals, even the fact that the natural topology on the real numbers coincides with the order topology that is induced by the natural order on $\mathbb{R}$; this interval notation will be used for any ordered space and not for $\mathbb{R}$ exclusively.

Definition 2.29 *If $(X, <)$ is a linearly ordered set, then we define the* order topology $\mathcal{T}_<$ *on X to be the topology that is generated by the subbase:*

$$\{(-\infty, a) : a \in X\} \cup \{(b, \infty) : b \in X\}$$

Definition 2.30 *Let X be a set, let $<$ be a linear order on X and let $\mathcal{T}_<$ be the* order topology *on X. Then $(X, \mathcal{T}_<)$ is called a* linearly ordered topological space *or LOTS, for abbreviation.*

In the literature the term *orderable* corresponds to a space, with the property that there exists a linear order on the underlying set, such that the order topology coincides with the original topology of the space. In this thesis we will use the term LOTS, instead of orderable space. The term *suborderable* is used as a synonym for GO-space, which refers to a subspace of a LOTS. GO-spaces, LOTS and ordinals (for ordinals see next section) are naturally occurring topological objects, and are canonical building blocks for topological examples.

Example 2.31 *A GO-space is not necessarily LOTS, i.e. LOTS is not a hereditary property. For example, if we consider the set of real numbers with its natural order* $<$*, then* $\mathbb{R}$ *is LOTS and the subset* $X = (1,2) \cup \{3\}$ *is therefore a GO-space but not a LOTS under the natural order, with the subspace topology inherited from* $\mathbb{R}$*. Indeed, one can show that no linear order on* X *induces this topology on* X*. To see this suppose that* $\lhd$ *is a linear order on* X *that does generate this topology. Note that in the space* X*, the point* 3 *is isolated and the set* $I = (1,2)$ *is connected. There are three cases to consider. (1)* $3 \lhd x$ *for all* $x \in I$*. (2)* $x \lhd 3$ *for all* $x \in I$*. (3) for some* $a, b \in I$*,* $x \lhd 3 \lhd y$*. Case (3) is impossible since* $\{x \in I : x \lhd 3\}$ *and* $\{x \in I : 3 \lhd x\}$ *are non-empty open sets that disconnect* I*. Case (2) is identical to case (1), so we assume that* $3 \lhd x$ *for all* $x \in I$*. In this case, since* 3 *is isolated, there is some* $a \in I$ *such that the* $\lhd$*-open interval* $J = \{x \in X : x \lhd a\}$ *has the property that* $X \cap J = \{3\}$*. This implies that the* $\lhd$*-open interval* $K = (3, a)$ *contains no points of* I*. Hence* a *is the least element of* I *and therefore* $I - \{a\}$ *is connected. But this is a contradiction since* $(1,2) - \{a\}$ *is not a connected subset of* $\mathbb{R}$*.*

LOTS and GO-spaces have strong topological properties. In particular, in [11], the authors show that a LOTS is a monotonically normal space. So, since any subspace of a monotonically normal space is monotonically normal (see [16] and [11]), we conclude that a GO-space will be monotonically normal, too. One can find interesting discussions on the topic, including a proof (by Henno Brandsma) that LOTS implies monotone normality, in the webpage “Ask a Topologist”, which is linked to the Topology Atlas.

The problem of characterising arbitrary LOTS and GO-spaces topologically was solved by van Dalen and Wattel [24]. Previously, a number of characterisations of particular LOTS had been given. There are, for example, characterisations of $\mathbb{Q}$, $[0,1]$, $\mathbb{R} - \mathbb{Q}$ and compact LOTS. There is a survey of such characterisations, that we also mentioned in our historical overview (Chapter 1), written by S. Purish [21]. David J. Lutzer has written a survey specifically for LOTS and GO-spaces [17].

2.3 Ordinals and Some of Their Topological Properties

Our main reference here will be Kunen's book [14].

Definition 2.32 *A set, X, is* transitive*, if every element of X is simultaneously a subset of X.*

Examples of transitive sets are $0, \{0\}, \{0, \{0\}\}$, where we define $0 = \emptyset$, but $\{\{0\}\}$ is not transitive.

Definition 2.33 *A set X is an* ordinal*, if and only if it is transitive and well-ordered by $\in$.*

From now on we will denote ordinals using lower case Greek letters, and we will divide them into three categories:

Definition 2.34

1. *the zero ordinal, denoted by 0,*
2. *the successor ordinals, that are of the form: $\alpha + 1 = \alpha \cup \{\alpha\}$ and*
3. *the limit ordinals, which are ordinals that are neither 0, nor successor ordinals.*

In particular, a *limit ordinal*, λ, is an ordinal which satisfies the following property:

$$\lambda = \bigcup_{\alpha < \lambda} \alpha$$

The set of natural numbers, ω, is extended to $\in$ −well-ordered sets, the ordinals, such that every well-ordered set is isomorphic to a unique ordinal. Furthermore, each natural number corresponds to an ordinal. For example, $0 = \emptyset$, $1 = 0 + 1 = 0 \cup \{0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}$, $2 = 1+1 = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0,1\}$, $3 = 2+1 = 2 \cup \{2\} = \{0,1\} \cup \{2\} = \{0,1,2\}$ etc.

The symbol we use to denote the least infinite ordinal is ω (the set of natural numbers). The first uncountable ordinal is denoted by ω_1. As a set, ω_1 consists of all countable ordinals. In general, we denote the α-th infinite initial ordinal by ω_α, for each ordinal α, where an initial ordinal is an ordinal having strictly greater cardinality than all of its predecessors.

Definition 2.35 *Two ordered sets X and Y are said to have the same* order type, *if they are order-isomorphic, i.e. there exists a bijection f, from X to Y, so that f and f^{-1} are order preserving.*

In the case where X is linearly ordered, the monotonicity of f implies the monotonicity of f^{-1}. Furthermore, every well-ordered set is order-isomorphic to a unique ordinal.

Definition 2.36 *Let $(\alpha, \leq)$ be a well-ordered set. Then, the set $\alpha_\xi = \{x \in \alpha : x \in \xi\}$ is called an* initial segment *of α.*

The proof of the statements, of the following proposition, is technical and follows straight from the definitions. These statements can be found in any introductory book in set theory, and we mention them as we will assume their knowledge in the next chapters.

Proposition 2.37

1. *Every element of an ordinal is an ordinal, too.*
2. *No ordinal is an element of itself.*
3. *Let α be an ordinal. Then, for every $\xi \in \alpha$, $\alpha_\xi = \xi$.*
4. *For every pair of ordinals, α and β, the following holds:*

$$\alpha \in \beta \Leftrightarrow \alpha \subsetneq \beta$$

5. *Two well-ordered sets are either isomorphic to one another or the one is isomorphic to an initial segment of the other.*

6. *Every well-ordered set is isomorphic to exactly one ordinal.*

7. *Two isomorphic ordinals are equal to each other.*

Let us now have a look at some topological properties of ordinals.

First, for defining a topology on an ordinal, we consider an arbitrary ordinal, ε, as a LOTS (ordinals are obviously linearly ordered sets, as they are well-ordered, too). We suppose $\alpha \in \varepsilon$ and we pick $\beta < \alpha$ and $\gamma > \alpha$, if α is not maximal and also not minimal in ε. Then, a neighbourhood of α will be of the form (β, γ). But, $\alpha < \gamma$. So, $\alpha + 1$ will be the least ordinal greater than α, so that $\alpha + 1 \leq \gamma$. So, $\alpha \in (\beta, \alpha + 1)$, which is equivalent to saying that $\alpha \in \{\delta : \beta < \delta < \alpha + 1\} = \{\delta : \beta < \delta \leq \alpha\} = (\beta, \alpha]$ (because there is nothing between α and $\alpha + 1$). Finally, the family $\{(\beta, \alpha] : \beta < \alpha\}$ forms a neighbourhood base for α, for every non-zero $\alpha \in \varepsilon$.

Theorem 2.38 *Let ε be an ordinal, with the usual order topology. Then, the following hold:*

1. *If $\alpha \in \varepsilon$, such that $\alpha = 0$ or α is a successor ordinal, then α is an isolated point.*

2. *ε is a scattered LOTS.*

Proof. If α is a successor ordinal, then $\alpha = \beta + 1$, for some $\beta < \alpha$. So,

$$(\beta, \alpha] = \{\delta : \beta < \delta \leqslant \alpha\} = \{\alpha\},$$

because there exists nothing between β and α. So, if α is a successor ordinal, or the zero ordinal, then α will be an isolated point.

In addition, if ε is any type of ordinal, and if $\emptyset \neq A \subseteq \varepsilon$, then A has a least element, α say (because a subset of an ordinal is a subset of a well-ordered set). Then, either $\alpha = 0$ (which is isolated) or $\alpha > 0$. So, $(0, \alpha] \cap A = \{\alpha\}$, which is an open set. So, α is isolated in A, which implies that ε is a scattered space. $\square$

Proposition 2.39 *A non-zero ordinal, ε, is a successor ordinal, if and only if it is compact with respect to the order topology.*

Proof. ($\Rightarrow$) We first prove that if ε is a successor ordinal, then ε is compact. For this, let $\varepsilon = \{\alpha : \alpha < \varepsilon\}$ be a successor ordinal, i.e. $\varepsilon = \beta + 1$, for some β. Let also $\mathcal{U}$ be an open cover for ε, where $\mathcal{U} = \{U_i : i \in I\}$.

Let $\mathcal{A}_1 = \{x \in \varepsilon : (x, \beta]$ is contained in some $U \in \mathcal{U}\}$. Then, $\mathcal{A}_1$ has a least element, α say. That is, there exists $U_1 \in \mathcal{U}$, such that $(\alpha_1, \beta] \subset U_1$.

If $\alpha_1 \neq 0$, let $\mathcal{A}_2 = \{x \in \varepsilon : (x, \alpha_1]$ is contained in some $U \in \mathcal{U}\}$. Then, $\mathcal{A}_2$ has a least element. That is, there exists $\alpha_2 \in \varepsilon$ and $U_2 \in \mathcal{U}$, such that $(\alpha_2, \alpha_1] \subset U_2$.

Continuing like this, there will be a $U_n \in \mathcal{U}$, such that $\alpha_n = 0$, for some n, since, otherwise, we would have formed a sequence $\alpha_1 > \alpha_2 > \ldots$, which would contradict the well-ordering of ε.

So, $\{U_1, U_2, \cdots, U_n\}$ is a subfamily of $\mathcal{U}$, which covers ε except, possibly, zero. Thus, a subfamily of $\mathcal{U}$, with $n + 1$ elements, covers ε, and so ε is compact.

($\Leftarrow$) For proving the converse, i.e. that if ε is compact, then ε is a successor ordinal, we prove that the negation of this statement is false. More specifically, we consider $\mathcal{U} = \{[0, \alpha] : \alpha < \lambda\}$ to be an open cover for a limit ordinal, λ.

Suppose $\mathcal{U}$ has a finite subcover:

$$[0, \alpha_1], [0, \alpha_2], \cdots, [0, \alpha_n]$$

Let $\max\{\alpha_1, \cdots, \alpha_n\} = \gamma < \lambda$. Then, $[0, \alpha_i] \subseteq [0, \gamma]$, $\forall\, i$. So, $\cup_{i \leqslant n}[0, \alpha_i] = [0, \gamma]$. But,

$[0,\gamma] \neq \lambda$, which leads to a contradiction. $\square$

2.4 Club-sets, Stationary Sets and the Pressing Down Lemma

Our main reference here will be Kunen [14] and the Handbook of Set Theoretic Topology [16].

Definition 2.40 *The* cardinality *of a set X is defined to be the least ordinal α, such that there exists a one-to-one and onto mapping between X and α.*

Ordinals, generally speaking, show the *order*, the position of an element, in a list of elements, and cardinals show how many elements there are in a set.

Definition 2.41 *The* cofinality *of an ordinal β, denoted by cf(β), is the least α, such that there exists a mapping $f : \alpha \to \beta$, such that $\beta = \sup\{f(\gamma) : \gamma \in \alpha\}$.*

In other words, the cofinality of an ordinal β is the least ordinal α, which is the order type of a cofinal subset of β.

Since a set, κ, is a *cardinal*, if and only if κ is the smallest ordinal of this size, we can clearly see the connection between ordinals and cardinals. In particular, if α is an ordinal, such that $|\alpha| = |\kappa|$, then $\alpha \geqslant \kappa$. Thus, for any ordinal, β, the cofinality of β is a cardinal and is always less than or equal to β.

Definition 2.42 *A cardinal, κ, is said to be* regular*, if and only if $\kappa =$cf(κ).*

Remark 2.43 *If $A \subseteq \lambda$, where λ is a limit ordinal, then A is* cofinal *in λ, if and only if $\lambda = \bigcup A = \sup A$, if and only if A is* unbounded *in λ, if and only if for every $\alpha \in \lambda$, there exists $\delta \in A$, such that $\delta > \alpha$.*

Definition 2.44 *Let λ be a limit ordinal. A set $C \subset \lambda$ is* closed *in λ, if and only if it is closed with respect to the order topology in λ.*

Definition 2.45 *A subset C, of a limit ordinal λ, is called* club, *if it is closed and unbounded in λ.*

As we have also mentioned in Chapter 1, Section 2, in van Douwen's paper [26] club sets are referred to as cubs. In addition, an ordered topological space X is called bear, if it is noncompact, and has no disjoint cubs. So, an ordinal $\alpha = [0, \alpha)$, in the order topology, is bear, if the cofinality of α is greater than or equal to ω_1 (see [26]).

Definition 2.46 *Let κ be a regular cardinal. A subset, A, of κ, is said to be* stationary, *if and only if $A \cap C \neq \emptyset$, for any club set C, of κ.*

Lemma 2.47 *Let λ be a limit ordinal. If $f : \lambda \to \lambda$ is a function, such that if $\beta < \mu$ then $f(\beta) \leq f(\mu)$, $\beta, \mu \in \lambda$, i.e. f is non-decreasing, then f is continuous with respect to the order topology, if and only if for every limit ordinal μ, $f(\mu) = \sup\{f(\beta) : \beta < \mu\}$.*

Proof. "$\Rightarrow$" Suppose f is continuous and $f(\mu) \geq \sup\{f(\beta) : \beta < \mu\}$. If $\beta < \mu$, then $f(\beta) \leq f(\mu)$. Suppose $f(\mu) > \sigma = \sup\{f(\beta) : \beta < \mu\}$. Then, $(\sigma, f(\mu)]$ is open in the ordinal topology, so $f^{-1}(\sigma, f(\mu)]$ is open, by continuity, and $\mu \in f^{-1}(\sigma, f(\mu)]$. But $U \cap (0, \mu) = \emptyset$, some open neighbourhood U, of μ; a contradiction, because μ is a limit ordinal.

"$\Leftarrow$" Let f be not continuous. Then, there exists an open set U, such that $f^{-1}(U)$ is not open. So, there must be a $\mu \in f^{-1}(U)$, such that for every $\beta < \mu$, $(\beta, \mu] \nsubseteq f^{-1}(U)$. Also, μ must be a limit ordinal and f is non-decreasing. So, there exists $\beta < \mu$, such that $(\beta, \mu] \cap f^{-1}(U) = \{\mu\}$. So, $f(\beta, \mu) \cap U = \emptyset$ and, finally, $\sup f(\beta, \mu) < \mu$. $\square$

Lemma 2.48 *Let κ be an uncountable regular cardinal. Let C_α be a club in κ, under the order topology in κ, for each $\alpha \in \kappa$. Let also $g : \kappa \to \kappa$, be 1-1 and non-decreasing in the* *sense that f is non-decreasing in Lemma 2.47, where g is defined by $g(\gamma) = \inf \bigcap_{\alpha \leqslant \gamma} C_\alpha$, for any $\gamma \in \kappa$. Then, g is continuous, with respect to the usual order topology for κ.*

Proof. Suppose that $\gamma = \sup\{\gamma_i : i \in \lambda,\ \lambda \leq \kappa\}$. Suppose also that $\delta = \sup\{g(\gamma_i) : i \in \lambda\}$. Then,

$$g(\gamma_i) \in \bigcap_{\alpha \leq \gamma_i} C_\alpha$$

and

$$\delta \in \bigcap_{\alpha \leq \gamma_i} C_\alpha$$

gives that

$$\delta \in \bigcap_{\alpha \leq \gamma} C_\alpha\,;$$

so, $\delta \geqslant g(\gamma)$.

Suppose that $\delta > g(\gamma)$. Then, there exists an i, such that $g(\gamma) \leqslant g(\gamma_i) < \delta$.

Since:

$$g(\gamma_i) \notin \bigcap_{\alpha \leq \gamma_{i+1}} C_\alpha$$

and

$$\delta = \inf \bigcap_{\alpha \leq \gamma} C_\alpha,$$

we have that

$$\delta = g(\gamma),$$

which gives continuity, according to Lemma 2.47. □

Lemma 2.49 *Let κ be an uncountable regular cardinal. Then, every increasing and continuous function, $f : \kappa \to \kappa$, that is $f(a) \geqslant a$, for all $a \in \kappa$, has a club set of fixed points.*

Proof. Let f be a continuous map (with respect to the usual order topology on κ) and let $f(a) \geqslant a$, for all $a \in \kappa$. If we choose $a_0 \in \kappa$, then $a_1 > f(a_0)$, $a_2 > f(a_1)$, etc. So, by continuity, if $a = \lim_{i\in\omega} a_i$, $f(a) = \lim f(a_i)$. But $a_i \leq f(a_i) \leq a_{i+1}$. So, the limit of $f(a_i)$ is equal to a, i.e. a is a fixed point. Since a_0 was chosen arbitrarily, the set of fixed points of κ will be unbounded. But the set of fixed points of any continuous function (at least for T_2-spaces), is closed. So, we get a club set of fixed points. □

We will make a clear use of the Pressing Down Lemma, that we state and prove below, in Example5.14, and we will link it to our Corollary 5.8 in the final section where we state open problems.

Theorem 2.50 (Pressing Down Lemma) *Let κ be an uncountable regular cardinal. If f is a strictly decreasing function on a stationary set $S \subset \kappa$, then there exists a stationary subset $S' \subset S$, with f constant on S'.*

(Note that this theorem is known as the Pressing Down Lemma, because it says that if we map each $a \in \kappa$ to something smaller than it, β_a say, then a whole stationary set will be mapped into one particular β. So, the whole stationary set presses down onto this β.)

Proof. Let f be a strictly decreasing function on S, where S is stationary in κ. Let us also suppose that the statement of the theorem is not true. Then, we have the negation of the logical sentence: there exists a stationary subset $S' \subset S$, with f constant in S'. Thus, for every $\alpha \in \kappa$, $f^{-1}(\alpha)$ is non-stationary, i.e. for all $\alpha \in \kappa$, there exists a club set C_α, such that $C_\alpha \cap f^{-1}(\alpha) = \emptyset$.

Let:

$$C = \{\beta : \forall\, \alpha \leq \beta,\, \beta \in C_\alpha\}$$

We show that C contains a club set.

We define a map:

$$g : \kappa \to \kappa,$$

with

$$g(\gamma) = \inf \bigcap_{\alpha \leq \gamma} C_\alpha.$$

Then, g is continuous, as we have already seen in Lemma 2.48. But, according to Lemma 2.49, every increasing and continuous function has a club set of fixed points. Let D be such a club set for g. Then, $D \subset C$, and since C contains a club set and S is stationary, there is a nonzero β, such that $\beta \in C \cap S$. But, since $\beta \in C_\alpha$, $\forall\, \alpha \leq \beta$, we get that $f(\beta) \geqslant \beta$, which leads into a contradiction. $\square$

Chapter 3

A Topological Solution to the Orderability Problem

In 1973, J. van Dalen and E. Wattel (see [24]) gave complete topological characterisations of orderable and generalised ordered spaces, by the existence of special subbases consisting of the union of nests. In this chapter we look again at van Dalen and Wattel's characterisation, from a more order-theoretic point of view. The results of the first two sections appear in [7].

3.1 An Ordering Relation via Nests

The notion of *nest* will play a dominant role in our characterisations of LOTS and ordinals, that will follow in the next chapters, as there is obviously a close link between nests and linear orders. Here we present a few results that appear in article [7], as a preliminary section to the characterisations of ordered spaces that will follow.

Definition 3.1 *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$. We call $\mathcal{L}$ a* nest*, if and only if $\mathcal{L}$ is linearly ordered by inclusion.*

Definition 3.2 *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$. We define $\lhd_{\mathcal{L}}$ on X by declaring that $x \lhd_{\mathcal{L}} y$, if and only if there exists some $L \in \mathcal{L}$, such that $x \in L$ and $y \notin L$.*

The close link between nests and linear orders can be seen in Theorem 3.3, as follows.

Theorem 3.3 *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$. Then, the following hold:*

1. *If $\mathcal{L}$ is a nest, then $\lhd_{\mathcal{L}}$ is a transitive relation.*

2. *$\mathcal{L}$ is a nest, if and only if for every $x, y \in X$, either $x = y$ or $x \ntriangleleft_{\mathcal{L}} y$ or $y \ntriangleleft_{\mathcal{L}} x$.*

3. *$\mathcal{L}$ is T_0-separating, if and only if for every $x, y \in X$, either $x = y$ or $x \lhd_{\mathcal{L}} y$ or $y \lhd_{\mathcal{L}} x$.*

4. *$\mathcal{L}$ is a T_0-separating nest, if and only if $\lhd_{\mathcal{L}}$ is a linear order.*

Proof. 1. is immediate from the definition of $\lhd_{\mathcal{L}}$.

For 2., suppose first that $\mathcal{L}$ is a nest. If $x \neq y$ and both $x \lhd_{\mathcal{L}} y$ and $y \lhd_{\mathcal{L}} x$, then there are M and N, in $\mathcal{L}$, such that $x \in M$ and $y \notin M$ and also $y \in N$ and $x \notin N$, so that M is not a subset of N and N is not a subset of M, contradicting to the fact that $\mathcal{L}$ is a nest. Conversely, suppose that M and N are elements of $\mathcal{L}$. If M is not a subset of N and if N is not a subset of M, then there are $x \in M - N$ and $y \in N - M$, so that both $x \lhd_{\mathcal{L}} y$ and $y \lhd_{\mathcal{L}} x$.

For 3., if $\mathcal{L}$ is T_0-separating and if $x \neq y$, then there is $N \in \mathcal{L}$, such that either $x \in N$ and $y \notin N$, so that $x \lhd_{\mathcal{L}} y$, or $y \in N$ and $x \notin N$, so that $y \lhd_{\mathcal{L}} x$. Conversely, if $x \neq y$, then without loss of generality $x \lhd_{\mathcal{L}} y$, so that there exists $N \in \mathcal{L}$, such that $x \in N$ and $y \notin N$.

4. follows from 1., 2. and 3. $\square$

Theorem 3.4 *Let X be a set. Suppose that $\mathcal{L}$ and $\mathcal{R}$ are two nests on X. Then, $\mathcal{L} \cup \mathcal{R}$ is T_1-separating, if and only if $\mathcal{L}$ and $\mathcal{R}$ are both T_0-separating and $\lhd_{\mathcal{L}} = \rhd_{\mathcal{R}}$.*

Proof. Suppose that $\mathcal{L} \cup \mathcal{R}$ is T_1-separating. If $x, y \in X$, $x \neq y$, then there are N and M, in $\mathcal{L} \cup \mathcal{R}$, such that $x \in N$ and $y \notin N$ and also $y \in M$ and $x \notin M$. Without loss of generality, let $N \in \mathcal{L}$. Since $\mathcal{L}$ is a nest, we have that $M \notin \mathcal{L}$ and so $M \in \mathcal{R}$. Hence,

$x \lhd_{\mathcal{L}} y$ and $x \lhd_{\mathcal{R}} y$. Since x and y were arbitrary, it follows that $\mathcal{L}$ and $\mathcal{R}$ are T_0-separating, respectively, and $\lhd_{\mathcal{L}} = \rhd_{\mathcal{R}}$.

Conversely, suppose that $\mathcal{L}$ and $\mathcal{R}$ are two T_0-separating nests, such that $\lhd_{\mathcal{L}} = \rhd_{\mathcal{R}}$. If $x, y \in X$, $x \neq y$, then there is $L \in \mathcal{L}$, such that, without loss of generality, $x \in L$ and $y \notin L$. Hence, $x \lhd_{\mathcal{L}} y$, so that $y \lhd_{\mathcal{R}} x$, which implies that there is some $R \in \mathcal{R}$, such that $y \in R$ and $x \notin R$. Hence, $\mathcal{L} \cup \mathcal{R}$ is T_1-separating. $\square$

Having in mind what we have discussed in this Chapter so far, we introduce two nests $\mathcal{L}$ and $\mathcal{R}$, on a set X, whose union is T_1-separating. Then, topologically speaking, if the elements of $\mathcal{L}$ and $\mathcal{R}$ are open sets in the topology that is generated by $\mathcal{L} \cup \mathcal{R}$, it is relatively simple to show that the order topology generated by $\lhd_{\mathcal{L}}$ is coarser than the topology on X, which is generated by $\mathcal{L} \cup \mathcal{R}$. As we shall see in Theorem 3.11, the following notion of interlocking, due to van Dalen and Wattel [24], is the key idea in ensuring that the topology induced by the order $\lhd_{\mathcal{L}}$ coincides with the topology generated by the subbase $\mathcal{L} \cup \mathcal{R}$.

Definition 3.5 *Let X be a set and let $\mathcal{S} \subset \mathcal{P}(X)$. We say that $\mathcal{S}$ is interlocking, if and only if for each $T \in \mathcal{S}$, such that:*

$$T = \bigcap \{S : T \subset S, S \in \mathcal{S} - \{T\}\}$$

we have that:

$$T = \bigcup \{S : S \subset T, S \in \mathcal{S} - \{T\}\}.$$

By Lemma 3.6 that follows, we clarify the relationship between an interlocking nest and the properties of its induced order.

Lemma 3.6 *Let X be a set and let $\mathcal{L}$ be a T_0-separating nest on X. Then, the following hold for $L \in \mathcal{L}$:*

1. *$L = \bigcap\{M \in \mathcal{L} : L \subsetneq M\}$, if and only if $X - L$ has no $\lhd_{\mathcal{L}}$-minimal element.*

2. *$L = \bigcup\{M \in \mathcal{L} : M \subsetneq L\}$, if and only if L has no $\lhd_{\mathcal{L}}$-maximal element.*

Proof. By Theorem 3.3, we get that $\lhd_{\mathcal{L}}$ is a linear order, on X.

For 1., if x is the $\lhd_{\mathcal{L}}$-minimal element of $X - L$, then for all $M \in \mathcal{L}$, such that $L \subsetneq M$, we have that $x \in M$, so that $L \neq \bigcap\{M \in \mathcal{L} : L \subsetneq M\}$. Conversely, if $X - L$ has no $\lhd_{\mathcal{L}}$-minimal element, then for all $x \notin L$, there is some $y \lhd_{\mathcal{L}} x$, such that $y \notin L$. Since $\mathcal{L}$ is a T_0-separating nest, there is some $M \in \mathcal{L}$, such that $y \in M$ and $x \notin M$. Since $y \in M - L$, we have that $L \subsetneq M$, so that $x \notin \bigcap\{M \in \mathcal{L} : L \subsetneq M\}$ and $L = \bigcap\{M \in \mathcal{L} : L \subsetneq M\}$. Then, 1. follows easily.

The proof of 2. is similar to the proof of 1. $\square$

It is immediate, from Definition 3.5, that a collection $\mathcal{L}$ is interlocking, if and only if for all $L \in \mathcal{L}$, either $L = \bigcup\{N \in \mathcal{L} : N \subsetneq L\}$ or $L \neq \bigcap\{N \in \mathcal{L} : L \subsetneq N\}$. Theorem 3.3 and Lemma 3.6 therefore imply the following.

Theorem 3.7 *Let X be a set and let $\mathcal{L}$ be a T_0-separating nest on X. The following are equivalent:*

1. *$\mathcal{L}$ is interlocking;*

2. *for each $L \in \mathcal{L}$, if L has a $\lhd_{\mathcal{L}}$-maximal element, then $X - L$ has a $\lhd_{\mathcal{L}}$-minimal element;*

3. *for all $L \in \mathcal{L}$, either L has no $\lhd_{\mathcal{L}}$-maximal element or $X - L$ has a $\lhd_{\mathcal{L}}$-minimal element.*

Lemma 3.8 *Let $<$ be a linear order on X. Let $\mathcal{L}_< = \{(-\infty, a) : a \in X\}$ and $\mathcal{R}_< = \{(a, \infty) : a \in X\}$. Then, $\mathcal{L}_<$ and $\mathcal{R}_<$ are T_0-separating, interlocking nests, such that $\mathcal{L}_< \cup \mathcal{R}_<$ is T_1-separating and $\lhd_{\mathcal{L}} = \rhd_{\mathcal{R}} =<$. Moreover, $\mathcal{L}_< \cup \mathcal{R}_<$ forms a subbase of order open sets, for the order topology on X.*

Proof. Clearly, $\mathcal{L}$ and $\mathcal{R}$ are T_0-separating nests, whose union is T_1-separating. By Theorem 3.4, $\lhd_{\mathcal{L}} = \rhd_{\mathcal{R}}$. If $x < y$, then $L = (-\infty, y) \in \mathcal{L}$ and also $x \in L$ and $y \notin L$; so that $x \lhd_{\mathcal{L}} y$.

On the other hand, if $x \lhd_{\mathcal{L}} y$, then for some $z \in X$, $x \in (-\infty, z)$ and $y \notin (-\infty, z)$, so that $x < z$ and $z \leqslant y$, which implies that $x < y$.

It remains to show that $\mathcal{L}$ and $\mathcal{R}$ are interlocking. Suppose $L = (-\infty, a) \in \mathcal{L}$ have a $<$-maximal element, m. Then, $m < a$ and, if $m < x \leq a$, $x = a$, so that a is the $<$-minimal element of $X - L$. By Theorem 3.7, $\mathcal{L}$ is interlocking. Using a similar argument, we find that $\mathcal{R}$ is interlocking, too. Finally, from the definition of the order topology that is induced from $<$, on X, $\mathcal{L} \cup \mathcal{R}$ forms a subbase of order open sets. $\square$

The definition, below, will help us add a few more comments on the properties of nests and their relation to order theory.

Definition 3.9 *Let X be a set and $\mathcal{L} \subset \mathcal{P}(X)$. Then,*

1. $\mathcal{L}$ *is* closed under (finite, countable, etc.) unions*, if and only if for all (finite, countable, etc.)* $\mathcal{M} \subset \mathcal{L}$, $\bigcup \mathcal{M} \in \mathcal{L}$.

2. $\mathcal{L}$ *is* closed under (finite, countable, etc.) intersections*, if and only if for all (finite, countable, etc.)* $\mathcal{M} \subset \mathcal{L}$, $\bigcap \mathcal{M} \in \mathcal{L}$.

Suppose that $\mathcal{L}$ and $\mathcal{N}$ are two nests, on X, such that $\lhd_{\mathcal{L}} = \lhd_{\mathcal{N}}$. How do $\mathcal{L}$ and $\mathcal{N}$ relate?

Proposition 3.10 *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$. Then, the following are true:*

1. *If $\mathcal{L} \subset \mathcal{L}_\cup$ and each element of $\mathcal{L}_\cup$ is a union of elements from $\mathcal{L}$, in particular if $\mathcal{L}_\cup$ is the closure of $\mathcal{L}$ under arbitrary unions, then $\lhd_{\mathcal{L}} = \lhd_{\mathcal{L}_\cup}$.*

2. *If $\mathcal{L} \subset \mathcal{L}_\cap$ and each element of $\mathcal{L}_\cap$ is an intersection of elements from $\mathcal{L}$, in particular if $\mathcal{L}_\cap$ is the closure of $\mathcal{L}$ under arbitrary intersections, then $\lhd_{\mathcal{L}} = \lhd_{\mathcal{L}_\cap}$.*

3. *If $\mathcal{L}$ is an interlocking nest, $\mathcal{L} \subset \mathcal{L}_\cup$ and each element of $\mathcal{L}_\cup$ is a union of elements from $\mathcal{L}$, then $\mathcal{L}_\cup$ is an interlocking nest.*

4. *If $\mathcal{L}$ is a T_0-separating nest and $\mathcal{L}' = \{(-\infty, a) : a \in X\}$ is a nest of left-infinite $\lhd_{\mathcal{L}}$-intervals, then $\mathcal{L}'$ is an interlocking nest and $\lhd_{\mathcal{L}} = \lhd_{\mathcal{L}'}$.*

Proof. For 1. and 2., we note first that, if $L \in \mathcal{L}$, then L is in both $\mathcal{L}_\cup$ and $\mathcal{L}_\cap$, so that $x \lhd_{\mathcal{L}_\cup} y$ and $x \lhd_{\mathcal{L}_\cap} y$, whenever $x \lhd_{\mathcal{L}} y$. If $x \lhd_{\mathcal{L}_\cup} y$, then for some $\mathcal{M} \subset \mathcal{L}$, $x \in \bigcup \mathcal{M}$ and $y \notin \bigcup \mathcal{M}$, so that for any $M \in \mathcal{M}$, $x \in M$ and $y \notin M$ and so $x \lhd_{\mathcal{L}} y$. If $x \lhd_{\mathcal{L}_\cap} y$, then for some $\mathcal{M} \subset \mathcal{L}$, $x \in \bigcap \mathcal{M}$ and $y \notin \bigcap \mathcal{M}$, so that for some $M \in \mathcal{M}$, $x \in M$ and $y \notin M$ and so $x \lhd_{\mathcal{L}} y$. 3. is immediate from Definition 3.5 of interlocking. This is because, if every element in the nest is the union of strictly smaller elements of the nest, then the whole nest will be interlocking. 4. is straightforward, given the proof of Lemma 3.8; the elements of $\mathcal{L}'$ are the elements of the nest $\mathcal{L}_<$ of 3.8. $\square$

3.2 A Characterisation of LOTS via Nests: van Dalen and Wattel revisited

In their paper [24], van Dalen and Wattel do not mention anything about T_0-separating nests in their characterisations. In particular, they use the notation $S_{x,-y}$, in order to say that x belongs to an open set $S \in \mathcal{S}$ and y does not belong to S, where $\mathcal{S}$ is a nest. Using the tools that we introduced in Chapter 2, and using the notation that we have introduced so far, we are now in position to give a slightly different and more direct proof of van Dalen and Wattel's characterisation of GO-spaces and LOTS. These results appear in [7].

Theorem 3.11 (van Dalen & Wattel) *Let $(X, \mathcal{T})$ be a topological space. Then:*

1. *If $\mathcal{L}$ and $\mathcal{R}$ are two nests of open sets, whose union is T_1-separating, then every $\lhd_{\mathcal{L}}$-order open set is open, in X.*

2. *X is a GO space, if and only if there are two nests, $\mathcal{L}$ and $\mathcal{R}$, of open sets, whose union is T_1-separating and forms a subbase for $\mathcal{T}$.*

3. *X is a LOTS, if and only if there are two interlocking nests $\mathcal{L}$ and $\mathcal{R}$, of open sets, whose union is T_1-separating and forms a subbase for $\mathcal{T}$.*

Proof. 1. Clearly, for any $a \in X$, the $\triangleleft_{\mathcal{L}}$-interval $(-\infty, a) = \bigcup\{L \in \mathcal{L} : a \notin L\}$ and the $\triangleleft_{\mathcal{R}}$-interval $(a, \infty) = \bigcup\{R \in \mathcal{R} : a \notin R\}$. It follows immediately that, if the sets in $\mathcal{L}$ and $\mathcal{R}$ are open in X, then every order-open set is open in X, so that 1. holds.

For 3., if X is a LOTS, with linear order $<$, then the existence of such two nests follows by Lemma 3.8. Conversely, suppose that there are two interlocking nests $\mathcal{L}$ and $\mathcal{R}$, of open sets, whose union is T_1-separating, and forms a subbase for the topology on X. By Theorem 3.3, $\triangleleft_{\mathcal{L}}$ is a linear order on X and, by 1., every order open set is open. It remains to show that every open set is order-open. Since $\mathcal{L} \cup \mathcal{R}$ forms a subbase for the topology $\mathcal{T}$, on X, and since $\mathcal{L}$ and $\mathcal{R}$ are both nests, then every $U \in \mathcal{T}$ can be written as a union of sets of the form $L \cap R$, where $L \in \mathcal{L}$ and $R \in \mathcal{R}$. It suffices, then, to show that each $L \in \mathcal{L}$ and each $R \in \mathcal{R}$ is order-open. So, suppose that $L \in \mathcal{L}$. If L has no $\triangleleft_{\mathcal{L}}$-maximal element, then there is some $A \subset L$ that is cofinal in L, with respect to the order $\triangleleft_{\mathcal{L}}$. But then, $L = \bigcup_{a \in A}(-\infty, a)$, so that L is order open. On the other hand, if L does have a $\triangleleft_{\mathcal{L}}$-maximal element, m, then since $\mathcal{L}$ is interlocking, $X - L$ has a $\triangleleft_{\mathcal{L}}$-minimal element, m', and $L = (-\infty, m] = (-\infty, m')$ is also order open. That each $R \in \mathcal{R}$ is order open follows in exactly the same way.

To see 2., we should have in mind that X is a GO-space, if $X \subset Y$, for some LOTS Y. Since Y is a LOTS, it has two interlocking nests of open sets, $\mathcal{L}$ and $\mathcal{R}$, whose union forms a T_1-separating subbase for the topology on Y. Setting $\mathcal{L}' = \{L \cap X : L \in \mathcal{L}\}$ and $\mathcal{R}' = \{R \cap X : R \in \mathcal{R}\}$, we obtain two nests of sets open in X, whose union forms a T_1-separating subbase for the topology on X. For the converse, suppose that the space X has two nests $\mathcal{L}$ and $\mathcal{R}$, whose union forms a T_1-separating subbase for X. We will construct a LOTS Y, such that X is a subspace of Y. Let $\mathcal{L}^*$ be the set of all L, in $\mathcal{L}$, such that L has a $\triangleleft_{\mathcal{L}}$-maximal element, but $X - L$ has no $\triangleleft_{\mathcal{L}}$-minimal element. Let $\mathcal{R}^*$ be the set of all $R \in \mathcal{R}$, such that R has a $\triangleleft_{\mathcal{R}}$-maximal element (i.e. a $\triangleleft_{\mathcal{L}}$-minimal element), but $X - R$ has no $\triangleleft_{\mathcal{R}}$-minimal element (i.e. no $\triangleleft_{\mathcal{L}}-$maximal element). For each $L \in \mathcal{L}^*$, let x_L denote the $\triangleleft_{\mathcal{L}}$-maximal element of L and, for each $R \in \mathcal{R}^*$, let y_R denote the $\triangleleft_{\mathcal{L}}$-minimal element of R. For each $L \in \mathcal{L}^*$ and $R \in \mathcal{R}^*$ choose two distinct points x_L^+ and y_R^-, respectively, such that they both do not belong to X. Let $Y = X \cup \{x_L^+ : L \in \mathcal{L}^*\} \cup \{y_R^- : R \in \mathcal{R}^*\}$. Define $\pi : Y \to X$, by:

$$\pi(x) = \begin{cases} x & \text{if } x \in X, \\ x_L & \text{if } x = x_L^+, \\ y_R & \text{if } x = y_R^-. \end{cases}$$

Define also the linear order $<$, on Y, by declaring $x < y$, if and only if either $\pi(x) \neq \pi(y)$ and $\pi(x) \triangleleft_{\mathcal{L}} \pi(y)$ or $x = x_L$ and $y = x_L^+$ or $x = y_R^-$ and $y = y_R$. Clearly, $X \subset Y$, and the restriction of $<$, to X, is equal to $\triangleleft_{\mathcal{L}}$. It remains to show that the topology $\mathcal{T}$, on X, coincides with the subspace topology on X, that is inherited from the order topology on Y. As in the argument for 3., since $\mathcal{L} \cup \mathcal{R}$ is a subbase for $\mathcal{T}$, consisting of two nests, every U, in $\mathcal{T}$, can be written as a union of sets of the form $L \cap R$, where $L \in \mathcal{L}$ and $R \in \mathcal{R}$. It suffices, therefore, to show that every $L \in \mathcal{L}$ and $R \in \mathcal{R}$ can be written as the intersection between an order-open set and X. If $L \notin \mathcal{L}^*$, then $L = X \cap \pi^{-1}(L)$ and $\pi^{-1}(L)$ is order-open. On the other hand, if L is in $\mathcal{L}^*$, with $\mathcal{L}^*$-maximal element x_L, then $L = X \cap (-\infty, x_L^+)$. The argument for $R \in \mathcal{R}$ is the same. $\square$

As van Dalen and Wattel point out ([24], Corollary 2.9), if X is a compact space and if the two nests $\mathcal{L}$ and $\mathcal{R}$ form a T_1-separating subbase for X, then both $\mathcal{L}$ and $\mathcal{R}$ are interlocking, corresponding to the fact that a compact GO-space is a LOTS. In fact, more is true.

Theorem 3.12 *Let X be a space and let $\mathcal{L}$ and $\mathcal{R}$ be two nests of open sets, whose union forms a T_1-separating subbase for X. Suppose $\mathcal{L}$ has the property that for all $L \in \mathcal{L}$, there is a compact set C, such that $L \subset C$. Then, the following are true:*

1. *$\mathcal{L}$ is interlocking.*

2. *If $\mathcal{R}$ is not interlocking, then this is only because there is a singleton $R_0 \in \mathcal{R}$, such that $R_0 = \bigcap\{R \in \mathcal{R} : R_0 \subsetneq R\}$.*

Proof. 1. Suppose that $\mathcal{L}$ is not interlocking, for a contradiction. By Theorem 3.7, there is some $L \in \mathcal{L}$, such that L has a $\lhd_{\mathcal{L}}$-maximal element, say x_L, but $L = \bigcap\{M \in \mathcal{L} : L \subsetneq M\}$. Choose some $N \in \mathcal{L}$ and a compact set C, such that $L \subsetneq N \subset C$. Then, there is an infinite decreasing subset $\mathcal{M}$ ("decreasing": $\mathcal{M}$ is a nest, so it is a family of sets, which is linearly ordered via inclusion $\subset$ and $M_j \subset M_i, i < j$) of $\{M \in \mathcal{L} : L \subsetneq M \subset N \subset C\}$, such that $\bigcap \mathcal{M} = L$. Since $\mathcal{L} \cup \mathcal{R}$ is T_1-separating, for each M and M', in $\mathcal{M}$, such that $M \subsetneq M'$, there is $x_M \in M$, $y_M \in M'$ and $R \in \mathcal{R}$, such that $x_M \notin R \cap M$ and $y_M \in R \cap M'$. But then, there exists an infinite increasing subset $\mathcal{S}$ ("increasing": $\mathcal{M}$ is a nest, so it is a family of sets, which is linearly ordered via inclusion $\subset$ and $M_i \subset M_j, i < j$), of $\mathcal{R}$, that covers $X - L$. It follows that $\{L\} \cup \mathcal{S}$ is an open cover of C, with no finite subcover. This contradiction proves 1.

2. Suppose now that $\mathcal{R}$ is not interlocking, so that for some $R \in \mathcal{R}$, R has a $\lhd_{\mathcal{L}}$-minimal element x_R, but $R = \cap\{S \in \mathcal{R} : R \subsetneq S\}$. If R is not a singleton (and thus R is not the least element of $\mathcal{R}$), then there is some $y \in R$, such that $x_R \lhd_{\mathcal{L}} y$. Let $M \in \mathcal{L}$ be such that $x_R \in M$ and $y \notin M$. Let also C be a compact set, such that $M \subset C$. Then, as for 1., $\{R\} \cup \{L \in \mathcal{L} : L \cap R = \emptyset\}$ is a cover of C by sets open in X, that has no finite subcover. $\square$

Remark 3.13 *Let $X = [0, 1) \cup \{2\}$, $\mathcal{L}' = \{(-\infty, a) : a \in \mathbb{R}\}$ and $\mathcal{L} = \{L \cap X : L \in \mathcal{L}'\}$. We remark that the set X, together with its subspace topology inherited from the topology of* *the real line, is a non-compact GO-space, but the ordering* $\triangleleft_{\mathcal{L}}$ *cannot "spot" the difference between* X *and the space* $Y = [0, 2]$*, because the order cannot tell whether there is a gap between* $[0, 1)$ *and* 2 *or not. The property of interlocking (Lemma 3.6) comes to the rescue, as we shall see in the example that follows.*

Example 3.14 *Let* $<$ *be the usual order on* $\mathbb{R}$*. We recall that the* Sorgenfrey line *is* $\mathbb{R}$*, together with the topology generated by the base of half-open* $<$*-intervals* $\{(a, b] : a < b\}$*. Clearly, the two nests* $\mathcal{L} = \{(-\infty, a] : a \in \mathbb{R}\}$ *and* $\mathcal{R} = \{(a, \infty) : a \in \mathbb{R}\}$ *form a* T_1*-separating subbase for the Sorgenfrey line. Also,* $\mathcal{R}$ *is interlocking, but* $\mathcal{L}$ *is not interlocking. On the other hand,* $< = \triangleleft_{\mathcal{L}}$*.*

The Michael line *is formed from the real line, by defining the topology of the set of real numbers declaring each irrational number to be an isolated point on the real line. The nests:*

$$\mathcal{N} = \{(-\infty, q) : q \in \mathbb{Q}\} \cup \{(-\infty, r] : r \notin \mathbb{Q}\}$$

and

$$\mathcal{M} = \{(q, \infty) : q \in \mathbb{Q}\} \cup \{[r, \infty) : r \notin \mathbb{Q}\}$$

form a T_1*-separating subbase for the Michael line. We notice that the nests* $\mathcal{L}$*,* $\mathcal{N}$ *and* $\mathcal{Q} = \{(-\infty, q) : q \in \mathbb{Q}\}$ *are all distinct. Indeed,* $\mathcal{L}$ *and* $\mathcal{Q}$ *are disjoint, yet all three generate the usual order on* $\mathbb{R}$*.*

3.3 Connectedness and Orderability

In this section we give a characterisation of interlockingness via connectedness. This will give a condition for a connected space to be LOTS.

Definition 3.15 *A partial order* $<$*, on a set* X*, is said to be dense if, for all* x *and* y *in* X *for which* $x < y$*, there exists some* z *in* X*, such that* $x < z < y$*.*

So, given Definition 3.15, the next lemma follows naturally.

Lemma 3.16 *Let X be a set and let $\mathcal{L}$ be a nest on X. Then, the ordering $\lhd_{\mathcal{L}}$ is dense in X, if and only if for every $x, y \in X$, $x \neq y$, there exist $L, M \in \mathcal{L}$, $L \subsetneq M$, such that $x \in L$ and $y \notin M$ or $y \in L$ and $x \notin M$.*

Proposition 3.17 *Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests of open sets on X, such that $\mathcal{L} \cup \mathcal{R}$ creates a T_1-separating subbase for a topology on X. If X is connected, with respect to the topology that is induced by the union of $\mathcal{L}$ and $\mathcal{R}$, then $\lhd_{\mathcal{L}}$ is dense in X.*

Proof. Suppose $\lhd_{\mathcal{L}}$ is not dense. Then, there exist $x, y \in X$, such that $(x, y) = \emptyset$. So, there exists $L \in \mathcal{L}$, such that $x \in L$ and $y \notin L$ and there also exists $R \in \mathcal{R}$, such that $x \notin R$ and $y \in R$ and also $L \cap R = \emptyset$ and $L \cup R = X$. So, X is not connected.

In Theorem 3.7 we described interlocking nests, in terms of maximal and minimal elements. Here we use this result, in order to give a characterisation of connected spaces via nests.

Theorem 3.18 *Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests of open sets on X, such that $\mathcal{L} \cup \mathcal{R}$ creates a T_1-separating subbase for a topology on X. If X is connected, with respect to the topology with subbase $\mathcal{L} \cup \mathcal{R}$, then $\mathcal{L}$ and $\mathcal{R}$ are interlocking nests.*

Proof. If $\mathcal{L}$ is not interlocking then, according to Theorem 3.7, there exists $L \in \mathcal{L}$, such that $L = (-\infty, x]$, but $X - L$ has no minimal element. The set L is open, as a subbasic element for the topology that is generated by $\mathcal{L} \cup \mathcal{R}$. So, for every $z \in X - L$, there exists z', such that $x \lhd_{\mathcal{L}} z' \lhd_{\mathcal{L}} z$. But, there exists $R_z \in \mathcal{R}$, such that $z' \notin R_z$ and $z \in R_z$. So, $X - L = \bigcup_{z \notin L} R_z$, i.e. $R_z \cap L = \emptyset$. Thus, $X - L$ is open and L is open, hence X is not connected. In a similar way, $\mathcal{R}$ is interlocking, too. $\square$

Theorem 3.18 permits us now to view LOTS, in the light of connectedness.

Corollary 3.19 *Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests of open sets on X, such that $\mathcal{L} \cup \mathcal{R}$ creates a T_1-separating subbase for a topology on X. If X is connected with respect to the topology with subbase $\mathcal{L} \cup \mathcal{R}$, then X is a LOTS.*

Proof. The proof follows immediately from the statements of Theorem 3.11 and Theorem 3.18. □

3.4 Some Order Theoretic Implications stemming from the Interval Topology

In this section we use properties of nests in order to examine order-theoretic properties of linearly ordered sets via the interval topology.

3.4.1 A Close Up to the Interval Topology via $\trianglelefteq_{\mathcal{L}}$, when $\mathcal{L}$ is T_0-separating.

Definition 3.20 *Let $(X, <)$ be a partially ordered set and $A \subset X$. We define $\uparrow A \subset X$, to be the set:*

$$\uparrow A = \{x : x \in X \textit{ and there exists } y \in A, \textit{ such that } y \leq x\}.$$

We also define $\downarrow A \subset X$, to be the set:

$$\downarrow A = \{x : x \in X \textit{ and there exists } y \in A, \textit{ such that } x \leq y\}.$$

We remind that the *upper topology* $\mathcal{T}_U$ is generated by the subbase $\mathcal{S} = \{X - \downarrow x : x \in X\}$ and the *lower topology* $\mathcal{T}_l$ is generated by the subbase $\mathcal{S} = \{X - \uparrow x : x \in X\}$. The *interval topology* $\mathcal{T}_{in}$ is defined as $\mathcal{T}_{in} = \mathcal{T}_U \vee \mathcal{T}_l$, where $\vee$ stands for supremum.

We will now construct the interval topology in terms of nests, and use our observations in the next subsection, in order to examine more closely properties of the line via the interval topology. Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests on X, such that $\mathcal{L} \cup \mathcal{R}$ T_1-separates X. According to Theorem 3.4 each of $\mathcal{L}$ and $\mathcal{R}$ is T_0-separating, so $\lhd_{\mathcal{L}} = \rhd_{\mathcal{R}}$ is a linear order.

Construction 3.21 *We consider the lower topology on X, with respect to $\unlhd_{\mathcal{L}}$. We denote this topology by $\mathcal{T}_l^{\unlhd_{\mathcal{L}}}$. Then, for each $y \in X$, $\uparrow y = \{x \in X : y \unlhd_{\mathcal{L}} x\}$. So, $X - \uparrow y = \{x \in X : x \lhd_{\mathcal{L}} y\}$. This happens, because $\mathcal{L}$ is a T_0-separating nest. Thus, a subbase for the lower topology on X, which is generated by $\unlhd_{\mathcal{L}}$, will be of the form:*

$$\mathcal{S}(\mathcal{T}_l^{\unlhd_{\mathcal{L}}}) = \{X - \uparrow y : y \in X\}.$$

We now consider the upper topology on X, with respect to $\unlhd_{\mathcal{L}}$. We denote this topology by $\mathcal{T}_U^{\unlhd_{\mathcal{L}}}$. Then, for each $y \in X$, $\downarrow y = \{x \in X : x \unlhd_{\mathcal{L}} y\}$. So, $X - \downarrow y = \{x \in X : y \lhd_{\mathcal{L}} x\}$. Thus, a subbase for the upper topology on X, that is generated by $\unlhd_{\mathcal{L}}$, is of the form:

$$\mathcal{S}(\mathcal{T}_U^{\unlhd_{\mathcal{L}}}) = \{X - \downarrow y : y \in X\}$$

We construct the interval topology which is generated by $\mathcal{L}$, denoted by $\mathcal{T}_{in}^{\mathcal{L}}$, as follows:

$$\mathcal{T}_{in}^{\mathcal{L}} = \mathcal{T}_U^{\unlhd_{\mathcal{L}}} \vee \mathcal{T}_l^{\unlhd_{\mathcal{L}}}.$$

A subbase for this topology will be:

$$\mathcal{S}_{in} = \mathcal{S}(\mathcal{T}_U^{\unlhd_{\mathcal{L}}}) \cup \mathcal{S}(\mathcal{T}_l^{\unlhd_{\mathcal{L}}}).$$

Remark 3.22 *(We remind that $\mathcal{L}$, throughout this subsection, T_0-separates X.)*

1. *In our construction of the interval topology we used a reflexive order $\trianglelefteq_{\mathcal{L}}$, rather than a non-reflexive one $\triangleleft_{\mathcal{L}}$. This is because the non-reflexive $\triangleleft_{\mathcal{L}}$ will generate an interval topology equal to the discrete topology on X (a trivial case to study). Indeed, $\downarrow a = \{x \in X : x \triangleleft_{\mathcal{L}} a\}$ and so $X - \downarrow a = \{x \in X : a \trianglelefteq_{\mathcal{L}} x\} = (-\infty, a]$. In a similar fashion, $x - \uparrow a = [a, \infty)$ and so $(-\infty, a] \cap [a, \infty) = \{a\}$.*

2. *The sets in $\mathcal{T}_U^{\trianglelefteq_{\mathcal{L}}}$ form a nest and the sets in $\mathcal{T}_l^{\trianglelefteq_{\mathcal{L}}}$ also form a nest. It will be particularly useful to remember this, whenever we compare $\mathcal{T}_{in}^{\mathcal{L}}$ with $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$, in the next subsection. It will be also useful to bear in mind that in the set of real numbers, equipped with its usual topology, $\mathcal{T}_{in}^{\mathcal{L}} = \mathcal{T}_{\mathcal{L}\cup\mathcal{R}} = \mathcal{T}_{\triangleleft_{\mathcal{L}}}$, where $\mathcal{L} = \{(-\infty, a) : a \in \mathbb{R}\}$ and $\triangleleft_{\mathcal{L}} = \triangleright_{\mathcal{R}}$.*

3.4.2 Order Theoretic Properties of the Line via the Interval Topology.

Consider the set of real numbers $\mathbb{R}$, equipped with its usual topology. Let $\mathcal{L} = \{(-\infty, a) : a \in \mathbb{R}\}$. We remark that for each $(-\infty, a) \in \mathcal{L}$, $\sup L = a \notin L$. We also remark that for each $k \in \mathbb{R}$, there exists $L = (-\infty, k) \in \mathcal{L}$, such that $\sup L = k$. We will now generalise this remark to arbitrary sets. In particular, we will use the following three conditions, namely (C1), (C2), (C3), in order to investigate the relationship between the topologies $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}}$ and $\mathcal{T}_{in}^{\mathcal{L}}$; this relationship will be a measure of linearity, that is, it will show how close -or not- is a space from a LOTS, regarding its structure. From now on, sup will be used for abbreviating the term *supremum* and inf will abbreviate the term *infimum*.

Let $\mathcal{L}$ be a nest on a set X. We introduce the following three conditions:

(C1) For each $L \in \mathcal{L}$, there exists $\sup L$ with respect $\trianglelefteq_{\mathcal{L}}$.

(C2) For each $L \in \mathcal{L}$, there exists $\sup L$ with respect to $\trianglelefteq_{\mathcal{L}}$, such that $\sup L \in X - L$.

(C3) For each x, there exists $L \in \mathcal{L}$, such that there exists $\sup L = x \in X - L$ and also property (C2) holds.

We deduce the following relations between (C1), (C2) and (C3).

Proposition 3.23

1. *(C3) implies (C2).*
2. *(C2) implies (C1).*
3. *(C1) does not always imply (C2).*
4. *(C2) does not always imply (C3).*
5. *(C3) implies that $\mathcal{L}$ is T_0-separating.*
6. *$\mathcal{L}$ T_0-separating implies neither (C1) nor (C2) nor (C3).*
7. *Neither (C1) nor (C2) imply that $\mathcal{L}$ is T_0-separating.*

Proof. The statement that (C3) implies (C2) follows immediately from the definition of (C3). Similarly, (C2) implies (C1) by the definition of (C2). Example 3.24 shows that (C1) does not always imply (C2) or T_0-separation. Example 3.25 shows that (C2) does not always imply (C3) or T_0-separation. Proposition 3.30 shows that (C3) implies T_0-separation. Examples 3.29, 3.28 and 3.27 show that the T_0-separation of $\mathcal{L}$ does not necessarily imply property (C1) or (C2) or (C3). $\square$

Example 3.24 *Let $X = (0, 1)$ and consider the nest $\mathcal{L} = \{(0, a] : a \in \mathbb{R}, \frac{1}{2} \leq a < 1\}$, on X. We remark that condition (C1) is satisfied, but (C2) is not satisfied. This is because for each $L \in \mathcal{L}$, $\sup L = a \in L$. This counterexample shows that (C1) does not always imply (C2). We also see that $\mathcal{L}$ is not T_0-separating, because there does not exist $L \in \mathcal{L}$ that T_0-separates, say, $\frac{1}{4}$ and $\frac{1}{8}$. This shows that condition (C1) does not always imply T_0-separation.*

Example 3.25 *Let $X = (0, 1)$ and consider the nest $\mathcal{L} = \{(0, a) : a \in \mathbb{R}, \frac{1}{2} \leq a < 1\}$, on X. We remark that condition (C2) is satisfied, but condition (C3) is not satisfied. This is because for each $L \in \mathcal{L}$, $\sup L = a \notin L$; this shows that (C2) is satisfied. But we also see that there does not exist $L \in \mathcal{L}$, such that $\sup L = \frac{1}{4} \in X - L$. This counterexample shows that (C2) does not always imply (C3) and also (C2) does not always imply that $\mathcal{L}$ is T_0-separating. Indeed, there does not exist $L \in \mathcal{L}$ that T_0-separates $\frac{1}{4}$ and $\frac{1}{8}$.*

Remark 3.26 *The results in both Examples 3.25 and 3.24 permit us to make some conclusions on the connection between T_0-separating nests and linear orders. It follows from the definition of nest and T_0-separation that a nest is T_0-separating, if and only if $\trianglelefteq_{\mathcal{L}}$ is a linear order. In addition, in Lemma 9 from [7], we get that if $<$ is a linear order on a set X, and $\mathcal{L}_< = \{(-\infty, a) : a \in X\}$, then $\mathcal{L}_<$ is T_0-separating. Why isn't the nest $\mathcal{L}$, in both of the above examples 3.25 and 3.24, T_0-separating? The answer lies on the fact that in the mentioned lemma from [7], the elements of the nest $\mathcal{L}_<$ satisfy an 1-1 correspondence with the elements of the set X, something that does not happen in our examples. So, the set X, in Examples 3.25 and 3.24 is not linearly ordered via $\trianglelefteq_{\mathcal{L}}$.*

Example 3.27 *Let $X = \{a, b\}$ and consider the nest $\mathcal{L} = \{\{a\}\}$, on X. We remark that $\mathcal{L}$ is T_0-separating. Indeed, since $a \neq b$, there exists $L = \{a\} \in \mathcal{L}$, such that $a \in \{a\}$ and $b \notin \{a\}$. We remark that (C3) is not satisfied though. Indeed, for $b \in X$, there does not exist $L \in \mathcal{L}$, such that $\sup L = b$. We observe that $L = \{a\} \in \mathcal{L}$ and that $\sup L = a$.*

Example 3.28 *Consider $X = \mathbb{R}$ and the nest $\mathcal{L} = \{(-\infty, a] : a \in \mathcal{R}\}$, on $\mathbb{R}$. One can easily see that $\mathcal{L}$ T_0-separates $\mathbb{R}$. But, for each $L \in \mathcal{L}$, we have that $\sup(-\infty, a] = a \in L$. So, property (C2) is not satisfied. With this example we see that the T_0-separation property of $\mathcal{L}$ does not necessarily imply property $(C2)$.*

Example 3.29 *Let $X = \mathbb{Q}$. For each $r \in \mathbb{R}$, let $L_r = (-\infty, r) \cap X$ and let $\mathcal{L} = \{L_r : r \in \mathbb{R}\}$. Certainly $\mathcal{L}$ is T_0-separating and $\mathcal{L}$ generates the usual order on $\mathbb{Q}$. But $L_{\sqrt{2}}$ does not have a supremum in X.*

We will now prove that property (C3) implies the T_0-separation of $\mathcal{L}$.

Proposition 3.30 *Let X be a set and let $\mathcal{L}$ be a nest on X that satisfies property (C3). Then, $\mathcal{L}$ T_0-separates X.*

Proof. Let $x \neq y \in X$. By (C3), there exists $L_x \in \mathcal{L}$, such that $\sup L_x = x$ and there also exists $L_y \in \mathcal{L}$, such that $\sup L_y = y$. Since $\mathcal{L}$ is a nest on X, we have that either $L_x \subset L_y$ or $L_y \subset L_x$. If $L_x \subset L_y$, then $\sup L_x \trianglelefteq_{\mathcal{L}} \sup L_y$, which implies that $x \triangleleft_{\mathcal{L}} y$. If $L_y \subset L_x$, we have that $\sup L_y \trianglelefteq_{\mathcal{L}} \sup L_x$, which implies that $y \triangleleft_{\mathcal{L}} x$. So, either $x \triangleleft_{\mathcal{L}} y$ or $y \triangleleft_{\mathcal{L}} x$, proving that $\mathcal{L}$ T_0-separates X. □

Lemma 3.31 *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$ be a nest.*

1. *If condition (C1) is satisfied and $\sup L = k$, then $L \supset X - \uparrow k$.*
2. *If condition (C2) is satisfied and $\sup L = k$, then $L \subset X - \uparrow k$.*

Proof. 1. Let $L \in \mathcal{L}$ and let $k = \sup L \in X$. Then, for each $x \in L$, $x \trianglelefteq_{\mathcal{L}} k$. Let $y \in X - L$. Since $x \in L$ and $y \notin L$, we have that $x \triangleleft_{\mathcal{L}} y$, for each x. So, $k \trianglelefteq_{\mathcal{L}} y$, and so $y \in \uparrow k$. Thus, for each $y \in X - L$, we have that $y \in \uparrow k$. The latter gives that $X - L \subset \uparrow k$, which implies that $L \supset X - \uparrow k$.

2. For each $x \in L$, we have $x \triangleleft_{\mathcal{L}} k$, so $k \ntrianglelefteq x$ [1], which implies that $x \in X - \uparrow k$. Thus, $L \subset X - \uparrow k$. □

[1]Indeed, if $k = x$ we get a contradiction. If $x \triangleleft_{\mathcal{L}} k$, then there exists $L_1 \in \mathcal{L}$, such that $x \in L_1$ and $k \notin L_1$. If $k \triangleleft_{\mathcal{L}} x$, then there exists $L_2 \in \mathcal{L}$, such that $k \in L_2$ and $x \notin L_2$. But $\mathcal{L}$ is a nest. If $L_1 \subset L_2$, then $x \notin L_1$ and $x \in L_1$, a contradiction. If $L_2 \subset L_1$ we get a contradiction in a similar way.

From now on, $\mathcal{T}_{\mathcal{L}}$ will denote the topology generated by the nest $\mathcal{L}$, on X, and $\mathcal{T}_l$ the lower topology on X.

Proposition 3.32 *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$ be a nest. If condition (C2) is satisfied, then:*

1. *$L = X - \uparrow k$, where $k = \sup L$, with respect to $\triangleleft_{\mathcal{L}}$, for each $L \in \mathcal{L}$.*

2. *$\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_l$.*

Proof. 1. follows by Lemma 3.31.

2. As we have seen in section 3, a subbase for $\mathcal{T}_l$ is of the form $\mathcal{S} = \{X - \uparrow k : k \in X\}$. Let $L \in \mathcal{L}$. Part 1. gives that $L = X - \uparrow k$, so $L \in \mathcal{T}_l$ and the result follows. $\square$

Theorem 3.33 *Let X be a set and let $\mathcal{L} \subset \mathcal{P}(X)$ be a nest on X, such that condition (C3) is satisfied. Then, $\mathcal{T}_{\mathcal{L}} = \mathcal{T}_l$.*

Proof. Proposition 3.32 gives that $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_l$. We now consider a subbasic open set of $\mathcal{T}_l$ of the form $X - \uparrow x$. Then, there exists $L \in \mathcal{L}$, such that $\sup L = x$. But, according to Proposition 3.32, $L = X - \uparrow x$. So, $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_l$ and the statement of the theorem follows. $\square$

Remark 3.34 *Let $\mathcal{L}$ be a nest on a set X. Let $\mathcal{R}$ be another nest on X, such that there exists a mapping from $\mathcal{L}$ to $\mathcal{R}$, so that $x \triangleleft_{\mathcal{L}} y$, if and only if $y \triangleleft_{\mathcal{R}} x$. So, $x \triangleleft_{\mathcal{L}} y$, if and only if there exists $L \in \mathcal{L}$, such that $x \in L$ and $y \notin L$, if and only if there exists $R \in \mathcal{R}$, such that $y \in R$ and $x \notin R$.*

Note that Theorem 3.4, from [7] requests $\mathcal{L} \cup \mathcal{R}$ to form a T_1-separating subbase for X; here we do not demand this, so neither $\mathcal{L}$ nor $\mathcal{R}$ will necessarily T_0-separate X. We keep only the dual order-theoretic properties of these two nests, but we do not necessarily keep the property that restricts them on a line. So, we are now able to rewrite for $\mathcal{R}$, in a dual way, the properties that hold for $\mathcal{L}$.

Definition 3.35 *Let X be a set and let $\mathcal{L}$ and $\mathcal{R}$ be two nests on X, that satisfy the properties of Remark 3.34. We call such nests* dual nests. *$\mathcal{L}$ will be called* dual to *$\mathcal{R}$ and $\mathcal{R}$ dual to $\mathcal{L}$.*

Let X be a set and let $\mathcal{R}$ be dual to the nest $\mathcal{L}$, where $\mathcal{L}$ satisfies properties (C1),(C2),(C3). In a similar fashion, we define the following properties for $\mathcal{R}$:

(C1)* For each $R \in \mathcal{R}$, there exists $\sup R$ with respect to $\trianglerighteq_{\mathcal{R}}$.
(Equivalently, for each $R \in \mathcal{R}$, there exists $\inf R$ with respect to $\trianglelefteq_{\mathcal{L}}$.)

(C2)* For each $R \in \mathcal{R}$, there exists $\sup R$ with respect to $\trianglerighteq_{\mathcal{R}}$, such that $\sup R \in X - R$.
(Equivalently, for each $R \in \mathcal{R}$ there exists $\inf R$ with respect to $\trianglelefteq_{\mathcal{L}}$, such that $\inf R \in X - R$).

(C3)* For each $x \in X$, there exists $R \in \mathcal{R}$, such that there exists $\sup R \in X - R$ with respect to $\trianglerighteq_{\mathcal{R}}$ and also property (C2)* holds.
(Equivalently, for each $x \in X$, there exists $R \in \mathcal{R}$, such that there exists $\inf R \in X - R$, with respect to $\trianglelefteq_{\mathcal{L}}$ and also property (C2)* holds).

One easily observes that Proposition 3.23 holds, too, if we substitute (C1)*, (C2)*, (C3)* in the place of (C1), (C2),(C3), respectively.

Proposition 3.32 can be also stated with respect to $\mathcal{R}$ in a dual way.

Proposition 3.36 *Let X be a set and let $\mathcal{R} \subset \mathcal{P}(X)$ be a nest. If condition (C2)* is satisfied, then:*

1. *$R = X - \uparrow k$, where $k = \sup R$ with respect to $\trianglerighteq_{\mathcal{R}}$ for each $R \in \mathcal{R}$ (or, equivalently, $R = X - \downarrow k$, where $k = \inf R$ with respect to $\trianglelefteq_{\mathcal{L}}$).*

2. *$\mathcal{T}_{\mathcal{R}} \subset \mathcal{T}_U$.*

In a similar way, we can restate Theorem 3.33, with respect to $\mathcal{R}$.

Theorem 3.37 *Let X be a set and let $\mathcal{R} \subset \mathcal{P}(X)$ be a nest on X, such that condition (C3)* is satisfied. Then $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_U$.*

We can now sum up Theorems 3.33 and 3.37, in the following theorem.

Theorem 3.38 *Let X be a set and let $\mathcal{L}$ and $\mathcal{R}$ be two dual nests on X.*

1. *If $\mathcal{L}$ satisfies (C2) and if $\mathcal{R}$ satisfies (C2)*, then $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}} \subset \mathcal{T}_{in}^{\mathcal{L}}$.*

2. *If $\mathcal{L}$ satisfies (C3) and if $\mathcal{R}$ satisfies (C3)*, then $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}} = \mathcal{T}_{in}^{\mathcal{L}}$.*

As we can see in the two examples that follow, the conditions of statements 1. and 2. from Theorem 3.38 are sufficient but not necessary.

Example 3.39 *Let $X = \{x_1, x_2\}$ and let $\mathcal{L} = \{\{x_1\}\}$. Then, $\mathcal{T}_{\mathcal{L}} = \{\{x_1\}, \{x_1, x_2\}, \emptyset\}$ is the topology on X which is generated by $\mathcal{L}$. We observe that $x_1 \lhd_{\mathcal{L}} x_2$. Then, $\uparrow x_1 = \{x_1, x_2\}$, $X- \uparrow x_1 = \emptyset$, $\uparrow x_2 = \{x_2\}$ and $X- \uparrow x_2 = \{x_1\}$. So, the lower topology $\mathcal{T}_l = \{\emptyset, \{x_1\}, \{x_1, x_2\}\} = \mathcal{T}_{\mathcal{L}}$. Now, we define $\mathcal{R} = \{\{x_2\}\}$ and $x_2 \rhd_{\mathcal{R}} x_1$, if and only if there exists $R \in \mathcal{R}$, such that $x_2 \in R$ and $x_1 \notin R$. So, $x_1 \lhd_{\mathcal{L}} x_2$ if and only if $x_2 \rhd_{\mathcal{R}} x_1$. Then, $\mathcal{T}_{\mathcal{R}} = \{\{x_2\}, \{x_1, x_2\}\}$ is the topology on X which is induced by $\mathcal{R}$. Also, $\downarrow x_1 = \{x_1\}$, $\downarrow x_2 = \{x_1, x_2\}$, $X- \downarrow x_1 = \{x_2\}$ and $X- \downarrow x_2 = \emptyset$. So, the upper topology $\mathcal{T}_U = \{\emptyset, \{x_2\}, \{x_1, x_2\}\} = \mathcal{T}_{\mathcal{R}}$.*

From the above, we conclude that $\mathcal{T}_{\mathcal{L}\cup\mathcal{R}} = \mathcal{T}_{in}^{\mathcal{L}}$ is equal to the discrete topology, although property (C3) is not satisfied. This is because x_2 is not the supremum of any element of $\mathcal{L}$.

Example 3.40 *Let $X = \{x_1, x_2, x_3, x_4\}$ and let $\mathcal{L} = \{\{x_1, x_2\}, \{x_1, x_2, x_3, x_4\}\}$. Then, one can easily see that $x_2 \lhd_{\mathcal{L}} x_3$, $x_2 \lhd_{\mathcal{L}} x_4$, $x_1 \lhd_{\mathcal{L}} x_3$ and $x_1 \lhd_{\mathcal{L}} x_4$. Also, $\uparrow x_1 = \{y \in X : x_1 \trianglelefteq_{\mathcal{L}} y\} = \{x_1, x_3, x_4\}$ and $X- \uparrow x_1 = \{x_2\}$. Similarly, $\uparrow x_2 = \{x_2, x_3, x_4\}$ and $X- \uparrow x_2 = \{x_1\}$; $\uparrow x_3 = \{x_3\}$ and $X- \uparrow x_3 = \{x_1, x_2, x_4\}$; $\uparrow x_4 = \{x_4\}$ and $X- \uparrow x_4 = \{x_1, x_2, x_3\}$.*

The lower topology now takes the form $\mathcal{T}_l = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_3, x_4\}\}$ *and* $\mathcal{T}_{\mathcal{L}} = \{\emptyset, \{x_1, x_2\}, \{x_1, x_2, x_3, x_4\}\}$*. So,* $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_l$*, but* $\mathcal{L}$ *is not* T_0*-separating, because* $x_3 \neq x_4$ *and there is no* $L \in \mathcal{L}$ *that* T_0*-separates* x_3 *and* x_4*. Also,* $\mathcal{L}$ *does not satisfy property (C2), because* $\sup\{x_1, x_2\}$ *does not exist.*

Now, we consider $\mathcal{R} = \{\{x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}$*, and we observe that* $x_3 \triangleright_{\mathcal{R}} x_1, x_3 \triangleright_{\mathcal{R}} x_2, x_4 \triangleright_{\mathcal{R}} x_2$ *and* $x_4 \triangleright_{\mathcal{R}} x_3$*. So, there exists a mapping between the nests* $\mathcal{L}$ *and* $\mathcal{R}$*, and their duality can be seen from the fact that* $x_3 \triangleright_{\mathcal{R}} x_1$ *iff* $x_1 \triangleleft_{\mathcal{L}} x_3$*,* $x_3 \triangleright_{\mathcal{R}} x_2$ *iff* $x_2 \triangleleft_{\mathcal{L}} x_3$*,* $x_4 \triangleright_{\mathcal{R}} x_2$ *iff* $x_2 \triangleleft_{\mathcal{L}} x_4$ *and* $x_4 \triangleright_{\mathcal{R}} x_1$ *iff* $x_1 \triangleleft_{\mathcal{L}} x_4$*. It can be easily deduced that* $\mathcal{T}_{\mathcal{R}} = \{\emptyset, \{x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}$ *and that the upper topology is* $\mathcal{T}_U = \{\emptyset, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, \{x_3\}, \{x_4\}, \{x_3, x_4\}, \{x_1, x_2, x_3, x_4\}\}$*. Also,* $\mathcal{R}$ *is not* T_0*-separating, neither satisfies property (C2)* and we deduce that* $\mathcal{T}_{\mathcal{R}} \subset \mathcal{T}_U$*. Last, but not least, we see that* $\mathcal{T}_{in}^{\mathcal{L}}$ *is the discrete topology, thus* $\mathcal{T}_{\mathcal{L} \cup \mathcal{R}} \subset \mathcal{T}_{in}^{\mathcal{L}}$*.*

In Remark 3.22 we stated that a non-reflexive order that is induced by a nest $\mathcal{L}$ makes $\mathcal{T}_{in}^{\mathcal{L}}$ equal to the discrete topology, so it will automatically be finer than $\mathcal{T}_{\mathcal{L} \cup \mathcal{R}}$. If the order is reflexive, then Theorem 3.38 shows that there is a case where $\mathcal{T}_{in}^{\mathcal{L}}$ is equal to $\mathcal{T}_{\mathcal{L} \cup \mathcal{R}}$, and this is when properties (C3) and (C3)* are both satisfied. But (C3) (resp. (C3)*) implies that $\mathcal{L}$ (resp. $\mathcal{R}$) is T_0-separating, while in Example 3.27 (and Proposition 3.23) we see that $\mathcal{L}$ can be T_0-separating, without (C3) being satisfied. So, the two topologies coincide in certain type of spaces that are T_0-separating under properties (C3) and (C3)*.

The real line, with its natural topology that is generated by the nests $\mathcal{L} = \{(-\infty, a) : a \in \mathbb{R}\}$ and $\mathcal{R} = \{(a, \infty) : a \in \mathbb{R}\}$ is a specific example of a space of the type that is described in Theorem 3.38 2. Question: are there other LOTS, apart from the real line with its natural order, such that 2. from Theorem 3.38 is satisfied? The answer is positive. Consider, for example, sum of copies of the real line. Other spaces admitting such nests are connected orderable spaces with no minimal and maximal elements (for instance the long line).

Furthermore, we remark that if property (C2) alone is satisfied, then for each $L \in \mathcal{L}$ we have that $\sup L \in X - L$, so that $\sup L \notin L$. So, for each $L \in \mathcal{L}$ there is no $\trianglelefteq_{\mathcal{L}}$-maximal element in L, because for each $L \in \mathcal{L}$ there exists $k \in L$, $x \trianglelefteq_{\mathcal{L}} k$, for each $x \in L$, so that $k = \sup L$. In a similar fashion, we can obtain a dual property for the dual nest $\mathcal{R}$, with the ordering $\trianglerighteq_{\mathcal{R}}$. We will use this remark in order to find conditions which imply the orderability problem that was introduced by J. van Dalen and E. Wattel, in [24].

Theorem 3.41 *Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests on X, such that $\triangleleft_{\mathcal{L}} = \triangleright_{\mathcal{R}}$. Let also properties (C3) and (C3)* be satisfied. Then, X is a LOTS.*

Proof. In section 1. we stated the characterization of LOTS that was introduced by van Dalen and Wattel. We observe that property (C3) (similarly (C3)*) implies T_0-separation and interlocking (see Theorem **??**), so that the conditions of van Dalen and Wattel follow immediately and so X is a LOTS. $\square$

Property (C3) (resp. (C3)*) implies naturally T_0-separation and interlocking. Property (C2) (resp. (C2)*) implies interlocking, if we add T_0-separation. So, we can restate Theorem 3.41 as follows:

Theorem 3.42 *Let X be a set and let $\mathcal{L}, \mathcal{R}$ be two nests on X, such that $\triangleleft_{\mathcal{L}} = \triangleright_{\mathcal{R}}$ and each of $\mathcal{L}$ and $\mathcal{R}$ T_0-separates X, respectively. Let also properties (C2) and (C2)* be satisfied. Then, X is a LOTS.*

Question: what is the difference between LOTS that are implied by Theorem 3.41 from LOTS being implied by Theorem 3.42? The answer is that the two theorems claim the same result. Namely, for a nest $\mathcal{L}$ of subsets of X, (C3) follows by (C2), provided that $\trianglelefteq_{\mathcal{L}}$ is a linear order on X. Indeed, suppose $\trianglelefteq_{\mathcal{L}}$ is a linear order on X and $\mathcal{L}$ satisfies (C2). Then, the nest $\mathcal{H} = \{\{x \in X : x \triangleleft_{\mathcal{L}} y\} : y \in X\}$ satisfies (C3) and $\triangleleft_{\mathcal{H}} = \triangleleft_{\mathcal{L}}$. To see this, take a point $y \in X$. If y is the $\trianglelefteq_{\mathcal{L}}$-first element of X, then it is the $\trianglelefteq_{\mathcal{L}}$-supremum of the empty set. Suppose there exists $x \in X$, with $x \lhd_{\mathcal{L}} y$ and let $H = \{x \in X : x \lhd_{\mathcal{L}} y\} \in \mathcal{H}$. If y is not the $\unlhd_{\mathcal{L}}$-supremum of H, then H has a $\unlhd_{\mathcal{L}}$-maximal element, namely z. Since $z \lhd_{\mathcal{L}} y$, there exists $L_z \in \mathcal{L}$, such that $z \in L_z$ and $y \notin L_z$. If $x \in H$ and $x \neq z$, it follows that $x \lhd_{\mathcal{L}} z$ and, by the same reason, $x \in L_x$, for some $L_x \in \mathcal{L}$ for which $z \notin L_x$. Since $\mathcal{L}$ is a nest, $x \in L_x \subset L_z$. Thus, $H \subset L_z$ and, in fact, $H = L_z$, because $y \notin L_z$. By (C2), the $\lhd_{\mathcal{L}}$-supremum of $H = L_z$ does not belong to H, a contradiction, because $z \in H$ and $x \lhd_{\mathcal{L}} z$, for every $x \in H$.

The following example shows that both properties (C2) and (C2)* do not necessarily imply T_0-separation. So, Theorem 3.42 without the T_0-separation property of $\mathcal{L}$ and $\mathcal{R}$ generates spaces that are not necessarily linearly ordered, but carry analogous order theoretic properties to linearly ordered sets.

Example 3.43 *Consider the set of real numbers* $\mathbb{R}$ *and the nests* $\mathcal{L} = \{(-\infty, n) : n \in \mathbb{N}\}$ *and* $\mathcal{R} = \{(n, \infty) : n \in \mathbb{N}\}$ *on* $\mathbb{R}$. *Then,* $\mathcal{L}$ *and* $\mathcal{R}$ *satisfy conditions (C2) and (C2)*, respectively. Indeed, for each* $L = (-\infty, n) \in \mathcal{L}$, $\sup L = n \notin L$ *and for each* $R \in \mathcal{R}$, $\inf(n, \infty) = n \notin R$. *We also remark, from the definition of* T_0*-separation, that neither* $\mathcal{L}$ *nor* $\mathcal{R}$ *is* T_0*-separating.*

Chapter 4

FUN-Spaces

Motivated by van Dalen and Wattel's topological characterisation of LOTS, we used two nests (Chapter 3) in order to characterise LOTS in simpler, more order-theoretic terms. In this chapter, we observe that a subbase, which is given by a union of more than two nests, generates spaces that are not of high order-theoretic interest. In particular, we give an example of a countable space, X, which is generated by the union of three nests $\mathcal{L}, \mathcal{R}, \mathcal{P}$, each T_0-separating X, such that their union T_1-separates X, but does not T_2-separate X.

4.1 Neight and Dimension

Looking at the example of particular vector spaces X, e.g. $X = \mathbb{R}^2$, we see that the natural product topology X^2 can be given by four nests which intersect in basic-open squares, but it can be also given by three nests, intersecting in basic-open triangles. Is there a pattern, for more abstract spaces, which gives the minimum number of nests, that generate the natural product topology? This question has been answered by Will Brian, from the University of Oxford, who is working independently in these ideas, and he is producing results of high interest. In particular, W.B. is currently examining properties of the *neight* of a topological space X, that is, the *nested weight* of X.

Definition 4.1 *The* neight *(nested weight), of a topological space X, is the smallest number of nests, on X, whose union provides a subbase for the topology on X.*

Obviously, for every open set A, of a topological space X, the set $\{A\}$ is a nest, so one can easily conclude that for every topological space there exists a subbase which can be written as a union of nests. So, for every topological space there is a least cardinal, κ, so that the topology on X is generated by a subbase which can be written as the union of κ nests on X. This cardinal is called the *neight* of X, denoted by $\mathfrak{n}(X)$. The neight cardinal function was first introduced by A.M. Yurovetskiĭ, in [29].

Following Theorem 3.11, we see that if X is LOTS, then $\mathfrak{n} = 2$. The fact that GO-spaces are subspaces of LOTS permits us to say that any GO-space has at most dimension one. A large part of W.B.'s work, on nests, is examining the extent to which subbases consisting of nests generalise the concept of GO-space and the extent to which the $\mathfrak{n}$-function, or rather the function $X \mapsto \mathfrak{n}(X) - 1$, acts like a measure of dimension. In both cases, the focus is on spaces of finite neight, which are called *FUN-spaces* (Finite Union of Nests - Spaces).

Definition 4.2 *If the neight of a topological space X is finite, then the space is called a* FUN-space.

We will now recall the definition of the *small inductive dimension*, in order to give a summary of some basic results of W. Brian's work.

Definition 4.3 *A space X is said to be n-dimensional, if it admits a base of sets with $(n-1)$-dimensional boundaries.*

Formally, one starts with $\text{ind}(\emptyset) = -1$. A space X satisfies $\text{ind}(X) \leq n$, if and only if there exists a base $\mathcal{B}$, of X, such that each $B \in \mathcal{B}$ satisfies $\text{ind}(\partial B) \leq n-1$. We say that the *small inductive dimension* of X is equal to n, whenever it is true that $\text{ind}(X) \leq n$, but it is not true that $\mathrm{ind}(X) \leq m$, for $m < n$. If, for any n, it is not true that $\mathrm{ind}(X) \leq n$, we then write $\mathrm{ind}(X) = \infty$.

W. Brian is dealing only with spaces for which $\mathrm{ind}(X) \neq \infty$, i.e. with spaces of *finite small inductive dimension.* Here we highlight some of his results, recalling that the *Sum Theorem* holds for a space X if, whenever A, B are closed subsets of X:

$$\mathrm{ind}(A \cup B) = \max\{\mathrm{ind}(A), \mathrm{ind}(B)\}$$

Theorem 4.4 (W.B.) *If X is a FUN separable metric space, then ind$(X) \leq \mathfrak{n}(X) - 1$.*

Corollary 4.5 (W.B.) *For every $n \in \mathbb{N}$, $\mathfrak{n}(\mathbb{R}^n) - 1 = n$.*

The proof of Corollary 4.5 is interesting: for every $n \in \mathbb{N}$, the union of a collection of $n+1$ nests forms a subbase for $\mathbb{R}^n$. Denoting such a collection by V, where $|V| = n + 1$, one can observe that the points of V are vertices of an n-dimensional polyhedron, whose faces are, respectively, $(n-1)$-dimensional hypersurfaces: each of these $n+1$ hypersurfaces defines a nest in $\mathbb{R}^n$.

Corollary 4.6 (W.B.) *Let X be a regular FUN space, and suppose that the sum theorem holds in X. Then, ind$(X) \leq \mathfrak{n}(X)$.*

Lemma 4.7 (Yurovetskiĭ) *Let X and Y be topological spaces and let X be a FUN-space. Then:*

$$\mathfrak{n}(X \times Y) - 1 \leq [\mathfrak{n}(X) - 1] + [\mathfrak{n}(Y) - 1] + 1.$$

Theorem 4.8 (W.B.) *There are compact LOTS X and Y, such that $\mathfrak{n}(X \times Y) - 1 = [\mathfrak{n}(X) - 1] + [\mathfrak{n}(Y) - 1] + 1$.*

Theorem 4.8 states that $\mathfrak{n} - 1$ is pathological as a measure of dimension, at least for compact LOTS, because it shows that the inequality in Lemma 4.7 cannot be sharpened, by omitting the "+1". The particular example that W. Brian gives, is the following: $X = \{0\} \times (\omega + 1) \cup \{1\} \times \omega_1$ is a space, which is ordered as $(i, \alpha) < (j, \beta)$, if and only if $i < j$, or $i = j = 0$ and $\alpha < \beta$, or $i = j = 1$ and $\beta < \alpha$, and its topology is induced by $<$, i.e. X is the ordered space obtained by concatenating a copy of $\omega + 1$ with a reversed copy of ω_1. Similarly, define $Y = \{0\} \times (\omega + 2) \cup \{1\} \times \omega_3$. It is then easy to see that both X and Y are compact LOTS. Therefore, as LOTS, both X and Y have neight 2, so that $\mathfrak{n}(X) - 1 = \mathfrak{n}(Y) - 1 = 1$. W. Brian shows that $\mathfrak{n}(X \times Y) - 1 = 3$.

4.2 Two Nests Give Strong Topological Properties

As we have seen in Chapter 3, two nests whose union generates a T_1-separating subbase, for a space X, give very strong topological properties. For example, they generate GO-spaces and LOTS. What if we had more than two nests, whose union is T_1-separating? Will the space X still carry strong topological properties?

Theorem 4.9 *Suppose X be a space, with a subbase $\mathcal{L} \cup \mathcal{M} \cup \mathcal{N}$ consisting of three nests $\mathcal{L}, \mathcal{M}, \mathcal{N}$ of open sets, such that $\mathcal{L} \cup \mathcal{M}$, $\mathcal{L} \cup \mathcal{N}$ and $\mathcal{M} \cup \mathcal{N}$ are T_1-separating, respectively. Let $x \sim_{\mathcal{L}} y$, if and only if $\{L \in \mathcal{L} : x \in L\} = \{L \in \mathcal{L} : y \in L\}$. Then:*

1. *$\sim_{\mathcal{L}}$ is an equivalence relation.*
2. *The quotient space $X/\sim_{\mathcal{L}}$, of the equivalence classes, is linearly ordered by $\lhd_{\mathcal{L}}$, where $[x] \lhd_{\mathcal{L}} [y]$, if and only if $x \in L$ and $y \notin L$, for some $L \in \mathcal{L}$.*
3. *For each $x \in X$, the $\sim_{\mathcal{L}}$ equivalence class, $[x]$, is a GO-space.*

Proof. 1. Obvious.

2. For each $L \in \mathcal{L}$, let $L' = \{[x] : x \in L\}$. Then, $\mathcal{L}' = \{L' : L \in \mathcal{L}\}$ is clearly a T_0-separating nest, on $X/\sim_{\mathcal{L}}$. Also, $x \in L$ and $y \notin L$, if and only if $[x] \in L'$ and $[y] \notin L'$. So, the results follows by Theorem 3.3 (4).

3. The nests $\mathcal{M}_x = \{M \cap [x] : M \in \mathcal{M}\}$ and $\mathcal{N}_x = \{N \cap [x] : N \in \mathcal{N}\}$ form a subspace topology on $[x]$. Also, $\mathcal{M}_x$ and $\mathcal{N}_x$ have a T_1-separating union. So, the result follows from Theorem 3.11 (2).

Proposition 4.10 *Let X be a space, which has a subbase that is formed from the union of n nests $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_n$, of open sets. Suppose, further, that for each $i \leq n$, there exist some $j_i \leq n$, such that $\mathcal{N}_i \cup \mathcal{N}_{j_i}$ is T_1-separating. Then, X is T_3.*

Proof. Let, for each i, there exist some j_i, such that $\mathcal{N}_i \cup \mathcal{N}_{j_i}$ is T_1-separating. Therefore, $\mathcal{N}_i \cup \mathcal{N}_{j_i}$ generates a GO-space topology on X; let us call it $\mathcal{T}_i$. The induced topology by $\mathcal{N}_i \cup \cdots \cup \mathcal{N}_n$ is the same as $\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$. Each $\mathcal{T}_i$ is T_3, so $\mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$ will be T_3, too.□

Before we move on, we recall the definition of *intersection topology.*

Definition 4.11 *If $\mathcal{T}_1, \mathcal{T}_2$ are topologies on a set X, then the* intersection topology, *with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$, is the topology $\mathcal{T}$, on X, such that the set $\{U_1 \cap U_2 : U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2\}$ forms a base for $(X, \mathcal{T})$.*

Example 4.12 *Let $X \subset \mathbb{R}$, $|X| = \omega_1$ and let $f : X \to \omega_1$ be a bijection. Let $\mathcal{T}$ be the subspace topology on X, inherited from $\mathbb{R}$. Let $\mathcal{T}'$ be the topology on X, which makes f a homeomorphism. Let $\mathcal{S}$ be the intersection topology, generated by $\mathcal{T} \cup \mathcal{T}'$. Then, since $X \subset \mathbb{R}$ and ω_1 are GO-spaces, there are nests $\mathcal{L}_1, \mathcal{L}_2$, whose union is T_1-separating, and which generates $\mathcal{T}$, and nests $\mathcal{L}_3$ and $\mathcal{L}_4$, whose union is T_1-separating, and which generates $\mathcal{T}'$. $\mathcal{S}$, therefore, has a subbase of nests $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{L}_4$, each of which has a twin, together with they form a T_1-separating union. So, $\mathcal{S}$ is T_3. But, $\mathcal{S}$ may or may not be T_4. That $\mathcal{S}$ may or may not be T_4 comes from [15] (Theorems 1,2,3), where the author considers the intersection topology formed by the ordinal topology, on ω_1, and a separable metric topology. Such a space can never be perfectly normal, but it can be either normal or perfect, depending on the model of set theory and the choice of the metric topology.*

Example 4.13 *Consider $\mathbb{Z} \times \mathbb{Z}$. For each $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$, let:*

$$L_n = \{(i, j) \in \mathbb{Z}^2 : i + j < n\},$$

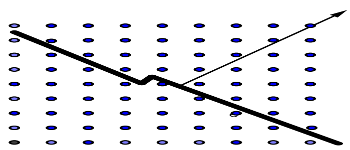
$$L_{n,k} = L_n \cup \{(i, j) : i + j = n, j < k\}$$

and

$$R_n = \{(i, j) \in \mathbb{Z}^2 : j - i < n\},$$

$$R_{n,k} = R_n \cup \{(i, j) : j - i = n + 1, j > k\}$$

So, $L_{n,k}$, for example, looks like:



The little arrow, in the picture above, which is placed on a particular set of the nest (containing points of $\mathbb{Z}^2$), shows the direction of the nests, which "move" upwards, as an increasing sequence of sets, ordered via inclusion.

We note that, $\mathcal{L} = \{L_n : n \in \mathbb{Z}\} \cup \{L_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}\}$ and $\mathcal{R} = \{R_n : n \in \mathbb{Z}\} \cup \{R_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}\}$ are T_0-separating nests, such that $\mathcal{L} \cup \mathcal{R}$ is T_1-separating.

Hence, $\mathcal{L} \cup \mathcal{R}$ makes $\mathbb{Z} \times \mathbb{Z}$ a GO-space (not homeomorphic to $\mathbb{Z}$, with the discrete topology). But, we remark that the (canonical) projection of any $L \in \mathcal{L}$ or $R \in \mathcal{R}$, to the first or second factor, is equal to $\mathbb{Z}$.

4.3 More than Two Nests Degenerate van Dalen and Wattel's Construction

We will now defend our argument, that more than two nests "destroy" the structure that we got from van Dalen and Wattel's characterisation of GO-spaces and LOTS, and carry weaker topological properties. In particular, we will give a counterexample, introducing a case where three nests, whose union is T_1-separating, generate a space which is not Hausdorff.

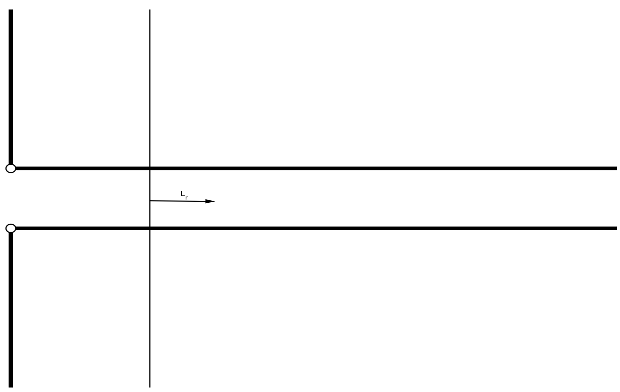
Proposition 4.14 *There is a countable set, X, with three T_0-separating nests, whose union is T_1-separating, but which do not generate a T_2 topology.*

Proof. Consider the subset

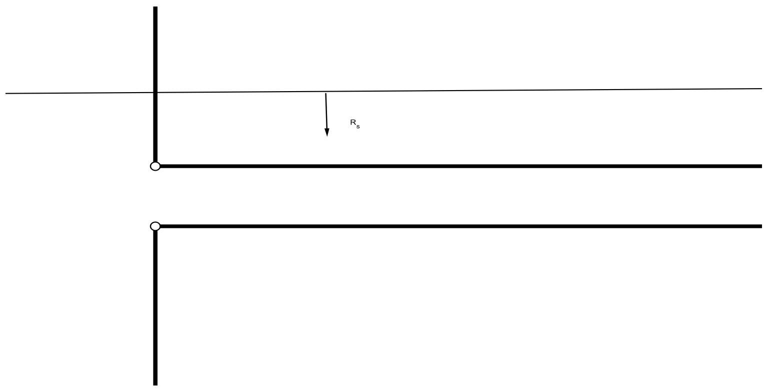
$$\mathcal{S} = \{[(0,\infty) \times (-\infty,-1)] \cup [(0,\infty) \times (1,\infty)] \cup \{(0,-1),(0,1)\},$$

of $\mathbb{R}^2$. For $r > 0, s > 1, 0 < \theta < \pi/2$, let:

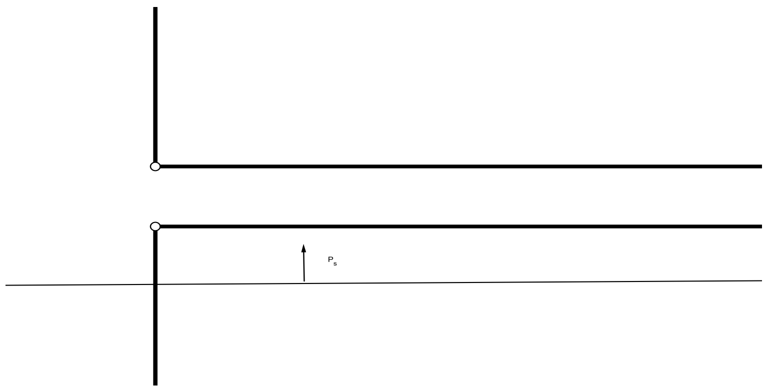
$$L_r = \{(x,y) \in \mathcal{S} : 0 < x < r\} :$$



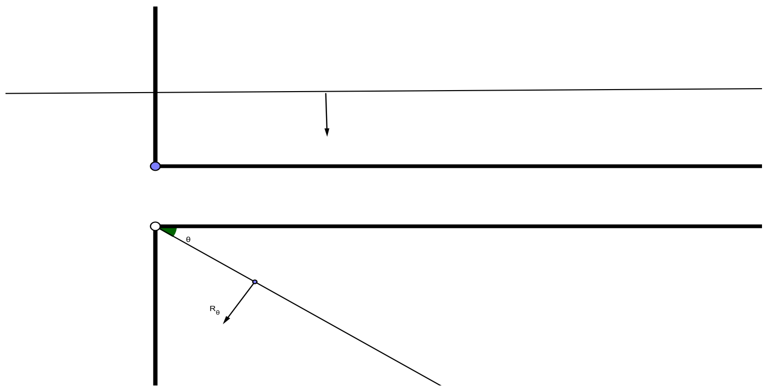
$$R_s = \{(x,y) \in \mathcal{S} : s < y\} :$$



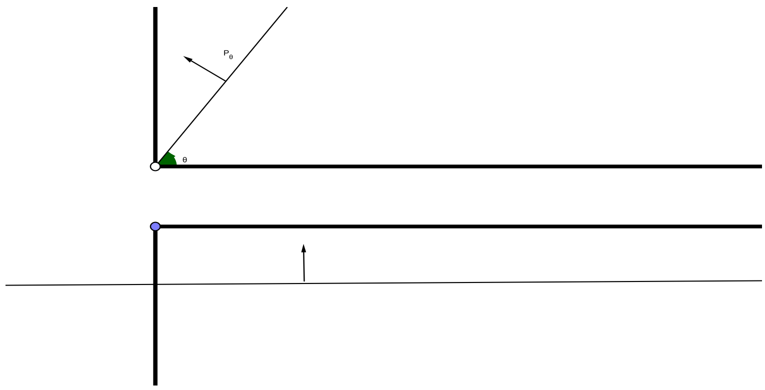
$$P_s = \{(x,y) \in \mathcal{S} : y < -s\} :$$



$$R_\theta = \{(x,y) \in \mathcal{S} : s < y\} \cup \{(x,y) \in \mathcal{S} : -1 - x\,tan\theta < y\} \cup \{(0,1)\} :$$



$$P_\theta = \{(x,y) \in \mathcal{S} : y < -s\} \cup \{(x,y) \in \mathcal{S} : y < 1 + x\,tan\theta\} \cup \{(0,-1)\} :$$



Note that $y = x\,tan\theta + 1$ is the equation of the line passing through $(0, 1)$, making an angle θ with the x-axis and $y = -x\,tan\theta - 1$ is the line passing through $(0, -1)$, making an angle $-\theta$ with the x-axis.

Now, let $X = \{(0, 1), (0, -1)\} \cup \{x_n : n \in \mathbb{N}\}$, where x_n is chosen as follows: let $\{B_n : n \in \mathbb{N}\}$ be a base for the usual topology on $[(0, \infty) \times (1, \infty)] \cup [(0, \infty) \times (-\infty, -1)]$. Given $x_1, x_2, \cdots, x_n \in X$, let x_1 be in B_1. Let S_n denote the set of straight lines, that are either:

1. horizontal, and pass through x_i, for some $i < n$;
2. vertical, and pass through x_i, for some $i < n$;
3. pass through $(0, 1)$ and x_i, for some $i < n$ or
4. pass through $(0, -1)$ and x_i, for some $i < n$.

Note that S_n is a finite set of lines. Now, choose $x_n \in B_n$, such that x_n does not lie on any line in S_n. Let:

$$\mathcal{L} = \{L_s \cap X : 0 < r\},$$

$$\mathcal{R} = \{R_s \cap X : 1 < s\} \cup \{R_\theta \cap X : 0 < \theta < \pi/2\},$$

$$\mathcal{P} = \{P_s \cap X : 1 > s\} \cup \{P_\theta \cap X : 0 < \theta < \pi/2\}. \qquad \square$$

Clearly, $\mathcal{L}, \mathcal{R}, \mathcal{P}$ are nests. Also, each of $\mathcal{L}, \mathcal{R}, \mathcal{P}$ is clearly T_0-separating.

For example, suppose $x, y \in X$. Suppose $x = (0, -1)$. Then, any $R \in \mathcal{R}$, containing y, will T_0-separate x and y.

Suppose $x = (0, 1)$. If $y = (y_1, y_2)$, with $y_2 > 1$, then $R_{y_2/2}$ T_0-separates x and y.

If, on the other hand, $y_2 < -1$, then there is an angle θ, such that $y \notin R_\theta$, but $x \in R_\theta$, for all $0 < \theta < \pi/2$.

Now, assume neither x nor y is equal to $(0,1)$ or $(0,-1)$. Hence, $x = x_n$ and $y = x_m$, for some $n, m \in \mathbb{N}$.

If $x = (x_1, x_2), y = (y_1, y_2)$, without loss of generality we have three cases:

1. $1 < x_2 < y_2$. Note that $x_2 \neq y_2$, since x and y do not lie on the same horizontal line.

2. $1 < x_2$, $y_2 < -1$

3. x lies on the line $y = -1 - x\, tan\theta$, y lies on the line $y = -1 - x\, tan\phi$ and $0 < \theta < \phi < \pi/2$.

Note that $\theta \neq \phi$, because x, y do not lie on the same line passing through $(0,-1)$. The points x and y are now separated by the sets $R_{\frac{x_2+y_2}{2}}$ in case 1., $R_{\frac{x_1+y_1}{2}}$ in case 2. and $R_{\frac{\theta+\phi}{2}}$, in case 3. Similar proofs hold for $\mathcal{L}$ and $\mathcal{P}$.

To see that $\mathcal{L} \cup \mathcal{R} \cup \mathcal{P}$ is T_1-separating, let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. It is enough (by symmetry), to consider the following cases:

1. $x = (0,1)$, $y = (0,-1)$;

2. $x = (0,1)$, $y_2 > 1$;

3. $x = (0,1), y_2 < -1$;

4. $1 < x_2 < y_2$: a) $x_1 > y_1$, b) $x_1 < y_1$ (note: the equality of x_2 and y_2 is not possible, by choice of the points in X);

5. $1 < x_2$, $y_2 < -1$: a) $x_1 > y_1$, b) $x_1 < y_1$.

Given the nests $\mathcal{L}, \mathcal{R}, \mathcal{P}$, one can show T_1-separation. In particular, for 1. there exists $R_\theta \in \mathcal{R}$, such that $x \in R_\theta$ and $y \notin R_\theta$ and also there exists $P_\theta \in \mathcal{P}$, such that $y \in P_\theta$ and $x \notin P_\theta$. For 2. there exists $R_s \in \mathcal{R}$, such that $y \in R_s$ and $(0,1) \notin R_s$ and also there exists $P_s \in \mathcal{P}$, such that $(0,1) \in P_s$ and $y \notin P_s$. For 3. there exists $R_\theta \in \mathcal{R}$, such that $(0,1) \in R_\theta$ and $y \notin R_\theta$ and also there exists $P_\theta \in \mathcal{P}$, such that $y \in P_\theta$ and $(0,1) \notin P_\theta$. For 4. a) and b), there exists, for both cases respectively, $R_s \in \mathcal{R}$, containing (y_1, y_2) but not (x_1, y_1) and there also exists $P_s \in \mathcal{P}$, containing (x_1, x_2), but not (y_1, y_2). For 5. a) and b), there exists, for both cases respectively, $R_\theta \in \mathcal{R}$, such that $(x_1, x_2) \in R_\theta$ but $(y_1, y_2) \notin R_\theta$ and there also exists $P_\theta \in \mathcal{P}$, such that $(y_1, y_2) \in P_\theta$, but $(x_1, x_2) \notin P_\theta$.

X is not T_2-separating, since $(0,1)$ and $(0,-1)$ cannot be T_2-separated by $\mathcal{L}, \mathcal{R}$ and $\mathcal{P}$.

We note that the nests in the above example are T_0-separating, but the space is not T_2.

On the other hand, consider the three nests consisting of subsets of $\mathbb{R}^2$, $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{R}$ defined as follows. For each $r \in \mathbb{R}$ let $P_r = \{(x,y) : x - y < r\}$, $L_r = \{(x,y) : y < r\}$ and $R_r = \{(x,y) : x > r\}$. In other words, P_r is the set of points above the line $y = x - r$, L_r is the set of points below the line $y = r$ and R_r is the set of points to the right of the line $x = r$. Then let $\mathcal{P} = \{P_r : r \in \mathbb{R}\}$, $\mathcal{L} = \{L_r : r \in \mathbb{R}\}$, and $\mathcal{R} = \{R_r : r \in \mathbb{R}\}$. These generate the usual Euclidean topology on $\mathbb{R}^2$, but none of them is T_0-separating. For example $\mathcal{R}$ cannot separate points on the same vertical line.

Chapter 5

Characterisations of Ordinals

5.1 Ordinals and Scattered Spaces

In this chapter we turn our attention to ordinal spaces. The results of the first section appear in [7].

Motivated, in particular, by Reed's "misnamed intersection topology" (see Section 1.2 and [22] and also [25], [8] and [13]), we ask whether it is possible to characterise ordinal spaces in purely topological terms. There are other essentially internal characterisations of certain ordinals, and subspaces of ordinals, due to Baker [1], van Douwen [26], Purisch [20], for example. However, these characterisations tend not to be as general or so simply stated as our own one. There are also external characterisations in terms of selections: see for example Section 5.3 and [12], [5] and [4].

We have already stated, in Definition 2.2, what it means for a family $\mathcal{S} \subset \mathcal{P}(X)$ to *scatter* a set X. We will make use of this definition for the characterisations that will follow. We will also need to define when a set X is right-separated.

Definition 5.1 *A topological space X is* right-separated*, if and only if there exists a well-order $<$, on X, such that $\{y \in X : y < x\}$ is open, for every $x \in X$.*

In other words, X is said to be right-separated, if and only if there is a well-order on

X, for which the initial segments are open (see [23]).

The following lemma is now obvious.

Lemma 5.2 *Let X be a topological space. Then, the following are equivalent:*

1. *X is scattered.*

2. *X is right-separated.*

3. *X is scattered by a nest of open subsets of X.*

Proof. "2. $\Rightarrow$ 3." Let X be right-separated. Then, there exists a well-order $<$, on X, such that the set $L_x = \{y : y < x\}$ is open, for all $x \in X$. We claim that the set $\{L_x : x \in X\}$ scatters X. Indeed, pick $A \subset X$, such that $A \neq \emptyset$. Let a be the $<$-least element in A. Let a^+ be the $<$-least element in $A - \{a\}$. Then, $L_{a^+} \cap A = \{a\}$. Thus, X is right-separated implies that X is scattered by a nest of open sets.

"3. $\Rightarrow$ 1." Obviously, if X is scattered by a nest of open sets, then X is scattered.

"1. $\Rightarrow$ 2." Let X be scattered and let x_0 be an isolated point of X. At stage α, if $X - \{x_\xi : \xi < \alpha\} \neq \emptyset$, let x_α be an isolated point of $X - \{x_\xi : \xi < \alpha\} \neq \emptyset$. For some η, $X - \{x_\xi : \xi \in \eta\} = \emptyset$, which gives that $X = \{x_\xi : \xi \in \eta\}$ is right-separated. $\square$

Theorem 5.3 *Let X be a set and let $\mathcal{L}$ be a nest on X. Then, the following are equivalent:*

1. *$\mathcal{L}$ scatters X.*

2. *$\lhd_{\mathcal{L}}$ is a well-ordering on X.*

3. *$\mathcal{L}$ is T_0-separating and well-ordered by $\subset$.*

4. *$\mathcal{L}$ is T_0-separating and, for every non-empty subset A of X, there is an $a \in A$, such that for any $x \in A$ and any $L \in \mathcal{L}$, if $x \in L$, then $a \in L$.*

Proof. 1. implies 2., because if $A \subset X$ and $L \cap A = \{a\}$, then a is clearly the $\lhd_{\mathcal{L}}$-least element of A.

2. implies 3., because if $\lhd_{\mathcal{L}}$ is a well-order on X, then it is also a linear order on X. Hence, $\mathcal{L}$ is a T_0-separating nest on X, by Theorem 3.3. Suppose that $L_1 \supsetneq L_2 \supsetneq L_3 \supsetneq \cdots$ be an infinite decreasing chain in $\mathcal{L}$. Then, there are points $x_i \in L_i - L_{i+1}$, which form an infinite decreasing $\lhd_{\mathcal{L}}$-chain, contradicting statement 2.

3. implies 4.: to see this, suppose that $A \subset X$, where $A \neq \emptyset$. Let L be the $\subset$-least element of $\mathcal{L}$, such that $L \cap A$ is non-empty. Since $\mathcal{L}$ is T_0-separating and well-ordered by $\subset$, we have that $L \cap A = \{a\}$, for some a. Then, if $M \in \mathcal{L}$ and $a \neq x \in A \cap M$, $L \subset M$, so that $a \in M$.

4. implies 1.: for proving this, consider $A \subset X$, where $A \neq \emptyset$. Let a be the point furnished by 4., for A, and let b be the point furnished by 4., for $A - \{a\}$. Since $\mathcal{L}$ is T_0-separating, there is $L \in \mathcal{L}$, which T_0-separates a and b. By 4., we have that $a \in L$, if $b \in L$. So, $a \in L$ and $b \notin L$. If $x \neq a$ and $x \in L$, then $b \in L$. So, $L \cap A = \{a\}$. $\square$

Theorem 5.4 *Let X be a set. Let also $\mathcal{L}$ and $\mathcal{R}$ be two nests, on X, that are each T_0-separating. Then, the following statements are true:*

1. *Suppose that for all non-empty $A \subset X$, there is some $a \in A$, such that if $a \in R \in \mathcal{R}$, then $A \subset R$. Then, $\rhd_{\mathcal{R}}$ is a well-order, on X, and $\mathcal{R}$ is well-ordered by $\supset$.*

2. *Let $\mathcal{L} \cup \mathcal{R}$ T_1-separates X. $\mathcal{L}$ is well-ordered by $\subset$, if and only if $\mathcal{R}$ is well-ordered by $\supset$.*

Proof. Clearly, if $A \subset X$, where $A \neq \emptyset$ and if a is as in the statement 1. of the Theorem, then a is the $\lhd_{\mathcal{R}}$-maximal element of A. Hence, it is the $\rhd_{\mathcal{R}}$-minimal element of A, and so 1. holds.

2. is an immediate consequence of Theorem 3.4 and Theorem 5.3. $\square$

Lemma 5.5 *Let X be a set and let $\mathcal{L}$ and $\mathcal{R}$ be subsets of $\mathcal{P}(X)$. Suppose that the nest $\mathcal{L}$ scatters X, i.e. for every $A \subset X$, there exists $L \in \mathcal{L}$, such that $|A \cap L| = 1$.*

1. *The collection $\mathcal{L}$ is T_0-separating.*
2. *If $\mathcal{L}$ is a nest, then $\mathcal{L}$ is interlocking.*
3. *If $\mathcal{L}$ and $\mathcal{R}$ are nests, such that $\mathcal{L} \cup \mathcal{R}$ forms a T_1-separating subbase for X, then there is a subset $\mathcal{M}$, of $\mathcal{L}$, that T_0-separates and scatters X, consisting of clopen sets.*

Proof. For 1., given $x \neq y$, there is $L \in \mathcal{L}$ such that $L \cap \{x, y\}$ is a singleton.

For 2., pick $L \in \mathcal{L}$. Since $\mathcal{L}$ scatters X, there is some $M \in \mathcal{L}$ such that $M \cap (X - L) = \{x\}$. Since $\mathcal{L}$ is a nest, $L \subsetneq M = L \cup \{x\}$ and, whenever $L \subsetneq M' \in \mathcal{L}$, $M \subseteq M'$. Hence $L \neq \bigcap\{M \in \mathcal{L} : L \subsetneq M\}$, so that $\mathcal{L}$ is interlocking.

To see that 3. holds, note first by Theorem 5.3 that $\lhd_{\mathcal{L}}$ is a well-order. Let $x \neq y$ and let x^+ denote the immediate $\lhd_{\mathcal{L}}$-successor of x, so that $x^+ \unlhd_{\mathcal{L}} y$. Since $\mathcal{L} \cup \mathcal{R}$ T_1-separates X, there are $L \in \mathcal{L}$ and $R \in \mathcal{R}$ such that $x \in L \not\ni x^+$ and $x \notin R \ni x^+$. Since the interval (x, x^+) is empty, we have that $X - L = R$. Since R is open, L is clopen, $x \in L \not\ni y$ and $y \in R \not\ni x$. $\square$

Lemma 5.6 *Let $\mathcal{L}$ be a nest of subsets of a set X and let $\mathcal{R}$ be the nest $\mathcal{R} = \{X - L : L \in \mathcal{L}\}$.*

1. *The nest $\mathcal{R}$ is interlocking if and only if, for all $L \in \mathcal{L}$, $L = \bigcap\{M \in \mathcal{L} : L \subsetneq M\}$ whenever $L = \bigcup\{M \in \mathcal{L} : M \subsetneq L\}$.*
2. *If X is a topological space and each $L \in \mathcal{L}$ is compact and open, then $\mathcal{R}$ is interlocking.*
3. *If $\mathcal{L}$ is T_0-separating, in particular if $\mathcal{L}$ scatters X, then $\mathcal{L} \cup \mathcal{R}$ is T_1-separating.*

Proof. That 1. holds is an immediate consequence of the de Morgan's Laws. For 2., suppose that $L = \bigcup\{M \in \mathcal{L} : M \subsetneq L\}$. Since L is compact, we have $L = M_1 \cup \cdots \cup M_k$, for some $M_i \subsetneq L$. But $\mathcal{L}$ is a nest, so we have $L = M_j$ for some $M_j \in \mathcal{L}$, such that $M_j \subsetneq L$, which is impossible. So the condition in 1. holds vacuously. Given Theorem 5.4, 3. follows immediately. □

Theorem 5.7 *Let X be a space. The following are equivalent:*

1. *X is homeomorphic to an ordinal.*
2. *X has two interlocking nests $\mathcal{L}$ and $\mathcal{R}$, of open sets, whose union is a T_1-separating subbase, such that $\mathcal{L}$ scatters X.*
3. *X has two interlocking nests $\mathcal{L}$ and $\mathcal{R}$, of open sets, whose union is a T_1-separating subbase, one of which is well-ordered by $\subset$ or $\supset$.*
4. *X is scattered by a nest $\mathcal{L}$, of clopen sets, such that:*
   (a) *$L \neq \bigcup\{M : M \subsetneq L\}$, for any $L \in \mathcal{L}$ and*
   (b) *$\{L - M : L, M \in \mathcal{L}\}$ is a base for X.*
5. *X is scattered by a nest of compact clopen sets.*

Proof. The equivalence of statements 1., 2. and 3. follows immediately from Theorem 5.3, Theorem 3.11 and Lemma 5.5.

1. implies both 4. and 5., because if α is an ordinal, then $\{[0, \beta] : \beta < \alpha\}$ is a nest of compact clopen subsets that scatters α, and satisfies conditions 4.(a) and 4.(b).

Lemmas 5.5 and 5.6 imply that, if either 4. or 5. holds, then both $\mathcal{L}$ and $\mathcal{R} = \{X - L : L \in \mathcal{L}\}$ are interlocking nests of open sets, whose union T_1-separates X.

If 4.(b) holds, then $\mathcal{L} \cup \mathcal{R}$ is a subbase for X, and we see that 4. implies 2.

To see that 5. implies 1., we argue as follows. We have that $\lhd_{\mathcal{L}}$ is a well-order on X and that the order topology induced by $\lhd_{\mathcal{L}}$ is coarser than the topology on X, by Theorem 3.11. If X is compact, then we note that the order topology is Hausdorff and coarser than the compact topology on X. Hence, the two topologies coincide. If X is not compact, then since the elements of $\mathcal{L}$ are clopen and compact, X is locally compact, and we may form the one-point compactification X^*, of X. But then, $\mathcal{L} \cup \{X^*\}$ is a nest of compact clopen sets that scatters X^*, so it is homeomorphic to an ordinal. Clearly, X is a $\lhd_{\mathcal{L}}$-initial segment of X^*, so that X is also homeomorphic to an ordinal. $\square$

The following corollary is now immediate.

Corollary 5.8 *X is homeomorphic to a cardinal, if and only if X is scattered by a nest $\mathcal{L}$, of compact clopen sets, such that $|L| < |X|$, for each $L \in \mathcal{L}$.*

In particular, X is homeomorphic to ω_1, if and only if X is uncountable and is scattered by a nest of compact, clopen, countable sets.

As in Theorem 3.11, we observe the following.

Proposition 5.9 *Let X be a space. Then, the following statements are true:*

1. *X admits a continuous bijection onto an ordinal, if and only if it is scattered by a nest of clopen sets.*

2. *X is homeomorphic to a subspace of an ordinal, if and only if it is scattered by a nest of clopen sets $\mathcal{L}$, and $\{L - M : L, M \in \mathcal{L}\}$ forms a subbase for X.*

Proof. The one direction is obvious, in each case.

For 1., we note that the order $\lhd_{\mathcal{L}}$ is a well-order and that every order-open set is open in X, by Theorem 3.11.

For 2., if $\mathcal{L}$ is a nest of clopen sets that scatters X, then $\lhd_{\mathcal{L}}$ is a well-order and is T_0-separating and interlocking, by Lemma 5.5. Let $\mathcal{R} = \{X - L : L \in \mathcal{L}\}$. Then, as $\mathcal{L}$ is

T_0-separating, $\mathcal{L} \cup \mathcal{R}$ is T_1-separating, so that the result follows by the proof of Theorem 3.11, statement 2. To see this, note that, by Lemma 5.6, interlocking fails in $\mathcal{R}$, for elements $X - L$, where $L = \bigcup\{M \in \mathcal{L} : M \subsetneq L\}$, but $L \neq \bigcap\{M \in \mathcal{L} : L \subsetneq M\}$. Let $\mathcal{L}'$ be the set of all such L. For each such L, we introduce a new point $x_L \notin X$, and we define an order $<$, on $X^* = X \cup \{x_L : L \in \mathcal{L}'\}$, by declaring:

$$x < y \Leftrightarrow \begin{cases} x, y \in X \text{ and } x \triangleleft_{\mathcal{L}} y, \\ x = x_L, y \in X \text{ and } y \notin L, \\ x \in X, y = x_L \text{ and } x \in L, \\ x = x_L, y = y_M \text{ and } L \subsetneq M. \end{cases}$$

Then, it is easy to see that $<$ is a well-order on X^* that agrees with $\triangleleft_{\mathcal{L}}$, on X, and that X is a subspace of X^*. $\square$

Lemma 5.2 shows that the existence of a nest of open sets, that scatters a space, is equivalent to right-separation. We exploit this in the following theorem.

Definition 5.10 *Let X be a space and let $<$ be a well-order on X. We say that $<$* left-separates *X, if and only if $\{y : y < x\}$ is closed, for all $x \in X$. In addition, $<$* weakly left-separates *X, if and only if $\{y \in X : y \leq x\}$ is closed, for every $x \in X$.*

Theorem 5.11 *Let X be a space. Then, the following statements hold:*

1. *X admits a continuous bijection onto an ordinal, if and only if it is right-separated and weakly left-separated, by the same well-order.*

2. *X is homeomorphic to a subspace of an ordinal, if and only if it is right-separated and weakly-left separated by a well-order, whose order-open intervals form a subbase for X.*

3. *X is homeomorphic to an ordinal, if and only if it is right-separated and weakly left-separated by the order $<$, so that if $C = \{x_\alpha : \alpha \in \lambda\}$ is an $<$-increasing sequence indexed by a limit ordinal λ, and C is closed, then C is $<$-cofinal in X.*

Proof. 1. and 2. follow easily from Proposition 5.9.

For 3., note first that the order topology induced by $<$ is coarser than the topology on X. But then, if the topology of X is strictly finer than the order topology on X, there is some order limit point x that is not a limit point in X, which contradicts the condition of the theorem. □

We finish this chapter by giving a few supporting examples, where the above results are applied.

Example 5.12 *The space $\omega_1 + 1 + \omega^*$, where ω^* denotes ω with the reverse order, is a compact scattered LOTS, that is not scattered by a nest of clopen sets. S. Purish, has shown, in [20], that every scattered GO-space is LOTS. Hence, the isolated points of ω_1 form a locally compact LOTS, and this space has a subbase consisting of two interlocking nests, whose union is T_1-separating. In particular, there is no nest of clopen countable sets that scatters the space, but there is a nest of compact clopen sets but it does not scatter the set. This nest consists of all sets of the form $[0, \beta]$ for $\beta < \omega_1$ and all sets of the form $[0, \omega_1] \cup \{k \in \omega^* : n \leq k\}$, $n \in \omega^*$. This nest does not scatter the space, because -for example- there is no one of these sets that picks a single element of ω^*.*

Example 5.13 *Let $\Psi = \omega \cup \{x_\alpha : \alpha \in \kappa\}$ denote Mrowka's Ψ-space and Ψ^* denote its one point compactification. Consider $L_\alpha = \omega \cup \{x_\beta : \beta < \alpha\}$ and let $\mathcal{L} = \{[0, n] : n \in \omega\} \cup \{L_\alpha : \alpha \in \kappa\}$ be a T_0-separating nest of open sets that scatters Ψ^*. Note that $L_0 = \bigcap\{M \in \mathcal{L} : L_0 \subsetneq M\}$, but $L_0 \neq \bigcup\{M \in \mathcal{L} : M \subsetneq L_0\}$, so that $\mathcal{L}$ is not interlocking. It follows that there need to be two nests whose union is T_1-separating, for the conclusion* *of Theorem 3.12, to hold. We conclude that Ψ^* is both right-separated and left-separated, but not by the same order.*

Example 5.14 *Let $X = \omega_1 \cup \{(\alpha, n) : n \in \omega, \alpha \in \omega_1 \text{ a limit}\}$. Let $<$ be the usual order on ω_1, and define a linear order $\lhd$, on X, by declaring $\alpha \lhd \beta$, if and only if $\alpha < \beta$; $\alpha \lhd (\beta, n)$, if and only if $\alpha \leq \beta$; $(\beta, n) \lhd \alpha$, if and only if $\beta < \alpha$; $(\alpha, n) \lhd (\beta, m)$, if and only if $\alpha < \beta$ or $\alpha = \beta$ and $m < n$. Then, with the order topology, which is generated by $\lhd$, X is a locally countable, locally compact scattered LOTS, which has a nest of compact clopen countable sets, that T_0-separates X, but X is not homeomorphic to an ordinal space. In particular, the nest is all sets of the form $[0, \alpha]$ where α is not a limit and all sets of the form $[0, (\alpha, n)]$ where α is a limit and $n \in \omega$. These are all compact clopen countable sets that T_0-separate X. But X is not homeomorphic to an ordinal. To see this, (α, n) converges to α as n tends to infinity so if X is homeomorphic to an ordinal there must be some order $<$ that well orders X. But then, there is k_α, for each α, such that $(\alpha, n) < \alpha$ for each $n \geq k_\alpha$. Define $f(\alpha) = (\alpha, k_\alpha)$. Then, by the Pressing Down Lemma, there is some stationary set S and an α_0, such that for each $\alpha \in S, f(\alpha) = (\alpha_0, k_{\alpha_0})$. But this is a contradiction as if $\alpha \neq \beta$, then $f(\alpha) \neq f(\beta)$.*

5.2 A Characterisation of Ordinals via Neighbourhood Assignments

Here we give a characterisation of ordinals, which is entirely topological, with no mention of order, using neighbourhood assignments as a primary tool.

Definition 5.15 *A* neighbourhood assignment*, for a space X, is a collection of neighbourhoods $\mathcal{U} = \{U_x : x \in X\}$, such that $x \in U_x$.*

Definition 5.16 *A collection of sets $\mathcal{A}$ is* linked*, if for every $A, B \in \mathcal{A}$, there is some $C \in \mathcal{A}$, which meets both A and B.*

Let $\mathcal{U}$ be a neighbourhood assignment for a space X. An element x is said to be a *linking point* of $\mathcal{U}$, if the set $\{\overline{U_y} : y \in \overline{U_x} - \{x\}\}$ is not linked.

We recall that a set A is a *strong* G_δ, if it is an intersection of countably many clopen sets.

Theorem 5.17 *A space X is homeomorphic to an ordinal, if and only if there exists a neighbourhood assignment $\mathcal{U} = \{U_x : x \in X\}$, which satisfies the following conditions:*

1. *U_x is compact, clopen, for every $U_x \in \mathcal{U}$;*
2. *for every non-empty $A \subset X$, there exists $x \in X$, such that $A \cap U_x = \{x\}$;*
3. *$\mathcal{U}$ is of rank-1, i.e. if $U_x \cap U_y \neq \emptyset$, then either $U_x \subset U_y$ or $U_y \subset U_x$;*
4. *If x is a linking point of $\mathcal{U}$, then $\{x\}$ is a strong G_δ;*
5. *there exists a pairwise disjoint collection $\mathcal{V}$, of open sets, such that the set Λ, of linking points, is covered by $\mathcal{V}$ and, for each $V \in \mathcal{V}$, $V \cap \Lambda$ is countable;*
6. *$\mathcal{U} = \{U_x : x \in X\}$ either has a countable subcover or a subcover of the form $\mathcal{U}_1 \cup \mathcal{U}_2$, where $\mathcal{U}_1$ is finite and $\mathcal{U}_2$ is a nest.*

Proof. "$\Rightarrow$" Suppose X is homeomorphic to an ordinal, with order $<$. For each x, let:

$$y_x = \begin{cases} \sup\{y < x : \mathrm{lt}(x) \leq \mathrm{lt}(y)\}, & \text{if } \exists\, y < x, \mathrm{lt}(y) \geq \mathrm{lt}(x), \\ 0 & \text{otherwise.} \end{cases}$$

Consider $U_x = (y_x, x]$. Then, properties 1. to 6. follow trivially. In particular, for an ordinal α, define $U_\beta = [0, \beta]$, for all $\beta < \alpha$. Then, there are no linking points; so 4. and 5. hold vacuously. As for 6. the collection of such U_β forms a nest.

"$\Leftarrow$" Conversely, first note that a countable union of compact ordinals is homeomorphic to an ordinal and a union of a finite number of compact ordinals, with any ordinal, is also homeomorphic to an ordinal. Therefore, we may assume that either $U_x = \{x\}$, for some x, or that $\mathcal{U}$ has a subcover $\mathcal{U}'$, which is a nest.

Define $x \trianglelefteq y$, if and only if $x \in U_y$. Pick z, such that $U_z \cap \{x\} = \{x\}$. Consider $\{y : z \trianglelefteq_{\mathcal{L}} y\} = \Gamma_{\mathcal{U}}$. Then, $\Gamma_{\mathcal{U}}$ is well-ordered by $\trianglelefteq_{\mathcal{L}}$ and is $\trianglelefteq_{\mathcal{L}}$-cofinal in X. We use induction on the order type of $\Gamma_{\mathcal{U}}$; we call the order type of $\Gamma_{\mathcal{U}}$ the *height* of $\mathcal{U}$.

Suppose that $\mathcal{W} \subset \mathcal{U}$ is a neighbourhood assignment of rank-1, for some subset Y, of X. Define $x \sim_{\mathcal{W}} y$, if and only if there exists z, such that $x, y \in U_z$. Then, $\sim_{\mathcal{W}}$ is an equivalence relation, whose equivalence classes are unions of elements from $\mathcal{W}$, and they partition Y.

1. Suppose that $\Gamma_{\mathcal{U}}$ has order type $\alpha + 1$ (a successor ordinal), for some α. Then, $\Gamma_{\mathcal{U}}$ has a maximal element, x', and $U_{x'} = X$.

If x' is not a linking point, then $\mathcal{U} - \{U_{x'}\}$ satisfies the conditions of the theorem, with height α, so $X - \{x'\}$ is homeomorphic to an ordinal, γ. X is then homeomorphic to $\gamma + 1$, by making $x' \geq y$, for all $y \in X$.

If x' is a linking point, then $X - \{x'\}$ is partitioned into disjoint sets, by $\sim_{\mathcal{U}_{x'}}$; we call these sets X_i, $i \in I$. Note that $\{U_x : x \in X_i\}$ satisfies the conditions of the theorem, with height less than or equal to α. Hence, each X_i is homeomorphic to an ordinal, α_i. Since x' is a strong G_δ, there are countably many clopen sets D_n, such that $\{x'\} = \bigcap D_n$. Since $U_{x'}$ is compact clopen, we can assume $D_0 = U_x$ and $D_{n+1} \subset D_n$. Since $D_n - D_{n+1}$ is compact clopen, there can be only finitely many X_i, with $X_{i_n} = X_i \cap (D_n - D_{n+1}) \neq \emptyset$. Hence, I is countable. So, without loss of generality, $I = \mathbb{N}$.

Since X_i is homeomorphic to an ordinal, each X_{i_n} is clopen and compact, so it is homeomorphic to an ordinal α_{i_n}, with order $<_{i_n}$. Define a well order on X, by declaring $y < x'$, for all $y \neq x'$, $x < y$, if $x, y \in X_{i_n}$ and $x <_{i_n} y$, $x < y$, if $x \in X_{i_n}$, $y \in X_{j_m}$ and $m > n$ or $m = n$ and $i < j$. Then, $<$ well orders X, and the order topology agrees with the topology on X, since U_x is compact.

2. Now, suppose that $\mathcal{U}$ has height λ, a limit ordinal. Consider $L = \{x \in \mathit{\Gamma}_{\mathcal{U}} : x \text{ is a linking point}\}$. In this case, X is scattered in the sense of property 2. of the theorem, in height λ, by the neighbourhood assignment $\mathcal{U} = \{\overline{U_x} : x \in X\}$, with the properties of the statement of the theorem. X is partitioned into a co-tree, without "top point". We consider $x_0 \in X$ and we look at its cofinal numbers, $S(x_0)$.

Since $\mathcal{U}$ satisfies condition 5. of the Theorem, the set of linking points, L, can be covered by a pairwise disjoint collection of open sets, $\mathcal{V}$, such that $V \cap L$ is countable, for every $V \in \mathcal{V}$. So, $L \cap V \cap S(x_0) = (a_\alpha, b_\alpha]$ will be countable, $a_\alpha, b_\alpha \in S(x_0)$, i.e. in each $(a_\alpha, b_\alpha]$ there are countably many linking points. We look at neighbourhoods $(a_\alpha, a_{\alpha+1})$, which obviously contain countably many linking points, too.

We observe that $U_{a_{\alpha+1}} - U_{a_\alpha}$ is compact, so there exists a finite subcover homeomorphic to an ordinal. We look at $\{U_y : y \in U_{a_{\alpha+1}} - \{a_{\alpha+1}\}, y \notin U_{a_\alpha}\}$, following the same process for $a_{\alpha+2}, \cdots, a_{\alpha+n}$, and observing that the $\sup a_{\alpha+n} = a$ cannot be a linking point, but an ordinal; if it were a linking point, it would then belong to a set $V_a \in \mathcal{V}$, but this cannot happen, because the sets V_a are disjoint. So, we stop, and apply case **1.**. We then continue the same process for ordinals greater than a.

Now, we define an ordering $\lhd$ in X, such that $x \lhd y$, if and only if $x \in U_y$, which gives a well ordering in X: $x_1 \lhd x_2 \lhd \cdots \lhd x_n \lhd x_{n+1} \lhd \cdots \lhd x_\omega \lhd \cdots$, where x_i is homeomorphic to $a_{\alpha+1}$, so that x_ω is homeomorphic to L.

It now remains to glue the ordinal-like sticks, that we considered in cases 1. and 2. in order to construct ordinals. For this, we define the ordering:

$$x \lhd y = \begin{cases} x \in X_i,\, y \in X_j,\, i < j & \text{if we have finitely many sticks (compact ordinal),} \\ x <_i y,\, x <_\omega y & \text{if we have infinitely many sticks (limit ordinal),} \\ x \in X_i,\, y \in X_\omega & \text{if at most one of the sticks is a limit ordinal.} \end{cases}$$

□

5.3 Selections and Orderability

5.3.1 LOTS and Ordinals via Selections: An Account of Results

In this section we state, without proof, the most important results, which lead to characterisations of LOTS and ordinals via selections. In the next section, we talk particularly about weakly orderable spaces, via weak selections. The notation in the literature differs from author to author, so we will stick to the notation of van Mill and Wattel (see [27]).

Let X be a space. Let $\mathcal{F}(X)$ denote the hyperspace of nonempty closed subsets of X, endowed with the *Vietoris topology.* A base for the Vietoris topology consists of sets:

$$< V_0, V_1, \cdots, V_n >= \{F \in \mathcal{F}(X) : F \subset \bigcup_{i \leq n} V_i \text{ and } F \cap V_i \neq \emptyset, \forall\, i \leq n\},$$

where V_i are open subsets of X.

Definition 5.18 *A* selection*, for X, is a continuous mapping $F : \mathcal{F}(X) \to X$, such that $F(A) \in A$, for all $A \in \mathcal{F}(X)$.*

Definition 5.19 *A* weak selection*, for X, is a continuous mapping $s : X^2 \to X$, such that for every $x, y \in X$:*

1. *$s(x, y) = s(y, x)$ and*

2. *$s(x, y) \in \{x, y\}$.*

Definition 5.20 *A weak selection $s : X^2 \to X$ is called* locally uniform*, provided that for all $x \in X$ and for each neighbourhood U, of x, there is a neighbourhood V, of x, which is contained in U, such that for all $p \in X - U$ and $y \in V$, $s(p, y) = p$, if and only if $s(p, x) = p$.*

In other words, the definition of locally uniform selection says that the behaviour of a weak selection s, at a point x, determines the behaviour of s in some small neighbourhood of x.

The theorem that follows characterises LOTS via selections and was introduced in 1981 by J. van Mill and E. Wattel, in [27].

Theorem 5.21 (van Mill and Wattel) *Let X be a compact space. Then, the following statements are equivalent:*

1. *X is LOTS.*

2. *X has a weak selection.*

3. *X has a selection.*

That 1. implies 3. follows trivially, if we consider a mapping $F : \mathcal{F}(X) \to X$, defined by $F(A) = \min(A)$. That 3. implies 2. is again a trivial statement, following from the definitions of selection and weak selection. For proving that 2. implies 1., the authors of Theorem 5.21 construct, in a quite lengthy and inspired proof, two nests of open sets, leading to a similar statement to the characterisation of Theorem 3.11.

J. van Mill and E. Wattel introduced another paper in 1981, a few months later than the paper we have just mentioned, where they presented another solution to the orderability problem, using again selections (see [28]). In particular, they characterised GO-spaces via locally uniform weak selections, without mentioning compactness. We state their main theorem, without proof.

Theorem 5.22 (van Mill and Wattel) *Let X be a space. Then, the following statements are equivalent:*

1. *X has a locally uniform weak selection.*

2. *X is a GO-space.*

In 1997, S. Fujii and T. Nogura introduced a characterisation of compact ordinal spaces, via continuous selections (see [4]). Their main theorem is the following:

Theorem 5.23 (Fujii and Nogura) *Let X be a compact Hausdorff space. The following are equivalent:*

1. *X is homeomorphic to an ordinal space.*

2. *There exists a continuous selection $F : 2^X \to X$, such that $F(A)$ is an isolated point of A, for every $A \in 2^X$.*

We can easily see the relationship between Theorem 5.23 and Theorem 5.7; they are both topological characterisations, where isolated points correspond to minimal elements. In Theorem 3.4, we linked well-ordering in a space with a nest scattering it. In Theorem 5.23, the authors describe this "scattering" in the space, via continuous selections.

S. Fujii generalised the above theorem to the case of local compactness of X, stating the following:

Theorem 5.24 (Fujii) *Let X be a Hausdorff space. The following are equivalent:*

1. *X is homeomorphic to an ordinal space.*

2. *There exists a continuous selection $F : 2^X \to X$, such that (i) $F(A)$ is an isolated point of A, for every $A \in 2^X$ and (ii) X is locally compact and F is continuously extendable to X^*, if X is not compact.*

By X^* the author denotes the one-point compactification $\alpha X = X \cup \{\infty\}$, of X. By F^*, the author denotes a selection on X^*, such that for every $A \in \mathcal{F}(X^*)$, $F^*(A) = F(A \cap X)$, if $A \cap X \neq \emptyset$ and $F^*(A) = \infty$, if $A = \{\infty\}$. He calls F^* the *extension* of F to X^*.

If a continuous selection F, on X, admits a continuous extension F^*, then F is called *continuously extendable* to X^*.

We believe that the following statement, which is a variation of Fujii's just mentioned characterisation, is true.

Conjecture A space X is homeomorphic to an ordinal, if and only if:

1. there exists a continuous selection $F : 2^X \to X$, such that $F(A)$ is isolated in A, for each A, and

2. either X is σ-compact or, if A and B are disjoint closed sets, then at least one of them is compact.

Chapter 6

Open Problems

In this final Chapter, we will propose a few open problems, which are based on some research results that we have introduced in this thesis. We will hopefully work on these problems in the near future.

6.1 The Suslin Line

Suslin's problem (Souslin, M. (1920). "Probléme 3". Fundamenta Mathematicae 1: 223)) refers to linearly ordered sets, and it has been shown to be independent of ZFC. Let X be a linearly ordered set, with the properties:

1. X has neither maximal nor minimal elements;
2. the order on X is dense (between any two comparable elements there exists another element);
3. the order on X is complete (every nonempty bounded subset of X has a supremum and an infimum in X) and
4. every collection of mutually disjoint nonempty open intervals, in X, is countable (this property is also known as the Countable Chain Condition, abbreviated as CCC).

Is X order-isomorphic to $\mathbb{R}$? If condition 4. is replaced by the condition that X contains a countable dense subset (i.e. X is separable), then the answer is positive. On the other hand, any linearly ordered set, which is not isomorphic to $\mathbb{R}$, but satisfies properties 1.-4. is known as the *Suslin line*.

An interesting problem will be to characterise the Suslin line exclusively via nests. For such a characterisation one will need to rewrite each property 1.-4., in the language of nests, given a T_0-separating nest $\mathcal{L}$, on X, and the ordering $\triangleleft_{\mathcal{L}}$, which we defined in Section 3.1. By such a characterisation we expect that properties 1.-4. will lead to a topological characterisation, referring to the subsets of X and not immediately to its elements. This might reduce the number of properties needed to describe the Suslin line and it will give a shorter and more dense characterisation.

6.2 The Pressing Down Lemma

We have stated and have given a proof for the Pressing Down Lemma (Theorem 2.50), and have noticed a relationship of this statement with our theorem which characterises ω_1 via nests (Corollary 5.8). Is there a simpler proof of the Pressing Down Lemma, given a characterisation of it via nests? In other words, one should characterise club sets and stationary sets via nests, and apply properties of nests in order to simplify the statement and, perhaps, the proof of this theorem.

6.3 Selections

Here we propose three problems:

1. Jan Mill and and E. Wattel's characterisation of LOTS via selections (Theorem 5.21) is restricted to compact spaces X; how can this problem be extended to any space X?

2. Our Conjecture, at the end of Section 5.3.1, will hopefully lead into a new external

characterisation of ordinals.

6.4 Nests and Sets with a Given Structure

Proposition 3.10 refers to arbitrary families of sets which are not necessarily nests. A question that arises is the following: when do the notions of nested family and family closed with respect to unions coincide? We believe that this question has a definite answer in the case of sets with some given structure, for example in the case of groups.

We will now have a look at an example of a family $\mathcal{L}$, which is closed under unions, but is not a nest, and then an example of a group, where we observe that the group structure gives a positive answer to the connection between being closed under unions and being a nested family.

Example 6.1 *Let* (X, d) *be a metric space and let* $\mathcal{L} = \{L : L \subset X, \, \delta(L) < \infty\}$*, where* $\delta(L)$ *is the* diameter *of the set* L*,* $\delta(L) = \sup\{d(x, y) : x, y \in L\}$*. If* $L_1, L_2 \in \mathcal{L}$*, then* $L_1 \cup L_2 \in \mathcal{L}$*, too, but this does not imply that* $L_1 \subset L_2$ *or* $L_2 \subset L_1$*. For example, let* $X = \mathbb{R}$ *and* $\mathcal{L} \subset \mathcal{P}(X)$*. Let* $L_1 = (1, 2)$ *and* $L_2 = (3, 4)$*. Then,* $L_1 \cup L_2 = (1, 2) \cup (3, 4) \in \mathcal{L}$*, but neither* $L_1 \subset L_2$*, nor* $L_2 \subset L_1$*.*

Proposition 6.2 *Let* $(G, *)$ *be a group and let* $\mathcal{L} = \{H : H \text{ is a subgroup of } G\}$*. If* $\mathcal{L}$ *is closed under unions, then* $\mathcal{L}$ *is a nest.*

Proof. Let us assume that $\mathcal{L}$ is closed under unions. Let $L_1 \in \mathcal{L}$ and $L_2 \in \mathcal{L}$. Then, $L_1 \cup L_2 \in \mathcal{L}$. We will prove that either $L_1 \subset L_2$ or $L_2 \subset L_1$. Indeed, let $L_1 \nsubseteq L_2$ and $L_2 \nsubseteq L_1$. Then, there exists $\phi \in L_1$, such that $\phi \notin L_2$ and there exists $\psi \in L_2$, such that $\psi \notin L_1$. But then, $\phi \in L_1 \subset L_1 \cup L_2$ and $\psi \in L_2 \subset L_1 \cup L_2$ imply that $\phi, \psi \in L_1 \cup L_2$. But $L_1 \cup L_2$ is a subgroup, under the operation $*$, so $\phi * \psi \in L_1 \cup L_2$, implying that $\phi * \psi \in L_1$ or $\phi * \psi \in L_2$.

When $\phi * \psi \in L_1$, we get that $\phi^{-1} \in L_1$ Thus, $\phi^{-1}(\phi * \psi) = \psi \in L_1$, which is a contradiction.

When $\phi * \psi \in L_2$, then $\psi \in L_2$ implies that $(\phi * \psi) * \psi^{-1} = \phi \in L_2$, which is again a contradiction.

Thus, either $L_1 \subset L_2$ or $L_2 \subset L_1$. $\square$

6.4.1 Order Theoretic Properties of the Line and Topological Implications

In Chapter 3, Section 4, we have introduced (see Theorem 3.38) spaces which satisfy order-theoretic properties very similar to those ones of the real line. **Question**: are there other LOTS, apart from the real line with its natural order, such that 2. from Theorem 3.38 is satisfied?

Question: What is the difference between LOTS that are implied by Theorem 3.41 from LOTS being implied by Theorem 3.42? Are there distinct examples of such spaces, spotting the difference between these properties?

6.4.2 Nests, Groups and Topological Groups

We consider the ordering $\lhd_{\mathcal{L}}$, on a group $(G, *)$, which is generated by a T_0-separating nest of sets, in G, and we give conditions which will make the order compatible with the group operation, $*$.

Let $(G, *)$ be a group, with operation $*$, and let $\mathcal{L}$ be a T_0-separating nest, on G. For every $x, y \in G$, $x \lhd_{\mathcal{L}} y$, if and only if there exists $L \in \mathcal{L}$, such that $x \in L$ and $y \notin L$. The ordering $\lhd_{\mathcal{L}}$ is said to be *compatible* with the group operation $*$, if and only if for every a, b and g, in G, the following hold:

$$a \lhd_{\mathcal{L}} b \Leftrightarrow$$

$$a * g \lhd_{\mathcal{L}} b * g$$

and

$$g * a \lhd_{\mathcal{L}} g * b.$$

Proposition 6.3 *Let $(G, *)$ be a group and let $\mathcal{L}$ be a T_0-separating nest on G. If for every $g \in G$, for every $L \in \mathcal{L}$:*

$$g * L \in \mathcal{L}$$

and

$$L * g \in \mathcal{L};$$

equivalently, if the maps:

$$t : \mathcal{L} \times G \to \mathcal{L}, \text{ where } t(L, g) = L * g$$

and

$$s : G \times \mathcal{L} \to \mathcal{L}, \text{ where } s(g, L) = g * L$$

are well-defined, then $\lhd_{\mathcal{L}}$ is compatible with $*$.

Proof. Let $e \in G$ denote the identity element of G, with respect to $*$. Let, for every $g \in G$ and for every $L \in \mathcal{L}$, $g * L \in \mathcal{L}$ and $L * g \in \mathcal{L}$. Let $a, b \in G$, such that $a \lhd_{\mathcal{L}} b$, and let also $g \in G$. We prove that $a * g \lhd_{\mathcal{L}} b * g$. But, since $a \lhd_{\mathcal{L}} b$, there exists $L \in \mathcal{L}$, such that $a \in L$ and $b \notin L$. Furthermore, $a \in L$ implies that $a * g \in L * g$ and $b \notin L$ implies that $b * g \notin L * g$, because if $b * g$ belonged to $L * g$, then $(b * g) * g^{-1} \in (L * g) * g^{-1}$, which would imply that $b * e \in L$, which would then imply that $b \in L$, a contradiction. Finally, $a * g \lhd_{\mathcal{L}} b * g$. In a similar way we prove that $g * a \lhd_{\mathcal{L}} g * b$. □

Example 6.4 *Let $(\mathbb{R}, +)$ be the group of the real numbers, under addition. Then, $\mathcal{L} = \{(-\infty, a) : a \in \mathbb{R}\}$ is obviously a T_0-separating nest on $\mathbb{R}$, and we observe that for every $b \in \mathbb{R}$, $b + (-\infty, a) = (-\infty, a + b) \in \mathcal{L}$. So, $\lhd_{\mathcal{L}}$ is compatible, with respect to $+$.*

Example 6.5 *Consider the abelian group* $(\mathbb{R}-\{0\},\times)$, *of the non-zero real numbers, endowed with the operation of multiplication. Obviously,* $\mathcal{L}=\{(-\infty,a):a\in\mathbb{R}\}$ *is a* T_0*-separating nest, on* $\mathbb{R}$. *We remark that if* $b\in\mathbb{R}$, *such that* $b\rhd_{\mathcal{L}}0$, *then* $(-\infty,a)\times b=(-\infty,a\times b)\in\mathcal{L}$, *but if* $b\lhd_{\mathcal{L}}0$, *then* $(-\infty,a)\times b=(a\times b,\infty)\notin\mathcal{L}$. *So,* $\lhd_{\mathcal{L}}$ *is not compatible, with respect to* $\times$.

Suggestion 1: It might be interesting to investigate more complex examples, preferably of non-abelian groups, and see what topological properties does $\lhd_{\mathcal{L}}$ bring to the structure of the group.

We will now make the problem a bit more difficult.

Proposition 6.6 *Let* $(G,*)$ *be a group. Let also* $\mathcal{L}$ *and* $\mathcal{R}$ *be families of subsets of* G. *Suppose that the following two conditions are satisfied:*

1. *For every* $L\in\mathcal{L}$, $L^{-1}\in\mathcal{R}$.
2. *For every* $R\in\mathcal{R}$, $R^{-1}\in\mathcal{L}$.

If we consider the topology generated by $\mathcal{L}\cup\mathcal{R}$, *then the map* $f:G\to G$, *where* $f(x)=x^{-1}$, *will be continuous.*

Proof. Let $L\in\mathcal{L}$. Then:

$$\begin{aligned} f^{-1}(L) \quad &= \{x\in G: f(x)\in L\} \\ &= \{x\in G: x^{-1}\in L\} \\ &= L^{-1}\in\mathcal{R}. \end{aligned}$$

Similarly, if $R\in\mathcal{R}$, then $f(R)=R^{-1}\in\mathcal{L}$. □

Proposition 6.7 *Let* $(G,*)$ *be a group. Let also* $\mathcal{L}$ *and* $\mathcal{R}$ *be families of subsets of* G. *Suppose that the following two conditions are satisfied:*

1. *If* $x * y \in L \in \mathcal{L}$*, then there exist* $L_x, L_y \in \mathcal{L}$*, such that* $x \in L_x, y \in L_y$ *and* $L_x * L_y \subset L$*.*

2. *If* $x * y \in R \in \mathcal{R}$*, then there exist* $R_x, R_y \in \mathcal{R}$*, such that* $x \in R_x, y \in R_y$ *and* $R_x * R_y \subset R$*.*

If we consider the topology generated by $\mathcal{L} \cup \mathcal{R}$*,then the map* $f : G \times G \to G$*, where* $f(x, y) = x * y$*, will be continuous.*

Proof. Let $L \in \mathcal{L}$. Then, $f^{-1}(L) = \{(x, y) \in G \times G : x * y \in L\}$. Statement 1. gives that for every $(x, y) \in G \times G$, such that $x * y \in L$, there exist $L_x, L_y \in \mathcal{L}$, such that $x \in L_x$, $y \in L_y$ and $L_x * L_y \subset L$, which implies that:

$$L_x \times L_y \subset f^{-1}(L). \qquad (1)$$

Indeed:

$$\begin{aligned} (a, b) \in L_x \times L_y &\quad \Rightarrow \\ a \in L_x, b \in L_y &\quad \Rightarrow \\ a * b \in L_x * L_y &\quad \Rightarrow \\ a * b \in L. & \end{aligned}$$

It is also true that:

$$\pi_1^{-1}(L_x) \cap \pi_2^{-1}(L_y) \subset L_x \times L_y \qquad (2),$$

where $\pi_1^{-1}(L_x)$ and $\pi_2^{-1}(L_y)$ are the inverse projections, which give the usual product topology, in $G \times G$.

Indeed:

$$\begin{aligned}(a,b)\in \pi_1^{-1}(L_x)\cap \pi_2^{-1}(L_y) &\quad\Rightarrow\\ a\in L_x, b\in L_y &\quad\Rightarrow\\ (a,b)\in L_x\times L_y &\end{aligned}$$

So, (1) and (2) give that $\pi_1^{-1}(L_x)\cap \pi_2^{-1}(L_y)\subset f^{-1}(L)$. The latter implies that:

$$\bigcup_{x*y\in L}[\pi_1^{-1}(L_x)\cap \pi_2^{-1}(L_y)\subset f^{-1}(L)] \qquad (3).$$

But, it also holds that:

$$f^{-1}(L)\subset \bigcup_{x*y\in L}\pi_1^{-1}(L_x)\cap \pi_2^{-1}(L_y) \qquad (4).$$

Indeed:

$$\begin{aligned}(a,b)\in f^{-1}(L) &\quad\Rightarrow\\ f(a,b)\in L &\quad\Rightarrow\\ a*b\in L &\quad\Rightarrow\\ \exists\, L_a, L_b\in \mathcal{L},\ a\in L_a, b\in L_b,\ L_a*L_b\subset L &\quad\Rightarrow\\ (a,b)\in \pi_1^{-1}(L_a)\cap \pi_2^{-1}(L_b)\subset \bigcup_{x*y\in L}[\pi_1^{-1}(L_x)\cap \pi_2^{-1}(L_y).] &\end{aligned}$$

So, (3) and (4) finally give that:

$$f^{-1}(L)=\bigcup_{x*y\in L}[\pi_1^{-1}(L_x)\cap \pi_2^{-1}(L_y)].$$

and we conclude that $f^{-1}(L)$ is open in $G\times G$. In a similar way, $f^{-1}(R)$ is open in $G\times G$, too. □

Suggestion 2 Proposition 6.7 refers to any family of subsets of a set X. Will it be possible to prove it by considering properties of nests? This will hopefully give a characterisation of topological groups, with the involvement of order-theoretic and topological properties nests.

Suggestion 3 We will finally summarise a list of problems worth looking at, concerning ordered groups. The author of this thesis would like to thank Dr. Rolf Suabedissen (University of Oxford) for kindly offering these interesting ideas, during our BOATS (Birmingham-Oxford Analytic Topology Seminar).

1. Suppose (X, τ) be a topological space. If (and only if) τ satisfies a condition P then there is a group operation $*$ on X such that $(X, *, \tau)$ is a topological group.

2. Suppose (X, τ) be a topological space. If (and only if) τ satisfies P then there is a group operation $*$, on X, and an order $<$, on X, such that $(X, *, <, \tau)$ is an ordered topological group with $<$ inducing τ.

3. Suppose X be a set, $*$ a group operation on X and τ a topology (coming from an order?) on X. Are there "easy" conditions to see that $*$ is continuous with respect to τ?

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