ABSTRACT TOPOLOGICAL DYNAMICS

by

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A thesis submitted to
The University of Birmingham
for the degree of
Doctor of Philosophy

School of Mathematics
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February 2012

Abstract

Let $T : X \to X$ be a function from a countably infinite set X to itself. We consider the following problem: can we put a structure on X with respect to which T has some meaning? In this thesis, the following questions are addressed: when can we endow X with a topology such that X is homeomorphic to the rationals $\mathbb{Q}$ and with respect to which T is continuous? We characterize such functions on the rational world. The other question is: can we put an order on X with respect to which X is order-isomorphic to the rationals $\mathbb{Q}$, naturals $\mathbb{N}$ or integers $\mathbb{Z}$ with their usual orders and with respect to which T is order-preserving (or order-reversing)? We give characterization of such bijections, injections and surjections on the rational world and of arbitrary maps on the naturals and integers in terms of the orbit structure of the map concerned.

Acknowledgements

I would like to thank my supervisor Dr Chris Good for his help, support and encouragement. I am also so grateful to my husband, Abdelgader, and my parents for their endless support.

Contents

Introduction

Dynamical systems arise in many areas of study, but from a purely theoretical viewpoint one may study the action of a function on an abstract topological space. Let $T : X \to X$ be a function on a non-empty set X. D. Ellis in [6] asked a question: is there a non-trivial topology on X so that T will be continuous? Contributions towards solving the problem were made by both Ellis [6] and Powderly and Tong [25]. The answer to this question was given by De Groote and De Vries [2], they showed that if X is infinite, then there is always a non-discrete metrizable topology on X such that T is continuous. De Vries in [3] proved that the Continuum Hypothesis is equivalent to: If X has cardinality $\mathfrak{c}$ and T is a bijection, there is a compact metric topology on X under which X is a homeomorphism. Good et al. in [11] suggested a fundamental question: If $T : X \to X$ is a function and $\mathcal{P}$ is some topological property, when can we put a topology on X that satisfies $\mathcal{P}$ and makes T continuous? They showed that there is a complete answer in the case when $\mathcal{P}$ is taken to mean compact and Hausdorff. This answer is in terms of orbit structures of the map T. Iwanik in [16] had earlier given an answer of this problem but for bijections. Another result by Good and Greenwood in [12] showed that there exists a separable metrizable topology or a hereditarily Lindelöf topology on X with respect to which T is continuous; this answer depends on the cardinality of the set X. Several authors have studied model theoretic aspects of structures of the form $T : X \to X$ (see for example [18]), we do not touch on this in this thesis.

In this thesis we consider $\mathcal{P}$ to mean "homeomorphic to the rational space $\mathbb{Q}$"; so the question is: when can one endow X with a topology such that X is homeomorphic to the rational space $\mathbb{Q}$ and with respect to which T is continuous? Mekler [22], Neumann [23] and Truss [31] studied the situation when G is a group of permutations acting on a countably infinite set X.

This problem leads us to a more general question: Let $T : X \to X$ be a function on a non-empty set X, can one put an order on X with respect to which T is order-preserving (or order-reversing) and X has some property $\mathcal{P}$. We study the cases when $\mathcal{P}$ means "order isomorphic to the natural numbers $\mathbb{N}$, integers $\mathbb{Z}$ or the rationals $\mathbb{Q}$" with their usual orders.

In Chapter 1, we introduce fundamental definitions and results of abstract topological dynamical systems that are useful throughout the thesis. Next, we discuss autohomeomorphisms of the rational world as given by Mekler [22], Neumann [23] and Truss [31]. Then we go through some conditions on a group of permutations as given by Truss in [31].

In Chapter 2, Theorem 2.3 provides the necessary and sufficient conditions for a countable set X with self-map T to be endowed with a topology with respect to which T is continuous and X is homeomorphic to $\mathbb{Q}$. The same question is answered for a number of self-maps defined on X (see Theorem 2.10 and Theorem 2.11). This study is in terms of the inverse images of certain subsets of X. In this chapter we also state a number of other related results and examples in terms of the orbit structures of the maps.

In Chapter 3, we study the cases in which we can put a linear order on a set with self-map T so that T is an order-preserving map and X is order-isomorphic to the rationals $\mathbb{Q}$, integers $\mathbb{Z}$ or naturals $\mathbb{N}$ with their usual order. This study is in terms of the orbit structure of the map concerned. We start in the first section with giving results and properties of order-preserving self-maps on any set, we also prove in Theorem 3.17

that any set with self-map T can be linearly ordered such that T is order-preserving provided that T has no cycles of length greater than 1. Then we give characterization of order-preserving bijective and injective self-maps on the rationals $\mathbb{Q}$. Our main theorem in Section 3.2 is Theorem 3.33 which describes the orbit structure of order-preserving surjective self-maps on $\mathbb{Q}$. Necessary conditions on the orbit structure of order-preserving surjections on $\mathbb{Q}$ are discussed followed by various results required in the proof of the main theorem of order-preserving surjections. We end this section with some examples of the general case and give some orbit structure of maps which cannot be order-preserving on $\mathbb{Q}$. In the final two sections we give the orbit structure of order-preserving self-maps on sets that are order-isomorphic to the naturals $\mathbb{N}$ (Theorem 3.53) or the integers $\mathbb{Z}$ (Theorem 3.65).

Chapter 4 is devoted to the investigation of order-reversing self-maps. The structure of this chapter is similar to Chapter 3, we start with proving a number of results and properties of order-reversing self-maps on any set, the main theorem of this section is Theorem 4.14. The orbit structure of order-reversing bijections, injections on $\mathbb{Q}$ was given in Section 4.2. Our main theorem in this section is Theorem 4.29 which describes the orbit structure of surjections on the rational world. In the last two sections we give the orbit structure of order-reversing self-maps on the naturals $\mathbb{N}$ (Theorem 4.37) and the integers $\mathbb{Z}$ (Theorem 4.45).

Chapter 1

Preliminaries

1.1 Basic Definitions and Results

This section contains some preliminary definitions and results which mostly can be found in [11] and will be used throughout our thesis.

Let $T : X \to X$ be a function and let $\sim$ be a relation on X defined as:

$$x \sim y \Leftrightarrow \text{ there exist } m, n \in \mathbb{N} \text{ with } T^m(x) = T^n(y).$$

The relation $\sim$ is an equivalence relation, whose equivalence classes are the orbits of T, or T-orbits [11].

Definition 1.1. [11],[12] Let $T : X \to X$ be a function and O be an orbit of the function T.

(1) O is an *n-cycle*, for some $n \in \mathbb{N}$, if there are distinct points $x_0, \ldots, x_{n-1}$ in O such that $T(x_{i-1}) = x_i$, where i is taken modulo n.

(2) O is a *$\mathbb{Z}$-orbit* if there are distinct points $\{x_i : i \in \mathbb{Z}\} \subseteq O$ such that $T(x_{i-1}) = x_i$ for all $i \in \mathbb{Z}$.

(3) O is an $\mathbb{N}$-*orbit* if it is not a $\mathbb{Z}$-orbit and there are distinct points $\{x_i : i \in \mathbb{N}\} \subseteq O$ such that $T(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$. In other words, O is an $\mathbb{N}$-orbit if it is neither an n-cycle for some $n \in \mathbb{N}$, nor a $\mathbb{Z}$-orbit.

The set $S = \{x_i : i \in \mathbb{M}\}$ which witnesses that O is an n-cycle, $\mathbb{Z}$-orbit or $\mathbb{N}$-orbit, where $\mathbb{M}$ is an appropriate indexing set, is called a *spine* for O [11].

The three different kinds of orbits in Definition 1.1 are illustrated in the following figures. In Figure 1.1, we give a picture of a 3-cycle.

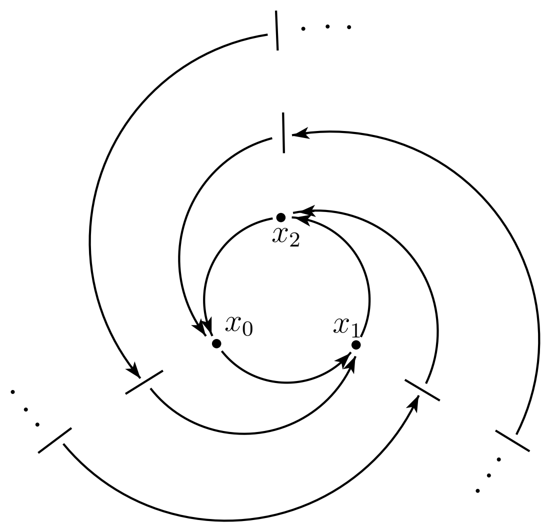

Figure 1.1: A 3-cycle.

In Figure 1.2 we have a picture of a $\mathbb{Z}$-orbit and Figure 1.3 gives an example of an $\mathbb{N}$-orbit.

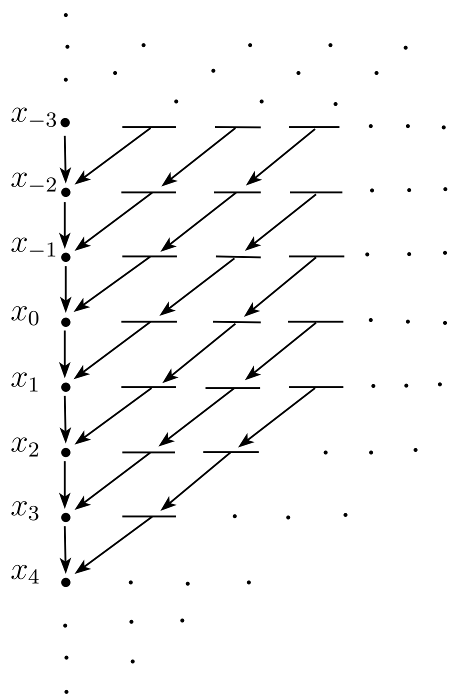

Figure 1.2: A $\mathbb{Z}$-orbit.

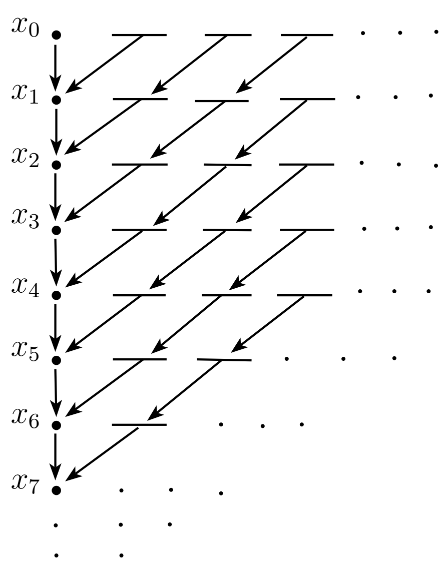

Figure 1.3: An example of $\mathbb{N}$-orbit.

Example 1.2. [12] Let $X = \{x_n : n \in \mathbb{N}\} \cup \{x_{ij} : 0 \leq j \leq i \in \mathbb{N}\}$. Let $T : X \to X$ be the map defined by

$$T(x) = \begin{cases} x_{ij-1}, & x = x_{ij}, 1 \leq j \leq i \in \mathbb{N}, \\ x_{n+1}, & x = x_n, n \geq 0, \\ x_0, & x = x_{i0}, i \in \mathbb{N}. \end{cases}$$

Then T has exactly one $\mathbb{N}$-orbit with infinitely many possible spines. Figure 1.4 illustrates the orbit of T.

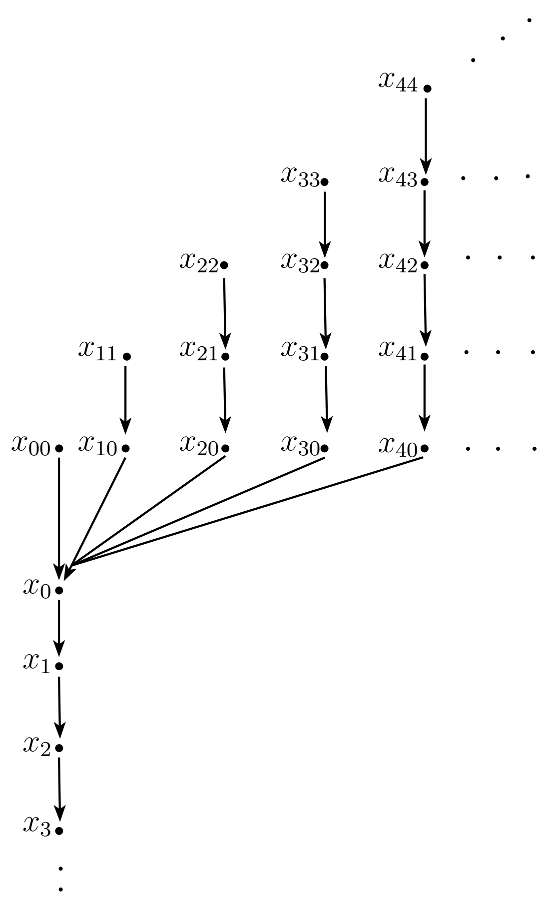

Figure 1.4: The Orbit of T.

Example 1.3. [4, 3.12, 3.4] Let S^1 denote the unit circle in the plane. We denote a point in S^1 by its angle θ measured in radians in the standard manner. So, a point is determined by any angle of the form $\theta + 2\pi n$, $n \in \mathbb{Z}$. Let $\alpha \in \mathbb{R}$ and let $T_\alpha : S^1 \to S^1$ be the map defined by $T_\alpha(\theta) = \theta + 2\pi\alpha$. If $\alpha \in \mathbb{Q}$, say $\alpha = p/q$ with $p, q \in \mathbb{Z}$ and p, q are coprime, then each point in S^1 lies in a cycle of length q. But when $\alpha \notin \mathbb{Q}$, then T_α has no cycles and the orbit of every point in S^1 is dense (the second case is known as Jacobi's Theorem).

Proof. [4, 3.13] Let $\alpha = p/q \in \mathbb{Q}$. Clearly, for each $\theta \in S^1$ we have $T_\alpha^q(\theta) = \theta + 2\pi p = \theta$, thus, every point in S^1 lies in a cycle of length q.

Let $\alpha \notin \mathbb{Q}$ and $\theta \in S^1$. If we suppose that $T_\alpha^n(\theta) = T_\alpha^m(\theta)$, for some $n, m \in \mathbb{N}$, then $(n - m)\alpha \in \mathbb{Z}$, so $n = m$, hence, the points on the orbit of θ are distinct. Now, since any infinite set on the circle must have a limit point, so given any $\epsilon > 0$, then there exist $n, m \in \mathbb{Z}$ for which $|T_\alpha^n(\theta) - T_\alpha^m(\theta)| < \epsilon$. If $l = n - m$, then $|T_\alpha^l(\theta) - \theta| < \epsilon$. Since T_α preserves lengths in S^1, then T_α^l maps the arc between θ and $T_\alpha^l(\theta)$ to the arc between $T_\alpha^l(\theta)$ and $T_\alpha^{2l}(\theta)$ which has length less than ϵ. It follows that $\theta, T_\alpha^l(\theta), T_\alpha^{2l}(\theta), ...$ partitions S^1 into arcs of length less than ϵ. Since ϵ was arbitrary, the orbit is dense. Thus, every orbit is a $\mathbb{Z}$-orbit. $\square$

Definition 1.4. [11] Let $T : X \to X$ be a function. The *orbit spectrum* of T is defined to be the sequence

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$$

of cardinals, where ν is the number of $\mathbb{N}$-orbits, ζ is the number of $\mathbb{Z}$-orbits and σ_n is the number of n-cycles.

Definition 1.5. [11] Let O be an orbit of a function $T : X \to X$. We say that O is a *simple* orbit if $T \restriction O$ is one-to-one, so that O consists only of a spine. A *semi-simple*

n-cycle is an orbit $O = \{x_i : 0 \leq i < n\} \cup \{y_j : j \in \mathbb{N}\}$, $x_i \neq y_j$ for all $0 \leq i < n$ and $j \in \mathbb{N}$, such that $T(x_i) = x_{i+1}$ for $i < n-1$, $T(x_{n-1}) = x_0$, $T(y_j) = y_{j-1}$, $j \neq 0$ and $T(y_0) = x_0$.

The semi-simple n-cycle is illustrated in Figure 1.5.

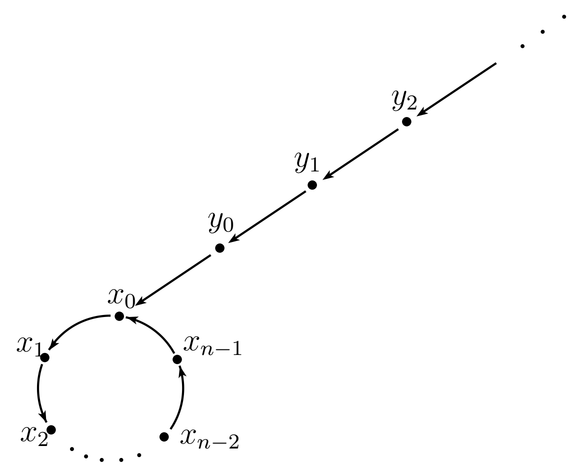

Figure 1.5: A semi-simple n-cycle.

Definition 1.6. [11] The *canonical representation* of a sequence of cardinals $\sigma = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ is the function $T : X \to X$ with $\sigma(T) = \sigma$, each of whose orbits is simple. A *semi-canonical representation* of σ is a function $T : X \to X$ with $\sigma(T) = \sigma$, each of whose orbits is simple except for one semi-simple n-cycle.

Definition 1.7. Let $T : X \to X$ be a function and let $\sim$ be an equivalence relation on X defined by:

$$x \sim y \Leftrightarrow \text{ there exists } i \in \mathbb{N} \text{ with } T^i(x) = T^i(y).$$

Consider $E_\sim$ to be the family of all equivalence classes of X under $\sim$. Let O be an orbit of T with spine S indexed as $\{x_i : i \in M\}$, where M is either $\mathbb{N}$, $\mathbb{Z}$ or $\{0, \dots, n-1\}, 0 \neq$

$n \in \mathbb{N}$, as appropriate. Consider $E_\sim$ to be indexed as follows. For each $x_i \in S$, let $L_i(O)$ denote the equivalence class of x_i; that is,

$$L_i(O) = \bigcup_{m \in \mathbb{N}} \{x \in O : T^m(x) = T^m(x_i)\}$$

and if O is an $\mathbb{N}$-orbit with $T^{-1}(L_0(O)) \neq \emptyset$, then for each $k < 0$, let $L_k(O) = T^{-1}(L_{k+1}(O))$.

The following lemma collects a number of facts about $L_i(O)$ we will need to use later. The proof follows directly from the definition above.

Lemma 1.8. *Let $T : X \to X$ be a function and let O be an orbit of T with spine S.*

(1) *For each $i \in N$, where N is either $\mathbb{Z}$ or $\{0, \ldots, n-1\}, 0 \neq n \in \mathbb{N}$, we have $T^{-1}(L_i(O)) = L_{i-1}(O)$, where $i-1$ is taken module n when O is an n-cycle.*

(2) *$O = \bigcup_{i \in N} L_i(O)$ and $L_i(O) \cap L_j(O) = \emptyset$ whenever $i \neq j$.*

Now we give a proof of the following result that we will use later.

Lemma 1.9. *Let $T : X \to X$ be a function and let O be an orbit of T. Let S be a spine of O indexed as $\{x_i : i \in M\}$, where M is either $\mathbb{N}$, $\mathbb{Z}$ or $\{0, \ldots, n-1\}, 0 \neq n \in \mathbb{N}$.*

(1) *If O is a $\mathbb{Z}$-orbit (or an $\mathbb{N}$-orbit), then $T^2 \restriction O$ has, in total, two $\mathbb{Z}$-orbits (or two $\mathbb{N}$-orbits). Moreover, the orbits of $T^2 \restriction O$ are $O_1 = \bigcup_{k \in \mathbb{Z}} L_{2k}(O)$ and $O_2 = \bigcup_{k \in \mathbb{Z}} L_{2k+1}(O)$. Spines may be chosen so that for all $i \in \mathbb{Z}$, $L_i(O_1) = L_{2i}(O)$ and $L_i(O_2) = L_{2i+1}(O)$.*

(2) *If O is an n-cycle, then $T^2 \restriction O$ has, in total, two $n/2$-cycles if n is even, and one n-cycle if n is odd. If n is even, the orbits of $T^2 \restriction O$ are $O_1 = \bigcup_{k=0}^{\frac{n}{2}-1} L_{2k}(O)$ and $O_2 = \bigcup_{k=0}^{\frac{n}{2}-1} L_{2k+1}(O)$. The spines may be indexed so that for all i, $L_i(O_1) = L_{2i}(O)$ and $L_i(O_2) = L_{2i+1}(O)$. If n is odd, the orbit of $T^2 \restriction O$ is O itself. The spine may*

be indexed so that for all i, writing $L_i^{T^2}(O)$ for $L_i(O)$ as evaluated with respect to the map T^2, $L_i^{T^2}(O) = L_j(O)$ where $j \equiv 2i(mod\ n)$.

(3) *O is simple if and only if all orbits of $T^2 \restriction O$ are simple.*

Proof. (1) Let O be a $\mathbb{Z}$-orbit with spine S indexed as $\{x_i : i \in \mathbb{Z}\}$ so that $T(x_i) = x_{i+1}$ for each $i \in \mathbb{Z}$. Since for each $i \in \mathbb{Z}$, $T(x_i) = x_{i+1}$, then $T^2(x_i) = T(x_{i+1}) = x_{i+2}$. This means that all points of the set $S_1 = \{x_i : i \text{ is even}\}$ lie in the same orbit under T^2, say O_1; and all points of the set $S_2 = \{x_i : i \text{ is odd}\}$ lie in the same orbit under T^2, say O_2. So, $S_1 \cap S_2 = \emptyset$ and $O_1 \neq O_2$. Hence, T^2 has two $\mathbb{Z}$-orbits O_1 and O_2 with spines S_1 and S_2 respectively.

For each $i \in \mathbb{Z}$, if $x \in L_i(O)$, so $T^m(x) = T^m(x_i)$ for some $m \in \mathbb{N}$, then we have two cases: if $m = 2k$, $k \in \mathbb{N}$, then $(T^2)^k(x) = (T^2)^k(x_i)$. If $m = 2k+1$, $k \in \mathbb{N}$, then $T^{m+1}(x) = T^{m+1}(x_i)$ so $(T^2)^{k+1}(x) = (T^2)^{k+1}(x_i)$; hence, x and x_i lie in the same orbit under T^2. Hence, all points in the set $L_i(O)$ lie in O_1 if i is even and lie in O_2 if i is odd. Hence, $O_1 = \bigcup_{k \in \mathbb{Z}} L_{2k}(O)$ and $O_2 = \bigcup_{k \in \mathbb{Z}} L_{2k+1}(O)$ with $O_1 \cup O_2 = O$, $O_1 \cap O_2 = \emptyset$ and each of O_1 and O_2 is a $\mathbb{Z}$-orbit with spines S_1 and S_2 respectively. Spines of O_1 and O_2 may be chosen so that for all $i \in \mathbb{Z}$, $L_i(O_1) = L_{2i}(O)$ and $L_i(O_2) = L_{2i+1}(O)$. The case when O is an $\mathbb{N}$-orbit follows in the same way.

(2) Let O be an n-cycle with spine S indexed as $\{x_0, x_1, \ldots, x_{n-1}\}$ so that $T(x_i) = x_{i+1}$ for each $0 \leq i < n$ and i is taken module n. Under the action of T^2 we have $T^2(x_i) = x_{i+2}$. So, if n is odd then $n-1$ is even and $T^2(x_{n-1}) = x_1$, so we have

$$x_0 \mapsto x_2 \mapsto \ldots \mapsto x_{n-1} \mapsto x_1 \mapsto x_3 \mapsto \ldots \mapsto x_{n-2} \mapsto x_0.$$

so T^2 has an n-cycle with spine S. If n is even, then $n-1$ is odd, so $T^2(x_{n-1}) = x_1$ and

$T^2(x_{n-2}) = x_0$. So, under the action of T^2 we have

$$x_0 \mapsto x_2 \mapsto \ldots \mapsto x_{n-2} \mapsto x_0$$

and

$$x_1 \mapsto x_3 \mapsto \ldots \mapsto x_{n-1} \mapsto x_1.$$

But this means that $S = S_1 \cup S_2$, where $S_1 = \{x_i \in S : i \text{ is even}\}$ and $S_2 = \{x_i \in S : i \text{ is odd}\}$ so $S_1 \cap S_2 = \emptyset$. So there is two cycles each of length $n/2$. The proof of the second part follows in the same way as in (1).

(3) The proof of this statement follows from the proof of (1) and (2) above. □

Neumann in [23] gave a definition of boundedly periodic orbit spectrum (or type as he called it) for bijections, the following definition is the same as he gave but for any function rather than bijections.

Definition 1.10. Let $\sigma = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$ be a sequence of cardinals. σ is said to be *boundedly periodic* if $\nu = 0$, $\zeta = 0$ and there exists $m \in \mathbb{N}$ such that $\sigma_k = 0$ for all $k \geq m$.

Definition 1.11. [12] Let $T : X \to X$ be a function and let $Y \subseteq X$. For any ordinal α and any limit ordinal β, define $T^{\alpha+1}(Y) = T(T^\alpha(Y))$ and $T^\beta(Y) = \bigcap_{\alpha\in\beta} T^\alpha(Y)$.

Now, the following definition comes from [12].

Definition 1.12. Let $T : X \to X$ be a function. The *rank of* $x \in X$ *under* T is defined as

$$\|x\| = \begin{cases} \alpha & \text{if } x \in T^\alpha(X) \setminus T^{\alpha+1}(X), \\ \infty & \text{if } x \in T^\mu(X), \end{cases}$$

where μ is the least ordinal such that $T^\mu(X) = T(T^\mu(X))$.

Now we give the following result that we need to use later, the result can be found with its proof in [12].

Lemma 1.13. *Let $T : X \to X$ be a function and $x \in X$. $\|x\| = \infty$ if and only if there exists a sequence $x_0, x_1, x_2, \dots,$ such that $x_0 = x$ and $T(x_{i+1}) = x_i$. In particular, if x lies in the spine of a $\mathbb{Z}$-orbit or of an n-cycle, then $\|x\| = \infty$, and if $\|x\| = \infty$, then x is not in an $\mathbb{N}$-orbit.*

Proof. Obviously, if there is a sequence $x_0, x_1, x_2, \dots$ such that $x_0 = x$ and $T(x_{i+1}) = x_i$, then $x_0 \in T^{\alpha}(X)$ for all ordinals α, hence $\|x\| = \infty$. Now, let $\|x\| = \infty$, in other words $x \in T^{\alpha}(X)$ for all ordinals α. If we suppose that $\|y\| < \infty$ for each $y \in T^{-1}(x)$, then $\|x\| = \sup\{\|y\| + 1 : y \in T^{-1}(x)\} < \infty$, so there is some $x_1 \in T^{-1}(x)$ with $\|x_1\| = \infty$. Therefore, there is an infinite sequence $x_0, x_1, x_2, \dots$ with $x_0 = x$, and $x_i = T(x_{i+1})$, as required. $\square$

The following definition was first introduced in [17].

Definition 1.14. [17],[11] Let $T : X \to X$ be a function. The orbit spectrum $\sigma(T)$ is said to be *finitely based* if the set $s = \{n \in \mathbb{N} : \sigma_n \neq 0\}$ has the following property: there exists a finite subset $\{n_1, n_2, \dots, n_k\}$ of s such that every $j \in s$ is a multiple of some n_i, $1 \leq i \leq k$.

We end this section with state a number of related theorems and results. Good et al. in [11] gave a characterization of continuous functions on compact Hausdorff spaces as follows.

Theorem 1.15. [11] *Let X be an infinite set and $T : X \to X$ be a function with orbit spectrum*

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \dots).$$

There is a compact Hausdorff topology on X with respect to which T is continuous if and only if $T^{\omega+1}(X) = T^{\omega}(X) \neq \emptyset$ and one of the following holds:

(1) $\zeta + \sum_{n\in\mathbb{N}} \sigma_n \geq \mathfrak{c}$ ($\mathfrak{c}$ *is the cardinality of the continuum); or*

(2) *both* $\zeta \neq 0$ *and* $\sum_{n\in\mathbb{N}} \sigma_n \neq 0$*; or*

(3) $\zeta = 0$ *and either*

   (a) $\sigma(T)$ *is finitely based; or*

   (b) $T \mid T^{\omega}(X)$ *is not one-to-one.*

Iwanik [16] had earlier studied the situation when $T : X \to X$ is a bijections and given the following result.

Theorem 1.16. [16] *Let* $T : X \to X$ *be a bijection. There is a compact Hausdorff topology on* X *under which* T *is continuous if and only if none of the following holds:*

(1) $|X| < \mathfrak{c}$ *and all orbits are infinite;*

(2) $|X| < \mathfrak{c}$*, all orbits are finite and the orbit spectrum is not finitely based.*

Homeomorphisms on compact metric spaces are characterized in [11] as follows.

Theorem 1.17. [11] *Let* $T : X \to X$ *be a bijection with orbit spectrum*

$$\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \dots).$$

There is a compact metric topology on X *with respect to which* T *is a homeomorphism iff* ζ *and each* $\sigma_n, n \in \mathbb{N}$*, is either countable or has cardinality* $\mathfrak{c}$*, and either:*

(1) $|X| = \mathfrak{c}$*; or*

(2) $\zeta \neq 0$ *and* $\sum_{n\in\mathbb{N}} \sigma_n \neq 0$*; or*

(3) $\sigma(T)$ *is finitely based.*

The following result comes from [12].

Theorem 1.18. *Let $T : X \to X$ be a function. There is a (zero-dimensional) Tychonoff, Lindelöf topology on X with respect to which T is continuous provided either:*

(1) $T^{\mathfrak{c}^+}(X) = T^{\mathfrak{c}^+ + 1}(X) \neq \emptyset$; *or*

(2) $T^{\alpha}(X) = \emptyset$ *for some* $\alpha < \mathfrak{c}^+$.

Good et al. in [12] shown that the existence of separable metrizable topology or a hereditarily Lindelöf topology on a set X with respect to which $T : X \to X$ is continuous depends on the cardinality of the set X. The next theorem comes from Theorem 1.8 in [12].

Theorem 1.19. *Let $T : X \to X$ be a function, the following are equivalent:*

(1) $|X| \leq \mathfrak{c}$.

(2) *There is a (zero-dimensional) Hausdorff, hereditarily Lindelöf topology on X with respect to which T is continuous.*

(3) *There is a (zero-dimensional) first countable, Hausdorff, Lindelöf topology on X with respect to which T is continuous.*

(4) *There is a (zero-dimensional) first countable, Hausdorff, separable topology on X with respect to which T is continuous.*

(5) *There is a (zero-dimensional) separable metrizable topology on X with respect to which T is continuous.*

1.2 Autohomeomorphisms in the Rational World

In this section we discuss autohomeomorphisms of the rational world as given by Neumann [23], Mekler [22] and Truss [31]. We start with giving a theorem of Sierpinski which describes which space is homeomorphic to the rationals. Neumann called such spaces the rational world and he modified what Sierpinski published in 1920 (see [28]) and gave a proof of Sierpinski's Theorem (see [23, p.440], [28], see also [29, p. 142]).

Theorem 1.20. (Sierpinski, Neumann [23]) *Let X be a second countable, 0-dimensional and T_1 topological space. Then X is homeomorphic to a subspace of $\mathbb{R}$. If moreover X is countable and has no isolated points then X is homeomorphic with $\mathbb{Q}$.*

Let G be a group of permutations acting on a countably infinite set X. Let $H(\mathbb{Q})$ denote the group of all autohomeomorphisms of $\mathbb{Q}$. Mekler [22] and Truss [31] studied the cases in which the group G can be embedded in $H(\mathbb{Q})$; that is, when a bijection from X to $\mathbb{Q}$ induces an embedding of G into $H(\mathbb{Q})$. Neumann [23] described the orbit structure of autohomeomorphisms of $\mathbb{Q}$.

Recall that if $T \in G$ then support $T=\{x \in X : T(x) \neq x\}$. Now, from Truss [31], we have the following four definitions.

Definition 1.21. We say that a group G satisfies NC, or *Neumann's Criterion*, if for every non-identity member T of G, support T is infinite.

Definition 1.22. A group G is said to have MC, or *Mekler's Criterion*, if for every finitely many members $T_1, \ldots, T_n \in G$, then $\cap\{$ support $T_i : 1 \leq i \leq n\}$ is empty or infinite.

Definition 1.23. We say that G satisfies SMC, or *Strong Mekler's Criterion*, if for every finitely many non-identity members of G, $T_1, \ldots, T_n$, we have $\cap\{$ support $T_i : 1 \leq i \leq n\}$ is infinite.

Definition 1.24. We say that G satisfies SH, *Sharp*, if for any non-identity member $T \in G$, then $X \setminus$ support T is finite.

Observation 1.25. [31] For any group G, then SH $\Rightarrow$ SMC $\Rightarrow$ MC $\Rightarrow$ NC.

The condition MC is a reformulation of the property that Mekler gave in [22] (this reformulation as MC is due to Neumann, ([31])). Mekler called it the mimicking property and he defined it as follows.

Definition 1.26. [22] We say that a group G has the *mimicking property* if for all $T_1, T_2, \ldots, T_n \in G$ and $x \in X$ there are infinitely many y such that : for all i and j, $T_i(y) = T_j(y)$ implies that $T_i(x) = T_j(x)$.

The following result comes from [22], but we give here another proof using MC property.

Lemma 1.27. *$H(\mathbb{Q})$ has the mimicking property.*

Proof. We will show that $H(\mathbb{Q})$ satisfies MC, which means that it has the mimicking property. Suppose that $T_1, \ldots, T_n \in H(\mathbb{Q})$. Since $\mathbb{Q}$ is a Hausdorff space, then we have each $Y_i = \{x : T_i(x) = x\}$ is closed, and then $\bigcup_{i=1}^{n} Y_i$ is also closed. So, the complement $\bigcap_{i=1}^{n} Y_i^c$ is open and hence it is either empty or infinite since $\mathbb{Q}$ has no isolated points. Consequently, since $Y_i^c = \{x : T_i(x) \neq x\} =$ support T_i, we have the required. $\square$

Mekler in [22] gave the following theorem which give the necessary and sufficient conditions for G to be embedded in $H(\mathbb{Q})$. Truss in [31] gave a different proof of the same theorem.

Theorem 1.28. (Mekler [22]) *Any countable group G has the mimicking property if and only if it can be embedded in $H(\mathbb{Q})$.*

Proof. See [22, 1.5]. $\square$

Neumann in [23] characterized autohomeomorphisms in the rational world in terms of the orbit structure of the map concerned.

Theorem 1.29. [23] *The canonical representation $T : X \to X$ of any sequence of cardinals $\sigma = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ is an autohomeomorphism with X is homeomorphic to $\mathbb{Q}$ except the case when σ is boundedly periodic and does not satisfy the following condition:*

$$\text{if } \sigma_m \neq 0 \text{ then there exists } n \text{ such that } m \text{ divides } n \text{ and } \sigma_n = \omega.$$

Proof. See [23, Prop. 2, Prop. 3, Prop. 4, Theorem p.446]. □

The next theorems give characterization of functions that satisfy NC, MC, SMC or SH in terms of their orbit structures.

Theorem 1.30. [31] *Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ be a sequence of cardinals. The canonical representation $T : X \to X$ of σ is a member of a group satisfying* NC *if and only if one of the following holds.*

(1) *$\zeta \neq 0$; or*

(2) *$\zeta = 0$ and σ is not boundedly periodic; or*

(3) *σ is boundedly periodic and if $L = LCM\{n : \sigma_n = \omega\}$ then $\sigma_n \neq 0 \to n$ divides L.*

Proof. See [31, 4.1.]. □

Theorem 1.31. [31] *Suppose $\sigma = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ is a sequence of cardinals. The canonical representation $T : X \to X$ of σ is a member of a group satisfying* SMC *if and only if one of the following holds.*

(1) *$\zeta \neq 0$; or*

(2) *$\zeta = 0$ and σ is not boundedly periodic; or*

(3) *σ is boundedly periodic, and if $L = LCM\{n \in \mathbb{N} : \sigma_n \neq 0\}$ then $\sigma_L = \omega$.*

Proof. See [31, 4.3.]. □

Theorem 1.32. [31] *A sequence of cardinals $\sigma = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ is an orbit spectrum of a member of a group satisfying* MC *if and only if one of the following holds.*

(1) $\zeta \neq 0$*; or*

(2) $\zeta = 0$ *and* σ *is not boundedly periodic; or*

(3) σ *is boundedly periodic and if* $\sigma_n \neq 0$ *for some* n*, then there is* m *such that* $\sigma_m = \omega$ *and* n *divides* m*.*

Proof. The proof of this theorem follows immediately from Theorem 1.29 and Theorem 1.28. Also, Truss in [31, 4.2] has given a different proof of this result. □

The proof of the following theorem follows along similar lines to the previous three theorems (Truss [31]).

Theorem 1.33. [31] *Suppose $\sigma = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ is a sequence of cardinals. The canonical representation $T : X \to X$ of σ is a member of a group satisfying* SH *if and only if one of the following holds.*

(1) $\sigma_n < \omega$ *for every* $n \in \mathbb{N}$*; or*

(2) *for some finite* n*,* $\sigma_n = \omega, \sum_{m \neq n} \sigma_m$ *is finite,* $\zeta = 0$ *and* $\sigma_m = 0$ *for all* m *not dividing* n*.*

Next we will give two examples (quoted in [31]), the first one shows that there is a group $\langle T_1 \rangle$ satisfying NC which cannot be embedded in $H(\mathbb{Q})$, and the second shows that there is a group $\langle T_2 \rangle$ satisfying MC and hence is embeddable in $H(\mathbb{Q})$ but fails to have SMC and SH properties. This illustrates that the necessary and sufficient condition for a group G to be embedded in $H(\mathbb{Q})$ is MC, the mimicking property. Mekler in [22] had earlier given an example of a permutation group which satisfies NC but not MC.

Example 1.34. [31] Let $T_1 : X \to X$ be a bijection with countably infinitely many 12-cycles, countably infinitely many 18-cycles, one 36-cycle and no other cycles [23]. Then by Theorem 1.30, $\langle T_1 \rangle$ satisfies NC. But from Theorem 1.32 and Theorem 1.28, it cannot be embedded in $H(\mathbb{Q})$.

Example 1.35. [31] Let $T_2 : X \to X$ be a bijection and let $\sigma(T_2) = (0, 0, \sigma_1, \sigma_2, \dots)$ such that $\sigma_{12} = \omega$, $\sigma_{18} = \omega$ and $\sigma_n = 0$ otherwise. According to Theorem 1.32 and Theorem 1.28, $\langle T_2 \rangle$ satisfies MC, so it can be embedded in $H(\mathbb{Q})$. But from Theorem 1.31 and Theorem 1.33, it does not satisfy SMC nor SH.

Chapter 2

Continuity in the Rational World

2.1 Proof of the Main Theorem of Continuity in the Rational World

In this section, we give a proof of the main theorem of this chapter, Theorem 2.3, which is a generalization of Mekler's Theorem. If $T : X \to X$ is an arbitrary map on the countable set X, this theorem provides the necessary and sufficient conditions for X to have a topology with respect to which T is continuous and X is homeomorphic to the rational space $\mathbb{Q}$.

Let X be a countably infinite set and $T : X \to X$ be any function. In the following, we consider that $\mathbb{N}$ includes 0. For each $n \in \mathbb{N}$ and $x \in X$, let $C_{n,x}$ be the set defined by

$$C_{n,x} = T^{-n}(x),$$

and for every $l, m \in \mathbb{N}$, consider $D_{l,m}$ to be the set defined as

$$D_{l,m} = \{x \in X : T^l(x) = T^m(x)\}.$$

Now, we can define the families C and D as follows:

$$C = \{C_{n,x} : n \in \mathbb{N},\, x \in X\};$$
$$D = \{D_{l,m} : l, m \in \mathbb{N}\}.$$

Now, let $\mathcal{B}' = \{X \setminus E : E \in C \cup D\}$ and then define $\mathcal{B}$ to be the family that consists of X and all finite intersections of sets in $\mathcal{B}'$, namely

$$\mathcal{B} = \{X\} \cup \{\bigcap_{i \leq k} B_i : B_1, \ldots, B_k \in \mathcal{B}',\, k \in \mathbb{N}\}.$$

This means that $\mathcal{B}$ consists of X and sets of the form

$$X \setminus (\bigcup_{(r,x)\in F} T^{-r}(x) \cup \bigcup_{(l,m)\in J} \{x : T^l(x) = T^m(x)\}),$$

for some finite sets $F \subset \mathbb{N} \times X$ and $J \subset \mathbb{N} \times \mathbb{N}$.

Observation 2.1. If X is a Hausdorff space and $T : X \to X$ is continuous then every element of $\mathcal{B}$ is an open subset of X.

If we take X to be homeomorphic to the rational space and consider $T : X \to X$ to be a continuous function, then we have the following.

Proposition 2.2. *Let $T : \mathbb{Q} \to \mathbb{Q}$ be a continuous function, then any member of $\mathcal{B}$ is either empty or infinite.*

Proof. Since $\mathbb{Q}$ is a Hausdorff space and from the Observation above, every member of $\mathcal{B}$ is an open subset of $\mathbb{Q}$. Thus, the proof follows immediately from the fact that $\mathbb{Q}$ has no isolated points. $\square$

Truss in [31] gave another proof of Mekler's Theorem, or equivalently Theorem 1.28 (see [31, 2.3, p. 339]). Here, we will assume that $T : X \to X$ is any function not necessarily be a bijection and we will use a similar technique that Truss has used in his proof. So, if $T : X \to X$ is any function, then we have the following theorem.

Theorem 2.3. *Let X be a countably infinite set and $T : X \to X$ be a function. Then there is a topology on X with respect to which T is continuous and X is homeomorphic to $\mathbb{Q}$ if and only if every element of $\mathcal{B}$ is either empty or infinite.*

Proof. If $T : X \to X$ is continuous and X is homeomorphic to $\mathbb{Q}$, then, by Proposition 2.2, every element of $\mathcal{B}$ is either empty or infinite.

Conversely, suppose that for each $B \in \mathcal{B}$, B is either infinite or empty. First, list all the distinct pairs of elements of X as $\{(x_i, y_i) : i \in \mathbb{N}\}$. Next, we will show that there is a collection $\{\tau_i : i \in \mathbb{N}\}$ of topologies, each is generated by a countable base $\mathcal{C}_i$, such that $\tau_0 \subseteq \tau_1 \subseteq \tau_2 \subseteq \dots$ and this collection satisfies the following:

(1) if $U \in \tau_i$, then $T^{-1}(U) \in \tau_i$;

(2) for every $U \in \mathcal{C}_i$, $X \setminus U \in \mathcal{C}_i$;

(3) there is $U \in \tau_{i+1}$ such that $x_i \in U$ and $y_i \notin U$; and

(4) if $U \in \tau_i$ then $U \cap B$ is either empty or infinite for all $B \in \mathcal{B}$.

Let τ be the topology generated by $\bigcup_{i \in \mathbb{N}} \tau_i$. It follows that (X, τ) is second countable, from (2) it is 0-dimensional, and condition (3) implies that X is a T_1 topological space. Also, from condition (4) all open sets in τ are empty or infinite since $X \in \mathcal{B}$, hence X has no isolated points. Thus, by Theorem 1.20 (Sierpinski's Theorem), (X, τ) is homeomorphic to $\mathbb{Q}$ and from (1) we have $T : X \to X$ is continuous with respect to the topology τ. Thus, the proof will be completed if we can find a collection $\{\tau_i : i \in \mathbb{N}\}$ satisfying the conditions above.

We will define τ_i by induction. Let $\tau_0 = \{\emptyset, X\}$, then τ_0 satisfies the conditions above immediately. Now, suppose that τ_n has been chosen to be generated by a countable base

$\mathcal{C}_n$, satisfying (1)-(4) such that $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_n$. Then τ_{n+1} will be the topology generated by the countable family:

$$\mathcal{C}_n \cup \{T^{-k}(X_n) : k \in \mathbb{N}\} \cup \{X \setminus T^{-k}(X_n) : k \in \mathbb{N}\},$$

where X_n is an infinite subset of X containing x_n but not y_n. Obviously, conditions (1), (2) and (3) hold. We have to define X_n in such a way that all members of τ_{n+1} satisfy (4). So, we have to ensure that all sets of the form

$$U_{n,k} \cap \bigcap_{i \in R} T^{-i}(X_n) \cap \bigcap_{j \in K} X \setminus T^{-j}(X_n) \cap B,$$

where $U_{n,k} \in \mathcal{C}_n, k \in \mathbb{N}$, $B \in \mathcal{B}$ and $R, K \subset \mathbb{N}$ are finite sets, are empty or infinite.

We will construct the set X_n in stages. Firstly, consider all quadruples of the form (m, I, J, B), where $m \in \mathbb{N}$ and $I, J \subset \mathbb{N}$ are finite sets and $B \in \mathcal{B}$, to be listed in a sequence such that each occurs infinitely many times. Next, we choose finite sets $Y_{n,0} \subseteq Y_{n,1} \subseteq Y_{n,2} \subseteq \ldots$ by induction so that $X_n = \bigcup_{k \in \mathbb{N}} Y_{n,k}$.

Let $Y_{n,0} = \{x_n\}$. Suppose that $Y_{n,k}$ has been chosen such that it does not contain y_n, and then let (m, I, J, B) be the kth quadruple in the sequence of quadruples above. Let

$$B_k(m, I, J, B) = U_{n,m} \cap \bigcap_{i \in I} T^{-i}(Y_{n,k}) \cap \bigcap_{j \in J} X \setminus T^{-j}(Y_{n,k}) \cap B.$$

If $B_k(m, I, J, B)$ is empty or infinite, let $Y_{n,k+1} = Y_{n,k}$. If $B_k(m, I, J, B)$ is a non-empty finite set, then there is some $y \in B_k(m, I, J, B)$. This means that $y \in \bigcap_{i \in I} T^{-i}(Y_{n,k})$ and $y \notin \bigcup_{j \in J} T^{-j}(Y_{n,k})$, in other words, $T^i(y) \in Y_{n,k}$ for all $i \in I$ and $T^j(y) \notin Y_{n,k}$ for all

$j \in J$. Hence $y \in \bigcap_{\substack{i\in I\\ j\in J}} \{x : T^i(x) \neq T^j(x)\}$. It follows that $y \in D_k(m, I, J, B)$, where

$$D_k(m, I, J, B) = U_{n,m} \cap \bigcap_{\substack{i\in I\\ j\in J}} \{x : T^i(x) \neq T^j(x)\} \cap \bigcap_{j\in J} X \setminus T^{-j}(Y_{n,k}) \cap B$$

which is infinite since $\bigcap_{\substack{i\in I\\ j\in J}} \{x : T^i(x) \neq T^j(x)\} \in \mathcal{B}$ and

$$\bigcap_{j\in J} X \setminus T^{-j}(Y_{n,k}) = \bigcap_{\substack{j\in J\\ p\in Y_{n,k}}} X \setminus T^{-j}(p),$$

which is also in $\mathcal{B}$. Hence, if $d \in D_k(m, I, J, B)$ then $T^i(d) \neq T^j(d)$ and $T^j(d) \notin Y_{n,k}$ for all $i \in I$ and $j \in J$. So, we can choose d' such that $d' \in D_k(m, I, J, B)$ but not in $B_k(m, I, J, B)$ such that $T^i(d') \neq y_n$ for all $i \in I$. Put

$$Y_{n,k+1} = Y_{n,k} \cup \{T^i(d') : i \in I\}.$$

So, $Y_{n,k+1}$ is a finite set that contains $Y_{n,k}$ and $y_n \notin Y_{n,k+1}$.

To see how this construction works let us suppose that the set

$$C = U_{n,k} \cap \bigcap_{i\in I} T^{-i}(X_n) \cap \bigcap_{j\in J} X \setminus T^{-j}(X_n) \cap B$$

is non-empty for some $k \in \mathbb{N}$, $B \in \mathcal{B}$ and finite sets $I, J \subset \mathbb{N}$, then it contains an element which must appear at some step of the construction of X_n. So, a new element will be added to C every time the quadruple (k, I, J, B) appears in the enumeration above whenever it is finite. Since every element occurs infinitely many times, then the set C must be infinite.

Obviously, $x_n \in X_n$ but $y_n \notin X_n$. Consequently, we have each set of the form

$$U_{n,m} \cap \bigcap_{i \in I} T^{-i}(X_n) \cap \bigcap_{j \in J} X \setminus T^{-j}(X_n) \cap B$$

is empty or infinite so condition (4) holds. Consequently, every member of τ_{n+1} satisfies conditions (1)-(4), as was required. $\square$

2.2 Examples and Applications

In this section we give some related results and examples in terms of the orbit structure of the function $T : X \to X$.

We start with the following result from [23], but we generalize it to any continuous function rather than homeomorphisms.

Lemma 2.4. *Let $\{T_i : X_i \to X_i\}_{i \in I}, I \subseteq \mathbb{N}$, be a collection of continuous maps, where X_i is homeomorphic to $\mathbb{Q}$ for each $i \in I$ and $X_i \cap X_j = \emptyset$ whenever $i \neq j$. If $X = \bigcup_{i \in I} X_i$, then the map $T : X \to X$ defined as $T \restriction X_i = T_i$ is continuous and X is homeomorphic to $\mathbb{Q}$.*

Proof. Since X_i is homeomorphic to $(i, i+1) \cap \mathbb{Q}$ for each $i \in I$, then $X = \bigcup_{i \in I} X_i$ is homeomorphic to $[\bigcup_{i \in I}(i, i+1)] \cap \mathbb{Q}$, so X is homeomorphic to $\mathbb{Q}$. The continuity of T follows from the continuity of each $T \restriction X_i = T_i$. $\square$

Now we give the following result of the case when T has a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit or a semi-simple cycle, then we will give some examples when T has only cycles.

Theorem 2.5. *Let X be a countable set and $T : X \to X$ be a function with orbit spectrum $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$. There is a topology on X with respect to which T is continuous and X is homeomorphic to $\mathbb{Q}$ in each of the following cases:*

(1) *T has a $\mathbb{Z}$-orbit or an $\mathbb{N}$-orbit;*

(2) *there is $S \subseteq X$ such that $T \restriction S$ is a semi-simple cycle;*

(3) *$\sigma(T)$ is not boundedly periodic.*

Proof. By Theorem 2.3, it is sufficient to prove that

$$X \setminus (\bigcup_{(r,x)\in F'} C_{r,x} \cup \bigcup_{(l,m)\in J'} D_{l,m})$$

is either infinite or empty for any finite sets $F' \subset \mathbb{N} \times X$ and $J' \subset \mathbb{N} \times \mathbb{N}$. Let

$$Y = X \setminus (\bigcup_{(r,x)\in B} C_{r,x} \cup \bigcup_{(l,m)\in J} D_{l,m}),$$

for some finite sets $B \subset \mathbb{N} \times X$ and $J \subset \mathbb{N} \times \mathbb{N}$, be a non-empty subset of X. Let $F = \{x : (r, x) \in B\}$ and $I = \{r \in \mathbb{N} : (r, x) \in B\}$.

(1) Without loss of generality, by Lemma 2.4, we can assume that T has a unique $\mathbb{Z}$-orbit or $\mathbb{N}$-orbit, O. Let $C = X \setminus O$, so C consists of points of cycles of T. Notice that for each $l, m \in \mathbb{N}$ we have $D_{l,m} \subseteq C$. Now we have two cases: if $x \in C$ for all $x \in F$, then we have $C_{r,x} \subseteq C$ for each $r \in I$. Since $D_{l,m} \subseteq C$ for each $l, m \in \mathbb{N}$, so we have

$$O \cap (\bigcup_{(r,x)\in B} C_{r,x} \cup \bigcup_{(l,m)\in J} D_{l,m}) = \emptyset,$$

hence $O \subseteq Y$ and Y is infinite. The other case is if there is $A \subseteq F$ such that $x \in O$ so $C_{n,x} \subseteq O$ for all $x \in A$ and $n \in J' \subseteq I$. Let $A = \{x_1, x_2, \ldots, x_k\}$, since all elements of $C_{n,x}$ lie in the same orbit O, then we have $T^{n_1}(x_1) = T^{n_2}(x_2) = \cdots = T^{n_k}(x_k) = z_0$ for some natural numbers $n_1, n_2, \ldots, n_k$. So we have

$$\bigcup_{0\neq j\in\mathbb{N}} T^j(z_0) \cap \bigcup C_{n,x_i} = \emptyset.$$

Since for each $l, m \in \mathbb{N}$, $D_{l,m} \subseteq C$ then we have

$$\bigcup_{0 \neq j \in \mathbb{N}} T^j(z_0) \cap (\bigcup_{(r,x) \in B} C_{r,x} \cup \bigcup_{(l,m) \in J} D_{l,m}) = \emptyset;$$

hence, $\bigcup_{0 \neq j \in \mathbb{N}} T^j(z_0) \subseteq Y$. Since $\bigcup_{j \in \mathbb{N}} T^j(z_0) \subseteq O$ is infinite, it follows that Y is also infinite and the proof is complete.

(2) Suppose that $S \subseteq X$ is such that $T \restriction S$ is a semi-simple n-cycle. Let S be indexed as $\{x_i : 0 \leq i < n\} \cup \{y_j : j \in \mathbb{N}\}$ with $T(x_i) = x_{i+1}$ for $i < n-1$, $T(x_{n-1}) = x_0$, $T(y_j) = y_{j-1},\ j \neq 0$ and $T(y_0) = x_0$.

Let $A = F \cap S$ and let $J' \subseteq J$ be such that $D_{l,m} \subset O$ for each $(l, m) \in J'$. If $J' \cup A = \emptyset$ then immediately S is a subset of Y and Y is infinite; so assume that $J' \cup A \neq \emptyset$. Let

$$D = \bigcup_{(r,x) \in B} C_{r,x} \cup \bigcup_{(l,m) \in J'} D_{l,m}.$$

Notice that for any $l < m \in \mathbb{N}$, $y_k \notin D_{l,m}$ for all $k > l$. Let $p = \max(\{0\} \cup \{n : y_n \in D\})$. So, we have $\bigcup_{0 \neq j \in \mathbb{N}} \{y_{p+j}\} \cap D = \emptyset$, which implies that $\bigcup_{0 \neq j \in \mathbb{N}} \{y_{p+j}\} \subseteq Y$; so Y is infinite.

(3) Suppose that $\sigma(T)$ is not boundedly periodic. Since

$$X \setminus \bigcup_{(l,m) \in J} D_{l,m} = \bigcap_{(l,m) \in J} \{x \in X : T^n(x) \neq T^m(x)\},$$

so it contains all cycles of length greater than $n - m$, and since each $C_{r,x}$ is a subset of a cycle, it follows immediately that Y is infinite. Hence, by Theorem 2.3, X can be endowed with a topology that makes X homeomorphic to $\mathbb{Q}$ and T a continuous function, as was required. $\square$

Now, we will give some examples of the case when T has, in total, countably many cycles.

Example 2.6. Let X be a countable set and $T : X \to X$ have a single n-cycle with spine $\{x_0, x_1, \ldots, x_{n-1}\}$, $n > 1$, such that $T^{-1}(x_i) = x_{i-1}$ for all except for x_0, where $T^{-1}(x_0) = \{x_{n-1}\} \cup C_0$, C_0 is infinite, and $T^{-2}(x_0) = x_{n-2}$ (see Figure 2.1). By applying Theorem 2.3, we find that $X \setminus T^{-1}(x_0) = \{x_0, x_1, \ldots, x_{n-2}\}$, so there is no topology on X such that T is continuous and X is homeomorphic with $\mathbb{Q}$.

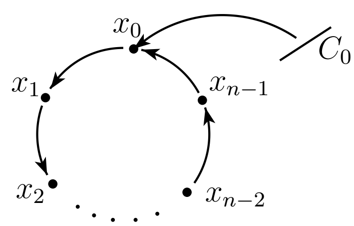

Figure 2.1: The Orbit of $T : X \to X$.

Example 2.7. Let $T_1 : X \to X$ have orbit spectrum

$$\sigma(T_1) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$$

with $\zeta = \nu = 0$, $\sigma_4 = \omega$, and $\sigma_n = 0$ otherwise. Let all 4-cycles be simple except for one 4-cycle which consists of a spine $\{x_0, x_1, x_2, x_3\}$ and a point y such that $T_1(y) = x_0$ (see Figure 2.2). According to Theorem 2.3, since $X \setminus \{x : T_1^4(x) = x\} = \{y\}$, X cannot be endowed with a topology so that T_1 is continuous and X is homeomorphic to $\mathbb{Q}$.

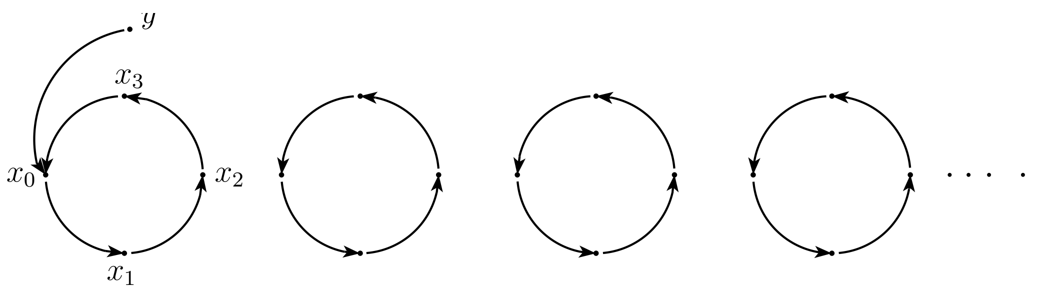

Figure 2.2: The Orbits of T_1.

Example 2.8. Let $T_2 : X \to X$ have only a single 3-cycle with spine $S = \{x_0, x_1, x_2\}$ such that $|T_2^{-1}(x)| = \omega$ for all $x \in S$ and $T_2^{-1}(x) = \emptyset$ otherwise, as in the following figure:

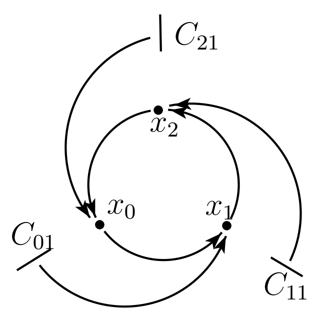

Figure 2.3: The Orbit of T_2.

In this example, all conditions of Theorem 2.3 hold, so we can find a topology on X so that T_2 is continuous and X is homeomorphic with $\mathbb{Q}$. Simply, we can choose homeomorphisms $h_i : C_{i1} \cup \{x_i\} \to [2i, 2i+1) \cap \mathbb{Q}$, $0 \leq i < 3$ with $h_i : x_i \mapsto 2i$. Then the map $h : \bigcup_{i=0}^{2}[2i, 2i+1) \cap \mathbb{Q} \to \bigcup_{i=0}^{2}[2i, 2i+1) \cap \mathbb{Q}$ defined as

$$h(x) = \begin{cases} 2 & \text{if } x \in [0,1) \cap \mathbb{Q}, \\ 4 & \text{if } x \in [2,3) \cap \mathbb{Q}, \\ 0 & \text{if } x \in [4,5) \cap \mathbb{Q}, \end{cases}$$

is continuous and has the same orbits of T_2.

We end this section with giving a proof of a the following related theorem.

Theorem 2.9. *Let T be a function on a countable set X. Then X can be always endowed with a topology with respect to which T is continuous and X is either homeomorphic to $\mathbb{Q}$ or homeomorphic to a compact subset of $\mathbb{Q}$.*

Proof. It is a well known fact that every countable compact Hausdorff space is metrizable and every countable compact metrizable space is homeomorphic to a compact subset of the rationals with their usual topology (see [8]). So, by Theorem 1.15, there is a topology on X with respect to which T is continuous and X is homeomorphic to a compact subset of $\mathbb{Q}$ if $T^{\omega+1}(X) = T^{\omega}(X) \neq \emptyset$ and either:

(a) both $\zeta \neq 0$ and $\sum_{n\in\mathbb{N}} \sigma_n \neq 0$; or

(b) $\zeta = 0$ and $\sigma(T)$ is finitely based; or

(c) $\zeta = 0$ and $T \restriction T^{\omega}(X)$ is not 1-1.

Also, by Theorem 2.5, if T has a $\mathbb{Z}$-orbit or $\mathbb{N}$-orbit, then there is a topology on X with respect to which T is continuous and X is homeomorphic to $\mathbb{Q}$. So, we are left with the following cases to consider:

(1) $T^{\omega+1}(X) \neq T^{\omega}(X) \neq \emptyset$.

(2) $T^{\omega}(X) = \emptyset$.

(3) $\zeta = 0$, $\sigma(T)$ is not finitely based and $T \restriction T^{\omega}(X)$ is 1-1.

Case (2) is equivalent to saying that all orbits of T are $\mathbb{N}$-orbits, so this case follows immediately from Theorem 2.5, part (1), so there is a topology that makes X homeomorphic to $\mathbb{Q}$ and T is continuous.

To see Case (3), since $\sigma(T)$ is not finitely based, this implies that $s = \{n \in \mathbb{N} : \sigma_n \neq 0\}$ is infinite, so $\sigma(T)$ is not boundedly periodic. Hence, the proof follows from Theorem 2.5, part (3).

Finally we will deal with Case (1) as follows. Suppose that T has no $\mathbb{Z}$ orbit nor $\mathbb{N}$-orbit (otherwise Theorem 2.5, part (1) implies that there is a topology on X such that X is homeomorphic to $\mathbb{Q}$ and T is continuous as we mentioned above). Suppose, for a contradiction, that there is no topology that makes X homeomorphic to $\mathbb{Q}$ and T a continuous map, so by Theorem 2.3, there exist finite sets $F \subset \mathbb{N} \times X$ and $J \subset \mathbb{N} \times \mathbb{N}$ such that

$$Y = X \setminus (\bigcup_{(r,x)\in F} C_{r,x} \cup \bigcup_{(l,m)\in J} D_{l,m})$$

is a non-empty finite set. Let $Y = \{y_1, y_2, \ldots, y_p\}$, so

$$X = \{y_1, y_2, \ldots, y_p\} \cup \bigcup_{(r,x)\in F} C_{r,x} \cup \bigcup_{(l,m)\in J} D_{l,m}.$$

Clearly, if $x \in D_{l,m}$, $l < m$, then $S_{l,m} = \bigcup_{l\leq q\leq m} \{T^q(x) : x \in D_{l,m}\}$ is a spine of some cycle in X. Let $A = \{i \in \mathbb{N} : (i, m) \in J \text{ or } (l, i) \in J \text{ or } (i, x) \in F\}$ and let $a = max\{i \in \mathbb{N} : i \in A\}$, then we have

$$T^a(X) \subseteq \{T^a(y_j) : 1 \leq j \leq p\} \cup \bigcup_{(r,x)\in F} T^{a-r}(x) \cup \bigcup_{(l,m)\in J} S_{l,m}.$$

Since $T^{k+1}(X) \subseteq T^k(X)$ for each $k \in \mathbb{N}$, it follows that $T^{\omega+1}(X) = T^\omega(X) \neq \emptyset$, which is a contradiction. So, Y is either empty or infinite, i.e.,

$$X \setminus \Big(\bigcup_{(r,x)\in F'} C_{r,x} \cup \bigcup_{(l,m)\in J'} D_{l,m} \Big)$$

is either infinite or empty for any finite sets $F' \subset \mathbb{N} \times X$ and $J' \subset \mathbb{N} \times \mathbb{N}$. Hence, by Theorem 2.3, there is a topology on X with respect to which X is homeomorphic to $\mathbb{Q}$ and T is continuous. $\square$

2.3 Continuity of a Countable Collection of Maps on the Rational World

Let $T_1 : X \to X$ and $T_2 : X \to X$ be arbitrary functions on a countably infinite set X. In this section we study conditions under which one can endow X with a topology with respect to which T_1 and T_2 are continuous and X is homeomorphic to $\mathbb{Q}$.

Consider $W(T_1, T_2)$ to be the family that consists of the identity map and all functions generated by T_1 and T_2; namely, obtained as compositions with finitely many factors of T_1 and T_2.

For each $T \in W(T_1, T_2)$ and $x \in X$, let $C_{T,x} = T^{-1}(x)$, and for every $T, F \in W(T_1, T_2)$, let $D_{T,F} = \{x \in X : T(x) = F(x)\}$. Now let

$$C = \{C_{T,x} : T \in W(T_1, T_2), x \in X\} \cup \{D_{T,F} : T, F \in W(T_1, T_2)\}.$$

Define $\mathcal{B}' = \{X \setminus E : E \in C\}$ and then define $\mathcal{B}$ to be the family that consists of X and all finite intersections of sets in $\mathcal{B}'$, namely

$$\mathcal{B} = \{X\} \cup \{\bigcap_{i\leq k} B_i : B_1, \ldots, B_k \in \mathcal{B}',\ k \in \mathbb{N}\}.$$

Now, we have the following theorem which is a generalization of Theorem 2.3. We use the same technique which we use in the proof of Theorem 2.3.

Theorem 2.10. *Let T_1 and T_2 be functions defined on the countably infinite set X. There is a topology on X with respect to which T_1 and T_2 are continuous and X is homeomorphic to $\mathbb{Q}$ if and only if every element of $\mathcal{B}$ is either empty or infinite.*

Proof. If $T_1 : \mathbb{Q} \to \mathbb{Q}$ and $T_2 : \mathbb{Q} \to \mathbb{Q}$ are continuous then each $F \in W(T_1, T_2)$ is continuous. Since $\mathbb{Q}$ is a Hausdorff space, then every element of $\mathcal{B}$ is open set, so it is either empty or infinite, since $\mathbb{Q}$ has no isolated points.

Conversely, suppose that for each $B \in \mathcal{B}$, B is either infinite or empty. First, as in Theorem 2.3, list all the distinct pairs of elements of X as $\{(x_i, y_i) : i \in \mathbb{N}\}$. Next, we will show that there is a collection $\{\tau_i : i \in \mathbb{N}\}$ of topologies, each is generated by a countable base $\mathcal{C}_i$, such that $\tau_0 \subseteq \tau_1 \subseteq \tau_2 \subseteq \dots$ and this collection satisfies that:

(1) if $U \in \tau_i$, then $T^{-1}(U) \in \tau_i$ for each $T \in W(T_1, T_2)$;

(2) for every $U \in \mathcal{C}_i$, $X \setminus U \in \mathcal{C}_i$;

(3) there is $U \in \tau_{i+1}$ such that $x_i \in U$ and $y_i \notin U$;

(4) if $U \in \tau_i$ then $U \cap B$ is either empty or infinite for all $B \in \mathcal{B}$.

If τ is the topology generated by $\bigcup_{i\in\mathbb{N}} \tau_i$ then conditions (2),(3) and (4) imply that X satisfies the conditions of Sierpinski's Theorem (Theorem 1.20), so (X, τ) is homeomorphic to $\mathbb{Q}$. Condition (1) implies that T_1 and T_2 are continuous with respect to τ. Thus, the proof will be completed if we can find a collection $\{\tau_i : i \in \mathbb{N}\}$ satisfying the conditions above.

We will define τ_i by induction. Let $\tau_0 = \{\emptyset, X\}$, clearly, τ_0 satisfies the conditions (1)-(4). Suppose that τ_n has been chosen to be generated by a countable base $\mathcal{C}_n$, satisfying

(1)-(4) such that $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_n$. Then we will choose τ_{n+1} to be the topology generated by the countable family:

$$\mathcal{C}_n \cup \{T^{-1}(X_n), X \setminus T^{-1}(X_n) : T \in W(T_1, T_2)\},$$

where X_n is an infinite subset of X containing x_n but not y_n. Clearly, conditions (1)-(3) hold so we have to choose X_n in such a way that all members of τ_{n+1} satisfy (4); i.e., all sets of the form

$$U_{n,k} \cap \bigcap_{T \in R} T^{-1}(X_n) \cap \bigcap_{T_3 \in K} X \setminus T_3^{-1}(X_n) \cap B,$$

where $U_{n,k} \in \mathcal{C}_n$, $k \in \mathbb{N}$, $B \in \mathcal{B}$ and R, K are finite subsets of $W(T_1, T_2)$, must be either empty or infinite.

We will build up the set X_n step by step as follows. List all quadruples of the form (m, R, M, B), $m \in \mathbb{N}$, $R, M \subset W(T_1, T_2)$ are finite sets and $B \in \mathcal{B}$, in a sequence such that each occurs infinitely many times. Next, we define finite sets $Y_{n,0} \subseteq Y_{n,1} \subseteq Y_{n,2} \subseteq \dots$ by induction so that $X_n = \bigcup_{k \in \mathbb{N}} Y_{n,k}$.

Let $Y_{n,0} = \{x_n\}$. Suppose that $Y_{n,k}$ has been chosen such that $Y_{n,k}$ does not contain y_n. Let (m, R, M, B) be the kth quadruple in the sequence of quadruples defined above. Let

$$B_k(m, R, M, B) = U_{n,m} \cap \bigcap_{T \in R} T^{-1}(Y_{n,k}) \cap \bigcap_{T_3 \in M} X \setminus T_3^{-1}(Y_{n,k}) \cap B.$$

If $B_k(m, R, M, B)$ is empty or infinite, let $Y_{n,k+1} = Y_{n,k}$. If $B_k(m, R, M, B)$ is a non-empty finite set, then there is some $y \in B_k(m, R, M, B)$ so $y \in \bigcap_{T \in R} T^{-1}(Y_{n,k})$ and $y \notin \bigcup_{T_3 \in M} T_3^{-1}(Y_{n,k})$. Hence, $T(y) \in Y_{n,k}$ for all $T \in R$ and $T_3(y) \notin Y_{n,k}$ for all $T_3 \in M$. So, we have $y \in \bigcap_{\substack{T \in R \\ T_3 \in M}} \{x : T(x) \neq T_3(x)\}$. It follows that $y \in D_k(m, R, M, B)$, where

$$D_k(m, R, M, B) = U_{n,m} \cap \bigcap_{\substack{T \in R \\ T_3 \in M}} \{x : T(x) \neq T_3(x)\} \cap \bigcap_{T_3 \in M} X \setminus T_3^{-1}(Y_{n,k}) \cap B.$$

Since

$$\bigcap_{T_3 \in M} X \setminus T_3^{-1}(Y_{n,k}) = \bigcap_{\substack{T_3 \in M \\ p \in Y_{n,k}}} X \setminus T_3^{-1}(p),$$

so both of $\bigcap_{\substack{T \in R \\ T_3 \in M}} \{x : T(x) \neq T_3(x)\}$ and $\bigcap_{T_3 \in M} X \setminus T_3^{-1}(Y_{n,k})$ belong to $\mathcal{B}$; hence, $D_k(m, R, M, B)$ is infinite.

Hence, if $d \in D_k(m, R, M, B)$ then $T(d) \neq T_3(d)$ and $T(d) \notin Y_{n,k}$ for all $T \in R$ and $T_3 \in M$. So, we can choose d' such that $d' \in D_k(m, R, M, B)$ but not in $B_k(m, R, M, B)$ such that $T(d') \neq y_n$ for all $T \in R$. Put

$$Y_{n,k+1} = Y_{n,k} \cup \{T(d') : T \in R\}.$$

So, $Y_{n,k+1}$ is a finite set with $Y_{n,k} \subseteq Y_{n,k+1}$ and $y_n \notin Y_{n,k+1}$.

Now we show how this construction works. Suppose that the set

$$C = U_{n,k} \cap \bigcap_{T \in R} T^{-1}(X_n) \cap \bigcap_{T_3 \in M} X \setminus T_3^{-1}(X_n) \cap B$$

is non-empty for some $k \in \mathbb{N}$, $B \in \mathcal{B}$ and finite sets $R, M \subset W(T_1, T_2)$, then C contains an element which must appear at some step of the construction of X_n. So a new element will be added to C every time the quadruple (k, R, M, B) appears in the list defined above whenever it is finite. Since every element occurs infinitely many times, then the set C must be infinite.

Clearly, $x_n \in X_n$ and $y_n \notin X_n$. Thus, each set of the form

$$U_{n,m} \cap \bigcap_{T \in R} T^{-1}(X_n) \cap \bigcap_{T_3 \in M} X \setminus T_3^{-1}(X_n) \cap B$$

is either empty or infinite so condition (4) holds. Consequently, every member of τ_{n+1} satisfies conditions (1)-(4), as required. $\square$

Let $\mathcal{T} = \{T_i : X \to X\}_{i \in I}$, with $I \subset \mathbb{N}$ is finite, be a collection of functions on a countably infinite set X. Consider $\mathcal{F}$ to be the collection that consists of the identity map and all functions generated by $\mathcal{T}$; namely, obtained as compositions with arbitrary many factors of functions in $\mathcal{T}$.

For each $f \in \mathcal{F}$ and $x \in X$, let $C_{f,x} = f^{-1}(x)$, and for every $f_1, f_2 \in \mathcal{F}$, let $D_{f_1,f_2} = \{x \in X : f_1(x) = f_2(x)\}$. Now let

$$C = \{C_{f,x} : f \in \mathcal{F}, x \in X\} \cup \{D_{f_1,f_2} : f_1, f_2 \in \mathcal{F}\}.$$

Define $\mathcal{B}' = \{X \setminus E : E \in C\}$ and let

$$\mathcal{B} = X \cup \{\bigcap_{i \leq k} B_i : B_1, \ldots, B_k \in \mathcal{B}',\ k \in \mathbb{N}\}.$$

Now we have the following result, the proof of this result follows in the same way of the proof of Theorem 2.10.

Theorem 2.11. *Let $\mathcal{T} = \{T_i : X \to X\}_{i \in I}$, with $I \subset \mathbb{N}$ is finite, be a collection of functions on a countably infinite set X. There is a topology on X that makes X homeomorphic to $\mathbb{Q}$ and each T_i a continuous function if and only if every element of $\mathcal{B}$ is either infinite or empty.*

Chapter 3

Order-Preserving Maps on Countable Linear Orders

In this chapter, we study cases in which we can put a linear order on a set X with self-map T so that T is an order-preserving map and X is order-isomorphic to the rationals $\mathbb{Q}$, integers $\mathbb{Z}$ or naturals $\mathbb{N}$ with their usual order. This study is in terms of the orbit structure of T. We start in the first section with giving results and properties of order-preserving self-maps on arbitrary sets, we prove that any set with self-map T can be linearly ordered so that T is order-preserving provided that T has no cycles of length greater than 1. Then we give characterization of order-preserving bijective and injective self-maps on the rationals $\mathbb{Q}$. The main theorem in Section 3.2 is Theorem 3.33 which describes the orbit structure of order-preserving surjective self-maps on $\mathbb{Q}$. We also give some examples when T is an arbitrary map rather than a bijection or surjection. In the final two sections we give the orbit structure of order-preserving self-maps on the naturals $\mathbb{N}$ and the integers $\mathbb{Z}$.

There are many previous studies on order-preserving maps $T : X \to Y$ between countable sets. For example, Orr in [24] showed that for every countable linearly ordered set A there is an order preserving surjection $h : A \to A$ such that the cardinality of $h^{-1}(x)$ is at least $f(x)$, where $f : A \to \mathbb{Z}^+$, for all but finitely many $x \in A$. Farley et al. in [9] constructed a lattice L with the property that every interval has finite height, but there exists no strictly order-preserving map from L to $\mathbb{Z}$. So he answered a 1979 problem of Erné (posed at the 1981 Banff Conference on Ordered Sets [26]). Consider the finite set $X_n = \{1, 2, ..., n\}$ with the usual order. Let $\mathcal{T}_n$ be the full transformation semigroup on X_n. Higgins in [15] investigated combinatorial properties of the semigroup of all order-preserving mappings on X_n, and of its subsemigroup that consists of all decreasing and order-preserving mappings.

A number of authors have studied the group of order-preserving permutations of $\mathbb{Q}$. Medvedev et al. [21] proved that the only lattice orders of the group of all automorphisms that are order-preserving permutations of $\mathbb{Q}$ are the pointwise order and its inverse. A. V. Zenkov [32] studied maximal and minimal partial orders of the group of all order automorphisms of a linearly ordered set of $\mathbb{Q}$. See also [20], [10] and [19]. Finally, we mention that theses studies are not quite relevant to our study which focuses on the orbit structure of self-maps on a set.

3.1 Preliminaries of Order-Preserving Self-maps

In this section, we study some preliminaries and results related to order-preserving self-maps on arbitrary set. Some of these lemmas may be known, we include proofs for completeness.

We start with the following basic definition (see for example [13]).

Definition 3.1. Let $(X, \preceq_1)$ and $(Y, \preceq_2)$ be ordered sets. A map $T : X \to Y$ is said to be *order-preserving*, or OP, if $x \preceq_1 y$ implies that $T(x) \preceq_2 T(y)$.

For terminology that we will use, let $(X, \preceq)$ be a linearly ordered set, if $a, b \in X$ with

$a \preceq b$ then

$$(a, b) = \{x \in X : a \prec x \prec b\},$$

$$[a, b] = \{x \in X : a \preceq x \preceq b\},$$

$$[a, b) = \{x \in X : a \preceq x \prec b\},$$

$$(a, b] = \{x \in X : a \prec x \preceq b\}.$$

If $x \in X$ and $Y \subseteq X$, we say $x \preceq Y$ if and only if $x \preceq y$ for all $y \in Y$. If $Y_1, Y_2 \subseteq X$, we say that $Y_1 \preceq Y_2$ if and only if for every $x \in Y_1$, $x \preceq y$ for all $y \in Y_2$.

Now we give definition of the ordered sum of a collection of linearly ordered sets (see [14]).

Definition 3.2. Let $(I, \preceq)$ be a linearly ordered set and let $\{(Y_i, \preceq_i)\}_{i \in I}$ be a collection of pairwise disjoint linearly ordered sets. The *sum-order* $\preceq_+$ of the $\preceq_i$ over $(I, \preceq)$ is defined on $Y = \bigcup_{i \in I} Y_i$ as follows: for $x, y \in Y$, where $x \in Y_r, y \in Y_t$ for some $r, t \in I$ then

$$x \preceq_+ y \Leftrightarrow (r = t \text{ and } x \preceq_r y) \text{ or } (r \neq t \text{ and } r \preceq t).$$

The ordered set $(Y, \preceq_+)$ is called the *ordered sum* of $(Y_i, \preceq_i)$ over $(I, \preceq)$.

Lemma 3.3. *Let $T : X \to X$ be an OP map on a linearly ordered set X and let*

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots).$$

Then $\sigma_n = 0$ for all $n > 1$.

Proof. Suppose, for a contradiction, that $\sigma_k \neq 0$ for some $k > 1$, so T has a k-cycle O. Let S be a spine of O indexed as $\{x_0, x_1, \dots, x_{k-1}\}$ so that $T(x_i) = x_{i+1}$, where $i + 1$ is taken module k. If $x_0 \preceq x_1$, then $x_1 = T(x_0) \preceq T(x_1) = x_2$ so we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_{k-1},$$

since T is an OP map. But then $x_1 = T(x_0) \preceq T(x_{k-1}) = x_0$, which is a contradiction. The case when $x_1 \preceq x_0$ follows in the same way. Hence, $\sigma_n = 0$ for all $n > 1$, as was required. $\square$

Lemma 3.4. *Let $(I, \preceq')$ be a linearly ordered set, $\{(X_i, \preceq_i)\}_{i \in I}$ be a collection of linearly ordered pairwise disjoint sets and $\{T_i : X_i \to X_i\}_{i \in I}$ be a collection of OP maps. If $(X, \preceq)$ is the ordered sum of $(X_i, \preceq_i)$ over $(I, \preceq')$, then the map $T : X \to X$ defined as $T \upharpoonright X_i = T_i$ is an OP map.*

Proof. Let $x, y \in X$ with $x \preceq y$. If $x, y \in X_i$ for some $i \in I$, then $T(x) = T_i(x) \preceq_i T_i(y) = Y(y)$, so $T(x) \preceq Y(y)$. If $x \in X_i, y \in X_j$, since $T_i(x) \in X_i$ and $T_j(y) \in X_j$, then immediately we have $T(x) \preceq T(y)$. Thus, T is an OP map under $\preceq$. $\square$

Lemma 3.5. *Let $(X, \preceq)$ be a linearly ordered set and $T : X \to X$ be an OP map. For each $x \in X$, if $y, z \in T^{-k}(x)$ for some $k \in \mathbb{N}$ with $y \preceq z$ and if $y \preceq t \preceq z$ for some $t \in X$, then $t \in T^{-k}(x)$.*

Proof. Let $x \in X$ and let $y, z \in T^{-k}(x)$ for some $k \in \mathbb{N}$ such that $y \preceq z$. Suppose that there is a $t \in X$ with $y \preceq t \preceq z$. Since T is OP then we have $T(y) \preceq T(t) \preceq T(z)$ and then $T^k(y) \preceq T^k(t) \preceq T^k(z)$. But this means that $x \preceq T^k(t) \preceq x$, i.e., $T^k(t) = x$. Hence, $t \in T^{-k}(x)$, as was required. $\square$

Lemma 3.6. *Let $(X, \preceq)$ be a linearly ordered set and $T : X \to X$ be an OP function. Let O_1 and O_2 be orbits of T. If $x, y \in O_1$, $z \in O_2$, $x \preceq z \preceq y$ and O_2 is a 1-cycle then $O_1 = O_2$.*

Proof. Let $x, y \in O_1$, then $T^m(x) = T^n(y)$ for some $n, m \in \mathbb{N}$. Without loss of generality, let $m = n + r$. Suppose that $T^k(z) = x_0$ is a fixed point for some $k \in \mathbb{N}$. Since $x \preceq z \preceq y$ then $T^k(x) \preceq x_0 \preceq T^k(y)$, so $T^{k+n}(x) \preceq x_0 \preceq T^{k+n}(y)$. But $T^{k+n}(x) \preceq x_0$ implies that

$$T^{k+n+r}(x) \preceq T^r(x_0) = x_0 \preceq T^{k+n}(y),$$

i.e., $T^{k+m}(x) \preceq x_0 \preceq T^{k+n}(y)$, since $m = n + r$. But $T^{k+m}(x) = T^{k+n}(y)$, so we have $x, y \in O_2$. Hence, $O_1 = O_2$. □

Theorem 3.7. *Let $(X, \preceq)$ be a linearly ordered set and $T : X \to X$ be an OP map. Suppose that x, y are in the same orbit O, where O is a $\mathbb{Z}$-orbit or an $\mathbb{N}$-orbit. Suppose that $n = min\{j \in \mathbb{N} : T^j(x) \in \bigcup_{r\in\mathbb{N}} T^r(y)\}$ and $m = min\{j \in \mathbb{N} : T^j(y) \in \bigcup_{r\in\mathbb{N}} T^r(x)\}$ so that $T^n(x) = T^m(y)$. Suppose further that $x \preceq T(x)$.*

(1) *If $x \preceq y \preceq T(x)$ then either $n = m$ or $n = m + 1$.*

(2) *If $y \preceq x \preceq T(x)$ then $n \leq m$.*

(3) *If $x \preceq T(x) \preceq y$ then $n \geq m + 1$.*

Proof. (1) Suppose that $x \preceq y \preceq T(x)$. Since T is an OP map then we have $T^m(x) \preceq T^m(y) = T^n(x) \preceq T^{m+1}(x)$. Since O is not a cycle then we have $m \leq n \leq m+1$, so either $n = m$ or $n = m + 1$.

(2) Suppose that $y \preceq x \preceq T(x)$. Since T is an OP map then we have $T^m(y) = T^n(x) \preceq T^m(x) \preceq T^{m+1}(x)$. Hence we have $n \leq m$.

(3) Similarly we have $x \preceq T(x) \preceq y$ implies that $T^m(x) \preceq T^{m+1}(x) \preceq T^m(y) = T^n(x)$ so $m + 1 \leq n$, i.e., $n \geq m + 1$, as required. □

The following corollary follows from the previous theorem, Theorem 3.7, and from Definition 1.7.

Corollary 3.8. *Let $(X, \preceq)$ be a linearly ordered set and $T : X \to X$ be an OP map. Let O be an orbit of T, where O is either an $\mathbb{N}$-orbit or a $\mathbb{Z}$-orbit. Let S be a spine of O indexed as $\{x_i : i \in M\}$, where M is either $\mathbb{Z}$ or $\mathbb{N}$. Then for each $x_i \in S$, if $x_i \preceq y \preceq x_{i+1}$ then either $y \in L_i(O)$ or $y \in L_{i+1}(O)$.*

Corollary 3.9. *Let $T : X \to X$ be an OP map on a linearly ordered set X consisting of a single $\mathbb{Z}$-orbit O, then O has no endpoints.*

Theorem 3.10. *Let $(X, \preceq)$ be a linearly ordered set and $T : X \to X$ be an OP map. Let O and O' be orbits of T, where neither O nor O' is a cycle. If $x \in O$ with $x \preceq T(x)$ and if $y \in O'$ with $x \preceq y \preceq T(x)$, then for any $z \in O'$ we have*

$$x \preceq z \preceq T(x) \text{ iff } T^n(z) = T^n(y) \text{ for some } n \in \mathbb{N}.$$

Similarly, if $T(x) \preceq x$ and if $y \in O'$ with $T(x) \preceq y \preceq x$, then for any $z \in O'$ we have $T(x) \preceq z \preceq x$ iff $T^n(z) = T^n(y)$ for some $n \in \mathbb{N}$.

Proof. Let $y, z \in O'$ and suppose, for a contradiction, that $x \preceq y \preceq T(x)$, $x \preceq z \preceq T(x)$ and $T^n(z) = T^m(y)$ with $m \neq n$. Since T is an OP map, then the assumption $x \preceq z \preceq T(x)$ implies that $T^n(x) \preceq T^n(z) \preceq T^{n+1}(x)$ so $T^n(z) \in [T^n(x), T^{n+1}(x)]$. On the other hand we have $x \preceq y \preceq T(x)$ implies that $T^m(x) \preceq T^m(y) = T^n(z) \preceq T^{m+1}(x)$, so $T^n(z) \in [T^m(x), T^{m+1}(x)]$, which is a contradiction, since $T^n(x) \prec T^{n+1}(x) \preceq T^m(x) \prec T^{m+1}(x)$ if $n < m$ and $T^m(x) \prec T^{m+1}(x) \preceq T^n(x) \prec T^{n+1}(x)$ if $n > m$. Hence, $m = n$.

Conversely, let $y \in O'$ with $x \preceq y \preceq T(x)$ and let $z \in O'$ with $T^n(z) = T^n(y)$ for some $n \in \mathbb{N}$. If $z \preceq x$ then $z \preceq x \preceq y$ which is a contradiction by Lemma 3.5, since $T^n(z) = T^n(y)$. If $T(x) \preceq z$ then $y \preceq T(x) \preceq z$, which again is a contradiction by Lemma 3.5. Hence, $x \preceq z \preceq T(x)$. Finally, the proof of the second statement follows in the same way. $\square$

The proof of the next corollary follows from Theorem 3.10, and from Definition 1.7.

Corollary 3.11. *Let $(X, \preceq)$ be a linearly ordered set and $T : X \to X$ be an OP map. Let O_1 and O_2 be orbits of T with spines S_1 and S_2 respectively, where each O_i, $i = 1, 2$ is either an $\mathbb{N}$-orbit or a $\mathbb{Z}$-orbit. Let S_i be a spine of O_i indexed as $\{x_{i,j} : j \in M\}$, where M is either $\mathbb{Z}$ or $\mathbb{N}$. Then one of the following holds:*

(1) *$O_1 \preceq O_2$ or $O_2 \preceq O_1$; or*

(2) *$L_r(O_1) \preceq L_{j+r}(O_2) \preceq L_{r+1}(O_1)$ for some $j \in \mathbb{Z}$ and for all $r \in \mathbb{Z}$.*

Moreover, if $x \preceq T(x)$ for all $x \in O_1$ and $T(y) \preceq y$ for all $y \in O_2$ then either $O_1 \preceq O_2$ or $O_2 \preceq O_1$.

Proof. Suppose that neither $O_1 \preceq O_2$ nor $O_2 \preceq O_1$. So, there is $y \in O_2$ and $z_1, z_2 \in O_1$ such that $z_1 \preceq y \preceq z_2$. By Lemma 1.8, there are $k, j \in \mathbb{Z}$ so that $y \in L_j(O_2)$ and $z_1 \in L_k(O_1)$. By Lemma 3.5, Theorem 3.10 and Lemma 1.8 we have $L_i(O_1) \preceq L_j(O_2) \preceq L_{i+1}(O_1)$, where $i = \max\{t \in \mathbb{Z} : t \geq k, L_t(O_1) \preceq L_j(O_2)\}$. Since T is OP then

$$L_{i+r}(O_1) \preceq L_{j+r}(O_2) \preceq L_{i+r+1}(O_1)$$

for all $r \in \mathbb{Z}$, i.e.,

$$L_r(O_1) \preceq L_{j'+r}(O_2) \preceq L_{r+1}(O_1),$$

$j' = j - i$, for all $r \in \mathbb{Z}$.

Now, suppose that $x \preceq T(x)$ for all $x \in O_1$ and $T(y) \preceq y$ for all $y \in O_2$. Suppose, for a contradiction, that neither $O_1 \preceq O_2$ nor $O_2 \preceq O_1$. So O_1 and O_2 satisfy that $L_r(O_1) \preceq L_{j+r}(O_2) \preceq L_{r+1}(O_1)$ for some $j \in \mathbb{Z}$ and for all $r \in \mathbb{Z}$. So, there is $y \in O_2$ and $z \in O_1$ such that

$$z \preceq y \preceq T(z) \preceq T(y),$$

which is a contradiction of being $T(y) \preceq y$. Thus, we have either $O_1 \preceq O_2$ or $O_2 \preceq O_1$. $\square$

Theorem 3.12. *Let T be an injection on a countably infinite set X with orbit spectrum*

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots).$$

Suppose that $\sigma_n = 0$ for all $n \in \mathbb{N}$. If $0 \neq \zeta + \nu < \omega$, then any linear order $\preceq$ on X that makes T an OP map satisfies the following: for any $x \in X$, there is either $y \succ x$ such that $(x, y) = \emptyset$ or $z \prec x$ such that $(z, x) = \emptyset$.

Proof. Suppose that $\preceq$ is a linear order on X that makes T an OP map. Suppose first that T has one $\mathbb{Z}$-orbit indexed as $\{x_i : i \in \mathbb{Z}\}$ so that $T(x_i) = x_{i+1}$ for each $i \in \mathbb{Z}$. If $x \preceq T(x)$ then Theorem 3.7 (1) implies that $x_i \preceq x_{i+1}$ for each $i \in \mathbb{Z}$, so $(x, T(x)) = \emptyset$. If $T(x) \preceq x$ then again from Theorem 3.7 (1) we have $x_{i+1} \preceq x_i$ for each $i \in \mathbb{Z}$ and then $(T(x), x) = \emptyset$, so the proof is complete.

Now, suppose that T has, in total, n $\mathbb{Z}$-orbits $O_k, 1 \leq k \leq n$, $n \in \mathbb{N}$ and let $x \in O_t$ for some $t \in \{1, \dots, n\}$. If $x \preceq T(x)$ then by the previous case and Theorem 3.10, there are only m possible elements $y_i \in O_i$, $i \neq t$, $0 \leq m < n$ such that

$$Y = \{y_i \in O_i : 1 \leq i \leq m, i \neq t\} \subseteq (x, T(x)).$$

Choose y to be the least element of Y if $Y \neq \emptyset$ and $y = T(x)$ otherwise. So we have $(x, y) = \emptyset$, as required. The case when $T(x) \preceq x$ follows in the same way. Finally, if T has $\mathbb{N}$-orbits, the proof follows along the same lines as above. $\square$

Let $T : X \to X$ be a function and let O be an orbit of T. Let S be a spine of O indexed as $\{x_i : i \in M\}$, where M is either $\mathbb{N}$, $\mathbb{Z}$ or $\{0, \dots, n-1\}$ for some $n \in \mathbb{N}$ as appropriate, such that $T(x_i) = x_{i+1}$ for each $i \in M$ and i is taken module n when O is an n-cycle. For each $i \in M$, and $k \in \mathbb{N}$, let $C_{i,0} = \{x_i\}$ and let $C_{i,1} = T^{-1}(x_{i+1}) \setminus \{x_i\}$.

Now, for every $i \in N$, where N is either $\mathbb{Z}$ or $\{0, \ldots, n-1\}$, and $k \in \mathbb{N}$, let

$$C_{i,k} = T^{-1}(C_{i+1,k-1}) = T^{-k+1}(C_{i+k-1,1}).$$

Similar construction can be found in [11].

Observation 3.13. For each $i \in N$, if $L_i(O)$ is the set defined in Definition 1.7, then immediately we have $L_i(O) = \bigcup_{k \in \mathbb{N}} C_{i,k}$.

Theses terminologies are illustrated in Figure 3.1 and Figure 3.2 (which gives an example of a 1-cycle).

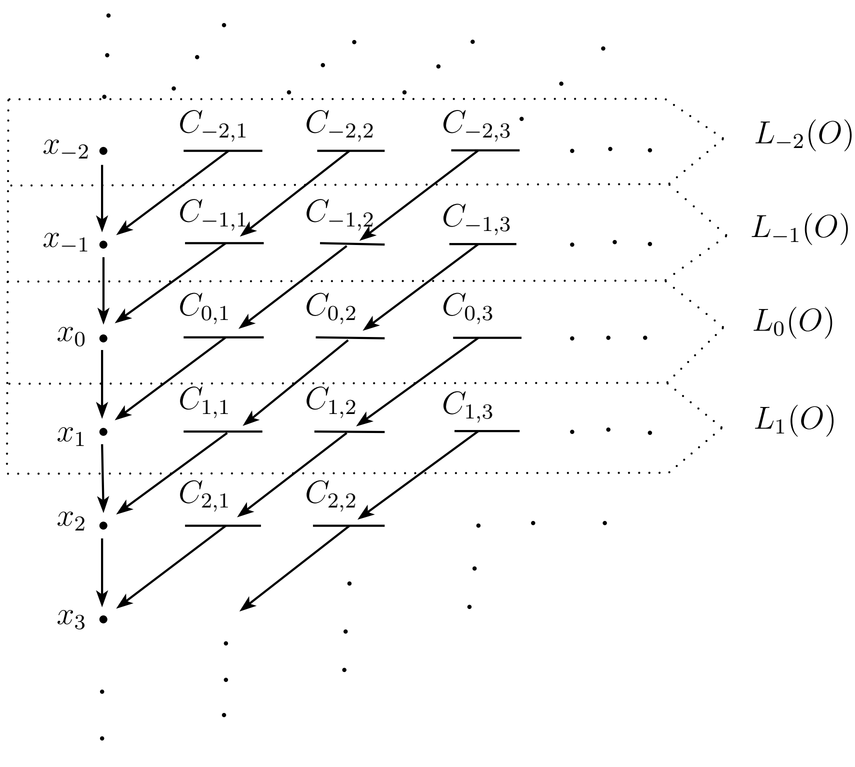

Figure 3.1: A $\mathbb{Z}$-orbit O.

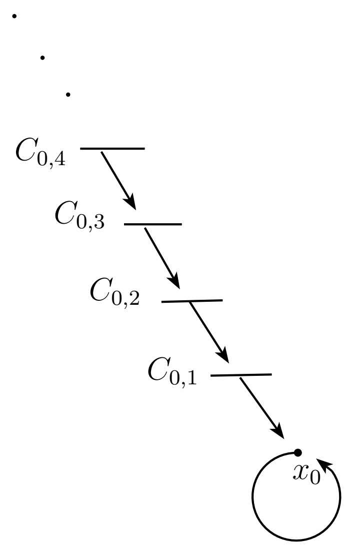

Figure 3.2: A 1-cycle O.

Lemma 3.14. *Let $T : X \to X$ be a function and let O be an orbit of T. Suppose that S is a spine of O. There is a family of linear orders on the sets $C_{i,k}$ ($i \in N, 0 \neq k \in \mathbb{N}$, where N is $\mathbb{N}$, $\mathbb{Z}$, or $\{0, \ldots, n-1\}, 0 < n \in \mathbb{N}$, according to the nature of the orbit) with respect to which $T \restriction C_{i,k}$ is an OP map.*

Proof. For each $i \in N$, choose any linear order $\preceq_{i1}$ on $C_{i,1} \neq \emptyset$ (where Axiom of Choice implies that any set can be linearly ordered). Clearly, each $T \restriction C_{i,1}$ is OP, since $T(C_{i,1}) = x_{i+1}$.

Now, suppose that for each $j \in N$ and $0 < k' < k$ we have defined a linear order $\preceq_{jk'}$ on $C_{j,k'} \neq \emptyset$ such that $T \restriction C_{j,k'}$ is an OP map. Now we will define a linear order $\preceq_{ik}$ on $C_{i,k}$ as follows. First, for every $x \in C_{i+1,k-1}$, if $|T^{-1}(x)| \neq 0$, choose any linear order $\preceq_x$ on $T^{-1}(x)$. Since

$$C_{i,k} = \bigcup\{T^{-1}(x) : x \in C_{i+1,k-1}\}$$

so we can define $(C_{i,k}, \preceq_{ik})$ to be the ordered sum of $T^{-1}(x)$ over $C_{i+1,k-1}$. It follows immediately from the construction of $\preceq_{ik}$ that each $T \restriction C_{i,k}$ is an OP map. $\square$

Corollary 3.15. *Let $T : X \to X$ be a function and let O be an orbit of T with spine S, where O is either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit or a 1-cycle. There is a family of linear orders on the sets $L_i(O) \neq \emptyset$ ($i \in N$, where N is $\mathbb{N}$, $\mathbb{Z}$ or $\{0\}$ according to the nature of the orbit) with respect to which $T \restriction L_i(O)$ is an OP map.*

Proof. Suppose that O is either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit or a 1-cycle with spine S. By Lemma 3.14, there is a family of linear orders on the sets $C_{i,k}$ ($i \in N, 0 \neq k \in \mathbb{N}$) with respect to which $T \restriction C_{i,k}$ is an OP map. Let $\preceq_{ik}$ denote the order on $C_{i,k}$. Since $L_i(O) = \bigcup_{k\in\mathbb{N}} C_{i,k}$, we can define $\preceq_i$ on $L_i(O)$ to be the sum-order of $\preceq_{ik}$ over $\mathbb{N}$; so $L_i(O)$ is the ordered sum of $C_{i,k}$ over $\mathbb{N}$. It follows immediately that

$$T \restriction L_i(O) : L_i(O) \to L_{i+1}(O)$$

is an OP map under this order. $\square$

Theorem 3.16. *Let $T : X \to X$ be a function and let O be an orbit of T with spine S, where O is either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit or a 1-cycle. Then there is a linear order on O with respect to which $T \restriction O$ is an OP map.*

Proof. Let O be a 1-cycle of T. By Corollary 3.15, since $L_0(O) = X$, then immediately we have $T \restriction O$ is an OP map. If O is either a $\mathbb{Z}$-orbit or an $\mathbb{N}$-orbit, then by Corollary 3.15, there is a family of linear orders $\preceq_i$ on the sets $L_i(O) \neq \emptyset$, $i \in \mathbb{Z}$, such that $T \restriction L_i(O)$ is an OP map. Let O be the ordered sum of $L_i(O)$ over $\mathbb{Z}$, so immediately we have $T \restriction O$ is an OP map. $\square$

Theorem 3.17. *Let X be arbitrary set and $T : X \to X$ be a map with orbit spectrum*

$$\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots).$$

There is a linear order on X with respect to which T is an OP map if and only if $\sigma_n = 0$ for all $n > 1$.

Proof. If $T : X \to X$ is an OP map, then by Lemma 3.3 we have $\sigma_n = 0$ for all $n > 1$. Conversely, let $\{O_i\}_{i \in I}$ be the collection of all orbits of T. By Theorem 3.16, for every $i \in I$, there is a linear order $\preceq_i$ on O_i such that $T \restriction O_i$ is an OP map. Choose a linear order on I and define $\preceq$ on X to be the sum-order of $\preceq_i$ over I; so by Lemma 3.4 we have T is an OP map on X. ◻

3.2 Order-Preserving Maps on the Rational World

In this section we give the necessary and sufficient conditions for bijective, injective or surjective map $T : X \to X$ to be an order-preserving map on X such that X is order-isomorphic to the rationals $\mathbb{Q}$. This study is in terms of the orbit structure of the map concerned.

It is well known by Cantor's Theorem (see for example [27] and [7]) that a countable linearly ordered set X is order-isomorphic to $\mathbb{Q}$ if X is densely ordered and has no end-points (i.e., no least or greatest element). If X, Y are any sets and if X is order-isomorphic to Y we will write $X \approx Y$. It is also well known, again by Cantor's Theorem, that

$$\bigcup_{j \in \mathbb{Z}} (a_j, b_j] \cap \mathbb{Q} \approx \bigcup_{j \in \mathbb{Z}} [a_j, b_j) \cap \mathbb{Q} \approx \bigcup_{i \in M} (c_i, d_i) \cap \mathbb{Q} \approx \mathbb{Q},$$

for any real numbers $a_j < b_j < a_{j+1}, j \in \mathbb{Z}$, $c_i < d_i, i \in M$ and $M \subseteq \mathbb{N}$.

The following is a number of facts about order-preserving maps on $\mathbb{Q}$.

Lemma 3.18. *Let $\{T_i : X_i \to X_i\}_{i \in \mathbb{N}}$ be a collection of OP maps, where $X_i \approx \mathbb{Q}$ for each $i \in \mathbb{N}$ and $X_i \cap X_j = \emptyset$ whenever $i \neq j$. Then the ordered sum of $(X_i, \preceq_i)$ over $\mathbb{N}$ is order-isomorphic to $\mathbb{Q}$ and the map $T : X \to X$ defined as $T \restriction X_i = T_i$ is an OP map.*

Proof. It is clear that $X = \bigcup_{i \in \mathbb{N}} X_i$ is order-isomorphic to $\mathbb{Q}$, since $X_i \approx \mathbb{Q}$ for each $i \in \mathbb{N}$. Also, by Lemma 3.4, we have T is an OP map on $X = \bigcup_{i \in \mathbb{N}} X_i$. $\square$

Lemma 3.19. *Let $T : X \to X$ be an OP map, where $X \approx \mathbb{Q}$. Then for every $x \in X$ we have $|T^{-k}(x)| =$ 0, 1 or ω for each $k \in \mathbb{N}$.*

Proof. Suppose that there is $x \in X$ with $|T^{-k}(x)| > 1$ for some $k \in \mathbb{N}$, then $|T^{-k}(x)| = \omega$ follows immediately from Lemma 3.5 and from the assumption that X is order-isomorphic to $\mathbb{Q}$ so X is densely ordered. $\square$

The proof of the following result follows from Lemma 3.5 and the fact that $\mathbb{Q}$ is densely ordered.

Lemma 3.20. *Let $T : \mathbb{Q} \to \mathbb{Q}$ be an OP map. Then for every $x \in \mathbb{Q}$, if $|T^{-k}(x)| = \omega$ for some $k \in \mathbb{N}$ then $T^{-k}(x)$ is densely ordered.*

3.2.1 Order-Preserving Bijections in the Rational World

In this section we give a characterization of order-preserving bijections on the rational world $\mathbb{Q}$. We start with this Proposition.

Proposition 3.21. *Let T be a bijection on a countably infinite set X and let*

$$\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots).$$

If $0 \neq \zeta < \omega$ then there is no linear order on X such that X is order-isomorphic to $\mathbb{Q}$ and T is an OP bijection.

Proof. This proof follows immediately from Theorem 3.12, Lemma 3.6 and the fact that $\mathbb{Q}$ is densely ordered. $\square$

Theorem 3.22. *Let X be a countably infinite set and let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \ldots)$ be a sequence of cardinals with $\zeta + \sum_{n \in \mathbb{N}} \sigma_n = \omega$. The canonical representation $T : X \to X$ of σ is an OP map with $X \approx \mathbb{Q}$ if and only if $\sigma_n = 0$ for all $n > 1$ and ζ is either 0 or ω.*

Proof. Suppose that $T : X \to X$ is an OP bijection and X is order-isomorphic to $\mathbb{Q}$, then by Lemma 3.3 we have $\sigma_n = 0$ for all $n > 1$, and by Proposition 3.21 we have ζ is either 0 or ω.

Conversely, suppose that $\sigma_n = 0$ for all $n > 1$ and ζ is either ω or 0, so we have the following cases:

Case (1): $\sigma_1 = \omega$ and $\zeta = 0$. This case is obvious since the identity map on any ordered set is OP.

Case (2): $\zeta = \omega$ and $\sigma_1 = 0$. Let $T : \mathbb{Q} \to \mathbb{Q}$ be the map defined as $T(x) = x + 1$. Clearly, T has, in total, infinitely many $\mathbb{Z}$-orbits, so $\sigma(T) = \sigma$, also T is an OP bijection.

Case (3): $\zeta = \omega$ and $0 < \sigma_1 < \omega$. By Lemma 3.18, we can assume that $\sigma_1 = 1$. Let $X = (0, 2) \cap \mathbb{Q}$. By Case (2), there are an OP bijection T_1 on $I_1 = (0, 1) \cap \mathbb{Q}$ with $\zeta = \omega$ and an OP bijection T_2 on $I_2 = (1, 2) \cap \mathbb{Q}$ with $\zeta = \omega$. So, let $T : X \to X$ be the bijection defined as : $T \restriction I_1 = T_1$, $T \restriction I_2 = T_2$ and $T(1) = 1$. Clearly, T has infinitely many $\mathbb{Z}$-orbits and one fixed point and T is an OP bijection.

Case (4): $\zeta = \omega$ and $\sigma_1 = \omega$. This case follows from Case (1), Case (2) and Lemma 3.18.

Therefore, if $\sigma_n = 0$ for all $n > 1$ and ζ is either ω or 0, then the canonical representation of σ is an OP bijection on a set X with X is order-isomorphic to $\mathbb{Q}$. $\square$

3.2.2 Order-Preserving Injections in the Rational World

In this section we study the situation when $T : X \to X$ is an injection, in this case the map T may have a number of $\mathbb{N}$-orbits.

Theorem 3.23. *Let* $\sigma = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$ *be a sequence of cardinals with* $\nu + \zeta + \sum_{n\in\mathbb{N}} \sigma_n = \omega$*. Then the canonical representation* $T : X \to X$ *of* σ *on the countable set* X *is an OP injection with* X *is order-isomorphic to* $\mathbb{Q}$ *if and only if* $\sigma_n = 0$ *for all* $n > 1$ *and* $\zeta + \nu$ *is either* ω *or 0.*

Proof. Suppose that X is order-isomorphic to $\mathbb{Q}$ and T is OP, then by Lemma 3.3 we have $\sigma_n = 0$ for all $n > 1$. By Theorem 3.12 and Lemma 3.6 we have $\zeta + \nu$ is either ω or 0.

Conversely, suppose $\sigma_n = 0$ for all $n > 1$ and $\zeta + \nu$ is either ω or 0. By Theorem 3.22 for bijections and Lemma 3.18, it is sufficient to consider the following cases:

Case (1): $\nu = \omega$, $\zeta + \sigma_1 = 0$. Simply, let $X = (0, \infty) \cap \mathbb{Q}$ and let $T : X \to X$ be the map defined by $T(x) = x + 1$. Obviously, T has only infinitely many $\mathbb{N}$-orbits and T is an OP injection.

Case (2): $\nu = \omega$, $\sigma_1 = 0$ and $0 < \zeta = k < \omega$. Let $f : \mathbb{Q} \to \mathbb{Q}$ be the function defined by $f(x) = x + 1$. List the elements in the set $[0, 1) \cap \mathbb{Q}$ as $\{q_i : 0 \neq i \in \mathbb{N}\}$ and then let $O_i = \bigcup_{m\in\mathbb{Z}} T^m(q_i)$. Let $A = \{1, \ldots, k\}$ and for each $i > 0$ let

$$P_i = \begin{cases} O_i & \text{if } \; i \in A, \\ O_i \cap [-i, \infty) & \text{if } \; i \notin A. \end{cases}$$

Let $X_k = \bigcup_{i>0} P_i$. Since

$$X_k \cap [-i, \infty) = (\mathbb{Q} \cap [-i, \infty)) \setminus F,$$

where $F \subset \mathbb{Q}$ is a non-empty finite set, then X_k is order-isomorphic to $\mathbb{Q}$. Let $T = f \restriction X_k$ then T is an OP injection with $\nu = \omega$ and $\zeta = k$.

Case (3): $\zeta = \omega$, $\sigma_1 = 0$ and $0 < \nu = n < \omega$. Let $T : \mathbb{Q} \to \mathbb{Q}$ be the function defined by $T(x) = x + 1$. Choose rational numbers $q_1, q_2, \ldots, q_n \in (0, 1) \cap \mathbb{Q}$ and let

$$I = \{x \in \mathbb{Q} : 0 < x < 1, x \neq q_i, 1 \leq i \leq n\}.$$

Let

$$Y = \bigcup_{i=1}^{n} [\bigcup_{k>0} T^{-k}(q_i)].$$

Let $X = \mathbb{Q} \setminus Y$, it follows that $X \approx \mathbb{Q}$. Hence, $T \restriction X$ is an OP injection having infinitely many $\mathbb{Z}$-orbits and n $\mathbb{N}$-orbits, as required.

Case (4): $\nu + \zeta = \omega$ and $0 < \sigma_1 < \omega$. By Lemma 3.18, we can assume that $\sigma_1 = 1$. Let $X = (0, 2) \cap \mathbb{Q}$. By Case (1), Case (2), Case (3) and Theorem 3.22, there are OP injections T_1 on $I_1 = (0, 1) \cap \mathbb{Q}$ and T_2 on $I_2 = (1, 2) \cap \mathbb{Q}$ each has, in total, infinitely many $\mathbb{N}$-orbits and $\mathbb{Z}$-orbits. So, let $T : X \to X$ be the injection defined as : $T \restriction I_1 = T_1$, $T \restriction I_2 = T_2$ and $T(1) = 1$. Clearly, $\nu + \zeta = \omega$, $\sigma_1 = 1$ and T is an OP injection. $\square$

3.2.3 Characterizing Order-Preserving Surjections in the Rational World

This section is devoted to giving a characterization of order-preserving surjections on the rational world in terms of the orbit structure of the map concerned. First, we give some terminologies that we will use during this section.

Let $T : X \to X$ be a surjection having a $\mathbb{Z}$-orbit or a 1-cycle O with spine S indexed as $\{x_i : i \in M\}$, where M is either $\mathbb{Z}$ or $\{0\}$ as appropriate. We use similar terminology as in Figure 3.1 and Figure 3.2, but we need to avoid $L_i(O)$ having an endpoint, so we split each C_{ik} into two disjoint infinite subsets. So, for each $x_i \in S$, let $C'_{i,0} = \{x_i\}$ and for each $x_i \in S$ with $|T^{-1}(x_i)| = \omega$, write

$$T^{-1}(x_i) = C'_{i-1,1} \cup \{x_{i-1}\} \cup C'_{i-1,-1},$$

where $C'_{i-1,1}$ and $C'_{i-1,-1}$ are disjoint infinite subsets of $T^{-1}(x_i)$ and $i-1$ is taken module 1 when O is a 1-cycle. Now, for each $k \in \mathbb{Z}$, let $C'_{i,k} = T^{-1}(C'_{i+1,k-1})$ if $k \geq 0$ and $C'_{i,k} = T^{-1}(C'_{i+1,k+1})$ if $k < 0$.

Again for each $i \in \mathbb{Z}$ notice that $L_i(O) = \bigcup_{k\in\mathbb{Z}} C'_{i,k}$, where $L_i(O)$ is defined in Definition 1.7. These terminologies are illustrated in the following figures.

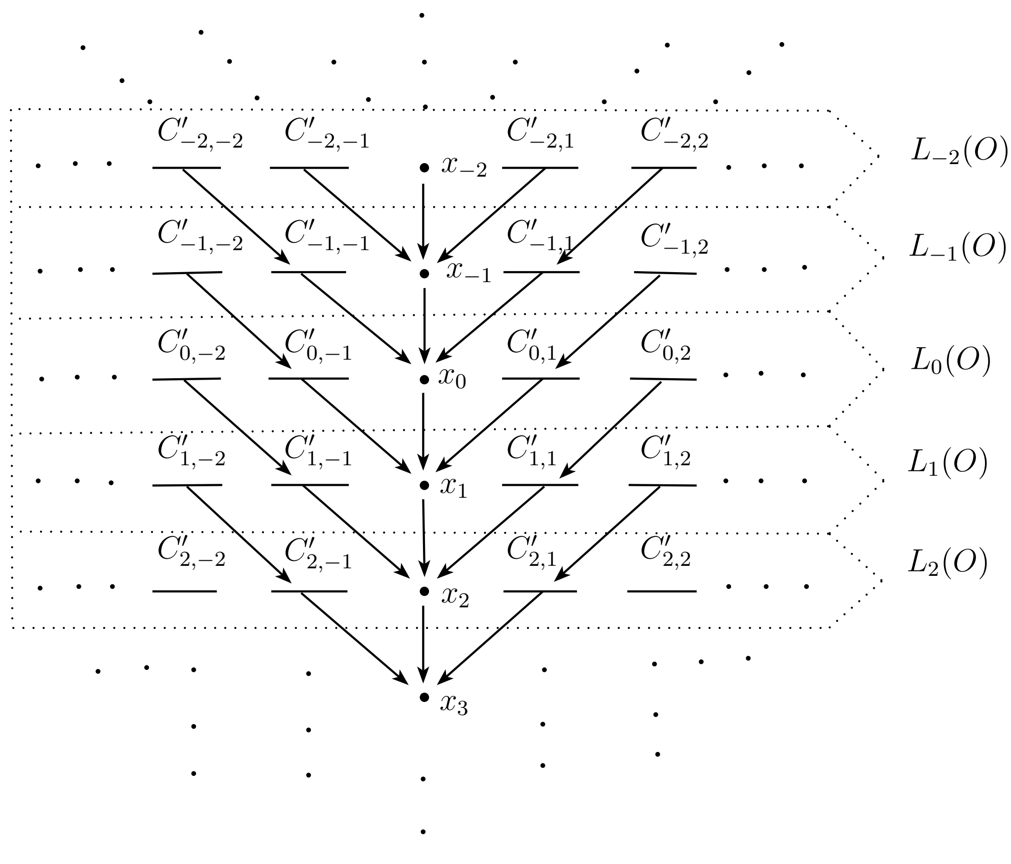

Figure 3.3: A $\mathbb{Z}$-orbit O.

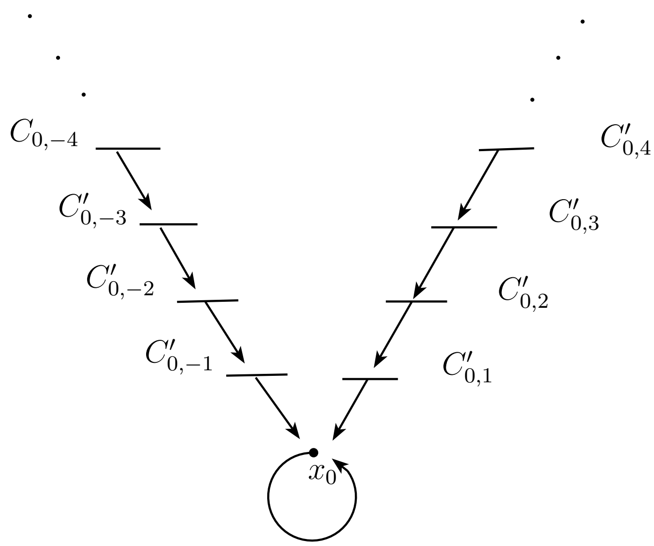

Figure 3.4: A 1-cycle.

Necessary Conditions

Lemma 3.24. *Let $T : X \to X$ be an OP surjection, where X is order-isomorphic to $\mathbb{Q}$. Then for every $x \in X$, we have $|T^{-k}(x)| = 1$ or ω for each $k \in \mathbb{N}$.*

Proof. The proof follows immediately from Lemma 3.19 and the assumption that T is a surjection. □

Let $T : X \to X$ be a surjection and let O be a $\mathbb{Z}$-orbit of T. We will consider the following condition for O :

(*) for all $x \in O$ there is $i \in \mathbb{N}$ such that $|T^{-1}(T^i(x))| = \omega$

Notice that if O is a $\mathbb{Z}$-orbit with spine S and O has (*), then $|L_i(O)| = \omega$ for each $i \in \mathbb{Z}$. Now we have the following result.

Theorem 3.25. *Let $T : X \to X$ be a surjection on the countable set X consisting of a single $\mathbb{Z}$-orbit O. Suppose that $|T^{-1}(x)| = 1$ or ω for all $x \in X$. If O does not satisfy the* *condition (*) then there is no linear order on X such that X is order-isomorphic to $\mathbb{Q}$ and T is an OP map.*

Proof. Suppose that O does not have (*) and suppose, for a contradiction, that there exists a linear order $\preceq$ on X so that $X \approx \mathbb{Q}$ and T is an OP map. Suppose that $x \preceq T(x)$ for each $x \in O$. Since O does not satisfy (*), then there exists $x_0 \in X$ such that $|T^{-1}(y)| = 1$ for each $y \in \bigcup_{k \in \mathbb{N}} T^k(x_0)$. By Theorem 3.7, if $x_0 \prec t \prec T(x_0)$ for some $t \in X$, then either $T^n(t) = T^n(x_0)$ or $T^n(t) = T^{n+1}(x_0)$ for some $n \in \mathbb{N}$. But if $T^n(t) = T^n(x_0)$ then $t \in T^{-n}(T^n(x_0))$ which means that there is an integer $r \in (0, n]$ with $|T^{-1}(T^r(x_0))| > 1$, which is a contradiction. Similarly, if $T^n(t) = T^{n+1}(x_0)$ then there is an integer $r \in (1, n+1]$ with $|T^{-1}(T^r(x_0))| > 1$, which again is a contradiction. Hence, there is no $y \in X$ with $x_0 \prec y \prec T(x_0)$ so X is not densely ordered, which is a contradiction of being $X \approx \mathbb{Q}$. The case when $z \succeq T(z)$ follows in the same way. Hence, there is no linear order on X such that X is order-isomorphic to $\mathbb{Q}$ and T is an OP map. $\square$

Theorem 3.26. *Let T be a surjection on a countably infinite set X and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$. Suppose that $\sigma_n = 0$ for all $n > 1$ and suppose that $|T^{-1}(x)| = 1$ or ω for all $x \in X$. If $\zeta < \omega$ and if the number of orbits which satisfy condition (*) is less than the number of other $\mathbb{Z}$-orbits then there is no linear order on X such that $X \approx \mathbb{Q}$ and T is an OP map.*

Proof. Let us first assume that $\sigma_1 = 0$. Assume that T has, in total, three orbits O_1, O_2 and O_3 such that O_1 satisfies (*) and both O_2 and O_3 do not have (*). Index the spine of O_m as $S_m = \{x_{m,n} : n \in \mathbb{Z}\}$ and for each $i \in \mathbb{Z}$, let $L_i(O_m)$ be defined. Since $O_j, j = 2, 3$ do not have (*), then there is $n_j \in \mathbb{Z}$ such that $T^{-1}(T(x_{j,r})) = x_{j,r}$ for each $r > n_j$. Suppose, for a contradiction, that there is a linear order $\preceq$ that makes $X \approx \mathbb{Q}$ and T an OP map. By Theorem 3.25 we cannot have $O_1 \preceq O_2$ nor $O_2 \preceq O_1$; so by Corollary 3.11, we have $L_r(O_1) \preceq L_{k+r}(O_2) \preceq L_{r+1}(O_1)$ for some $k \in \mathbb{Z}$ and for all $r \in \mathbb{Z}$. Without loss of generality, let $k = 0$. Since O_2 does not have (*) then we have $L_p(O_2) = x_{2,p}$ for all $p > n_2$, so $L_r(O_1) \preceq x_{2,r} \preceq L_{r+1}(O_1)$ for all $r > n_2$. Also, for O_3 we cannot have $O_1 \preceq O_3$ nor $O_3 \preceq O_1$, so by Corollary 3.11, we have $L_r(O_1) \preceq L_{k'+r}(O_3) \preceq L_{r+1}(O_1)$ for some $k' \in \mathbb{Z}$ and for all $r \in \mathbb{Z}$. Again let $k' = 0$, then we have $L_r(O_1) \preceq x_{3,r} \preceq L_{r+1}(O_1)$ for all $r > n_3$. But if $n' > \max\{n_2, n_3\}$, then we have either $L_r(O_1) \preceq x_{2,r} \preceq x_{3,r} \preceq L_{r+1}(O_1)$ or $L_r(O_1) \preceq x_{3,r} \preceq x_{2,r} \preceq L_{r+1}(O_1)$ for all $r > n'$; so in both of the cases, there is no $z \in X$ with $x_{2,r} \prec z \prec x_{3,r}$, which is a contradiction of being $\mathbb{Q}$ densely ordered.

Now, if T has, in addition, q $\mathbb{Z}$-orbits $O'_1, \ldots, O'_q$, that do not have (*). Index a spine of O'_m as $S_m = \{x'_{m,n} : n \in \mathbb{Z}\}$, $1 \leq m \leq q$, since O'_m does not have (*), then there is $n'_m \in \mathbb{Z}$ such that $T^{-1}(T(x'_{m,r})) = x'_{m,r}$ for each $r > n'_m$. Similarly, following the same steps as above, for all $1 \leq m \leq q$, we will have $L_r(O_1) \preceq x'_{m,r} \preceq L_{r+1}(O_1)$ for all $r > n'_m$. So at some stage, there is $m' \in \mathbb{Z}$ such that no element lie between any x, y, where $x, y \in \{x_{2,m'}, x_{3,m'}, x'_{1,m'}, \ldots, x'_{q,m'}\}$, a contradiction. Therefore, for the orbit O_1 which have (*), there is a unique orbit O which do not have (*) so that O, O_1 satisfy (2) in Corollary 3.11 and any other orbit O' without condition (*) must be either $O' \preceq O_1$ or $O_1 \preceq O'$.

Now, suppose that T has n $\mathbb{Z}$-orbits and let $\{O_1, O_2, \ldots, O_n\}$ be the collection of all orbits of T. Suppose that $O_1, \ldots, O_m$ have condition (*) and let $k = n - m$, so we have $m < k$. Suppose, for a contradiction, that there is a linear order $\preceq$ that makes $X \approx \mathbb{Q}$ and T an OP map. Then, as we showed above, for every $O_j, 1 \leq j \leq m$ either $O_j \preceq O_p$ or $O_p \preceq O_j$ for all $p \in \{m+1, \ldots, n\}$ except for a unique orbit $O_p, p \in \{m+1, \ldots, n\}$ such that O_j, O_p satisfy (2) in Corollary 3.11. Since $k > m$, there is O_q, $q \in \{m+1, \ldots, n\}$ with either $O_q \preceq O_r$ or $O_r \preceq O_q$ for all $r \in \{1, \ldots, m\}$; so by Theorem 3.25, we have O_q is not densely ordered and then X is also not dense, which is a contradiction. Hence, there is no linear order on X such that $X \approx \mathbb{Q}$ and T is an OP map.

Finally, if $\sigma_1 \neq 0$, so T has, in addition, 1-cycles, then the proof follows from what we proved above and from Lemma 3.6. $\square$

Theorem 3.27. *Let T be a surjection on a countable set X with, in total, finitely many orbits. Suppose that $\sigma_n = 0$ for all $n > 1$ and that $|T^{-1}(x)| = 1$ or ω for all $x \in X$. If the number of simple 1-cycles is greater than or equal to the number of other orbits except orbits which do not have (*), then there is no linear order on X that makes X order-isomorphic to $\mathbb{Q}$ and T an OP map.*

Proof. Suppose, for a contradiction, that there is a linear order on X that makes X order-isomorphic to $\mathbb{Q}$ and T an OP map. Let T have n orbits with k simple 1-cycles and m $\mathbb{Z}$-orbits that do not satisfy (*); so by Theorem 3.25 and Theorem 3.26 there are at most $l = n - (k+m)$ orbits, $O_i, 1 \leq i \leq l$, each is homeomorphic to $\mathbb{Q}$, so $O_i \approx (i, i+1) \cap \mathbb{Q}$, and $T \restriction O_i$ is OP. By Lemma 3.6, if $\{x_0\}$ is a simple 1-cycle and if $x \preceq x_0 \preceq y$ for some $x, y \in X$ then x, y cannot lie in the same orbit. Hence, since $O_i \approx (i, i+1) \cap \mathbb{Q}$ for at most l orbits $O_i, 1 \leq i \leq l$, then there can only be one fixed point between any two intervals (orbits). But $k > l$, so we will have either endpoints of X or X is not densely ordered, which is a contradiction. $\square$

Ordering Finitely Many Orbits

In this subsection we show how to order a set X such that a surjective map $T : X \to X$ with finitely many orbits is order-preserving and X is order-isomorphic to $\mathbb{Q}$.

Lemma 3.28. *Let T be a surjection on the countably infinite set X and let O be either a $\mathbb{Z}$-orbit or an n-cycle of T with spine S. Suppose that for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω. There is a family of linear orders on the sets $C'_{i,k}$ ($i \in N, 0 \neq k \in \mathbb{Z}$, where N is $\mathbb{Z}$ or $\{0, \ldots, n-1\}, 0 < n \in \mathbb{N}$, according to the nature of the orbit) with respect to which $T \restriction C'_{i,k}$ is an OP map and $C'_{i,k} \approx \mathbb{Q}$ when $C'_{i,k} \neq \emptyset$.*

Proof. Let S be indexed as $\{x_m : m \in N\}$ so that $T(x_m) = x_{m+1}$ for all $m \in N$. For each $i \in N$ with $C'_{i,1} \neq \emptyset$, choose a bijection $f_i : C'_{i,1} \to (0,1) \cap \mathbb{Q}$ and define an order $\preceq_{i1}$ on $C'_{i,1}(O)$ as follows: for each $x, y \in C'_{i,1}(O)$ put $x \preceq_{i1} y$ iff $f_i(x) \leq f_i(y)$, so $C'_{i,1} \approx \mathbb{Q}$. Clearly, each $T \restriction C'_{i,1}(O)$ is OP since $T(C'_{i,1}(O)) = x_{i+1}$.

Now, we deal with $C'_{j,k}$ with $k > 0$, the case when $k < 0$ follows in the same way. Suppose that for each $j \in N$ and $0 < k' < k$ we have defined a linear order $\preceq_{jk'}$ on $C'_{j,k'} \neq \emptyset$ such that $C'_{j,k'} \approx \mathbb{Q}$ and $T \restriction C'_{j,k'}$ is OP. Now we will define a linear order $\preceq_{i,k}$ on $C'_{i,k}$ as follows. First, for each $x \in C'_{i+1,k-1}$, if $|T^{-1}(x)| = \omega$ define a linear order $\preceq_x$ on $T^{-1}(x)$ such that $T^{-1}(x) \approx (0,1) \cap \mathbb{Q}$. If $|T^{-1}(x)| = 1$, let $\preceq_x$ be the unique order on $T^{-1}(x)$. Now, $C_{i,k} = \bigcup\{T^{-1}(x) : x \in C_{i+1,k-1}\}$ so we can define $(C_{i,k}, \preceq_{ik})$ to be the ordered sum of $T^{-1}(x)$ over $C_{i+1,k-1}$.

Now, we will prove that $C'_{i,k} \approx \mathbb{Q}$. Let $x, y \in C'_{i,k}$ with $x \prec_{ik} y$, if $x, y \in T^{-1}(z)$ for some $z \in C'_{i+1,k-1}$, then the proof is clear since $T^{-1}(z) \approx \mathbb{Q}$. If $x \in T^{-1}(y_1)$ and $y \in T^{-1}(y_2)$, $y_1 \neq y_2$ then there is $z \in C'_{i+1,k-1}$ such that $y_1 \prec_{i+1,k-1} z \prec_{i+1,k-1} y_2$, since $C'_{i+1,k-1}$ is densely ordered. From the definition of $\preceq_{ik}$ and since T is a surjection we have $x \prec_{ik} T^{-1}(z) \prec_{ik} y$, so $C'_{i,k}$ is densely ordered. Finally, since $C'_{i+1,k-1}$ has no endpoints and from the construction of $\preceq_{ik}$, it follows immediately that $C'_{i,k}$ has no endpoints and that each $T \restriction C'_{i,k}$ is OP. $\square$

The following result follows immediately from Lemma 3.28.

Corollary 3.29. *Let T be a surjection on the countably infinite set X and let O be either a $\mathbb{Z}$-orbit or a 1-cycle of T with spine S. Suppose that for every $x \in X$, $|T^{-1}(x)|$ is 1 or ω. For each $i \in M$ (where M is $\mathbb{Z}$ or $\{0\}$ according to the nature of the orbit), if $(L_i(O), \preceq_i)$ is the ordered sum of $(C'_{i,k} \preceq_{ik})$ over $\mathbb{Z}$, then $L_i(O) \approx \mathbb{Q}$ when $|L_i(O)| = \omega$. Moreover, $T \restriction L_i(O)$ is OP with respect to this family of linear orders on the sets $L_i(O)$, $i \in M$.*

Corollary 3.30. *Let X be a countable set and $T : X \to X$ be a surjection with a single*

1-cycle. Suppose that for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω. Then there is a linear order on X such that X is order-isomorphic to $\mathbb{Q}$ and T is an OP map.

Proof. The proof follows immediately by Corollary 3.29 and the fact that $X = L_0(O)$ for any 1-cycle O. □

The following result shows how to order a collection of finitely many $\mathbb{Z}$-orbits.

Theorem 3.31. *Let $T : X \to X$ be a surjection on the countable set X and let T have, in total, finitely many $\mathbb{Z}$-orbits. Suppose that for any $x \in X$, $|T^{-1}(x)|$ is either 1 or ω. If the number of $\mathbb{Z}$-orbits which satisfy condition (*) is no less than the number of other $\mathbb{Z}$-orbits, then there is a linear order on X with respect to which T is OP and $X \approx \mathbb{Q}$.*

Proof. Suppose that T has, in total, finitely many $\mathbb{Z}$-orbits. Without loss of generality, by Lemma 3.18, we might assume that T has, in total, two orbits, O_1 and O_2 such that O_1 satisfies (*) and O_2 does not have (*). Index a spine of $O_r, r = 1, 2$ as $S_r = \{x_{r,n} : n \in \mathbb{Z}\}$ so that $T(x_{r,n}) = x_{r,n+1}$ for each $n \in \mathbb{Z}$. By Corollary 3.29, since for each $i \in \mathbb{Z}$ we have $|L_i(O_1)| = \omega$, then there is a family of linear orders on the sets $L_i(O_1), i \in \mathbb{Z}$ such that $L_i(O_1) \approx \mathbb{Q}$ and $T \restriction L_i(O_1)$ is OP. Let $(O_1, \preceq_1)$ be the ordered sum of $(L_i(O), \preceq_i)$ over $\mathbb{Z}$. Since O_2 does not have (*), so there is $x_{2,m} \in S$ such that $|L_n(O_2)| = \omega$ for all $n < m$ and $L_n(O_2) = \{x_{2,n}\}$ for all $n \geq m$. So, again by Corollary 3.29, since for each $n < m$ we have $|L_n(O_1)| = \omega$, then there is a family of linear orders on $L_n(O_1)$, $n \in \mathbb{Z}$ such that $L_i(O_1) \approx \mathbb{Q}$ for $i < m$ and $T \restriction L_n(O_1)$, $n \in \mathbb{Z}$ is an OP map. Let $(O_2, \preceq_2)$ be the ordered sum of $(L_i(O_2), \preceq_i)$ over $\mathbb{Z}$; so we have $T \restriction O_2$ is OP.

Now, let $\preceq$ be the linear order defined on X as follows: for all $x, y \in X$ with $x \in L_i(O_r)$, $y \in L_j(O_p)$, $i, j \in \mathbb{Z}$ and $r, p \in \{1, 2\}$, then

$$x \preceq y \Leftrightarrow (r = p \text{ and } x \preceq_r y) \text{ or } (r \neq p \text{ and } i < j) \text{ or } (r = 2, p = 1 \text{ and } i = j).$$

Now we will show that X is order-isomorphic to $\mathbb{Q}$. Since for each $i \in \mathbb{Z}$ then $L_i(O_1) \approx \mathbb{Q}$ so there is an order isomorphism $h_i : L_i(O_1) \to (2i, 2i+1) \cap \mathbb{Q}$. Define $h : X \to \mathbb{Q}$ such that $h \restriction L_i(O_1) = h_i$, $h : L_j(O_2) \mapsto (2j-1, 2j) \cap \mathbb{Q}$ if $j < m$ and $h : L_j(O_2) \mapsto 2j-1$ if $j \geq m$. Clearly, h is an order isomorphism from X to a dense subset of $\mathbb{Q}$ with no endpoints.

Finally, we will prove that T is an OP map under this order. If $x \preceq y$ and x, y are in the same orbit then as we showed above $T \restriction O_r, r = 1, 2$ is an OP map. If $x \preceq y$, $x \in L_l(O_r)$, $y \in L_k(O_p)$, $l, k \in \mathbb{Z}$ and $r, p \in \{1, 2\}$ then we have two cases: if $l < k$ then $l + 1 \leq k + 1$ which implies that $T(x) \preceq T(y)$, since $T(x) \in L_{l+1}(O_r)$, $T(y) \in L_{k+1}(O_p)$. The other case is $r = 2$, $p = 1$ and $l = k$, so $l + 1 = k + 1$ and $T(x) \preceq T(y)$, since $T(x) \in L_{l+1}(O_2)$ and $T(y) \in L_{k+1}(O_1)$. Hence, T is an OP map. $\square$

Proof of the Main Theorem of Order-Preserving Surjections on $\mathbb{Q}$

In this subsection, we prove the main theorem of this section, Theorem 3.33, which describe the orbit structure of order-preserving surjections on $\mathbb{Q}$. We start with the following proposition.

Proposition 3.32. *Let X be a countably infinite set and $T : X \to X$ be a surjection. Suppose that for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω. There is a linear order on X with respect to which $X \approx \mathbb{Q}$ and T is an OP map in each of the following cases:*

(1) *T has, in total, infinitely many 1-cycles; or*

(2) *T has, in total, infinitely many $\mathbb{Z}$-orbits.*

Proof. Let $O = \{O_l\}_{l \in \mathbb{N}}$ be the collection of all orbits of T. For each $l \in \mathbb{N}$, index a spine of O_l as $S_l = \{x_{l,n} : n \in M\}$, where M is either $\mathbb{Z}$ or $\{0\}$, and let $S = \bigcup_{l \in \mathbb{N}} S_l$. Since $T \restriction S$ is a bijection, then by Theorem 3.22, there is a linear order $\preceq_s$ on S such that $S \approx \mathbb{Q}$ and $T \restriction S$ is an OP bijection.

Suppose that there is at least one orbit which is not simple. For each $l \in \mathbb{N}$ and $i \in M$, let $L_i(O_l)$ be defined, so by Corollary 3.29 there is a linear order $\preceq_{li}$ on $L_i(O_l)$ with $|L_i(O_l)| = \omega$ such that $L_i(O_l) \approx \mathbb{Q}$ and $T \restriction L_i(O_l)$ is OP. For every $l \in \mathbb{N}$ and $i \in M$, write $L_i(O_l)$ as $L(x_{l,i})$, where $L_i(O_l) \ni x_{l,i}$. Now, let $(X, \preceq)$ be the ordered sum of $L(x_{l,i})$ over S. Since S and each $L_k(O_l)$ with $|L_k(O_l)| = \omega$ is order-isomorphic to $\mathbb{Q}$ and from the construction of $\preceq$, it follows that X is order-isomorphic to $\mathbb{Q}$ and T is OP, as was required. $\square$

Let $T : X \to X$ be a surjection with orbit spectrum $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. Recall that a $\mathbb{Z}$-orbit O of T has (*) property if:

(*) for all $x \in O$ there is $i \in \mathbb{N}$ such that $|T^{-1}(T^i(x))| = \omega$.

If $\zeta < \omega$, let ζ_1 be the number of $\mathbb{Z}$-orbits that have the condition (*) and $\zeta_2 = \zeta - \zeta_1$. If $\sigma_1 < \omega$, let $\sigma_1 = \sigma_1' + \sigma_1''$, where σ_1' is the number of simple 1-cycles. Using these terminologies we have the following theorem, the main theorem of this section.

Theorem 3.33. *Let X be a countably infinite set and $T : X \to X$ be a surjection with orbit spectrum $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. There is a linear order on X with respect to which X is order-isomorphic to $\mathbb{Q}$ and T is an OP map if and only if $\sigma_n = 0$ for all $n > 1$, for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω and one of the following holds:*

(1) *$\zeta = \omega$; or*

(2) *$\zeta < \omega$ with the property that $\zeta_2 \leq \zeta_1$ and either:*

   (i) *$\sigma_1 = \omega$; or*

   (ii) *$\sigma_1 < \omega$ and $\sigma_1' < \zeta_1 + \sigma_1''$.*

Proof. Suppose that X is order isomorphic to $\mathbb{Q}$ and T is an OP surjection, then by Lemma 3.3 we have $\sigma_n = 0$ for all $n > 1$. Lemma 3.24 implies that for every $x \in X$,

$|T^{-1}(x)|$ is either 1 or ω. If (1) is false, so $\zeta < \omega$, then by Theorem 3.26 we have $\zeta_2 \leq \zeta_1$. If $\sigma_1 < \omega$ then by Theorem 3.27 we have $\sigma_1' < \zeta_1 + \sigma_1''$.

Conversely, suppose that condition (1) holds, so $\zeta = \omega$. If $\sigma_1 = 0$ then the proof follows by Proposition 3.32 and if $\sigma_1 = \omega$, the proof follows from Proposition 3.32 and Lemma 3.18. So, we are left with the following cases to consider:

(a) $0 \neq \sigma_1'' < \omega$ and $\sigma_1' = 0$.

(b) $0 \neq \sigma_1' < \omega$ and $\sigma_1'' = 0$.

(c) $0 \neq \sigma_1' < \omega$ and $0 \neq \sigma_1'' < \omega$.

Case (a) follows from Proposition 3.32, Corollary 3.30 and Lemma 3.18. We now deal with Case (b), let $\{x_i : 1 \leq i \leq n\}$ be the set of all fixed points of T and let $\{O_l : l \in \mathbb{N}\}$ be the collection of all $\mathbb{Z}$-orbits of T. For each $0 \leq j \leq n$ define $F_j = \bigcup_{l \in \mathbb{N}} O_{(n+1)l+j}$; so

$$X = \bigcup_{j=0}^{n} F_j \cup \{x_1, \ldots, x_n\}.$$

Now by Proposition 3.32, for each $0 \leq j \leq n$ there is a linear order on F_j such that $T \restriction F_j$ is OP and $F_j \approx \mathbb{Q}$; so there is an order isomorphism $h_j : F_j \to (j, j+1) \cap \mathbb{Q}$. Let h be the map defined by

$$h : X \to (0, n+1) \cap \mathbb{Q}$$

such that $h \restriction F_j = h_j$ for each $0 \leq j \leq n$, and $x_i \mapsto i$ for each $1 \leq i \leq n$, so h is an order isomorphism and T is OP under this order. Case (c) follows immediately from Case (a), Case (b) and Lemma 3.18.

Now, suppose that condition (2) holds. If (i) holds then the proof follows from Theorem 3.31, Proposition 3.32 and Lemma 3.18. Suppose that (ii) holds. Let $\{x_n : 0 < n \leq \sigma_1'\}$ be the set of all fixed points. Let $C_1, C_2, \ldots, C_{\zeta_2}$ be the $\mathbb{Z}$-orbits that do not satisfy (*) and $C_1', \ldots, C_{\zeta_2}', \ldots, C_{\zeta_1}'$ be the $\mathbb{Z}$-orbits that satisfy (*). For each $1 \leq m \leq \zeta_2$, let

$O_m = C_m \cup C'_m$ and for each $\zeta_2 < k \leq \zeta_1$, let $O_k = C'_k$. Then, by Theorem 3.31, for each $0 < l \leq \zeta_1$ there is a linear order on O_l such that O_l is order-isomorphic to $\mathbb{Q}$ and $T \restriction O_l$ is OP. Also, if $\{O'_m : 0 < m \leq \sigma''_1\}$ is the collection of all 1-cycles of T which is non-simple, then by Corollary 3.30, for each $0 < m \leq \sigma''_1$ there is a linear order on O'_m such that $O'_m \approx \mathbb{Q}$ and $T \restriction O'_m$ is OP. So, for each $0 < l \leq \zeta_1$ and $0 < m \leq \sigma''_1$, let $f_l : O_l \to I_l$ and $g_m : O'_m \to I_{\zeta_1+m}$ be these order isomorphisms, where I_k is the set $(k, k+1) \cap \mathbb{Q}$, and define an order isomorphism:

$$f : X \to [\{i : 1 < i \leq \sigma'_1 + 1\} \cup \bigcup_{r=1}^{\zeta_1+\sigma''_1} I_r]$$

such that $f \restriction O_l = f_l$, $f \restriction O'_m = g_m$ and $f : x_n \mapsto n+1$ for each $0 < n \leq \sigma'_1$, so $X \approx \mathbb{Q}$. Since for each $0 < l \leq \zeta_1$ and $0 < m \leq \sigma''_1$, $T \restriction O_l$ and $T \restriction O'_m$ are OP, then T is OP under this order, as required. $\square$

3.2.4 Examples of the General Case

In this section we consider $T : X \to X$ to be any map rather than injections or surjections and give examples of some structures of self-maps which cannot be order-preserving on $\mathbb{Q}$ as well as examples of some order-preserving maps on $\mathbb{Q}$.

Example 3.34. Let T be an arbitrary function on a countably infinite set X. Suppose that for each $x \in X$, $|T^{-1}(x)|$ is either 0, 1 or ω. Let $y \in X$ and $z \in \bigcup_{i \in \mathbb{N}} T^{-i}(y)$. If $T^{-n}(z) \subseteq T^{-m}(y)$ for some $n < m \in \mathbb{N}$ and if $2 < |T^{-m}(y) - T^{-n}(z)| < \omega$, then there is no linear order on X with respect to which X is order-isomorphic to $\mathbb{Q}$ and T is OP.

Proof. Suppose, for a contradiction, that there is a linear order $\preceq$ on X that makes X order-isomorphic to $\mathbb{Q}$ and T an OP map. Let $T^{-m}(y) - T^{-n}(z) = \{x_0, x_1, \ldots, x_r\}$, $r > 1$ with $x_i \preceq x_{i+1}$ for all $i \in \{0, \ldots, r-1\}$. By Lemma 3.5, if $x_0 \prec t \prec x_1$ for some $t \in X$ then $t \in T^{-m}(y)$ so $t \in T^{-n}(z)$; hence, by Lemma 3.5, we have $x_0 \prec T^{-n}(z) \prec x_1$. But this means that there is no $t' \in T^{-m}(y)$ which satisfies that $x_1 \prec t' \prec x_2$, again by Lemma 3.5, which is a contradiction of being X is densely ordered. Consequently, there is no linear order on X with respect to which $X \approx \mathbb{Q}$ and T is an OP map. $\square$

Example 3.35. Let $T : X \to X$ be a function on the countable set X. Let $x_0 \in X$ and let $T^{-n}(x_0)$, $n = 1, 2, 3$ be defined as in the following figure, where $|T^{-1}(x_0)| = |T^{-1}(x_2)| = |T^{-1}(x_4)| = \omega$.

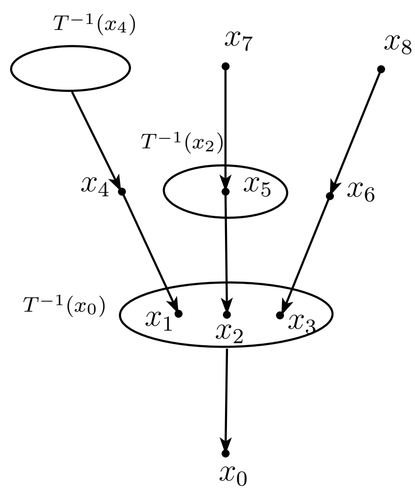

Figure 3.5: $T^{-n}(x_0)$, $n = 1, 2, 3$.

Then there is no linear order on X with respect to which $X \approx \mathbb{Q}$ and T is an OP map.

Proof. Suppose, for a contradiction, that there is a linear order $\preceq$ on X that makes $X \approx \mathbb{Q}$ and T an OP map. By Lemma 3.20, both of $T^{-2}(x_0)$ and $T^{-3}(x_0)$ are densely ordered. Hence, by Lemma 3.5, we have either $x_7 \preceq T^{-1}(x_4) \preceq x_8$ or $x_8 \preceq T^{-1}(x_4) \preceq x_7$. If $x_7 \preceq T^{-1}(x_4) \preceq x_8$, then $x_5 \preceq x_4 \preceq x_6$, since T is an OP map. But this implies, by

Lemma 3.5, that $T^{-1}(x_2) \preceq x_4 \preceq x_6$; so from Lemma 3.5 again, there is no $z \in X$ with $x_4 \prec z \prec x_6$. The case when $x_7 \preceq T^{-1}(x_4) \preceq x_8$ follows in the same way. Hence, X cannot be endowed with a linear order that makes $X \approx \mathbb{Q}$ and T an OP map. $\square$

Example 3.36. Let X be a countable set and $T : X \to X$ be a function with, in total, one $\mathbb{Z}$-orbit O with spine $\{x_i : i \in \mathbb{Z}\}$ so that $T(x_i) = x_{i+1}$ for each $i \in \mathbb{Z}$. The structure of O is illustrated in Figure 3.6, where $|C_i| = \omega$, $T^{-1}(C_i) = y_{i-1}$, $T^{-1}(y_i) = z_{i-1}$ and $T^{-1}(z_i) = \emptyset$ for all $i \in \mathbb{Z}$.

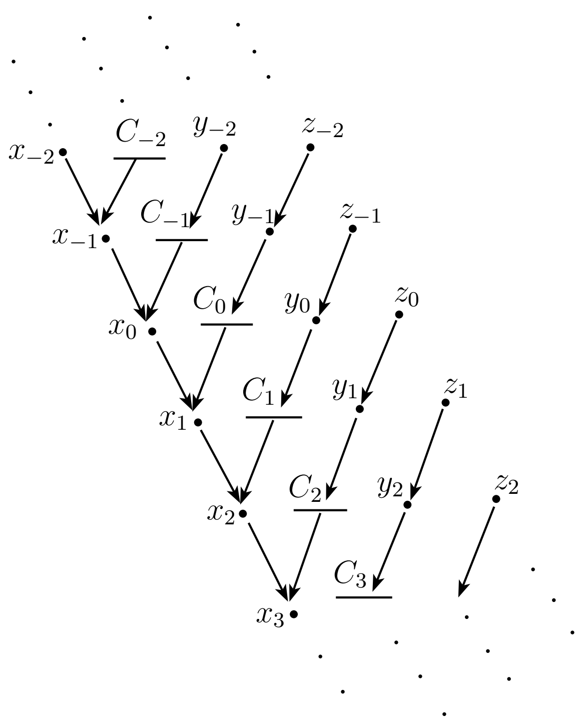

Figure 3.6: Structure of O.

Suppose that there is a linear order $\preceq$ on X that makes $X \approx \mathbb{Q}$ and T an OP map. Suppose that $x \preceq T(x)$ for all $x \in O$. By Lemma 3.20 we have $X_i = C_i \cup \{x_i, y_i, z_i\}$

is densely ordered for all $i \in \mathbb{Z}$, since $X_i = T^{-3}(x_{i+3})$. Also, by Lemma 3.5 we have $X_i \preceq X_{i+1}$ for all $i \in \mathbb{Z}$. Let $i = 0$, by Lemma 3.5 we have either $y_0 \preceq C_0 \cup \{x_0\} \preceq z_0$ or $z_0 \preceq C_0 \cup \{x_0\} \preceq y_0$. If $z_0 \preceq C_0 \cup \{x_0\} \preceq y_0$ then $y_1 = T(z_0) \preceq C_1 \cup \{x_1\}$, since T is OP. But then we have no $t \in X$ with $y_0 \prec t \prec y_1$ since $X_i \preceq X_{i+1}$ for all $i \in \mathbb{Z}$, which is a contradiction. Similarly, if $y_0 \preceq C_0 \cup \{x_0\} \preceq z_0$ then $C_1 \cup \{x_1\} \preceq y_1$, so $z_1 \preceq C_1 \cup \{x_1\} \preceq y_1$; which means that there is no $t \in X$ with $z_0 \prec t \prec z_1$. Therefore, X cannot be endowed with a linear order that makes $X \approx \mathbb{Q}$ and T an OP map.

Example 3.37. Let $T : X \to X$ be a function with, in total, one $\mathbb{Z}$-orbit O with spine $\{x_i : i \in \mathbb{Z}\}$. The structure of O is illustrated in Figure 3.7, where $|C_i| = \omega$, $T^{-1}(C_i) = y_{i-1}$, $T^{-1}(y_i) = \emptyset$ for each $i \in \mathbb{Z}$.

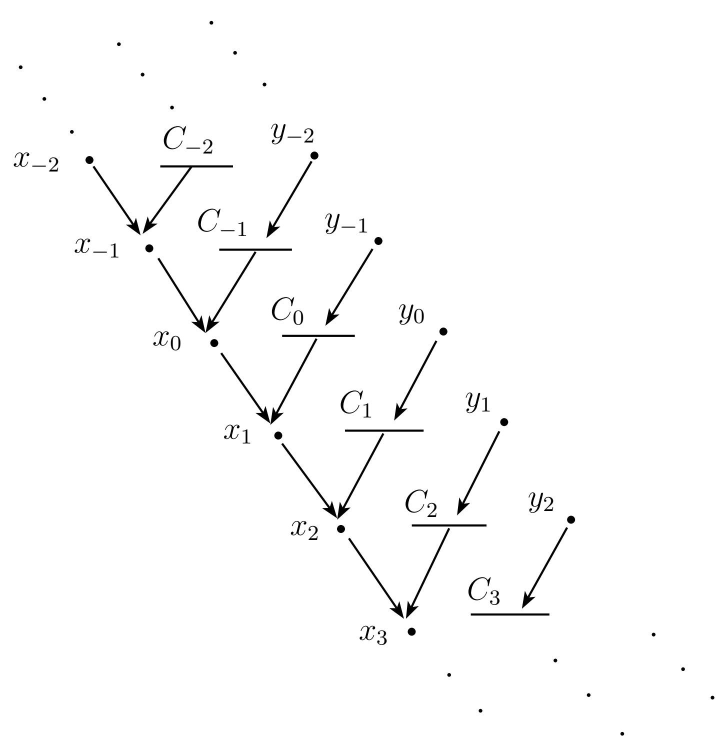

Figure 3.7: Orbit of T.

Let $X_i = C_i \cup \{x_i, y_i\}$ and choose an order isomorphism $h_i : (X_i, \preceq_i) \to (i, i+1] \cap \mathbb{Q}$ in such a way that $y_i \mapsto i+1$ and $x_i \preceq_i T(y_{i-1})$. Now consider X to be the ordered sum of $(X_i, \preceq_i)$ over $\mathbb{Z}$; so $X \approx \bigcup_{i \in \mathbb{Z}} (i, i+1] \cap \mathbb{Q}$. Hence, $X \approx \mathbb{Q}$ and T is an OP map under this order.

3.3 Orbit Structure of Order-Preserving Maps on the Integers and Naturals

In this section we give characterization of order-preserving maps on sets that are order isomorphic to the naturals $\mathbb{N}$ or to the integers $\mathbb{Z}$ with their usual orders, this characterization is in terms of the orbit structure of the map concerned.

It is a well known fact that the naturals have a least element and the integers has no endpoints. If $x, y \in \mathbb{M}$, $\mathbb{M}$ is either $\mathbb{N}$ or $\mathbb{Z}$, then $|(x, y)| < \omega$. So, a countably infinite linearly ordered set X is order-isomorphic to $\mathbb{N}$ iff X has a least element and $|(a, b)| < \omega$ for all $a, b \in X$. X is order-isomorphic to $\mathbb{Z}$ iff X has no endpoints and $|(a, b)| < \omega$ for all $a, b \in X$.

Let $T : X \to X$ and let O be a 1-cycle of T with spine $\{x_0\}$. Let $x \in T^{-1}(x_0) \setminus \{x_0\}$, we say that x has

(P1) if there is a unique $y \in T^{-k+1}(x)$ for some $0 \neq k \in \mathbb{N}$ with $|T^{-1}(y)| = \omega$ and x satisfies that $|T^{-k-1}(x)| = 0$.

(P2) if $T \restriction \cup_{k \in \mathbb{N}} T^{-k}(x)$ is finite-to-one and $| \cup_{k>0} T^{-k}(x)| = \omega$.

(P3) if $T \restriction \cup_{k \in \mathbb{N}} T^{-k}(x)$ is finite-to-one and $| \cup_{k>0} T^{-k}(x)| < \omega$.

Let O be either a 1-cycle with spine S_1 or an $\mathbb{N}$-orbit with spine $S_{\mathbb{N}}$. We say that O has property (C1) if

(C1) for each $x \in O \setminus S_1$, if $|T^{-k}(x)| = \omega$ for some $k \in \mathbb{N}$ then $|T^{-k-1}(x)| = 0$; and if $x \in S_1$ with $|T^{-1}(x)| = \omega$ then $|T(T^{-2}(x))| < \omega$.

Theorem 3.38. *Let $T : X \to X$ be an OP map, where X is either $\mathbb{N}$ or $\mathbb{Z}$. Then each $\mathbb{N}$-orbit and 1-cycle of T has property (C1).*

Proof. Let S be the set of all fixed points of T. Suppose, for a contradiction, that there is a $y \in X \setminus S$ with $|T^{-k}(y)| = \omega$ for some $k \in \mathbb{N}$ and $|T^{-k-1}(y)| \neq 0$. Let $z \in T^{-k-1}(y)$ so $T(z) \in T^{-k}(y)$ and $T^{k+1}(z) = y$. Since $|T^{-k}(y)| = \omega$, then by Lemma 3.5 we have either $T^{-k}(y) = (-\infty, a)$ or $T^{-k}(y) = (a, \infty)$ for same $a \in \mathbb{Z}$ when $X = \mathbb{Z}$ and $T^{-k}(y) = (a, \infty)$ for same $a \in \mathbb{N}$ when $X = \mathbb{N}$. Without loss of generality let $T^{-k}(y) = (a, \infty)$ for some $a \in \mathbb{N}$ in both of the cases. Since $y \notin S$ then $z \notin S$, so we have either $z < T(z) < T^{k+1}(z)$ or $T^{k+1}(z) < T(z) < z$ since T is an OP map. So, from Lemma 3.5 we have either $z < (a, \infty) < T^{k+1}(z) = y$ or $T^{k+1}(z) = y < (a, \infty) < z$, which is a contradiction. Hence, $|T^{-k-1}(y)| = 0$.

Now, let $|T^{-1}(x_0)| = \omega$ for some $x_0 \in S$. If $X = \mathbb{N}$ then by Lemma 3.5 we have $T^{-1}(x_0) = [a, \infty)$ for some $a \in \mathbb{N}$, then $T^{-2}(x_0) \setminus T^{-1}(x_0) \subseteq [0, a)$, so it is finite. Since $T(T^{-2}(x_0)) = T(T^{-2}(x_0) \setminus T^{-1}(x_0)) \cup \{x_0\}$ then we have $|T(T^{-2}(x_0))| < \omega$. If $X = \mathbb{Z}$ then either $T^{-1}(x_0) = [a, \infty)$ or $T^{-1}(x_0) = (-\infty, a]$ for some $a \in \mathbb{Z}$, without loss of generality let $T^{-1}(x_0) = [a, \infty)$, so $a \leq x_0$. Let $t \in T^{-2}(x_0)$, if $t \in [a, \infty)$ then $T(t) = x_0$. If $t < a$ then $T(t) \leq T(a) = x_0$. Hence, $a \leq T(T^{-2}(x_0)) \leq x_0$ so $|T(T^{-2}(x_0))| < \omega$, as required. ☐

Theorem 3.39. *Let $T : X \to X$ be an OP map where X is either $\mathbb{N}$ or $\mathbb{Z}$. Let O be a 1-cycle of T with spine $\{x_0\}$. Then for each $x \in T^{-1}(x_0) \setminus \{x_0\}$, x has one of the properties (P1), (P2) or (P3).*

Proof. Let $x \in T^{-1}(x_0) \setminus \{x_0\}$ have neither (P2) nor (P3). Hence, there is $y \in \cup_{k \in \mathbb{N}} T^{-k}(x)$ with $|T^{-1}(y)| = \omega$, say $y \in T^{-m+1}(x)$, so $|T^{-m}(x)| = \omega$. But Theorem 3.38 implies that

$|T^{-m-1}(x)| = 0$, so x has property (P3). $\square$

Observation 3.40. Let $T : X \to X$, where X is a countable set, and let O be an $\mathbb{N}$-orbit of T with spine S having property (C1). From now on, if $|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$ except for at most a unique point y_0 with $|T^{-1}(y_0)| = \omega$, we choose a spine S_N of O in such a way that $T^{-1}(L_0(O)) = \emptyset$ and $y_0 \in L_1(O)$ if there exists y_0 with $|T^{-1}(y_0)| = \omega$ which is possible since O has property (C1). See the following Figure.

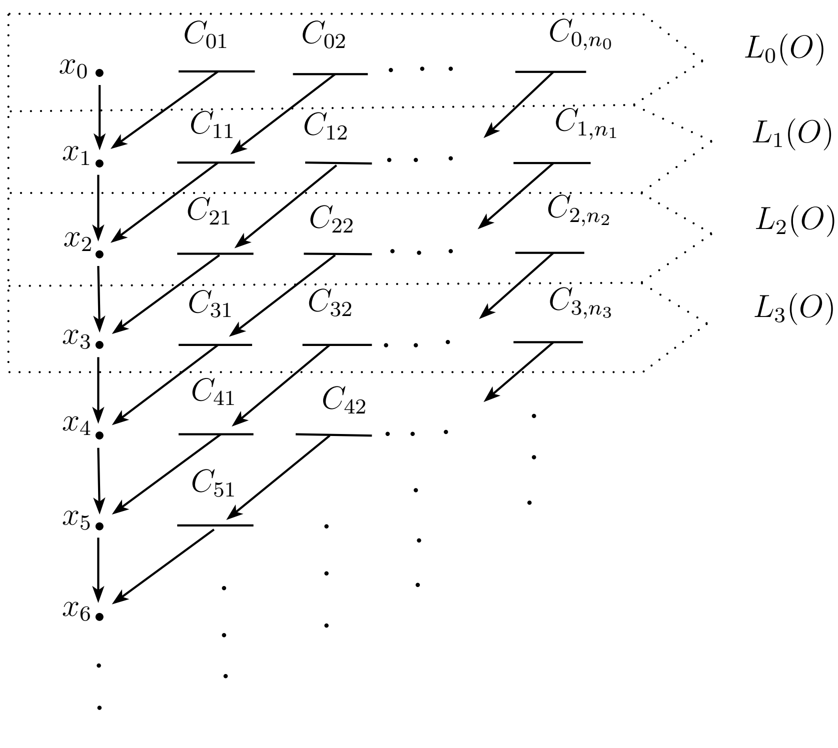

Figure 3.8: An $\mathbb{N}$-Orbit of T.

The proof of the following lemma is routine, it follows from the fact that $|(x, y)| < \omega$ for any $x, y \in \mathbb{M}$, where $\mathbb{M}$ is either $\mathbb{N}$ or $\mathbb{Z}$.

Lemma 3.41. *Let $\{(A_i, \preceq_i)\}_{i\in\mathbb{M}}$ be a collection of finite linearly ordered sets with $A_i \cap A_j = \emptyset$ whenever $i \neq j$, where $\mathbb{M}$ is either $\mathbb{N}$ or $\mathbb{Z}$. Then the ordered sum of $\{(A_i, \preceq_i)\}_{i\in\mathbb{M}}$ over $\mathbb{M}$ is order isomorphic to $\mathbb{M}$.*

Lemma 3.42. *Let $T_1 : (X_1, \preceq) \to (X_1, \preceq)$ and $T_2 : (X_2, \preceq') \to (X_2, \preceq')$ be OP maps where $X_1 \approx X_2 \approx \mathbb{N}$. Suppose further that either $X_1 \cap X_2 = \emptyset$ or $X_1 \cap X_2 = \{x_0\}$ with $T_1(x_0) = T_2(x_0) = x_0$ and x_0 is the least element of both of X_1 and X_2. If $X = X_1 \cup X_2$ is the ordered sum of $(X_1, \preceq^{-1})$ and $(X_2, \preceq')$, then $X \approx \mathbb{Z}$ and the map $T : X \to X$ defined as $T \restriction X_i = T_i, i = 1, 2$ is an OP map.*

Proof. Clearly, since $(X_1, \preceq) \approx \mathbb{N}$ then $(X_1, \preceq^{-1}) \approx (-\infty, 0]$. If $X_1 \cap X_2 = \emptyset$ then the ordered sum $X_1 \cup X_2 \approx (-\infty, 0] \cup [1, \infty) = \mathbb{Z}$, so by Lemma 3.4 we have T is an OP map. If $X_1 \cap X_2 = \{x_0\}$ with $T_1(x_0) = T_2(x_0) = x_0$ and x_0 is the least element of X_1 then x_0 is the greatest element of $(X_1, \preceq^{-1})$ and $(X_1, \preceq^{-1}) \approx (-\infty, 0]$, so the ordered sum $X_1 \cup X_2 \approx (-\infty, 0] \cup [0, \infty) \approx \mathbb{Z}$. Finally, by Lemma 3.4 and since $T(x_0) = x_0$, then immediately we have T is an OP map. $\square$

Now we have the following result. We choose spines of $\mathbb{N}$-orbits to be S_N as we described in Observation 3.40.

Theorem 3.43. *Let $T : X \to X$ be a map on the countably infinite set X. Let $\mathcal{O}$ be the collection of all orbits of T and S be the set of all spine points of orbits of T. Suppose further that:*

(1) *each orbit of T is either a 1-cycle, an $\mathbb{N}$-orbit or a $\mathbb{Z}$-orbit,*

(2) *$|L_k(O)| < \omega$ for each $O \in \mathcal{O}$ and $k \in N$, where N is either $\mathbb{Z}$, $\mathbb{N}$ or $\{0\}$, and $T^{-1}(L_0(O)) = \emptyset$ for each $\mathbb{N}$-orbit O, and*

(3) *$(S, \preceq_s)$ is linearly ordered such that $S \approx \mathbb{M}$, where $\mathbb{M}$ is either $\mathbb{N}$ or $\mathbb{Z}$, and $T \restriction S$ is an OP map.*

Then there is a linear order on X such that T is an OP map and $X \approx \mathbb{M}$.

Proof. Let $\mathcal{O} = \{O_i\}_{i \in I}, I \subseteq \mathbb{N}$. Let $(S, \preceq_s)$ be indexed as $\{x_r : r \in \mathbb{M}\}$ so that $x_r \preceq_s x_{r+1}$ for all $r \in \mathbb{M}$. For each $i \in I$ and $k \in N$, let $L_k(O_i)$ be defined, so by Corollary 3.15, there is a family of linear orders on the sets $L_k(O_i)$, $k \in N$ such that $T \restriction L_k(O_i)$ is an OP map. For each $i \in I$ and $k \in N$, write $L_k(O_i)$ as $L(x_r)$ whenever $L_k(O_i) \ni x_r$ for some $r \in \mathbb{M}$, (where each $L_k(O_i) \neq \emptyset$ contains a unique $x \in S$). Now, let X be the ordered sum of $\{L(x_r)\}_{x_r \in S}$ over S. Hence, by Lemma 3.41 we have $X \approx \mathbb{M}$.

Finally, we prove that T is an OP map. Let $x, y \in X$ with $x \preceq y$. If $x, y \in L(x_r)$, then $T(x) \preceq T(y)$, since $T \restriction L_(x_r)$ is an OP map as we mentioned above. If $x \in L(x_r), y \in L(x_j)$ for some $r, j \in \mathbb{M}, r \neq j$ then $T(x) \in L(T(x_r)), T(y) \in L(T(x_j))$ and $x_r \preceq_s x_j$. Since $T \restriction S$ is OP, then $T(x_r) \preceq_s T(x_j)$, hence $T(x) \preceq T(y)$. $\square$

3.4 Characterizing Order-Preserving Maps on the Naturals

3.4.1 Order-Preserving Bijections, Injections and Surjections on the Naturals

Lemma 3.44. *Let $T : \mathbb{N} \to \mathbb{N}$ be an OP map, then T has no $\mathbb{Z}$-orbits.*

Proof. The proof follows by Corollary 3.9 and the fact that $\mathbb{N}$ has a least element. $\square$

Theorem 3.45. *Let $\sigma = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ be a sequence of cardinals. Then a countable set X can be linearly ordered so that $X \approx \mathbb{N}$ and the canonical representation $T : X \to X$ of σ is an OP injection if and only if $\zeta = \sigma_n = 0$ for each $n > 1$ and either:*

(1) *$\nu = 0$ and $\sigma_1 = \omega$; or*

(2) *$\nu \neq 0$ and $\sigma_1 < \omega$.*

Proof. Let T be an OP injection on $\mathbb{N}$, then by Lemma 3.3 we have $\sigma_n = 0$ for each $n > 1$ and by Lemma 3.44 we have $\zeta = 0$. Now, let $\mathcal{O} = \{O_i\}_{i \in I}, I \subseteq \mathbb{N}$, be the collection of all $\mathbb{N}$-orbits of T and let $O = \bigcup_{i \in I} O_i$. If $\nu \neq 0$ so $O \neq \emptyset$, then by Lemma 3.6 we have $O = (a, \infty)$ for some $a \in \mathbb{N}$. Hence, $X \setminus O = [0, a]$, so $\sigma_1 < \omega$. If $\sigma_1 = \omega$, then again by Lemma 3.6 we have $\nu = 0$.

Conversely, let $\sigma = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ be a sequence of cardinals with $\zeta = \sigma_n = 0$ for each $n > 1$. If $\nu = 0$ and $\sigma_1 = \omega$, let T be the identity map on $\mathbb{N}$, so it is an OP map. Now, suppose that $\sigma_1 = 0$ and $\nu \neq 0$. Suppose that $\nu = \omega$ and enumerate the naturals $\mathbb{N}$ as follow:

$$\{a_{00} \mid a_{10},\ a_{11} \mid a_{20},\ a_{21},\ a_{22}, \mid a_{30},\ a_{31},\ a_{32},\ a_{33} \mid \cdots \mid a_{k0},\ a_{k1},\ a_{k2},\ a_{k3}, \dots a_{kk} \mid \dots\},$$

then $a_{00} = 0, a_{10} = 1$ and $a_{ij} = \frac{i(i+1)}{2} + j$, for $i > 0$ and $0 \leq j \leq i$. For each $k \in \mathbb{N}$ let

$$B_k = \{a_{m,k} : m \geq k\},$$

so $\mathbb{N} = \bigcup_{k \in \mathbb{N}} B_k$. Let $T : \mathbb{N} \to \mathbb{N}$ be the map defined by $T(a_{i,j}) = a_{i+1,j}$, so for each $k \in \mathbb{N}$, we have B_k is an $\mathbb{N}$-orbit of T. Since we have $a_{i,j} < a_{i',j'}$ iff $i < i'$ or $i = i'$ and $j < j'$, so it is simple to verify that T is an OP injection. Hence, T has, in total, countably infinitely many $\mathbb{N}$-orbits. If $\nu = p < \omega$, then we can take $B = B_0 \cup \cdots \cup B_{p-1}$, so $T \restriction B$ is an OP injection with p $\mathbb{N}$-orbits and $B \approx \mathbb{N}$.

Finally, let $\sigma_1 = n < \omega$ and $\nu \neq 0$. As above, if $0 \neq \nu = k$ then there is an OP map T on $[n, \infty)$ with, in total, k $\mathbb{N}$-orbits. Let $T_1 : \mathbb{N} \to \mathbb{N}$ be the map defined as $T_1 \restriction [n, \infty) = T$ and $T(a) = a$ for each $a \in [0, n)$, then T has, in total, n 1-cycles and k $\mathbb{N}$-orbits, and T is an OP injection. $\square$

If $T : X \to X$ is a bijection, then we have the following well-known result.

Theorem 3.46. *Let T be a bijection on a countably infinite set X. There is a linear order on X with respect to which T is an OP bijection and $X \approx \mathbb{N}$ if and only if T is the identity map.*

Now we study the same problem but for surjective maps. We start with giving the following two results.

Lemma 3.47. *Let $T : \mathbb{N} \to \mathbb{N}$ be an OP map and T has only 1-cycles.*

(1) *If there exists $y \in X$ with $|T^{-1}(y)| = \omega$ then $|X \setminus T^{-1}(y)| < \omega$.*

(2) *If there exists an orbit $O \subseteq X$ with $|O| = \omega$ then $|X \setminus O| < \omega$.*

(3) *If T has infinitely many 1-cycles then $|O| < \omega$ for each orbit O of T.*

Proof. (1) Clearly, by Lemma 3.5, we have $T^{-1}(y) = (n, \infty)$ for some $n \in \mathbb{N}$, so we have $X \setminus T^{-1}(y) = [0, n)$; hence it is a finite set.

(2) Let $O \subseteq X$ with $|O| = \omega$. By Lemma 3.6 we have $O = (n, \infty)$ for some $n \in \mathbb{N}$, so $X \setminus O = [0, n)$ which is finite.

(3) Suppose, for a contradiction, that there is an orbit C with $|C| = \omega$, then from (2) we have $|X \setminus C| < \omega$; which is a contradiction. Hence, $|O| < \omega$ for each orbit O of T. $\square$

Lemma 3.48. *Let T be a finite-to-one map on the countable set X with a unique 1-cycle O with spine $\{x_0\}$. Then there is a linear order $\preceq$ on X in such a way that x_0 is the least element of X and with respect to which T is an OP map. Moreover, if $|X| = \omega$ then $X \approx \mathbb{N}$.*

Proof. By Lemma 3.14, there is a family of linear orders on the sets $C_{0,k} \neq \emptyset$, $0 \neq k \in \mathbb{N}$, with respect to which $T \restriction C_{0,k}$ is an OP map. Let $(X, \preceq)$ be the ordered sum of $(C_{0,k}, \preceq_k)$ over $\mathbb{N}$, so $x_0 = C_{0,0}$ is the least element of X. Now, suppose that $|X| = \omega$. Since for each $k \in \mathbb{N}$ we have $|C_{0,k}| < w$, then Lemma 3.41 implies that X is order-isomorphic to $\mathbb{N}$. $\square$

Theorem 3.49. *Let $T : X \to X$ be a surjection on a countably infinite set X and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. There is a linear order on X with respect to which T is an OP surjection and $X \approx \mathbb{N}$ iff $\zeta = \sigma_n = 0$ for all $n > 1$, T is finite-to-one and either T is the identity map or $\sigma_1 < \omega$ with a unique 1-cycle O with $|O| = \omega$.*

Proof. Let T be an OP map on $\mathbb{N}$, then by Lemma 3.3 we have $\sigma_n = 0$ for each $n > 1$ and by Lemma 3.44 we have $\zeta = 0$. Lemma 3.47 (1) implies that the surjection T is finite-to-one. By Lemma 3.47 (3), if $\sigma_1 = \omega$ then all 1-cycles are simple, so T is the identity map. Lemma 3.47 (2) implies that if $\sigma_1 < \omega$, then there is a unique cycle O with $|O| = \omega$.

Conversely, suppose that $\zeta = \sigma_i = 0$ for all $i > 1$ and T is finite-to-one. If $\sigma_1 = \omega$ then directly T is the identity map, so it is an OP map. If $\sigma_1 = n < \omega$ and O is the unique 1-cycle with $|O| = \omega$, then by Lemma 3.48, there is a linear order on O that makes $T \restriction O$ an OP map and $O \approx \mathbb{N}$, so $O \approx [n, \infty)$. Since $X \setminus O \approx [0, n)$ and each 1-cycle different from O is simple, then we have $X \approx \mathbb{N}$ and T is an OP surjection. □

3.4.2 Orbit Structure of Order-Preserving Self-Maps on $\mathbb{N}$

Let $T : X \to X$ be an arbitrary map. In this section we give the necessary and sufficient conditions for a countable set with self-map to be endowed with a linear order with respect to which $X \approx \mathbb{N}$ and T is an OP map.

The proof of the following lemma follows immediately from Lemma 3.47.

Lemma 3.50. *Let $T : \mathbb{N} \to \mathbb{N}$ be an OP map and T has only 1-cycles.*

(1) *T has at most one point $y \in X$ with $|T^{-1}(y)| = \omega$.*

(2) *If there exists $S \subseteq O$ for some 1-cycle O such that $T \restriction S$ is a semi-simple 1-cycle, then $|T^{-1}(x)| < \omega$ for each $x \in X$.*

Lemma 3.51. *Let $T : \mathbb{N} \to \mathbb{N}$ be an OP map and let $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. If $\nu \neq 0$, then $|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$ and an $\mathbb{N}$-orbit O. Moreover, $\sigma_1 < \omega$ with all 1-cycles are finite.*

Proof. Suppose, for a contradiction, that there is an $\mathbb{N}$-orbit O with spine S with $|L_k(O)| = \omega$ for some $k \in \mathbb{Z}$, then by Lemma 3.5, we have $L_k(O) = (a, \infty)$ for some $a \in \mathbb{N}$. But this means that $S \setminus \{x_k\} \subseteq [0, a]$, which is a contradiction since the spine S is an infinite set.

Now, let $\{O_i\}_{i \in I}$, $I \subseteq \mathbb{N}$, be the collection of all $\mathbb{N}$-orbits of T, then $O = \bigcup_{i \in I} O_i$ is an infinite subset of $\mathbb{N}$. By Lemma 3.6 we have $O = (a, \infty)$ for some $a \in \mathbb{N}$. Hence, $X \setminus O = [0, a]$, so $\sigma_1 < \omega$ and each 1-cycle must be finite. $\square$

Proposition 3.52. *Let T be a map on a countably infinite set X consisting of a single 1-cycle O with spine $S = \{x_0\}$ and suppose that O has property (C1). Suppose further that T has a unique point y with $|T^{-1}(y)| = \omega$. If either T is the constant map or there is $z \in T^{-1}(S) \setminus S$ with $\|z\| = max\{\|t\| : t \in T^{-1}(S) \setminus S\} < \infty$ and $y \in \bigcup_{i \in \mathbb{N}} T^{-i}(z)$, then there is a linear order $\preceq$ on O in such a way that x_0 is the least element of O and $X \approx \mathbb{N}$ and with respect to which T is an OP map.*

Proof. If T is the constant map, the proof is trivial. So let $y \neq x_0$ with $|T^{-1}(y)| = \omega$. Let $z \in T^{-1}(S) \setminus S$ with $\|z\| = \max\{\|t\| : t \in T^{-1}(S) \setminus S\} = m < \infty$ and let $y \in T^{-m+1}(z)$ be the unique point with $|T^{-1}(y)| = \omega$. For each $0 \leq i < m$, let $y_i = T^i(y)$. For each $k \in \mathbb{N}$, let $C_{0,k}$ be defined and choose a linear order $\preceq_{01}$ on $C_{0,1}$ in such a way that $y_{m-1} = z$ is the greatest element of $C_{0,1}$. Next, for each $x \in O \setminus S$, $x \neq y$, if $|T^{-1}(x)| \neq 0$ choose a linear order $\preceq_x$ on $T^{-1}(x)$ such that if $|T^{-1}(y_i)| > 1$ choose $\preceq_{y_i}$ so that y_{i-1} is the greatest element of $T^{-1}(y_i)$. If $x = y$ so $|T^{-1}(y)| = \omega$, choose $\preceq_y$ such that $T^{-1}(y) \approx \mathbb{N}$. Now, use the same construction in Lemma 3.14, so for each $0 < k \leq m + 1$, the order $\preceq_{0k}$ on $C_{0,k}$ is defined to be the sum-order of the $\preceq_x$ over $C_{0,k-1}$. So, each $T \restriction C_{0,k}$ is OP. Let $(X, \preceq)$ be the ordered sum of $(C_{0,k}, \preceq_{0k})$ over $\mathbb{N}$, so we have T is an OP map under $\preceq$.

Since $C_{0,0} = \{x_0\}$, hence x_0 is the least element of X. Since

$$m = \max\{\|t\| : t \in T^{-1}(S) \setminus S\},$$

then we have $C_{0,m+2} = \emptyset$. Since z is the greatest element of $C_{0,1}$ and for each $0 \leq i < m$, y_{i-1} is the greatest element of $T^{-1}(y_i)$ (hence the greatest element of $C_{0,m-i}$), so we have $T^{-1}(y) \succeq T^{-m}(z) \succeq C_{0,m+1}$, i.e., $T^{-1}(y) \succeq O$, hence $X \approx \mathbb{N}$. $\square$

Now we give a proof of the main theorem of this section.

Theorem 3.53. *Let T be a map on a countably infinite set X with orbit spectrum $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ and let $\mathcal{O}$ be the collection of all 1-cycles of T. There is a linear order on X with respect to which $X \approx \mathbb{N}$ and T is an OP map iff $\sigma_n = 0$ for all $n > 1$, $\zeta = 0$, each $O \in \mathcal{O}$ has property (C1) and either:*

(1) *$\nu = 0$ and $\sigma_1 = \omega$ with $|O| < \omega$ for all $O \in \mathcal{O}$; or*

(2) *$\nu = 0$ and $\sigma_1 < \omega$ with $|O| < \omega$ for all $O \in \mathcal{O}$ except for a unique 1-cycle O such that either T is finite-to-one or T has a unique $y \in X$ with $|T^{-1}(y)| = \omega$ such that $|X \setminus T^{-1}(y)| < \omega$; or*

(3) *$\nu \neq 0$ with $|L_i(O)| < \omega$ for each $i \in \mathbb{N}$ and $\mathbb{N}$-orbit O, and $\sigma_1 < \omega$ with $|O'| < \omega$ for all $O' \in \mathcal{O}$.*

Proof. Suppose that T is an OP map on $\mathbb{N}$, then by Lemma 3.3 we have $\sigma_n = 0$ for all $n > 1$ and by Lemma 3.44 we have $\zeta = 0$. Also, Theorem 3.38 implies that O has property (C1) for each $O \in \mathcal{O}$. If $\nu = 0$ and $\sigma_1 = \omega$ then, by Lemma 3.47 (3), we have $|O| < \omega$ for each $O \in \mathcal{O}$. If $\nu = 0$ and $\sigma_1 < \omega$ then, by Lemma 3.47 (2), we have $|O'| < \omega$ for each $O' \in \mathcal{O}$ except for a unique 1-cycle O with $|O| = \omega$. If T is not finite-to-one, then by Lemma 3.50 (1), there is a unique $y \in X$ with $|T^{-1}(y)| = \omega$ and from Lemma 3.47 (1), we have $|X \setminus T^{-1}(y)| < \omega$. If $\nu \neq 0$, then by Lemma 3.51 we have $\sigma_1 < \omega$ with $|O| < \omega$ for all $O \in \mathcal{O}$ and $|L_i(O)| < \omega$ for each $i \in \mathbb{N}$ and an $\mathbb{N}$-orbit O.

Conversely assume that $\sigma_n = 0$ for all $n > 1$, $\zeta = 0$ and each 1-cycle has property (C1).

(1) Let $\nu = 0$ and $\sigma_1 = \omega$. Let $\mathcal{O} = \{O_i\}_{i \in \mathbb{N}}$ be the collection of all 1-cycles with $|O_i| = n_i < \omega$ for each $i \in \mathbb{N}$. By Lemma 3.48, for each $i \in \mathbb{N}$, there is a linear order on O_i such that $T \restriction O_i$ is an OP map. So, by Lemma 3.41, the ordered sum of $(O_i, \preceq_i)$ over $\mathbb{N}$ is order isomorphic to $\mathbb{N}$ and from Lemma 3.4, T is an OP map.

(2) Let $\nu = 0$ and $\sigma_1 < \omega$ with $|O'| < \omega$ for all $O' \in \mathcal{O}$ except for a unique 1-cycle O with $|O| = \omega$. Now, let $S = \{x_0\}$ be the spine of O. If T is finite-to-one then by Lemma 3.48, there is a linear order on O such that $T \restriction O$ is OP and $O \approx \mathbb{N}$. Now, let O have a unique $y \in O$ with $|T^{-1}(y)| = \omega$ and $|X \setminus T^{-1}(y)| < \omega$. Let

$$A = \{x \in T^{-1}(S) \setminus S : \|x\| \geq 1, |\bigcup_{k \in \mathbb{N}} T^{-k}(x)| < \omega\}$$

and let $C = \{x_0\} \cup \bigcup_{k \in \mathbb{N}} T^{-k}(A)$. Let $B = \{x_0\} \cup (O \setminus C)$. So we have $T \restriction B$ satisfies the conditions of Proposition 3.52, so there is a linear order $\preceq$ on B in such a way that x_0 is the least element and with respect to which $T \restriction B$ is an OP map and $B \approx \mathbb{N}$. Also, since C is finite, then $T \restriction C$ is finite-to-one, so by Lemma 3.48 there is a linear order $\preceq'$ on C in such a way that x_0 is the least element and $T \restriction C$ is OP. Hence, since $B \cap C = \{x_0\}$, x_0 is the least element of B and x_0 is the greatest element of $(C, \preceq'^{-1})$, then the ordered sum of $(C \setminus \{x_0\}, \preceq'^{-1})$, $(B, \preceq)$ is order isomorphic to $\mathbb{N}$ and $T \restriction O$ is an OP map.

Finally, let $|X \setminus O| = n < \omega$, then by Lemma 3.48 and Lemma 3.4 we have $|X \setminus O| \approx [0, n)$ and $T \restriction (X \setminus O)$ is OP. Since $O \approx \mathbb{N}$ as we showed above, then $O \approx [n, \infty)$, so we have X as an ordered sum of $(X \setminus O)$ and O is order-isomorphic to $\mathbb{N}$. Also, from Lemma 3.4, T is an OP map.

(3) Let $\nu \neq 0$ with $|L_i(O)| < \omega$ for each $i \in \mathbb{N}$ and an $\mathbb{N}$-orbit O. Let S be the set of spine points of orbits of T, then by Theorem 3.45 there is a linear order on S so that $S \approx \mathbb{N}$ and $T \restriction S$ is an OP map. Since $|O'| < \omega$ for all $O' \in \mathcal{O}$, then we have $|L_i(O)| < \omega$ for all orbits of T. Hence, the proof follows immediately from Theorem 3.43. $\square$

3.5 Order-Preserving Self-Maps on the Integers

In this section we characterize order-preserving maps on sets that are order-isomorphic to the integers with their usual orders. This characterization is in terms of the orbit structure of the map concerned.

First, we prove the following result that is useful and we need to use in this section.

Lemma 3.54. *Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OP map and let $\mathcal{O}$ be the collection of all orbits of T. Suppose that T has no $\mathbb{Z}$-orbits and that $|\mathcal{O}| > 1$. Suppose further that $|L_i(O)| < \omega$ for each $i \in \mathbb{N}$ and an $\mathbb{N}$-orbit $O \in \mathcal{O}$. Then $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ where $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ and each of $\mathcal{O}_1$ and $\mathcal{O}_2$ is an orbit structure of an OP map on $\mathbb{N}$.*

Proof. By Lemma 3.3 we have $\sigma_n = 0$ for each $n > 1$. If $\sigma_1 = 0$, then T has only $\mathbb{N}$-orbits; since $|L_i(O)| < \omega$ for each $O \in \mathcal{O}$ and $|\mathcal{O}| > 1$, then the proof follows by Theorem 3.53. If $\sigma_1 \neq 0$ with at least a 1-cycle O is finite, then by Lemma 3.6, $O = [a, b]$ for some $a \leq b \in \mathbb{Z}$, so for each orbit O of T, either $O < [a, b]$ or $[a, b] < O$. So, take $\mathcal{O}_1$ to be the collection of all orbits of $T \restriction (b, \infty)$ and $\mathcal{O}_2$ to be the collection of all orbits of $T \restriction (-\infty, b]$. If there is a 1-cycle O with $|O| = \omega$, then again by Lemma 3.6 and Lemma 3.5, either $O = (-\infty, a)$ or $O = (a, \infty)$ for some $a \in \mathbb{N}$. Without loss of generality, let $O = (-\infty, a)$, so $O \approx \mathbb{N}$. Hence, any orbit different from O is a subset of $[a, \infty)$ and hence $X \setminus O = [a, \infty) \approx \mathbb{N}$. $\square$

3.5.1 Order-Preserving Injections, Bijections and Surjections on the Integers

Lemma 3.55. *Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OP map. If T has a $\mathbb{Z}$-orbit then*

(1) *T has no cycles;*

(2) *$|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$ and orbit O; and*

(3) *the number of $\mathbb{Z}$-orbits is finite.*

Proof. Let O be a $\mathbb{Z}$-orbit of T with spine S indexed as $\{x_i : i \in \mathbb{Z}\}$ so that $T(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$. Let $x < T(x)$, then we have $x_i < x_{i+1}$ for all $i \in \mathbb{Z}$ since T is an OP map.

(1) By Lemma 3.3, T has no cycles of length $n > 1$. Suppose, for a contradiction, that T has a 1-cycle with spine $\{y_0\}$. It follows from Lemma 3.6 that either $y_0 \preceq O$ or $O \preceq y_0$, say $y_0 \preceq O$. But then $|[y_0, x_i)| = \omega$ for all $i \in \mathbb{Z}$, which is a contradiction, since $|(a, b)| < \omega$ for any $a, b \in \mathbb{Z}$. Thus, T has no 1-cycles.

(2) Suppose, for a contradiction, that $|L_i(O)| = \omega$ for some $i \in \mathbb{Z}$. By Lemma 3.5 we have either $L_i(O) = (a, \infty)$ or $L_i(O) = (-\infty, a)$ for some $a \in \mathbb{Z}$, say $L_i(O) = (-\infty, a)$. But this implies, by Theorem 3.5, that $x_j < L_i(O) = (-\infty, a)$ for all $j < i$ which is a contradiction. Similarly, if $L_i(O) = (a, \infty)$ then, by Theorem 3.5, $L_i(O) = (a, \infty) < x_j$ for all $j > i$, again it is a contradiction. Hence, $|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$.

(3) Suppose, for a contradiction, that T has countably infinitely many $\mathbb{Z}$-orbits. Let $\{O_i\}_{i \in \mathbb{N}}$ be the collection of all $\mathbb{Z}$-orbits of T. For each $i \in \mathbb{N}$, index a spine of O_i as $\{z_{i,j} : j \in \mathbb{Z}\}$. Let $z_{1,0} < z_{1,1}$, so the set $(z_{1,0}, z_{1,1})$ must be finite. But this implies that there are $k \in \mathbb{N}$ and $j \in \mathbb{Z}$ with $z_{k,j} < z_{1,0}$ and $z_{1,1} < z_{k,j+1}$, which is a contradiction of being T is an OP map. Consequently, T has finitely many $\mathbb{Z}$-orbits. $\square$

The following theorem describes the orbit structure of order-preserving injections on the integers $\mathbb{Z}$.

Theorem 3.56. *Let $\sigma = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ be a sequence of cardinals. Then the canonical representation $T : X \to X$ of σ on the countable set X is an OP injection with X is order-isomorphic to $\mathbb{Z}$ if and only if $\sigma_n = 0$ for all $n > 1$ and either:*

(1) *$\zeta = 0$ and $\sigma_1 = \omega$; or*

(2) *$\zeta = 0$, $\sigma < \omega$ and $\nu > 1$; or*

(3) *$0 \neq \zeta < \omega$ and $\sigma_1 = 0$.*

Proof. Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OP map, then by Lemma 3.3 we have $\sigma_n = 0$ for all $n > 1$ and by lemma 3.55 (3) we have $\zeta < \omega$. If $\zeta = 0$ and $\sigma_1 < \omega$ so there are finitely many fixed points then by Lemma 3.54 we have $\nu > 1$. If $\zeta \neq 0$ then, by lemma 3.55 (1), we have $\sigma_1 = 0$.

Conversely, (1) and (2) follow immediately from Theorem 3.53 and Lemma 3.42. Suppose that (3) holds. If $\zeta = k < \omega$ and $\nu = 0$, let $T : \mathbb{Z} \to \mathbb{Z}$ be the map defined as $T(x) = x + k$, then T has, in total, k $\mathbb{Z}$-orbits and T is an OP injection. If $\zeta = k < \omega$ and $\nu \neq 0$, enumerate the naturals $\mathbb{N}$ as

$$\{a_{k-1,0}, a_{k-1,1}, \dots a_{k-1,k-1} \mid a_{k,0}, a_{k,1}, \dots a_{k,k} \mid \cdots \mid a_{k+n,0}, a_{k+n,1}, a_{k+n,2}, \dots a_{k+n,k+n} \mid \dots\},$$

then $a_{k-1,0} = 0, a_{k-1,1} = 1$ and $a_{ij} = \frac{i(i+1)}{2} + j - \frac{k(k-1)}{2}$, for all $i \geq k-1$ and $0 \leq j \leq i$. Now list $\mathbb{Z} \setminus \mathbb{N}$ as

$$\{\dots, \mid a_{n,0}, a_{n,1}, \dots,\ a_{n,k-1} \mid \dots, \mid a_{-2,0}, a_{-2,1}, \dots,\ a_{-2,k-1} \mid a_{-1,0}, a_{-1,1}, \dots,\ a_{-1,k-1}\},$$

then $a_{i,m} = ki + m$, for each $i \in \mathbb{Z} \setminus \mathbb{N}$ and $m \in \{0, 1, \dots, k-1\}$.

For each $n \geq k$, let $B_n = \{a_{r,n} : r \geq n\}$ and for each $m \in \{0, 1, \dots, k-1\}$, let

$$C_m = \{a_{i,m} : i \in \mathbb{Z} \setminus \{0, \dots, k-2\}\}.$$

Define T such that $T(a_{i,j}) = a_{i+1,j}$ if $i \neq -1$ and $T(a_{-1,j}) = a_{k-1,j}$. Clearly, since $a_{i,j} < a_{i',j'}$ iff $i < i'$ or $i = i'$ and $j < j'$, so it is simple to verify that T is an OP injection. Also, T has m $\mathbb{Z}$-orbits $\{C_m : 0 \leq m \leq k-1\}$, and infinitely many $\mathbb{N}$-orbits, $\{B_n\}_{n \geq k}$. If $\nu = p < \omega$, let $B = B_k \cup B_{k+1} \cup \cdots \cup B_{k+p}$ and $C = \bigcup_{0 \leq m < k} C_m \cup B$, so $T \restriction C$ is also OP with m $\mathbb{Z}$-orbits and p $\mathbb{N}$-orbits. $\square$

Now, if $T : X \to X$ is a bijection on the countably infinite set X then we have the following result which follows from Theorem 3.56.

Theorem 3.57. *Let $T : X \to X$ be a bijection on the countably infinite set X. Then X can be ordered so that X is order-isomorphic to $\mathbb{Z}$ and T is an OP bijection if and only if either T is the identity map or T has, in total, finitely many $\mathbb{Z}$-orbits.*

The following theorem gives the answer of the same question but for surjective maps.

Theorem 3.58. *Let $T : X \to X$ be a surjection on a countably infinite set X with $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$. Let $A = \{O : O$ is a cycle and $|O| = \omega\}$. Then X can be ordered so that $X \approx \mathbb{Z}$ and $T : X \to X$ is an OP surjection iff T is finite-to-one, $\sigma_n = 0$ for all $n > 1$ and either:*

(1) *$\zeta < \omega$, $\sigma_1 = 0$ and $|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$ and a $\mathbb{Z}$-orbit O; or*

(2) *$\zeta = 0$ and either:*

   (a) *$\sigma_1 = \omega$ and $|A| \leq 1$; or*

   (b) *$1 < \sigma_1 < \omega$ and $|A| = 2$; or*

   (c) *T has a unique 1-cycle with spine S such that $|T^{-1}(S) \setminus S| > 1$.*

Proof. Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OP map, then by Lemma 3.3 we have $\sigma_n = 0$ for each $n > 1$. Suppose, for a contradiction, that T is not finite-to-one, so there is $y \in X$ with $|T^{-1}(y)| = \omega$ so $|T^{-k}(y)| = \omega$ for all $k \in \mathbb{N}$, since T is a surjection. But Lemma 3.5 implies that either $T^{-1}(y) = (-\infty, a)$ and $T^{-2}(y) = (b, \infty)$ for some $a, b \in \mathbb{Z}$ or vice versa. But then $\bigcup_{k>2} T^{-k}(y) \in [a, b]$ which is a contradiction, so T is finite-to-one. If $\zeta \neq 0$ then Lemma 3.55 implies that $\zeta < \omega$, $\sigma_1 = 0$ and $|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$ and a $\mathbb{Z}$-orbit O. Now let $\zeta = 0$. Lemma 3.6 implies that if O is a 1-cycle with $|O| = \omega$, then we have either $O = (-\infty, a)$ or $O = (a, \infty)$ for some $a \in \mathbb{Z}$ so $|A| \leq 2$. Hence, if $\sigma_1 = \omega$ then $|A| \leq 1$ and if $1 < \sigma_1 < \omega$ then $|A| = 2$.

Let T have a unique 1-cycle with spine $S = \{x_0\}$. Suppose, for a contradiction, that $T^{-1}(S) \setminus S = \{y\}$. If $y < T(y)$ then we have

$$\cdots < T^{-n}(y) < T^{-n+1}(y) \cdots < T^{-2}(y) < T^{-1}(y) < y < T(y) = x_0,$$

since T is OP, so x_0 is the greatest element of X. Similarly, if $y > T(y)$, we have x_0 is the least element of X. But both of the cases is a contradiction since $\mathbb{Z}$ has no endpoints; so $|T^{-1}(S) \setminus S| > 1$.

Conversely, case (1) follows from Theorem 3.56 and Theorem 3.43. Cases (2) (a) and (b) follow from Theorem 3.53 and Lemma 3.42. Now, suppose that (c) holds. Let $S = \{x_0\}$. Let $y_1, y_2 \in T^{-1}(S) \setminus S$ so $\|y_1\| = \|y_2\| = \infty$. Let

$$X_1 = \{x_0\} \cup \bigcup_{k \in \mathbb{N}} T^{-k}(y_1)$$

and $X_2 = \{x_0\} \cup (O \setminus X_1)$. Then $X_1 \cap X_2 = \{x_0\}$ and from Lemma 3.48, there is linear orders $\preceq_1$ and $\preceq_2$ on X_1 and X_2 respectively in such a way that x_0 is the least element of both of them so that $X_1 \approx X_2 \approx \mathbb{N}$, $T \restriction X_1$ and $T \restriction X_2$ are OP maps. Hence, X_1 and X_2 satisfy conditions of Lemma 3.42, so X as the ordered sum of $(X_1, \preceq_1^{-1})$ and $(X_2, \preceq_2)$ is order-isomorphic to $\mathbb{Z}$ and T is an OP surjection. $\square$

3.5.2 Order-Preserving Maps with no Cycles on $\mathbb{Z}$

Let $T : X \to X$ be an arbitrary map. In this section, we assume that T has no cycles and give the orbit structure of T which allows to order X in such a way that T is order-preserving map and $X \approx \mathbb{Z}$.

Now we have the following Lemma, where spines of $\mathbb{N}$-orbits are chosen as in Observation 3.40 so that $T^{-1}(L_0(O)) = \emptyset$, which is possible by Lemma 3.55 (2).

Lemma 3.59. *Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OP map with only $\mathbb{N}$-orbits. Suppose that there is $y_0 \in X$ with $|T^{-1}(y_0)| = \omega$.*

(1) *For all $y \in X$ such that $|T^{-1}(y)| = \omega$, $y = y_0$.*

(2) *$|L_i(O) \setminus T^{-1}(y_0)| < \omega$ for each $i \in \mathbb{N}$ and an orbit O.*

Proof. (1) Assume that there exists $y \in X$ with $|T^{-1}(y)| = \omega$. Let y be in an orbit O with spine S. By Lemma 3.5, if $y \neq y_0$ then we have either $T^{-1}(y_0) = (a, \infty)$ and $T^{-1}(y) = (-\infty, b)$ for some $a > b \in \mathbb{Z}$ or vice versa. But this implies that $S \setminus (T^{-1}(y) \cup T^{-1}(y_0)) \subseteq [a, b]$, which is a contradiction since S is infinite. Hence, $y = y_0$.

(2) Let y_0 lie in an $\mathbb{N}$-orbit O. Suppose, for a contradiction, that $|L_i(O) \setminus T^{-1}(y_0)| = \omega$ for some $i \in \mathbb{N}$ and an orbit O. By Lemma 3.5 we have either $T^{-1}(y_0) = (-\infty, a]$ and $L_i(O) \setminus T^{-1}(y_0) = [b, \infty)$ or $T^{-1}(y_0) = [b, \infty)$ and $L_i(O) \setminus T^{-1}(y_0) = (-\infty, a]$ for some $a, b \in \mathbb{N}$ (where $a = b$ if $y_0 \in L_{i+1}(O)$). But then

$$S \setminus (T^{-1}(y_0) \cup L_i(O)) \subseteq (a, b),$$

which is a contradiction. Consequently, $|L_i(O) \setminus T^{-1}(y_0)| < \omega$ for each $i \in \mathbb{N}$ and an orbit O. $\square$

The orbit structure of order-preserving self-maps that have no cycles is described in the following theorem.

Theorem 3.60. *Let T be a function on the countably infinite set X and suppose T has no cycles. Let $A = \{x \in X : |T^{-1}(x)| = \omega\}$. There is a linear order on X with respect to which T is an OP map and $X \approx \mathbb{Z}$ iff each $\mathbb{N}$-orbit has property (C1), $|L_i(O) \backslash T^{-1}(A)| < \omega$ for each $i \in \mathbb{N}$ and an orbit O and either:*

(1) $0 \neq \zeta < \omega$ *and* $|A| = 0$*; or*

(2) $\zeta = 0$ *and* $|A| \leq 1$ *provided that* $|A| = 1$ *when* $\nu = 1$*.*

Proof. Let T be an OP map on $\mathbb{Z}$. Theorem 3.38 implies that each $\mathbb{N}$-orbit has property (C1). Lemma 3.59 implies that $|L_i(O) \setminus T^{-1}(A)| < \omega$ when $\zeta = 0$, Lemma 3.55 implies that $\zeta < \omega$ and if $\zeta \neq 0$ then $|A| = 0$ and $|L_i(O)| < \omega$ for every $i \in \mathbb{Z}$ and orbit O. If $\zeta = 0$, so T has only $\mathbb{N}$-orbits, then by Lemma 3.59 (1) we have $|A| \leq 1$. Now, let $\nu = 1$ and choose a spine of the $\mathbb{N}$-orbit as in Observation 3.40 so that $T^{-1}(L_0(O)) = \emptyset$. Suppose, for a contradiction, that $|A| = 0$, so we have $|L_i(O)| < \omega$ for each $i \in \mathbb{N}$. If the spine S_N of O is indexed as $\{x_i : i \in \mathbb{N}\}$ and if $x_0 < x_1$, then Theorem 3.7 implies that $y < x_0$ holds only if $y \in L_0(O)$; which means that there is a least element of $\mathbb{Z}$, so we have a contradiction. Similarly, if $x_0 > x_1$ we will have a greatest element of $\mathbb{Z}$, which also is a contradiction. Hence, $|L_0(O)| = \omega$ and $|A| = 1$.

Conversely, let $\zeta < \omega$ and $|A| = 0$. Let $\{O_i\}_{i \in I}, I \subseteq \mathbb{N}$, be the collection of all orbits of T. Choose spines of $\mathbb{N}$-orbits as in Observation 3.40 so that $L_i(O) = \emptyset$ for all $i < 0$. Let S be the set of all spine points of orbits of T. By Theorem 3.56, there is a linear order on S with respect to which $S \approx \mathbb{Z}$ and T is an OP map. Since $|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$ and an orbit O, then the proof follows immediately by Theorem 3.43.

Suppose that (2) holds. The case $|A| = 0$ and $\nu > 1$ is included in the proof above, so let $|A| = 1$ and let $y \in A$. Let $\{O_i\}_{i \in I}, I \subseteq \mathbb{N}$, be the collection of all $\mathbb{N}$-orbits of T. Let $y \in O_0$ and choose a spine of O_0 indexed as $\{x_i : i \in \mathbb{N}\}$ so that $T(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$ in such a way that $x_0 \in T^{-1}(y)$ (which is possible since O has property (C1)). By

Theorem 3.53 there is linear orders $\preceq$ on $O = (\bigcup_{i\in I} O_i) \setminus (T^{-1}(y) \setminus \{x_0\})$ so that $O \approx \mathbb{N}$ and T is an OP map. This means that there is $k \in I$ so that $L_0(O_k) \approx [0, n)$ for some $n \in \mathbb{N}$; without loss of generality let $k = 0$, so $x_0 \approx 0$. So, choose an order isomorphism from $T^{-1}(y) \setminus \{x_0\}$ to $(-\infty, 0)$, then X as the ordered sum of $T^{-1}(y) \setminus \{x_0\}$ and O is order isomorphic to $\mathbb{Z}$ and T is OP. $\square$

3.5.3 Order-Preserving Maps with 1-cycles on $\mathbb{Z}$ and the Main Theorem of OP Maps on $\mathbb{Z}$

In this section the main theorem which describes the orbit structure of order-preserving maps on $\mathbb{Z}$ is given. First, we study the same question but we assume that T has 1-cycles which, by Lemma 3.55, means that T has no $\mathbb{Z}$-orbits.

Lemma 3.61. *Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OP map with, in total, one 1-cycle O with spine S. Let $A = \{x \in X : |T^{-1}(x)| = \omega\}$, then $|A| \leq 2$. Moreover, if $|A| = 2$ then $|X \setminus T^{-1}(A)| < \omega$ and if $k = \max\{\|x\| : x \in T^{-1}(S) \setminus S\}$ then $\|y\| = k$ for some $y \in T^{-1}(S) \setminus S$ with $|\bigcup_{k\in\mathbb{N}} T^{-k}(y)| = \omega$.*

Proof. Let $x_1, x_2 \in A$, by Lemma 3.5 we have $T^{-1}(x_1) = (-\infty, a]$ and $T^{-1}(x_2) = [b, \infty)$ for some $a \leq b \in \mathbb{Z}$. Hence, $X \setminus (T^{-1}(x_1) \cup T^{-1}(x_2))$ is finite, so $|A| \leq 2$.

Now, let $S = \{x_0\}$ and suppose that $T^{n_i}(x_i) = x_0, i = 1, 2$, where $n_i = \min\{r \in \mathbb{N} : T^r(x_i) = x_0\}$. Let $n_1 \geq n_2$ and let $y_1 = T^{n_1-1}(x_1)$, so $\|y_1\| = n_1$. Suppose that $t' \in T^{-1}(S) \setminus S, t' \neq y_1, t' \neq T^{n_2-1}(x_2)$ with $\|t'\| = p > 0$, so there is $t \in T^{-p}(t')$ and $T^{p+1}(t) = x_0$. By Lemma 3.5 we have $T^{-1}(x_1) = (-\infty, a]$ and $T^{-1}(x_2) = [b, \infty)$ for some $a < b \in \mathbb{Z}$. So we have $a < t < b$, which implies that $T^{n_1+1}(a) \leq T^{n_1+1}(t) \leq T^{n_1+1}(b)$, i.e., $x_0 \leq T^{n_1+1}(t) \leq x_0$. So, $T^{n_1+1}(t) = x_0$ which means that $p + 1 \leq n_1 + 1$, i.e., $p \leq n_1$. Thus $\|y_1\| = n_1 = \max\{\|x\| : x \in T^{-1}(S) \setminus S\}$, as required. $\square$

Theorem 3.62. *Let $T : \mathbb{Z} \to \mathbb{Z}$ be a finite-to-one OP map with, in total, one 1-cycle with spine $S = \{x_0\}$. Then $|\{x \in T^{-1}(S) \setminus S : \|x\| = \infty\}| > 1$.*

Proof. Suppose, for a contradiction, that there is a unique $y \in T^{-1}(S) \setminus S$ with $\|y\| = \infty$. Let $B = \bigcup_{k \in \mathbb{N}} T^{-k}(y)$, so $|B| = \omega$ and $|X \setminus B| < \omega$. If $y < T(y)$ then we have

$$x_0 = T(y) > y > T^{-1}(y) > T^{-2}(y) > \cdots > T^{-n}(y) > T^{-n-1}(y) > \dots,$$

since T is an OP map. Since $|X \setminus B| < \omega$, then we will have some $z \in \{x_0\} \cup (X \setminus B)$ is the greatest element of X, which is a contradiction. The case $y > T(y)$ follows in the same way. Hence, $|\{x \in T^{-1}(S) \setminus S : \|x\| = \infty\}| > 1$. □

Let O be a 1-cycle with $|O| = \omega$ we say that O satisfies:

(C2) if $T \restriction O$ is either finite-to-one or there is a unique point $y \in O$ with $|T^{-1}(y)| = \omega$ and O satisfies that $|O \setminus T^{-1}(y)| < \omega$.

Also we say that O satisfies:

(C3) if $T^{-1}(S) \setminus S = A_1 \cup A_2$ such that $S \cup \bigcup_{k \in \mathbb{N}} T^{-k}(A_i), i = 1, 2$ has (C2) provided that if $|\{x \in T^{-1}(S) \setminus S : \|x\| = \infty\}| = 0$ and if $n = \max\{\|x\| : x \in T^{-1}(S) \setminus S\}$ then $\|y\| = n$ for some $y \in T^{-1}(S) \setminus S$ with $|\bigcup_{k \in \mathbb{N}} T^{-k}(y)| = \omega$.

Theorem 3.63. *Let $T : X \to X$ be a function on the countably infinite set X and let $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. Let $\mathcal{O}$ be the collection of all cycles of T. If $\nu = 0$ and $\mathcal{O} \neq \emptyset$, then there is a linear order on X with respect to which T is OP and $X \approx \mathbb{Z}$ iff $\zeta = \sigma_n = 0$ for all $n \geq 2$, each $O \in \mathcal{O}$ has (C1), $|\{x : |T^{-1}(x)| = \omega\}| \leq 2$ and either:*

(1) *$\sigma_1 = \omega$ with $|O| < \omega$ for all $O \in \mathcal{O}$ except for at most one orbit O which has* (C2)*; or*

(2) $1 < \sigma_1 < \omega$ *with* $|O| < \omega$ *for all* $O \in \mathcal{O}$ *except for two cycles each has* (C2)*; or*

(3) $\sigma_1 = 1$ *with* O *has* (C3).

Proof. Let T be an OP map on $\mathbb{Z}$, then by Theorem 3.38 we have each $O \in \mathcal{O}$ has (C1). Lemma 3.55 (1) implies that $\zeta = 0$ and by Lemma 3.3 we have $\sigma_n = 0$ for all $n \geq 2$. Also, by Lemma 3.61, we have $|\{x : |T^{-1}(x)| = \omega\}| \leq 2$. If $\sigma_1 \neq 1$, then by Lemma 3.54 and Theorem 3.53 we have either (1) or (2) holds. Let $\sigma_1 = 1$, so T has one 1-cycle O with spine $S = \{x_0\}$. If T is finite-to-one, then by Theorem 3.62, we have $|\{x \in T^{-1}(S) \setminus S : \|x\| = \infty\}| > 1$; so O has (C3). Let $A = \{x : |T^{-1}(x)| = \omega\}$. If $|A| = 1$, say $y_0 \in A$, then by Lemma 3.5 we have either $T^{-1}(y_0) = (a, \infty)$ or $T^{-1}(y_0) = (-\infty, a)$ for some $a \in \mathbb{Z}$, so $O \setminus T^{-1}(y_0)$ is either $[a, \infty)$ or $(-\infty, a]$. So, $O \setminus T^{-1}(y_0)$ is infinite and since O has (C1), then $|\{x \in T^{-1}(S) \setminus S : \|x\| = \infty\}| \neq 0$. Let $A_1 = \{x \in T^{-1}(S) \setminus S : \|x\| = \infty\}$ and $A_2 = (T^{-1}(S) \setminus S) \setminus A_1$. So, if $X_i = \bigcup_{k \in \mathbb{N}} T^{-k}(A_i) \cup S, i = 1, 2$ then $T \restriction X_1$ is finite-to-one and $y_0 \in X_2$ with $|X_2 \setminus T^{-1}(y_0)| < \omega$, so both of X_1, X_2 has (C2); hence O has (C3).

If $|A| = 2$, say $y, y' \in A$, then by Lemma 3.61 we have $|X \setminus T^{-1}(A)| < \omega$; hence, $|\{x \in T^{-1}(S) \setminus S : \|x\| = \infty\}| = 0$. Also, if $n = \max\{\|x\| : x \in T^{-1}(S) \setminus S\}$, then Lemma 3.61 implies that $\|z\| = n$ for some $z \in T^{-1}(S) \setminus S$ with $|\bigcup_{k \in \mathbb{N}} T^{-k}(z)| = \omega$. So let $A_1 = \{z\}$ and $A_2 = (T^{-1}(S) \setminus S) \setminus A_1$ then $X_i = \bigcup_{k \in \mathbb{N}} T^{-k}(A_i) \cup S, i = 1, 2$ has (C2), so O has (C3).

Conversely, (1) and (2) follow immediately from Lemma 3.42 and Theorem 3.53. Suppose that (3) holds, so T has a single 1-cycle O with spine $S = \{x_0\}$ and $T^{-1}(S) \setminus S = A_1 \cup A_2$ such that $S \cup \bigcup_{k \in \mathbb{N}} T^{-k}(A_i), i = 1, 2$ has (C2). By Lemma 3.42, it is sufficient to find two sets $X_1, X_2 \subset X$ with linear ordered $\preceq_1, \preceq_2$ respectively such that $X_1 \cap X_2 = \{x_0\}$, $X_1 \approx X_2 \approx \mathbb{N}$, x_0 is the least element of both of X_1 and X_2, $T \restriction X_1$ and $T \restriction X_2$ are OP maps. So, if X_1 and X_2 were found, then X as the ordered sum of $(X_1, \preceq_1^{-1})$ and $(X_2, \preceq_2)$ is order-isomorphic to $\mathbb{Z}$ and T is OP. So, we have the following cases:

Case (1): T is finite-to-one, let $X_i = S \cup \bigcup_{k\in\mathbb{N}} T^{-k}(A_i), i = 1, 2$ so both of $T \restriction X_1, T \restriction X_2$ is finite-to-one and $X_1 \cap X_2 = \{x_0\}$. By Lemma 3.48, there is linear orders on each of X_1, X_2 in such a way that x_0 is the least element of both of them, and with respect to which $X_1 \approx X_2 \approx \mathbb{N}$, $T \restriction X_1$ and $T \restriction X_2$ are OP maps.

Case (2): $|\{x : |T^{-1}(x)| = \omega\}| = 1$, so $T \restriction (S \cup \bigcup_{k\in\mathbb{N}} T^{-k}(A_1))$ is finite-to-one and $S \cup \bigcup_{k\in\mathbb{N}} T^{-k}(A_2)$ contains a point y with $|T^{-1}(y)| = \omega$. Let

$$X_1 = S \cup \bigcup_{k\in\mathbb{N}} T^{-k}(A_1) \cup \{x \in T^{-1}(S) \setminus S : \|x\| \geq 1, |\bigcup_{k\in\mathbb{N}} T^{-k}(x)| < \omega\}$$

and $X_2 = S \cup (X \setminus X_1)$. Since $T \restriction X_1$ is finite-to-one, then, by Lemma 3.48, there is a linear order on X_1 in such a way that x_0 is the least element of X_1, $X_1 \approx \mathbb{N}$ and $T \restriction X_1$ an OP map. Since X has (C1), then X_2 also has (C1) and $T \restriction X_2$ satisfies conditions of Proposition 3.52, hence there is a linear order on X_2 in such a way that x_0 is the least element of X_2 such that $X_2 \approx \mathbb{N}$ and $T \restriction X_2$ is OP.

Case (3): $|\{x : |T^{-1}(x)| = \omega\}| = 2$ and there is $y_1 \in T^{-1}(S) \setminus S$ with $|\bigcup_{k\in\mathbb{N}} T^{-k}(y_1)| = \omega$ and $\|z\| \leq \|y_1\|$ for each $z \in T^{-1}(S) \setminus S$. Let

$$X_1 = S \cup \bigcup_{k\in\mathbb{N}} T^{-k}(y_1) \cup \{x \in T^{-1}(S) \setminus S : \|x\| \geq 1, |\bigcup_{k\in\mathbb{N}} T^{-k}(x)| < \omega\}$$

and $X_2 = S \cup (O \setminus X_1)$. Hence, each of $T \restriction X_1$ and $T \restriction X_2$ satisfies conditions of Proposition 3.52, so the proof is complete. $\square$

Now, the following theorem describes the situation when $T : X \to X$ has both a 1-cycle and an $\mathbb{N}$-orbit so, by Lemma 3.55, T has no $\mathbb{Z}$-orbits.

Theorem 3.64. *Let T be a function on a countably infinite set X with orbit spectrum $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. If $\sigma_1 \neq 0$ and $\nu \neq 0$, then there is a linear order on X with respect to which T is OP and $X \approx \mathbb{Z}$ iff $\zeta = \sigma_n = 0$ for each $n \geq 2$, $|L_i(O)| < \omega$ for each*

$i \in \mathbb{N}$ and $\mathbb{N}$-orbit O, each orbit has property (C1) and either:

(1) *$\sigma_1 = \omega$ with $|O| < \omega$ for each 1-cycle O;or*

(2) *$\sigma_1 < \omega$ and either:*

 (a) *$|O| < \omega$ for all 1-cycles except for a unique 1-cycle which has* (C2)*;*

 (b) *$|O| < \omega$ for all 1-cycles and $\nu \neq 1$.*

Proof. Let T be an OP map on $\mathbb{Z}$, by Lemma 3.3 and Lemma 3.55 (1) we have $\zeta = \sigma_n = 0$ for all $n \geq 2$. By Lemma 3.38 we have each orbit has property (C1). Since $\sigma_1 \neq 0$, then by Lemma 3.6 we have a spine of a 1-cycle is equal to n for some $n \in \mathbb{Z}$; hence any $\mathbb{N}$-orbit O of T must be either a subset of $(-\infty, n)$ or a subset of (n, ∞). Hence, by Theorem 3.53 we have that $|L_i(O)| < \omega$ for all $i \in \mathbb{N}$ and an $\mathbb{N}$-orbit O. Also, by Lemma 3.54 and Theorem 3.53 we have either (1), (2) (a) or (2) (b) holds.

Conversely, let $\mathcal{O} = \{O'_i\}_{i\in I}, I \subseteq \mathbb{N}$, be the collection of all $\mathbb{N}$-orbits, then by Theorem 3.53, there is a linear order $\preceq_1$ on $O' = \bigcup_{i\in I} O'_i$ such that $O' \approx \mathbb{N}$ and $T \restriction O'$ is an OP map. Let $\{O_j\}_{j\in J}, J \subseteq \mathbb{N}$, be the collection of all 1-cycles with $|O_j| < \omega$. If either (1) or (2), (a) holds, then by Theorem 3.53, there is a linear order $\preceq_2$ on $O = \bigcup_{j\in J} O_j$ such that $O \approx \mathbb{N}$ and $T \restriction O$ is an OP map. Hence, by applying Lemma 3.42, we have X as the ordered sum of $(O', \preceq_1^{-1}), (O, \preceq_2)$ is order isomorphic to $\mathbb{Z}$ and T is an OP map.

Now suppose that (b) holds, so $\sigma_1 < \omega$ with $|O| < \omega$ for all 1-cycles and $\nu \neq 1$. Let $O'_1 \in \mathcal{O}$, so $O' \setminus O'_1$ has a linear order $\preceq'_1$ which is the restriction of $\preceq_1$ on $O' \setminus O'_1$. Also, by Theorem 3.53, there is a linear order $\preceq$ on $C = O'_1 \cup \bigcup_{j\in J} O_j$ so that C is order-isomorphic to $\mathbb{N}$ and $T \restriction C$ is OP. Hence, again by Lemma 3.42, we have X as the ordered sum of $(C, \preceq^{-1})$ and $(O' \setminus O'_1, \preceq'_1)$ is order isomorphic to $\mathbb{Z}$ and T is an OP map. $\square$

Finally, we combine Theorem 3.60, Theorem 3.63 and Theorem 3.64 together, so we have the following main theorem which describes the orbit structure of maps on the sets that are order-isomorphic to the integers $\mathbb{Z}$ with their usual order.

Theorem 3.65. *Let T be a function on a countable set X with orbit spectrum $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. Let $A = \{x \in X : |T^{-1}(x)| = \omega\}$ and $\mathcal{O}$ be the collection of all 1-cycles of T. There is a linear order on X with respect to which T is an OP map and $X \approx \mathbb{Z}$ iff $\sigma_n = 0$ for all $n > 1$, each 1-cycle and $\mathbb{N}$-orbit has property* (C1) *and either:*

(1) $\sigma_1 = 0$ *and either:*

  (a) $0 \neq \zeta < \omega$ *and* $|A| = 0$*; or*

  (b) $\zeta = 0$ *and* $|A| \leq 1$ *provided that* $|A| = 1$ *when* $\nu = 1$.

(2) $\zeta = \nu = 0$ *and either:*

  (a) $\sigma_1 = \omega$ *with* $|O| < \omega$ *for all* $O \in \mathcal{O}$ *except for at most one orbit* O *which has* (C2)*;*

  (b) $1 < \sigma_1 < \omega$ *with* $|O| < \omega$ *for all* $O \in \mathcal{O}$ *except for two cycles each has* (C2)*; or*

  (c) $\sigma_1 = 1$ *with* O *has* (C3).

(3) $\zeta = 0$, $\nu \neq 0$, $\sigma_1 \neq 0$ *and either:*

  (a) $\sigma_1 = \omega$ *with* $|O| < \omega$ *for all* $O \in \mathcal{O}$ *;*

  (b) $\sigma_1 < \omega$ *with* $|O| < \omega$ *for all* $O \in \mathcal{O}$ *except for a unique 1-cycle with* (C2)*; or*

  (c) $\sigma_1 < \omega$ *with* $|O| < \omega$ *for all* $O \in \mathcal{O}$ *and* $\nu \neq 1$.

Chapter 4

Order-Reversing Maps on Countable Linear Orders

In this chapter, we study the orbit structure of order-reversing map on rationals $\mathbb{Q}$, integers $\mathbb{Z}$ and naturals $\mathbb{N}$ with their usual order. In the first section with give some results and properties of order-reversing self-maps on arbitrary sets. Next we give characterization of order-reversing bijections and injections on the rationals $\mathbb{Q}$. The main theorem of the second section is order-reversing surjections on the rationals $\mathbb{Q}$. In the final two sections we describe the orbit structure of order-reversing self-maps on sets that are order-isomorphic to $\mathbb{N}$ or $\mathbb{Z}$.

A number of authors have studied order-reversing maps. For example, Björner [1] gave conditions under which order-reversing map of a complete lattice into itself has a unique fixed point, where A. Tarski in [30] had earlier shown that every order-preserving map of a complete lattice into itself has a fixed point. In [5], among other results the maximal regular subsemigroup of the ideals of the semigroup of all order-preserving or order-reversing transformations on the chain $\{1, \ldots, n\}$ under the natural order was characterized.

4.1 Preliminaries of Order-Reversing Self-Maps

In this section we study some preliminaries and results of order-reversing, or OR, self-maps on arbitrary sets. Some of these lemmas and results may be known, we include proofs for completeness. We start with the following basic definition (see for example [13]).

Definition 4.1. Let $(X, \preceq_1)$ and $(Y, \preceq)$ be ordered sets. A map $T : X \to Y$ is said to be *order-reversing* (or *antitone* map) if $x \preceq_1 y$ implies that $T(x) \succeq T(y)$.

The following result is a well known fact about order-reversing and order-preserving maps which we need to use during this chapter.

Lemma 4.2. *Let $T : (X, \preceq_1) \to (Y, \preceq_2)$ be a map, where $X \cap Y = \emptyset$.*

(1) *If T is OP then each of $T : (X, \preceq_1) \to (Y, \preceq_2^{-1})$ and $T : (X, \preceq_1^{-1}) \to (Y, \preceq_2)$ is an OR map.*

(2) *If T is OR then each of $T : (X, \preceq_1) \to (Y, \preceq_2^{-1})$ and $T : (X, \preceq_1^{-1}) \to (Y, \preceq_2)$ is an OP map.*

Lemma 4.3. *Let X be a linearly ordered set and $T : X \to X$ be an OR map with orbit spectrum $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. Then*

(1) *T has at most one fixed point.*

(2) *$\sigma_n = 0$ for all $n > 2$.*

Proof. (1) Clearly, if we suppose that T has two fixed points x_0 and x_1 such that $x_0 \preceq x_1$, then we will have $T(x_0) \succeq T(x_1)$, i.e. $x_0 \succeq x_1$; hence $x_0 = x_1$. Consequently, T has at most one fixed point.

(2) Suppose, for a contradiction, that T has a k-cycle, $k > 2$. Since T is an OR map then T^2 is an OP map so, by Lemma 1.9, if k is odd then T^2 has a k-cycle, and if k is even

then T^2 has two cycles each of length $k/2$. But from Lemma 3.3, this is a contradiction since OP maps have no cycles of length greater than 1. $\square$

Lemma 4.4. *Let $(I, \preceq_I)$ be a linearly ordered set, $\{(X_i, \preceq_i)\}_{i \in I}$ be a collection of linearly ordered pairwise disjoint sets and $\{T_i : X_i \to X_i\}_{i \in I}$ be a collection of OR maps. There is a linear order $\preceq$ on $X = \bigcup_{i \in I} X_i$ such that the map T defined as $T \restriction X_i = T_i$ is an OR map provided that T has at most one 1-cycle. Moreover, for each $i \in I$ we have $\preceq_i$ is the restriction of $\preceq$ on X_i.*

Proof. Assume that T_t, $t \in I$ has at most 1-cycle. For each $i \in I \setminus \{t\}$, let

$$X_{i,1} = \{x \in X_i : T_i(x) \preceq_i x\}$$

with the restriction of $\preceq_i$ on $X_{i,1}$ and

$$X_{i,2} = \{x \in X_i : x \preceq_i T_i(x)\}$$

with the restriction of $\preceq_i$ on $X_{i,2}$. So, $X_i = X_{i,1} \cup X_{i,2}$, $T(X_{i,1}) \subseteq X_{i,2}$, $T(X_{i,2}) \subseteq X_{i,1}$ and $X_{i,1} \cap X_{i,2} = \emptyset$. Let $(Y_1, \preceq')$ be the ordered sum of $X_{i,1}$ over $I \setminus \{t\}$ and $(Y_2, \preceq)$ be the ordered sum of $X_{i,2}$ over $(I \setminus \{t\}, \preceq_I^{-1})$. Now consider $(X, \preceq_X)$ to be the ordered sum of $(Y_2, \preceq)$, $(X_t, \preceq_t)$ and $(Y_1, \preceq')$ respectively.

Now, we prove that T is an OR map under $\preceq_X$. Let $x, y \in X$ with $x \preceq_X y$, if $x, y \in X_t$ then $T(y) \preceq_t T(x)$ so $T(y) \preceq_X T(x)$. If $x, y \in Y_1$, so $x \in X_{i,1}, y \in X_{j,1}$ for some $i \preceq_I j \in I \setminus \{t\}$, then $T(x) \in X_{i,2}, T(y) \in X_{j,2}$. Since $j \preceq_I^{-1} i$, then $T(y) \preceq T(x)$ so $T(y) \preceq_X T(x)$. If $x, y \in Y_2$, then the proof follows in the same way. If $x \in Y_2, y \in X_t$, then $T(x) \in Y_1, T(y) \in X_t$, so $T(y) \preceq_X T(x)$. Similarly, if $x \in X_t, y \in Y_1$, then $T(y) \preceq_X T(x)$. If $x \in Y_2, y \in Y_1$, then $T(x) \in Y_1$ and $T(y) \in Y_2$, so $T(y) \preceq_X T(x)$. Hence, T is an OR map. $\square$

Lemma 4.5. *Let $(X, \preceq)$ be a linearly ordered set and $T : X \to X$ be an OR map. Then for all $x \in X$, if $y, z \in T^{-k}(x)$ for some $k \in \mathbb{N}$ with $y \preceq z$ and if $y \preceq t \preceq z$ for some $t \in X$ then $t \in T^{-k}(x)$.*

Proof. Let $x \in X$ and $y, z \in T^{-k}(x)$ for some $k \in \mathbb{N}$ with $y \preceq z$. Suppose that there is a $t \in X$ such that $y \preceq t \preceq z$. Since T is an OR map, then we have $T(z) \preceq T(t) \preceq T(y)$ so $T^k(z) \preceq T^k(t) \preceq T^k(y)$ if k is odd and $T^k(y) \preceq T^k(t) \preceq T^k(z)$ if k is even. But in both of the cases, this means that $x \preceq T^k(t) \preceq x$, i.e., $T^k(t) = x$. Hence, $t \in T^{-k}(x)$, as was required. $\square$

Lemma 4.6. *Let T be an OR map on a linearly ordered set $(X, \preceq)$. Let O be a 2-cycle of T with spine $\{x_0, x_1\}$ and $x_0 \preceq x_1$. Then for each orbit O' of T, $O' \neq O$, we have either $O' \subseteq (x_0, x_1)$ or $O' \cap (x_0, x_1) = \emptyset$. Moreover, $O' \subseteq (x_0, x_1)$ when O' is a 1-cycle.*

Proof. Let O' be an orbit of T and suppose that $z \in O'$. If $x_0 \preceq z \preceq x_1$, then we have

$$x_0 = T(x_1) \preceq T(z) \preceq T(x_0) = x_1,$$

since T is OR. So, for each $k \in \mathbb{N}$ we have $x_0 \preceq T^k(z) \preceq x_1$, i.e. $T^k(z) \in (x_0, x_1)$. Now, suppose, for a contradiction, that there is $t \in O'$ with $t \notin (x_1, x_2)$. If $t \preceq x_0$, then we have $T^k(t) \preceq T^k(x_0) = x_0$ whenever k is even and $T^k(y) \succeq T^k(x_0) = x_1$ whenever k is odd; so for each $k \in \mathbb{N}$ we have $T^k(t) \notin (x_0, x_1)$. Since t and z are in the same orbit then we have $T^n(z) = T^m(t)$ for some $n, m \in \mathbb{N}$. But $T^n(z) \in (x_0, x_1)$ and $T^m(t) \notin (x_0, x_1)$, which is a contradiction. The case when $t \succeq x_1$ follows in the same way, so $t \in (x_0, x_1)$. Hence, $x_0 \preceq y \preceq x_1$ for each $y \in O'$. Finally, if O' is a 1-cycle with spine $\{y_0\}$ then immediately we have $x_0 \preceq y_0 \preceq x_1$ and from Lemma 4.5 we have $O' \subseteq (x_0, x_1)$. $\square$

Corollary 4.7. *Let T be an OR map on a linearly ordered set $(X, \preceq)$. Let O be a 2-cycle of T with spine $\{x_0, x_1\}$ and $x_0 \preceq x_1$. If T has a 2-cycle $O' = \{y_0, y_1\}$ with $y_0 \preceq y_1$ and if $O' \cap (x_0, x_1) = \emptyset$ then $y_0 \preceq x_0$ and $y_1 \succeq x_1$.*

Theorem 4.8. *Let T be an OR map on a linearly ordered set $(X, \preceq)$, where T is either an injection with a single orbit or X is a semi-simple cycle of T. If $x \in X$ with $x, T(x)$ do not lie in a spine of a cycle and $x \preceq T(x)$, then either $\bigcup_{k>0}\{T^k(x)\} \subseteq (x, T(x))$ and $(x, T(x)) \cap \bigcup_{k>0} T^{-k}(x) = \emptyset$ or $\bigcup_{k>0} T^{-k}(x) \subseteq (x, T(x))$ and $(x, T(x)) \cap \bigcup_{k>0}\{T^k(x)\} = \emptyset$. Moreover, if O is a 1-cycle, then $\bigcup_{k>0}\{T^k(x)\} \subseteq (x, T(x))$ and $(x, T(x)) \cap \bigcup_{k>0} T^{-k}(x) = \emptyset$.*

Proof. By Lemma 4.3, T has no n-cycles with $n > 2$. Let O be either a $\mathbb{N}$-orbit, $\mathbb{Z}$-orbit or a semi-simple n-cycle, $n = 1, 2$. Let $x \in X$ with $x, T(x)$ are not in a spine of a cycle and let $x \preceq T(x)$. From the assumption that $x \preceq T(x)$ we have $T^2(x) \preceq T(x)$ since T is an OR map, so we have the following cases:

Case (1): $x \preceq T^2(x) \preceq T(x)$. We claim that for each $2 < k \in \mathbb{N}$, $x \preceq T^k(x) \preceq T(x)$. For $k = 3$ we have $T(x) \succeq T^3(x) \succeq T^2(x)$, since T is OR, hence from the assumption we have $x \preceq T^2(x) \preceq T^3(x) \preceq T(x)$. Now, suppose that $x \preceq T^m(x) \preceq T(x)$ for each $m \leq k$, then we have $T(x) \succeq T^{k+1}(x) \succeq T^2(x)$. From the assumption that $x \preceq T^2(x)$ we have $x \preceq T^2(x) \preceq T^{k+1}(x) \preceq T(x)$. Hence, $T^k(x) \in (x, T(x))$ for each $0 < k \in \mathbb{N}$.

Now we deal with $T^{-1}(x)$ in the case when $T^{-1}(x) \neq \emptyset$. $x \preceq T(x)$ implies that $x \preceq T^{-1}(x)$, also the assumption $x \preceq T^2(x)$ implies that $T(x) \preceq T^{-1}(x)$, so $T^{-1}(x) \notin (x, T(x))$. Now suppose that $T^{-m}(x) \notin (x, T(x))$ for all $m \leq k$, so either $T^{-k}(x) \succeq T(x)$ which implies that $T^{-k-1}(x) \preceq T^{-1}T(x) = x$, or $T^{-k}(x) \preceq x$ which implies that $T^{-k-1}(x) \succeq T^{-1}(x)$, so as we showed above $T^{-k-1}(x) \succeq T^{-1}(x) \succeq T(x)$. Hence, in both cases $T^{-k-1}(x) \notin (x, T(x))$. Thus, $T^{-k}(x) \notin (x, T(x))$ for each $0 < k \in \mathbb{N}$, as required.

Case (2): $T^2(x) \preceq x \preceq T(x)$ with $T^2(x) \neq T(x)$. So, $T^2(x) \preceq x$ implies that $T^3(x) \succeq T(x)$. Suppose that $T^m(x) \notin (x, T(x))$ for all $m \leq k$. If $T^k(x) \preceq x$ then $T^{k+1}(x) \succeq T(x)$ (provided that $T^{k+1}(x) \neq T^k(x)$). If $T^k(x) \succeq T(x)$, then $T^{k+1}(x) \preceq T^2(x)$ (provided that $T^{k+1}(x) \neq T^k(x)$), but from the assumption that $T^2(x) \preceq x$ we have $T^{k+1}(x) \preceq T^2(x) \preceq x$. Hence, both of the cases imply that $T^{k+1}(x) \notin (x, T(x))$. Thus,

$T^k(x) \notin (x, T(x))$ provided that $T^{k+1}(x) \neq T^k(x)$ for all $1 < k \in \mathbb{N}$, i.e., O is not a 1-cycle.

Now, $T^2(x) \preceq x \preceq T(x)$ implies that $T(x) \succeq T^{-1}(x) \succeq x$. Suppose that $x \preceq T^{-m}(x) \preceq T(x)$ for all $m \leq k$. Since T is an OR map, then this implies that $T^{-1}(x) \succeq T^{-k-1}(x) \succeq x$. Hence, $T^{-k}(x) \in (x, T(x))$ for each $0 \neq k \in \mathbb{N}$, as required. $\square$

From the proof of the previous theorem we immediately have the following corollary.

Corollary 4.9. *Let T be an OR map on a linearly ordered set $(X, \preceq)$, where T is either an injection with, in total, one orbit or X is a semi-simple cycle of T. Let $x, T(x) \in X$ do not lie in a spine of a cycle and $x \preceq T(x)$.*

(1) *If $(x, T(x)) \cap \bigcup_{k>0}\{T^k(x)\} = \emptyset$, then $\{T^{2m}(x) : m \in \mathbb{N}\} \preceq x$ and $\{T^{2m+1}(x) : m \in \mathbb{N}\} \succeq T(x)$.*

(2) *If $(x, T(x)) \cap \bigcup_{k>0} T^{-k}(x) = \emptyset$, then $\{T^{-2m}(x) : m \in \mathbb{N}\} \preceq x$ and $\{T^{-2m-1}(x) : m \in \mathbb{N}\} \succeq T(x)$.*

Theorem 4.10. *Let T be an injection on a countably infinite set X and let $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. If $\sigma_n = 0$ for all $n \in \mathbb{N}$ and $0 \neq \zeta + \nu < \omega$, then any linear order $\preceq$ on X with respect to which T is an OR map satisfies the following: for any $x \in X$, there is either $y \succ x$ such that $(x, y) = \emptyset$ or $z \prec x$ such that $(z, x) = \emptyset$.*

Proof. Let $0 \neq \zeta + \nu = n < \omega$. Suppose, for a contradiction, that T is an OR map on $(X, \preceq)$ and $\preceq$ does not satisfy the condition above. Let $\sigma(T^2) = (\nu', \zeta', \sigma'_1, \sigma'_2, \sigma'_3, \dots)$, then Lemma 1.9 implies that $\zeta' + \nu' = 2n < \omega$. Since T is an OR map, then T^2 is an OP map on $(X, \preceq)$ with $\zeta' + \nu' = 2n < \omega$, which is a contradiction by Theorem 3.12. Thus, for every $x \in X$, there is either $y \succ x$ such that $(x, y) = \emptyset$ or $z \prec x$ such that $(z, x) = \emptyset$. $\square$

Let $T : X \to X$ be a function and let O be either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit, a 1-cycle or a 2-cycle of T. Let S be a spine of O indexed as $\{x_j : j \in M\}$, where M is either $\mathbb{N}$, $\mathbb{Z}$, $\{0\}$ or $\{0, 1\}$ as appropriate, such that $T(x_j) = x_{j+1}$ for each $j \in M$ and j is taken module n when O is an n-cycle. For all $i \in N$, where N is either $\mathbb{Z}$, $\{0\}$ or $\{0, 1\}$, let $C_{i,k}$ be defined as before (see Figure 3.1 and Figure 3.2 for 1-cycles and $\mathbb{Z}$-orbits). For all $i \in N$, let $L_i(O)$ be the set defined in Definition 1.7 (also $L_i(O) = \bigcup_{k\in\mathbb{N}} C_{i,k}$). Terminology of 2-cycles is illustrated in Figure 4.1 .

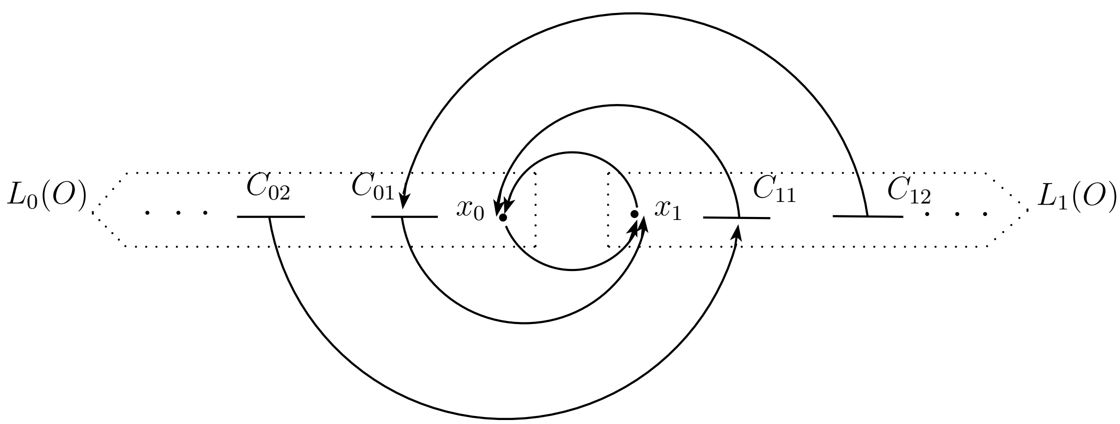

Figure 4.1: A 2-cycle O.

Lemma 4.11. *Let $T : X \to X$ be a function and let O be an orbit of T, where O is either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit, a 1-cycle or a 2-cycle. Suppose that S is a spine of O. There is a family of linear orders on the sets $C_{i,k}$ ($i \in N, 0 \neq k \in \mathbb{N}$, where N is $\mathbb{Z}$, $\mathbb{N}$, $\{0\}$ or $\{0, 1\}$ according to the nature of the orbit) with respect to which $T \restriction C_{i,k}$ is an OR map.*

Proof. By Lemma 3.14, there is a family of linear orders $\preceq_{ik}$ on the sets $C_{i,k}$ with respect to which $T \restriction C_{i,k}$ is an OP map. For each $0 \neq k \in \mathbb{N}$, if k is even define an order on $C_{i,k}$ to be the same order $\preceq_{i,k}$; and if k is odd take the order on $C_{i,k}$ to be $\preceq_{i,k}^{-1}$. Since for each even number $j = 2k$, $k \in \mathbb{N}$ we have

$$T \restriction C_{i,j} : (C_{i,j}, \preceq_{ij}) \to (C_{i+1,j-1}, \preceq_{i+1,j-1})$$

is OP, then by Lemma 4.2 we have the map

$$T \restriction C_{i,j} : (C_{i,j}, \preceq_{ij}) \to (C_{i+1,j-1}, \preceq_{i+1,j-1}^{-1})$$

is OR. Also, it follows that for each odd number $j = 2k+1$, $k \in \mathbb{N}$, we have

$$T \restriction C_{i,j} : (C_{i,j}, \preceq_{ij}^{-1}) \to (C_{i+1,j-1}, \preceq_{i+1,j-1})$$

is also OR. So the proof is complete. $\square$

Corollary 4.12. *Let $T : X \to X$ be a function and let O be an orbit of T with spine S, where O is either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit or a 2-cycle. There is a family of linear orders on the sets $L_i(O) \neq \emptyset$ ($i \in N$, where N is $\mathbb{Z}$, $\mathbb{N}$ or $\{0, 1\}$ according to the nature of the orbit) with respect to which $T \restriction L_i(O)$ is an OR map.*

Proof. Let O be an orbit of T, where O is either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit or a 2-cycle. By Lemma 4.11, there is a family of linear orders on the sets $C_{i,k}$, $0 \neq k \in \mathbb{N}$, with respect to which $T \restriction C_{i,k}$ is an OR map. Let $\preceq_{ik}$ denoted the order on $C_{i,k}$. For each $i \in N$, let $L_i(O) = \bigcup_{n \in \mathbb{N}} C_{i,n}$ be defined. Let $E = \{k \in \mathbb{Z} : k \text{ is even}\}$ and $E' = \{k \in \mathbb{Z} : k \text{ is odd}\}$. For each $i \in N$, if $i \in E$ define $(L_i(O), \preceq_i)$ to be the ordered sum of $(C_{i,k}, \preceq_{ik})$ over $(\mathbb{N}, \leq^{-1})$, and if $i \in E'$, define $(L_i(O), \preceq_i)$ to be the ordered sum of $(C_{i,k}, \preceq_{ik})$ over $\mathbb{N}$.

Now, we prove that $T \restriction L_i(O)$ is OR. Notice that if $x \in C_{i,n}, y \in C_{i,m}, m, n > 0$ and $i \in E$, then $T(x) \in C_{i+1,n-1}$ and $T(y) \in C_{i+1,m-1}$, so $i+1 \in E'$ (where $i+1$ is taken module n when O is n-cycle). Hence, if $x \preceq_i y$ then $n \geq m$ so $n-1 \geq m-1$; hence $T(x) \succeq_{i+1} T(y)$ and $T \restriction L_i(O)$ is an OR map. The case when $i \in E'$ follows in the same way. Thus, $T \restriction L_i(O)$ is an OR map. $\square$

Theorem 4.13. *Let $T : X \to X$ be a function with an orbit O, where O is either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit, a 1-cycle or a 2-cycle. Then there is a linear order on O with respect to which $T \restriction O$ is an OR map.*

Proof. Let O be either a $\mathbb{Z}$-orbit, an $\mathbb{N}$-orbit or a 2-cycle. By Corollary 4.12, there is a family of linear orders on the sets $L_i(O) \neq \emptyset$ ($i \in N$, where N is $\mathbb{Z}$, $\mathbb{N}$ or $\{0,1\}$ according to the nature of the orbit) with respect to which $T \restriction L_i(O)$ is an OR map. Let $E = \{k \in \mathbb{Z} : k \text{ is even}\}$ and $E' = \{k \in \mathbb{Z} : k \text{ is odd}\}$. Let $(X_1, \preceq_1)$ be the ordered sum of $(L_{2m}(O), \preceq_{2m})$ over $(N, \leq^{-1})$ and let $(X_2, \preceq_2)$ be the ordered sum of $(L_{2k+1}(O), \preceq_{2k+1})$ over N. Let $(X, \preceq)$ be the ordered sum of X_1 and X_2 respectively. We prove that $T \restriction O$ is an OR map under $\preceq$. Let $x \in L_i(O), y \in L_k(O)$ for some $i, k \in N$, then we have $T(x) \in L_{i+1}(O), y \in L_{k+1}(O)$, where $i+1, k+1$ is taken module 2 when O is a 2-cycle. If $i \in E$ and $k \in E'$ then $i+1 \in E'$ and $k+1 \in E$, so $T(x) \succeq T(y)$. If $i, k \in E$ then $i \geq k$, hence $i+1 \geq k+1$ and $i+1, k+1 \in E'$, so $T(x) \succeq T(y)$. The case when $i, k \in E'$ follows in a the same way. Consequently, $T \restriction O$ is an OR map.

Now, if O is a 1-cycle then, by Lemma 4.11, there is a family of linear orders $\preceq_{0k}$ on the sets $C_{0,k} \neq \emptyset$, $0 \neq k \in \mathbb{N}$, with respect to which $T \restriction C_{0,k}$ is an OR map. Let $(X_1, \preceq_1)$ be the ordered sum of $(C_{0,2m}, \preceq_{0,2m})$ over $(\mathbb{N}, \leq^{-1})$ and let $(X_2, \preceq_2)$ be the ordered sum of $(C_{0,2k+1}, \preceq_{0,2k+1})$ over $\mathbb{N}$. Let $(X, \preceq)$ be the ordered sum of X_1 and X_2 respectively.

Finally, we prove that $T \restriction O$ is OR under $\preceq$. Let $x \in C_{0,i}, y \in C_{0,k}$ for some $0 < i, k \in \mathbb{N}$, then we have $T(x) \in C_{0,i+1}, y \in C_{0,k+1}$. If $i = k$ then the proof is clear. If $i \in E$ and $k \in E'$ then $i+1 \in E'$ and $k+1 \in E$, so $T(x) \succeq T(y)$. If $i, k \in E$ then $i \geq k$, so $i+1 \geq k+1$ and $i+1, k+1 \in E'$, hence $T(x) \succeq T(y)$. The case when $i, k \in E'$ follows in a similar way. Consequently, $T \restriction O$ is an OR map. $\square$

Theorem 4.14. *Let $T : X \to X$ be a map with orbit spectrum $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$.*

There is a linear order on X such that T is an OR map if and only if $\sigma_n = 0$ for all $n > 2$ and $\sigma_1 \leq 1$.

Proof. If $T : X \to X$ is an OR map then by Lemma 4.3 we have $\sigma_n = 0$ for all $n > 2$ and $\sigma_1 \leq 1$. Conversely, let $\mathcal{O} = \{O_i\}_{i\in I}$ be the collection of all orbits of T, then by Theorem 4.13, for each $O \in \mathcal{O}$, there is a linear order $\preceq_o$ on O such that $T \restriction O$ is an OR map. Choose a linear order on I. So, by Theorem 4.4, since $\{T \restriction O_i\}_{i\in I}$ is a collection of OR maps with at most one 1-cycle, then there is a linear order $\preceq$ on $X = \bigcup_{i\in I} O_i$ with respect to which T is OR on X, as required. $\square$

We end this section with a proof of the following result that we will use later during this chapter.

Lemma 4.15. *Let T be a function on a finite set X consisting of a single n-cycle with spine S_n, $n = 1, 2$. Suppose that there are $r \leq r' \in \mathbb{N}$ such that $C_{i,r} \neq \emptyset$, $C_{i+1,r'} \neq \emptyset$ and $C_{i,k} = C_{i+1,m} = \emptyset$ for each $k > r$ (with $k \neq r'$ if $n = 1$) and $m > r'$, where $i \in \{0, n-1\}$ and $i + 1$ is taken module n. If $y, z \in X$, then there is a linear order on O in such a way that z, y are endpoints of X and T is OR in each of the following cases:*

1. *$r = r' - 1$, $z \in C_{i,r}$ and $y \in C_{i+1,r'}$ provided that $y \in T^{-k-1}(T^k(z))$ whenever $|T^{-k-1}(T^k(z)| \neq 0$ and $T^k(z) \notin S_1$.*
2. *$n = 2$, $r = r'$, $z \in C_{i,r}$ and $y \in C_{i+1,r'}$.*
3. *$n = 1$, $|C_{0,1}| > 1$ and $z, y \in C_{0,r'}$ such that $r' = min\{n : T^n(y) = T^n(z)\}$.*

Moreover, If $n = 2$ and $S_2 = \{x_0, x_1\}$, then $(x_0, x_1) = \emptyset$.

Proof. Let $z \in C_{i,r}$ and $y \in C_{i+1,r'}$. For each $0 \leq j \leq r$, let $z_j = T^{r-j}(z)$ and for each $0 \leq k \leq r'$, let $y_k = T^{r'-k}(y)$. We will build up an order on X step by step as follow.

For each $x \in O \setminus S$, if $x \neq z_j$, $0 < j \leq r$, $x \neq y_k$, $0 < k \leq r'$ and $|T^{-1}(x)| \neq 0$, choose any linear order $\preceq_x$ on $T^{-1}(x)$. For each $0 < j \leq r$, if $x = z_j \in C_{i',j}$, for some

$i' \in \{0, n-1\}$ and $|T^{-1}(z_j)| \neq 0$, choose $\preceq_{z_j}$ in such a way that z_{j+1} is the greatest element of $T^{-1}(z_j)$. For each $0 < k \leq r$, if $x = y_k \in C_{i',k}$, for some $i' \in \{0, n-1\}$, $y_k \neq z_k$ and $|T^{-1}(y_k)| \neq 0$, choose $\preceq_{y_k}$ in such a way that y_{k+1} is the greatest element of $T^{-1}(y_k)$. Now, define a linear order on $C_{i',1}$, $i' \in \{0, n-1\}$ as follow.

(1) If $r = r' - 1$ so $z_1, y_1 \in C_{i',1}$ for some $i' \in \{0, n-1\}$; so in this case there is $0 \leq p \leq r$ such that $z_k = y_k$ for all $0 \leq k \leq p$. Define a linear order such that z_1 is the greatest element of $C_{i',1}$ and also y_1 is the greatest element of $C_{i,1} \setminus \{z_1\}$ if $p = 0$.

(2) If $n = 2$ and $r = r'$, then $z_1 \in C_{i',1}, y_1 \in C_{i'+1,1}$, $i' \in \{0, 1\}$. Choose a linear order such that z_1 is the greatest elements of $C_{i',1}$ and y_1 is the greatest element of $C_{i'+1,1}$.

Now, for each $i \in \{0, 1\}$ and $1 < l \leq r'$, define a linear order $\preceq_{il}$ on $C_{i,l}$ so that $T \restriction C_{i,l}$ is OP as in Lemma 3.14, so $C_{i,l} = \bigcup\{T^{-1}(x) : x \in C_{i+1,l-1}\}$ is the ordered sum of $T^{-1}(x)$ over $C_{i+1,l-1}$.

Now, we prove that $z = z_r$ is the greatest element of $C_{i,r}$ and $y = y_{r'}$ is the greatest element of $C_{i,r'}$. Suppose, for a contradiction, that $z \preceq_{ir} t$ for some $t \in C_{i,r}$, then from the construction of orders above we will have $T^{r-1}(z) \preceq_{i'1} T^{r-1}(t) \in C_{i',1}$. But this is a contradiction of being $T^{r-1}(z) = z_1$ the greatest element of $C_{i',1}$. Now for the point y we have two cases: if $|T^{-k-1}(T^k(z))| = 0$ whenever $T^k(z) \notin S$ then the proof that y is the greatest element of $C_{1,r'}$ follows in the same way of z. If $y \in T^{-k-1}(T^k(z))$ for some $k \in \mathbb{N}$, let $k' = \min\{k \in \mathbb{N} : y \in T^{-k-1}(T^k(z))\}$ so $p = r - k'$ and $T^{r-p}(z) = z_p = y_p = T^{r'-p}(y)$. Suppose, for a contradiction, that $y \preceq_{i+1,r'} t$ for some $t \in C_{i+1,r'}$, then $y_r = T(y) \preceq_{ir} T(t) \preceq_{ir} z = z_r$ (where t cannot be in $T^{-1}(y_r)$ since $y_{r'}$ is the greatest element of $T^{-1}(y_r)$). But then

$$T^{r+1-p}(y) \preceq_{i',r-p} T^{r+1-p}(t) \preceq_{i',r-p} T^{r-p}(z),$$

i.e., $y_p \preceq_{i',r-p} T^{r-p+1}(t) \preceq_{i',r-p} z_p$, which is, by Lemma 4.5, a contradiction since $z_p = y_p$.

Hence, z is the greatest element of $C_{i,r}$ and y is the greatest element of $C_{i,r'}$.

Now, if O is a 2-cycle, let $\preceq$ be defined on X as in Lemma 4.11; so for each $k \in \mathbb{N}$, if $i = 1$ define an order on $C_{i,k}$ to be the same order $\preceq_{i,k}$ and if $i = 0$ take the order on $C_{i,k}$ to be $\preceq_{i,k}^{-1}$. Let $(L_0(O), \preceq_0)$ be the ordered sum of $(C_{0,r}, \preceq_{0,r})$ over $(\mathbb{N}, \leq^{-1})$ and $(L_1(O), \preceq_1)$ be the ordered sum of $(C_{1,r}, \preceq_{1r})$ over $\mathbb{N}$. Let $(X, \preceq)$ be the ordered sum of $L_0(O), L_1(O)$. If O is a 1-cycle, for each $k < r'$, take $\preceq_{0,2k}^{-1}$ to be the order on $C_{0,2k}$ and $\preceq_{0,2k+1}$ to be the order on $C_{0,2k+1}$. Let $(X_1, \preceq_1)$ be the ordered sum of $(C_{0,2m}, \preceq_{0,2m})$ over $(\mathbb{N}, \leq^{-1})$ and $(X_2, \preceq_2)$ be the ordered sum of $(C_{0,2k+1}, \preceq_{0,2k+1})$ over $\mathbb{N}$. Let $(X, \preceq)$ be the ordered sum of X_1 and X_2 respectively. Clearly, from the construction of $\preceq$ we have $C_{0,q} \preceq X$, where q is either r or r'; and $X \preceq C_{1,q'}$, where $q \neq q' \in \{r, r'\}$. Hence, y, z are the endpoints of X. Finally, from the construction of $\preceq$ we have $(x_0, x_1) = \emptyset$.

(3) Let $z, y \in C_{0,r'}$. Let $t_0 \in C_{0,1}$ be such that $t_0 = T^{r'-1}(z)$ and $t_1 \in C_{0,1}$ be such that $t_1 = T^{r'-1}(y)$. Write $C_{0,1}$ as $C_{0,1} = C_{0,1}^0 \cup C_{0,1}^1$ with $t_i \in C_{0,1}^i, i = 0, 1$. For each $1 < k \leq r'$, let $C_{0,k}^0 = T^{-1}(C_{0,k-1}^0)$ and $C_{0,k}^1 = T^{-1}(C_{0,k-1}^1)$. Let $O_0 = \bigcup_{k>0} C_{0,k}^0 \cup S$ and $O_1 = \bigcup_{k>0} C_{0,k}^1 \cup S$. Define linear orders $\preceq_i$ on $O_i, i = 0, 1$ as in (1) in such a way that $z, T^{-1}(z)$ are endpoints of O_0 such that $z \in C_{0,r'}^0$ is the least element; and $y, T^{-1}(y)$ are endpoints of O_1 such that $y \in C_{0,r'}^1$ is the greatest element. Let $\preceq_{ik}$ be the restriction of $\preceq_i$ on $C_{0,k}^i$. Let

$$X_0 = \bigcup_{j=1}^{r'} C_{0,j}^{0+j} \quad \text{and} \quad X_1 = \bigcup_{j=1}^{r'} C_{0,j}^{1+j},$$

where $0 + j, 1 + j$ are taken module 2. Let $(X_0, \preceq_0)$ be the ordered sum of $C_{0,j}^{0+j}$ over $\{1, \ldots, r'\}$ and $(X_1, \preceq_1)$ be the ordered sum of $C_{1,j}^{0+j}$ over $(\{1, \ldots, r'\}, \geq)$. Let X be the ordered sum of $X_0, \{x_0\}$ and X_1 respectively. Notice that if $x \in C_{0,i}^p$, then $T(x) \in C_{0,i-1}^p$, so from the construction of $\preceq$, it is simple to verify that T is an OR map. Also, from the construction of $\preceq$, it follows that $C_{0,r'}^p \preceq X$ and $X \preceq C_{0,r'}^{p+1}$, where $p \in \{0, 1\}$ and $p + 1$ is taken module 2. Hence, y, z are endpoints of O. $\square$

4.2 Order-Reversing Maps on The Rational World

In this section we give the necessary and sufficient conditions for a set X with a self-map T to be linearly ordered in such a way that $T : X \to X$ is order-reversing and X is order-isomorphic to $\mathbb{Q}$.

Lemma 4.16. *Let $\{X_i\}_{i \in \mathbb{N}}$, be a collection of pairwise disjoint sets, where $X_i \approx \mathbb{Q}$ for each $i \in \mathbb{N}$, and $\{T_i : X_i \to X_i\}_{i \in \mathbb{N}}$ be a collection of OR maps. There is a linear order on $X = \bigcup_{i \in \mathbb{N}} X_i$ such that $X \approx \mathbb{Q}$ and the map $T : X \to X$ defined as $T \restriction X_i = T_i$ is an OR map provided that T has at most one fixed point.*

Proof. Let $X = \bigcup_{i \in \mathbb{N}}(X_i, \preceq_i)$, where each $(X_i, \preceq_i)$ is order-isomorphic to $\mathbb{Q}$ and $X_i \cap X_j = \emptyset$ whenever $i \neq j$. Assume that T_t, $t \in \mathbb{N}$ has at most 1-cycle. For each $i \in \mathbb{N} \setminus \{t\}$, let

$$X_{i,1} = \{x \in X_i : x \preceq_i T_i(x)\}$$

with the restriction of $\preceq_i$ on $X_{i,1}$ and

$$X_{i,2} = \{x \in X_i : T_i(x) \preceq_i x\}$$

with the restriction of $\preceq_i$ on $X_{i,2}$; so $X_i = X_{i,1} \cup X_{i,2}$, $T(X_{i,1}) \subseteq X_{i,2}$, $T(X_{i,2}) \subseteq X_{i,1}$ and $X_{i,1} \cap X_{i,2} = \emptyset$. Moreover, we have $X_{i,1} \approx X_{i,2} \approx \mathbb{Q}$. So, following the same proof of Lemma 4.4 we have $T : X \to X$ is OR and $X \approx \mathbb{Q}$ since $X_t \approx X_{i,1} \approx X_{i,2} \approx \mathbb{Q}$ for each $i \in \mathbb{N}$. $\square$

Lemma 4.17. *Let $T : X \to X$ be a function on the countably infinite set X. If there is a linear order on X that makes $X \approx \mathbb{Q}$ and T an OR map then for every $x \in X$, we have $|T^{-k}(x)|$ is 0, 1 or ω for each $k \in \mathbb{N}$.*

Proof. Suppose that there is $x \in X$ with $|T^{-k}(x)| > 1$ for some $k \in \mathbb{N}$. Since X is order-isomorphic to $\mathbb{Q}$, then X is densely ordered, so by Lemma 4.5, it follows immediately that

$T^{-k}(x)$ is infinite, hence $|T^{-k}(x)| = \omega$. $\square$

4.2.1 Order-Reversing Bijections in the Rational World

Proposition 4.18. *Let $T : X \to X$ be a bijection and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. If $0 \neq \zeta < \omega$ then there is no linear order on X with respect to which X is order-isomorphic to $\mathbb{Q}$ and T is an OR bijection.*

Proof. This proof follows immediately from Theorem 4.10, Lemma 4.6 and the fact that $\mathbb{Q}$ is densely ordered. $\square$

Theorem 4.19. *Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \dots)$ be a sequence of cardinals with $\zeta + \sum_{n\in\mathbb{N}} \sigma_n = \omega$. Then the canonical representation $T : X \to X$ of σ on the countable set X is an OR map with $X \approx \mathbb{Q}$ if and only if $\sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$ and ζ is either 0 or ω.*

Proof. Suppose that X is order-isomorphic to $\mathbb{Q}$ and T is an OR bijection on X, then by Lemma 4.3, we have $\sigma_n = 0$ for all $n > 2$ and $\sigma_1 \leq 1$. By Proposition 4.18 and Lemma 4.6, we have ζ is either 0 or ω.

Conversely, suppose $\sigma_n = 0$ for all $n > 1$, $\sigma_1 \leq 1$ and ζ is either 0 or ω. Suppose first that $\sigma_1 = 0$, se we have the following cases to consider:

Case (1): $\sigma_2 = \omega$ and $\zeta = 0$. Let $X = \mathbb{Q} \setminus \{0\}$ and let $T : X \to X$ be the map defined as $T(x) = -x$. It is clear that T is an OR bijection. Since for all $a \in X$ we have $T^2(a) = a$, so T has, in total, countably infinitely many 2-cycles.

Case (2): $\zeta = \omega$ and $\sigma_2 = 0$. Let $X = \mathbb{Q} \setminus \{0\}$ and let $T : X \to X$ be the bijection defined as $T(x) = -2x$. Clearly, T is an OR bijection and T has only infinitely many $\mathbb{Z}$-orbits.

Case (3): $\zeta = \omega$ and $0 < \sigma_2 < \omega$. By Lemma 4.16, we can assume that $\sigma_2 = 1$. Let $X = (0, 3) \cap \mathbb{Q}$. By Case (2), there are OR bijections T_1 on $I_1 = (1, 2) \cap \mathbb{Q}$ and T_2 on $I_2 = [(0, 1) \cup (2, 3)] \cap \mathbb{Q}$ and each has, in total, infinitely many $\mathbb{Z}$-orbits. So, let $T : X \to X$

be the bijection defined as : $T \restriction I_1 = T_1$, $T \restriction I_2 = T_2$, $T(1) = 2$ and $T(2) = 1$. Clearly, T has infinitely many $\mathbb{Z}$ orbits and one 2-cycle and T is an OR bijection.

Case (4): $\zeta = \omega$ and $\sigma_2 = \omega$. This case follows immediately from Case(1), Case (2) and Lemma 4.16.

Therefore, if $\sigma_n = 0$ for all $n \neq 2$ and ζ is either 0 or ω, then there is a linear order on X that makes $X \approx \mathbb{Q}$ and T an OR bijection.

Finally, suppose that $\sigma_1 = 1$. Let $X = [(0,1) \cup (1,2)] \cap \mathbb{Q}$, so in each of the four cases above there is an OR bijection T on X. Let

$$T_1 : X \cup \{1\} \to X \cup \{1\}$$

be defined such that $T_1 \restriction X = T$ and $T_1(1) = 1$, it follows immediately that T_1 is an OR bijection with a unique fixed point $x_0 = 1$. $\square$

4.2.2 Order-Reversing Injections in the Rational World

Theorem 4.20. *Let $\sigma = (\nu, \zeta, \sigma_1, \sigma_2, \dots)$ be a sequence of cardinals with $\nu + \zeta + \sum_{n \in \mathbb{N}} \sigma_n = \omega$. Then the canonical representation $T : X \to X$ of σ on the countable set X is OR with $X \approx \mathbb{Q}$ if and only if $\sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$ and $\zeta + \nu$ is either 0 or ω.*

Proof. Suppose that T is an OR injection on X which is order-isomorphic to $\mathbb{Q}$, then by Lemma 4.3 we have $\sigma_n = 0$ for all $n > 2$ and $\sigma_1 \leq 1$. Also, from Theorem 4.10, we have $\zeta + \nu$ is either 0 or ω.

Conversely, suppose that $\sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$ and $\zeta + \nu$ is either 0 or ω. Suppose first that $\sigma_1 = 0$. By Theorem 4.19 and Lemma 4.16, it is sufficient to prove the following cases:

Case (1): $\nu = \omega$ and $\zeta + \sigma_2 = 0$. Let $X = \mathbb{Q} \setminus \{0\}$ and let $T : X \to X$ be the bijection $T(x) = -\frac{1}{2}x$. So, immediately T is OR with infinitely many $\mathbb{Z}$-orbits. Since for each $a \in \mathbb{Q}$, $T^n(a) \to 0$ as $n \to \infty$, so we can take $Y = \mathbb{Q} \cap [(-1, 0) \cup (0, 1)]$. Hence, $T_1 = T \restriction Y$ has infinitely many $\mathbb{N}$-orbits and no others.

Case (2): $\nu = \omega$, $\sigma_2 = 0$ and $0 < \zeta = k < \omega$. Let $f : \mathbb{Q} \setminus \{0\} \to \mathbb{Q} \setminus \{0\}$ be the function defined by $f(x) = -\frac{1}{2}x$. List the elements in the set $(0, 1] \cap \mathbb{Q}$ as $\{q_i : 0 \neq i \in \mathbb{N}\}$ in such a way that q_m, q_n do not lie in the same orbit whenever $0 < m, n \leq k$. Let $O_i = \bigcup_{m \in \mathbb{Z}} T^m(q_i)$ and let $C = \{1, \ldots, k\}$. Now for each $i > 0$ let

$$O'_i = \begin{cases} O_i & \text{if } \ i \in C, \\ O_i \cap [-i, i] & \text{if } \ i \notin C. \end{cases}$$

Let $X_k = \bigcup_{i>0} O'_i$. Notice that $X_k \cap [-i, i] = (\mathbb{Q} \cap [-i, i]) \setminus F$, where F is a non-empty finite subset of $\mathbb{Q}$, so X_k is order-isomorphic to $\mathbb{Q}$. Let $T = f \restriction X_k$, then T is an OR injection with infinitely many $\mathbb{N}$-orbits and k $\mathbb{Z}$-orbits.

Case (3) : $\zeta = \omega$, $\sigma_2 = 0$ and $0 < \nu = n < \omega$. Let $T : \mathbb{Q} \to \mathbb{Q}$ be the OR bijection defined by $T(x) = -2x$. Choose rational numbers $q_1, q_2, \ldots, q_n \in (1, 2) \cap \mathbb{Q}$ and let

$$I = \{x \in \mathbb{Q} : 1 < x < 2, x \neq q_i, 1 \leq i \leq n\}.$$

Let

$$Y = \bigcup_{i=1}^{n} \bigcup_{k>0} T^{-k}(q_i).$$

Let $X = \mathbb{Q} \setminus Y$, so $X \approx \mathbb{Q}$. Hence, $T \restriction X$ is an OR injection with, in total, infinitely many $\mathbb{Z}$-orbits and n $\mathbb{N}$-orbits, as required.

Case (4): $\nu + \zeta = \omega$ and $0 < \sigma_2 < \omega$. By Lemma 4.16, we can assume that $\sigma_2 = 1$. Let $X = (0, 3) \cap \mathbb{Q}$. By Case (1), Case (2), Case (3) and Theorem 4.19, there are OP injections T_1 on $I_1 = (1, 2) \cap \mathbb{Q}$ and T_2 on $I_2 = [(0, 1) \cup (2, 3)] \cap \mathbb{Q}$ and each has, in total, infinitely many $\mathbb{N}$-orbits and $\mathbb{Z}$-orbits. So, let $T : X \to X$ be the injection defined as

$T \upharpoonright I_1 = T_1$, $T \upharpoonright I_2 = T_2$, $T(1) = 2$ and $T(2) = 1$. Clearly, T has infinitely many $\mathbb{Z}$ orbits and one 2-cycle and T is an OR injection.

Finally, suppose that $\sigma_1 = 1$. If the canonical representation of $\sigma = (\nu, \zeta, 0, \sigma_2, \sigma_3, \dots)$ is an OR injection on $\mathbb{Q}$, then there is an OR injection T on $X = [(0,1) \cup (1,2)] \cap \mathbb{Q}$. Let $T_1 : X \cup \{1\} \to X \cup \{1\}$ be defined such that $T_1 \upharpoonright X = T$ and $T_1(1) = 1$; clearly T_1 is an OR injection having one fixed point. □

4.2.3 Order-Reversing Surjections in the Rational World

In this section we study the same problem but for surjections, we give a characterization of order-reversing surjections on the rational world in terms of their orbit structure.

Necessary Conditions

Recall that if T has a $\mathbb{Z}$-orbit, O, we say that O satisfies condition (*) if:

(*) for all $x \in O$ there is $i \in \mathbb{N}$ such that $|T^{-1}(T^i(x))| = \omega$.

Notice that if O is a $\mathbb{Z}$-orbit with spine S and O has (*), then $|L_i(O)| = \omega$ for each $i \in \mathbb{Z}$. Now we have the following result.

Lemma 4.21. *Let T be a surjection on a countable set X with, in total, a $\mathbb{Z}$-orbit O. Then O has condition (*) if and only if each orbit of T^2 has condition (*).*

Proof. Let O be the unique orbit of T and let S be a spine of O indexed as $\{x_i : i \in \mathbb{Z}\}$ so that $T(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$. By Lemma 1.9, T^2 has, in total, two $\mathbb{Z}$-orbits, $O_1 = \bigcup_{k\in\mathbb{Z}} L_{2k}(O)$ and $O_2 = \bigcup_{k\in\mathbb{Z}} L_{2k+1}(O)$.

Suppose that O has (*), then $|L_i(O)| = \omega$ for each $i \in \mathbb{Z}$. Hence, both of O_1 and O_2 satisfy that for all $i \in \mathbb{Z}$, $|L_i(O_j)| = \omega, j = 1, 2$. Therefore, each orbit of T^2 has condition (*).

Conversely, suppose each orbit of T^2 has condition (*). Suppose, for a contradiction, that O does not have (*). So, there is $m \in \mathbb{Z}$ such that $T^{-1}(x_i) = x_{i-1}$ for each $i > m$. Hence, both of O_1 and O_2 satisfy that $(T^2)^{-1}(x_i) = x_{i-2}$ for each $i > m+1$. Hence, neither O_1 nor O_2 have condition (*), which is a contradiction. Thus, O has condition (*). □

Theorem 4.22. *Let $T : X \to X$ be a surjection on the countable set X and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. Suppose that $|T^{-1}(x)| = 1$ or ω for all $x \in X$. If $\zeta < \omega$ and if the number of orbits which satisfy condition (*) is less than the number of other $\mathbb{Z}$-orbits then there is no linear order on X such that $X \approx \mathbb{Q}$ and T is an OR map.*

Proof. Let T have n $\mathbb{Z}$-orbits with condition (*) and m $\mathbb{Z}$-orbits which do not have (*). Suppose that $n < m$ and suppose, for a contradiction, that there is a linear order $\preceq$ that makes $X \approx \mathbb{Q}$ and T an OR map. By Lemma 1.9, T^2 has, in total, $2(n+m)$ $\mathbb{Z}$-orbits and by Lemma 4.21 there are $2n$ $\mathbb{Z}$-orbits which have (*) and $2m$ $\mathbb{Z}$-orbits that do not have (*). Since T is OR then T^2 is OP with $2n < 2m$; which is a contradiction by Theorem 3.26. Hence, X cannot admit a linear order such that X is order-isomorphic to $\mathbb{Q}$ and T is OR. □

Theorem 4.23. *Let $T : X \to X$ be a surjection with, in total, finitely many orbits. Suppose that $\sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$ and $|T^{-1}(x)|$ is either 1 or ω for all $x \in X$. If the number of simple 2-cycles is greater than or equal to the number of orbits that have infinite cardinalities except $\mathbb{Z}$-orbits which do not have (*), then there is no linear order on X that makes $X \approx \mathbb{Q}$ and T an OR map.*

Proof. Let T have k simple 2-cycles and n infinite orbit with m $\mathbb{Z}$-orbits that do not satisfy (*), so we have $k \geq n - m$. Suppose, for a contradiction, that there is a linear order on X that makes X order-isomorphic to $\mathbb{Q}$ and T an OR map. Since T is an OR surjection then T^2 is an OP surjection. Lemma 1.9 implies that T^2 has $2k$ simple 1-cycles and $2m$ $\mathbb{Z}$-orbits which, by Lemma 4.21, do not satisfy (*). So, if T has a non-simple 1-cycle, then we have $2k > 2n - 1 - 2m$ (since $2k \geq 2n - 2m$), if T has no 1-cycle then $2k \geq 2n - 2m$ and if T has a simple 1-cycle then $2k + 1 > 2n - 2m$. But in each of these cases we have the number of simple 1-cycles is greater than or equal to the number of other orbits except orbits which do not have (*), which is a contradiction by Theorem 3.27. Hence, there is no linear order on X with respect to which $X \approx \mathbb{Q}$ and T an OR map. ☐

Ordering Finitely Many Orbits

In this section we show how to order a set X such that a surjective map $T : X \to X$ with finitely many orbits is order-reversing and X is order-isomorphic to $\mathbb{Q}$.

In this section we will use the same terminology that was used in Section 3.2 (see Figure 3.3 and Figure 3.4).

Lemma 4.24. *Let T be a surjection on the countably infinite set X and let O be an orbit of T with spine S. Suppose that for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω. There is a family of linear orders on the sets $C'_{i,k}$ ($i \in N$, $0 \neq k \in \mathbb{Z}$, where N is $\mathbb{Z}$ or $\{0, \ldots, n-1\}, 0 < n \in \mathbb{N}$, according to the nature of the orbit) with respect to which $T \restriction C'_{i,k}$ is an OR map and $C'_{i,k} \approx \mathbb{Q}$ when $C'_{i,k} \neq \emptyset$.*

Proof. By Lemma 3.28, there is a family of linear orders on the sets $C'_{i,k}$ with respect to which $T \restriction C'_{i,k}$ is an OP map and $C'_{i,k} \approx \mathbb{Q}$ when $C'_{i,k} \neq \emptyset$. Now, we deal with $C'_{j,k'} \neq \emptyset$ with $k > 0$ and the case $k < 0$ follows in the same way. For each $k > 0$, if k is even define an order on $C'_{i,k}$ to be the same $\preceq_{i,k}$ and if k is odd take the order on $C'_{i,k}$ to be $\preceq_{i,k}^{-1}$. Since for each even number $j = 2k$, $k \in \mathbb{N}$ we have

$$T \restriction C'_{i,j} : (C'_{i,j}, \preceq_{ij}) \to (C'_{i+1,j-1}, \preceq_{i+1,j-1})$$

is OP, then by Lemma 4.2 we have

$$T \restriction C'_{i,j} : (C_{i,j}, \preceq_{ij}) \to (C'_{i+1,j-1}, \preceq^{-1}_{i+1,j-1})$$

is OR. Also , it follows that for each odd number $j > 0$, we have

$$T \restriction C'_{i,j} : (C'_{i,j}, \preceq^{-1}_{ij}) \to (C'_{i+1,j-1}, \preceq_{i+1,j-1})$$

is also OR. So the proof is complete. □

Corollary 4.25. *Let T be a surjection on the countably infinite set X. Let O be either a $\mathbb{Z}$-orbit or a 2-cycle of T with spine S. Suppose that for every $x \in X$, $|T^{-1}(x)|$ is 1 or ω. There is a family of linear orders on the sets $L_i(O)$ ($i \in N$, where N is $\mathbb{Z}$ or $\{0, 1\}$ according to the nature of the orbit) with respect to which $T \restriction L_i(O)$ is an OR map and $L_i(O)$ with $|L_i(O)| = \omega$ is order isomorphic to $\mathbb{Q}$.*

Proof. By Lemma 4.24, there is a family of linear orders on the sets $C'_{i,k}$ ($i \in N, 0 \neq k \in \mathbb{Z}$) with respect to which $T \restriction C'_{i,k}$ is an OR map and $C'_{i,k} \approx \mathbb{Q}$ for $C'_{i,k} \neq \emptyset$. Now we define linear orders on $L_i(O)$ as follows. We have two cases, either $i = 2m, m \in \mathbb{Z}$ or $i = 2m+1, m \in \mathbb{Z}$. If $i = 2m$, let $(L_i(O), \preceq_i)$ be the ordered sum of $C'_{i,k}$ over $(\mathbb{Z} \leq^{-1})$. If $i = 2m+1$, let $(L_i(O), \preceq_i)$ be the ordered sum of $C'_{i,k}$ over $\mathbb{Z}$.

Obviously, for each $i \in \mathbb{Z}$, since $C'_{i,0} = x_i$ and since for each $k \in \mathbb{Z}$, $C'_{i,k} \approx \mathbb{Q}$, then $L_i(O) \approx \mathbb{Q}$. Since $T \restriction C'_{i,k}$ is OR and from the construction of each $\preceq_i$, it is simple to verify that $T \restriction L_i(O)$ is an OR map. □

Theorem 4.26. *Let X be a countably infinite set and $T : X \to X$ be a surjection on the set X. Suppose that for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω. If T has, in total, one n-cycle, for $n = 1$ or 2, then there exists a linear order on X such that $X \approx \mathbb{Q}$ and T is an OR map.*

Proof. Suppose that T has a 2-cycle O with spine $\{x_0, x_1\}$. By Corollary 4.25, there is a family of linear orders $\preceq_i$ on the sets $L_i(O)$, $i \in \{0,1\}$ such that $L_i(O) \approx \mathbb{Q}$ and $T \restriction L_i(O)$ is OR. Let $(X, \preceq_o)$ be the ordered sum of $(L_0(O), \preceq_0)$ and $(L_1(O), \preceq_1)$. Since $L_0(O) \approx L_1(O) \approx \mathbb{Q}$, then $X \approx \mathbb{Q}$. From the construction of $\preceq_o$, we have T is an OR surjection.

Now, suppose that T has a 1-cycle with spine $\{x_0\}$. By Lemma 4.24, there is a family of linear orders $\preceq_{0,k}$ on the sets $C'_{0,k}$ such that $C'_{0,k} \approx \mathbb{Q}$ and $T \restriction C'_{0,k}$ is OR. For each $k \in \mathbb{N}$, let $(C_{0,k}, \preceq_{0,k})$ be the ordered sum of $C'_{0,k}, C'_{0,-k}$. Let $(Y_1 = \bigcup_{i\in\mathbb{N}} C_{0,2i}, \preceq_1)$ be the ordered sum of $C_{0,2i}$ over $(\mathbb{N}, \leq^{-1})$ and let $(Y_2 = \bigcup_{i\in\mathbb{N}} C_{0,2i+1}, \preceq_2)$ be the ordered sum of $C_{0,2i+1}$ over $\mathbb{N}$. Let $\preceq$ be defined on X as the sum-order of $\preceq_1$ and $\preceq_2$ respectively.

Clearly, we have $X \approx \mathbb{Q}$ since $C_{0,k} \approx \mathbb{Q}$ for all $k \in \mathbb{N}$. Since $T \restriction C_{i,k}$ is OR and from the construction of each $\preceq_i$, it follows that T is OR. □

Theorem 4.27. *Let $T : X \to X$ be a surjection on the countably infinite set X with, in total, finitely many $\mathbb{Z}$-orbits. Suppose further that for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω. If the number of orbits which satisfy condition (*) is greater than or equal to the number of other orbits then there exists a linear order on X such that $X \approx \mathbb{Q}$ and T is an OR map.*

Proof. Without loss of generality, by Lemma 4.16, we might assume that T has, in total, two orbits O_1 and O_2 such that O_1 satisfies (*) and O_2 does not have (*). Index a spine of $O_r, r = 1, 2$ as $S_r = \{x_{r,n} : n \in \mathbb{Z}\}$ so that $T(x_{r,n}) = x_{r,n+1}$ for each $n \in \mathbb{Z}$. By Corollary 4.25, for each $r = 1, 2$, there is a family of linear orders $\preceq_{ir}$ on the sets $L_i(O_r)$, $i \in \mathbb{Z}$ such that $L_i(O_r) \approx \mathbb{Q}$ when $|L_i(O_r)| = \omega$ and $T \restriction L_i(O_r)$ is an OR map. So, we can define a linear order on $O_r, r = 1, 2$ as in Theorem 4.13; so $T \restriction O_r$ is OR and $O_1 \approx \mathbb{Q}$ (since O_1 has (*), so for each $j \in \mathbb{Z}$, $|L_j(O_1)| = \omega$). Let $E = \{k \in \mathbb{Z} : k \text{ is even}\}$ and $E' = \{k \in \mathbb{Z} : k \text{ is odd}\}$. Now, define $\preceq$ to be the linear order defined on X as follows:

for all $x, y \in X$ with $x \in L_p(O_r), y \in L_q(O_t), p, q \in \mathbb{Z}$ and $r, t \in \{1, 2\}$, then

$$x \preceq y \Leftrightarrow (r = t \text{ and } x \preceq_r y) \text{ or } (r = 1, t = 2 \text{ and } p, q \in E) \text{ or } (r = 2, t = 1 \text{ and } p, q \in E')$$
$$\text{or } (r \neq t, p \in E \text{ and } q \in E').$$

We now prove that $X \approx \mathbb{Q}$. Let $x, y \in X$ and $x \preceq y$. If $x, y \in O_1$, then there is $z \in X$ such that $x \prec z \prec y$, since $O_1 \approx \mathbb{Q}$. Let $x, y \in O_2$, so $x \in L_p(O_2), y \in L_q(O_2)$ for some $p, q \in \mathbb{Z}$. If $p \in E, q \in E'$ then $x \prec L_{q-1}(O_1) \prec y$. If $p, q \in E'$ then $x \preceq L_q(O_1) \preceq y$ and if $p, q \in E$ then $x \preceq L_p(O_1) \preceq y$. Hence, X is densely ordered. Also, from the construction of $\preceq$, X has no endpoints, so $X \approx \mathbb{Q}$. Since $T \upharpoonright O_r$ is OR for each $r = 1, 2$ and from the construction of $\preceq$, we have T is an OR map. □

Proof of The Main Theorem of Order-Reversing Surjections on $\mathbb{Q}$

Proposition 4.28. *Let X be a countably infinite set and $T : X \to X$ be a surjection. Suppose that for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω and T has at most a 1-cycle. There is a linear order on X such that $X \approx \mathbb{Q}$ and T is an OR map in each of the following cases:*

(1) *T has, in total, infinitely many 2-cycles; or*

(2) *T has, in total, infinitely many $\mathbb{Z}$-orbits.*

Proof. Suppose first that T has no 1-cycle. Let $\{O_i\}_{i \in \mathbb{N}}$ be the collection of all orbits of T, where either O_i is a $\mathbb{Z}$-orbit for all $i \in \mathbb{N}$ or O_i is a 2-cycle for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, index a spine of O_i as $S_i = \{x_{i,m} : m \in M\}$, where M is either $\mathbb{Z}$ or $\{0, 1\}$ as appropriate, and let $S = \bigcup_{i \in \mathbb{N}} S_i$. By Theorem 4.19, there is a linear order $\preceq_s$ on S such that $S \approx \mathbb{Q}$ and $T \upharpoonright S$ is an OR map. For each $i \in \mathbb{N}$ and $m \in M$, let $L_m(O_i)$ be defined, so by Corollary 4.25, there is a family of linear orders $\preceq_{im}$ on the sets $L_m(O_i)$ such that

$L_m(O_i) \approx \mathbb{Q}$ when $|L_m(O_i)| = \omega$ and $T \restriction L_m(O_i)$ is OR. Write $L_m(O_i)$ as $L(x_{i,m})$ (where $L_m(O_i) \ni x_{i,m}$), and let $(X, \preceq)$ be the ordered sum of $L(x_{i,m})$ over S. Since for each $i \in \mathbb{N}$ and $j \in M$, $S \approx L_j(O_i) \approx \mathbb{Q}$, then we have $X \approx \mathbb{Q}$. Clearly, since $T \restriction S$ and $T \restriction L_m(O_i)$ are OR for all $i \in \mathbb{N}$ and $m \in M$ and from the construction of $\preceq$ we have T is an OR map.

Finally, if T has, in addition, a 1-cycle O' with spine $\{x_0\}$, then we have two cases: if O' is non-simple, then the proof follows by Theorem 4.26 and Lemma 4.16. If O' is simple, define $\preceq$ as above and further put $y \preceq x_0$ for each $y \in \{x \in X : x \preceq T(x)\}$ and $x_0 \preceq y$ for each $y \in \{x \in X : x \succeq T(x)\}$, so the proof is complete. □

Let $T : X \to X$ have orbit spectrum $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$. Recall that if T has a $\mathbb{Z}$-orbit, O, we say that O satisfies condition (*) if:

(*) for all $x \in O$ there is $i \in \mathbb{N}$ such that $|T^{-1}(T^i(x))| = \omega$.

If $\zeta < \omega$, let ζ_1 be the number of $\mathbb{Z}$-orbits that have the condition (*) and $\zeta_2 = \zeta - \zeta_1$. Also, if $\sigma_2 < \omega$, consider $\sigma_2 = \sigma_2' + \sigma_2''$, where σ_2' is the number of simple 2-cycles. Using these terminologies we have the following theorem, the main theorem of this section.

Theorem 4.29. *Let T be a surjection on a countably infinite set X with orbit spectrum $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$. There is a linear order on X with respect to which X is order-isomorphic to $\mathbb{Q}$ and T is an OR surjection if and only if $\sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$, for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω and either:*

(1) *$\zeta = \omega$; or*

(2) *$\zeta < \omega$ provided that $\zeta_2 \leq \zeta_1$ and either:*

  (a) *$\sigma_2 = \omega$; or*

  (b) *$\sigma_2 < \omega$, T has no non-simple 1-cycle and $\sigma_2' < \zeta_1 + \sigma_2''$; or*

(c) *$\sigma_2 < \omega$, T has a non-simple 1-cycle and $\sigma_2' \leq \zeta_1 + \sigma_2''$.*

Proof. Suppose that there is a linear order on X such that $X \approx \mathbb{Q}$ and T is an OR map, then by Lemma 4.3 we have $\sigma_n = 0$ for all $n > 2$ and $\sigma_1 \leq 1$. By Lemma 4.17 we have for every $x \in X$, $|T^{-1}(x)|$ is either 1 or ω. Now suppose that (1) is false, then $\zeta < \omega$, so by Theorem 4.22 we have $\zeta_2 \leq \zeta_1$. If $\sigma_2 < \omega$ then by Theorem 4.23 we have either (b) or (c) holds.

Conversely, suppose that condition (1) holds, so $\zeta = \omega$. Assume first that $\sigma_1 = 0$. If σ_2 is either 0 or ω, then the proof follows from Proposition 4.28 and Lemma 4.16. If $\sigma_2' = 0$ then the proof follows by Proposition 4.28, Lemma 4.26 and Lemma 4.16, so we are left with the following case to consider: $0 \neq \sigma_2' = n < \omega$ and $\sigma_2'' < \omega$.

Suppose first that $\sigma_2'' = 0$. Index the n simple 2-cycles as $O_k' = \{x_{k,0}, x_{k,1}\}$, $0 < k \leq n$. Let $\mathcal{O} = \{O_i\}_{i \in \mathbb{N}}$ be the collection of all $\mathbb{Z}$-orbits of T. For each $0 \leq j \leq n$, let $\mathcal{O}_j = \{O_{(n+1)m+j} : m \in \mathbb{N}\}$, then by Proposition 4.28 for each j there is a linear order $\preceq_j$ on $O_j = \bigcup_{m \in \mathbb{N}} O_{(n+1)m+j}$ such that $O_j \approx \mathbb{Q}$ and $T \restriction O_j$ is OR. For each $j \in \mathbb{N}$ let

$$I_j = [(-j-1, -j) \cup (j, j+1)] \cap \mathbb{Q}.$$

So, for each $0 \leq j \leq n$, there are order-isomorphisms $h_j : O_j \to I_j$. Now define and order isomorphism

$$h : X \to [(-(n+1), n+1) \setminus \{0\}] \cap \mathbb{Q}$$

such that $h \restriction O_j = h_j$, $x_{k,0} \mapsto -k$ and $x_{k,1} \mapsto k$, $1 \leq k \leq n$. Hence, h is order isomorphism and it is obvious that T is an OR map. The case $0 \neq \sigma_2'' < \omega$ follows by the previous case, Lemma 4.16 and Theorem 4.26. Thus, if $\zeta = \omega$ and $\sigma_1 = 0$ then there is a linear $\preceq$ on X with respect to which $X \approx \mathbb{Q}$ and T is OR.

Now, if $\sigma_1 = 1$, then by Lemma 4.16 and Theorem 4.26, it is sufficient to prove the case when the 1-cycle is simple. Let $O = \{x_0\}$ be the 1-cycle and define $\preceq$ as above and further put $y \preceq x_0$ for each $y \in \{x \in X : x \preceq T(x)\}$ and $x_0 \preceq y$ for each $y \in \{x \in X : x \succeq T(x)\}$.

Suppose that condition (2) holds, so $\zeta < \omega$ with $\zeta_2 \leq \zeta_1$. If $\sigma_2 = \omega$ then the proof follows immediately by Theorem 4.27, Theorem 4.28 and Lemma 4.16.

To see case (b), suppose first that $\sigma_1 = 0$. Let $C_1, C_2, \ldots, C_{\zeta_2}$ be the $\mathbb{Z}$-orbits of T that do not satisfy (*) and $C'_1, \ldots, C'_{\zeta_2}, \ldots, C'_{\zeta_1}$ be the $\mathbb{Z}$-orbits of T that satisfy (*). For each $1 \leq p \leq \zeta_2$, let $O_p = C_p \cup C'_p$ and for every $\zeta_2 < k \leq \zeta_1$, let $O_k = C'_k$. Then by Theorem 4.27 we have each $O_l, 0 < l \leq \zeta_1$, can be ordered so that it is order-isomorphic to $\mathbb{Q}$ and $T \upharpoonright O_l$ is OR. Let $O'_m, 0 < m \leq \sigma''_2$, be the 2-cycles of T which are infinite, then by Theorem 4.26, for each m, there is a linear order on O'_m such that $O'_m \approx \mathbb{Q}$ and $T \upharpoonright O'_m$ is OR. For each $0 < l \leq \zeta_1$ and $0 < m \leq \sigma''_2$, let $f_l : O_l \to I_l$ and $f'_m : O'_m \to I_{\zeta_1+m}$ be the order isomorphisms concerned. Let $\{x_{n,0}, x_{n,1}, 0 < n \leq \sigma'_2\}$ be the simple 2-cycles of T. Since $\sigma'_2 < \zeta_1 + \sigma''_2$, then we can define an order isomorphism

$$f : X \to [\bigcup_{r=1}^{\zeta_1+\sigma''_2} I_r \cup \{i : 1 < i \leq \sigma'_2 + 1\}]$$

such that $f \upharpoonright O_l = f_l$, $f \upharpoonright O'_m = f'_m$, $f : x_{n,0} \mapsto -n-1$ and $f : x_{n,1} \mapsto n+1$ for each $0 < n \leq \sigma'_2$. It follows that $X \approx \mathbb{Q}$ and T is an OR map under this order. If T has, in addition, a simple 1-cycle, $\{x_0\}$, define

$$g : X \to [\bigcup_{r=1}^{\zeta_1+\sigma''_2} I_r \cup \{i : 1 < i \leq \sigma'_2 + 1\} \cup \{0\}]$$

such that $g \upharpoonright X \setminus \{x_0\} = f$ and $x_0 \mapsto 0$.

Finally let (c) holds and let O be the non-simple 1-cycle of T. If $\sigma'_2 < \zeta_1 + \sigma''_2$, then by Case (b), Theorem 4.26 and Lemma 4.16 the proof is complete. If $\sigma'_2 = \zeta_1 + \sigma''_2$, then by Theorem 4.26, O can be ordered so that $O \approx \mathbb{Q}$ and $T \upharpoonright O$ is OR. So, there is an order isomorphism $f_o : O \to I_0 \cup \{0\}$. Define an order isomorphism

$$h : X \to [\bigcup_{r=0}^{\zeta_1+\sigma_2''} I_r \cup \{i : 1 < i \leq \sigma_2' + 1\} \cup \{0\}]$$

such that $h \upharpoonright O_l = f_l, 0 < l \leq \zeta_1$, $h \upharpoonright O'_m = f'_m, 0 < m \leq \sigma''_2$, $h : x_{n,0} \mapsto -n$ and $f : x_{n,1} \mapsto n$ for each $0 < n \leq \sigma'_2$, and $h \upharpoonright O = f_o$. So the proof is complete. $\square$

4.3 Orbit Structure of Order-Reversing Maps on the Naturals and Integers

In this section we characterize order-reversing maps on sets which are order-isomorphic to the naturals or integers with their usual orders. This characterization is in terms of their orbit structure.

First we prove some results that are useful and we will use later.

Lemma 4.30. *Let $T_1 : (X_1, \preceq_1) \to (X_1, \preceq_1)$ and $T_2 : (X_2, \preceq_2) \to (X_2, \preceq_2)$ be OR maps with X_2 is finite and $X_1 \approx \mathbb{M}$, where $\mathbb{M}$ is either $\mathbb{N}$ or $\mathbb{Z}$. Suppose further that either $X_1 \cap X_2 = \emptyset$ or $X_1 \cap X_2 = \{x_0, x_1\}$ such that:*

(1) *T_1 has no 1-cycle;*

(2) *$x_0 \preceq_i x_1$, $T_i(x_0) = x_1$ and $T_i(x_1) = x_0$, $i = 1, 2$;*

(3) *x_0 and x_1 are the endpoints of X_2 and*

(4) *$\{y \in X_1 : x_0 \preceq_1 y \preceq_1 x_1\} = \emptyset$.*

If $X = X_1 \cup X_2$, then there is a linear order on X such that $X \approx \mathbb{M}$ and the map $T : X \to X$ defined as $T \upharpoonright X_i = T_i, i = 1, 2$ is an OR map.

Proof. Let $Y_1 = \{x \in X_1 : x \preceq_1 T_1(x)\}$ and $Y_2 = \{x \in X_1 : T_1(x) \preceq_1 x\}$. Since $x_0 \preceq_1 x_1$, $T_1(x_0) = x_1$, $T_1(x_1) = x_0$ and $\{y \in X_1 : x_0 \preceq_1 y \preceq_1 x_1\} = \emptyset$, then $x_0 \in Y_1$ is the greatest element of Y_1 and $x_1 \in Y_2$ is the least element of Y_2. Let $\preceq_1'$ be the restriction of $\preceq_1$ on Y_1 and $\preceq_1''$ be the restriction of $\preceq_1$ on Y_2. If $X_1 \cap X_2 = \emptyset$, define a linear order $\preceq$ on X to be the sum-order of $\preceq_1'$, $\preceq_2$ and $\preceq_1''$, so X is the ordered sum of Y_1, X_2 and Y_2 respectively. If $X_1 \cap X_2 = \{x_0, x_1\}$, let X be the ordered sum of $Y_1, X_2 \setminus \{x_0, x_1\}$ and Y_2 respectively. If $\mathbb{M} = \mathbb{N}$, then we have $|Y_1| < \omega$ and $|Y_2| = \omega$, hence we have $X \approx \mathbb{N}$ since X_2 is finite. If $\mathbb{M} = \mathbb{Z}$, then $|Y_1| = |Y_2| = \omega$, hence $X \approx \mathbb{Z}$ since X_2 is finite.

Finally we prove that T is an OR map under this order. Let $x, y \in X$ with $x \preceq y$. If $x, y \in X_i, i = 1, 2$, then immediately we have $T(x) \succeq T(y)$. If $x \in Y_1, y \in X_2$ then $T(x) \in Y_2, T(y) \in X_2$; so $T(x) \succeq T(y)$. Similarly, if $x \in X_2, y \in Y_2$ then $T(y) \in Y_1$; so $T(x) \succeq T(y)$. Hence, T is an OR map. $\square$

Lemma 4.31. *Let T be a map on a finite set X with, in total, a 2-cycle O with spine $\{x_0, x_1\}$. Then there is a linear order $\preceq$ on O in such a way that x_0 and x_1 are the endpoints of X and with respect to which T is an OR map.*

Proof. By Lemma 4.11, there is a family of linear orders $\preceq_{ik}$ on the sets $C_{i,k} \neq \emptyset$, $i \in \{0, 1\}$ and $k \in \mathbb{N}$ with respect to which $T \restriction C_{i,k}$ is an OR map. Let $L_0(O) = \bigcup_{k \in \mathbb{N}} C_{0,k}$ and let $\preceq_0$ be the order defined on $L_0(O)$ to be the sum-order of $\preceq_{0,k}$ over $\mathbb{N}$; so $x_0 = C_{0,0}$ is the least element of $L_0(O)$. Let $L_1(O) = \bigcup_{k \in \mathbb{N}} C_{1,k}$ and define $\preceq_1$ on $L_1(O)$ as the sum-order of $\preceq_{0,k}$ over $(\mathbb{N}, \leq^{-1})$; so $x_1 = C_{1,0}$ is the greatest element of $L_1(O)$. Let $(X, \preceq)$ be the ordered sum of $(L_0(O), \preceq_0)$ and $(L_1(O), \preceq_1)$. It follows that x_0 is the least element of X and x_1 is the greatest element of X. Also, from the construction of $\preceq$, it is simple to verify that T is an OR map. $\square$

Corollary 4.32. *Let T be a map on a countably infinite set X and let O be a 2-cycle of T with spine $S = \{x_0, x_1\}$. Let $n \in \mathbb{N}$ and let*

$$A = \{x \in T^{-1}(S) \setminus S : \|x\| \geq n, |\bigcup_{k\in\mathbb{N}} T^{-k}(x)| < \omega\}.$$

If $|A| < \omega$ *and if* $O' = X \setminus (\bigcup_{k\in\mathbb{N}} T^{-k}(A))$ *is ordered in such a way that* $(x_0, x_1) = \emptyset$ *so that* $T \restriction O'$ *is OR and* $O' \approx \mathbb{M}$*, where* $\mathbb{M}$ *is either* $\mathbb{N}$ *or* $\mathbb{Z}$*, then* X *can be ordered so that* $X \approx \mathbb{M}$ *and* T *is an OR map.*

Proof. By Lemma 4.31, if $A \neq \emptyset$, then there is a linear order on $C = \bigcup_{k\in\mathbb{N}} T^{-k}(A) \cup S$ so that $T \restriction C$ is an OR map in such a way that x_0, x_1 are the endpoints of C. Hence, using Lemma 4.30, the proof is complete. □

We end with the following result. Spines of $\mathbb{N}$-orbits are chosen as in Observation 3.40.

Theorem 4.33. *Let* $T : X \to X$ *be a map on the countably infinite set* X*. Let* $\mathcal{O}$ *be the collection of all orbits of* T *and* S *be the set of all spine points of orbits of* T*. Suppose further that:*

(1) *each orbit of* T *is either a 1-cycle, a 2-cycle, an* $\mathbb{N}$*-orbit or a* $\mathbb{Z}$*-orbit,*

(2) $|L_k(O)| < \omega$ *for each* $O \in \mathcal{O}$ *and* $k \in N$*, where* N *is either* $\mathbb{N}$*,* $\mathbb{Z}$*,* $\{0\}$ *or* $\{0, 1\}$*, and* $T^{-1}(L_0(O)) = \emptyset$ *for each* $\mathbb{N}$*-orbit* O*, and*

(3) $(S, \preceq_s)$ *is ordered such that* $S \approx \mathbb{M}$*, where* $\mathbb{M}$ *is either* $\mathbb{N}$ *or* $\mathbb{Z}$*, and* $T \restriction S$ *is an OR map.*

Then there is a linear order on X *such that* T *is an OR map and* $X \approx \mathbb{M}$*.*

Proof. Let $\{O_i\}_{i\in I}, I \subseteq \mathbb{N}$, be the collection of all orbits of T. Let S be indexed as $\{x_n : n \in \mathbb{M}\}$ so that $x_n \preceq_s x_{n+1}$ for each $n \in \mathbb{M}$. By Corollary 4.12, for each $i \in I$, there is a family of linear orders on the sets $L_k(O_i)$, $k \in N$ such that $T \restriction L_k(O_i)$ is an OR map. For each $i \in I$ and $k \in N$, write $L_k(O_i)$ as $L(x_n)$ whenever $x_n \in L_k(O_i)$, so each $L_k(O_i)$ contains a unique $x \in S$. Let $(X, \preceq)$ be the ordered sum of $\{L(x_n)\}_{x_n\in S}$ over

S. Since $|L_k(O_i)| < \omega$ for each $i \in I$ and $k \in N$, then by Lemma 3.41 we have $X \approx \mathbb{M}$. Finally, we prove that T is an OR map under this order. Let $x, y \in X$ with $x \preceq y$. If $x, y \in L(x_n)$ then $T(x), T(y) \in L(T(x_n))$, so $T(x) \succeq T(y)$. If $x \in L(x_n), y \in L(x_j)$ for some $n, j \in \mathbb{M}, n \neq j$, then $T(x) \in L(T(x_n)), T(y) \in L(T(x_j))$ and $x_n \preceq_s x_j$. Since $T \restriction S$ is OR then $T(x_j) \preceq_s T(x_n)$; hence $T(y) \preceq T(x)$. Thus, T is OR. $\square$

4.4 Orbit Structure of Order-Reversing Maps on the Naturals

Lemma 4.34. *Let $T : \mathbb{N} \to \mathbb{N}$ be an OR map and let $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. Then*

(1) $\zeta = \nu = 0$ *and* $\sigma_2 < \omega$.

(2) T *has a unique orbit* O *with* $|O| = \omega$.

(3) $|T^{-1}(x)| < \omega$ *for all* $x \in X$ *except for a unique point* $y_0 \in X$ *with* $|T^{-1}(y_0)| = \omega$ *such that* $|X \setminus T^{-1}(y_0)| < \omega$. *Moreover, if* T *has a 1-cycle* O *and* $y_0 \in O$, *then* $O = X$.

Proof. (1) Let $T : \mathbb{N} \to \mathbb{N}$ be an OR map, then Theorem 4.8 implies that $\zeta = 0$. Also, Corollary 4.9 and Theorem 4.8 imply that $\nu = 0$. Lemma 4.6 together with Corollary 4.7 imply that $\sigma_2 < w$, since $\mathbb{N}$ has a least element.

(2) The proof of this case follows immediately from (1) above and from Lemma 4.6.

(3) Let O be either a 1-cycle or a 2-cycle of T with spine S and let $|O| = \omega$. Suppose, for a contradiction, that T is finite-to-one. So, there is $C \subseteq O$ such that $T \restriction C$ is a semi-simple cycle. But this is a contradiction, according to Theorem 4.8 and Corollary 4.9. Consequently, if $|O| = \omega$ then there is no $C \subseteq O$ such that $T \restriction C$ is a semi-simple cycle, hence there is $y_0 \in X$ with $|T^{-1}(y_0)| = \omega$. By Lemma 4.5 we have $T^{-1}(y_0) = (n, \infty)$ for some $n \in \mathbb{N}$ so $|X \setminus T^{-1}(y_0)| < \omega$.

Now, if T has a 1-cycle O and $y_0 \in O$, then by Lemma 4.6 and Lemma 4.5, we have $O = X$, as was required. $\square$

From Lemma 4.34 and Lemma 4.3 we immediately have the following theorem.

Theorem 4.35. *Let $T : X \to X$ be either a bijection, an injection or a surjection on the countable set X. There is no linear order on X with respect to which X is order-isomorphic to $\mathbb{N}$ and T is an OR map.*

Before giving a proof of the main theorem of this section, we have the following result.

Proposition 4.36. *Let T be a map on the countably infinite set X with, in total, an n-cycle with spine S_n, $n = 1$ or 2. Suppose further that*

(1) *T has a unique point y_0 with $|T^{-1}(y_0)| = \omega$ and $|X \setminus T^{-1}(y_0)| < \omega$; and*

(2) *for every $y \in X$ with $y \notin S_2$, if $|T^{-k}(y)| = \omega$ for some $0 \neq k \in \mathbb{N}$ then $|T^{-k-1}(y) \setminus T^{-k}(y)| < \omega$ and $|T^{-k-2}(y) \setminus T^{-k-1}(y)| = 0$.*

Then there is a linear order $\preceq$ on X with respect to which T is an OR map and $X \approx \mathbb{N}$.

Proof. Let $T : X \to X$ be a map on a countably infinite set X. Let X be either a 1-cycle or a 2-cycle of T. By Corollary 4.32 and from condition (1), if X is a 2-cycle with spine $S_2 = \{x_0, x_1\}$, it is sufficient to order the set O in such a way that $(x_0, x_1) = \emptyset$, where $O = X \setminus (\bigcup_{k\in\mathbb{N}} T^{-k}(C))$ and $C = \{x \in T^{-1}(S_2) \setminus S_2 : \|x\| \geq 1, |\bigcup_{k\in\mathbb{N}} T^{-k}(x)| < \omega\}$ (where C is finite). If $n = 1$, let $O = X$.

Now, for each $i \in N$ and $j \in \mathbb{N}$, where N is either $\{0\}$ or $\{0, 1\}$, let $C_{i,j}$ be defined. Let $y_0 \in X$ be the unique point with $|T^{-1}(y_0)| = \omega$ and let $A = \{x \in T^{-1}(y_0) : |T^{-1}(x)| \neq 0\}$. Condition (1) implies that there is $r \in \mathbb{N}$ such that $C_{i,r} \neq \emptyset$ and $C_{i,j} = \emptyset$ for each $j > r$ (with also $C_{i+1,j} = \emptyset$ for each $j \geq r$ when $n = 2$). Condition (2) implies that one of the following holds:

Case (1): $|C_{i,r}| < \omega$ and $|C_{i+1,r-1}| = \omega$; so $y_0 \in C_{i+2,r-2}$. In this case, if $|A| \neq 0$ choose $y \in A$ and let $z = T(y)$. If $|A| = 0$ choose $z \in T^{-1}(y_0)$ and choose $y \in C_{i,r}$ satisfying (1) of Lemma 4.15. Let $T^{-1}(y_0) \approx \mathbb{N}$ with z is the least element of $T^{-1}(y_0)$.

Case (2) : $|C_{i,r}| = \omega$, so $y_0 \in C_{i+1,r-1}$. Let $y = y_0$ and choose $z \in T^{-1}(y_0)$. Let $T^{-1}(y_0) \approx \mathbb{N}$ with z is the least element of $T^{-1}(y_0)$.

Now, z, y satisfy conditions of Lemma 4.15 (1), so there is a linear order $\preceq_1$ on $O_1 = (O \setminus T^{-1}(y_0)) \cup \{z\}$ in such a way that z is the greatest element and y is the least element of O_1 and T is OR (with $(x_0, x_1) = \emptyset$ when $n = 2$). Let O be the ordered sum of $O_1 \setminus \{z\}$ and $T^{-1}(y_0)$ respectively, then immediately we have X is order-isomorphic to $\mathbb{N}$ and T is an OR map. $\square$

Now we give a proof of the main theorem of this section which describes order-reversing self-maps on sets which are order-isomorphic to the naturals $\mathbb{N}$ with their usual order.

Theorem 4.37. *Let $T : X \to X$ be a function on the countably infinite set X and let $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. There is a linear order on X with respect to which $X \approx \mathbb{N}$ and T is an OR map iff $\zeta = \nu = \sigma_n = 0$ for each $n > 2$, $\sigma_1 \leq 1$, $\sigma_2 < \omega$ and T has a unique cycle O with $|O| = \omega$ and O satisfies the following:*

(a) *there is a unique $y_0 \in O$ with $|T^{-1}(y_0)| = \omega$ and $|X \setminus T^{-1}(y_0)| < \omega$;*

(b) *for each $y \in O$ with y is not in a spine of a 2-cycle, if $|T^{-k}(y)| = \omega$ for some $0 < k \in \mathbb{N}$ then $|T^{-k-1}(y) \setminus T^{-k}(y)| < \omega$ and $|T^{-k-2}(y) \setminus T^{-k-1}(y)| = 0$; and*

(c) *if O is a 1-cycle, then $O = X$.*

Proof. Suppose that T is an OR map on $\mathbb{N}$, then by Lemma 4.34 (1) we have $\zeta = \nu = 0$ and $\sigma_2 < \omega$. By Lemma 4.3, we have $\sigma_1 \leq 1$ and $\sigma_n = 0$ for each $n > 2$. Lemma 4.34 (2) implies that there is a unique cycle O with $|O| = \omega$. Also Lemma 4.34 (3) implies that (a) and (c) hold. To see condition (b), let S_n be the spine of the n-cycle O for $n = 1$ or 2 and suppose, for a contradiction, that there is $z \in O$, $z \notin S_2$ with $|T^{-k}(z)| = \omega$ for some $0 < k \in \mathbb{N}$, $|T^{-k-1}(z) \setminus T^{-k}(z)| < \omega$ and $|T^{-k-2}(z) \setminus T^{-k-1}(z)| \neq 0$. So, there is $t \in T^{-k-2}(z) \setminus T^{-k-1}(z)$, hence $T^2(t) \in T^{-k}(z)$ and $T(t) \notin T^{-k}(z)$. But Lemma 4.5 implies that $T^{-k}(z) = [a, \infty)$ for some $a \in \mathbb{N}$, so $T^2(t) \in [a, \infty)$. Now we have two cases:

Case (1): $z \in S_1$. Since $T^{-k}(z) = [a, \infty)$ and $T^{-k}(z) \subseteq T^{-k-1}(z)$ so we have $T^{-k-1}(z) = [b, \infty)$ for some $b \in \mathbb{N}, b \leq a$. Since $T(t) \notin T^{-k}(z)$ then $T(t) \in [b, a)$ so $T(t) \leq T^2(t)$. On the other hand, since $t \notin T^{-k-1}(z) = [b, \infty)$, then $t < T(t)$ so $T^2(t) \leq T(t)$ since T is OR, so $T(t) = T^2(t)$, which is a contradiction since, as we showed above, $T^2(t) \in T^{-k}(z)$ and $T(t) \notin T^{-k}(z)$.

Case (2): $z \notin S_1$. Since $t < a$ (because $t \notin T^{-k}(z)$ and $t \notin S_2$), then $T(t) \geq T(a)$ and $T^2(t) \leq T^2(a)$, since T is an OR map. But this means that $[a, \infty) \leq T^2(a)$, by Lemma 4.5, so $T^2(a) \in [a, \infty)$. But this holds only if z lies in a spine of 2-cycle so $z \in S_2$, which is a contradiction since from the assumption $z \notin S_2$. Thus, $|T^{-k-2}(z) \setminus T^{-k-1}(z)| = 0$.

Conversely, suppose that T has in total, finitely many cycles and O is the unique cycle with $|O| = \omega$. Then the proof follows by Lemma 4.30, Lemma 4.31 and Proposition 4.36. $\square$

4.5 Characterizing Order-Reversing Maps on the Integers

Let $T : X \to X$ be an arbitrary map. In this section we give the necessary and sufficient conditions for a countable set with self-map to be linearly ordered with respect to which $X \approx \mathbb{Z}$ and T is an OR map.

4.5.1 Order-Reversing Injections and Surjections on the Integers

Lemma 4.38. *Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OR map. Then T has no $\mathbb{Z}$-orbits.*

Proof. The proof follows immediately from Theorem 4.8 and from the fact that $|(a, b)| < \omega$ for any $a, b \in \mathbb{Z}$. $\square$

Now we give the following theorem which describes the orbit structure of order-reversing injections on the integers $\mathbb{Z}$ with their usual order.

Theorem 4.39. *Let $\sigma = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ be a sequence of cardinals. The canonical representation $T : X \to X$ of σ on the countable set X is an OR injection with X is order-isomorphic to $\mathbb{Z}$ if and only if $\zeta = \sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$ and either:*

(1) *$\nu \neq 0$ and $\sigma_2 < \omega$; or*

(2) *$\nu = 0$ and $\sigma_2 = \omega$.*

Proof. Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OR map, then by Lemma 4.3 we have $\sigma_n = 0$ for all $n > 2$ and $\sigma_1 \leq 1$. By Lemma 4.38 we have $\zeta = 0$. Lemma 4.6, Corollary 4.9 and Corollary 4.7 imply that either (1) or (2) holds.

Conversely, suppose that (1) holds, so $\nu \neq 0$ and $\sigma_2 < \omega$. Suppose first that $\sigma_2 = \sigma_1 = 0$. Let $\nu = \omega$ and enumerate the natural numbers $\mathbb{N}$ as:

$$\{a_{0,0,0} \mid a_{1,0,0}, a_{1,1,0} \mid a_{2,0,0}, a_{2,1,0}, a_{2,2,0} \mid \cdots \mid a_{k,0,0}, a_{k,1,0}, a_{k,2,0} \dots a_{k,k,0} \mid \dots\},$$

then $a_{0,0,0} = 0$ and $a_{i,j,0} = \frac{i(i+1)}{2} + j$, for all $i > 0$ and $0 \leq j \leq i$. Also we enumerate the set $\mathbb{Z} \setminus \mathbb{N}$ as follows:

$$\{\cdots \mid a_{k,k,1}, \dots a_{k,2,1}, a_{k,1,1} \dots a_{k,0,1} \mid \cdots \mid a_{2,2,1}, a_{2,1,1}, a_{2,0,1} \mid a_{1,1,1}, a_{1,0,1}\}$$

then $a_{1,0,1} = -1$ and $a_{i,j,1} = -a_{i,j,0} = -(\frac{i(i+1)}{2} + j)$, for all $i > 0$ and $0 \leq j \leq i$.

For each $m \in \mathbb{N}$, let

$$B_m = \{a_{i,m,k} : i \geq m, k \in \{0,1\}\},$$

so we have $\mathbb{Z} = \bigcup_{m \in \mathbb{N}} B_m$. Let $T : \mathbb{Z} \to \mathbb{Z}$ be the map defined as

$$T(a_{i,j,k}) = \begin{cases} a_{1,0,0} & \text{if } i = j = k = 0, \\ a_{i+1,j,0} & \text{if } k = 1, \\ a_{i,j,1} & \text{if } k = 0. \end{cases}$$

So, for each $m \in \mathbb{Z}$ we have B_m is an $\mathbb{N}$-orbit of T, so T has countably infinitely many $\mathbb{N}$-orbits.

Now we prove that T is an OR injection. Let $x = a_{i,j,k}, y = a_{i',j',k'}$, $j \leq i \in \mathbb{N}, j' \leq i' \in \mathbb{N}$ and $k, k' \in \{0,1\}$, with $x < y$. If $k = 1$, $k' = 0$, so $x < 0$ and $y \geq 0$, then $T(x) = a_{i+1,j,0} > 0$ and $T(y) = a_{i',j',1} < 0$, so $T(x) \geq T(y)$. If $k = k' = 1$, then $T(x) = a_{i+1,j,0}$ and $T(y) = a_{i'+1,j',0}$. But $a_{i,j,1} < a_{i',j',1}$ means that either $i > i'$ or $i = i'$ and $j > j'$. If $i > i'$ then $i+1 > i'+1$, so $T(x) = a_{i+1,j,0} \geq a_{i'+1,j',0} = T(y)$, and if $i = i'$, $j > j'$ then $i + 1 = i' + 1$ so $T(x) \geq T(y)$ (since $a_{i,j,0} < a_{i',j',0}$ means that either $i < i'$ or $i = i'$ and $j < j'$). The case when $k = k' = 0$ follows in the same way. Hence, T is OR.

If $\nu = n < \omega$, then we can take $X = B_0 \cup \cdots \cup B_{n-1}$. Since for each $0 \leq r < n$, $B_r \subseteq \mathbb{Z}$ is infinite and has no endpoints, so we have $B \approx \mathbb{Z}$ and $T \restriction B$ is an OP injection with n $\mathbb{N}$-orbits. Finally, if $0 \neq \sigma_1 + \sigma_2 < \omega$, then the proof follows by Lemma 4.30 and Lemma 4.31.

Now, let (2) hold, so $\sigma_2 = \omega$ and $\nu = 0$. Let $T : \mathbb{Z} \to \mathbb{Z}$ be the map $T(x) = -x$. Clearly T is an OR map with, in total, countably infinitely many 2-cycles and a unique 1-cycle $\{0\}$. If $\sigma_1 = 0$, let $T_1 = T \restriction (\mathbb{Z} \setminus \{0\})$. $\square$

If $T : X \to X$ is a bijection on the countably infinite set X then we have the following result which follows from the previous theorem.

Theorem 4.40. *Let $\sigma = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$ be a sequence of cardinals. Then X can be ordered so that X is order-isomorphic to $\mathbb{Z}$ and the canonical representation $T : X \to X$ of σ is an OR bijection if and only if $\zeta = \sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$ and $\sigma_2 = \omega$.*

Now we deal with the case when $T : X \to X$ is a surjective map, so we have the following theorem.

Theorem 4.41. *Let $T : X \to X$ be a surjection on a countable set X and let $\sigma(T) = (0, \zeta, \sigma_1, \sigma_2, \sigma_3, \dots)$. Then X can be linearly ordered so that $X \approx \mathbb{Z}$ and T is an OR surjection iff T is finite-to-one, $\zeta = \sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$ and either:*

(1) *$\sigma_2 = \omega$ and each orbit is simple; or*

(2) *$\sigma_2 < \omega$ and all orbits are simple except for a unique cycle O with $|O| = \omega$ provided that $O = X$ when O is a 1-cycle.*

Proof. Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OR map, then by Lemma 4.3 we have $\sigma_n = 0$ for all $n > 2$ and $\sigma_1 \leq 1$. Also, by Lemma 4.38, we have $\zeta = 0$. Suppose, for a contradiction, that T is not finite-to-one, so there is $y \in X$ with $|T^{-1}(y)| = \omega$ so $|T^{-m}(y)| = \omega$ for all $m \in \mathbb{N}$, since T is a surjection. Lemma 4.5 implies that either $T^{-1}(y) = (-\infty, a)$ and $T^{-2}(y) = (b, \infty)$ for some $a < b \in \mathbb{Z}$ or vice versa. But then $\bigcup_{m>2} T^{-m}(y) \in [a, b]$; which is a contradiction, so T is finite-to-one. If $\sigma_2 = \omega$, then Lemma 4.6 and Corollary 4.7 imply that all cycles must be simple. If $\sigma_2 < \omega$, then Lemma 4.6 implies that there is a unique cycle O with $|O| = \omega$. If O is a 1-cycle then Lemma 4.6 and Lemma 4.5 imply that $O = X$.

Conversely, suppose that (1) holds, then the proof follows by Theorem 4.39. If (2) holds, let $C_{i,k}$ be defined for each $i \in N$ and $k \in \mathbb{N}$, where N is either $\{0\}$ or $\{0, 1\}$, and let $\preceq$ be the linear order defined in Theorem 4.13, so T is an OR map under $\preceq$. Since T

is finite-to-one then for each $i \in N$ and $k \in \mathbb{N}$, we have $|C_{i,k}| < \omega$, then immediately we have $X \approx \mathbb{Z}$ by Lemma 3.41. $\square$

4.5.2 Orbit Structure of Order-Reversing Self-Maps on the Integers

Lemma 4.42. *Let T be an OR map on $\mathbb{Z}$. Then $|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$ and an $\mathbb{N}$-orbit O.*

Proof. Suppose, for a contradiction, that $|L_i(O)| = \omega$ for some i and an $\mathbb{Z}$-orbit O. Let the spine of O indexed as $\{x_j : j \in \mathbb{Z}\}$ so that $T(x_j) = x_{j+1}$ for all $j \in \mathbb{Z}$. By Lemma 4.5 we have either $L_i(O) = (a, \infty)$ or $L_i(O) = (-\infty, a)$ for some $a \in \mathbb{Z}$. Without loss of generality, let $L_i(O) = (-\infty, a)$. Let $Y = \{x_k : k \in \mathbb{N}, k > i\}$, so Y is T-invariant, Y is an $\mathbb{N}$-orbit of $T \restriction Y$ and $T \restriction Y$ is OR. But $Y \subseteq [a, \infty)$ is infinite so $Y \approx \mathbb{N}$, which is a contradiction by Theorem 4.37. Hence, $|L_i(O)| < \omega$ for each $i \in \mathbb{N}$ and an $\mathbb{N}$-orbit O. $\square$

Lemma 4.43. *Let T be an OR map on $\mathbb{Z}$ with, in total, finitely many 2-cycles and at most a 1-cycle. Let S be the set of all spine points of T and let $\mathcal{O}$ be the collection of all orbits of T. Then*

(1) $|\{O \in \mathcal{O} : |O| = \omega\}| = 1$ *and if O is a 1-cycle with $|O| = \omega$, then $X = O$.*

(2) $|\{x : |T^{-1}(x)| = \omega\}| \leq 2$ *and if $|\{x : |T^{-1}(x)| = \omega\}| \neq 0$ then $\|y\| < \infty$ for each $y \in X \setminus S$.*

(3) *if $|\{x : |T^{-1}(x)| = \omega\}| = 1$ then either T is the constant map or there is $y \in X$ with $|T^{-1}(y)| = |T^{-2}(y) \setminus T^{-1}(y)| = \omega$.*

Proof. (1) The proof of the first statement follows immediately from Lemma 4.6 and Corollary 4.7. If O is a 1-cycle with $|O| = \omega$ then $O = X$ follows by Lemma 4.6 and Lemma 4.5.

(2) Suppose that $|\{x : |T^{-1}(x)| = \omega\}| = 2$, so there is $y, z \in X$ with $|T^{-1}(y)| = |T^{-1}(z)| = \omega$. It follows by Lemma 4.5 that $T^{-1}(y) = (-\infty, a)$ and $T^{-1}(z) = (b, \infty)$ for some $a < b \in \mathbb{Z}$, so

$$|O \setminus (T^{-1}(y) \cup T^{-1}(z))| < \omega.$$

Hence, $|T^{-1}(x)| < \omega$ for each $x \in X \setminus \{y, z\}$.

Now, let $|\{x : |T^{-1}(x)| = \omega\}| \neq 0$, so there is $y_0 \in O$ for some orbit O of T with $|T^{-1}(y_0)| = \omega$. Lemma 4.5 implies that either $T^{-1}(y_0) = (a, \infty)$ or $T^{-1}(y_0) = (-\infty, a)$ for some $a \in \mathbb{Z}$; say $T^{-1}(y_0) = (-\infty, a)$. Hence, $X \setminus T^{-1}(y_0) = [a, \infty)$. Suppose, for a contradiction, that there is a $z \in X \setminus S$ with $\|z\| = \infty$. From (1) above we have $z \in O$ since O is the unique orbit of T with $|O| = \omega$. Let S' be the spine of O. Without loss of generality let $z \in T^{-1}(S') \setminus S'$ (otherwise, $T^k(z) \in T^{-1}(S') \setminus S'$ for some $k \in \mathbb{N}$). So, by Lemma 1.13, there is $C \subseteq \bigcup_{k\in\mathbb{N}} T^{-k}(z)$ such that $T \restriction (C \cup S')$ is a semi-simple cycle. Since T is an OR map, then $T \restriction (C \cup S')$ is also an OR map, $C \subseteq [a, \infty)$ and $C \cup S'$ is a T-invariant infinite set. But then $(C \cup S') \approx \mathbb{N}$, which is a contradiction by Theorem 4.37. Hence, $\|y\| < \infty$ for each $y \in X \setminus S$.

(3) Suppose that $|\{x : |T^{-1}(x)| = \omega\}| = 1$, so there is $z \in O$ for some orbit O of T with $|T^{-1}(z)| = \omega$. If T is not the constant map then $O \setminus T^{-1}(z) \neq \emptyset$, so by Lemma 4.5 we have either $T^{-1}(z) = (-\infty, a)$ or $T^{-1}(z) = (a, \infty)$ for some $a \in \mathbb{Z}$, say $T^{-1}(z) = (-\infty, a)$. From (2) we have $\|y\| < \infty$ for each $y \in X \setminus S$. But $X \setminus T^{-1}(z) = [a, \infty)$, so it must be infinite. Hence, we must have $|T^{-2}(z) \setminus T^{-1}(z)| = \omega$, since z is the unique point of X with $|T^{-1}(z)| = \omega$. $\square$

Lemma 4.44. *Let $T : \mathbb{Z} \to \mathbb{Z}$ be an OR map. Let O be an n-cycle of T with spine S_n, for $n = 1$ or 2. Then for each $y \in O$, if $|T^{-k}(y)| = \omega$ for some $k \in \mathbb{N}$, then $|T^{-k-1}(y) \setminus T^{-k}(y)|$ is either ω or 0 and if $y \notin S_1$ then $T^{-k+1}(y)$ contains a unique point z with $|T^{-1}(z)| = \omega$.*

Proof. Let $y \in O$ with $|T^{-k}(y)| = \omega$ for some $k \in \mathbb{N}$, then by Lemma 4.5 we have either

$T^{-k}(y) = (-\infty, a)$ or $T^{-k}(y) = (a, \infty)$ for some $a \in \mathbb{Z}$, say $T^{-k}(y) = (a, \infty)$; so we have $T^{-k-1}(y) \setminus T^{-k}(y) \subseteq (-\infty, a]$. Suppose, for a contradiction, that $0 \neq |T^{-k-1}(y) \setminus T^{-k}(y)| < \omega$. Let $z \in T^{-k-1}(y) \setminus T^{-k}(y)$ be the least element of this set. Since $|T^{-k-1}(y) \setminus T^{-k}(y)| < \omega$, then there is $t \in \mathbb{Z}$ with $t < z$; so $t \notin T^{-k-1}(y)$. But $t < z$ implies that $T(t) \geq T(z)$. Since $T(z) \in T^{-k}(y)$ and from Lemma 4.5, we have $T(t) \geq (a, \infty)$, i.e., $T(t) \in (a, \infty) = T^{-k}(y)$. But this means that $t \in T^{-k-1}(y)$, which is a contradiction. Hence, $T^{-k-1}(y) \setminus T^{-k}(y)$ is infinite.

Now suppose that $y \notin S_1$ and suppose, for a contradiction, that there are two points $x_1, x_2 \in T^{-k+1}(y)$ with $|T^{-1}(x_1)| = |T^{-1}(x_2)| = \omega$, then $T^{-1}(x_1) = (a, \infty)$ and $T^{-1}(x_2) = (-\infty, b)$ for some $b < a \in \mathbb{Z}$. But then we have $T^{-k}(y) = (-\infty, b) \cup (a, \infty)$ and $T^{-k+1}(y) \subseteq [a, b]$. By Lemma 4.5, this is only true if $T^{-k+1}(y) \subseteq T^{-k}(y)$, i.e., $y \in S_1$, which is a contradiction. Hence, $x_1 = x_2$. $\square$

Now we give a proof of the main theorem of this section which provides conditions in which a countable set X with self-map T can be linearly ordered with respect to which T is OR and $X \approx \mathbb{Z}$. Before that we give the following property of a cycle of T.

Let T have an n-cycle O with spine S_n, $n \in \{1, 2\}$, we say that O has (C4) property if O satisfies the following:

(C4) For each $y \in O$, if $|T^{-k}(y)| = \omega$ for some $k \in \mathbb{N}$ then $|T^{-k-1}(y) \setminus T^{-k}(y)| = \omega$ or 0 and if $y \notin S_1$ then $T^{-k+1}(y)$ contains a unique point z with $|T^{-1}(z)| = \omega$.

Theorem 4.45. *Let T be a function on a countably infinite set X with orbit spectrum $\sigma(T) = (\nu, \zeta, \sigma_1, \sigma_2, \sigma_3, \ldots)$. Let $\mathcal{O}$ be the collection of all n-cycles of T and $A = \{x \in X : |T^{-1}(x)| = \omega\}$. There is a linear order with respect to which $X \approx \mathbb{Z}$ and T is an OR map iff $\zeta = \sigma_n = 0$ for all $n > 2$, $\sigma_1 \leq 1$, each $O \in \mathcal{O}$ has property (C4)and either:*

(1) *$\nu \neq 0$, $\sigma_2 < \omega$ with $|O| < \omega$ for all $O \in \mathcal{O}$ and $L_i(O') < \omega$ for all $i \in \mathbb{Z}$ and $\mathbb{N}$-orbit O'; or*

(2) $\nu = 0$, $\sigma_2 = \omega$ *and* $|O| < \omega$ *for all* $O \in \mathcal{O}$*; or*

(3) $\nu = 0$, $\sigma_2 < \omega$, $|O'| < \omega$ *for all* $O' \in \mathcal{O}$ *except for a unique* n*-cycle* O, $n \in \{1, 2\}$ *with spine* S *such that either* T *is finite-to-one or* O *satisfies the following:*

   (a) $\|z\| < \infty$ *for each* $z \in O \setminus S$*;*

   (b) $0 \neq |A| \leq 2$ *and if* $|A| = 1$ *then either* T *is the constant map or* $|T^{-2}(y) \setminus T^{-1}(y)| = \omega$ *for* $y \in A$*; and*

   (c) *if* O *is a 1-cycle then* $O = X$*.*

Proof. Suppose that T is an OR map on $\mathbb{Z}$, then by Lemma 4.38 we have $\zeta = 0$ and from Lemma 4.3 we have $\sigma_n = 0$ for each $n > 2$ and $\sigma_1 \leq 1$. By Lemma 4.44 we have each $O \in \mathcal{O}$ satisfies (C4).

If $\nu \neq 0$, then by Lemma 4.42 we have $|L_i(O)| < \omega$ for each $i \in \mathbb{Z}$ and an $\mathbb{N}$-orbit O. If $\nu = 0$ and $\sigma_2 = \omega$ then Lemma 4.6 and Corollary 4.7 imply that $|O| < \omega$ for all $O \in \mathcal{O}$. If $\nu = 0$ and $\sigma_2 < \omega$, then by Lemma 4.43 (1), we have $|O| = \omega$ for a unique cycle $O \in \mathcal{O}$. If T is not finite-to-one, then Lemma 4.43 implies that (a), (b) and (c) holds.

Conversely, if either (1) or (2) holds, then the proof follows by Theorem 4.39 and Theorem 4.33, since $|O| < \omega$ for all $O \in \mathcal{O}$ and $|L_i(O')| < \omega$ for all $i \in \mathbb{N}$ and $\mathbb{N}$-orbit O' (where spines of $\mathbb{N}$-orbits are chosen as in Observation 3.40).

Now, suppose that (3) holds, so $\sigma_2 < \omega$ and T has a unique cycle O with spine S with $|O| = \omega$. By Lemma 4.30 and Lemma 4.31, it is sufficient to order O such that $O \approx \mathbb{Z}$ and $T \restriction O$ is OR. If T is finite-to-one, let $\preceq$ be the linear order defined in Theorem 4.13, so T is an OR map under $\preceq$. Since T is finite-to-one then for each $i \in N$, where N is either $\{0\}$ or $\{0, 1\}$, and $k \in \mathbb{N}$, we have $|C_{i,k}| < \omega$, then we have $X \approx \mathbb{Z}$ by Lemma 3.41.

Now let $\|z\| < \infty$ for each $z \in O \setminus S$. By Corollary 4.32 and from condition (b) above, if O is a 2-cycle with spine $S_2 = \{x_0, x_1\}$, it is sufficient to order the set O' in such a way that $(x_0, x_1) = \emptyset$, where $O' = X \setminus (\bigcup_{k\in\mathbb{N}} T^{-k}(C))$ and $C = \{x \in T^{-1}(S_2) \setminus S_2 : \|x\| \geq 2, |\bigcup_{k\in\mathbb{N}} T^{-k}(x)| < \omega\}$ (where C is finite). If $n = 1$, let $O' = X$.

For each $i \in N$ and $j \in \mathbb{N}$, let $C_{i,j}$ be defined, so from condition (a), there is $r \leq r' \in \mathbb{N}$ such that $C_{i,r} \neq \emptyset$, $C_{i+1,r'} \neq \emptyset$ and $C_{i,k} = C_{i+1,m} = \emptyset$ for each $k > r$ (with $k \neq r'$ if $n = 1$) and $m > r'$. Let T has two points x_1, x_2 with $|T^{-1}(x_1)| = |T^{-1}(x_2)| = \omega$. Choose $t_j \in T^{-1}(x_j)$, $j = 1, 2$ and let $\preceq_j$ be a linear order defined on $T^{-1}(x_j) \setminus \{t_j\}$ in such a way that $T^{-1}(x_j) \setminus \{t_j\} \approx \mathbb{N}$. Since O has (C4), we have the following cases:

(1′) $r = r' - 1$, so $|C_{i,r'}| = |C_{i+1,r'-1}| = \omega$, $t_1 \in C_{i,r'}$ and $t_2 \in C_{i+1,r'-1}$ with either $t_1 \in T^{-m-1}(T^m(t_2))$ for some $m \in \mathbb{N}$ or $|T^{-k-1}(T^k(t_2))| = 0$ whenever $T^k(t_2)$ is not a fixed point.

(2′) O is a 2-cycle, $r = r'$, $|C_{i,r}| = |C_{i+1,r}| = \omega$, $t_1 \in C_{i,r}$ and $t_2 \in C_{i+1,r}$.

(3′) O is a 1-cycle and $t_1, t_2 \in C_{0,r'}$.

Let

$$O_1 = (O' \setminus (T^{-1}(x_1) \cup T^{-1}(x_2)) \cup \{t_1, t_2\}.$$

Now, t_1, t_2 satisfy conditions in Lemma 4.15 with respect to O_1, so there is a linear order $\preceq'$ on O_1 in such a way that t_1 is the least element of O_1 and t_2 is the greatest element of O_1 and $T \restriction O_1$ is OR. Now, let $(O', \preceq)$ be the ordered sum of $(T^{-1}(x_1)\setminus\{t_1\}, \preceq_1^{-1}), (O_1, \preceq')$ and $(T^{-1}(x_2) \setminus \{t_2\}, \preceq_2)$ respectively, so immediately we have T is OR and $X \approx \mathbb{Z}$.

Finally, if T has a unique $y_0 \in O'$ with $|T^{-1}(y_0)| = |T^{-2}(y_0) \setminus T^{-1}(y_0)| = \omega$, so $y_0 \in C_{i+1,r-1}$ and $|C_{i,r}| = |C_{i+1,r'}| = \omega$, where $r' = r + 1$. By Theorem 4.37, there is a linear order $\preceq$ on $X_1 = O' \setminus C_{i+1,r'}$ so that $X_1 \approx \mathbb{N}$ and $T \restriction X_1$ is OR. By Lemma 4.11 there is a linear order $\preceq_1$ on $C_{i+1,r'}$ so that $T \restriction C_{i+1,r'} : C_{i+1,r'} \to C_{i,r}$ is OR. Since $C_{i,r} \approx \mathbb{N}$, then $C_{i+1,r'} \approx \mathbb{Z} \setminus \mathbb{N}$. Hence, O' as the ordered sum of $C_{i+1,r'}$ and X_1 is order-isomorphic to $\mathbb{Z}$ and T is an OR map. $\square$

Concluding Remarks

Let $T : X \to X$ be a function on the countably infinite set X. In this thesis, we have given answers to particular cases of the following question: Can we put a structure on the set X with respect to which the function T has some meaning? We give characterization of continuous function on the rational world, this characterization is in terms of the inverse image of certain subsets of X. Also, we characterize order-preserving and order-reversing bijections, injections and surjections on the rational world in terms of the orbit structure of the map T. We deal with more countable sets and give cases in which we can put an order on X such that X is order-isomorphic to the naturals or the integers with their usual order and with respect to which T is order-preserving or order-reversing map; the answer was in terms of the orbit structure of T.

In further work, there are many questions we can possibly ask such as the following:

Question 1. Let $T : X \to X$ be arbitrary function on the countably infinite set X. Is there an order on X with respect to which T is order-preserving and X is order-isomorphic to $\mathbb{Q}$? What about order-reversing maps?

Question 2. What is the orbit structure of continuous functions on the rationals?

Question 3. If $T_1 : X \to X$ and $T_2 : X \to X$ are functions on the countably infinite set X. Is there an order on X with respect to which T_1 and T_2 are order-preserving maps and X is order-homeomorphic to $\mathbb{Q}$, $\mathbb{N}$ or $\mathbb{Z}$? What about three maps or a countable collection of maps? What about order-reversing maps?

Question 4. Let $T : X \to X$ be a function on the countably infinite set X. Is there an order $\preceq$ on X with respect to which X is order-isomorphic to $\mathbb{Q}$, T is order-preserving and T is continuous with respect to the order topology generated by $\preceq$?

Question 5. A number of authors have studied the group of order-preserving permutations of $\mathbb{Q}$ (see, for example, [21], [32], [20] and [10]). Can we characterize countable subgroups of order-preserving permutations on $\mathbb{Q}$ like Truss in [31] and Mekler in [22] who characterize countable subgroups of autohomeomorphisms of $\mathbb{Q}$?

Question 6. What is the orbit structure of order-preserving bijections on $\mathbb{R}$? What about the orbit structure of order-preserving maps and order-reversing maps on $\mathbb{R}$, Cantor set, or irrationals?

Question 7. If $T : X \to X$ is a bijection, can we find set-theoretic conditions on X so that T is a homeomorphism and X is homeomorphic to $\mathbb{R}$?

Question 8. If $T : X \to X$ is a function, can we find set-theoretic conditions on X so that T is order-preserving and X is order-isomorphic to $\mathbb{Q}$?

Question 9. If $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, what is the orbit structure of a homeomorphism of S^1?

Question 10. If $T_1 : X \to X$ and $T_2 : X \to X$ are functions on a countable set X. When is there a compact, Hausdorff topology on X with respect to which T_1 and T_2 are continuous? (Rolf Suabedissen, DPhil Thesis, Oxford, studied the case when X is an arbitrary set and $T_1T_2 = T_2T_1$)

Question 11. Let $T : X \to Y$ be a function on arbitrary sets X, Y and $X \cap Y = \emptyset$. Are there metrizable, compact topologies on X and Y such that T is continuous?

Question 12. Can we generalize the (NC), (SMC) and (SH) properties for any map rather than bijections? What is the orbit structure of maps which have one of these properties?

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