OPERADS AND SPECIAL ALGEBRAS

by

FELIX GABRIEL REHREN

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School of Mathematics
College of Engineering and Physical Sciences
University of Birmingham

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To Astronauts

Abstract

We review the basic theory of operads, some variations thereof and other relevant constructions, including algebras over operads. The free operad is constructed from first principles.

We define spheres with tubes and the sewing product, a way of composing the spheres with tubes. This has the structure of an operad and several subsets have other algebraic structures that we consider.

Spheres with tubes were used by Yi-Zhi Huang to study vertex operator algebras. We cover some of the relevant work to look at the relation they hold to the operad of spheres with tubes and consider generalisations both of this operad and of operads generally.

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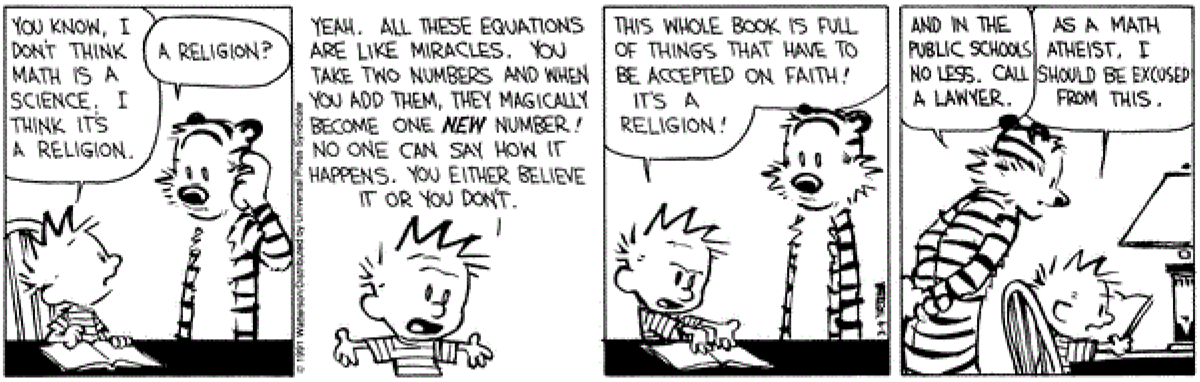


0.1 Introduction

Operads abstract the notion of the composition of functions, and put coherence conditions on families of functions.

An introduction to the subject of operads that concerns itself with their history must start with M. Boardman and R. Vogt, who first defined operads [BV68] in 1968 for their work in homotopy theory. Namegiving was P. May some years later [M72].

We do not concern ourselves with this history in much further detail. It should be mentioned, however, that there was a great renaissance of operads in the 90s (leading to the publication of [O97]) when some papers appeared which stimulated new interest. Since then, there has been so much work done on and with operads that it is not possible to give it all fair mention. Instead, with great bias towards ideas which come up in this work, we present here a small, incomplete choice of topics.

Operads have been given thorough treatment in category treatment, seen as a special kind of algebraic theory. An important reference is T. Leinster's [L03].

In an early renaissance development, operads were seen to capture the combinatorial structure of moduli spaces of stable algebraic curves. This complex geometry side was notably developed by E. Getzler and M. Kapranov, in papers including [GK95] and [GK98].

Operads also found many applications in mathematical physics, in diverse topics which include vertex operator algebras. This is the link we investigate. In particular, we examine the sewing product – a composition of Riemann surfaces with tubes, which are similar to surfaces with parametrised boundaries – defined by Y. Huang in [H97].

We start this text with a simple exposition of the theory of operads. There are many good texts giving introductions; we have borrowed heavily from that of M. Markl, S. Shnider and J. Stasheff [MSS02] (and also from T. Leinster [L03]) throughout Chapter 1. We then explore the elementary relation between operads and algebras (which is not in any sense completely represented here, as it includes, for example, Lie algebras [O97]). The approach to the free operad is also from [MSS02].

We switch hymn sheets to Y. Huang [H97] in Chapter 2, where we use his definitions and exposition as a starting point as we lay some groundwork for the material on the sewing product. Some propositions from [H97] are stated and given proofs, some of which were not previously perspicuous. (If there is doubt, [H97] should be considered authoritative.)

Chapter 3 sees a treatment of the sphere operad considered primarily as an operad. The sphere operad can be seen as a substructure of a more general construction, and we develop the relation to this ambient background. We also investigate the structure of the sphere operad, which turns out to be very analytic. From [H97], we point out some further substructures.

In the final Chapter 4, we first give some scant details of the isomorphism between the sphere operad and vertex operator algebras that Y. Huang demonstrates in [H97]. From [GK95] we get an interesting development of so-called *cyclic* operads that exploits the symmetry of outputs and inputs of operations; we explain the startling relation this has to algebras (and later show that it holds for the sphere operad). There is also a serious attempt made at motivating the investigation of operads in relation to algebraic surfaces, given by expanding on sections of an early paper [V87] on conformal field theories by C. Vafa. Our ultimate work is to generalise the sewing product, and, although the generalisation does not satisfy in the way the original does, we find some relations of interest to investigate.

We concisely review the contents and ordering of this text.

In this Section 0.1, there is a history of operads, followed by an overview of the work accomplished here, and finally the contents and ordering of this text is reviewed. There is also a mention of conventions this author likes, and an acknowledgement of debt.

In Chapter 1, the theory of operads is reviewed. Trees are a motivating example; then, in Section 1.2, operads are defined. These, like other category-theoretic notions, admit many variations and definitions, and we will introduce some intermittently throughout the text. A strong relation between algebras and operads is exhibited in Section 1.4. The main achievement this chapter can boast is the construction and proof of the free operad, in Section 1.6.

Chapter 2 covers material requisite for the sphere operad. Spheres with tubes and their automorphisms are introduced (Sections 2.1, 2.2). Spheres with tubes can be 'sewn' together via their tubes. This composition is discussed over the next three sections, simply to show its elementary properties.

This pays off in Chapter 3, where it is now easy to show that spheres with tubes have the structure of an operad. We discuss some of the substructures present, on the levels of tubes, spheres with tubes and the space of these, in Sections 3.2 through 3.4.

Finally, in Chapter 4, we consider the relationship the sphere operad has with vertex operator algebras (Section 4.1), its origins in theoretical physics (Section 4.3) and look at generalisations of operads in general (Sections 4.2) and of the sphere operad in particular (Section 4.4).

Lastly, there is a view towards possible future developments.

There is a small amount of material in Appendix C on category theory, to provide some background and define the terms used in this work. It ranges from the elementary definitions and standard abstractions, in C.1, to categories with more structure in C.2.

Lemmas exist to support the statements of definitions or proofs of propositions, or to shore up some claim. Propositions are small results. Definitions and unproven statements are ended with a $\square$; in other places, a $\square$ marks the end of a proof.

$\mathbb{N}$ is the set of nonnegative integers. The letters i through n mostly denote whole numbers. For $n \geq 1$, $[n] = \{1, \ldots, n\}$. The symmetric group S_n on n letters is $\mathrm{Aut}[n]$, and $S_0 = S_1$. Permutations and most group actions are written as exponentiation, and these therefore act from the right. The multiplicative group of a ring R is $R^{\times}$.

I owe warm thanks to many people – colleagues, friends and family – for ideas, time and hospitality. Special thanks are due to Cornelius Hoffman, Adam James, and Sergey Shpectorov, who have been of particular help with this work.

CHAPTER 1

OPERADS

The indeterminates i, j, k are positive integers and l, m, n are natural numbers. When dealing with sets, we assume that everything is small, that we are working in some universe.

1.1 Trees

We aim to motivate the subsequent Definition 1.4 of operads with this opening example.

Definition 1.1. A *magma* is a set X with a binary operation $X \times X \to X$. □

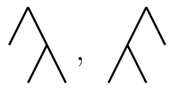
Let X be the free magma generated by $\{x\}$, whose binary operation is written as juxtaposition. An element of X is a word: a sequence of x with parentheses. For illustration, the shortest elements of X are $\{x, (xx), x(xx), (xx)x, \dots\}$.

There is a simple inductively defined pictorial representation of words of X. The word x is drawn as a point •. Words are drawn from left to right in the same order as its constituent subwords are written. Two juxtaposed subwords, if already drawn, are connected by drawing a point above both and connecting it by a line to the top point of each of the two elements. In this way every word of X is uniquely drawn as a tree.

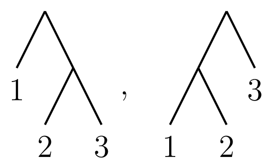
Definition 1.2. A plane rooted *binary tree* (in this section, a *tree*) is a finite directed connected loop-free graph, with a set of points and a set of lines. If points c and p are connected by a line directed towards p, p is the *parent* of c and c the *child* of p. Every point has either 0 or 1 parent point(s), and 0 or 2 ordered child points. Points without children are *leaves*. Let $\tau(n)$ be the set of trees with n leaves, and $\tau = \bigcup_{i\in\mathbb{N}} \tau(i)$ the set of all trees. Also write $\tau(\geq n) = \bigcup_{i\geq n} \tau(i)$. $\square$

Since every tree is finite and loop-free, every tree has a point without a parent; since it is connected, this point is unique. This is the *root*.

We draw trees with the root at the top, and if a child of a point is before the other child in the ordering, it is drawn to the left. For illustration, the first few trees are

$\tau(0) = \emptyset, \quad \tau(1) = \{\,\bullet\,\}, \quad \tau(2) = \Big\{$  $\Big\}, \quad \tau(3) = \Big\{$  $\Big\}.$

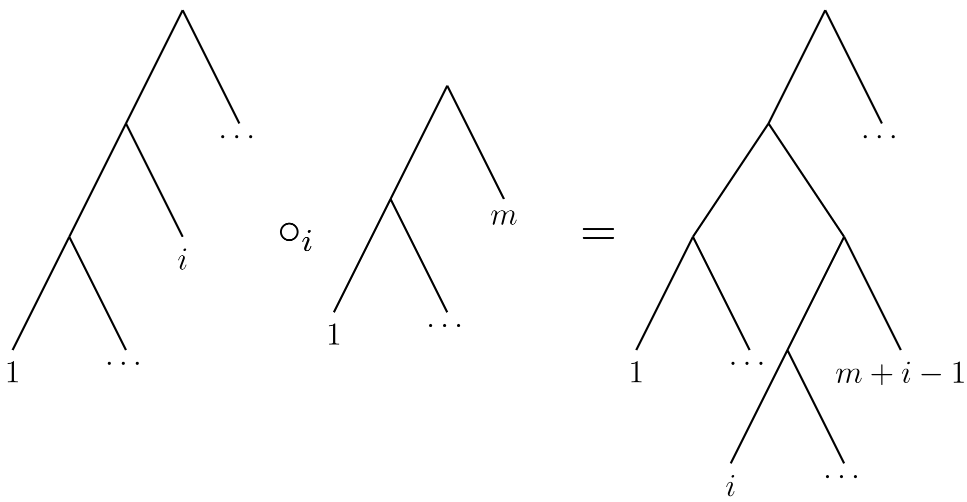
Considering trees as the unique diagrammatic representation of elements of the magma X establishes that two trees are equal precisely if their corresponding words are equal in X. It also fixes the ordering of the leaves (and of the edges directed towards a point) of a tree. Each leaf of a tree t corresponds to an instance of the letter x in the word of X corresponding to t, and the letters are ordered from left to right; we give leaves the same ordering, indexed by $\{1,\dots\}$. We draw the trees so that the leaves are ordered from left to right. Thus

$\tau(3) = \Bigg\{$  $\Bigg\}.$

We have enough structure to introduce composition on these trees: a family of composition operations

$$\circ_i \colon \tau(\geq i) \times \tau \to \tau, \text{ written } \circ_i (t, u) = t \circ_i u.$$

Let ℓ be the i^{th} leaf of t. Then $t \circ_i u$ is defined as the tree formed by substituting ℓ with the root of u. (In the magma, the word of u is substituted in place of the i^{th} instance of x in the word of t.) The ordering of the tree $t \circ_i u$ is adjusted to accomodate the leaves newly added. In a diagram:



This composition has a parameter i. As the leaves of trees are numbered from $\{1, \dots\}$, implicitly from the definition $i \geq 1$. The compositions therefore form a family $\{\circ_i\}_{i \geq 1}$.

We look at some of the elementary properties of the families of trees $\{\tau(n)\}_{n \in \mathbb{N}}$ and compositions $\{\circ_i\}_{i \geq 1}$, which will be indicative of the subsequent definition of operads.

The only tree $\bullet$ in $\tau(1)$ acts as identity: for all $i \geq 1$ and trees t with at least i leaves, $\bullet \circ_1 t = t = t \circ_i \bullet$.

The composition is almost associative: the order in which several trees are composed is irrelevant – the product is always the same – if we allow (purely bookkeeping) changes in the indices of composition. The precise expression is written here. For trees $t \in \tau(l)$, $i \leq l$, $u \in \tau(m)$, $j \leq l+m-1$, $v \in \tau(n)$, we have that

$$(t \circ_i u) \circ_j v = \begin{cases} (t \circ_j v) \circ_{i+n-1} u & \text{if } j < i, \\ t \circ_i (u \circ_{j-i+1} v) & \text{if } i \leq j \leq i+m-1, \\ (t \circ_{j+m-1} v) \circ_i u & \text{if } i+m-1 < j \leq l+m-1. \end{cases}$$

The composition allows us to generate all trees with at least 2 leaves from the unique binary tree $\Lambda = \underset{1\ 2}{\wedge} \in \tau(2)$. (The only tree not generated by Λ is $\bullet$, the unique unary tree.) In fact, by judicious use of the 'associativity' every tree can be written in these terms using only $\circ_1$ and $\circ_2$.

Lemma 1.3. Every tree can be written as a sequence beginning with $\bullet$ and continuing in finitely many uses of $\Lambda, \circ_1, \circ_2$.

Proof. It is simple to prove by induction. The base cases are vacuously $\bullet = \bullet$ and then $\Lambda = \bullet \circ_1 \Lambda$. Now suppose it is true for trees of at most n leaves. Let t with $n+1$ leaves be arbitrary, and let p_1 and p_2 be the two children of its root. For $i = 1, 2$, let t_i be the subtree of t downward closed from p_i, so any point from which there is a path to p_i directed to p_i is in t_i.

Both t_1 and t_2 have at most n leaves; by induction, there exists an expression for each satisfying the claim. Then $t = (\bullet \circ_2 t_2) \circ_1 t_1$ is the desired expression for t. $\square$

The identification of trees with words from the magma (and the induced ordering on the children of any point) gives a fixed ordering on the leaves. If t is a tree with n leaves $\ell_1, \ldots, \ell_n$ (such that ℓ_i is ordered before ℓ_{i+1}), we can therefore give it a labeling $l_t \colon i \mapsto \ell_i$. Then a permutation $\sigma \in S_n$ has an action on t by acting on the labeling: t^σ is the tree with labeling $\sigma \circ l_t$.

It is possible that t^σ is the same as another tree with a 'vanilla' labeling matching the ordering. In general this is not true – for example, (23) acting on the tree of $(xx)x$. So we formally distinguish t^σ from any other tree t' with which it may coincide, for example, $\Lambda^{(12)} \neq \Lambda$. A tree with vanilla labeling represents a word in X and t^σ is not necessarily a word of X.

So we forget the magma X now and define extended sets of trees $\bar{\tau}(n) = \{t^\sigma \mid t \in \tau(n), \sigma \in S_n\}$. Again $\bar{\tau} = \bigcup_{i\in\mathbb{N}} \bar{\tau}(i)$ and $\bar{\tau}(\geq n) = \bigcup_{i\geq n} \bar{\tau}(i)$. Now the composition operation $t^\sigma \circ_i u$ replaces the leaf labeled i with the root of u rather than replacing the i^{th} leaf in the ordering. We have the same family of compositions $\{\circ_i\}_{i\geq 1}$.

For $t \in \tau(n \geq i)$, $u \in \tau(m)$, $\sigma \in S_n$ we can calculate

$$t^\sigma \circ_i u = (t \circ_{\sigma(i)} u)^{\sigma'}$$

where σ' is the permutation given by σ on a modified set of n letters $\{1, \dots, i-1, (i, \dots, i+m-1), i+m, \dots, n+m-1\}$ – *i.e.,* the leaves of u are permuted as a block. Of course there exists $\sigma' \in S_{n+m-1}$ with the same action. Now instead for $t \in \tau(n \geq i), v \in \tau(m)$ and $\sigma \in S_m$, there exists $\sigma' \in S_{m+n-1}$ such that

$$t \circ_i v^\sigma = (t \circ_i v)^{\sigma'}.$$

Here σ' acts with the translated natural action of σ on the set of m letters $\{i, i+1, \dots, i+m-1\}$, and fixes all other letters of $\{1, \dots, m+n-1\}$.

Finally we consider the compositions $\{\circ_i\}_{i\geq 1}$ in a different way. Each $\circ_i$ is a binary function that outputs a tree. They too can be composed. What we are getting at is that, if $\circ_i$ is written $\_\circ_i\_$, then $\_\circ_i(\_\circ_j\_)$ is another composition function of trees, for any i, j, and it has three inputs. Of course $(\_\circ_i\_)\circ_j\_$ is a different composition with three inputs. Like trees, we can compose any compositions. (Observe that a version of Lemma 1.3 also holds in this case.) So there is a composition of compositions:

$$\circledcirc_k\colon \{\circ_i\}_{i\geq k} \times \{\circ_i\}_{i\geq 1} \to \{\circ_i\}_{i\geq 1}.$$

This situation is very close to that considered so far. In effect, we can consider a composition $\circ_i$ as Λ with an extra datum: the index i. Modulo the index, this gives the same algebraic structure that the trees formed. It is quick to see that the compositions of compositions *also* admit composition. In this case, the composition of $\circledcirc$ is algebraically the same as the composition of $\circ$.

1.2 Operads

(Elsewhere this definition may be called *circle-i*, *pseudo* or *reduced*.)

Definition 1.4. An *operad* C consists of a family $\{C(n)\}_{n\in\mathbb{N}}$ of sets $C(n)$ of *operations* $c \in C(n)$ with *arity* n that admits a family of binary compositions $\{\circ_i\}_{i\geq 1}$. Let $1 \leq i \leq m$, and $n \in \mathbb{N}$; then, for $c \in C(m)$ and $d \in C(n)$, compositions are

$$\circ_i\colon C(m) \times C(n) \to C(m+n-1), \text{ written } \circ_i(c,d) = c \circ_i d \in C(m+n-1).$$

Moreover composition satisfies

i. **composition associativity** (*c.f.* Figure 1.1): for $1 \leq i \leq l$ and $1 \leq j \leq l+m-1$, $a \in C(l)$, $b \in C(m)$, $c \in C(n)$,

$$(a \circ_i b) \circ_j c = \begin{cases} (a \circ_j c) \circ_{i+n-1} b & \text{if } j < i \\ a \circ_i (b \circ_{j-i+1} c) & \text{if } i \leq j \leq i+m-1 \\ (a \circ_{j-m+1} c) \circ_i b & \text{if } i+m < j \leq m+n-1. \end{cases}$$

ii. **identity**: there exists $\mathrm{id} \in C(1)$ such that for $1 \leq i \leq n$, $c \in C(n)$,

$$\mathrm{id} \circ_1 c = c, \qquad c \circ_i \mathrm{id} = c.$$

□

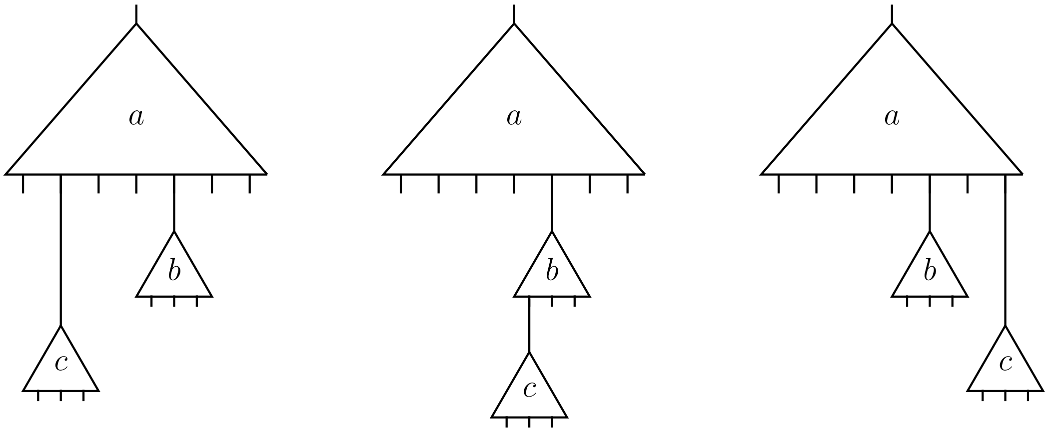


Figure 1.1: Associativity of operads in three cases of i. in Definition 1.4. Here, operads look like lampshades, their composition, mobiles.

Features of the trees introduced earlier are apparent in these definitions. Indeed the system T_2 of trees $\{\tau(n)\}_{n\in\mathbb{N}}$ and compositions $\{\circ_i\}_{i\geq 1}$, described from Definition 1.2 onwards, is an operad. Thus we emphasise that the 'operations' in this definition can be abstract objects without immediate recognisability as operations or functions on a domain, say. The concrete point of view of functions on a domain is the source of the term *arity*, which is in analogy to the number of inputs of a function.

We also introduced permutations of the trees' leaves. Following this, we define a symmetric operad as an operad on which there is a permutation action under which composition is equivariant.

Definition 1.5. A *symmetric* operad (C, α) is an operad C together with an action $\alpha = \{\alpha_n\}$ of S_n on $C(n)$ for all $n \in \mathbb{N}$, such that

iii. **symmetries**: for m, $n \in \mathbb{N}$, $1 \leq i \leq m$, $\sigma \in S_m$ and $\tau \in S_n$, for all $c \in C(m)$,

$d \in C(n)$

$$c^\sigma \circ_i d^\tau = (c \circ_{i^{\sigma^{-1}}} d)^{\sigma \circ_i \tau},$$

where $\sigma \circ_i \tau$ acts on $\{1, \dots, m+n-1\}$ as given by τ acting on $\{i, i+1, \dots, i+n-1\}$ as though it were $\{1, \dots, n\}$, and σ acting on $\{1, \dots, i-1, (i, i+1, \dots, i+n-1)^\tau, i+n, \dots, m+n-1\}$ as on $\{1, \dots, m\}$. □

As algebraically fairly elementary objects, perhaps with a flavour of abstract nonsense, a category theoretic understanding of operads is desirable. There is an equivalent definition in terms of multicategories. These are otherwise ordinary categories admitting extended hom sets, such that each arrow has a sequence of sources rather than a unique source. Recall their Definition C.15.

Now an operad is a multicategory with one object. The requirements of composition associativity and identity are obviously the same in the definition of each object, and it is straightforward to construct one from the other:

Suppose that $\{C(n)\}_{n \in \mathbb{N}}$ with $\{\circ_i\}_{i \geq 1}$ is an operad. Then C with sole fixed object $x \in C$ and morphisms $\hom_C(x, \dots, x;\ x) = C(n)$ is a multicategory. Similarly, given a multicategory C with sole object x, $\{\hom_C(x, \dots, x;\ x)\}_{n \in \mathbb{N}}$ with the compositions $\{\circ_i\}_{i \geq 1}$ is an operad.

The hom sets of a category are elements of the category Set. This can be generalised. In an enriched category (Definition C.14), the hom sets are elements of some symmetric monoidal category (Definition C.13). Looking at operads as enriched multicategories is useful for a more general definition. We single out one particular case of interest and use. It is easy to abstract this one to other (strict) symmetric monoidal categories.

Fix a commutative unital ring k. Then $\mathrm{k}\mathrm{Mod}$ is the strict symmetric monoidal category of k-modules with tensor product $\otimes_{\mathrm{k}}$ (usually written $\otimes$) and unit k. Note that the correct analogy of the empty set (as initial element in the category Set) in $\mathbf{k}\mathrm{Mod}$ is 0.

When we consider a $\mathbf{k}\mathrm{Mod}$-enriched multicategory with one object, we have an operad with more structure.

Definition 1.6. A $\mathbf{k}$-*operad* (read 'copperad') K is a collection $\{K(n)\}_{n\in\mathbb{N}}$ of $\mathbf{k}$-modules with $\mathbf{k}$-linear binary compositions $\circ_i\colon K(m \geq i) \otimes K(n) \to K(m+n-1)$ satisfying commutativity of the following diagrams:

i. **composition associativity**: for $1 \leq l, m$ and $n \in \mathbb{N}$, $1 \leq h, j \leq l$, $1 \leq i \leq m$, $1 \leq k \leq l+m-1$,

$$\begin{tikzcd}
K(l) \otimes K(m) \otimes K(n) \arrow[r, "\mathrm{id} \otimes \circ_i"] \arrow[d, "\circ_h \otimes \mathrm{id}"'] & K(l) \otimes K(m+n-1) \arrow[d, "\circ_j"] \\
K(l+m-1) \otimes K(n) \arrow[r, "\circ_k"'] & K(l+m+n-2)
\end{tikzcd}$$

ii. **identity**: there exists $\eta\colon \mathbf{k} \to K(1)$ such that

$$\begin{tikzcd}
K(1) \otimes K(n) \arrow[r, "\circ_1"] & K(n) \\
\mathbf{k} \otimes K(n) \cong K(n) \arrow[u, "\eta \otimes \mathrm{id}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

iii. and **symmetry**: if K is a $\mathbf{k}$-operad with $\mathbf{k}$-linear action α satisfying Definition 1.5. $\square$

Equivalently to iii., we see that the $\mathbf{k}$-operad is symmetric precisely when each $K(n)$ is a $\mathbf{k}[S_n]$-module, where $\mathbf{k}[S_n]$ is the $\mathbf{k}$-module $\mathbf{k} \otimes_{\mathbf{k}} S_n$.

1.3 Morphisms

Definition 1.7. An operad homomorphism or *omomorphism* $f\colon C \to D$ consists of a family of maps $\{f_n\colon C(n) \to D(n)\}_{n\in\mathbb{N}}$ equivariant under the compositions. So for $c \in C(m)$, $c' \in C(n)$, if the compositions of C and D are written $\circ_i$ and $*_i$ respectively,

$$f_{m+n-1}(c \circ_i c') = f_m(c) *_i f_n(c').$$

If f is a k(-linear)-omomorphism, kernel $\ker f$ is the collection $\{\ker f_n\}_{n\in\mathbb{N}}$, and $\operatorname{im} f$ &c is likewise.

An operad D is a *factor* operad of C if there exists an omomorphism $f\colon C \to D$ whose every component f_m is surjective. It is a *sub*operad of C, written $D \leq C$, if there exists an omomorphism $f\colon D \to C$ whose every component f_m is an embedding. □

Since we have now defined operads and their morphisms, we have everything we need to define the environment in which we are working.

Definition 1.8. The *category of operads* Op (*category of* k-*operads* kOp respectively) is the subcategory of Set (kMod) whose objects are operads (k-operads) as in Definition 1.4 (Definition 1.6) and whose morphisms are (k-linear) omomorphisms. □

To look at some factor operads, we first introduce three simple operads: the natural number operad, the trivial operad and the symmetric group operad. As in Definition 1.8, these all admit analoguous k-operads.

The *natural number operad* N is the collection $\{\{n\}\}_{n\in\mathbb{N}}$ together with the compositions $\circ_i = \circ$ for all $1 \leq i$ that act as $m \circ n = m + n - 1$ (and are not defined on $(0,0)$, so $0 \circ 0$ is never evaluated). By the obvious identification $n \leftrightarrow \{n\}$, as sets $\mathbb{N} \cong N$ and so this puts an operad structure on the natural numbers.

Suppose that C is an operad and $f\colon C \to N$ an omomorphism. The map $f_m\colon C(m) \to N(m)$ is the obvious constant map onto m, for each $m \in \mathbb{N}$. Indeed, since the omomorphism is labeled with the arity of the operations, we can build a different omomorphism with domain C by replacing m with some object $*$ in this definition; $f_m\colon c \mapsto *$ for each $c \in C(m)$ can also be an omomorphism.

This leads us to define the *trivial operad* I: it is the collection $\{\{*\}\}_{n\in\mathbb{N}}$, with the composition $\circ\colon \{*\} \times \{*\} \to \{*\}$.

The omomorphism $f\colon N \to I$ given by $f_m\colon \{m\} \to \{*\}$ is an isomorphism of operads, since every f_m is a bijection of sets and – precisely because each component f_m is labeled with arity m, even though $*$ is not – it preserves composition. We therefore see that composition of $*$ is well-behaved even though it does not have an intrinsic arity.

In this way the natural number operad N and the trivial operad I are isomorphic as operads. However the behaviour of N is what we intuitively expect from operads, and I is in some sense pathological. Clearly they are not isomorphic as sets (*i.e.,* there is no bijection in the category Set from N to I). So we can formalise this difference. An operad C has an *intrinsic arity* if there exists a function $f\colon \bigcup_{n\in\mathbb{N}} C(n) \to N$ such that $\{f|_{C(n)}\colon C(n) \to \{n\}\}_{n\in\mathbb{N}}$ is an omomorphism. This exists precisely when the sets $C(m)$ are pairwise disjoint. Note that intrinsic arity can distinguish between isomorphic operads.

The natural number operad N admits many suboperads. For example, $\{\emptyset, \{1\}, \emptyset, \dots\}$ (with no operations of arity other than 1) is an operad under the composition $\circ$, for $1 \circ 1 = 1 + 1 - 1 = 1$; also $\{\{1\}\} \cup \{\{n\}\}_{n\geq 3}$ is closed under $\circ$ and is therefore an operad. Indeed any subset of $\mathbb{N}$ closed under $\circ\colon (m,n) \mapsto m+n-1$ induces a suboperad of N; this classifies all suboperads of N.

N is a terminal object of the category Op, because the one-point set is a terminal object in the category Set. We also see that N is very often a factor operad of an operad C; there always exists an omomorphism $f\colon C \to N$, but $f_m\colon C(m) \to N(m)$ is surjection if and only if $C(m)$ is nonempty. So N is a factor operad precisely if $C(n)$ is nonempty for all $n \in \mathbb{N}$. And, for any operad C, there exists a suboperad M of N which is a factor operad of C. We call a suboperads M of N an *arity shape*, as each such M describes the pattern of arities in any operad of which M is a factor operad.

The family $\{S_n\}_{n\in\mathbb{N}}$ of symmetric groups has a natural operad structure, named the *symmetric group operad* S. In fact a description of this structure is already in the axiom iii. for symmetric operads in Definition 1.5, and S can therefore be described by requiring that iii. holds and $\mathrm{id}_m \circ_i \mathrm{id}_n = \mathrm{id}_{m+n-1}$ for $1 \le i \le m$, $1 \le n$.

We also give an explicit description. Composition $\circ_i$ of two operations $\sigma \in S_m$ and $\tau \in S_n$ gives the following element of S_{m+n-1}, in two-line notation with permutation denoted by exponentiation:

$$\begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & i+n-1 & i+n & \dots & m+n-1 \\ 1^\sigma & \dots & (i-1)^\sigma & i^\sigma+1^\tau-1 & i^\sigma+2^\tau-1 & \dots & i^\sigma+n^\tau-1 & (i+1)^\sigma+n & \dots & m^\sigma+n \end{pmatrix}.$$

So the action of τ is translated, and the entire set of letters on which τ acts is acted on as a single block by σ. This $\circ_i$ composition is induced by the ordinary permutation composition in the sense that it embeds σ and τ into S_{m+n-1} and takes their permutation composition.

Now recall the symmetric tree operad $\bar{T}_2$, constructed by extending the tree operad T_2 by relabeling its leaves using permutations. There are at least two factor operads: $f\colon \bar{T}_2 \to T_2$ to the tree operad, which forgets the labelings, and $g\colon \bar{T}_2 \to S$ to the symmetric group operad, which forgets the trees. Both arise from the unique presentation as t^σ that exists for each operation in $\bar{T}_2$, where $\sigma \in S_m$ and $t \in \tau(m)$ if $t^\sigma \in \bar{\tau}(m)$. Then $f\colon t^\sigma \mapsto t$ and $g\colon t^\sigma \mapsto \sigma$. Every element t^σ of $\bar{T}_2$ can be written (t, σ).

It is clear that this is related to the product of sets, written $\times$ (or $\times_{\mathrm{Set}}$, for emphasis). We define also a product on operads. We omit the analoguous definition for k-operads, because it is quite analoguous.

Definition 1.9. Suppose that C and D are operads with compositions $\circ_i$ and $\star_i$, that is, both are elements of the category Op. Then define $\times_o \colon \mathrm{Op} \times \mathrm{Op} \to \mathrm{Op}$ so that $C \times_o D = \{C(m) \times_{\mathrm{Set}} D(m)\}_{m \in \mathbb{N}}$, where the compositions $\{(\circ_i, \star_j)\}_{1 \leq i,j}$ map as (for $c \in C(m)$, $d \in D(m)$, $c' \in C(n)$, $d' \in D(n)$ and $1 \leq i, j \leq m$) $(c, d)(\circ_i, \star_j)(c', d') = (c \circ_i c', d \star_j d')$. (To match Definition 1.4, we can identify $1 \leq i, j$ with a linearly ordered set.) $\square$

Observe that this $\times_o$ is a tensor product in the category of operads, and this feature comes from the tensor product $\times_{\mathrm{Set}}$ of Set.

It is now clear that $\bar{T}_2 = T_2 \times_o S$. This $T_2 \times_o S$ is said to be the *symmetrisation* of T_2, which is defined for any operad C as $C \times_o S$. This is equivalent to taking T_2 together with the faithful permutation action on its labels to make a symmetric operad, as per Definition 1.5.

In the next two lemmas, we see that, for operads related by omomorphisms, additional set-morphisms of the operads as families of sets can be enough to fully determine the operad structure.

Lemma 1.10. Suppose that X is an operad with composition $\circ_i$ and $f \colon X \to C$, $g \colon X \to D$ make C and D into factor operads. If there is a family h of set bijections $h_n \colon C(n) \to D(n)$ such that $h \circ f = g$, then C and D are isomorphic as operads.

Proof. Since f and g are omomorphisms, so is h, and therefore h is an isomorphism of operads. $\square$

Lemma 1.11. Suppose that X and f, C, g, D are as in Lemma 1.10. If, for all $n \in \mathbb{N}$, for all pairs $c \in C(n)$, $d \in D(n)$ there exists a unique $x \in X(n)$ such that $f(x) = c$ and $g(x) = d$, then $X \cong C \times_o D$.

Proof. Since f and g are surjections, we see that $X(m) = C(m) \times_{\mathrm{Set}} D(m)$ for all m; so $h \colon x \mapsto (f(x), g(x))$ is a bijection. The compositions on C and D are completely determined by those from X, so we also denote them $\circ_i$. Then $(f(x_1 \circ_i x_2), g(x_1 \circ_i x_2)) \leftrightarrow x_1 \circ_i x_2 \leftrightarrow (f(x_1), g(x_1))(\circ_i, \circ_i)(f(x_2), g(x_2))$. Thus h is the desired isomorphism. $\square$

Here is another way a suboperad could arise. Suppose that we are given a binary operation c and compositions $\circ_1$ and $\circ_2$ from an operad C. Then we can write two 3-ary, tertiary, operations: $c \circ_1 c$ and $c \circ_2 c$. These can be composed further.

In the end, we have an entire family of operations and compositions: a suboperad of C. (This is an idea converse to Lemma 1.3, which shows that the tree operad T is generated this way.) We generalise this notion in the next definition, as an *operad closure* or *completion*.

Definition 1.12. Let C be an operad $\{C(n)\}_{n \in \mathbb{N}}$ with compositions $\{\circ_i\}_{1 \leq i}$. Let X be a family of subsets $\{X(n) \subseteq C(n)\}_{n \in \mathbb{N}}$, and $\{\circ_i\}_{i \in I \subseteq \{1, \dots\}}$ a subset of the compositions of C. Then X generates a suboperad $\langle X \rangle_C$ of C, usually written just $\langle X \rangle$, inductively defined as

$$\langle X \rangle(n) = \{c \circ_i d \mid c \in \langle X \rangle(l) \cup X(l), d \in \langle X \rangle(m) \cup X(m), 1 \leq i \leq l \text{ such that } l + m - 1 = n\}$$

with all the compositions $\{\circ_i\}$ that can be generated from the given ones. If there is no unit $\mathrm{id} \in X(1)$, we adjoin one. $\square$

With an abuse of notation, we make this definition more flexible. For an operation c of arity n from C, define $\langle c \rangle$ as $\langle X \rangle$ for X the family $\{\emptyset, \dots, \emptyset, \{c\}, \emptyset, \dots\}$, and allow all compositions from C. Similarly any incomplete collection of sets can be extended to a family of subsets by adjoining empty sets.

Rather than defining the operad closure $\langle X \rangle$ inductively 'upwards', we can characterise it as a minimal (operadic) hull. To this end, we prove

Lemma 1.13. Suppose that X, Y are suboperads of an operad C. Then $D = X \cap Y = \{X(n) \cap Y(n)\}_{n \in \mathbb{N}}$ is a suboperad of C.

Proof. Since X and Y are each operads, the composition $\circ_i$ of C is also a composition $D(m) \times D(n) \to D(m+n-1)$ on D. Composition associativity follows since it holds in all of C, and as the identity element id of C is in $X(1)$ and $Y(1)$, $\mathrm{id} \in D(1)$ as required. $\square$

Now we can prove

Proposition 1.14. *Let X be a family of subsets $\{X(n) \subseteq C(n)\}_{n\in\mathbb{N}}$ together with all compositions $\{\circ_i\}_{1\leq i}$ of C. Then $\langle X\rangle_C = \bigcap\{D \leq C \mid X(n) \subseteq D(n) \quad \forall\, n \in \mathbb{N}\}$.*

Proof. Denote $Z = \bigcap\{D \leq C \mid X(n) \subseteq D(n) \quad \forall\, n \in \mathbb{N}\}$. From Lemma 1.13, it is an operad. By Y denote the expression for $\langle X\rangle$ in Definition 1.12. Clearly Y is a suboperad of C containing X componentwise and therefore $Z \subseteq Y$. Now suppose that $y \in Y(m)$. Therefore y can be written as a composition of operations from X. Thus any operad containing X must also contain y, so $y \in Z(m)$. So $Y(m) = Z(m)$ for all m. As their operad composition is the same, they are equal as operads. $\square$

1.4 Algebras

The case of omomorphisms whose codomain is an endomorphism operad (which we introduce now) is special. The endomorphism operads play a somewhat similar rôle to that of the general linear groups in representation theory – in a heliocentric view of operads, they form the sun. Such an omomorphism gives enough structure to induce an associative algebra, that is, a module with a multiplication. (Throughout, we will only consider associative algebras.) *A priori* this multiplication could have any arity.

Throughout this section k is a fixed commutative unital ring, V is a finitely generated k-module and v and v_i are indeterminate vectors from V, for $i \in \mathbb{N}$ or i any Latin letter. Also $\mathrm{End_V(n)}$ is the module $\hom_{\mathbf{k}}(V^{\otimes n}\ V)$ of k-linear maps $V^{\otimes n} \to V$.

Definition 1.15. The *endomorphism* k-*operad* End_V of a given k-module V is the family of sets of operations $\{\mathrm{End_V(n)}\}_{\mathrm{n}\in\mathbb{N}}$ with operadic composition determined by the usual composition of multilinear maps. So

$$f \circ_i g = f(\mathrm{id}_V{}^{\otimes i-1} \otimes g \otimes \mathrm{id}_V{}^{\otimes m-i})$$

for the now-familiar choices $f\colon V^{\otimes m} \to V$, $1 \leq i \leq m$, $g\colon V^{\otimes n} \to V$ for $n \in \mathbb{N}$.

To allow this to be considered a symmetric operad, we point out that it admits a faithful permutation action: for $f \in \mathrm{End}_V(m)$ and $\sigma \in S_m$, let $f^\sigma(v_1 \otimes \cdots \otimes v_m) = f(v_{1^\sigma} \otimes \cdots \otimes v_{m^\sigma})$. □

In analogy to representation theory, a representation of an operad is an omomorphism into an endomorphism operad. With some foresight, we term this an algebra.

Definition 1.16. An *algebra* over a k-operad K is a k-module V together with a *representation* omomorphism $\rho\colon K \to \mathrm{End}_V$, for End_V the endomorphism operad. □

This in fact captures the full ordinary meaning of algebras, although this interpretation is not apparent from this rather concise definition. We start to rectify this with the following lemma, which states that algebras over operads put a certain structure on the module V.

Lemma 1.17. An algebra $(V, \rho\colon K \to \mathrm{End}_V)$ is equivalently $(V, \mathrm{ev}_\rho = \mathrm{ev})$: the k-module V with k-linear *evaluation maps* $\mathrm{ev}\colon K(n) \otimes V^{\otimes n} \to V$ for $n \in \mathbb{N}$, such that the diagrams of the following properties commute:

i. **associativity**: for $1 \leq i \leq m$ and $n \in \mathbb{N}$,

$$\begin{tikzcd}
K(m) \otimes K(n) \otimes V^{\otimes n} \otimes V^{\otimes m-1} \arrow[r, "\circ_i \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mathrm{ev} \otimes \mathrm{id}"'] & K(m+n-1) \otimes V^{\otimes m+n-1} \arrow[d, "\mathrm{ev}"] \\
K(m) \otimes V^{\otimes m} \arrow[r, "\mathrm{ev}"] & V
\end{tikzcd}$$

ii. **identity**: for the unit map $\eta\colon \mathbf{k} \to K(1)$ of K (of ii. in Definition 1.6)

$$\begin{tikzcd}
\mathbf{k} \otimes V \arrow[r, "\cong"] \arrow[d, "\eta \otimes \mathrm{id}"'] & V \\
K(1) \otimes V \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

iii. and, if the k-operad is symmetric, **symmetries**: for $m \geq 1$ and $\sigma \in S_m$,

$$\begin{tikzcd}
K(m) \otimes V^{\otimes m} \arrow[r, "\sigma \otimes \sigma^{-1}"] \arrow[dr, "\mathrm{ev}"'] & K(m) \otimes V^{\otimes m} \arrow[d, "\mathrm{ev}"] \\
& V
\end{tikzcd}$$

Proof. We give the constructions. Let $\rho\colon K \to \mathrm{End}_V = \{\rho_m\colon K(m) \to \mathrm{End_V(m)}\}_{\mathrm{m}\in\mathbb{N}}$ be an algebra over the k-operad K. Then let $\mathrm{ev}\colon K(n) \otimes V^{\otimes n} \to V$ be the map

$$\mathrm{ev}\colon c \otimes \bar{v} \mapsto (\rho(c)\colon V^{\otimes n} \to V)(\bar{v}) \in V,$$

for $c \in K(n)$ and $\bar{v} \in V^{\otimes n}$.

Suppose instead we have k-linear $\mathrm{ev}\colon K(n) \otimes V^{\otimes n} \to V$ satisfying points i. and ii. Then for every $c \in K(n)$ this defines ('curries') a map $\mathrm{ev}(c)\colon V^{\otimes n} \to V$ as $\bar{v} \mapsto (\mathrm{ev}(c))(\bar{v}) = \mathrm{ev}(c \otimes \bar{v}) \in V$ for $\bar{v} \in V^{\otimes n}$. This $\mathrm{ev}(c)$ is necessarily also k-linear and therefore $\mathrm{ev}(c) \in \mathrm{End}_V(n)$. So define $\rho\colon K \to \mathrm{End}_V$ as the map $c \mapsto \mathrm{ev}(c)$.

The image of ρ satisfies operad associativity. Let $c \in K(m)$ and $d \in K(n)$. Then, for $1 \leq i \leq m$, $\mathrm{ev}(c) \circ_i \mathrm{ev}(d) = \mathrm{ev}(c \circ_i d)$ as from i., the two sides of the equality are pointwise equal for all $\bar{v} \in V^{\otimes m+n-1}$. In this way each of the axioms for operads implies that the related axiom for the evaluation map is satisfied – and vice versa. $\square$

What can we say about the evaluation maps crucial to this Lemma 1.17? For simplicity of exposition, we suppose that $K = \langle K(1), K(2) \rangle$, although everything is the same for K of any arity shape (as defined in Section 1.3).

Suppose that $\rho\colon K \to \mathrm{End}_V$ is an algebra. For every $c \in K(2)$, $\mu_2^c(v_1, v_2) = \mathrm{ev}(c \otimes (v_1 \otimes v_2))$ is a binary product on V. Since $c \circ_1 c$, $c \circ_2 c \in K(3)$, by i. in the lemma this defines also $\mathrm{ev}(c \circ_i c \otimes (v_1 \otimes v_2 \otimes v_3))$ for $i = 1, 2$. Indeed, this defines ev on all $\langle c \rangle(n) \otimes V^{\otimes n}$ by induction, for $\langle c \rangle$ as in Definition 1.12.

We conclude that if K as an operad is equal to $\langle K(1), K(2) \rangle$ then ev is inductively completely determined by its values on $K(1) \otimes V$ and $K(2) \otimes V^{\otimes 2}$.

In our view, even though a binary product also induces n-ary products for all $n \geq 2$, the binary product is the crucial object. Therefore let us say that a family of products 'is' a single product μ (of some arity n) if it can be generated by (various compositions of) the product μ. (We can phrase this in terms of operads. It corresponds to taking some minimal family $\mathcal{C}$, with $\mathcal{C}(m) \subseteq C(m)$ for all m, such that $C = \langle \mathcal{C} \rangle$.)

1.5 Associative algebras

We now seek to characterise associative and commutative associative algebras as algebras over k-operads.

Define $A(0) = 0$ and $A(n) = \mathbf{k}$ for all $n \geq 1$. Then $\{A(n)\}_{n \in \mathbb{N}}$ with compositions $\otimes_{\mathbf{k}} = \otimes$ is an operad A, for, as a k-module, $\mathbf{k}^{\otimes n} \cong \mathbf{k}$ and therefore $A(m) \otimes A(n) = \mathbf{k} \otimes \mathbf{k} \cong \mathbf{k} = A(m + n - 1)$ for all $m, n \geq 1$. Observe that this is a suboperad of $\mathrm{End}_{\mathbf{k}}$ (analoguous to the trivial operad I of Section 1.3).

Now let $\rho\colon A \to \mathrm{End}_V$ be a representation of A. By the preceding Lemma 1.17, this is equivalent to the k-module V and evaluation maps ev. Take $1_n = 1 \in \mathbf{k}$ as the unit of k considered as an element of $A(n)$. Then the evaluation map induces n-products written $\mu_n\colon V^{\otimes n} \to V$ for all $n \in \mathbb{N}$ on V, as $\mu_n(v_1, \ldots, v_n) = \mathrm{ev}(1_n, v_1, \ldots, v_n)$, for indeterminates $v_1, \ldots, v_n$ from V. By the associativity i. of ev, μ_n can be written as a product of several μ_i for i of lower degree. It is obvious that other products are also induced by defining, for $a \in A(n) \cong \mathbf{k}$, ${\mu^a}_n = \mathrm{ev}(a, v_1, \ldots, v_n)$. By the $\mathbf{k}$-linearity of ev, these are merely scaled versions of the same product: ${\mu^a}_n = a\mu_n$.

The diagram of identity, ii. in the preceding lemma, forces $\mu_1(v) = v$, *i.e.,* $\mu_1 = \mathrm{id}$. If we bias our point of view to bilinear multiplication, written as juxtaposition, then the associativity of ev forces $(v_1v_2)v_3 = v_1(v_2v_3)$. (This bias is arbitrary and emphatically does not come from the use of operads to describe algebras.)

Thus any algebra over this operad is an associative algebra.

Now A is a non-symmetric operad. Let τ be the family of trivial actions for A; then (A, τ) is (trivially!) a symmetric operad. So consider instead a representation $\rho\colon (A, \tau) \to \mathrm{End}_V$. Then iii. forces $\mu_n(v_1, \ldots, v_n) = \mu_n(v_{1^\sigma}, \ldots, v_{n^\sigma})$ for all $n \in \mathbb{N}$ and $\sigma \in S_n$. In particular, $v_1v_2 = v_2v_1$ and therefore any algebra over the trivially symmetrised $\mathbf{k}$-operad (A, τ) is commutative.

A is a suboperad of $\mathrm{End}_{\mathbf{k}}$. Any algebra over $\mathrm{End}_{\mathbf{k}}$ is also an algebra over A. In general we expect that algebras over an operad are more restricted than those over its suboperads (since the larger operad encodes more information). In this case, since $\mathrm{End}_{\mathbf{k}}(0) = \mathbf{k}$ is nonempty, a representation $\mathrm{End}_{\mathbf{k}} \to \mathrm{End}_V$ also induces maps $\mu_0 : V^{\otimes 0} \to V$. As $V^{\otimes 0} = \mathbf{k}$, this endows the algebra with a multiplicative identity: there exists $1_V = \mu_0(1_{\mathbf{k}})$ for which $1_V \cdot v = v = v \cdot 1_V$ for all $v \in V$. The symmetrization $\mathrm{End}_{\mathbf{k}} \times_o S$ is a symmetric operad and so its algebras are unital commutative algebras.

We have a description of algebras over these operads. Now we can show that all algebra products on a given module can be induced from the algebra of an operad. We prove only the easy-to-describe binary case.

Proposition 1.18. *Let A be the operad $\{0\} \cup \{\mathrm{End}_{\mathbf{k}}(n)\}_{n\geq 1}$ as described above, and V a* k*-module. For every associative binary product on V, there exists a representation $\rho\colon A \to \mathrm{End}_V$ inducing that product, and there is a unique representation inducing* *only that product.*

First, a simple case. The ring $\mathbf{k}$ is a $\mathbf{k}$-module, so the algebra of $\rho\colon A \to \mathrm{End}_{\mathbf{k}}$ is in the scope of this statement. One such algebra is the map $A(n) \to \mathrm{End}_{\mathbf{k}}(n)$, $a \mapsto \mathrm{ad}_a$, which is an omomorphism. For all representations $\rho\colon A \to \mathrm{End}_{\mathbf{k}}$, the induced multiplication on $\mathbf{k}$ is the ordinary product of $\mathbf{k}$, by $\mathbf{k}$-linearity of the terms.

Proof. Let $\cdot$ be some fixed product $V \times V \to V$. First we show existence.

As before, v and v_i for all i are indeterminate vectors from V. We define $\mathrm{ev}(a \otimes v) = av$ for $a \in A(1) = \mathbf{k}$ in parallel to the argument preceding this proposition. Then define $\mathrm{ev}\colon A(2) \otimes V^{\otimes 2} \to V$, for some $a \in A(2) = \mathbf{k}$ and vectors v_1, v_2 in V, as $\mathrm{ev}(a \otimes v_1 \otimes v_2) = av_1 \cdot v_2$. Then also $\mu_2(v_1, v_2) = v_1 \cdot v_2$ for all v_1, v_2 in V, which is the special case $a = 1_{\mathbf{k}}$.

It remains only to verify that this defines ev for all $A(n) \otimes V^{\otimes n}$ and that this satisfies Lemma 1.17. Fix $n > 2$. Assume that ev is defined and that associativity i. holds for all $m < n$ and therefore both are true for $2 < n$ and $n - 1 < n$.

Take $a \in A(n) = \mathbf{k}$ and $v_1, \ldots, v_n \in V$. This $a \in \mathbf{k} = A(n-1)$ and $a = a \otimes 1_2$ for $A(2) \ni 1_2 = 1_{\mathbf{k}} \in \mathbf{k}$. Thus $a \otimes (v_1 \otimes \ldots \otimes v_n) \in A(n) \otimes V^{\otimes n}$ is equal to $a \otimes 1_2 \otimes (v_1 \otimes v_2) \otimes (v_3 \otimes \ldots \otimes v_n) \in A(n-1) \otimes A(2) \otimes V^{\otimes 2} \otimes V^{\otimes n-1}$. From this ev is defined inductively for n and simultaneously we see that associativity holds for all n.

The $\mathbf{k}$-linearity is evident: for $a, b \in \mathbf{k}$, $\mathrm{ev}(b(a \otimes v_1 \otimes v_2)) = \mathrm{ev}(ab \otimes v_1 \otimes v_2) = abv_1 \cdot v_2 = b\,\mathrm{ev}(a \otimes v_1 \otimes v_2)$.

The identity property ii. holds, for $\mathbf{k}$ comes already with its multiplicative identity 1 and so the unit map $\mathbf{k} \to A(1) = \mathbf{k}$ is just id. Then the requisite diagram commutes immediately, and this completes the proof of existence.

For uniqueness, we use the usual argument; suppose that there is another evaluation map $\mathrm{ev}'\colon A(n) \otimes V^{\otimes n}$ that induces the product $\cdot$ on V. If ev and ev' induce only this product, they are fully determined by their values on $A(1) \otimes V^{\otimes 1}$ and $A(2) \otimes V^{\otimes 2}$. Now $\mathrm{ev}'(1_2 \otimes (v_1 \otimes v_2)) = \mu_2'(v_1, v_2) = v_1 \cdot v_2 = \mu_2(v_1, v_2) = \mathrm{ev}(1_2 \otimes (v_1 \otimes v_2))$ for all $v_1, v_2 \in V$. By $\mathbf{k}$-linearity, ev and ev' also coincide for all other operations from $A(1)$ and $A(2)$ and therefore they are equal on this domain. Thus there is no difference between ev and ev'. $\square$

Proposition 1.19. *Suppose that* $\rho\colon K \to \mathrm{End}_V$ *induces exactly one product* μ *on* V *and this has arity* n. *Then there exists* $c \in K(n)$ *such that* $\langle c \rangle \cong \rho^{-1}(\mathrm{End}_V)$*, the preimage of* End_V *in* K.

Proof. We have long discussed how elements of the operad induce products on the algebra. No operation d of K can give rise to a different product (*i.e.,* one not generated by μ) since μ is the unique induced product. Thus either d induces the same product μ, or d induces a trivial product $\mu^d\colon \bar{v} \mapsto 0$ for $\bar{v} \in V^{\otimes m}$ and therefore $d \in \ker \rho$. Take c as any operation of arity $\min\{m \mid \exists d \in K(m) \text{ inducing } \mu\} = n$ such that $\mathrm{ev}(c \otimes \bar{v}) = \mu(\bar{v})$ for all $\bar{v} \in V^{\otimes n}$. $\square$

So the term 'algebra' for omomorphisms to the endomorphism operad is justified, in that any algebra can be thus represented using the operad A. That operad is trivial, and all information about the algebra is in the representation omomorphism. There is a way to reverse this situation, using the isomorphism between an algebra and its dual. Note that a k-algebra $\mathcal{A}$ is also a k-module, written $\mathcal{A}^\circ$.

Lemma 1.20. Let $\mathcal{A}$ be a unital associative algebra over k, with product $\cdot$. Then, for $A(1) = \mathcal{A}^\circ$ and $A(n) = 0$ for all $n \neq 1$, $A = \{A(n)\}_{n \in \mathbb{N}}$ with compositions $\{\cdot\}_{i \geq 1}$ is a k-operad. There exists a representation $\rho\colon A \to \mathrm{End}_{\mathcal{A}^\circ}$ whose algebra is $\mathcal{A}$.

Proof. Clearly $A(1)$ is closed under $\cdot$. Associativity for the operad composition is inherited from the associativity of the algebra product $\cdot$. The identity comes from the multiplicative unit $1 \in \mathcal{A}$.

Then the familiar adjoint representation $\rho\colon a \mapsto \mathrm{ad}_a$ is an omomorphism $A \to \mathrm{End}_{\mathcal{A}^\circ}$ (which is verified practically as soon as thought of). Thus by Lemma 1.17, the algebra of ρ is the k-module $\mathcal{A}^\circ$ with $\mathrm{ev}\colon A(1) \otimes \mathcal{A}^\circ \to \mathcal{A}^\circ$. The evaluation maps ev give multiplication to the module and this is precisely the algebra $\mathcal{A}$. $\square$

In fact, using the notations of this proof, $\rho A = \mathrm{ad}\, A$ and $\mathrm{ad}\, A(1) \cong \mathcal{A}^{\circ *}$, and the evaluation maps endowed the module $\mathcal{A}^{\circ *}$ with an action on $\mathcal{A}^{\circ}$, thereby making $\mathcal{A}^{\circ}$ into the algebra $\mathcal{A}$ again. The operad A here is a thinly disguised version of the algebra $\mathcal{A}$.

Consider instead the following operad B, for $\mathcal{A}$ an associative algebra. Let $B(0) = 0$, $B(1) = \mathbf{k}\,\mathrm{id}$ and $B(n) = \mathcal{A}^{\otimes n}$ for $n > 2$. Composition $\circ_i$ for elements $\bar{b} \in B(m)$ and $\bar{c} \in B(n)$ is

$$\bar{b} \circ_i \bar{c} = (b_1 \otimes \ldots \otimes b_{i-1} \otimes b_i \cdot c_1 \otimes \ldots \otimes b_i \cdot c_n \otimes b_{i+1} \otimes \ldots \otimes b_m) \in B(m+n-1).$$

Now $B = \langle B(2) \rangle$ if and only if there exists $a \in \mathcal{A}$ such that $\mathrm{ad}_a \colon \mathcal{A} \to \mathcal{A}$ is a surjection (such as there must be in a unital algebra). If there is no such a, any representation into $\mathrm{End}_{\mathcal{A}^{\circ}}$ induces only a 4-ary multiplication. If there exists $a \in \mathcal{A}$ with multiplicative inverse, we can define a 'twisting' binary multiplication, by conjugating the first term by a.

Observe that there is a wealth of additional information in these representations. For example, $\rho\colon \{B(1), B(2)\} \to \mathrm{End}_{\mathbf{k}}$ is a bilinear form on $\mathcal{A}$ which is invariant exactly if $B = \langle B(2) \rangle$.

1.6 Free operads

Free operads are interesting as the most general operads to study for a given set of relations. We take this opportunity to generalise operads one more time.

Definition 1.21. A *nonunital* (possibly k-)operad is an object C satisfying the Definition 1.4 of operads (respectively, Definition 1.6 of k-operads) except perhaps part ii. of the axiom of identity. The word operad implicitly refers to those *unital* operads that do satisfy ii. □

Any (unordered) combination of the following adjectives therefore makes for a distinct type of operad: k-, nonunital, symmetric.

So we are equipped to define now the free operad for any kind of operad, beginning with nonunital nonsymmetric k-operads. This first definition is simply the typical category theory definition of a free functor as adjoint (Definition C.8) to a forgetful functor.

By a family of $\mathbb{N}$-indexed k-modules, we mean a set $F = \{F(n)\}_{n\in\mathbb{N}}$ such that $F(n)$ is a k-module for all $n \in \mathbb{N}$.

Definition 1.22. Let kModF and kOp be the category of families of $\mathbb{N}$-indexed k-modules and category of k-operads respectively. The free operad functor $\Phi\colon \mathrm{kModF} \to \mathrm{kOp}$ is the left adjoint of the forgetful functor $\mathrm{kOp} \to \mathrm{kModF}$. $\square$

A construction is more useful and we provide one in Theorem 1. Here is how we will proceed.

Fundamental is that we can use families of modules (objects from ModF) to put the same 'family of modules'-structure onto arbitrary finite sets, as explained in Lemma 1.23. The subsequent improvement on this is inducing module structures onto trees (which are defined anew in Definition 1.24). In fact, inducing this structure is functorial. Finally the colimit of this functor is the free operad functor.

This argument incidentally confirms the intuition that trees are somehow fundamental to operads.

Lemma 1.23. Let $C \subseteq \mathrm{Set}_f$ be the category with objects finite sets and only bijective morphisms. Suppose that $M = \{M(n)\}_{n\in\mathbb{N}}$ is an object of kModF. Then the induced map that sends $S \in C$ to $M(m) \times \mathrm{Bij}([m], S)$ is a functor $\hat{M}\colon C \to \mathrm{ModF}$.

Proof. The claim is that $h = \hat{M}(f)$ makes the following diagram commute:

$$\begin{tikzcd}
S \arrow[rr, "f"] \arrow[d, "\hat{M}"'] & & T \arrow[d, "\hat{M}"'] \\
M(m) \times \mathrm{Bij}([m], S) \arrow[rr, "h = \hat{M}(f)"'] & & M(m) \times \mathrm{Bij}([m], T)
\end{tikzcd}$$

for $f \colon S \to T$ a bijection of sets of size m. This is clear as the action of h is identity on the module $M(m)$ and postcomposition on the bijections $\mathrm{Bij}([m], S)$. $\square$

Definition 1.24. A reduced *tree* t is all of the following. A finite connected graph without loops and multiple edges. Exactly one vertex, which is adjacent to at most one edge, is marked the *root*. All other vertices adjacent to one edge are *leaves* $l \in \mathrm{L}(t)$. No vertex lies on only two edges. All edges are directed towards the root. Given a vertex $v \in \mathrm{V}(t)$, its *legs* are the edges $e \in \mathrm{Edges}(t)$ adjacent to and oriented towards v, and l_v is the set of such. A *planar embedding* of t is a bijection v_ρ given for each vertex v, from the m legs v_l of v to $[m]$. A labeling of the n leaves is a bijection $\ell \colon L \to [n]$.

Suppose that t is an ℓ-labeled tree with leaf l_i such that $\ell(l_i) = i$, and that s is a tree with root r. Their composition $t \circ_i s$ is the tree constructed by deleting l_i from t and r from s and connecting the parent of l_i with the child of r. The new labeling is given by the obvious composition of labelings. $\square$

Lemma 1.25. Every finite tree can be written as a $\{\circ_i\}_{1 \leq i}$ composition of corollae from $\{\{c_n\}\}_{n \in \mathbb{N}}$, where c_n is the tree with one root, n leaves and one other vertex.

This use of corollae to build trees is equivalent to putting a formal composition on the natural number operad N (introduced in Section 1.3).

Proof. An easy upgrade (in allowed edges on each vertex) of Lemma 1.3, which makes the same claim for finite binary trees only. $\square$

We now use Lemma 1.23 in the case of trees. Let $T(n)$ be the category of labeled trees with exactly n leaves, whose morphisms are again only bijections. (It is a subcategory of the category of labeled trees, where morphisms are graph morphisms that map edges to edges and preserve labelings.) Denote the set of vertices which are not leaves or roots by In. For $t \in T(n)$ define

$$\tilde{M}(t) = \bigotimes_{v \in \mathrm{In}(t)} \hat{M}(v_l).$$

(Note that this is isomorphic to $\bigotimes_{v \in \mathrm{V}(t)} \hat{M}(v_l)$.) If X is a set (or even a set of sets) of trees, we define the action of $\tilde{M}$ on X as acting on each of the trees in X individually.

Lemma 1.26. If $\mathcal{T}$ is an operad of trees, $\tilde{M}\mathcal{T}$ is also an operad.

Proof. The composition (also written $\circ_i$), and its associativity, is inherited from the composition on trees by

$$\begin{aligned}\tilde{M}(t) \circ_i \tilde{M}(s) &= \bigotimes_{v \in \mathrm{In}(t)} \hat{M}(v_l) \otimes \bigotimes_{v \in \mathrm{In}(s)} \hat{M}(v_l) \\ &\cong \bigotimes_{v \in \mathrm{In}(t) \cup \mathrm{In}(s)} \hat{M}(v_l) \cong \bigotimes_{v \in \mathrm{In}(t \circ_i s)} \hat{M}(v_l) \cong \tilde{M}(t \circ_i s).\end{aligned}$$

□

Finally we exhibit the free functor $\Phi\colon \mathrm{ModF} \to \mathrm{kOp}$ of Definition 1.22. (Recall perhaps Definition C.10 of colimits.)

Theorem 1. *The free nonunital nonsymmetric* k*-operad for some arity-indexed family of* k*-modules $M = \{M(n)\}_{n \in \mathbb{N}}$ is given by the functor Φ as*

$$\Phi M(n) = \operatorname*{colim}_{t \in T(n)} \tilde{M}(t),$$

with $\tilde{M}$ the upgrade for trees of the functor $\hat{M}$ from Lemma 1.23.

Proof. For an element M of ModF, $\Phi M = \{\Phi M(n)\}_{n \in \mathbb{N}}$. We first verify that this indeed defines a k-operad. By Lemma 1.26, the $\tilde{M}\mathcal{T}$ have operad structure for $\mathcal{T}$ a tree operad. In particular, $\{T(n)\}_{n\in\mathbb{N}}$ is such a tree operad with the composition of trees. This is preserved by colimits; we also write the composition this puts on ΦM as $\circ_i$. It follows naturally that ΦM is an operad.

The remainder of the claim is that Φ is the left-adjoint functor (Definition C.8) to the forgetful functor from Definition 1.22, which we will write $\mathcal{F}$. This means there exist natural morphisms $\alpha\colon \Phi\circ\mathcal{F}\to \mathrm{id}_{\mathbf{k}\mathrm{Op}}$ and $\beta\colon\ \mathrm{id}_{\mathrm{ModF}}\to\mathcal{F}\circ\Phi$, satisfying the triangle diagrams of Definition C.8. Equivalently, there exists a bijection $\gamma\colon\ \hom_{\mathbf{k}\mathrm{Op}}(\Phi M\ C)\to\hom_{\mathrm{ModF}}(M\ \mathcal{F}C)$, with inverse δ, defined by

$$\gamma(f)=(\mathcal{F}f\circ\beta_M)\colon M\to\mathcal{F}(f(\Phi M))\text{ and }\delta(g)=\alpha_C\circ\Phi g$$

for $f\in\hom_{\mathbf{k}\mathrm{Op}}(\Phi M\ C)$ and $g\in\hom_{\mathrm{ModF}}(M\ \mathcal{F}C)$.

We can reduce the problem. By the discussion after Definition C.10, $\Phi(M)(n)=\bigotimes_{t\in T(n)}\tilde{M}(t)/\sim$, where $\sim$ is the equivalence arising from $\tilde{M}(t_1)\ni e\sim\big(\tilde{M}(f)\big)(e)\in\tilde{M}(t_2)$ if $f\in\hom_{T(n)}(t_1\ t_2)$. Thus every set of operations $\Phi M(n)$ is a coproduct of $\tilde{M}t$ for trees t with n leaves. Therefore the action of some $f\in\hom_{\mathbf{k}\mathrm{Op}}(\Phi M\ C)$ is completely determined by its action on $\tilde{M}t$; this action on m-leaf trees is denoted by f_m.

Moreover any morphism f must make commute

$$\begin{tikzcd}
\Phi M_m\otimes\Phi M(n) \arrow[r, "f_m\otimes f_n"] \arrow[d, "\circ_i"'] & C_m\otimes C(n) \arrow[d, "\circ_i"] \\
\Phi M(m+n-1) \arrow[r, "f_{m+n-1}"'] & C(m+n-1).
\end{tikzcd}$$

In words, f respects the operadic composition.

By Lemma 1.25, any tree can be written as a composition of corollae; therefore f is defined by its action on these. Let X_m be the set of all possible ways of writing a tree with m leaves as a composition of corollae. In particular, the action of $(\delta\circ\gamma(f))_m$

is given by $\bigotimes X_m$. Therefore, by the commuting diagram, the two actions coincide and $f = \delta(\gamma(f))$ as required. $\square$

More simply, ignoring some of the subtlety, we could have taken equivalence classes of trees. So if $\mathcal{T}(n)$ is the category of n-leaf trees up to isomorphism without any morphisms (the skeleton category of $T(n)$ as per Definition C.16), $\Phi(M)(n) = \bigotimes_{[t]\in\mathcal{T}(n)} M(t)$.

1.7 The circle operad

In anticipation of the sphere operad, we consider a simpler construction, as a lower-dimensional analogy. This circle operad appears to have more structure at first than it eventually does. This first appeared in a contribution by Yi-Zhi Huang to [O97].

Definition 1.27. Let A and B be topological spaces, $f\colon A \to B$ and $p \in A$. The *germ* of f at p is the equivalence class of functions $g\colon A \to B$ for which there exists an open set $U \ni p$ on which f and g are pointwise equal. $\square$

The arity-n operations of the *circle operad* C are given by $c = (S^1, f_0, \bar{p}, \bar{f})$. In this definition S^1 is the real projective line, considered geometrically as a circle. f_0 is the germ of some differentiable invertible-around-∞ formal series $S^1 \to \mathbb{R}$ in indeterminate $\frac{1}{x}$, which vanishes at ∞. The n-tuples $\bar{f} = (f_1, \ldots, f_n)$ and $\bar{p} = (p_1, \ldots, p_n)$ give n germs of differentiable invertible-around-p_i formal series $f_i\colon S^1 \to \mathbb{R}$, each in indeterminate x and vanishing at marked point p_i. The indexing of these points from $[n]$ is understood from the ordered tuple.

The family of compositions is given by composing these circles via their germs. Take circles $c \in C(m)$, $d \in C(n)$ and $i \leq m$, where the i^{th} germ and point on c is f_i at p_i, and g_0 is the germ vanishing at ∞ on d. Suppose that there exists $r > 0$ such that ${f_i}^{-1}(B^r)$ does not contain any marked point, or ∞, other than p_i, and that likewise ${g_0}^{-1}(B^{1/r})$ does not contain any marked point of d. Then the composition $c \circ_i d$ is

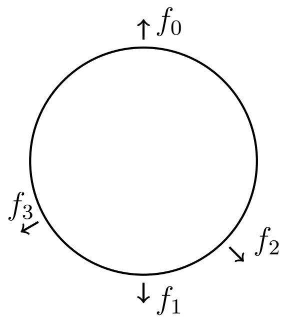


Figure 1.2: A circle with four germs

$(c' \sqcup d', f_0, \bar{h}, \bar{q})$, for $c' = c \setminus f_i^{-1}(B^r)$, $d' = d \setminus g_0^{-1}(B^{1/r})$ and $c' \sqcup d'$ is topologically S^1 when we identify the boundary points of c' with those of d' via $g_0^{-1} \circ (x \mapsto \frac{1}{x}) \circ f_i$. Also $\bar{h}$ and $\bar{q}$ are an $m+n-1$-tuples of germs and locations respectively, where $\bar{h} = (f_1, \ldots, f_{i-1}, g_1, \ldots, g_n, f_{i+1}, \ldots, f_m)$ and $\bar{q}$ is similarly arranged.

The effect of this circle composition is in effect just to remove one germ and point from circle c and insert the bulk of the points from d in their suitably-transformed places in that place on c. Note that if there is no $r > 0$, for a particular choice of circles and germs and points, to satisfy the above, then their composition is not defined. The operad is therefore partial:

Definition 1.28. A *partial* operad P is a family $\{P(n)\}_{n\in\mathbb{N}}$ of sets of operations and a family of compositions $\{\circ_i\}_{i\geq 1}$ satisfying Definition 1.4 of operads, except that possibly for some $c \in P(m)$, $d \in P(n)$ and $1 \leq i \leq m$, $c \circ_i d$ is not defined, in which case i. the composition associativity does not apply. □

Moreover this operad is unital; $(S^1, \frac{1}{x}, \{x\}, \{0\})$ is the obvious identity element from $C(1)$. Each $C(n)$ can also be endowed with an action of the symmetric group S_n on n letters, permuting the order of the germs and punctures by the natural action on their indices. Thus C is also a symmetric operad.

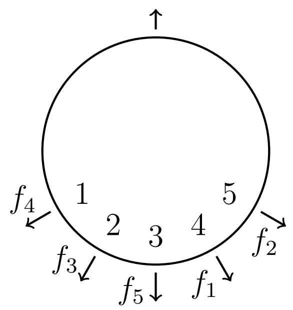
Lemma 1.29. Let c and d be two circles with n germs and points: $\bar{f}$, $\bar{p}$ for c and $\bar{g}$, $\bar{q}$ for d. Suppose that $p_i < p_j$ precisely when $q_i < q_j$ for all $1 \leq i, j \leq n$. Then there exists a diffeomorphism $\delta\colon c \to d$ that maps each p_i to q_i and satisfies $g_i = \delta \circ f_i$. $\square$

Thus we have two conclusions for these circles, up to diffeomorphism. Firstly, $c \circ_i d$ is defined for all c and d, where c is of arity greater than i, for any germ can be dilated to accommodate any value of $r > 0$ – so, up to diffeomorphism, it is not partial. Secondly, there are precisely $n!$ elements in each set $C(n)$ of circles of arity n modulo diffeomorphisms, since those are the possible orderings on a set of size n.

That statement suggests the following.

Consequence. *The circle operad C up to diffeomorphism is isomorphic to the symmetric group operad S.*

Proof. By the lemma, the composition of the circles is always defined. Moreover, up to diffeomorphism, every set of circles of arity n is a single orbit under S_n. $\square$



$\leftrightarrow \quad (1\ 4)\ (2\ 3\ 5)$

Figure 1.3: An identification of circles with permutations

Thus when the analytic structure of the projective real line is taken into account, this circle operad is relatively trivial. We shall see in what follows that in the complex analytic case there is a great deal more subtlety.

CHAPTER 2

SPHERES AND SEWING

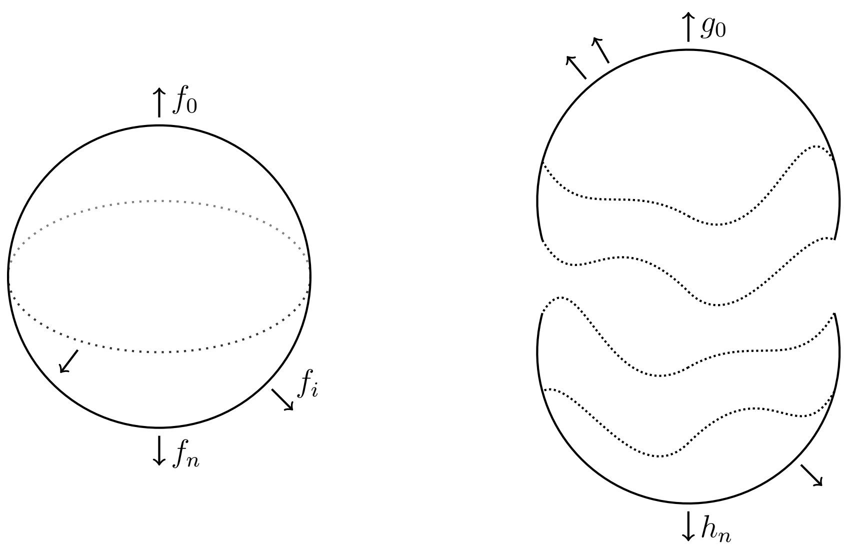


Figure 2.1: On the left: a sphere with tubes.
On the right: sewing two spheres with tubes.

In this chapter we consider spheres with tubes and the sewing product. This is the requisite material for the construction of the sphere operad. Its structure carries information that other operads so far have not, so it reflects an algebraically more complicated situation.

We work from the exposition in Yi-Zhi Huang's monograph *Two-Dimensional Conformal Geometry and Vertex Operator Algebras* [H97]. In it, he proves an isomorphism between a geometric and the algebraic formulation of vertex operator algebras. The geometric aspect is encoded in spheres with tubes and the sewing product. We examine this part of the theory in some detail, proving foundational results on the behaviour of these surfaces and their composition via the sewing product.

In some parts of that monograph, the definitions in use were unclear to this author. Here there is an attempt to make explicit definitions, but it may be that these are not the definitions which lead to the results proven in the monograph.

In the first section, we define the algebro-topological spheres with tubes, then consider their automorphisms in the next section. In Section 2.3 we define the sewing product. In the subsequent two sections it is analysed further and some of its properties are proven.

For clarity, before we start we make explicit here some things used later. Firstly some notations.

- $\mathbb{C}^\times$ is the set of nonzero finite complex numbers.
- $\mathbb{C}^\infty$ is the collection of infinite sequences taking values from $\mathbb{C}$.
- $\mathbb{C}[z]$ is the polynomial ring in indeterminate z.
- $f \in \mathbb{C}[z]$ is a polynomial f in z. So $f = \sum_{i\in\mathbb{N}} a_i z^i$ for $a_i \in \mathbb{C}$, $(a_i)_{i\in\mathbb{N}} \in \mathbb{C}^\infty$ and z an indeterminate for $\mathbb{C}[z]$. Moreover there exists $N \in \mathbb{N}$ such that $i > N$ implies that $a_i = 0$. For the polynomial $z \in \mathbb{C}[z]$, $f(z)$ is a function $\mathbb{C} \to \mathbb{C}$. If $z \in \mathbb{C}$, $f(z) \in \mathbb{C}$.
- $\mathbb{C}[[z]]$ is a ring of formal power series in indeterminate z. So $f = \sum_{i\in\mathbb{N}} a_i z^i$, where $(a_i)_{i\in\mathbb{N}}$ is not necessarily eventually zero and in any case considered an infinite sequence.
- $\mathbb{C}[z, z^{-1}]$ and $\mathbb{C}[[z, z^{-1}]]$ are the rings of polynomials and formal power series respectively, where elements of the latter must have finitely many nonzero terms in z or z^{-1}.

Now, definitions we will commonly use without reference.

- A *conformal equivalence*, or conformal map, is an analytic (that is, holomorphic) isomorphism with analytic inverse. Objects are conformally equivalent if there exists a conformal equivalence between them.
- A *chart* is a function with codomain $\mathbb{C}$ which is analytic on an open contractible set whose image includes 0.
- The open disk of radius r around a point p is $B^r(p) = \{z \in \mathbb{C} \mid |z - p| < r\}$. When p is zero we may simply write B^r. Its topological closure is $\bar{B}^r$.
- The Riemann sphere is the extended complex numbers $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with positive orientation given by i.
- The *inversion map* $\hat{J}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ acts as $0 \mapsto \infty$, $\infty \mapsto 0$, $z \mapsto \frac{1}{z}$.
- The complex structure of $\hat{\mathbb{C}}$ is defined by the charts id on $\mathbb{C} \subseteq \hat{\mathbb{C}}$ $\hat{J}$ on $\check{\mathbb{C}} = \mathbb{C}^\times \cup \{\infty\}$.
- A *sphere* is a genus-zero compact connected one-dimensional complex manifold.

Also the following lemmata are well-known and important.

Lemma. All spheres are conformally equivalent to the Riemann sphere. □

Lemma. A conformal map has, at each nonsingular point of its domain, a unique power series representation in $\mathbb{C}[[z^{-1}]]$ or $\mathbb{C}[[z]]$. □

2.1 Spheres with tubes

We introduce further terminology before we can define the objects of interest. A *puncture* on a sphere is a marked point. Its *orientation* is an arbitrarily fixed choice of sign. To a puncture p we assign a tube, which is defined (as an *ansatz*) as a neighbourhood U of p and chart $\varphi\colon U \to \mathbb{C}$ analytic on U and centered at p: $\varphi(p) = 0$. Note that φ necessarily has a nonzero radius of convergence.

The use of a distinguished neighbourhood is redundant here. The chart φ of a given tube with puncture p of course has a Laurent expansion around p. That is, there exists a power series $f \in \mathbb{C}[[z^{-1}, z]]$ such that there exists an open neighbourhood of p on which f and φ agree. This is a special case of the *germ* of a function. In fact, $f \in \mathbb{C}[[z]]$ unless φ is centered at ∞, in which case $f \in \mathbb{C}[[z^{-1}]]$. This form of the charts is amenable to our calculations.

We say that a *tube* (p, f) is a puncture p and the germ or power series f of a chart centered at p. In particular, we do not distinguish any neighbourhood in this definition. The *gradient* of this tube is the value of the first derivative of f at p, and we require that all tubes have nonzero gradient. The image of a tube (p_i, f_i) under conformal equivalence F is defined in the natural way as $(F(p_i), f_i \circ F^{-1})$.

Definition 2.1. A *sphere with* $(1+n)$ *tubes* is a sphere with ordered tubes $(p_i, f_i)_{0 \leq i \leq n}$ that have pairwise distinct punctures, and the 0^{th} tube has negative orientation and all others are positively oriented.

A *conformal equivalence* between spheres with tubes also maps tubes to tubes, preserving their ordering. □

To endow this definition with some imaginative sense, we suggest that each sphere carries the notion of a flow, of complex-analytic plumbing, from several inputs to one output.

The collection $\mathcal{K}$ of all spheres with tubes is a structure over the moduli space of spheres with punctures. We name it a *space*: the space of spheres with tubes. The subset $\mathcal{K}(n)$ contains all spheres with $1 + n$ tubes and $\mathcal{K}(\geq n)$ contains all with at least $1 + n$ tubes.

2.2 Conformal equivalences

The appropriate settings for us for spheres with tubes is up to conformal equivalence, since conformal equivalences are the isomorphisms in the category of spheres with tubes. ($\mathcal{K}$ is this category with additional algebraic structure.)

The space $\mathcal{K}$ admits a partition into conformal equivalence classes, since conformal equivalence is an equivalence relation. In the coming proposition and lemma, we prove that there is a well-defined way of picking class representatives, which are termed canonical. Moreover a canonical sphere is understood to be a canonical sphere with tubes.

Definition 2.2. A sphere with at least $1 + 1$ tubes, with tubes $(p_i, f_i)_{0 \leq i \leq n}$, is *canonical* if the underlying sphere is the Riemann sphere $\hat{\mathbb{C}}$ and its tubes satisfy that $p_0 = \infty$, $p_n = 0$ and f_n has gradient 1 at ∞. So

$$\left.\frac{df_0(w)}{d(\frac{1}{w})}\right|_{w=\infty} = \lim_{w \to \infty} w \cdot f_0(w) = 1.$$

□

Note that the requirement we make that the gradient of every tube is nonzero is

$$\left.\frac{df_i(w)}{dw}\right|_{w=p_i} = \lim_{w \to p_i} \frac{f_i(w)}{w - p_i} \neq 0.$$

Proposition 2.3. *Every sphere with tubes is conformally equivalent to a unique canonical sphere (with tubes).*

Proof. Firstly, every sphere is conformally equivalent to the Riemann sphere and we may thus assume without loss that the underlying sphere of the sphere with tubes is indeed $\hat{\mathbb{C}}$. Now recall that Möbius transformations are conformal equivalences and form the automorphism group $PGL_2(\mathbb{C})$ of the sphere.

$$PGL_2(\mathbb{C}) \ni M(z) = \frac{az + b}{cz + d} \quad \text{for } a, b, c, d \in \mathbb{C} \text{ such that } ad - bc \neq 0.$$

This is well-known to be sharply, *i.e.,* simply, 3-transitive on the points of the sphere.

Let S be an arbitrary sphere with $1+n$ tubes $(p_i, f_i)_{0\le i\le n}$. Then let M be a Möbius transformation such that $M(p_0) = \infty$ and $M(p_n) = 0$. Thus $ap_n = -b$ and $cp_0 = -d$. We can assume now without loss that indeed $p_0 = \infty$ and $p_n = 0$, and therefore that M fixes these points.

The transformation M would be completely determined if we choose a third point's image; instead, we wish to fix the gradient of $f_0 \circ M^{-1}$ at ∞ to be 1. This is in fact equivalent.

The gradient of the tube (∞, f_0) is $\frac{df_0(z)}{d(\frac{1}{z})}$ evaluated at $z = \infty$, the coefficient of $\frac{1}{z}$, the highest power of z in f_0. So let $\lim_{w\to\infty} w \cdot f_0(w) = g_1 \in \mathbb{C}$. Then

$$\begin{aligned}
1 = \frac{d}{d\frac{1}{z}} f_0(M^{-1}(z))\bigg|_{z=M(p_0)=\infty} &= \lim_{z\to\infty} z \cdot f_0(M^{-1}(z)) \\
= \lim_{z\to\infty} z \cdot \left(\frac{g_1}{\frac{dz-b}{-cz+a}} + \sum_{i\ge 2} \frac{g_i}{(M^{-1}(z))^i} \right) & \\
= g_1 \cdot \lim_{z\to\infty} \frac{z(a-cz)}{dz-b}. &
\end{aligned}$$

The last inequality follows as terms in higher powers of $\frac{1}{z}$ vanish in the limit $z \to \infty$. From this, $c = 0$ and $\frac{a}{d} = \frac{1}{g_1}$. Our requirement is that the gradient, and so the coefficient of the leading term in $f_0 \circ M^{-1}$, is 1. So we set $d = 1$ and $M = \frac{z}{g_1}$, which gives the desired conformal equivalence. □

One case has been so far neglected. To find a canonical representation for a sphere with $1+0$ tubes, it is clear that so far we have only made two requirements: that the negatively oriented tube be at ∞ with gradient 1. This lemma, and its implicit definition, assert the third choice we may make to uniquely define the canonical class representative of a sphere with $1+0$ tubes. It therefore completes the proof of Proposition 2.3.

Lemma 2.4. A sphere S with $1+0$ tubes is conformally equivalent to a unique canonical sphere with tubes whose power series has no term in $\frac{1}{z^2}$.

Proof. By the proof of Proposition 2.3, S is conformally equivalent to at least one sphere with its only tube at ∞ and of gradient 1. Let S' be any such sphere, with tube (∞, f). Of course, $f \in \mathbb{C}[[z^{-1}]]$ is centered at ∞. Thus $f = \frac{1}{z} + \sum_{i\geq 2} \alpha_i \frac{1}{z^i}$ for $(\alpha_i) \in \mathbb{C}^\infty$. Again the statement of the proposition is equivalent to fixing the third parameter of the Möbius transformation.

There is a related problem which is easier to solve. Given a power series in $\mathbb{C}[[z]]$, we can find a conformal equivalence under which the power series has no term in z^2.

Now $g = f \circ \hat{J} \in \mathbb{C}[[z]]$ is a power series vanishing at 0. Let $M = \frac{az+b}{cz+d}$ be the Möbius transformation satisfying $M(0) = 0$ (so $b = 0$) and preserving the gradient of g at 0. Thus M has determinant $1 = ad - bc = ad$. We wish to find the other parameters a, c, d of M such that $f \circ \hat{J} \circ M^{-1} \in \mathbb{C}[z]$ has no term in z^2. We have the power series expansion about $z = 0$:

$$\begin{aligned} M^{-1} &= \frac{dz}{a-cz} = \sum_{i\in\mathbb{N}} z^i \frac{M^{-1(i)}(0)}{i!} \\ &= \frac{dz}{a-cz}\bigg|_{z=0} + \frac{d}{dz}\frac{dz}{a-cz}\bigg|_{z=0} z + \frac{1}{2}\frac{d^2}{dz^2}\frac{dz}{a-cz}\bigg|_{z=0} z^2 + \dots \\ &= 0 + \frac{d}{a}z - \frac{dc}{a^2}z^2 + \dots \end{aligned}$$

where all other terms are of higher order.

We see that $f \circ \hat{J} \circ M^{-1}$ has the following expansion, again up to terms of second order:

$$\begin{aligned} f \circ \hat{J} \circ M^{-1} &= \left(\frac{d}{a}z - \frac{dc}{a^2}z^2 + \dots\right) + \alpha_2\left(\frac{d}{a}z - \frac{dc}{a^2}z^2 + \dots\right)^2 + \dots \\ &= \frac{d}{a}z + \left(\alpha_2\frac{d^2}{a^2} - \frac{dc}{a^2}\right)z^2 + \dots \end{aligned}$$

Since the gradient of this function is the coefficient of z and must be 1, $\frac{d}{a} = 1 = ad$ and thus $d = a^2 d$ whence $a = d = 1$ or $a = d = -1$. Now for the coefficient of z^2 to be zero, $c = \alpha_2 d$. Thus $M = \frac{z}{\alpha_2 z+1} = \frac{-z}{-\alpha_2 z-1}$. Finally, we compose this with $\hat{J}$, whence the starting sphere S is conformally equivalent to $S' \circ \hat{J} \circ M \circ \hat{J}$ with power series

$f \circ \hat{J} \circ M^{-1} \circ \hat{J} \in \mathbb{C}[[z^{-1}]]$ vanishing at ∞ with gradient 1 and, by the preceding discussion, no term in z^{-2}. $\square$

We refer to the collection of canonical spheres with tubes as K. It is isomorphic to $\mathcal{K}/PGL_2(\mathbb{C})$.

K is also the skeleton category (Definition C.16) of the category of spheres with tubes. Its elements are stable, in that they are invariant under conformal equivalences.

2.3 Sewing spheres

Spheres with tubes become interesting when considered with the following partial multiplication on the space of spheres with tubes. Two spheres with tubes (now assumed canonical for the remainder of this chapter) are multiplied or *sewn* together by removing an open disk from each and using an equivalence relation to relate the overlapping boundaries of the remaining disks, and then made, via a conformal equivalence, into another sphere with tubes.

Definition 2.5. Let S be a sphere with $1+m$ tubes, one of which is the i^{th} positive tube (p, f). Let T be a another sphere with $1+n$ tubes, with its negative tube (∞, g). The product $S_i\infty_0 T$ is defined if there exist open sets $U \subset S$ and $V \subset T$ such that f and g are invertible on $f(U)$, $g(V)$ respectively, p and q are the only punctures in U and V respectively, and there are real positive numbers $r < R$ to satisfy

$$\bar{B}^R = \bar{B}^R(0) \subset f(U), \qquad \bar{B}^{1/r} \subset g(V),$$

and we define the sewn sphere $S_i\infty_0 T$ as follows.

Let $\sqcup$ be the disjoint union and A_b^a (for $b < a$) the half-open half-closed annulus $\bar{B}^a \setminus \bar{B}^b$. Define $E = \{w_1 \in S \mid f(w_1) \in A_r^R\} \sqcup \{w_2 \in T \mid g(w_2) \in A_{1/R}^{1/r}\}$, which gives the area in which the spheres overlap. It is the domain of the equivalence relation $\sim$, under which $w_1 \sim w_2$ if

$$f(w_1) = \frac{1}{g(w_2)}.$$

The first stage of this construction is the *partly-sewn* sphere, given by

$$\big(S \setminus f^{-1}(\bar{B}^r)\big) \sqcup \big(T \setminus g^{-1}(\bar{B}^{1/R})\big)$$

up to the equivalence relation $\sim$, so every point in E is replaced by its equivalence class $\{w_1, w_2 \mid w_1 \sim w_2\}$.

Topologically, this is a sphere. Let F be a map acting as identity on all points except on E, where it identifies w_1 with w_2 whenever $w_1 \sim w_2$. It is a well-defined map to a sphere. In fact, it has two constituent parts (whose domains overlap, in a sense, on E). These are the conformal maps id on $S \setminus f^{-1}(\bar{B}^r)$, and id on $T \setminus g^{-1}(\bar{B}^{1/R})$. Now the image under F is a sphere. This is the second stage of the construction.

The image under F becomes a sphere with $1 + (m + n - 1)$ tubes when we add to it all tubes from S and T, except those involved in sewing. The new ordering on the tubes is given by inserting, with respect to the ordering, the positive tubes from T in the place of the positive i^{th} tube in the ordering of the tubes of S. We refer to this as the *pre-canonical sewn sphere*, the third stage.

Now by Proposition 2.3, there exists a conformal equivalence G from this sphere with tubes to a canonical sphere with tubes, and this is the sewing product $S_i\infty_0 T$.

$\square$

Observe the radii, which were used crucially in sewing the two spheres together, are not referred to in the product. We immediately show that the product is independent of the choice of radii, which can therefore be neglected – so long as a suitable choice exists. This is ensured by the equivalence relation $\sim$ on the partly-sewn sphere: the domain E of the equivalence relation is a function of the radii, with the same image under the conformal equivalence to a sewn sphere regardless of the radii.

Lemma 2.6. Any choice of (viable) r and R for sewing $S_i\infty_0 T$ yields the same sphere with tubes.

Proof. It suffices to show that the new coordinates on $S_i\infty_0 T$ do not change, and thus no other puncture locations or power series are affected.

Consider the underlying sets. For a change from sewing with r, R to r', R', it is clear that the location of points in

$$S \setminus f^{-1}(\bar{B}^{r}) \cap S \setminus f^{-1}(\bar{B}^{r'}) = S \setminus f^{-1}(\bar{B}^{r} \cup \bar{B}^{r'}) = S \setminus f^{-1}(\bar{B}^{\max\{r,r'\}})$$

and $T \setminus g^{-1}(\bar{B}^{1/R} \cup \bar{B}^{1/R'})$ is not affected. By the restriction on the choice of r, R and r', R', punctures are not affected. Suppose that $r' < r$ and $R' = R$. Then $S \setminus f^{-1}(\bar{B}^{r'}) \subset S \setminus f^{-1}(\bar{B}^{r})$ but $A_r^R \subset A_{r'}^R$ and $A_{1/R}^{1/r} \subset A_{1/R}^{1/r'}$. Now for all points x_1 in $A_{r'}^R \setminus A_r^R$ there is exactly one point x_2 in $A_{1/R}^{1/r'} \setminus A_{1/R}^{1/r}$ such that $f^{-1}(x_1) \sim g^{-1}(x_2)$. Thus up to the relation $\sim$, the partly-sewn $S_i\infty_0 T$ using r', R and given by

$$\left(S \setminus f^{-1}(\bar{B}^{r'})\right) \sqcup \left(T \setminus g^{-1}(\bar{B}^{1/R})\right)$$

has the same image under the map to a sphere with tubes as that given by using r and R. The same argument for $r' > r$ and $R' \neq R$ holds in all cases. $\square$

Notice that in the equivalence relation, $w_1 \sim w_2$ if $f(w_1) = \frac{1}{g(w_2)}$. We would like to draw attention to this use of the map $\hat{J}$ in $f = \hat{J} \circ g$.

Suppose that we are sewing spheres with tubes S and T. Refer to the earlier Definition 2.5 and again let E be the domain of the equivalence relation $\sim$. Then the conformal equivalence F makes sense if and only if for each $\{w_1, w_2\} \in E$, w_1 and w_2 have comparable magnitudes. That is, informally, w_1 should be large on S precisely when w_2 is large on T, and likewise for smallness.

Figure 2.2 illustrates this using the two disjoint boundaries δ_1, δ_2 of the domain E. Values of w_1 are largest at δ_1 and smallest at δ_2, as projected onto S. They have to be aligned with the projections of δ_1, δ_2 onto T. This is the meaning of the inversion map $\hat{J}$: it preserves orientation.

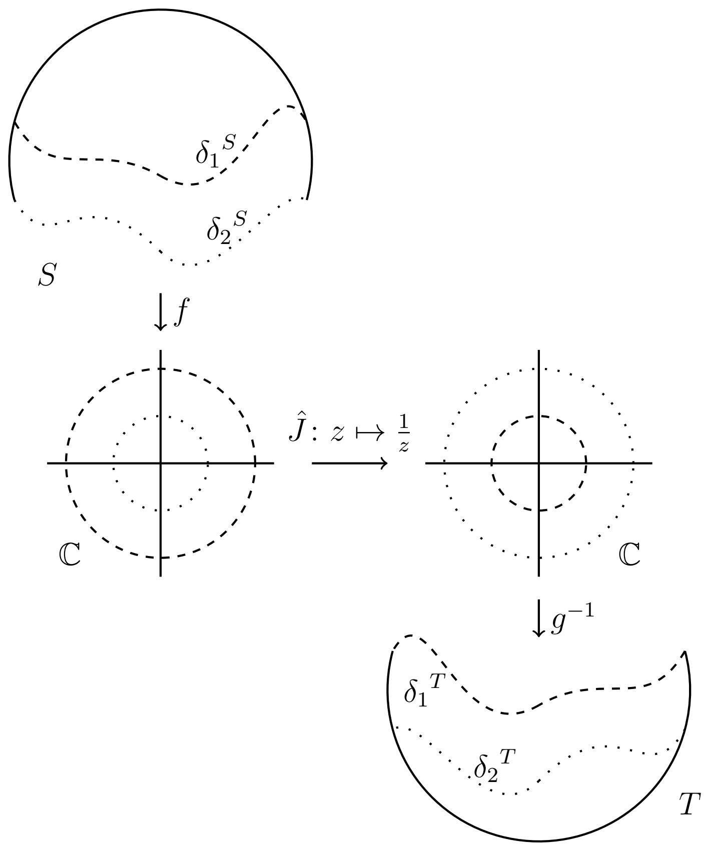


Figure 2.2: The boundaries δ_1, δ_2 of the domain E of the equivalence relation $\sim$ have images $\delta_1{}^S$, $\delta_2{}^S$ on S, as indicated here using dashed and dotted lines respectively. Their successive images under the chart f, inversion map $\hat{J}$ and inverse chart g are illustrated. The final images $g^{-1} \circ \hat{J} \circ f(\delta_i{}^S)$ for $i = 1, 2$ of these boundaries on T match the image $\delta_i{}^T$ of δ_i under a projection of E onto T.

We would also draw attention to the role of spheres with $1 + 0$ tubes in sewing. They act as caps, that is, if S has $1 + m$ tubes and T has $1 + 0$ tubes then $S_i\infty_0 T$ has $1 + (m - 1)$ tubes, and it is the image of the i^{th} tube of S which is removed. In a way, this tube has been 'sealed'.

2.4 The sewing equations

We are interested in the conformal equivalence from a pre-canonical sewn sphere to a canonical sphere and therefore make explicit some of the constraints on it.

Let S be a sphere with $1+m$ tubes for m at least 1. Let T be a sphere with $1+n$ tubes and, if $n=0$, let $1 \leq i < m$; otherwise, if $n \neq 0$, we allow also $i = m$. Let the i^{th} tube of S be (p, f) and the 0^{th} tube of T be (∞, g). Consider the pre-canonical sewn sphere $S_i\infty_0 T$, simply written $S_i\infty_0 T$ in this section. We can endow this with another 'coordinate system' $\{h_S \colon W_S \to S, h_T \colon W_T \to T\}$ for $W_S \cup W_T$ covering $S_i\infty_0 T$. Then the intention is that W_X contain the image of X under the sewing composition, for X one of the spheres with tubes S or T.

The same Riemann sphere underlies S and T. For clarity, we write the Riemann sphere of the sphere with tubes X as $\hat{\mathbb{C}}(X)$. We will use the identification $\hat{\mathbb{C}}(S) = \hat{\mathbb{C}}(T)$, and so h_S and h_T are similar to charts. This viewpoint is somewhat an abuse of coordinates, and is stated more precisely as follows.

Given two positive real numbers $\rho < P$, there exist open sets W_S, W_T such that $S_i\infty_0 T \subseteq W_S \cup W_T$, with conformal equivalences h_S and h_T as before, such that

$$h_S(W_S) \subseteq \hat{\mathbb{C}} \setminus f^{-1}(\bar{B}^{\rho}) \subseteq \hat{\mathbb{C}}(S), \qquad h_T(W_T) \subseteq \mathbb{C} \setminus g^{-1}(\bar{B}^{1/P} \setminus \{0\}) \subseteq \hat{\mathbb{C}}(T).$$

Now for some r and R congenial to sewing $S_i\infty_0 T$, we may and do choose ρ and P such that necessarily for such $\{(W_S, h_S), (W_T, h_T)\}$,

$$W_S \supseteq \{w_S \in S_i\infty_0 T \mid f(w_S) \in A_r^R\}, \quad W_T \supseteq \{w_T \in T \mid g(w_T) \in A_{1/R}^{1/r}\}$$

(where, in the predicates, w_S and w_T is each taken as a point on S and T for which f and g are respectively defined).

Thus the domain E of the equivalence relation $\sim$ on the partly-sewn sphere can be identified with subsets of both W_S and W_T. This coordinate system must also agree with this equivalence relation. For $w_S \sim w_T$, we have that $f(w_S) = \frac{1}{g(w_T)}$, and so if $w_S, w_T \in W_S \cap W_T$ then $h_S(w_S) = h_T(w_T)$. Thus for $w \in W_S \cap W_T$ we require

$$g^{-1}\left(\frac{1}{f(w)}\right) = {h_T}^{-1} \circ h_S(w).$$

Let F be the unique conformal equivalence from the pre-canonical sewn sphere to $S_i\infty_0 T$. For $w \in W_S \cap W_T$, we therefore require

$$F(w) = F\left(g^{-1}\left(\frac{1}{f(w)}\right)\right).$$

Using the coordinate system we may examine F in two constituents. Let F_X: $h_X(W_X) \to \hat{\mathbb{C}}$, $w \mapsto F \circ {h_X}^{-1}(w)$ for X one of S and T. Since F is the map to the canonical sphere, we have the following equations.

Lemma 2.7. Suppose that S and T are spheres with tubes and take everything as in the preceding discussion; then, for $w \in W_S \cap W_T$,

$$F_S \circ h_S(w) = F_T \circ h_T\left(g^{-1}\left(\frac{1}{f(w)}\right)\right) \tag{2.1}$$

and, for X the sphere with tubes from which the last tube (in the ordering of tubes of $S_i\infty_0 T$) originates:

$$F_S(\infty) = \infty, \quad F_X(0) = 0, \quad \lim_{z\to\infty} \frac{F_S(z)}{z} = 1. \tag{2.2}$$

□

The product of the sewing operation, a canonical sphere, is the image of this F. Thus it is determined uniquely by F_S and F_T, whence determined by the equations in this Lemma 2.7. The conformal equivalence F must be unique, since it has three points with fixed images and comes from a sharply 3-transitive group. Therefore any solution to these equations must be unique.

There is one case remaining. Suppose that T has $1+0$ tubes, S has $1+m$ tubes. Then the last positive puncture of $S_m\infty_0 T$ comes from S, but is not at 0. Let the last tube of $S_m\infty_0 T$ be (p'_{m-1}, f'_{m-1}), the image of the penultimate tube (p_m, f_m) on S.

The conformal equivalence F to a canonical sphere involves a translation by $-p'_{m-1}$, to move the last tube from its position at p'_{m-1} to 0. So for positive tubes (p_i, f_i) with $1 \leq i \leq m-1$, $F(p_i) = p_i - p'_{m-1}$ and $F \circ f_i = f_i \circ (z \mapsto z + p'_{m-1})$. The location of the negative tube is fixed. Also

$$F \circ f_0 = f_0 \circ \left(z \mapsto \frac{z}{1 + p'_{m-1} z} \right).$$

The former inequality follows from F fixing the gradient of f_0.

So the last set of the equations in Lemma 2.7 becomes

$$F_S(\infty) = \infty, \quad F_S(p_{m-1}) = 0, \quad \lim_{z \to \infty} \frac{F_S(z)}{z} = 1. \tag{2.3}$$

2.5 The sewing functions

With some elementary methods, we can make some statements about sewing, considered as functions.

Proposition 2.8. *The sewing multiplications are associative.*

There are two facets to this claim to consider. Let A, B, C be three spheres with tubes such that $(A_i\infty_0 B)_j\infty_0 C$ is defined. C might either be sewn to the part of the sphere $A_i\infty_0 B$ originating with A, or to the part originating with B. Thus in the first case if the j^{th} tube on $A_i\infty_0 B$ originates from some j'^{th} tube on A, we require that there is also an i'^{th} tube on sphere B and $(A_i\infty_0 B)_j\infty_0 C = (A_{j'}\infty_0 C)_{i'}\infty_0 B$. Otherwise – in the second case – this j^{th} tube comes from the j'^{th} on B and $(A_i\infty_0 B)_j\infty_0 C = A_i\infty_0(B_{j'}\infty_0 C)$. Note that any change in tube number, from i, j to i', j', is a purely cosmetic change, reflecting nothing more than the reindexing of tubes. This situation is illustrated in Figure 2.3.

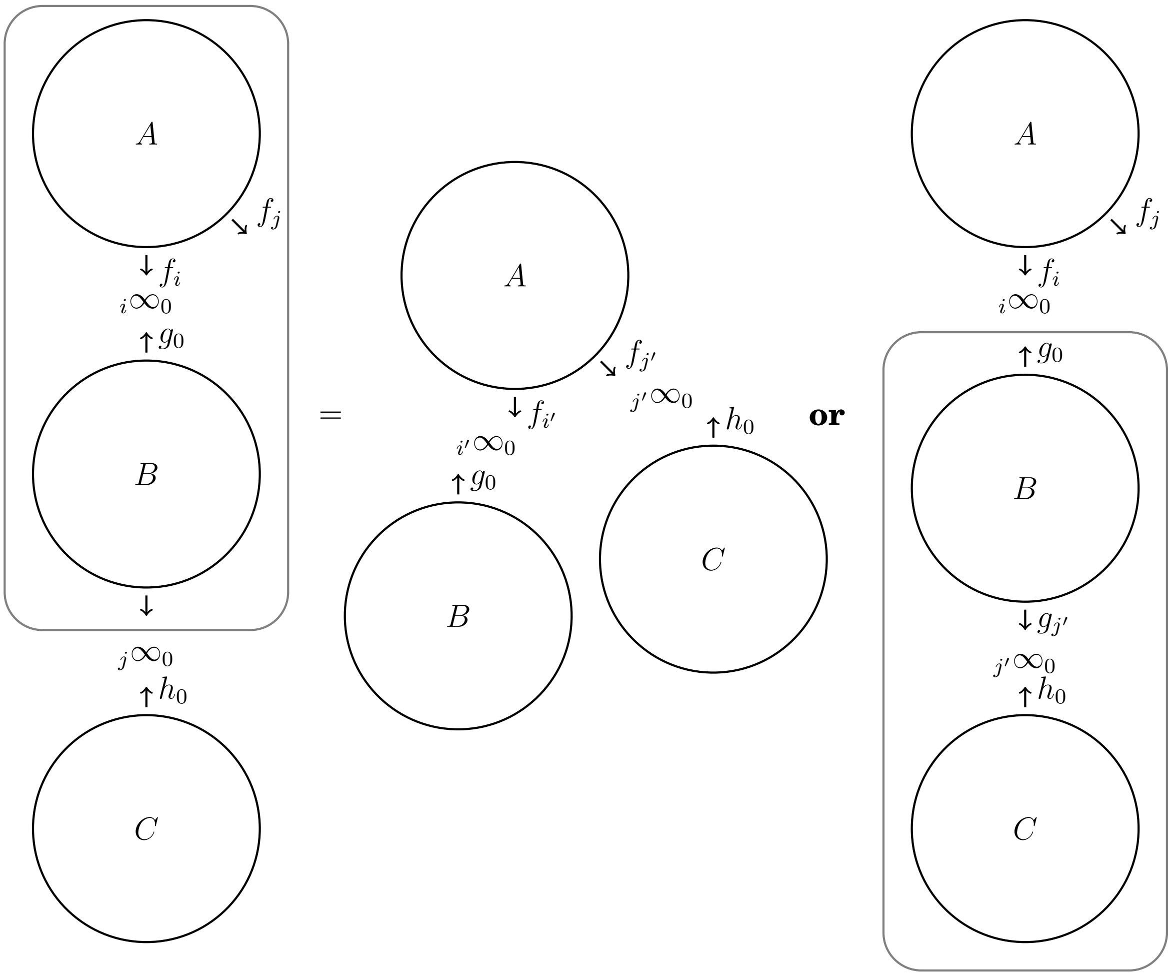


Figure 2.3: the different possible associative arrangements.
Either both B and C are sewn to A, or C can be sewn to B first.

To prove that one side is defined exactly when the other is, we use the idea of the proof of Lemma 2.6. Essentially, the equivalence relation always 'absorbs' exactly what is necessary to produce a new sphere in the right way. To prove equality, we exploit the conformal equivalences used in the sewing operation, which overlap sufficiently and have inverses, to force the result.

Proof. We omit the proof for the first case, to instead give a detailed account of the second case. The first case is no different.

Let r_1, R_1 and r_2, R_2 be suitable radii for sewing B to A and C to $A_i\infty_0 B$ respectively. Now $B \setminus {g_j}'^{-1}(B^{r_2})$ is conformally equivalent to a subset b of $B_{j'}\infty_0 C$, and the four radii can be arranged such that this b contains the preimage of B^{1/R_1} under g_0. The set b is necessarily also free of punctures and singularities. Since r_1, R_1 were suitable for sewing A to B, there are suitable radii to sew A to $B_{j'}\infty_0 C$. This shows the existence of $A_i\infty_0(B_{j'}\infty_0 C)$. To start with $A_i\infty_0(B_{j'}\infty_0 C)$ and deduce the existence of $(A_i\infty_0 B)_j\infty_0 C$ can be done in the same way.

Now we intend to show the desired equality of the two bracketings. Let A' be A setminus the disk to be removed around the i^{th} tube of A, and C' be C without the disk around its negative tube. There is a conformal equivalence F between the partly-sewn sphere and the sewn sphere $S = A_i\infty_0 B$, and a conformal equivalence G between the partly-sewn sphere and $S_j\infty_0 C$.

Let E be the domain of the equivalence relation $\sim$ on the partly-sewn S. Enrich $S_j\infty_0 C$ by replacing each point p in $G \circ F(E)$ with $\{x, y\}$ to make a new object U, where x and y both have coordinate p but on the spheres A or $B_{j'}\infty_0 C$ respectively. (This is a partition much like that of the equivalence relation $\sim$, prior to being 'smoothed out' by a map to a complex analytic sphere.)

Now $F|_{A'}{}^{-1} \circ G|_S{}^{-1}$ is defined on $G \circ F(A') \subset U$ and we thus have an obvious conformal equivalence from A' in $(A_i\infty_0 B)_j\infty_0 C$ to A' in $A_i\infty_0(B_{j'}\infty_0 C)$. More significantly, we have a conformal equivalence of a subset of B in the former with a subset of $B_{j'}\infty_0 C$, via $G \circ F\colon B \to U$ and $G|_S{}^{-1}\colon (U' \subset U) \to B_{j'}\infty_0 C$. There is also such an equivalence from C' to a subset of $B_{j'}\infty_0 C$. It proceeds along $G_S{}^{-1} \circ G_C\colon C' \to U \to B_{j'}\infty_0 C$.

Finally, all the above-mentioned conformal equivalences overlap and coincide on an open region, whence there is a single conformal equivalence H effecting all this. There is exactly one negative tube on every sphere and no matter in which order our expression is sewn, in this case it has to come from A. Therefore H acts as identity on this tube. This is enough to fix H as identity on a neighbourhood of this tube and thus H is identity on the entire complex structure, too. So $\mathrm{id} = H\colon (A_i\infty_0 B)_j\infty_0 C \to$

$A_i\infty_0(B_{j'}\infty_0 C)$ and this is it. $\square$

We have already seen conformal equivalence as a way of simplifying the space of spheres with tubes. We will now prove that all sewing functions ${}_i\infty_0$ preserve the distinctions made by conformal equivalence. In particular, conformal equivalence provides invariants under the sewing functions. Thus sewing is a class function with respect to conformal equivalence.

Proposition 2.9. *If A, B are conformally equivalent to A' and B' respectively, then $A_i\infty_0 B$ is conformally equivalent to $A'{}_i\infty_0 B'$.*

Proof. Let the i^{th} tube of A be given by power series f at puncture location p, and the 0^{th} tube of B have power series g at q. Suppose that $M\colon A \to A'$, $N\colon B \to B'$ are conformal equivalences of spheres with tubes. Of course they are continuous. As they also act on the power series of the tubes, there exist radii such that the f- and g- preimages of the disks on the complex plane are puncture-free on A and B precisely when there are such radii for the $f \circ M^{-1}$- and $g \circ N^{-1}$-preimages on A', B'. Thus if $A_i\infty_0 B$ is defined for A and B, $A'{}_i\infty_0 B'$ is defined for all A', B' conformally equivalent to A, B respectively.

By abuse of their domains, M and N are defined on the subsets A and B of the partly-sewn sphere $A_i\infty_0 B$. Let F and F' be the conformal equivalence from the partly-sewn sphere to the canonical sphere, as used in the last step of the sewing operation 2.5 for $A_i\infty_0 B$ and $A'{}_i\infty_0 B'$. Then there exists a conformal equivalence $\varphi\colon A_i\infty_0 B \to A_i\infty_0 B'$:

$$\varphi(x) = \begin{cases} F'(M(F^{-1}(x))) & \text{if } F^{-1}(x) \text{ is a point on } A \\ F'(N(F^{-1}(x))) & \text{if } F^{-1}(x) \text{ is a point on } B \end{cases}$$

Clearly their domains overlap, because of the equivalence relation used on the partly-sewn sphere during sewing. By the sewing equation (1) of Lemma 2.7 this is nevertheless well-defined. Since the functions overlap and agree, φ is the desired conformal equivalence relation. $\square$

Consequence. *Sewing with ${}_i\infty_0$, for each $i \in \mathbb{N}$, is a partial product $K \times K \to K$.* $\square$

We can now consider how closely conformal equivalence controls sewing.

Proposition 2.10. *The product ${}_i\infty_0$ on $K(m) \times K(n)$ is injective in each parameter, for $1 \leq m, n$.*

Proof. Suppose that there is a conformal equivalence $H\colon (A_i\infty_0 B) \to (C_j\infty_0 D)$. Let F be the conformal equivalence from the partly-sewn sphere to $A_i\infty_0 B$ and G that from the partly-sewn sphere to $C_j\infty_0 D$. By abuse of domain (in particular, treating the overlap E of the partly sewn sphere as a subset of A only), let F_A be the maximal restriction of F to a subset of A and G_C similar for G and C.

There is precisely one negative tube on each of A and C, and necessarily ${G_C}^{-1} \circ H \circ F_A$ maps the former to the latter, and in fact extends to a conformal equivalence of A to C. Therefore also $i = j$. There is no negative tube on B or D, and $B \nsubseteq A$ and $D \nsubseteq C$, so the conformal equivalence ${G_D}^{-1} \circ H \circ F_B$ maps an open subset of B to an open subset of D. It therefore also extends to a conformal equivalence $B \to D$. $\square$

That the sewing functions are not injective if $n = 0$ can be deduced from Proposition 3.3. It is also not injective in both parameters; suppose that, for $S \in K(m)$ and $T \in K(n)$ for $1 \leq m, n$, $S_k\infty_0 T$ is defined. As usual, take the tubes of S to be $(p_i, f_i)_i$, and those of T as $(q_j, g_j)_j$. Then let S' be equal to S on all tubes except that the k^{th} tube of S' is $(p_i, \pi f_i)$, where $|\pi| = 1$, and likewise T' be like T except with 0^{th} tube $(\infty, \pi^{-1} g_0)$. Then ${S'}_k\infty_0 T' = S_k\infty_0 T$, but S' and T' are conformally equivalent to S and T respectively if and only if $\pi = 1$.

From the following lemma we can prove that every sphere with tubes can be sewn from two spheres with tubes. Thus it shows that the sewing function ${}_i\infty_0$ is a surjection for $K(\geq i)$.

Lemma 2.11. The sphere I with $1+1$ tubes and power series $\frac{1}{z}$, z for these tubes acts as identity for sewing.

Proof. This lemma is a consequence of the future Proposition 3.3. In short, the sewing product I with any other sphere is always defined, since we can use arbitrary radii on I, and the rest is a simple consequence of the sewing equations 2.7. $\square$

These last results in short:

Theorem 2. *Sewing is a family*

$$\{{}_i\infty_0\colon K(\geq i)\times K\to K\}_{i\geq 1}$$

of associative surjective products ${}_i\infty_0$ on K the space of canonical spheres with tubes. Each ${}_i\infty_0$ is injective on one parameter of $K(m)\times K(n)$ for $1\leq m,n$. $\square$

CHAPTER 3

THE SPHERE OPERAD

In this chapter we continue the notation of Chapter 2. In particular, $K(n)$ is the space of canonical spheres with $1+n$ tubes (*i.e.,* a space of class representatives of $\mathcal{K}(n)$ modulo $PGL_2(\mathbb{C})$), and ${}_i\infty_0$ is the sewing operation that joins two spheres via their tubes.

3.1 Sphere operad

This following lemma is perhaps overdue.

Lemma 3.1. $\{K(n)\}_{n\in\mathbb{N}}$ with $\{{}_i\infty_0\}_{1\leq i}$ is a partial operad: the *sphere operad*.

Proof. The $\{K(n)\}_{n\in\mathbb{N}}$ are obviously the sets of 'operations' from the Definition 1.4 of operads, and $\{{}_i\infty_0\}_{1\leq i}$ the compositions. Point i. is satisfied by Proposition 2.8, giving the associativity of the compositions $\{{}_i\infty_0\}_{1\leq i}$ when they exist (therefore it is a partial operad, as per Definition 1.28) and ii. by Lemma 2.11, showing the existence of an identity element with respect to sewing. $\square$

It now seems appropriate to discuss the further properties of operads the sphere operad may satisfy. We have just seen that it is unital. It also satisfies iii., the property of symmetries.

The action of the symmetric group S_n on spheres with $1+n$ tubes, written as exponentiation, is on the ordering of the n positive tubes. This action is for the most part very simple – unless it relabels the n^{th} tube, so the sphere might then no longer be canonical.

Proposition 3.2. *$\{K(n)\}_{n\in\mathbb{N}}$ with $\{{}_i\infty_0\}_{1\leq i}$ admits a faithful action of the symmetric groups; in particular, is a nontrivial (partial) symmetric operad.*

Proof. The symmetric group S_n is generated by $\mathrm{Stab}_{S_n}(n) \cong S_{n-1}$, acting on the first $n-1$ letters, and the permutation $(n-1, n)$.

Let S be a sphere with $1+n$ tubes. The action of S_n on the n positive tubes is induced via the natural action on the ordering of the tubes. In other words, S^σ has tubes $(p_{i^\sigma}, f_{i^\sigma})_{1\leq i\leq n}$. ($S_n$ fixes the negative tube $(p_0 = \infty, f_0)$.)

A permutation $\sigma \in \mathrm{Stab}_{S_n}(n)$ fixes the n^{th} tube (and of course the 0^{th} tube). Therefore the location and gradient of the 0^{th} and last tubes of S are unaffected by σ and so S^σ is canonical. Therefore S and S^σ are conformally equivalent exactly when equal, which is if and only if σ acts trivially. And if σ has trivial action, as the tubes have pairwise distinct locations then $\sigma = \mathrm{id}$. Thus $\mathrm{Stab}_{S_n}(n)$ acts faithfully.

We calculate the action of $\tau = (n-1, n)$ on S. To be explicit, let S have positive tubes $(p_i, f_i)_{1\leq i\leq n}$ and the tubes of the so far noncanonical sphere $S^{\tau\prime}$ are written $(p_{\tau i}, f_{\tau i})_{1\leq i\leq n}$. A conformal equivalence F from $S^{\tau\prime}$ to the canonical sphere S^τ is a translation by $-p_{\tau n}$ (where both conformal equivalence and canonical sphere are unique). So for positive tubes with $1 \leq i \leq n-1$, $Fp_{\tau i} = p_{\tau i} - p_{n-1}$ and $Ff_{\tau i} = f_{\tau i} \circ (z \mapsto z + p_{n-1})$. For the negative tube, $Fp_{\tau 0} = Fp_0 = F\infty = \infty$. Also

$$Ff_0 = f_0 \circ \left(z \mapsto \frac{1}{\frac{1}{z} + p_{n-1}} \right) = f_0 \circ \left(z \mapsto \frac{z}{1 + p_{n-1} z} \right).$$

The former inequality follows from F fixing the gradient of f_0. □

It is easy to see in its own right, and moreover as a consequence of the faithful action of S_n on $K(n)$, that the sphere operad has intrinsic arity.

We can prove that the operad K is generated by a subset G of $K(\leq 2)$.

Certain power series are especially simple in their own right, as ordinary polynomials. This proposition shows that they can give rise to tubes on spheres that behave simply, too.

Proposition 3.3. *Let S be a sphere with tubes $(p_i, f_i)_{0\leq i\leq m}$, T be a sphere with tubes $(q_j, g_j)_{0\leq j\leq n}$, and $1 \leq k \leq m$. Suppose also that if T has $1+0$ tubes,* i.e., *$n = 0$, then either S has $1+1$ tubes or $k \neq m$.*

If T has 0^{th} tube $(\infty, \hat{J})$ then the solution to the sewing equations for $S_k\infty_0 T$ is $F_S = \mathrm{id}$ and $F_T = {f_i}^{-1}$.

If S has an k^{th} tube $(p_k, a(\mathrm{id} - p_k))$, then the solution to the sewing equations for $S_k\infty_0 T$ is $F_S = \frac{1}{a}{g_0}^{-1} \circ \hat{J} \circ f_k$ and $F_T = \frac{1}{a}\,\mathrm{id}$.

As an aside: in each case, the tubes of $S_i\infty_0 T$ are given by $(F_X(p), f \circ {F_X}^{-1})$ for a tube (p, f) arising from the sphere $X = S$ or T.

Also, as an indicative example, the equality $F_T = {f_k}^{-1}$ should be taken to mean that F_T and ${f_k}^{-1}$ are pointwise equal where there is a natural identification between points in the domain of F and the subset of the codomain of f_k where f_k is invertible. Everywhere else, F is an analytic continuation of this restricted inverse.

Observe that the caveat ("Suppose also ...") of the statement is the same as that for the sewing equations of Lemma 2.7. If T has $1+0$ tubes, sewing $S_i\infty_0 T$ eliminates the i^{th} tube of S. Therefore if $i = m \geq 2$ and S has $1+m$ tubes, then the tube of $S_i\infty_0 T$ which is last in the ordering is not at 0, and thus $S_i\infty_0 T$ is not canonical with the solutions given above. There is of course a conformal equivalence to a canonical sphere, but the solutions will not be as given here. See instead the proof of Proposition 3.2 which deals with just this.

Proof. Now, any solution to the sewing equations Lemma 2.7 is unique; if we exhibit that the given F_S and F_T are solutions, we are done. This is obvious at least in the first instance. In the second part, observe also the normalisation conditions, equations (2), of Lemma 2.7. The scaling $\frac{1}{a}$ is present in F_S to preserve the gradient of the negative tube, and this is reflected also in F_T. $\square$

We can now show that any sphere is generated using a very limited subset of spheres with tubes. The claim here is very much along the lines of Lemma 1.3, although the situation is more complicated. The idea is this: using spheres with $1+2$ tubes, we can arrange the locations of tubes on a sphere however we want; using some spheres with $1+1$ tubes, we can change the power series to our whim. As is to be expected, the preceding Proposition 3.3 is crucial.

Proposition 3.4. *Any sphere with tubes can be expressed as a finite sewing product of spheres from the set G of the following spheres:*

1. *The unique sphere U with $1+0$ tubes $(p_0 = \infty, \hat{J})$.*

2. *The spheres Q^g_{id} and $Q^{\hat{J}}_f$, where Q^g_f is a sphere with $1+1$ tubes (∞, g) and $(0, f)$.*

3. *The spheres $P(w)$ with $1+2$ tubes $(\infty, \hat{J})$, $(w, \mathrm{id} - w)$ and $(0, \mathrm{id})$, for $w \in \mathbb{C}$.*

So G suffices to write all spheres from K, and thus in the notation of Definition 1.12, $K = \langle G \rangle$.

Proof. By Proposition 3.3 we have that $Q^g_f = Q^g_{\mathrm{id}}{}_1\infty_0 Q^{\hat{J}}_f$, so G affords us all spheres of the form Q^g_f. Observe that this is all of $K(1)$.

Let A be an arbitrary sphere with tubes. Our argument to explicitly find an expression for A is by induction.

If A has $1+0$ tubes and sole tube (∞, g), then $A = Q^g_{\mathrm{id}}{}_1\infty_0 U$ is the required expression. If A has $1+1$ tubes then as $A \in K(1)$ it is one of the spheres already in G.

These are the base cases. Suppose a fixed sphere with tubes A has $1+n$ tubes, for n at least 2. Its tubes are (∞, f_0), (p_1, f_1), $\ldots$, (p_{n-1}, f_{n-1}) and $(0, f_n)$. Fix i such that $|p_i| = min_{1 \leq j \leq n} |p_j|$.

Let B be the sphere with the $1+(n-1)$ tubes (∞, f_0), (p_1, f_1), $\ldots$, (p_{i-1}, f_{i-1}), (p_{i+1}, f_{i+1}), $\ldots$, (p_{n-1}, f_{n-1}) and $(0, \mathrm{id})$. In other words, it is A with the i^{th} tube removed and the last tube replaced. By induction, the claim holds for B.

Now we can add the tubes (p_i, f_i) and $(0, f_n)$ of A to B by use of the spheres in the claim. If p_i is the unique puncture of minimal size, let $X = B_n\infty_0 P(p_i)$. Otherwise, it is possible to find $\varepsilon \in \mathbb{C}$ such that

$$X = \left(\left(B_n\infty_0 P(\frac{1}{2}p_i + \varepsilon)\right)_{(n-1)}\infty_0 P(\frac{1}{2}p_i - \varepsilon)\right)_n\infty_0 U$$

is defined. In either case, the desired expression for A is

$$A = \left(X_{(n-1)}\infty_0 Q^{\hat{j}}_{f_{n-1}}\right)_n\infty_0 Q^{\hat{j}}_{f_n}.$$

Reading from the left (which matches the order of evaluation), X has n punctures in the same order and locations as the punctures of A; the sewing product $_{(n-1)}\infty_0$ changes its $(n-1)^{\text{th}}$ power series to f_{n-1}; the last use of $_n\infty_0$ changes the ultimate power series to f_n. $\square$

The generating set G is not minimal, in that some functions $f$may be written as the composition of other functions and thus Q^f_g or Q^g_f for all other g need not be included in 2. However it is not possible to only admit spheres of type Q^g_{id} or $Q^{\hat{j}}_f$ to 2. Similarly some elements included in 3. are unnecessary, as (for example) $P(2) = (P(1)_2\infty_0 P(1))_2\infty_0 U$. But there is no significant reduction to G to be made.

We see that $K(n)$ can be made into $\mathbb{C}$ modules in several different ways. For example, $c \in \mathbb{C}$ can act on a sphere S in $K(n)$ by multiplication of the locations of tubes. In other words, if S has tubes $(p_i, f_i)_{0\le i\le n}$ then $c.S$ has tubes $(cp_i, f_i \circ (z \mapsto \frac{z}{c}))_{1\le i\le n}$.

In fact this action does not make K a $\mathbb{C}$-operad, for

$$c.(S_i\infty_0 T) = c.S_i\infty_0 c.T$$

and thus the sewing operation is not $\mathbb{C}$-linear in this way. Indeed K is not a $\mathbb{C}$-operad at all.

Lemma 3.5. The only action of a ring k that makes each $K(n)$ into a k-module under which the sewing product ${}_i\infty_0$ is equivariant (*i.e.,* k-linear) is the trivial action.

Proof. We see that the action of k on K is completely determined by its action on I, the sphere with $1+1$ tubes $(\infty, \hat{J})$ and $(0, \mathrm{id})$. For if $c \in$ k and S is some sphere with tubes, then $c.S = c.(I_1\infty_0 S) = (c.I)_1\infty_0 S$, and also $c.S = S_i\infty_0 c.I$ for all i.

Let $R = c.I$, and write its $1+1$ tubes as (∞, r_0) and $(0, r_1)$. Now, in the notation of Proposition 3.4, we see that $P(w)_1\infty_0 R = P(w)_2\infty_0 R$, and by the second statement of Proposition 3.3, we have that $r_0 \circ \hat{J} = \mathrm{id}$. Thus $r_0 = \hat{J}$. From $R_1\infty_0 P(w) = P(w)_2\infty_0 R$ we arrive at $r_1 = \mathrm{id}$. This of course means that $R = I$. □

3.2 Power series

We now look at an alternative parametrisation of power series. A power series $f = \sum_{i\in\mathbb{N}} a_i x^i \in \mathbb{C}[x]$ can be naturally represented by the sequence of its coefficients $(a_i)_{i\in\mathbb{N}} \in \mathbb{C}^\infty$. Indeed, the bijection between the two is obvious. The following bijection, or representation, of a power series will emphasize its use as an operator.

Definition 3.6. The exponential operator is, for $X \in \operatorname{End}\mathbb{C}[[x]]$,

$$e^X = \exp(X) = \sum_{i\in\mathbb{N}} \frac{1}{i!} X^i.$$

The logarithm operator is

$$l_{\mathrm{id}+X} = \log(\mathrm{id}+X) = \sum_{i\geq 1} \frac{(-1)^{i+1}}{i} X^i. \qquad \square$$

Definition 3.7. For a sequence $\bar{a} = (a_i)_{i\geq 2} \in \mathbb{C}^\infty$, we define the map $L\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ as $L\colon \bar{a} \mapsto (L(\bar{a})_i)_{i\geq 2}$ such that, as functions $\mathbb{C} \to \mathbb{C}$,

$$x + \sum_{i\geq 2} L(\bar{a})_i x^i = (\exp\left(\sum_{i\geq 2} a_i x^i \frac{\partial}{\partial x}\right))x.$$

Then $L(\bar{a}) = (L(\bar{a})_i)_{i\geq 2} \in \mathbb{C}^\infty$. We call $\bar{a}$ the $\exp$-*coefficient sequence* and let

$$H = \{(a_i)_{i\geq 2} \in \mathbb{C}^\infty \mid (\exp\left(\sum_{i\geq 2} a_i x^i \frac{\partial}{\partial x}\right))x \text{ converges in some neighbourhood of } 0\},$$

i.e., H is the set of sequences mapped by L to coefficient sequences of power series absolutely convergent in some neighbourhood of 0. Also define $\bar{L}\colon \bar{a} \mapsto x + \sum_{i\geq 2} L(\bar{a})_i x^i$, a function taking an $\exp$-coefficient sequence to a power series. □

Lemma 3.8. The definition just given, Definition 3.7, is well-defined. Moreover L is an invertible endomorphism, so

$$f(x) = x + \sum_{i\geq 2} a_i x^i = (\exp\left(\sum_{i\geq 2} L^{-1}(\bar{a})_i x^i \frac{\partial}{\partial x}\right))x$$

for (infinite) formal series $f \in \mathbb{C}[[x]]$ and $L^{-1}(\bar{a}) = (L^{-1}(\bar{a})_i)_{i\geq 2} \in \mathbb{C}^\infty$.

Proof. Firstly, the righthand side of Definition 3.7 is well-defined. Each term $\big(\sum_{i\geq 2} a_i x^i \frac{\partial}{\partial x}\big)^k x$ has lowest-order term x^{k+1} so it is in $x^{k+1}\mathbb{C}[[x]]$. Therefore the entire expression is an element of $x + x^2\mathbb{C}[[x]]$. Calculating the terms is essentially easy, although as the terms increase in order, the number of (recursing) infinite sums grows very rapidly. Note that the coefficient $L(\bar{a})_i$ of x^i is a_i plus some polynomial p in $a_1, \ldots, a_{i-1}$.

Secondly, taking some fixed $\bar{a} \in \mathbb{C}^\infty$, for an indeterminate $\bar{\alpha} \in \mathbb{C}^\infty$ we may solve $L(\bar{\alpha}) = \bar{a}$. By what we have just observed, $\alpha_1 = a_1$, and for $i \geq 2$, $\alpha_i =$

$a_i + p(a_1, \ldots, a_{i-1})$. So it is simple to solve for $\bar{\alpha}$ by induction. Then $L^{-1}(\bar{a}) = \bar{\alpha}$, and

$$\begin{aligned} f(x) &= x + \sum_{i\geq 2} a_i x^i = x + \sum_{i\geq 2} L(L^{-1}(\bar{a})_i)_i x^i \\ &= (\exp\left(\sum_{i\geq 2} L^{-1}(\bar{a})_i x^i \frac{\partial}{\partial x}\right))x. \end{aligned}$$

□

It should be emphasised that these equalities only hold for the particular case of $f(x)$, that is, evaluation of the operator $\exp\left(\sum_{i\geq 2} a_i x^i \frac{\partial}{\partial x}\right)$ at x specifically is crucial. We give here a formula of the terms $L(\bar{a})_i$, which could be made explicit but is not of use to us computationally.

$$\begin{aligned} f = \bar{L}(\bar{a}) = x + \sum_{i\geq 2} L(\bar{a})_i x^i = (\exp\left(\sum_{i\geq 2} a_i x^i \frac{\partial}{\partial x}\right))x &= x + \sum_{k=1} \frac{1}{k!}\left(\sum_{i\geq 2} x^i \vartheta(i,k)\right) \\ &= x + \sum_{i\geq 2} x^i \left(\sum_{k=1}^{i-1} \frac{1}{k!}\vartheta(i,k)\right) \end{aligned}$$

where $\vartheta(i,k)$ is the sum of all distinct ordered k-tuples $a_{m_1} a_{m_2} \cdots a_{m_k}$ such that all $m_1, \ldots, m_k \geq 2$ and $m_1 + m_2 + \cdots + m_k = i + k - 1$ (and each term has some polynomially bounded integer coefficient). In other words there is a combinatorial nature to these terms. Here the first few are computed.

$$\begin{aligned} f = x + \sum_{i\geq 2} L(\bar{a})_i x^i &= x + x^2 a_2 + x^3\big(a_3 + \frac{1}{2}2a_2a_2\big) + x^4\big(a_4 + \frac{1}{2}5a_2a_3 + \frac{1}{3!}6a_2a_2a_2\big) \\ &\quad + x^5\big(a_5 + \frac{1}{2}(6a_2a_4 + 3a_3a_3) + \frac{1}{3!}26a_2a_2a_3 + \frac{1}{4!}20a_2a_2a_2a_2\big) + \cdots . \end{aligned}$$

Proposition 3.9. *The set H of sequences of locally convergent* exp*-coefficients (whose associated functions have no constant term and gradient one) is a subvector space of $\mathbb{C}^\infty$. Moreover it is an ideal for sequences of polynomially bounded growth.*

Proof. Firstly, H is stable under scalar multiplication. Let $c \in \mathbb{C}$, and $\bar{a} \in H$. Then $\bar{L}(\bar{a})$ is absolutely convergent. The coefficient of x^n in $\bar{L}(\bar{a})$ is a linear sum of terms of the form $\frac{1}{i!}a_{j_1}a_{j_2}\cdots a_{j_i}$ for $1 \leq i < n$. Thus for the coefficient of x^n in $\bar{L}(c\bar{a})$ we have $\frac{c^i}{i!}a_{j_1}a_{j_2}\cdots a_{j_i}$. In the limit $i \to \infty$, the growth of c^i is dominated by the growth of $i!$ for any fixed c, so the behaviour of $\frac{c^i}{i!}$ in this limit is comparable to that of $\frac{1}{i!}$. Therefore if the sum of terms of the form $\frac{1}{i!}a_{j_1}a_{j_2}\cdots a_{j_i}$ converges, the sum of terms of the form $\frac{c^i}{i!}a_{j_1}a_{j_2}\cdots a_{j_i}$ also converges. Thus absolute convergence of $L(c\bar{a})$ follows from the absolute convergence of $L(\bar{a})$. Observe that multiplication by a scalar is equivalent to pointwise multiplication by a constant sequence.

It does not take much to upgrade this argument. Let $\bar{a} \in H$ and $p \in \mathbb{C}[n]$. Then $(a_i p(i))_{i\geq 2} \in H$, as in place of c^i we have finite products of $p(i)$ and its derivatives in at most i terms. This also grows more slowly than $i!$ as $i \to \infty$, which is the last claim.

Secondly, H is closed under addition. This claim follows because binomial coefficients also grow much more slowly than factorials. Let $\bar{a}, \bar{b} \in H$. Then, in the expression above, taking coefficients from $a + b$ rather than a, we see that the coefficient of x^n in $\bar{L}(\bar{a} + \bar{b})$ is in terms of $\frac{1}{i!}(a_{j_1} + b_{j_1})(a_{j_2} + b_{j_2})\cdots(a_{j_i} + b_{j_i})$. Expanding this product gives coefficients of products of a_j, b_j of the form $\frac{1}{i!}\binom{i}{k}$. Again the absolute convergence is assured by the factorial growth dominating all other terms as $i \to \infty$. $\square$

We introduce a new term. A given power series f with no constant term and gradient 1 induces some operators on power series. Firstly, by precomposition f induces $\varphi\colon g(x) \mapsto g \circ f(x)$, and $\varphi \in \operatorname{End}\mathbb{C}[[x]]$. Clearly, since the lowest order term of f is x, $\varphi = \mathrm{id} + xX$ for id the identity from $\operatorname{End}\mathbb{C}[[x]]$ and $X \in \operatorname{End}\mathbb{C}[[x]]$. Recalling Definition 3.6, we now define l_f as $l_{\mathrm{id}+xX}$. That is the logarithm of f *as an endomorphism of power series* acting by precomposition. Thus its exponent e^{l_f} is again an endomorphism of $\mathbb{C}[[x]]$. We can show some results on the relation it has to f.

Proposition 3.10. *For a power series f with gradient 1, $f(x) = e^{l_f}(x)$.*

Note that $f \neq e^{l_f}$: the two coincide only if evaluated on the specific polynomial x. To be clear, φ and e^{l_f} are two distinct endomorphisms of $\mathbb{C}[[z]]$, while $f(x)$ and $e^{l_f}(x)$ are the same function $\mathbb{C} \to \mathbb{C}$.

The proof is achieved by making the appropriate identifications between the power series and the endomorphisms.

Proof. So fix $f = x + \sum_{i \geq 2} a_i x^i$ (and therefore also the series $\bar{a} = (a_i)_{i \geq 2}$) as in the hypothesis. We use the notation $\varepsilon = \sum_{i \geq 2} L^{-1}(\bar{a})_i x^i \dfrac{\partial}{\partial x}$ for the element of $\operatorname{End} \mathbb{C}[[x]]$.

By the definition in Lemma 3.8, $f(x) = \exp(\varepsilon)(x)$. We can view $f(x)$ as the endomorphism $\varphi \in \operatorname{End} \mathbb{C}[[x]]$ as in the preceding discussion. Then $\log f(x)$ is defined and equal to $l_f(x)$. Since, as endomorphisms of power series, $\log$ and $\exp$ are inverses (where defined), we have that $\log \circ \exp(\varepsilon)(x) = \varepsilon(x)$.

Finally we again take exponentials to find $e^{l_f}(x) = \exp(l_f)(x) = \exp(\varepsilon)(x) = f(x)$. □

Let us now generalise this to power series of arbitrary gradient. Let O be an invertible linear operator of $\mathbb{C}[x^{-1}, x]$. Then it induces another linear operator $O^{x(d/dx)}$ by $x^n \mapsto O^n x^n$. In particular, multiplication by $a_1 \in \mathbb{C}^\times$ is an invertible linear operator ad_{a_1}, and so we have ${\operatorname{ad}_{a_1}}^{x(d/dx)}$, which extends to an operator on $\mathbb{C}[[x^{-1}, x]]$.

We now consider power series with arbitrary nonzero gradient. Following [H97], we prefer to write

$$f(x) = a_1 \left(x + \sum_{i \geq 2} a_i x^i \right)$$

rather than $f(x) = a_1 x + \sum_{i \geq 2} a_i' x^i$. We can thus improve Proposition 3.10 to the statement: *for an arbitrary power series f with gradient a_1, $f(x) = e^{l_f} {\operatorname{ad}_{a_1}}^{x(d/dx)} x$.*

This expression of functions as operators on functions finds use in [H97] to find explicit solutions to the sewing equations of Lemma 2.7; we will simply cite this work in Section 4.1.

3.3 Structure of $K(n)$

For each $n \in \mathbb{N}$, the space $K(n)$ is more than just a set.

Recall that a conformal sphere with $1+n$ tubes is given by $n-1$ puncture locations and $n+1$ power series. The power series for the 0^{th} puncture has to have gradient 1. All others may take any gradient from $\mathbb{C}^\times$.

In Section 3.2, we defined H as the set of $\exp$-coefficients of locally convergent sequences with no constant term. An element $\bar{h} = (h_i)_{i\in\mathbb{N}} \in H$ defines a power series $f = z + \sum_{i\geq 2} E(\bar{h})_i z^i$ which is centered on and converges in some neighbourhood of 0. By simply applying a translation $z \mapsto z - p$, we get a power series of the same $\bar{h} \in H$, but centered on and convergent in some neighbourhood of $p \in \mathbb{C}$. This is just a change of coordinates. Thus all tubes on a sphere can be recorded as a location on the sphere, a gradient and an element of H.

Lemma 3.11. The spaces $K(n)$, $\mathcal{K}(n)$ of spheres with $1+n$ tubes can be identified with

$$K(n) \cong P^{n-1} \times H \times (\mathbb{C}^\times \times H)^n, \text{ a subspace of } \hat{P}^{n+1} \times (\mathbb{C}^\times \times H)^{n+1} \cong \mathcal{K}(n),$$

for P^n the set of ordered n-tuples from $\mathbb{C}^\times$ which are pairwise distinct, and $\hat{P}^n$ the moduli space of n punctures on a sphere.

Proof. We exhibit the embedding. It is clear that $P^n \subset \hat{P}^n$. Then $K(n) \hookrightarrow \mathcal{K}(n)$ by

$$(p_1, \ldots, p_{n-1}, \bar{h}_0, a_1, \bar{h}_1, \ldots, a_n, \bar{h}_n) \mapsto (\infty, p_1, \ldots, p_{n-1}, 0, 1, \bar{h}_0, \ldots, a_n, \bar{h}_n),$$

where we have just filled in the details of a canonical sphere with tubes. $\square$

We can therefore write a tube (p, f) (for $p \in \hat{\mathbb{C}}$ and $f \in \mathbb{C}[[z]]$ or $\mathbb{C}[[z^{-1}]]$) also as $(p, a, \bar{h})$, where $\bar{h} \in H$ such that if $f = a\left(x + \sum_{i\geq 2} b_i x^i\right)$ then $\bar{h} = E^{-1}(\bar{b})$, $\bar{b} = (b_i)_{i\geq 2}$.

Observe that $K(n)$ is infinite-dimensional for all values of n, and therefore is not an algebraic variety in the usual sense.

What analytic structure does $\mathcal{K}(n)$ admit? Clearly the crux of an analytic structure is a norm on the tubes.

As a preliminary, note that $\hat{P}^1 = \hat{\mathbb{C}}$ admits the spherical norm, that is, for a nonzero point $p \in \hat{\mathbb{C}}$ and $\hat{\mathbb{C}}$ considered embedded in $\mathbb{R}^3$, there is a unique plane containing 0, p and the center O of the sphere $\hat{\mathbb{C}}$ in $\mathbb{R}^3$, so we define $\|p\|_\circ$ as the minimum angle between the lines O to p and O to 0 in this plane. Also define $\|0\|_\circ = 0$, as expected. Then the norm on $\hat{P}^2$ is just the product norm given by $\hat{P}^2 \subset \hat{P} \times \hat{P}$ (subset since $\hat{P}$ admits only tuples of pairwise disjoint values), and the norm on P^n is defined similarly as the product norm.

Now to 'measure' tubes.

In [H97], Huang uses the supremum norm to define an increasing sequence of Banach spaces of power series. Namely, let $f = \sum_{i\geq 1} a_i x^i$ be an arbitrary power series; then we define B_n as the vector spaces of power series absolutely convergent on $\bar{B}^{1/n}$, the closed disk of radius $\frac{1}{n}$. This is a subset of $L^1(\bar{B}^{1/n})$, the space of Lebesgue-measurable functions with domain $\bar{B}^{1/n}$. Note that this is a nested sequence, *i.e.,* $B_m \subset B_n$ if $m < n$. Then $\|f\|_{B_n} = \sup_{|z|\leq \frac{1}{n}} |f(z)|$ is a complete norm.

Since the functions $f \in B$ are holomorphic in some neighbourhood of 0, they are measurable on this neighbourhood, and we see that we could use instead any p-norm $\|\cdot\|_p$ for $p \in [1, \infty)$ (the norm $\|\cdot\|_{B_n}$ being the $p = \infty$-norm). By the well-known Riesz-Fischer theorem, this too is a complete normed vector space.

Let B be the vector space of all formal series $\sum_{i\geq 1} a_i x^i$ absolutely convergent in some neighbourhood of 0. Then it turns out that B is linearly isomorphic to the colimit of the sequence $(B_n)_{1\leq n}$. The colimit of an increasing sequence of Banach spaces is an *(LB)-space*. The embedding $H \hookrightarrow B$ is given by $\bar{E}$ of Definition 3.7. This embedding and the linear isomorphism give holomorphic transition functions from $\mathcal{K}(n)$ to an (LB)-space, making $\mathcal{K}(n)$ an *(LB)-manifold*.

By definition, H is a space of sequences. So alternatively to the L^p spaces we can consider an ℓ_p norm on H for $p \in [1, \infty)$. Of course the space ℓ_1, defined as the space of absolutely convergent sequences, can be viewed as the space of power series absolutely convergent on all of $\bar{B}^1$. For $\bar{a} \in \ell_1$ means that $\sum_{i\geq 1}|a_i|$ converges, and thus $|x| + \sum_{i\geq 1}|a_i x^{i+1}|$ is convergent for all $|x| \leq 1$.

We can only conjecture the following.

Conjecture. Suppose $f = x + \sum_{i\geq 1} a_i x^i$ is absolutely convergent in some neighbourhood of 0. Then for $\bar{a} = (a_i)_{i\geq 1}$, the power series $x + \sum_{i\geq 1} E^{-1}(\bar{a})_i x^i$ converges absolutely on $\bar{B}^1$. □

This, if true, would mean that $H \subseteq \ell_1$.

Now we consider the relationship the canonical spheres with tubes have to this analytic structure. Let $c : \mathcal{K}(n) \to \mathcal{K}(n)$ take spheres with tubes to conformally equivalent canonical spheres with tubes. It is therefore an idempotent. On the equivalence classes $\mathcal{K}(n)/PGL_2(\mathbb{C})$, c is constant.

We call any function $c\colon \mathcal{K}(n) \to \mathcal{K}(n)$ constant on the $PGL_2(\mathbb{C})$-equivalence classes a canonical choice function or *canon*. Note that, when we identify $K(n)$ with a subset of $\mathcal{K}(n)$, a canon is an idempotent.

We see that canons can be composed with elements of $PGL_2(\mathbb{C})$, and that since $PGL_2(\mathbb{C})$ is a group, the relation $c_1 \sim c_2$, meaning there exists $M \in PGL_2(\mathbb{C})$ such that $M \circ c_1 = c_2$, is an equivalence relation on canons. Within the equivalence class of Huang's canon (which has an implicit definition in Definition 2.2), the sewing equations of Lemma 2.7 are invariant under choice of canon in the Definition 2.5 of the sewing operation.

It is clear that a canon is never compatible with a metric structure on $\mathcal{K}(n)$. Indeed, any such function will break some symmetry in $\mathcal{K}(n)$. So we see that $\mathcal{K}(n)/PGL_2(\mathbb{C})$ is a nontrivial bundle of which a canon c is a section. If c is in the equivalence class of Huang's canon, we see that its image is a trivial bundle since

$K(n)$ is a manifold.

3.4 Subsets of $K(1)$

We focus here on the space $K(1)$ of canonical spheres with $1+1$ tubes. It is a rich source of substructures.

We first saw in Section 2.5 that the sewing product is a partial product on the space K of spheres with tubes. For a subset of $K(1)$, it becomes a full product: let $K^{\circ}(1)$ denote the set of spheres with $1+1$ tubes whose functions are entire, that is, holomorphic on all of $\mathbb{C}$ or possibly $\check{\mathbb{C}} = \hat{\mathbb{C}} \smallsetminus \{0\}$.

Lemma 3.12. For all $S, T \in K^{\circ}(1)$, $S_1\infty_0 T$ is defined.

Proof. Let S have 1^{th} tube $(0, f)$ and T have 0^{th} tube (∞, g). Now for arbitrary $r < R \in \mathbb{R}$, $f^{-1}(\bar{B}^R) \ni 0$ and as $f \in \mathbb{C}[[z]]$, $\infty \notin f^{-1}(\bar{B}^R)$. Likewise, since $\infty \in g^{-1}(\bar{B}^{1/r})$, $0 \notin g^{-1}(\bar{B}^{1/r})$. Thus neither preimage contains another puncture. Since there are no singularities, by Definition 2.5, this is sufficient. $\square$

Of course we already know that ${}_1\infty_0$ maps $K(1) \times K(1)$ into $K(1)$. By collecting some previous results, we can therefore identify some algebraic structures $K^{\circ}(1)$ admits.

Consequence. *$K^{\circ}(1)$ is a monoid.*

Proof. The binary operation ${}_1\infty_0$ on $K^{\circ}(1)$ is associative by Proposition 2.8 and has an identity element I as in Lemma 2.11. $\square$

To make $K^{\circ}(1)$ into a group, we need inverses. We call a tube (p, f) *entire*, or *invertible*, if f is entire, or invertible on $\mathbb{C}$ or $\check{\mathbb{C}}$, respectively.

Proposition 3.13. *Let $K^{-1}(1) \subset K(1)$ be the set of spheres with $1+1$ entire invertible tubes; with ${}_1\infty_0$, it is a group.*

Proof. By Lemma 3.12, ${}_1\infty_0$ is always defined on $K(1) \times K(1)$ and in particular it is a nonpartial function on $K^{-1}(1) \times K^{-1}(1)$.

Moreover it is a product $K^{-1}(1) \times K^{-1}(1) \to K^{-1}(1)$, for the only modification to the tubes during sewing is effected by composition with the conformal equivalence from the pre-canonical to the canonical sphere. Any conformal equivalence is invertible, and the composition of invertible functions is invertible, so the sewing product is an element of $K^{-1}(1)$.

Associativity was proven in Proposition 2.8, and the identity element I is as in Lemma 2.11. It only remains to describe the inverses.

By assumption, for every tube (p, f) there exists a function f^{-1} inverse to f. So let A be an arbitrary sphere in $K^{-1}(1)$ with tubes (∞, f_0) and $(0, f_1)$. Then let $B \in K^{-1}(1)$ have tubes $(\infty, \hat{J})$ and $(0, {f_1}^{-1})$, and $C \in K^{-1}(1)$ have tubes $(\infty, {f_0}^{-1} \circ \hat{J})$ and $(0, \mathrm{id})$. By Proposition 3.3, we see that

$$B_1\infty_0(C_1\infty_0 A) = I = (A_1\infty_0 B)_1\infty_0 C$$

which (using associativity) makes $(B_1\infty_0 C)$ into the desired inverse. $\square$

There are more ways of finding full sub- or factor-operads of the sphere operad. First, we should consider how close the sphere operad is to being full. Clearly the sewing product $S_i\infty_0 T$ is not defined exactly when there are no suitable radii r, R for sewing the spheres; this is the converse statement to the Definition 2.5. 'Suitable' means that the germ preimages of B^R and $B^{1/r}$ contain other punctures. So changing – in particular, dilating – the germ can change the situation and enable sewing. We have already discussed that this cannot be effected by conformal equivalence. Instead, the sewing multiplication can modify a particular germ.

Lemma 3.14. Let R be a subgroup of the multiplicative group $\mathbb{C}^\times$. Let also $r \in R$ and $U = U(r)$ be the sphere with $1 + 1$ tubes $(\infty, \hat{J})$ and $(0, r\,\mathrm{id})$. Then for any sphere

T with i^{th} power series f, $T_i\infty_0 U$ has i^{th} power series rf and is the same as T in all other features.

Proof. By Lemma 3.3. $\square$

Certainly this applies in the case of sewing together $U(s)$ and $U(t)$ for $s, t \in R \leq \mathbb{C}^\times$. The reasonable expectation of a group structure similar to that of R on $K_R(1) = \{U(s) \mid s \in R\}$ can be satisfied; indeed $U(s)_1\infty_0 U(t) = U(st)$ and $K_R(1)$ with product ${}_1\infty_0$ is isomorphic as a group to R. The ultimate purpose of the lemma is its following consequence.

We say than a partial operad C is *rescalable* when there exists a subgroup G of $C(1)$, such that modulo composition with elements of G, the operad composition is always defined.

Consequence. *The sphere operad is a $K_{\mathbb{C}^\times}(1) \cong \mathbb{C}^\times$-rescalable operad.*

Proof. From the lemma, for any sphere S with $1 + m$ tubes and $1 \leq i \leq m$, if S has i^{th} germ f at location p then $S_i\infty_0 U$ has i^{th} germ sf, for $U = U(s)$ with $s \in \mathbb{R}$ arbitrary. Let T be any other sphere, with germ g at ∞. Choose arbitrary $r > 0$ such that $g^{-1}(B^{1/r})$ contains no other punctures. Now there exists an open neighbourhood V of p on S not containing any other puncture. Finally let $R > r$. There exists $s > 0$ such that $(sf)^{-1}(B^R) = f^{-1}(B^{R/s}) \subseteq V$, as $\lim_{s\to\infty}(sf)^{-1}(B^1) = \{p\} \subset V$. Thus r, R are radii suitable for sewing $S_i\infty_0 U(s)$ and T at the i^{th} puncture, and therefore $(S_i\infty_0 U(s))_i\infty_0 T$ is defined.

By the previous proposition, we see that $\{U(s) \mid s \in \mathbb{R}\} \cong \mathbb{R}$ is a group using the sewing multiplication, and isomorphic to $\mathbb{R}$ as a group. We can also rotate as well as dilate tubes, by introducing a phase factor. Thus $\{U(s) \mid s \in \mathbb{C}^\times\} \cong \mathbb{C}^\times$ is a subgroup of $K(1)$ satisfying the definition of a rescaling group. $\square$

It is clear that $\mathbb{C}^\times$ is not the smallest ring sufficient to rescale K; in fact, for any subring $R \leq \mathbb{C}^\times$ which is not contained in the set of complex numbers of unit length, the sphere operad is $K_R \cong R$-rescalable.

We now widen our view back to $K(n)$ for arbitrary n.

Here we can recover a full suboperad of the (partial) sphere operad. A sphere with tubes is *uniformly sewable* if the ${f_i}^{-1}(\bar{B}^1)$ are pairwise disjoint, for f_i the power series of the tubes. Denote by $\mathfrak{K}(n)$ the subset of uniformly sewable spheres of $K(n)$, the set of spheres with $1+n$ tubes.

Proposition 3.15. *$\{\mathfrak{K}(n)\}_{n\in\mathbb{N}}$ with the sewing compositions is a full operad $\mathfrak{K}$.*

Proof. Let S and $T \in \mathfrak{K} = \bigcup_{n\in\mathbb{N}} \mathfrak{K}(n)$ be uniformly sewable spheres. Suppose that S has $1+n$ tubes. Then for all $i \leq n$, $S_i\infty_0 T$ is defined, as $r = 1 = R$ demonstrates the existence of a suitable pair of radii to sew S to T, by Definition 2.5. Moreover $S_i\infty_0 T$ is uniformly sewable. The other properties are induced from Lemma 3.1. □

When a sphere with tubes is uniformly sewable, we see that each tube can be identified with an analytically parametrized boundary component of the sphere. Therefore the operad $\mathfrak{K}$ can also be interpreted as the operad of spheres with ordered, analytically parametrized boundary components.

The rescaling group G of the operad K from Lemma 3.14 indicates another suboperad of K.

Lemma 3.16. For R a subgroup of $\mathbb{C}^\times$, Let $K_R(n)$ be the space of spheres with $1+n$ tubes such that the 0^{th} tube is $(\infty, \hat{J})$ and for every other tube (p, f) there is some $r \in R$ such that it can be written $(p, z \mapsto r(\mathrm{id} - p))$. Then $K_R = \{K_R(n)\}_{n\in\mathbb{N}}$ is a (partial) suboperad of K.

Proof. This is clear by Proposition 3.3, which gives solutions to the sewing equations in simple cases like this. □

By Lemma 1.13 on the intersections of operads, $K_R \cap \mathfrak{K}$ is also an operad. It is evidently isomorphic to the well-known little disks operad, which we describe now.

Let $\bar{B}^1$ be the closed unit disk in $\mathbb{C}$. Let $D(m)$ be the set of ordered m-tuples of maps $f\colon \bar{B}^1 \to \bar{B}^1$ with pairwise disjoint images, which each act only as a combination of translations, dilations and rotations. The composition of $\bar{f} = (f_1, \ldots, f_m) \in D(m)$ and $\bar{g} \in D(n)$ is, for $1 \leq i \leq m$,

$$\bar{f} \circ_i \bar{g} = (f_1, \ldots, f_{i-1}, g_1 \circ f_i, \ldots, g_n \circ f_i, f_{i+1}, \ldots, f_m) \in D(m+n-1).$$

Then $D = \{D(n)\}_{n\in\mathbb{N}}$ with compositions $\{\circ_i\}_{i\geq 1}$ is the *little 2-disks operad* (where 2 refers to the dimension – it is easily defined for any positive dimension). It is an example of a *configuration space operad*.

Then the isomorphism from $K_R \cap \mathfrak{K}$ to D is just a relabeling, writing a positive tube $(p, f\colon z \mapsto r(z-p))$ as the function f^{-1} and a sphere with tubes (p_i, f_i) for $0 \leq i \leq m$ as $({f_1}^{-1}, \ldots, {f_m}^{-1})$.

These suboperads are by no means exhaustive. It is particularly easy to find more by considering classes of functions closed under composition. We take this no further.

CHAPTER 4

SPHERE ALGEBRAS

4.1 Vertex operator algebras

The sphere operad was conceived to study vertex operator algebras.

Namely, Huang describes a special algebra over the sphere operad [H97], called a geometric vertex operator algebra, and proves the *isomorphism* between the categories of geometric vertex operator algebras and vertex operator algebras.

We pick up this study of geometric vertex operator algebras. A further result on spheres with tubes and another operad from linear algebra are introduced beforehand.

Beyond the associativity of the sewing product, there is another associativity in the order in which tubes are sewn onto spheres.

Recall the notation $P(w)$ (from Proposition 3.4), which is the sphere with $1+2$ tubes $(\infty, \hat{J})$, $(w, \mathrm{id} - w)$ and $(0, id)$. Furthermore, I again refers to the identity sphere with $1+1$ tubes $(\infty, \hat{J})$ and $(0, \mathrm{id})$, and in Definition 4.4 we will refer to the sphere I_a, which has $1+1$ tubes $(\infty, \hat{J})$ and $(0, a\,\mathrm{id})$.

Lemma 4.1. Let v, $w \in \mathbb{C}$ such that $|v| > |w| > |v-w| > 0$. Then

$$P(v)_2\infty_0 P(w) = P(w)_1\infty_0 P(v-w).$$

In words, we can first place a tube at v and then at w onto the sphere I, or we can modify I and then place both tubes into their correct locations at once.

Proof. The assumptions on v and w are such that the sewing product is always defined, since suitable radii exist in each case. Now by the solutions to Proposition 3.3, we can evaluate the lefthand side and righthand side of the claimed equality individually; in each case, the result is the canonical sphere with tubes $(\infty, \hat{J})$, $(v, \mathrm{id} - v)$, $(w, \mathrm{id} - w)$ and $(0, \mathrm{id})$. $\square$

In the Definition 1.15 of the endomorphism operad, we restricted ourselves to finite-dimensional k-modules. We can generalise this to an infinite-dimensional but well-behaved case.

Let I be an index set. An *I-graded* k*-module* V is $\bigoplus_{i \in I} V_i$, the direct sum of I-indexed k-modules V_i. The submodule V_i is said to have *rank* i, as are its elements (written $\operatorname{rk} v = i$ for $v \in V_i$). A submodule is homogeneous if it can be linearly spanned by a set of elements, all of which have the same rank.

Definition 4.2. Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be a $\mathbb{Z}$-graded, not necessarily finitely spanned, k-module, where each V_n is a finitely-spanned module (*i.e.,* homogeneous subspaces can be finitely spanned).

Let $V' = \bigoplus_{n \in \mathbb{Z}} {V_n}^*$ be the graded dual of V, where ${V_n}^*$ is the dual module of V_n (and therefore has a natural pairing $[\cdot,\cdot]\colon {V_n}^* \times V_n \to \mathbf{k}$, $[f, v] = f(v)$). Let $\bar{V} = \Pi_{n \in \mathbb{Z}} V_n = V'^*$ be the algebraic completion of V. The natural pairings on the modules V_n extend to a natural pairing $[,]\colon V' \times \bar{V} \to \mathbf{k}$. Define also $H(n) = \hom_{\mathbf{k}}(V^{\otimes n}\ \bar{V})$.

There is a partially-defined *contraction*

$$
\begin{aligned}
{}_i{*}_0 \colon H(m) \times H(n) &\to H(m+n-1) \\
(f,g) &\mapsto f\,{}_i{*}_0\, g
\end{aligned}
$$

$$
\begin{aligned}
&(f\,{}_i{*}_0\, g)(v_1 \otimes \cdots v_{m+n-1}) = \\
&\quad = \sum_{k\in\mathbb{Z}} f(v_1 \otimes \cdots \otimes v_{i-1} \otimes \mathrm{pr}_k\, g(v_i \otimes \cdots \otimes v_{i+n-1}) \otimes v_{i+n} \otimes \cdots \otimes v_{m+n-1}),
\end{aligned}
$$

where pr_k is the usual projection onto V_k. The contraction $f\,{}_i{*}_0\, g$ is defined precisely when the series on the righthand side is absolutely convergent. □

Lemma 4.3. $\{H(n)\}_{n\in\mathbb{N}}$ with contractions $\{{}_i{*}_0\}_{i\in\mathbb{N}}$ is a partial k-operad, denoted H.

Proof. We show that H satisfies Definition 1.6. The first thing to verify is the associativity of contractions. Suppose that $f \in H(l)$, $g \in H(m)$ and $h \in H(n)$; for $1 \le i \le l$ and $1 \le j \le l+m-1$, that $(f\,{}_i{*}_0\, g)\,{}_j{*}_0\, h$ is equal to $(f\,{}_{j'}{*}_0\, h)\,{}_{i'}{*}_0\, g$ (for either $j' = j$ and $i' = i+n-1$ or $j' = j-m+1$ and $i' = i$) or $f\,{}_i{*}_0(g\,{}_{j-i+1}{*}_0\, h)$, depending on i, j, is obvious by the associativity of function substitution and the k-linearity (in particular, the distributive property) of the terms involved.

Secondly, the identity map $\mathrm{id} \in H(1)$ is the required unit. For, if $f \in H(m)$ and $1 \le i \le m$, we have $\mathrm{id}\,{}_1{*}_0 f = \sum_{k\in\mathbb{Z}} \mathrm{pr}_k\, f = f$ and $f\,{}_i{*}_0\,\mathrm{id} = \sum_{k\in\mathbb{Z}} f(v_1 \otimes \cdots \otimes pr_k v_i \otimes \cdots \otimes v_m) = f(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_m) = f$ by linearity. □

Now we wish to describe geometric vertex operator algebras as an algebra, in the sense of Definition 1.16, using the generalisation H of the endomorphism operad End_V of Definition 1.15.

This requires a great deal of work in areas not alluded to in this text. Of course, this work has been done in [H97]. There is in the end a single term (of fundamental importance) that we need to reference in this calculation. We do so to proceed with our description, but we treat it as a 'black box'.

Recall the notation of Section 3.2 and suppose that $\bar{a} = (a_i)_{i\ge 2}, b \in H$ and $a_1 \in \mathbb{C}^\times$,

$c \in \mathbb{C}$. Denote by γ^c the term $\exp(-\Gamma(a, b, a_1)c) \in \hat{\mathbb{C}}$ from Lemma 5.2.1 in [H97] (for Γ a formal power series as defined in [H97], Proposition 4.2.1, to satisfy equations analoguous to the sewing equations) which is finite when a sphere with tubes S that has an i^{th} tube whose $\exp$-coefficients are $\bar{a} \in H$ and whose gradient is a_1 can be sewn to a sphere with tubes T that has 0^{th} tube with $\exp$-coefficients $\bar{b} \in H$. It turns out that γ^c is precisely the factor by which we need to scale, so that the sewing of spheres corresponds to the contraction of elements of H.

By a *projective* omomorphism $\rho\colon C \to D$ for operads C and D with compositions $\circ_i$ and $*_i$, we mean that the components $D(n)$ of D have the structure of modules and that $\rho(c_1 \circ_i c_2)$ is in the span of but not necessarily equal to $\rho(c_1) *_i \rho(c_2)$.

Definition 4.4. A *geometric vertex operator algebra* over $\mathbb{C}$ is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ over $\mathbb{C}$ with finite-dimensional homogeneous subspaces and $V_n = 0$ for sufficiently small n (the *positive energy* requirement), together with a projective omomorphism $\nu = \{\nu_n\colon K(n) \to H(n)\}_{n\in\mathbb{N}}$ (for K the sphere operad of Lemma 3.1) satisfying also

i. **grading**: For $v' \in V'$, $v \in V_n$ and $a \in \mathbb{C}^\times$, $[v', \nu_1(I_a)(v)] = a^{-n}[v', v]$.

ii. **meromorphicity**: For all $n > 0$, $S \mapsto [\cdot, \nu_n(S)]$ is meromorphic. Moreover, if (p_i, f_i) and (p_j, f_j) are two distinct tubes on S, then the order of the pole of $[\cdot, \nu_n(S)]$ at $v' \otimes \bigotimes_k v_k$ in the limit $p_i \to p_j$ is finitely bounded, the bound depending on v_i and v_j.

iii. **sewing**: There exists a unique $c \in \mathbb{C}$, the *central charge* or *rank*, such that if, for $S \in K(m)$ and $T \in K(n)$, $S_i\infty_0 T$ is defined then $\nu_m(S)_i{*_0}\nu_n(T)$ exists (*i.e.,* $[\cdot, \nu_m(S)_i{*_0}\nu_n(T)]$ is absolutely convergent on its domain) and moreover $\nu_{m+n-1}(S_i\infty_0 T) = \gamma^c \nu_m(S)_i{*_0}\nu_n(T)$, for γ^c depending on S and T. □

Note that in [H97] there is an additional axiom of equivariance under the sewing equation: for $S \in K(n)$ and $\sigma \in S_n$, $\nu_n(S)^\sigma = \nu_n(S^\sigma)$. For us this follows from the requirement that ν is an omomorphism of symmetric operads.

We see that ν is a (projective) representation of the sphere operad.

We give also the definition of vertex operator algebras.

Definition 4.5. A *vertex operator algebra* over $\mathbb{C}$ is a $\mathbb{Z}$-graded vector space V with finite-dimensional homogenous subspaces and $V_n = 0$ for sufficiently small n, distinguished elements $1 \in V_0$ and $\omega \in V_2$ called the *vacuum* and *Virasoro element*, and a *central charge* or *rank* $c \in \mathbb{C}$, together with a linear map $Y\colon V \to \mathrm{End}_V[[x, x^{-1}]]$ that maps $v \in V$ to the *vertex operator* $Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}$ (for $v_n \in \mathrm{End}_V$), such that we have

i. for $u, v \in V$, $u_n v = 0$ for sufficiently large n (depending on u and on v),

ii. **creation property**: $Y(v, x)1 \in V[[x]]$ and $\lim_{x\to 0} Y(v, x)1 = v$

iii. **Jacobi identity**:

$$\frac{1}{x}\delta\left(\frac{y-z}{x}\right) Y(u, y)Y(v, z) - \frac{1}{x}\delta\left(\frac{z-y}{-x}\right) Y(v, z)Y(u, y) = \\ = \frac{1}{z}\delta\left(\frac{y-x}{z}\right) Y(Y(u, x)v, z)$$

where $\delta(x)$ is formally $\sum_{n\in\mathbb{Z}} x^n$,

iv. **Virasoro relations**: for $L(n)$ such that $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$ and that for $v \in V_n$, $L(0)v = nv$, and δ the Kronecker delta,

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{n+m}c$$

v. $L(-1)$**-derivative property**: $\frac{d}{dx}Y(v, x) = Y(L(-1)v, x)$. □

Now we look at the isomorphism of the categories of geometric vertex operator algebras to the category of vertex operator algebras. Suppose that (V, ν) is a geometric vertex operator algebra.

Let $U \in K(0)$ again be the sphere with $1 + 0$ tube $(\infty, \hat{J})$, *i.e.,* the sphere such that sewing $S_i\infty_0 U$ for any sphere with tubes S removes the i^{th} tube from S but leaves the other tubes invaried. Note that ν_0 maps a sphere S with $1+0$ tubes to a function in $H(0) = \hom_{\mathbb{C}}(V^{\otimes 0}\ \bar{V}) = \hom_{\mathbb{C}}(\mathbb{C}\ \bar{V})$; we can therefore identify $\nu_0(S)$ with the vector $\nu_0(S)(1) \in \bar{V}$. Define $1_\nu = \nu_0(U)(1) \in \bar{V}$.

We can show that $1_\nu \in V_0$, using the grading axiom of ν. Observe that for $S \in K(1)$ and $T \in K(0)$,

$$\nu_1(S)\big(\nu_0(T)(1)\big) = \sum_{k\in\mathbb{Z}} \nu_1(S)\big(\operatorname{pr}_k \nu_0(T)(1)\big) = \big(\nu_1(S)_i *_0 \nu_0(T)\big)(1) = \\ = \frac{1}{\gamma^c}\nu_0(S_1\infty_0 T)(1) = \gamma^{-c}\nu_0(T)(1).$$

Thus the functions $\nu_1(S) \circ \nu_0(T)$ and $\gamma^{-c}\nu_0(T)$ coincide, for γ depending on S and T.

Then

$$[\cdot, \nu_1(I_a)\big(\nu_0(U)(1)\big)] = \gamma^{-c}[\cdot, \nu_0(U)(1)] = a^{-\operatorname{rk}\nu_0(U)(1)}[\cdot, \nu_0(U)(1)],$$

whence $\gamma^c = a^{\operatorname{rk}\nu_0(U)(1)}$. From Proposition 4.2.1 in [H97], we see that in this case $\Gamma = 0$ and therefore $\gamma^c = 1$. The equation holds for all $a \in \mathbb{C}$ and thus $\operatorname{rk}\nu_0(U)(1) = 0$.

Define also $W_\varepsilon \in K(0)$ as the sphere with tube (∞, f), for $f = \frac{1}{z} + \frac{\varepsilon}{z^3}$. Then let $\omega_\nu = -\frac{d}{d\varepsilon}\nu_0(W_\varepsilon)|_{\varepsilon=0}$.

Finally, let $Y_\nu(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}$ for $v_n \in \operatorname{End}_V$ satisfying

$$[v', v_n(w)] = \operatorname{Res}_z z^n [v', \nu_2(P(z))(v \otimes w)]$$

for all $v' \in V'$ and $w \in V$, $z \in \mathbb{C}$. Then Proposition 5.4.4 of [H97] asserts that V with Y_ν, 1_ν and ω_ν is a vertex operator algebra (of rank equal to $\operatorname{rk}(V,\nu)$).

From Proposition 3.4 stating that the sphere operad $K = \langle G \subset K(\leq 2)\rangle$ (and a remark in Section 1.4), we see that the isomorphism can be entirely proven using only spheres from G.

Then, for example, Lemma 4.1 translates into a statement on associativity in vertex operator algebras:

Consequence. *Let* $\imath_{ab}\colon \mathbb{C}(x_a, x_b) \to \mathbb{C}[[x_a, {x_a}^{-1}, x_b, {x_b}^{-1}]]$ *be the operation sending rational polynomials to the (binomial-expansion) corresponding formal series. For a vertex operator algebra* $(V, Y, 1, \omega)$*,*

$$\imath_{12}{}^{-1}[v', Y(v_1, x_1)Y(v_2, x_2)v] = \imath_{20}{}^{-1}[v', Y(Y(v_1, x_1 - x_2)v_2, x_2)v].$$

There is therefore a strong correspondence between the sphere operad and vertex operator algebras. The grading of a vertex operator algebra corresponds to the arity-grading of spheres, compositions are partial in a vertex operator algebra exactly when partial in the associated operad of spheres, and so on.

4.2 Cyclic operads

The distinction between positive and negative tubes on a sphere with tubes is formal. The operad structure is preserved when the orientation of tubes is interchanged.

So suppose that S is a sphere with $1 + m$ tubes given by $(p_i, f_i)_{0\le i\le m}$. Then consider the sphere S' with tubes

$$(p_m, f_m), (p_0, f_0), (p_1, f_1), \ldots, (p_{m-1}, f_{m-1}),$$

where the ordering on the tubes is indicated by the order in which they are written. The 0^{th} tube (p_m, f_m) is regarded as negative and all others positive. S' is not a canonical sphere with tubes (since the 0^{th} tube is at $p_m = 0$), but is certainly conformally equivalent to one.

In effect, we only changed the ordering of tubes from S to S'. As a permutation, it would be written $(0, 1, \ldots, m)$. This gives an action of the symmetric group on the ordering of tubes, an action extended from that of the prior Proposition 3.2.

Definition 4.6. Let S_m^+ be the symmetric group $\mathrm{Aut}\{0, 1, \ldots, m\}$. The action of

$\sigma \in S_m^+$ on a sphere S with $1+m$ tubes gives S^σ, the canonical sphere conformally equivalent to the sphere with tubes $(p_{\sigma i}, f_{\sigma i})$. S_m is identified with $Stab_{S_m^+}(0)$. $\square$

The significant statement of course is the action exchanging the negative tube with a positive tube. Part of the claim that the operad structure is preserved by S_n^+ is that the sewing product is still defined under this action, and that in fact the sewing product is independent of any orientation of tubes.

We will discuss this generalised sewing product in Section 4.4; in this section, our focus is on abstracting the exchange of inputs and outputs.

We had already shown that the symmetric group has an action on spheres with tubes; the new information is that the labeling of negative and positive tubes can be permuted. Since in this view all positive tubes are equal, what we have introduced is a cyclic permutation of $(-,+,+,\dots,+)$.

So given a sphere with $1+n$ tubes, Definition 4.6 describes especially the action of the cyclic group C_n^+ on $1+n$ letters. Together with Definition 3.2 of the action of S_n on the n positive tubes, this generates the action of S_n^+.

Considering spheres as operations, the number of positive tubes is the operation's arity. We need to be careful of a number of compatibility conditions, but we can abstract this property of the sphere operad to operads generally as follows. This theory was first developed by Getzler and Kapranov in [GK95], and axiom ii. supplemented by P. van der Laan.

The definition is given for k-operads, because the meaning for ordinary operads can be easily read off from the diagrams.

Definition 4.7. Let $\tau_n = (0,1,\dots,n)$ and $C_n^+ = \langle \tau_n \rangle$. A *cyclic* k-operad K is an operad as in Definition 1.6 with an extended family of operations $\{\circ_i\}_{i\in\mathbb{N}}$ which admits a C_n^+ action on operations of arity n, such that (for $B_{x,y}\colon x\otimes y \mapsto y\otimes x$ the braiding of Definition C.12) all the following diagrams commute:

i. (*c.f.* Figure 4.1) and ii., for $1 < i < m$,

$$
\begin{tikzcd}
K(m) \otimes K(n) \arrow[r, "\circ_m"] \arrow[d, "B_{K(m),K(n)}"'] & K(m+n-1) \arrow[dd, "\tau_{m+n-1}"] \\
{} \arrow[d, "\tau_n \otimes \tau_m"'] & \\
K(n) \otimes K(m) \arrow[r, "\circ_1"'] & K(m+n-1)
\end{tikzcd}
\qquad
\begin{tikzcd}
K(m) \otimes K(n) \arrow[r, "\circ_i"] \arrow[d, "\tau_m \otimes \mathrm{id}"'] & K(m+n-1) \arrow[d, "\tau_{m+n-1}"] \\
K(m) \otimes K(n) \arrow[r, "\circ_{i+1}"'] & K(m+n-1)
\end{tikzcd}
$$

iii.

$$
\begin{tikzcd}
K(1) \arrow[r, "\tau_1"] & K(1) \\
\mathbf{k} \arrow[u, "\eta"] \arrow[ur, "\eta"'] &
\end{tikzcd}
$$

$\square$

4.7 i.

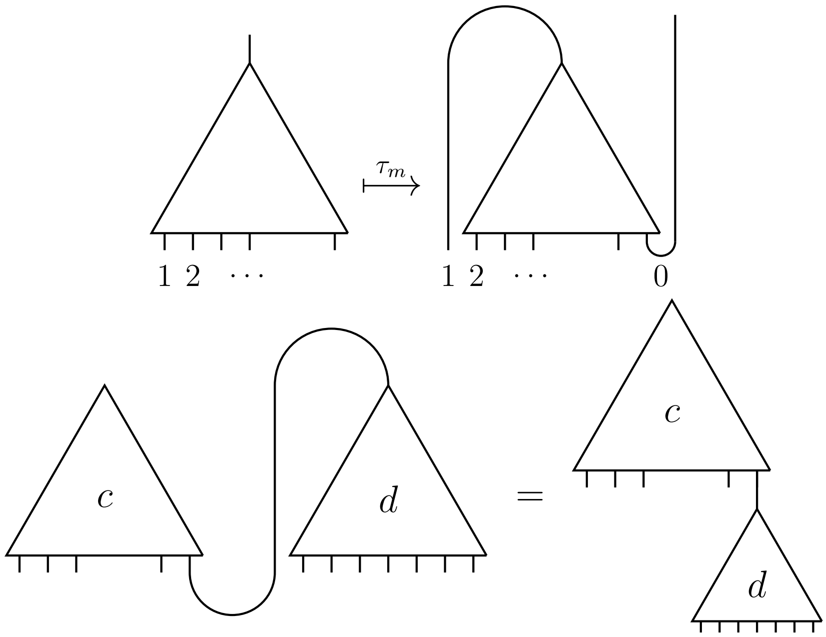


Figure 4.1: The action of τ_m on some operation from K, followed by an illustration of i. from Definition 4.7.

If K is a cyclic operad, then ii. would read, for $1 < i < m$ and $c \in K(m)$, $d \in K(n)$, as $\tau_m(c) \circ_i d = \tau_{m+n-1}(c \circ_i d)$. Also iii. expresses that $\tau_1(\mathrm{id}) = \mathrm{id}$ for $\mathrm{id} \in K(1)$.

We consider some previously defined operads in light of this new definition.

Proposition 4.8. *The tree operad T (for trees as in Definition 1.24) and the endomorphism operad for finitely spanned vector spaces (over* k *a field) with a nondegenerate symmetric bilinear form are cyclic operads.*

Proof. In Definition 1.24, the root is just a distinguished leaf. The action of C_n^+ reallocating the distinguishedness makes T a cyclic operad, for i. and ii. are satisfied by all graphs, and iii. is obvious since τ_1 has trivial action on the identity tree.

Fix a k-vector space V and a nondegenerate bilinear form $\langle , \rangle$ on V. Recall the Definition 1.15 of the endomorphism operad. It has components $\mathrm{End}_V(n) = \hom_{\mathbf{k}}(V^{\otimes n}\ V)$ the vector space of k-linear maps $V^{\otimes n} \to V$. We use the form for the isomorphism $\hom_{\mathbf{k}}(V^{\otimes n}\ V) \cong \hom_{\mathbf{k}}(V^{\otimes 1+n}\ \mathbf{k}) = h(n)$ by identifying $f \in \hom_{\mathbf{k}}(V^{\otimes n}\ V)$ with $\lambda(f) = \langle f, \rangle$. Then $\langle f, \rangle$ is an element of $h(n)$ by the k-linearity of $\langle , \rangle$, and the identification is an isomorphism by nondegeneracy. For $f \in \mathrm{End}_V(m)$ and $g \in \mathrm{End}_V(n)$, we define $\langle f, \rangle \circ_i \langle g, \rangle$ for $0 \leq i \leq m$ as $\langle f, \rangle(v_0 \otimes \cdots \otimes v_{i-1} \otimes g(v_i \otimes \cdots \otimes v_{i+n-1}) \otimes v_{i+n} \otimes \cdots \otimes v_{m+n-1}$. The collection $\{h(n)\}_{n\in\mathbb{N}}$ together with the compositions $\{\circ_i\}_{i\in\mathbb{N}}$ is an operad h, and λ is an invertible omomorphism.

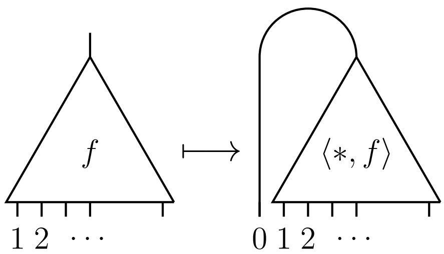


Figure 4.2: The identification in the isomorphism $\hom_{\mathbf{k}}(V^{\otimes n}\ V) \cong \hom_{\mathbf{k}}(V^{\otimes 1+n}\ \mathbf{k})$

Now h (and therefore End_V) is cyclic. Axiom i. becomes, under the isomorphism

λ,

$$\begin{tikzcd}
h(m) \otimes h(n) \arrow[r, "\circ_m"] \arrow[d, "\tau_m \otimes \tau_n"'] & h(m+n-1) \arrow[dd, "\tau_{m+n-1}"] \\
{} \arrow[d, "B_{h(m),h(n)}"'] & \\
h(n) \otimes h(m) \arrow[r, "\circ_0"'] & h(m+n-1)
\end{tikzcd}$$

and indeed we see that (where v_i and w_j are variables for all i, j)

$$\begin{aligned}(\langle f, \rangle \circ_m \langle g, \rangle)^{\tau_{m+n-1}} &= \langle f(v_1 \otimes \cdots \otimes v_m), g(v_{m+1} \otimes \cdots \otimes v_0)\rangle = \\ &= \langle g(v_1 \otimes \cdots \otimes v_n), v_0\rangle \circ_0 \langle f(w_1 \otimes \cdots \otimes w_m), w_0\rangle\end{aligned}$$

as required, using of course the symmetry of $\langle , \rangle$.

Then ii. practically verifies itself, and as for iii., $\eta\colon 1_{\mathbf{k}} \mapsto \langle , \rangle$. □

We additionally show that the sphere operad is cyclic in Proposition 4.12.

It is clear that the bilinear form $\langle , \rangle$ on V was the motor of the preceding proof of the cyclicity of End_V. However, we only considered End_V, and we still have the opportunity to consider representations.

If $\mathcal{A}$ is an algebra in the ordinary sense, with product $\cdot$, it is common to call a bilinear form $\langle , \rangle$ on $\mathcal{A}$ invariant if $\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle$. We would like to upgrade this definition to an k-operad K. Therefore we call a bilinear form $\langle , \rangle$ on a K-algebra V *invariant* if for all products μ (of arity m) induced on V by K we have that $\langle \mu(v_1 \otimes \cdots \otimes v_m), v_{m+1}\rangle = \langle v_1, \mu(v_2 \otimes \cdots \otimes v_{m+1})\rangle$.

When we consider algebras of operads, we see – in the following proposition – that invariant bilinear forms exactly characterise cyclicity.

Proposition 4.9. *Suppose that the module V admits a nondegenerate bilinear form,* *and that K is an operad with faithful representation $\rho : K \to \mathrm{End}_V$. Then ρ is a cyclic representation (that is, equivariant with respect to the axioms of cyclic operads) exactly if the form is invariant.*

Proof. Exactly if ρ is invariant we have that for $c \in K(m)$, $\lambda\rho(c) = \lambda\rho(c^{\tau_m}) = (\lambda\rho(c))^{\tau_m}$. Thus we see that, using Lemma 1.17's evaluation maps ev, that $\langle , \rangle$ is invariant. $\square$

4.3 Mathematical physics

This section is at best a philosophical look at the meaning of this area of work.

We start with a very concise description of the sphere operad's *raison d'être*. Let $T\hat{\mathbb{C}}$ be the tangent bundle of the Riemann sphere $\hat{\mathbb{C}}$. By the hairy ball theorem, this is a nontrivial tangent bundle and thus there is no smooth section for all of $\hat{\mathbb{C}}$. To cover the sphere, several sections are necessary. Then the sphere operad is a theory of how different vector fields on the sphere can interact.

It is perhaps felt that this theory deserves more explanation that just that. Here we make one such attempt. One of the origins of this study of Riemann surfaces under sewing operations is string theory. Vaguely stated, strings are described by path integrals over all 'worldsheets' on which a conformal field theory lives.

We attempt an overview of the motivation given in a paper by Vafa [V87] for studying Riemann surfaces in this context. We thank A. James for explaining this argument to us. Moreover his thesis [J11] is the source of definitions.

Let $M(g, n) = M$ be the moduli space of Riemann surfaces of genus g with n punctures. It is well-known that all manifolds in $M(g, n)$ are topologically equivalent.

The *Teichmüller* space of a given topological space S is the set of all pairs (X, φ) of surfaces X with a complete hyperbolic metric and *marking* homeomorphism $\varphi \colon S \to X$, up to isotopy (*i.e.,* (X, φ) and (X', φ') are the same if there exists a continuous path of homeomorphisms between φ and φ'). It is a topological invariant. Therefore the Teichmüller space of a Riemann surface in M is identical for all Riemann surfaces in M, and we denote it T.

The mapping class group Γ of a surface S is the group of isotopy classes of orientation preserving homeomorphisms of S that act as identity on the boundary of S. It is another topological invariant, and therefore identical for all Riemann surfaces in M. It is a fundamental result that $M = T/\Gamma$ and we therefore consider this quotient.

In all but a handful of cases the mapping class group for surfaces of finite genus is generated by elements of finite order. Any finite order element of the mapping class group has fixed points in its action on the Teichmüller space, since there exists an element in the moduli space for which it is an automorphism. (Indeed fixed points in the Teichmüller space correspond exactly to automorphisms of Riemann surfaces.)

The mapping class group acts properly discontinuously on the Teichmüller space, which is a contractible topological $\mathbb{R}$-manifold, whence the moduli space is an orbifold. In particular, the moduli space $M = M(g,n)$ has orbifold singularities if the number of marked points $n \leq 2g+2$, since in this case there exist Riemann surfaces whose automorphisms fix the marked points.

For the understanding of worldsheets we require flat vector bundles, *i.e.,* vector bundles with affine connections of zero curvature. The flat vector bundles over a manifold are classified by the representations of its fundamental group. If we specialise to line bundles, the representations of its first homology, which is the abelianisation of the fundamental group, suffice to classify the flat line bundles. For orbifolds, flat bundles are classified via the fundamental group of its resolved space. So in the case of M, we need T.

Instead of attempting to directly understand the orbifolds, we introduce a more complicated moduli space $P = P(g,n)$ of genus g Riemann surfaces with n ordered tubes. Tubes are marked points with a local coordinate function vanishing at that point. Every nontrivial automorphism of the sphere with marked points has a nontrivial effect on the local coordinate functions, whence the mapping class group acts freely, that is, fixed-point free. Now the moduli space P can be obtained by modding out its (contractible) Teichmüller space by the mapping class group, written $\Gamma = \Gamma(g, n)$. Because this has no orbifold singularities, it is a more accessible object to study. As Γ acts freely, in fact the fundamental group $\pi_1(P) = \Gamma$.

Riemann surfaces of genus 0 provide a *tree-level* (approximation to the) theory, which are generally better understood. However 'a well-defined conformal theory should make sense on an arbitrary-genus Riemann surface' [V87].

4.4 Surface composition

We look at a generalisation of the sphere operad here, namely generalising its sewing composition (first in Definition 2.5). In Section 4.2, we already mentioned that the orientation of tubes is not significant. In Section 3.3 we discussed Huang's canonical choice function c and how it relates $K(n)$ to $\mathcal{K}(n)$. These are guidelines for the next definition.

For a set $C \subset \mathbb{C}$, define $1/C$ as $\mathbb{C} \setminus \overline{\hat{J}(C)}$, where $\hat{J}(C) = \{\hat{J}(c) \mid c \in C\}$ is the usual pointwise image of C under the inversion map $\hat{J}$.

Definition 4.10. Let S and T be Riemann surfaces with ordered tubes (p_i, f_i) and (q_i, g_i) respectively (assumed without loss to be labeled from $\mathbb{N}$). We define their *generalised sewing product* $S\,{}_i\!\bowtie_j\, T$ if there exists an open contractible $C \subset \mathbb{C}$ containing 0 such that there exist open contractible $U \subset S$ and $V \subset T$ such that the only tubes centered in U and V are (p_i, f_i) and (q_j, g_j) respectively and f_i and g_j are invertible on $f_i(U) \supset \overline{C}$ and $g_j(V) \supset \overline{1/C}$ respectively.

Then $S\,{}_i\!\bowtie_j\, T$ is $(S \setminus {f_i}^{-1}(\bar{B}^r)) \sqcup (T \setminus {g_j}^{-1}(\bar{B}^{1/r}))$ modulo $S \ni w_1 \sim w_2 \in T$ if $f(w_1) = \frac{1}{g(w_2)}$, under the image of the conformal equivalence which replaces equivalence classes (w_1, w_2) with single points, with all tubes but the i^{th} of S and the j^{th} of T. $\square$

In our description of the (ordinary) sewing product, we will now switch to considering a single radius $r \in \mathbb{R}$ and disk $B^r \subset \mathbb{C}$, rather than a pair of radii. This is more convenient to discuss, and we will not need the more precise statements using r and R for the sequel.

Observe that we are able to recover the original sewing function of Definition 2.5, as we will explain now. Firstly, if S and T are as just described, the existence of $r \in \mathbb{R}$ such that $f_i(U) \supset \bar{B}^r$ and $g_j(V) \supset \bar{B}^{1/r}$ demonstrates the existence of a $C \subset \mathbb{C}$ satisfying the above. (In particular, if $C = B^r$ then $1/C = B^{1/r}$.) Thus if S and T are *circle-sewable*, that is, sewable using ${}_i\infty_0$, they are certainly *region-sewable*: using ${}_i\bowtie_0$.

Now by ${}_i\overset{c}{\bowtie}_j$ we refer to the function that maps surfaces S and T to $c(S\,{}_i\bowtie_j\,T)$. Then, if c is Huang's canon, when we compare definitions it is clear that $S\ {}_i\overset{c}{\bowtie}_0\ T = S_i\infty_0 T$.

We should now show that the use of an open contractible region C in this generalised sewing definition leads to some new results.

Proposition 4.11. *Suppose that $S \in \mathcal{K}(m)$ has punctures $\{p_j\}_j$ and i^{th} tube (p_i, f_i), and $T \in \mathcal{K}(n)$ has punctures $\{q_k\}_k$ and 0^{th} tube (q_0, g_0). Suppose also that f_i and g_j are rational functions (that is, meromorphic everywhere) whose sets of singularities are Σ_S and Σ_T respectively, and let $\Pi_S = \{p_j\}_{j\neq i}$ and $\Pi_T = \{q_k\}_{k\neq 0}$. If $X = \Sigma_S \cup \Pi_S$ and $Y = f_i^{-1} \circ \hat{J} \circ g_0(\Sigma_T \cup \Pi_T)$ are pairwise disjoint, then $S\,{}_i\bowtie_0\,T$ is defined.*

In other words, tubes with rational functions are always sewable, as long as punctures or singularities on the spheres do not coincide in an unfortunate way.

Proof. Since X and Y are pairwise disjoint, any $x \in X$ and $y \in Y$ are seperable by open sets. Thus there exists an open contractible neighbourhood $U \supset Y$, containing also p_i, which is disjoint to X. Then $f_i(U)$ satisfies the requirements for $C \subset \mathbb{C}$ in Definition 4.10. $\square$

Consequence. *There exist spheres $S \in \mathcal{K}(m)$ and $T \in \mathcal{K}(n)$ and $1 \leq i \leq m$ such that S and T are region-sewable but not circle-sewable.*

Proof. It suffices to furnish an example of spheres with tubes S and T for which $S_i\infty_0 T$ is not defined, whose i^{th} and 0^{th} tubes have rational functions. This is simple. In fact, it is always possible to modify sewable spheres such that they are not sewable; here is a description of this in one case:

Suppose that spheres with at least $1+2$ tubes A and T are circle-sewable at the i^{th} tube of A and that f_i on A and g_0 on T are rational functions. Let $r = \inf\{\rho \mid A_i\infty_0 B \text{ can be sewn using radius } \rho\}$. Since A and T have at least $1+2$ tubes, $r \in (0,\infty)$. Then let S be A, with some additional tube placed at distance $\frac{1}{2}r$ from p_i. Since r is the infinum of suitable radii for A, there can be no suitable radius now to sew S to T, and thus they are not circle-sewable. $\square$

We now prove a previously-promised result. It is in fact independent of the generalised sewing product given above, but it exhibits the fact that allowing an action of $\tau_m = (01\cdots m)$ on a sphere with $1+m$ tubes is equivalent to generalising the sewing product to sew spheres by any two tubes.

Proposition 4.12. *The sphere operad is cyclic.*

Proof. The action of τ_m on a sphere $S \in K(m)$ is by permutation on the ordering of tubes, and then mapping to a canonical sphere with tubes. Again c is the canon as in Definition 2.2. We have three statements from Definition 4.7 to verify.

Firstly we approach iii.; $I \in K(1)$ is invariant under $\tau_1 = (01)$, for permuting the tubes of I gives a sphere with (ordered!) tubes $(0, \mathrm{id})$ and $(\infty, \hat{J})$ in $\mathcal{K}(1)$, and the conformal equivalence $\hat{J}$ makes it canonical. Clearly then I^{τ_1} has tubes $(\infty, \hat{J})$ and $(0, \mathrm{id})$, whence $I^{\tau_1} = I$.

For i., Figure 4.1 is the correct visualisation. Let $S \in K(m)$ and $T \in K(n)$ for $1 \leq m, n$. Keeping track of the labeling, in the compositions $S_m \infty_0 T$ and $T^{\tau_n}{}_1 \infty_0 S^{\tau_m}$ the same pair of tubes are sewn together. Since S and S^{τ_m}, T and T^{τ_n} are conformally equivalent (ignoring the tube ordering), by Proposition 2.9 we have the desired equality.

By the same argument, for $T \in K(n)$ with $n \in \mathbb{N}$, we see that $(S \circ_i T)^{\tau_{m+n-1}} = S^{\tau_m} \circ_{i+1} T$ if $1 < i < m$, satisfying ii. $\square$

We now return to the generalised sewing product ${}_i\bowtie_j$, to remark that there is also no *a priori* need to sew two *distinct* surfaces together.

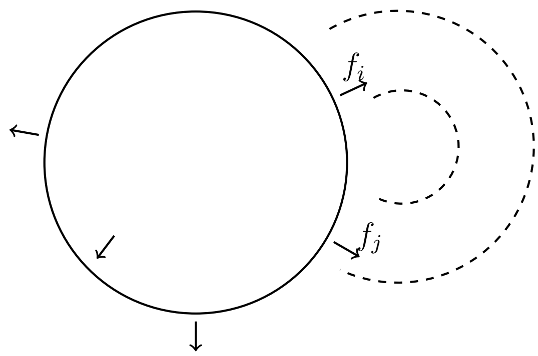


Figure 4.3: Sewing a sphere to itself, making a genus 1 surface (with tubes).

Definition 4.13. Let S be a Riemann surface with at least two tubes (p_i, f_i) and (p_j, f_j). The *generalised sewing product* $S8^i_j$ *of* S *with* S is defined if there exists $r \in \mathbb{R}$ and open contractible and *disjoint* $U, V \subset S$ such that the only tubes in U and V are (p_i, f_i) and (p_j, f_j) respectively and f_i and f_j are invertible on $f_i(U) \supset \bar{B}^r$ and $g_j(V) \supset \bar{B}^{1/r}$ respectively.

Then we define $S8^i_j$ as the surface $S \setminus f_i{}^{-1}(\bar{B}^r) \setminus f_j{}^{-1}(\bar{B}^{1/r})$, modulo the identification of w_1 with w_2 if $f_i(w_1) = \frac{1}{f_j(w_2)}$, considering equivalence classes as single points, recoordinatising the space with a conformal equivalence F and putting all other tubes on the new surface as $(Fp_k, f \circ F^{-1})_{k \neq i,j}$. $\square$

If S is a sphere, then $S8^i_j$ is a torus, and in general the genus of the surface $S8^i_j$ is greater by one than the genus of S.

Recall the remarks around Figure 2.2, discussing the meaning of $\hat{J}$ in $w_1 \sim w_2$ if $f(w_1) = \hat{J} \circ (w_2)$. In the self-sewn product $S8^i_j$, its role is different.

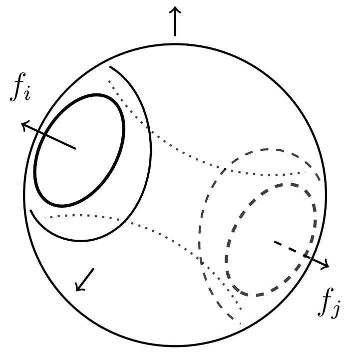


Figure 4.4: Sewing a sphere to itself,
under the identification $w_1 \sim w_2$ if $f_i(w_1) = f_j(w_2)$.

This can be best-formulated in terms of loops. Suppose that S is a sphere with tubes and that $S8^i_j$ exists. Then suppose that T is $S8^i_j$, sewn as in Definition 4.13, illustrated in Figure 4.3, and that U is $S8^i_j$ sewn using the identification $f_i(w_1) = f_j(w_2)$, as in Figure 4.4. Then there exists an open path ℓ_1 on S such that its image on T is a simple closed noncontractible path around the hole in T. There also exists an open path ℓ_2 on S whose image on U is a simple closed noncontractible path *through* the hole on U.

So there is a duality to Figures 4.3 and 4.4, in the way that loops around the hole and through the hole are dual.

Now denote by $\mathcal{K}_g(n)$ the space of Riemann surfaces of genus g with n tubes. (Then $\mathcal{K}(n) = \mathcal{K}_0(n)$.) For $S \in \mathcal{K}_g(n)$, $S8^i_j \in \mathcal{K}_{g+1}(n)$.

In analogy with the work of Chapters 2 and 3, we make some basic statements about the generalised (self-)sewing product.

Proposition 4.14. *The generalised sewing product associates in the following ways:*

For surfaces S, T and U,

i. $(S \bowtie T) \bowtie U = (S \bowtie U) \bowtie T$ *or* $= S \bowtie (T \bowtie U)$ *(for appropriately labeled tubes)*

ii. $(S\,{}_i\!\bowtie_j T)8^k_l = S\,{}_i\!\bowtie_{j'} (T8^{k'}_{l'})$ *or* $= (S8^{k'}_{l'})\,{}_{i'}\!\bowtie_j T$,

iii. $(S\,{}_i\!\bowtie_j T)8^k_l = (S\,{}_{k'}\!\bowtie_{l'} T)8^{i'}_{j'}$,

iv. $S8^i_j 8^k_l = S8^{k'}_{l'} 8^{i'}_{j'}$,

where by the i'^{th} tube i', we mean the tube i under a change in labeling.

Note that the first point is similar to Proposition 2.8, which makes the same claim for canonical spheres with tubes and the ordinary sewing operation.

Proof. In all cases, since there is no canonical sphere with tubes, it suffices that if the lefthand side exists, suitable radii to sew the righthand side exist; this is obvious. Since the other tubes not referenced in the sewing product ${}_i\!\bowtie_j$ are invariant, the equalities follow. $\square$

Lemma 4.15. The generalised sewing product is a conformal equivalence class function. $\square$

This result about to be proven is in analogy with the well-known fact that any Riemann surface of genus at least two has a pair-of-pants decomposition.

Proposition 4.16. *An arbitrary Riemann surface S of genus g with n punctures can be written as a generalised sewing product of spheres from the $PGL_2(\mathbb{C})$-orbit of G from Proposition 3.4.*

Proof. The proof goes via two claims. An arbitrary surface can be written as the generalised sewing product of some surface of the same genus and some spheres with tubes, *i.e.,* if we are given any surface of genus g we can express all surfaces of genus g using the given one and spheres with tubes. Secondly, we can sew spheres with tubes to give a surface of some arbitrary genus.

We proceed by an induction in two variables that interleaves both claims. The induction hypothesis is that all surfaces with any finite number of tubes of genus at most g can be expressed as in the statement of the proposition.

The base case – that any sphere with tubes can be written as such a product – comes quickly from Proposition 3.4. Suppose that the induction hypothesis holds for genus $g \geq 1$. There certainly exists some $S \in \mathcal{K}_g(3)$ (with $1+3=4$ tubes) such that $S8^3_4 \in \mathcal{K}_{g+1}(1)$ exists. The generalised sewing product preserves orientation, and by the classification theorem of orientable Riemann surfaces, up to conformal equivalence there is a unique surface of each genus. Thus $S8^3_4$ is this surface, with two tubes. Using spheres of type $P(w) \in G$, we can rearrange the positions of tubes (and increase their number); then, using spheres $Q^f_g \in G$, we can change the power series at each tube. $\square$

In Section 3.3, we discussed the role of the canon function $c\colon \mathcal{K}(n) \to K(n)$. Clearly this is a source of interesting complexity for the sewing composition ${}_i\infty_0$. While this can be recovered on the generalised sewing equation (as ${}_i\overset{c}{\bowtie}_0$ or even ${}_i\overset{c}{\bowtie}_j$) on spheres, an obvious generalisation of such a non-trivial section of $\mathcal{K}_g/\operatorname{Aut} S_g$ (where S_g is any Riemann surface of genus g) is missing. It is well-known that as the genus increases, the automorphism group changes; S_0 has automorphism group $PGL_2(\mathbb{C})$ and $\operatorname{Aut} S_1$ of the torus is $SL_2(\mathbb{Z})$, but for any higher genus the automorphism group is finite and occasionally trivial. Therefore results analoguous to those of the sphere operad are unclear until there is an understanding of families of genus-indexed mutually compatible canons.

We make some last remarks.

If S and S' are two distinct spheres, each with 1 tube, then $S\,{}_1\!\bowtie_1 S'$ has 0 tubes. Therefore the sewing operation ${}_i\infty_0$, generalised sufficiently to make the sphere operad cyclic, leads us to consider $K(-1) = \mathcal{K}(-1) = \{\hat{\mathbb{C}}\}$, the space of spheres with *no* tubes.

The theory of cyclic operads on Riemann surfaces lead Getzler and Kapranov to develop the theory of *modular operads* in [GK98]. The restrictions are stronger, in that modular operads are full operads (the starting point of Getzler and Kapranov is spheres with invertible, disjoint tubes), but otherwise it is an abstraction of the generalised sewing operation, including the genus-increasing map 8^i_j.

We recover an operad structure on Riemann surfaces of arbitrary genus with tubes in $\{\bigoplus_{g\in\mathbb{N}} \mathcal{K}_g(n)\}_{-1\leq n}$, with compositions $\{{}_i\bowtie_j\}_{i,j\in\mathbb{N}}$.

We will end this chapter again generalising operads by generalising the composition of operations. We follow Markl's definitions [M06].

Definition 4.17. A k-PROP P is a symmetric strict monoidal category (with product $\otimes$ and unit 1) enriched over kMod such that

i. the objects of P are identified with $\mathbb{N}$
ii. $m \otimes n = m + n$ for any $m, n \in \mathbb{N}$. $\square$

We remarked in Section 1.2 that an operad comes from the $\hom$-sets of an enriched multicategory with one object. For PROPs, we also need to look to the $\hom$-sets $P(m, n) = \hom_P(m\ n)$. We identify $m \cong 1^{\otimes m}$ and thus each $P(m, n)$ is a (S_m, S_n)-bimodule, that is, S_m has a left action and S_n a right action, and these commute.

Now the tensor product $\otimes$ induces a *horizontal* composition

$$\otimes\colon P(a, b) \otimes P(c, d) \to P(a + b, c + d),$$

and from the categorial composition we have a *vertical* composition

$$\circ\colon P(a, b) \otimes P(b, c) \to P(a, c).$$

This may be induced by a more general composition, for $1 \leq m \leq b$,

$$\circ_{i_1,\ldots,i_m} \colon P(a,b) \otimes P(m,c) \to P(a, b+c-m),$$

when the outputs i_j for $1 \leq j \leq m$ are composed with the m inputs of elements of $P(m,c)$.

Observe that the generalised sewing composition gives a PROP structure to Riemann surfaces with tubes when each tube has an orientation.

5.1 Further Work

The last chapter is intended as a springboard for further work.

From the isomorphism between vertex operator algebras and the sphere operad, a certain vertex operator algebra could be chosen and its sphere operad analysed; for example, the celebrated moonshine vertex operator algebra construction (of I. Frenkel, J. Lepowsky and A. Meurman [FLM88]) should be interesting.

A better understanding of the role of the sphere operad – and generalising it to higher genus – must involve an insightful characterisation of the sewing function.

There has already been considerable work on different kinds of algebras and dualities between them via operads. We have seen that, for example, bilinear forms can be expressed operadically; is there any property of algebras not characterisable using representations of operads? Perhaps operads are the correct notion with which to express algebras and their relations. Many groups have constructions as symmetries of an algebra with several forms and products. Is it possible to recast these constructions operadically?

APPENDIX C

CATEGORY THEORY

We present a very brief introduction to the theory of categories. This chapter will be full of definitions, like the subject itself.

For simplicity, we take a naive approach to categories. Essentially it is assumed that everything is a set; in the scope of this work, we do not encounter the contradictions of this assumption.

It is hoped that the very elementary parts of category theory are familiar. Nevertheless, everything is at least outlined. To start, we have the definition of categories, then of diagrams, duality, functors, natural morphisms, universality and adjoints.

Definition C.1. A *category* C is a set of objects $x \in C$, and a ('hom'-)set $\hom_C(x\ y)$ of *morphisms* f (with source x, target y, written $f\colon x \to y$) for any $x, y \in C$. It satisfies

i. **composition**: for $g \in \hom_C(x\ y)$, $h \in \hom_C(y\ z)$, there exists $h \circ g \in \hom_C(x\ z)$.
ii. **associativity**: $(h \circ g) \circ f = h \circ (g \circ f)$ for $f \in \hom_C(w\ x)$, $g \in \hom_C(x\ y)$, $h \in \hom_C(y\ z)$.
iii. **identity**: for all $x \in C$ there is an identity morphism $\mathrm{id}_x \in \hom_C(x\ x)$, such that for $g \in \hom_C(w\ x)$, $h \in \hom_C(x\ y)$, $\mathrm{id}_x \circ g = g$ and $h \circ \mathrm{id}_x = h$. □

A category D is a subcategory of C if for all $x \in D$ also $x \in C$, $\hom_D(x\ y) \subseteq \hom_C(x\ y)$ for all $x, y \in D$, and composition in D is the same as in C (in particular, identity morphisms are the same). In cases where we are not considering several categories or context gives sufficient precision, we may drop the subscript C from $\hom_C(x\ y)$.

Certain features and statements can often be best-represented diagramatically. This is part of the basic language of categories. The first example of this comes in Definition C.6.

Definition C.2. A *diagram* is a directed graph whose vertices are objects in a category C and directed edges between vertices x, y are elements $f \in \hom_C(x\ y)$ drawn as $x \xrightarrow{f} y$. We also allow, but omit to draw, composition of edges, so if there are edges $g\colon x \to y$, $h\colon y \to z$ then there is an edge between x and z given by $h \circ g \in \hom_C(x\ z)$. $\square$

Definition C.3. A diagram *commutes* if, for any vertices x, z, for all edges $x \xrightarrow{f_1, f_2} z$, $f_1 = f_2$. In particular, for $g_i\colon x \to y_i$, $h_i\colon y_i \to z$ for $i = 1, 2$ it is required that $h_1 \circ g_1 = h_2 \circ g_2$. Usually any diagram we write is required to commute; if indicated otherwise, a subdiagram may be indicated commutative by a $\bigcirc$ in its middle. $\square$

Definition C.4. Given a diagram, its *dual diagram* has the same objects and direction-reversed edges between them. $\square$

By *the category of groups* we refer to the category whose elements are groups and whose morphisms are group homomorphisms. Similarly, for most mathematical objects, there is such a category. There is also a category of categories, written Cat.

Definition C.5. A *functor* is a morphism in the category Cat of categories. Explicitly, it is a morphism φ from one category $x \in \mathrm{Cat}$ to another $y \in \mathrm{Cat}$ respecting the morphisms in x and y: if $f \in \hom_x(a\ b)$ then $\varphi(f) \in \hom_y(\varphi(a)\ \varphi(b))$; for all $a \in x$, $\varphi(\mathrm{id}_a) = \mathrm{id}_{\varphi(a)}$; and for $g \in \hom_x(a\ b)$ and $h \in \hom_x(b\ c)$, $\varphi(h \circ g) = \varphi(h) \circ \varphi(g)$. $\square$

Definition C.6. For categories C and D, let $F_{CD} = \hom_{\mathrm{Cat}}(C\ D)$ be the category of functors $C \to D$. Suppose that φ, ψ are in F_{CD}. A *natural morphism* from φ to ψ is an

$$\alpha = \{\alpha_x \in \hom(\varphi(x)\ \psi(x)) \mid x \in C\} \in \hom_{F_{CD}}(\varphi\ \psi)$$

such that for all $f \in \hom_C(x\ y)$,

$$\begin{tikzcd}
\varphi(x) \arrow[r, "\alpha_x"] \arrow[d, "\varphi(f)"'] & \psi(x) \arrow[d, "\psi(f)"] \\
\varphi(y) \arrow[r, "\alpha_y"] & \psi(y).
\end{tikzcd}$$

For $x \in C$, α_x is a *component* of α. □

One of the most remarkable things about category theory is its excellent abstraction of the distinguishedness of certain objects in algebra.

Definition C.7. An object $x \in C$ is *initial* (*terminal*) in C if, for all $y \in C$, there exists exactly one $f \in \hom_C(x\ y)$ ($f \in \hom_C(y\ x)$, respectively) up to isomorphism. Then x is also called *universal*. □

One part of the category theoretic manifesto is to always prefer to work up to isomorphism. In many cases, requiring identities or equalities is too harsh. In fact even isomorphism between categories is often too strong. We can relax this by allowing isomorphisms up to isomorphism, referred to as adjoints.

Definition C.8. Given functors $\lambda\colon C \to D$ and $\rho\colon D \to C$, λ is *left adjoint* to ρ (which is *right adjoint* to λ) if there exist natural morphisms $\alpha\colon \lambda \circ \rho \to \mathrm{id}_D$ and

$\beta\colon \mathrm{id}_C \to \rho \circ \lambda$ such that the triangle identities commute:

$$\begin{tikzcd} \lambda = \lambda \,\mathrm{id}_C \arrow[r, "\mathrm{id}_\lambda \beta"] \arrow[dr, "\mathrm{id}_\lambda"'] & \lambda\rho\lambda \arrow[d, "\alpha\, \mathrm{id}_\lambda"] \\ & \lambda \end{tikzcd} \qquad \begin{tikzcd} \rho\lambda\rho \arrow[d, "\mathrm{id}_\rho \beta"'] & \mathrm{id}_D \rho = \rho \arrow[l, "\alpha\, \mathrm{id}_\rho"'] \arrow[dl, "\mathrm{id}_\rho"] \\ \rho & \end{tikzcd}$$

□

C.1 Constructions

Familiar algebraic operations, such as the product of sets, have clear categorical generalisations if abstracted.

We give these for products, coproducts and then limits.

Definition C.9. Elements $x, y \in C$ have a *product* $x \times y$ with $\mathrm{pr}_x\colon x \times y \to x$, $\mathrm{pr}_y\colon x \times y \to y$, if for all $a \in C$ with $g \in \hom_C(a\ x)$, $h \in \hom_C(a\ y)$ there exists a unique $f\colon a \to x \times y$ such that the following diagram commutes:

$$\begin{tikzcd} & a \arrow[dl, "g"'] \arrow[d, "f"] \arrow[dr, "h"] & \\ x & x \times y \arrow[l, "\mathrm{pr}_x"'] \arrow[r, "\mathrm{pr}_y"] & y \end{tikzcd}$$

□

Now $x \times y$ is universal in the following sense. Let the category C_{xy} have objects (a, g, h) for $a \in C$, $g \in \hom_C(a\ x)$ and $h \in \hom_C(a\ y)$, and morphisms $f_{12}\colon (a_1, g_1, h_1) \to (a_2, g_2, h_2)$ making this diagram commute:

$$\begin{tikzcd} & a_1 \arrow[dl, "g_1"'] \arrow[d, "f_{12}"] \arrow[dr, "h_1"] & \\ x & a_2 \arrow[l, "g_2"'] \arrow[r, "h_2"] & y \end{tikzcd}$$

Then in this category C_{xy}, $(x \times y, \mathrm{pr}_x, \mathrm{pr}_y)$ is a terminal object.

We say that a category C *has products* if for all $x, y \in C$ there exists a product $x \times y$. The category of categories has products. That is, if $C, D \in \mathrm{Cat}$ are two categories, the product category $C \times D$ has objects (c, d) for $c \in C$, $d \in D$, and morphisms (f, g) for f a morphism in C, g a morphism in D, and these are composed componentwise.

There is a notion of a *coproduct* $x + y$, dual to the product construction. In particular, the coproduct of x and $y \in C$ is an initial object $(x + y, \mathrm{pr}^x \colon x \to x + y, \mathrm{pr}^y \colon y \to x + y)$ in the category C^{xy} of objects (a, g, h), for $a \in C$, $g \in \hom_C(x\ a)$ and $h \in \hom_C(y\ a)$, and the obvious morphisms.

We introduce *comma categories* to facilitate the subsequent definition. Consider $A \xrightarrow{\varphi} C \xleftarrow{\psi} B$. The comma category $(\varphi \downarrow \psi)$ has objects

$$f \in \hom_C(\varphi(a)\ \psi(b)), \qquad \text{for } a \in A, b \in B,$$

and arrows that are pairs $(g, h) \in \hom_A(a\ a') \times \hom_B(b\ b')$ satisfying

$$\begin{tikzcd}
\varphi(a) \arrow[r, "f"] \arrow[d, "\varphi(g)"'] & \psi(b) \arrow[d, "\psi(h)"] \\
\varphi(a') \arrow[r, "f'"] & \psi(b').
\end{tikzcd}$$

We will also need the one-point category $\{\bullet\}$, which has exactly one object (written $\bullet$) and exactly one morphism: the identity morphism $\mathrm{id}_\bullet$.

Also, for categories C, D and some object $d \in D$, we define the constant functor $\Delta_x \colon C \to D$ as mapping every object of C to the object d, and every morphism of C to id_d.

Definition C.10. The *limit* of a functor $\varphi \colon J \to C$ is the terminal object in the category $(\Delta \downarrow \dot{\varphi})$, where $\dot{\varphi}$ is a functor $\{\bullet\} \to \hom_{\mathrm{Cat}}(J\ C)$ acting as $\bullet \mapsto \varphi$, and Δ is the functor $C \to \hom_{\mathrm{Cat}}(J\ C)$ that maps $x \mapsto \Delta_x$. Its *colimit* is the initial object in $(\dot\varphi \downarrow \Delta)$. $\square$

It should be seen that comma categories are really not so bad in the case of limits and colimits. We illustrate this for colimits. Recall Definition C.6, where F_{JC} is given as the category of functors from J to C. The objects of $(\dot\varphi \downarrow \Delta)$ are $f_x \in \hom_{F_{JC}}(\dot\varphi(\bullet)\ \Delta(x))$, in other words, $f_x \in \hom_{F_{JC}}(\varphi\ \Delta_x)$, for $x \in C$. As only x can vary, the morphisms are (id, Δ_g) (which we identify with Δ_g) for $g \in \hom_C(x\ y)$. The terminal object t in F_{JC} is f_z for some $z \in C$ such that for each other object f_y there is a unique morphism $\Delta_h\colon f_y \to f_z$.

Now an object f_x is a natural morphism from $\varphi \to \Delta_x$. Therefore f_x has components for each object j in J. For all $j \in J$, $\Delta_x(j) = x$. So finally, for $h \in \hom_J(j\ k)$, we require f_x to make the following diagram commute:

$$\begin{tikzcd}
\varphi(j) \arrow[r, "f_{xj}"] \arrow[d, "\varphi(h)"'] & x \\
\varphi(k) \arrow[ur, "f_{xk}"'] &
\end{tikzcd}$$

Then f_{xj} is just a morphism $\varphi(j) \to x$, and f_x gives a family of these compatible with all morphisms of the form $\varphi(h)$. Then the terminal f_z is a 'least such', *i.e.,* in the diagram above it identifies all $a \in \varphi(j)$ with $a \in \varphi(k)$.

Limits encompass products and coproducts among other things. In particular, let J be the category with two objects and no (nonidentity) morphisms, written $\{\ \bullet_1 \quad \bullet_2\ \}$. Then a functor $\varphi\colon J \to C$ that maps $\bullet_1$ to $x \in C$ and $\bullet_2$ to $y \in C$ has limit $x \times y$ and colimit $x + y$.

C.2 Adjectives

Definition C.11. A *monoidal category* is a category M with

i. a *tensor product*, that is, a functor $\otimes\colon M \times M \to M$,
ii. a *unit* object $1 \in M$,

and the following natural isomorphims (which, if they are all identity, makes it a *strict* monoidal category):

iii. associator α with components given by $\alpha_{xyz}\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$,
iv. left unitor λ with $\lambda_x\colon 1 \otimes x \to x$,
v. right unitor ρ with $\rho_x\colon x \otimes 1 \to x$,

such that the diagrams of the *triangle* and *pentagon identities* commute:

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "\alpha_{xIy}"] \arrow[dr, "\rho_x \otimes \mathrm{id}_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "\mathrm{id}_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}$$

$$\begin{tikzcd}
& (w \otimes x) \otimes (y \otimes z) \arrow[dr, "a_{w,x,y\oplus z}"] & \\
((w \otimes x) \otimes y) \otimes z \arrow[ur, "a_{w\otimes x,y,z}"] \arrow[d, "a_{w,x,y} \otimes \mathrm{id}_z"'] & & w \otimes (x \otimes (y \otimes z)) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr, "a_{w,x\otimes y,z}"'] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "\mathrm{id}_w \otimes a_{x,y,z}"']
\end{tikzcd}$$

□

A monoidal category is an abstraction and generalisation of tensor products. Then the triangle and pentagon identities simply express associativity of the tensor product; in the strict case, this gives exactly the usual properties of the tensor product. We can make further explicit demands on the tensor product, in particular as regards commutativity.

Definition C.12. A *braiding* on a monoidal category is a natural isomorphism $B_{x,y}\colon x \otimes y \to y \otimes x$ satisfying the *hexagon* identity:

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{x,y,z}"] \arrow[d, "B_{x,y} \otimes \mathrm{id}_z"'] & x \otimes (y \otimes z) \arrow[r, "B_{x,y\otimes z}"] \arrow[d, phantom, "\bigcirc"] & (y \otimes z) \otimes x \arrow[d, "a_{y,z,x}"] \\
(y \otimes x) \otimes z \arrow[r, "a_{y,x,z}"'] & y \otimes (x \otimes z) \arrow[r, "\mathrm{id}_y \otimes B_{x,z}"'] & y \otimes (z \otimes x)
\end{tikzcd}$$

□

Definition C.13. A *symmetric monoidal category* is a braided monoidal category where $B_{x,y}B_{y,x} = 1_{x\otimes y}$. □

The category Set of sets is a strict symmetric monoidal category. Usually we require that the $\hom$-sets of a category are sets. We can generalise this.

Definition C.14. A category C is *enriched* over the symmetric monoidal category M if, for x and $y \in C$, $\hom_C(x\ y)$ is an object of M. □

Suppose that a certain strict monoidal category C has an object x. Then also $x \otimes x$ and indeed $x^{\otimes n} \in C$ for all n. Now a map $f \in \hom_C(x^{\otimes n}\ x)$ has, formally, a single source $x^{\otimes n}$. Intuition identifies this with a map with n sources – namely n copies of x. The following definition can be seen as formalising multilinear algebra in categories.

Definition C.15. A category C is a *multicategory* if, for all objects $x_1, \ldots, x_n, y \in C$ it also admits extended sets $\hom_C(x_1, \ldots, x_n;\ \ y)$ of morphisms (where an $f \in \hom_C(x_1, \ldots, x_n;\ \ y)$ has sources $x_1, \ldots, x_n$ and target y), and it has a family of compositions $\{\circ_i\}_{i\geq 1}$, such that $\circ_i \colon C^{m\geq i} \times C^n \to C^{m+n-1}$ acting in $f \circ_i g$ replaces the i^{th} source of f by the image of g and satisfies the usual associativity and identity. $\square$

Admittedly the following goes against the category theory manifesto, but it is just too damn handy.

Definition C.16. A *skeleton* category is one in which the only isomorphisms are identity. The skeleton category of a category C is C modulo its isomorphisms. $\square$

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