Lie Algebras and Incidence Geometry

by

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Abstract

An element x of a Lie algebra L over the field F is extremal if $[x, [x, L]] \subseteq Fx$. One can define the extremal geometry of L whose points $\mathcal{E}$ are the projective points of extremal elements and lines $\mathcal{F}$ are projective lines all of whose points belong to $\mathcal{E}$. We prove that any finite dimensional simple Lie algebra L is a classical Lie algebra of type A_n if it satisfies the following properties: L contains no elements x such that $[x, [x, L]] = 0$, L is generated by its extremal elements and the extremal geometry $\mathcal{E}$ of L is a root shadow space of type $A_{n,\{1,n\}}$.

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Contents

Introduction

Lie algebras were independently introduced by Lie and Killing in the late nineteenth century. Between 1888 and 1890, Killing published a series of four papers on Lie algebras and in the second paper [17] he classifies simple Lie algebras over the complex numbers. In this remarkable paper, Killing introduces important concepts such as root systems, Cartan integers and Cartan subalgebras. Killing associates to every simple Lie algebra an irreducible root system and concludes that the Lie algebra can be recovered from the root system thus reducing the problem to classifying irreducible root systems. Shortly after, Cartan refined the proof of the classification and published it in his PhD thesis [4]. Half a century later, Dynkin provided a combinatorial proof for the classification of irreducible root systems by assigning to each such root system a certain graph known as a Dynkin diagram (c.f. [14]). In the second half of the twentieth century there was an interest in simple Lie algebras over fields of characteristic $p > 0$ culminating in a classification result by Premet and Strade which states that every finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ is known. Details of the development of the classification can be found in [30] and the result can be found in the papers [20, 21, 22, 23, 24, 25].

In the last decade there has been an effort, spearheaded by Cohen, to provide a geometric proof of the aforementioned classification allowing the result to be extended to all fields. A focal point of the project is a special class of elements called extremal elements.

An element x of a Lie algebra L over a field F of odd characteristic is *extremal* if $[x,[x,y]] \subseteq Fx$ for all elements y of L and an extremal element is a *sandwich* if $[x,[x,y]] = 0$ for all $y \in L$. For even characteristic additional conditions are required. Extremal elements span one dimensional inner ideals as defined in [1]. In [26, 27], Premet shows that every Lie algebra over an algebraically closed field with characteristic $p \neq 2, 3$ contains an extremal element. Furthermore, he shows that when L is simple it is either a classical Lie algebra or contains a sandwich element. An interesting result given by Cohen, Ivanyos and Roozemond in [10] is that if L is simple over a field of characteristic $p \neq 2, 3$ and contains a nonsandwich extremal element, then L is generated by its extremal elements or is the (non-classical) Witt algebra $W_{1,1}(5)$. For Lie algebra L generated by its extremal elements E containing no sandwich element, Cohen and Ivanyos define the *extremal geometry* $\mathcal{E}(L)$, a point-line space whose points are the projective points spanned by extremal elements and prove the following result.

Theorem 1 *If L is a finite dimensional simple Lie algebra and satisfies the above properties, then $\mathcal{E}(L)$ is isomorphic to the root shadow space of an irreducible spherical building or $\mathcal{E}(L)$ has no lines.*

One of the questions posed is whether L can be recovered from $\mathcal{E}(L)$ and we provide a partial answer. The problem is best approached from a local point of view by investigating what happens inside an apartment. We suppose that $\mathcal{E}$ is a root shadow space of a spherical building Δ. The intersection $\mathcal{E}_\Sigma = \mathcal{E} \cap \Sigma$ is a root shadow space of the (thin) building Σ and inherits its structure from $\mathcal{E}$. We can, therefore, bypass the more complicated structure of a building and instead focus on the familiar object Σ since the observations we make in $\mathcal{E}_\Sigma$ can be lifted to $\mathcal{E}$. The points of $\mathcal{E}_\Sigma$ are in a bijective correspondence with the long roots. Such a correspondence naturally leads to the construction of a Chevalley basis for a subalgebra L' of L. One now only needs to establish that L and L' coincide.

Before embarking on the principal goal of the thesis we build the foundations by introducing the relevant notions and results concerning *spherical buildings* with a particular emphasis on the *Coxeter chamber system*. We present a geometric interpretation of the Coxeter chamber system whose construction is not only pivotal to achieving our goal but highlights the central theme of the thesis, namely, the translation of algebraic into geometric notions and geometric into algebraic notions. In Chapter 2 we concentrate on a specific structure defined on spherical buildings, namely the point-line space known as a *root shadow space* $\mathcal{P}_\Delta$ *of type* $X_{n,J}$. The set J is defined to be the set of vertices in the Dynkin diagram of type X_n that connect to the new vertex of the corresponding extended Dynkin diagram and is called the *root set*. The points of $\mathcal{P}_\Delta$ are $(I \setminus J)$-residues called J*-shadows* and a j*-line* is the union of J-shadows that intersect a given j-panel for $j \in J$. We show that there is a bijective correspondence between the *polar regions* of the half-apartments of a given an apartment Σ and the long roots of the root system of type X_n. This leads to the creation of a trilingual dictionary between the collinearity graph of $\mathcal{P}$, roots α and the corresponding half-apartments of Σ. In particular, a natural angle forms between J-shadows of the building Δ and it proves to be particularly fruitful for the construction of a Chevalley basis in Chapter 5.

The main purpose of the thesis is to classify the finite dimensional simple Lie algebras which are generated by its extremal elements. Accordingly, in Chapter 3 we acquaint ourselves with basic notions of Lie algebras before focusing on the simple Lie algebras over the complex numbers. We describe the classification of complex simple Lie algebras and present a theorem of Chevalley that states that every such Lie algebra has a special basis indexed by the corresponding root system and is known as a *Chevalley basis*. Such a basis $\mathcal{B}$ has integral structure constants and we use this property to construct a new Lie algebra over an arbitrary field F by taking the tensor product of the $\mathbb{Z}$-span of $\mathcal{B}$ and F. A theorem of Steinberg reveals that such a Lie algebra, modulo its centre, is simple. The new Lie algebras are called *Chevalley algebras* and the quotients of Chevalley algebras by their centres are called the *classical Lie algebras* providing such a quotient is simple. An exhaustive treatment of the extremal elements of the classical Lie algebra $\mathfrak{g} = \mathfrak{psl}(n+1, F)$ is given in the final two sections of Chapter 3. We show that, up to a generic exception, that the set of extremal elements of $\mathfrak{g}$ consists of all the rank 1 matrices and verify that the extremal geometry $\mathcal{E}(\mathfrak{g})$ is a root shadow spaces of type $A_{n,\{1,n\}}$.

Theorem 1 is a special case of a general theory developed by Cohen and Ivanyos in the two papers [8] and [9]. The point-line space $\mathcal{E}(L)$ is an example of a root filtration space which, more generally, is a point-line space that admits a filtration of five symmetric relations $\{\mathcal{E}_i\}_{i=-2}^{2}$ on its points. Aptly, we formally define a *root filtration space* and state a number of useful properties of such spaces in Chapter 4. A consequence of these properties is that symmetric relations can be characterised by the collinearity graph of $\mathcal{E}$ for any nondegenerate root filtration space. The fundamental result from [9] is that any nondegenerate root filtration space of finite singular rank is isomorphic to the root shadow space of a spherical buildings or has no lines. In Section 4.3, extremal elements are formally defined as elements x of L such that $[x, [x, y]] = 2g_x(y)x$ for all elements y of L where $g_x : L \to F$ is a (linear) map and two additional conditions where F has even characteristic. The extremal geometry of a Lie algebra is formally defined as a point-line space with point set $\mathcal{E}$ consisting of the projective points Fx for all extremal elements x in L and lines $\mathcal{F}$ are projective lines $F\langle x, y\rangle$ such that $0 \neq \lambda x + \mu y$ is an extremal element for all scalars λ and μ of F. Theorem 1 is crucial consequence from the results in [8] and [9] that we verify.

Combining all the result we begin the process of recovering the Lie algebra L from $\mathcal{E}(L) = \mathcal{E}$. We concentrate on the case where $\mathcal{E}$ is a root shadow space of type $X_{n,J}$ where X is simply laced and $p = \mathrm{char}(F)$ is not equal to 2. The dictionary established in Section 2.2 induces an injective map from Φ to $\mathcal{E}$, denoted by $\alpha \mapsto E_\alpha$ and we examine the relationship between E_α and E_β and the corresponding angles formed between α and β. For the subalgebra $L' = \langle E_\alpha \mid \alpha \in \Phi \rangle$ one can appropriately choose an $x_\alpha \in E_\alpha$ for each root α and set $h_i = [x_{\alpha_i}, x_{-\alpha_i}]$ for each simple root α_i such that $B = \{x_\alpha, h_i \mid \alpha \in \Phi, i \in \{1, \ldots, n\}\}$ forms a spanning set for L' exhibiting the same structure constants as a Chevalley basis of type X_n. Hereinafter, the root system Φ is assumed to be of type A_n and $p \neq 2$ and we conclude that L' is isomorphic to $\mathfrak{psl}(n+1, F)$. In particular, the extremal geometry $\mathcal{E}(L')$ is a root shadow space of type $A_{n,\{1,n\}}$ and $\mathcal{E}(L')$ is a subspace of $\mathcal{E}(L)$. In the final step, we reconstruct the building $\Delta(\mathcal{E})$ of type A_n from the two different types of lines of its root shadow space $\mathcal{E}$. A similar building $\Delta(\mathcal{E}')$ can be reconstructed from the root shadow space $\mathcal{E}' = \mathcal{E}(L')$ in terms of the lines from $\mathcal{E}'$. Such a construction induces a natural embedding of $\Delta(\mathcal{E}')$ into $\Delta(\mathcal{E})$ and by comparing ranks, we conclude that $\Delta(\mathcal{E}')$ and $\Delta(\mathcal{E})$ are isomorphic which forces L and L' to coincide. We summarise the main result in the following theorem.

Theorem 2 *Let L be a finite dimensional Lie algebra over a field F of characteristic p. Suppose that L is generated by its extremal elements, contains no sandwich elements and has extremal geometry isomorphic to the root shadow space of type $A_{n,\{1,n\}}$. Let L' be the aforementioned subalgebra of L. Then*

(i) *the subalgebra L' is a classical Lie algebra and,*

(ii) *if $p \neq 2$ and $(n, p) \neq (2, 3)$, then L and L' coincide.*

In particular, L is a classical Lie algebra.

Chapter 1

Coxeter chamber systems and their geometric realisations

In this chapter we introduce the notion of chamber systems and focus on a particular class of chamber systems, namely Coxeter chamber systems. An important example of a chamber system is a building. We briefly discuss spherical buildings and state the classification of irreducible spherical buildings up to type.

1.1 Chamber systems

A graph $\mathit{\Gamma}$ is a pair (V, E) where V is a set whose members are called *vertices* and E is a set of subsets of size 2 of V whose members are called *edges*. Two vertices x and y are said to be *joined by an edge* if $\{x, y\} \in E$ and in such a case we write $x \sim y$ and say 'x is *adjacent* to y'. The *valency* of a vertex x is the number of edges that pass through x, that is, $|\{\{u, v\} \in E \mid x \in \{u, v\}\}|$. A graph $\mathit{\Gamma}' = (V', E')$ is a *subgraph* of $\mathit{\Gamma}$ if $V' \subseteq V$ and $E \subseteq E'$. A subgraph $\mathit{\Gamma}'$ is an *induced subgraph* if, for $x, y \in V'$, $\{x, y\} \in E'$ whenever $\{x, y\} \in E$. A *path* in $\mathit{\Gamma}$ is a sequence of vertices $x_0, x_1, \ldots, x_k$ such that $x_i \sim x_{i+1}$ for all $i \in \{0, \ldots, k-1\}$. An *edge-coloured graph* is a graph $\mathit{\Gamma}$ such that each edge is labelled with a colour from a set I. The set I is called the *index set* of $\mathit{\Gamma}$ and is typically given by

$\{1, \ldots, n\}$ for some natural number n. If an edge $\{x, y\}$ is labelled with the colour i, then we write $x \sim_i y$ and say 'x is *i-adjacent* to y'. For any subset $J \subseteq I$, we define a *J-residue* (or *residue*) to be a connected component of Γ after deleting all the edges labelled with a colour not in J. If $J = \{j\}$, then we say a J-residue is a *j-panel* (or *panel*). The *rank* and *corank* of a residue of type J are $|J|$ and $|I \setminus J|$, respectively. Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be two edge-coloured graphs with index set I. Then the pair (ϕ, σ) is an *isomorphism* from Γ to Γ' if $\phi : V \to V'$ and $\sigma : I \to I$ are bijections such that x and y are i-adjacent if and only if $\phi(x)$ and $\phi(y)$ are $\sigma(i)$-adjacent for all vertices x and y in V. An isomorphism is *special* if σ is the identity map.

Definition 1.1.1 *A* chamber system *is an edge-coloured graph Δ with index set I such that, for each $i \in I$, every i-panel is a complete graph with at least two vertices.*

The vertices of Δ are called *chambers*. The chamber system is called *thick* if each panel has at least three chambers and *thin* if each panel has exactly two chambers. The *rank* of Δ is defined to be the cardinality of I. A path in a chamber system is called a *gallery*. The distance between two chambers x and y is denoted $\mathrm{dist}(x, y)$ and is defined as the length of a minimal gallery between x and y. A subset of chambers X is said to be *convex* if every minimal gallery between every pair of chambers $x, y \in X$ is contained in X. The *diameter* of a convex set X is $\mathrm{diam}(X) := \sup\{\mathrm{dist}(x, y) \mid x, y \in X\}$. A *subchamber system* of Δ is an edge-coloured induced subgraph Δ' (in which colours are preserved) which is also a chamber system in its own right.

1.2 Coxeter chamber systems

The purpose of this section is to introduce the language of Coxeter chamber systems and to state their well known properties. It is not a detailed exposition and any proofs that are omitted can be found in [34].

Definition 1.2.1 *A* Coxeter group *is a group given by a presentation*

$$W = \langle r_i \mid (r_i r_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \rangle$$

where $I = \{1, \ldots, n\}$ *is an index set,* $m_{ii} = 1$ *and* $m_{ij} = m_{ji} \geq 2$ *for all* $i, j \in I$*.*

The finite Coxeter groups, called *Weyl groups*, were classified by Coxeter in 1935 (c.f. [12]). In this paper, the author assigns to each such group a graph called a Coxeter diagram. The *Coxeter diagram* of the Coxeter group W has vertex set $I = \{1, \ldots, n\}$ and the edge $\{i, j\}$ is labelled m_{ij}. We deleted edges labelled 2 and drop the label 3 from the corresponding edges. We denote the Coxeter diagram by Π and we sometimes add the subscript Π to W to indicate that it is associated with Π and to eliminate any ambiguity. The Coxeter group W with a set of generators $S = \{r_i \mid i \in I\}$ and Coxeter diagram uniquely determine each other (up to an automorphism of the diagram) and are usually considered as a pair. We say that W is the *Coxeter group of type* Π and (W, S) is a *Coxeter System* of type Π. For a subset J of I, we define W_J to be the subgroup of W generated by $S_J = \{r_j \mid j \in J\}$ and define Π_J to be the subgraph of Π obtained by deleting the vertices $I \setminus J$. The pair (W_J, S_J) is a Coxeter system of type Π_J.

The chamber system of a Coxeter system plays a pivotal role in the construction of buildings and it corresponds to the following chamber system.

Definition 1.2.2 *Let* (W, S) *be a Coxeter system of type* Π*. Let* Σ_Π *be the chamber system whose chambers are the elements of* W *and two chambers* x *and* y *are* i*-adjacent if and only if* $xr_i = y$ *for* $r_i \in S$*. We call* Σ_Π *the* Coxeter chamber system of type Π.

The chamber system Σ_Π is thin and we drop the index Π when the context is clear. Let $\mathrm{Aut}^\circ(\Sigma)$ denote the group of special automorphisms of Σ. Every such automorphism can be viewed as left multiplication by an element of W and thus $\mathrm{Aut}^\circ(\Sigma) \cong W$. A *reflection* is an element s of order 2 of W that interchanges two chambers of an edge. For a reflection s, the set M_s consisting of the edges that are fixed setwise by s is called *the wall of* s. The graph $\Sigma \setminus M_s$ has two connected components of equal size called *half-apartments*. We denote the half-apartments by α and $-\alpha$. In this context, we write M_α for M_s and $\mathrm{refl}_\Sigma(\alpha)$ for s. For a half-apartment α, the set $\partial\alpha$ consisting of the chambers in α on edges from M_α is called *the boundary of* α. Note that $\mathrm{refl}_\Sigma(\alpha) = \mathrm{refl}_\Sigma(-\alpha)$ and $M_\alpha = M_{-\alpha}$. For an edge $\{x, y\}$, let $\Sigma_{(x\backslash y)} := \{z \in \Sigma \mid \mathrm{dist}(x, z) < \mathrm{dist}(y, z)\}$. We state some properties about half-apartments.

Lemma 1.2.3 *Let α be the half-apartment corresponding to the reflection s that fixes the edge $\{x, y\}$ by interchanging x and y. Then α is convex and, up to permuting x and y, $\alpha = \Sigma_{(x\backslash y)}$ and $-\alpha = \Sigma_{(y\backslash x)}$. Furthermore, for any element w of W*

(i) *$\alpha^w = \Sigma_{(wx\backslash wy)}$ is a half-apartment corresponding to the reflection wsw^{-1}, and*

(ii) *$M_{\alpha^w} = M_\alpha^w$ and $\partial(\alpha^w) = (\partial\alpha)^w$.*

In particular, W acts on the set of half-apartments of Σ.

Proof. The first two assertions are Corollary 3.15 and Proposition 3.19 in [34]. Write $\alpha = \Sigma_{(x\backslash y)}$ and let w be in W. Using that w is an automorphism of Σ that preserves distance, we have

$$\begin{aligned} w\alpha &= \{wz \in \Sigma \mid \mathrm{dist}(x, z) < \mathrm{dist}(y, z)\} \\ &= \{z \in \Sigma \mid \mathrm{dist}(wx, z) < \mathrm{dist}(wy, z)\} \\ &= \Sigma_{(wx\backslash wy)}. \end{aligned}$$

Since wx and wy are adjacent, $w\alpha$ is a half-apartment. The element wsw^{-1} has order 2 since s has order 2 and $wsw^{-1}(wx) = wy$. Thus $wsw^{-1} = \mathrm{refl}_\Sigma(w\alpha)$ and this proves (i). Part (ii) follows from the property that $su = v$ if and only if $wsw^{-1}(wu) = wv$. □

The residues of chamber systems provide a great insight into its overall structure. We state a few basic properties of residues in Coxeter chamber systems.

Lemma 1.2.4 *Let J be a subset of I. The following assertions hold.*

(i) *All residues are convex.*

(ii) *The Coxeter group W acts transitively on the set of J-residues.*

(iii) *Let R be a J-residue, x be a chamber of R, x_j be the unique chamber j-adjacent to x and s_j the reflection determined by $\{x, x_j\}$. Then $V = \langle s_j \mid j \in J \rangle$ acts transitively on the chambers of R.*

Proof. Part (i) is Proposition 3.24 in [34]. Part (ii) follows from the fact that W preserves colours and acts transitively on the chambers of Σ. For part (iii), we can write for any $y \in R$, $y = xr_{j_1} \dots r_{j_k}$, for $j_i \in J$. Note that $s_{j_i} = xr_{j_i}x^{-1} \in V$ and that $s_{j_1} \dots s_{j_k}(x) = y$. Thus V acts transitively on R. □

Residues in Coxeter chamber systems possess a 'gatedness' property. This property is expressed in the following result, that is Theorem 3.22 of [34].

Theorem 1.2.5 *For every residue R and every chamber x in Σ, there exists a unique closest element y in R such that* $\mathrm{dist}(x, z) = \mathrm{dist}(x, y) + \mathrm{dist}(y, z)$ *for all chambers z in R.* □

In this result, we call the chamber y the *projection of x onto R* and denote it by $\mathrm{proj}_R(x)$. It is clear that y is unique. Indeed, suppose, by contradiction, that y' is another chamber with the same property. We have that $\mathrm{dist}(x, y) = \mathrm{dist}(x, y') + \mathrm{dist}(y', y) = \mathrm{dist}(x, y) + 2 \cdot \mathrm{dist}(y,' y)$. In particular, $\mathrm{dist}(y, y') = 0$ and $y = y'$.

If W is a finite Coxeter group, then $\mathrm{diam}(\Sigma) = \max\{\mathrm{dist}(x, y) \mid x, y \in \Sigma\}$ is finite. We say that chambers x and y are *opposite* each other in Σ if and only if $\mathrm{dist}(x, y) = \mathrm{diam}(\Sigma)$. The following result is taken from Proposition 5.2 and 5.4 in [34].

Lemma 1.2.6 *Suppose that Σ has a finite diameter. Every chamber x has a unique opposite chamber denoted by $\mathrm{op}_\Sigma(x)$. Furthermore, for opposite chambers x and y, the following assertions hold.*

(i) *We have $|\{x, y\} \cap \alpha| = 1$ for any half-apartment α.*

(ii) *Every chamber of Σ lies on a minimal gallery from x to y.* $\square$

Let Π be the the Coxeter diagram of Σ with vertex set I and, for $J \subseteq I$, let Π_J denote the subdiagram of Π obtained by deleting the vertices in $I \setminus J$. If R is a J-residue, then R is a Coxeter subchamber system of type Π_J. In particular, the chambers in W_J correspond to a J-residue R_0 containing the chamber 1 and W acts transitively on all J-residues, that is, there exists $w \in W$ such that $R_0^w = R$. In particular, R is isomorphic to the Coxeter chamber system Σ_{Π_J}. Lemma 1.2.6 tells us that for each chamber x in R, there is a unique chamber denoted by $\mathrm{op}_R(x)$ such that $\mathrm{dist}(x, \mathrm{op}_R(x)) = \mathrm{diam}(R)$. The chambers x and $\mathrm{op}_R(x)$ are said to be *opposite* in R.

1.3 Spherical buildings

There are numerous ways to define buildings and the definition below is taken from [28] and [35].

Definition 1.3.1 *Let W be a Coxeter group of type* Π *and let I be the vertex set of* Π. *A* building of type Π with index set I *is a chamber system Δ with index set I with a collection of subchamber systems $\mathcal{A}$ called* apartments *such that*

(B1) *Each Σ in $\mathcal{A}$ is isomorphic to the Coxeter chamber system Σ_Π.*

(B2) *Each pair of chambers x,y is contained in a common apartment.*

(B3) *For each pair of chambers x, y and each pair of apartments Σ, Σ' containing both x and y, there exists a special isomorphism from Σ to Σ' that fixes x and y.*

(B4) *For every chamber x and each pair of apartments Σ, Σ' that contain x and each panel P such that $P \cap \Sigma$ and $P \cap \Sigma'$ are nonempty, there exists a special isomorphism that fixes x and sends $P \cap \Sigma$ to $P \cap \Sigma'$.*

The *rank* of Δ is defined to be the cardinality of its index set I. We say that a building is *spherical* if the apartments have finite diameter, *thick* (*thin*, respectively) if its underlying chamber system is thick (thin, respectively), *irreducible* if the corresponding diagram Π is connected and *reducible* if Π is not connected. We state, without proof, that all apartments are convex (c.f. Corollary 8.9 of [34]).

Definition 1.3.2 *A set of chambers α is a* half-apartment *of the building Δ if it is a half-apartment of one of the apartments of Δ.*

From axioms (B1) and (B2) we can define a distance in W between any pair of chambers. Let x and y be chambers in Δ. By (B2), there exists an apartment Σ such that $x, y \in \Sigma$. Let ρ be a special isomorphism from Σ to Σ_Π. Define the distance $\delta_\Sigma(x, y) := w$ where w is the element of W such that $w = (x^\rho)^{-1}y^\rho$. Suppose x and y are contained in another apartment Σ'. By (B3), there exists a special isomorphism φ from Σ' to Σ that fixes x and y. Note that $\varphi\rho$ is a special isomorphism from Σ' to Σ_Π. In particular, $\delta_{\Sigma'}(x, y) = (x^{\varphi\rho})^{-1}(y^{\varphi\rho}) = (x^\rho)^{-1}y^\rho = w$. Thus we can drop the subscript of the apartment and define a map $\delta : \Delta \times \Delta \to W$ called the *Weyl distance map*. We often write Δ as the pair (Δ, δ). If $\delta(x, y) = x^{-1}y = r_{i_1} \dots r_{i_k}$, then $(x, xr_{i_1}, xr_{i_1}r_{i_2}, \dots, xr_{i_1}r_{i_2} \dots r_{i_k} = y)$ is a minimal gallery from x to y and we say its *type* is $f = i_1i_2 \dots i_k$. We often write $r_f = r_{i_1i_2\dots i_k}$.

Remark 1.3.3 *Let Σ be a Coxeter chamber system of type Π. Then Σ is a thin building of type Π whose collection of apartments is $\{\Sigma\}$. Properties (B1)-(B4) follow easily since there is only one apartment in the building.*

Definition 1.3.4 *Let (Δ, δ) be a building of type Π with index set I. A subbuilding is a subchamber system Δ' with index set $J \subseteq I$ with the property that (Δ', δ') is a building of type Π_J where δ' is δ restricted to $\Delta' \times \Delta'$, in particular $\delta'(\Delta', \Delta') \subseteq W_J$.*

Remark 1.3.5 *Any apartment Σ of a building Δ is a subbuilding of the same type and rank. The residues of Σ correspond to the nonempty $R \cap \Sigma$ where R is a residue of Δ.*

Residues in buildings have similar properties to residues in Coxeter chamber systems. In particular, residues of buildings are convex and if R is a J-residue, then R is a subbuilding of type Π_J. We state a useful result that highlights the high level of symmetry of residues contained in apartments. The result is Proposition 8.20 of [34].

Proposition 1.3.6 *Let R and Q be residues that both intersect the apartments Σ_1 and Σ_2. Then there exists a special isomorphism from Σ_1 to Σ_2 that sends $R \cap \Sigma_1$ to $R \cap \Sigma_2$ and $Q \cap \Sigma_1$ to $Q \cap \Sigma_2$.* $\square$

Furthermore, residues of buildings also possess the 'gatedness' property. The following result is analogous to Lemma 1.2.5 and is Corollary 7.21 in [34].

Lemma 1.3.7 *For every residue R and every chamber x in Δ, there exists a unique closest element y in R, denoted by $\text{proj}_R x$, such that*

$$\text{dist}(x, z) = \text{dist}(x, y) + \text{dist}(y, z),$$

for all chambers z in R, and y is contained in every apartment containing x and some chamber of R.$\square$

A useful property is that the intersection of residues with apartments leaves the projection of a chamber (inside the apartment) in that residue invariant. We state it more precisely.

Lemma 1.3.8 *Suppose that x is a chamber contained in an apartment Σ and $\Sigma \cap R$ is nonempty. Then* $\text{proj}_R x = \text{proj}_{\Sigma\cap R} x$.

Proof. By the convexity of Σ, $\text{proj}_R x$ must be contained in $\Sigma \cap R$. By the uniqueness of $\text{proj}_R x$, we have that $\text{proj}_{R\cap\Sigma} x = \text{proj}_R x$. □

The diameter of the spherical building is finite allowing us to consider opposite chambers.

Lemma 1.3.9 *Let Δ be a spherical building. Then*

(i) $\text{diam}(\Delta) = \text{diam}(\Sigma)$ *for any apartment Σ of Δ,*

(ii) *Each pair of opposite chambers in x and y are contained in a unique apartment Σ.*

(iii) *The apartment in (ii) consists of all the chambers which lie on a minimal gallery from x to y.*

Proof. Part (i) is true since all apartments are isomorphic as chamber systems and by (B2). Part (ii) and (iii) are exactly Theorem 9.2 in [34]. □

Theorem 1.3.10 (Classification of the types of irreducible spherical buildings) *Let Δ be a thick irreducible building of type Π and rank at least 3. Then Π is A_n for $n \geq 2$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4 or G_2.* □

This is a well known result and can be found in [34], for example. We often write that Δ is a thick irreducible building of type X_n where X is either A, B, C, D, E, F or G and n is the corresponding rank. Tits gives a full classification of thick irreducible spherical buildings of rank at least three in [32] and a simpler proof was later given by Tits and Weiss in [33].

Buildings can be viewed as both chamber systems and geometries. It is sometimes more convenient to study the geometry of a building. We recall definitions stated in Appendix A without going into detail.

Let $I = \{1, \ldots, n\}$. A *geometry* $\mathit{\Gamma}$ over I is a triple $(V, \sim, \tau)$ where V is the set consisting of *objects*, $\sim$ is a reflexive symmetric relation on V and τ is a surjective map from V to I such that no two incident objects have the same image under τ.

Definition 1.3.11 *The* flag geometry associated to the building Δ *is the geometry* $(V, \sim, \tau)$ *where* V *is the set of corank one residues of* Δ*,* $R \sim S$ *if and only if* R *and* S *have a nonempty intersection as sets of chambers of* Δ *and* τ *is defined to be the cotype of such a residue. The flag geometry is denoted by* $\mathrm{Gm}(\Delta)$*.*

Definition 1.3.12 *The* Chamber system associated to the geometry $\mathit{\Gamma}$ *is the chamber system* $(\mathcal{C}, \{\sim\}_{i \in I})$ *where* $\mathcal{C}$ *is the set of maximal flags of* $\mathit{\Gamma}$ *and* $C \sim_i D$ *if and only if* C *and* D *share a common flag of cotype* i*. The chamber system is denoted by* $\mathrm{Ch}(\mathit{\Gamma})$

The following result follows from Proposition A.0.1 and A.0.4.

Proposition 1.3.13 *For a building* Δ*, we have that* $\mathrm{Ch}(\mathrm{Gm}(\Delta)) \cong \Delta$*.*

1.4 Root systems

Root systems were first introduced by Killing [17] in order to classify the finite dimensional complex simple Lie algebras. This section deals with the role of root systems in spherical buildings which are later used to give a geometrical interpretation of Coxeter chamber systems. We introduce root systems in an abstract sense but root systems originally arose from the study of semisimple Lie algebras. The material in this section is taken from [2] and [18].

Let V be a finite dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(,)$ and let α be a nonzero vector of V. A *reflection in the hyperplane perpendicular to* α is the map

$$s_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha,$$

We define and denote the *length* of a vector α by $|\alpha| = (\alpha, \alpha)$.

Definition 1.4.1 *A* root system Φ of V *is a finite spanning set of non-zero vectors of* V *such that for each* $\alpha, \beta \in \Phi$*:*

(R1) $t\alpha \in \Phi$ *if and only if* $t \in \{-1, 1\}$.

(R2) $s_\alpha(\Phi) = \Phi$.

(R3) $n_{\alpha,\beta} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

The *rank* of Φ is the dimension of V. We say that Φ is *reducible* if there Φ can be written as the disjoint union of nonempty subsets Φ_1 and Φ_2 such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. We say that Φ is the *orthogonal sum* of root systems Φ_1 and Φ_2. A root system is *irreducible* if it is not reducible. A subset Φ' of Φ is a *subroot system* if it is a root system in its own right.

Lemma 1.4.2 *Every root system is the disjoint union of irreducible root systems.* $\square$

For each $\alpha \in \Phi$, define $H_\alpha = \{v \in V \mid (v, \alpha) = 0\}$ to be the *root hyperplane orthogonal to* α. Choose $t \in V \setminus \cup_{\alpha\in\Phi} H_\alpha$. Then, for each root α, $(t, \alpha) > 0$ or $(t, \alpha) < 0$. Define $\Phi^+ = \{\alpha \in \Phi \mid (\alpha, t) > 0\}$ and $\Phi^- = \{\alpha \in \Phi \mid (\alpha, t) < 0\}$. Notice that $\Phi^- = -\Phi^+$ and $\Phi = \Phi^+ \sqcup \Phi^-$. We say that $\Phi = \Phi^+ \sqcup \Phi^-$ is a *polarisation* of Φ with respect to t. The subsets Φ^+ and Φ^- are the *positive* and *negative* roots, respectively. There exists a unique subset $B = \{\alpha_1, \ldots, \alpha_n\}$ of Φ^+ such that every positive root α has the form

$$\sum_{i=1}^{n} m_i\alpha_i \text{ for some } m_i \in \mathbb{Z}_0^+ \tag{1.4.3}$$

where n is the rank of Φ (c.f. Corollary 7.18 of [18]). The set B is the *basis of the root system* Φ with respect to the polarisation and the roots in B are called *simple roots.* The set B is also a basis of the vector space in V in the usual sense.

Let $W = W_\Phi$ be the group generated by the reflections $\{s_\alpha \mid \alpha \in \Phi\}$. The group W is called the *Weyl group of* Φ. The group W is a finite subgroup of the orthogonal group $O(V)$. Furthermore, W acts on the set of roots and preserves the angles between roots. The Weyl groups are finite Coxeter groups and the significance of this emerges later. The group W_Φ acts sharply transitively on the set of bases of Φ and thus is in one-to-one correspondence with the polarisations of Φ (c.f. Lemma 7.24 and Corollary 7.38 in [18]).

Let α be given by (1.4.3). The *height of* α *with respect to the basis* B is $\sum_{i=1}^{n} m_i$. For each basis B there is a unique root called the *longest root* which has maximal height. We state a few results about the longest roots of irreducible root systems. The result can be found in Chapter VI, Section 1.8, Proposition 25 of [2].

Lemma 1.4.4 *Let Φ be an irreducible root system and let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for B. Let $\widehat{\alpha} = \sum_{i=1}^{n} m_i\alpha_i$ be the longest root with respect to B. Then the following statements hold.*

(i) *Each integer m_i is greater than zero.*

(ii) *Any root β has the form $\sum_{i=1}^{n} p_i\alpha_i$ where $p_i \leq m_i$ for all $i \in \{1, \ldots, n\}$.*

(iii) *We have that $|\widehat{\alpha}| \geq |\beta|$ for all roots β and $n_{\beta,\widehat{\alpha}} \in \{0, 1\}$ for all $\beta \neq \pm\widehat{\alpha}$.*

The proof of the following result can be found in Chapter VI, Section 1.3 in [2].

Lemma 1.4.5 *Fix a polarisation of Φ. The angle between any two distinct simple roots is obtuse and is contained in $\{\pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}$.*

Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of Φ. Let $I = \{1, \ldots, n\}$ be the vertex set and let two vertices i and j be joined by m and the integer m is determined by the angle θ_{ij} between α_i and α_j: $m = 0$ if $\theta_{ij} = \pi/2$, $m = 1$ if $\theta_{ij} = 2\pi/3$, $m = 2$ if $\theta_{ij} = 3\pi/4$, and $m = 3$ if $\theta_{ij} = 5\pi/6$. By convention, the edge labelled $m = 0$ is not visible. Furthermore, whenever $m > 1$ then the two roots corresponding to the edge are of different length and we direct the edge with an arrow pointing towards the root with shorter length. This graph is called the *Dynkin diagram* of Φ and is denoted by Π_Φ. The Dynkin diagram is called *simply laced* $m \in \{0, 1\}$ and is called *nonsimply laced* otherwise. The Dynkin diagram of Φ is unique up to automorphism since W acts sharply transitively on the set of bases of Φ and preserves angles between roots. Additionally, note that these are very similar to the Coxeter diagrams corresponding to W_Φ but the labelling is different.

The root system Φ is irreducible if and only if its Dynkin diagram is connected. Let Φ be the union of disjoint irreducible root systems $\Phi_1 \sqcup \ldots \sqcup \Phi_k$. Let Π_i be the Dynkin diagram corresponding to Φ_i and let $W_i = \langle s_\alpha \mid \alpha \in \Phi_i \rangle$. Then $W_\Phi = W_1 \times \ldots \times W_k$ and W_i acts on Φ_j for each $i, j \in I$ and this action is trivial if and only if $i \neq j$.

We now state the classification of irreducible root systems and provide a realisation of each root system in Appendix B.

Theorem 1.4.6 (Classification of irreducible root systems) *Let Φ be an irreducible root system of rank n. Then Φ is of type A_n for $n \geq 2$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4 or G_2.*

1.5 Perpendicular roots

By abuse of notation we denote the root system of type X_n by X_n. We fix a simply laced irreducible root system Φ of rank $n \geq 3$ and let W be the corresponding Weyl group. This section determines the number orbits of pairs of perpendicular roots (α, β) under the action of the W and we show that for every such pair there exist roots γ_1

and γ_2 such that the angles between α and γ_j, and between β and γ_j are both $\pi/3$ for $j = 1, 2$. Let (α, β) and (α', β') be two pairs of perpendicular roots. Since the Weyl group acts transitively on the roots, there exists an element of W that sends the pair (α, β) to (α', β'') for some root β''. If we show there exists another element of W that sends (α', β'') to (α', β'), then there exists an element $w \in W$ such that $(\alpha, \beta)^w = (\alpha', \beta')$. We are thus interested in finding elements of W that send (α, β) to (α, β') for any fixed α and all roots $\beta, \beta' \in H_\alpha$. Let $\Psi = \Phi \cap H_\alpha$. It is clear that (R1) and (R3) hold for Ψ. If $\beta, \gamma \in \Psi$, then $s_\beta(\gamma) = \gamma - 2(\gamma, \beta)/(\beta, \beta)\beta$. But since $(\alpha, \gamma) = (\alpha, \beta) = 0$, then $(s_\beta(\gamma), \alpha) = 0$ and thus $s_\beta(\gamma) \in \Psi$. It follows that Ψ is a subroot system of Φ. We write $\Psi = \Psi_1 \perp \ldots \perp \Psi_k$ as the orthogonal sum of disjoint irreducible root systems. Let $W_\Psi = \langle s_\beta \mid \beta \in \Psi \rangle$. Then

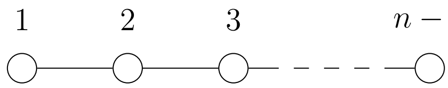
$$W_\Psi = W_1 \times \ldots \times W_k$$

where W_i is the Weyl group of the root system Ψ_i and W_i acts trivially on Ψ_j whenever $i \neq j$. Note that W_Ψ fixes α since α and β are perpendicular for all $\beta \in \Psi$. The subgroup W_i acts transitively on the roots in Ψ_i and thus acts transitively on the pairs of roots $\{(\alpha, \beta) \mid \beta \in \Psi_i\}$. The number of orbits of pairs of perpendicular roots in Φ under W is at most k since W_Ψ is contained in the stabiliser of α. If there exists $w \in W$ such that $(\alpha, \beta)^w = (\alpha, \beta')$ and there exist roots γ_1 and γ_2 such that $(\alpha, \gamma_j) = (\beta, \gamma_j) = 1$ for $j = 1, 2$, then $(\alpha, \gamma_j^w) = (\beta', \gamma_j^w) = 1$ for $j = 1, 2$. Therefore, to show that each pair of perpendicular roots form an angle of $\pi/3$ with two distinct common roots it suffices to show that there exists $\beta_i \in \Psi_i$ such that $(\alpha, \gamma_j) = (\beta_i, \gamma_j) = 1$ for some roots γ_j for $j = 1, 2$ and for all $i \in \{1, \ldots, k\}$.

Root system of type A_n

Assume that $n \geq 3$. The root system A_n is embedded in a hyperplane of $(n + 1)$-dimensional Euclidean space and consists of the vectors $\{\pm(e_i - e_j) \mid 1 \leq i < j \leq n + 1\}$.

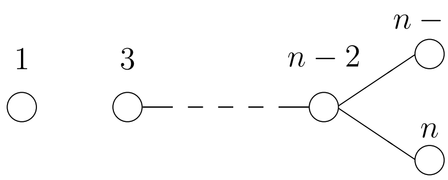
Let $\alpha = e_n - e_{n+1}$. The set of roots perpendicular to α is $\Psi = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n-1\}$. In particular, Ψ is the irreducible root system A_{n-2} and there is only one orbit on pairs of perpendicular roots $\{(\alpha, \beta) \mid \beta \in \Psi\}$. Take $\beta = e_1 - e_2$. Then $\gamma_1 = e_1 - e_{n+1}$ and $\gamma_2 = e_n - e_2$ form angles of $\pi/3$ with both α and β. The diagram of Ψ is the following.

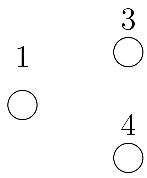


Root system of type D_n

The root system D_n is embedded in n-dimensional Euclidean space and consists of the vectors $\{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$. Choose $\alpha = e_{n-1} - e_n$. The set of roots perpendicular to α is $\Psi = \{\pm(e_{n-1} + e_n)\} \perp \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n-2\}$.

The roots $\pm(e_{n-1} + e_n)$ form the root system A_1. The roots $\{\pm e_i \pm e_j \mid 1 \leq i < j \leq n-2\}$ form the root system $A_1 \perp A_1$ if $n = 4$ and D_{n-2} if $n > 4$ where D_3 is A_3. Therefore there are at most three and two orbits of pairs of perpendicular roots $\{(\alpha, \beta) \mid \beta \in \Psi\}$ when $n = 4$ and $n > 4$, respectively. From the first component $\{\pm(e_{n-1} + e_n)\}$, choose $\beta = e_{n-1} + e_n$. Then $\gamma_1 = e_1 + e_{n-1}$ and $\gamma_2 = e_2 + e_{n-1}$ form angles of $\pi/3$ with both α and β. If $n = 4$, then $\{\pm e_i \pm e_j \mid 1 \leq i < j \leq 2\} = \{\pm(e_1 - e_2)\} \perp \{\pm(e_1 + e_2)\}$. In the first component choose $\beta = e_1 - e_2$. Then $\gamma_1 = e_1 + e_3$ and $\gamma_2 = -(e_2 + e_4)$ form angles of $\pi/3$ with both α and β. In the second component choose $\beta = e_1 + e_2$. Then $\gamma_1 = e_1 - e_4$ and $\gamma_2 = e_1 + e_3$ form angles of $\pi/3$ with both α and β. For $n > 4$, $\{\pm e_i \pm e_j \mid 1 \leq i < j \leq n-2\}$ is the root system D_{n-2} and we choose $\beta = e_1 - e_2$. Then $\gamma_1 = e_1 + e_{n-1}$ and $\gamma_2 = -(e_2 + e_n)$ form angles of $\pi/3$ with both α and β. The two possible diagrams of Ψ for $n > 4$ and $n = 4$ are the following.





Root system of type E_6

The root system E_6 is embedded in 8-dimensional Euclidean space and consists of the roots $D_5 \cup X$ where

$$X = \left\{ \pm\frac{1}{2}\left(\sum_{i=1}^{5}(-1)^{v_i}e_i - e_6 - e_7 + e_8\right) \mid \sum_{i=1}^{5} v_i \text{ is even} \right\}.$$

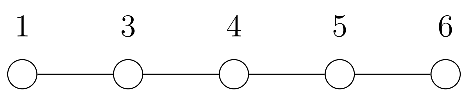
Choose $\alpha = \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 - e_8)$. The set of roots in D_5 perpendicular to α is

$$\Psi_1 = \{\pm(e_1 + e_i) \mid 2 \le i \le 5\} \cup \{\pm(e_i - e_j) \mid 2 \le i < j \le 5\}$$

Clearly Ψ_1 is an irreducible root system with 20 roots and thus Ψ_1 is A_4. Any additional orthogonal component must have rank 1. The roots Ψ_2 in X perpendicular to α are the eight roots of the form

$$\left\{ \pm\frac{1}{2}\left(e_1 + \sum_{i=2}^{5}(-1)^{v_i}e_i + e_6 + e_7 - e_8\right) \mid \sum_{i=2}^{5} v_i = 3 \right\}$$

and the two roots of the form $\{\pm\frac{1}{2}(1, 1, 1, 1, 1, -1, -1, 1)\}$. None of the 10 roots in Ψ_2 is perpendicular to every root in Ψ_1. Thus $\Psi = \Psi_1 \cup \Psi_2 = E_6 \cap H_\alpha$ is an irreducible root system with $20 + 10 = 30$ roots and thus Ψ is A_5. There is only one orbit on pairs of perpendicular roots $\{(\alpha, \beta) \mid \beta \in \Psi\}$. Choose $\beta = e_1 + e_2$. Then $\gamma_1 = e_2 + e_3$ and $\gamma_2 = e_2 + e_4$ form angles of $\pi/3$ with both α and β. The diagram for Ψ is the following.



Root system of type E_7

The root system E_7 is embedded in a hyperplane of 8-dimensional Euclidean space and consists of the roots $D_6 \cup \{\pm(e_7 - e_8)\} \cup X$ where

$$X = \left\{ \pm\frac{1}{2}\left(\sum_{i=1}^{6}(-1)^{v_i}e_i + e_7 - e_8\right) \mid \sum_{i=1}^{6} v_i \text{ is odd } \right\}.$$

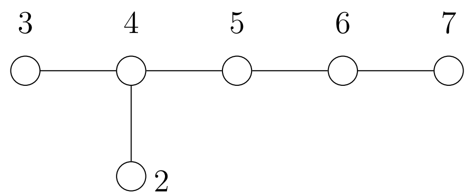
Choose $\alpha = \frac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 - e_8)$. The set of roots perpendicular to α from $D_6 \cup \{\pm(e_7 - e_8)\}$ is $\Psi_1 = \{\pm(e_1 + e_i) \mid 2 \leq i \leq 6\} \cup \{\pm(e_i - e_j) \mid 2 \leq i < j \leq 6\}$ and there are 30 such roots. It is clear that Ψ_i forms an irreducible root system with 30 and thus Ψ_1 is A_5. Any additional orthogonal component must have rank 1. The roots Ψ_2 from X perpendicular α are those of the form

$$\left\{\pm\frac{1}{2}\left(e_1 + \sum_{i=2}^{6}(-1)^{v_i}e_i + e_7 - e_8\right) \mid \sum_{i=2}^{5} v_i = 3\right\}$$

and

$$\left\{\pm\frac{1}{2}\left(-e_1 + \sum_{i=2}^{6}(-1)^{v_i}e_i + e_7 - e_8\right) \mid \sum_{i=2}^{6} v_i = 4\right\}.$$

There are $20 + 10 = 30$ such roots and no root in Ψ_2 is perpendicular to every root in Ψ_1. Thus $\Psi = \Psi_1 \cup \Psi_2 = E_7 \cap H_\alpha$ is an irreducible root system with 60 roots and thus Ψ is D_6. There is only one orbit on pairs of perpendicular roots $\{(\alpha, \beta) \mid \beta\Psi\}$. Choose $\beta = e_1 + e_2$. Then $\gamma_1 = e_2 + e_3$ and $\gamma_2 = e_2 + e_4$ are perpendicular to both α and β. The diagram for Ψ_1 is the following.



Root system of type E_8

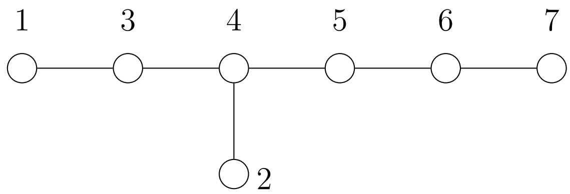
The root system E_8 is embedded into 8-dimensional Euclidean space and consists of the roots $D_8 \cup X$ where

$$X = \left\{ \frac{1}{2} \left(\sum_{i=1}^{8} (-1)^{v_i} e_i \right) \mid \sum_{i=1}^{8} v_i \text{ is even} \right\}.$$

Choose $\alpha = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8)$. The set of roots from D_8 which are perpendicular to α is $\Psi_1 = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 8\}$ which forms A_7. Any further orthogonal component would give rank at least 8 but E_8 has rank 8 and so that is not possible. The set of roots Ψ_2 in X that are perpendicular to α is

$$\left\{ \pm\frac{1}{2} \left(\sum_{i=1}^{8} (-1)^{v_i} \right) \mid \sum_{i=1}^{8} v_i = 4 \right\}$$

and there are 70 such roots. In total we have 126 roots and thus $\Psi = \Psi_1 \cup \Psi_2 = E_8 \cap H_\alpha$ is E_7. Choose $\beta = e_1 - e_8$. Then $\gamma_1 = e_1 + e_2$ and $\gamma_2 = e_1 + e_3$ form angles of $\pi/3$ with both α and β. The diagram of Ψ is the following.



For a root α of Φ, let $\alpha^{\perp_\Phi}$ denote the subroot system $\alpha^\perp \cap \Phi$.

Lemma 1.5.1 *Let α and β be two roots of Φ. Then $\alpha^{\perp_\Phi}$ and $\beta^{\perp_\Phi}$ do not coincide unless $\beta = \pm\alpha$.*

Proof. Let V be the vector space spanned by Φ. Note that for any vectors u and v such that $u^\perp = v^\perp$ we have that $u^\perp = v^\perp = \langle u, v\rangle^\perp$. In particular, $V = \langle u, v\rangle^\perp \oplus \langle u, v\rangle = u^\perp \oplus \langle u, v\rangle$. In particular, $\langle u, v\rangle$ is one dimensional and thus $\langle u\rangle = \langle v\rangle$. If $\alpha^{\perp_\Phi} = \beta^{\perp_\Phi}$, then $\alpha^\perp = \beta^\perp$ and hence α and β are multiples of one another. In particular, $\beta = \pm\alpha$. $\square$

1.6 Coxeter chamber systems in Euclidean space

Fix an irreducible root system Φ of rank n embedded in a Euclidean space V and let $I = \{1, \dots, n\}$. For each root $\alpha \in \Phi$, let H_α be the root hyperplane orthogonal to α.

Definition 1.6.1 *A* Weyl chamber *is a connected component of* $V \setminus \cup_{\alpha\in\Phi} H_\alpha$.

The two *sides of* H_α are the regions $\{v \in V \mid (v, \alpha) > 0\}$ and $\{v \in V \mid (v, \alpha) < 0\}$. Two vectors $u, v \in V$ are said to be *separated by* H_α if they lie on different sides of H_α. For each $\alpha \in \Phi$, the vectors of a Weyl chamber lie on the same side of H_α and every Weyl chamber can be given by a system of inequalities $\pm(\alpha, v) > 0$. Each Weyl chamber C is an open set and can be written as $C = \{v \in V \mid \varepsilon_{C,\alpha}(v, \alpha) > 0\}$ for $\varepsilon_{C,\alpha} \in \{-1, 1\}$. The closure and boundary of C are given by $\overline{C} = \{v \in V \mid \varepsilon_{C,\alpha}(v, \alpha) \geq 0\}$ and $\partial C = \{v \in V \mid (v, \alpha) = 0\}$, respectively. We can write $\partial C = \cup_{\alpha\in\Phi} F_\alpha$ where $F_\alpha = \{v \in \overline{C} \mid (\alpha, v) = 0\}$. The *wall of* C is the set of hyperplanes H_α that contains a nonzero F_α. Let $M(C)$ denote the set of walls of C.

Note that every vector t that determines a polarisation of Φ can be found in some Weyl chamber C and, since C is open, any two vectors in C give the same polarisations. In particular, given a Weyl chamber C, then

$$\Phi^+ = \{\alpha \in \Phi \mid (\alpha, t) > 0\} \text{ for any } t \in C \tag{1.6.2}$$

is a polarisation of Φ and determines a unique basis denoted by $B(C)$. Conversely, if $\Phi = \Phi^+ \sqcup \Phi^-$ is a polarisation with basis $B = \{\alpha_1, \dots, \alpha_n\}$, then

$$C = \{v \in V \mid (v, \alpha_i) > 0 \text{ for all } i \in I\} \tag{1.6.3}$$

is a Weyl chamber.

Theorem 1.6.4 *There is a one-to-one correspondence between Weyl chambers and polarisations of Φ as given by (1.6.2) and (1.6.3)* $\Box$.

The Weyl group W_Φ acts on Φ, and thus it acts on the set of root hyperplanes and the set of Weyl chambers. By the above correspondence, W acts sharply transitively on the Weyl chambers and thus there is a one-to-one correspondence between Weyl chambers and elements of the Weyl group (c.f. Corollary 7.38 of [18]). Fix a Weyl chamber C and let $B(C) = \{\alpha_1, \ldots, \alpha_n\}$ be the corresponding basis of Φ. Any Weyl chamber can be written in the form $s_{i_1} \ldots s_{i_k}(C)$ where $s_{i_j} = s_{\alpha_{i_j}}$ for some $i_j \in I$. The following result is Corollary 7.28 of [18].

Lemma 1.6.5 *Let C be as in (1.6.3). Then C has n walls, namely H_{α_i} for all $i \in I$, and thus every Weyl chamber has n walls. In particular, the Weyl chamber $w(C)$ has walls $H_{w(\alpha_i)}$ for $i \in I$.*

Lemma 1.6.6 *Let C be a Weyl chamber and $u, v \in \overline{C}$. Then $(u, v) \geq 0$.* $\Box$

Proof. The subset $\Phi' := \Phi \cap \langle u, v \rangle$ is a subroot system of rank 2. Indeed, (R1) and (R3) clearly hold and for $\alpha, \beta \in \Phi'$, $s_\alpha(\beta) = \beta - 2(\beta, \alpha)/(\alpha, \alpha)\alpha \in \langle u, v \rangle$ and thus (R2) holds. In particular Φ' is either $A_1 \perp A_1$, A_2, B_2 or G_2. If $(u, v) < 0$, then the angle between u and v is greater than $\pi/2$ and thus u and v must lie on different sides of some hyperplane in Φ'. This contradicts that u and v lie in the closure of a Weyl chamber. Hence $(u, v) \geq 0$. $\Box$

Definition 1.6.7 *Let Γ_Φ be a graph with the set of all Weyl chambers as vertices and two Weyl chambers C and C' are joined by an edge if and only if C and C' share a common wall H_α and the reflection s_α interchanges C and C'.*

We call Γ_Φ the *Weyl graph* of the root system of Φ. Note that Γ_Φ is connected and each vertex has valency n. The aim is to colour the edges of Γ_Φ so that it is isomorphic to a

Coxeter chamber system. To obtain the corresponding Coxeter diagram from the Dynkin diagram, we replace multiple edges with single edges: a single edge, double edge and triple edge are replaced with the labels 3, 4 and 6, respectively. We denote the diagram by Π_Φ and call it the *Coxeter diagram associated to the root system* Φ. Consequently, the Coxeter diagram for C_n and B_n are equal and and their Weyl groups are isomorphic. This Coxeter diagram is denoted by $(BC)_n$. The results from this section are taken from Chapter 2 of [35] and the omitted proofs can be found in [35] and Chapter VI Section 1.5 of [2].

Let Φ be a root system and let Π be the corresponding Coxeter diagram with vertex set I, W_Π be the Coxeter group of Π and W_Φ be the Weyl group of Φ. Fix a Weyl chamber C and let $B(C) = \{\alpha_1, \ldots, \alpha_n\}$ be the basis of Φ corresponding to C. In particular, $W_\Phi = \langle s_{\alpha_i} \mid i \in I\rangle$ and $W_\Pi = \langle r_i \mid (r_i r_j)^{m_{ij}} = 1 \text{ for all } i, j \in I\rangle$. The following result is Proposition 2.8 from [35].

Proposition 1.6.8 *There exists a bijection* $\phi : I \to B(C)$ *such that the map* $r_i \mapsto s_{\phi(i)}$ *extends to an isomorphism from* W_Π *to* W_Φ *and* ϕ *is unique up to an automorphism of* Π. *Furthermore, the map* $g \mapsto \pi(g)(C)$ *is a graph isomorphism from* Σ_Π *to* $\mathit{\Gamma}_\Phi$.

In particular, $xr_i = y$ if and only if $s_\alpha(\pi(x)(C)) = \pi(y)(C)$ where $\alpha = \pi(x)^{-1}s_{\alpha_i}$. Recall that C has n walls and thus it is adjacent to n Weyl chambers. The chamber 1 of Σ_Π corresponds to the Weyl chamber C of $\mathit{\Gamma}_\Phi$ with respect to this map. A colouring of the edges of C is a colouring of all edges since W_Φ acts sharply transitively on the set of Weyl chambers. Thus there is a unique way, up to an automorphism of Π, to colour $\mathit{\Gamma}_\Phi$ so that it is isomorphic (preserving colours) to Σ_Π. In particular, for each $i \in I$, the edge $\{C, S_{\alpha_i}(C)\}$ is coloured i. Let C' be any other Weyl chamber and let w be the unique element of W_Φ such that $w(C) = C'$. Then $B(C') = \{w(\alpha_1), \ldots, w(\alpha_n)\}$. The group W preserves angles and for it to preserve colours we require that the unique Weyl chamber that is i-adjacent to C' is $s_{w(\alpha_i)}(C')$. Let Σ_Φ denote the graph $\mathit{\Gamma}_\Phi$ with the above edge-colouring. We state this formally in a result whose proof can be found in [35].

Theorem 1.6.9 *The edge-coloured graph Σ_Φ is a Coxeter chamber system isomorphic to Σ_Π.* $\square$

For each $\alpha \in \Phi$, let $\Sigma_\Phi(\alpha)$ denote the set of Weyl chambers that are contained in the region $\{v \in V \mid (v, \alpha) > 0\}$. The reflections, in the chamber system sense, of Σ_Φ are the reflections from the Weyl group W_Φ.

Lemma 1.6.10 *Let α be a root in Φ. Let C and C' be two adjacent chambers in $\Sigma_\Phi(\alpha)$ and $\Sigma_\Phi(-\alpha)$, respectively. Then s_α interchanges C and C'.*

Lemma 1.6.11 *Let α be a root in Φ. The two half-apartments of Σ_Φ associated with the reflection s_α are $\Sigma_\Phi(\alpha)$ and $\Sigma_\Phi(-\alpha)$. Conversely, a half-apartment A of Σ_Φ corresponds to a unique root α such that $\Sigma_\Phi(\alpha) = A$.*

Proof. It suffices to show that every gallery from $\Sigma_\Phi(\alpha)$ to $\Sigma_\Phi(-\alpha)$ passes through a pair of adjacent chambers that are interchanged by s_α. Indeed, by deleting the edges that are fixed setwise by s_α we would get the two connected components $\Sigma_\Phi(\alpha)$ and $\Sigma_\Phi(-\alpha)$ and thus, by definition, they are the half-apartments corresponding to s_α. This follows from Lemma 1.6.10.

Let A be a half-apartment in Σ_Φ, then $\Sigma_\Phi = A \sqcup -A$. In particular, there exists a reflection s_α that interchanges two chambers C_1 and C_2 in A and $-A$, respectively. We assume, without loss of generality, that $(C_1, \alpha) > 0$. Then $(C_2, \alpha) = (s_\alpha(C_1), \alpha) = (C_1 - 2(\alpha, C_1)/(\alpha, \alpha)\alpha, \alpha) = -(C_1, \alpha) < 0$. Therefore the boundary of A, ∂A, consists of the panels with two chambers fixed by s_α on different sides of H_α. Let C be a Weyl chamber such that $(C, \alpha) > 0$. Any minimal gallery γ from C to C_2 must pass through ∂A exactly once. In particular, γ must pass through C_1. In particular, as half-apartments are convex, C is in A. Similarly, one can show that if $(C, \alpha) < 0$, then $C \in -A$. Therefore $\Sigma_\Phi(\alpha) = A$. □

We see that there is a one-to-one correspondence between roots and half apartments of a given apartment. For this reason, half-apartments are often called roots. This geometrical interpretation of Coxeter chamber systems is important for later chapters.

Chapter 2

Root Shadow Spaces

2.1 Root shadow spaces of spherical buildings

Let Δ be an irreducible spherical building with Coxeter diagram Π. Let Φ be the root system of the corresponding Dynkin diagram. In the case of $\Pi = (BC)_n$, we take the root system of type B_n. We fix a basis $B = \{\alpha_1, \ldots, \alpha_n\}$ for the root system and denote the longest root with respect to B by α. Define a set $J \subseteq I$ to consist of all $i \in I$ such that $(-\alpha, \alpha_i) \neq 0$. This corresponds to the set of vertices that are joined to the new vertex in the corresponding extended Dynkin diagram. We call J the *root set*. The root set is $J = \{1, n\}$ for A_n, $J = \{2\}$ for $(BC)_n, D_n, E_6$ and G_2, $J = \{1\}$ for E_7 and F_4, and $J = \{8\}$ for E_8. We introduce a point-line space on the building.

Definition 2.1.1 *Let Δ be an irreducible spherical building of type X_n and let J be the root set of Δ. A J-*shadow* in Δ is a $(I \setminus J)$-residue. For each $j \in J$, a j-*line* is the set of all J-shadows that contain chambers from a given j-panel. Let $\mathcal{P}$ be the set of J-shadows and $\mathcal{L}$ be the set of j-lines for $j \in J$. We say that $(\mathcal{P}, \mathcal{L})$ is the root shadow space of type $X_{n,J}$ and we denote it by $\mathcal{P}_\Delta$. If $|J| = 1$, then we write $X_{n,j}$ instead of $X_{n,J}$.*

Let Δ be an irreducible spherical building and recall from Appendix A that the building geometry is given by $\mathrm{Gm}(\Delta) = (V, \sim, \tau)$ where V is the set of corank one residues of Δ, two such residues are incident under $\sim$ if and only if they have a nonempty intersection as sets of chambers and $\tau(R)$ is defined to be the cotype of the residue. The *root shadow space of* $\mathrm{Gm}(\Delta)$ is as follows. The J-shadows are the flags of type J and, for each $j \in J$, fix a flag F of cotype j and take the set of all flags of type J whose union with F are also flags and we denote this by l_F.

Remark 2.1.2 *Let F be a flag of type J of $\mathrm{Gm}(\Delta)$ and let C and C' be chambers that contain the flag F, namely, $C = \{R_i \mid i \in I \setminus J\} \cup F$ where R_i is a flag of cotype i and $C' = \{S_i \mid i \in I \setminus J\} \cup F$. The chambers C and C' are chambers of $\mathrm{Ch}(\mathrm{Gm}(\mathrm{Ch})) \cong \Delta$ and correspond to the chambers $c := \cap_{i \in I \setminus J} R_i \cap F$ and $c' := \cap_{i \in I \setminus J} S_i \cap F$ in Δ under the isomorphism. It is clear that there exists a gallery c and c' whose type is a word in $I \setminus J$ and thus c and c' are in a common J-shadow of Δ.*

Proposition 2.1.3 *The root shadow space of Δ and $\Gamma := \mathrm{Gm}(\Delta)$, denoted by $\mathrm{RSh}(\Delta)$ and $\mathrm{RSh}(\Gamma)$, are isomorphic as point-line spaces.*

Proof. Let $F = \{R_j \mid j \in J\}$ be a flag of type J of $\mathrm{Gm}(\Delta)$ and let $\widehat{F} = \{R_1, \ldots, R_n\}$ be any chamber contains F. The chamber $\widehat{F}$ is a chamber of $\mathrm{Ch}(\mathrm{Gm}(\Delta))$ and by Proposition 1.3.13, $\mathrm{Ch}(\mathrm{Gm}(\Delta)) \cong \Delta$. Let $c := \cap_{i \in I} R_i$ be the corresponding chamber in Δ under the isomorphism and let R_F be the J-shadow of Δ containing the chamber c. Note that, by Remark 2.1.2, R_F is independent of which particular chamber is chosen to contain F. Define a map from $\mathrm{RSh}(\Gamma)$ to $\mathrm{RSh}(\Delta)$ by sending F to R_F. Let R be a J-shadow in Δ and take a chamber c in R and let R_i be the unique residue of cotype i for each $i \in I$ that contains c. If we let $F = \{R_j \mid j \in J\}$ be the flag of type J, then $R_F = R$. Suppose that $R_{F_1} = R_{F_2}$. Then we can extend both F_1 and F_2 to the same chamber and thus $F_1 = F_2$. Therefore the map $F \mapsto R_F$ is a bijection. It suffices to show that it preserves lines.

Fix a flag G and take two flags F_1 and F_2 of type J such that $F_i \cup G$ is a (maximal) flag for $i = 1, 2$. Let $\widehat{F_1} = F_1 \cup G$ and $\widehat{F_2} = F_2 \cup G$ be chambers containing F_1 and F_2, respectively. Then $\widehat{F_1} = \{R_1, \ldots, R_n\}$ and $\widehat{F_2} = \{S_1, \ldots, S_n\}$ such that $R_i = S_i$ for all $i \neq j$ and $R_j \neq S_j$. Let $c_1 := \cap_{i \in I} R_i$ and $c_2 := \cap_{i \in I} S_i = \cap_{i \neq j} R_i \cap_{j \in J} S_j$. Then clearly c_1 and c_2 are joined by an edge of type j and thus R_{F_1} and R_{F_2} are J-shadows which have chambers that lie in a common j-panel. $\square$

The *distance* between two J-shadows R and Q is the infimum of $\{\mathrm{dist}(x, y) \mid x \in R, y \in Q\}$. We write $\mathrm{dist}(R, Q)$ to denote this distance. The context is always clear and should not be confused with distance between two chambers. We state an important result about residues of buildings that is taken from [13].

Lemma 2.1.4 *Let Δ be a building with Weyl-distance map δ. Let R and Q be J-residues, respectively and Set $R' = \mathrm{proj}_Q R$ and $Q' = \mathrm{proj}_R Q$. Then*

(i) *The compositions $\mathrm{proj}_R \mathrm{proj}_Q$ and $\mathrm{proj}_Q \mathrm{proj}_R$ are the identity maps on R' and Q', respectively,*

(ii) $\mathrm{dist}(R, R) = \mathrm{dist}(x, y)$ *for $x \in R$ abd $y \in Q$ if and only if* $\mathrm{proj}_R(y) = x$ *and* $\mathrm{proj}_Q(x) = y$, *and*

(iii) *for x and y satisfying (ii), there exists $w \in W$ such that $\delta(x, y) = w$.* $\square$

Let R and Q be J-shadows in Δ where J is the root set of Δ. Let $W_0 = \langle r_j | j \in I \setminus J \rangle$ be a subgroup of W. Let $x \in R$ and $y \in Q$ and let $x' = \mathrm{proj}_R y$ and let $y' = \mathrm{proj}_Q x'$. By two applications of Lemma 1.3.7, $\mathrm{dist}(x, y) = \mathrm{dist}(x, x') + \mathrm{dist}(x', y) = \mathrm{dist}(x, x') + \mathrm{dist}(x', y') + \mathrm{dist}(y', y)$. In particular, there exists a minimal gallery γ from x to y that passes through x' and y'. By 2.1.4 (i), we have $\mathrm{proj}_R \mathrm{proj}_Q$ restricted to R' is the identity. Therefore, $\mathrm{proj}_R y' = \mathrm{proj}_R \mathrm{proj}_Q x' = x'$. By 2.1.4 (ii), $\mathrm{dist}(x', y') = \mathrm{dist}(R, Q)$. By Lemma 2.1.4 (iii), $\delta(x', y') = w$ for some $w \in W$. In particular, $\delta(x, y) \in W_0 w W_0$. By Section 2.7 of [5] and using that γ is minimal, w is the unique minimal double coset representative of the class $W_0 w W_0$. We summarise this in the following result.

Lemma 2.1.5 *Let R and Q be J-shadows and x and y be arbitrary chambers of R and Q, respectively. Let x', y' and W_0 be defined as above. If γ is a minimal gallery from x to y that passes through x' and y', then there exists a unique element w of W such that $\delta(x,y) = w_0 w w_0'$ for some w_0, w_0' in W_0.*$\square$

The element w is independent of x and y and we call it the *Weyl distance between R and Q*.

2.2 Polar regions

In this section we introduce the notion of polar regions of roots and establish their correspondence with J-shadows. In a given apartment, there is a one-to-one correspondence between long roots and half-apartments. Given a root α, we denote the corresponding half-apartment $\Sigma_\Phi(\alpha)$ by $[\alpha]$. Let Φ be an irreducible root system of rank at least 2 with a simply laced Dynkin diagram. Let Σ_Φ be the associated Coxeter chamber system.

Definition 2.2.1 *The* polar region of a root *α of Φ is the set of Weyl chambers C such that α is contained in a wall of C. The polar region of α is denoted by $P(\alpha)$.*

The definition of a polar region is motivated by and equivalent to the definition given in [34] (see Definition 6.4 of [34]) and are studied in more detail by the same authors in [19].

Lemma 2.2.2 *A chamber C is contained in the polar region $P(\alpha)$ if and only if α is the longest root with respect to the basis $B(C) = \{\alpha_1, \ldots, \alpha_n\}$ associated with C.*

Proof. Let $I = \{1, \ldots, n\}$. Suppose that α is the longest root with respect to $B(C) = \{\alpha_1, \ldots, \alpha_n\}$. This means that $C = \{v \in V \mid (v, \alpha_i) > 0 \text{ for all } i \in I\}$ and $\overline{C} = \{v \in V \mid (v, \alpha_i) \geq 0 \text{ for all } i \in I\}$. Let J be the root set of Φ. By Lemma 1.4.4 (iii) $(\alpha, \alpha_i) \geq 0$ for all $i \in I$ and $(\alpha, \alpha_j) = 0$ for all $j \in J$. Thus $\alpha \in \overline{C} \setminus C = \partial C$ and thus α is contained in a wall of C.

Suppose that C is in the polar region $P(\alpha)$. Then $\alpha \in \overline{C}\backslash C$. Let $B(C) = \{\alpha_1, \ldots, \alpha_n\}$. Thus $\alpha \in \overline{C} = \{v \in V \mid (v, \alpha_i) \geq 0 \text{ for all } i \in I\}$. There exists i_1 and i_2 such that $(\alpha, \alpha_{i_1}) \neq 0$ since Φ is irreducible and $(\alpha, \alpha_{i_2}) = 0$ since $\alpha \notin C$. Let β be the longest root with respect to $B(C) = \{\alpha_1, \ldots, \alpha_n\}$. It is clear that $\alpha \in \Phi^+$ since $(\alpha, t) > 0$ if you take t in C arbitrarily close to α and $\alpha \in \overline{C} \setminus C$. By Lemma 1.4.4 (i), we have that $\beta = \sum_{i=1}^{n} m_i \alpha_i$ for some positive integers m_i. By Lemma 1.4.4 (iii), $(\alpha, \beta) \in \{0, 1, 2\}$. If $(\alpha, \beta) = 2$, then $\alpha = \beta$ as required. If $(\alpha, \beta) = 0$, then

$$0 = (\alpha, \beta) = (\alpha, \sum_{i=1}^{n} m_i \alpha_i) = \sum_{i=1}^{n} m_i (\alpha, \alpha_i).$$

But $m_i \geq 1$ and thus $(\alpha, \alpha_i) = 0$ for all $i \in I$. This is a contradiction. If $(\alpha, \beta) = 1$, then $1 = (\alpha, \beta) = \sum_{i=1}^{n} m_i (\alpha, \alpha_i)$. Thus there exists a unique $j \in J$ such that $(\alpha, \alpha_j) = 1$ and $m_j = 1$ and $(\alpha, \alpha_i) = 0$ for all $i \neq j$. That is, the root system perpendicular to α has rank $n - 1$. By comparing the results from Section 1.5, we firstly note that Φ cannot be A_n because $\alpha^{\perp}$ would have rank $n - 2$. For all other cases, we can determine exactly which $j \in I$ has the property $(\alpha, \alpha_j) = 1$. By reading off the data from Section 1.5, we conclude that if Φ is D_n, E_6, E_7, E_8, then j is 2, 2, 1, 8, respectively and such values of j correspond to the root sets of such root systems. In particular, as β is the longest root with respect to $B(C)$ we conclude that $(\alpha, \alpha_i) = (\beta, \alpha_i)$ for all $i \in I$. Thus $\alpha = \beta$ since $(,)$ is nondegenerate and this contradicts that $(\alpha, \beta) = 1$. After considering all cases we yield that $\alpha = \beta$ is the longest root with respect to $B(C)$ as required. □

Lemma 2.2.3 *Every polar region $P(\alpha)$ of a long root α is a J-shadow and every J-shadow is the polar region of a long root.*

Proof. Let α be a long root. Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for which α is the longest root and let J be the root set. Let $C = \{v \in V \mid (v, \alpha_i) > 0 \text{ for all } i \in I\}$ be a Weyl chamber. In particular, $B = B(C)$. Note that $\overline{C} = \{v \in V \mid (v, \alpha_i) \geq 0 \text{ for all } i \in I\}$ and

$(\alpha, \alpha_i) \in \{0, 1\}$. We have that $\alpha \in \overline{C}$ and thus $(\alpha, \alpha_j) \neq 0$ for all $j \in J$ and $(\alpha, \alpha_i) = 0$ for all $i \in I \setminus J$. Thus $\alpha \in \overline{C} \setminus C$ and $C \in P(\alpha)$.

We fix a colouring on Σ_Φ by letting $s_{\alpha_i}(C)$ be the unique Weyl chamber i-adjacent to C. Note that if $i \in I \setminus J$, then $s_{\alpha_i}(\alpha) = \alpha$. So $s_{\alpha_i}(C)$ has a wall that contains α and $s_{\alpha_i}(C) \in P(\alpha)$. Let $V = \langle s_{\alpha_i} \mid i \in I \setminus J \rangle$ and let R be the unique $(I \setminus J)$-residue containing C. By Lemma 1.2.4 (iii), the group V acts transitively on R and for $s \in V$, $s(\alpha) = \alpha$ and so $R \subseteq P(\alpha)$.

Suppose that $D \in P(\alpha) \setminus R$ is a chamber adjacent to a chamber $C' \in R$. Then D and C' are j-adjacent for some $j \in J$. Let $s \in V$ be the unique element that sends C to C'. Then

$$B(C') = \{\alpha'_1, \ldots, \alpha'_n\}$$

$\alpha'_i = s(\alpha_i)$. We know that V fixes α and preserves angles, so $(\alpha, \alpha'_j) \neq 0$ for all $j \in J$. The group V also preserves colours, so the unique i-adjacent chamber to C' is $s_{\alpha'_i}(C')$. In particular, there exists $j \in J$ such that $D = s_{\alpha'_j}(C')$. Note that $s_{\alpha'_j}(\alpha) = \alpha - (\alpha, \alpha'_j)\alpha'_j = \alpha - \alpha'_j \neq \alpha$. But α is contained in a wall of D and thus there exists a root β that is contained in a wall of C' such that $s_{\alpha'_j}(\beta) = \alpha$. Note that $\alpha = \sum_{i=1}^n m_i\alpha_i$ for some positive integers m_i and $\beta = \sum_{i=1}^n p_i\alpha_i$ for some nonnegative integers p_i such that $p_i \leq m_i$ for all $i \in I$. We know that $\alpha \neq \beta$, thus there exists $k \in I$ such that $p_k < m_k$. By Lemma 2.2.2, α is the longest root with respect to $\{\alpha'_1, \ldots, \alpha'_n\}$ and then by Lemma 1.6.6 (iii), $(\alpha'_j, \beta) \geq 0$ since $\beta, \alpha'_j \in \overline{C'}$. Then $s_{\alpha'_j}(\beta) = \beta - (\alpha_j, \beta)\alpha'_j$ has at most the same height as β and thus has smaller height than α and so $s_{\alpha'_j}(\beta) \neq \alpha$. This is a contradiction and thus $P(\alpha) \subseteq R$. Thus $P(\alpha) = R$ and $P(\alpha)$ is a J-shadow.

Suppose that R is a J-shadow. Let C be a chamber of R and let $B(C) = \{\alpha_1, \ldots, \alpha_n\}$. We relabel α_i so that $s_{\alpha_i}(C)$ is the unique i-adjacent chamber to C. Let α be the longest root with respect to $B(C)$. We know that $P(\alpha)$ is a J-shadow. The group $V = \langle s_{\alpha_i} \mid i \in I \setminus J \rangle$ acts transitively on R and fixes α and so $R \subseteq P(\alpha)$. Hence $R = P(\alpha)$ as required.□

The next result is taken from Lemma 6.8 in [34] and its proof is omitted.

Lemma 2.2.4 *Let $P(\alpha)$ be a polar region in an apartment Σ. Then* $\mathrm{op}_\Sigma(P(\alpha)) = P(-\alpha)$.

For the remainder of this section we fix a root α, a Weyl chamber C of the polar region $P(\alpha)$, a basis $B(C) = \{\alpha_1, \ldots, \alpha_n\}$ of Φ and colouring of $\Sigma = \Sigma_\Phi$ by letting $s_{\alpha_i}(C)$ be the unique i-adjacent Weyl chamber of C. Let J be the root set of Φ.

Lemma 2.2.5 *Let β be a root. Then the following are equivalent.*

(i) *α and β form an angle of $\pi/3$.*

(ii) *$P(\alpha) \subseteq [\beta]$ and $P(\beta) \subseteq [\alpha]$.*

(iii) *$P(\alpha)$ and $P(\beta)$ are incident as J-shadows of Σ.*

Proof. We show that (i) is equivalent to (ii) and (iii). Suppose that (i) holds, that is, $(\alpha, \beta) = 1$. We search for root hyperplanes which separate α and β. Let γ be an arbitrary root and H_γ be the hyperplane with respect to γ. Let X be the subroot system of type A_2 containing α and β. If $(\gamma, \alpha) = (\gamma, \beta) = 0$, then H_γ contains the plane X and so does not separate α and β. If $(\gamma, \alpha) = 0 (\neq 0)$ and $(\gamma, \beta) \neq 0 (= 0)$, then the intersection of H_γ and X is the line spanned by α (β, respectively). In particular, H_γ does not separate α and β. The remaining case to examine is that when $(\gamma, \alpha), (\gamma, \beta) \neq 0$. This divides into two additional cases. In the first case we suppose that $(\gamma, \alpha) = (\gamma, \beta) = \pm 1$, then $(\gamma, \pm(\alpha - \beta)) = 0$. Then the intersection of H_γ and X is the line spanned by $\alpha - \beta$ and this does not separate α and β. In the second case we suppose that $(\gamma, \alpha) = -(\gamma, \beta) = \pm 1$, then $(\delta, \pm(\alpha + \beta)) = 0$. The intersection of H_γ and X is the line spanned by $\alpha + \beta$ and this line does separate α and β. We conclude that the only hyperplanes that divide α

and β are those H_γ for which $(\gamma, \alpha) = -(\gamma, \beta) = \pm 1$. Without loss of generality, suppose that $(\gamma, \alpha) = -(\gamma, \beta) = 1$. The Gram matrix with respect to $(,)$ restricted to the basis $\{\alpha, \beta, \gamma\}$ is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

This matrix has determinant zero and γ lies on the plane spanned by α and β. Thus the unique choice of root for γ is $\alpha - \beta$ (or, $\beta - \alpha$). In particular, the only hyperplane that separates α and β is the hyperplane $H_{\alpha-\beta}$. If $P(\alpha)$ and $P(\beta)$ were not incident, then they would be separated by at least two hyperplanes but this is clearly not possible. Hence, $P(\alpha)$ and $P(\beta)$ are incident. Therefore (i) implies (iii). Suppose that (iii) holds. Let C' and C'' be chambers of $P(\alpha)$ and $P(\beta)$ that are adjacent. Let w be the unique element of V such that $w(C) = C'$ and thus $B(C') = \{w(\alpha_1), \ldots, w(\alpha_n)\}$. There exists $j \in J$ such that $s_{w(\alpha_j)}(C') = C''$ and $\beta = s_{w(\alpha_j)}(\alpha) = \alpha - (\alpha, w(\alpha_j))w(\alpha_j)$. Thus $(\alpha, \beta) = (\alpha, \alpha - (\alpha, w(\alpha_j))w(\alpha_j)) = (\alpha, \alpha) - (\alpha, w(\alpha_j))^2$. Either $(\alpha, \beta) = 1$ or $(\alpha, \beta) = 2$ but $\alpha \neq \beta$ and so $(\alpha, \beta) = 1$, This proves that (iii) implies (i).

Suppose that (i) holds. Let C' be any chamber in $P(\alpha)$. Since $(,)$ is continuous, there exists a point $v \in C'$ close enough to α such that $(\beta, v) > 0$. In particular v is on the positive side of H_β and so the entire chamber C' lies on the positive side of H_β. Thus $P(\alpha) \subseteq [\beta]$. By symmetry, $P(\beta) \subseteq [\alpha]$. Suppose that (ii) holds. If $(\alpha, \beta) \leq 0$, then α will be in the boundary of some chamber in $[-\beta]$. In particular, $P(\alpha)$ is not contained in $[\beta]$. Therefore $(\alpha, \beta) = 1$. This proves (ii) implies (i). □

Lemma 2.2.6 *Let α and β be roots. The angle formed between α and β is*

(i) $\pi/2$ *if and only if $P(\alpha) \cap [\beta] \neq \emptyset$ and $P(\alpha)$ is not contained in $[\beta]$,*

(ii) $2\pi/3$ *if and only if* $P(\alpha) \subseteq [-\beta]$, *and*

(iii) π *if and only if* $P(\beta) = P(-\alpha)$.

Proof. We start with (ii). By Lemma 2.2.5, $(\alpha, \beta) = -1$ if and only if $(\alpha, -\beta) = 1$, if and only if $P(\alpha) \subseteq [-\beta]$. Part (iii) is obvious since $\beta = -\alpha$. By (ii), (iii) and Lemma 2.2.5, and by process of elimination, (i) must hold. □

Lemma 2.2.7 *Let* α *and* β *be roots. The angle formed between* α *and* β *is*

(i) $\pi/2$ *if and only if* $P(\alpha)$ *and* $P(\beta)$ *have at least two common neighbours,*

(ii) $2\pi/3$ *if and only if* $P(\alpha)$ *and* $P(\beta)$ *have the unique common neighbour* $P(\alpha+\beta)$, *and*

(iii) π *if and only if* $P(\alpha)$ *and* $P(\beta)$ *have no common neighbour.*

Proof. By Lemma 2.2.5, two J-shadows $P(\alpha)$ and $P(\beta)$ have a common neighbour $P(\gamma)$ if and only if γ forms an angle of $\pi/3$ with α and β. Suppose α and β form an angle of $\pi/2$. By Section 1.5, α and β have at least two common neighbours. Suppose α and β form an angle of $2\pi/3$. Then $\alpha+\beta$ is a root that forms an angle of $\pi/3$ with both α and β and thus $P(\alpha+\beta)$ is a common neighbour of $P(\alpha)$ and $P(\beta)$. Note that a $P(\gamma)$ is a common neighbour of $P(\alpha)$ and $P(\beta)$ if and only if γ forms an angle of $\pi/3$ with α and β, if and only if γ lies in the same plane as α and β since α and β form an angle of $2\pi/3$. Thus $P(\alpha+\beta)$ is the unique common neighbour. Suppose that $\beta = -\alpha$. Then there exists no root γ that forms an angle of $\pi/3$ with both α and β and so $P(\alpha)$ and $P(\beta)$ has no common neighbour. We have proved the forward direction of (i), (ii) and (iii) and the other direction follows from this and Lemma 2.2.5. □

2.3 Dictionary between J-shadows and root systems

Let Δ be an irreducible spherical building of type X_n and let $\mathcal{P}$ be the corresponding point-line root shadow space. Let Φ be the root system corresponds to Δ. Let $\theta_{\alpha,\beta}$ be the angle formed between α and β. Let Σ be an apartment of Δ and let $\mathcal{P}_\Sigma$ be the point shadow space of Σ where Σ is viewed as a thin building of type X_n. It can be easily seen that $\mathcal{P}_\Sigma = \mathcal{P} \cap \Sigma$. We summarise the results from the previous section.

Theorem 2.3.1 *Let R and Q be J-shadows in $\mathcal{P}_\Sigma$. Thus $R = P(\alpha)$ and $Q = P(\beta)$ for some roots α and β in Φ. The follow assertions hold.*

(i) $\theta_{\alpha,\beta} = 0$ *if and only if* $P(\alpha) = P(\beta)$.

(ii) $\theta_{\alpha,\beta} = \pi/3$ *if and only if* $P(\alpha)$ *and* $P(\beta)$ *are incident.*

(iii) $\theta_{\alpha,\beta} = \pi/2$ *if and only if* $P(\alpha)$ *and* $P(\beta)$ *have at least two common neighbours.*

(iv) $\theta_{\alpha,\beta} = 2\pi/3$ *if and only if* $P(\alpha)$ *and* $P(\beta)$ *have a unique common neighbour, namely* $P(\alpha + \beta)$.

(v) $\theta_{\alpha,\beta} = \pi$ *if and only if* $P(\alpha)$ *and* $P(\beta)$ *have no common neighbour.* $\square$

We need to measure the 'angle' between J-shadows that are not contained in a common apartment and we do this by intersecting J-shadows with apartments. If Σ is an apartment such that $R' = R \cap \Sigma$ and $Q' = Q \cap \Sigma$, for J-shadows R and Q, are nonempty, then R' and Q' are J-shadows of the (thin) building Σ and thus R' and Q' are polar regions with respect to roots. We show that the angle between the two roots is independent of the choice of Σ. We want to replicate Theorem 2.3.1 but for J-shadows in the building. The first lemma is a consequence of Lemma 2.2.5 and Lemma 2.2.6 and shows that the angle does not depend on the choice of apartment.

Lemma 2.3.2 *Let R and Q be two J-shadows. Let Σ_i intersect R and Q, let $R_i = \Sigma_i \cap R$ and $Q_i = \Sigma_i \cap Q$, write $R_i = P(\alpha_i)$ and $Q_i = P(\beta_i)$ and let $\theta_{\alpha_i,\beta_i}$ be the angle between α_i and β_i for $i = 1, 2$. Then $\theta_{\alpha_1,\beta_1} = \theta_{\alpha_2,\beta_2}$.*

Proof. By Proposition 1.3.6, there exists a special isomorphism g from Σ_1 to Σ_2 that maps R_1 to R_2 and Q_1 to Q_2. In particular, g maps $P(\alpha_1)$ to $P(\alpha_2)$ and $P(\beta_1)$ to $P(\beta_2)$. It follows from the bijective correspondence between roots and J-shadows (of a fixed apartment) that g maps the half-apartments $[\alpha_1]$ to $[\alpha_2]$ and $[\beta_1]$ to $[\beta_2]$. By Lemma 2.2.5 and Lemma 2.2.6, the angles between roots can be characterised by simple inclusion and intersection properties of the corresponding half-apartments and polar regions. Clearly g, as a bijection, preserves such set theoretic properties. We illustrate this the case of $\pi/3$. The angle $\theta_{\alpha_1,\beta_1} = \pi/3$, if and only if $P(\alpha_1) \subseteq [\beta_1]$, if and only if $P(\alpha_1)^g \subseteq [\beta_1]^g$, if and only if $P(\alpha_2) \subseteq [\beta_2]$, if and only if $\theta_{\alpha_2,\beta_2} = \pi/3$. The other cases can be shown in a similar way. □

Lemma 2.3.3 *Let Δ be a building and Σ be an apartment. Let R and Q be two J-shadows that intersect Σ and let $R' = R \cap \Sigma$ and $Q' = Q \cap \Sigma$. Let $\mathrm{dist}_\Sigma(R, Q)$ be the distance between R and Q in the root shadow space of the (thin) building Σ. Then $\mathrm{dist}_\mathcal{P}(R, Q) = \mathrm{dist}_\Sigma(R, Q)$. In particular, $\mathrm{dist}_\Sigma(R, Q) = \mathrm{dist}_{\Sigma'}(R, Q)$ for any two apartments Σ and Σ'.*

Proof. Let $d_\Sigma = \mathrm{dist}_\Sigma(R, Q)$ and $d_\mathcal{P} = \mathrm{dist}_\mathcal{P}(R, Q)$. It is clear that $0 \leq d_\mathcal{P} \leq d_\Sigma \leq 3$ and that $d_\mathcal{P} = 0$ if and only if $d_\Sigma = 0$. Suppose that $d_\mathcal{P} = 1$ and let $x \in R'$ and $y \in Q'$. The Weyl distance between R and Q in Δ is r_j for a $j \in J$. Then Σ contains the gallery from x to y that passes through $\mathrm{proj}_R Q$ and $\mathrm{proj}_Q R$ and thus $d_\Sigma = 1$. Therefore, $d_\mathcal{P} = 1$ if and only if $d_\Sigma = 1$. Let $d_\Sigma = 3$. Then $d_\mathcal{P} \in \{2, 3\}$. For a contradiction, suppose that $d_\mathcal{P} = 2$. Let T be a J-shadow that is collinear to both R and Q. Let $R_1 = \mathrm{proj}_R T$, $T_1 = \mathrm{proj}_T R$, $T_2 = \mathrm{proj}_T Q$ and $Q_1 = \mathrm{proj}_Q T$. Let γ be the shortest gallery from R_1 to Q_1 that passes through T_1 and T_2, let x and y be the initial and final chambers of γ, respectively, and let $u = \text{proj}_T x$ and $v = \text{proj}_T y$. In particular, the subgallery of γ from u to v is minimal.

Suppose that γ is minimal. Let Σ' be an apartment containing x and y. Then Σ' contains the gallery γ. Let $R'' = R \cap \Sigma$ and $Q'' = Q \cap \Sigma$. Then $\text{dist}_{\Sigma'}(R, Q) = 2$. By Proposition 1.3.6, there exists a special automorphism of Δ that maps Σ to Σ', R' to R'' and Q' to Q''. We assumed that $\text{dist}_{\Sigma}(R, Q) = 3$ and, by Lemma 6.8 of [34] R' and Q' are opposite residues in Σ. Thus R'' and Q'' are opposite residues in Σ' contradicting that $\text{dist}_{\Sigma'}(R, Q) = 2$. Thus γ is not minimal.

Let γ_1 and γ_2 be the subgalleries from x to v and u to y, respectively. Suppose that γ_1 is not minimal. Let $x' = \text{proj}_R v$ and $v' = \text{proj}_T x'$. Then, by applying Lemma 1.3.7 twice, we have that $\text{dist}(x, v) = \text{dist}(x, x') + \text{dist}(x', v') + \text{dist}(v', v)$. Let γ_1' denote the gallery from x to v that passes through x' and v'. Then the gallery (γ_1', r_j) is a gallery from R_1 to Q_1 that passes through T_1 and T_2 and is of shorter length than γ. This contradicts the choice of γ and thus γ_1 is minimal. A similar argument shows that γ_2 is minimal.

Let $w \in W$ with length m, r_{j_1} and r_{j_2} be generators of W with $j_1, j_2 \in J$ such that $uw = v$, $xr_{j_1} = u$ and $vr_{j_2} = y$. Then both γ_1 and γ_2 have length $m + 1$ and $xr_{j_1}wr_{j_2} = y$. We know that the length of $r_{j_1}wr_{j_2}$ is either m or $m + 2$. If the latter were true, then γ would be minimal but this is not the case. Thus $r_{j_1}wr_{j_2}$ has length m. Thus $\text{dist}(x, y) = \text{dist}(u, v) = m$. Let P and S be the j_1-panel containing x and u and the j_2-panel containing y and v, respectively. Let $P_1 = \text{proj}_P S$ and $S_1 = \text{proj}_S P$. As P_1 and S_1 are residues of P and S, respectively, either both P_1 and S_1 contain a single chamber or coincide with P and S, respectively. The chambers u and v are contained in T and thus any minimal path between u and v contained in T. The word of W corresponding to such a minimal path contains no r_j for $j \in J$. Thus, by the gatedness property of residues, $u \in P_1$ and $v \in S_1$. By Lemma 2.1.4 (iii) and that $\text{dist}(x, y) = \text{dist}(u, v) = m$, we conclude that $P = P_1$ and $S = S_1$ By applying Lemma 2.1.4 (iii), $xw = y$ and thus

x and y are contained in the same J-shadow and this contradicts that $d_{\mathcal{P}} = 2$. Thus $d_{\mathcal{P}} = 3$. If $d_{\mathcal{P}} = 3$, then $d_{\Sigma} = 3$. Therefore, $d_{\mathcal{P}} = 3$ if and only if $d_{\Sigma} = 3$. By process of elimination, $d_{\mathcal{P}} = 2$ if and only if $d_{\Sigma} = 2$. □

We say that the root shadow space $\mathcal{P}$ of a building Δ *possesses the property* (UCN) if:

(UCN) *For every two J-shadows R and Q and apartment Σ that intersects both R and Q, R and Q have a unique common neighbour in $\mathcal{P}_{\Delta}$ if and only if $R \cap \Sigma$ and $Q \cap \Sigma$ have a unique common neighbour in $\mathcal{P}_{\Sigma}$.*

Note that one direction holds in all cases. If $R \cap \Sigma$ and $Q \cap \Sigma$ have at least two common neighbours in $\mathcal{P}_{\Sigma}$, then R and Q have at least two common neighbours in $\mathcal{P}_{\Delta}$. In particular, a unique common neighbour in $\mathcal{P}_{\Delta}$ implies a unique common neighbour in $\mathcal{P}_{\Sigma}$. For the rest of the chapter and Chapter 5, we assume that the root shadow spaces in question have the property (UCN) and we prove that this property holds case by case on demand.

Definition 2.3.4 *Let R and Q be J-shadows in Δ and let Σ be an apartment that has an nonempty intersection with R and Q denoted by R' and Q'. Let α and β be roots such that $R' = P(\alpha)$ and $Q' = P(\beta)$ and let θ be the angle formed between α and β. We say that* R and Q form an angle of θ.

We now state a lemma that describes the relationship between distances and angles in the root shadow space of the building. This result is analogous to Lemma 2.3.1. Recall that we assume that the (UCN) property holds in the following theorem.

Theorem 2.3.5 *Let R and Q be J-shadows in $\mathcal{P}$. Let Σ be an apartment that intersects R and Q and set $R' = R \cap \Sigma$ and $Q' = Q \cap \Sigma$. Then write $R' = P(\alpha)$ and $Q' = P(\alpha)$. Let $\theta_{\alpha,\beta}$ be the angle between R and Q. The follow assertions hold.*

(i) $\theta_{\alpha,\beta} = 0$ *if and only if* $R = Q$

(ii) $\theta_{\alpha,\beta} = \pi/3$ *if and only if* R *and* Q *are incident.*

(iii) $\theta_{\alpha,\beta} = \pi/2$ *if and only if* R *and* Q *have at least two common neighbours.*

(iv) $\theta_{\alpha,\beta} = 2\pi/3$ *if and only if* R *and* Q *have a unique common neighbour.*

(v) $\theta_{\alpha,\beta} = \pi$ *if and only if* R *and* Q *have no common neighbour.*

Proof. The assertions (i),(ii),(v) follow from Lemma 2.3.1, Lemma 2.3.3 and Definition 2.3.4. For assertion (iv) we assume that property (UCN) holds. Assertion (iii) follows by the process of elimination. □

Chapter 3

Classical Lie Algebras

3.1 Complex semisimple Lie algebras

A Lie algebra is a vector space L over a field F equipped with a bilinear form $[,] : L \times L \to L$ such that for all x, y and z in L we have

(L1) $[x, y] = -[y, x]$ and

(L2) $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

The properties (L1) and (L2) are called *antisymmetry* and the *Jacobi identity*, respectively. The bilinear form $[,]$ is called the *Lie bracket* of L. For subspaces H and K of L, we define $[H, K]$ to be the linear span of $\{[x, y] \mid x \in H, y \in K\}$. A subspace H is a *subalgebra* of L if $[H, H] \subseteq H$. A subspace I is an *ideal* of L if $[I, L] \subseteq I$. A *subalgebra* H is called *commutative* if $[H, H] = \{0\}$. The *centre* of L is $Z(L) = \{y \in L \mid [x, y] = 0 \text{ for all } x \in L\}$. The Lie algebra L is *simple* if it is noncommutative and its only ideals are the zero space and itself. The Lie algebra L is *semisimple* if it is the direct sum of simple Lie algebras. For an element $x \in L$, the *adjoint action of x on L* is the map $\mathrm{ad}_x : L \to L$ given by $\mathrm{ad}_x(y) = [x, y]$. A *Cartan subalgebra* of L is a commutative subalgebra H such that ad_x is diagonalisable for all $x \in H$ and $C_L(H) = \{x \in H \mid [x, h] = 0 \textit{ for all } h \in H\} = H$.

For the rest of this section we assume that F is the field of complex numbers $\mathbb{C}$ and L is semisimple. We call a Lie algebra over $\mathbb{C}$ a *complex Lie algebra*. It is known that Cartan subalgebras exist for L and are all isomorphic under the action of a subgroup of automorphisms of L (c.f. Section 6.7 of [18]). Thus, let H be a Cartan subalgebra of L and H^* be the dual of the vector space H. For each $\alpha \in H^*$ define $L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x \text{ for all } h \in H\}$.

Lemma 3.1.1 *Let L be a semisimple Lie algebra and H be a Cartan subalgebra of L. Then L exhibits the following decomposition*

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where $\Phi = \{\alpha \in H^* \mid L_\alpha \neq 0\}$.□

The decomposition in Lemma 3.1.1 is called the *Cartan decomposition* of L. The set Φ is called *the root set of L with respect to H*. The next result is a composition of a number of results taken from Section 6.6 of [18] whose proof is omitted.

Lemma 3.1.2 *Let $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ be a Cartan decomposition.*

(i) *The subspace L_α is one-dimensional for all $\alpha \in \Phi$.*

(ii) *The subspace $H_\alpha = [L_\alpha, L_{-\alpha}]$ is one-dimensional for all $\alpha \in \Phi$.*

(iii) *If $\beta \neq -\alpha$, then*

$$[L_\alpha, L_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi \\ L_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi. \end{cases}$$

□

The root set Φ forms a root system as described in Section 1.4. There is a one-to-one correspondence between the root systems defined in Section 1.4 and complex semisimple

Lie algebras. Furthermore, there is a one-to-one correspondence between irreducible root systems and complex simple Lie algebras. Two different Cartan subalgebras yield two different but isomorphic root systems. Thus a simple Lie algebra L and its root system Φ determine each other. Therefore, to classify the complex simple Lie algebras, one only needs to determine the irreducible root systems and this is stated in Theorem 1.4.6. We restate this result and give details about the infinite families of complex simple Lie algebras.

Theorem 3.1.3 (Classification of complex simple Lie algebras) *Let L be a complex simple Lie algebra with corresponding Dynkin diagram Π. Then Π is one of $A_n(n \geq 1)$, $B_n(n \geq 2)$, $C_n(n \geq 3)$, $D_n(n \geq 4)$, E_6, E_7, E_8, F_4, or G_2. The four infinite families of simple Lie algebras A_n, B_n, C_n and D_n correspond to $\mathfrak{sl}(n+1, \mathbb{C})$, $\mathfrak{so}(2n+1, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$ and $\mathfrak{so}(2n, \mathbb{C})$, respectively.* $\square$

The Lie algebra of type A_n is described in Section 3.3 and descriptions of the other three infinite families can be found in Appendix A of [18]. The original classification was carried out by Killing in in 1888 in his paper [17] in which he introduced the root systems. It was revised by Cartan in his PhD thesis [4] submitted in 1894. In the mid-twentieth century Dynkin reproved the classification theorem in a completely novel way by introducing Dynkin diagrams and classifying such diagrams using graph theoretical arguments (c.f. [14]).

3.2 Chevalley algebras

We return to an arbitrary field F of characteristic p. We first set up some notation that is taken from [29]. Define $c_{\alpha,\beta} = 2(\alpha, \beta)/(\beta, \beta)$ where $(,)$ is the standard inner product on Euclidean space. Define $r_{\alpha,\beta}$ to be 0 if $\alpha + \beta$ is not a root and to be the smallest positive integer r such that $\alpha - r\beta$ is not a root. The first result due to Chevalley yields a huge class of simple Lie algebras and can be found in [7].

Theorem 3.2.1 *Let L_0 be a simple Lie algebra over $\mathbb{C}$ and H_0 be a Cartan subalgebra with corresponding root system Φ and let the set $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for Φ. Then there exists a basis $\{x_\alpha; h_i \mid \alpha \in \Phi\, 1 \leq i \leq n\}$ of L_0 such that*

(i) $[h_i, h_j] = 0$ *for all* $1 \leq i, j \leq n$,

(ii) $[h_i, x_\alpha] = c_{\alpha,\alpha_i} x_\alpha$ *for all* $\alpha \in \Phi$ *and* $1 \leq i \leq n$,

(iii) $[x_\alpha, x_{-\alpha}]$ *is an integral combination of* $h_1, \ldots, h_n$.

(iv) *If* $\alpha \neq \pm\beta$, *then* $[x_\alpha, x_\beta] = \pm r_{\alpha,\beta} x_{\alpha+\beta}$.

Such a basis is called a *Chevalley basis* for L_0. From this basis we can construct Lie algebras over arbitrary fields using tensor products. Let B be a Chevalley basis with respect to the root system Φ and let $(L_0)_\mathbb{Z}$ be algebra generated by B over $\mathbb{Z}$. Let F be an arbitrary field of characteristic p. Define $\overline{L} := (L_0)_\mathbb{Z} \otimes_\mathbb{Z} F$. The vector space $\overline{L}$ has elements of the form $\sum_{(x,\lambda)\in((L_0)_\mathbb{Z},F)} x \otimes \lambda$ and for all $x, y \in L_0$, $a \in \mathbb{Z}$ and $\lambda \in F$ we have

$$\begin{aligned}(x+y)\otimes\lambda &= x\otimes\lambda + y\otimes\lambda,\\ x\otimes(\lambda+\mu) &= x\otimes\lambda + x\otimes\mu,\\ ax\otimes\lambda &= x\otimes a\lambda.\end{aligned}$$

Scalar multiplication over the field F is defined as $\mu(x\otimes\lambda) = x\otimes\mu\lambda$. By abuse of notation, we define a bilinear form on $\overline{L}$ by the linear expansion of $[x\otimes\lambda, y\otimes\mu] = [x, y]\otimes\lambda\mu$.

Lemma 3.2.2 *The vector space $\overline{L}$ is a Lie algebra over F under the bilinear form defined above. In particular, if $\{y_1, \ldots, y_t\}$ is a basis for $(L_0)_\mathbb{Z}$, then $\{y_1\otimes 1, \ldots, y_t\otimes 1\}$ is a basis for $\overline{L}$.*

Proof. The Lie bracket is clearly a bilinear form by definition. We need it to satisfy the antisymmetric property and the Jacobi identity. Let $x, y, z \in L_0$ and $\lambda, \mu, \nu \in F$. The antisymmetric property holds: $[x\otimes\lambda, y\otimes\mu] = [x, y]\otimes\lambda\mu = -[y, x]\otimes\mu\lambda = -[y\otimes\mu, x\otimes\lambda]$. Observe that using property that L_0 is a Lie algebra we have

$$\begin{aligned}
[x\otimes\lambda, [y\otimes\mu, z\otimes\nu]] &= [x\otimes\lambda, [y, z]\otimes\mu\nu] \\
&= [x, [y, z]]\otimes\lambda(\mu\nu) \\
&= ([[x, y], z] + [y, [x, z]])\otimes\lambda(\mu\nu) \\
&= [[x, y], z]\otimes(\lambda\mu)\nu + [y, [x, z]]\mu(\lambda\nu) \\
&= [[x\otimes\lambda, y\otimes\mu], z\otimes\nu] + [y\otimes\mu, [x\otimes\lambda, z\otimes\nu]]
\end{aligned}$$

Thus the Jacobi identity holds for $\overline{L}$ and so $\overline{L}$ is a Lie algebra over F. Suppose that $x = \sum_{i=1}^{t} a_i y_i$. Then $x\otimes\lambda = \sum_{i=1}^{t} a_i y_i\otimes\lambda = \sum_{i=1}^{t} a_i\lambda(y_i\otimes 1)$. As any element of $\overline{L}$ has the form $\sum_{(x,\lambda)\in((L_0)_{\mathbb{Z}},F)} x\otimes\lambda$, we have that $\{y_1\otimes 1, \dots, y_t\otimes 1\}$ is a basis for $\overline{L}$. □

The Lie algebra $\overline{L}$ is called a *Lie algebra of Chevalley type* Φ (or simply a *Chevalley algebra*). Additionally, $\overline{H} = H_0\otimes_{\mathbb{Z}} F$ is a commutative subalgebra of $\overline{L}$. We carry forward the notation of $\{x_\alpha \mid \alpha\in\Phi\}\cup\{h_i \mid 1\le i\le n\}$ as a basis for $\overline{L}$ and drop the tensor notation. For each $\alpha,\beta\in\Phi$, we define $\overline{c}_{\alpha,\beta}$ to be $c_{\alpha,\beta}$ if $p = 0$ and the unique c such that $0\le c\le p-1$ and $c_{\alpha\beta}\equiv c \mod p$. Let $\overline{Z}$ be the centre of the Lie algebra $\overline{L}$.

Lemma 3.2.3 *The centre $\overline{Z}$ is contained in the subalgebra $\overline{H}$ and consists of the $h = \sum_{i=1}^{n}\mu_i h_i$ such that $\overline{\alpha(h)} := \sum_{i=1}^{n}\mu_i\overline{c_{\alpha,\alpha_i}} = 0$ for all $\alpha\in\Phi$.*

Proof. Suppose that $x = h + \sum_{\beta\in\Phi}\lambda_\beta x_\beta$ is $\overline{Z}$. Then $0 = [x, x_{-\alpha}] = \lambda_\alpha h_\alpha + \sum_{\beta\in\Phi}\lambda'_\beta x_\beta$. In particular $\lambda_\alpha = 0$ for each $\alpha\in\Phi$. Furthermore, $[x, x_\alpha] = [h, x_\alpha] = \sum_{i=1}^{n}\mu_i\overline{c_{\alpha,\alpha_i}}$ for all $\alpha\in\Phi$. In particular, $h\in\overline{Z}$ if and only if $\overline{\alpha(h)} = 0$ for all $\alpha\in\Phi$. □

The Chevalley algebras are not always simple. The next result we state is (2.6) from [29] and provides a list of simple Lie algebras of Chevalley type. We are only interested in the simply laced diagrams and so we simplify the statement.

Theorem 3.2.4 *Let $\overline{L}$, $\overline{H}$ and $\overline{Z}$ be as above and let Φ be simply laced. Assume that $p \neq 2$ if $\Phi = A_1$. Then $L := \overline{L}/\overline{Z}$ is a simple Lie algebra with Cartan subalgebra $H := \overline{H}/\overline{Z}$.* □

Definition 3.2.5 *A Lie algebra of Chevalley type is a* classical Lie algebra *if it is simple.*

With a few minor restrictions to the rank and characteristic, the dimensions of the centres of all Chevalley algebras are computed in [15]. We explicitly compute the centre of the Chevalley algebra of type A_n.

Lemma 3.2.6 *Let $\overline{L}$ be the Lie algebra of Chevalley type A_n. Then the centre $\overline{Z}$ is the 1-dimensional ideal spanned by the vector $\sum_{i=1}^{n} ih_i$ if p divides $n+1$ and otherwise it is the zero space.*

Proof. Let $h = \sum_{i=1} \lambda_i h_i$ be in the centre. By Lemma 3.2.3, $\sum_{i=1}^{n} \overline{\lambda_i \alpha(h_i)} = 0$ for all $\alpha \in \Phi$ where $\alpha(h_i) = (\alpha, \alpha_i)$. Each positive α can be written as $\sum_{i=1}^{n} n_i \alpha_i$ where $n_i \in \{0, 1\}$. In particular, for each $\alpha \in \Phi$, x_α can be written as a Lie bracket of $\{\pm x_{\alpha_i} \mid 1 \leq i \leq n\}$ and thus $\overline{L}$ is generated by $\{\pm x_{\alpha_i} \mid 1 \leq i \leq n\}$. To show that h is in the centre it suffices to show that $\overline{\alpha_i(h)} = 0$ for all $i = 1, \ldots, n$.

We claim that $\lambda_k = k\lambda_1$ for all $k \in \{2, \ldots, n\}$ and proceed by induction. By inspecting the Dynkin diagram, we can see that $\alpha_j(h) = -\lambda_{j-1} + 2\lambda_j - \lambda_{j+1}$. Note that $0 = \alpha_1(h) = 2\lambda_1 - \lambda_2$ implies that $\lambda_2 = 2\lambda_1$. Assume that $k \geq 2$ and that $\lambda_j = j\lambda_1$ for all $j \leq k$. Then $\alpha_k(h) = -\lambda_{k-1} + 2\lambda_k - \lambda_{k+1} = 0$. By rearranging and applying the inductive hypothesis, we obtain $\lambda_{k+1} = (-(k-1) + 2k)\lambda_1 = (k+1)\lambda_k$. We finally require that $\overline{\alpha_n(h)} = 0$. By inspecting the Dynkin diagram, we have that $\alpha_n(h) = -\lambda_{n-1} + 2\lambda_n = (-(n-1)+2n)\lambda_1 = (n+1)\lambda_1 = 0$ if and only if $n+1 = 0$ or $\lambda_1 = 0$. Thus the centre is the zero space if p does not divide $n+1$ and the 1-dimensional ideal $\langle \sum_{i=1}^{n} ih_i \rangle$ if p divides $n+1$. □

We end this section by defining a certain class of automorphism groups of the Chevalley algebras. The *Chevalley group of type $X_n(F)$* is the group generated by $\{\exp(t\mathrm{ad}_{x_\alpha}) \mid \alpha \in \Phi, t \in F\}$ where Φ is the root system of type X_n.

3.3 Extremal and quasiextremal elements in $\mathfrak{sl}(V)$

Let V be an $(n+1)$-dimensional vector space over a field F of characteristic p and assume that $p \neq 2$. Let $\mathfrak{gl}(n+1, F)$ be the set of $(n+1) \times (n+1)$ matrices over the field F. For simplicity, we denote $\mathfrak{gl}(n+1, F)$ by $\mathfrak{gl}(V)$. The vector space $\mathfrak{gl}(V)$ is isomorphic to the space of linear maps from V to V denoted by $\mathrm{End}(V)$. Indeed, fix a basis $\{v_1, \ldots, v_{n+1}\}$ and let f be a linear map in $\mathrm{End}(V)$ and write $f(v_i) = \sum_{j=1}^{n+1} a_{ij} v_j$. Then the map that sends f to $A = (a_{ij})$ is an isomorphism from $\mathrm{End}(V)$ to $\mathfrak{gl}(V)$. Furthermore, $\mathrm{End}(V)$ and $\mathfrak{gl}(V)$ are isomorphic as F-algebras with multiplication as usual map composition and matrix multiplication, respectively. The subalgebra $\mathfrak{sl}(V)$ is defined as the set of matrices A of $\mathfrak{gl}(V)$ with trace 0. A standard Chevalley basis for $\mathfrak{sl}(V)$ is $\{E_{jk}; H_i \mid 1 \leq j, k \leq n+1, j \neq k, 1 \leq i \leq n\}$ where E_{ij} is an elementary matrix which has 1 in position (i, j) and 0 in all other positions and H_i is the diagonal matrix $\mathrm{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0)$ where the diagonal entries i and $i+1$ are 1 and -1, respectively. The Lie bracket is defined as $[A, B] = AB - BA$.

Proposition 3.3.1 *The basis $\mathcal{B} = \{E_{jk}; H_i \mid 1 \leq j, k \leq n+1, j \neq k, 1 \leq i \leq n\}$ is a Chevalley basis for $\mathfrak{sl}(V)$.*

Proof. Let $\Phi = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\}$ be the root system of type A_n. Let E_{ij} be the root element corresponding to $\alpha_{ij} = e_i - e_j$. Firstly observe that $[H_i, H_j] = 0$ for all i, j. Note that $E_{ij}E_{kl}$ is 0 if $j \neq k$ and E_{il} if $j = k$. From this we compute

$$[E_{ij}, E_{kl}] = \begin{cases} 0 & \text{if } j \neq k \text{ and } i \neq l \\ E_{il} & \text{if } j = k \text{ and } i \neq l \\ -E_{kj} & \text{if } j \neq k \text{ and } i = l \\ E_{ii} - E_{jj} & \text{if } j = k \text{ and } i = l \end{cases}$$

In the first case, $(\alpha_{ij}, \alpha_{kl}) = 0$ and thus $\alpha_{ij} + \alpha_{kl}$ is not a root. In the second and third case, $(\alpha_{ij}, \alpha_{jl}) = (\alpha_{ij}, \alpha_{ki}) = -1$. In the final case, $(\alpha_{ij}, \alpha_{ji}) = -2$ and $E_{ii} - E_{jj} = H_i + H_{i+1} + \ldots + H_{j-1}$ assuming that $i \leq j$. Finally, we wish to consider the brackets of the form $[H_i, E_{jk}]$. By noting that $H_i = E_{ii} - E_{i+1,i+1}$ we can compute all seven cases

$$[H_i, E_{jk}] = \begin{cases} -2E_{jk} & \text{if } i \neq j, i+1 \neq k, i = k \text{ and } i+1 = j \\ -E_{jk} & \text{if } i \neq j, i+1 \neq k, i = k \text{ and } i+1 \neq j \\ 0 & \text{if } i \neq j, i+1 \neq k, i \neq k \text{ and } i+1 \neq j \\ -E_{jk} & \text{if } i \neq j, i+1 \neq k, i \neq k \text{ and } i+1 = j \\ E_{jk} & \text{if } i = j \text{ and } i+1 \neq k \\ E_{jk} & \text{if } i \neq j \text{ and } i+1 = k \\ 2E_{ik} & \text{if } i = j \text{ and } i+1 = k \end{cases}$$

For this to satisfy the condition of being a Chevalley basis, we need to check that $[H_i, E_{jk}] = (\alpha_{i,i+1}, \alpha_{jk})E_{jk}$. But this can easily be checked and we show just the first case. Note that $\alpha_{jk} = \alpha_{i+1,i}$ and so $(\alpha_{i,i+1}, \alpha_{jk}) = -2$ as required. The other cases can be done in a similar way. By comparing the constant structures computed above with the conditions in Theorem 3.2.1, one can deduce that $\mathcal{B}$ is a Chevalley basis for $\mathfrak{sl}(V)$. □

The Lie algebra $\mathfrak{sl}(V)$ is simple unless p divides $n+1$. In such a case, the centre is generated by $\sum_{i=1}^{n} ih_i$ and corresponds to the space generated by the matrix $\text{diag}(1, 1, \ldots, 1, -n)$. By observing that $-n \equiv 1 \mod p$, we conclude that the centre is generated by the identity matrix. The next result is Theorem 11.3.2 of [6].

Theorem 3.3.2 *The Chevalley group of type $A_n(F)$ is isomorphic to* $\text{PSL}(n+1, q)$. □

Let V^* be the dual vector space of V. Then the tensor product of V and V^*, denoted $V \otimes V^*$, is linearly spanned by $\{v \otimes \varphi \mid v \in V, \varphi \in V^*\}$. The space $V \otimes V^*$ and $\text{End}(V)$ are vector spaces over F of dimension $(n+1)^2$ and the map $v \otimes \varphi \mapsto f_v^\varphi$ where $f_v^\varphi(w) = \varphi(w)v$ induces a linear map $\Psi : V \otimes V^* \to \text{End}(V)$. Note that for $\lambda \in F$ we have $\Psi(\lambda(v\otimes\varphi)) = \Psi(\lambda v\otimes\varphi) = f^{\varphi}_{\lambda v} = \lambda f^{\varphi}_{v} = \lambda\Psi(v\otimes\varphi)$. The kernel of Ψ is the subspace $\langle v \otimes \varphi \mid \varphi(w)v = 0$ for all $w \in V\rangle$ and the condition that $\varphi(w)v = 0$ for all $w \in V$ implies that $\varphi = 0$ or $v = 0$. In particular, the kernel of Ψ is zero and thus Ψ is a linear isomorphism from $V \otimes V^*$ to $\mathrm{End}(V)$. Define a product on $V \otimes V^*$ by

$$(v \otimes \varphi) \circ (u \otimes \psi) = \varphi(u)v \otimes \psi$$

turning $V \otimes V^*$ into an F-algebra and Ψ into an F-algebra isomorphism. In order to verify this we must show that $\Psi((v \otimes \varphi) \circ (u \otimes \psi)) = \Psi(v \otimes \varphi)\Psi(u \otimes \psi)$. We show that both sides act identically on an arbitrary vector $w \in V$. Indeed, $\Psi((v\otimes\varphi)\circ(u\otimes\psi))(w) = \Psi(\varphi(u)v\otimes\psi)(w) = \varphi(u)\psi(w)v$ and $\Psi(v\otimes\varphi)\Psi(u\otimes\psi)(w) = \Psi(v\otimes\varphi)\psi(w)u = \psi(w)\varphi(u)v$. Therefore we have three different ways of describing the same F-algebra, namely $\mathrm{End}(V)$, $\mathfrak{gl}(V)$, and $V \otimes V^*$. For each of the F-algebras, we define the usual Lie bracket on an associative algebra, namely $[x, y] = xy - yx$. Thus $\mathfrak{gl}(V)$, $\mathrm{End}(V)$ and $V \otimes V^*$ are isomorphic Lie algebras.

Let $\{e_1, \ldots, e_{n+1}\}$ be a basis for V and let $\{\varphi_1, \ldots, \varphi_{n+1}\}$ be its dual basis, that is, $\varphi_i(e_j) = \delta_{ij}$. Then $\{e_i \otimes \varphi_j \mid 1 \le i, j \le n+1\}$ is a basis for $V \otimes V^*$. A linear map $f : V \to V$ has *rank one* if $\dim(\mathrm{Im}(f)) = 1$. Using the isomorphism between $\mathrm{End}(V)$ and $V \otimes V^*$ we characterise the rank one elements of $V \otimes V^*$.

Lemma 3.3.3 *The rank* 1 *elements of* $V\otimes V^*$ *are precisely the elements of the form* $v\otimes\varphi$ *for* $v \in V$ *and* $\varphi \in V^*$ *such that* $v \neq 0$ *and* $\varphi \neq 0$. *Such elements are also known as* split tensors.

Proof. Let $v\otimes\varphi$ be a nonzero element of $V\otimes V^*$. Then $\dim(\mathrm{Im}(\Psi(v\otimes\varphi))) = \dim(\mathrm{Im}(f^{\varphi}_{v}))$ and $\mathrm{Im}(f^{\varphi}_{v}) = \{f^{\varphi}_{v}(w) \mid w \in V\} = \{\varphi(w)v \mid w \in V\} = Fv$. Then $\dim(\mathrm{Im}(f^{\varphi}_{v})) = 1$ and thus $v \otimes \varphi$ has rank 1.

Take an arbitrary element $x = \sum_{i,j} \lambda_{ij} e_i \otimes \varphi_j$ of $V \otimes V^*$ and note that its rank is the rank of its image $\Psi(\sum_{i,j} \lambda_{ij} e_i \otimes \varphi_j) = \sum_{i,j} \lambda_{i,j} f_{e_i}^{\varphi_j}$ where $f_{e_i}^{\varphi_j}(v) = \varphi_j(v) e_i$ for all $v \in V$. The rank of such a linear map is the dimension of its image in V. The image of $\sum_{i,j} \lambda_{i,j} f_{e_i}^{\varphi_j}$ is the subspace $\{\lambda_{i,j}\varphi_j(v)e_i \mid v \in V, 1 \leq i, j \leq n+1\} = \langle \lambda_{i,j} e_i \mid 1 \leq i \leq n+1 \rangle$. Therefore, if the rank of x is 1, then $\lambda_{i_0,j} = 1$ for a single i_0 and for all j, and $\lambda_{ij} = 0$ for all $i \neq i_0$ and for all j. In particular, $x = \sum_j \lambda_{i_0,j} e_{i_0} \otimes \varphi_j = e_{i_0} \otimes (\sum_j \lambda_{i_0,j} \varphi_j)$ and thus x has the form $v \otimes \varphi$. □

The trace of an arbitrary element (a_{ij}) of $\mathfrak{gl}(V)$ is given by $\mathrm{Tr}((a_{ij})) = \sum_i a_{ii}$. Let $x = \sum_{i,j} \lambda_{ij} e_i \otimes \varphi_j$ be an arbitrary element of $V \otimes V^*$. Choose the particular basis $\{e_1 \ldots, e_{n+1}\}$ to compute the matrix of f_x. For a fixed $1 \leq k \leq n+1$, we have that $f_x(e_k) = \sum_i \lambda_{ik} e_i$ and thus the matrix corresponds to matrix (λ_{ji}). The trace of such a matrix is $\sum_i \lambda_{ii} = \sum_{i,j} \lambda_{i,j} \varphi_j(e_i)$. More generally, given an element x which is the sum of split tensors $v \otimes \varphi$, the trace of x is given by $\sum \varphi(v)$. We identify $\mathfrak{sl}(V)$ with the subspace of $V \otimes V^*$ consisting of elements whose traces are zero. A rank one element $v \otimes \varphi$ is in $\mathfrak{sl}(V)$ if and only if $\varphi(v) = 0$. We recall the definition of an extremal element for the case when the characteristic of the field is not even.

Definition 3.3.4 *An* extremal element *of any subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is an element x of $\mathfrak{g}$ such that $[x, [x, y]] = a_y x$ for all y in $\mathfrak{g}$ and some scalar a_y in F that depends on y.*

The next result is the first half of the classification of extremal elements but first we define an extremal element in

Lemma 3.3.5 *All rank 1 elements of $\mathfrak{sl}(V)$ are extremal.*

Proof. Let $u \otimes \varphi$ be an arbitrary rank 1 element of $V \otimes V^*$ and for any other element $v \otimes \psi$ note that $\varphi(u) = \psi(v) = 0$. The rest is a simple computation:

$$
\begin{aligned}
[v \otimes \varphi, [v \otimes \varphi, u \otimes \psi]] &= [v \otimes \varphi, \varphi(u)v \otimes \psi - \psi(v)u \otimes \varphi] \\
&= \varphi(u)[v \otimes \varphi, v \otimes \psi] - \psi(v)[v \otimes \varphi, u \otimes \varphi] \\
&= \varphi(u)(\varphi(v)v \otimes \psi - \psi(v)v \otimes \varphi) - \psi(v)(\varphi(u)v \otimes \psi - \varphi(v)u \otimes \varphi \\
&= -\varphi(u)\psi(v)v \otimes \varphi - \psi(v)\varphi(u)v \otimes \varphi \\
&= -2\varphi(u)\psi(v)v \otimes \varphi
\end{aligned}
$$

as required. □

We aim to show that all the extremal elements of $\mathfrak{sl}(V)$ have rank one or are scalar matrices. Let $x = \sum_{i,j} \lambda_{ij} e_i \otimes \varphi_j$ be an extremal element of $V \otimes V^*$. Then x can be written as the following $\sum_i (\sum_j \lambda_{ji} e_j) \otimes \varphi_i$. In particular, we can write $x = \sum_{i=1}^k u_i \otimes \varphi_i$ where $u_i = \sum_j \lambda_{ij} e_j$ such that $\{u_1, \ldots, u_k\}$ and $\{\varphi_1, \ldots, \varphi_k\}$ are both linearly independent sets of vectors in V and V^*, respectively. Let f_x be the corresponding linear map defined by $f_x(v) = \sum_i \varphi_i(v) u_i$. We compute $f_x(u_l) = \sum_i \varphi_i(u_l) u_i = \sum_i \varphi(\sum_j \lambda_{lj} e_j) u_i = \sum_i (\lambda_{li} e_i) u_i = \sum_i \lambda_{li} u_i$ and note that $\varphi_j(u_i) = \varphi_j(\sum_l \lambda_{il} e_l) = \sum_l \lambda_{il} \varphi_j(e_l) = \lambda_{ij}$. In particular, the matrix representing f_x is given by $A = (a_{ij} = \varphi_j(u_i))$.

Lemma 3.3.6 *Suppose that $(a_1, \ldots, a_k)$ is a right λ-eigenvector of A and $(b_1, \ldots, b_k)$ is a left μ-eigenvector of A and $\lambda \neq \mu$. Then x is a multiple of $u \otimes \varphi$ where $u = \sum_{i=1}^k a_i u_i$ and $\varphi = b_j \varphi_j$. In particular, x has rank 1.*

Proof. Firstly notice that for all $j \in \{1, \ldots, k\}$, we have that $\lambda a_j = \sum_i \varphi_j(u_i) a_i$ and $\mu b_j = \sum_i \varphi_i(u_j) b_i$. Set $u = \sum_i a_i u_i$ and $\varphi = \sum_j b_j \varphi_j$. Let $t = \mathrm{Tr}(u \otimes \varphi)$. Then $t = \sum_{i,j} a_i b_j \mathrm{Tr}(u_i \otimes \varphi_j) = \sum_{i,j} a_i b_j \varphi_j(u_i)$. On one hand, $t = \sum_i a_i (\sum_j \varphi_j(u_i) b_j) = \mu \sum_i a_i b_i$. On the other hand, $t = \sum_j b_j (\sum_i a_i \varphi_j(u_i)) = \lambda \sum_j a_j b_j$. Since $\lambda \neq \mu$, then $t = 0$. In particular, $u \otimes \varphi$ is in $\mathfrak{sl}(V)$. Observe that $[x, u \otimes \varphi] = \sum_i [u_i \otimes \varphi_i, u \otimes \varphi] = \sum_i (\varphi_i(u) u_i \otimes \varphi - \varphi(u_i) u \otimes \varphi_i)$. We do each sum separately:

$$\begin{aligned}
\textstyle\sum_i \varphi_i(u)u_i \otimes \varphi &= \textstyle\sum_{i,j} a_j\varphi_i(u_j)u_i \otimes \varphi \\
&= \textstyle\sum_i \lambda a_i u_i \otimes \varphi \\
&= \lambda(u \otimes \varphi)
\end{aligned}$$

and

$$\begin{aligned}
\textstyle\sum_i \varphi(u)u \otimes \varphi_i &= \textstyle\sum_{i,j} b_j\varphi_j(u_i)u \otimes \varphi_i \\
&= \textstyle\sum_i b_i\mu u \otimes \varphi_i \\
&= \mu(u \otimes \varphi).
\end{aligned}$$

In particular, $[x, u \otimes \varphi] = (\lambda - \mu)u \otimes \varphi$ and so $[x, [x, u \otimes \varphi]] = (\lambda - \mu)^2 u \otimes \varphi \neq 0$. Thus x is a multiple of $u \otimes \varphi$ since x is extremal and so x has rank 1. □

Before we carry out the complete classification we introduce a more general notion of extremal element.

Definition 3.3.7 *An element x of $\mathfrak{sl}(V)$ is* quasiextremal *if $[x, [x, y]] = a_y x + b_y I$ for all $y \in \mathfrak{sl}(V)$ where I is the identity matrix and a_y and b_y are scalars in F that depend on y.*

Firstly, note that all extremal elements are quasiextremal. Secondly, the identity matrix is an element of $\mathfrak{sl}(V)$ if and only if p divides $n + 1$ and thus any quasiextremal element is an extremal element if p does not divide $n + 1$. The quasiextremal elements are only interesting when p divides $n+1$ and correspond to extremal elements of quotient of $\mathfrak{sl}(V)$ by the unique nontrivial ideal $Z(\mathfrak{sl}(V))$.

Let $f \in \mathfrak{sl}(V)$ and let U be a proper f-invariant subspace of V. By extending a basis of U to V, f has block matrix representation $A = \left(\begin{smallmatrix} A_U & \mathbf{0} \\ A_1 & A_2 \end{smallmatrix}\right)$.

Lemma 3.3.8 *Let A be an (quasi)extremal element of $\mathfrak{sl}(V)$ and let U be a proper A-invariant subspace of V. Then A_U is an (quasi)extremal element in $\mathfrak{gl}(U)$.*

Proof. Let B_1 be an element of $\mathfrak{gl}(U)$. Define B to be the block matrix $\left(\begin{smallmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{smallmatrix}\right)$ such that $\mathrm{Tr}(B_2) = -\mathrm{Tr}(B_1)$. Let $k = \dim(U)$ and let I be the block diagonal matrix $\mathrm{diag}(I_k, I_{n+1-k})$. Then $B \in \mathfrak{sl}(V)$ and $[A,[A,B]] = A^2B - 2ABA + BA^2 = \lambda_B A$ $(= \lambda_B A + \mu_B I)$ for some scalar λ_B (and μ_B). It is easy to compute that A^2B is

$$\begin{pmatrix} A_U^2 B_1 & \mathbf{0} \\ (A_1A_U + A_2A_1)B_1 & A_2^2 B_2 \end{pmatrix}$$

and similarly ABA and BA^2 have forms $\left(\begin{smallmatrix} A_UB_1A_U & \mathbf{0} \\ * & * \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} B_1A_U^2 & \mathbf{0} \\ * & * \end{smallmatrix}\right)$ where $*$ is any block matrix. In particular, $A^2B - 2ABA + BA^2$ has form

$$\begin{pmatrix} A_U^2B_1 - 2A_UB_1A_U + B_1A_U^2 & \mathbf{0} \\ * & * \end{pmatrix}$$

But since A is (quasi)extremal, we have that $A_U^2B_1 - 2A_UB_1A_U + B_1A_U^2 = \lambda A_U$ $(= \lambda A_U + \mu_B I_k)$. Since B_1 was arbitrary, we have that A_U is (quasi)extremal in $\mathfrak{gl}(U)$. $\square$

In Lemma 3.3.6, we assume that the extremal element x has at least two eigenvalues. Therefore it suffices to deal with the case where x has at most one eigenvalue. To ensure that x has at least one eigenvalue we must extend the field appropriately and such a field extension motivates the next construction.

Let $\mathfrak{g}$ denote $\mathfrak{gl}(V)$ or $\mathfrak{sl}(V)$. Let $\widehat{F}$ be a field extension of F and define a new Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_F \widehat{F}$ which is spanned by $\{x \otimes \lambda \mid x \in \mathfrak{g}, \lambda \in \widehat{F}\}$ and its Lie bracket is induced by $[x \otimes \lambda, y \otimes \mu] = [x, y] \otimes (\lambda\mu)$. One can verify this is a Lie algebra much in the way the Chevalley algebras are constructed in Section 3.2. An element $x \otimes \lambda$ of $\widehat{\mathfrak{g}}$ is quasiextremal if $[x \otimes \lambda, [x \otimes \lambda, y \otimes \mu]] = a_y(x \otimes \lambda) + b_y(I \otimes 1)$ for all y, for some $a_y, b_y \in \widehat{F}$ and extremal if $b_y = 0$.

Lemma 3.3.9 *Let $x \in \mathfrak{g}$. If x is (quasi)extremal in $\mathfrak{g}$, then $x \otimes \lambda$ is (quasi)extremal in $\widehat{\mathfrak{g}}$ for all $\lambda \in \widehat{F}$. In particular, if $x \otimes \lambda$ is not (quasi)extremal in $\widehat{\mathfrak{g}}$, then x is not (quasi)extremal in $\mathfrak{g}$.*

Proof. If x is quasiextremal in $\mathfrak{g}$, then $[x,[x,y]] = a_y x + b_y I$ for all $y \in \mathfrak{g}$. Take arbitrary $\lambda, \mu \in \widehat{F}$ and observe that $[x \otimes \lambda, [x \otimes \lambda, y \otimes \mu]] = [x,[x,y]] \otimes (\lambda^2 \mu) = (a_y x + b_y I) \otimes (\lambda^2 \mu) = a_y x \otimes (\lambda^2 \mu) + b_y I \otimes (\lambda_2 \mu) = (a_y \lambda \mu) x \otimes \lambda + (b_y \lambda^2 \mu) I \otimes 1$. In particular, $x \otimes \lambda$ is quasiextremal in $\widehat{\mathfrak{g}}$. If x is extremal in $\mathfrak{g}$, then we repeat the above argument with $b_y = 0$ to conclude that $x \otimes \lambda$ is extremal in $\widehat{\mathfrak{g}}$. □

Let $\widehat{V}$ be $(n+1)$-dimensional vector space over $\widehat{F}$. If we view $\mathfrak{g}$ as the F-algebra of $(n+1) \times (n+1)$ matrices, then it is clear that $\widehat{\mathfrak{g}}$ is isomorphic to $\mathfrak{gl}(\widehat{V})$ and such an isomorphism can be induced by the map $x \otimes \lambda \mapsto \lambda x$ where x is identified as a $(n+1) \times (n+1)$ with entries from F. In particular, if x is in $\mathfrak{g}$, then x has rank one in $\mathfrak{g}$ if and only if $x \otimes \lambda$ has rank one in $\widehat{\mathfrak{g}}$. We can also drop the tensor product notation and consider x to be an element of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$.

One of the main objectives in this section is to show that if x in $\mathfrak{sl}(V)$ has rank at least 2, then up to some minor exceptions it cannot be extremal. We show in Lemma 3.3.6 that if an extremal element x has two eigenvalues, then it has rank 1 and therefore from this point onwards, unless stated otherwise, we may assume that x has at most one eigenvalue. Suppose that $\widehat{F}$ corresponds to the algebraic closure of F. Recall that that the rank of x in $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ coincide. Suppose that x has rank at least 2. If x is not (quasi)extremal in $\mathfrak{sl}(\widehat{V})$, then x is not (quasi)extremal in $\mathfrak{sl}(V)$. For the purpose of showing that only rank one elements can be (quasi)extremal, up to some minor exceptions, we assume that F is algebraically closed. One advantage of having an algebraically closed field is that any element of $\mathfrak{sl}(V)$ has at least one eigenvalue and has a Jordan normal form. Since we are considering elements with at most one eigenvalue, we can assume that every element has exactly one eigenvalue.

One can easily see that the conjugate of an extremal element is an extremal element in $\mathfrak{sl}(V)$ and thus it is harmless to assume that all elements are written in Jordan normal form. We state six examples of elements which are not (quasi)extremal in $\mathfrak{sl}(U)$ for some subspace U of V and in particular not extremal in $\mathfrak{sl}(V)$.

Example 3.3.10 *Let $U = F^2$ Let $\lambda \neq 0$ and consider the matrix $x = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$. Let $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and by a simple computation one can check that $[x,[x,y]]$ is $\left(\begin{smallmatrix} 0 & -2 \\ 0 & 0 \end{smallmatrix}\right)$. Therefore x is not extremal in $\mathfrak{sl}(2,F)$ if $p \neq 2$.*

Example 3.3.11 *Let $U = F^4$ be a four dimensional subspace of V. Let $\{e_1, e_2, e_3, e_4\}$ and $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ be dual bases for U and U^*, respectively. Consider the block matrix $x = \left(\begin{smallmatrix} J & \mathbf{0} \\ \mathbf{0} & J \end{smallmatrix}\right)$ where $J = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. This corresponds to $e_1 \otimes \varphi_2 + e_3 \otimes \varphi_4$ in $U \otimes U^*$. Let $y = e_2 \otimes \varphi_3$. Then, using the definition of $[,]$ defined on $V \otimes V^*$, $[x,[x,y]] = -2e_1 \otimes \varphi_4$. In particular, x is not an extremal element of $\mathfrak{sl}(U)$ if $p \neq 2$.*

Example 3.3.12 *Let $U = F^3$ be a three dimensional subspace of V. Let $\{e_1, e_2, e_3\}$ and $\{\varphi_1, \varphi_2, \varphi_3\}$ be dual bases for U and U^*, respectively. Consider the matrix $x = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. This corresponds to $e_1 \otimes \varphi_2 + e_2 \otimes \varphi_3$ in $U \otimes U^*$.*

(i) *Let $y_1 = e_3 \otimes \varphi_2$. One can compute that $[x,[x,y_1]] = e_1 \otimes \varphi_2 - 2e_2 \otimes \varphi_3$ and this is a multiple of x if and only if F has characteristic 3. Therefore x is not an extremal element of $\mathfrak{sl}(U)$ if $p \neq 3$.*

(ii) *Let $y_2 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. One can compute that $[x,[x,y_2]]$ is the usual 3×3 elementary matrix $2E_{1,3}$. Therefore x is not an extremal element of $\mathfrak{gl}(U)$ if $p \neq 2$.*

Example 3.3.13 *Let $U = F^4$ be a four dimensional subspace of V. Let x be the matrix $\mathrm{diag}(J_\lambda, J_\mu)$ where J_λ is a Jordan block of size 3 with eigenvalue λ and J_μ is a Jordan block of size 1 with eigenvalue μ. Take $y = \mathrm{diag}(E_{31}, 0)$ where E_{31} is the usual 3×3 elementary matrix. Then $[x,[x,y]] = \mathrm{diag}(1,-2,1,0)$. It is clear that x is not quasiextremal in $\mathfrak{sl}(U)$.*

Example 3.3.14 *Let $U = F^4$ be a four dimensional subspace of V. Let x be the matrix* $\mathrm{diag}(J_\lambda, J_\mu)$ *where J_λ and J_μ are Jordan blocks of size 2 with eigenvalues λ and μ, respectively. Let $y = \mathrm{diag}(E_{21}, \mathbf{0})$ where E_{21} is the usual 2×2 elementary matrix. Then $[x,[x,y]] = \mathrm{diag}(-2E_{12}, \mathbf{0})$. It is clear that x is not quasiextremal in $\mathfrak{sl}(U)$.*

Example 3.3.15 *Let $U = F^4$ be a four dimensional subspace of V. Let x be the Jordan block J_λ of size 4 with eigenvalue λ. Let y be the usual 4×4 elementary matrix E_{41}. Then $[x,[x,y]]$ corresponds to the matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. It is clear that x is not a quasiextremal element in $\mathfrak{sl}(U)$.*

Example 3.3.16 *Let $U = F^2$ be a two dimensional subspace of V. Suppose that $x = diag(\lambda, \mu)$ and $\lambda \neq \mu$. Let y be the usual 2×2 elementary matrix E_{21}. Then $[x,[x,y]]$ is the matrix $\begin{pmatrix} 0 & 0 \\ (\mu-\lambda)^2 & 0 \end{pmatrix}$. It is clear that x is not a quasiextremal element in $\mathfrak{sl}(U)$.*

Example 3.3.17 *Let $U = F^3$ be a three dimensional subspace of V. Suppose that $x = diag(J_\lambda, \mu)$ where J_λ is the Jordan block of size 2 with eigenvalue λ and $\lambda \neq \mu$. Let y be the usual 3×3 elementary matrix E_{31}. Then $[x,[x,y]]$ is the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\lambda_2-\lambda_1)^2 & 0 & 0 \end{pmatrix}$. It is clear that x is not a quasiextremal element in $\mathfrak{sl}(U)$.*

Lemma 3.3.18 *Suppose that x is an element that has exactly one eigenvalue and has Jordan normal form the block diagonal matrix* $\mathrm{diag}(J_1, \ldots, J_s)$ *where each J_i is a Jordan block with eigenvalue λ. Furthermore, suppose that $(n,p) \neq (2,3)$. If x has rank at least 2 and is not a scalar, then x is not extremal. In particular, if x is extremal, then it is either has rank 1 or is a scalar element.*

Proof. For any subspace U of V, let x_U be the restriction of x to U. Suppose that $\lambda = 0$. If every Jordan block has size 1, then x is the zero matrix and this case is ruled out. Suppose that x has at least one Jordan block of size 2, of the form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and none of size greater than 2. If x has exactly one Jordan block of size 2, then x has rank one. Suppose that x has at least two Jordan Blocks of size 2 and let U be the subspace of V that is generated by the four basis elements corresponding to the two Jordan blocks of size 2. Then U is an x_U-invariant subspace. The matrix x_U is the element considered in Example 3.3.11 and thus is not extremal $\mathfrak{sl}(U)$. By Lemma 3.3.8, x is not extremal $\mathfrak{sl}(V)$.

Suppose that x has a Jordan block of size at least 3. Let U be the subspace generated by the last 3 vectors of the basis on this Jordan block. Then U is an x_U-invariant subspace. If $p = 3$, then $n \geq 3$ and thus U is a proper subspace of V and the matrix x_U is the element considered in Example 3.3.12 (ii) and thus not extremal in $\mathfrak{gl}(U)$. By Lemma 3.3.8, x is not extremal in $\mathfrak{sl}(V)$. If $p \neq 3$, then the matrix x_U is the element considered in the Example 3.3.12 (i) and thus not extremal in $\mathfrak{sl}(U)$. By Lemma 3.3.8, x is not extremal in $\mathfrak{sl}(V)$.

Suppose that $\lambda \neq 0$. If every Jordan block has size 1, then x is a scalar matrix. Suppose that x has at least one Jordan block of size at least 2. Let U be generated by the last two basis elements corresponding to the Jordan block of size at least 2. Then U is an x_U-invariant subspace. The matrix x_U is element considered in Example 3.3.10 and thus is not extremal in $\mathfrak{sl}(U)$. By Lemma 3.3.8, x is not extremal $\mathfrak{sl}(V)$.

We have exhausted all the possible Jordan forms of x and thus proved the result. □

We summarise the results in the following theorem.

Theorem 3.3.19 *Let x be an element of $\mathfrak{sl}(V)$ and $(n, p) \neq (2, 3)$. Then x is extremal if and only if x has rank 1 or is a scalar matrix.*

In a similar way we classify the quasiextremal elements. The next result is the first half of the classification.

Lemma 3.3.20 *Suppose that x is an extremal element. Then for any $(a, b) \in F^{2^*}$, $ax+bI$ is a quasiextremal element.*

Proof. Take any element y of $\mathfrak{sl}(V)$ and note that $[y, I] = 0$. Then $[ax + bI, [ax + bI, y]] = a^2[x, [x, y]] = a^2\lambda_y x = a\lambda_y(ax + bI) - ab\lambda_y I$ for some $\lambda_y \in F$. Thus $ax + bI$ is quasiextremal. □

Lemma 3.3.21 *Suppose that x is an arbitrary element of $\mathfrak{sl}(V)$ and that $(n, p) \neq (2, 3)$. If x is not the sum of a rank one and a scalar element, then x is not quasiextremal. In particular, if x is quasiextremal, then x is the sum of a rank one and a scalar element.*

Proof. Let x have Jordan normal form $\operatorname{diag}(J_1, \ldots, J_s)$ where each J_i is a Jordan block. For any subspace U of V, let x_U be the restriction of x to U. Suppose each J_i has size 1. If there are two J_λ and J_μ such that $\lambda \neq \mu$, then take U be the two dimensional subspace corresponding to the Jordan blocks J_λ and J_μ. Then $x_U = \operatorname{diag}(\lambda, \mu)$ is the element considered in Example 3.3.16 and thus is not quasiextremal in $\mathfrak{sl}(U)$. Otherwise, each J_i corresponds to the same eigenvalue and thus x is a scalar matrix. Suppose that exactly one J_i has size 2 and all other Jordan blocks have size 1. If there exists an Jordan block of size 1 corresponding to eigenvalue μ such $\lambda \neq \mu$ where J_λ is the Jordan block of size 2, then take U to be the three dimensional subspace corresponding to the Jordan blocks J_λ and μ. Then $x_U = \operatorname{diag}(J_\lambda, \mu)$ is the element considered in Example 3.3.17 and thus is not quasiextremal in $\mathfrak{sl}(U)$. If $\lambda = \mu$, then x is the sum of a rank 1 and scalar matrix. Suppose that exactly one J_i has size at least 3 and all other Jordan blocks have size 1. Let U correspond to the subspace generated by the last three elements of J_i and the basis element of any Jordan block of size 1. Then $x_U = \operatorname{diag}(J_\lambda, \mu)$ where J_λ and μ correspond to a Jordan blocks of size 3 and 1 with eigenvalues λ and μ, respectively. Then x_U is the element considered in Example 3.3.13 and thus is not quasiextremal in $\mathfrak{sl}(U)$. By Lemma 3.3.8, x is not quasiextremal in $\mathfrak{sl}(V)$. We may assume that at least two of the Jordan blocks have size at least 2 or x has exactly one Jordan block of size at least 3. Suppose that x has two Jordan blocks of size at least 2 and let U be the subspace generated by the last two basis elements of two such Jordan blocks. Then $x_U = \operatorname{diag}(J_\lambda, J_\mu)$ where

J_λ and J_μ correspond to Jordan blocks of size 2 with eigenvalues λ and μ, respectively. Then x_U is the element considered in Example 3.3.14 and thus is not quasiextremal in $\mathfrak{sl}(U)$. By Lemma 3.3.8, x is not quasiextremal in $\mathfrak{sl}(V)$. Finally, suppose that x is a Jordan block of size at least 3. If $p = 3$, then $n \geq 3$ and thus we can assume that x is a Jordan block of size at least 4. Let U be the subspace generated by the last four elements of the basis corresponding to the Jordan block. Then x_U is the elementary considered in Example 3.3.15 and thus is not quasiextremal in $\mathfrak{sl}(U)$. By Lemma 3.3.8, x is not quasiextremal in $\mathfrak{sl}(V)$. Suppose that $p \neq 3$. Take U to be the subspace generated by the last three elements of the basis corresponding to the x. Then x_U is the matrix $\left(\begin{smallmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{smallmatrix}\right)$. Set y to be the 3×3 elementary matrix E_{31}. From previous examples, it is easy to see that $[x_U, [x_U, y]] = \mathrm{diag}(1, -2, 1)$. Since $p \neq 3$, it is clear that x_U is not quasiextremal in $\mathfrak{sl}(U)$ because $[x_U, [x_U, y]]$ is not a scalar. In particular x is not quasiextremal in $\mathfrak{sl}(V)$. This exhausts all the cases and proves the result. $\square$

We summarise the results in the following theorem.

Theorem 3.3.22 *Let x be an element of $\mathfrak{sl}(V)$ and $(n, p) \neq (2, 3)$. Then x is quasiextremal if and only if x has the form $ay + bI$ where y is a rank 1 element of $\mathfrak{sl}(V)$ and $(a, b) \in F^{2^*}$.*

3.4 The extremal geometry of $\mathfrak{sl}(V)$

Let V be an $(n+1)$-dimensional vector space over the field F and let p be the characteristic of F . Throughout this section it is assumed that $(n, p) \neq (2, 3)$ and $p \neq 2$. The extremal geometry of a class of Lie algebras is introduced more generally in Chapter 4 but we describe the extremal geometry of $\mathfrak{sl}(V)$ by introducing points, collinearity between points and lines.

Definition 3.4.1 *The* extremal geometry $\mathcal{E}(\mathfrak{sl}(V))$ *of* $\mathfrak{sl}(V)$ *consists of* points Fx *where x is extremal in $\mathfrak{sl}(V)$. Two points Fx and Fy are* collinear *if $Fx \neq Fy$, $[x, y] = 0$ and*

$0 \neq \lambda x + \mu y$ is extremal for all $\lambda, \mu \in F$. The lines of $\mathcal{E}(\mathfrak{sl}(V))$ are the projective lines $\{F(\lambda x + \mu y) \mid \lambda, \mu \in F\}$ for pairs of collinear points Fx and Fy.

Lemma 3.4.2 *Let $\mathfrak{g} = \mathfrak{sl}(V)$ and let Z be the centre of $\mathfrak{g}$ and $\overline{\mathfrak{g}} = \mathfrak{g}/Z$. Consider the map $f : \mathfrak{g} \to \overline{\mathfrak{g}}$ by $x \mapsto x + Z$. An element $x \in L$ is quasiextremal in $\mathfrak{g}$ if and only if its image is extremal in $\overline{\mathfrak{g}}$. Furthermore, f bijectively maps the rank 1 extremal elements of $\mathfrak{g}$ onto the extremal elements of $\overline{\mathfrak{g}}$.*

Proof. If $Z = 0$, then $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ are the same Lie algebra and all quasiextremal elements in $\mathfrak{g}$ are extremal since the identity is not in $\mathfrak{g}$. We may suppose that $Z = \langle I \rangle$. By Theorem 3.3.22, an element of $\mathfrak{g}$ is quasiextremal if and only if it has the form $ax + bI$ for some scalars $a, b \in F$ where x is a rank 1 element and I is the identity matrix. An element $x + Z$ of $\overline{\mathfrak{g}}$ is extremal if and only if $[x + Z, [x + Z, y + Z]] = [x, [x, y]] + Z = ax + Z$ if and only if $[x, [x, y]] = ax + bI$ for some $b \in F$ and for all $y \in \mathfrak{g}$. Additionally, $[x + cI, [x + cI, y]] = a(x + cI) + (b - ac)I$ for any $c \in F$. This means that $x + Z$ is extremal in $\overline{\mathfrak{g}}$ if and only if any preimage of $x + Z$ in $\mathfrak{g}$ is quasiextremal.

If $x + Z$ is an extremal element, then x can be chosen as a rank 1 extremal element in $\mathfrak{g}$. Suppose that $x + Z = y + Z$ for rank 1 extremal elements x and y. Then $x - y = \lambda I$ for a scalar $\lambda \in F$. Observe that $\mathrm{rk}(\lambda I)$ is $n + 1$ if $\lambda \neq 0$ and 0 if $\lambda = 0$. Then $\mathrm{rk}(\lambda I) = \mathrm{rk}(x - y) \leq \mathrm{rk}(x) + \mathrm{rk}(y) = 2$. We already assume that $n \geq 2$ and thus $\lambda = 0$. In particular, $x = y$ and so f bijectively maps the extremal rank 1 elements of $\mathfrak{g}$ onto the extremal elements of $\overline{\mathfrak{g}}$. □

We define a point-line space $\mathcal{E}_1$ on the set of rank 1 elements of $\mathfrak{sl}(V)$ viewed as the trace zero elements of $V \otimes V^*$. We define the points $\mathcal{P}(\mathcal{E}_1)$ to be the set $\{F(v \otimes \varphi) \mid v \in V \setminus \{0\}, \varphi \in V^* \setminus \{0\}\}$. Two points $F(v \otimes \varphi)$ and $F(u \otimes \psi)$ of $\mathcal{E}_1$ are defined to be *collinear* if and only if $[v \otimes \varphi, u \otimes \psi] = 0$ and $\lambda(v \otimes \varphi) + \mu(u \otimes \psi)$ is a rank one element for all $(\lambda, \mu) \in F^{2*}$. The first condition holds if and only if $\varphi(u)v \otimes \psi - \psi(v)u \otimes \varphi = 0$, if and only if, $\varphi(u) = 0$ and $\psi(v) = 0$, if and only if $u \otimes \varphi$ and $v \otimes \psi$ are rank one elements. The second condition holds if and only if $\lambda v = \mu u$ or $\lambda\varphi = \mu\psi$, if and only if $Fv = Fu$ or $F\varphi = F\psi$. The lines $\mathcal{F}(\mathcal{E}_1)$ are defined as different types of lines, namely $\{(\lambda v + \mu u) \otimes \varphi \mid \lambda, \mu \in F\}$ and $\{v \otimes (\lambda\varphi + \mu\psi) \mid \lambda, \mu \in F\}$. Note that $\mathcal{E}_1$ is a subspace of the extremal geometry $\mathcal{E}(\mathfrak{sl}(V))$.

Suppose $\mathfrak{g} = \mathfrak{sl}(V)$, $Z = \langle I \rangle$ is the centre of $\mathfrak{sl}(V)$ and $\overline{\mathfrak{g}} = \mathfrak{g}/Z$. Then we can define a similar geometry $\mathcal{E}_2$ whose points $\mathcal{P}(\mathcal{E}_2)$ consist of $Fx + Z$ where x is a rank 1 element of $\mathfrak{g}$. Two points $Fx + Z$ and $Fy + Z$ are collinear if and only if $[x + Z, y + Z] = Z$ and $F(\lambda x + \mu y) + Z \in \mathcal{P}(\mathcal{E}_2)$. The first condition holds if and only if $[x, y] \in Z$. The lines $\mathcal{F}(\mathcal{E}_2)$ are defined as $\{F(\lambda x + \mu y) + Z \mid (\lambda, \mu) \in F^{2*}\}$. By Lemma 3.4.2, $\mathcal{E}_2$ is the extremal geometry $\mathcal{E}(\overline{\mathfrak{g}})$.

Lemma 3.4.3 *The point-line space $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic.*

Proof. Let $f : \mathfrak{g} \to \overline{\mathfrak{g}}$ be given as $x \mapsto x + Z$. By Lemma 3.4.2, f induces a bijection between the points of $\mathcal{E}_1$ and $\mathcal{E}_2$. If $\{F(\lambda x + \mu y) \mid \lambda, \mu \in F\}$ is a line in $\mathcal{E}_1$, then $[x, y] = 0 \in Z$. Thus $\{F(\lambda x + \mu y) + Z \mid \lambda, \mu \in F\}$ since all $\lambda x + \mu y$ are rank 1 in $\mathfrak{sl}(V)$. Suppose that $\{F(\lambda x + \mu y) + Z \mid \lambda, \mu \in F\}$ is a line in $\mathcal{E}_2$. Then $\lambda x + \mu y$ is rank 1 and $[x, y] \in Z$. But $\mathrm{rk}([x, y]) = \mathrm{rk}(xy - yx) \leq \mathrm{rk}(xy) + \mathrm{rk}(yx) = 2$. In particular, $[x, y] = 0$ because every nonzero element of Z has rank $n+1 \geq 3$. Therefore $\{F(\lambda x+\mu y) \mid \lambda, \mu \in F\}$ is a line in $\mathcal{E}_1$. Thus $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic. $\square$

Theorem 3.4.4 *The point line space $\mathcal{E}_1$ of $\mathfrak{g}$ and $\mathcal{E}_2$ of $\mathfrak{g}/Z$ are isomorphic to the root shadow space of a building of type A_n.*

Proof. We construct the standard realisation of a root shadow space of type A_n. Let V be an $(n+1)$-dimensional vector space over a field F. Let the set of points $\mathcal{P}$ be the set of pairs $(\langle v \rangle, W)$ where $\langle v \rangle$ is a 1-space contained in the hyperplane W. Two points $(\langle v \rangle, W)$ and $(\langle u \rangle, U)$ are collinear if and only if $\langle v \rangle = \langle w \rangle$ or $W = U$. The lines $\mathcal{F}$ are of two types, namely, $\{(\langle \lambda v + \mu w\rangle, W) \mid (\lambda, \mu) \in F^{2^*}\}$ and $\{(\langle v\rangle, \ker(\lambda\varphi + \mu\psi)) \mid (\lambda, \mu) \in F^{2^*}\}$ where $W = \ker(\varphi)$ and $U = \ker(\psi)$ for linear maps $\varphi, \psi : V \to F$. Then it is clear that the map $F(v \otimes \varphi) \mapsto (\langle v\rangle, \ker(\varphi))$ is a bijection between points that preserves lines. In particular, $\mathcal{E}_1$ is a root shadow space of type A_n. By Lemma 3.4.3, $\mathcal{E}_2$ is a root shadow space of type A_n. □

We state an important proposition describing the action of the Chevalley group of type $A_n(F)$ on the set of rank one elements of $\mathfrak{sl}(V)$.

Proposition 3.4.5 *The group* PSL(V) *acts transitively on the set* $\{F(v \otimes \varphi) \mid 0 \neq v \in V, \neq 0\varphi \in V^*, \varphi(v) = 0\}$. *In particular, it acts transitively on the point set of* $\mathcal{E}_1$ *and* $\mathcal{E}_2$.

Proof. Let $v \otimes \varphi$ and $w \otimes \psi$ be two arbitrary rank one elements with trace zero. Firstly observe that a map $g \in$ GL(V) induces an action on the 1-spaces of V^*. In particular, we define $\langle\varphi\rangle^g = \langle\psi\rangle$ if and only if $\ker(\varphi)^g = \ker(\psi)$. Indeed, if $\ker(\varphi)^g = \ker(\psi) = \ker(\psi')$, then ψ and ψ' are scalar multiples of one another. This induces an action of g on $\mathcal{E}$, namely $F(v \otimes \varphi)^g = F(v^g \otimes \varphi^g)$.

Observe that $\ker(\varphi)$ and $\ker(\psi)$ are hyperplanes and there exists $u_1, u_2 \in V$ such that $V = \ker(\varphi) \oplus \langle u_1\rangle = \ker(\psi) \oplus \langle u_2\rangle$. By assumption $\varphi(v) = 0$ and $\psi(w) = 0$. Extend a basis $\{v, v_2, \ldots, v_n\}$ of $\ker(\varphi)$ to a basis $\{v, v_2, \ldots, v_n, u_1\}$ of V and a basis $\{w, w_2, \ldots, w_n\}$ for $\ker(\psi)$ to a basis $\{w, w_2, \ldots, w_n, u_2\}$ of V. There exists $g \in \mathrm{GL}(V)$ induced by $v \mapsto w$, $v_i \mapsto w_i$ and $u_1 \mapsto u_2$. Then g has matrix representation (a_{ij}). Suppose that the determinant of g is λ. Then modify g by mapping $u_1 \mapsto 1/\lambda u_2$. Then the new matrix representation of g is $(a_{ij})\mathrm{diag}(1, \ldots, 1, 1/\lambda)$ and has determinant 1. In particular, g is an element of $\mathrm{SL}(V)$ that maps $\ker(\varphi)$ to $\ker(\psi)$ and v to w. Then for any $\lambda \in F$, there exists $\mu \in F$ such that $(\lambda(v \otimes \varphi))^g = (v \otimes (\lambda\varphi))^g = w \otimes \mu\psi$ and hence $F(v \otimes \varphi)^g = F(w \otimes \psi)$. The group $\mathrm{PSL}(V)$ is defined to be $\mathrm{SL}(V)/Z(\mathrm{SL}(V))$ and $Z(\mathrm{SL}(V)) = \{\lambda I_n \mid \lambda \in F\}$ is the kernel of the action of $\mathrm{SL}(V)$ on $\mathcal{E}$. Thus $\mathrm{PSL}(V)$ acts transitively on $\mathcal{E}$. □

Chapter 4

Root Filtration Spaces

Root filtration spaces were introduced by Cohen and Ivanyos in [8]. They were inspired by the study of long root geometries, which are examples of root filtration spaces. In this chapter we survey the standard results in [8, 9] and provide some interesting examples of root filtration spaces. The most important example concerning this text is the extremal geometry of a Lie algebra.

4.1 Point-line spaces

We use this section to familiarise ourselves with the standard definitions and properties of point-line spaces.

Definition 4.1.1 *A* point line space *(or simply a* space*) is a pair* $(\mathcal{P}, \mathcal{L})$ *where* $\mathcal{P}$ *is a set whose members are* points *and* $\mathcal{L}$ *is a collection of subset of* $\mathcal{P}$ *whose members are* lines.

A space is called a *partial linear* if every pair of distinct points lie on at most one common line, a *singular space* if every two points lie on a common line, and a *linear space* if it is both a singular and a partial linear space.

A subset $\mathcal{X}$ of $\mathcal{P}$ is a *subspace* if it contains all the points of a line l of $\mathcal{L}$ whenever $\mathcal{X} \cap l$ contains at least two points. Every point p is a subspace and we call it the *trivial* *subspace*. Every line l is a subspace of $\mathcal{P}$. A subspace $\mathcal{H}$ of $\mathcal{P}$ is a *hyperplane* if every line of $\mathcal{P}$ has a nonempty intersection with $\mathcal{H}$, in particular, every line is either contained in $\mathcal{H}$ or intersects $\mathcal{H}$ in a singleton. For any subset $\mathcal{X}$, we denote $\langle \mathcal{X} \rangle$ to be the intersection of all subspaces that contain $\mathcal{X}$, that is, the smallest subspace of $\mathcal{P}$ that contains $\mathcal{X}$. For any collection of subsets $\{\mathcal{X}_i\}_{i=1}^m$, $\langle \mathcal{X}_1, \ldots, \mathcal{X}_m \rangle = \cap_{i=1}^m \langle \mathcal{X}_i \rangle$. If x and y lie in a unique line, then we denote the line by xy.

The *rank* of a linear space $\mathcal{X}$ is the length of a maximal chain of proper nontrivial subspaces and we denote it by $\mathrm{rk}(\mathcal{X})$. The rank of a trivial subspace is 0 and the rank of a line is 1. The *singular rank* of a space is the supremum of the ranks of all maximal singular spaces.

Definition 4.1.2 *A linear space $(\mathcal{P}, \mathcal{L})$ is called a projective geometry if*

(PG1) *every line contains at least three points, and*

(PG2) *if, whenever x, y, and z are noncollinear points and a line l meets xy and xz in distinct points, we have that l meets yz.*

The second axiom (PG2) is called *Pasch's axiom* and it is equivalent to the axiom

(PG2)' for distinct points a, b, c, d, then ab and cd meet in a point if and only if ac and bd meet in a point.

We state a standard result whose proof can be found in [16].

Lemma 4.1.3 *Let $\mathcal{P}$ be a projective geometry. Let p be a point and $\mathcal{X}$ be a subspace not containing p. Then the subspace $\langle \mathcal{X}, p \rangle$ consists of the set of lines that pass through p and a point of $\mathcal{X}$.* □

Lemma 4.1.4 *Let $\mathcal{P}$ be a projective geometry, p be a point and $\mathcal{X}$ be a subspace not containing p. Then $\mathrm{rk}(\langle \mathcal{X}, p \rangle) \geq \mathrm{rk}(\mathcal{X}) + 1$. In particular, any subspace of $\mathcal{P}$ which has the same rank as $\mathcal{P}$ must coincide with $\mathcal{P}$.*

Proof. Let $\mathcal{X}_m = \mathcal{X} \supset \mathcal{X}_{m-1} \supset \ldots \supset \mathcal{X}_1$ be a maximal chain of proper subspaces. The rank of $\mathcal{X}$ is m. Then $\langle \mathcal{X}, p \rangle \supset \mathcal{X} = \mathcal{X}_m \supset \ldots \supset X_1$ is a chain of length $m+1$. We do not, however, know if this chain is maximal. Thus $\mathrm{rk}(\langle \mathcal{X}, p \rangle) \geq m + 1 = \mathrm{rk}(\mathcal{X}) + 1$. $\square$

4.2 Root filtration spaces

We introduce some notation. Let $\mathcal{R}$ be a relation on $\mathcal{P}$, that is, $\mathcal{R}$ is subset of $\mathcal{P} \times \mathcal{P}$. If $x \in \mathcal{R}$, then $\mathcal{R}(x)$ denotes the set of points $y \in \mathcal{P}$ such that $(x, y) \in \mathcal{R}$. If $x, y, \ldots, z$ are points in $\mathcal{P}$, then $\mathcal{R}(x, y, \ldots, z) = \mathcal{R}(x) \cap \mathcal{R}(y) \cap \ldots \cap \mathcal{R}(z)$.

Definition 4.2.1 *Let $(\mathcal{E}, \mathcal{F})$ be a partial linear space and $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ be a set of disjoint symmetric relations on $\mathcal{E}$. Let $\mathcal{E}_{\leq j} = \cup_{i=-2}^{j} \mathcal{E}_i$. We say that $(\mathcal{E}, \mathcal{F})$ is a* root filtration space with filtration $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ *if the following hold.*

(A) *$\mathcal{E}_{-2}$ is equality between points in $\mathcal{E}$.*

(B) *$\mathcal{E}_{-1}$ is collinearity between distinct points in $\mathcal{E}$.*

(C) *There exists a map $\mathcal{E}_1 \to \mathcal{E}$, denoted by $(x, y) \mapsto [x, y]$, such that $[x, y] \in \mathcal{E}_{\leq i+j}(z)$ whenever $z \in \mathcal{E}_i(x) \cap \mathcal{E}_j(y)$.*

(D) *If $(x, y) \in \mathcal{E}_2$, then $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y)$ is empty.*

(E) *For each $x \in \mathcal{E}$, $\mathcal{E}_{\leq -1}(x)$ and $\mathcal{E}_{\leq 0}(x)$ are subspaces of $(\mathcal{E}, \mathcal{F})$.*

(F) *For each $x \in \mathcal{E}$, $\mathcal{E}_{\leq 1}(x)$ is a hyperplane of $(\mathcal{E}, \mathcal{F})$.*

Note that for each $i \in \{-2, \ldots, 2\}$ and $x \in \mathcal{E}$, $\mathcal{E}_{\leq i}(x)$ is a subspace since $\mathcal{E}_{-2}(x) = \{x\}$ is a trivial subspace. The *collinearity graph* of $(\mathcal{E}, \mathcal{F})$ is the graph whose vertices are the points in $\mathcal{E}$ and two vertices are joined by an edge if they are contained in a common line in $\mathcal{F}$. The collinearity graph of $\mathcal{E}$ is denoted $(\mathcal{E}, \mathcal{E}_{-1})$. We say that two points are *neighbours* if there is an edge in $(\mathcal{E}, \mathcal{E}_{-1})$ joining them.

We say that a root filtration space $(\mathcal{E}, \mathcal{F})$ is *nondegenerate* if the following two conditions hold.

(G) For each $x \in \mathcal{E}$, $\mathcal{E}_2(x)$ is nonempty.

(H) The collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ is connected.

Example 4.2.2 *Let G be a group and $\mathcal{E}$ be a set of nontrivial abelian subgroups of G such that no such subgroup is contained in any other such that $G = \langle \mathcal{E} \rangle$, $\mathcal{E}^g = \mathcal{E}$ for all $g \in G$ and for each pair $a, b \in \mathcal{E}$ with $X = \langle a, b \rangle$ one of the following holds*

(-2) *$a = b$ and $X = a = b$.*

(-1) *$[a, b] = 0$ and $a \neq b$ and $X \setminus \{1\}$ is partitioned by $\{c \setminus \{1\}\}_{c \in \mathcal{E}, c \leq X}$. In this case, we call the* line *ab the set of subgroups $c \in \mathcal{E}$ with $c \leq ab$.*

(0) *$[a, b] = 0$ and $X \setminus \{1\}$ is not partitioned by $\{c \setminus \{1\}\}_{c \in \mathcal{E}, c \leq X}$.*

(1) *$[a_0, b] = [a, b_0] = [a, b] \leq Z(X)$ for all $a_0 \in a$ and for all $b_0 \in b$ and $[a, b] \in \mathcal{E}$.*

(2) *For each nontrivial $a_0 \in a$ there exists a nontrivial $b_0 \in b$ such that $a^{b_0} = b^{a_0}$ and similarly when a and b are interchanged.*

Such a pairing $(G, \mathcal{E})$ is called a set of abstract root subgroups*. Let $\mathcal{F}$ denote the set of lines. Suppose $\mathcal{E}_{-2} \cup \mathcal{E}_{-1} \cup \mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are nonempty, $(\mathcal{E}, \mathcal{E}_2)$ is connected and the largest solvable normal subgroup of G is trivial. Then, by Lemma 13 in [8], $(\mathcal{E}, \mathcal{F})$ is a nondegenerate root filtration.*

Any nondegenerate root filtration space with finite singular rank can be identified with a root shadow space. This important result is stated and proven by Cohen and Ivanyos in [9] as follows.

Theorem 4.2.3 *A nondegenerate root filtration space with finite singular rank is isomorphic to the root shadow space of type $A_{n,\{1,n\}}$, $BC_{n,2}$, $D_{n,2}$, $E_{6,2}$, $E_{7,1}$, $E_{8,8}$, $F_{4,1}$, or $G_{2,2}$.*

Example 4.2.4 *Let Φ be a simply laced irreducible root system of type X_n. Set $\mathcal{E} = \Phi$ and for each $i \in \{-2,-1,0,1,2\}$ define $\mathcal{E}_i = \{(\alpha,\beta) \mid \langle \alpha,\beta \rangle = -i\}$ where $\langle , \rangle$ is the standard inner product on Euclidean space. We define collinearity by $\mathcal{E}_{-1}$, that is, α and β are collinear if $\langle \alpha,\beta \rangle = 1$. The line containing α and β is simply $\{\alpha,\beta\}$. Note that $\alpha \in \mathcal{E}_{\leq i}(\beta)$ if and only if $\langle \alpha,\beta \rangle \geq -i$. The properties (A) and (B) are clearly satisfied. For (C), if $(\alpha,\beta) \in \mathcal{E}_1$, then $\alpha+\beta \in \Phi$ and thus we define a map from $\mathcal{E}_1$ to $\mathcal{E}$ by $(\alpha,\beta) \mapsto \alpha+\beta$. Suppose $\gamma \in \mathcal{E}_{\leq i}(\alpha) \cap \mathcal{E}_{\leq j}(\beta)$. We have that $\langle \alpha,\gamma \rangle \geq -i$ and $\langle \beta,\gamma \rangle \geq -j$ and thus $\langle \alpha+\beta,\gamma \rangle \geq -(i+j)$. In particular, $\alpha+\beta \in \mathcal{E}_{\leq i+j}(\gamma)$. If $(\alpha,\beta) \in \mathcal{E}_2$, then $\langle \alpha,\beta \rangle = -2$ and $\beta = -\alpha$. For any $\gamma \in \mathcal{E}_{\leq 0}(\alpha) \cap \mathcal{E}_{\leq -1}(-\alpha)$, we have that $\langle \gamma,\alpha \rangle \geq 0$ and $\langle \gamma,-\alpha \rangle \geq 1$. The second inequality can be written as $\langle \gamma,\alpha \rangle \leq -1$ and thus γ does not exist. In particular $\mathcal{E}_{\leq 0}(\alpha) \cap \mathcal{E}_{\leq -1}(\beta)$ is empty and this proves that property (D) holds. Property (E) holds trivially since every line is thin, that is, contains precisely two points. For $\alpha \in \mathcal{E}$, $\mathcal{E}_{\leq 1}(\alpha) = \mathcal{E} \setminus \{-\alpha\}$. Thus it follows that any line of $\mathcal{E}$ is either entirely contained in $\mathcal{E}_{\leq 1}(\alpha)$ or intersects it in a unique point. Thus $\mathcal{E}_{\leq 1}(\alpha)$ is a hyperplane in $\mathcal{E}$. Note that this root filtration space is isomorphic to the root shadow space of the thin building of type X_n.*

We can describe nondegenerate root filtration space in terms of points and lines by forgetting the five relations stated above. Before doing that we state a few technical results that can be found in [8].

Lemma 4.2.5 *Let $(\mathcal{E},\mathcal{F})$ be a root filtration space. If $(x,y) \in \mathcal{E}_1$, then $[x,y]$ is the unique common neighbour of x and y.*

Proof. For (i), note $x \in \mathcal{E}_{-2}(x) \cap \mathcal{E}_1(y)$. By (C), $[x, y] \in \mathcal{E}_{\leq -1}(x)$. By symmetry, $[x, y] \in \mathcal{E}_{\leq -1}(y)$. It is clear that $[x, y] \notin \{x, y\}$, thus $[x, y] \in \mathcal{E}_{-1}(x) \cap \mathcal{E}_{-1}(y)$. If $z \in \mathcal{E}_{-1}(x) \cap \mathcal{E}_{-1}(y)$, then by (C) we have $[x, y] \in \mathcal{E}_{-2}(z) = z$. Therefore $[x, y]$ is the unique common neighbour of x and y. □

Lemma 4.2.6 *Let $(\mathcal{E}, \mathcal{F})$ be a nondegenerate root filtration space.*

(i) *If $(\mathcal{E}, \mathcal{F})$ is thick and $(\mathcal{E}, \mathcal{E}_{-1})$ is connected, then $(\mathcal{E}, \mathcal{E}_2)$ is connected and any two points $(x, y) \in \mathcal{E}_2$ are distance 2 apart in $(\mathcal{E}, \mathcal{E}_{-1})$.*

(ii) *If $(\mathcal{E}, \mathcal{E}_2)$ is connected and $\mathcal{F}$ is nonempty, then $(\mathcal{E}, \mathcal{E}_{-1})$ is connected.* □

Lemma 4.2.7 *Suppose $(\mathcal{E}, \mathcal{F})$ is nondegenerate. Let $(x, y) \in \mathcal{E}_0$ and $u \in \mathcal{E}_{-1}(x, y)$. There exists $v \in \mathcal{E}_{-1}(x, y)$ that is not equal nor collinear to u. In particular, x and y have at least two common neighbours.* □

Lemma 4.2.8 *Let $(\mathcal{E}, \mathcal{F})$ be a nondegenerate root filtration space. Then its defining relations can be characterised by the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ in the following.*

(-2) *$(x, y) \in \mathcal{E}_{-2}$ if and only if $x = y$.*

(-1) *$(x, y) \in \mathcal{E}_{-1}$ if and only if x and y are distinct collinear points.*

(0) *$(x, y) \in \mathcal{E}_0$ if and only if x and y have at least two common neighbours.*

(1) *$(x, y) \in \mathcal{E}_1$ if and only if x and y have a unique common neighbour.*

(2) *$(x, y) \in \mathcal{E}_2$ if and only if x and y have no common neighbours.*

Furthermore, pairs of points in $\mathcal{E}_{-2}$, $\mathcal{E}_{-1}$, $\mathcal{E}_0 \cup \mathcal{E}_1$, and $\mathcal{E}_2$ have a distance between them in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ of 0, 1, 2 and 3, respectively.

Proof. It is clear that the five relations are disjoint symmetric relations on $\mathcal{E}$. Relations $\mathcal{E}_{-2}$ and $\mathcal{E}_{-1}$ follow from the definition of a root filtration space. Let $(x, y) \in \mathcal{E}_0$. Then Lemma 4.2.7 tells us that x and y have at least two common neighbours. Let $(x, y) \in \mathcal{E}_1$. Lemma 4.2.5 tells us that x and y have a unique common neighbour denoted by $[x, y]$. Let $(x, y) \in \mathcal{E}_2$. By property (D), the intersection of $\mathcal{E}_{-1}(x)$ and $\mathcal{E}_{-1}(y)$ is empty and thus x and y have no common neighbours.

The final assertion is obvious for $\mathcal{E}_{-2}$, $\mathcal{E}_{-1}$, and $\mathcal{E}_0 \cup \mathcal{E}_1$. The last assertion for $\mathcal{E}_2$ is found in Lemma 4.2.6 (i). □

4.3 Lie algebras and extremal elements

In this section we show that every simple Lie algebra has an associated root filtration space whose points are one dimensional spaces generated by extremal elements. Informally, an element x is an extremal element if $[x, [x, L]] \subseteq Fx$ where L is the Lie algebra of a field F. Extremal elements are explored in [11] and equivalent to the elements that generate 1-dimensional inner ideals of L as defined in [1]. We have seen extremal elements in Chapter 3 but here they are formally defined for all fields.

Definition 4.3.1 *Let L be a Lie algebra over a field F. An element $x \in L$ is said to be* extremal *if there is a map $g_x : L \to F$ such that*

$$[x, [x, y]] = 2g_x(y)x, \tag{4.3.2}$$

$$[[x, y], [x, z]] = g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z] \tag{4.3.3}$$

and

$$[x, [y, [x, z]]] = g_x([y, z])x - g_x(z)[x, y] - g_x(y)[x, z]. \tag{4.3.4}$$

The Jacobi identity gives:

$$[[x,y],[x,z]] = [x,[y,[x,z]]] - [y,[x,[x,z]]] \tag{4.3.5}$$

Proposition 4.3.6 *If the F has odd characteristic, then (4.3.3) and (4.3.4) follow from (4.3.2).*

Proof. Let $x \in L$ be extremal and y and z be any elements of L. Suppose that (4.3.2) holds. Using (4.3.5) and (4.3.2) we get $[[x,y],[x,z]] = [x,[y,[x,z]]]+2g_x(z)[x,y]$. Therefore it suffices to show that

$$\begin{aligned} [x,[y,[x,z]]] &= g_x([y,z])x - g_x(z)[x,y] - g_x(y)[x,z] \\ &= \tfrac{1}{2}([x,[x,[y,z]]] + [y,[x,[x,z]]] + [z,[x,[x,y]]]). \end{aligned}$$

Using the Jacobi identity on the last two terms of the right hand side of the equation gives

$$[y,[x,[x,z]]] + [z,[x,[x,y]]] = [x,[y,[x,z]]] + [x,[z,[x,y]]].$$

Therefore it suffices to show that

$$2[x,[y,[x,z]]] = [x,[x,[y,z]]] + [x,[y,[x,z]]] + [x,[z,[x,y]]],$$

and this is clear since $[x,[x,[y,z]]] + [x,[z,[x,y]]] = [x,[y,[x,z]]$ as required. Therefore (4.3.3) holds. Equation (4.3.4) follows from (4.3.2) and (4.3.3). To see this note that (4.3.5)

$$\begin{aligned} [x,[y,[x,z]]] &= [[x,y],[x,z]] + [y,[x,[x,z]] \\ &= g_x([y,z])x - g_x(z)[x,y] - g_x(y)[x,z] \end{aligned}$$

as required. ☐

Let E be the set of nonzero extremal elements of L. We state two properties whose proofs can be found in [8].

Proposition 4.3.7 *Let $x \in E$ and $y \in L$. The follow assertions hold.*

(i) *g_x is a uniquely determine linear map.*

(ii) *If $y \in L$, then $g_x(y) = g_y(x)$ and $g_x([y,z]) = -g_y([x,z])$*

Lemma 4.3.8 *Let x, y and z be extremal elements in L. Then*

$$[[x,y],[y,[x,z]]] = g_y(x)[y,[x,z]] + g_x([y,z])[y,x] - g_x(z)g_y(x)y$$

.

Proof. We first rewrite $[[x,y],[y,[x,z]]]$ using the Jacobi identity as $[[[x,y],y],[x,z]] + [y,[[x,y],[x,z]]]$. Using (4.3.2) and (4.3.3) it can be written as

$$2g_y(x)[y,[x,z]] + [y, g_x([y,z])x + g_x(z)[x,y] - g_x(y)[x,z]]$$

Expanding this gives

$$2g_y(x)[y,[x,z]] + g_x([y,z])[y,x] + g_x(z)[y,[x,y]] - g_x(y)[y,[x,z]]$$

and applying Proposition 4.3.7 (ii) allows us to simply it to

$$g_y(x)[y,[x,z]] + g_x([y,z])[y,x] - g_x(z)g_y(x)y$$

as required. ☐

We say that an element $x \in L$ is a *sandwich* if $\mathrm{ad}_x^2 = 0$ and $\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x = 0$ for all $y \in L$. Thus, if x is a sandwich, then (4.3.2) follows from $\mathrm{ad}_x^2 = 0$, (4.3.4) follows from $\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x = 0$, and (4.3.3) follows from (4.3.2) and (4.3.4) with $g_x = 0$. If F has odd characteristic and $\mathrm{ad}_x^2 = 0$, then x is a sandwich. By convention we set g_x to be the zero map whenever x is a sandwich. This follows from Proposition 4.3.6.

We define five symmetric binary relations E_i for $i = -2, -1, 0, 1, 2$ on E as follows:

(-2) $(x, y) \in E_{-2}$ if and only if x and y are linearly dependent.

(-1) $(x, y) \in E_{-1}$ if and only if x and y are linearly independent, $[x, y] = 0$ and $\lambda x + \mu y \in E$ for all $(\lambda, \mu) \in F^{2*}$.

(0) $(x, y) \in E_0$ if and only if $[x, y] = 0$ and (x, y) is not in $E_{-2} \cup E_{-1}$.

(1) $(x, y) \in E_1$ if and only if $g_x(y) = 0$ and $[x, y] \neq 0$.

(2) $(x, y) \in E_2$ if and only if $g_x(y) \neq 0$ and $[x, y] \neq 0$.

Remark 4.3.9 *If $(x, y) \in E_{-1}$, then the corresponding linear functional is $g_{\lambda x + \mu y} := \lambda g_x + \mu g_y$. Furthermore, if L is generated by E containing no sandwiches, then E_{-1} has another characterisation, namely, $(x, y) \in E_{-1}$ if x and y are linearly independent, $[x, y] = 0$ and for every $z \in L$, $[x, [y, z]] = g_y(z)x + g_x(z)y$. The proof of this equivalence is Lemma 24 and Lemma 27 of [8].*

Lemma 4.3.10 *Suppose that F does not have even characteristic. If $(x, y) \in E_2$, then $[x, y]$ is not an extremal element.*

Proof. We know that $[x, y] \neq 0$ and $g_x(y) \neq 0$. If $[x, y]$ is extremal, then $[[x, y], [[x, y], z]] = \lambda_z [x, y]$ for all $z \in L$. Choose $z = x$, then $[[x, y], [[x, y], x]] = 4 g_x(y)^2 x \neq 0$ and not a multiple of $[x, y]$. Therefore $[x, y]$ is not extremal. □

Let $\mathcal{E}$ be the projective points spanned by the extremal elements of L. That is, $\mathcal{E} = \{Fx \mid x \in E\}$. Let $\mathcal{F}$ be the set of projective lines $F\langle x, y\rangle$ for $(x, y) \in E_{-1}$. We call $(\mathcal{E}, \mathcal{F})$ the *extremal geometry* of the Lie algebra L. By the definition of E_{-1}, the unique line in $\mathcal{F}$ that contains the two incident points Fx and Fy is $\{F(\lambda x + \mu y) \mid (\lambda, \mu) \in F^{2^*}\}$. The symmetric relations $\{\mathcal{E}_i\}_{i=-2}^{2}$ correspond to $\{E_i\}_{i=-2}^{2}$ in the natural way, namely, $(Fx, Fy) \in \mathcal{E}_i$ if and only if $(x, y) \in E_i$ for each i. In particular, $(\mathcal{E}, \mathcal{F})$ is a partial linear space and $\{\mathcal{E}_i\}_{i=-2}^{2}$ are five disjoint symmetric relations on $\mathcal{E}$ where $\mathcal{E}_{-2}$ is equality and $\mathcal{E}_{-1}$ is collinearity. The following result is Theorem 28 in [8]. It is followed by a corollary.

Theorem 4.3.11 *If L does not contain sandwiches, then $(\mathcal{E}, \mathcal{F})$ is a root filtration space. Let $\mathcal{B}_i$ be the connected components of $(\mathcal{E}, \mathcal{E}_2)$ and let L_i be the Lie subalgebra generated by $\mathcal{B}_i$ of L. Then either each $\mathcal{B}_i$ is a nondegenerate root filtration space or a root filtration space without lines, L is the direct sum of the Lie subalgebras L_i and $[L_i, L_j] = 0$ whenever i and j are distinct. In particular, L_i is an ideal of L.*

Corollary 4.3.12 *Let L be a simple Lie algebra generated by its extremal elements E and contain no sandwiches. Then $(\mathcal{E}, \mathcal{F})$ is a nondegenerate root filtration space with thick lines or a root filtration space without lines.*

Proof. The simple Lie algebra L has no ideals except for $\{0\}$ and L. By Theorem 4.3.11, $(\mathcal{E}, \mathcal{E}_2)$ is connected. Suppose that $\mathcal{F}$ is nonempty. Using these two facts, Lemma 4.2.6 (ii) implies that $(\mathcal{E}, \mathcal{E}_{-1})$ is connected. Hence by definition, $(\mathcal{E}, \mathcal{F})$ is a nondegenerate root filtration space. Suppose Fx and Fy are collinear. By the definition of E_{-1}, the entire set of points

$$\{F(\lambda x + \mu y) \mid (\lambda, \mu) \in F^{2^*}\}.$$

lies in the same line as Fx and Fy. In particular, the line containing Fx and Fy also contains $F(x + y)$. Therefore the lines in $\mathcal{F}$ are thick. □

Lemma 4.3.13 *The space $(\mathcal{E}, \mathcal{F})$ has finite singular rank if L is finite dimensional.*

Proof. It suffices to show that the rank of any singular subspace of $\mathcal{E}$ is finite. Lemma 4.4.2 tells us that L is linearly spanned by a subset of the extremal elements. Let $\mathcal{X} = \{Fx \mid x \in E' \subseteq E\}$ be a singular subspace of $\mathcal{E}$. Let $\{x_1, \ldots, x_k\}$ be a minimal set of extremal elements such that E' is contained in its span. Suppose $\mathcal{X}'$ is a subspace of $\mathcal{X}$ that contains $\{Fx_1, \ldots, Fx_k\}$. Note that $\mathcal{X}'$ is also a singular subspace of $\mathcal{E}$. We show by induction that $\mathcal{X} = \mathcal{X}'$. For $x \in E'$, write $x = \lambda_1 x_1 + \ldots + \lambda_k x_k$ and $\text{supp}(x) = \mid \{\lambda_i \mid \lambda_i \neq 0\} \mid$. We show that $Fx \in \mathcal{X}'$ for all $x \in E'$ by induction on $\text{supp}(x)$. If $\text{supp}(x) = 1$, then $Fx = Fx_i \in \mathcal{X}'$ for some i. If $\text{supp}(x) = 2$, then, without loss of generality, $x = \lambda_1 x_1 + \lambda_2 x_2$. As $\mathcal{X}$ is singular, Fx_1 and Fx_2 are collinear and the unique line containing both points is $\{F(\mu_1 x_1 + \mu_2 x_2) \mid (\mu_1, \mu_2) \in F^{2*}\}$. As $\mathcal{X}'$ is a subspace, it must contain this line and thus it must contain Fx. Suppose that $j = \text{supp}(x) > 2$. Let $x = \lambda_1 x_1 + \ldots + \lambda_{j-1} x_{j-1} + \lambda_j x_j$ and let $x' = x - \lambda_j x_j$. By the induction, Fx' and $F(\lambda_j x_j) = Fx_j$ are points of $\mathcal{X}' \subseteq \mathcal{X}$ and the unique line containing Fx' and Fx_j is $\{F(\mu_1 x_1 + \ldots + \mu_j x_j) \mid (\mu_1, \ldots, \mu_j) \in F^j \setminus \{\mathbf{0}\}\}$ and this line must be contained in $\mathcal{X}'$. Thus Fx is contained in $\mathcal{X}'$. By induction, we have that that $\mathcal{X} = \mathcal{X}'$. Therefore any proper subspace $\mathcal{X}'$ of $\mathcal{X}$ must contain only a proper subset of $\{Fx_1, \ldots, Fx_k\}$. By applying this argument to inductively we can deduce that any chain of proper subspaces of $\mathcal{X}$ has finite length. Therefore the singular rank of $\mathcal{E}$ is finite. □

Corollary 4.3.14 *The extremal geometry $(\mathcal{E}, \mathcal{F})$ of a simple Lie algebra generated by its extremal elements that contains no sandwiches is isomorphic to the root shadow space $A_{n,\{1,n\}}$, $BC_{n,2}$, $D_{n,2}$, $E_{6,2}$, $E_{7,1}$, $E_{8,8}$, $F_{4,1}$, or $G_{2,2}$.*

Proof. This follows from Theorem 4.2.3, Corollary 4.3.12 and Lemma 4.3.13. □

4.4 Root subgroups

Automorphisms of Lie algebras preserve its structure and not surprisingly the extremal geometry of a Lie algebra. In this section we prove several technical results concerning the action of $\mathrm{Aut}(L)$ on $\mathcal{E}$. For simplicity, the automorphisms of L are written on the right to avoid parenthesis.

Lemma 4.4.1 *Let φ be an automorphism of L and x be an extremal element. Then $x\varphi$ is an extremal element with $g_{x\varphi}(z) = g_x(z\varphi^{-1})$. Furthermore, φ sends sandwiches to sandwiches.*

Proof. We have to verify identities (4.3.2), (4.3.3) and (4.3.4) for $x\varphi$. For each $y \in L$, we have $[x\varphi, [x\varphi, y]] = \varphi([x, [x, y\varphi^{-1}]]) = \varphi(2g_x(y\varphi^{-1})x) = 2g_x(y\varphi^{-1})(x\varphi) = 2g_{x\varphi}(y)(x\varphi)$. Thus (4.3.2) is satisfied. For each $y, z \in L$, we have

$$\begin{aligned}
[[x\varphi, y], [x\varphi, z]] &= \varphi([[x, y\varphi^{-1}], [x, z\varphi^{-1}]]) \\
&= \varphi(g_x([y, z]\varphi^{-1})x + g_x(z\varphi^{-1})[x, y\varphi^{-1}] - g_x(y\varphi^{-1})[x, z\varphi^{-1}]) \\
&= g_x([y, z]\varphi^{-1})x\varphi + g_x(z\varphi^{-1})[x\varphi, y] - g_x(y\varphi^{-1})[x\varphi, z] \\
&= g_{x\varphi}([y, z])x\varphi + g_{x\varphi}(z)[x\varphi, y] - g_{x\varphi}(y)[x\varphi, z]
\end{aligned}$$

Thus (4.3.3) is satisfied. It can be similarly shown that (4.3.4) holds. Thus $x\varphi$ is extremal with $g_{x\varphi}(z) = g_x(z\varphi^{-1})$. It can be easily shown that if $\mathrm{ad}_x^2 = \mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_x = 0$ for all $y \in L$, then $\mathrm{ad}_{x\varphi}^2 = \mathrm{ad}_{x\varphi}\mathrm{ad}_y\mathrm{ad}_{x\varphi} = 0$ for all $y \in L$. Thus φ acts on the set of sandwich extremal elements. □

Lemma 4.4.2 *Every Lie algebra generated by extremal elements is linearly spanned by extremal elements.*

Proof. Every element z can be written as a combination of Lie brackets applied to the extremal elements. We proceed by induction on the length of this bracket. If $z = [x_2, x_1]$, then $x := \exp(x_2, 1)x_1 = x_1 + [x_2, x_1] + g_{x_2}(x_1)x_2$ is an extremal element. Thus $[x_2, x_1] = x - x_1 - g_{x_2}(x_1)x_2$ is the linear span of extremal elements. Suppose that the length of the bracket is $k > 2$. Then z has the form $[x_k, [\ldots, [x_3, [x_2, x_1]], \ldots]]$. Set x as before and observe that $[x_2, x_1] = x - x_1 - g_{x_2}(x_1)x_2$. Then z can be rewritten as the linear combination of elements with shorter bracket lengths, namely

$$[x_k, [\ldots, [x_3, x], \ldots]] - [x_k, [\ldots, [x_3, x_1]], \ldots] - g_{x_2}(x_1)[x_k, [\ldots, [x_3, x_2], \ldots]].$$

The rest follows from induction. □

Lemma 4.4.3 *Every automorphism of the Lie algebra L is an automorphism of its extremal geometry $\mathcal{E}$.*

Proof. Let φ be an automorphism of L. We know that for extremal element x, $x\varphi$ is an extremal element with $g_{x\varphi}(z) = g_x(z\varphi^{-1})$. It suffices to show that φ sends lines to lines. Two points Fx and Fy lie in a common line if x and y are linearly independent, $[x, y] \neq 0$ and for all $z \in L$, $[x, [y, z]] = g_y(z)x + g_x(z)y$. It is clear that $x\varphi$ and $y\varphi$ are linearly independent and $[x\varphi, y\varphi] = 0$. Let $z \in L$. Then $[x\varphi, [y\varphi, z]] = ([x, [y, z\varphi^{-1}]])\varphi = (g_y(z\varphi^{-1})x + g_x(z\varphi^{-1})y)\varphi = g_{y\varphi}(z)(x\varphi) + g_{x\varphi}(z)(y\varphi)$. Thus $x\varphi$ and $y\varphi$ are collinear. The line containing Fx and Fy is $\{F(\lambda x + \mu y) \mid (\lambda, \mu) \in F^{2^*}\}$ and this is mapped to the line $\{F(\lambda(x\varphi) + \mu(y\varphi)) \mid (\lambda, \mu) \in F^{2^*}\}$. This completes the proof. □

For each extremal element x define the following map $\exp(x, s)$ from L to L given by $y \mapsto y + s[x, y] + s^2 g_x(y)x$ for a scalar $s \in F$.

Lemma 4.4.4 *The map $\exp(x, s)$ is an automorphism of L.*

Proof. A tedious computation shows that $\exp(x, s)[y, z] = [\exp(x, s)y, \exp(x, s)z]$. It suffices to show that $\exp(x, s)$ has a zero kernel. Let y be in the kernel of $\exp(x, s)$. Then $y + s[x, y] + s^2 g_x(y)x = 0$. By applying $[x,]$ on both sides we obtain

$$[x, y] + s[x, [x, y]] = 0. \tag{4.4.5}$$

If F has characteristic 2, then $0 = [x, y] + 2sg_x(y)x = [x, y]$. Thus $y = 0$. We may then suppose that the characteristic of F is odd. If $s = 0$ or $[x, y] = 0$, then $y = 0$. We assume that $s \neq 0$ and $[x, y] \neq 0$. By (4.4.5), $[x, y]$ and x do not commute and $[x, y] = -s[x, [x, y]] = -2sg_x(y)x$. But then $[x, y]$ and x are multiples of one another and thus commute. This contradicts what is stated above. Thus, we either have that $[x, y] = 0$ or $s = 0$ and in both cases $y = 0$. Therefore the kernel of $\exp(x, s)$ is zero and $\exp(x, s)$ is an automorphism of L. $\square$

Chapter 5

Decomposition of Simply Laced Lie Algebras

Let L be a simple Lie algebra over the field F with characteristic p that contains no sandwiches and is generated by its extremal elements E such that its extremal geometry (with lines) $(\mathcal{E}, \mathcal{F})$ is isomorphic to the root shadow space $\mathcal{P} = \mathcal{P}_{n,J}$ of a building Δ of type X_n. We suppose that $p \neq 2$ and $n \geq 2$. Let Φ be the root system of type X_n where X is assumed to be either A, D, or E and let Σ be a fixed apartment of Δ. Recall from Chapter 2 that there is a one-to-one correspondence between J-shadows contained in Σ and the (long) roots of Φ, namely, every root α corresponds to the J-shadow (polar region) $P(\alpha)$. In particular, there is an injection from Φ to $\mathcal{E}$, namely $\alpha \mapsto P(\alpha) \mapsto \widehat{P}(\alpha) \mapsto E_\alpha$ where E_α is a 1-dimensional subspace generated by an extremal element and corresponds to $\widehat{P}(\alpha)$ with respect to the isomorphism between $\mathcal{E}$ and $\mathcal{P}$ where $\widehat{P}(\alpha)$ is the unique J-shadow of Δ corresponding to the J-shadow $P(\alpha)$ of Σ. Let $0 \leq \theta_{\alpha\beta} \leq \pi$ denote the angle formed between α and β. Until stated otherwise we assume that x_δ is an arbitrary vector from E_δ for each root δ in Φ. We later choose such vectors more carefully.

Throughout this chapter roots of the root system Φ are denoted by $\alpha, \beta, \gamma, \delta, \varepsilon$ and γ, and elements of the field F are denoted by λ, μ, ν and ξ. We drop the hat from $\widehat{P}(\alpha)$ for simplicity.

5.1 Construction of a subalgebra of Chevalley type

Lemma 5.1.1 *With respect to the bijection from $\mathcal{E}$ to $\mathcal{P}$ described above, we have the following correspondence:*

(i) *If $\theta_{\alpha\beta} = 0$, then $Fx_\alpha = Fx_\beta$.*

(ii) *If $\theta_{\alpha\beta} = \pi/3$, then $[x_\alpha, x_\beta] = 0$ and there exists $\gamma \in \Phi$ such that $[x_\alpha, x_\gamma] = \lambda x_\beta$ for some $\lambda \in F$.*

(iii) *If $\theta_{\alpha\beta} = \pi/2$, then $[x_\alpha, x_\beta] = 0$.*

(iv) *If $\theta_{\alpha\beta} = 2\pi/3$, then $[x_\alpha, x_\beta] \neq 0$, $[x_\alpha, x_\beta] = \lambda x_{\alpha+\beta}$ for some $\lambda \in F \setminus \{0\}$.*

(v) *If $\theta_{\alpha\beta} = \pi$, then $[x_\alpha, x_\beta] \neq 0$, $[x_\alpha, x_\beta]$ is not extremal and $\langle x_\alpha, x_\beta, [x_\alpha, x_\beta]\rangle \cong \mathfrak{sl}(2, F)$.*

Proof. Part (i) follows from the isomorphism from $\mathcal{E}$ to $\mathcal{P}$ and Theorem 2.3.1. Suppose that $\theta_{\alpha\beta} = 2\pi/3$. By Theorem 2.3.1, this occurs if and only if $(P(\alpha), P(\beta)) \in \mathcal{P}_1$, if and only if $P(\alpha + \beta)$ is the unique common neighbour of $P(\alpha)$ and $P(\beta)$ in the collinearity graph $(\mathcal{P}, \mathcal{P}_{-1})$. By the established isomorphism between $\mathcal{E}$ and $\mathcal{P}$, this occurs if and only if $(x_\alpha, x_\beta) \in \mathcal{E}_1$. By Remark 4.3.9, we have that $F[x_\alpha, x_\beta] = Fx_{\alpha+\beta}$, if and only if $[x_\alpha, x_\beta] = \lambda x_{\alpha+\beta}$ for some $\lambda \in F \setminus \{0\}$. This proves part (iv). We use part (iv) to prove part (ii). The angle $\theta_{\alpha\beta}$ is $\pi/3$ if and only if $\theta_{\alpha,\beta-\alpha} = 2\pi/3$. By (iv), we have that $[x_\alpha, x_{\beta-\alpha}] = \lambda x_\beta$ for some $\lambda \in F$. This proves (ii) by setting $\gamma = \beta - \alpha$. We have $\theta_{\alpha,\beta} = \pi/2$ if and only if $(P(\alpha), P(\beta)) \in \mathcal{P}_0$, if and only if $(Fx_\alpha, Fx_\beta) \in \mathcal{E}_0$. In particular $[x_\alpha, x_\beta] = 0$. The first two claims of (v) follow from the isomorphism between $\mathcal{E}$ and $\mathcal{P}$ and Lemma 4.3.10. Set $x = x_\alpha$, $y = x_\beta$ and $h = [x, y]$. Then $[h, x] = -2g_x(y)x$ and $[h, y] = 2g_y(x)y$. If we replace y with $(-1/g_x(y))y$, then $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$. Therefore $\langle x, y, h\rangle \cong \mathfrak{sl}(2, F)$. $\square$

Proposition 5.1.2 *Let α and β be roots. Then $[[x_\alpha, x_{-\alpha}], x_\beta] = \lambda x_\beta$ and $[x_\alpha, [x_{-\alpha}, x_\beta]] = \mu x_\beta$ for some scalars λ and μ in F. The scalar λ is zero when $\theta_{\alpha,\beta} = \pi/2$ and otherwise nonzero. The scalar μ is zero when $\theta_{\alpha,\beta} \in \{\pi/2, 2\pi/3, \pi\}$ and otherwise zero. In particular, $[[x_\alpha, x_{-\alpha}], x_\beta]$ and $[x_\alpha, [x_{-\alpha}, x_\beta]]$ are both multiples of x_β.*

Proof. If $\alpha = \beta$, then $[[x_\alpha, x_{-\alpha}], x_\alpha] = -[x_\alpha, [x_\alpha, x_{-\alpha}]] = -2g_{x_\alpha}(x_{-\beta})x_\alpha \neq 0$. A similar argument can be made if $\beta = -\alpha$. Suppose now that $\theta_{\alpha\beta} \in \{\pi/3, \pi/2, 2\pi/3\}$. Using the Jacobi identity we obtain

$$[[x_\alpha, x_{-\alpha}], x_\beta] = [x_\alpha, [x_{-\alpha}, x_\beta]] + [[x_\alpha, x_\beta], x_{-\alpha}]. \tag{5.1.3}$$

If $\theta_{\alpha\beta} = \pi/2$, then $\theta_{-\alpha,\beta} = \pi/2$ and $[[x_\alpha, x_{-\alpha}], x_\beta] = 0$. If $\theta_{\alpha\beta} = \pi/3$, then $\theta_{-\alpha,\beta} = 2\pi/3$ and, by Lemma 5.1.1 (ii) and (iv), $[[x_\alpha, x_{-\alpha}], x_\beta] = \lambda'[x_\alpha, x_{-\alpha+\beta}]$ for some $\lambda' \in F$. Since $\theta_{\alpha,-\alpha+\beta} = 2\pi/3$, $\lambda[x_\alpha, x_{-\alpha+\beta}] = \lambda x_\beta \neq 0$ for some $\lambda \in F$ by Lemma 5.1.1 (iv). A similar argument can be made when $\theta_{\alpha\beta} = 2\pi/3$.

We now look at the second statement. If $\theta_{\alpha\beta} \in \{\pi/2, 2\pi/3, \pi\}$, then $[x_\alpha, [x_{-\alpha}, x_\beta]] = 0$. If $\alpha = \beta$, then $[x_\alpha, [x_{-\alpha}, x_\beta]] = -[x_\alpha, [x_\alpha, x_{-\alpha}]] = -2g_{x_\alpha}(x_{-\alpha})x_\alpha \neq 0$. If $\theta_{\alpha\beta} = \pi/3$, then $[x_\alpha, [x_{-\alpha}, x_\beta]] = \lambda'[x_\alpha, x_{-\alpha+\beta}] = \lambda x_\beta \neq 0$ for some λ in F. □

For each root $\alpha \in \Phi$, let $h_\alpha = [x_\alpha, x_{-\alpha}]$ for arbitrary x_α in E_α and $x_{-\alpha}$ in $E_{-\alpha}$. Let $B = \{\alpha_1, \dots, \alpha_n\}$ be a basis for Φ and let $I = \{1, \dots, n\}$ be the index set. Let H be the subalgebra of L generated by the elements $\{h_{\alpha_i} \mid i \in I\}$. Note that the angle between any two distinct roots from B is either $\pi/2$ or $2\pi/3$.

Lemma 5.1.4 *The Lie algebra H is commutative and $h_\alpha \in H$ for all $\alpha \in \Phi$.*

Proof. Let α and β be roots in B. Let $x = [x_\alpha, x_{-\alpha}]$ and $y = [x_\beta, x_{-\beta}]$. Using the Jacobi identity we write $z := [x, y]$ in two different ways:

$$\begin{aligned} [[x_\alpha, x_{-\alpha}], [x_\beta, x_{-\beta}]] &= [[[x_\alpha, x_{-\alpha}], x_\beta], x_{-\beta}] + [x_\beta, [[x_\alpha, x_{-\alpha}], x_{-\beta}]] \\ &= [x_\alpha, [x_{-\alpha}, [x_\beta, x_{-\beta}]]] + [[x_\alpha, [x_\beta, x_{-\beta}]], x_{-\alpha}]. \end{aligned}$$

Applying Lemma 5.1.2 to the first expression gives $z = [\lambda x_\beta, x_{-\beta}] + [x_\beta, \mu x_{-\beta}] = \nu[x_\beta, x_{-\beta}]$. Applying Lemma 5.1.2 to the second expression gives us $z = [x_\alpha, \lambda' x_{-\alpha}] + [\mu' x_\alpha, x_{-\alpha}] = \nu'[x_\alpha, x_{-\alpha}]$. Hence $\nu x = \nu' y$. If $\nu' = 0$, then $z = 0$ and H is commutative. If $\nu' \neq 0$, then $y = \xi x$ and $z = [x, y] = [x, \xi x] = 0$. In all cases, H is commutative.

We prove the second statement by induction on the height of α. Suppose that α has height at least two, that is, $\alpha = \alpha' + \alpha_i$ where α' is some root that forms an angle of $2\pi/3$ with α_i. In particular, by Lemma 5.1.1, $x_{\alpha'+\alpha_i} = \lambda[x_{\alpha'}, x_{\alpha_i}]$ and $x_{(-\alpha')+(-\alpha_i)} = \mu[x_{-\alpha'}, x_{-\alpha_i}]$. Using the Jacobi identity and Proposition 5.1.2, we have

$$\begin{aligned} [x_\alpha, x_{-\alpha}] &= \lambda\mu[[x_{\alpha'}, x_{\alpha_i}], [x_{-\alpha'}, x_{-\alpha_i}]] \\ &= [[[x_{\alpha'}, x_{\alpha_i}], x_{-\alpha'}], x_{-\alpha_i}] + [x_{-\alpha'}, [[x_{\alpha'}, x_{\alpha_i}], x_{-\alpha_i}]] \\ &= \nu[x_{\alpha_i}, x_{-\alpha_i}] + \xi[x_{-\alpha'}, x_{\alpha'}] \end{aligned}$$

The first summand is in H since it is a generating element of H and the second summand is in H by the inductive hypothesis. Hence $[x_\alpha, x_{-\alpha}] \in H$. □

Later in this chapter we choose x_α and $x_{-\alpha}$ more carefully so that h_α is fixed. Note that H is linearly spanned by $\{h_{\alpha_i} \mid i \in I\}$ since it is commutative. Let K be the subspace linearly spanned by $\{x_\alpha \mid \alpha \in \Phi\}$. By Lemma 5.1.2, for each $h \in H$ and a fixed α we have that $[x_\alpha, h] = f_\alpha(h) x_\alpha$ where f_α is a linear functional from H^*.

Lemma 5.1.5 *The set of vectors $\{x_\alpha \mid \alpha \in \Phi\}$ is linearly independent whenever F does not have even characteristic.*

Proof. Suppose for a contradiction that this is not true. Let

$$\sum_{\alpha \in \Phi} \lambda_\alpha x_\alpha = 0 \tag{5.1.6}$$

be a linear combination in which not all coefficients λ_α are zero and has minimal number of nonzero coefficients. In this case, at least two coefficients, say λ_β and $\lambda_{\beta'}$, are nonzero. Suppose that $\beta' = -\beta$ and that all other coefficients are zero. Then $\langle x_\beta \rangle = \langle x_{-\beta} \rangle$ and this is a clear contradiction. Therefore, we can assume that $\beta' \neq \pm\beta'$. By Lemma 1.5.1, there exists a root γ that is perpendicular to β but not perpendicular to β'. Apply ad_{-h_γ} to both sides of (5.1.6) to get

$$\sum_{\alpha \in \Phi} a_\alpha f_\alpha(h_\gamma) x_\alpha = 0.$$

This sum loses at least one term when $\alpha = \beta$ and retains at least one term, namely $\lambda_{\beta'} f_{\beta'}(h_\gamma) x_{\beta'}$ and this contradicts the minimality of (5.1.6). We use the fact that $f_\alpha(h_\alpha) = 0$ if and only if (α, β) by Lemma 5.1.2. □

From this result we can conclude that $\{x_\alpha \mid \alpha \in \Phi\}$ is a basis for K and K has dimension $|\Phi|$.

Lemma 5.1.7 *Let L' be the subspace $H + K$. Then L' is a Lie algebra.*

Proof. Let x_α and x_β be elements of the basis for K. Then

$$[x_\alpha, x_\beta] = \begin{cases} 0 & \text{if } \theta_{\alpha\beta} \leq \pi/2, \\ A_{\alpha,\beta} x_{\alpha+\beta} & \text{if } \theta_{\alpha\beta} = 2\pi/3, \\ h_\alpha & \text{if } \beta = -\alpha. \end{cases}$$

In particular, $[x_\alpha, x_\beta] \in K + H$ and by the bilinearity of the Lie bracket and that K is spanned by $\{x_\alpha \mid \alpha \in \Phi\}$ we have that $[K, K] \subseteq K + H$. By Lemma 5.1.2, $[K, H] \subseteq K$. By Lemma 5.1.4, $[H, H] = 0$. Hence $[H+K, H+K] \subseteq H+K$ and so L' is a Lie algebra.□

We previously defined h_α as the Lie bracket $[x_\alpha, x_{-\alpha}]$. However, x_α and $x_{-\alpha}$ can be arbitrarily chosen up to scalars and the same results above follow. Our aim is to choose the appropriate scalars of x_α and $x_{-\alpha}$ so that we can form a Chevalley basis for the Lie subalgebra L'. Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for Φ. For each $i \in I$, choose an arbitrary vector x_{α_i} from E_{α_i}. For each positive root γ, let $\gamma = \delta_\gamma + \varepsilon_\gamma$ be a fixed decomposition of γ into the sum of two positive roots. We define x_γ inductively on the height of γ by $x_\gamma = [x_{\delta_\gamma}, x_{\varepsilon_\gamma}]$. Therefore extremal elements corresponding to positive roots are well defined.

Lemma 5.1.8 *Let α and β be two positive roots such that $\alpha + \beta = \gamma$ and $\gamma = \delta_\gamma + \varepsilon_\gamma$ be the fixed decomposition of γ. Let $\delta = \delta_\gamma$ and $\varepsilon = \varepsilon_\gamma$ for simplicity. Thus we have $[x_\alpha, x_\beta] = A_{\alpha\beta} x_\gamma = A_{\alpha\beta}[x_\delta, x_\varepsilon]$ for some $A_{\alpha\beta} \in F$. Then $A_{\alpha\beta} = \pm 1$.*

Proof. We do induction on the height l of γ. If $l = 1$, then γ is simple and so there is nothing to check. If $l = 2$, then $\gamma = \alpha_i + \alpha_j = \alpha_j + \alpha_i$ are the only two possible decompositions of γ and $[x_{\alpha_i}, x_{\alpha_j}] = -[x_{\alpha_j}, x_{\alpha_i}]$ as required. Suppose that $l > 2$ and that the lemma holds for all roots of smaller height. Note we have

$$[x_\alpha, x_\beta] = A_{\alpha\beta}[x_\delta, x_\varepsilon]. \tag{5.1.9}$$

Let Ψ be the root subsystem spanned by $\{\alpha, \beta, \delta, \varepsilon\}$. Then Ψ has rank at most 3 since $\alpha+\beta = \delta+\varepsilon$. Suppose Ψ has rank 2. Then Ψ cannot be $A_1 \perp A_1$ because the angles do not match. If Ψ is A_2, then $\alpha, \beta \in \{\delta, \varepsilon\}$ since $\alpha+\beta = \delta+\varepsilon$. Thus $[x_\alpha, x_\beta] \in \{[x_\delta, x_\varepsilon], -[x_\delta, x_\varepsilon]\}$ and this satisfies the conclusion of the lemma.

Suppose that Ψ is rank 3. There are three possible Dynkin diagrams for Ψ, namely, $A_1 \perp A_1 \perp A_1$, $A_1 \perp A_2$ and A_3. Firstly, Ψ cannot be $A_1 \perp A_1 \perp A_1$ for similar reasons above. If Ψ is $A_1 \perp A_2$, then at least three of the four roots, say α, β, δ, are in A_2 but then all four must be because the angle between δ and ε is $2\pi/3$ (and not $\pi/2$). Then, without loss of generality, $\alpha = \delta$ and $\beta = \varepsilon$ since A_2 has only two positive roots and clearly $[x_\alpha, x_\beta] = \pm[x_\delta, x_\varepsilon]$. Suppose that Ψ is A_3. We can assume that $\{\alpha, \beta, \delta, \varepsilon\}$ are distinct roots. We fix a basis for Ψ and let ht_Ψ denote the height function for a root in Ψ. If $\mathrm{ht}_\Psi(\gamma) = 2$, then all $\{\alpha, \beta, \delta, \varepsilon\}$ must be simple and this contradicts that A_3 only has 3 simple roots. Therefore $\mathrm{ht}_\Psi(\gamma) = 3$ and this gives rise to the following possibilities: (i) α and δ are simple and β and ε have height 2, (ii) α and ε are simple and β and δ have height 2, (iii) β and δ are simple and α and ε have height 2 and (iv) β and ε are simple and α and δ have height 2. In all cases, let κ be the third simple root. Suppose that case (i) holds and that the angle between α and δ is $2\pi/3$. The Dynkin diagram, up to reordering α and β, is one of

Then $\beta = \delta + \kappa$ and $\varepsilon = \alpha + \kappa$. In the first diagram, α and κ are perpendicular and so ε is not a root. Thus α and δ are perpendicular and both form an angle of $2\pi/3$ with κ and $\beta = \delta + \kappa$ and $\varepsilon = \alpha + \kappa$. Furthermore, $x_\beta = A_{\kappa\delta}[x_\kappa, x_\delta]$ and $x_\varepsilon = A_{\alpha\kappa}[x_\alpha, x_\kappa]$. The height of β and ε in Φ are strictly less than the height of γ since $\gamma = \alpha + \beta = \delta + \varepsilon$. Thus by the induction hypothesis $A_{\kappa\delta}, A_{\alpha\kappa} \in \{-1, 1\}$. We obtain

$$\begin{aligned}
[x_\alpha, x_\beta] &= [x_\alpha, A_{\kappa\delta}[x_\kappa, x_\delta]] \\
&= A_{\kappa\delta}[[x_\alpha, x_\kappa], x_\delta] \\
&= A_{\kappa\delta}A_{\alpha\kappa}[x_\varepsilon, x_\delta] \\
&= -A_{\kappa\delta}A_{\alpha\kappa}[x_\delta, x_\varepsilon]
\end{aligned}$$

By comparing this with (5.1.9), $A_{\alpha\beta} = -A_{\kappa\delta}A_{\alpha\kappa} \in \{-1, 1\}$ as desired. The other cases are very similar and we omit the details. $\square$

Corollary 5.1.10 *Let α and β be two positive roots. Then*

$$[x_\alpha, x_\beta] = \begin{cases} 0 & \text{if } (\alpha,\beta) \geq 0 \\ \pm x_{\alpha+\beta} & \text{if } (\alpha,\beta) = -1 \end{cases}$$

□

For each positive root α, we can choose $x_{-\alpha}$ such that $[[x_\alpha, x_{-\alpha}], x_\alpha] = 2x_\alpha$. Indeed, $[[x_\alpha, x_{-\alpha}], x_\alpha] = -2g_{x_\alpha}(x_{-\alpha})x_\alpha \neq 0$. Replacing $x_{-\alpha}$ with $(2/\lambda)x_{-\alpha}$ where $\lambda = -2g_{x_\alpha}(x_{-\alpha})$ gives the desired result. Note that in this case we have $g_{x_\alpha}(x_{-\alpha}) = -1$.

Lemma 5.1.11 *Let α and β be two roots that form an angle of $2\pi/3$. Then*

(i) $[x_{-\alpha}, [x_\alpha, x_\beta]] = x_\beta$,

(ii) $[h_\alpha, x_\beta] = -x_\beta$, *and*

(iii) $[h_\alpha, [x_\alpha, x_\beta]] = [x_\alpha, x_\beta]$.

Proof. By writing $h_\alpha = [x_\alpha, x_{-\alpha}]$ and using equation (4.3.3) from Section 4.3, we obtain

$$[h_\alpha, [x_\alpha, x_\beta]] = g_{x_\alpha}([x_{-\alpha}, x_\beta])x_\alpha + g_{x_\alpha}(x_\beta)h_\alpha - g_{x_\alpha}(x_{-\alpha})[x_\alpha, x_\beta].$$

Note that $[x_{-\alpha}, x_\beta] = 0$ and $g_{x_\alpha}(x_\beta) = 0$. The above equation reduces to $[h_\alpha, [x_\alpha, x_\beta]] = -g_{x_\alpha}(x_{-\alpha})[x_\alpha, x_\beta] = [x_\alpha, x_\beta]$. This proves part (iii). By Lemma 5.1.3, we know that $[x_{-\alpha}, [x_\alpha, x_\beta]] = \mu x_\beta$. Using equation (4.3.4) from Section 4.3, we obtain

$$[x_\alpha, [x_{-\alpha}, [x_\alpha, x_\beta]]] = g_{x_\alpha}([x_{-\alpha}, x_\beta])x_\alpha - g_{x_\alpha}(x_\beta)h_\alpha - g_{x_\alpha}(x_{-\alpha})[x_\alpha, x_\beta].$$

This equation reduces to $[x_\alpha, [x_{-\alpha}, [x_\alpha, x_\beta]]] = -g_{x_\alpha}(x_{-\alpha})[x_\alpha, x_\beta] = [x_\alpha, x_\beta]$. On the other hand, $[x_\alpha, [x_{-\alpha}, [x_\alpha, x_\beta]]] = [x_\alpha, \mu x_\beta]$. Therefore $\mu = 1$ and this proves part (i). By Lemma

5.1.2, $[h_\alpha, x_\beta] = \lambda x_\beta$. Applying $x = x_\alpha$, $y = x_{-\alpha}$ and $z = x_\beta$ to the identity given in Lemma 4.3.8 we obtain

$$\begin{aligned}[h_\alpha, [x_{-\alpha}, [x_\alpha, x_\beta]]] &= g_{x_{-\alpha}}(x_\alpha)[x_{-\alpha}, [x_\alpha, x_\beta]] \\ &= -[x_{-\alpha}, [x_\alpha, x_\beta]] \\ &= -x_\beta.\end{aligned}$$

On the other hand, $[h_\alpha, [x_{-\alpha}, [x_\alpha, x_\beta]]] = [h_\alpha, x_\beta] = \lambda x_\beta$. Therefore $\lambda = -1$ and this proves part (ii). $\square$

Note that α and β do not necessarily need to be positive roots in the hypothesis of the previous lemma.

Lemma 5.1.12 *Let γ be a positive root and $\gamma = \alpha + \beta$ be the fixed decomposition of γ into the sum of two positive roots so that $x_\gamma = [x_\alpha, x_\beta]$. Then $x_{-\gamma} = [x_{-\beta}, x_{-\alpha}]$ and $h_\gamma = h_\alpha + h_\beta$.*

Proof. Note that $x_{-\gamma}$ is the unique vector of $E_{-\gamma}$ such that $[h_\gamma, x_\gamma] = 2x_\gamma$ where $h_\gamma = [x_\gamma, x_{-\gamma}]$. Suppose that $\mathrm{ht}(\gamma) = 2$. Then $\gamma = \alpha_i + \alpha_j = \alpha_j + \alpha_i$. Assume the former equality is the fixed decomposition. Using the Jacobi identity and that $[x_{\alpha_i}, x_{-\alpha_j}] = [x_{-\alpha_i}, x_{\alpha_j}] = 0$ we obtain

$$\begin{aligned}[[x_{\alpha_i}, x_{\alpha_j}], [x_{-\alpha_j}, x_{-\alpha_i}]] &= [[[x_{\alpha_i}, x_{\alpha_j}], x_{-\alpha_j}], x_{-\alpha_i}] + [x_{-\alpha_j}, [[x_{\alpha_i}, x_{\alpha_j}], x_{-\alpha_i}]] \\ &= [[x_{\alpha_i}, [x_{\alpha_j}, x_{-\alpha_j}]], x_{-\alpha_i}] + [x_{-\alpha_j}, [[x_{\alpha_i}, x_{-\alpha_i}], x_{\alpha_j}]] \\ &= [[x_{\alpha_i}, h_{\alpha_j}], x_{-\alpha_i}] + [x_{-\alpha_j}, [h_{\alpha_i}, x_{\alpha_j}]].\end{aligned}$$

By Lemma 5.1.11 (ii), this reduces to $[[x_{\alpha_i}, x_{\alpha_j}], [x_{-\alpha_j}, x_{-\alpha_i}]] = [x_{\alpha_i}, x_{-\alpha_i}] + [x_{\alpha_j}, x_{-\alpha_j}] = h_{\alpha_i} + h_{\alpha_j}$. Finally, using Lemma 5.1.11 (iii), we obtain

$$\begin{aligned}[h_{\alpha_i}+h_{\alpha_j},[x_{\alpha_i},x_{\alpha_j}]] &= [h_{\alpha_i},[x_{\alpha_i},x_{\alpha_j}]]+[h_{\alpha_j},[x_{\alpha_i},x_{\alpha_j}]]\\ &= 2[x_{\alpha_i},x_{\alpha_j}]\end{aligned}$$

Therefore, with $\gamma = \alpha_i + \alpha_j$, we have $x_{-\gamma} = [x_{-\alpha_j}, x_{-\alpha_i}]$ and $h_\gamma = h_{\alpha_i} + h_{\alpha_j}$. Suppose that $\mathrm{ht}(\gamma) > 2$. Let $\gamma = \alpha + \beta$ be the fixed decomposition. We do exactly as we did in the case where $\mathrm{ht}(\gamma) = 2$. Using the Jacobi identity, that $[x_\alpha, x_{-\beta}] = [x_{-\alpha}, x_\beta] = 0$, the inductive hypothesis and finally Lemma 5.1.11 (ii), we obtain

$$\begin{aligned}[[x_\alpha,x_\beta],[x_{-\beta},x_{-\alpha}]] &= [[[x_\alpha,x_\beta],x_{-\beta}],x_{-\alpha}]+[x_{-\beta},[[x_\alpha,x_\beta],x_{-\alpha}]]\\ &= [[x_\alpha,[x_\beta,x_{-\beta}]],x_{-\alpha}]+[x_{-\beta},[[x_\alpha,x_{-\alpha}],x_\beta]]\\ &= [[x_\alpha,h_\beta],x_{-\alpha}]+[x_{-\beta},[h_\alpha,x_\beta]]\\ &= [x_\alpha,x_{-\alpha}]+[x_\beta,x_{-\beta}]\\ &= h_\alpha+h_\beta\end{aligned}$$

Finally, using Lemma 5.1.11 (iii), we obtain

$$\begin{aligned}[h_\alpha+h_\beta,[x_\alpha,x_\beta]] &= [h_\alpha,[x_\alpha,x_\beta]]+[h_\beta,[x_\alpha,x_\beta]]\\ &= 2[x_\alpha,x_\beta]\end{aligned}$$

Therefore, with $\gamma = \alpha + \beta$, we have $x_{-\gamma} = [x_{-\beta}, x_{-\alpha}]$ and $h_\gamma = h_\alpha + h_\beta$ as required. □

The next two results are analogous to Lemma 5.1.10 for a combination of negative and positive roots.

Corollary 5.1.13 *Let α and β be positive roots such that $\gamma = \alpha + \beta$. Then $x_{-\gamma} = \pm[x_{-\alpha}, x_{-\beta}]$.*

Proof. In Lemma 5.1.12 we show that if $x_\gamma = [x_\alpha, x_\beta]$, then $x_{-\gamma} = [x_{-\beta}, x_{-\alpha}]$. Similarly, if $x_\gamma = -[x_\alpha, x_\beta]$, then $x_{-\gamma} = [x_{-\alpha}, x_{-\beta}]$ □

Corollary 5.1.14 *Let α and β be two positive roots such that α and $-\beta$ form an angle of $2\pi/3$. Then $[x_\alpha, x_{-\beta}] = \pm x_{\alpha-\beta}$.*

Proof. Let $A = A_{\alpha,-\beta}$. Then $[x_\alpha, x_{-\beta}] = Ax_{\alpha-\beta}$. Suppose that $\alpha - \beta$ is positive. By Lemma 5.1.11 and Corollary 5.1.10, $x_\alpha = [x_\beta, [x_{-\beta}, x_\alpha]] = -A[x_\beta, x_{\alpha-\beta}] = \pm Ax_\alpha$. Therefore $A = \pm 1$. Suppose $\alpha - \beta$ is negative. By Lemma 5.1.11 and Corollary 5.1.13, $x_{-\beta} = [x_{-\alpha}, [x_\alpha, x_{-\beta}]] = A[x_{-\alpha}, x_{\alpha-\beta}] = \pm Ax_{-\beta}$. Therefore $A = \pm 1$. This completes the proof. □

We summarise the result from this section in the following theorem.

Theorem 5.1.15 *Let α and β be two roots. Then*

$$[x_\alpha, x_\beta] = \begin{cases} 0 & \text{if } (\alpha,\beta) \geq 0 \\ \pm x_{\alpha+\beta} & \text{if } (\alpha,\beta) = -1 \\ \pm h_\alpha & \text{if } (\alpha,\beta) = -2 \end{cases}$$

and in the latter case, h_α is an integral sum of $h_{\alpha_1}, \ldots, h_{\alpha_n}$.

Lemma 5.1.16 *Let α and β be two roots. Then $[h_\alpha, x_\beta] = (\alpha, \beta)x_\beta$.*

Proof. We do case by case analysis. If $\alpha = \beta$, then $(\alpha, \beta) = 2$ and by construction, $[h_\alpha, x_\alpha] = 2x_\alpha$. If $(\alpha, \beta) = 1$, that is, α and β form an angle of $\pi/3$, then by Lemma 5.1.11 (ii) $[h_\alpha, x_\beta] = -[h_{-\alpha}, x_\beta] = x_\beta$. If $(\alpha, \beta) = 0$, then $[h_\alpha, x_\beta] = [x_\alpha, [x_{-\alpha}, x_\beta]] + [[x_\alpha, x_\beta], x_{-\alpha}] = 0$. If $(\alpha, \beta) = -1$, that is α and β form an angle of $2\pi/3$, then by Lemma 5.1.11 $[h_\alpha, x_\beta] = -x_\beta$. Finally, if $(\alpha, \beta) = -2$, that is $\beta = -\alpha$, then $[h_\alpha, x_\beta] = [[x_\alpha, x_{-\alpha}], x_{-\alpha}] = [x_{-\alpha}, [x_{-\alpha}, x_\alpha]] = 2g_{x_{-\alpha}}(x_\alpha)x_{-\alpha} = -2x_{-\alpha}$. □

Theorem 5.1.17 *The spanning set $\{x_\alpha; h_{\alpha_i} \mid \alpha \in \Phi, i \in I\}$ of L' satisfies conditions (i)-(iv) of a Chevalley basis given in Theorem 3.2.1.*

Proof. Part (i) holds since H is commutative as shown in Lemma 5.1.4. Lemma 5.1.16 is exactly condition (ii) and Lemma 5.1.15 is exactly (iii) and (iv). □

5.2 Recovering the classical Lie algebra of type A_n

We firstly establish a homomorphism from $\mathfrak{sl}(V)$ to the Lie algebra L' corresponding to the root system of type A_n. Identify the root system of type A_n with $\Phi = \{\alpha_{ij} = \pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\}$. Let V be the $(n+1)$-dimension vector space over the field F with characteristic p. We assume that $(n, p) \neq (2, 3)$. For a fixed decomposition of $\gamma = \alpha_i + (\alpha_{i+1} + \ldots + \alpha_{j-2} + \alpha_{j-1})$ into two positive root, we inductively define $x_\alpha = [x_{\alpha_i}, x_{\alpha_{i+1}+\ldots+\alpha_{j-1}}]$.

Proposition 5.2.1 *Let $\{H_i, E_{jk} \mid 1 \leq j, k \leq n+1, j \neq k, 1 \leq i \leq n\}$ be a Chevalley basis for $\mathfrak{sl}(V)$ as given in Section 3.3. Then the map $\varphi : \mathfrak{sl}(V) \to L'$ given by $H_i \mapsto h_i$ and $E_{jk} \mapsto x_{\alpha_{jk}}$ is a homomorphism of Lie algebras.*

Proof. By comparing the constant structures in Proposition 3.3.1 with those from Lemma 5.1.4, 5.1.15 and 5.1.16, we obtain the desire result. □

Let Z be the kernel of φ. If $Z \neq \{0\}$, then Z is a nontrivial ideal of $\mathfrak{sl}(V)$ and thus it must be the unique nontrivial ideal of $\mathfrak{sl}(V)$, namely $Z = \langle I \rangle$ the space generated by the identity matrix.

Proposition 5.2.2 *The Lie algebra L' is either isomorphic to $\mathfrak{sl}(V)$ or $\mathfrak{sl}(V)/Z$.* □

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the geometries defined in Section 3.4. Suppose that $\mathfrak{sl}(V)$ is isomorphic to L' via φ. It is clear that φ sends extremal 1-spaces to extremal 1-spaces, that is, φ sends points of $\mathcal{E}(\mathfrak{sl}(V))$ to points of $\mathcal{E}(L')$. Furthermore, if x and y are extremal in $\mathfrak{sl}(V)$ such that $[x, y] = 0$ and $\lambda x + \mu y$ is extremal for all $\lambda, \mu \in F$, then $[\varphi(x), \varphi(y)] = 0$ and $\lambda\varphi(x) + \mu\varphi(y)$ is extremal for all $\lambda, \mu \in F$. In particular, φ sends lines of $\mathcal{E}(\mathfrak{sl}(V))$ to lines of $\mathcal{E}(L')$. Therefore $\mathcal{E}(L') = \varphi(\mathcal{E}(\mathfrak{sl}(V))$. Set $\tilde{\mathcal{E}}_1 = \varphi(\mathcal{E}_1)$. Then $\tilde{\mathcal{E}}_1$ is a subspace of $\mathcal{E}(L')$. Suppose that $\mathfrak{sl}(V)/Z$ is isomorphic to L' and let $\widehat{\varphi} : \mathfrak{sl}(V)/Z \to L'$ be the induced isomorphism. It is clear that $\mathcal{E}(L') = \widehat{\varphi}(\mathcal{E}(\mathfrak{sl}(V)/Z)) = \widehat{\varphi}(\mathcal{E}_2)$. Set $\tilde{\mathcal{E}}_2 = \widehat{\varphi}(\mathcal{E}_2)$. Then $\tilde{\mathcal{E}}_2$ is the extremal geometry $\mathcal{E}(L')$. It is clear that $L' = \langle \tilde{\mathcal{E}}_i \rangle$ in the appropriate cases.

Let $G = \langle exp(x_\alpha, t) \mid \alpha \in \Phi, t \in F \rangle$ be the Chevalley group of type $A_n(F)$. Then by Theorem 3.3.2, $G \cong PSL(V)$ and $G \leq \mathrm{Aut}(L)$.

Proposition 5.2.3 *The space $\tilde{\mathcal{E}}_i$ is a subspace of the extremal geometry $\mathcal{E}$ of L.*

Proof. We firstly observe that $Fx_\alpha \in \tilde{\mathcal{E}}_i$ for all $\alpha \in \Phi$ since its preimage has rank 1. By Proposition 3.4.5, G acts transitively on the points of $\mathcal{E}_i$ and thus G acts transitively on the points of $\tilde{\mathcal{E}}_i$. From this we deduce that $(Fx_\alpha)^G = \tilde{\mathcal{E}}_i$. The group G is a subgroup of the automorphism group of L and acts on the set of points of the extremal geometry $\mathcal{E}$ of L. Note that Fx_α is a point of $\mathcal{E}$ and thus $\mathcal{E}_i \subseteq \mathcal{E}$. We have already seen that $F(\lambda x + \mu y) \in \mathcal{E}_1$ and $F(\lambda x + \mu y) + Z \in \mathcal{E}_2$ for all $\lambda, \mu \in F$ and thus the lines of $\mathcal{E}_i$ are full lines of $\mathcal{E}$ because we work over the same field. In particular, $\mathcal{E}_i$ is a subspace of $\mathcal{E}$.□

Consider the building of type A_n whose chambers are maximal chains of subspaces $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n$ where $\dim(V_i) = i$. Two chambers are i-adjacent if they differ in precisely the ith subspace. The root shadow space $\mathcal{P}$ of type $A_{n,\{1,n\}}$ consists of points $(\langle v \rangle, W)$ where $v \in W, W \subseteq V$ and $\dim(W) = n$. The apartment Σ with respect to the basis $\{e_1, \ldots, e_{n+1}\}$ of V consists of the chambers of the form

$$\{\langle e_{\sigma(1)} \rangle, \langle e_{\sigma(1)}, e_{\sigma(2)} \rangle, \ldots, \langle e_{\sigma(1)}, \ldots, e_{\sigma(n)} \rangle\}$$

for $\sigma \in \mathrm{Sym}(n+1)$.

Lemma 5.2.4 *The root shadow space $\mathcal{P}$ possesses the property (UCN).*

Proof. For $j \in \{1, \dots, n+1\}$, let $W_j = \langle e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{n+1} \rangle$. Take a point $(\langle e_1 \rangle, W_{n+1})$ of $\mathcal{P}$. A point $(\langle e_i \rangle, W_j)$, with $\langle e_1 \rangle \neq \langle e_i \rangle$ and $W_{n+1} \neq W_j$, has a unique common neighbour in $\mathcal{P}_\Sigma$ with $(\langle e_1 \rangle, W_{n+1})$ if and only if (i) $\langle e_1 \rangle \subseteq W_j$ and $\langle e_i \rangle \nsubseteq W_{n+1}$ in which case $i = n+1$, or (ii) $\langle e_1 \rangle \nsubseteq W_j$ and $\langle e_i \rangle \subseteq W_{n+1}$ in which case $j = 1$. The unique common neighbour is $(\langle e_1 \rangle, W_j)$ in case (i) and $(\langle e_i \rangle, W_{n+1})$ in case (ii). Suppose that we now consider $(\langle e_1 \rangle, W_{n+1})$ and $(\langle e_i \rangle, W_j)$ as J-shadows in the entire building and that $(\langle v \rangle, W)$ is a common neighbour. In particular, $\langle v \rangle = \langle e_1 \rangle$ and $W = W_j$ or $\langle v \rangle = \langle e_i \rangle$ and $W = W_{n+1}$. In case (i), $i = n+1$ and thus $(\langle v \rangle, W)$ is $(\langle e_1 \rangle, W_j)$. In case (ii), $j = 1$ and thus $(\langle v \rangle, W)$ is either $(\langle e_i \rangle, W_{n+1})$. In all cases we show that two points in $\mathcal{P}_\Sigma$ have a unique common neighbour, then they also have a unique common neighbour in $\mathcal{P}$ and this is property (UCN). $\square$

Inside the root shadow space of type $A_{n,\{1,n\}}$ there are two types of lines, namely red and blue lines. Two collinear points lie in a common red line if they are of the form $(\langle v \rangle, W)$ and $(\langle v \rangle, U)$ and a common blue line if they are of the form $(\langle v \rangle, W)$ and $(\langle u \rangle, W)$. We make the following interesting observation about the relationship between the intersection and colour of lines.

Remark 5.2.5 *Let $(\langle v \rangle, W)$ be the intersection of two lines l and m. Suppose, without loss of generality, that l is a blue line. Thus all other points on that line have the form $(\langle u \rangle, W)$ for some $\langle u \rangle \subseteq W$. If m is blue, then it also has points of the form $(\langle u \rangle, W)$ and thus every point on l is collinear with every point on m. If m is red, then it has points of the form $(\langle v \rangle, U)$ for some $U \supset \langle v \rangle$ and thus no point on l is collinear to a point on m.*

The next result implies that lines in subspaces of $\mathcal{P}$ remain the same colour in $\mathcal{P}$ providing that the subspace is also a root shadow space of type same. For two lines l and m, we say that l and m *strongly intersect* if l and m intersect and $m \subseteq l^\perp$ and $l \subseteq m^\perp$.

Lemma 5.2.6 *Suppose that $\mathcal{P}_1$ is a subspace of $\mathcal{P}$ that is also a root shadow space of the same type. Then either*

(i) *all red and blue lines in $\mathcal{P}_1$ are red and blue lines of $\mathcal{P}$, respectively, or*

(ii) *all red and blue lines in $\mathcal{P}_1$ are blue and red lines of $\mathcal{P}$, respectively.*

In particular, up to relabelling of the colours, every red and blue line of $\mathcal{P}_1$ is a red and blue line of $\mathcal{P}$, respectively.

Proof. Let l and m be two intersecting lines in $\mathcal{P}_1$. If l and m have different colours, then $l^\perp \cap m^\perp$ is a unique point and if they have the same colour, then l and m strongly intersect. This observation follows from Remark 5.2.5. In both cases, all such lines lie in $\mathcal{P}_1$. If we were to view l and m as lines in $\mathcal{P}$, then lines between the points on l and m would remain unchanged. Thus the colours of l and m would be predetermined. In particular, either l and m remain the same colour or they both change colour.

Consider an arbitrary pair of lines l and m. Since the collinearity graph of a root shadow space is connected and has finite diameter, there exists a sequence of lines $(l_0 = l, l_1, l_2, \ldots, l_s = m)$ such that l_{i-1} and l_i intersect. By the above observation, if l changes colour, then l_1 changes colour, l_2 changes colour and continue in this way until we reach m and this is forced to change colour. If the colour of l remains unchanged, then in a similar manner as above, the colour of m is forced to remain unchanged. The lines l and m were arbitrary and thus the result is proven. $\square$

We define two relations $\mathcal{R}$ and $\mathcal{B}$ on $\mathcal{P}$ in the following way: $x\mathcal{R}y$ (respectively, $x\mathcal{B}y$) if and only if x and y lie in a common red line (respectively, blue line).

Lemma 5.2.7 *The relations $\mathcal{R}$ and $\mathcal{B}$ are equivalence relations.*

Proof. We prove that $\mathcal{R}$ is an equivalence relation. Let x, y and z be points of $\mathcal{P}$. It is clear that $\mathcal{R}$ is reflexive and symmetric. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then with $x = (\langle v \rangle, W)$ we require that $y = (\langle v \rangle, U)$ and similarly $z = (\langle v \rangle, T)$. Thus $x\mathcal{R}z$. Thus $\mathcal{R}$ is an equivalence relation. Similarly, one can show that $\mathcal{B}$ is an equivalence relation. □

The $\mathcal{R}$-classes and $\mathcal{B}$-classes are called *red* and *blue classes*, respectively. The red class and blue class containing $(\langle v \rangle, W)$ are denoted by $\langle v \rangle_{\mathcal{R}} = \{(\langle v \rangle, U) \mid U \supset \langle v \rangle\}$ and $W_{\mathcal{B}} = \{(\langle u \rangle, W) \mid \langle u \rangle \subseteq W\}$, respectively. We state following observations as a lemma.

Lemma 5.2.8 *Let $\langle u \rangle$ and $\langle v \rangle$ be 1-dimensional and U and W be n-dimensional subspaces of V. Then we have*

(i) *$\langle u \rangle = \langle v \rangle$ if and only if $\langle u \rangle_{\mathcal{R}} = \langle v \rangle_{\mathcal{R}}$,*

(ii) *$U = W$ if and only if $U_{\mathcal{B}} = W_{\mathcal{B}}$, and*

(iii) *$\langle v \rangle_{\mathcal{R}} \cap W_{\mathcal{B}}$ is nonempty if and only if $\langle v \rangle_{\mathcal{R}} \cap W_{\mathcal{B}} = \{(\langle v \rangle, W)\}$ if and only if $\langle v \rangle \subseteq W$.*

Proof. It is clear that $\langle u \rangle = \langle v \rangle$ implies $\langle u \rangle_{\mathcal{R}} = \langle v \rangle_{\mathcal{R}}$. Suppose that $\langle u \rangle_{\mathcal{R}} = \langle v \rangle_{\mathcal{R}}$. Then $\{(\langle u \rangle, W) \mid W \supset \langle u \rangle\} = \{(\langle v \rangle, W) \mid W \supset v\}$. But the set of all hyperplanes containing $\langle v \rangle$ cannot also contain $\langle u \rangle$ unless $\langle v \rangle = \langle u \rangle$ as required. This proves (i), and (ii) is achieved in a similar way. For part (iii), it is obvious that $\langle v \rangle_{\mathcal{R}} \cap W_{\mathcal{B}}$ is nonempty if and only if the intersect consists of the single pairs $(\langle v \rangle, W)$ and, by definition, this occurs if and only if $\langle v \rangle \subseteq W$. □

Definition 5.2.9 *A pair (R, B) with $R \subseteq \mathcal{R}$ and $B \subseteq \mathcal{B}$ subsets of red and blue classes is called a* dual pair *if $\langle v \rangle_{\mathcal{R}} \cap W_{\mathcal{B}}$ is nonempty for all $\langle v \rangle_{\mathcal{R}} \in R$ and $W_{\mathcal{B}} \in B$.*

Define a symmetric relation $\sim_{\mathcal{P}}$ on the set of dual pairs by $(R, B) \sim_{\mathcal{P}} (R', B')$ if and only if $R \subseteq R'$ and $B' \subseteq B$, or $R' \subseteq R$ and $B \subseteq B'$. A dual pair (R, B) is *maximal* if whenever (R', B') is another dual pair and $R \subseteq R'$ and $B' \subseteq B$, or $R' \subseteq R$ and $B \subseteq R'$, we have that $R = R'$ and $B = B'$. Let $\mathit{\Gamma}$ denote the set of maximal dual pairs of $\mathcal{P}$.

If we choose $R_0 \subseteq \mathcal{R}$ arbitrarily, then we can choose B maximal such that (R_0, B) is a dual pair. We can then take R to be the largest subset of $\mathcal{R}$ that contains R_0 such that (R, B) is a dual pair. The dual pair (R, B) is maximal and every maximal dual pair can be obtained in this way. Indeed, suppose that $(R \cup r, B)$ is a dual pair for some $r \in \mathcal{R}$. Then $r \cap b$ is nonempty for all $b \in B$. But R was chosen maximally to have this property and thus $r \in R$. Suppose that $(R, B \cup b)$ is a dual pair for $b \in \mathcal{B}$. Then $b \cap r$ is nonempty for all $r \in R_0 \subseteq R$. But B was chosen maximally to have this property and thus $b \in B$.

To illustrate this process, let $R_0 = \{\langle v_a \rangle_{\mathcal{R}} \mid a \in \Lambda\}$ where Λ is an indexing set. Take B to be as large as possible such that (R_0, B) is a dual pair, that is, $B = \cap_{a\in\Lambda}\{W_{\mathcal{B}} \mid W_{\mathcal{B}} \cap \langle v_a \rangle \neq \emptyset\}$. By Lemma 5.2.8, $W_{\mathcal{B}} \cap \langle v_a \rangle_{\mathcal{R}}$ is nonempty for all $a \in \Lambda$ if and only if $\langle v_a \rangle \subseteq W$ for all $a \in \Lambda$. Thus $\langle \sum_{a\in\Lambda} \lambda_a v_a \rangle \subseteq W$ for all $\lambda_a \in F$ and $a \in \Lambda$ and this holds if and only if $W_{\mathcal{B}} \cap \langle \sum_{a\in\Lambda} \lambda_a v_a \rangle_{\mathcal{R}}$ is nonempty for all $\lambda_a \in F$. Let $\{v_{i_1}, \ldots, v_{i_k}\}$ be a basis for the subspace $\langle v_a \mid a \in \Lambda \rangle$. Set R to be the maximal subset of $\mathcal{R}$ containing R_0 such that (R, B) is a dual pair, that is,

$$\begin{aligned} R &= \left\{\langle \textstyle\sum_{a\in\Lambda} \lambda_a v_a \rangle_{\mathcal{R}} \mid \lambda_a \in F, a \in \Lambda\right\} \\ &= \left\{\langle \textstyle\sum_{j=1}^{k} \lambda_j v_{i_j} \rangle_{\mathcal{R}} \mid \lambda_j \in F\right\}, \end{aligned}$$

where $B = \cap_{j=1}^{k}\{W_{\mathcal{B}} \mid W_{\mathcal{B}} \cap \langle v_{i_j} \rangle_{\mathcal{R}} \neq \emptyset\}$. In particular, every maximal dual pair has the form (R, B).

Lemma 5.2.10 *There is an incidence-preserving bijection ϕ from Γ to $PG(V)$, namely, $(R, B)^{\phi} = \langle v_{i_1}, \ldots, v_{i_k} \rangle$ where (R, B) is defined as above.*

Proof. We first check that this map is well defined. In the construction of (R, B), we choose a basis $\{v_{i_1}, \ldots, v_{i_k}\}$ of $\langle v_a \mid a \in \Lambda \rangle$ but if we were to choose another basis $\{w_{i_1}, \ldots, w_{i_k}\}$, then $\langle v_{i_1}, \ldots, v_{i_k} \rangle = \langle v_a \mid a \in \Lambda \rangle = \langle w_{i_1}, \ldots, w_{i_k} \rangle$. Thus the image of (R, B) under ϕ is well defined. Let $R' = \{\langle \sum_{j=1}^{k} \mu_j w_{i_j} \rangle_{\mathcal{R}} \mid \mu_j \in F\}$ and $B' \subseteq \mathcal{B}$ such that (R', B') is a maximal pair. Suppose $(R, B)^{\phi} = (R', B')^{\phi}$. Then $\langle v_{i_1}, \ldots, v_{i_k}\rangle = \langle w_{i_1}, \ldots, w_{i_k}\rangle$ and this holds if and only if

$$\left\{\langle\sum_{j=1}^{k}\lambda_j v_{i_j}\rangle \mid \lambda_j \in F\right\} = \left\{\langle\sum_{j=1}^{k}\mu_j w_{i_j}\rangle \mid \mu_j \in F\right\}.$$

By Lemma 5.2.8 (i), this holds if and only if $R = R'$ and $B = B'$. That is, $(R, B)^{\phi} = (R', B')^{\phi}$ if and only if $(R, B) = (R', B')$. Thus the map ϕ is injective. It is clear that ϕ is surjective and we only need to show that it preserves incidence. Suppose that $(R, B) \sim_{\mathcal{P}} (R', B')$ and, without loss of generality, $R \subseteq R'$ and $B' \subseteq B$. Let (R, B) be as before and $R' = \{\langle\sum_{j=1}^{k+l}\lambda_j v_{i_j}\rangle_{\mathcal{R}} \mid \lambda_j \in F\}$ and $B' \subseteq \mathcal{B}$ such that (R', B') is a maximal dual pair. Then $(R, B)^{\phi} = \langle v_{i_1}, \ldots, v_{i_k}\rangle \subseteq \langle v_{i_1}, \ldots, v_{i_k}, \ldots, v_{i_{k+l}}\rangle = (R', B')^{\phi}$ as required. □

Under this isomorphism, objects of type 1 correspond to the pairs $(\langle v\rangle_{\mathcal{R}}, B_v)$ where $B_v = \{U_{\mathcal{B}} \mid U \supset \langle v\rangle\}$ and objects of type n correspond to the pairs $(R_W, W_{\mathcal{B}})$ where $R_W = \{\langle u\rangle_{\mathcal{R}} \mid \langle u\rangle \subseteq W\}$. Two such objects are incident if and only if $\langle v\rangle_{\mathcal{R}} \in R_W$ and $W_{\mathcal{B}} \in B_v$, if and only if $\langle v\rangle_{\mathcal{R}} \cap W_{\mathcal{B}} = \{(\langle v\rangle, W)\}$ if and only if $\langle v\rangle \subseteq W$. Therefore there exists a natural map from $\mathrm{RSh}(\mathit{\Gamma})$ to $\mathcal{P}$, namely, $((\langle v\rangle_{\mathcal{R}}, B_v), (R_W, W_{\mathcal{B}})) \mapsto (\langle v\rangle, W)$. A blue line in $\mathrm{RSh}(\mathit{\Gamma})$ between two points $(\langle v_1\rangle_{\mathcal{R}}, B_{v_1})$ and $(\langle v_2\rangle_{\mathcal{R}}, B_{v_2})$ is $\{(\langle\lambda_1 v_1 + \lambda_2 v_2\rangle_{\mathcal{R}}, B) \mid \lambda_i \in F\}$ for some uniquely determined B. In particular, this maps onto the line $\{(\langle\lambda_1 v_1 + \lambda_2 v_2\rangle, W) \mid \lambda_i \in F\}$ where W is a hyperplane determined by B. Similarly, one can bijectively map red lines to red lines. In particular, $\mathrm{RSh}(\mathit{\Gamma})$ and $\mathcal{P}$ are isomorphic root shadow spaces.

Lemma 5.2.11 *Let Δ be the building of type A_n as a projective space and $\mathcal{P}$ be the corresponding root shadow space. Then a blue line in $\mathcal{P}$ determines a unique line in Δ and every line in Δ is determined by a (non-unique) blue line in $\mathcal{P}$.*

Proof. Let W be a hyperplane in V. For $\langle v_1\rangle, \langle v_2\rangle \subseteq W$, consider the blue line through the points $(\langle v_1\rangle, W)$ and $(\langle v_2\rangle, W)$, namely the line $\{(\langle\lambda_1 v_1 + \lambda_2 v_2\rangle, W) \mid \lambda_i \in F\}$. This corresponds to the line $\langle v_1, v_2 \rangle$ in Δ. On the other hand, any other hyperplane U incident to both $\langle v_1 \rangle$ and $\langle v_2 \rangle$ yields the same line. Conversely, it is clear that any line $\langle v_1, v_2 \rangle$ in Δ corresponds to a blue in $\mathcal{P}$. □

Remark 5.2.12 *The building Γ is a projective space and thus is determined by its points and lines, that is, its objects of type* 1 *and* 2. *The points are of the form $(\langle v \rangle_{\mathcal{R}}, B)$ and the unique line between the two points $(\langle v_1 \rangle_{\mathcal{R}}, B_1)$ and $(\langle v_2 \rangle_{\mathcal{R}}, B_2)$ is $(\langle v_1, v_2 \rangle_{\mathcal{R}}, B)$ where $B = B_1 \cap B_2 = \{W_{\mathcal{B}} \mid W \supset \langle v_1, v_2 \rangle\}$. We can identify $(\langle v_i \rangle_{\mathcal{R}}, B_i)$ with $\langle v_i \rangle_{\mathcal{R}}$ and $(\langle v_1, v_2 \rangle_{\mathcal{R}}, B)$ with $B = \{W_{\mathcal{B}} \mid W \supset \langle v_1, v_2 \rangle\}$. In particular, points and lines are identified with collections of red and blue lines, respectively and thus giving us another description of the building Γ in terms of red and blue lines.*

We have recovered a building of type A_n from the root shadow space $\mathcal{P}$, namely the set of maximal dual pairs and we have described everything in terms of red and blue lines. However, we can define maximal dual pairs in the extremal geometry. There is no natural way of defining red and blue lines in the $\mathcal{E}$ but we can call a line red (blue, respectively) if its image under the (unknown) isomorphism given in [9] is a red line (blue line, respectively) in the root shadow space as constructed from the vector space. In particular, from the extremal geometry $(\mathcal{E}, \mathcal{F})$, which is isomorphic to $\mathcal{P}$, we can obtain a building of type A_n denoted by $\Delta(\mathcal{E})$, namely the set of maximal dual pairs defined on $\mathcal{E}$. We can define dual pairs more explicitly in terms of the Lie algebra. Recall that $(\mathcal{E}, \mathcal{F})$ is the extremal geometry of L, that is, $\mathcal{E} = \{Fx \mid x \in E\}$ where E is the set of nonzero extremal elements of L. We define the relations $\mathcal{R}$ and $\mathcal{B}$ as before and let $x_{\mathcal{R}}$ and $x_{\mathcal{B}}$ denote the red and blue classes of the point Fx, respectively.

Definition 5.2.13 *A pair (R, B) with $R \subseteq \mathcal{R}$ and $B \subseteq \mathcal{B}$ is a* dual pair of the extremal geometry $\mathcal{E}$ *if $x_{\mathcal{R}} \cap y_{\mathcal{B}}$ is nonempty for all $x_{\mathcal{R}} \in R$ and $y_{\mathcal{B}} \in B$.*

We define $\Delta(\mathcal{E})$ to be the set of maximal dual pairs of $\mathcal{E}$ and a symmetric relation on $\Delta(\mathcal{E})$ as usual inclusion. We state a result which follows from [9] and Lemma 5.2.10.

Theorem 5.2.14 *The set $\Delta(\mathcal{E})$ is a building of type A_n.* $\square$

As before, we can identify the root shadow space of $\Delta(\mathcal{E})$ with $\mathcal{E}$ in the following way. The points and hyperplanes of $\Delta(\mathcal{E})$ correspond to the pairs (r, B_r) and (R_b, b), respectively and they are incident if and only if $r \in R_b$ and $b \in B_r$, if and only if $r \cap b$ is nonempty. The intersection $r \cap b$ corresponds to a unique point of $\mathcal{E}$ whenever it is nonempty. In particular, we identify points $((r, B_r), (R_b, b))$ of the root shadow space $\Delta(\mathcal{E})$ with the point $r \cap b$ of $\mathcal{E}$. Let $P_\Delta(\mathcal{E})$ be the description of the point-line space of $\Delta(\mathcal{E})$ as given in Remark 5.2.12. In particular, a point of $P_\Delta(\mathcal{E})$ is identified as the collection of red lines through a given extremal 1-space and a line is given as a collection of blue lines.

Theorem 5.2.15 *Let L be a finite dimensional Lie algebra over a field F of characteristic p. Suppose that L is generated by its extremal elements, contains no sandwich elements and has extremal geometry isomorphic to the root shadow space of type $A_{n,\{1,n\}}$. If $p \neq 2$ and $(n,p) \neq (2,3)$, then L is a classical Lie algebra or type or the quotient of a Chevalley algebra by its unique 1-dimensional ideal.*

Proof. It suffices to show that L and L' coincide. Consider the subspace $\tilde{\mathcal{E}}_i$ of $\mathcal{E}(L')$. By Theorem 3.4.4, $\tilde{\mathcal{E}}_i$ is a root shadow space of a building of type A_n. In particular, we can construct the building $\Delta(\tilde{\mathcal{E}}_i)$ as in Theorem 5.2.14. By Lemma 5.2.6, we can assume that red lines and blue lines of $\tilde{\mathcal{E}}_i$ are red lines and blue lines of $\mathcal{E}$, respectively. In particular, the collection of red lines in $\tilde{\mathcal{E}}_i$ going through a point in $\tilde{\mathcal{E}}_i$ is a subcollection of the red lines in $\mathcal{E}$ going through the same point and blue lines of $\tilde{\mathcal{E}}_i$ are blue lines of $\mathcal{E}$. This allows for the projective space $P_\Delta(\tilde{\mathcal{E}}_i)$ to be embedded into the projective space $P_\Delta(\mathcal{E})$. Then $P_\Delta(\tilde{\mathcal{E}}_i)$ can be viewed as a subspace of $P_\Delta(\mathcal{E})$. But $P_\Delta(\tilde{\mathcal{E}}_i)$ has the same rank as

$P_\Delta(\mathcal{E})$. By Lemma 4.1.4, we conclude that $P_\Delta(\mathcal{E}) = P_\Delta(\tilde{\mathcal{E}}_i)$. Hence $\Delta(\mathcal{E})$ and $\Delta(\tilde{\mathcal{E}}_i)$ are isomorphic buildings and thus $\mathcal{E} = \tilde{\mathcal{E}}_i$. But then $L = \langle \mathcal{E} \rangle = \langle \tilde{\mathcal{E}}_i \rangle = L'$ as required. $\square$

Appendix A
Chamber Systems and Geometries

Throughout this chapter the set I is $\{1, \ldots, n\}$. A *geometry* $\mathit{\Gamma}$ over I is a triple $(V, \sim, \tau)$ where V is the set consisting of *objects*, $\sim$ is a reflexive symmetric relation on V and τ is a surjective map from V to I such that no two incident objects have the same image under τ. This map is called the *type map*. A *flag* is a subset of pairwise incident objects of V. In particular, τ restricted to a flag is injective. The *type* (*cotype*) and *rank* (*corank*) of a flag F is $\tau(F)$ $(I \setminus \tau(F))$ and $|\tau(F)|$ $(|I \setminus \tau(F)|)$, respectively. Another property we require a geometry to satisfy is that every maximal flag has type I and that every flag is contained in a maximal flag. A flag of type I is called a *chamber*. The *rank* of $\mathit{\Gamma}$ is $|I|$. The *residue* of a flag F is the set of objects in $V \setminus F$ that are incident to every object of F and is denoted by $\mathrm{Res}_{\mathit{\Gamma}}(F)$. A geometry is *residually connected* if and only if every residue of a flag of corank at least two and every residue of a flag of corank one are connected and nonempty, respectively. In particular, if $\mathit{\Gamma}$ is a residually connected geometry of rank at least two, then $\mathit{\Gamma}$ is the residue of the empty flag and thus connected.

A *chamber system* Δ *over* I is a pair $(\mathcal{C}, \{\sim_i\}_{i \in I})$ such that is an edge-coloured graph where $\mathcal{C}$ is a the set of vertices called *chambers* and $x \sim_i y$ if and only if x and y are joined by an edge labelled (or coloured) i. For a subset $J \subseteq I$, a *residue of type* J or a J*-residue* is a connected component of the graph obtained from Δ by deleting all the edges with labels $i \in I \setminus J$. If $J = \{j\}$, then a J-residue is called a j*-panel*. For a J-residue R, we say that R has rank $|J|$, cotype $I \setminus J$ and corank $|I \setminus J|$. Another property we require a chamber to satisfy is that, for each $i \in I$, every i-panel is a complete graph. The chamber system Δ is *residually connected* if, for every subset J of I and every collection of cotype j residues R_j one for each $j \in J$ with the property that any two have a nonempty intersection, we have that $\cap_{j \in J} R_j$ is a residue of type $I \setminus J$.

Proposition A.0.1 *Every building is a residually connected chamber system.*

Proof. Let $J \subseteq I$ and for each $j \in J$ let R_j be a collection of cotype j residues with the property that any two have a nonempty intersection. We know that either $\cap_{j \in J} R_j$ is empty or a residue of type $I \setminus J$. It suffices to show that it is nonempty. We do induction on $|J|$. If $|J|$, then the result holds trivially. Suppose that $|J| > 2$. Choose $j_0 \in J$ and set $J' = J \setminus \{j_0\}$. Then, by the inductive hypothesis, $R := \cap_{j \in J'} R_j$ is nonempty and hence a residue. Let $x \in R_{j_0}$ and, for each $j \in J'$, let $x_j = \mathrm{proj}_{R_j} x$ and let $x' = \mathrm{proj}_R x$.

Since $R_{j_0} \cap R_j$ is nonempty and R_{j_0} is convex, we have that $x_j \in R_{j_0}$ for each $j \in J'$. By Lemma 1.3.7, for every $y \in R \subseteq R_j$, x_j is the unique chamber in R_j such that $\text{dist}(x, y) = \text{dist}(x, x_j) + \text{dist}(x_j, y)$. However, $x' \in R \subseteq R_j$ and, by the Lemma 1.3.7, $\text{dist}(x, y) = \text{dist}(x, x') + \text{dist}(x', y)$. Hence by the uniqueness of x_j, we have that $x' = x_j$ for all $j \in J'$. But then $x' \in R_{j_0}$ and thus $x' \in \cap_{j \in J} R_j$ as required. □

Let $\Delta = (\mathcal{C}, \{\sim_i\}_{i \in I})$ be a chamber system over a set $I = \{1, \ldots, n\}$.

Definition A.0.2 *The* flag geometry associated to Δ *is the geometry* $\text{Gm}(\Delta) = (V, \sim, \tau)$ *where V is the set of corank one residues of Δ, $R \sim S$ if and only if R and S have a nonempty intersection as sets of chambers of Δ and τ is defined to be the cotype of such a residue.*

Note that if $\tau(S) = \tau(T)$, then R and S are both residues of cotype i and they are incident if and only if they have a nonempty intersection, if and only if $R = S$. Thus τ is a well defined type-map. Let $\mathit{\Gamma} = (V, \sim, \tau)$ be a geometry over I.

Definition A.0.3 *The* Chamber system associated to the geometry $\mathit{\Gamma}$ *is the chamber system* $\text{Ch}(\mathit{\Gamma}) = (\mathcal{C}, \{\sim\}_{i \in I})$ *where $\mathcal{C}$ is the set of maximal flags of $\mathit{\Gamma}$ and $C \sim_i D$ if and only if C and D share a common flag of cotype i.*

Proposition A.0.4 *If Δ is residually connected, then* $\text{Gm}(\Delta)$ *is residually connected and* $\Delta \cong \text{Ch}(\text{Gm}(\Delta))$.

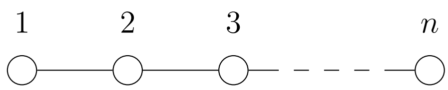
Proof. The first claim follows from Proposition 3.6.6. of [3]. Objects of $\text{Gm}(\Delta)$ are the residues R_i of cotype i for all $i \in I$. We have that $\tau(R_i) = i$ and $R_i \sim R_j$ if and only if $R_i \cap R_j$ is nonempty. Chambers of $\text{Ch}(\text{Gm}(\Delta))$ are the maximal flags $\{R_1, \ldots, R_n\}$ such that $R_i \cap R_j$ is nonempty for all $i, j \in I$. As Δ is residually connected, $\cap_{i \in I} R_i$ is a residue of type $I \setminus I$, that is, $\cap_{i \in I} R_i$ is a chamber. Two chambers $\{R_1, \ldots, R_n\}$ and $\{S_1, \ldots, S_n\}$ are j-adjacent if $R_i = S_i$ for all $i \neq j$ and $R_j \neq S_j$. Let c be a chamber of Δ and let R_i be the unique residue of cotype i containing c. Then $c = \cap_{i \in I} R_i$. Define a map from Δ to $\text{Ch}(\text{Gm}(\Delta))$ by $\cap_{i \in I} R_i \mapsto \{R_1, \ldots, R_n\}$. This map is clearly bijective. Let $c = \cap R_i$ and $d = \cap S_i$ be two chambers in Δ. Then $c \sim_j d$, if and only if the edge joining c and d lies in each R_i and S_i for all $i \neq j$, if and only if $R_i = S_i$ for all $i \neq j$ (by the uniqueness of the residue of cotype i containing c) and $R_j \neq S_j$, if and only if $\{R_1, \ldots, R_n\} \sim_j \{S_1, \ldots, S_n\}$ as required. □

Appendix B
Irreducible root systems

At the end of Section 1.3 and 1.4 we stated classifications, by type, of spherical buildings and root systems, respectively and each type was of the form X_n. In this appendix we construct the Dynkin diagrams and the corresponding irreducible root systems. The constructions can be found in [2].

Root system of type A_n

Let $V = \mathbb{R}^{n+1}$ and define the root system to be $\Phi = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n+1\}$ and let the simple roots be $B = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq n\}$. The Dynkin diagram of type A_n is

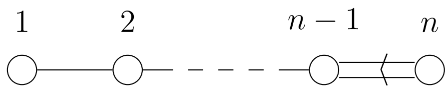


Root system of type B_n

Let $V = \mathbb{R}^n$ and define the root system to be $\Phi = \{\pm e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid i \leq i < j \leq n\}$ and let the simple roots be $B = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = e_n\}$. The Dynkin diagram of type B_n is

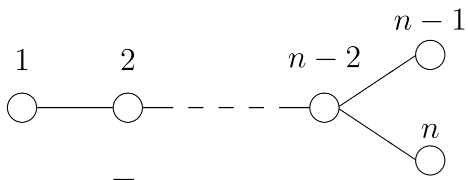
Root system of type C_n

Let $V = \mathbb{R}^n$ and define the root system to be $\Phi = \{\pm 2e_i \mid 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \mid i \leq i < j \leq n\}$ and let the simple roots be $B = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\alpha_n = 2e_n\}$. The Dynkin diagram of type B_n is



Root system of type D_n

Let $V = \mathbb{R}^n$ and define the root system to be $\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$ and let the simple roots be $B = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$. The Dynkin diagram of type D_n is

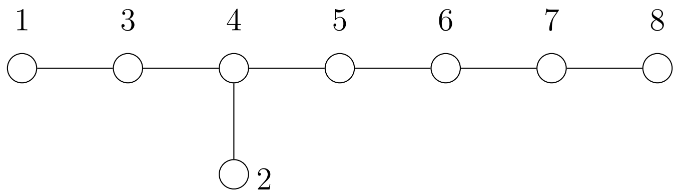


Root system of type E_8

Let $V = \mathbb{R}^8$ and define a root system as $\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 8\} \cup X$ where

$$X = \left\{ \frac{1}{2} \left(\sum_{i=1}^{8} (-1)^{v_i} e_i \right) \mid \sum_{i=1}^{8} v_i \text{ is even} \right\}.$$

The simple roots are $B = \{\alpha_1 = \frac{1}{2}(e_1+e_8) - \frac{1}{2}(e_2+e_3+e_4+e_5+e_6+e_7), \alpha_2 = e_1+e_2, \alpha_3 = e_2 - e_1, \alpha_4 = e_3 - e_2, \alpha_5 = e_4 - e_3, \alpha_6 = e_5 - e_4, \alpha_7 = e_6 - e_5, \alpha_8 = e_7 - e_6\}$. The Dynkin diagram is

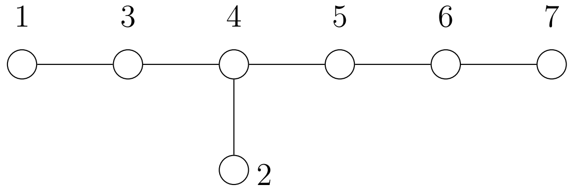


Root system of type E_7

The vector space V spans a 7-dimensional subspace of $\mathbb{R}^8$ and the root system is a subroot system of E_8. The root system of type E_7 is $\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 6\} \cup \{e_7 - e_8\} \cup X$ where

$$X = \left\{ \pm\frac{1}{2} \left(\sum_{i=1}^{6} (-1)^{v_i} e_i + e_7 - e_8 \right) \mid \sum_{i=1}^{6} v_i \text{ is odd} \right\}.$$

The simple roots are $B = \{\alpha_i \mid i \in [1, 7]\}$ where α_i is defined in the root system of type E_8. The Dynkin diagram is



Root system of type E_6

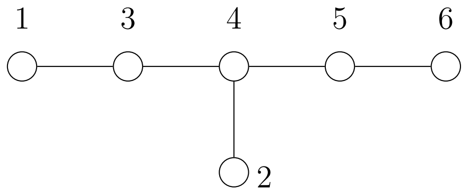
The vector space V spans a 6-dimensional subspace of $\mathbb{R}^8$ and the root system is a subroot system of E_8. The root system of type E_6 is $\Phi = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 5\} \cup X$ where

$$X = \left\{ \pm\frac{1}{2}\left(\sum_{i=1}^{6}(-1)^{v_i}e_i + e_7 - e_8\right) \mid \sum_{i=1}^{6} v_i \text{ is odd } \right\}.$$

The simple roots are $B = \{\alpha_i \mid i \in [1,7]\}$ where α_i is defined in the root system of type E_8. The Dynkin diagram is

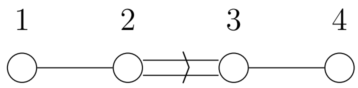
$$X = \left\{ \pm\frac{1}{2}\left(\sum_{i=1}^{5}(-1)^{v_i}e_i - e_6 - e_7 + e_8\right) \mid \sum_{i=1}^{5} v_i \text{ is even } \right\}.$$

The simple roots are $B = \{\alpha_i \mid i \in [1,6]\}$ were α_i is defined in the root system of type E_8. The Dynkin diagram is



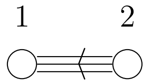
Root system of type F_4

Let $V = \mathbb{R}^4$ and define the root system as $\Phi = \{\pm e_i \mid 1 \leq i \leq 4\} \cup \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$. The simple roots are $B = \{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$. The Dynkin diagram of type F_4 is



Root system of type G_2

The vector space V is a hyperplane of $\mathbb{R}^3$ and the root system is $\Phi = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \pm(2e_1 - e_2 - e_3), \pm(-e_1 + 2e_2 - e_3), \pm(-e_1 - e_2 + 2e_3)\}$. The simple roots are $B = \{\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3\}$. The Dynkin diagram of type G_2 is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