A bitopological point-free approach to compactifications

by

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Abstract

This thesis extends the concept of compactifications of topological spaces to a setting where spaces carry a partial order and maps are order-preserving. The main tool is a Stone-type duality between the category of *d-frames*, which was developed by Jung and Moshier, and bitopological spaces. We demonstrate that the same concept that underlies d-frames can be used to do recover short proofs of well-known facts in domain theory. In particular we treat the upper, lower and double powerdomain constructions in this way.

The classification of order-preserving compactifications follows ideas of B. Banaschewski and M. Smyth. Unlike in the categories of spaces or locales, the lattice-theoretic notion of normality plays a central role in this work. It is shown that every compactification factors as a normalisation followed by the maximal compactification, the Stone-Čech compactification. Sample applications are the Fell compactification and a stably compact extension of algebraic domains.

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Chapter 0

Introduction and Motivation

Order theory pervades everyday life, through the order on natural numbers as a concept of quantities as well as the order on the real numbers as a concept of the flow of time and causality. Topological spaces are one of the two branches that real analysis, the study of smooth real-valued functions, developed into at the beginning of the 20th century (the other branch is measure spaces). Thanks to its generality, topology has found numerous applications throughout mathematical sciences, theoretical computer science being no exception. Compactifications are, in a certain sense, ways of making a large space larger in order to make it appear small. In this thesis we explore the tools necessary to extend the theory of compactifications of topological spaces to a setting where spaces are ordered and maps between them are order-preserving.

To be more precise about the line of thought in this thesis, we abstract the compactification problem of ordered spaces to a more general setting. The techniques we employ are borrowed from domain theory and locale theory. In the first abstraction step we replace the two items of data, the topology and the order on the underlying set, by two new items of data. The ordered topological space becomes a set with two topologies (a *bitopological* space) where we think of the two collections as the opens that are downward and upward closed opens in the given order, respectively. Thus the original problem becomes the compactification problem for bitopological spaces. In the second abstraction step we move to the localic setting and consider algebraic objects that are related to bitopological spaces via a Stone-type duality. These algebraic objects we call *d-frames*. Once an appropriate notion of compactification is formulated for d-frames, it becomes apparent that compactifications are obtained by a familiar technique that is called *round ideal completion* in the domain theory literature.

There are two recurring themes throughout this work: The first one is the use of dualities, where we not only make use of Stone-type dualities between categories of spaces and categories of algebraic objects, but also extensive use of self-dualities such as the order-dual of partially ordered sets. The second theme is the decomposition of an order relation on a single set into two more fundamental relations between a pair of sets.

In the following sections we motivate the topic and the tools of this thesis and review where the techniques that we use originate from.

0.1 Organisation of this document

0.1.1 What is where

The remaining sections of this chapter are not meant to serve as a comprehensive introduction. Instead, the reader may refer to the more detailed introductions at the beginning of each chapter. The notes concluding each chapter contain historical notes, references, list the author's contributions and hint at open problems and ongoing work. The appendix lists definitions and known facts about the various structures we intend to present in Chapters 1, 2 and 3. It is best consulted as supplementary reference from within the main chapters. In order to aid the reader in coping with the idiosyncrasies of this thesis, lists of symbols (page 199) and an index (page 208) are provided.

0.1.2 What it is built on

Our methods rely heavily on techniques that were developed around continuous lattices and domains, the core of which is now gathered in the Compendium [22]. In one way or another, practically every result in this thesis can be considered as a contribution to domain theory. As reference on bitopology, Nachbin's monograph [45] or Kopperman's paper [38] may be used, although our point-free approach has much in common with the *biframes* that Banaschewski and others [6] have developed. The very definition of the structure we employ, *d-frames*, is due to Jung and Moshier, and their technical report [33] may serve the reader as additional reference. Special cases of order-preserving compactifications have been considered in the domain-theoretic community, instances are [50, 26, 28, 23]. In domain theory, *round ideal completions* have been a tool since the early stages, see [17, 49, 23, 48, 1, 60]. It has also been known for long that round ideal completions are intimately related to compactifications, a fact that we shall exploit in Chapter 4. We assume a modest amount of category theory as a prerequisite for reading this thesis, however, not much more than functors, adjunctions and monads are used here. In the appendix, these notions can be seen instantiated on partially ordered sets. One of the standard references for category theory is [43]. Another important tool we use is *Stone duality*, originally conceived by M. H. Stone [51, 52] and developed further by Isbell, Priestley [46], Johstone [30] and many others. Stone duality allows one to switch between the spatial/geometric and the logical/algebraic point of view. D-frames are Stone duals of bitopological spaces and, as discussed in Section 3.1, have some advantages over biframes.

0.1.3 What is new

Chapter 1 recovers well-knownand not-so-well-known facts about *domains* (see Subsection 6.1.7) in a novel way. It can be seen as combining the presentation of a domain by abstract bases with the presentation by information systems (see Section 6.2). Our approach is tuned so that (1) proofs can be done in a natural deduction style and never involve infinite operations, (2) dualities are built into the structures rather explicitly, and therefore (3) switching from one point of view to the dual is effortless. Each of the first few sections of Chapter 1 comes equipped with a minimal set of axioms necessary to characterise an associated subcategory of domains. Although our morphisms are slightly more complicated to work with than the morphisms between information systems, we retain all the advantages. For instance, the duality functors such as Stone duality or Lawson duality leave the data that describes a morphism invariant, only reverses its direction. At the same time, Chapter 1 lays the conceptual foundations for the subsequent chapters, and indeed in Section 2.4 we meet a proper subcategory of the category explored in Chapter 1. While the results of Chapters 2, 3 and 4 are mostly independent of those in Chapter 1, the proofs share the same techniques.

Chapter 2 mainly builds up auxiliary results for Chapter 3, but the structures considered there do not appear in other published work except [36] and are interesting in their own right. A particular aspect worth mentioning is a seemingly innocent generalisation of normal lattices which behaves rather differently in this work.

In Chapter 3 we present the concrete Stone duality for bitopological spaces that was developed by Jung and Moshier. We demonstrate that said Stone duality completes the three categories Top, Frm and BiTop to a commutative square of (dual) adjunctions. Other novel results involve a bitopological analogue of the Heyting negation of frames and some useful facts about regularity.

Chapter 4 sets out with a bitopological version of completely regular frames, that we derive from the analogous concepts for locales and bitopological spaces in a straightforward manner. Once a notion of point-free bitopological compactification is agreed upon, we prove a classification theorem very much like the ones known for spaces or locales. Surprisingly, the compactifications of the structures that we consider all admit a certain factorisation, which is not known of spaces or locales.

Compared to the journal paper [36], the definition of proximity ([36, Definition 13] vs. Definition 4.3.1) is simplified, and Chapter 1 as well as the applications in Section 4.5 are new. Some of the notation was unified, compare [36, Lemma 3] against Lemma 2.3.6.

0.1.4 How to read this document

The reader interested in compactifications alone may first read the introduction of Chapter 4 to get an overview of the previously existing techniques, and then start with Chapter 2. The Stone duality-minded person may find the introduction to Chapter 3 a good entry point, although numerous references to Chapter 2 are made in Chapter 3. The domain theorist and computer scientist may enjoy Chapter 1, where we provide new proofs involving Stone duality, Lawson duality and the Smyth, Hoare and double power constructions.

0.2 Why compactness?

Historically, the archetype of a compact space is a closed bounded interval in the real line. The Heine-Borel Theorem states that whenever such an interval is covered by a collection of open intervals then there exists a finite sub-collection of open intervals that still cover the closed interval. This theorem has many useful consequences. To list a few, any real-valued function on such a closed bounded interval is uniformly continuous and attains its infimum and supremum. The latter property is a consequence of the general fact that the image of a compact set under a continuous map is again compact. Compactness is also used in more involved constructions in analysis: Holomorphic functions defined on a subset of the complex plane are extended analytically along *paths*, that are continuous images of the unit interval. Again the Heine-Borel covering property is of central importance here. One approach to Lebesgue integration is via real-valued functions with compact support. Such a function is constant zero outside a compact subset. This approach works because the real line is *locally compact*, whence any real-valued function can be approximated by functions with compact support.

The Heine-Borel property can easily be generalised to arbitrary topological spaces by declaring a space to be compact if any covering of the space by open sets has a finite sub-cover.

In a wide variety of cases compactness is just as good as being finite (we make this more precise below). In a suitable setting one can show that (1) the intersection of compactly many open sets is open[1], and more generally, (2) the point-wise infimum of compactly many continuous functions is continuous[2].

In computer science compact spaces enter the scene via denotational semantics. Domain theory gives a model of functional programming languages such as the lambda calculus and PCF that associates each data type with an ordered topological space. The topology is a redundant bit of information since it is determined by the order alone. However, the denotation of a program term yields a *continuous* order-preserving map between the denotations of its source and target type. This fact is exploited when proving that a certain mathematical function between denotations can not arise as the denotation of a program. For example, the naive approach to exact real number computation via infinite sequences of binary digits is flawed, because even addition of two real numbers is not a

[1]There is a paper by Escardó with that name, submitted for publication.

[2]This holds for a collection of continuous functions $X \to L$ where X is exponentiable and L is a continuous lattice.

continuous operation on the denotational model.

As Martín Escardó phrased it once, the compact spaces are those spaces which continuous functions defined on them believe to be finite. This has rather surprising consequences. Given a suitable representation of a compact set[3] as a subset of a data type, one can search it in finite time, even if the compact subset is infinite. What this means is that for any continuous predicate on the data type (i.e. a boolean-valued function), it can be decided in finite time whether the predicate is true for every element of the compact subset. One says that the compact subsets admit *continuous universal quantification.* If we disregard for a moment that not every continuous function between types is computable, then the Hofmann-Mislove Theorem states that the compact subspaces are precisely the ones that admit continuous universal quantification.

If a continuous function $f : X \to Y$ is defined on a non-compact space where Y is compact, then one is interested in the *compactification problem*, that is to embed the space X into a compact space βX (called a compactification of X) such that f can be uniquely extended to a continuous map $\beta f : \beta X \to Y$. Classically one requires βX to be compact Hausdorff, whence X itself must be Hausdorff as well. In that case uniqueness of βf can be achieved by requiring X to be dense as a subspace of βX. By dropping the requirement that the map $X \to \beta X$ must be a topological embedding one obtains a more general notion of compactification.

An instance which has naturally attracted much attention is the compactifications of the real line. It admits a smallest compactification, the Alexandrov compactification. The real line is homeomorphic to the open unit interval $(0, 1)$ which embeds densely into the unit circle S^1 via $t \mapsto e^{2\pi t}$. A largest compactification always exists and its construction is due to Stone and Čech, who independently published it in 1937. The Stone-Čech compactification is functorial and in fact provides the left adjoint to the inclusion functor from compact Hausdorff spaces to topological spaces.

In case of the real line there is a third compactification that appears to be rather natural. One amends the real line by two points at infinity (say $-\infty$ and ∞) and thus obtains the *extended real line* $[-\infty, \infty]$ which is homeomorphic to the closed unit interval. This thesis is partly motivated by the question as to what extent this compactification is canonical.

0.3 Why bitopology?

A typical example of a bitopological space in disguise is the real line. Endowed with the Euclidean topology it is an ordered topological space where the usual order $\leq$ is a closed subset of the product space $\mathbb{R} \times \mathbb{R}$. It is a topological group where addition is continuous and order-preserving in each argument. For any ordered set the collections of upper- or lower closed subsets are closed under arbitrary unions and intersections. It follows that

[3] Escardó [20] uses *selection functions* `(X → 2) → X`

the upper- or lower closed open subsets of an ordered topological space each form another topology that is coarser than the original one. On the real line the two topologies thus constructed are known as the topology of lower- and upper semicontinuity. Addition is both upper- and lower semicontinuous in each argument, which captures monotonicity. Negation, however, is neither upper- nor lower semicontinuous because it reverses the order. Thus it must be understood as a map from the reals with the topology of upper semicontinuity to the reals with the topology of lower semicontinuity (or the other way around). In fact negation provides a homeomorphism between these two topological spaces. Observe that the order can be recovered from the topology alone as the specialisation order of the topology of lower semicontinuity. Another fact about the reals that is not true for arbitrary ordered topological spaces is that the Euclidean topology is the coarsest topology that contains both topologies of semicontinuity. This is why bitopological constructions are often not identified as such.

Moving from the Euclidean reals to the reals with the topology of lower semicontinuity can be understood as an act of symmetry breaking. It is a well-known fact that in every Hausdorff space compact subsets are closed. If the Hausdorff space is itself compact, then the dual statement is also true: Every closed subset is compact, resulting in the concepts of "closed" and "compact" being identical. For an asymmetric version of compact Hausdorff spaces one considers sets with two topologies where the closed subsets with respect to the first topology are precisely the compact saturated subsets with respect to the other topology. In that situation one says that the concepts of "closed" and "compact" are dual.

The class of asymmetric topological spaces with the self-duality described above is the class of stably compact spaces. These include many important instances, such as the real unit interval with the topology of lower semicontinuity, the coherent spaces that Stone used in his representation theorem for bounded distributive lattices as well as virtually every class of domains that are relevant to semantics of programming languages. Many theorems about compact Hausdorff spaces can be generalised to stably compact spaces. For example, there is a well-established theory of power constructions including the probabilistic powerspace. The Vietoris powerspace (a compact Hausdorff topology on the set of closed subsets of a compact Hausdorff space) can be generalised in two ways, considering either the closed or the compact subsets of a stably compact space.

A (not entirely) different reason for considering bitopological spaces is the question how to represent contradicting or incomplete knowledge in logic. Vickers [59] gives an account of how a logic where the law of excluded middle fails leads to topological spaces (or rather locales). Similarly, Jung and Moshier motivate d-frames via Belnap's four-valued logic [10], a logic where a statement can be true and false at the same time.

0.4 Why interaction algebras?

Interaction algebras arose initially as an abstraction of normal d-lattices which we introduce in Section 2.4. In particular the question was: What is the most liberal kind of morphism that gives rise to a continuous map between stably compact spaces? Various related answers were given by Jung, Sünderhauf and Jung, Kegelmann and Moshier in the form of algebras that carry the signature of bounded distributive lattices and certain relations between them. In each approach, an element of the bounded distributive lattice can be interpreted either as a basic open set or a basic compact set of the stably compact space. This dual interpretation reflects the self-dual nature of the category of stably compact spaces. However, in order to make the self-duality more tangible and the translation from spaces to algebraic structures more direct, one may choose to keep the opens and the compact sets separate, resulting in a structure where two bounded distributive lattices *interact* via relations; hence the name "interaction algebras".

Another – and in the author's opinion more compelling – reason for interaction algebras is the way in which the internal structure of a stably compact topology is represented. Apart from unions and intersections of open sets the key ingredient is the way-below relation between open sets that is *witnessed* by compact sets. Effectively the way-below relation is broken down into a composition of two more fundamental relations between opens and compact sets. These fundamental relations are hidden in the previous approaches because opens and compacts were indistinguishable. Once one knows what to look for, a multitude of order relations used in mathematics are witnessed relations in the sense that they can be decomposed into two other relations. In fact the original definition of way-below relation on a domain that was given by Dana Scott is of this form.

Chapter 1

Interaction algebras

This chapter serves two purposes. It provides a supercategory for categories we define in Chapters 2 and 3. At the same time, it presents the synthesis of Abramsky and Jung's *abstract bases*, which are identical to Smyth's concept of *R-structures*, and Vickers' *information systems*. Both concepts have the purpose of specifying a domain without having to define *all* of its elements. Let us call the elements of an abstract basis or an information system "tokens". While every token of an abstract basis gives rise to an element of the domain presented, there is no such assignment for tokens of an information system. Instead, every token yields a Scott open set of the presented domain. This reflects the localic viewpoint of information systems.

One advantage, and historically possibly the main reason for considering abstract bases at all, is a reduction in cardinality. For instance, the algebraic domains in the standard Scott model of PCF can be of uncountable cardinality, but they all have a countable basis of compact elements. Likewise, a topological space might have an uncountable lattice of open sets, but it could still have a countable basis. This is the case for the Euclidean topology on the real numbers, to mention just one important example.

Knowing the action of a continuous map on a set of basic elements is enough to determine its action on arbitrary elements. However, once one has chosen bases for presenting two domains, most continuous maps between the two structures will fail to map basic elements to basic elements. Thus one is forced to consider relations between the bases. When working with abstract bases, the fundamental question one asks about a continuous map $f : D \to E$ is: *Is y way below $f(x)$?* The answer is recorded in a relation between tokens. One says that x is related to y whenever $y \ll f(x)$. The approach for information systems is quite similar. Here, the question one asks about a continuous map is: *When is the Scott open V completely below $f^{-1}(U)$?* A basic open V is related to a basic open U whenever $V \lll f^{-1}(U)$. Why do these relations contain all information about the map f? Any element of the domain D is the directed join of tokens from the abstract basis, and f preserves this directed join. Therefore, $f(x)$ equals the join of the set $\{f(x') \mid x' \text{ is a token way below } x\}$. Now the elements $f(x')$ might themselves not be tokens, but our relation knows all tokens y that are way below $f(x')$. Then $f(x)$ equals the join of the set $\{y \mid y \text{ is a token related to some token } x' \ll x\}$. The same idea applies to information systems.

We wish to combine the two approaches and use the best of both. The main idea becomes clear when we formulate the fundamental question to ask about a continuous map f: *When is $f(x)$ contained in the Scott open U?* Observe that one could equivalently formulate the question as *When is x contained in the preimage $f^{-1}(U)$?* It is obvious how to fit abstract bases into this scheme: Let x range over all tokens of the abstract basis, and let U range over all Scott open sets of the form $\Uparrow y$. With information systems, the situation is slightly less obvious. The completely-below relation on Scott open sets is *witnessed* by elements of the domain as follows (see Proposition 6.1.20). U' is completely below U precisely when there exists an element $x \in U$ which is a lower bound of U'. Therefore V is completely below $f^{-1}(U)$ precisely when there exists a lower bound x of V which is an element of $f^{-1}(U)$. We have taken up the thread which will guide us through this work:

A relation between two entities is witnessed by a third.

When dealing with interpolative relations, such as the way-below relation between elements of a domain, or the completely-below relation between elements of a completely distributive lattice, the witness can always be chosen of the same kind. Indeed, the interpolation property states that $x \ll y$ precisely when there exists a z with $x \ll z \ll y$. But suppose we want more freedom in what kind of witness we choose. For example, a more natural characterisation of the way-below relation (and in fact this was Scott's original definition) is that x is way below y if there exists a Scott open set U such that x is a lower bound of U and y is an element of U. Now the way-below relation is decomposed into two other relations, the is-a-lower-bound-of relation and the is-an-element-of relation between points and Scott opens. The interpolation property of the way-below relation is now hidden in the relationship between y and U. Indeed, for a domain it is true that $y \in U$ precisely when there exists an element y' of U and another Scott open U' such that y is an element of U', the open U' has y' as a lower bound and y' is an element of U.

Let us collect the situation in a picture. For a domain D, denote its Scott topology by σD. For the sake of neutrality, we write $U \mathrel{\bigcirc} x$ whenever x is an element of U. Likewise, we write $x \mathrel{\succ\!\!\prec} U$ whenever x is a lower bound of U. Thus we obtain the following diagram of relations[1].

$$\sigma D \underset{\succ\!\!\prec}{\overset{\bigcirc}{\rightleftarrows}} D$$

If we forget for a moment that the two sets in the diagram are just two sides of the same

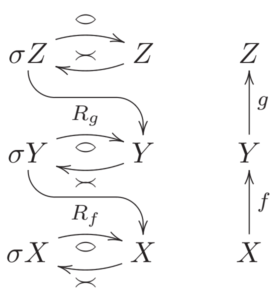
[1]Think of D as the domain of open upper rays on the real line and σD as being represented by open lower rays. The idea behind the symbols is that $\bigcirc$ represents two overlapping open intervals $-\!\!\leftarrow\!\!\rightarrow\!\!-$ and $\succ\!\!\prec$ represents disjoint intervals $\rightarrow\!\!-\!\!\leftarrow$.

medal, then all we have is a pair of sets interacting via two relations, thus our name *interaction algebra*. We write composition of relations as ";" and from left to right. With this, the way-below relation on D is recovered as $\succ\!\prec; \bigcirc$. The interpolation property of $\ll$, as explained above, is now a consequence of an interpolative law for the relation $\bigcirc$: it reads $\bigcirc = \bigcirc; \succ\!\prec; \bigcirc$. Indeed, using this law we can deduce

$$\ll = \succ\!\prec; \bigcirc = \succ\!\prec; \bigcirc; \succ\!\prec; \bigcirc = \ll; \ll .$$

But the diagram tells us even more. The composition $\bigcirc; \succ\!\prec$ is a relation on σD, and by now it is an old friend: $U \bigcirc; \succ\!\prec U'$ precisely when U' is completely below U. Now the same property of $\bigcirc$ that allowed us to deduce the interpolation property of the way-below relation on D allows us to deduce the interpolation property for the completely-below relation on σD. The author believes that the interpolative law for $\bigcirc$ is the more fundamental one.

Bearing in mind our basic questions about a continuous map, it is readily seen that the identity function is represented by the relation $\bigcirc$. This suggests that if we wish to turn our construction into a category, we should make the relation $\bigcirc$ the identity morphism of an object. But what would composition of morphisms look like? Again, the interpolative law contains the answer. Recall that the identity morphisms of objects X and Y have to satisfy $\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X$ for any morphism $f : X \to Y$. In particular $\mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}_X$. The latter equation is exactly what the interpolative law for $\bigcirc$ provides. Let us make this precise. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous maps of domains. When is $(g \circ f)(x)$ in a Scott open $U \in \sigma Z$? Note that $g(f(x)) \in U$ precisely when $f(x) \in g^{-1}(U)$. We claim that $g(f(x)) \in U$ is equivalent to the assertion that $U \bigcirc g(y)$, $y \succ\!\prec V$ and $V \bigcirc f(x)$ for some $y \in Y$ and $V \in \sigma Y$. Indeed, if the assertion is true, then because of $y \in g^{-1}(U)$ we must have $f(x) \in g^{-1}(U)$ and so $g(f(x)) \in U$. Conversely, if $f(x) \in g^{-1}(U)$ then we can find some $y \in g^{-1}(U)$ with $y \ll f(x)$. Now a witness V for the relation $y \ll f(x)$ yields what we desired. As we have demonstrated, the relation $\succ\!\prec$ plays a role in the composition of two functions. Let $U R_g y$ whenever $g(y) \in U$ and $V R_f x$ whenever $f(x) \in V$. Then the relation $R_{g \circ f}$ is the composition $R_g; \succ\!\prec; R_f$. With this it becomes clear that the interpolative law for the relation $\bigcirc : \sigma X \to X$ says nothing but $\mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}_X$, and that a relation R_f derived from a continuous function f must satisfy $R_f = \bigcirc; \succ\!\prec; R_f; \succ\!\prec; \bigcirc$. The diagram below is an illustration of how composition works.



Role of tokens	*Fundamental question about functions*
Information system	
Basic Scott open set	Is V completely below $f^{-1}(U)$?
Abstract basis	
Basic point of the domain	Is y way below $f(x)$?
Interaction algebra	
Token: basic point; Witness: basic Scott open set	Is $f(x)$ an element of the Scott open U?

Table 1.1: Presenting domains and their continuous functions

It should be mentioned that the direction of the arrows $\multimap$ and $\rightarrowtail$ is chosen somewhat arbitrary and forces the assignment $f \mapsto R_f$ to be contravariant. We find it convenient nevertheless because we can write the way-below relation as $\rightarrowtail; \multimap$. Table 1.1 compares the three concepts we encountered above.

Even without the interpolative law, the idea of decomposing an order relation into two other relations is by no means limited to domain theory. There are numerous examples from topology and algebra. We list a few.

- For opens U' and U of a topological space, say that U' is *well inside* U and write $U' \lhd U$ if the closure of U' is contained in U. This relation has a *canonical witness*, namely the closure of U'. But we can express the fact that U' is well inside U more neutrally by saying that there exists another open V such that $U' \cap V$ is empty and $V \cup U$ is the entire space. Here the relation $\lhd$ is decomposed into the is-disjoint-from ("consistency", con) and the covers-the-space relation ("totality", tot) between opens. Observe that the separation axiom *regularity* is equivalent to requiring every open U being the union of all the opens well inside it. Moreover, the separation axiom *normality* is equivalent to the interpolative law $\mathsf{tot} = \mathsf{tot}; \mathsf{con}; \mathsf{tot}$.

- For opens U' and U of a topological space, say that U' is *really inside* U and write $U' \lessdot U$ whenever there exists a continuous function f into the unit interval separating them, meaning that f is constant 0 on U' and constant 1 outside U. Here the witnesses are bounded real-valued functions and the relation is broken down into $U' \cap f^{-1}(0, 1] = \emptyset$ and $f^{-1}[0, 1) \subseteq U$. The separation axiom *complete regularity* is equivalent to requiring that every open is the union of all opens really inside it. The really-inside relation is interpolative for all spaces.

- If one restricts $\lhd$ using only witnesses which are simultaneously open and closed, then the concept of zero-dimensionality amounts to the assertion that every open is the union of opens below it in this sense.

- A continuum is a compact, connected metrizable space. For opens U' and U of a topological space write $U' \prec U$ whenever $U' \subseteq C \subseteq U$ for some subcontinuum of the space. The assertion that every open U is the union of all the opens U' with $U' \prec U$ is known as *connectedness im Kleinen*. Evidently $\prec$ is stronger than the way-below relation between opens, whence every space which is connected im Kleinen is locally compact.

- If L is a bounded lattice, consider relations between the set $\operatorname{Filt} L$ of filters of L and the set $\operatorname{Idl} L$ of ideals of L. One says $\operatorname{Filt} L \ni F \leq I \in \operatorname{Idl} L$ whenever the filter F intersects the ideal I, and $I \leq F$ whenever all elements of the ideal I are lower bounds of the filter F. This defines a structure known as the *intermediary structure* of the lattice L, the MacNeille completion of which is the canonical extension of the lattice L.

1.1 Interaction algebras for completely distributive frames

Our main goal is to treat domain theory the way we proposed in the introduction above. But it turns out that it is convenient to examine interaction algebras for completely distributive frames first. This section provides the backbone category of this chapter and "Stage 0" in our hierarchy of continuous posets. Many results proved in this section can be specialised or extended in subsequent sections. Be warned that from Section 1.2 on we change the semantics of interaction algebras (round ideals vs. round lower sets). Therefore, moving from Section 1.1 to Section 1.2 does not mean restricting to more special structures on the semantic side.

Completely distributive frames have every feature one needs for an interaction algebra (see subsection 6.1.10 of the Appendix):

- There is an interpolative auxiliary relation which is of central importance to the structure, namely the completely-below relation $\lll$.

- Any completely distributive frame is order-isomorphic to the set of *round lower sets* with respect to $\lll$.

- For every completely distributive frame, the auxiliary relation $\lll$ is witnessed by another lattice, namely its order dual.

- There is a well-behaved self-duality $(-)^{\wedge\!\!\wedge}$ on completely distributive frames which transforms tokens into witnesses and vice versa.

Recall that completely prime upper sets of a complete lattice L are defined using the structure map of the lattice: $U \subseteq L$ is completely prime if

$$\bigvee S \in U \Leftrightarrow S \cap U \neq \emptyset$$

for any lower set S. If x is a lower bound for such a set U and y is an element of it, then x is completely below y. Indeed, if S is a lower set with $y \leq \bigvee S$ then in particular $\bigvee S \in U$ whence $S \cap U \neq \emptyset$. Then x –being a lower bound for U– must be below some element of S, and since S is a lower set, it must contain x. Completely distributive frames have the property that the converse also holds: If $x \lll y$ then there exists some completely prime upper set U which contains y and has x as a lower bound.

Although the lattice of completely prime upper sets is order-isomorphic to the order dual of a complete lattice (every completely prime upper set of a complete lattice L is of the form $L \setminus {\downarrow}x$ for some $x \in L$), keeping the viewpoint of subsets is sometimes convenient. In many aspects the completely prime upper sets provide the functionality for completely distributive frames that we assigned to the Scott open sets of a domain. Analogous to the way outlined in the introduction for the way-below relation, we decompose the completely-below relation into two relations. Write $L^{\wedge\!\!\wedge}$ for the completely prime upper sets of a completely distributive frame L, ordered by inclusion. Consider the structure

$$L^{\wedge\!\!\wedge} \underset{\succ\!\prec}{\overset{\circ}{\rightleftarrows}} L \tag{1.1}$$

where, just as for domains, the relation $\circ$ is the *contains*-relation between upper sets and points, and $\succ\!\prec$ is the *lower-bound-of*-relation between points and upper sets. Armed with this intuition about information systems for completely distributive frames, we embark upon the axiomatic approach.

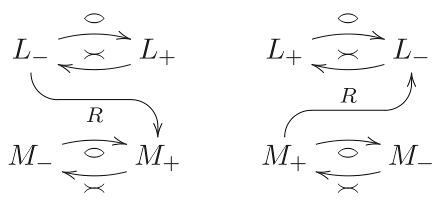
Definition 1.1.1. An *interaction algebra* consists of two sets L_+ and L_- which we call *tokens* and *witnesses*, respectively. These interact by two relations $\circ : L_- \to L_+$ and $\succ\!\prec : L_+ \to L_-$ where the former satisfies the *interpolative law* $\circ \;=\; \circ;\succ\!\prec;\circ$. A *morphism between interaction algebras* $(L_-, L_+, \succ\!\prec, \circ)$ and $(M_-, M_+, \succ\!\prec, \circ)$ is a relation $R : L_- \to M_+$ which satisfies $R \;=\; \circ;\succ\!\prec;R;\succ\!\prec;\circ$. The composition of two morphisms is defined as $R ⨾ S \;:=\; R;\succ\!\prec;S$ and the identity morphism on every object is $\circ$. Emphasising the role of the token set, the category of interaction algebras is denoted by Tok_0. With later refinements in mind, interaction algebras are also called Stage 0 interaction algebras and their morphisms Stage 0 morphisms.

Notation. We usually denote interaction algebras with upper case script letters, e.g. $\mathcal{L}$, $\mathcal{M}$,... The set of tokens and witnesses are then denoted by the same letter in standard typeface. For example, the interaction algebra $\mathcal{L}$ has components $(L_-, L_+, \succ\!\prec, \bigcirc)$. Witnesses are usually given lower case Greek letters, e.g. ϕ, ψ,... and tokens lower case Roman letters.

1.1.1 Basic facts and constructions

Self-duality of interaction algebras

Probably the first observation that strikes the reader is that there is no apparent difference between the witnesses and the tokens, apart from the fact that we write the former set with a negative subscript and the latter with a positive subscript. Moreover, as both the internal structure and the morphisms are relations, one might just as well reverse all arrows and obtain the same kind of structure. If we start with the depiction of a morphism as on the left in the diagram below, then reversal of the arrows makes the relation R now have type $M_+ \to L_-$. The only way to make this a morphism again is to swap the roles of tokens and witnesses, that is, to flip the entire diagram at the vertical axis.



(1.2)

Clearly now the diagram on the right represents another morphism between interaction algebras. We arrive at

Proposition 1.1.1. *There is a contravariant functor* Flip : $\mathsf{Tok}_0 \to \mathsf{Tok}_0$ *which takes an interaction algebra* $(L_-, L_+, \succ\!\prec, \bigcirc)$ *to* $(L_+, L_-, \succ\!\prec^{-1}, \bigcirc^{-1})$ *and a morphism* R *to* R^{-1}. *The category* Tok_0 *is self-dual, as* Flip $\circ$ Flip *is the identity functor.*

The functor Flip will save us some work below because for many constructions on interaction algebras there are four possible definitions. The involution Flip usually reduces the possibilities to two.

Orders on tokens and witnesses

With information systems in mind, one can re-assemble the internal structure $\succ\!\prec$ and $\bigcirc$ of an interaction algebra into a binary relation $\succ\!\prec; \bigcirc$ on tokens. On the witnesses one can do the same: The composition $\bigcirc; \succ\!\prec$ is a binary relation on witnesses.

Definition 1.1.2. For any interaction algebra, we write $\prec$ for the composition $\succ\!\prec; \bigcirc$ and call this the *auxiliary order on tokens.* Likewise, we abbreviate the composite $\bigcirc; \succ\!\prec$ by $\succ$ and call its relational inverse the *auxiliary order on witnesses.*

As the symbol $\prec$ suggests, we understand this relation as a sort of *less-than*-relation, whereas $\succ$ is to be understood as a *greater-than*-relation.

Warning! Deviating from standard notation of order theory, the relation $\succ$ is not just the relational inverse of $\prec$. The convention of writing Greek symbols for witnesses and Roman symbols for tokens should help to avoid confusion. One nice feature about interaction algebras is that in fact one is never forced to turn one of those relations around, and proofs can be written rather elegantly keeping the order on witnesses in the greater-than-style.

We record a few easy observations.

Lemma 1.1.2. *For any interaction algebra, the relations $\succ$ and $\prec$ are idempotent. Hence both $(L_-, \succ)$ and $(L_+, \prec)$ are information systems. The interpolative law for the relation $\frown$ can be written as $\frown = \frown; \prec = \succ; \frown$. A relation $R : L_- \to M_+$ between witnesses and tokens of two interaction algebras is a Stage 0 morphism precisely when $\succ; R; \prec = R$. Moreover, any relation $R : L_- \to M_+$ can be turned into a morphism by the idempotent operation $R \mapsto (\succ; R; \prec)$.*

It should be noted that apart from being transitive, the relations $\succ$ and $\prec$ have little in common with partial orders, for they are neither reflexive nor antisymmetric in general. This can be turned into one of the strengths of the information system approach: One can present a domain using a convenient set of (possibly redundant) tokens and the relation $\succ$ collapses several tokens into the same basic Scott open set.

The involution Flip of Proposition 1.1.1 is, as far as the order on tokens and witnesses is concerned, an order-preserving operation.

Lemma 1.1.3. *Let $\mathcal{L}$ be an interaction algebra, $\phi \succ \psi$ be witnesses and $a \prec b$ be tokens. Then in the interaction algebra* Flip $\mathcal{L}$ *the witnesses a and b are related by $b \succ a$ and the tokens ϕ and ψ are related by $\psi \prec \phi$.*

Proof. We have $a \prec b$ in $\mathcal{L}$ iff $a \asymp; \frown b$ which is equivalent to $b(\frown^{-1}); (\asymp^{-1})a$. Recall that $(\frown^{-1}); (\asymp^{-1})$ is the relation $\succ$ on the witnesses of Flip $\mathcal{L}$. A similar argument applies to ϕ and ψ. □

When presented with a preorder, one natural question to ask is what its upper and lower sets are. Given a preorder $(P, \leq)$, one calls a subset $U \subseteq P$ an upper set if $U \ni u \leq x$ implies $x \in U$. Although the relations $\prec$ and $\succ$ are technically not preorders, we want a suggestive notation for upper and lower sets.

Notation. For any subset $U \subseteq L_+$ of tokens of an interaction algebra, we write $\uparrow U$ for the upper closure of the set U with respect to $\prec$, meaning that $a \in \uparrow U$ if there exists some $u \in U$ with $u \prec a$. Dually, the lower closure of $\downarrow U$ is the set of tokens a for which $a \prec u$ for some $u \in U$. Likewise, one writes $\uparrow \Phi$ for the set of witnesses ψ with $\psi \succ \phi$ for some $\phi \in \Phi$ and calls this set the upper closure of the witness set Φ. The lower closure of a set of witnesses is defined accordingly. Upper and lower closures of singletons are abbreviated as $\uparrow \{a\} = \uparrow a$ etc.

The reader should be aware that the upper and lower closure are not a closure operators in the strict sense, because for example the upper closure $\uparrow\{a\}$ does not contain a unless the token satisfies $a \prec a$. However, upper and lower closure are idempotent operations because the relations $\prec$ and $\succ$ are idempotent. For the same reason every closed set $\uparrow U$ or $\downarrow U$ is *round* in the sense of Definition 1.1.3 below. More results on upper and lower closed sets are gathered in Proposition 1.1.5.

Definition 1.1.3. Let X be a set with with a transitive binary relation $<$. We call a subset $U \subseteq X$ a *round upper set* with respect to $<$ if it satisfies

$$x \in U \Leftrightarrow \exists u \in U.\, u < x.$$

The collection of round upper sets of X, ordered by inclusion, is denoted by $\mathrm{Up}^{<} X$. Dually, a subset $U \subseteq X$ is called a *round lower set* with respect to $<$ if

$$x \in U \Leftrightarrow \exists u \in U.\, x < u$$

holds. The collection of round lower sets of X, ordered by inclusion, is denoted by $\mathrm{Lo}^{<} X$.

One checks that both $\mathrm{Up}^{<} X$ and $\mathrm{Lo}^{<} X$ are complete lattices, where joins are computed as set union (in particular, the empty set is both round lower and round upper).

Remark. Consider subsets of X as binary relations $U \subseteq X \times \{*\}$ Then, in the language of relational composition, the round lower sets of X are those morphisms $U : X \to \{*\}$ of Rel which satisfy $<; U = U$. Dually, if one regards subsets of X as morphisms $U : \{*\} \to X$ in Rel then round upper sets are those which satisfy $U = U; <$. The interaction algebra version of this observation is the content of Proposition 1.1.5.

So far we have not specified any requirements on the size of the relations $\circ$ and $\rightarrowtail$. In fact, even the empty relation between witnesses and tokens satisfies the interpolative law. But it should be obvious that the fewer tokens and witnesses are related by $\rightarrowtail$ and $\circ$, the more uninteresting the interaction algebra becomes. As it turns out, any token or witness which is not "bounded above" by some other element can be considered as superfluous.

Definition 1.1.4. We call a token a of an interaction algebra *bounded* if there exists another token b with $a \prec b$. Likewise, a witness ϕ is called bounded if $\psi \succ \phi$ for some witness ψ. The tokens and witnesses which are bounded by themselves, i.e. $a \prec a$ or $\phi \succ \phi$, are called *compact.*

Proposition 1.1.4. *If* $\mathcal{L} = (L_-, L_+, \rightarrowtail, \circ)$ *is an interaction algebra, let* $\overline{L_-}$ *and* $\overline{L_+}$ *denote its bounded witnesses and tokens. With the relations restricted accordingly, the tuple* $(\overline{L_-}, \overline{L_+}, \rightarrowtail, \circ)$ *is an interaction algebra which is isomorphic to* $\mathcal{L}$.

Proof. The interpolative law for $\circ$ implies that $\phi \circ a$ if and only if there are bounded ψ and b such that $\phi \succ \psi \circ b \prec a$. Let R be the restriction of $\circ$ to $L_- \times \overline{L_+}$ and S the

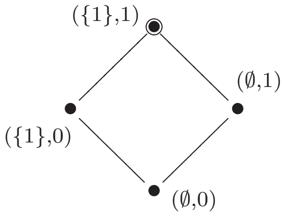


Figure 1.1: The interaction algebra of the lattice $2 = \{0, 1\}$ pictured as the product of witnesses and tokens. A bullet • is a pair related by $\asymp$ and a circled element is a pair related by $\circ$.

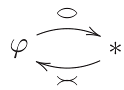


Figure 1.2: The interaction algebra **1** with only one witness and one token. It is the bounded part of the interaction algebra depicted in Figure 1.1.

restriction of $\circ$ to $\overline{L_-} \times L_+$. Then it is easy to see that both R and S are morphisms and $R \mathbin{;} S = S \mathbin{;} R = \circ$. □

Example 1. Consider the two-chain $2 = \{0, 1\}$ which is obviously a completely distributive frame where $1 \lll 1$. It has two completely prime upper sets, namely $\emptyset$ and $\{1\}$. This yields the interaction algebra depicted in Figure 1.1. The sets of tokens and witnesses have only one bounded member each, which happens to be compact: $1 \asymp \{1\} \circ 1$. By the previous proposition, this interaction algebra is isomorphic to the interaction algebra comprising only one witness and one token where the relations $\asymp$ and $\circ$ are maximal. We call this reduced interaction algebra **1**, write its token as $*$ and its witness as φ.

1.1.2 Duality for interaction algebras

Interaction algebras are no good if we do not know what they represent. In the following we develop a contravariant duality between the categories Tok_0 and CDFrm. The style of presentation has the flavour of a Stone-type duality. This means that each contravariant functor is presented using a dualising object.

Round upper and round lower sets

In the category Rel of sets and relations, the set of morphisms between any two sets is a complete lattice when ordered by inclusion. Furthermore, composition of relations is monotone and preserves arbitrary unions. The same applies to the category Tok_0 and the composition $\mathbin{;}$. Indeed, the only non-trivial fact to check is that the union of a set of Stage 0 morphisms is again a Stage 0 morphism. Suppose $\mathfrak{R}$ is a subset of $\mathsf{Tok}_0(\mathcal{L}, \mathcal{M})$. We have $\phi\, (\bigcup \mathfrak{R})\, a$ if and only if $\phi R a$ for some $R \in \mathfrak{R}$. If $\psi \succ \phi$ and $a \prec b$ then because R is a morphism we also have $\psi R b$ and thereby $\psi\, (\bigcup \mathfrak{R})\, b$. Further, $\phi R a$ implies that $\phi \succ \theta R c \prec a$ for some θ and c. This shows that $\bigcup \mathfrak{R}$ is a Stage 0 morphism.

Proposition 1.1.5. *Let $\mathcal{L}$ be an interaction algebra and **1** be the interaction algebra with token set $\{*\}$ and witness set $\{\varphi\}$ as depicted in Figure 1.2.*

1. *There is an order-isomorphism between the lattice of round upper sets of tokens (sets satisfying $U = \uparrow U$) and the complete lattice of morphisms $\mathbf{1} \to \mathcal{L}$.*

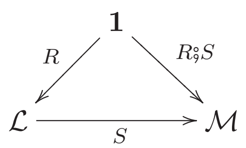
2. *There is an order-isomorphism between the round upper sets of witnesses (sets satisfying $\Phi = \uparrow \Phi$) and the complete lattice of morphisms $\mathcal{L} \to \mathbf{1}$.*

3. *Both the complete lattices $\mathsf{Tok}_0(\mathbf{1}, \mathcal{L})$ and $\mathsf{Tok}_0(\mathcal{L}, \mathbf{1})$ are completely distributive frames, with U completely below V if and only if there exists an element of V which is a lower bound for U.*

Proof. (1) If $R : \mathbf{1} \to \mathcal{L}$ is a Stage 0 morphism then the set $U = \{a \in L_+ \mid \varphi R a\}$ is easily seen to be a round upper set with respect to $\prec$ because of $R = R; \prec$. Conversely, any round upper set U generates a morphism $R : \mathbf{1} \to \mathcal{L}$ via $\varphi R a \Leftrightarrow a \in U$. Clearly the two constructions are mutually inverse and preserve the inclusion order. One proves (2) by applying the contravariant functor Flip to the situation of (1).

(3) Follows from (1) and (2) and the fact that $U = \bigcup_{u \in U} \uparrow u$ holds for any round upper set. □

The proposition tells us that there are two ways of extracting a completely distributive frame from an interaction algebra. In fact, "general categorical nonsense" tells us even more: The assignment $\mathcal{L} \mapsto \mathrm{Up}^{\prec} L_+$ which is presented as $\mathsf{Tok}_0(\mathbf{1}, -)$ is covariantly functorial, where the action of the functor on morphisms is just post-composition with the morphism:



Dually, the construction presented as $\mathsf{Tok}_0(-, \mathbf{1})$ is contravariantly functorial where the action on morphisms is pre-composition. Since we know that composition of relations preserves all unions, the functors transform Stage 0 morphisms to join-preserving maps between completely distributive frames. But which functor is the one we are after, the functor that extends to the duality with CDFrm? Consider again the interaction algebra (1.1) we constructed from a completely distributive frame L. We know that L is isomorphic to the round lower sets of L with respect to the completely-below relation. This seems to match none of the functors above. To see that the contravariant functor $\mathsf{Tok}_0(-, \mathbf{1})$ is the one we want, we invoke a lemma which is so useful for the rest of this chapter that we dare call it the Fundamental Lemma of interaction algebras.

Lemma 1.1.6 (The Fundamental Lemma of interaction algebras)**.** *For any interaction algebra, the complete lattice of round lower sets of tokens is order-isomorphic to the* *complete lattice of round upper sets of witnesses via the following operations.*

$$U^{\circ} = \{\phi \in L_- \mid \exists u \in U.\, \phi{\circ}u\} \tag{1.3}$$

$$\Phi_{\asymp} = \{a \in L_+ \mid \exists \phi \in \Phi.\, a{\asymp}\phi\} \tag{1.4}$$

Proof. Fist observe that a set U of tokens is a round lower set if and only if it coincides with its lower closure.

The operations (1.3) and (1.4) can be defined for arbitrary subsets $U \subseteq L_+$ and $\Phi \subseteq L_-$. Clearly both maps are monotone with respect to inclusion. Using the definition of $\prec$ and $\succ$ it is easy to see that $U = (U^{\circ})_{\asymp}$ precisely when U is a round lower set and likewise $\Phi = (\Phi_{\asymp})^{\circ}$ precisely when Φ is a round upper set. Thus, by the very definition, the composites $(-)_{\asymp} \circ (-)^{\circ}$ and $(-)^{\circ} \circ (-)_{\asymp}$ restrict to the identities on round lower sets of tokens and round upper sets of witnesses, respectively. Observe that both $(-)^{\circ}$ and $(-)_{\asymp}$ preserve round sets. Indeed, ${\circ};{\prec} = {\succ};{\circ}$ whence $(\downarrow U)^{\circ} = \uparrow(U^{\circ})$, and likewise for sets $\Phi \subseteq L_-$. □

Theorem 1.1.7. 1. *There is a contravariant functor $\Omega : \mathsf{Tok}_0 \to \mathsf{CDFrm}$ which is presentable as $\mathsf{Tok}_0(-, \mathbf{1})$. It maps a morphism $R : \mathcal{L} \to \mathcal{M}$ to a join-preserving map $\Omega(R) : \mathrm{Lo}^{\prec} M_+ \to \mathrm{Lo}^{\prec} L_+$ between round lower sets of tokens. The functor Ω is order-preserving on hom-sets.*

2. *Suppose $h : L \to M$ is a join-preserving map between completely distributive frames and $S : M^{\Lambda\!\!\Lambda} \to L$ is the relation between interaction algebras as in the diagram (1.1) derived from L and M, where a completely prime upper set $U \subseteq M$ is related to an element $x \in L$ if and only if $h(x) \in U$. Then the map $\Omega(S) : \mathrm{Lo}^{\prec} L \to \mathrm{Lo}^{\prec} M$ is isomorphic to h.*

Proof. (1) From Proposition 1.1.5 and the observations following it, we know that $\mathsf{Tok}_0(-, \mathbf{1})$ presents a contravariant functor which sends an interaction algebra to the completely distributive frame of round upper sets of witnesses. The Fundamental Lemma 1.1.6 states that we can equivalently describe that functor as producing maps between round lower sets of tokens. As the functor $\mathsf{Tok}_0(-, \mathbf{1})$ as pre-composition, and composition is monotone, the functor is monotone on hom-sets. (2) Recall that in the interaction algebra depicted in diagram (1.1) the relation $\prec$ on tokens coincides with the completely-below relation. Further recall that any round lower set with respect to $\lll$ must be the lower set of a point (see Lemma 6.1.18 of the Appendix). Let $x \in L$ and consider the round lower set $\downdownarrows x = \{y \in L \mid y \lll x\}$. The map $(-)^{\circ}$ defined in equation (1.3) of the Fundamental Lemma 1.1.6 transforms the set $\downdownarrows x$ to the round upper set $\mathcal{V} = \left\{V \in L^{\Lambda\!\!\Lambda} \,\middle|\, \exists y \ll x.\, y \in V\right\}$ which is the same as $\left\{V \in L^{\Lambda\!\!\Lambda} \,\middle|\, x \in V\right\}$. Thus, the point $x \in L$ is represented by the Stage 0 morphism $R : L^{\Lambda\!\!\Lambda} \to \mathbf{1}$ where $VR*$ iff $V \ni x$. Using the relation $USy \Leftrightarrow h(y) \in U$

and its image under the contravariant functor $\mathsf{Tok}_0(-,\mathbf{1})$ we obtain the relation

$$\begin{aligned} U(S\mathbin{;}R)* &\Leftrightarrow \exists y, U.\, USy \succ\!\!\prec VR* \\ &\Leftrightarrow h(y)\in U,\ y \lll x \\ &\Leftrightarrow h^{-1}(U) \ni y \lll x \\ &\Leftrightarrow h(x)\in U \end{aligned}$$

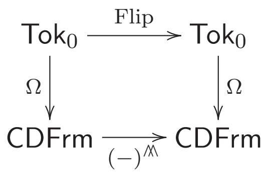
which results in the round upper set $\mathcal{U} = \left\{U \in M^{\wedge\!\!\wedge} \,\middle|\, h(x)\in U\right\}$. Now apply the isomorphism defined in equation (1.4) of the Fundamental Lemma and obtain the round lower set

$$\left\{m \in M \,\middle|\, \exists U \in M^{\wedge\!\!\wedge}.\, m \succ\!\!\prec U \ni h(x)\right\}$$

which, as the completely prime upper sets witness the completely-below relation, can be written as $\twoheaddownarrow h(x)$. We conclude that $\Omega(S)(\twoheaddownarrow x) = \twoheaddownarrow h(x)$ whence the map $\Omega(S)$ is isomorphic to h. □

Having found a use for the contravariant functor $\mathsf{Tok}_0(-,\mathbf{1})$, we do the same for the covariant functor $\mathsf{Tok}_0(\mathbf{1},-)$. Observe that $\mathsf{Tok}_0(\mathbf{1},-)$ can be expressed as the functor Flip followed by $\mathsf{Tok}_0(-,\mathbf{1})$. We arrive at

Proposition 1.1.8. *The following diagram of contravariant functors commutes (up to isomorphism).*



Here $(-)^{\wedge\!\!\wedge}$ *is the self-duality on completely distributive frames (see Theorem 6.1.21).*

Proof. Using the Fundamental Lemma 1.1.6 and Proposition 1.1.5, we reformulate the object part of the diagram as the assertion "The lattice of completely prime upper sets of $\mathrm{Lo}^{\prec} L_+$ is isomorphic to the lattice of of round upper sets of tokens." For any round upper set $U \subseteq L_+$ we define

$$U^{\sharp} = \left\{A \in \mathrm{Lo}^{\prec} L_+ \,\middle|\, U \cap A \neq \emptyset\right\}.$$

Since arbitrary joins of round lower sets are computed as set union, it is trivial to prove that any such $U^{\sharp}$ is a completely prime upper set in $\mathrm{Lo}^{\prec} L_+$. We claim that the inverse to the operation $(-)^{\sharp}$ is given by the assignment

$$\mathcal{C}_{\sharp} = \{a \in A \,|\, \downarrow\! a \in \mathcal{C}\}.$$

Observe that $a \prec b$ implies that $\downarrow\! a \lll \downarrow\! b$ because of the characterisation of $\lll$ we gave in Proposition 1.1.5 (3). Hence the set $\mathcal{C}_{\sharp}$ is an upper set in L_+. It is also round with

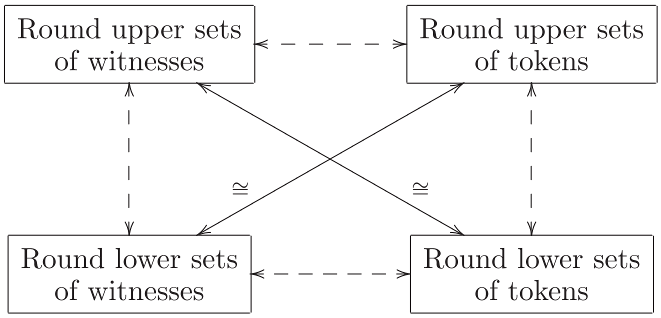


Figure 1.3: The relationships between round upper and lower sets of an interaction algebra. The solid lines indicate isomorphism; the dashed lines stand for one lattice being the order dual of the other.

respect to $\prec$. Indeed, if $\downarrow a$ is an element of the completely prime upper set $\mathcal{C}$, then there must be some round lower set $B \in \mathcal{C}$ with $B \lll \downarrow a$. This means that $B \subseteq \downarrow b$ for some $b \prec a$, whence also $\downarrow b \in \mathcal{C}$.

It remains to show that the two operations $(-)^\sharp$ and $(-)_\sharp$ are mutually inverse. Starting with a round upper set of tokens $U \in \mathrm{Up}^\prec L_+$, the tokens a with $\downarrow a \cap U \neq \emptyset$ are precisely the elements of U. Indeed, the inclusion $(U^\sharp)_\sharp \subseteq U$ is trivial, and if $a \in U$ then $b \in U$ for some $b \prec a$ because U is round. Now suppose $\mathcal{C}$ is a completely prime upper set of round lower sets. If $\downarrow a \in \mathcal{C}$ and $a \in A$ for some round lower set A then certainly $\downarrow a \subseteq A$ whence A itself must be an element of $\mathcal{C}$. Thus $(\mathcal{C}_\sharp)^\sharp$ is contained in $\mathcal{C}$. For an element A of $\mathcal{C}$ use complete primality and obtain $\mathcal{C} \ni B \lll A$ for some B. This means that $B \subseteq \downarrow a$ for some $a \in A$. Hence a is a witness for $A \cap \mathcal{C}_\sharp \neq \emptyset$.

Note that we can read what we just showed the other way around: The lattice of completely prime upper sets of $\mathrm{Up}^\prec L_+$ is isomorphic to the lattice of round lower sets of tokens.

Now we turn towards morphisms. What we have to show is the fact which corresponds to the equivalence $h(x) \in U \Leftrightarrow x \in h^{-1}(U)$ in the category of completely distributive frames. Let $S \in \mathsf{Tok}_0(\mathcal{L}, \mathcal{M})$ be a Stage 0 morphism. It is convenient to apply the Fundamental Lemma once more and regard $\Omega(S)$ as a map from round lower sets of tokens of $\mathcal{M}$ to round upper sets of witnesses of $\mathcal{L}$ sending a round lower set $A \in \mathrm{Lo}^\prec M_+$ to the round upper set $\{\phi \in L_+ \mid \exists a \in A.\, \phi S a\}$. Similarly, we regard $(\Omega \circ \mathrm{Flip})(S)$ as a map sending round lower sets of L_- to round upper sets of M_+ via $\Phi \mapsto \{a \in M_+ \mid \exists \phi \in \Phi.\, \phi S a\}$. We need to show that $\Omega(S)(A)$ intersects Φ if and only if $(\Omega \circ \mathrm{Flip})(S)(\Phi)$ intersects A. But both statements are easily seen to be equivalent to $\Phi \times A$ intersecting the relation S. $\square$

Our studies of round upper and lower sets are summarised in Figure 1.1.2.

The canonical interaction algebra

We now turn towards the problem of finding an adjoint to the functor Ω presented in Theorem 1.1.7. Looking at equation (1.1), there is an obvious candidate:

Definition 1.1.5. If L is a completely distributive frame, let $\mathrm{Ialg}\, L$ denote the interaction algebra where the set of tokens is L and the set of witnesses is $L^{\mathbb{M}}$, the completely prime upper sets of L. The relations $\bigcirc$ and $\succ\!\!\prec$ are defined as $U \bigcirc x$ iff $x \in U$ and $x \succ\!\!\prec U$ iff x is a lower bound for U. If $h : L \to M$ is a join-preserving map between completely distributive frames then let $\mathrm{Ialg}(h) : M^{\mathbb{M}} \to L$ be the relation $U\, \mathrm{Ialg}(h) x \Leftrightarrow h(x) \in U$.

One needs to check that the construction $\mathrm{Ialg}\, L$ indeed yields a Stage 0 interaction algebra and $\mathrm{Ialg}(h)$ indeed a Stage 0 morphism. The interpolative law for $\bigcirc$ holds because $\succ\!\!\prec; \bigcirc$ coincides with the interpolative relation $\lll$ and because the completely prime upper sets are precisely the round upper sets with respect to $\lll$. The relation $\mathrm{Ialg}(R)$ is indeed a morphism: If $h(x) \in U \in M^{\mathbb{M}}$ then there exists some $m \lll h(x)$ which is still in U. As the set $U' = \{y \in M \mid m \lll y\}$ is completely prime, we get $U \bigcirc m \succ\!\!\prec U' \bigcirc h(x)$ and thereby $\mathrm{Ialg}(h) \subseteq \succ; \mathrm{Ialg}(h)$. On the other side we have $x = \bigvee \Downarrow x$ and h preserves this join, whence because of U being completely prime we must have $h(x') \in U$ for some $x' \lll x$. Thus $\mathrm{Ialg}(h) \subseteq \mathrm{Ialg}(h); \prec$. The converse inclusion $\succ; \mathrm{Ialg}(h); \prec \subseteq \mathrm{Ialg}(h)$ is trivial.

Now we have all ingredients for our contravariant duality.

Theorem 1.1.9. *The functors* $\mathsf{CDFrm} \underset{\Omega}{\overset{\mathrm{Ialg}}{\rightleftarrows}} \mathsf{Tok}_0$ *constitute a contravariant Stone-type duality of categories. The dualising object in* CDFrm *is the two-chain* 2 *and the dualising object in* Tok_0 *is the interaction algebra* $\mathbf{1}$ *with one witness and one token.*

Proof. Lemma 6.1.22 and Theorem 1.1.7 tell us that Ω is presented as $\mathsf{Tok}_0(-, \mathbf{1})$ and Ialg is presented as $\mathsf{CDFrm}(-, 2)$. Furthermore, from Theorem 1.1.7 (2) we know that $\Omega \circ \mathrm{Ialg}$ is equivalent to the identity functor on CDFrm. It remains to show that $\mathrm{Ialg} \circ \Omega$ is equivalent to the identity functor on Tok_0.

Using Proposition 1.1.8 and the Fundamental Lemma, we can express the interaction algebra $\mathrm{Ialg}\, \Omega\mathcal{L}$ as comprising the round lower sets of L_+ as tokens and the round lower sets of L_- as witnesses. A round lower set of tokens Φ corresponds to the completely prime upper set $\mathcal{U} := \left\{B \in \mathrm{Lo}^{\prec} L_+ \,\middle|\, B^{\bigcirc} \cap \Phi \neq \emptyset\right\}$ whence we have $\Phi \bigcirc A$ in $\mathrm{Ialg}\, \Omega\mathcal{L}$ if and only if the product $\Phi \times A$ intersects the relation $\bigcirc$ in $\mathcal{L}$. We claim that A is a lower bound for the completely prime upper set $\mathcal{U}$ that corresponds to Φ precisely when $A \times \Phi$ is contained in $\succ\!\!\prec; \bigcirc; \succ\!\!\prec$. Suppose $A \times \Phi \subseteq \succ\!\!\prec; \bigcirc; \succ\!\!\prec$ and the round lower set $B \in \mathrm{Lo}^{\prec} L_+$ is an element of the completely prime upper set $\mathcal{U}$. This means that $\phi \bigcirc b$ for some $\phi \in \Phi$ and $b \in B$. By hypothesis we have $a \succ\!\!\prec; \bigcirc; \succ\!\!\prec \phi \bigcirc b$ for all $a \in A$, whence $A \lll B$. Hence A is a lower bound for $\mathcal{U}$. Now suppose A is a lower bound for $\mathcal{U}$. Fix an element $\phi \in \Phi$. Since Φ is a round lower set of witnesses, there exists some $\psi \in \Phi$ with $\psi(\bigcirc; \prec) b \succ\!\!\prec \phi$. This means that the round lower set $\downarrow b$ is an element of the completely prime upper set $\mathcal{U}$. By hypothesis A is contained in $\downarrow b$ whence for all $a \in A$ we have $a \prec b \succ\!\!\prec \phi$. As ϕ was chosen arbitrary, we deduce $A \times \Phi \subseteq \succ\!\!\prec; \bigcirc; \succ\!\!\prec$ and the claim is proved. Observe that $A \succ\!\!\prec; \bigcirc B$ in the interaction algebra $\mathrm{Ialg}\, \Omega\mathcal{L}$ if and only if $A \lll B$ in $\mathrm{Lo}^{\prec} L_+$.

Next we construct the isomorphism between $\mathcal{L}$ and $\mathrm{Ialg}\, \Omega\mathcal{L}$. Define a relation R between witnesses and round lower sets of tokens by $\phi R A$ iff $\phi \bigcirc a$ for some $a \in A$. Likewise define a relation S between round lower sets of witnesses and tokens by $\Phi S a$ iff $\phi \bigcirc a$ for some $\phi \in \Phi$. Clearly $\succ; R = R$ and $S = S; \prec$. Observe that $\phi R \downarrow a$ if and only if $\phi \bigcirc a$, and likewise $\downarrow \phi\, S a$ precisely when $\phi \bigcirc a$. With this and the characterisation of the completely-below relation on round lower sets, one shows $R = R; \lll$ and $\ggg; S = S$. Hence R is a Stage 0 morphism $\mathcal{L} \to \mathrm{Ialg}\,\Omega\mathcal{L}$ and S is a Stage 0 morphism $\mathrm{Ialg}\,\Omega\mathcal{L} \to \mathcal{L}$. Finally, by iterating the interpolative law for $\bigcirc$ one finds that $\phi \bigcirc a$ is equivalent to $\phi R \downarrow b \rightarrowtail \downarrow \psi\, S a$ for suitable round lower sets of tokens and witnesses. Dually, $\Phi \bigcirc A$ if and only if $\Phi \ni \phi \bigcirc a \in A$ for some ϕ and a, which we can expand to $\Phi \ni \phi \bigcirc; \rightarrowtail; \bigcirc a \in A$ using the interpolative law. Notice that the latter relation says $\Phi S \fatsemi RA$. This concludes the proof of $\mathcal{L} \cong \mathrm{Ialg}\,\Omega\mathcal{L}$.

It remains to consider the action of $\mathrm{Ialg} \circ \Omega$ on morphisms. Let $Q : \mathcal{L} \to \mathcal{M}$ be a morphism between interaction algebras. We claim that $\phi Q a$ is equivalent to $\downarrow \phi\, (\mathrm{Ialg}(\Omega(Q))) \downarrow a$. We know that $\Omega(Q)$ maps the round lower set $\downarrow a \in \mathrm{Lo}^{\prec} M_+$ to the round upper set of tokens $\{\psi \in L_- \mid \exists b \prec a.\, \psi Q b\}$. Because of $Q = Q; \prec$ we have the simpler description $\Omega(Q)(\downarrow a) = \{\psi \in L_- \mid \psi Q a\}$. Obviously, a round lower set of witnesses Φ intersects this set if any only if $\Phi \ni \phi Q a$ for some member ϕ. In particular, using $\succ; Q = Q$ we conclude that $\downarrow \phi$ intersects $\Omega(Q)(\downarrow a)$ precisely when $\phi Q a$. □

1.1.3 Special morphisms

For the remainder of this section we study special kinds of morphisms. An important tool will be the concept of adjoint pairs of morphisms, which we derive from the corresponding notion in the category Rel. There, a relation $R : X \to Y$ is called right adjoint to a relation $S : Y \to X$ if $x(R; S)x$ for every $x \in X$ and $y(S; R)y'$ implies $y = y'$ for any pair $y, y' \in Y$. Using the identity relations Δ_X and Δ_Y, we can write the conditions more concisely as $\Delta_X \subseteq R; S$ and $S; R \subseteq \Delta_Y$. Observe that the right adjoints in Rel are precisely the defined and single-valued relations, i.e. the functions.

In the context of interaction algebras and their duality with completely distributive frames, however, the adjoint pairs of relations serve a different purpose. As we shall see, the existence of an adjoint is intimately related to compact tokens and maps preserving $\lll$.

Adjoints

The definition of adjoint pairs of relations between interaction algebras is straightforward. All we have to do is to replace relational composition with $\fatsemi$ and the diagonal relation with the identity relation $\bigcirc$. Interestingly, the definition makes sense even if the relations involved are not Stage 0 morphisms. We borrow notation from locale theory, where the frame homomorphism associated with a locale map f is commonly denoted by f^* and its right adjoint by f_*.

Definition 1.1.6. Let $\mathcal{L}$ and $\mathcal{M}$ be interaction algebras, $R^* : L_- \to M_+$ and $R_* : M_- \to L_+$

be a pair of relations. Then (R^*, R_*) is an adjoint pair if

$$\begin{aligned} \subset\!\!\supset &\subseteq R_* \fatsemi R^*, \\ R^* \fatsemi R_* &\subseteq \subset\!\!\supset. \end{aligned}$$

Here, R^* is called the left adjoint and R_* is called the right adjoint. One writes $R^* \dashv R_*$ to indicate the fact that R^* is left adjoint to R_*. A morphism R which has a (Stage 0) right adjoint is called *semi-open*.

Using the monotonicity of relational composition, it is easy to prove that if one of the relations in an adjoint pair is a Stage 0 morphism, then one can produce an adjoint pair of Stage 0 morphisms: Suppose $R_* \fatsemi R^* = R_* \fatsemi \subset\!\!\supset \fatsemi R^*$ and $R^* \fatsemi R_* = R^* \fatsemi \subset\!\!\supset \fatsemi R_*$. Then the defining inequalities of the adjunction can be extended to

$$\begin{aligned} \subset\!\!\supset &\subseteq \subset\!\!\supset \fatsemi R_* \fatsemi \subset\!\!\supset \fatsemi R^* \fatsemi \subset\!\!\supset \\ \subset\!\!\supset \fatsemi R^* \fatsemi \subset\!\!\supset \fatsemi R_* \fatsemi \subset\!\!\supset &\subseteq \subset\!\!\supset \end{aligned}$$

whence the relation $\subset\!\!\supset \fatsemi R^* \fatsemi \subset\!\!\supset$ is left adjoint to $\subset\!\!\supset \fatsemi R_* \fatsemi \subset\!\!\supset$. Moreover, a Stage 0 adjoint is unique: Suppose $R^* \dashv R_*$ is an adjoint pair. Let S be the union of all right adjoints to R^*. Then clearly $R_* \subseteq S$ and therefore $\subset\!\!\supset \fatsemi R_* \fatsemi \subset\!\!\supset \subseteq \subset\!\!\supset \fatsemi S \fatsemi \subset\!\!\supset$. For the reverse inclusion, use $\subset\!\!\supset \subseteq R_* \fatsemi R^*$ and $R^* \fatsemi S \subseteq \subset\!\!\supset$ to deduce $\subset\!\!\supset \fatsemi S \subseteq R_* \fatsemi \subset\!\!\supset$ from the tautology $R_* \fatsemi R^* \fatsemi S \subseteq R_* \fatsemi R^* \fatsemi S$.

Observe that the contravariant involution Flip preserves the order on relations, whence it transforms left adjoints to right adjoints and vice versa. The contravariant functor Ω also preserves the order on hom-sets and therefore preserves adjoints:

$$\begin{aligned} & \subset\!\!\supset \subseteq R_* \fatsemi R^*, \quad R^* \fatsemi R_* \subseteq \subset\!\!\supset \\ \Rightarrow \quad & \mathrm{id} \leq \Omega(R_*) \circ \Omega(R^*), \quad \Omega(R^*) \circ \Omega(R_*) \leq \mathrm{id} \\ \Rightarrow \quad & \Omega(R^*) \dashv \Omega(R_*) \end{aligned}$$

A first application of adjoints is a characterisation of compact tokens and witnesses.

Proposition 1.1.10. *There is a bijection between*

1. *Equivalence classes of compact tokens of an interaction algebra $\mathcal{L}$,*
2. *Morphisms $R : \mathcal{L} \to \mathbf{1}$ which have a right adjoint,*
3. *Round lower sets of tokens $A \in \mathrm{Lo}^{\prec} L_+$ which are completely compact, meaning $A \lll A$.*

Proof. Suppose $a \prec a$ is a compact token in L_+. Then $a \!\succ\!\!\prec\! \phi \subset\!\!\supset a$ for some compact witness ϕ. Define relations $\mathcal{L} \underset{R_*}{\overset{R}{\rightleftarrows}} \mathbf{1}$ as $\psi R * \Leftrightarrow \psi \succ \phi$ and $\varphi R_* b \Leftrightarrow a \prec b$. Because of

$a \prec a$ and $\phi \succ \phi$ both relations are Stage 0 morphisms. We have $\bigcirc\ \subseteq R_* \,{}^{\circ}_{9}\, R$ because of $\varphi R_* a \succ\!\!\prec \phi R*$. If $\psi(R \,{}^{\circ}_{9}\, R_*)b$ then $\psi \succ \phi \bigcirc a \prec b$ whence $\psi \bigcirc b$. Thus R_* is right adjoint to R. If a' is equivalent to a, meaning $a' \prec a \prec a'$, then we have $a' \succ\!\!\prec \phi' \bigcirc a'$ for some compact witness ϕ'. It is easy to show that $\phi' \succ \phi \succ \phi'$ also holds and the pair (ϕ, a) generates the same pair of morphisms (R, R_*). The functor Ω sends the morphism R to the round lower set $\downarrow a$. This is completely compact in $\mathrm{Lo}^{\prec} L_+$, because a union of round lower sets contains $\downarrow a$ precisely when a is an element of one of the round lower sets already. Conversely, if A is a completely compact round lower set of tokens, then by the characterisation of the completely-below relation we have $A \subseteq \downarrow a$ for some $a \in A$ whence $A = \downarrow a$ and in particular $a \in \downarrow a$ which means $a \prec a$. □

The proposition above is a special case of a more general fact. Notice that the set $\{*\}$ is completely compact as a round lower set, meaning $\{*\} \lll \{*\}$. Therefore, any map into the round lower sets of L_+ which preserves the completely-below relation will map $\{*\}$ to a set of the form $\downarrow a$ where a is a compact token. We arrive at the interaction algebra version of Proposition 6.1.24.

Theorem 1.1.11. *The following are equivalent for a Stage 0 morphism R.*

1. *The morphism R is semi-open, i.e. has a right adjoint.*
2. *The homomorphism $\Omega(R)$ between round lower sets of tokens preserves the completely-below relation.*

Proof. Suppose $R : \mathcal{L} \to \mathcal{M}$ has the right adjoint R_* and A, B are round lower sets of tokens of $\mathcal{M}$ where $A \subseteq \downarrow b$ for some $b \in B$. Since B is round we have $b \succ\!\!\prec \psi_0 \bigcirc b'$ for some $b' \in B$. For our convenience we employ the Fundamental Lemma 1.1.6 once more and write $\Omega(A) = \{\phi \in L_- \mid \exists a \in A.\, \phi R a\}$. From $b \succ\!\!\prec \psi_0 \bigcirc b' \in B$ and $\bigcirc\ \subseteq R_* \,{}^{\circ}_{9}\, R$ we obtain $\psi_0 R_*; \succ\!\!\prec \phi_0 R b'$. Notice that in particular $\phi_0 \in \Omega(B)$. We claim that $\Omega(A) \subseteq \uparrow \phi_0$. Indeed, if $\phi R a$ for some $a \in A$ then $\phi R a \prec b \succ\!\!\prec \psi_0 R_*; \succ\!\!\prec \phi_0$ which because of $R \,{}^{\circ}_{9}\, R_* \subseteq\ \bigcirc$ implies $\phi \succ \phi_0$. This shows that $\Omega(R)$ preserves the completely-below relation.

For the reverse implication, suppose that the map $\Omega(R)$ preserves the completely-below relation. We construct a right adjoint to R as follows. Let $S : M_+ \to L_-$ be the relation

$$m S \phi :\Leftrightarrow \forall \phi' \in L_-.\, (\phi' R m \Rightarrow \phi' \succ \phi).$$

We claim that the right adjoint to R is $R_* := \bigcirc; S; \bigcirc$.

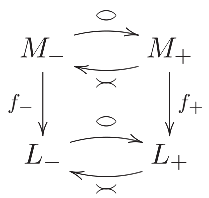
$$\begin{aligned} \phi'(R \,{}^{\circ}_{9}\, R_*)a &\Leftrightarrow \phi'(R; \prec) m S \phi \bigcirc a \\ &\Rightarrow \phi' R m S \phi \bigcirc a \\ &\Rightarrow \phi' \succ \phi \bigcirc a \\ &\Rightarrow \phi' \bigcirc a. \end{aligned}$$

Hence $R \fatsemi R_*$ is contained in $\bigcirc$. The fact that $\Omega(R)$ preserves the completely-below relation has the following consequence. Whenever $m' \prec m$ in M_+ then the round lower sets generated by these tokens have $\downarrow m' \lll \downarrow m$ and thus $\Omega(R)(\downarrow m') \lll \Omega(R)(\downarrow m)$, meaning that there exist a witness $\phi R m$ with the property that $\phi' R m'$ implies $\phi' \succ \phi$. We use this to show $\bigcirc \subseteq R_* \fatsemi R$. Suppose $\psi \bigcirc m$ in $\mathcal{M}$. We interpolate to $\psi \bigcirc m' \prec m$ and use the observation we just made: There exists a witness $\phi R m$ which, by definition of the relation S above, has $m' S \phi$. Hence $\psi \bigcirc m' S \phi R m$ which, using $\bigcirc \fatsemi R = R$, we can expand to $\psi(R_* \fatsemi R)m$. This finishes the proof that R_* is right adjoint to R. □

Token maps

In some cases we are lucky, and the particular sets of tokens and witnesses we chose to represent a completely distributive frame allow us to express a homomorphism in CDFrm as a *function* between tokens rather than a relation. This situation typically occurs when representing functors and monads, such as the powerdomain monads of sections 1.10, 1.11 and 1.12.

Definition 1.1.7. A *token map* between interaction algebras $\mathcal{M}$ and $\mathcal{L}$ is a pair of functions



which preserve the structure of the interaction algebra, meaning that $f_-(\phi) \bigcirc f_+(a)$ whenever $\phi \bigcirc a$ and $f_+(a) \asymp f_-(\phi)$ whenever $a \asymp \phi$.

The central fact about token maps is the existence of a functorial assignment to adjoint pairs of morphisms:

Proposition 1.1.12. *There is a contravariant functor from the category of Stage 0 interaction algebras and token maps to the category of Stage 0 interaction algebras and semi-open morphisms.*

Proof. Given a token map $(f_-, f_+) : \mathcal{M} \to \mathcal{L}$ define relations $R_+ : L_- \to M_+$ and $R_- : M_- \to L_+$ as $\phi R_+ m$ iff $\exists m' \prec m.\, \phi \bigcirc f_+(m')$ and $\psi R_- a$ iff $\exists \psi'.\, \psi \succ \psi'$ and $f_-(\psi') \bigcirc a$. First observe that preservation of $\asymp$ and $\bigcirc$ implies that both f_+ and f_- are monotone with respect to $\prec$ and $\succ$, respectively.

It is easy to see that both R_+ and R_- are Stage 0 morphisms, whence we concentrate on showing that R_- is right adjoint to R_+. Suppose $\phi \bigcirc f_+(m')$, $m' \prec m \asymp \psi \succ \psi'$ and $f_-(\psi') \bigcirc a$. Then in particular $f_+(m') \asymp ; \bigcirc ; \asymp f_-(\psi')$ and therefore $\phi \bigcirc a$. Hence $R_+ \fatsemi R_- \subseteq \bigcirc$. Now suppose $\psi \bigcirc m$. Then there exist ψ' and m' such that $\psi \succ \psi' \bigcirc m' \prec m$. Consequently $f_-(\psi') \bigcirc f_+(m')$ with which we can conclude $\psi R_- \fatsemi R_+ m$. Thus $\bigcirc \subseteq R_- \fatsemi R_+$.

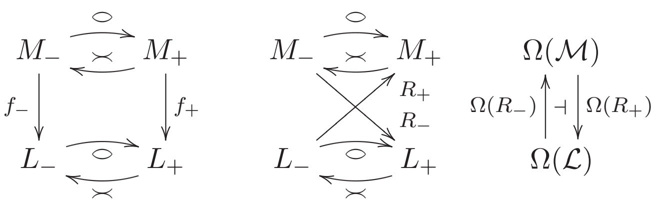


Figure 1.4: Token maps give rise to semi-open morphisms, which in turn yield adjoint pairs of linear maps between completely distributive frames.

Observe that in case (f_-, f_+) is the pair of identity maps on $L_- \times L_+$ then the resulting relations are both equal to $\circ$. Let again be $(f_-, f_+) : \mathcal{M} \to \mathcal{L}$, $(g_-, g_+) : \mathcal{L} \to \mathcal{K}$ and S_- and S_+ be the relations generated by g_- and g_+. We have $\theta S_+ \mathbin{;} R_+ m$ iff $\theta \circ g_+(a')$, $a' \prec a \asymp \phi \circ f_+(m')$ and $m' \prec m$. Then $a' \prec f_+(m')$ and so $g_+(a') \prec (g_+ \circ f_+)(m')$ which shows that (θ, m) is in the relation generated by $g_+ \circ f_+$.

Conversely, suppose that $\theta \circ (g_+ \circ f_+)(m')$ for some $m' \prec m$. Interpolate to $m' \prec m'' \prec m$ and note $f_+(m') \prec f_+(m'')$. Define $a' = f_+(m')$. Then $\theta \circ g_+(a')$, $a' \prec a \asymp \phi \circ f_+(m'')$ for suitable a and ϕ whereby $\theta(S_+ \mathbin{;} R_+)m$. The proof for $S_- \mathbin{;} R_-$ is analogous. □

In some cases the token maps themselves satisfy some sort of a continuity condition, in which case the definition of the resulting adjoint pair of relations becomes easier.

Lemma 1.1.13. *If $(f_-, f_+) : \mathcal{M} \to \mathcal{L}$ is a token map such that additionally*

1. *$\psi \circ f_+(m)$ implies that there exists $m' \prec m$ with $\psi \circ f_+(m')$,*

2. *$f_-(\psi) \circ a$ implies that there exists $\psi \succ \psi'$ with $f_-(\psi') \circ a$*

Then the resulting adjoint pair of relations has the simpler definition $\phi R_+ m$ iff $\phi \circ f_+(m)$ and $\psi R_- a$ iff $f_-(\psi) \circ a$.

Proof. Straightforward. □

Corollary 1.1.14. *Consider the following diagram of interaction algebras, where R is a Stage 0 morphism and (g_-, g_+) and (f_-, f_+) are token maps.*

$$\mathcal{J} \xleftarrow{(g_-,g_+)} \mathcal{L} \xrightarrow{R} \mathcal{M} \xleftarrow{(f_-,f_+)} \mathcal{K}$$

Suppose the component maps f_+ and g_+ satisfy the additional condition of Lemma 1.1.13.

1. *The composition of R with the Stage 0 morphism defined by f_+ is given as $\phi(R \mathbin{;} f_+)k$ iff $\phi R f_+(k)$.*

2. *The composition of R with the Stage 0 morphism defined by g_+ is given as $\theta(g_+ \mathbin{;} R)m$ iff $\theta \succ g_-(\phi)$, $\phi R m$ for some witness $\phi \in L_-$.*

Proof. (1) The map f_+ defines the relation $\psi F k \Leftrightarrow \psi \bigcirc f_+(k)$. Thus, $\phi(R \,; F)k$ iff $\phi R m \curlyeqprec \psi \bigcirc f_+(k)$ which is equivalent to $\phi R f_+(k)$.

(2) The map g_- defines the relation $\theta G a \Leftrightarrow \theta \bigcirc g_+(a)$. Thus $\theta(G \,; R)m$ holds by definition if and only if $\theta \bigcirc g_+(a)$ and $a \curlyeqprec \phi R m$ for some token $a \in L_+$ and some witness $\phi \in L_-$. Since (g_-, g_+) preserves $\curlyeqprec$ we deduce $\theta \bigcirc g_+(a) \curlyeqprec g_-(\phi)$. Conversely, if $\theta \succ g_-(\phi)$ and $\phi R m$ then $\phi \bigcirc a \curlyeqprec ; R m$ for some token a whereby $\theta \succ ; \bigcirc g_+(a)$ and therefore $\theta(G \,; R)m$. □

1.2 Interaction algebras for domains

Having established the core category Tok_0 of interaction algebras and having studied its basic properties, we turn towards interaction algebras of the kind we saw in the introduction to this chapter. The tokens are now thought of as elements of a domain. There are two available dualities for domains: Lawson duality puts a domain into correspondence with the poset of its Scott open filters. However, this duality is not functorial for Scott continuous maps. One needs to ensure that the preimage of a Scott open filter is again a filter. The most suitable category to consider Lawson duality on is the category of *preframes* and Scott continuous maps preserving finite meets. Hence we postpone the study of Lawson duality until we reach this stage.

Stone duality puts domains and Scott continuous maps into dual equivalence with completely distributive frames and frame homomorphisms. This matches our intuition about the type of witnesses we want to use. However, there are some obstacles we have to overcome:

- The Stage 0 morphisms between interaction algebras correspond to maps preserving all joins. Hence we need a way of expressing when such a map also preserves finite meets.
- If the set of tokens is to be an abstract basis for the domain we present, then we need to ensure that the round lower set of any token is an ideal with respect to the relation $\prec$.

It turns out that one can achieve both of the above goals neatly by closing the basis of Scott open sets that makes up the set of witnesses under binary meets. This does not do any harm to the cardinality of the information system, as for example the set of finite subsets of a countable set is still countable. The binary meet of Scott open sets then obeys some simple rules. For example, a point of the domain is contained in the intersection of two Scott open sets if and only if it is contained in both opens. Another fact which is always true is that any point has at least one Scott open set it is a member of. (Observe that the corresponding fact is not true for completely distributive frames and completely prime upper sets: The least element $0 \in L$ of a completely distributive frame is only member of the upper set $L = \Uparrow 0$. This is not completely prime, as $0 = \bigvee \emptyset \in L$ but clearly $\emptyset$ does

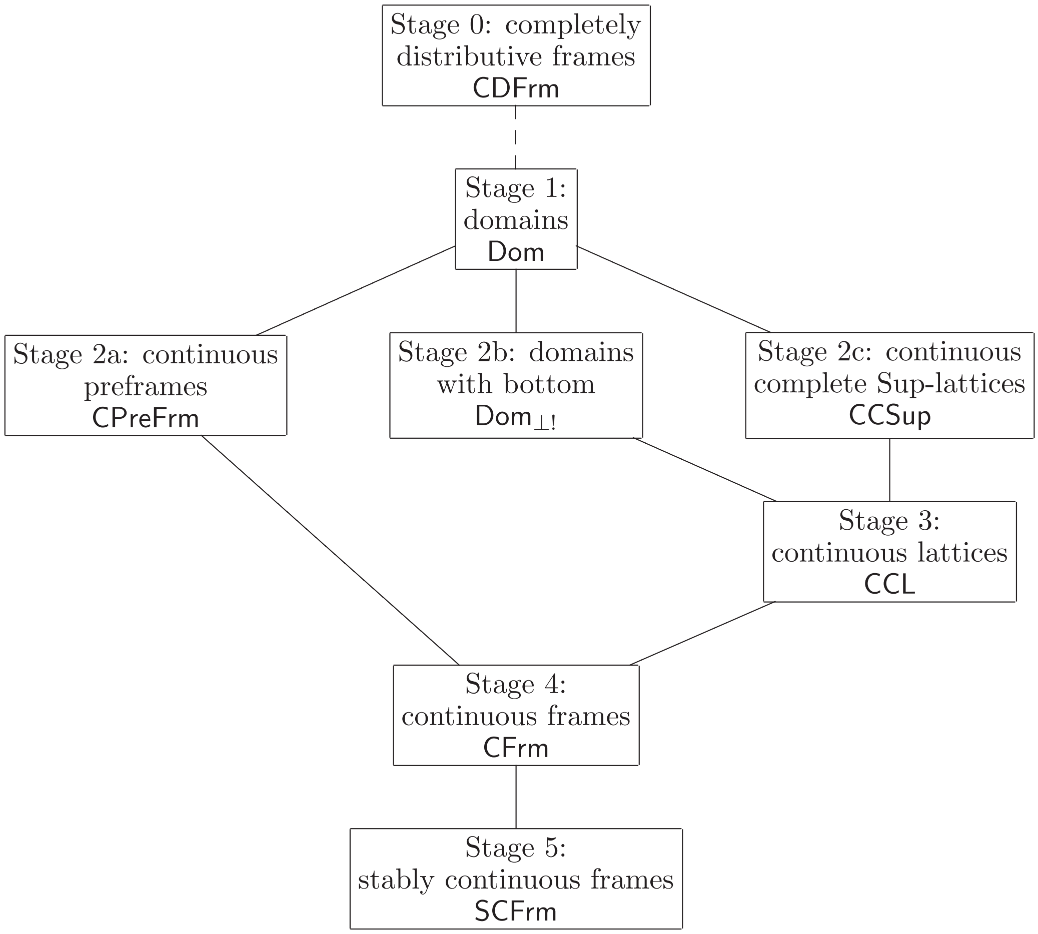


Figure 1.5: Hierarchy of categories and stages of interaction algebras. Solid lines signify inclusion of subcategories.

not intersect L.) Now the round lower set $\downarrow a = \downarrow a$ of a token is an ideal with respect to $\prec = \ll$ by the very definition of a continuous dcpo. Conveniently, the binary meet of Scott open sets will also enable us to express frame homomorphisms between the Scott topologies in a manner slightly easier than the *approximable mappings* between Vickers' information systems.

The subcategories of domains we consider are all constructed by postulating additional finitary structure on the domains. For example, a continuous lattice is a domain which, in addition to directed joins, has finite joins as well. A continuous preframe is a domain which in addition to directed joins has finite meets, and finite meets distribute over directed joins. Distributive laws of this kind can be enforced with a simple trick: We know that the set of all round lower sets of tokens is completely distributive, hence satisfies all distributive laws we could ever want. If we present a preframe as a certain subset of round lower sets, and show that this subset is closed under directed joins and finite meets in that frame, then it inherits the distributive law from the ambient completely distributive frame.

The hierarchy of categories we study is depicted in Figure 1.5. For each stage, we will add binary or nullary operations to either tokens or witnesses. The prime example is the binary meet of Scott open sets. However, there is no need for the binary operation to be a real semilattice operation. While it is convenient to assume that the operation is commutative and associative, it is not required to be idempotent. The distinguished elements, although we use them as if they were neutral elements or units for the binary operation, need not obey the corresponding algebraic laws in the strict sense. Instead, it suffices if the algebraic laws hold up to a certain equivalence. For example, a binary operation $\sqcap$ on witnesses may have $\phi \sqcap \phi \neq \phi$, but we require $\phi \sqcap \phi \bigcirc a$ to be equivalent to $\phi \bigcirc a$. A distinguished witness 1 need not have $1 \sqcap \phi = \phi \sqcap 1 = \phi$, but it suffices if $1 \sqcap \phi \bigcirc a$ precisely when $\phi \bigcirc a$.

The algebraic axioms we add to either tokens or witnesses are listed in Table 1.2, written in the deduction-rule style. By convention, all axioms which hold for a morphism R in particular hold for the identity morphism $\bigcirc$. The axioms accumulate throughout the hierarchy. For example, at Stage 2a all axioms of Stage 1 hold, too. Since Stage 3 is just the union of the Stage 2b and Stage 2c axioms, we omitted the Stage 3 axioms from the table.

Notation. If n is either 0, 1, 2a, 2b, 2c, 3 or 4 then by Tok_n we denote the category where objects are interaction algebras that satisfy all Stage n axioms, and morphisms are relations R that satisfy all Stage n axioms involving R. We refer to these structures as Stage n interaction algebras and Stage n morphisms.

1.2.1 Semilattice-like structure

If a set L is equipped with a semilattice operation $\sqcap$ then one derives a partial order from it by defining $x \sqsubseteq y$ whenever $x \sqcap y = x$. In this order, $x \sqcap y$ is the binary meet of x and y or infimum of the set $\{x, y\}$. Notice that the partial order $\sqsubseteq$ is reflexive because $\sqcap$ is idempotent, transitive because $\sqcap$ is associative, and antisymmetric because we used the antisymmetric relation $=$ in the definition. Since equality of witnesses is too strong a requirement, the above definition of partial order will have little meaning for Stage 1 interaction algebras. Instead, let us study how the binary operation $\sqcap$ on witnesses behaves with respect to the relation $\succ$.

Definition 1.2.1. We say that a token a of an interaction algebra $\mathcal{L}$ is *weakly below* a token b and write $a \preccurlyeq b$ if the round lower set $\downarrow a$ is contained in the round lower set $\downarrow b$. The tokens a and b are said to be *lower equivalent*, written $a \simeq b$, if $a \preccurlyeq b \preccurlyeq a$. Similarly, a witness ψ is weakly below ϕ if $\downarrow\psi$ is contained in $\downarrow\phi$. We write this relation as $\phi \succcurlyeq \psi$ and say that the witnesses are lower equivalent if $\phi \succcurlyeq \psi \succcurlyeq \phi$.

It is immediate that both $\preccurlyeq$ and $\succcurlyeq$ are preorders. From the duality with completely distributive frames we know that these preorders have a natural interpretation: The former

Stage 0 (Completely distributive frame)	$\bigcirc ; \succ\!\prec ; R ; \succ\!\prec ; \bigcirc = R$
Stage 1 (Domain)	$(L_-, \sqcap)$
binary meet	$\dfrac{\phi R a \qquad \psi R a}{\phi \sqcap \psi R a}$
weakening rules	$\dfrac{\phi \sqcap \psi R a}{\phi R a} \qquad \dfrac{a \succ\!\prec \phi}{a \succ\!\prec \phi \sqcap \psi}$
definedness	$\dfrac{a \in L_+}{\exists \phi \in L_- .\, \phi R a}$
Stage 2a (Continuous preframe)	$(L_+, \sqcap)$
binary meet	$\dfrac{\phi R a \qquad \phi R b}{\phi R a \sqcap b}$
weakening rules	$\dfrac{\phi R a \sqcap b}{\phi R a} \qquad \dfrac{a \succ\!\prec \phi}{a \sqcap b \succ\!\prec \phi}$
dual definedness	$\dfrac{\phi \in L_-}{\exists a \in L_+ . \phi R a}$
Stage 2b (Continuous cpo and strict maps)	$1 \in L_-$, $0 \in L_+$
empty meet	$\dfrac{}{1 R a} \qquad \dfrac{}{0 \succ\!\prec 1}$
strictness	$\dfrac{\phi R 0}{\phi \bigcirc 0}$
Stage 2c (Continuous complete Sup-lattice)	$(L_+, \sqcup)$
binary join	$\dfrac{a \succ\!\prec \phi \qquad b \succ\!\prec \phi}{a \sqcup b \succ\!\prec \phi}$
weakening rules	$\dfrac{\phi R a}{\phi R a \sqcup b} \qquad \dfrac{a \sqcup b \succ\!\prec \phi}{a \succ\!\prec \phi}$
join-strength	$\dfrac{\phi R a \sqcup b}{\exists \psi R a,\, \theta R b .\, \phi \succ \psi \sqcap \theta}$
Stage 3 (Continuous lattice)	combine Stages 2b and 2c
Stage 4 (Continuous frame)	$\sqcap$ and $\sqcup$ distribute (Definition 1.7.1)

Table 1.2: Axioms for interaction algebras

preorder is derived from the inclusion $\downarrow\colon L_+ \to \Omega\mathcal{L}$ of tokens into round lower sets, whereas the latter is derived from the inclusion $\downarrow\colon L_- \to (\Omega\mathcal{L})^{\curlywedge\!\!\!\curlywedge}$ of witnesses into completely prime upper sets.

Observe that $a \preccurlyeq b$ can be defined in terms of $\bigcirc$ alone. Indeed, using the interpolative law it is easy to show that $x \prec a$ implies $x \prec b$ precisely when $\phi \bigcirc a$ implies $\phi \bigcirc b$ for all witnesses ϕ. Likewise, the relation $\phi \succcurlyeq \psi$ holds if and only if $\psi \bigcirc a$ implies $\phi \bigcirc a$ for all tokens a.

We claim that for witnesses ϕ and ψ of a Stage 1 interaction algebra, the witness $\phi \sqcap \psi$ is a greatest lower bound[2] with respect to the preorder $\succcurlyeq$. Indeed, the weakening rule for $\sqcap$ yields that $\phi, \psi \succcurlyeq \phi \sqcap \psi$. If θ is another lower bound for both ϕ and ψ, then $\theta \bigcirc a$ implies $\phi \bigcirc a$ and $\psi \bigcirc a$, whence using the meet rule $\phi \sqcap \psi \bigcirc a$. Therefore $\phi \sqcap \psi \succcurlyeq \theta$ which proves the claim. In particular, ϕ is lower equivalent to $\phi \sqcap \phi$.

One way of interpreting our findings is that in case one quotients the set of witnesses by the equivalence relation $\succcurlyeq \cap (\succcurlyeq)^{-1}$ then $\sqcap$ becomes a true semilattice operation.

1.2.2 The round ideal functor

In the category of Stage 0 interaction algebras, we presented the duality with completely distributive frames as the hom-set functor $\mathsf{Tok}_0(-, \mathbf{1})$. As we shall see, the same approach can be taken in the category Tok_1 of Stage 1 interaction algebras and Stage 1 morphisms.

Definition 1.2.2. A *round ideal* of tokens of an interaction algebra $\mathcal{L}$ is a round lower set $I \subseteq L_+$ which is directed with respect to $\prec$, meaning that I is not empty and whenever $a, b \in I$ then $a, b \prec c$ for some $c \in I$. The collection of all such round ideals, ordered by inclusion, is denoted by $\mathrm{Idl}^{\prec} L_+$.

Vickers shows in [60] that for an information system $(X, <)$ the completely distributive frame $\mathrm{Up}^{\prec} X$ of round upper sets is the Stone dual of $\mathrm{Idl}^{\prec} X$, where a basis for the topology is given by sets of the form $\{I \in \mathrm{Idl}^{\prec} X \mid a \in I\}$ for tokens $a \in X$. We will derive a similar result for interaction algebras. Note that the interaction algebra $\mathbf{1}$ satisfies all Stage 1 axioms if we declare a binary operation on the witness set $\{\varphi\}$ by $\varphi \sqcap \varphi = \varphi$.

Lemma 1.2.1. *Let $\mathcal{L}$ be a Stage 1 interaction algebra. The following sets are order-isomorphic.*

1. *The set $\mathrm{Idl}^{\prec} L_+$ of round ideals of tokens,*
2. *The set of round filters of witnesses, that is non-empty round upper sets which are closed under $\sqcap$,*
3. *Stage 1 morphisms $\mathcal{L} \to \mathbf{1}$.*

[2] In a preorder, greatest lower bounds are not always unique.

Proof. The isomorphism $(-)^{\circ}$ which we defined in equation (1.3) of the Fundamental Lemma 1.1.6 restricts to an isomorphism between round ideals of tokens and round filters of witnesses. Knowing this, the other claims follow from Proposition 1.1.5. □

An immediate consequence of the lemma above is that the round ideal operator is contravariantly functorial on Tok_1. What is the class of objects thus constructed? Round ideals are closed under directed unions, because round lower sets are closed under all unions, and using the isomorphism with round filters of witnesses it becomes apparent that the directed union of such filters is again closed under $\sqcap$.

Lemma 1.2.2. *For a Stage 1 interaction algebra $\mathcal{L}$, the map $\downarrow$ takes tokens to round ideals of tokens and transforms the relation $\prec$ into $\ll$.*

Proof. First observe that for a token of a Stage 1 interaction algebra the round lower set $\downarrow a$ is a round ideal. Indeed, the Stage 1 axioms for the relation $\circ$ say that for any token a the set $\{a\}^{\circ} = \{\phi \in L_- \mid \phi \circ a\}$ is a round filter and by the lemma above corresponds to the round ideal $(\{a\}^{\circ})_{\asymp} = \downarrow a$. From Proposition 1.1.5 (3). we know that $\downarrow a \lll \downarrow b$ in the complete lattice of round lower sets whenever $a \prec b$. As the round ideals of tokens form a sub-dcpo, we now have $\downarrow a \ll \downarrow b$ in the dcpo of round ideals. □

Proposition 1.2.3. *The hom-set functor $\mathsf{Tok}_1(-, \mathbf{1})$ presents a contravariant functor pt_1 from Stage 1 interaction algebras to domains and Scott continuous maps.*

Proof. Let I be a round ideal of tokens. Since I is in particular a round lower set, we know that $I = \bigcup_{a \in I} \downarrow a$. This union is directed. Indeed, $a \prec b$ implies $\downarrow a \subseteq \downarrow b$, and I is directed with respect to $\prec$. From Lemma 1.2.2 above we know that $\downarrow a \ll I$ for any $a \in I$.

For a Stage 0 morphism $R : \mathcal{L} \to \mathcal{M}$ the map $\Omega(R) : \mathrm{Lo}^{\prec} M_+ \to \mathrm{Lo}^{\prec} L_+$ preserves arbitrary unions of round lower sets, and the functor which is presented by $\mathsf{Tok}_1(-, \mathbf{1})$ is just the restriction of Ω to round ideals, whence this restriction preserves directed unions. □

Remark. In the proof of the proposition above we showed that the structure $(L_+, \prec)$ is an abstract basis in the sense of Abramsky and Jung. The preceding lemma and proposition correspond to [1, Proposition 2.2.22].

We postpone the description of the dual adjoint of the functor pt_1 until we know more about the Scott topology of the domain $\mathrm{pt}_1 \mathcal{L}$.

1.2.3 Stone duality of domains via interaction algebras

Theorem 1.2.4. *Let $\mathcal{O}$ be the contravariant functor which takes a Scott continuous map between domains to the preimage operation between the Scott topologies. The following* *diagram of contravariant functors commutes (up to isomorphism).*

$$\begin{array}{ccc} \mathsf{Tok}_1 & \xrightarrow{\mathrm{pt}_1} & \mathsf{Dom} \\ {\scriptstyle \mathrm{Flip}}\downarrow & & \downarrow{\scriptstyle \mathcal{O}} \\ \mathsf{Tok}_0 & \xrightarrow[\Omega]{} & \mathsf{Frm} \end{array}$$

Proof. The proof follows the same pattern as the proof of Proposition 1.1.8. □

Frame homomorphisms between Scott topologies

A priori we only know that $\Omega \circ \mathrm{Flip}$ produces join-preserving maps between completely distributive frames. However, it can be shown directly that this functor maps Stage 1 morphisms to frame homomorphisms: Let $R : L_- \to M_+$ be as in the proof of the theorem above. The join-preserving map of interest sends a round lower set $\Phi \subseteq L_-$ to $h(\Phi) = \{m \in M_+ \mid \exists \phi \in \Phi.\, \phi R m\}$. First, notice that h preserves the top element. Indeed, because of the definedness axiom, the entire set M_+ is a round upper set. Use definedness of the relation R and obtain $\forall m \in M_+ \exists \phi \in L_-.\, \phi R m$. Interpolate to $\phi \succ \phi' R m$. Now observe that ϕ' is an element of the largest round lower set in L_-. Thus, the definedness axiom for R implies that the map h defined above preserves the empty meet.

For binary meets, because of monotonicity of h it suffices to check the inclusion $h(\Phi) \wedge h(\Psi) \subseteq h(\Phi \wedge \Psi)$. Suppose that $m \in h(\Phi) \wedge h(\Psi)$, that is, $\Phi \ni \phi R x$, $\Psi \ni \psi R y$ and $x, y \prec m$. From weakening it is immediate that $\phi, \psi R m$ whence by the meet rule $\phi \sqcap \psi R m$. Roundness of R now yields $\phi \sqcap \psi \succ \theta R m$. Observe that because of the weakening rules, $\phi \sqcap \psi$ is an element of $\Phi \cap \Psi$ and therefore θ is an element of the meet $\Phi \wedge \Psi$. This shows $m \in h(\Phi \wedge \Psi)$.

1.2.4 Duality with domains

By now it should be clear what the adjoint $\mathsf{Dom} \to \mathsf{Tok}_1$ to the functor pt_1 might be:

Definition 1.2.3. For a domain D, let $\mathrm{Ialg}_1\, D$ denote the interaction algebra where the witnesses are Scott open sets of D, tokens are elements of D, the relation $U \multimapboth x$ holds whenever $x \in U$ and the relation $x \rightarrowtail\!\!\!\!\!\prec U$ holds if x is a lower bound for U. If $f : D \to E$ is a Scott continuous map between domains, then $\mathrm{Ialg}(f)$ is the relation $\sigma E \to D$ with $U\, \mathrm{Ialg}_1(f) x$ iff $f(x) \in U$.

The axioms of Stage 1 are easily checked for the interaction algebra $\mathrm{Ialg}_1\, D$, where $\sqcap$ is binary intersection of Scott opens sets. In the introduction we convinced ourselves that, since the relation $\prec$ on the tokens of $\mathrm{Ialg}_1\, D$ coincides with the way-below relation, the domain $\mathrm{pt}_1\, \mathrm{Ialg}_1\, D$ is isomorphic to D and that the Scott continuous map $f : D \to E$ corresponds to the map $(\mathrm{pt}_1 \circ \mathrm{Ialg}_1)(f)$. The relations $\multimapboth$ and $\rightarrowtail\!\!\!\!\!\prec$ of the interaction algebra $\mathrm{Ialg}_1(D)$ behave well with respect to the complete structure on tokens and witnesses in the following sense. If $\mathcal{U}$ is a set of Scott opens sets of D and X is a directed set in D, then $\bigcup\mathcal{U} \bigcirc \bigsqcup X$ precisely when $U \bigcirc x$ for some pair $(U, x) \in \mathcal{U} \times X$ already. Dually, the relation $\bigsqcup X \succ\!\!\prec \bigcup \mathcal{U}$ holds if and only if $x \succ\!\!\prec U$ for all $x \in X$ and $U \in \mathcal{U}$.

Theorem 1.2.5. *The functors* $\mathsf{Dom} \underset{\mathrm{pt}_1}{\overset{\mathrm{Ialg}_1}{\rightleftarrows}} \mathsf{Tok}_1$ *constitute a contravariant equivalence of categories. The functor* Ialg_1 *is presented by* $\mathsf{Dom}(-, 2)$ *while the functor* pt_1 *is presented by* $\mathsf{Tok}_1(-, \mathbf{1})$.

Proof. Under the Scott topology, the domain 2 is homeomorphic to the Sierpinski space. It is well-known that the opens of any topological space X are in order-preserving bijection with the continuous maps into the Sierpinski space. If the Scott open $U \in \sigma E$ is presented by its characteristic map $\chi_U : E \to 2$ then for a map $f : D \to E$ the relation $\mathrm{Ialg}_1(f)$ defined above has the description $U \,\mathrm{Ialg}_1(f) x$ iff $(\chi_U \circ f)(x) = 1$.

It remains to show that the composite $\mathrm{Ialg}_1 \circ \mathrm{pt}_1$ is isomorphic to the identity functor on Tok_1. For this we modify the proof of Theorem 1.1.9. From Proposition 1.2.4 we know that the witnesses of the interaction algebra $\mathrm{Ialg}_1 \,\mathrm{pt}_1 \,\mathcal{L}$ are the round lower sets of L_-. In the same way as in the proof of Theorem 1.1.9 one can now show that $\mathrm{Ialg}_1 \circ \mathrm{pt}_1$ is indeed isomorphic to the identity functor, checking the Stage 1 axioms at the appropriate places in the proof. □

1.2.5 Duality with information systems and abstract bases

We know that both the categories of information systems and abstract bases are equivalent to the category of domains. Using Theorem 1.2.5 we can conclude that both categories are dually equivalent to Tok_1. The detour via Dom is a bit wasteful when considering the cardinalities of the sets involved. It is desirable to know whether the dualities can be described efficiently without blowing up the cardinalities too much. Figure 1.6 summarises how to extract an information system or an abstract basis from an interaction algebra.

Proposition 1.2.6. *There are contravariant faithful functors from the category* Tok_1 *of Stage 1 interaction algebras to the categories* $\mathsf{Infosys}$ *and* Abs.

Proof. If $\mathcal{L}$ is a Stage 1 interaction algebra, then its tokens together with the relation $\succ\!\!\prec ; \bigcirc \;= \prec$ form an abstract basis which presents the domain $\mathrm{Idl}^{\prec} L_+$. According to Vickers an information system $(X, <)$ presents the domain $\mathrm{Idl}^{<} X$ which has the round upper sets of X with respect to $<$ as Scott opens. Thus we may take the witnesses of $\mathcal{L}$ as tokens of an information system and let $<$ be the relation $\bigcirc ; \succ\!\!\prec \;= \succ$. Then the round upper sets with respect to $<$ are the round lower sets with respect to $\succ$, and from Theorem 1.2.4 we know that these are the Scott opens of $\mathrm{pt}_1 \,\mathcal{L}$.

It remains to consider morphisms. In Definitions 6.2.1 and 6.2.2 we chose a contravariant way of writing the morphisms between information systems and abstract bases. This

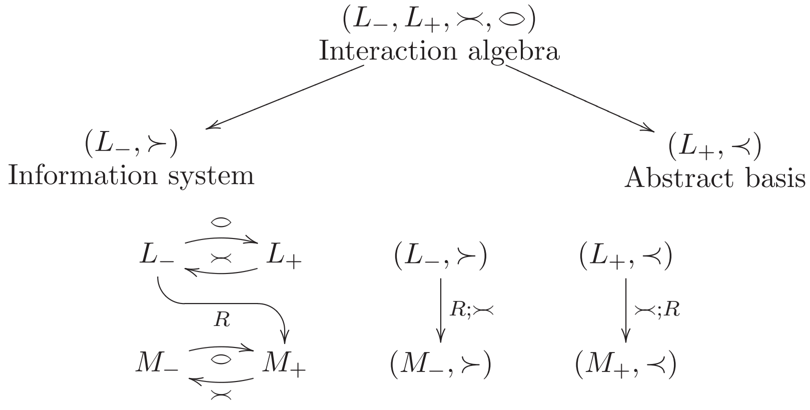


Figure 1.6: A Stage 1 interaction algebra contains an information system as well as an abstract basis.

makes it easier to construct the morphism parts of our functors. Given a Stage 1 morphism $R : \mathcal{L} \to \mathcal{M}$, define relations $R;\mathrel{\succ\!\!\!\prec} : L_- \to M_-$ and $\mathrel{\succ\!\!\!\prec};R : L_+ \to M_+$. We claim that the former relation is an approximable mapping, i.e. a morphism of the category Infosys, and the latter is an approximable relation, which are the morphisms in the category Abs. It is immediate that $\mathrel{\succ\!\!\!\prec};R$ satisfies the axiom (AM1) of Definition 6.2.1 because $R;\mathrel{\succ\!\!\!\prec};\succ = R;\prec;\mathrel{\succ\!\!\!\prec} = R;\mathrel{\succ\!\!\!\prec}$. Even more trivial to check is the axiom (AM2). For (AM3), suppose $\Phi \subseteq L_-$ is a finite set and $\psi \succ \psi'$ are elements of M_- such that for every element ϕ of Φ the relation $\phi(R;\mathrel{\succ\!\!\!\prec})\psi \succ \psi'$ holds. Let $m \in M_+$ be a token with $\psi \mathrel{\subset\!\!\supset} m \mathrel{\succ\!\!\!\prec} \phi'$. In case Φ is the empty set, use the definedness axiom of Stage 1 and obtain some witness $\phi' \in L_-$ with $\phi' Rm$ which implies $\phi R;\mathrel{\succ\!\!\!\prec}\psi'$. Otherwise, if Φ is not empty then for every $\phi \in \Phi$ we have $\psi R;\mathrel{\succ\!\!\!\prec};\mathrel{\subset\!\!\supset} m$ and thereby ϕRm. Now use the meet rule of Stage 1 and obtain $\bigsqcap \Phi Rm$. Using $R \subseteq \succ;R$ we obtain $\bigsqcap \Phi \succ \phi' Rm$ for some witness ϕ' whence by the weakening rules $\phi \succ \phi'$ for every $\phi \in \Phi$. Combining this with $m \mathrel{\succ\!\!\!\prec} \phi'$ yields the axiom (AM3).

The relation $\mathrel{\succ\!\!\!\prec};R$ clearly satisfies the axioms (AR1) and (AR2) of Definition 6.2.2. We check axiom (AR3). Let $A \subseteq L_-$ be a finite set, $m \in M_+$ and suppose for all $a \in A$ the relation $a(\mathrel{\succ\!\!\!\prec};R)m$ holds. In case A is the empty set we use the definedness axiom and obtain some witness ϕ such that ϕRm. Then also $\phi \mathrel{\subset\!\!\supset} a'(\mathrel{\succ\!\!\!\prec};R)m$ and we are done. If A is not empty then for any such $a \in A$ there is a witness ϕ_a with $a \mathrel{\succ\!\!\!\prec} \phi_a Rm$. Apply the meet rule and get $\bigsqcap_{a \in A} \phi_a Rm$. Using $R \subseteq \succ;R$ and weakening we get some witness ϕ' and a token a' with $a \mathrel{\succ\!\!\!\prec} \bigsqcap_{a \in A} \phi_a \mathrel{\subset\!\!\supset} a' \mathrel{\succ\!\!\!\prec} \phi' Rm$. Then a' is the desired token for the axiom (AR3).

For faithfulness observe that $R;\mathrel{\succ\!\!\!\prec} = S;\mathrel{\succ\!\!\!\prec}$ implies $R;\mathrel{\succ\!\!\!\prec};\mathrel{\subset\!\!\supset} = R = S = S;\mathrel{\succ\!\!\!\prec};\mathrel{\subset\!\!\supset}$. Similarly $\mathrel{\succ\!\!\!\prec};R = \mathrel{\succ\!\!\!\prec};S$ implies $R = S$. □

Given an abstract basis $(X, \prec)$ we know that the tokens also yield a basis for the Scott topology on $\mathrm{Idl}^{\prec} X$ via the assignment $x \mapsto \{ I \in \mathrm{Idl}^{\prec} X \mid x \in I \}$. In order to construct a Stage 1 interaction algebra out of this, we need to close this basis under binary meets.

We achieve this in the same manner as one constructs the free semilattice over a poset.

Proposition 1.2.7. *From an abstract basis $(X, \prec)$ construct an interaction algebra where the tokens are elements of X and the witnesses are finite subsets of X. The relations $\circ$ and $\rightarrowtail$ are defined as*

$$M \circ x \quad \Leftrightarrow \quad \forall m \in M.\, m \prec x$$
$$x \rightarrowtail M \quad \Leftrightarrow \quad \exists m \in M.\, x \prec m$$

With a binary operation $\sqcap$ defined as union of finite sets, the structure $(\mathrm{Fin}\, X, X, \rightarrowtail, \circ)$ is a Stage 1 interaction algebra where the relation $\rightarrowtail; \circ$ agrees with $\prec$ on X. This assignment extends to a functor $\mathsf{Abs} \to \mathsf{Tok}_1$ as follows. If $R : Y \to X$ is a relation constituting an approximable relation between abstract bases $(X, \prec)$ and $(Y, \prec)$ then $M\hat{R}x \Leftrightarrow \forall y \in M.\, yRx$ is a Stage 1 morphism.

Proof. We begin with checking the Stage 0 axioms for the relations $\circ : \mathrm{Fin}\, X \to X$ and $\rightarrowtail : X \to \mathrm{Fin}\, X$. If $M \subseteq X$ is a finite set and $M \circ x$ then using the axiom (AR3) of Definition 6.2.2 we find a token x' with $M \circ x' \prec x$. Use the interpolation property of $\prec$ to obtain $x' \prec b \prec x$. Then $M \circ x' \rightarrowtail \{b\} \circ x$. The inclusion $\circ; \rightarrowtail; \circ \subseteq \circ$ is a trivial application of transitivity of $\prec$. Notice that $x' \rightarrowtail M \circ x$ implies $x' \prec m \prec x$ for some $m \in M$ and thereby $x' \prec x$. Conversely, $x' \prec x$ is equivalent to $x' \prec m \prec x$ for some token m, whereby $x' \rightarrowtail \{m\} \circ x$. Thus the relation $\prec$ of the abstract basis agrees with $\rightarrowtail; \circ$.

The Stage 1 axioms are easily checked. The meet rule and the associated weakening rule for $\circ$ holds because $\circ$ was defined using universal quantification on the witness side. Similarly, the weakening rule for $\rightarrowtail$ follows because that relation was defined using existential quantification. The definedness axiom is trivial because $\emptyset \circ x$ for any token x.

Suppose $R : Y \to X$ is a relation which satisfies the axioms of Definition 6.2.2. Define a relation

$$\hat{R} : \mathrm{Fin}\, Y \to X, \quad M\hat{R}x \Leftrightarrow \forall y \in M.\, yRx$$

We claim this is a Stage 1 morphism. Indeed, this is proved is the same way as the Stage 1 axioms for $\circ$ and $\rightarrowtail$ above. Obviously the assignment $R \mapsto \hat{R}$ maps $\prec$ to $\circ$. It remains to show that $\widehat{S;R} = \hat{S} \,⨟\, \hat{R}$ for two approximable relations $Z \xrightarrow{S} Y \xrightarrow{R} X$. Suppose $M \subseteq Z$ is a finite set and $M\widehat{S;R}x$. That means for all $z \in M$ there exists some $y_z \in Y$ with zSy_zRx. Writing $N := \{y_z\}_{z \in Z}$ we have $N\hat{R}x$. Using the interpolative law for $\hat{R}$ we obtain $N \succ N'\hat{R}x$ for some finite set $N' \subseteq Y$, where the relation $N \succ N'$ is witnessed by a token $y' \in Y$. Observe that because of the way we defined N, the relation $N \circ y'$ implies $M\hat{S}y'$. Therefore $M\hat{S}y' \rightarrowtail N'\hat{R}x$ and we have shown the inclusion $\widehat{S;R} \subseteq \hat{S} \,⨟\, \hat{R}$. For the other inclusion, suppose $M\hat{S}y \rightarrowtail N'\hat{R}x$. Then $y \prec y'Rx$ for some $y' \in N'$ and we can conclude $M\widehat{S;R}x$. □

If R is an approximable relation between abstract bases, then using the terminology of the proposition above, the relation $\prec;\hat{R}$ is the same as the relation R itself. Thus post-composing the functor of Proposition 1.2.7 with the forgetful functor from Proposition 1.2.6 yields the identity functor on Abs. The other way around, notice that the identity morphism of an interaction algebra $\mathcal{L}$ can be regarded as a morphism from $\mathcal{L}$ to the interaction algebra $(\operatorname{Fin} L_+, L_+, \prec, \circ)$ as constructed in Proposition 1.2.7 above, and likewise the identity morphism $\circ$ of the interaction algebra $(\operatorname{Fin} L_+, L_+, \prec, \circ)$ serves as a morphism into $\mathcal{L}$. These two morphisms are readily seen to establish an isomorphism in Tok_1.

Interestingly, there seems to be no simple way of turning an information system into a Stage 1 interaction algebra while preserving countability of the sets involved. A construction proposed by Achim Jung goes a follows. If $(X, <)$ is an information system where X is countable, then the relation $<$ itself is a countable set. For any pair $x_0 < y$ of tokens, use countable dependent choice and the interpolation property of $<$ to construct a countable ascending chain $x_0 < x_1 < x_2 < \ldots < y$ which yields a round ideal $I(x_0, y) = \{x \in X \mid \exists n \in \mathbb{N}.\, x < x_n\}$ that contains x_0 and has y as an upper bound. The collection of all ideals thus constructed is still countable. Take, as before, the witnesses of an interaction algebra to be the finite powerset $\operatorname{Fin} X$ with set union as the binary operation $\sqcap$, and let the tokens of the interaction algebra be the set of ideals of the form $I(x_0, y)$. Let the relation $\circ$ between finite sets and round ideals be the set inclusion relation and let $I(x_0, y) \prec A$ whenever $I(x_0, y) \subseteq \downarrow a$ for some element a of the finite set A. One checks that the structure thus defined is indeed a Stage 1 interaction algebra and that the relation $\succ$ on the witnesses extends the relation $<$ of the information system in the sense that $\{x_0\} \succ \{y\}$ holds precisely when $x_0 < y$. Furthermore, since by construction $\{x_0\} \circ I(x_0, y) \prec \{y\}$, one can show that for any two finite sets $A \succ B$ implies that $A \succ \{y\} \succ B$ for some singleton. It follows that the round ideals of the information system $(\operatorname{Fin} X, \succ)$ are in bijective correspondence with the round ideals of $(X, <)$.

1.2.6 Semi-open morphisms and token maps

In some settings it is convenient to consider only a subclass of all Scott continuous maps between domains. One such subclass is formed by the maps which preserve the way-below relation. A *proper map* f between locally compact topological spaces is a continuous map where the preimage of a compact saturated set is compact again. (In the absence of local compactness, a proper map is further required to be closed, but for locally compact spaces the latter follows from the former, see [27, Remark 1.3] or [28, Prop. 3.3].) Such maps will feature later in this thesis (Sections 3.7, 4.5) and are intimately related to the *Patch topology* [27, 38, 18]. Now f is perfect if and only if the frame homomorphism $\mathcal{O}(f)$ preserves the way-below relation (see [22, V-5.18,V-5.19]). We shall prove a fact for Stage 1 interaction algebras corresponding to [22, IV-1.4], based on Theorem 1.1.11.

Theorem 1.2.8. *The following are equivalent for a Stage 1 morphism R.*

1. *The morphism R has a Stage 0 right adjoint.*

2. *The homomorphism $\mathrm{pt}_1(R)$ between round ideals of tokens preserves the way-below relation.*

Proof. Theorem 1.1.11 states that the morphism R, when considered as a Stage 0 morphism, has a Stage 0 right adjoint if and only if the induced map $\Omega(R)$ between round lower sets preserves the completely-below relation. Recall from the remark following Proposition 1.2.3 that the set of round ideals is a sub-dcpo of the complete lattice of round lower sets, and furthermore the way-below relation on round ideals is the restriction of the completely-below relation to this sub-dcpo. Therefore it suffices to prove the implication (2) $\Rightarrow$ (1).

If a Scott continuous map $f : D \to E$ between domains preserves the way-below relation, then the forward image operation $\mathrm{Up}(f)$ restricts to a join-preserving map between Scott open sets. Indeed, if U is a Scott open set of D then $\mathrm{Up}(f)(U) = \bigcup_{d \in U} \uparrow f(d)$. For any $d \in U$ there exists some $d' \ll d$ with $d' \in U$, and preservation of $\ll$ yields $f(d') \ll f(d)$. Therefore $\mathrm{Up}(f)(U)$ is Scott open. Clearly the map $\mathrm{Up}(U)$ is left adjoint to the preimage map $\mathcal{O}(f)$ between Scott topologies.

Apply the above observation to the case where $f = \mathrm{pt}_1(R)$. Stone duality for interaction algebras tells us that the frame homomorphism $\mathcal{O}(f)$ is represented by $\Omega(\mathrm{Flip}(R))$, whence this map having a left adjoint means that R must have a Stage 0 right adjoint. □

Remark. Be aware that the existence of a Stage 0 right adjoint to a Stage 1 morphism R does not mean that the Scott continuous map $\mathrm{pt}_1(R)$ has a right adjoint morphism in Dom. It is true, however, that the frame homomorphism corresponding to $\mathrm{pt}_1(R)$ will have a left adjoint.

Definition 1.2.4. A Scott continuous map between domains is called *semi-open* if it preserves the way-below relation.

Corollary 1.2.9. *Let $\mathcal{L}$ be a Stage 1 interaction algebra. There is an order-preserving bijection between*

1. *Stage 1 morphisms $\mathcal{L} \to \mathbf{1}$ which have a right adjoint,*

2. *Weak equivalence classes of compact tokens of $\mathcal{L}$,*

3. *Compact elements of the domain of round ideals $\mathrm{Idl}^{\prec} L_+$.*

Proof. The duality between Stage 1 interaction algebras and domains takes $\mathbf{1}$ to the one-element domain $\{*\}$ where $* \ll *$. Clearly the compact round ideals $I \ll I$ of $\mathrm{Idl}^{\prec} L_+$ are in bijection with the maps $f : \{*\} \to \mathrm{Idl}^{\prec} L_+$ which preserve the way-below relation. The theorem above and the duality with domains then yields the bijection between morphisms $\mathcal{L} \to \mathbf{1}$ that have a right adjoint and compact round ideals. Then apply Proposition 1.1.10 to finish the proof. □

Remark. A similar characterisation is used in locale theory: A locale X is compact if and only if the unique map $X \to 1$ is proper, meaning that the frame homomorphism $\mathcal{O}(f) : 2 \to \mathcal{O}X$ preserves $\ll$.

The theorem above demonstrates one of the strengths of the information system approach: A Scott continuous map between domains and the associated frame homomorphism between Scott topologies are presented by the same relation. Thus, switching between the point-set theoretic and the localic viewpoint requires no effort at all. We conclude our study of semi-open morphisms by extending Proposition 1.1.12 to the category Tok_1.

Proposition 1.2.10. *There is a contravariant functor from the category of Stage 1 interaction algebras and token maps to the category of Stage 1 interaction algebras and Stage 1 morphisms which possess a Stage 0 right adjoint.*

Proof. Given a token map $(f_-, f_+) : \mathcal{M} \to \mathcal{L}$, it suffices to check that the left adjoint $R_+ : L_- \to M_+$ is a Stage 1 morphism. Recall that $\phi R_+ m$ if there exists some $m' \prec m$ such that $\phi \circ f_+(m')$. Further recall that the map f_+ preserves the relation $\prec$. First we verify the meet rule for R_+. Suppose $\phi, \psi R_+ m$, meaning that there exist $m_\phi, m_\psi \prec m$ such that $\phi \circ f_+(m_\phi)$ and $\psi \circ f_+(m_\psi)$. Using the fact that $\downarrow m$ is an ideal, we obtain a token m' with $m_\phi, m_\psi \prec m' \prec m$. Then, as f_+ preserves the relation $\prec$, we have $\phi \circ f_+(m_\phi) \prec f_+(m')$ and $\psi \circ f_+(m_\psi) \prec f_+(m')$. Therefore $\phi, \psi \circ f_+(m')$ and with the meet rule for $\circ$ we obtain $\phi \sqcap \psi R_+ m$. The weakening rule for R_+ trivially follows from the weakening rule for $\circ$. Finally, to see that R_+ satisfies the definedness axiom, let $m \in M_+$ be any token. As $\downarrow m$ is an ideal, it is in particular not empty. Let $m' \prec m$. Then $f_+(m')$ is a token of $\mathcal{L}$ and by the definedness axiom of $\circ$ we have $\phi \circ f_+(m')$ for some witness ϕ, whereby $\phi R_+ m$. $\square$

1.3 Interaction algebras for continuous preframes

Beginning with this section, we keep adding finitary algebraic structure to domains. The addition of finite meets is chosen to be the first for the sake of Lawson duality. Observe that instead of using Scott open sets as witnesses in the Stage 1 interaction algebra presenting a domain, one could restrict to Scott open filters. A domain has a plentiful supply of Scott open filters; a fact that can be shown using the interpolation property of the way-below relation and countable dependent choice: Given $x \ll y_0$ one uses the interpolation property to successively build a countable descending chain $x \ll \ldots \ll y_2 \ll y_1 \ll y_0$ which gives rise to a Scott open filter $\phi = \{y \mid \exists n \in \mathbb{N}.\, y_n \sqsubseteq y\}$ that contains y_0 and has x as a lower bound. The set of all Scott open filters of a domain D, ordered by inclusion, is denoted by $D^\wedge$ and called the Lawson dual of D. In any poset the set of filters is closed under directed unions, whence $D^\wedge$ is a sub-dcpo of the completely distributive frame of Scott open sets and thereby a domain, too. Any domain D is isomorphic to its second

Lawson dual (see [41, Theorem 3.7],[22, IV-2.14]) where the isomorphism is given by the map $x \mapsto \{\phi \in D^\wedge \mid x \in \phi\}$. Unfortunately, this duality is not functorial, as the preimage of a Scott open filter under a Scott continuous map is Scott open, but not necessarily a filter. In the Compendium [22] and Lawons's paper [41], this is remedied by restricting to the subcategory DOMFILT of domains with Scott continuous maps having the property that the preimage of a Scott open filter is again a filter. A more elegant way is to further restrict to the subcategory of *continuous semilattices*, that is, meet-semilattices which are also domains, and Scott continuous maps which are semilattice homomorphisms. A pleasing property of continuous semilattices is that these automatically satisfy the *preframe distributive law*, which says that for every element x the meet operation $d \mapsto x \sqcap d$ is Scott continuous. Directed complete semilattices with this property (not necessarily domains) are also called *meet-continuous* for obvious reasons. We regard continuous semilattices as an intermediate step towards continuous frames. Therefore we call a meet-continuous semilattice with top element a *continuous preframe*, where the Scott continuous semilattice homomorphisms are referred to as preframe homomorphisms. On this category CPreFrm, Lawson duality is a contravariant involution [41, Theorem 7.4]. When looking at the axioms for Stage 2a in Table 1.2 is is not the least surprising that continuous preframes have a nice self-duality, as the token- and witness side have the same algebraic structure and axioms.

1.3.1 Duality for Stage 2a interaction algebras

As it was demonstrated above, the witnesses for the way-below relation on a continuous preframe can be taken to be Scott open filters instead of arbitrary Scott open sets. This motivates the following definition.

Definition 1.3.1. For a continuous preframe L we define an interaction algebra $\mathrm{Ialg}_{2a}\, L$ which has the set L as tokens and the Lawson dual $L^\wedge$ as witnesses. The relations $\circlearrowleft$, $\rightarrowtail$ and the morphism part of the functor Ialg_{2a} are the same as in $\mathrm{Ialg}_1\, L$, that is, $\phi \circlearrowleft x$ whenever $x \in \phi$, $x \rightarrowtail \phi$ whenever x is a lower bound for ϕ and $\phi\, \mathrm{Ialg}_{2a}(f) x$ whenever $f(x) \in \phi$. The binary operation $\sqcap$ on the tokens is given as binary meet.

Lemma 1.3.1. *Let $\mathcal{L}$ be a Stage 2a interaction algebra. The round lower set operation $\downarrow\colon L_+ \to \mathrm{Lo}^{\prec} L_+$ transforms $\sqcap$ into binary meet.*

Proof. The round lower set of a token $\downarrow a = \{a' \in L_+ \mid a' \prec a\}$ is an ideal in any Stage 1 interaction algebra, so in particular a round lower set. The round lower sets are closed under all meets which are computed as lower closure of set intersection. Therefore, if a and b are tokens of $\mathcal{L}$, the meet $\downarrow a \wedge \downarrow b$ is the set of tokens x which satisfy $x \prec x' \prec a, b$ for some token x'. If $x \prec a \sqcap b$ then $x \prec x' \rightarrowtail \phi \circlearrowleft a \sqcap b$ for some token x' and some witness ϕ. Using the weakening rule of Stage 2a we deduce $x' \prec a, b$ whence x is an element of the round lower set $\downarrow a \wedge \downarrow b$. Now suppose $x \in \downarrow a \wedge \downarrow b$, meaning $x \rightarrowtail \theta \circlearrowleft x'$, $x' \prec a$ and $x' \prec b$.

Then $\theta \bigcirc a, b$ whence by the meet rule $\theta \bigcirc a \sqcap b$ and therefore $x \prec a \sqcap b$. We have shown that $\downarrow(a \sqcap b) = \downarrow a \wedge \downarrow b$. □

Lemma 1.3.2. *Let $\mathcal{L}$ be a Stage 2a interaction algebra. The binary meet of round upper sets of witnesses is computed as set intersection.*

Proof. Let Φ, Ψ be round upper sets of witnesses. It suffices to show that $\Phi \cap \Psi$ is a round upper set. Let $\theta \in \Phi \cap \Psi$. Since both Φ and Ψ are round upper sets, we have $\theta \bigcirc a \succ\!\prec \phi \in \Phi$ and $\theta \bigcirc b \succ\!\prec \psi \in \Psi$ for some tokens a, b and some witnesses ϕ, ψ. Using the binary meet rule and the weakening rule of Stage 2a one deduces $\theta \bigcirc a \sqcap b \succ\!\prec \phi, \psi$ which we expand to $\theta \succ \theta' \bigcirc a \sqcap b$. Then clearly $\theta' \in \Phi \cap \Psi$ and therefore $\Phi \cap \Psi$ is a round upper set. □

Theorem 1.3.3. *The dual equivalence between the categories* Dom *and* Tok_1 *restricts to a dual equivalence between the categories* $\mathsf{CPreFrm}$ *of continuous preframes and preframe homomorphisms and the category* Tok_{2a} *of Stage 2a interaction algebras and Stage 2a morphisms.*

Proof. First observe that the interaction algebra $\mathrm{Ialg}_{2a} L$ constructed from a continuous preframe indeed satisfies all Stage 2a axioms. The binary meet rule holds because the witnesses are now closed under binary meets, and dual definedness holds because every witness is non-empty, as it contains the top element $1 \in L$, the neutral element for $\sqcap$. It is an easy exercise to check that the relation $\mathrm{Ialg}_{2a}(f)$ derived from a preframe homomorphism satisfies the Stage 2a axioms as well.

Every continuous preframe L is in particular a domain. Notice that for every such L the interaction algebras $\mathrm{Ialg}_1 L$ and $\mathrm{Ialg}_{2a} L$ are isomorphic as objects in Tok_1. This amounts to the fact that if a point x is contained in a Scott open set U then there is some Scott open filter completely below U which still contains x. Let pt_{2a} denote the restriction of the round ideal functor pt_1 to the subcategory Tok_{2a}. It suffices to verify that pt_{2a} produces continuous preframes and preframe homomorphisms. Once that is done, the duality Theorem 1.2.5 tells us that $\mathrm{pt}_{2a} \circ \mathrm{Ialg}_{2a}$ is isomorphic to the identity functor on $\mathsf{CPreFrm}$ and $\mathrm{Ialg}_{2a} \circ \mathrm{pt}_{2a}$ is isomorphic to the identity functor on Tok_{2a}.

Let $R : \mathcal{L} \to \mathcal{M}$ be a Stage 2a morphism between Stage 2a interaction algebras. Fist we show that $\mathrm{pt}_{2a} \mathcal{L} = \mathrm{Idl}^{\prec} L_+$ is a continuous preframe. Because of the dual definedness axiom of Stage 2a, the set L_- of witnesses is a round upper set: If $\phi \in L_-$ then $\phi \bigcirc a$ for some token a and therefore $\phi \succ \phi' \bigcirc a$ for some other witness ϕ'. The meet rule of Stage 1 now tells us that L_- is in fact a round filter, and clearly the largest such. Using the Fundamental Lemma 1.2.1 we conclude that the domain $\mathrm{Idl}^{\prec} L_+$ has a largest element, namely the ideal of all bounded tokens. Recall from Lemma 1.3.1 that the binary meet of round the ideals $\downarrow a$ and $\downarrow b$ in the frame of round lower sets of tokens is $\downarrow(a \sqcap b)$, which is again a round ideal. Further recall that any round ideal I can be written as the directed union $\bigcup_{a \in I} \downarrow a$. Therefore, using the preframe distributive law in the frame $\mathrm{Lo}^{\prec} L_+$, we can write the binary meet of two round ideals I and J as

$$I \wedge J = \bigcup \{ \downarrow\!(a \sqcap b) \mid a \in I, b \in J \} \,.$$

This union is directed, because $\downarrow$ transforms $\prec$ into $\ll$ and $\sqcap$ into binary meet, as we saw in Lemma 1.3.1. Hence $I \wedge J$ is again a round ideal. Now the preframe distributive law of $\mathrm{Idl}^{\prec} L_+$ follows from the preframe distributive law of $\mathrm{Lo}\, L_+$. This concludes the proof that $\mathrm{pt}_{2a}\, \mathcal{L}$ a continuous preframe.

It remains to show that the Stage 2a morphism R induces a preframe homomorphism $\mathrm{pt}_{2a}(R)$. Just as in the proof of Theorem 1.2.5 it is convenient to regard the map $\mathrm{pt}_{2a}(R)$ as a map from round ideals of tokens to round filters of witnesses. Since pt_{2a} is just the restriction of pt_1 to Stage 2a interaction algebras, it suffices to show that $\mathrm{pt}_{2a}(R)$ preserves finite meets. Because of the dual definedness axiom we know that every witness $\phi \in L_+$ has $\phi R m$ for some (bounded) token $m \in M_+$ which means that $\mathrm{pt}_{2a}(R)$ preserves the empty meet. The binary meet rule of Stage 2a tells us that

$$\{\phi \mid \phi R a\} \cap \{\phi \mid \phi R b\} = \{\phi \mid \phi R (a \sqcap b)\} \,.$$

Apply Lemma 1.3.2 and conclude that $\mathrm{pt}_{2a}(R)$ preserves the binary meet $\downarrow\! a \wedge \downarrow\! b$. The general case follows from the same argument we used above for the domain $\mathrm{Idl}^{\prec} L_+$. □

1.3.2 Lawson duality

Once more the functor Flip on interaction algebras proves to be very useful. Recall that in the category of Stage 1 interaction algebras it presented the Stone duality between domains and completely distributive frames. In the category of Stage 2a interaction algebras it presents a duality as well, but this time the points of the dual are Scott open filters instead of Scott open sets. The following theorem is the interaction algebra version of [55, Theorem 3.2.3].

Theorem 1.3.4. *Let $(-)^\wedge$ be the duality on preframes which takes a preframe homomorphism to the preimage operation between Scott open filters. The following diagram of contravariant functors commutes (up to isomorphism).*

$$\begin{array}{ccc} \mathsf{Tok}_{2a} & \xrightarrow{\mathrm{pt}_{2a}} & \mathsf{CPreFrm} \\ {\scriptstyle \mathrm{Flip}}\downarrow & & \downarrow{\scriptstyle (-)^\wedge} \\ \mathsf{Tok}_{2a} & \xrightarrow[\mathrm{pt}_{2a}]{} & \mathsf{CPreFrm} \end{array}$$

Proof. Notice that the axioms of a Stage 2a interaction algebra are symmetric in tokens and witnesses. Therefore the functor Flip restricts to a contravariant involution on Tok_{2a}. From Theorem 1.2.4 we know that $\mathrm{pt}_1 \circ \mathrm{Flip}$ takes a Stage 1 morphism to the preimage operation between Scott open sets. Therefore it suffices to show that for any Stage 2a interaction algebra $\mathcal{L}$ the domain $\mathrm{pt}_{2a}\,\mathrm{Flip}\,\mathcal{L}$ is the Lawson dual of $\mathrm{pt}_{2a}\,\mathcal{L}$. In the proof of Theorem 1.2.4 we showed that Scott open sets of $\mathrm{Idl}^{\prec} L_+$ are in order-preserving bijection with round upper sets of tokens. This order-isomorphism restricts to an order-isomorphism between Scott open filters in $\mathrm{Idl}^{\prec} L_+$ and round filter of tokens. □

Remark. The proof that round filters of tokens correspond to Scott open filters of round ideals does not require the Stage 2a axioms. Together with Theorem 1.1.8 we recover a result published by Lawson [41]: The Scott topology of the Lawson dual of a domain is order-isomorphic to the order dual of the domain's Scott topology. In short: $\sigma(D^\wedge) \cong (\sigma D)^\partial$.

Knowing that the preframe of round ideals of witnesses is the Lawson dual of the round ideals of tokens, one wonders what the interaction algebra $\mathrm{Ialg}_{2a}\,\mathrm{pt}_{2a}\,\mathcal{L}$ concretely looks like. In the proof of Theorem 1.1.9 we gave a conrete description of the Stage 0 interaction algebra derived from the completely distributive frame of round lower sets of tokens. The same argument can be modified to account for round ideals.

Proposition 1.3.5. *Let $\mathcal{L}$ be a Stage 2a interaction algebra.*

1. *A round ideal I of tokens is contained in the Scott open filter corresponding to the round ideal Φ of witnesses precisely when the set $\Phi \times I$ intersects the relation $\circ$.*
2. *A round ideal I of tokens is a lower bound for the Scott open filter corresponding to the round ideal Φ of witnesses if and only if the set $I \times \Phi$ is contained in the relation $\prec\!;\circ;\prec$.*

1.4 Interaction algebras for domains with least element

In this section we demonstrate how to add an empty join, i.e. a least element to a domain presented by an interaction algebra. The canonical interaction algebra we used to present a domain D has all Scott open sets as witnesses. There is always a largest witness, namely D. If we write the largest witness more neutrally as 1, then we have $1 \circ a$ for every token a of our canonical interaction algebra. The domain D has a least element precisely when there is some token $0 \in D$ which is a lower bound for the witness 1. Indeed, any token $d \in D$ then has $0 \prec 1 \circ d$ and thereby $0 \ll d$. In particular the least element 0 always satisfies $0 \ll 0$. It must therefore be an element of any set of tokens which we might use to present the domain. Phrased differently, the domain D has a least element if and only if the maximal witness 1 is compact in the sense that $1 \succ 1$.

A map $f : D \to E$ between domains with bottom is said to be *strict* if $f(0) = 0$. The category of domains with bottom and strict Scott continuous maps is denoted by $\mathsf{Dom}_{\perp!}$.

Theorem 1.4.1. *The duality between domains and Stage 1 interaction algebras restricts to a dual equivalence between Stage 2b interaction algebras and Stage 2b morphisms and the category $\mathsf{Dom}_{\perp!}$ of domains with least element and strict Scott continuous maps.*

Proof. If D is a domain with least element 0, let 1 denote the Scott open D. Then the interaction algebra $\mathrm{Ialg}_1 D$ with Scott opens as witnesses and points of the domain as tokens satisfies all axioms of Stage 2b listed in Table 1.2. If $f : D \to E$ is a strict map between domains with bottom, then certainly $f(a) \in E$ for any element $a \in D$, whence the relation $\mathrm{Ialg}_1(f)$ satisfies the empty meet rule of Stage 2b. The relation $\phi\,\mathrm{Ialg}_1(f)0$ holds by definition when $f(0)$ is an element of the Scott open set ϕ, but since the map f is strict this means that $0 \in \phi$. Therefore the functor Ialg_1 restricts to a functor $\mathrm{Ialg}_{2b} : \mathsf{Dom}_{\perp!} \to \mathsf{Tok}_{2b}$.

It remains to show that the restriction of pt_1 to the subcategory Tok_{2b} produces domains with bottom and strict maps. From the axioms of Stage 2b it follows that the postulated token 0 satisfies $0 \prec 0$, whence we know by Corollary 1.2.9 that $\downarrow 0$ is a compact round ideal of tokens. Moreover, any other token a has $0 \rightarrowtail 1 \circ a$, whence the round ideal $\downarrow 0$ is contained in any other round ideal. If R is a Stage 2b morphism then the image of $\downarrow 0$ under $\mathrm{pt}_1(R)$ is again the lower set of 0, as enforced by the strictness axiom. □

1.5 Interaction algebras for continuous Sup-lattices

In section 1.3 we added finite meets to a domain. Dually one might want to add finite joins to a domain and thus obtain a continuous (complete) lattice. But we already demonstrated in section 1.4 how to add the empty join to a domain, so it suffices to find a way of adding binary joins to a domain. Together with the directed joins such a domain will be a *complete Sup-lattice*, that is, a poset which has joins for all non-empty subsets. Since complete Sup-lattices can be described as the Eilenberg-Moore algebras for the non-empty-lowerset monad on Poset, the natural choice of morphisms between these structures are the monotone maps which preserve all non-empty joins.

In due course we will prove that the interaction algebras that present continuous complete Sup-lattices are precisely those satisfying the Stage 2c axioms. The binary join and weakening rules for Stage 2c listed in Table 1.2 look harmless, but the join-strength axiom might require explanation. Suppose L is a continuous complete Sup-lattice, $\phi \subseteq L$ is a Scott open set and $a \sqcup b \in \phi$. From continuity we know that $a = \bigsqcup \twoheaddownarrow a$ and $b = \bigsqcup \twoheaddownarrow b$. Since directed joins distribute over binary joins, we can write $a \sqcup b = \bigsqcup \{a' \sqcup b' \mid a' \ll a, b' \ll b\}$. Observe that this join is directed. Therefore, by ϕ being a Scott open set, we have $a' \sqcup b' \in \phi$ for some $a' \ll a$ and $b' \ll b$ already. Clearly then $a \in \twoheaduparrow a'$ and $b \in \twoheaduparrow b'$ and the intersection $\twoheaduparrow a' \cap \twoheaduparrow b'$ is completely below the Scott open ϕ, as witnessed by $a' \sqcup b'$. This can be turned into a *join-strength* axiom for the way-below relation. It reads

$$\frac{x \ll a \sqcup b}{\exists a' \ll a\, \exists b' \ll b.\, x \ll a' \sqcup b'}$$

The term "join-strength" was first mentioned to the author by Sam van Gool, but auxiliary relations which satisfy the join-strength axiom and an order-dual thereof have been examined by Smyth [50] and subsequently many others [16, 2, 35, 33].

In any domain the join of two points a and b, if it exists, has the property that $a \sqcup b \ll x$ whenever $a \ll x$ and $b \ll x$. For continuous Sup-lattices this means that it is possible to compute the binary meet of Scott opens by taking point-wise joins. Indeed, if U_1 and U_2 are Scott open sets then the set $U = \{a \sqcup b \,|\, a \in U_1, b \in U_2\}$ is clearly contained in $U_1 \cap U_2$. For the reverse inclusion, any $u \in U_1 \cap U_2$ has $a' \ll a \sqsubseteq u$ and $b' \ll b \sqsubseteq u$ for some $a', a \in U_1$ and $b', b \in U_2$. Thus $a' \sqcup b' \ll a \sqcup b \sqsubseteq u$ whence $U_1 \cap U_2 \subseteq U$. We begin our study of Stage 2c interaction algebras by exhibiting the analogues of the observations above.

Lemma 1.5.1. *Let $\mathcal{L}$ be a Stage 2c interaction algebra.*

1. *The relations $\succ\!\prec$ and any Stage 2c morphism (in particular $\bigcirc$) satisfy*

$$\frac{\phi R a \qquad \psi R b}{\phi \sqcap \psi R a \sqcup b} \qquad \frac{a \succ\!\prec \phi \qquad b \succ\!\prec \psi}{a \sqcup b \succ\!\prec \phi \sqcap \psi}$$

2. *The relation $\prec$ on tokens satisfies the join-strength axiom*

$$\frac{x \prec a \sqcup b}{\exists a' \prec a \, \exists b' \prec b.\, x \prec a' \sqcup b'}$$

3. *The relation $\prec$ on tokens satisfies*

$$\frac{a \prec x \qquad b \prec x}{a \sqcup b \prec x}$$

4. *Any token b is lower equivalent to $b \sqcup b$.*

5. *Any witness ϕ is lower equivalent to $\phi \sqcap \phi$.*

6. *The relation $\succ$ on witnesses satisfies*

$$\frac{\phi \succ \theta \qquad \psi \succ \theta}{\phi \sqcap \psi \succ \theta}$$

Proof. (1) If $\phi R a$ and $\psi R b$ then first use the weakening axiom to obtain $\phi, \psi R a \sqcup b$ and then use the meet rule of Stage 1 to get $\phi \sqcap \psi R a \sqcup b$. The rule for $\succ\!\prec$ is proved dually.

(2) Suppose $x \succ\!\prec \phi \bigcirc a \sqcup b$. Apply the join-strength axiom and obtain witnesses $\psi \bigcirc a$, $\theta \bigcirc b$ such that $\phi \succ \psi \sqcap \theta$. With the interpolative law for $\bigcirc$ we get $\psi \bigcirc a' \prec a$ and $\theta \bigcirc b' \prec b$ for some tokens a' and b'. Using (1) we deduce $\psi \sqcap \theta \bigcirc a' \sqcup b'$ and with $\phi \succ \psi \sqcap \theta$ this implies $\phi \bigcirc a' \sqcup b'$. Thus $x \prec a' \sqcup b'$.

(3) If $a \succ\!\prec \phi \bigcirc x$ and $b \succ\!\prec \psi \bigcirc x$ then because of the weakening rule of Stage 2c we have $a \sqcup b \succ\!\prec \phi, \psi \bigcirc x$ which using the axioms of Stage 1 implies $a \sqcup b \succ\!\prec \phi \sqcap \psi \bigcirc x$.

(4) The weakening rule of Stage 2c yields the implication $\phi \bigcirc b \Rightarrow \phi \bigcirc b \sqcup b$. For the reverse implication, apply the join-strength axiom to $\phi \bigcirc b \sqcup b$ and obtain $\psi \bigcirc b$ with $\phi \succ \psi$ whereby $\phi \bigcirc b$.

(5) Is true even for Stage 1 interaction algebras; see the observations following Definition 1.2.1.

(6) If $\phi \bigcirc a \succ\!\prec \theta$ and $\psi \bigcirc b \succ\!\prec \theta$ then use (1) and the weakening rule for $\succ\!\prec$ and obtain $\phi \sqcap \psi \bigcirc a \sqcup b \succ\!\prec \theta$. □

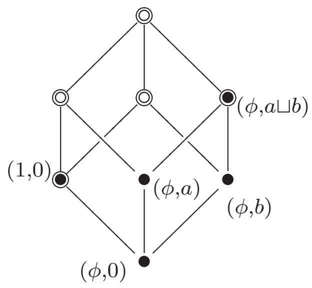


Figure 1.7: The interaction algebra of Example 2 depicted as the product $L_- \times L_+$. Filled dots are members of $\rightarrowtail\!\!\!\!\leftarrowtail$ and circled dots are members of $\circ$.

Corollary 1.5.2. *A round lower set of tokens of a Stage 2c interaction algebra is a round ideal if and only if it is non-empty and closed under the operation* $\sqcup$*.*

Proof. The rule (3) of the previous lemma is reversible. This can be shown easily using the weakening rule for $\rightarrowtail\!\!\!\!\leftarrowtail$ of Stage 2c. □

Corollary 1.5.3. *If* ϕ *is a witness of a Stage 2c interaction algebra, then the round upper set* $\uparrow\!\phi$ *is either empty or a round filter.*

Proof. Follows from the rule (6) of the previous lemma. □

The join-strength axiom is not derivable from other axioms of Stage 2c, as the following example demonstrates.

Example 2. Consider an interaction algebra where the witness set is the two-element meet-semilattice $L_- = \{\phi, 1\}$ where $\phi \sqcap 1 = \phi$. The token set is the four-element join-semilattice $L_+ = \{0, a, b, a \sqcup b\}$ where 0 is a unit for the binary operation $\sqcup$. The relations $\circ$ and $\rightarrowtail\!\!\!\!\leftarrowtail$ are depicted in Figure 1.7. One can think of the witness ϕ as the singleton set $\{a \sqcup b\}$ and of 1 as the set L_+. This interaction algebra satisfies all Stage 1 axioms and moreover all axioms of Stage 2b and Stage 2c except for the join-strength axiom. Indeed, $\phi \circ a \sqcup b$ but the only choice for witnesses $\psi \circ a$ and $\theta \circ b$ is $\psi = \theta = 1$. Since $1 \sqcap 1 = 1$ and 1 is not below ϕ, the join-strength axiom fails.

Observe that this interaction algebra has only two round ideals, namely the ones generated by the compact tokens 0 and $a \sqcup b$. Hence the tokens a and b are superfluous, and in fact omitting them yields a Stage 2c interaction algebra.

Remark. A dual to axiom (2) of Lemma 1.5.1 does not hold. If $\phi \succ \psi \sqcap \theta$ then there do not necessarily exist $\psi' \succ \psi$ and $\theta' \succ \theta$ such that $\phi \succ \psi' \sqcap \theta'$. An example where this fails is the complete Sup-lattice $\mathbb{N} + \{\infty\}$ where $\mathbb{N}$ carries the flat order and ∞ is above any $n \in \mathbb{N}$. The Scott open sets generated by the even and odd numbers intersect to the singleton $\{\infty\}$ which is completely below itself, but the former two Scott open sets are not bounded below.

Theorem 1.5.4. *The dual equivalence between the categories* Dom *and* Tok_1 *restricts to a dual equivalence between the categories* CCSup *of continuous complete Sup-lattices and Sup-lattice homomorphisms and the category* Tok_{2c} *of Stage 2c interaction algebras and Stage 2c morphisms.*

Proof. Let L be a continuous complete Sup-lattice. Then the tokens of the interaction algebra $\mathrm{Ialg}_1 L$ have binary joins. We already verified the join-strength axiom at the beginning of this section. The binary join rule of Stage 2c is valid because the lower bounds of any Scott open set are closed under all non-empty joins. Checking the weakening rules of Stage 2c requires no ingenuity. We conclude that the functor Ialg_1 restricts to a functor $\mathrm{Ialg}_{2c} : \mathsf{CCSup} \to \mathsf{Tok}_{2c}$.

It remains to show that the functor pt_1 maps Stage 2c interaction algebras to Sup-lattices and Stage 2c morphisms to Sup-lattice homomorphisms. Let $\mathcal{L}$ be a Stage 2c interaction algebra. Its domain of round ideals of tokens has binary joins which are computed just as one computes the join of ideals of any join-semilattice:

$$I \vee J = \downarrow\{a \sqcup b \mid a \in I,\, b \in J\}$$

The easiest way of seeing that the set thus defined is indeed the join of I and J in the domain of round ideals, is to consider the special case where $I = \downarrow a_0$ and $J = \downarrow b_0$. Using Lemma 1.5.1 (2) and (3) one shows that $\downarrow a_0 \vee \downarrow b_0 = \downarrow a_0 \sqcup b_0$, which is clearly the smallest possible ideal containing both $\downarrow a_0$ and $\downarrow b_0$. Hence the map $\downarrow : L_+ \to \mathrm{Idl}^{\prec} L_+$ not only transforms $\prec$ into $\ll$ but also $\sqcup$ into binary join. The case where I and J are arbitrary round ideals now follows using the basic round ideals. Therefore the set of round ideals of tokens has binary joins. The Fundamental Lemma 1.2.1 tells us that the round filters of witnesses have binary joins, too. These are computed in the way we would expect:

$$F \vee G = \uparrow\{\phi \sqcap \psi \mid \phi \in F,\, \psi \in G\}\ .$$

Let $R : \mathcal{L} \to \mathcal{M}$ be a Stage 2c morphism. Once again, we regard the map $\mathrm{pt}_1(R)$ as a function from round ideals of tokens in $\mathcal{M}$ to round filters of witnesses in $\mathcal{L}$. We already know that $\mathrm{pt}_1(R)$ is monotone, whence it suffices to check the inequality $\mathrm{pt}_1(\downarrow(a \sqcup b)) \subseteq \mathrm{pt}_1(\downarrow a) \vee \mathrm{pt}_1(\downarrow b)$. A witness $\phi \in L_-$ is an element of $\mathrm{pt}_1(\downarrow(a \sqcup b))$ precisely when $\phi R a \sqcup b$. Now the join-strength rule for R tells us that $\phi \succ \psi \sqcap \theta$ for some $\phi \in \mathrm{pt}_1(\downarrow a)$ and $\theta \in \mathrm{pt}_1(\downarrow b)$. We have shown that $\mathrm{pt}_1(R)$ preserves binary joins of basic round ideals, and this suffices to conclude that the function preserves all binary joins. □

One of the strengths of interaction algebras for domains is that the Lawson dual is "built in" to the presentation rather explicitly as round ideals of witnesses. Observe that in the proof of Theorem 1.3.4 we did not make use of the Stage 2a axioms to show that the Scott open filters of round ideals of tokens correspsond to round ideals of witnesses. In fact the Stage 2a axioms only come into play when showing that the Lawson dual is functorial.

Thus we may, even in the absence of the Stage 2a axioms, derive properties of the Lawson dual of a domain by inspecting the algebraic axioms of tokens and witnesses. An instance of this is provided by Lemma 1.5.1 (6):

Proposition 1.5.5. *For a Stage 2c interaction algebra $\mathcal{L}$, the Lawson dual of $\mathrm{pt}_1 \mathcal{L}$ is stably locally continuous, meaning that the way-upper set of any point is a filter, provided that it is not empty.*

Proof. Suppose Φ, Ψ and Θ are round ideals of witnesses of a Stage 2c interaction algebra with $\Phi \ll \Psi, \Theta$. Then $\Phi \subseteq\downarrow\psi$ and $\Phi \subseteq\downarrow\theta$ for some $\psi \in \Psi$ and $\theta \in \Theta$. Further we have $\psi'' \succ \psi' \succ \psi$ for some witnesses $\psi'', \psi' \in \Psi$, and likewise $\Theta \ni \theta'' \succ \theta' \succ \theta$. Using the weakening rule of Stage 1 we find that $\psi'', \theta'' \succ \psi' \sqcap \theta'$ whence $\psi' \sqcap \theta' \in \Psi \cap \Theta$. Now apply Lemma 1.5.1 (6) and conclude that $\downarrow\psi \sqcap \theta$ still contains the ideal Φ. Notice that the round lower set $\downarrow\psi \sqcap \theta$ is not necessarily a round ideal, but with countable dependent choice we can construct a round ideal of witnesses that contains $\psi \sqcap \theta$ and is bounded above by $\psi' \sqcap \theta'$. Thus we have shown that the way-upper set of the ideal Φ, provided that it is not empty, is a filter. □

Remark. The proposition above has a converse: The continuous complete Sup-lattices are precisely the Lawson duals of stably locally continuous domains.

1.6 Interaction algebras for continuous lattices

In order to present continuous lattices by interaction algebras, all we have to do is combine Theorem 1.4.1 and Theorem 1.5.4. This yields a dual equivalence between the category Tok_3 and the category CCL of continuous complete lattices and join-preserving maps. From the viewpoint of universal algebra, the join-preserving maps are by no means the canonical choice of morphisms. For example, in [22] continuous lattices are studied as the category CONT where the morphisms are the Scott continuous maps. Another approach we have taken previously is to look for distributive laws and require homomorphisms to preserve the meets and joins that are involved in the distributive law. Surprisingly, continuous lattices can be characterised by the *directed distributive law* [22, I-2.7]: They are those complete lattices where arbitrary meets distribute over directed joins. Although a direct proof is not too complicated, the abstract basis approach provides a neat alternative.

Proposition 1.6.1. *In the complete lattice of round ideals of tokens of a Stage 3 interaction algebra, arbitrary meets are computed as in the complete lattice of round lower sets, that is, as lower closure of set intersection. Consequently, directed joins of round ideals distribute over all meets.*

Proof. Let $\mathcal{L}$ be a Stage 3 interaction algebra and let $\mathcal{I}$ be a set of round ideals of tokens. The meet of $\mathcal{I}$ in the lattice of round lower sets is given as $\bigwedge \mathcal{I} = \{a \in L_+ \mid \exists b \in \bigcap \mathcal{I}.\, a \prec b\}$.

It suffices to show that this is a round ideal. First notice that $\bigwedge \mathcal{I}$ is not empty, because any round ideal of a Stage 2b interaction algebra contains the compact token 0, whence $0 \prec 0 \in \bigcap \mathcal{I}$. If $a \prec a' \in \bigcap \mathcal{I}$ and $b \prec b' \in \mathcal{I}$ then use the characterisation in Lemma 1.5.2 and Lemma 1.5.1 (3) to obtain $a \sqcup b \prec a' \sqcup b' \in \bigcap \mathcal{I}$. Hence $\bigwedge \mathcal{I}$ is a round ideal.

Recall that the complete lattice $\mathrm{Lo}^{\prec} L_+$ of round lower sets is a completely distributive frame. As $\mathrm{Idl}^{\prec} L_+$ shares directed joins and arbitrary meets with the frame $\mathrm{Lo}^{\prec} L_+$, the round ideals inherit the directed distributive law. □

A consequence of Proposition 1.6.1 is that any continuous lattice satisfies the preframe distributive law. Therefore we could have included the Stage 2a axioms into Stage 3. However, by convention this would mean that the homomorphisms between complete lattices must preserve finite meets. We conclude:

Proposition 1.6.2. *There is a dual equivalence between the category of continuous lattices with frame homomorphisms and the category of Stage 3 interaction algebras and morphisms which satisfy the Stage 2a axioms as well.*

1.7 Interaction algebras for continuous frames

The algebraic structure of continuous frames is best regarded as a distributive lattice which happens to be a domain. This is because the frame distributive law can be decomposed into the distributive law of lattices and the preframe distributive law. The latter distributive law, however, can be derived because any continuous meet-semilattice is meet-continuous. In the light of Proposition 1.6.2 it therefore comes at no surprise that in the step from continuous lattices to continuous frames we simply add a distributive law to the operations $\sqcup$ and $\sqcap$ we postulate on the tokens.

The Scott topology on any domain (and thereby every completely distributive frame) is a continuous frame, whence the Stage 4 interaction algebras could be considered as more general than the ones of previous stages. But, at least until Subsection 1.7.2, we are interested in a continuous frame itself as a domain.

1.7.1 A lattice distributive law without equality

Notation. By Tok_{3a} we denote the category of interaction algebras and morphisms that satisfy the axioms of Stage 2a, 2b and 2c.

A Stage 3a interaction algebra has two operations on tokens: There is the operation $\sqcap$ of Stage 2a and the operation $\sqcup$ of Stage 2c. Classically, a distributive law involving these operations would read $a \sqcap (b \sqcup c) = (a \sqcup b) \sqcap (a \sqcup c)$. However, in Section 1.2 we already pointed out that equality of tokens is way too strong and replaced it with the lower equivalence of Definition 1.2.1. As we shall see, lower equivalence is the right framework in which to define a distributive law.

Lemma 1.7.1. *In every Stage 3a interaction algebra, for any three tokens a, b, c the token $(a \sqcap b) \sqcup (a \sqcap c)$ is weakly below the token $a \sqcap (b \sqcup c)$.*

Proof. We have to show that for any witness ϕ the relation $\phi \bigcirc (a \sqcap b) \sqcup (a \sqcap c)$ implies that $\phi \bigcirc a \sqcap (b \sqcup c)$. Suppose the former relation holds. Using the join-strength rule we obtain witnesses $\psi \bigcirc a \sqcap b$ and $\theta \bigcirc a \sqcap c$ satisfying $\phi \succ \psi \sqcap \theta$. With the weakening rule of Stage 2a we can decompose the former two relations into $\psi \bigcirc a$, $\theta \bigcirc a$, $\psi \bigcirc b$ and $\theta \bigcirc c$. These four we re-gather into the two relations $\phi \sqcap \theta \bigcirc a$ and $\psi \sqcap \theta \bigcirc b \sqcup c$, where the first is derived using the meet rule of Stage 1 and the second using Lemma 1.5.1 (1). An application of the meet rule then yields $\psi \sqcap \theta \bigcirc a \sqcap (b \sqcup c)$ whence we can conclude $\phi \bigcirc a \sqcap (b \sqcup c)$. □

The lemma above has an analogue in lattices: It simply states that the map $x \mapsto a \sqcap x$ is monotone. This formulation makes no sense for tokens of an interaction algebra, because $\sqcap$ and $\sqcup$ do not define the same preorder. Nevertheless we define our distributive law as the reversal of Lemma 1.7.1.

Definition 1.7.1. We say that binary relations $\sqcap$ and $\sqcup$ of a Stage 3a interaction algebra *distribute* if for all tokens a,b and c the token $a \sqcap (b \sqcup c)$ is weakly below $(a \sqcap b) \sqcup (a \sqcap c)$. The category Tok_4 of Stage 4 interaction algebras is the full subcategory of Tok_{3a} where the operations $\sqcap$ and $\sqcup$ distribute.

It is merely a curious observation that the distributive law can be phrased without referring to the weakly-below relation. Instead, what distinguishes a continuous frame from a continuous lattice is a "bounded" variant of the join-strength axiom.

Lemma 1.7.2. *The join-strength rule of Stage 2c is reversible, meaning*

$$\frac{\psi R a \qquad \theta R b \qquad \phi \succ \psi \sqcap \theta}{\phi R a \sqcup b}$$

Proof. If $\psi R a$ and $\theta R b$ then using the idea of Lemma 1.5.1 (1) we obtain $\psi \sqcap \theta R a \sqcup b$. Composing this with the hypothesis $\phi \succ \psi \sqcap \theta$ yields $\phi R a \sqcup b$. □

Proposition 1.7.3. *A Stage 3a interaction algebra has distributive operations $\sqcap$ and $\sqcup$ in the sense of Definition 1.7.1 if the* bounded join-strength *rule*

$$\frac{\phi \bigcirc a \qquad \phi \bigcirc b \sqcup c}{\exists \psi \bigcirc b, \exists \theta \bigcirc c.\, \phi \succ \psi \sqcap \theta \bigcirc a}$$

holds.

Proof. Suppose $\phi \bigcirc a \sqcap (b \sqcup c)$. With the weakening rule we get $\phi \bigcirc a$ and $\phi \bigcirc b \sqcup c$. Then we can apply the bounded join-strength rule and obtain $\phi \succ \psi \sqcap \theta \bigcirc a$ for some witnesses $\psi \bigcirc b$ and $\theta \bigcirc c$. Notice that further $\psi \bigcirc a$ and $\theta \bigcirc a$ hold. Re-gather the last four relations into $\psi \bigcirc a \sqcap b$ and $\theta \bigcirc a \sqcap c$ and these two into $\psi \sqcap \theta \bigcirc (a \sqcap b) \sqcup (a \sqcap c)$ and deduce $\phi \bigcirc (a \sqcap b) \sqcup (a \sqcap c)$. We have shown that the bounded join-strength axiom implies that $a \sqcap (b \sqcup c)$ is weakly below $(a \sqcap b) \sqcup (a \sqcap c)$. □

For any lattice $(L, \sqcap, \sqcup)$ the distributive law, as we phrased it, is equivalent to the assertion that for any three elements $(a \sqcup b) \sqcap (a \sqcup c) \sqsubseteq a \sqcup (b \sqcap c)$. We will prove that our distributive law implies a similar fact. Our proof essentially follows the familiar lattice-theoretical proof that can be found in standard textbooks such as [13].

Lemma 1.7.4. *For any Stage 3a interaction algebra and tokens a and b, the tokens $a \sqcup (a \sqcap b)$ and $(a \sqcup b) \sqcap a$ are both lower equivalent to a.*

Proof. Clearly, if $\phi \frown a$ then $\phi \frown a \sqcup (a \sqcap b)$ by the weakening rule of Stage 2c. Conversely, if $\phi \frown a \sqcup (a \sqcap b)$ then use the join-strength rule to obtain $\psi \frown a$ and $\theta \frown a \sqcap b$ with $\phi \succ \psi \sqcap \theta$. From the weakening rule of Stage 2a we know $\theta \frown a$ whence we may apply the meet rule of Stage 1 and get $\psi \sqcap \theta \frown a$. Therefore $\phi \frown a$, which finishes the proof of the first claim.

The Stage 2a weakening rule tells us that $\phi \frown (a \sqcup b) \sqcap a$ implies $\phi \frown a$. For the reverse implication, use the weakening rule of Stage 2c to deduce $\phi \frown a \sqcup b$ from $\phi \frown a$, and then the meet rule of Stage 2a to get $\phi \frown (a \sqcup b) \sqcap a$. □

Proposition 1.7.5. *If the operations $\sqcap$ and $\sqcup$ of a Stage 3a interaction algebra distribute then the* dual distributive law *holds as well, meaning that for any tokens a, b and c the token $(a \sqcup b) \sqcap (a \sqcup c)$ is weakly below $a \sqcup (b \sqcap c)$.*

Proof. Suppose $\phi \frown (a \sqcup b) \sqcap (a \sqcup c)$. We show $\phi \frown a \sqcup (b \sqcap c)$. First apply the distributive law of Definition 1.7.1 to obtain $\phi \frown ((a \sqcup b) \sqcap a) \sqcup ((a \sqcup b) \sqcap c)$. Now we can use the join-strength rule and get witnesses $\psi \frown (a \sqcup b) \sqcap a$ and $\theta \frown (a \sqcup b) \sqcap c$ which satisfy $\phi \succ \psi \sqcap \theta$. Using Lemma 1.7.4 and the distributivity of $\sqcap$ and $\sqcup$ we obtain $\psi \frown a$ and $\theta \frown (a \sqcap c) \sqcup (b \sqcap c)$. With the help of Lemma 1.5.1 (1) the latter two relations can be assembled into $\psi \sqcap \theta \frown a \sqcup (a \sqcap c) \sqcup (b \sqcap c)$. To finish, apply Lemma 1.7.4 once more and obtain $\phi \frown a \sqcup (b \sqcap c)$. □

1.7.2 Duality with continuous frames and locally compact spaces

Lemma 1.7.6. *In a Stage 4 interaction algebra, binary meets of round ideals distribute over binary joins.*

Proof. Let $\mathcal{L}$ be a Stage 4 interaction algebra and I, J_1, J_2 be round ideals of L_+. By monotonicity of the meet operation $I \wedge -$ the inclusion $(I \wedge J_1) \vee (I \wedge J_2) \subseteq I \wedge (J_1 \vee J_2)$ holds, so it suffices to prove the reverse inclusion. Let $a \in I \wedge (J_1 \vee J_2)$. Then $a \prec \phi \frown a' \sqcap (b_1 \sqcup b_2)$ where $a' \in I$ and $b_i \in J_i$. Using distributivity of the operations $\sqcap$ and $\sqcup$ we obtain $a \prec \phi \frown (a' \sqcap b_1) \sqcup (a' \sqcap b_2)$. Apply Lemma 1.5.1 (2) and get $x_i \prec a' \sqcap b_i$ such that $a \prec x_1 \sqcup x_2$. This shows that $a \in (I \wedge J_1) \vee (I \wedge J_2)$. □

Theorem 1.7.7. *The contravariant duality between Stage 2a interaction algebras and continuous preframes restricts to a duality between the category* Tok_4 *and the category* CFrm *of continuous frames and frame homomorphisms.*

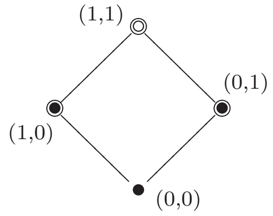


Figure 1.8: The interaction algebra that corresponds to the topology on the one-point space, depicted as the product of witnesses and tokens. Filled dots are members of $\succ\!\!\prec$ and circled dots are members of $\circ\!\!-$.

Proof. From Proposition 1.6.2 we know that the Stage 3a interaction algebras are dually equivalent to continuous lattices and frame homomorphisms. The previous lemma tells us that the Stage 4 interaction algebras correspond to continuous frames. If L is a continuous frame then the interaction algebra $\mathrm{Ialg}_{2a}\, L$ satisfies the axioms of Stage 2b and 2c as well. As the binary operations $\sqcap$ and $\sqcup$ on the tokens of $\mathrm{Ialg}_{2a}\, L$ are meet and join in L, the distributive law of Definition 1.7.1 holds. □

A continuous frame is spatial (assuming choice). The category CFrm of continuous frames is dually equivalent to the category lcSob of sober locally compact spaces, whence the category Tok_4 is equivalent to lcSob. Although we could use general Stone duality to present this equivalence, it is interesting to see how one can describe the equivalence directly.

Let X be a locally compact sober space. We build an interaction algebra $\mathcal{X} = \mathrm{Ialg}\, X$ where the tokens are bounded opens of X (Call an open set $U \in \mathcal{O}X$ *bounded* if it is contained in some compact subspace of X) and the witnesses are compact (saturated) subsets of X. On the token side X_+ the binary operations $\sqcap$ and $\sqcup$ are meet and join in the lattice $\mathcal{O}X$ and $0 \in X_+$ denotes the empty set. The bounded opens indeed form a (distributive) lattice with least element. On the witness side, let $\sqcap$ be binary union of compact sets and $1 \in X_-$ denote the empty set. Define relations $K \circ\!\!- U$ iff $K \subseteq U$ and $U \succ\!\!\prec K$ iff $U \subseteq K$ (The notation becomes more intuitive if one thinks of the witnesses as the complements of compact sets). Clearly then, the way-below relation on the tokens coincides with $\prec$. The algebraic axioms for Stage 4 all follow from the algebraic properties of the set inclusion relation and the fact that $\mathcal{O}X$ is a continuous frame. A continuous map $f : X \to Y$ translates to a Stage 4 morphism $\mathrm{Ialg}\, X \to \mathrm{Ialg}\, Y$ defining $K\, \mathrm{Ialg}(f) U$ iff $f(K) \subseteq U$ iff $K \subseteq f^{-1}(U)$. This definition matches with the one for the functor Ialg_{2a} because of the Hofmann-Mislove Theorem ([22, II-1.20],[29]): The Scott open filters of $\mathcal{O}X$ are in order-reversing bijection with the compact saturated subsets of X. Thus we can read $K \subseteq f^{-1}(U)$ as $\mathcal{O}(f)(U) \in \phi_K$ where ϕ_K denotes the Scott open filter in $\mathcal{O}X$ generated by K.

Going from interaction algebras to spaces, we employ the interaction algebra variant of Stone duality. The interaction algebra **2** shown in Figure 1.8 is the Stage 4 interaction

Compactness	$0 \in L_-$ $\qquad \dfrac{}{a \mathrel{\succ\!\!\prec} 0}$
Stable local continuity	$(L_-, \sqcup)$
binary join	$\dfrac{a \mathrel{\succ\!\!\prec} \phi \qquad a \mathrel{\succ\!\!\prec} \psi}{a \mathrel{\succ\!\!\prec} \phi \sqcup \psi}$
weakening rules	$\dfrac{a \mathrel{\succ\!\!\prec} \phi \sqcup \psi}{a \mathrel{\succ\!\!\prec} \phi} \qquad \dfrac{\phi \bigcirc a}{\phi \sqcup \psi \bigcirc a}$
dual join-strength	$\dfrac{\phi \sqcup \psi \bigcirc a}{\exists \phi \bigcirc b,\ \psi \bigcirc c.\, b \sqcap c \prec a}$

Table 1.3: Axioms for compactness and stable local continuity for Stage 2a

algebra generated from the one-point space and thereby corresponds to the two-element frame 2. Stone duality works as follows. Topologise the hom-set $\mathsf{Tok}_4(\mathbf{2}, \mathcal{L})$ using basic opens of the form $\{R : \mathbf{2} \to \mathcal{L} \mid 0Ra\}$ where a ranges over the tokens of $\mathcal{L}$. Notice that the algebraic operations $\sqcap$ and $\sqcup$ on tokens translate to binary intersection and union of basic opens.

1.8 Stable continuity and stable local continuity for preframes

So far we specialised the category Tok_1 presenting domains by adding algebraic structure to the set of tokens – binary meets for preframes, the empty join and binary joins for continuous lattices. It is a natural question to ask what happens if we add more algebraic structure to the set of witnesses. As the witnesses always have a binary operation we think of as meet, and adding a neutral element for this operation does not make any difference (recall that every interaction algebra of the form Ialg L has such a neutral element) we are left with investigating what domains arise if we add binary joins or empty joins to the set of witnesses. The categories of domains we thus obtain are interesting from a duality point of view, because the functor Flip will put them into dual equivalence with the well-known categories of Stage 2. Table 1.3 lists the rules we are about to study.

Definition 1.8.1. We call a preframe *compact* if the largest element 1 is way-below itself, or equivalently, if the way-upper set of any point is non-empty. A preframe is called *stably locally continuous* if the way-upper set of any point is closed under binary meets. If a preframe is both compact and stably locally continuous then we call it *stably continuous.* A Stage 2a interaction algebra is called stably locally continuous respectively compact if it satisfies the corresponding rules of Table 1.3.

1.8.1 Compactness for preframes

The compactness axiom listed in Table 1.3 has itself little influence on the shape of the domain of round ideals of tokens. This is because as long as the witness 0 has no token with $0 \bigcirc a$, one can safely omit this witness and obtain an isomorphic interaction algebra. The first stage at which we can guarantee that the witness 0 is meaningful is Stage 2a, where the dual definedness axiom implies the existence of some token a with $0 \bigcirc a$. In that case we immediately know that $(0, a)$ is a pair of compact elements because $0 \bigcirc a \mathrel{\succ\!\!\prec} 0$. Furthermore, for any token b we get $b \mathrel{\succ\!\!\prec} 0 \bigcirc a$ which tells us that every token is bounded by a. Dually, if ϕ is any witness then dual definedness yields $\phi \bigcirc b$ for some token b and thereby $\phi \bigcirc b \mathrel{\succ\!\!\prec} 0$ which tells us that 0 is below every witness. Let us therefore rename the token a to 1. This choice is justified, because every token b is lower equivalent to the token $1 \sqcap b$. Indeed, if $\phi \bigcirc 1 \sqcap b$ then $\phi \bigcirc b$ because of the weakening rule of Stage 1. Conversely, if $\phi \bigcirc b$ then because of $\phi \succ 0 \bigcirc 1$ we also have $\phi \bigcirc 1$ and therefore $\phi \bigcirc 1 \sqcap b$. Observe that the round ideal $\downarrow 1$ is the top element of the preframe of round ideals of tokens. Corollary 1.2.9 tells us that the top element $\downarrow 1$ is compact, i.e. way below itself.

Given a preframe L with top element 1 satisfying $1 \ll 1$, we clearly have a smallest Scott open filter, namely the singleton $\{1\}$. Evidently this filter is bounded below by every point of the preframe, whence $\mathrm{Ialg}_{2a}\, L$ satisfies the compactness axiom. We conclude:

Proposition 1.8.1. *The preframe of round ideals of a Stage 2a interaction algebra has a compact top element if any only if the interaction algebra is isomorphic to one which satisfies the compactness axiom of Table 1.3.*

Note that the name "compactness" for the axiom we just considered has nothing to do with the domain of round ideals being compact in the Scott topology. The name is justified by the fact that a topological space X is compact if and only if the frame of opens $\mathcal{O}X$ is a compact preframe. Therefore the Stage 4 interaction algebras which satisfy the compactness axiom are precisely the duals of compact locally compact sober spaces.

Observe that the functor Flip transforms the compactness axiom into the axioms of Stage 2b, except for the strictness axiom. This yields a proof of the following fact (see [41, Proposition 9.5]):

Proposition 1.8.2. *The Lawson duals of continuous preframes with bottom are precisely the continuous preframes with a compact top element.*

Proof. The strictness axiom of Stage 2b is only used to ensure that a Scott continuous map preserves the bottom element. □

We can not expect strictness from the morphism part of Lawson duality because the preimage of the smallest Scott open filter $\{1\}$ might be larger than just a singleton.

1.8.2 Stable local continuity for preframes

The domain of all filters of a preframe has finite joins. If F and G are filters of a preframe L, then the join $F \vee G$ is given as the upper set of $\{a \sqcap b \,|\, a \in F,\ b \in G\}$. Notice that even if both F and G are Scott open, the join $F \vee G$ does not need to be. But suppose that L is a continuous preframe which in addition is stably locally continuous. Then, if $a \in F$ and $b \in G$ we know that $F \ni a' \ll a$ and $G \ni b' \ll b$ because both filters were assumed to be Scott open, and from stable local continuity we can conclude $F \vee G \ni a' \sqcap b' \ll a \sqcap b$ which shows that $F \vee G$ is Scott open. Note that in this case we can describe $F \vee G$ as $\uparrow \{a \sqcap b \,|\, (a, b) \in F \times G\}$. Using the language of interaction algebras, we will also prove the converse: If the Lawson dual of a continuous preframe L has binary joins, then L itself must be stably locally continuous.

Lemma 1.8.3. *If a Stage 2a interaction algebra satisfies the stable local continuity axioms of Table 1.3, then the following rules hold.*

$$\frac{x \prec a \qquad x \prec b}{x \prec a \sqcap b} \qquad \frac{\phi \succ \psi \qquad \phi \succ \theta}{\phi \succ \psi \sqcup \theta}$$

Proof. Suppose $x \succ\!\!\prec \phi \frown a$ and $x \succ\!\!\prec \psi \frown b$. Then because of the binary join rule of stable local continuity and its associated weakening rule we know that $x \succ\!\!\prec \phi \sqcup \psi \frown a, b$. With the meet rule of Stage 2a conclude $x \succ\!\!\prec \phi \sqcup \psi \frown a \sqcap b$. The second rule is proved dually. □

Proposition 1.8.4. *Suppose a continuous preframe L is presented by a Stage 2a interaction algebra $\mathcal{L}$. Then L is stably locally continuous if and only if $\mathcal{L}$ is isomorphic to an interaction algebra that satisfies the stable local continuity axioms of Table 1.3.*

Proof. From the lemma above we know that for basic round ideals of tokens $\downarrow x$, $\downarrow a$ and $\downarrow b$ the implication $\downarrow x \ll \downarrow a, \downarrow b \Rightarrow \downarrow x \ll \downarrow a \sqcap b$ holds. This is enough to deduce stable local continuity for the entire preframe of round ideals, since $I \ll J$ holds if and only if $I \ll \downarrow x \ll J$ for some token x.

We already convinced ourselves that stable local continuity of a preframe L implies that the Lawson dual has binary joins. Defining the operation $\sqcup$ as binary join of Scott open filters, the interaction algebra $\mathrm{Ialg}_{2a}\, L$ is readily seen to satisfy all stable local continuity axioms of Table 1.3. Observe, for example, that the dual join-strength axiom holds because of the way binary joins of Scott open filters are computed. □

The reader will have noticed that the stable local continuity axioms are formally similar to the axioms of Stage 2c, except that there we imposed them on tokens and not on witnesses. Hence, the contravariant involution Flip swaps the Stage 2a axioms with Stage 1 axioms and the Stage 2c axioms with stable local continuity axioms. This yields a short proof of [41, Proposition 9.4]:

Proposition 1.8.5. *The stably locally continuous preframes are precisely the Lawson duals of continuous preframes which are complete Sup-lattices.*

Combining this with Proposition 1.8.2 we obtain [41, Corollary 9.6]:

Proposition 1.8.6. *The Lawson duals of stably compact preframes are precisely the continuous lattices.*

1.8.3 Stably locally continuous frames

A consequence of Proposition 1.8.5 is a dual of Lemma 1.5.1. Notice that Lemma 1.8.3 above already contains the dual rules to Lemma 1.5.1 (3) and (6).

Lemma 1.8.7. *Let $\mathcal{L}$ be a stably locally continuous Stage 2a interaction algebra.*

1. *The relations $\bigcirc$ and $\succ\!\prec$ satisfy*

$$\frac{\phi \bigcirc a \qquad \psi \bigcirc b}{\phi \sqcup \psi \bigcirc a \sqcap b} \qquad \frac{a \succ\!\prec \phi \qquad b \succ\!\prec \psi}{a \sqcap b \succ\!\prec \phi \sqcup \psi}$$

2. *The relation $\succ$ satisfies the join-strength rule*

$$\frac{\phi \sqcup \psi \succ \theta}{\exists \phi \succ \phi', \exists \psi \succ \phi'. \, \phi' \sqcup \psi' \succ \theta}$$

3. *Any token b is lower equivalent to $b \sqcap b$.*

4. *Any witness ϕ is lower equivalent to $\phi \sqcup \phi$.*

Lemma 1.8.8. *If $b \prec a$ in a Stage 2c interaction algebra then $\phi \bigcirc a \sqcup b$ implies $\phi \bigcirc a$.*

Proof. If $\phi \bigcirc a \sqcup b$ then the join-strength rule yields witnesses $\psi \bigcirc a$, $\theta \bigcirc b$ with $\phi \succ \psi \sqcap \theta$. Now $b \prec a$ implies $\theta \bigcirc a$ whence also $\phi \succ \psi \sqcap \theta \bigcirc a$. □

Proposition 1.8.9. *In a stably locally continuous Stage 4 interaction algebra the operations $\sqcup$ and $\sqcap$ on the set of witnesses distribute, meaning that for any witnesses ϕ, ψ and θ the witnesses $\phi \sqcap (\psi \sqcup \theta)$ is lower equivalent to $(\phi \sqcap \psi) \sqcup (\phi \sqcap \theta)$.*

Proof. In any stably locally continuous Stage 2a interaction algebra the witness $(\phi \sqcap \psi) \sqcup (\phi \sqcap \theta)$ is weakly below $\phi \sqcap (\psi \sqcup \theta)$. The proof is dual to that of Lemma 1.7.1.

Next we show that the dual distributive law of Proposition 1.7.5 implies that the witness $\phi \sqcap (\psi \sqcup \theta)$ is weakly below $(\phi \sqcap \psi) \sqcup (\phi \sqcap \theta)$. Suppose $\phi \sqcap (\psi \sqcup \theta) \bigcirc a$. With the weakening rule of Stage 1 we deduce $\phi \bigcirc a$ and $\psi \sqcup \theta \bigcirc a$. The dual join-strength rule yields tokens b and c such that $\psi \bigcirc b$, $\theta \bigcirc c$ and $b \sqcap c \prec a$. Use Lemma 1.5.1 (1) to get $\phi \sqcap \psi \bigcirc a \sqcup b$ and $\phi \sqcap \theta \bigcirc a \sqcup c$. The dual rule in Lemma 1.8.7 (1) then gives $(\phi \sqcap \psi) \sqcup (\phi \sqcap \theta) \bigcirc (a \sqcup b) \sqcap (a \sqcup c)$. Now apply the dual distributive law for tokens to obtain $(\phi \sqcap \psi) \sqcup (\phi \sqcap \theta) \bigcirc a \sqcup (b \sqcap c)$. With the preceding lemma we get $(\phi \sqcap \psi) \sqcup (\phi \sqcap \theta) \bigcirc a$. □

1.9 Interaction algebras for stably continuous frames

We already know two categories with a well-behaved self-duality: The completely distributive frames, presented by Stage 0 interaction algebras, and continuous preframes, presented by Stage 2a interaction algebras. Now we have arrived at another self-dual category: The stably continuous frames, presented by stably continuous Stage 4 interaction algebras. Indeed, in a stably continuous Stage 4 interaction algebra both the tokens and witnesses carry algebraic structure with the signature $(\sqcap, \sqcup, 0, 1)$ and the functor Flip reflects the rules that the algebraic structure is postulated to obey. In terms of domain theory we get:

Proposition 1.9.1. *The Lawson dual of a stably continuous frame is again a stably continuous frame. Lawson duality restricts to a contravariant endofunctor on the category* SCFrm *of stably continuous frames and frame homomorphisms.*

Proof. Proposition 1.8.9 allows us to specialise Proposition 1.8.6 to the desired result. □

The Stone duality between continuous frames and locally compact sober spaces we laid out in Section 1.7 restricts to a duality between stably continuous frames and *stably compact spaces*. The self-duality of stably compact spaces maps such a space to its *de Groot dual*. Its topology is an instance of the co-compact topology known in domain theory, whose subbasic open sets are the complements of compact saturated sets. The special property of stably compact spaces is that this subbasis is already a topology. For a comprehensive account of results about stably compact spaces, the reader may consult [40] or [35, Chapter 1]. As stably continuous frames and their Stone duals play an important role in the remaining chapters of this thesis, we christen the stably continuous Stage 4 interaction algebras and Stage 4 morphisms the ultimate Stage 5 of our hierarchy.

1.10 The Smyth powerdomain

The subcategories of domains we represented using interaction algebras of Stage 2a, 2b and 2c all have left adjoints to the inclusion functor. Just as every adjunction, those left adjoints give rise to monads on the category Dom which can be thought of as adding the appropriate algebraic structure to a domain in a "free" manner. In universal algebra one introduces algebraic operations on a carrier set by postulating a *structure map* from the free algebra over the carrier into the carrier. Typically this free algebra will be a subset of the powerset, but other power objects have been considered; the prime example being the probabilistic powerdomain whose points are probability distributions. A priori a single object in a category might have many such structure maps for a given monad. For example, a non-trivial set admits different monoid structures. Therefore it is noteworthy that all monads on Poset appearing in this thesis belong to a family of monads which allow at most one structure map per object.

The free continuous preframe over a domain is called the *Smyth powerdomain*. In contrast to the Smyth power-construction for arbitrary dcpos, the Smyth powerdomain of a domain D has a nice concrete description: Its points are the Scott compact saturated subsets of D, whence the Smyth power-construction is also referred to as the upper power-construction (Recall that saturated sets are upper sets in the specialisation order).

The reader should be warned that in most sources the Smyth powerdomain is defined as the free deflationary dcpo-semilattice, without the neutral element for the binary operation. This means to exclude the empty set from the compact saturated sets. In computer science contexts this is a sensible thing to do: when modelling nondeterminism, is is an obvious requirement that a process has *some* possible behaviour at any state. However, as our preframes always have a largest element, we include the empty set in our definition of Smyth powerdomain. For the classical construction, replace "finite set" by "non-empty finite set" in everything that follows.

Definition 1.10.1. If $R \subseteq L \times M$ is a binary relation then the *Smyth lifting* of R to the finite powersets of L and M is given as

$$AR_S B :\Leftrightarrow \forall b \in B \, \exists a \in A. \, aRb$$

Abramsky and Jung [1] prove that if $(L, \prec)$ is an abstract basis for a domain D then $(\mathrm{Fin}\, L, \prec_S)$ is an abstract basis for the Smyth powerdomain of D.

Interestingly, the finite meets we wish to add to the domain D now arise in the same way as one constructs the free meet-semilattice over a set: If L is a set, then the join operation $\cup$ is a binary operation on the finite powerset $\mathrm{Fin}\, L$ with the empty set as neutral element. On the tokens of the abstract basis $(\mathrm{Fin}\, L, \prec_S)$ we declare binary union to be a meet operation. This meet extends to round ideals of tokens.

We aim to present the Smyth powerdomain as a construction on Stage 1 interaction algebras. In doing so, we exhibit and exploit a few not-so-well-known facts about the Smyth powerdomain (only the statement about the unit is somewhat explicitly stated in [1]):

- The Smyth powerdomain of any domain is a stably continuous preframe.
- The Smyth powerdomain functor preserves semi-open maps.
- The unit and multiplication of the Smyth powerdomain monad are semi-open maps, and can be derived from the finite-powerset monad Fin on Set.

1.10.1 Relation lifting with algebraic operations

In what sense is the Smyth lifting a canonical construction? There is a canonical way of lifting a binary relation through a functor called *relation lifting*. Suppose T is a functor on Set and $R \subseteq L \times L$ is a binary relation. Let $L \xleftarrow{\pi_1} R \xrightarrow{\pi_2} L$ denote the projections of the set R onto its first and second coordinate. Then $T(\pi_i) : TR \to TL$ is the image of the i-th projection under the functor T. Define a relation $\overline{T(R)} \subseteq TL \times TL$ by declaring $a\overline{T(R)}b$ whenever there exists an element $x \in TR$ with the property that $a = T(\pi_1)(x)$ and $b = T(\pi_2)(x)$. In the case where $T = \mathrm{Fin}$ is the finite powerset functor and $R = \prec$ is the order relation of an abstract basis, one obtains the *Egli-Milner* lifting

$$A\overline{\mathrm{Fin}(\prec)}B \Leftrightarrow \forall b \in B\, \exists a \in A.\, a \prec b \text{ and } \forall a \in A\, \exists b \in B.\, a \prec b.$$

Notation. When lifting a relation R through the finite powerset functor we write R_{EM} instead of the more complicated $\overline{\mathrm{Fin}(R)}$.

The abstract basis $(\mathrm{Fin}\, L, \prec_{EM})$ gives rise to the Vietoris powerdomain. Vosmaer [61] uses relation lifting to generalise the Vietoris powerlocale construction to a power construction parametrised by a certain monad T on Set. However, the Smyth lifting of $\prec$ seems not to be of the kind $\overline{T(\prec)}$ for a functor T. We show how to remedy this using algebraic operations.

Suppose L is a set with a binary relation $\sqsubseteq$ we want to think of as a less-than-relation. Suppose further that we have a binary operation $\sqcap$ we regard as binary meet. Then one natural rule for how $\sqsubseteq$ and $\sqcap$ should interact is the weakening rule

$$\frac{x \sqsubseteq y}{x \sqcap x' \sqsubseteq y}\ .$$

One could read this as a production rule to enlarge the relation $\sqsubseteq$. And indeed: If $(L, \prec)$ is an abstract basis and $A \prec_{EM} B$ in the finite powerset of L, then for any finite set $A' \subseteq L$ we have $A \cup A' \prec_S B$. Conversely, suppose $A \prec_S B$. Let $A' = A \cap \downarrow B$. Then $A' \prec_{EM} B$ and clearly $A \cup A' = A$. We have shown that the Smyth lifting of $\prec$ is the relation lifting of $\prec$ through the finite powerset functor, followed by closure under the weakening rule above.

A binary algebraic operation can be lifted through the finite powerset functor as follows. For a set L there is a natural map $\mathrm{Fin}\, L \times \mathrm{Fin}\, L \to \mathrm{Fin}(L \times L)$ sending a pair (A, B) to the set $A \times B$. Thus an operation $\sqcap : L \times L \to L$ lifts to a binary operation $\mathrm{Fin}\, L \times \mathrm{Fin}\, L \to \mathrm{Fin}\, L$ by post-composing the natural map by $\mathrm{Fin}(\sqcap)$. This yields $A \sqcap B = \{a \sqcap b \,|\, (a, b) \in A \times B\}$.

Definition 1.10.2. For a Stage 1 interaction algebra $\mathcal{L}$ define the *Smyth poweralgebra* $\mathbb{P}_S\mathcal{L}$ as follows. Let the witness set be the finite powerset $\mathrm{Fin}\, L_-$ of the witnesses of $\mathcal{L}$. Further let $\mathbb{0}$ denote the empty set in $\mathrm{Fin}\, L_-$. Let the token set of $\mathbb{P}_S\mathcal{L}$ be the finite powerset of the tokens of $\mathcal{L}$, with the symbol $\mathbb{1}$ denoting the empty set of tokens. For finite sets of witnesses Φ, Ψ and finite sets of tokens A, B declare binary operations

$$\begin{aligned} \Phi \sqcap \Psi &:= \{\phi \sqcap \psi \,|\, (\phi, \psi) \in \Phi \times \Psi\} \\ \Phi \sqcup \Psi &:= \Phi \cup \Psi \\ A \sqcap B &:= A \cup B \end{aligned}$$

The relations $\prec\!\!\!\!\succ$, $\circ$ and all morphisms R are lifted to relations $\mathbb{P}_S(\prec\!\!\!\!\succ)$, $\mathbb{P}_S(\circ)$ and $\mathbb{P}_S(R)$ on poweralgebras by first lifting the relation through the functor Fin and then closing under the rules of Stage 2a and the rules of Table 1.3.

As we convinced ourselves above, the relation lifting thus defined produces the Smyth lifting we described earlier. It remains to show that the Smyth liftings of R and $\prec\!\!\!\!\succ$ are closed under all Stage 2a rules.

Stage 0. The Smyth lifting is easily seen to be functorial as an operation on morphisms of Rel. Hence equational axioms such as the Stage 0 axiom are preserved by Smyth lifting. One consequence of functoriality is that $\prec\!\!\!\!\succ_S ; \circ_S = \prec_S$ whence we can say that the domain $\mathrm{Idl}^{\prec_S} \mathrm{Fin}\, L_+$ presents the Smyth powerdomain of $\mathrm{Idl}^{\prec} L_+$. More generally the Smyth lifting is monotone with respect to inclusion of relations whence adjoint pairs of relations are preserved.

Stage 1. Let R be a Stage 1 morphism. If $\Phi R_S A$ and $\Psi R_S A$ then for all $a \in A$ there exist $\phi \in \Phi$, $\psi \in \Psi$ such that $\phi, \psi R a$. Since R is closed under the meet rule of Stage 1, we have $\Phi \sqcap \Psi \ni \phi \sqcap \psi R a$ whence $\Phi \sqcap \Psi R_S A$. Similarly, the weakening rule for $\sqcap$ on finite sets of witnesses follows because R satisfies that rule. For the empty set $\mathbb{1}$ of tokens the definedness axiom holds vacuously; for all non-empty sets A it follows from the definedness axiom for R. If $\forall \phi \in \Phi \exists a \in A.\, a \prec\!\!\!\!\succ \Phi$ then this certainly remains true for if we replace A with a set of the form $A \sqcap B := A \cup B$. Thus all Stage 1 rules hold.

Stage 2a. As the binary meet $\sqcap$ of finite sets of tokens is given as binary union, it is trivial to check that the meet rule of Stage 2a holds for the Smyth lifting of a relation R. The weakening rule for R_S holds because $\Phi R_S A$ implies that $\Phi R_S A'$ for all $A' \subseteq A$. We used the weakening rule for $\prec\!\!\!\!\succ$ to obtain $\prec\!\!\!\!\succ_S$ from the relation lifting $\prec\!\!\!\!\succ_{EM}$, so this rule holds by definition. The dual definedness rule is trivial because $\Phi R_S \mathbb{1}$ holds vacuously, as $\mathbb{1}$ is the empty set of tokens.

Stable continuity. The empty set of witnesses $\mathbb{0}$ satisfies the compactness rule $A \prec\!\!\!\!\succ_S \mathbb{0}$ vacuously. Notice that $\mathbb{0} \circ_S \mathbb{1}$. The binary join rule for the operation $\sqcup$ on witnesses holds because we used this rule to obtain the Smyth lifting of $\prec\!\!\!\!\succ$ from its relational lifting $\prec\!\!\!\!\succ_{EM}$. Checking the weakening rules is straightforward. The dual join-strength rule holds for all morphisms R: Suppose A is a finite set of tokens, Φ and Ψ are finite sets of witnesses and suppose $\Phi \sqcup \Psi R_S A$, meaning $\forall a \in A \exists \theta \in \Phi \cup \Psi.\, \theta R a$. Then there is another finite set $A' \prec_S A$ with $\Phi \sqcup \Psi R_S A'$. Define two sets $B := \{a \in A' \mid \exists \phi \in \Phi.\, \phi R a\}$ and $C := \{a \in A' \mid \exists \psi \in \Psi.\, \psi R a\}$. Then $\Phi R_S B$ and $\Psi R_S C$ and furthermore $B \sqcap C = B \cup C = A'$ whence $B \sqcap C \prec_S A$.

The names $\mathbb{0}$ and $\mathbb{1}$ for the empty set of witnesses and tokens are chosen purposefully. The empty set $\mathbb{0}$ satisfies $\mathbb{0} \sqcup \Phi = \Phi$ and $\mathbb{0} \sqcap \Phi = \mathbb{0}$ for all finite sets Φ of witnesses and

$A{\prec}_S\mathbb{0}$ for all finite sets A of tokens. Dually, the empty set of tokens $\mathbb{1}$ satisfies $\mathbb{1} \sqcap A = A$ for all finite sets A of tokens and $\Phi \circ_S \mathbb{1}$ for all finite sets Φ of witnesses. Moreover, the operations $\sqcap$ and $\sqcup$ on finite sets of witnesses distribute in the classical sense. For we have

$$\begin{aligned}\Phi \sqcap (\Psi \sqcup \Theta) &= \{\phi \sqcap \eta \,|\, \phi \in \Phi,\, \eta \in \Psi \cup \Theta\} \\ &= \{\phi \sqcap \psi \,|\, \phi \in \Phi,\, \psi \in \Psi\} \cup \{\phi \sqcap \theta \,|\, \phi \in \Phi,\, \theta \in \Theta\} \\ &= (\Phi \sqcap \Psi) \sqcup (\Phi \sqcap \Theta).\end{aligned}$$

This distributive law was expected because intuitively the Smyth powerdomain of a domain is the Lawson dual of the Scott topology. And in fact, Proposition 1.8.6 tells us that the Lawson dual of the domain presented by a Smyth poweralgebra is a frame. We need to show yet that this frame is in fact the Scott topology we expect it to be.

The Smyth poweralgebra construction preserves token maps in virtue of the finite powerset functor. A token map (f_-, f_+) between Stage 1 interaction algebras yields a pair $(\mathrm{Fin}(f_-), \mathrm{Fin}(f_+))$ between finite powersets, and it is easy to verify that the latter pair of functions preserves the relations $\circ_S$ and $\prec_S$ whenever the former preserves $\circ$ and $\prec$. Moreover, if we transform a token map into an adjoint pair of relations and lift these, then the result is the same as first lifting the token map and then transforming it into an adjoint pair of relations. We collect our findings in the following theorem.

Theorem 1.10.1. *The Smyth poweralgebra $\mathbb{P}_S$ is a functor from Tok_1 into the subcategory of stably continuous Stage 2a interaction algebras. It preserves token maps and adjoint pairs. If $D = \mathrm{Idl}^{\prec} L_+$ is the domain presented by an interaction algebra $\mathcal{L}$, then the interaction algebra $\mathbb{P}_S\mathcal{L}$ presents the Smyth powerdomain of D.*

To conclude the study of the Smyth poweralgebra functor, we give the interaction algebra proof for the domain-theoretic characterisation of the Smyth powerdomain [22, Theorem IV-8.10] we mentioned earlier.

Proposition 1.10.2. *For any Stage 1 interaction algebra $\mathcal{L}$ the following domains are isomorphic.*

1. *The domain of round ideals of witnesses of $\mathbb{P}_S\mathcal{L}$,*
2. *The domain of round lower sets of witnesses of $\mathcal{L}$.*

Consequently, the domain presented by $\mathbb{P}_S\mathcal{L}$ is the Lawson dual of the Scott topology of the domain presented by $\mathcal{L}$.

Proof. A round ideal $\mathcal{I}$ in $\mathrm{Fin}\, L_-$ yields a round lower set $\bigcup \mathcal{I}$ in L_-. Conversely, a round lower set $\Phi \subseteq L_-$ gives rise to a round ideal $\{\Psi \in \mathrm{Fin}\, L_- \,|\, \Psi \subseteq \Phi\}$. These two assignments are order-preserving and mutually inverse. □

One would expect that the order-isomorphism of the proposition above can be realised as an isomorphism of interaction algebras, for example between the Stage 1 interaction algebra $\mathrm{pt}_1\, \mathbb{P}_S\mathcal{L}$ and the Stage 0 interaction algebra $\Omega\mathcal{L}$. But recall that the relation $\succ$ on the witnesses of $\Omega\mathcal{L}$ is the completely-above relation, not the way-above relation. Hence such an isomorphism can not exist. The preceding proposition combined with Theorem 1.3.4 yields the interaction algebra presentation of the contravariant functor $\mathcal{O} : \mathsf{Dom} \to \mathsf{CFrm}$ as $\mathrm{Flip} \circ \mathbb{P}_S$.

1.10.2 The Smyth powerdomain monad

The finite powerset functor Fin extends to a monad on Set, with the singleton operation as unit and union of finite sets as multiplication. In due course we show that the Smyth poweralgebra functor is a comonad on the category Tok_1. Its co-unit and co-multiplication are in fact given by token maps derived from the unit and multiplication of the monad $(\mathrm{Fin}, \{-\}, \bigcup)$ on Set.

Let $\mathcal{L}$ be a Stage 1 interaction algebra. Observe that the Smyth liftings $\bigcirc_S$ and $\asymp_S$ extend the relations $\bigcirc$ and $\asymp$ in the sense that $\phi \bigcirc a$ holds if and only if $\{\phi\}\bigcirc_S\{a\}$, and likewise $a \asymp \phi$ holds precisely when $\{a\}\asymp_S\{\phi\}$. In particular this tells us that the pair of singleton maps $(\{-\},\{-\}) : L_- \times L_+ \to \mathrm{Fin}\, L_- \times \mathrm{Fin}\, L_+$ is a token map. It has some convenient additional properties: Because we defined the operation $\sqcap$ on finite sets of witnesses element-wise, the singleton map on witnesses preserves the operation $\sqcap$. If $\Phi \bigcirc_S \{a\}$ then $\phi \bigcirc a$ for some $\phi \in \Phi$ and therefore $\phi \bigcirc b \prec a$ for some token b. This means that the singleton map on tokens satisfies the continuity condition of Lemma 1.1.13 whereby its induced relation $\mathrm{Fin}\, L_- \to L_+$ has a simpler description. Let us write $E^-_{\mathcal{L}}$ and $E_{\mathcal{L}}$ for the relations induced by the singleton maps on witnesses and tokens, respectively. We have an adjoint pair $E_{\mathcal{L}} \dashv E^-_{\mathcal{L}}$ where

$$\begin{aligned} \Phi E_{\mathcal{L}} a &\Leftrightarrow \Phi \bigcirc_S \{a\} \\ &\Leftrightarrow \exists \phi \in \Phi . \phi \bigcirc a \\ \phi E^-_{\mathcal{L}} A &\Leftrightarrow \exists \psi .\, \phi \succ \psi,\, \forall a \in A .\, \psi \bigcirc a \\ &\Leftrightarrow \exists b .\, \phi \bigcirc b,\, \forall a \in A .\, b \prec a \end{aligned}$$

Note that not only $\bigcirc \subseteq E^-_{\mathcal{L}} \mathbin{;} E_{\mathcal{L}}$ but the stronger identity $\bigcirc = E^-_{\mathcal{L}} \mathbin{;} E_{\mathcal{L}}$ holds. In order to verify that the relation $E_{\mathcal{L}}$ extends to a natural transformation in the category of Stage 1 interaction algebras, we employ Corollary 1.1.14. The post-composition of a morphism $\mathbb{P}_S(R)$ with $E_{\mathcal{M}}$ is given as $\Phi(\mathbb{P}_S(R) \mathbin{;} E_{\mathcal{M}})a$ iff $\Phi \mathbb{P}_S(R)\{a\}$ which is equivalent to $\exists \phi \in \Phi .\, \phi R a$. The pre-composition of R with a relation $E_{\mathcal{L}}$ is given as $\Phi(E_{\mathcal{L}} \mathbin{;} R)a$ iff $\Phi \succ_S \{\psi\}, \psi R a$ for some witness ψ. But $\Phi \succ_S \{\psi\}$ is equivalent to $\exists \phi \in \Phi .\, \phi \succ \psi$ whence $\Phi(E_{\mathcal{L}} \mathbin{;} R)a$ iff $\exists \phi \in \Phi .\, \phi R a$. Thus $\mathbb{P}_S(R) \mathbin{;} E_{\mathcal{M}} = E_{\mathcal{L}} \mathbin{;} R$.

Let us now consider the pair of union maps $\bigcup : \mathrm{Fin}^2\, L_- \to \mathrm{Fin}\, L_-$ and $\bigcup : \mathrm{Fin}^2\, L_+ \to$

$\operatorname{Fin} L_+$. Observe that for sets of witnesses we have

$$\begin{aligned} \bigcup\{\Phi, \Psi\} &= \Phi \sqcup \Psi, \\ \bigcup\{\Phi\} &= \Phi, \\ \bigcup\emptyset &= \emptyset = \mathbb{0} \end{aligned}$$

Therefore, extending the arity of $\sqcup$ on $\operatorname{Fin}^2 L_-$ to finite sets, we can say that for $\Phi \in \operatorname{Fin}^2 L_-$ the union $\bigcup \Phi$ coincides with $\bigsqcup \Phi$. Dually, for finite sets of tokens we have

$$\begin{aligned} \bigcup\{A, B\} &= A \sqcap B, \\ \bigcup\{A\} &= A, \\ \bigcup\emptyset &= \emptyset = \mathbb{1} \end{aligned}$$

whence we can say that $\bigcup \mathbb{A} = \bigsqcap \mathbb{A}$ for an element $\mathbb{A} \in \operatorname{Fin}^2 L_+$.

Lemma 1.10.3. *For any stably continuous Stage 2a interaction algebra $\mathcal{L}$ the pair of maps $(\bigsqcup, \bigsqcap) : \mathbb{P}_S\mathcal{L} \to \mathcal{L}$ is a token map. It is natural in $\mathcal{L}$, meaning that the relation $S_\mathcal{L}$ associated with the map $\bigsqcap$ constitutes a natural transformation from the identity functor on stably continuous Stage 2a interaction algebras to $\mathbb{P}_S$.*

Proof. Let $\mathcal{L}$ be a stably continuous Stage 2a interaction algebra. Suppose $\Phi \circ_S A$ in $\mathbb{P}_S\mathcal{L}$. Then there exists a family $\{\phi_a\}_{a\in A} \subseteq \Phi$ with $\phi_a \circ a$ for all $a \in A$. The first rule in Lemma 1.8.7 (1) yields $\bigsqcup_{a\in A} \phi_a \circ \bigsqcap A$. Using the weakening rule for $\sqcup$ and Lemma 1.8.7 (4) we conclude $\bigsqcup \Phi \circ \bigsqcap A$. Now suppose $A \prec\!\!\succ_S \Phi$. A similar argument using Lemma 1.8.7 (1) and (3) yields $\bigsqcap A \prec\!\!\succ_S \bigsqcup \Phi$. Therefore $(\bigsqcup, \bigsqcap)$ is a token map.

We claim that the map $\bigsqcap : \operatorname{Fin} L_+ \to L_+$ satisfies the continuity condition of Lemma 1.1.13. Indeed, if $\phi \circ \bigsqcap A$ then $\phi \circ b \prec \bigsqcap A$ for some token b. With the weakening rule of Stage 2a we get $\forall a \in A.\, b \prec a$ whence $\{b\} \prec_S A$. Therefore the relation $S_\mathcal{L}$ defined by $\bigsqcap$ has the simple description $\phi S_\mathcal{L} A$ iff $\phi \circ \bigsqcap A$. Now suppose $R : \mathcal{L} \to \mathcal{M}$ is a Stage 1 morphism. Corollary 1.1.14 tells us that post-composing the morphism R with the relation $S_\mathcal{M}$ defined by $\bigsqcap : \operatorname{Fin} M_+ \to M_+$ gives the relation $\phi(R \mathbin{⨟} S_\mathcal{M})B$ iff $\phi R \bigsqcap B$. Pre-composing $\mathbb{P}_S(R)$ with $S_\mathcal{L}$ gives the relation $\phi(S_\mathcal{L} \mathbin{⨟} \mathbb{P}_S(R))B$ iff $\phi \succ \bigsqcup \Psi$ and $\Psi \mathbb{P}_S(R) B$ for some $\Phi \in \operatorname{Fin} L_-$. Recall from Lemma 1.8.3 and the weakening rule for $\sqcup$ that $\phi \succ \bigsqcup \Psi$ is equivalent to $\forall \psi \in \Psi.\, \phi \succ \psi$ whence $\phi(S_\mathcal{L} \mathbin{⨟} \mathbb{P}_S(R))B$ is equivalent to $\{\phi\}(\succ_S ; \mathbb{P}_S(R))B$ which in turn is seen to coincide with $R \mathbin{⨟} S_\mathcal{M}$. □

Remark. The lemma above does not hold in the absence of stable continuity. In domain-theoretic terms we can say: The map that takes a compact saturated subset of a continuous preframe to its infimum does not necessarily preserve the way-below relation, but for stably compact preframes it does.

Corollary 1.10.4. *The pair of union maps* $\bigcup : \mathrm{Fin}^2 L_- \to \mathrm{Fin}\, L_-$ *and* $\bigcup : \mathrm{Fin}^2 L_+ \to \mathrm{Fin}\, L_+$ *form a token map for any Stage 1 interaction algebra* $\mathcal{L}$*. Moreover, the relation* $U_{\mathcal{L}}$ *associated with the map* $\bigsqcup$ *on tokens is a natural transformation in* Tok_1 *from* $\mathbb{P}_S$ *to* $(\mathbb{P}_S)^2$*.*

Proof. The Smyth poweralgebra of a Stage 1 interaction algebra $\mathcal{L}$ is a stably continuous Stage 2a interaction algebra, whence the previous lemma applies. As we explained above, on $(\mathbb{P}_S)^2\mathcal{L}$ the union of sets of witnesses is computed via $\bigsqcup$ and the union of sets of tokens is computed via $\bigsqcap$. □

Now that we have established two natural transformations on the category of Stage 1 interaction algebras which are given by token maps, and knowing that the Smyth poweralgebra functor preserves token maps, we get the defining diagrams for a comonad on Tok_1 for free: They simply follow from the fact that Fin extents to a monad on Set. We summarise:

Theorem 1.10.5. *The Smyth poweralgebra functor on* Tok_1 *extends to a comonad where the co-unit and co-multiplication are given by token maps* $(\{-\},\{-\})$ *and* $(\bigcup,\bigcup)$*. In particular, the Smyth poweralgebra functor restricts to a monad on the category of Stage 1 interaction algebras and token maps.*

Corollary 1.10.6. *The Smyth powerdomain monad on the category of domains has semi-open unit and multiplication maps and preserves semi-open maps.*

1.10.3 Algebras for the Smyth powerdomain monad

For all the monads on the category of posets we encountered so far (for instance the lower set monad and ideal monad) the Eilenberg-Moore algebras played an important role. As these monads are KZ-monads (Kock-Zöberlein, [37]) where the multiplication is adjoint to the unit, every object in the underlying category admits at most one monad algebra. In this subsection we demonstrate, using interaction algebras, that the Eilenberg-Moore algebras for the Smyth powerdomain monad on Dom are precisely the continuous preframes (For a more general account of this fact, see [47]). Since the points of the Smyth powerdomain are the compact saturated subsets of a domain, we conclude that a continuous preframe must have infima for all compact subsets, not only for the finite ones as postulated in the definition of a preframe. Once more, this emphasises our credo that compact sets behave as if they were finite.

Explicit top elements

Although preframes in this thesis have a top element (the empty meet), we do not need to add a special token 1 to every interaction algebra presenting a preframe. Instead, the top element is represented by the round ideal of all bounded tokens, which contains every other round ideal. Sometimes, however, it is convenient to have a special token 1 which generates this largest round ideal. The stably continuous Stage 2a interaction algebras are an example, where we extended the binary operation $\sqcap$ on tokens to a finitary operation $\bigsqcap$ with $\bigsqcap\emptyset = 1$. In order to do this for any (not necessarily stably continuous) Stage 2a interaction algebra, we add a token 1 and a witness 1 which obey the following nullary weakening rules.

$$\frac{}{1Ra} \qquad \frac{}{\phi R1}$$

Notice that these rules imply the definedness and dual definedness rules. Any Stage 2a interaction algebra enhanced with these rules is isomorphic to the original one. Indeed, one can apply Proposition 1.1.4 because neither the token 1 nor the witness 1 are bounded.

Hence it is justified to use $\bigsqcap$ as if it was a map on finite sets of tokens, even if the original interaction algebra did not have a token 1. Observe that the argument of Lemma 1.10.3 still applies where we showed that the map $\bigsqcap$ satisfies the continuity condition of Lemma 1.1.13. Thus we can express the structure morphism $S_{\mathcal{L}}$ of a Stage 2a interaction algebra as $\phi S_{\mathcal{L}} A$ iff $\phi \bigcirc \bigsqcap A$ and use the first part of Corollary 1.1.14, even though $\bigsqcap$ does not extend to a token map.

Coalgebra morphisms

Lemma 1.10.7. *If $\mathcal{L}$ is a Stage 2a interaction algebra then the structure morphism $S_{\mathcal{L}}$ is right adjoint to the unit $E_{\mathcal{L}}$.*

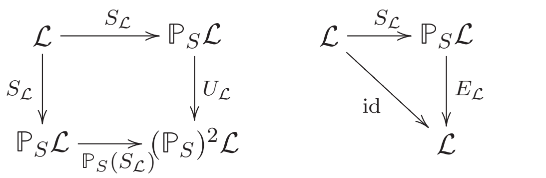
Proof. Under Stage 2a axioms the right adjoint $E_{\mathcal{L}}^{-}$ to $E_{\mathcal{L}}$ coincides with the structure map $S_{\mathcal{L}}$. In general $\phi E_{\mathcal{L}}^{-} A$ if there exists some witness ψ with $\phi \succ \psi$ and $\forall a \in A.\, \psi \bigcirc a$. With Stage 2a axioms this is equivalent to $\phi \succ \psi \bigcirc \bigsqcap A$ which in turn is equivalent to $\phi \bigcirc \bigsqcap A$. □

Remark. For continuous preframes the lemma above becomes manifest in the equivalence $x \leq \bigwedge K \Leftrightarrow \uparrow x \supseteq K$ where x is a point and K a compact saturated subset (Recall that $\uparrow$ is the unit of the Smyth powerdomain monad and that the order on compact saturated subsets is reverse set inclusion).

Lemma 1.10.8.

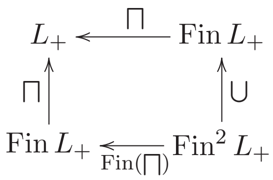
1. *If $\mathcal{L}$ is a Stage 2a interaction algebra then the morphism $S_{\mathcal{L}}$ derived from the structure map $\bigsqcap$ is a coalgebra for the Smyth poweralgebra comonad.*
2. *The Stage 1 morphisms between Stage 2a interaction algebras that are Smyth poweralgebra morphisms are precisely the Stage 2a morphisms.*

Proof. (1) First we verify the defining identities for $S_{\mathcal{L}}$ being a comonad coalgebra. This means we have to prove that the diagrams below commute.



(1.5)

We already observed above that the right-hand diagram commutes, as $S_{\mathcal{L}}$ coincides with the right adjoint $E_{\mathcal{L}}^{-}$. The left-hand square commutes because we assumed the operation $\sqcap$ to be associative. Indeed, all morphisms in the square above are given by maps on tokens, whence the square above commutes if and only if the square below commutes.



But this is precisely associativity of $\prod$.

(2) Now suppose $R : \mathcal{L} \to \mathcal{M}$ is a Stage 2a morphism. This is a Smyth poweralgebra morphism if $S_{\mathcal{L}} \mathbin{;} \mathbb{P}_S(R) = R \mathbin{;} S_{\mathcal{M}}$. But this identity is precisely naturality of the structure morphism which we showed in Lemma 1.10.3.

Finally, we show that R being a Smyth poweralgebra morphism implies that R satisfies the Stage 2a rules. The meet rule: Let $\phi R a$ and $\phi R a'$. Use $\succ ; R = R$ and get $\phi \bigcirc b \prec \psi R a$ and $\phi \bigcirc b' \prec \psi' R a'$. With $B := \{b, b'\}$ and $\Psi := \{\psi, \psi'\}$ we have $\phi S_{\mathcal{L}} B \prec_S \Psi \mathbb{P}_S(R) A$. Use the hypothesis $(S_{\mathcal{L}} \mathbin{;} \mathbb{P}_S(R)) \subseteq (R \mathbin{;} S_{\mathcal{M}})$ and obtain $\phi R x \prec \theta S_{\mathcal{L}}$ for some token x and some witness θ. Then $\mathcal{M}$ being a Stage 2a interaction algebra yields $\phi R x \prec \theta \bigcirc a \sqcap a'$ and so $\phi R a \sqcap a'$. The weakening rule for R follows from the weakening rule for the relation $\bigcirc$ in $\mathcal{M}$. Dual definedness: Let $\phi \in L_-$. By dual definedness for $\mathcal{L}$ we have $\phi \bigcirc b$ for some token $b \in L_+$. We can write this as $\phi S_{\mathcal{L}} \{b\}$. Trivially $\{b\} \prec_S \emptyset \mathbb{P}_S(R) \emptyset$. Then use the inclusion $(S_{\mathcal{L}} \mathbin{;} \mathbb{P}_S(R)) \subseteq (R \mathbin{;} S_{\mathcal{M}})$ and obtain $\phi R x$ for some token $x \in M_+$. □

Next we prove a converse to Lemma 1.5: If a Stage 1 interaction algebra admits a coalgebra for the Smyth poweralgebra comonad, then it is isomorphic to a Stage 2a interaction algebra. The structure morphism $S_{\mathcal{L}}$ of the lemma above tells us when a basic Scott open ϕ contains the infimum of a finite set A. As we will see below, this relation is the only possible way a morphism can be a Smyth poweralgebra coalgebra.

Lemma 1.10.9. *If $R : \mathcal{L} \to \mathbb{P}_S\mathcal{L}$ is a coalgebra map for the Smyth poweralgebra comonad, then the interaction algebra $\mathcal{L}$ presents a continuous preframe. Moreover, the morphism R is unique because it is right adjoint to the co-unit morphism $E_{\mathcal{L}}$.*

Proof. First let us examine what it means that R is a coalgebra for the Smyth poweralgebra comonad. The unit law states that $R \mathbin{;} E_{\mathcal{L}} = \bigcirc$ which means that $\phi \bigcirc a$ if and only if $\phi R\{a\}$.

In the interaction algebra $\mathbb{P}_S\mathcal{L}$ this means that $\Phi \bigcirc_S A$ if and only if $\Phi\mathbb{P}_S(R)\,\{\{a\}\mid a \in A\}$. Suppose $A \in \mathrm{Fin}\, L_+$ is a non-empty set of tokens and $\phi R A$. With $U_\mathcal{L}$ denoting the co-multiplication of the Smyth poweralgebra obtained from the map $\bigcup$, we can write

$$
\begin{aligned}
\phi R A &\Leftrightarrow \phi(R \mathbin{;} \bigcirc_S)A \\
&\Leftrightarrow \phi(R \mathbin{;} \bigcirc_S) \bigsqcap_{a\in A}\{a\} \text{ by definition of the meet operation} \\
&\Rightarrow \forall a \in A.\, \phi R\{a\} \text{ by the weakening rule for } \bigcirc_S \\
&\Rightarrow \{\phi\}\bigcirc_S A \text{ by the unit law.}
\end{aligned}
$$

Recall that by the Fundamental Lemma 1.2.1 the domain $\mathrm{pt}_1\,\mathcal{L}$ is isomorphic to the domain of round filters of witnesses of $\mathcal{L}$. Here, a token $a \in L_+$ corresponds to the round filter $F_a = \{\phi \in L_- \mid \phi \bigcirc a\}$. We claim that the binary meet of filters F_a and F_b is given as $F_{a\sqcap b} := \{\phi \in L_- \mid \phi R\{a,b\}\}$. We already convinced ourselves that $\phi R\{a,b\}$ implies $\phi R a$ and $\phi R b$, whence the filter $F_{a\sqcap b}$ is contained in both F_a and F_b. The meet of F_a and F_b as round upper sets is given as $F_a \wedge F_b = \{\phi \in L_- \mid \phi \succ \psi \bigcirc a, b\}$. For any $\phi \in F_a \wedge F_b$ we have $\phi \bigcirc c \prec \psi \bigcirc a, b$ for some token c and some witness ψ. One can write this as $\phi R\{c\} \prec_S \{\psi\} \bigcirc_S \{a, b\}$ whereby $\phi \in F_{a\sqcap b}$. Thus the meet $F_a \wedge F_b$ coincides with the round filter $F_{a\sqcap b}$ which makes the latter the infimum of F_a and F_b in the domain of round filters. Having binary meets for filters of the form F_a suffices to get binary meets for arbitrary round filters, as the filters F_a form a basis for the domain of round filters.

Recall that the empty set of tokens $\mathbb{1} \in \mathrm{Fin}\, L_+$ has $A \prec_S \mathbb{1}$ for any finite set A of tokens. Any round filter F of witnesses consists entirely of witnesses which are bounded below, as $\phi \in F$ implies that $\phi \succ \psi \in F$ for some other witness ψ. Now $\phi \succ \psi$ means $\phi \bigcirc a \prec \psi$ for some token a, whereby $\phi R\{a\}$ and so $\phi R \mathbb{1}$. Hence every round filter F of witnesses is contained in the round filter $\{\phi \in L_- \mid \phi R \mathbb{1}\}$ whereby the domain $\mathrm{pt}_1\,\mathcal{L}$ has a largest element.

It remains to show that R is right adjoint to the co-unit morphism $E_\mathcal{L}$. The unit law tells us that $\bigcirc = R \mathbin{;} E_\mathcal{L}$, so in particular $\bigcirc \subseteq R \mathbin{;} E_\mathcal{L}$. At the beginning of this proof we convinced ourselves that $\phi R A$ implies $\{\phi\}\bigcirc_S A$, which we can use to show that $E_\mathcal{L} \mathbin{;} R \subseteq \bigcirc_S$. □

The previous two lemmas combined yield:

Theorem 1.10.10. *Every Stage 1 interaction algebra admits at most one coalgebra for the Smyth poweralgebra comonad. The subcategory of Stage 2a interaction algebras is equivalent to the Eilenberg-Moore category of the Smyth poweralgebra comonad.*

For the category of domains we record (compare [47, Lemma 4.4, Theorem 7.16]):

Corollary 1.10.11. *The Eilenberg-Moore algebras for the Smyth powerdomain monad on* Dom *are precisely the continuous preframes. Moreover, the meet operation on compact saturated subsets of a stably continuous preframe is a semi-open map.*

Continuing from Theorem 1.10.10, the general theory of monads tells us that the Smyth powerdomain functor is the left adjoint to the inclusion functor $\mathsf{CPreFrm} \hookrightarrow \mathsf{Dom}$. In other words, the Smyth powerdomain of a domain is the free continuous preframe over that domain. Dually, the Smyth poweralgebra functor $\mathbb{P}_S$ is the right adjoint to the inclusion functor $\mathsf{Tok}_{2a} \hookrightarrow \mathsf{Tok}_1$. This means that for any Stage 2a interaction algebra $\mathcal{L}$ and any Stage 1 interaction algebra $\mathcal{M}$ there is a natural isomorphism of hom-sets $\mathsf{Tok}_{2a}(\mathcal{L}, \mathbb{P}_S\mathcal{M}) \cong \mathsf{Tok}_1(\mathcal{L}, \mathcal{M})$. Concretely this isomorphism goes as follows. For a Stage 1 morphism $R : L_- \to M_+$ define a relation $R^\dagger : L_- \to \mathrm{Fin}\, M_+$ by composing the structure map $S_\mathcal{L}$ with the relation $\mathbb{P}_S(R)$. This yields the relation

$$\phi R^\dagger A \text{ iff } \forall a \in A.\, \phi R a. \tag{1.6}$$

In particular that means $\phi R^\dagger \{a\}$ iff $\phi R a$. Going the other direction, given a Stage 2a morphism $R : L_- \to \mathrm{Fin}\, M_+$ one obtains a Stage 1 morphism $L_- \to M_+$ simply by restricting the relation R to singleton sets on the right.

1.10.4 Smyth powerdomains at other stages

We conclude our study of the Smyth poweralgebra with some preservation results, which can be found in [25, Theorems 5.1,6.1].

Proposition 1.10.12. *1. The Smyth poweralgebra of a Stage 2b interaction algebra is a Stage 2b interaction algebra. If R is a Stage 2b morphism, then so is $\mathbb{P}_S(R)$.*

2. The Smyth poweralgebra of a Stage 2c interaction algebra is a Stage 2c interaction algebra. If R is a Stage 2c morphism, then so is $\mathbb{P}_S(R)$.

Proof. (1) Let $\mathcal{L}$ be a Stage 2b interaction algebra with distinguished witness 1 and distinguished token 0 which satisfy the axioms of Stage 2b. Then it is easy to see that the Smyth liftings of the relations $\circ$ and $\rightarrowtail$ have $\{1\} \circ_S A$ for every finite set A of tokens and $\{0\} \rightarrowtail_S \Phi$ for every finite set Φ of witnesses. If R is a Stage 2b morphism and $\Phi \mathbb{P}_S(R)\{0\}$ then $\phi R 0$ for some witness $\phi \in \Phi$. Then the strictness rule for R yields $\phi \circ 0$ whence $\Phi \circ_S \{0\}$. Thus the morphism $\mathbb{P}_S(R)$ is strict.

(2) Let $\mathcal{L}$ be a Stage 2c interaction algebra. Lift the binary operation $\sqcup$ on tokens to $\mathrm{Fin}\, L_+$ element-wise: $A \sqcup B := \{a \sqcup b \mid (a, b) \in A \times B\}$. The binary join rule and weakening rules for $\sqcup$ on $\mathrm{Fin}\, L_+$ follow immediately from the corresponding rules of $\mathcal{L}$. Similarly, if R is a Stage 2c morphism then $\mathbb{P}_S(R)$ is easily seen to satisfy the binary join rule and the weakening rules of Stage 2c. It remains to check the join-strength rule. Let R be a Stage 2c morphism, Φ be a finite set of witnesses, A and B be finite sets of tokens with $\Phi \mathbb{P}_S(R) A \sqcup B$. First consider the case where either A or B is the empty set. Suppose $B = \mathbb{1}$ is the empty set. Then $\Phi \mathbb{P}_S(R) A \sqcup \mathbb{1}$ holds vacuously because $A \sqcup \mathbb{1}$ is the empty set. Using the definedness axiom of Stage 1 we can find a set $\Psi = \{\psi_a \mid a \in A\}$ with $\psi_a R a$ for all $a \in A$. For the set B we choose the empty set of witnesses $\mathbb{0}$ which also

satisfies $\mathbb{0}\mathbb{P}_S(R)\mathbb{1}$. Now $\Psi \sqcap \mathbb{0}$ is the empty set of witnesses and therefore $\Phi \succ_S \Psi \sqcap \mathbb{0}$ holds vacuously. This proves the join-strength axiom for the case that at least one of A, B is empty. Now suppose that both A and B are not empty. We have to find $\Psi\mathbb{P}_S(R)A$ and $\Theta\mathbb{P}_S(R)B$ such that $\Phi \succ_S \Psi \sqcap \Theta$. Recall that $\Phi\mathbb{P}_S(R)A \sqcup B$ means that for all pairs $(a, b) \in A \times B$ there exists a witness $\phi_{ab} \in \Phi$ with $\phi_{ab}Ra \sqcup b$. Fix a pair (a, b). The join-strength rule for R yields witnesses $\psi_{ab}Ra$ and $\theta_{ab}Rb$ with $\phi_{ab} \succ_S \psi_{ab} \sqcap \theta_{ab}$. Doing this for all pairs (a, b) yields two families $\{\psi_{ab} \,|\, (a, b) \in A \times B\}$ and $\{\theta_{ab} \,|\, (a, b) \in A \times B\}$. By the meet rule of Stage 1 we know that for any $a \in A$ the witness $\psi_a := \bigsqcap_{b \in B} \psi_{ab}$ satisfies $\psi_a Ra$. Likewise, $\theta_b := \bigsqcap_{a \in A} \theta_{ab}$ is a witness with $\theta_b Rb$ for any token $b \in B$. Furthermore, the weakening rule for $\rightarrowtail$ of Stage 1 implies that $\phi_{ab} \succ \psi_a \sqcap \theta_b$ for any pair (a, b), because ψ_a is the meet of ψ_{ab} with some more witnesses and likewise for θ_b. Form two finite sets $\Psi := \{\psi_a \,|\, a \in A\}$ and $\Theta := \{\theta_b \,|\, b \in B\}$. By construction we have $\Psi\mathbb{P}_S(R)A$ and $\Theta\mathbb{P}_S(R)B$. Moreover, the meet $\Psi \sqcap \Theta$ has the description $\{\psi_a \sqcap \theta_b \,|\, (a, b) \in A \times B\}$ and we have convinced ourselves above that any such pair has $\phi_{ab} \succ \psi_a \sqcap \theta_b$ for some $\phi_{ab} \in \Phi$. Therefore $\Phi \succ_S \Psi \sqcap \Theta$ and the proof of the join-strength rule is complete. □

Corollary 1.10.13. *The Smyth poweralgebra of a Stage 3 interaction algebra is a Stage 5 interaction algebra. If R is a Stage 3 morphism, then so is $\mathbb{P}_S(R)$.*

Proof. The only fact which does not follow from Proposition 1.10.12 is that the operations $\sqcap$ and $\sqcup$ on $\mathrm{Fin}\, L_+$ commute. The proof for this uses the same argument that we employed to show that $\sqcap$ and $\sqcup$ on $\mathrm{Fin}\, L_-$ commute. □

The corollary above is part of an even more pleasing fact: The adjunction between the categories Tok_1 and Tok_{2a} restricts to an adjunction between Tok_3 and Tok_4.

Theorem 1.10.14. *The Smyth poweralgebra functor is right adjoint to the inclusion functor $\mathsf{Tok}_3 \hookrightarrow \mathsf{Tok}_4$.*

Proof. Let $\mathcal{L}$ be a Stage 4 interaction algebra and $\mathcal{M}$ be a Stage 3 interaction algebra. We claim that the assignment $(-)^\dagger$ defined in equation (1.6) yields a natural isomorphism $\mathsf{Tok}_4(\mathcal{L}, \mathbb{P}_S\mathcal{M}) \cong \mathsf{Tok}_3(\mathcal{L}, \mathcal{M})$. By Proposition 1.10.12 the left-hand side of this identity is well-defined. To conclude the proof observe that the Stage 3 axioms are independent of those of Stage 2a whence $R^\dagger$ is a Stage 4 morphism precisely when R is a Stage 3 morphism. □

1.11 The Hoare powerdomain

In this section we address the problem of finding a left adjoint to the inclusion functor from continuous lattices to domains. Recall that the Scott topology of a domain is a completely distributive frame and the category of completely distributive frames is invariant under taking order-duals. As every completely distributive frame is in particular a continuous lattice, the lattice of Scott closed subsets of a domain is a continuous lattice. There is an obvious embedding from a domain into the lattice of Scott closed sets: It maps a point x to its principal ideal $\downarrow x$. To see more concretely that the lattice of Scott closed subsets of a domain is continuous, we show that it is isomorphic to the lattice of round lower sets with respect to the way-below relation: If $C \subseteq D$ is a Scott closed subset of a domain D, then the set $\twoheaddownarrow C = \bigcup_{x \in C} \twoheaddownarrow x$ is a round lower set with respect to $\ll$ and clearly C is its Scott closure. Conversely, if $U \in \mathrm{Lo}^{\ll} D$ is a round lower set, then U is certainly contained in the round lower set of the Scott closure of U. To see that the round lower set of the Scott closure of U is contained in U, let $I \subseteq U$ be an ideal and $x \ll \bigsqcup I$. Then $x \in I$ and therefore $x \in U$.

From the information system point of view, the observation we just made is not the least surprising. Since we know that the round upper sets of D with respect to $\ll$ are precisely the Scott opens, the round lower sets must be the Scott opens of the Lawson dual of D. It is easy to check that every Scott closed set $C \subseteq D$ yields a Scott open set of Scott open filters via $\{\phi \in D^\wedge \mid \phi \cap C \neq \emptyset\}$ and furthermore every Scott open set of $D^\wedge$ arises this way (compare Theorem 1.2.4).

A topology on the lattice of Scott closed sets of D is given by the following basis. For any finite set Φ of Scott open sets of D consider the collection of Scott closed sets which intersect every Scott open set in Φ. Notice that a Scott open $\phi \in \Phi$ intersects a Scott closed set C precisely when ϕ intersects $\twoheaddownarrow C$. For two finite collections Φ and Ψ of Scott opens, it is easy to see that the basic open set defined by $\Phi \cup \Psi$ is the intersection of the two basic opens defined by Φ and Ψ.

1.11.1 The Hoare order

It is a common technique in order theory to turn a join into a directed join by first forming finite subsets. For example, every set is the directed union of its finite subsets. Given a domain presented by an abstract basis $(L, \prec)$ one approximates a Scott closed set $C \subseteq D$ by finite subsets $A \subseteq \twoheaddownarrow C \cap L$. These form a round ideal when ordered in the *Hoare order*:

Definition 1.11.1. If $R \subseteq L \times M$ is a binary relation then the *Hoare lifting* of R to the finite powersets of L and M is given as

$$A R_H B :\Leftrightarrow \forall a \in A\, \exists b \in B.\, aRb$$

One checks that the Hoare lifting of relations preserves relational composition and inclusion of relations. If $(L, \prec)$ is an abstract basis for the domain D then $(\mathrm{Fin}\, L, \prec_H)$ is an abstract basis for the domain of Scott closed subsets of D, which is called the *Hoare powerdomain* or lower powerdomain.

It should be mentioned that some authors define the Hoare powerdomain excluding the empty set from the lattice of closed sets. That way one obtains the free inflationary dcpo-semilattice over a domain where the binary operation does not have a neutral element. Writing "non-empty finite" instead of "finite" in what follows, one obtains Stage 2c interaction algebras instead of Stage 3 interaction algebras. The reader may convince himself that the arguments we use below work independently for non-empty finite sets and for the empty set and that statements about the empty set typically require only Stage 2b axioms.

Definition 1.11.2. For a Stage 1 interaction algebra $\mathcal{L}$ define the *Hoare poweralgebra* $\mathbb{P}_H\mathcal{L}$ as follows. Let the witness set be the finite powerset $\mathrm{Fin}\, L_-$ of the witnesses of $\mathcal{L}$. Further let $\mathbb{1}$ denote the empty set in $\mathrm{Fin}\, L_-$. Let the token set of $\mathbb{P}_H\mathcal{L}$ be the finite powerset of the tokens of $\mathcal{L}$, with the symbol $\mathbb{0}$ denoting the empty set of tokens. For finite sets of witnesses Φ, Ψ and finite sets of tokens A, B declare binary operations

$$\begin{aligned} \Phi \sqcap \Psi &:= \Phi \cup \Psi \\ A \sqcup B &:= A \cup B \end{aligned}$$

The relations $\rightarrowtail\!\!\!\leftarrowtail$, $\circ$ and all morphisms R are lifted to relations $\mathbb{P}_H(\rightarrowtail\!\!\!\leftarrowtail)$, $\mathbb{P}_H(\circ)$ and $\mathbb{P}_H(R)$ on poweralgebras by first lifting the relation through the functor Fin and then closing under the rules of Stage 2b and Stage 2c.

We claim that the relation lifting one obtains from Definition 1.11.2 is precisely the Hoare lifting. If Φ is a finite set of witnesses and A is a finite set of tokens and Φ is related to A by the relation lifting of a Stage 1 morphism R, then by the weakening rule for $\sqcup$ of Stage 2c we have $\Phi \mathbb{P}_H(R) A \cup B$ for any other finite set B of tokens. This shows that the Hoare lifting of R is contained in the relation $\mathbb{P}_H(R)$. Conversely, if Φ is related to A by the Hoare lifting of R then let B be the set $\{a \in A \mid \exists \phi \in \Phi.\, \phi R a\}$. Now the set Φ is related to B by the relation lifting $\overline{\mathrm{Fin}(R)} = R_{EM}$ and the weakening rule for $\sqcup$ tells us that $\Phi \mathbb{P}_H(R) A = B \cup A$. Using the same argument, but the weakening rule involving $\rightarrowtail\!\!\!\leftarrowtail$ and $\sqcap$ one shows that the lifting $\mathbb{P}_H(\rightarrowtail\!\!\!\leftarrowtail)$ coincides with the Hoare lifting $\rightarrowtail\!\!\!\leftarrowtail_H$. It remains to check that the Hoare lifting of relations $\circ$, R and $\rightarrowtail\!\!\!\leftarrowtail$ is closed under all other rules of Stage 3.

The following lemma is very helpful because it allows us to transfer results about the Smyth poweralgebra to the Hoare poweralgebra.

Lemma 1.11.1. *If $R : L \to M$ is a binary relation then the inverse of the Smyth lifting of R to $\mathrm{Fin}\, L \times \mathrm{Fin}\, M$ is the Hoare lifting of the inverse of R.*

Proof. Let $A \subseteq L$ and $B \subseteq M$ be finite sets. Then $B(R_S)^{-1}A$ if and only if $\forall b \in B \exists a \in A.\, aRb$ which is clearly the same as saying that $B(R^{-1})_H A$. □

Armed with this Lemma, we can easily derive that the Hoare lifting of a Stage 1 morphism is a Stage 3 morphism. The Hoare lifting is a Stage 0 morphism if and only if its inverse is, and this follows from the Smyth lifting being a Stage 0 morphism. The

Stage 1 axioms of the Hoare lifting are the Stage 2a axioms for the inverse whence this, too, follows from what we know about the Smyth lifting. The Stage 2b axioms holds because every Smyth poweralgebra is compact and the Stage 2c axioms correspond to stable local continuity of the Smyth poweralgebra.

Remark. The Stage 1 axioms for R are not needed to show that the Hoare lifting R_H is a Stage 3 morphism. The Hoare powerdomain functor can be defined on the category Tok_0 instead.

The names $\mathbb{0}$ and $\mathbb{1}$ for the empty sets of tokens and witnesses, respectively, match their algebraic behaviour, as $\mathbb{0}$ is a neutral element for $\sqcup$ and $\mathbb{1}$ is a neutral element for $\sqcap$. Similar to the Smyth lifting of section 1.10 one finds that whenever a pair (f_-, f_+) is a token map between Stage 1 interaction algebras $\mathcal{L}$ and $\mathcal{M}$ then the pair $(\mathrm{Fin}(f_-), \mathrm{Fin}(f_+))$ is a token map between $\mathbb{P}_H\mathcal{L}$ and $\mathbb{P}_H\mathcal{M}$. We arrive at:

Theorem 1.11.2. *The Hoare poweralgebra $\mathbb{P}_H$ is a functor from Tok_1 into the subcategory of Stage 3 interaction algebras. It preserves token maps and adjoint pairs. If $D = \mathrm{Idl}^{\prec} L_+$ is the domain presented by an interaction algebra $\mathcal{L}$ then the interaction algebra $\mathbb{P}_H\mathcal{L}$ presents the Hoare powerdomain of D.*

Another application of Lemma 1.11.1 is the following.

Corollary 1.11.3. *For any interaction algebra $\mathcal{L}$, the round ideals of tokens of the Hoare poweralgebra $\mathbb{P}_H\mathcal{L}$ are in order-preserving bijection with the round ideals of witnesses of the Smyth poweralgebra of* $\mathrm{Flip}\,\mathcal{L}$.

Proof. The functor Flip sends all relations to their inverse and swaps tokens with witnesses. □

We just gave an almost trivial proof for the fact that the Lawson dual of the Hoare powerdomain of a domain is the Smyth powerdomain of the Lawson dual of the domain. In short $(\mathbb{P}_H D)^\wedge \cong \mathbb{P}_S(D^\wedge)$. This is not too surprising if we recall that a similar commutative law of functors holds for the Scott topology of domains: From Theorem 1.3.4 we deduced that $(\sigma D)^\partial = \sigma(D^\wedge)$.

The following is the interaction algebra proof for the fact that the Hoare powerdomain of a domain is isomorphic to the lattice of Scott closed subsets [22, Corollary IV-8.6].

Proposition 1.11.4. *For any Stage 1 interaction algebra $\mathcal{L}$ the following domains are isomorphic.*

1. *The domain of round ideals of tokens of $\mathbb{P}_H\mathcal{L}$,*
2. *The domain of round lower sets of tokens of $\mathcal{L}$.*

Consequently, the domain presented by $\mathbb{P}_H\mathcal{L}$ is the domain of Scott closed sets of the domain presented by $\mathcal{L}$.

Proof. Apply Lemma 1.11.1 to Proposition 1.10.2. □

1.11.2 The Hoare powerdomain monad

Co-unit and co-multiplication

One result we can transfer immediately from the Smyth poweralgebra is that the singleton and union maps extend to token maps $\mathcal{L} \to \mathbb{P}_H\mathcal{L}$ and $\mathbb{P}_H^2\mathcal{L} \to \mathbb{P}_H\mathcal{L}$. We have $\phi{\frown}a$ if and only if $\{\phi\}{\frown}_H\{a\}$ and $a{\succ\!\!\prec}\phi$ if and only if $\{a\}{\succ\!\!\prec}_H\{\phi\}$. The singleton map on witnesses satisfies the continuity condition of Lemma 1.1.13 because the singleton map on the tokens does for the Smyth poweralgebra. Not derivable from section 1.10 is the fact that the singleton map on tokens satisfies the continuity condition, too. If $\Phi{\frown}_H\{a\}$ then by definition $\phi{\frown}a$ for all $\phi \in \Phi$. In case that $\Phi = \mathbb{1}$ is the empty set, we can just use definedness to obtain $b \prec a$ with $\mathbb{1}{\frown}_H\{b\}$. Otherwise use the meet rule of Stage 1 to obtain $\bigsqcap\Phi{\frown}a$ and deduce $\bigsqcap\Phi{\frown}b \prec a$. Hence $\Phi{\frown}_H\{b\} \prec_H \{a\}$ and so both singleton maps satisfy the continuity condition of Lemma 1.1.13. For every Stage 1 interaction algebra the token map $(\{-\},\{-\})$ yields an adjoint pair $E_\mathcal{L} \dashv E_\mathcal{L}^-$ where

$$\begin{aligned}
\Phi E_\mathcal{L} a &\Leftrightarrow \Phi{\frown}_H\{a\} \\
&\Leftrightarrow \forall\phi \in \Phi.\phi{\frown}a \\
\phi E_\mathcal{L}^- A &\Leftrightarrow \{\phi\}{\frown}_H A \\
&\Leftrightarrow \exists a \in A.\, \phi{\frown}a.
\end{aligned}$$

Notice that for all non-empty sets of witnesses Φ we can write $\Phi E_\mathcal{L} a$ iff $\bigsqcap\Phi{\frown}a$. We show that the left adjoint $E_\mathcal{L}$ extends to a natural transformation from $\mathbb{P}_H$ to the identity (The corresponding argument for the Smyth poweralgebra shows that the right adjoint, too, is a natural transformation, but it is only a Stage 0 morphism in general). Given a Stage 1 morphism $R : \mathcal{L} \to \mathcal{M}$, Corollary 1.1.14 tells us that the relation $\Phi(\mathbb{P}_H(R) ⨾ E_\mathcal{M})a$ holds if and only if $\forall\phi \in \Phi.\, \phi Ra$. The relation $\Phi(E_\mathcal{L} ⨾ R)a$ holds whenever there is a witness ψ with $\Phi \succ_H \{\phi\}$ and ϕRa. Clearly the inclusion $(E_\mathcal{L} ⨾ R) \subseteq (\mathbb{P}_H(R) ⨾ E_\mathcal{M})$ holds. For the reverse inclusion, (assuming the non-trivial case where Φ is not empty) we deduce $\bigsqcap\Phi Ra$ from $\Phi\mathbb{P}_H(R)\{a\}$ and then use $\succ; R = R$ to get a witness ψ which satisfies $\Phi \succ_H \{\psi\}$ and ψRa.

Now let us turn attention towards the token map derived from the multiplication $\bigcup$ of the finitepowerset monad on Set. Just as in Lemma 1.10.3 we internalise the union maps as algebraic operations on an interaction algebra. For a Stage 3 interaction algebra we use the conventions $\bigsqcap\{\phi\} = \phi$ and $\bigsqcap\emptyset = 1$ for finite sets of witnesses and $\bigsqcup\{a\} = a$, $\bigsqcup\emptyset = 0$ for finite sets of tokens. First we prove an extension of Lemma 1.5.1 (1).

Lemma 1.11.5. *For all Stage 3 morphisms R (in particular for ${\frown}$) between Stage 3 interaction algebras, for all finite sets of witnesses Φ and all finite sets of tokens A the following rules hold.*

$$\frac{\Phi R_H A}{\bigsqcap\Phi R \bigsqcup A} \qquad \frac{A{\succ\!\!\prec}_H\Phi}{\bigsqcup A{\succ\!\!\prec}\bigsqcap\Phi}$$

Proof. We need to perform case analysis on the cardinality of the sets Φ and A. Suppose $\Phi = \mathbb{1}$ is the empty set. Then the premise of the first rule is trivial and the consequence $1R\bigsqcup A$ is the empty meet rule of Stage 2b. In case $\Phi = \{\phi\}$ is a singleton, the premise reads $\exists a \in A.\, \phi R a$. The weakening rule for Stage 2c then yields $\phi R \bigsqcup A$. In all other cases, $\Phi R_H A$ implies that for some subset $A' \subseteq A$ the stronger relation $\Phi R_{EM} A'$ holds. Then apply Lemma 1.5.1 (1) to obtain $\bigsqcap \Phi R \bigsqcup A'$ and with the weakening rule of Stage 2c finally $\bigsqcap \Phi R \bigsqcup A$.

Now consider the second rule. Again, in case $A = \mathbb{0}$ is the empty set, the premise of the second rule is trivial and the consequence $0 \rightarrowtail \bigsqcap \Phi$ follows from the Stage 2b axioms. In case A is a singleton, the rule is the same as the weakening rule for $\sqcap$ of Stage 1. In all other cases use the same trick as for the first rule above and employ Lemma 1.5.1 (1). □

Lemma 1.11.6. *For any Stage 3 interaction algebra $\mathcal{L}$ the pair of maps $(\bigsqcap, \bigsqcup) : \mathbb{P}_H \mathcal{L} \to \mathcal{L}$ is a token map. It constitutes a natural transformation from the identity on Tok_3 to $\mathbb{P}_H$.*

Proof. Let $\mathcal{L}$ be a Stage 3 interaction algebra, The fact that the pair $(\bigsqcap, \bigsqcup)$ is a token map is a consequence of Lemma 1.11.5. We claim that both $\bigsqcap$ and $\bigsqcup$ satisfy the continuity condition of Lemma 1.1.13. For the map $\bigsqcap : \mathrm{Fin}\, L_- \to L_-$ the proof is dual to the argument we used on the map $\bigsqcap : \mathrm{Fin}\, L_+ \to L_+$ in the proof of Lemma 1.10.3. Next suppose $\phi \circ \bigsqcup A$ where $A \in \mathrm{Fin}\, L_+$. In case $A = \mathbb{0}$ there is nothing to show because $\mathbb{0} \prec_H \mathbb{0}$. In case that $A = \{a\}$ is a singleton, simply use the interpolative law for $\circ$ and get $\phi \circ b = \bigsqcup \{b\}$ for some $\{b\} \prec_H \{a\}$. For the remaining cases, let us assume for simplicity that $A = \{a, b\}$ is a two-element set. Then $\phi \circ \bigsqcup A$ means $\phi \circ a \sqcup b$. Apply the join-strength rule of Stage 2c and obtain witnesses $\psi \circ a$, $\theta \circ b$ satisfying $\phi \succ \psi \sqcap \theta$. Then also $\psi \circ a' \prec a$ and $\theta \circ b' \prec b$ for some tokens a' and b'. With Lemma 1.5.1 (1) we obtain $\phi \succ \psi \sqcap \theta \circ a' \sqcup b'$. Thus we have $\phi \circ \bigsqcup \{a', b'\} \prec_H \{a, b\}$ and thereby the continuity condition of Lemma 1.1.13. The Stage 1 morphism $S_{\mathcal{L}} : L_- \to \mathrm{Fin}\, L_+$ corresponding to the structure map $\bigsqcup$ therefore has the characterisation $\phi S_{\mathcal{L}} A \Leftrightarrow \phi \circ \bigsqcup A$.

It remains to show that $S_{\mathcal{L}}$ is natural in the parameter $\mathcal{L}$. Let $R : \mathcal{L} \to \mathcal{M}$ be a Stage 3 morphism. From Corollary 1.1.14 we know that $\phi(R \fatsemi S_{\mathcal{M}})A$ precisely when $\phi R \bigsqcup A$, and $\phi(S_{\mathcal{L}} \fatsemi \mathbb{P}_H(R))A$ holds if and only if $\phi \succ \bigsqcap \Psi$ and $\Psi \mathbb{P}_H(R) A$ for some $\Psi \in \mathrm{Fin}\, L_-$. Notice that by Lemma 1.11.5 the relation $\Psi \mathbb{P}_S(R) A$ implies $\bigsqcap \Psi R \bigsqcup A$ whence the relation $(S_{\mathcal{L}} \fatsemi \mathbb{P}_S(R))$ is contained in $(R \fatsemi S_{\mathcal{M}})A$. The reverse inclusion is precisely the join-strength rule for R. □

Corollary 1.11.7. *The pair of union maps $\bigcup : \mathrm{Fin}^2\, L_- \to \mathrm{Fin}\, L_-$ and $\bigcup : \mathrm{Fin}^2\, L_+ \to \mathrm{Fin}\, L_+$ form a token map for any Stage 1 interaction algebra $\mathcal{L}$ natural in the parameter $\mathcal{L}$. This yields a natural transformation in Tok_1 from $\mathbb{P}_H$ to $(\mathbb{P}_H)^2$.*

As with the Smyth poweralgebra comonad, we get the unit and associative law for free, because the token maps of the co-unit and co-multiplication are derived from the unit and multiplication of the finite-powerset monad on Set.

Theorem 1.11.8. *The Hoare poweralgebra functor on* Tok_1 *extends to a comonad where the co-unit and co-multiplication are given by token maps* $(\{-\},\{-\})$ *and* $(\bigcup,\bigcup)$. *In particular, the Hoare poweralgebra functor restricts to a monad on the category of Stage 1 interaction algebras and token maps.*

Corollary 1.11.9. *The Hoare powerdomain monad on the category of domains has semi-open unit and multiplication maps and preserves semi-open maps.*

1.11.3 Algebras for the Hoare powerdomain monad

Just as the Eilenberg-Moore coalgebras for the Smyth poweralgebra comonad are precisely the Stage 2a interaction algebras, we show that the coalgebras for the Hoare poweralgebra comonad are precisely the Stage 3 interaction algebras. For the category Dom this means that the only Hoare powerdomain algebra maps are the join operations of continuous lattices. Again, such a structure map $\bigvee$ is adjoint to the unit: For any Scott closed set C and any point x we have $\bigvee C \leq x \Leftrightarrow C \subseteq {\downarrow} x$. In terms of interaction algebras, this equivalence becomes manifest in the following lemma.

Lemma 1.11.10. *If* $\mathcal{L}$ *is a Stage 3 interaction algebra, then the structure map* $S_{\mathcal{L}}$ *is left adjoint to the co-unit morphism* $E_{\mathcal{L}}$.

Proof. Recall from Lemma 1.11.6 that the structure map $S_{\mathcal{L}}$ derived from $\bigsqcup$ has the characterisation $\phi S_{\mathcal{L}} A$ iff $\phi \frown \bigsqcup A$ whereas the co-unit morphism is given as $\Phi E_{\mathcal{L}} a$ iff $\bigsqcap \Phi \frown a$. First we show $\frown_H \subseteq E_{\mathcal{L}} \mathbin{;} S_{\mathcal{L}}$. From the first rule of Lemma 1.11.5 we know that $\Phi \frown_H A$ implies $\bigsqcap \Phi \frown \bigsqcup A$. Use the interpolative law for $\frown$ and get $\bigsqcap \Phi \frown ; \succ\!\!\prec ; \frown \bigsqcup A$ which means $\Phi (E_{\mathcal{L}} \mathbin{;} S_{\mathcal{L}}) A$.

For the inclusion $S_{\mathcal{L}} \mathbin{;} E_{\mathcal{L}} \subseteq \frown$ suppose $\phi \frown \bigsqcup B$, $B \succ\!\!\prec_H \Psi$ and $\bigsqcap \Psi \frown a$. Use the second rule of Lemma 1.11.5 and get $\phi \frown \bigsqcup B \succ\!\!\prec \bigsqcap \Psi \frown a$, whence $\phi \frown a$ holds. Notice that in fact $S_{\mathcal{L}} \mathbin{;} E_{\mathcal{L}} = \frown$ because $\phi \frown a$ implies $\phi S_{\mathcal{L}} \{b\} \succ\!\!\prec_H \{\psi\} E_{\mathcal{L}} a$ for suitable b and ψ. □

Lemma 1.11.11.

1. *If* $\mathcal{L}$ *is a Stage 3 interaction algebra, then the morphism* $S_{\mathcal{L}}$ *derived from the structure map* $\bigsqcup$ *is a coalgebra for the Hoare poweralgebra comonad.*

2. *Those Stage 1 morphisms between Stage 3 interaction algebras which are Hoare poweralgebra morphisms are precisely the Stage 3 morphisms.*

Proof. (1) Let $\mathcal{L}$ be a Stage 3 interaction algebra and $\phi S_{\mathcal{L}} A$ iff $\phi \frown \bigsqcup A$. We have to verify that the diagrams (1.5) in the proof of Lemma 1.10.8 commute, where $\mathbb{P}_S$ is replaced by $\mathbb{P}_H$ and the definitions of the morphisms $S_{\mathcal{L}}$, $E_{\mathcal{L}}$ and $U_{\mathcal{L}}$ are adjusted accordingly. The unit law was verified in the preceding lemma. The associative law, again, is just associativity of the operation $\sqcup$ on tokens: Both the multiplication $U_{\mathcal{L}}$ and the structure map $S_{\mathcal{L}}$ are given by token maps, whence it suffices to check that the square below commutes.

$$\begin{array}{ccc} L_+ & \xleftarrow{\ \bigsqcup\ } & \operatorname{Fin} L_+ \\ {\scriptstyle \bigsqcup}\uparrow & & \uparrow{\scriptstyle \bigcup} \\ \operatorname{Fin} L_+ & \xleftarrow[\operatorname{Fin}(\bigsqcup)]{} & \operatorname{Fin}^2 L_+ \end{array}$$

(2) Let $R : \mathcal{L} \to \mathcal{M}$ be a Stage 3 morphism between Stage 3 interaction algebras. We show that R is a $\mathbb{P}_H$-coalgebra morphism, meaning $(S_\mathcal{L} \mathbin{⨾} \mathbb{P}_H(R)) = (R \mathbin{⨾} S_\mathcal{M})$. This identity holds because of naturality of $S_\mathcal{L}$ which we proved in Lemma 1.11.6.

Next we prove the converse: If $R : \mathcal{L} \to \mathcal{M}$ is a Stage 1 morphism which satisfies $(S_\mathcal{L} \mathbin{⨾} \mathbb{P}_H(R)) = (R \mathbin{⨾} S_\mathcal{M})$ then it satisfies all Stage 3 rules. For the empty meet rule of Stage 2b, let $a \in M_+$. Since $\mathcal{L}$ is a Stage 3 interaction algebra, we have $1 \mathrel{\bigcirc} 0$ in $\mathcal{L}$. Recall that $0 = \bigsqcup \mathbb{0}$ and observe that $\mathbb{1} R_H \{a\}$ trivially holds. Therefore $1 E_\mathcal{L} \mathbb{0} \mathrel{\succ\!\prec}_H \mathbb{1} \mathbb{P}_H(R) \{a\}$ and so by hypothesis $1Ra = \bigsqcup \{a\}$. For the strictness axiom of Stage 2b, suppose $\phi R 0$. Since $0 = \bigsqcup \mathbb{0}$ the hypothesis yields $1 E_\mathcal{L} B \mathrel{\succ\!\prec}_H \Psi \mathbb{P}_H(R) \mathbb{0}$ for some sets B and Ψ. Notice that $\Psi \mathbb{P}_H(R) \mathbb{0}$ can only be true if Ψ is empty, which in turn implies that B must be empty. Then $\phi \mathrel{\bigcirc} \bigsqcup \emptyset$ which gives the desired $\phi \mathrel{\bigcirc} 0$. The weakening rule for R of Stage 2c follows from the fact that $\mathcal{M}$ is a Stage 3 interaction algebra: If $\phi R a$ then $\phi (R \mathbin{⨾} \mathrel{\bigcirc}) a$. Now use the weakening rule for $\mathrel{\bigcirc}$. For the join-strength rule of Stage 2c, suppose $\phi R a_1 \sqcup a_2$. By hypothesis we have $\phi \mathrel{\bigcirc} \bigsqcup B$, $B \mathrel{\succ\!\prec}_H \Psi$ and $\Psi \mathbb{P}_H(R) \{a_1, a_2\}$ for some finite sets B and Ψ. For the rest of this proof, let i range over $\{1, 2\}$. Write $\Psi = \Psi_1 \sqcap \Psi_2$ where $\Psi_i = \{\psi \in \Psi \mid \psi R a_i\}$ (recall that $\sqcap$ is set union). We assumed that R is a Stage 1 morphism, so by the definedness rule there exist witnesses θ_1, θ_2 with $\theta_i R a_i$ for all i. By the meet rule of Stage 1 we have $\bigsqcap \Psi_i \sqcap \theta_i R a_i$ for all i. Further observe that $\phi \mathrel{\bigcirc} \bigsqcup B \mathrel{\succ\!\prec} \bigsqcap \Psi$ whence by the weakening rule also $\phi \succ (\bigsqcap \Psi_1 \sqcap \theta_1) \sqcap (\bigsqcap \Psi_2 \sqcap \theta_2)$. This concludes the proof that R satisfies the join-strength rule. □

The last ingredient for our characterisation of Tok_3 is to show that coalgebras for the Hoare poweralgebra comonad on Tok_1 all correspond to Stage 3 interaction algebras.

Lemma 1.11.12. *If $R : \mathcal{L} \to \mathbb{P}_H \mathcal{L}$ is a coalgebra for the Hoare powerdomain comonad then $\mathcal{L}$ presents a continuous lattice. Moreover, the morphism R is unique because it is left adjoint to the co-unit morphism $E_\mathcal{L}$.*

Proof. Let $R : \mathcal{L} \to \mathbb{P}_H \mathcal{L}$ be a coalgebra for the Hoare poweralgebra comonad. This means that R satisfies the unit law $\mathrel{\bigcirc} = (R \mathbin{⨾} E_\mathcal{L})$ and the associative law $(R \mathbin{⨾} U_\mathcal{L}) = (R \mathbin{⨾} \mathbb{P}_H(R))$. The former identity tells us that $\phi \mathrel{\bigcirc} a$ if and only if $\phi R \{a\}$ whereas the latter says that for any witness ϕ and any $\mathbb{A} \in \operatorname{Fin}^2 L_+$ we have $\phi R \bigcup \mathbb{A}$ if and only if for some $B \in \operatorname{Fin} L_+$ and $\Psi \in \operatorname{Fin} L_-$ the relations $\phi R B \mathrel{\succ\!\prec}_H \Psi \mathbb{P}_H(R) \mathbb{A}$ hold.

We show that $\mathrel{\bigcirc}_H \subseteq (E_\mathcal{L} \mathbin{⨾} R)$. For this, suppose $\Phi \mathrel{\bigcirc}_H A$. First consider the case where $\Phi = \mathbb{1}$ is the empty set. From the definedness axiom for R and $\succ ; R = R$ we know that for some witnesses ϕ, ϕ' and some token a we have $\phi \frown a \succ\!\!\prec \phi' RA$. The relation $\mathbb{1} E_{\mathcal{L}} a$ trivially holds, whereby $\mathbb{1}(E_{\mathcal{L}} \mathbin{;} R)A$. In particular this argument applies to $A = \mathbb{0}$. Here we get a token $0 \in L_+$ and a witness $1 \in L_-$ satisfying $\mathbb{1} E_{\mathcal{L}} 0 \succ\!\!\prec 1 R \mathbb{0}$. Since $\mathbb{0} \prec_H \{a\}$ for any token a, the relation $1 R \mathbb{0}$ together with the unit law implies that $1 \frown a$ for every token $a \in L_+$. Thus $\mathcal{L}$ is a Stage 2b interaction algebra and presents a domain with bottom. We use the convention $\bigsqcap \emptyset := 1$ to extend the binary operation $\sqcap$ on witnesses to finite arities. Next consider the case where $\Phi \frown_H A$ and both Φ and A are non-empty.

$$\begin{aligned}
\forall \phi \in \Phi \exists a \in A.\, \phi \frown a \quad &\Rightarrow \quad \forall \phi \in \Phi \exists a \in A.\, \phi R\{a\} \text{ by the unit law} \\
&\Rightarrow \quad \forall \phi \in \Phi \exists a \in A.\, \phi(R \mathbin{;} \frown_H)\{a\} \\
&\Rightarrow \quad \forall \phi \in \Phi.\, \phi RA \text{ by the weakening rule for } \frown_H \\
&\Rightarrow \quad \bigsqcap \Phi RA \text{ by the meet rule of Stage 1} \\
&\Rightarrow \quad \bigsqcap \Phi(\frown; \succ\!\!\prec; R)A \\
&\Rightarrow \quad \Phi(E_{\mathcal{L}} \mathbin{;} R)A.
\end{aligned}$$

We have shown the inclusion $\frown_H \subseteq (E_{\mathcal{L}} \mathbin{;} R)$. Together with the unit law we now know that R is left adjoint to the co-unit morphism $E_{\mathcal{L}}$. This establishes uniqueness of R.

Next we show that the Stage 1 interaction algebra $\mathcal{L}$ is isomorphic to a Stage 3 interaction algebra $\mathcal{L}_\diamond$. Intuitively, the relation R tells us when the join of a finite set of points is contained in a basic Scott open set. Let the witnesses of the interaction algebra $\mathcal{L}_\diamond$ be the witnesses of $\mathcal{L}$. As tokens are of $\mathcal{L}_\diamond$ take the finite subsets of the token set L_+. We interpret a finite set $A \subseteq L_+$ as its join, so we take as the relation $\frown$ of $\mathcal{L}_\diamond$ the relation R. The join of a finite set A is a lower bound of a basic Scott open if and only if all elements $a \in A$ are lower bounds of the Scott open set. Hence we define a relation $\succ\!\!\prec_\diamond : \operatorname{Fin} L_+ \to L_-$ by $A \succ\!\!\prec_\diamond \phi \Leftrightarrow \forall a \in A.\, a \succ\!\!\prec \phi \Leftrightarrow A \succ\!\!\prec_H \{\phi\}$. We claim that the interaction algebra $\mathcal{L}_\diamond = (L_-, \operatorname{Fin} L_+, \succ\!\!\prec_\diamond, R)$ satisfies all Stage 3 axioms and is isomorphic to $\mathcal{L}$.

Stage 0: Let ϕRA. Since R is a Stage 1 morphism from $\mathcal{L}$ to $\mathbb{P}_H \mathcal{L}$ we have $\phi \frown b \succ\!\!\prec \psi RA$ for some token b and witness ψ. By the unit law $\phi R\{b\} \succ\!\!\prec_\diamond \psi RA$, whence $R \subseteq (R \mathbin{;} R)$ in $\mathcal{L}_\diamond$. For the reverse inclusion, suppose $\phi RB \succ\!\!\prec_\diamond \psi RA$. Again use the identity $\prec; R = R$ and get $\phi RB \succ\!\!\prec_\diamond \psi \frown x \succ\!\!\prec \psi' RA$. Observe that now $\phi(R; \succ\!\!\prec_H; E_{\mathcal{L}})x$ whence we may apply the inclusion $R \mathbin{;} E_{\mathcal{L}} \subseteq \frown$ and obtain $\phi \frown x \succ\!\!\prec \psi' RA$ and thus ϕRA. Therefore $\mathcal{L}_\diamond$ is a Stage 0 interaction algebra.

Stage 1: The binary operation $\sqcap$ on witnesses of $\mathcal{L}_\diamond$ is the same as in $\mathcal{L}$. The definedness rule and meet rule of Stage 1 hold because R is a Stage 1 morphism.

Stage 2b: Above we already showed the existence of a witness $1 \in L_-$ with $1RA$ for all finite sets $A \subseteq L_+$. Together with the empty set $\mathbb{0} \in \operatorname{Fin} L_+$ which satisfies $\mathbb{0} \succ\!\!\prec_\diamond \phi$ for every witness ϕ, we know that $\mathcal{L}_\diamond$ is a Stage 2b interaction algebra.

Stage 2c: The Stage 2c weakening rule for the relation $\succ\!\!\prec_\diamond$ is trivial: If $a \succ\!\!\prec \phi$ for all $a \in A$ and $b \succ\!\!\prec \phi$ for all $b \in B$ then certainly $x \succ\!\!\prec \phi$ for all tokens $x \in A \cup B$. For the other weakening rule, suppose ϕRA and $B \in \mathrm{Fin}\, L_+$. Use $R = R; \prec_H$ and the weakening rule of the Stage 3 interaction algebra $\mathbb{P}_H\mathcal{L}$ to obtain $\phi R; \prec_H A \cup B$ and thereby $\phi RA \cup B$. Next we prove the join-strength rule. Suppose $\phi RA_1 \sqcup A_2$. With $\mathbb{A} = \{A_1, A_2\}$ we have $\phi(R \mathbin{⨟} U_{\mathcal{L}})\mathbb{A}$ which by the associative law implies $\phi RB \succ\!\!\prec_H \Psi \mathbb{P}_H(R)\mathbb{A}$ for some finite sets B and Ψ. For $i \in \{1,2\}$ let $\Psi_i = \{\psi \in \Psi \mid \psi RA_i\}$. By the Stage 1 meet rule we have $\bigsqcap \Psi_i RA_i$. Abbreviate $\psi_i = \bigsqcap \Psi_i$. Successive application of the Stage 1 weakening rule for $\succ\!\!\prec$ yields $\forall b \in B.\, b \succ\!\!\prec \psi_1 \sqcap \psi_2$. Therefore $\phi(R; \succ\!\!\prec_\diamond)\psi_1 \sqcap \psi_2$ and $\psi_i RA_i$. This finishes the proof that $\mathcal{L}_\diamond$ is a Stage 3 interaction algebra.

Finally we establish an isomorphism between $\mathcal{L}$ and $\mathcal{L}_\diamond$. Observe that the relation R can be considered as a morphism $\mathcal{L} \to \mathcal{L}_\diamond$. Likewise, $\circ$ serves as a morphism $\mathcal{L}_\diamond \to \mathcal{L}$. The identity $R; \succ\!\!\prec_\diamond; \circ = \circ$ holds because of $R; \succ\!\!\prec_H; E_{\mathcal{L}} = \circ$. The identity $\circ; \succ\!\!\prec; R = R$ holds because R is in particular a Stage 0 morphism. $\square$

Combining Lemma 1.11.11 and Lemma 1.11.12 yields:

Theorem 1.11.13. *Every Stage 1 interaction algebra admits at most one coalgebra for the Hoare poweralgebra comonad. The subcategory of Stage 3 interaction algebras is equivalent to the Eilenberg-Moore category of the Hoare poweralgebra comonad.*

Corollary 1.11.14. *The Eilenberg-Moore algebras for the Hoare powerdomain monad on* Dom *are precisely the continuous lattices. Moreover, the join operation on Scott closed subsets of a continuous lattice is a semi-open map.*

(Compare with [47, Lemma 4.3, Theorem 6.6].)

We have not used the binary operation $\sqcap$ on witnesses in the definition of the Hoare poweralgebra of a Stage 1 interaction algebra. One might lift this binary meet element-wise to finite sets of witnesses. The corresponding operation on the Hoare powerdomain of a continuous lattice is the point-wise join of Scott closed sets. It generalises the binary join from points to arbitrary Scott closed sets.

Once more, general category theory tells us that the Hoare powerdomain functor is left adjoint to the inclusion functor $\mathsf{CCL} \hookrightarrow \mathsf{Dom}$ and thus the Hoare powerdomain is the free continuous lattice of a domain. Dually the Hoare poweralgebra functor is the right adjoint to the inclusion functor $\mathsf{Tok}_3 \hookrightarrow \mathsf{Tok}_1$. Consequently we have a natural isomorphism of hom-sets $\mathsf{Tok}_3(\mathcal{L}, \mathbb{P}_S\mathcal{M}) \cong \mathsf{Tok}_1(\mathcal{L}, \mathcal{M})$ resulting from the same categorical construction as for the Smyth poweralgebra functor: If $R : L_- \to M_+$ is a Stage 1 morphism from a Stage 3 interaction algebra $\mathcal{L}$ to a Stage 1 interaction algebra $\mathcal{M}$ then let $R^\dagger$ be the composition of the structure map $S_{\mathcal{L}}$ with $\mathbb{P}_H(R)$. Concretely the resulting relation is given as

$$\phi R^\dagger A \text{ iff } \exists \Psi \in \mathrm{Fin}\, L_-.\, \phi \succ \bigsqcap \Psi,\ \Psi R_H A. \tag{1.7}$$

Notice that $\phi R^\dagger\{a\}$ iff ϕRa.

1.11.4 Hoare powerdomains at other stages

Just as the Smyth poweralgebra functor restricts to the subcategories Tok_{2b} and Tok_{2c}, the Hoare poweralgebra functor enjoys similar preservation properties (Compare with [25, Theorems 5.1,6.1]):

Proposition 1.11.15. *1. The Hoare poweralgebra of a Stage 2a interaction algebra is a Stage 2a interaction algebra. If R is a Stage 2a morphism, then so is $\mathbb{P}_H(R)$.*

2. The Hoare poweralgebra functor preserves stable local continuity.

3. The Hoare poweralgebra functor preserves compactness.

Proof. (1) We lift a binary operation $\sqcap$ on tokens to a binary operation on finite sets of tokens element-wise: Define $A \sqcap B = \{a \sqcap b \,|\, (a, b) \in A \times B\}$. Let R be a Stage 2a morphism. Using the Stage 2a rules for R, one shows that $\Phi R_H A \sqcap B$ holds precisely when ΦRA and ΦRB. Likewise, the weakening rule for $\succ\!\!\prec_H$ and $\sqcap$ follows the corresponding weakening rule for $\succ\!\!\prec$. In particular the above applies to $R = \circ$ whence the Hoare poweralgebra functor restricts to the subcategory Tok_{2a}.

(2) One lifts the binary operation $\sqcup$ on witnesses element-wise to $\mathrm{Fin}\, L_-$. The stable continuity rules of Table 1.3 lift to the Hoare poweralgebra in a straightforward manner.

(3) If a Stage 2a interaction algebra $\mathcal{L}$ has a witness 0 with $a \succ\!\!\prec 0$ for all tokens $a \in L_+$, then the witness $\mathbb{0} := \{0\} \in \mathrm{Fin}\, L_-$ has $A \succ\!\!\prec_H \mathbb{0}$ for all $A \in \mathrm{Fin}\, L_+$. Moreover, the token 1 with $0 \circ 1 \succ\!\!\prec 0$ lifts to a token $\mathbb{1} = \{1\}$ of the Hoare poweralgebra with the property $\mathbb{0} \circ_H \mathbb{1} \succ\!\!\prec_H \mathbb{0}$. Thus the functor $\mathbb{P}_H$ preserves compactness of Stage 2a interaction algebras. □

Corollary 1.11.16. *The Hoare poweralgebra of a Stage 2a interaction algebra is a Stage 4 interaction algebra.*

Proof. It is easy to show that the element-wise meet operation $\sqcap$ on finite sets of tokens distributes over set union. □

Theorem 1.11.17. *The Hoare poweralgebra functor is right adjoint to the inclusion functor* $\mathsf{Tok}_{2a} \hookrightarrow \mathsf{Tok}_4$.

Proof. Let $\mathcal{L}$ be a Stage 4 interaction algebra and $\mathcal{M}$ be a Stage 2a interaction algebra. We claim that the natural isomorphism $(-)^\dagger$ defined in equation (1.7) yields a natural isomorphism $\mathsf{Tok}_4(\mathcal{L}, \mathbb{P}_H\mathcal{M}) \cong \mathsf{Tok}_{2a}(\mathcal{L}, \mathcal{M})$. By the previous corollary the left-hand side of this identity is well-defined. To conclude the proof, observe that the Stage 2a axioms are independent of those of Stage 3 whence $R^\dagger$ is a Stage 4 morphism precisely when R is a Stage 3 morphism. □

1.12 The double powerdomain

Having studied the Smyth and Hoare powerdomains, we put both together in a neat result, proved in greater generality for arbitrary locales by Johnstone and Vickers in [31] and for arbitrary dcpos by Heckmann [25]: The Smyth and Hoare powerdomain functors commute, giving rise to another monad on the category Dom. In fact we already are in possession of all the necessary jigsaw pieces. According to Theorems 1.10.14 and 1.11.17 the Smyth powerdomain provides the free continuous frame over a continuous lattice and the Hoare powerdomain gives the free continuous frame over a continuous preframe. Thus, given any domain, one can construct a (stably) continuous frame out of it in two ways: As the Hoare powerdomain of the Smyth powerdomain or as the Smyth powerdomain of the Hoare powerdomain. Left adjoint functors compose, whence both composite functors construct the free frame over a domain (compare [25]). But left adjoints are unique up to isomorphism, so for general categorical reasons these functors must be naturally isomorphic.

1.12.1 Choices and the finite distributive law

It is an interesting exercise to see that at the core of this commutative law of functors is a familiar commutative law of semilattices. The reason is that while in one interaction algebra a finite set of finite sets represents a meet of joins, in the other interaction algebra a finite set of finite sets stands for a join of meets. Therefore an isomorphism between the two interaction algebras $\mathbb{P}_H\mathbb{P}_S\mathcal{L}$ and $\mathbb{P}_S\mathbb{P}_H\mathcal{L}$ must use the finite distributive law to transform a meet of joins into a join of meets. In a bounded distributive lattice this goes as follows. Let $\mathbb{A}$ be a finite set of finite subsets of a bounded distributive lattice. The lattice element $\bigwedge\{\bigvee A \mid A \in \mathbb{A}\}$ can be written as the join of meets

$$\bigvee\left\{\bigwedge \gamma(\mathbb{A}) \,\middle|\, \gamma : \mathbb{A} \to \bigcup \mathbb{A} \text{ choice function}\right\}.$$

Vickers notes in [57] that the formula for the distributive law remains true if one allows multi-valued choice functions. While this makes no difference classically, the use of choice relations rather than choice functions renders his argument valid in any topos. One can write the join of meets above as

$$\bigvee\left\{\bigwedge C \,\middle|\, C \in \mathrm{Ch}(\mathbb{A})\right\}$$

where Ch is the choice operator defined below[3].

Definition 1.12.1. For any set X define the *choice operator* $\mathrm{Ch} : \mathrm{Fin}^2 X \to \mathrm{Fin}^2 X$ which sends a finite set $\mathbb{A}$ of finite subsets of X to the collection of its choices, that are sets $C \in \bigcup \mathbb{A}$ with the property that for every $A \in \mathbb{A}$ there exists some $x \in A \cap C$.

[3] The choice operator could be defined on the full double powerset, but we do not need this full generality. It is easy to show that the general construction restricts to finite sets.

One finds that in the interaction algebra $\mathbb{P}_H\mathbb{P}_S\mathcal{L}$ the relation $\Phi(\bigcirc_S)_H\mathbb{A}$ between a witness and a token holds precisely when $\mathrm{Ch}(\Phi)(\bigcirc_H)_S\,\mathrm{Ch}(\mathbb{A})$ holds in the interaction algebra $\mathbb{P}_S\mathbb{P}_H\mathcal{L}$.

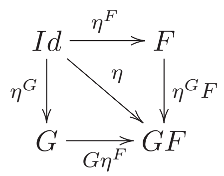
Theorem 1.12.1. *The functors $\mathbb{P}_H\mathbb{P}_S$ and $\mathbb{P}_S\mathbb{P}_H$ are naturally isomorphic via the token map* $(\mathrm{Ch}, \mathrm{Ch})$.

1.12.2 The double powerdomain monad and its algebras

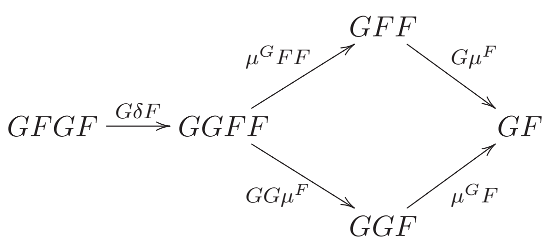
Given two monads on a category and a distributive law between the two functors, it is not necessarily the case that one can compose the functors and obtain another monad. In this subsection we show that one indeed obtains another monad when instantiating to the Smyth- and Hoare powerdomain monads on the category of domains. In order to keep the notation tidy we formulate most facts abstractly and only instantiate where the proof is not purely categorical. Interaction algebras do us a great service here, because thanks to token maps all calculations involving the natural transformations of the powerdomain monads can be reduced to calculations in the category of sets. The distinction between the two powerdomain monads becomes merely a question of typechecking.

Composing monads

Suppose (F, η^F, μ^F) and (G, η^G, μ^G) are two monads on a category and there is a natural isomorphism $\delta : FG \to GF$. There always is a natural transformation η from the identity functor to GF given by the commutative square[4]



This will serve as a unit for the new monad. In order to define a natural transformation μ from $GFGF$ to GF that becomes the multiplication of the monad, one uses the natural transformation δ in the commutative diagram:



If the natural transformation δ satisfies certain compatibility conditions with the unit and multiplication of the two monads, then (GF, η, μ) is another monad. These compatibil-

[4]The square commutes because of the naturality of η^G

ity conditions can equivalently be formulated as identities involving μ and the unit and multiplication of the monads. For an account of these conditions consult [9].

In the instance where the two monads are the Smyth and Hoare powerdomain monads on the category Dom, we know through the theory of interaction algebras that their units and multiplications are derived from the finite powerset monad on Set. Therefore, apart from some typechecking, the compatibility conditions amount to the fact that the free bounded distributive lattice over a set is constructed as the free join-semilattice over the free meet-semilattice. The distributive law between the two possible compositions of the two free semilattice functors is, of course, given by the choice operator we defined above. Thus we deduce the compatibility conditions for the two powerdomain monads from what is well-known in lattice theory. We record:

Theorem 1.12.2. *The double powerdomain functor extends to a monad on the category* Dom *of domains and Scott continuous maps. It has semi-open unit and multiplication maps and its functor preserves semi-open maps. The image of the double powerdomain monad consists of stably continuous frames.*

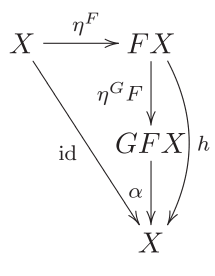
Proof. In terms of interaction algebras, a token set L_+ is mapped to the set $\mathrm{Fin}^2 L_+$ and likewise for witnesses. The token map presenting the unit sends a token a to $\{\{a\}\}$. The multiplication token map is described as follows. Given a set $\mathfrak{A} \in \mathrm{Fin}^4 L_+$, one first applies the choice operator to all its elements and obtains another set $\{\mathrm{Ch}(\mathcal{A}) \mid \mathcal{A} \in \mathfrak{A}\} \in \mathrm{Fin}^4 L_+$. Then one applies the union operation either on the outermost and then on the innermost level or the other way around. □

Algebras for the double powerdomain monad

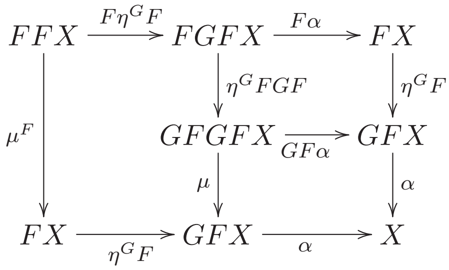
Suppose $\alpha : GFX \to X$ is an algebra for the monad (GF, η, μ). When does the object X have algebras for both of the component monads? There are obvious candidates. Define a map $h : FX \to X$ as

$$FX \xrightarrow{\eta^G F} GFX \xrightarrow{\alpha} X\,.$$

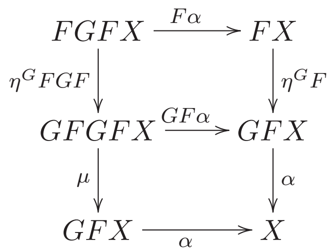
In order for this map to be an algebra for the monad (F, η^F, μ^F), the unit law requires that the diagram



commutes, which it does because it is just the unit law for α as a GF-algebra (recall $\eta = \eta^G F \circ \eta^F$). The associative law for the map h requires that the outer square in the following diagram commutes.



The small square in the top right commutes because of naturality of η^G, whereas commutativity of the square in the bottom right is just the associative law for the GF-algebra α. Since commutative squares compose, it remains to check that the square on the left commutes. In the case of the powerdomain monads we can perform this within the category of sets because all of the maps involved are token maps. Let us regard the object X as a set and both the functors F and G as the finite powerset functor. The unit maps η^F and η^G are singleton maps and the multiplications μ^F and μ^G set union. The action of μ is described in the proof of Theorem 1.12.2. Let $\mathbb{A} \in \mathrm{Fin}^2 X$ be a finite set of finite sets, which we regard as an element of FFX. Then the natural transformation $F\eta^G F$ instantiated to X sends $\mathbb{A}$ to the set $\{\{A\} \mid A \in \mathbb{A}\} \in \mathrm{Fin}^3 X$. This set is transformed by $\eta^G FGF$ into $\{\{\{A\} \mid A \in \mathbb{A}\}\} \in \mathrm{Fin}^4 X$. Observe that for any set B the set $\{\{b\} \mid b \in B\}$ has only one choice, namely B. Hence, when applying the natural transformation μ to the set above we first obtain $\{\{\mathbb{A}\}\}$, then $\bigcup\{\{\mathbb{A}\}\} = \{\mathbb{A}\}$ and finally $\{\bigcup \mathbb{A}\}$. Going the other way in the square, the transformation μ^F maps $\mathbb{A}$ to $\bigcup \mathbb{A}$ which under $\mu^G F$ goes to $\{\bigcup \mathbb{A}\}$. Hence the left square in the diagram above commutes in the case where the two monads are the Smyth- and Hoare powerdomains, which means that $\alpha \circ \eta^G F$ is a F-algebra structure map on X. Further observe that the GF-algebra α is in particular a F-algebra structure map, as can be seen from the composition of commutative squares



where μ is the GF-algebra structure on GFX. Notice that for the argument above it is irrelevant whether F is the Smyth- or Hoare powerdomain. We have shown:

Proposition 1.12.3. *If a domain has an algebra structure map for the double powerdomain monad, then it has algebra structure maps for both the Smyth- and Hoare powerdomain monads. Consequently, the category of Eilenberg-Moore algebras for the double powerdomain monad on* Dom *is a full subcategory of the category of continuous lattices and frame homomorphisms. In particular, any algebra structure map for the double pow-*

erdomain is a frame homomorphism.

Remark. For the construction of a G-algebra structure map from a GF-algebra structure map α one seems to have two possibilities:

$$\text{Either}\quad GX \xrightarrow{G\eta^F} GFX \xrightarrow{\alpha} X$$

$$\text{or}\quad GX \xrightarrow{\eta^G F} FGX \xrightarrow{\delta} GFX \xrightarrow{\alpha} X\,.$$

But one of the compatibility conditions for δ that Barr and Wells [9] list is that $\delta \circ \eta^F G = G\eta^F$.

Recall that algebras for the Smyth- and Hoare powerdomains are, insofar they exist, unique. We shall use this fact to show that algebras for the double powerdomain must also be unique. Suppose X is a continuous lattice presented by a Stage 3a interaction algebra $\mathcal{L}$ and suppose α is an algebra structure map for the double powerdomain on X that is presented by a Stage 3a morphism $R_\alpha : \mathcal{L} \to \mathbb{P}_H\mathbb{P}_S\mathcal{L}$. Without loss of generality we regard the token set L_+ as a basis of the domain X. As such, it is a dense subset if the continuous lattice X is endowed with the Lawson topology. Any token of $\mathbb{P}_H\mathbb{P}_S\mathcal{L}$ is a finite join of tokens from $\mathbb{P}_S\mathcal{L}$, since we can write $\mathbb{A} = \bigsqcup_{A\in\mathbb{A}}\{A\}$. According to Proposition 1.12.3 the algebra structure map α preserves joins (and meets) and the meet-semilattice structure on X is obtained by injecting elements of the Smyth powerdomain of X into the double powerdomain and then computing the value under α. Therefore α sends a generator $\mathbb{A} \in \mathrm{Fin}^2\, L_+$ of the double powerdomain to the lattice element $\bigvee_{A\in\mathbb{A}} \bigwedge A$. The map α is thus uniquely determined on a Lawson-dense subset of the double powerdomain of X (indexed by the tokens of $\mathbb{P}_H\mathbb{P}_S\mathcal{L}$), whence by Scott-continuity of α this map must be unique.

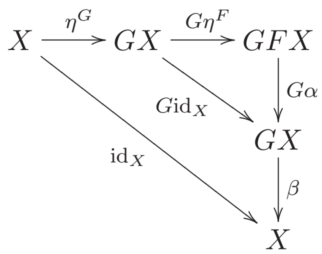
Dually one shows that in case the algebra structure map α is presented by a morphism $\mathcal{L} \to \mathbb{P}_S\mathbb{P}_H\mathcal{L}$ then a generator $\mathbb{A} \in \mathrm{Fin}^2\, L_+$ of the double powerdomain is mapped to the element $\bigwedge_{A\in\mathbb{A}} \bigvee A$. The isomorphism between the two representations of the double powerdomain is given by the choice operator on $\mathrm{Fin}^2\, L_+$. Therefore the continuous lattice enjoys the finite distributive law, since

$$\bigvee_{C\in \mathrm{Ch}(\mathbb{A})} \bigwedge C = \alpha(\mathbb{A}) = \bigwedge_{A\in\mathbb{A}} \bigvee A.$$

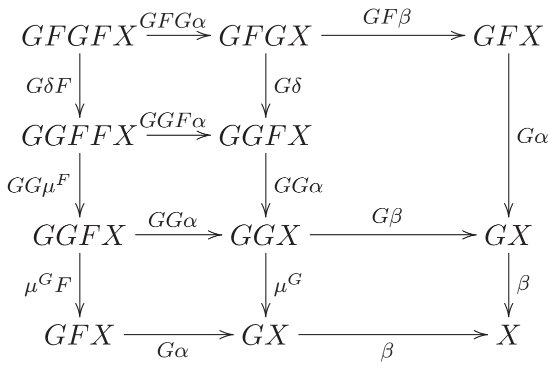
We have strengthened Proposition 1.12.3 to

Proposition 1.12.4. *Algebras for the double powerdomain are unique. The Eilenberg-Moore category of the double powerdomain monad on* Dom *is a full subcategory of the category* CFrm *of continuous frames and frame homomorphisms.*

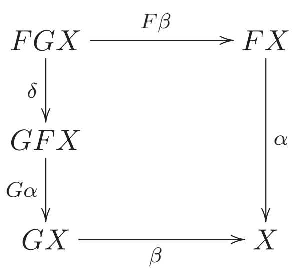
Next we turn towards the dual problem: If an object X has an F- as well as a G-algebra morphism, when can we use these to construct a GF-algebra morphism? Suppose $\alpha : FX \to X$ is a F-algebra and $\beta : GX \to X$ is a G-algebra. There is an obvious candidate for a GF-algebra map: We define $\gamma : GFX \to X$ as the composite $GFX \xrightarrow{G\alpha} GX \xrightarrow{\beta} X$. The unit law for the composite monad requires that the outer triangle in the following diagram commutes.



But this is obvious because the inner square and triangle commute. The associative law for γ requires that the outer square in the diagram below commutes.



On the left-hand side of the diagram, the top square commutes because δ is a natural transformation, the middle square commutes because α is an F-algebra and the bottom square commutes because μ^G is a natural transformation. The square in the bottom right commutes because β is a G-algebra. Thus it remains to check that the larger square in the top right corner commutes. By functoriality of G we may prove the simpler fact that the square below commutes.



(1.8)

Let us consider this situation in the instance where X is a domain presented by an interaction algebra $\mathcal{L}$, where (F, η^F, μ^F) is the Hoare powerdomain monad and (G, η^G, μ^G) is the

Smyth powerdomain monad. In Sections 1.10 and 1.11 we showed that the existence of the monad algebras α and β means that X is a continuous lattice (which is automatically a continuous preframe), so we may assume that the interaction algebra $\mathcal{L}$ presenting it is a Stage 3a algebra. Further recall from Section 1.10 that without loss of generality we may assume that the token set L_+ of the interaction algebra $\mathcal{L}$ has a special element 1 so that the structure map β is derived from a map $\bigsqcap : \mathrm{Fin}\, L_+ \to L_+$ where $\bigsqcap \emptyset = 1$. The square (1.8) above is instantiated (under reversal of all arrows) to a square in the category Tok_1 given by maps on tokens:

$$\begin{array}{ccc}
\mathrm{Fin}^2 L_+ & \xrightarrow{\mathrm{Fin}(\bigsqcap)} & \mathrm{Fin}\, L_+ \\
\downarrow{\scriptstyle \mathrm{Ch}} & & \\
\mathrm{Fin}^2 L_+ & & \downarrow{\scriptstyle \bigsqcup} \\
\downarrow{\scriptstyle \mathrm{Fin}(\bigsqcup)} & & \\
\mathrm{Fin}\, L_+ & \xrightarrow[\bigsqcap]{} & L_+
\end{array}$$

This square is precisely the finite distributive law for the operations $\sqcup$ and $\sqcap$ on the token set as formulated in Definition 1.7.1. Indeed, commutativity of the corresponding square of Tok_1 morphisms means that for any set of sets $\mathbb{A} \in \mathrm{Fin}^2 L_+$ the tokens $\bigsqcap \{\bigsqcup C \mid C \in \mathrm{Ch}(\mathbb{A})\}$ and $\bigsqcup \{\bigsqcap A \mid A \in \mathbb{A}\}$ are lower equivalent. We arrive at:

Proposition 1.12.5. *Any continuous frame has an algebra structure map for the double powerdomain monad. A frame homomorphism between continuous frames is a homomorphism of double powerdomain algebras.*

Together with Proposition 1.12.4 we get

Theorem 1.12.6. *The Eilenberg-Moore category of the double powerdomain monad on* Dom *is equivalent to the category* CFrm.

1.12.3 An alternative description of the double powerdomain

We conclude this section with a fact mentioned in [56, 58]. Recall that a domain D enjoys the identity $(\sigma D)^\partial \cong \sigma(D^\wedge)$ where σ is the Scott topology, $(-)^\partial$ is order dual and $(-)^\wedge$ the Lawson dual. We also showed in Proposition 1.10.2 and Proposition 1.11.4 that the Smyth powerdomain of D is isomorphic to $(\sigma D)^\wedge$ and the Hoare powerdomain of D is isomorphic to $(\sigma D)^\partial$. Therefore the Hoare powerdomain of the Smyth powerdomain of D is isomorphic to

$$\left(\sigma\left((\sigma D)^\wedge\right)\right)^\partial \cong \left((\sigma\sigma D)^\partial\right)^\partial \cong \sigma\sigma D.$$

1.13 Notes on Chapter 1

All of the domain-theoretic results presented in this chapter are in principle known, although some –to our knowledge– have not been stated explicitly in the literature, for example our results on semi-open maps and the powerdomains. To compare the proofs based on interaction algebras with the classical ones, the reader may consult the Compendium [22] or the Handbook [1]. The latter also contains an account of how to present a domain by an abstract basis, as well as further applications and bibliography on this subject. The Handbook approaches powerdomains via the more general dcpo-algebras, which can be constructed freely using abstract bases. Before the rather simple and "pure" presentation for continuous dcpos was developed, Scott [48] used information systems for the algebraic dcpos arising in programming semantics to solve fixed point equations for domains. These information systems feature a *covering relation* that captures the information when the join of a set of points is above a certain element. The concept of covering relations, or coverages, is also found in locale theory where it is used to describe the frame of opens of a locale using some set of generators.

The idea for the category of interaction algebras was developed by the author together with M. Andrew Moshier in 2010. It has numerous sources of inspiration: There are Chu spaces and the intermediary structure of a lattice, to mention more distantly related ones, as well as abstract bases and Vickers' information systems which we covered in 1.2.5 and 6.2. The Stage 5 interaction algebras have their ancestry in the multi-lingual sequent calculus [32] and strong proximity lattices [35].

Independently, a presentation for locally compact spaces bearing much similarity to interaction algebras is being developed by Paul Taylor. His presentation is motivated by open and compact intervals on the real line, where one can do away with finite joins of tokens because a compact interval is contained in the union of open intervals if and only if it is contained in a single interval already.

Another related structure are the *topological systems* of Vickers [59]. These comprise "points" and "opens" which are linked by a satisfaction relation $x \models U$ that can be understood as "The point x has the observable property U."

The proofs in Section 1.1 are quite similar to what can be found in [60] and [57]. The author thanks Reinhold Heckmann for providing unpublished notes on Vickers' information systems and what is called a *token map* in this work.

Future work

It should be straightforward to give presentations of categorical constructions on completely distributive frames and domains such as products and coproducts. For the category Tok_0 one may take products and coproducts of the token- and witness sets. For higher stages it may be necessary to generate the free algebra of the appropriate signature subject to some relations.

In particular we expect that Stage 1 interaction algebras give a hint at why the category of domains is not closed under function spaces. For a construction of mixed variance such as function space the witness set should be a mixture of tokens and witnesses from source and target. It seems to be impossible to define the binary meet of witnesses on such an algebra in a meaningful way.

The category of completely distributive frames and linear maps carries a symmetric monoidal closed structure. How this structure is presented by interaction algebras remains to be worked out.

We gave the translation between Stage 5 interaction algebras and stably compact spaces. The latter category has several possible notions of morphism other than the continuous maps we covered. Another translation that is of interest is that between the category Tok_5 and the multi-lingual sequent calculus.

Finally, the various stages of lattice-like structure are by no means the only possible way the base category Tok_0 could be enhanced. Other lines of work might be modal operators or probabilistic powerconstructions.

Chapter 2

D-lattices

Because of the origins of topology in the Euclidean geometry, spaces that do not enjoy the Hausdorff separation property are a curiosity in most circles of classical topologists. Theoretical computer science found great use for spaces which typically do not satisfy more than the T_0 separation axiom. The study of domains, continuous lattices and their most canonical topology, the Scott topology, their apparent connection to topological semigroups and semilattices, quickly lead to the consideration of refinements of the Scott topology which render said T_0 spaces Hausdorff. This is typically done by forming the common refinement of the Scott topology with a “dual” topology. Of most interest were those structures for which the common refinement of said two topologies yields a compact Hausdorff space. Examples include:

- Continuous lattices under the Lawson topology, the common refinement of the Scott topology and the weak lower topology,
- Priestley duality for bounded distributive lattices, where the compact Hausdorff topology on the spectrum of a lattice is the join of two (coherent) topologies,
- Power constructions of spaces where the set of closed subsets of a space is topologised using two different kinds of sub-basic opens. Important instances are the Vietoris topology on the set of closed subsets of a compact Hausdorff space and the Fell compactification [21] of a locally compact space.

In a compact Hausdorff space the concepts of “closed” and “compact” coincide. In all of the examples listed above it happens that the closed sets of one topology are precisely the compact sets of the other. Using open sets instead of closed sets one can state this connection between the two counterparts as: *The compact sets of one topology are precisely those whose complement is open in the other.*

Now there are two views on this statement: (1) One might want to stress the duality between compact and open. The spaces where this duality is absolutely symmetric are the stably compact spaces, and it is now well-known that virtually all kinds of domains

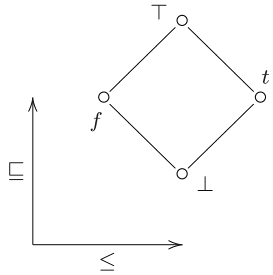


Figure 2.1: The lattice of truth values in Belnap's four-valued logic and the logical ($\leq$) and information ($\sqsubseteq$) orderings.

used in theoretical computer science are stably compact in their Scott topology. (2) One could regard the duality of compact and open as coincidental and retain the fact that one is dealing with a bitopological space.

In this chapter we shall assume the latter point of view, but abstract a little further. Let us motivate this abstraction by means of a particular logic, the four-valued logic of Belnap. An example Belnap gave for his logic was a database with bits of information entered by agents. A request to this database in form of a proposition may be answered with either of the following.

(t) "I have evidence for the truth of the proposition."

(f) "I have evidence for the falsehood of the proposition."

($\bot$) "I have no information available to decide the truth of the proposition."

($\top$) "I have various items of information available, some of them positive and some of them negative evidence for the truth of the proposition."

These four possibilities form the lattice of truth values depicted in Figure 2.1. Forming the Lindenbaum algebra of propositional logic over basic propositions $p, q, \ldots$ with connectives $\vee, \wedge, t, f, \bot, \top$ yields a bounded distributive lattice of (equivalence classes of) formulas. Its least and greatest elements are the formulas f and t while the formulas $\bot$ and $\top$ form a complemented pair, that is $\bot \vee \top = t$ and $\bot \wedge \top = f$. It is well-known from the theory of bilattices that this lattice of formulas can be written as the product of two sub-lattices: If $(L, \wedge, \vee, f, t, \bot, \top)$ is such a lattice, let L_- be the image of L under the map $\varphi \mapsto \varphi \wedge \bot$ and likewise L_+ be the image of L under $\varphi \mapsto \varphi \wedge \top$. Both of these are sub-lattices of L; The former is the interval $[f, \bot]$ in the logical order, whereas the latter is the interval $[f, \top]$. Using the fact that $(\bot, \top)$ is a complemented pair one finds that $\varphi = (\varphi \wedge \bot) \vee (\varphi \wedge \top)$.

In this lattice, two sets of formulas are of particular interest: First there are those formulas that are not self-contradictory in any model, meaning that under any assignment of truth values to the basic propositions the value of the formula can be either $\bot$, t, f but never $\top$. We call these formulas the *consistent* formulas because their true and false extents do not overlap in the space of all models. Secondly there are those formulas whose true and false extents cover the space of all models, meaning that under any assignment of truth values to the basic propositions the value of the formula can be t, f, $\top$ but never $\bot$. These formulas we call the *total* ones. Since both $\{\bot, t, f\}$ and $\{t, f, \top\}$ form sub-lattices of the lattice of truth values, both the consistent and the total formulas each form a sub-lattice of the lattice of all formulas. In the intersection of these two sub-lattices we find the formulas that behave like the formulas of ordinary propositional logic: They can be either true or false, but never self-contradictory.

Thus we arrive at an algebraic theory with the signature $(L, \wedge, \vee, t, f, \bot, \top, \mathsf{con}, \mathsf{tot})$ where $\wedge, \vee, t, f$ is a bounded distributive lattice structure on the set L and con and tot are two predicates on L comprising the consistent and total formulas.

2.1 The different faces of a d-lattice

There are several ways of defining the structure we call a d-lattice. In the bilattice style, a d-lattice consists of a set that carries two bounded distributive lattice structures and two unary predicates. By a well-known representation theorem for bilattices, the d-lattice thus represented is the product of two of its sub-lattices and the two predicates induce a pair of relations between the two sub-lattices.

2.1.1 D-lattices as bilattices

Definition 2.1.1. A *d-lattice* is a structure $(L, \sqcap, \sqcup, \wedge, \vee, \bot, \top, \mathbb{f}, \mathbb{t}, \mathsf{con}, \mathsf{tot})$ where the set L carries two bounded distributive lattice structures $(L, \sqcap, \sqcup, \bot, \top)$ and $(L, \wedge, \vee, \mathbb{f}, \mathbb{t})$. The partial order $\sqsubseteq$ induced by the former lattice structure is called the *information order* and the order $\leq$ induced by the latter is called the *logical order*. Further, $\mathsf{con}, \mathsf{tot} \subseteq L$ are two predicates called *consistency* and *totality*. The lattice structures are required to be connected as follows.

1. The pair $(\bot, \top)$ is complemented with respect to $\wedge, \vee$, meaning that $\bot \wedge \top = \mathbb{f}$ and $\bot \vee \top = \mathbb{t}$.

2. For any two elements $x, y \in L$ one has

$$\begin{aligned} x \sqcap y &= (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y) \\ x \sqcup y &= (x \wedge \top) \vee (y \wedge \top) \vee (x \wedge y) \end{aligned}$$

3. The pair $(\mathbb{f}, \mathbb{t})$ is complemented with respect to $\sqcap, \sqcup$.

4. All four lattice connectives distribute over one another.

The predicates con and tot are required to have the following properties.

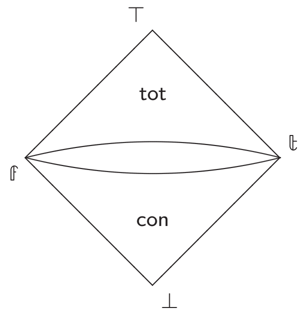


Figure 2.2: A generic d-lattice with predicates con and tot.

1. The predicate con is a lower set in information order.
2. The predicate tot is an upper set in information order.
3. Both con and tot are sub-lattices of the lattice $(L, \wedge, \vee, \mathbb{f}, \mathbb{t})$.
4. For any two elements $x \in \mathsf{con}$ and $y \in \mathsf{tot}$, whenever $x \sqcup y = x \wedge y$ or $x \sqcup y = x \vee y$ then $x \sqsubseteq y$.

The picture of a generic d-lattice we suggest the reader to keep in mind is the one given in Figure 2.2. Notice that the predicates con and tot by definition overlap in the set $\{\mathbb{f}, \mathbb{t}\}$ but the intersection may be larger. However, there may be elements of the d-lattice which are neither consistent nor total.

The 90-degree-lemma

The definition of a d-lattice we gave contains some redundancies. For the bilattice structure it suffices to specify the logical structure $(\wedge, \vee, \mathbb{f}, \mathbb{t})$ and a complemented pair $(\bot, \top)$. All other facts can be derived from the so-called 90-degree-lemma. It has its origin in a ternary operation studied by Birkhoff [11] and Grau [24].

Proposition 2.1.1 (90-degree-lemma)**.** *Let $(L, \vee, \wedge, \mathbb{f}, \mathbb{t})$ be a bounded distributive lattice and $(\top, \bot)$ a complemented pair, that is $\top \vee \bot = \mathbb{t}$ and $\top \wedge \bot = \mathbb{f}$. Then by the operations*

$$x \sqcap y \;:=\; (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y) \tag{2.1}$$

$$x \sqcup y \;:=\; (x \wedge \top) \vee (y \wedge \top) \vee (x \wedge y) \tag{2.2}$$

one obtains another bounded distributive lattice $(L, \sqcap, \sqcup, \bot, \top)$ in which $(\mathbb{f}, \mathbb{t})$ is a complemented pair. Moreover, the operations $\vee, \wedge, \sqcap, \sqcup$ distribute over each other. By substituting $\mathbb{f}$ for $\bot$, $\mathbb{t}$ for $\top$, $\sqcap$ for $\vee$ and $\sqcup$ for $\vee$ in (2.1) and (2.2) one one can recover the original lattice operations.

The name "90-degree-lemma" is chosen to reflect the understanding that the logical order and information order are "orthogonal" to each other. In the face of the 90-degree-lemma one may understand bounded distributive lattices as high-dimensional diamond-shaped objects, where every complemented pair can be made the top and bottom of the structure and thus lie on the "surface" of the lattice, while the non-complemented elements are situated in the "inside". Consequently, a boolean algebra is a sort of high-dimensional hollow object.

The bilattice representation theorem

Examining the identities (2.1) and (2.2) of the 90-degree-lemma, one finds that the interval $[\bot, \mathbb{f}]$ in the information order is a sub-lattice of L where the information order is dual to the logical order, that is, for $x, y \sqsubseteq \mathbb{f}$ one has $x \vee y = x \sqcap y$. Dually, whenever $x, y \sqsubseteq \mathbb{t}$ then $x \vee y = x \sqcup y$, so in the interval $[\bot, \mathbb{t}]$ the logical and information order agree. There are maps $x \mapsto x \sqcap \mathbb{f}$ and $x \mapsto x \sqcap \mathbb{t}$ which send a lattice element x to its projections onto the intervals $[\bot, \mathbb{f}]$ and $[\bot, \mathbb{t}]$, respectively. Using the distributive law and the fact that $(\mathbb{f}, \mathbb{t})$ is a complemented pair one finds that the join $(x \sqcap \mathbb{f}) \sqcup (x \sqcap \mathbb{t})$ is equal to x. This observation is the key ingredient to the following representation theorem. A proof can be found for example in [3].

Theorem 2.1.2. *The following are equivalent for a bilattice* $(L, \vee, \wedge, \sqcap, \sqcup, \mathbb{f}, \mathbb{t}, \bot, \top)$.

1. *The bilattice satisfies the hypothesis of the 90-degree-lemma 2.1.1.*

2. *The bilattice is isomorphic to the product of two bounded distributive lattices* $(L_-, \sqcap, \sqcup, 0, 1)$ *and* $(L_+, \sqcap, \sqcup, 0, 1)$ *where* $L \cong L_- \times L_+$, $\bot = (0, 0)$, $\top = (1, 1)$, $\mathbb{f} = (1, 0)$, $\mathbb{t} = (0, 1)$ *and the lattice operations on* L *are given as follows.*

$$\begin{aligned}
(x, y) \sqcap (x', y') &= (x \sqcap x', y \sqcap y') \\
(x, y) \sqcup (x', y') &= (x \sqcup x', y \sqcup y') \\
(x, y) \wedge (x', y') &= (x \sqcup x', y \sqcap y') \\
(x, y) \vee (x', y') &= (x \sqcap x', y \sqcup y')
\end{aligned}$$

There is a version of the bilattice representation theorem which does not require the bounds. Observe that in a bilattice of the form $L_- \times L_+$ the information join $(x, y) \sqcup (x', y')$ coincides with the logical meet $(x, y) \wedge (x', y')$ precisely when $y = y'$ and dually $(x, y) \sqcup (x', y')$ coincides with the information join $(x, y) \vee (x', y')$ precisely when $x = x'$. Thus one reconstructs the "layers" of the product even in the absence of the bounds. We shall use this fact to interpret the requirement (4) on the predicates con and tot in Definition 2.1.1 above.

Example 3. If $(L, \sqcap, \sqcup, \wedge, \vee, \bot, \top, \mathbb{f}, \mathbb{t})$ is a bilattice satisfying the distributive law of the definition above, then the minimal possible predicates that complete this structure to a d-lattice are $\mathsf{con} = {\downarrow}\{\mathbb{f}, \mathbb{t}\}$ and $\mathsf{tot} = {\uparrow}\{\mathbb{f}, \mathbb{t}\}$.

Example 4. Let $(A, \sqcap, \sqcup, 0, 1)$ be a bounded distributive lattice and $\prec$ be an auxiliary relation on A that is a sub-lattice of $A \times A$. Special cases of this are (1) when $\prec$ is the order of the lattice or (2) when $\prec$ is a quasi-proximity on A. Construct a d-lattice where L is the product of the order-dual of A with A. Let con be the lattice order $\sqsubseteq$ regarded as a subset of $A^\partial \times A$, and let tot be the auxiliary relation $\prec$ regarded as a subset of $A^\partial \times A$. One has $\mathbb{f} = (0,0)$, $\mathbb{t} = (1,1)$, $\bot = (1,0)$ and $\top = (0,1)$.

Example 5. Let $(A, \sqcap, \sqcup, 0, 1)$ be a bounded distributive lattice. On the product $A \times A$ let $(x, y) \in \mathsf{con}$ whenever $x \sqcap y = 0$ and $(x, y) \in \mathsf{tot}$ whenever $x \sqcup y = 1$. We call this the *symmetric d-lattice* associated with A.

Example 6. Suppose L_- and L_+ are two sub-lattices of a bounded distributive lattice. For instance, the ambient lattice could be the powerset of a set X. On $L_- \times L_+$ declare the predicate con to consist of the pairs (U, V) where $U \cap V = \emptyset$ (the consistent, or disjoint pairs) and the predicate tot to consist of pairs (U, V) which satisfy $U \cup V = X$ (the total pairs). It is this class of examples that the naming of the predicates originates from.

2.1.2 D-lattices as pairs of lattices with relations

Because of the bilattice representation theorem one can give an interaction algebra-style definition of a d-lattice. Although d-lattices enjoy an obvious symmetry, we shall distinguish notationally between the two sides by using lowercase Greek symbols for the one and lowercase Roman symbols for the other. Evidently, Definition 2.1.2 below is equivalent to the bilattice-style Definition 2.1.1.

Definition 2.1.2. A *d-lattice* is a structure

$$L_- \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} L_+$$

where $(L_-, \sqcap, \sqcup, 0, 1)$ and $(L_+, \sqcap, \sqcup, 0, 1)$ are bounded distributive lattices and con and tot are two relations satisfying the axioms of Table 2.1.

Notation. Similarly to the interaction algebras of Chapter 1 we denote a d-lattice by uppercase script letters $\mathcal{L}, \mathcal{M}, \ldots$ and their components by the same letter in standard font. For example, $\mathcal{L}$ stands for the d-lattice with components $(L_-, L_+, \mathsf{con}, \mathsf{tot})$.

The reader easily convinces himself that the upper- and lower set axioms amount to the fact that $\mathsf{con} \subseteq L_+ \times L_-$ is a lower set and $\mathsf{tot} \subseteq L_- \times L_+$ is an upper set. The meet and join axioms say that both relations define sub-lattices of the bounded distributive lattice ${L_-}^\partial \times L_+$ where $\mathbb{f} = (1,0)$ and $\mathbb{t} = (0,1)$. The last axioms require that in each "slice" of the form $L(x) = \{(\phi, x) \,|\, \phi \in L_-\}$ or $L(\phi) = \{(\phi, x) \,|\, x \in L_+\}$ any consistent pair is below any total pair.

lower set	$\dfrac{x\,\mathsf{con}\,\phi}{(x \sqcap y)\,\mathsf{con}\,(\phi \sqcap \psi)}$	(con- $\downarrow$)
binary meet and join	$\dfrac{x\,\mathsf{con}\,\phi \qquad y\,\mathsf{con}\,\psi}{(x \sqcap y)\,\mathsf{con}\,(\phi \sqcup \psi)}$ $\quad$ $\dfrac{x\,\mathsf{con}\,\phi \qquad y\,\mathsf{con}\,\psi}{(x \sqcup y)\,\mathsf{con}\,(\phi \sqcap \psi)}$	(con-$\wedge$), (con-$\vee$)
empty meet and join	$\dfrac{}{0\,\mathsf{con}\,1}$ $\quad$ $\dfrac{}{1\,\mathsf{con}\,0}$	(con-$\mathbb{f}$), (con-$\mathbb{t}$)
upper set	$\dfrac{\phi\,\mathsf{tot}\,x}{(\phi \sqcup \psi)\,\mathsf{tot}\,(x \sqcup y)}$	(tot- $\uparrow$)
binary meet and join	$\dfrac{\phi\,\mathsf{tot}\,x \qquad \psi\,\mathsf{tot}\,y}{(\phi \sqcap \psi)\,\mathsf{tot}\,(x \sqcup y)}$ $\quad$ $\dfrac{\phi\,\mathsf{tot}\,x \qquad \psi\,\mathsf{tot}\,y}{(\phi \sqcup \psi)\,\mathsf{tot}\,(x \sqcap y)}$	(tot-$\vee$), (tot-$\wedge$)
empty meet and join	$\dfrac{}{0\,\mathsf{tot}\,1}$ $\quad$ $\dfrac{}{1\,\mathsf{tot}\,0}$	(tot-$\mathbb{t}$), (tot-$\mathbb{f}$)
consistency vs. totality	$\dfrac{x\,\mathsf{con}\,\phi \qquad \phi\,\mathsf{tot}\,y}{x \sqsubseteq y}$ $\quad$ $\dfrac{\phi\,\mathsf{tot}\,x \qquad x\,\mathsf{con}\,\psi}{\phi \sqsupseteq \psi}$	(con-tot)

Table 2.1: Axioms for a d-lattice in the interaction algebra style. The third column lists the names used in [33].

2.1.3 Morphisms

The bilattice-style definition of a d-lattice suggests the following definition of d-lattice homomorphism.

Definition 2.1.3. A homomorphism of d-lattices is a function between the underlying sets that preserves the two lattice structures, their constants and the predicates con and tot. The category of d-lattices and d-lattice homomorphisms is denoted by dLat.

Remark. Because of the 90-degree-lemma it suffices to postulate that a d-lattice homomorphism is a homomorphism with respect to the lattice structure $(\sqcap, \sqcup, \bot, \top)$ and furthermore preserves the complemented pair $(\mathbb{f}, \mathbb{t})$ and the predicates.

When writing a d-lattice in the interaction algebra-style of Definition 2.1.2, a homomorphism becomes a pair of homomorphisms between the bounded distributive component lattices that preserve the relations con and tot. For example, if $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ and $(M_-, M_+, \mathsf{con}, \mathsf{tot})$ are d-lattices then a homomorphism is a pair of lattice homomorphisms $h_- : L_- \to M_-$ and $h_+ : L_+ \to M_+$ such that $\phi\,\mathsf{tot}\,x$ implies $h_-(\phi)\,\mathsf{tot}\,h_+(x)$ and $x\,\mathsf{con}\,\phi$ implies $h_+(x)\,\mathsf{con}\,h_-(\phi)$.

Remark. Notice the similarity between homomorphisms of d-lattices and token maps between interaction algebras.

2.1.4 Basic constructions

Due to the symmetry of the d-lattice structure there are two possible ways of generalising the order-dual operation $(-)^\partial$ of lattices to d-lattices. While one of them is merely useful for providing slick proofs, the other one will turn out to be significant once we know how to interpret d-lattices topologically.

The order-dual of a d-lattice

Definition 2.1.4. If $\mathcal{L}$ denotes the d-lattice $L_- \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} L_+$ then $\mathcal{L}^\partial$ denotes the structure $L_+^\partial \underset{\mathsf{tot}}{\overset{\mathsf{con}}{\rightleftarrows}} L_-^\partial$ which we call the *order-dual* of $\mathcal{L}$.

The assignment $\mathcal{L} \mapsto \mathcal{L}^\partial$ extends to a covariant involution on the category dLat that sends a homomorphism (h_-, h_+) to (h_+, h_-).

Swapping polarities

The more interesting variant of order-dual swaps the component lattices.

Definition 2.1.5. If $\mathcal{L}$ denotes the d-lattice $L_- \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} L_+$ then $\operatorname{Flip}\mathcal{L}$ denotes the structure $L_+ \underset{\mathsf{con}^{-1}}{\overset{\mathsf{tot}^{-1}}{\rightleftarrows}} L_-$ which we call the *dual* of $\mathcal{L}$.

The assignment $\operatorname{Flip}(h_-, h_+) := (h_+, h_-)$ turns Flip into another covariant endofunctor on dLat.

Remark. In Chapter 1 we already defined a functor Flip on interaction algebras. This dual use of names is justified because syntactically both functors perform the same operations on d-lattice homomorphisms or token maps.

2.2 The well-inside relation

The last two axioms in Table 2.1 are the only ones connecting the relations con and tot. These axioms are connected to the so-called *well-inside* relation from topology. In a topological space, a subset A is said to be well inside another subset B if the topological closure of A is contained in the interior of B. Equivalently, one can demand that there exist open sets U and V such that A is disjoint from U, V is contained in B and the union of U and V covers the space. Clearly the well-inside relation is stronger than the set inclusion relation. Note that in case B is open one may choose $V = B$.

Let $\mathcal{O}X$ denote the lattice of open sets of the topological space X. Then according to Example 6 one can build a d-lattice $(\mathcal{O}X, \mathcal{O}X, \mathsf{con}, \mathsf{tot})$ where the well-inside relation restricted to open sets coincides with the relational composition $\mathsf{con};\mathsf{tot}$.

Definition 2.2.1. Let $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ be a d-lattice. We say that an element $x \in L_+$ is *well inside* another element $y \in L_+$ and write $x \lhd y$ if there exists an element $\phi \in L_-$ with $x \mathsf{con} \phi \mathsf{tot} y$. The element ϕ is said to be a witness for the relation $x \lhd y$. Dually, we say that $\psi \in L_-$ is well inside $\phi \in L_-$ and write $\phi \rhd \psi$ if there exists a witnessing element $x \in L_+$ with $\phi \mathsf{tot} x \mathsf{con} \psi$.

Let us state some immediate properties of the well-inside relations.

Lemma 2.2.1. *The well-inside relation $\lhd$ on the component lattice L_+ of a d-lattice has the following properties.*

1. *The well-inside relation is stronger than the lattice order.*
2. *The relational composition $\sqsubseteq; \lhd; \sqsubseteq$ is identical to $\lhd$, that is, the well-inside relation is downward closed on the left and upward closed on the right.*
3. *The well-inside relation is transitive.*
4. *The well-inside relation is a sub-lattice of the product lattice $L_+ \times L_+$.*

Similar facts hold for the well-inside relation on the component lattice L_-.

Proof. The fact that $x \lhd y$ implies $x \sqsubseteq y$ is enforced by the last row of Table 2.1. To see that $\lhd$ is downward closed on the left, use the definition $\mathsf{con}; \mathsf{tot}$ and the fact that con is downward closed. Similarly, the fact that tot is upward closed implies that $\lhd$ is upward closed on the right. Transitivity of $\lhd$ is a consequence of the first two properties. Observe that $0\mathsf{con}1\mathsf{tot}0$ and $1\mathsf{con}0\mathsf{tot}1$, whence the relation $\lhd$ contains the bounds of the lattice ${L_+}^2$. For the algebraic properties, suppose $x\mathsf{con}\phi\mathsf{tot}y$ and $x'\mathsf{con}\psi\mathsf{tot}y'$. Then $(x \sqcap x')\mathsf{con}(\phi \sqcup \psi)\mathsf{tot}(y \sqcap y')$. Thus $\lhd$ is a sub-meet-semilattice. The proof for binary joins is dual. □

Lemma 2.2.2. *D-lattice homomorphisms preserve the well-inside relation.*

Proof. Follows from the fact that d-lattice homomorphisms preserve the relations con and tot. □

In general there are not many elements of a d-lattice that are well inside themselves. The only elements of this kind present in any d-lattice are the bounds, as $0 \lhd 0$ and $1 \lhd 1$ is always true. For the well-inside relation on the powerset of a topological space the sets well inside themselves are precisely the *clopen* sets. Indeed, a set U being well-inside itself by definition means that its closure is contained in its interior, so U is both open and closed. Each clopen set has an obvious counterpart witnessing clopenness: The complement of a clopen set is again clopen.

Definition 2.2.2. Let $\mathcal{L}$ be a d-lattice. An element of a component lattice is called *complemented* if it is well-inside itself. For example, $x \in L_+$ is complemented if $x\mathsf{con}\phi\mathsf{tot}x$ for some element $\phi \in L_+$. In the situation $x\mathsf{con}\phi\mathsf{tot}x$ we say that (ϕ, x) is a complemented pair.

Obviously complemented elements come in pairs. If $x \lhd x$ is a complemented element in the component lattice L_+ then any witness $x\mathsf{con}\phi\mathsf{tot}x$ is also complemented because $\phi \rhd \phi$ is witnessed by x. An easy corollary to Lemma 2.2.2 is that d-lattice homomorphisms preserve complemented elements.

2.3 Round sets and round ideal completion

In Chapter 1 we used *round ideals* to interpret interaction algebras as presentations of dcpos. We intend to do the same with d-lattices, even though a d-lattice is not an interaction algebra in general (but those d-lattices that *are* interaction algebras will be of particular importance). Let us begin with some definitions and facts.

2.3.1 Some topologies on d-lattices

Recall from Definition 1.1.3 that a round upper set of a set L with transitive binary relation $\lhd$ is a subset $U \subseteq L$ such that $x \in U$ if and only if $u \lhd x$ for some $u \in U$. Dually, a round lower set of L is a subset U such that $x \in U$ precisely when $x \lhd u$ for some $u \in U$. Thus a round lower set is a round upper set with respect to the relational inverse of $\lhd$.

Definition 2.3.1. Let $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ be a d-lattice. We call a subset $U \subseteq L_+$ an *open upper set* of the component lattice L_+ if it is a round upper set with respect to the well-inside relation. The complete lattice of open upper sets of L_+ is denoted by $\mathrm{Up}^{\lhd} L_+$. Dually, we call a subset U an *open lower set* of L_+ if it is a round lower set with respect to $\lhd$. The complete lattice of open lower sets is denoted by $\mathrm{Lo}^{\lhd} L_+$.

In the same manner one defines the open upper sets and open lower sets[1] $\mathrm{Up}^{\lhd} L_-$ and $\mathrm{Lo}^{\lhd} L_-$ of the component lattice L_-.

The open upper sets of a component lattice are indeed upper sets in the lattice order: If $U \ni u \sqsubseteq x$ then we know that there is some element $u' \in U$ well inside u and thereby well inside x. Hence x is also an element of U. Next we show that our choice to call round sets "open" is justified.

Proposition 2.3.1. *Let $\mathcal{L}$ be a d-lattice. The open upper sets of the component lattice L_+ form a compact topology on L_+. Similar statements holds for the component lattice L_- and the open lower sets of both component lattices.*

Proof. For general reasons the open upper sets form a complete lattice where arbitrary joins are computed by set union. We have to show that finite meets are also given as set intersection. First observe that the entire lattice L_+ is an open upper set because of $0 \lhd 0$. Thus the empty meet in $\mathrm{Up}^{\lhd} L_+$ is the empty set meet[2]. Now suppose U_1 and U_2 are open upper sets and x is an element of their intersection. Then there are elements $u_i \in U_i$ $(i \in \{1, 2\})$ well inside x. Since the well-inside relation is closed under finite joins on the left we get $U_1 \cap U_2 \ni u_1 \sqcup u_2 \lhd x$ which shows that the intersection $U_1 \cap U_2$ is an open upper set. Therefore $\mathrm{Up}^{\lhd} L_+$ is indeed a topology. It is a compact topology because any open cover of L_+ must in particular cover the bottom element.

[1]The notation might cause some confusion, because by convention we write the well-inside relation on L_- as its relational inverse $\rhd$. But if one interprets $\mathrm{Up}^{\rhd} L_-$ as in Definition 1.1.3 then this symbolises what we call open lower sets here.

[2]Recall that in general the largest round upper subset is not necessarily the entire set.

Apply what we just proved to the order-dual $\mathcal{L}^\partial$ and the dual $\operatorname{Flip}\mathcal{L}$ and obtain proofs for the other statements of this proposition. □

In general there might be very few open upper and lower sets. If the relations con and tot are minimal, meaning that $x \lhd y$ implies that either $x = 0$ or $y = 1$ then the only open upper sets of L_+ are the empty set, $\{1\}$ and L_+.

Remark. Observe that d-lattice homomorphisms give rise to frame homomorphisms between open upper and lower sets. Suppose (h_-, h_+) is a d-lattice homomorphism. For general reasons the map $\mathrm{Up}(h_+)$ between upper sets preserves all unions, and since h_+ preserves the well-inside relation, $\mathrm{Up}(h_+)$ maps open upper sets to open upper sets. Since finite meets of open upper sets are computed as element-wise join and h_+ preserves these, the map $\mathrm{Up}(h_+)$ preserves finite meets.

The following fact relating open upper and lower sets will be very useful in our development. Its proof is the same as the proof we gave for the Fundamental Lemma of interaction algebras 1.1.6.

Notation. If U is a subset of a component lattice of a d-lattice then we write U^{tot} for the set of elements of the other component lattice that are total with some element of U. Similarly, U_{con} denotes the set of elements consistent with some element of U. For singleton sets we write x^{tot} and x_{con} instead of $\{x\}^{\mathsf{tot}}$ and $\{x\}_{\mathsf{con}}$.

Lemma 2.3.2. *For any d-lattice $\mathcal{L}$ the frame of open lower sets of L_+ is order-isomorphic to the frame of open upper sets of L_-. The isomorphism sends an open lower set $U \in \mathrm{Lo}^{\lhd} L_+$ to*

$$U^{\mathsf{tot}} = \{\phi \in L_- \mid \exists u \in U.\, \phi \mathsf{tot} u\}$$

and an open upper set $\Phi \in \mathrm{Up}^{\lhd} L_-$ to

$$\Phi_{\mathsf{con}} = \{x \in L_+ \mid \exists \phi \in \Phi.\, x \mathsf{con} \phi\}\,.$$

2.3.2 The open ideal completion

Next we restrict the open upper and lower sets to filters and ideals. It is well-known from lattice theory that the ideal completion of a bounded distributive lattice is a frame where directed joins are computed as set union, arbitrary meets are computed as set intersection and binary joins are computed element-wise.

Definition 2.3.2. For a d-lattice $\mathcal{L}$, let $\mathrm{Idl}^{\lhd} L_+$ denote the *open ideals* of the component lattice L_+, that is, lattice ideals which are round lower sets with respect to the well-inside relation. Dually let $\mathrm{Filt}^{\lhd} L_+$ denote those filters of L_+ that are open upper sets.

Similarly one defines the open ideals and filters of the component lattice L_-.

From the algebraic properties of the consistency and totality relations one derives the following lemma.

Lemma 2.3.3. *The isomorphism of Lemma 2.3.2 restricts to open ideals and filters.*

Since open ideals share directed joins and all meets with the open lower sets, it is immediate that open ideals form a preframe. But even more is true:

Lemma 2.3.4. *The open ideals of a component lattice of a d-lattice form a compact sub-frame of the frame of all ideals. A dual statement applies to open filters.*

Proof. Is suffices to show that the binary join of open ideals is again open. If I and J are open ideals of the component lattice L_+ of a d-lattice $\mathcal{L}$ then any element of the join $I \vee J$ in the frame of all ideals can be written as $x \sqcup y$ where $x \in I$ and $y \in J$. By hypothesis there exist x' and y' with $x \lhd x' \in I$ and $y \lhd y' \in J$. Now use the fact that the well-inside relation is a sub-lattice of L_+^2 and obtain $x \sqcup y \lhd x' \sqcup y'$. Hence the binary join of open ideals is open. The frame of open ideals is compact because the largest open ideal is principal. □

Corollary 2.3.5. *A d-lattice homomorphism gives rise to frame homomorphisms between open ideals and filters.*

Proof. A component $h_+ : L_+ \to M_+$ of a d-lattice homomorphism $\mathcal{L} \to \mathcal{M}$ gives rise to a frame homomorphism via the ideal completion functor Idl. We already remarked that the maps $\mathrm{Up}(h_+)$ and $\mathrm{Lo}(h_+)$ derived from a d-lattice homomorphism preserve open sets. But the action of $\mathrm{Idl}(h_+)$ on ideals is just the restriction of the lower set map $\mathrm{Lo}(h_+)$ to ideals. Dually $\mathrm{Filt}(h_+)$ restricts to a frame homomorphism between open filters. □

Lemma 2.3.6. *Let* $\mathcal{L} = (L_-, L_+, \mathsf{con}, \mathsf{tot})$ *be a d-lattice,* $I \in \mathrm{Idl}^{\lhd} L_-$, $J \in \mathrm{Idl}^{\lhd} L_+$ *be open ideals and* $F \in \mathrm{Filt}^{\lhd} L_-$, $G \in \mathrm{Filt}^{\lhd} L_+$ *be open filters. Define the following consistency and totality relations.*

$$J \,\mathsf{con}_\circ\, I \quad :\Leftrightarrow \quad J \times I \subseteq \mathsf{con} \tag{2.3}$$

$$I \,\mathsf{tot}_\circ\, J \quad :\Leftrightarrow \quad (I \times J) \between \mathsf{tot} \tag{2.4}$$

$$F \,\mathsf{con}^\circ\, G \quad :\Leftrightarrow \quad F \times G \subseteq \mathsf{tot} \tag{2.5}$$

$$G \,\mathsf{tot}^\circ\, F \quad :\Leftrightarrow \quad (G \times F) \between \mathsf{con} \tag{2.6}$$

1. *Both the* open ideal completion $\mathrm{Idl}_\circ \mathcal{L} := (\mathrm{Idl}^{\lhd} L_-, \mathrm{Idl}^{\lhd} L_+, \mathsf{con}_\circ, \mathsf{tot}_\circ)$ *and the* open filter completion $\mathrm{Idl}^\circ \mathcal{L} := (\mathrm{Filt}^{\lhd} L_+, \mathrm{Filt}^{\lhd} L_-, \mathsf{con}^\circ, \mathsf{tot}^\circ)$ *are d-lattices.*

2. *The two d-lattices defined in (1) are isomorphic. That is, the open ideal completion of $\mathcal{L}$ is isomorphic to the open ideal completion of the order dual $\mathcal{L}^\partial$.*

Proof. The proof is straightforward. The techniques are identical to the ones used to prove Theorem 1.3.4, Proposition 1.3.5 and Theorem 1.7.7. If the d-lattice $\mathcal{L}$ was a Stage 2a interaction algebra then the content of assertion (2) is that any continuous preframe is isomorphic to its second Lawson dual. □

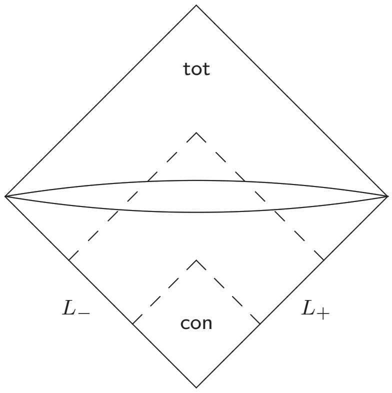


Figure 2.3: The relations $\mathsf{con}_\circ$ and $\mathsf{tot}_\circ$ of the open ideal completion of a d-lattice. Pictured is the product $L_- \times L_+$ of the component lattices together with the predicates con and tot on the product. The smaller dashed square depicts a consistent pair of ideals, whose product is contained in con. The larger square depicts a total pair of ideals, whose product intersects tot.

Lemma 2.3.7. *The open ideal completion of d-lattices is functorial.*

Proof. We already noted that a d-lattice homomorphism (h_-, h_+) lifts to a pair of frame homomorphisms $\mathrm{Idl}(h_-)$ and $\mathrm{Idl}(h_+)$ between the frames of open ideals. It remains to show that the pair $(\mathrm{Idl}(h_-), \mathrm{Idl}(h_+))$ preserves the relations $\mathsf{con}_\circ$ and $\mathsf{tot}_\circ$. If the product $J \times I$ of open ideal is contained in con, then so is the product of the forward images $h_+(J) \times h_-(I)$ because (h_-, h_+) preserves con. Therefore $(\mathrm{Idl}(h_-), \mathrm{Idl}(h_+))$ preserves $\mathsf{con}_\circ$. Suppose $I \ni \phi \mathsf{tot} x \in J$ is a witnessing pair for the relation $I \mathsf{tot}_\circ J$. Then $h_-(\phi) \mathsf{tot} h_+(x)$ is a witnessing pair for the relation $\mathrm{Idl}(h_-)(I) \mathsf{tot}_\circ \mathrm{Idl}(h_+)(J)$. □

We have gathered all the facts necessary to prove the existence of the *open ideal completion* endofunctor on d-lattices.

Theorem 2.3.8. *There is an endofunctor* $\mathrm{Idl}_\circ$ *on the category* dLat *that maps a d-lattice* $\mathcal{L}$ *to the d-lattice of open ideals as defined in Lemma 2.3.6 and a d-lattice homomorphism* $h = (h_-, h_+)$ *to the pair* $\mathrm{Idl}_\circ(h) = (\mathrm{Idl}(h_-), \mathrm{Idl}(h_+))$.

A d-lattice homomorphism extends to a homomorphism of open ideal completions. However, one can be more liberal and still obtain d-lattice homomorphisms between open ideal completions.

Proposition 2.3.9. *Let* $\mathcal{L}$ *and* $\mathcal{M}$ *be d-lattices and let* $h : L_+ \to M_+$ *be a lattice homomorphism that preserves the well-inside relation. Then the map* $\mathrm{Idl}(h) : \mathrm{Idl}^{\lhd} L_+ \to \mathrm{Idl}^{\lhd} M_+$ *can be extended to a d-lattice homomorphism between the open ideal completions.*

Proof. We define a homomorphism $f : \mathrm{Idl}^{\lhd} L_- \to \mathrm{Idl}^{\lhd} M_-$ by mapping an open ideal $I \subseteq L_-$ to

$$f(I) = \{\psi \in M_- \mid \exists \phi \in I \, \exists x \in L_+ . \, \phi \mathsf{tot} x, \, h(x) \mathsf{con} \psi\} .$$

In terms of the isomorphism between the open ideals and filters one can write $f(I) = (\mathrm{Filt}(h)(I^{\mathsf{tot}}))_{\mathsf{con}}$. From this expression it is evident that f is even a frame homomorphism. We claim that the pair $(f, \mathrm{Idl}(h))$ preserves the consistency and totality relations. Suppose the open ideal $J \subseteq L_+$ is consistent with I, that is, $J \times I$ is contained in the relation con. For any $j \in J$ and $\psi \in f(I)$ we have $j \mathsf{con} \phi \mathsf{tot} x$ and $h(x) \mathsf{con} \psi$ for some $\phi \in I$ and $x \in L_+$. Now use the assumption that h preserves the well-inside relation and deduce $h(j) \lhd h(x) \mathsf{con} \psi$ which shows that $\mathrm{Idl}(J) \times f(I)$ is contained in the consistency relation. Thus the pair $(f, \mathrm{Idl}(h))$ preserves the relation $\mathsf{con}_\circ$.

Now suppose J is an open ideal that is total with I, meaning $I \ni \phi \mathsf{tot} j \in J$ for some pair (ϕ, j). Since I is an open ideal, the element ϕ is well inside another element ϕ' of I. Then $\phi' \mathsf{tot} x \mathsf{con} \phi \mathsf{tot} j$ and so by hypothesis $h(x) \lhd h(j)$. This means that $h(x) \mathsf{con} \psi \mathsf{tot} h(j)$ for some $\psi \in M_-$. Observe that this ψ is a member of the open ideal $f(I)$ which demonstrates that the pair $(f, \mathrm{Idl}(h))$ preserves the relation $\mathsf{tot}_\circ$. □

2.4 Normal d-lattices

When examining Lemma 2.3.6 one finds that on a component frame of the open ideal completion the well-inside relation is contained in the way-below relation. Indeed, let L be any poset and D be a sub-dcpo of the ideal completion of L (Think of L as a component lattice and D as the frame of its open ideals). Whenever $I, J \in D$ are ideals with $I \subseteq {\downarrow} x$ for some $x \in J$ then I is way below J. This is because by hypothesis directed joins in D are computed as set union, so any directed union $\bigcup \mathbb{J}$ of ideals in D containing J must have a member that contains x and thereby the ideal I.

Below we exhibit a class of d-lattices that admit a notion of "principal open ideal". Their open ideal completions have the property that the frame of open ideals is a domain and moreover the well-inside relation on open ideals coincides with the way-below relation. Thus the way-below relation inherits the property of being closed under finite meets on the right from the well-inside relation. Such a frame we called stably continuous. As a hint towards why stable continuity is desirable, consider the d-lattice $(\mathcal{O}X, \mathcal{O}X, \mathsf{con}, \mathsf{tot})$ constructed from a locally compact sober topological space X in the fashion of Example 6. The way-below relation on the domain $\mathcal{O}X$ coincides with the well-inside relation of this d-lattice precisely when X is a compact Hausdorff space.

2.4.1 Normality for d-lattices

Definition 2.4.1. A d-lattice $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ is called *normal* if

$$(\mathsf{tot}; \mathsf{con}; \mathsf{tot}) = \mathsf{tot}.$$

The name of this property is justified, as it subsumes normal lattices as well as normal spaces (see Lemma 6.3.3).

Notation. For an element of a component of a d-lattice we write $\downarrow$ for the operator that returns the ideal of elements well-inside a given element. Dually, $\uparrow$ is the operator that returns the filter of elements that a given element is well-inside of.

Let us list some immediate consequences of this definition.

Lemma 2.4.1. *Let $\mathcal{L}$ be a normal d-lattice.*

1. *The well-inside relations on the component lattices have the interpolation property.*
2. *For any element $x \in L_+$ the filter $x^{\mathsf{tot}} = \{\phi \in L_- \mid \phi \mathsf{tot} x\}$ is open.*
3. *The ideal $\downarrow x$ of elements well-inside the element $x \in L_+$ is open. A similar statement holds for the component lattice L_-.*
4. *A bounded distributive lattice is normal if and only if the d-lattice derived from it as in Example 5 is normal. In particular, a topological space has the T_4 separation axiom if and only if the symmetric d-lattice $(\mathcal{O}X, \mathcal{O}X, \mathsf{con}, \mathsf{tot})$ is normal.*
5. *A normal d-lattice is a compact Stage 2a interaction algebra that also satisfies the Stage 2b rules.*

Proof. (1) By definition the well-inside relation $\lhd$ is the relational composition $\mathsf{con}; \mathsf{tot}$. By normality this relation is identical to $\mathsf{con}; \mathsf{tot}; \mathsf{con}; \mathsf{tot}$.

(2) If we write normality as $\mathsf{tot} = (\rhd; \mathsf{tot})$ then it is immediate that ϕ is total with x if and only if there is some ψ well inside ϕ still total with x.

(3) Observe that the ideal $\downarrow x$ can be written as $(x^{\mathsf{tot}})_{\mathsf{con}}$, so this ideal is open because of (2) and the Fundamental Lemma 2.3.2. Alternatively, use the fact that the well-inside relation is interpolative.

(4) In the symmetric d-lattice $(A, A, \mathsf{con}, \mathsf{tot})$ derived from a bounded distributive lattice $(A, \sqcap, \sqcup, 0, 1)$ one has $\phi \mathsf{tot} x$ whenever $\phi \sqcup x = 1$ and $x \mathsf{con} \phi$ whenever $x \sqcap \phi = 0$. Clearly then the lattice A is normal in the lattice-theoretical sense of Lemma 6.3.3 precisely when its symmetric d-lattice is normal.

(5) A normal d-lattice satisfies all axioms of Tables 1.2 and 1.3 except for the join-strength rules. □

A d-lattice can be normal for trivial reasons. Suppose the totality relation contains only those pairs (ϕ, x) where either $\phi = 1$ or $x = 1$ (or both). In this situation, whenever $\phi \mathsf{tot} x$ then one of the elements involved is 1 and so one can extend the relation either as $1 \mathsf{tot} 0 \mathsf{con} 1 \mathsf{tot} x$ or as $\phi \mathsf{tot} 1 \mathsf{con} 0 \mathsf{tot} 1$. More generally, if the relation tot is such that for any total pair (ϕ, x) there exists either $\psi \sqsubseteq \phi$ or $y \sqsubseteq x$ complemented, then the d-lattice is normal for trivial reasons. The interaction algebra of Example 2 is in fact a d-lattice that is normal for trivial reasons, so it demonstrates that the join-strength rules do not necessarily hold. The d-lattice of Example 2 is derived as follows. Let $X = \{x, y\}$ be a two-element set and let $L_- = \{\emptyset, X\}$ and $L_+ = \mathcal{P}X$ be sub-lattices of its powerset. Consistent pairs of sets are disjoint and total pairs of sets cover X.

A topological space is normal if and only if the well-inside relation on the powerset is interpolative. This equivalence breaks down if one considers the well-inside relation restricted to the lattice of open sets. For example, the natural numbers with the cofinite topology is not a normal[3] space but the well-inside relation on the lattice of open sets is interpolative for trivial reasons, since there are no non-trivial disjoint pairs of opens.

2.4.2 The normal coreflection

Certainly the most useful property of normal d-lattices is that the well-inside relation $\lhd$ is interpolative. In a general d-lattice only the inclusion $(\lhd;\lhd) \subseteq \lhd$ holds. Using induction one shows that iterating the operator $R \mapsto (R;R)$ on the well-inside relation produces a descending chain of binary relations. The properties listed in Lemma 2.2.1 are all preserved under arbitrary intersection of relations. Thus one may consider the intersection of the descending chain $\lhd, (\lhd;\lhd), \ldots$ which is the greatest fixed point of the operator $R \mapsto (R;R)$ contained in the well-inside relation. Being such a fixed point, this relation is interpolative. Surprisingly one can use this interpolative relation to turn any d-lattice into a normal d-lattice.

A construction enforcing normality

Lemma 2.4.2. *Let $\mathcal{L}$ be a d-lattice and $\prec$ be an interpolative auxiliary relation on L_+, stronger than the well-inside relation, that is a sub-lattice of L_+^2. (In other words, $\prec$ has all properties that the well-inside relation of a normal d-lattice would have.) Define a relation $\mathsf{tot}^{\prec} = \mathsf{tot};\prec$. The structure $\mathcal{L}^{\prec} = (L_-, L_+, \mathsf{con}, \mathsf{tot}^{\prec})$ is a normal d-lattice where the well-inside relation on L_+ agrees with $\prec$.*

Proof. First notice that by moving from tot to $\mathsf{tot};\prec$ we do not break any d-lattice axioms. The join- and meet rules still hold because $\prec$ has all necessary algebraic properties and below we will see that the new well-inside relation is even stronger than the old one. We claim that the identity $(\prec;\lhd) = \prec$ holds. Indeed, since $\lhd$ is stronger than the lattice order, the composite $\prec;\lhd$ is contained in $\prec;\sqsubseteq$ and the latter is identical to $\prec$ because this relation is upward closed on the right. For the reverse inclusion, use the fact that $\prec$ is the same as $\prec;\prec$ and the assumption that $\prec$ is contained in $\lhd$. Similarly one proves

[3] Because of the Urysohn Lemma and the Intermediate Value Theorem, any connected normal space with at least two points must be uncountable.

that $\lhd;\prec$ is identical to $\prec$. With the claim proved we can write

$$\begin{aligned}
\mathsf{tot}^{\prec};\mathsf{con};\mathsf{tot}^{\prec} &= \mathsf{tot};\prec;\mathsf{con};\mathsf{tot};\prec \\
&= \mathsf{tot};\prec;\lhd;\prec \\
&= \mathsf{tot};\prec;\prec \\
&= \mathsf{tot};\prec \\
&= \mathsf{tot}^{\prec}.
\end{aligned}$$

The identity we just proved is precisely normality of the d-lattice $\mathcal{L}^{\prec}$. Its well-inside relations compute to

$$\begin{aligned}
\mathsf{con};\mathsf{tot}^{\prec} &= \mathsf{con};\mathsf{tot};\prec \\
&= \lhd;\prec \\
&= \prec \\
\mathsf{tot}^{\prec};\mathsf{con} &= \mathsf{tot};\prec;\mathsf{con}.
\end{aligned}$$

□

We intend to instantiate the preceding lemma with the interpolative relation described earlier. But before we do so, let us give a more interesting and concrete description.

The really-inside relation

Notation. We write $\mathbb{D}$ for the set of *dyadic rationals*, that are rational numbers of the form $\frac{m}{2^n}$ where n is a natural number and m is a natural number between 0 and 2^n.

The dyadic rationals can be defined inductively as the least fixed point of an inflationary operator on the powerset of the unit interval: The set $\mathbb{D}$ is the smallest subset of $[0, 1]$ with the properties $\{0, 1\} \subseteq \mathbb{D}$ and whenever $d, e \in \mathbb{D}$ then their midpoint (arithmetic mean) $\frac{d+e}{2}$ is also in $\mathbb{D}$. In Theorem 4.1.5 we will see that the dyadic rationals are intimately related to real-valued functions. Notice that the dyadic rationals form a bounded distributive lattice in the usual order, as does any chain with least and greatest element.

Definition 2.4.2. A *scale* between elements x_0 and x_1 of the component lattice L_+ of a d-lattice is an extension of the set $\{x_0, x_1\}$ to a dyadic-indexed set $\{x_d\}_{d\in\mathbb{D}}$ such that whenever $d < e$ in $\mathbb{D}$ then x_d is well-inside x_e.

Definition 2.4.3. Let x_0 and x_1 be elements of the component lattice L_+ of a d-lattice. We say that x_0 is *really inside* x_1 and write $x_0 \lessdot x_1$ if there exists a scale between x_0 and x_1.

It is immediate from the definition that the really-inside relation is stronger than the well-inside relation, because $0 < 1$ in $\mathbb{D}$ implies $x_0 \lhd x_1$. Further, if $\{x_d\}_{d\in\mathbb{D}}$ is a scale then $x_0 \leqslant x_{\frac{1}{2}} \leqslant x_1$ because one can re-index the set of dyadic rationals between 0 and $\frac{1}{2}$ using the map $d \mapsto 2d$ and obtain a scale between x_0 and $x_{\frac{1}{2}}$. Hence the really-inside relation is interpolative. Let us show that it has all the algebraic properties required in the hypothesis of Lemma 2.4.2. Trivially an element well inside itself is also really inside itself, as one can extend $x \lhd x$ to a scale where $x_d = x$ for all dyadic rationals. Thus $0 \leqslant 0$ and $1 \leqslant 1$. The really-inside relation is downward closed on the left and upward closed on the right, because if $x \sqsubseteq x_0 \lhd x_d \lhd x_1 \sqsubseteq y$ holds for all dyadics d strictly between 0 and 1 then also $x \lhd x_d \lhd y$ holds. Finally, element-wise binary meet or join of scales preserves the property of being a scale, as the well-inside relation is a sub-lattice of L_+^2.

Lemma 2.4.3. *In a normal d-lattice the well-inside relation is identical with the really inside relation.*

Proof. If $x_0 \lhd x_1$ then use the interpolation property of the well-inside relation to inductively extend $\{x_0, x_1\}$ to a scale. □

Since d-lattice homomorphisms preserve the well-inside relation, it is evident that the image of a scale under a d-lattice homomorphism is again a scale, whence d-lattice homomorphisms preserve the really-inside relation. We arrive at:

Theorem 2.4.4. *The category of d-lattices coreflects into the full subcategory of normal d-lattices via the functor that maps a d-lattice $\mathcal{L}$ to the d-lattice $\mathcal{L}^{\leqslant}$ as defined in Lemma 2.4.2 and leaves a d-lattice homomorphism unchanged.*

Proof. The counit of the coreflection is the pair of identity lattice homomorphisms $\mathrm{id}_{\mathcal{L}} : \mathcal{L}^{\leqslant} \to \mathcal{L}$. Being the identity on elements, this family of maps parametrised by $\mathcal{L}$ is clearly a natural transformation. Since the relation $\mathsf{tot};\leqslant$ is contained in the totality relation of $\mathcal{L}$, the identity homomorphism preserves both consistency and totality. It remains to show that any d-lattice homomorphism $\mathcal{N} \to \mathcal{L}$ from a normal d-lattice $\mathcal{N}$ into $\mathcal{L}$ factors uniquely through the normal coreflection $\mathcal{L}^{\leqslant}$. By Lemma 2.4.3 the totality relation on the normal d-lattice $\mathcal{N}$ coincides with $\mathsf{tot};\leqslant$ and since d-lattice homomorphisms preserve both tot and $\leqslant$, any d-lattice homomorphism from $\mathcal{N}$ into $\mathcal{L}$ can be regarded as a homomorphism into the normal coreflection. Thus we obtain the desired unique factorisation: A d-lattice homomorphism $\mathcal{N} \xrightarrow{h} \mathcal{L}$ factors as $\mathcal{N} \xrightarrow{h} \mathcal{L}^{\leqslant} \xrightarrow{\mathrm{id}_{\mathcal{L}}} \mathcal{L}$. □

Remark. It may seem that the normal coreflection of a d-lattice is an asymmetric construction. This is not so: Suppose ϕ_1 and ϕ_0 are elements of the component lattice L_- of a d-lattice $\mathcal{L}$ where ϕ_0 is well-inside ϕ_1 in the normal coreflection. From the proof of Lemma 2.4.2 we know that this means that $\phi_1 \mathsf{tot} x_0 \leqslant x_1 \mathsf{con} \phi_0$ for some elements $x_0, x_1 \in L_+$. We construct a scale between ϕ_0 and ϕ_1 using a scale between x_0 and x_1 as follows.

$$
\begin{aligned}
& \phi_1 \mathsf{tot} x_0 \leqslant x_1 \mathsf{con} \phi_0 \\
\Rightarrow\quad & \phi_1 \mathsf{tot} x_0 \leqslant x_{\frac{1}{4}} \lhd x_{\frac{3}{4}} \leqslant x_1 \mathsf{con} \phi_0 \\
\Rightarrow\quad & \exists \phi_{\frac{1}{2}} .\, \phi_1 \mathsf{tot} x_0 \leqslant x_{\frac{1}{4}} \mathsf{con} \phi_{\frac{1}{2}} \mathsf{tot} x_{\frac{3}{4}} \leqslant x_1 \mathsf{con} \phi_0 \\
\Rightarrow\quad & \phi_1 (\mathsf{tot}; \lhd; \mathsf{con}) \phi_{\frac{1}{2}} (\mathsf{tot}; \lhd; \mathsf{con}) \phi_0 \\
\Rightarrow\quad & \phi_1 \rhd \phi_{\frac{1}{2}} \rhd \phi_0
\end{aligned}
$$

In the next step, apply same construction as shown above to $\phi_1 \mathsf{tot} x_0 \leqslant x_{\frac{1}{4}} \mathsf{con} \phi_{\frac{1}{2}}$ and $\phi_{\frac{1}{2}} \mathsf{tot} x_{\frac{3}{4}} \leqslant x_1 \mathsf{con} \phi_0$ and so on until ϕ_d is defined for all dyadic rationals. Hence the well-inside relation on the component lattice L_- of the normal coreflection is the really-inside relation.

2.4.3 The open ideal completion of a normal d-lattice

As we promised above we proceed to show that normal d-lattices are yet another way of presenting stably continuous frames.

Proposition 2.4.5. *Let $\mathcal{L}$ be a normal d-lattice. Then the open ideals of each component lattice form a stably continuous frame and the way-below relation on it coincides with the well-inside relation induced by the open ideal completion d-lattice* $\mathrm{Idl}_\circ \mathcal{L}$.

Proof. All statements below apply to both component lattices of a normal d-lattice by virtue of the endofunctor Flip that preserves normality. Recall from Lemma 2.4.1 (5) that a normal d-lattice is a compact Stage 2a interaction algebra. Thus by Theorem 1.3.3 the open ideals of a component lattice form a continuous preframe. The way-below relation on it is characterised as follows (See Lemma 1.2.2). An open ideal I is way-below an open ideal J if and only if J contains an upper bound of I. In Lemma 2.3.4 we showed that this preframe is in fact a compact frame. If we knew that the dual join-strength rule of Table 1.3 held for any normal d-lattice, then stable local continuity would follow from Proposition 1.8.4. However, stable local continuity can be shown without this rule[4]. Indeed, in Lemma 2.3.6 we showed that the well-inside relation in the d-lattice of open ideals has the same characterisation as the one we gave for the way-below relation above. Now the well-inside relation is always closed under binary meets on the right, whence the domain of open ideals is stably locally continuous. □

According to Theorem 1.3.4 the component frames of the open ideal completion of a normal d-lattice are Lawson duals of each other. Can we relate the consistency and totality relations $\mathsf{con}_\circ$ and $\mathsf{tot}_\circ$ to the Lawson dual? Yes, indeed: By Proposition 1.3.5 The relation $I \mathsf{tot}_\circ J$ holds precisely when the ideal J is contained in the Scott open filter

[4] Later we will see why this does not contradict Proposition 1.8.4.

corresponding to I, whereas the relation $J\mathsf{con}_\circ I$ holds if and only if the ideal J is a lower bound for said Scott open filter.

2.4.4 A topological characterisation of normality

If a d-lattice is normal, the topology of open upper sets on a component lattice has a basis of open filters. Indeed, If $U \subseteq L_+$ is an open upper set and $x \in U$ then by definition there is some $y \in U$ that is well inside x. Hence x is an element of the open filter $\Uparrow y$ (it is open by Lemma 2.4.1 (3)) which in turn is a subset of U.

This observation leads to a topological view on the normality axiom: It says that whenever a pair (ϕ, x) is total then there exist basic opens $\Uparrow\psi$ and $\Uparrow y$ such that $(\phi, x) \in (\Uparrow\psi \times \Uparrow y) \subseteq \mathsf{tot}$. In other words, a normal d-lattice has a totality relation that is open as a subset of the product space $(L_-, \mathrm{Up}^{\lhd} L_-) \times (L_+, \mathrm{Up}^{\lhd} L_+)$. The converse is also true: If the totality relation is open in that product space then $\phi\mathsf{tot}x$ implies that there are open sets $\phi \in \Phi \in \mathrm{Up}^{\lhd} L_-$ and $x \in U \in \mathrm{Up}^{\lhd} L_+$ with $\Phi \times U \subseteq \mathsf{tot}$. Consequently, there exists some $u \in U$ well inside x with the property $\phi\mathsf{tot}u \lhd x$, whence the d-lattice is normal.

2.5 Cut rules and the patch lattice

The intuition we suggest the reader to keep in mind when thinking about d-lattices is the situation of Example 6: There is an ambient bounded distributive lattice L, which the component lattices L_- and L_+ are sub-lattices of, such that an element $x \in L_+$ is consistent with an element $\phi \in L_-$ precisely when the meet $x \sqcap \phi$ in L is 0 and similarly ϕ is total with x if and only if $\phi \sqcup x = 1$ holds.

The question that we address in this section is: Does every d-lattice arise this way? The answer is "no" and in due course we shall derive two necessary conditions and point to examples violating these.

2.5.1 The finitary cut rules

Let ϕ, ψ, x and y be elements of a bounded distributive lattice L and suppose we have $\psi \sqcup x \sqcup y = 1 = \phi \sqcup \psi \sqcup y$ and $x \sqcap \phi = 0$. Then using the distributive law one finds

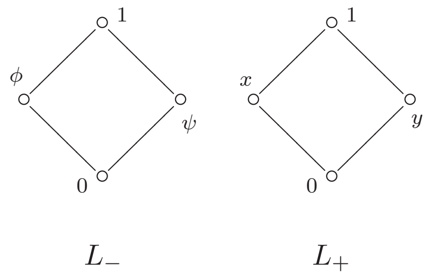
$$\begin{aligned} \psi \sqcup y &= (\psi \sqcup y) \sqcup (x \sqcap \phi) \\ &= (\psi \sqcup y \sqcup x) \sqcap (\psi \sqcup y \sqcup \phi) \\ &= 1 \sqcap 1 = 1. \end{aligned}$$

By considering the order-dual, one finds that $y \sqcap \psi = 0$ whenever $y \sqcap \phi \sqcap \psi = 0 = y \sqcap x \sqcap \psi$ and $x \sqcup \phi = 1$. Thus any d-lattice that arises from two sub-lattices of an ambient bounded distributive lattice must obey the *finitary cut rules*:

$$\frac{y\,\mathsf{con}(\psi \sqcap \phi) \qquad \phi\,\mathsf{tot}\,x \qquad (x \sqcap y)\,\mathsf{con}\,\psi}{y\,\mathsf{con}\,\psi}\ (\mathrm{cut}_{\mathsf{con}})$$

$$\frac{\psi\,\mathsf{tot}(y \sqcup x) \qquad x\,\mathsf{con}\,\phi \qquad (\phi \sqcup \psi)\,\mathsf{tot}\,y}{\psi\,\mathsf{tot}\,y}\ (\mathrm{cut}_{\mathsf{tot}})$$

Example 7. Consider a d-lattice $\mathcal{L} = (L_-, L_+, \mathsf{con}, \mathsf{tot})$ with the component lattices depicted below.



Let the consistency relation be the lower set of $\{(x, \phi), (0, 1), (1, 0)\}$ in the information order and the totality relation be the upper set of $\{(\phi, x), (0, 1), (1, 0)\}$ in the information order. This renders $\mathcal{L}$ a d-lattice where the premises of both finitary cut rules are satisfied but neither $y\,\mathsf{con}\,\psi$ nor $\psi\,\mathsf{tot}\,y$ hold.

The counterexample above is a rather esoteric one. Let us consider conditions under which the cut rules *do* hold.

Lemma 2.5.1. *D-lattices that arise from an auxiliary relation on a bounded distributive lattice in the way of Example 4 obey the cut rules.*

Proof. Suppose a d-lattice is derived from a lattice with relation $\prec$ as such:

$$L^{\partial} \underset{\sqsubseteq}{\overset{\prec}{\rightleftarrows}} L$$

Let x, y, ϕ, ψ be elements of a bounded distributive lattice L and suppose $\psi \prec (y \sqcup x)$, $x \sqsubseteq \phi$ and $(\phi \sqcup \psi) \prec y$. Then also $\psi \prec (y \sqcup \phi)$ holds, and knowing that $\prec$ is a sub-lattice of L^2 one deduces the rule $(\mathrm{cut}_{\mathsf{tot}})$ as

$$\psi = \psi \sqcap (\psi \sqcup \psi) \prec (y \sqcup \phi) \sqcap y = y.$$

Dually, suppose that $y \sqsubseteq (\psi \sqcap \phi)$, $\phi \prec x$ and $(x \sqcap y) \sqsubseteq \psi$. Since $\prec$ is stronger than the lattice order also $(\phi \sqcap y) \sqsubseteq \psi$ holds and therefore the rule $(\mathrm{cut}_{\mathsf{con}})$ is derivable as

$$y = y \sqcup (\phi \sqcap y) \sqsubseteq (\psi \sqcup \phi) \sqcup \psi = \psi.$$

□

Join-strong d-lattices

As we remarked in Section 2.4, a normal d-lattice is almost a Stage 5 interaction algebra, except that the join-strength rules (see Tables 1.2 and 1.3) might fail. Even without normality, join-strength is a useful concept.

Definition 2.5.1. A d-lattice is called *join-strong on the right* if it obeys the rule on the right below. Dually, the d-lattice is *join-strong on the left* if the rule on the left below holds.

$$\frac{(\phi \sqcup \psi)\mathsf{tot}x}{\exists \phi \mathsf{tot} a, \exists \psi \mathsf{tot} b.\, a \sqcap b \sqsubseteq x} \qquad \frac{\phi \mathsf{tot}(x \sqcup y)}{\exists \psi \mathsf{tot} x, \exists \theta \mathsf{tot} y.\, \phi \sqsupseteq \psi \sqcap \theta}$$

A d-lattice that is join-strong both on the left and on the right is called join-strong.

One can give a characterisation of join-strength in terms of the map $(-)^{\mathsf{tot}}$ on ideals of a component lattice. Recall that the binary join of ideals of a bounded distributive lattice can be computed as element-wise join and dually the binary join of filters of a bounded distributive lattice can be computed as element-wise meet. Let $I, J \subseteq L_+$ be ideals and consider the filter

$$(I \vee J)^{\mathsf{tot}} = \{\phi \in L_- \mid \exists (x, y) \in I \times J.\, \phi \mathsf{tot}(x \sqcup y)\}\,.$$

If the d-lattice is join-strong on the right then the filter above equals the join of the filters I^{tot} and J^{tot}. By considering principal ideals one finds that the d-lattice is join-strong on the right precisely when the map $(-)^{\mathsf{tot}} : \mathrm{Idl}\, L_+ \to \mathrm{Filt}\, L_-$ preserves binary joins.

Lemma 2.5.2. *A d-lattice that is either join-strong on the left or on the right satisfies the cut rule* ($\mathrm{cut}_{\mathsf{tot}}$)*.*

Proof. Without loss of generality assume that the d-lattice is join-strong on the right. Suppose the premise of the rule ($\mathrm{cut}_{\mathsf{tot}}$) is satisfied, meaning $\psi \mathsf{tot}(x \sqcup y)$, $x \mathsf{con} \phi$ and $(\phi \sqcup \psi)\mathsf{tot} y$. By the join-strength rule there exist $\theta \mathsf{tot} x$ and $\zeta \mathsf{tot} y$ with $\psi \sqsupseteq \theta \sqcap \zeta$. Together with $x \mathsf{con} \phi$ we deduce that ϕ is well inside θ which in particular means that $\theta \sqsupseteq \phi$. Since the relation tot is upward closed we obtain $(\theta \sqcup \psi)\mathsf{tot} y$, and together with $\zeta \mathsf{tot} y$ we conclude $(\theta \sqcup \psi) \sqcap \zeta \mathsf{tot} y$. Using the distributive law we can write $(\theta \sqcup \psi) \sqcap \zeta = (\theta \sqcap \zeta) \sqcup (\psi \sqcap \zeta)$ and because of $\psi \sqsupseteq \theta \sqcap \zeta$ we obtain $\psi = \psi \sqcup (\psi \sqcap \zeta)\mathsf{tot} y$. □

D-lattices with closed consistency

Consider the component lattices of a d-lattice as topological spaces endowed with the topologies of open upper sets described in Definition 2.3.1. We characterised the normal d-lattices as those d-lattices where the totality relation is open as a subset of the product space $L_- \times L_+$. So it is natural to consider d-lattices where the consistency relation is closed as a subset of the product $L_+ \times L_-$.

Definition 2.5.2. We say that a d-lattice $\mathcal{L}$ has *closed consistency* if the relation $\mathsf{con} \subseteq L_+ \times L_-$ is closed when the component lattices are endowed with their topologies of open upper sets according to Definition 2.3.1. Concretely, a d-lattice has closed consistency if whenever x is not consistent with ϕ then there exists some y well inside x and some ψ well inside ϕ such that y is still not consistent with ψ.

Lemma 2.5.3. *A d-lattice with closed consistency satisfies the cut rule* ($\text{cut}_{\mathsf{con}}$)*.*

Proof. Let $y\mathsf{con}(\psi \sqcap \phi)$, $\phi\mathsf{tot}x$ and $(x \sqcap y)\mathsf{con}\psi$. According to the cut rule ($\text{cut}_{\mathsf{con}}$) we should have $y\mathsf{con}\psi$. We assume that y is not consistent with ψ and derive a contradiction under the hypothesis that the consistency relation is closed. Because of closed consistency, we may assume that there exist elements $b\mathsf{con}\beta\mathsf{tot}y$ and $\psi\mathsf{tot}a\mathsf{con}\alpha$ such that b is not consistent with α.

Our first claim is that $\beta \sqsupseteq \psi$. From $\beta\mathsf{tot}y$ and $\phi\mathsf{tot}x$ deduce that $\beta \sqcup \phi$ is total with $x \sqcap y$, whence by $(x \sqcap y)\mathsf{con}\phi$ we obtain that ψ is well inside $\beta \sqcup \phi$. Consequently $(\psi \sqcap \beta) \sqcup (\psi \sqcap \phi)$ is larger than ψ. Observe that because of $\beta\mathsf{tot}y\mathsf{con}(\psi \sqcap \phi)$ we have $\beta \sqsupseteq (\psi \sqcap \phi)$ whereby we obtain $(\beta \sqcap \psi) \sqcup \beta \sqsupseteq \psi$. Apply the absorption law $\beta = (\beta \sqcap \psi) \sqcup \beta$ to finish the proof of the claim.

From $b\mathsf{con}\beta$ and $\beta \sqsupseteq \psi$ one derives $b\mathsf{con}\psi$. But then $b\mathsf{con}\psi \rhd \alpha$ implies $b\mathsf{con}\alpha$ in contradiction to the choice of b and α. Therefore the assumption that y is not consistent with ψ must be false and so the cut rule ($\text{cut}_{\mathsf{con}}$) holds. □

As an example of a d-lattice that has no closed consistency consider Example 7. Here y is not consistent with ψ but the only elements well inside y or ψ are the least elements of the component lattices. As 0 is always consistent with 0 we conclude that the consistency relation is not closed.

2.5.2 The patch lattice

Definition 2.5.3. Given a bounded distributive lattice L, let $L_=$ denote the *symmetric d-lattice* over L whose component lattices are both identical to L and where $\phi\mathsf{tot}_=x$ holds whenever $\phi \sqcup x = 1$ in L and $x\mathsf{con}_=\phi$ if $x \sqcap \phi = 0$. A homomorphism h of bounded distributive lattices lifts to a d-lattice homomorphism (h, h) between symmetric d-lattices.

Clearly the assignment $(-)_=$ constitutes a functor from the category of bounded distributive lattices to the category of d-lattices. This functor has a left adjoint that can be seen as the canonical way of embedding the component lattices into an ambient lattice.

Definition 2.5.4. The *patch lattice* $\operatorname{Patch}\mathcal{L}$ of a d-lattice $\mathcal{L}$ has generators $\ulcorner\phi\urcorner^-$ and $\ulcorner x\urcorner^+$ where ϕ ranges over elements of the component lattice L_- and x ranges over the component lattice L_+. One quotients the free bounded distributive lattice over these generators by the rules enforcing that that the pair $(\ulcorner-\urcorner^-, \ulcorner-\urcorner^+)$ is a d-lattice homomorphism from $\mathcal{L}$ to $(\operatorname{Patch}\mathcal{L})_=$.

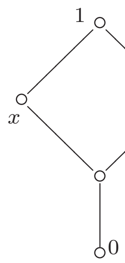
Concretely the relations on the patch lattice say that $\ulcorner - \urcorner^- : L_- \to \operatorname{Patch} \mathcal{L}$ and $\ulcorner - \urcorner^+ : L_+ \to \operatorname{Patch} \mathcal{L}$ are lattice homomorphisms and further the rules

$$\frac{x \mathsf{con} \phi}{\ulcorner x \urcorner^+ \sqcap \ulcorner \phi \urcorner^- = 0} \qquad \frac{\phi \mathsf{tot} x}{\ulcorner \phi \urcorner^- \sqcup \ulcorner x \urcorner^+ = 1}$$

hold. Given a d-lattice homomorphism $h = (h_-, h_+)$ one obtains a map between generators of the patch lattices by letting $\operatorname{Patch}(h)(\ulcorner \phi \urcorner^-) = \ulcorner h_-(\phi) \urcorner^-$ and $\operatorname{Patch}(h)(\ulcorner x \urcorner^+) = \ulcorner h_+(x) \urcorner^+$. Evidently the function $\operatorname{Patch}(h)$ preserves all relations by which the free lattice over the generators is factored. Thus the assignment $\mathcal{L} \mapsto \operatorname{Patch} \mathcal{L}$ extends to a functor from d-lattices to lattices. The same calculation that was used at the beginning of Subsection 2.5.1 to motivate the cut rules shows that whenever the premise of the rule ($\mathrm{cut}_{\mathsf{con}}$) holds in a d-lattice then $\ulcorner y \urcorner^+ \sqcap \ulcorner \psi \urcorner^- = 0$ holds in the patch lattice, and similarly for the rule ($\mathrm{cut}_{\mathsf{tot}}$).

The composite functor $\operatorname{Patch} \circ (-)_=$ is not always the identity on objects, as for example the three-element chain is transformed to the free bounded distributive lattice over two generators. Another negative result about the patch lattice is that it may destroy normality. This was to be expected, as the normal coreflection construction for d-lattices demonstrates that normality of d-lattices is much easier to achieve than normality for lattices. Here is the possibly simplest counterexample.

Example 8. Consider the bounded distributive lattice L_+ pictured below.



It is not normal in the sense of Definition 6.3.7, as $x \sqcup y = 1$ but there are no non-trivial pairs whose meet is 0. Complete this lattice to a d-lattice where the component L_- is the two-element chain. Observe that the consistency and totality relations on such a structure are uniquely determined by the axioms. Now the d-lattice we defined is normal for trivial reasons. Its patch lattice, however, is isomorphic to L_+ and thereby not normal.

Theorem 2.5.4. *The patch functor is left adjoint to the symmetric d-lattice functor.*

Proof. The unit η of the adjunction is the pair $(\ulcorner - \urcorner^-, \ulcorner - \urcorner^+)$ from a d-lattice to the symmetric d-lattice over the patch. The patch lattice was defined so that this pair is indeed a d-lattice homomorphism.

For a lattice A and a d-lattice homomorphism $f : \mathcal{L} \to A_=$ define a lattice homomorphism $\ulcorner f \urcorner$ from $\operatorname{Patch} \mathcal{L}$ to A by its action on the generators: $\ulcorner \phi \urcorner^- \mapsto f_-(\phi)$ and $\ulcorner x \urcorner^+ \mapsto f_+(x)$. This extends to a well-defined lattice homomorphism precisely because

f preserves con and tot. Clearly $(\ulcorner f \urcorner)_= \circ \eta_{\mathcal{L}}$ equals f. For uniqueness, observe that any lattice homomorphism $h : \mathrm{Patch}\,\mathcal{L} \to A$ with $h_= \circ \eta_{\mathcal{L}} = f$ must coincide with $\ulcorner f \urcorner$ on the generators of $\mathrm{Patch}\,\mathcal{L}$. □

Lemma 2.5.5. *1. A complemented pair of a d-lattice gives rise to a complemented pair of the patch lattice.*

2. If an element x of a bounded distributive lattice has a complement, then the generators $\ulcorner x \urcorner^-$ and $\ulcorner x \urcorner^+$ of the patch lattice $\mathrm{Patch}(L_=)$ *are equal.*

Proof. (1) If $\phi \mathsf{tot} x \mathsf{con} \phi$ is a complemented pair in a d-lattice then by definition of the patch lattice $\ulcorner \phi \urcorner^- \sqcup \ulcorner x \urcorner^+ = 1$ and $\ulcorner x \urcorner^+ \sqcap \ulcorner \phi \urcorner^- = 0$.

For the proof of (2) it is convenient to use a characterisation of the lattice distributive law that can for example be found in [13, Exercise 6.6]. Suppose x and y are complements in the bounded distributive lattice L, that is, $x \sqcap y = 0$ and $x \sqcup y = 1$. Then in the symmetric d-lattice over L we have $x \mathsf{tot}_= y$ and $y \mathsf{con}_= x$ but also $x \sqcap y = 0$ and $x \sqcup y = 1$ in the component lattice L_+. Therefore in the patch lattice the generators satisfy the identities

$$\begin{aligned} \ulcorner x \urcorner^- \sqcap \ulcorner y \urcorner^+ &= 0 = \ulcorner x \urcorner^+ \sqcap \ulcorner y \urcorner^+ \\ \ulcorner x \urcorner^- \sqcup \ulcorner y \urcorner^+ &= 1 = \ulcorner x \urcorner^+ \sqcup \ulcorner y \urcorner^+ \end{aligned}$$

whereby the generators $\ulcorner x \urcorner^-$ and $\ulcorner x \urcorner^+$ are identical in the patch lattice. □

Corollary 2.5.6. *The composite functor* $\mathrm{Patch} \circ (-)_=$ *is the identity on boolean algebras.*

As a neat application, let us demonstrate that the patch lattice can be used to compute the free boolean algebra over a bounded distributive lattice.

Proposition 2.5.7. *If $(L, \sqsubseteq)$ is a bounded distributive lattice then the patch lattice of the d-lattice*

$$L^{\partial} \underset{\sqsubseteq}{\overset{\sqsubseteq}{\rightleftarrows}} L$$

is the free boolean algebra over L.

Proof. First we show that the patch lattice of the d-lattice $\mathcal{L}$ shown in the statement is indeed a boolean algebra. For every element x of the lattice L we have two generators $\ulcorner x \urcorner^-$ and $\ulcorner x \urcorner^+$. Let us write $\ulcorner x \urcorner$ for the latter generator and $\neg x$ for the former. The relations imposed by consistency and totality say that $x \sqsubseteq y$ implies that $\neg x \sqcup \ulcorner y \urcorner = 1$ and $\ulcorner x \urcorner \sqcap \neg y = 0$. In particular this means that the generator $\neg x$ is the complement of the generator $\ulcorner x \urcorner$. Hence every generator has a complement. Now every element of the free distributive lattice over a set is a finite join of finite meets of generators (see Section 1.12), and since complemented elements are closed under finite joins and meets it follows that every element of the patch lattice is complemented. Thus $\mathrm{Patch}\,\mathcal{L}$ is a boolean algebra.

Given a bounded distributive lattice L, the generator map $x \mapsto \ulcorner x \urcorner$ is a lattice homomorphism from L into $\mathrm{Patch}\,\mathcal{L}$. Obviously this assignment is natural in the parameter L. We have to show that any lattice homomorphism $h : L \to B$ into a boolean algebra factors uniquely through the generator embedding. We construct a map $\mathrm{Patch}\,\mathcal{L} \to B$ as follows. Using $\neg$ for the negation on the boolean algebra B and the de-Morgan laws, one finds that the map $\neg \circ h$ is a lattice homomorphism from the order dual L^{∂} to B. Moreover, the pair $(\neg \circ h, h)$ is a d-lattice homomorphism from $\mathcal{L}$ to the symmetric d-lattice $B_=$. The image of this homomorphism under the functor Patch is a lattice homomorphism from $\mathrm{Patch}\,\mathcal{L}$ to $\mathrm{Patch}(B_=)$. By construction the image of the generator $\ulcorner x \urcorner$ under this map is $h(x)$, so we have a commutative triangle:

$$\begin{array}{ccc} L & \xrightarrow{\ulcorner \_ \urcorner} & \mathrm{Patch}\,\mathcal{L} \\ & \searrow^{h} & \downarrow{\scriptstyle \mathrm{Patch}(\neg\circ h, h)} \\ & & B \cong \mathrm{Patch}(B_=) \end{array}$$

For uniqueness of this factorisation observe that any boolean homomorphism $\mathrm{Patch}\,\mathcal{L} \to B$ making the triangle commute must coincide with $\mathrm{Patch}(\neg \circ h, h)$ on the generators. □

The patch lattice can be used to compute the coproduct in the category of bounded distributive lattices.

Proposition 2.5.8. *Let L_- and L_+ be bounded distributive lattices. Then the coproduct $L_- + L_+$ in the category of bounded distributive lattices and homomorphisms is the patch lattice of the d-lattice $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ where the relations con and tot are minimal.*

Proof. The construction of coproducts in a category such as Lat follows a general pattern of universal algebra: In order to compute the coproduct of structures L_- and L_+ one forms the free object over the set $L_- + L_+$ (here the operation $+$ denotes the coproduct in Set, i.e. disjoint union) and then quotients by the relations that enforce the injection of generators to be homomorphisms with respect to the algebraic structure. In the case of bounded distributive lattices the coproduct thus becomes

$$\mathsf{Lat}\left\langle \ulcorner \phi \urcorner^-, \ulcorner x \urcorner^+ ; \phi \in L_-, x \in L_+ \,\middle|\, \ulcorner - \urcorner^-, \ulcorner - \urcorner^+ \text{ are lattice homomorphisms} \right\rangle$$

which is the same as the patch lattice of the d-lattice with components L_- and L_+ and minimal consistency and totality. □

2.6 Notes on Chapter 2

Although the concept was known to the authors, Jung and Moshier focused in their technical report [33] almost entirely on *d-frames* rather than d-lattices. The usefulness of d-lattices became apparent with the discovery of the open ideal completion and its relevance to compactifications.

While most of the results about normal d-lattices are new, conceptually the proofs are standard domain theory. For example, Proposition 2.4.5 is, modulo nomenclature, the same as [50, Theorem 1] or [23, 5.6]. The really-inside relation was used by Johnstone [30] and independently by Banaschewski [8, 4] to define completely regular locales and their Stone-Čech compactifications. The normal coreflection Theorem 2.4.4 is a new result that is believed to set apart the theory of d-lattices from other point-free treatments of topology. Although we can not provide a good example at the moment, we believe that the normal coreflection in general breaks the cut rules of Section 2.5. The cut rules have their name from Gentzen's cut rule in logic. The definition of the patch lattice is a straightforward analogue of Jung and Moshier's construction (Definition 3.4.1). Theorem 2.5.4 and the applications following it are new.

Future work

Little is known about the categorical structure of dLat. While the categorical product is straightforward, the existence of coproducts is not obvious. A reasonable starting point is to take the coproduct of component lattices in the category Lat and define the consistency and totality relations inductively. However, the axiom (con-tot) might force one to take quotients of the component lattices eventually.

There is a spectral theory of d-lattices that subsumes Priestley duality and yields interesting bitopological variants carrying two orders. The adept reader will be able to reconstruct this spectral theory, knowing the spectral theory of d-frames laid out in Section 3.2. While the condition *join-strength* is known under the name *Wilker property* among topologists and often used as a technical condition, the connection between normal d-lattices and Stage 5 interaction algebras are still somewhat mysterious. We do not know whether there exists a way of transforming any normal d-lattice into a join-strong normal d-lattice.

Chapter 3

D-frames

3.1 Introduction

It is our intention to develop a concrete Stone duality for the category of bitopological spaces and bicontinuous maps. Locale theory tells us that in the vast majority of cases one can recover the points of a topological space just from the algebraic structure of its frame of opens. However, there are interesting examples of bitopological spaces where each of the two topologies, when considered for itself, tells little about how many points the space really has. For a finite example that will gain importance later the reader may consult Figure 3.1. Knowing that it does not suffice to record the two frames of open sets, one wonders how much more information one needs in order reconstruct the points of the underlying set. A solution that Banaschewski, Hardie and Brümmer [6] offer is to record the common refinement of the two topologies and how the two original topologies are embedded into it. As Jung and Moshier remarked in their report [33], the resulting duality between the categories of bitopological spaces and the category BiFrm of *biframes* and biframe homomorphisms cannot be concrete over Set, essentially because the free biframe over the one-point space has the wrong shape.

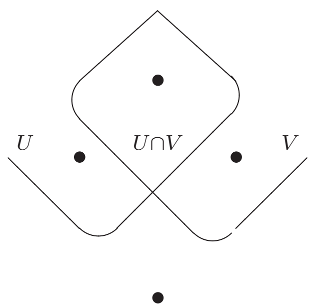


Figure 3.1: A four-element bitopological space where each topology is not T_0, but the common refinement is. One topology has U as the only non-trivial open set, while the other topology's only non-trivial open set is V.

Can there be a more efficient way of recording how the opens of one topology interact with the opens of the other? Let us consider a generic bitopological space (X, τ_-, τ_+) and a point x in it. Consider the (completely prime) filters $F_- \subseteq \tau_-$ and $F_+ \subseteq \tau_+$ of open neighbourhoods of the point x. For any pair $(U, V) \in F_- \times F_+$ we know for certain that U and V can not be disjoint, as both sets contain x. Moreover, given any disjoint pair (U, V) of opens, where $U \in \tau_-$ and $V \in \tau_+$, the point x evidently can be an element of at most one of them. Phrased differently, if $U \cap V = \emptyset$ and $x \in U$ then V can not be a neighbourhood of x. Dually, is there a condition that forces an open V to be a neighbourhood of x? Indeed, suppose the union of opens $U \cup V$ covers the space X and suppose further that the point x is not an element of U. Clearly then V must be a neighbourhood of x, because otherwise we had a contradiction to the hypothesis that U and V cover the space. Thus we arrive at the following two conditions that the pair of neighbourhood filters (F_-, F_+) satisfies.

- Whenever two opens $U \in \tau_-$ and $V \in \tau_+$ are disjoint, then either $U \notin F_-$ or $V \notin F_+$.

- Whenever two opens $U \in \tau_-$ and $V \in \tau_+$ cover the space, then either $U \in F_-$ or $V \in F_+$.

Let us assume for the moment that the two conditions above distinguish those pairs of neighbourhood filters (F_-, F_+) that are neighbourhood filters of the same point among the set of all pairs of neighbourhood filters. In order be able to phrase these conditions we need to record when a pair $(U, V) \in \tau_- \times \tau_+$ is disjoint and when it covers the space. Observe that for a fixed open $U \in \tau_-$ the set of all opens $V \in \tau_+$ disjoint from U has a largest element, namely the interior of the complement of U with respect to the topology τ_+. Dually, the set of opens $V \in \tau_+$ that cover the space together with U is a filter, namely the filter of open neighbourhoods of the τ_--closed set $X \setminus U$.

A bicontinuous map $f : (X, \tau_-, \tau_+) \to (Y, \nu_-, \nu_+)$ is separately continuous as a map $(X, \tau_-) \to (Y, \nu_-)$ and $(X, \tau_+) \to (Y, \nu_+)$. Therefore the preimage map f^{-1} gives rise to a pair of frame homomorphisms $\nu_- \to \tau_-$ and $\nu_+ \to \tau_+$. Furthermore, if U and V are disjoint then the preimage $f^{-1}(U)$ is disjoint from $f^{-1}(V)$. Similarly, whenever a pair of opens cover the space Y then the pair of preimages under f will cover the space X.

Let us record the observations made into a working hypothesis that we are to validate in the next section.

Working hypothesis A bitopological space (X, τ_-, τ_+) is described completely by the pair of frames of opens (τ_-, τ_+) together with the binary relations between them that record disjointness and covering of the space. A bicontinuous map is entirely described by a pair of frame homomorphisms between the topologies that preserves the disjointness and covering relations.

3.2 Concrete Stone duality for bitopological spaces

We begin by casting the observations and the working hypothesis made in the introduction into a definition.

Definition 3.2.1. The category dFrm of *d-frames* has objects of the form

$$L_- \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} L_+$$

where L_- and L_+ are frames and con and tot are two relations satisfying the axioms of Table 3.1. We call con the *consistency relation* and say that $x \in L_-$ is consistent with $\phi \in L_-$ if (x, ϕ) is an element of the consistency relation. We call tot the *totality relation* and say that ϕ is total with x if (ϕ, x) is an element of the totality relation. A homomorphism of d-frames is a pair of frame homomorphisms

$$\begin{array}{ccc} L_- & \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} & L_+ \\ {\scriptstyle f_-}\downarrow & & \downarrow{\scriptstyle f_+} \\ M_- & \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} & M_+ \end{array}$$

that preserve the relations, meaning that $x\mathsf{con}\phi$ implies $f_+(x)\mathsf{con}f_-(\phi)$ and $\phi\mathsf{tot}x$ implies $f_-(\phi)\mathsf{tot}f_+(x)$. Composition of homomorphisms is done coordinate-wise in the obvious way.

Notation. We typically denote d-frames with uppercase script letters $\mathcal{L}, \mathcal{M}, \dots$ and their component frames by the same letter in standard font. The elements of the first component frame are denoted by Greek letters $\phi, \psi, \dots$ (except for the frame constants 0 and 1) whereas the elements of the second component frame are referred to by Roman letters $x, y, \dots$ D-frame homomorphisms commonly have Roman letters $f, g, h \dots$ and their components are subscripted accordingly. For example, f denotes a pair (f_-, f_+) of frame homomorphisms between component frames.

3.2.1 D-frames are d-lattices

The only axiom of Table 3.1 that can not be found in Table 2.1 listing the axioms of a d-lattice is the axiom (con-$\bigsqcup$) stating that every element of each component frame has a maximal counterpart in the other component frame that it is consistent with. As every frame is in particular a bounded distributive lattice, the category of d-frames is a subcategory of the category of d-lattices. If $\mathcal{L}$ is a d-lattice where both component lattices are finite, then the rules (con-$\vee$) and (con-$\wedge$) entail the rule (con-$\bigsqcup$) whence finite d-lattices and finite d-frames are the same thing.

lower set	$\dfrac{x \mathsf{con} \phi}{(x \sqcap y) \mathsf{con} (\phi \sqcap \psi)}$		(con- ↓)
binary meet and join	$\dfrac{x \mathsf{con} \phi \qquad y \mathsf{con} \psi}{(x \sqcap y) \mathsf{con} (\phi \sqcup \psi)}$	$\dfrac{x \mathsf{con} \phi \qquad y \mathsf{con} \psi}{(x \sqcup y) \mathsf{con} (\phi \sqcap \psi)}$	(con-∧), (con-∨)
arbitrary join	$\dfrac{\forall \phi \in \Phi.\, x \mathsf{con} \phi}{x \mathsf{con} \bigsqcup \Phi}$	$\dfrac{\forall x \in X.\, x \mathsf{con} \phi}{\bigsqcup X \mathsf{con} \phi}$	(con-$\bigsqcup$)
empty meet and join	$\overline{0 \mathsf{con} 1}$	$\overline{1 \mathsf{con} 0}$	(con-𝕗), (con-𝕥)
upper set	$\dfrac{\phi \mathsf{tot} x}{(\phi \sqcup \psi) \mathsf{tot} (x \sqcup y)}$		(tot- ↑)
binary meet and join	$\dfrac{\phi \mathsf{tot} x \qquad \psi \mathsf{tot} y}{(\phi \sqcap \psi) \mathsf{tot} (x \sqcup y)}$	$\dfrac{\phi \mathsf{tot} x \qquad \psi \mathsf{tot} y}{(\phi \sqcup \psi) \mathsf{tot} (x \sqcap y)}$	(tot-∨), (tot-∧)
empty meet and join	$\overline{0 \mathsf{tot} 1}$	$\overline{1 \mathsf{tot} 0}$	(tot-𝕥), (tot-𝕗)
consistency vs. totality	$\dfrac{x \mathsf{con} \phi \qquad \phi \mathsf{tot} y}{x \sqsubseteq y}$	$\dfrac{\phi \mathsf{tot} x \qquad x \mathsf{con} \psi}{\phi \sqsupseteq \psi}$	(con-tot)

Table 3.1: Axioms for a d-frame in the interaction algebra style. The third column lists the names used in [33].

A frame can be regarded as a bounded distributive lattice which is directed complete and satisfies the preframe distributive law. And indeed one may state the rule (con-$\bigsqcup$) for directed sets only, as the arbitrary case then follows together with the rules (con-∨) and (con-∧). A fact that is well-known in domain theory is that a directed subset of the product of two dcpos is essentially the same thing as a pair of directed subsets of the factors. Hence one may phrase the axiom (con-$\bigsqcup$) of Table 3.1 as: "The consistency relation con of a d-frame $\mathcal{L}$ is Scott closed as a subset of the dcpo $L_+ \times L_-$."

Proposition 3.2.1. *The following are equivalent for a d-lattice $\mathcal{L}$.*

1. *The d-lattice $\mathcal{L}$ is a d-frame.*
2. *The component lattices L_- and L_+ are preframes and the consistency relation* con *is a Scott closed subset of $L_+ \times L_-$.*

Furthermore, those d-lattice homomorphisms between d-frames that are d-frame homomorphisms are precisely the pairs (f_-, f_+) whose component maps are Scott continuous.

3.2.2 The forgetful functor and free d-frames

As promised, we are going to develop a duality between bitopological spaces and d-frames as a concrete duality. Therefore we have to specify a forgetful functor dFrm → Set and show that it has a left adjoint. As with d-lattices, one can regard consistency and totality relations of a d-frame $\mathcal{L}$ as predicates on the product frame $L_- \times L_+$ and d-frame homomorphisms as frame homomorphisms between product frames of this kind that preserve these predicates and the constants $\mathbb{f} = (1, 0)$ and $\mathbb{t} = (0, 1)$. Hence the obvious choice of a forgetful functor from d-frames to sets is the following.

Definition 3.2.2. The forgetful functor $U_{\mathsf{dFrm}} : \mathsf{dFrm} \to \mathsf{Set}$ takes a d-frame homomorphism $f = (f_-, f_+)$ between d-frames $\mathcal{L}$ and $\mathcal{M}$ to the product map between sets $f_- \times f_+ : (L_- \times L_+) \to (M_- \times M_+)$.

The definition above indeed defines a functor which is, as required, faithful. For our concrete Stone duality we need a left adjoint to this functor where we are particularly interested in the free d-frame over the one-point set. Recall that the free frame $F_{\mathsf{Frm}}1$ over the one-point set is the three-element chain. Indeed, given a frame L the set of functions from the one-point set into L is isomorphic to L as a set. Now, a frame homomorphism $F_{\mathsf{Frm}}1 \to L$ must preserve the constants 0 and 1 but may map the middle element a of the three-chain to any element of the frame L.

Given a d-frame $\mathcal{L}$, the universal property of the cartesian product in Set tells us that a function $X \to L_- \times L_+$ is the same as a pair of functions $X \to L_-$ and $X \to L_+$. This suggests that the components of the free d-frame over the set X should be two copies of the free frame $F_{\mathsf{Frm}}X$ over X. What should the consistency and totality relations be? These must be the minimal relations allowed by the axioms of a d-frame, because otherwise they constrain what pairs of frame homomorphisms from $F_{\mathsf{Frm}}X$ to L_- and L_+ qualify as d-frame homomorphisms.

Proposition 3.2.2. *The free d-frame $F_{\mathsf{dFrm}}X$ over a set X has as component frames two copies of the free frame over X. The consistency and totality relations on the free d-frame are minimal. This means that x is consistent with ϕ precisely when $x = 0$ or $\phi = 0$, and dually ϕ is total with x if and only if either $\phi = 1$ or $x = 1$ holds.*

The free d-frame over the one-point set is pictured in Figure 3.2.

3.2.3 The d-frame of a bitopological space

In Chapter 2 we came across numerous d-lattices that are actually d-frames. In particular, Example 6 provides the contravariant functor from bitopological spaces to d-frames.

Definition 3.2.3. The functor $\mathcal{O} : \mathsf{BiTop} \to \mathsf{dFrm}$ maps a bitopological space (X, τ_-, τ_+) to the structure

$$\tau_- \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} \tau_+$$

where the consistency relation records when two opens are disjoint and the totality relation records when two opens cover the space. A bicontinuous map between bitopological spaces maps to the pair of frame homomorphisms that is obtained by restricting the preimage

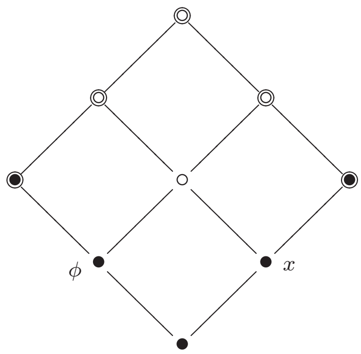


Figure 3.2: The free d-frame over the one-point set, pictured as the product of the component frames. Filled elements are consistent pairs and circled element are total.

operation f^{-1} to the individual topologies. We call the d-frame $\mathcal{O}X$ the Stone dual of the bitopological space X.

In the introduction we convinced ourselves that the definition above indeed produces d-frames. As an example, consider the bitopological space $\mathbb{S}.\mathbb{S}$ of Figure 3.1. Its two topologies are both isomorphic to the three-element chain and there are no non-trivial disjoint or total pairs of opens. We conclude that the free d-frame over the one-point set pictured in Figure 3.2 is the d-frame $\mathcal{O}\mathbb{S}.\mathbb{S}$ associated with the four-element bitopological space of Figure 3.1.

If the Stone duality is to work as expected, the functor $\mathcal{O}$ from bitopological spaces to d-frames must be presentable as the hom-set functor $\mathsf{BiTop}(-, \mathbb{S}.\mathbb{S})$. And indeed, given a bitopological space (X, τ_-, τ_+) and a pair of opens $U' \in \tau_-$, $V' \in \tau_+$ one defines a map $\chi_{U',V'} : X \to \mathbb{S}.\mathbb{S}$ as follows. Map a point x to the unique element of $U \cap V$ if and only if the point x is a member of $U' \cap V'$. If x is an element of U' but not V' then map it to the unique element of $U \setminus V$ and so forth. As Jung and Moshier described it, the dualising object $\mathbb{S}.\mathbb{S}$ represents the four possible ways a point of a bitopological space can be related to a pair of opens. One can recover the pair of opens (U', V') from the map $\chi_{U',V'}$ by considering the preimages of U and V under this map. Moreover, the opens U' and V' are disjoint if and only if the image of $\chi_{U',V'}$ does not intersect the set $U \cap V$, and dually the union $U' \cup V'$ covers the space precisely when the image of $\chi_{U',V'}$ is contained in $U \cup V$.

3.2.4 The spectrum of a d-frame

The obvious forgetful functor U_{BiTop} from bitopological spaces to sets has a left adjoint F_{BiTop} that takes a set S and endows it with two copies of the discrete topology. In particular, the free bitopological space $F_{\mathsf{BiTop}}1$ over the one-point set has two lattices of open sets each isomorphic to the two-element chain $2 = \{0, 1\}$. The d-frame $\mathcal{O}F_{\mathsf{BiTop}}1$ is an old friend of ours, namely the dualising object $\mathbf{2}$ we employed in Section 1.7 to compute the locally compact sober space associated with a Stage 4 interaction algebra.

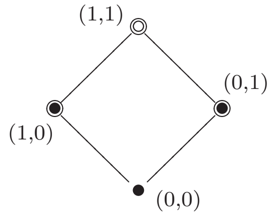


Figure 3.3: The dualising object $\mathbf{2}$ of d-frames, depicted as the product 2×2. Filled dots are members of con and circled dots are members of tot.

Definition 3.2.4. Define a contravariant functor pt from d-frames to bitopological spaces as follows. For a d-frame $\mathcal{L}$ the *spectrum* $\mathrm{pt}\,\mathcal{L}$, also referred to as its Stone dual, is the set of its d-points, that are d-frame homomorphisms into the dualising object $\mathbf{2}$ depicted in Figure 3.3. The spectrum is endowed with two topologies, whose opens are of the form

$$\begin{aligned} \ulcorner\phi\urcorner^- &= \{(h_-, h_+) \in \mathsf{dFrm}(\mathcal{L}, \mathbf{2}) \mid h_-(\phi) = 1\}\,; \quad \phi \in L_-, \\ \ulcorner x\urcorner^+ &= \{(h_-, h_+) \in \mathsf{dFrm}(\mathcal{L}, \mathbf{2}) \mid h_+(x) = 1\}\,; \quad x \in L_+. \end{aligned}$$

The action of the functor pt on d-frame homomorphisms is pre-composition: For a d-frame homomorphism $f : \mathcal{L} \to \mathcal{M}$ one has

$$\mathrm{pt}(f) = (- \circ f) : \mathsf{dFrm}(\mathcal{M}, \mathbf{2}) \to \mathsf{dFrm}(\mathcal{L}, \mathbf{2}).$$

Observe that the assignment from elements of a component frame to opens of the spectrum is indeed a frame homomorphism, for the same reasons that make this true in locale theory. Further observe that a consistent pair of a d-frame gives rise to a disjoint pair of opens on the spectrum and likewise for total pairs. Most of the time we will work with an alternative description of the spectrum of a d-frame in terms of pairs of completely prime filters. Observe that every d-point (h_-, h_+) gives rise to a pair of completely prime filters $(h_-^{-1}(1), h_+^{-1}(1))$ of the component frames.

Lemma 3.2.3. *A pair of completely prime filters $F_- \subseteq L_-$ and $F_+ \subseteq L_+$ corresponds to a d-point of the d-frame $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ if and only if it satisfies the following two rules.*

$$\frac{x \mathsf{con} \phi}{x \notin F_+ \ \textit{or}\ \phi \notin F_-} \quad (\mathrm{dp}_{\mathsf{con}})$$

$$\frac{\phi \mathsf{tot} x}{\phi \in F_- \ \textit{or}\ x \in F_+} \quad (\mathrm{dp}_{\mathsf{tot}})$$

Proof. Let (h_-, h_+) be a d-point of the d-frame $\mathcal{L}$. The component frame homomorphisms give rise to completely prime filters

$$\begin{aligned} F_- &= \{\phi \in L_- \mid h_-(\phi) = 1\}\,, \\ F_+ &= \{x \in L_+ \mid h_+(x) = 1\}\,. \end{aligned}$$

A consistent pair (ϕ, x) of $\mathcal{L}$ can not be mapped to $(1,1)$ because this is not a consistent pair of $\mathbf{2}$. Thus the rule $(\mathrm{dp}_{\mathsf{con}})$ holds. Dually, a total pair (ϕ, x) can not be mapped to $(0,0)$ because this is not a total pair of $\mathbf{2}$, so the rule $(\mathrm{dp}_{\mathsf{tot}})$ holds.

If (F_-, F_+) is a pair of prime filters satisfying the rules $(\mathrm{dp}_{\mathsf{con}})$ and $(\mathrm{dp}_{\mathsf{tot}})$ then by letting $h_-(\phi) = 1$ iff $\phi \in F_-$ and $h_+(x) = 1$ iff $x \in F_+$ one obtains a d-frame homomorphism $\mathcal{L} \to \mathbf{2}$. □

Lemma 3.2.4. *Let $F_+ \subseteq L_+$ be a completely prime filter of a component frame of the d-frame $\mathcal{L}$ and let I_+ denote the complement of F_+. A completely prime filter $F_- \subseteq L_-$ extends F_+ to a d-point if and only if F_- separates the filter $(I_+)^{\mathsf{tot}}$ from the ideal $(F_+)_{\mathsf{con}}$. (We used notation from Lemma 2.3.2.)*

Proof. Suppose (F_-, F_+) is a d-point. By $(\mathrm{dp}_{\mathsf{con}})$we have $F_+ \ni x \mathsf{con} \phi \Rightarrow \phi \notin F_-$ whence F_- is disjoint from $(F_+)_{\mathsf{con}}$. By $(\mathrm{dp}_{\mathsf{tot}})$we have $\phi \mathsf{tot} x \in I_+ \Rightarrow \phi \in F_-$ whence F_- contains $(I_+)^{\mathsf{tot}}$. Conversely, if $F_+ = L_+ \setminus I_+$ is a completely prime filter then $(F_+)_{\mathsf{con}} \cap (I_+)^{\mathsf{tot}} = \emptyset$ because otherwise there is some element of the filter F_+ well inside some element of I_+ and so I_+ can not be the complement of F_+. It remains to show that any completely prime filter $F_- \subseteq L_-$ separating the ideal $(F_+)_{\mathsf{con}}$ from the filter $(I_+)^{\mathsf{tot}}$ is satisfies the axioms $(\mathrm{dp}_{\mathsf{con}})$ and $(\mathrm{dp}_{\mathsf{tot}})$. If $x \mathsf{con} \phi$ then either $x \in F_+$, in which case $\phi \in (F_+)_{\mathsf{con}}$ and by hypothesis $\phi \notin F_-$, or $x \in I_+$ in which case $(\mathrm{dp}_{\mathsf{con}})$ evidently holds. If $\phi \mathsf{tot} x$ then either $x \in F_+$ in which case there is nothing to show, or $x \in I_+$ whence $\phi \in (I_+)^{\mathsf{tot}}$ and so by hypothesis $\phi \in F_+$. Thus the rule $(\mathrm{dp}_{\mathsf{tot}})$ holds as well. □

As promised, let us now prove the existence of the concrete Stone duality between the categories dFrm and BiTop.

Theorem 3.2.5. *The functors $\mathcal{O}$ and* pt *of Definitions 3.2.3 and 3.2.4 constitute a concrete Stone duality between the categories* BiTop *and* dFrm.

Proof. Most of the items that are to check follow from the well-known Stone duality between the categories Top and Frm. The extra bit of work arises from the consistency and totality predicates. As we remarked, consistent pairs of a d-frame translate to disjoint opens and total pairs translate to pairs of opens that cover the spectrum. This yields the counit

$$\varepsilon_{\mathcal{L}} = (\ulcorner - \urcorner^-, \ulcorner - \urcorner^+) : \mathcal{L} \to \mathcal{O}\,\mathrm{pt}\,\mathcal{L}.$$

A d-frame homomorphism $f : \mathcal{L} \to \mathcal{O}X$ is a pair of frame homomorphisms $f_- : L_- \to \tau_-$ and $f_+ : L_+ \to \tau_+$ into the topologies of the bitopological space X. Classical Stone duality tells us how to turn such frame homomorphisms into continuous maps from X into the spaces of completely prime filters of L_- and L_+ in a unique way: A point $x \in X$ maps to the pair of frame homomorphisms

$$L_- \xrightarrow{f_-} \tau_- \xrightarrow{\chi_-(x)} 2$$

$$L_+ \xrightarrow[f_+]{} \tau_+ \xrightarrow[\chi_+(x)]{} 2$$

where $\chi_-(x)$ and $\chi_+(x)$ are the characteristic functions of the neighbourhood filters of x in the two topologies. The pair $(\chi_-(x), \chi_+(x))$ is always a d-frame homomorphism from $\mathcal{O}X$ to $\mathbf{2}$. From the diagram above one can see that the construction takes a point of X and returns a d-point of $\mathcal{L}$ precisely because the pair (f_-, f_+) is a d-frame homomorphism. It follows that every d-frame homomorphism $f : \mathcal{L} \to \mathcal{O}X$ factors through the counit map $\epsilon_{\mathcal{L}}$ in a unique way. □

Let us list some examples from Chapter 2 that are actually d-frames.

Example 9. The open ideal completion functor $\mathrm{Idl}_\circ$ on the category of d-lattices (see Theorem 2.3.8) takes values in the subcategory of d-frames.

Example 10. Given a d-lattice $\mathcal{L}$ one can form a d-frame with component frames $\mathrm{Idl}\, L_-$ and $\mathrm{Idl}\, L_+$ where consistency and totality relations are defined as for the open ideal completion in Lemma 2.3.6. The spectrum of the d-frame thus constructed can be understood as the spectrum of the d-lattice. A special case of this is Priestley duality for bounded distributive lattices: Given a lattice L, construct a d-lattice in the fashion of Example 4. The resulting d-frame has as component frames the ideal- and filter completions of L and its spectrum is the set of prime filters of L. An ideal of L is consistent with a filter of L as opens of the spectrum if and only if the ideal consists of lower bounds of the filter. Dually, the ideal is total with the filter precisely when the ideal intersects the filter.

Example 11. Lemma 2.4.2 applies to d-frames. Whenever $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ is a d-frame and $\prec$ is a binary interpolative relation on the frame L_+ that satisfies all the properties listed in the lemma, then $\mathcal{L}^{\prec} = (L_-, L_+, \mathsf{con}, \mathsf{tot}^{\prec})$ is a d-frame where $(\mathsf{tot}^{\prec}; \mathsf{con}; \mathsf{tot}^{\prec}) = \mathsf{tot}^{\prec}$. In particular, the normal coreflection functor of Theorem 2.4.4 restricts to the subcategory of d-frames.

3.2.5 Sobriety and spatiality

As with every adjunction, a natural question to ask is that of the images of the functors.

Definition 3.2.5. A bitopological space (X, τ_-, τ_+) is called *d-sober* if it is bihomeomorphic to the spectrum of some d-frame $\mathcal{L}$. A d-frame is called *spatial* if it is isomorphic to a d-frame of the form $\mathcal{O}X$ for some bitopological space X.

A d-frame $\mathcal{L}$ is spatial if and only if four conditions are satisfied:

1. If $x \not\sqsubseteq y$ are distinct elements of the component frame L_+ then there exists a d-point $f : \mathcal{L} \to \mathbf{2}$ such that the component map f_+ maps x to 1 and y to 0,
2. If $\phi \not\sqsubseteq \psi$ are distinct elements of the component frame L_- then there exists a d-point $f : \mathcal{L} \to \mathbf{2}$ such that the component map f_- maps ϕ to 1 and ψ to 0,
3. If x is not consistent with ϕ then there exists a d-point $f : \mathcal{L} \to \mathbf{2}$ such that $f_-(\phi) = 1$ and $f_+(x) = 1$,
4. If ϕ is not total with x then there exists a d-point $f : \mathcal{L} \to \mathbf{2}$ such that $f_-(\phi) = 0$ and $f_+(x) = 0$.

The notorious counterexample 7 demonstrates that, in contrast to the category of frames, not even finite d-frames need to be spatial. Indeed, the spectrum of the d-frame in that example is a two-element bitopological space with two copies of the discrete topology. The d-frame derived from this bitopology has the same component frames as the original d-frame, but richer consistency and totality relations. In other words, the last two of the four conditions above are violated. However, the example was constructed to violate the finitary cut rules. And in fact for finite d-frames the cut rules characterise the spatial d-frames.

Theorem 3.2.6. *A finite d-frame is spatial if and only if it satisfies the finitary cut rules.*

Proof. Every spatial d-frame satisfies the finitary cut rules. Let $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ be a finite d-frame. We have to verify the four properties of spatiality. The first two state that one can separate elements in either frame by d-points. This is proved using Lemma 3.2.4 and the Prime Ideal Theorem; in Proposition 3.6.5 we give a more abstract, different proof. Therefore only the last two properties of spatiality need to be checked.

To show the third property, let $(x, \phi) \in (L_+ \times L_-) \setminus \mathsf{con}$ be a non-consistent pair. By finiteness of the d-frame we may assume that the pair (x, ϕ) is minimal in the complement of con. Observe that the element x is not in the ideal $\phi_{\mathsf{con}} := \{x' \in L_+ \mid x' \mathsf{con} \phi\}$ whence we can find a completely prime filter $F_+ = L_+ \setminus {\downarrow} p$ (here p is a meet-prime element) with $x \in F_+$ and $\phi_{\mathsf{con}} \subseteq {\downarrow} p$. As the frame L_+ is finite the filter F_+ has a least element, say y. For the same reason the filter p^{tot} of elements in L_- that are total with p must have a smallest element, say ψ. From Lemma 3.2.4 we know that a completely prime filter $F_- \subseteq L_-$ extends F_+ to a d-point if and only if it contains ψ and is disjoint from the ideal y_{con}, that is, it contains no element that is consistent with y. There is a largest element in the ideal y_{con} which we call θ. Now there are two cases. **Case 1:** $\phi \sqcap \psi \not\sqsubseteq \theta$. In that case we can find a completely prime filter F_- containing $\phi \sqcap \psi$ but not θ, so we have found a d-point that witnesses the fact that x is not consistent with ϕ. **Case 2:** $\phi \sqcap \psi \sqsubseteq \theta$. In that case notice that $p \sqcap y$ is strictly smaller than y because p is not in the filter $F_+ = {\uparrow} y$. It follows that $p \sqcap y$ is also strictly smaller than x because x is an element of the filter F_+. Now (x, ϕ) was assumed to be a minimal non-consistent pair whence $(p \sqcap y) \mathsf{con} \phi$. Notice that by definition $\psi\mathsf{tot}p$, as ψ was defined to be the smallest element with that property. As $\phi \sqcap \psi \sqsubseteq \theta$ and θ is the largest element consistent with y, we can apply the cut rule

$$\frac{y\mathsf{con}(\psi \sqcap \phi) \qquad \psi\mathsf{tot}p \qquad (p \sqcap y)\mathsf{con}\phi}{y\mathsf{con}\phi}$$

which contradicts the choice of the filter $F_+ = \Uparrow y$ being disjoint from all elements that are consistent with ϕ. In summary, the axiom (cut$_{\mathsf{con}}$) rules out Case 2 and therefore Case 1 applies, where we constructed a d-point with the desired properties.

The fourth property of spatiality states that whenever ϕ is not total with x then we can find a d-point (F_-, F_+) witnessing this fact, meaning $\phi \notin F_-$ and $x \notin F_+$. We dualise the proof of the third property: Recall that under the order-dual operation the component lattices of a d-lattice reverse their order and the roles of con and tot are swapped. Therefore, using the cut rule (cut$_{\mathsf{tot}}$) one finds a pair of prime ideals (P_-, P_+) with $\phi \in P_-$ and $x \in P_+$. Then the complements $F_- = L_- \setminus P_-$ and $F_+ = L_+ \setminus P_+$ form a d-point with the required properties. □

There is much to be said about d-sober spaces. However, we want to focus on the d-frame side rather than the spatial side and so we are content with demonstrating that d-sobriety is a much more inclusive concept than sobriety.

Example 12. Consider the open unit interval $(0, 1)$ with the bitopology generated by sets of the form $(x, 1)$ for the "upper" topology τ_+ and $(0, x)$ for the "lower" topology τ_- where x ranges over elements of the unit interval $[0, 1]$. Thus both topologies are order-isomorphic to the unit interval. The consistency relation, considered as a subset $\mathsf{con} \subseteq [0, 1] \times [0, 1]$ is nothing but the usual order relation. Indeed, $x \leq y$ is equivalent to $(0, x) \cap (y, 1) = \emptyset$. Similarly, the totality relation is characterised by the strict order $<$ with the additional elements added by the rules (tot-$\mathbb{t}$) and (tot-$\mathbb{f}$). The open unit interval with either topology considered on its own is not a sober space. Instead, the soberification of the open unit interval with the upper topology yields $(0, 1]$ whereas the soberification of the unit interval with the lower topology produces $[0, 1)$. This is because the set of non-empty opens in either topology is a completely prime filter. Let us examine whether the end-point 0 could be a d-point of the d-frame we described. In the lower topology, the neighbourhoods of 0 are all non-empty opens. In order for this completely prime filter to extend to a d-point, we need a completely prime filter of upper opens that does not contain any opens which are consistent with a lower neighbourhood of 0, which rules out every open but the maximal $(0, 1)$. But the singleton $\{(0, 1)\}$ is not a completely prime filter in the frame of upper opens. A dual argument applies to the end-point 1 arising in the soberification of the upper topology.

Example 13. The previous example can be tweaked into a d-frame that has spatial component frames but no d-points at all. Observe that the unit interval $[0, 1]$ in its natural order $\leq$ is a frame, and so is its order dual. One forms a d-lattice according to

Example 4. The resulting structure

$$[0,1]^{\partial} \underset{\leq}{\overset{\leq}{\rightleftarrows}} [0,1]$$

is not only a d-lattice but a d-frame, with the pseudocomplement map given as $x \mapsto 1-x$. Any completely prime filter of the component frame $[0,1]$ is of the form $F_x := (x,1]$ for some $0 \leq x < 1$. The ideal of elements in $[0,1]^{\partial}$ that is consistent with some element of this filter is the set $\{y \in [0,1] \mid \exists x' > x.\, x' \leq y\} = (x,1]$. The complement of the completely prime filter F_x is the ideal $[0,x]$. The filter of elements that are total with some element of this ideal is the set $\{y \in [0,1] \mid \exists x' \leq x.\, y \leq x'\} = [0,x]$. We know that any completely prime filter of the frame $[0,1]^{\partial}$ that completes F_x to a d-point must contain the set $[0,x]$ and be disjoint from the set $(x,1]$. This leaves only one choice, namely $[0,x]$. But this is not a completely prime filter in $[0,1]^{\partial}$. We conclude that the d-frame we described has no d-points at all.

Notice that because of Lemma 2.5.1 the d-frame of this example satisfies the finitary cut rules. Even stronger, since its component frames are chains, it trivially satisfies the join-strength rules of Definition 2.5.1, and since the well-inside relation on each component frame coincides with the frame order (every element is complemented) the d-frame has a closed consistency relation in the sense of Definition 2.5.2.

3.3 Pseudocomplements

A frame is a complete Heyting algebra and is therefore equipped with a Heyting arrow. Recall that the Heyting arrow $\to$ in a Heyting algebra A is a binary operation such that for any $a \in A$ the map $b \mapsto (a \to b)$ is right adjoint to the map $b \mapsto (a \sqcap b)$. In particular, any frame has a *Heyting negation* defined as $\neg a = a \to 0$. Because of completeness of the frame A one can write the Heyting negation as $\neg a = \bigsqcup\{b \in A \mid a \sqcap b = 0\}$, whence one also calls the element $\neg a$ the *pseudocomplement* of a. For topological spaces the pseudocomplement of an open set has a concrete interpretation: It is the interior of the (closed) complement of the open set, that is the largest open set disjoint from the given open.

Obviously the structure of a d-frame allows us to define an analogue to the pseudocomplement of frames: Recall that by axiom ($\mathsf{con}\text{-}\bigsqcup$) for every element of one component frame there is a largest element in the other component frame that it is consistent with.

Definition 3.3.1. Let $\mathcal{L}$ be a d-frame. Then there is a Galois connection between the component frames

$$L_- \underset{\neg}{\overset{\neg}{\rightleftarrows}} L_+$$

given by the assignments $\phi \mapsto \bigsqcup\{x \in L_+ \mid x\mathsf{con}\phi\}$ and $x \mapsto \bigsqcup\{\phi \in L_- \mid x\mathsf{con}\phi\}$. We call the element $\neg\phi$ the *pseudocomplement* of ϕ and likewise $\neg x$ the pseudocomplement of x.

The pair of pseudocomplement maps is indeed a Galois connection, as $x \sqsubseteq \neg\phi$ is equivalent to $x\mathsf{con}\phi$ which is in turn equivalent to $\phi \sqsubseteq \neg x$. Therefore pseudocomplementation transforms all joins to meets. Observe that in particular $\neg 0 = 1$ and $\neg 1 = 0$. The pseudocomplement serves as a canonical witness for the well-inside relation. Indeed, recall that $x \lhd y$ holds in the component frame L_+ of a d-frame $\mathcal{L}$ if there exists a witness $\phi \in L_-$ with the property $x\mathsf{con}\phi\mathsf{tot}y$. By definition then $\phi \sqsubseteq \neg x$ whence also $x\mathsf{con}\neg x\mathsf{tot}y$ holds.

Let us record a useful fact about the pseudocomplement and the well-inside relations.

Lemma 3.3.1. *Suppose x is well inside y in the component frame L_+ of a d-frame $\mathcal{L}$. Then the pseudocomplement $\neg y$ is well-inside the pseudocomplement of x in the component frame L_-. A dual statement holds for the well-inside relation of the other component frame.*

$$\frac{x \lhd y}{\neg x \rhd \neg y} \qquad \frac{\phi \rhd \psi}{\neg\phi \lhd \neg\psi}$$

Proof. By definition $x \lhd y$ holds in the component frame L_+ if and only if $x\mathsf{con}\phi\mathsf{tot}y$ for some witness $\phi \in L_-$. Note that $x \sqsubseteq \neg\phi$ and, equivalently, $\neg x \sqsupseteq \phi$. For the element $y \in L_+$ we have $y\mathsf{con}\neg y$ and therefore $\phi \rhd \neg y$. We conclude $\neg x \rhd \neg y$. □

Let us turn towards properties of the pseudocomplementation of d-frames that distinguish it from the Heyting complement of frames. Although the Heyting negation on a frame A does only in the rarest instances transform meets to joins, it is always true that the double negation $a \mapsto \neg\neg a$ preserves finite meets[1]. The double negation on, say, the component frame L_+ of a d-frame $\mathcal{L}$ is a closure operator, meaning it is inflationary and idempotent, but it does not always preserve finite meets. This is because any any Galois connection between two frames can be turned into the pseudocomplement maps of a d-frame. For the sake of the example one may assume the totality relation to be trivial.

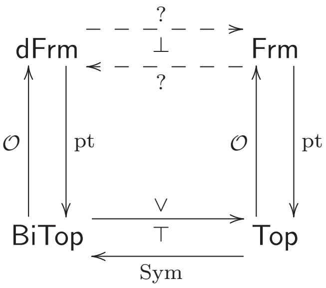
3.4 The patch frame of a d-frame

Possibly the most common way to relate the theories of topological spaces with that of bitopological spaces is via the join topology, sometimes also called the patch topology. If τ_- and τ_+ are two topologies on the same set X then their join $\tau_- \vee \tau_+$ is the coarsest topology on X that renders the opens sets of both topologies open. A bicontinuous map between two bitopological spaces is automatically continuous with respect to the join topologies (the converse does not hold in general). Thus one obtains a functor $\vee : \mathsf{BiTop} \to \mathsf{Top}$. It has an easy-to-describe left adjoint: Any topological space (X, τ) can be regarded as the bitopological space (X, τ, τ). Let us call such a bitopological space *symmetric* and

[1] A remarkable consequence of this is that any locale has a smallest dense sublocale.

write Sym : $\mathsf{Top} \to \mathsf{BiTop}$ for the functor that maps a topological space to its symmetric bitopological space and acts as the identity on morphisms.

Can one lift this adjunction to the point-free setting? So far we can draw a diagram of adjunctions as this:



(3.1)

Here, the vertical arrows are the two contravariant Stone dualities. If the adjunction indicated in dashed arrows is to represent the adjunction at the bottom of the square, we expect that the composite $\mathsf{dFrm} \dashrightarrow \mathsf{Frm} \xrightarrow{\text{pt}} \mathsf{Top}$ is the same as the spectrum functor of d-frames followed by $\vee$ and dually the composite functor $\mathsf{Top} \xrightarrow{\mathcal{O}} \mathsf{Frm} \dashrightarrow \mathsf{dFrm}$ is the same as the functor Sym followed by the functor from bitopological spaces to d-frames. There is an obvious candidate for the functor from frames to d-frames, namely the symmetric d-lattice functor of Definition 2.5.3 which maps a lattice L to the d-lattice $(L, L, \mathsf{con}, \mathsf{tot}) = L_{=}$ where $x\mathsf{tot}y$ iff $x \sqcup y = 1$ and $x\mathsf{con}y$ iff $x \sqcap y = 0$. It is not hard to see that this functor maps the subcategory of frames and frame homomorphisms to the subcategory of d-frames and d-frame homomorphisms. And indeed we have the identity $(-)_{=} \circ \mathcal{O} = \mathcal{O} \circ \text{Sym}$, as required. From Section 2.5 we also know what the left adjoint to the symmetric d-frame functor should be:

Definition 3.4.1. The *patch frame* $\text{Patch}\,\mathcal{L}$ of a d-frame $\mathcal{L}$ has generators $\ulcorner\phi\urcorner^-$ and $\ulcorner x\urcorner^+$ where ϕ ranges over elements of the component frame L_- and x ranges over the component frame L_+. One quotients the free frame over these generators by the rules enforcing that that the pair $(\ulcorner-\urcorner^-, \ulcorner-\urcorner^+)$ is a d-frame homomorphism from $\mathcal{L}$ to $(\text{Patch}\,\mathcal{L})_{=}$.

Theorem 3.4.1. *The patch functor on d-frames is left adjoint to the symmetric d-frame functor.*

Proof. Analogous to the proof of Theorem 2.5.4. □

Corollary 3.4.2. *For every d-frame, the spectrum of its patch frame is homeomorphic to the bitopological spectrum of the d-frame endowed with the join topology.*

Proof. Once more we make use of the fact that the dualising object $\mathbf{2}$ of d-frames is the symmetric d-frame $2_{=}$ of the dualising object of the category of frames. Because of the adjunction Patch $\dashv (-)_{=}$ we have a natural isomorphism of hom-sets

$$\mathsf{dFrm}(\mathcal{L}, \mathbf{2}) \cong \mathsf{Frm}(\text{Patch}\,\mathcal{L}, 2).$$

Therefore the spectrum of the patch frame has the same points as the bitopological spectrum. It remains to show that the two sets have the same topology. Observe that every element of the patch frame is a (possibly infinite) join of elements of the form $\ulcorner\phi\urcorner^- \sqcap \ulcorner x\urcorner^+$ where $(\phi, x) \in L_- \times L_+$. A frame homomorphism $f : \mathrm{Patch}\,\mathcal{L} \to 2$ maps such a meet to 1 precisely if it maps both $\ulcorner\phi\urcorner^-$ and $\ulcorner x\urcorner^+$ to 1. Consequently the basic open of $\mathrm{pt}\,\mathrm{Patch}\,\mathcal{L}$ that is given by $\ulcorner\phi\urcorner^- \sqcap \ulcorner x\urcorner^+$ is the same as the intersection of basic opens $\ulcorner\phi\urcorner^-$ and $\ulcorner x\urcorner^+$ of the bitopological spectrum $\mathrm{pt}\,\mathcal{L}$, and these intersections from a basis of the join topology. □

Example 14. Consider the d-frame of Example 12. The patch frame of this has a basis of the form $\ulcorner y\urcorner^- \sqcap \ulcorner x\urcorner^+$ where $x, y \in [0, 1]$. The relations between these kinds of frame elements match the standard presentation of the Euclidean topology by generators and relations. To see this, identify the closed unit interval with the set of extended reals $[-\infty, \infty]$ and an element $\ulcorner y\urcorner^- \sqcap \ulcorner x\urcorner^+$ of the patch frame with the open interval (y, x) of $\mathbb{R}$.

Moving to Example 13 one finds that its patch frame, considered as a locale, constitutes a sublocale of the locale of real numbers. The generator $(-\infty, x) \cup (x, \infty)$ is identified with the maximal element of the frame of opens. This is in fact a standard trick to produce a non-trivial sublocale of the locale of reals that has no points.

The patch construction was originally used by Jung and Moshier to provide an adjunction between the category of d-frames and Banaschewski and Hardie's category BiFrm of biframes. Given a biframe (L, L_-, L_+) where L_- and L_+ are sub-frames of L generating it, one defines a d-frame with components frames L_- and L_+ where $x \mathsf{con} \phi$ if and only if $x \sqcap \phi = 0$ in L and dually $\phi \mathsf{tot} x$ precisely when $\phi \sqcup x = 1$ in L. The adjoint to this "forgetful functor" $\mathsf{BiFrm} \to \mathsf{dFrm}$ is given by the patch frame of a d-frame together with the two sub-frames that arise as the images of the component frames under the generator maps $\ulcorner - \urcorner^-$ and $\ulcorner - \urcorner^+$. Related patch constructions are also associated with adjunctions, but all rely on an ambient frame. Banaschewski and Brümmer [5] and Escardó [18, 19] construct the compact regular coreflection of a stably compact locale by means of congruences resp. nuclei on a frame.

3.5 Regular d-frames

Just as with topological spaces, the bare definition of d-frames does not lead to a wealth of interesting results. This is because the link between the two component frames is so weak that it is easy to construct counterexamples to all sorts of conjectures. The motivating examples of bitopological spaces, such as the order topologies on the real line or the de Groot dual of stably compact spaces, are all instances of a bitopological version of regular spaces.

In many situations the T_2 (Hausdorff) separation axiom implies the T_3 separation axiom. For example, any locally compact Hausdorff space is regular[2], any compact Hausdorff space is locally compact and therefore regular. These statements even hold for locales, as Vermeulen [54] showed.

The T_3 separation axiom can be formulated without explicit reference to points. Indeed, suppose U is an open set of a T_3 space and x an element of U. Then x is not in the closed complement of U, whence one can find disjoint open neighbourhoods $x \in U'$ and $(X \setminus U) \subseteq V$. Observe that the open set V is a witness for the fact that U' is well inside U. Hence for T_3 one simply requires that every open set U is the union of opens sets well inside U.

There is an obvious bitopological generalisation of T_3 that can be found for instance in [38]. Given a bitopological space (X, τ_-, τ_+) one requires that any open $U \in \tau_+$ is the union of opens $U' \in \tau_+$ whose closure *with respect to* τ_- is contained in U, and likewise for opens of the other topology. Again, there is a simple description in terms of the well-inside relation.

Lemma 3.5.1. *A bitopological space X is T_3 in the sense of [38] if and only if the Stone dual $\mathcal{O}X$ has approximating well-inside relations.*

In the lemma above, the notion of well-inside relation is that of Definition 2.2.1. The term *approximating* is borrowed from domain theory and shorthand for saying that every element x is the join of elements y that satisfy $y \lhd x$. We turn this observation into a definition.

Definition 3.5.1. A d-frame $\mathcal{L}$ is *regular* if its well-inside relations are approximating, meaning that for all $\phi \in L_-$ and $x \in L_+$ one has

$$\begin{aligned} \phi &= \bigsqcup \{\psi \in L_- \mid \phi \rhd \psi\} \\ x &= \bigsqcup \{y \in L_+ \mid y \lhd x\} \end{aligned}$$

Using the notation $\downarrow\!x$ to denote the ideal of elements well inside x and similarly for ϕ, one can phrase regularity as $x = \bigsqcup \downarrow\!x$ and $\phi = \bigsqcup \downarrow\!\phi$ for all $(\phi, x) \in L_- \times L_+$.

Let us list some examples.

Example 15. A finite d-frame is regular if and only if it consists entirely of complemented pairs. Indeed, suppose x is an element of the finite component frame L_+. If the d-frame is regular then x is the join of the ideal $\downarrow\!x$ which, as any ideal of a finite frame, must be principal. Consequently $x \lhd x$ which means that x is complemented in the sense of Definition 2.2.2. Conversely, any d-frame that consists entirely of complemented elements, whether it is finite or not, is trivially regular.

[2] By local compactness, any open is the union the interior of its compact subsets. In a Hausdorff space, all compact subsets are closed.

Example 16. The d-frame of Example 12 is regular. Indeed, the upper open set $(x, 1)$ is the join of opens $(x', 1)$ where $x < x'$. Observe that $(x, 1)$ is total with $(0, x')$ which in turn is consistent with $(x', 1)$.

A curious property of regular d-frames is that the consistency relation is redundant.

Proposition 3.5.2. *In a regular d-frame $\mathcal{L}$ the pseudocomplement of an element $x \in L_+$ can be computed as $\neg x = \bigsqcap x^{\mathsf{tot}}$.*

Proof. Recall that because of the axiom (con-tot) in every d-frame the element $\neg x$ is a lower bound of the filter x^{tot}. Hence it suffices to show the inequality $\bigsqcap x^{\mathsf{tot}} \sqsubseteq \neg x$. Because of regularity we can write x as the join $\bigsqcup \mathrel{\downarrow\!\!\!\!\cdot} x$ of elements well inside it. The pseudocomplement map $\neg : L_+ \to L_-$, being part of a Galois connection, transforms all joins to meets. Therefore

$$\neg x = \neg \bigsqcup \mathrel{\downarrow\!\!\!\!\cdot} x = \bigsqcap \{\neg y \mid y \lhd x\}\,.$$

To finish the proof, recall that an element y is well inside x if and only if $\neg y$ is total with x. Therefore the set $\{\neg y \mid y \lhd x\}$ is contained in x^{tot}. ☐

Recall that the consistency relation of any d-frame is entirely described by the Galois connection of pseudocomplement maps associated with it. Consequently, knowing that a d-frame is regular one is able to reconstruct the consistency relation from the totality relation alone. Another way of interpreting the proposition above is to say that the regular d-frames are those where the consistency relation is as large as the axiom (con-tot) permits.

Checking for regularity

In many instances one defines a topology on a set by specifying a basis or even a subbasis only. An arbitrary open set then might be a lot harder to describe than a subbasic open. Therefore it comes handy that in order to determine whether a d-frame is regular, it suffices to check a subbasis of each component frame only.

Lemma 3.5.3. *Let $\mathcal{L}$ be a d-frame and B_-, B_+ be subbases of the component frames L_- and L_+, respectively. (That is, every frame element is a join of finite meets of subbasic elements.) If every element of a each subbasis is the join of elements well inside it, then the d-frame is regular.*

Proof. The statement of the lemma follows from a general fact about auxiliary relations on frames. Suppose $\lhd$ is an auxiliary relation on a frame L such that for every element x of the frame the lower set $\mathrel{\downarrow\!\!\!\!\cdot} x = \{y \in L \mid y \lhd x\}$ is an ideal and further suppose that the relation $\lhd$ is closed under finite meets on the right. Then the assignment $x \mapsto \mathrel{\downarrow\!\!\!\!\cdot} x : L \to \mathrm{Idl}\, L$ preserves finite meets. Now suppose x, y are elements of a subbasis of L and $x = \bigsqcup \mathrel{\downarrow\!\!\!\!\cdot} x$

and $y = \bigsqcup \Downarrow y$ hold. Then

$$\bigsqcup \Downarrow(x \sqcap y) = \bigsqcup (\Downarrow x \cap \Downarrow y) = \left(\bigsqcup \Downarrow x\right) \sqcap \left(\bigsqcup \Downarrow y\right) = x \sqcap y.$$

Here we exploited the fact that the join operation on ideals of a frame is a frame homomorphism and therefore preserves finite meets. We showed that if the relation $\lhd$ is approximating on a subbasis, then it is approximating on the basis generated by the subbasis. Now let x be an arbitrary element of the frame and let B be a set of basic elements with $x = \bigsqcup B$. Suppose that on B the relation $\lhd$ is approximating. Then we write

$$x = \bigsqcup B = \bigsqcup \left\{ \bigsqcup \Downarrow b \,\middle|\, b \in B \right\} = \bigsqcup \bigcup_{b \in B} \Downarrow b$$

to see that x, too, is approximated by elements of $\Downarrow x$. □

As an application of the preceding lemma let us prove that an important class of bitopological spaces is regular.

Proposition 3.5.4. *For a continuous poset D, let σD denote the Scott topology and ωD denote the weak lower topology. Then the bitopological space $(D, \sigma D, \omega D)$ is regular.*

Proof. It is well-known that the Scott topology on a continuous poset is itself a domain where a Scott open U_0 is way below a Scott open U_1 if and only if there is a finite set $A \subseteq U_1$ such that $U_0 \subseteq {\uparrow} A$. As the finitely generated upper sets are the basic closed sets of the weak lower topology, it follows that the well-inside relation of the bitopology in concern is approximating on σD.

Now consider a subbasic open $D \setminus {\uparrow} d$ of the weak lower topology. We claim that this open set is the union of subbasic opens $D \setminus {\uparrow} y$ where y is way below d. This is the same as claiming that ${\uparrow} d$ is the intersection of the upper sets ${\uparrow} y$ where y ranges over $\Downarrow d$. If $d \not\sqsubseteq d'$ then there exists some $x \ll d$ with $d' \notin \Uparrow x$ (see [22, I-1.6]). Using the interpolation property we find $x \ll y \ll d$ whereby $d' \notin {\uparrow} y$. This proves the claim. Observe that for $y \ll d$ the lower open $D \setminus {\uparrow} d$ is total with the Scott open set $\Uparrow y$ which in turn is consistent with $D \setminus {\uparrow} y$. With Lemma 3.5.3 is follows that the well-inside relation on ωD is approximating. □

3.5.1 Spectra of regular d-frames

Let us check that our definition is sound in the sense that spectra of regular d-frames enjoy the corresponding separation axiom.

Lemma 3.5.5. *In a regular d-frame, any Scott open subset of a component frame is open in the sense of Definition 2.3.1, that is, a round upper set with respect to the well-inside relation.*

Proof. If $\bigsqcup \twoheaddownarrow x = x \in U \in \sigma L_+$ then by definition of the Scott topology the ideal $\twoheaddownarrow x$ intersects the Scott open U. □

In particular we have a relationship between the way-below relation of a component frame and its well-inside relation.

Lemma 3.5.6. *If $x \ll y$ holds in the component frame L_+ of a regular d-frame then x is well inside y.*

Proof. $x \ll y = \bigsqcup \twoheaddownarrow y$ implies $x \in \twoheaddownarrow y$ by definition of the way-below relation. □

Proposition 3.5.7. *The spectrum of a regular d-frame is bitopologically T_3.*

Proof. Consider a d-point of the d-frame $\mathcal{L}$ in the guise of a pair of completely prime filters (F_-, F_+). Suppose that this point is disjoint from the closed set whose complement is the open $\ulcorner \phi \urcorner^-$ where $\phi \in L_-$. In other words ϕ is an element of the completely prime filter F_-. This filter is in particular Scott open, whence by the preceding lemma there exists an element $\psi \in F_-$ that is well inside ϕ. By definition of the well-inside relation we have $\phi \mathsf{tot} x \mathsf{con} \psi$ for some witness $x \in L_+$. Observe that because of $\psi \in F_-$ the point under consideration is an element of the open set $\ulcorner \psi \urcorner^-$. Furthermore, the closed set $\operatorname{pt} \mathcal{L} \setminus \ulcorner \phi \urcorner^-$ is contained in the open set $\ulcorner x \urcorner^+$ because of $\phi \mathsf{tot} x$ and the latter open is disjoint from $\ulcorner \psi \urcorner^-$ because of $x \mathsf{con} \psi$. We conclude that in the spectrum of $\mathcal{L}$, any point not contained in a closed set with respect to the topology τ_- can be separated from the closed set by a pair of disjoint opens, where the open neighbourhood of the τ_--closed set is τ_+-open and the neighbourhood of the point is τ_--open. By symmetry, a similar statement holds for sets that are closed with respect to the topology τ_+. □

In due course we will show that not only is the spectrum of a regular d-frame T_3, but also pairwise Hausdorff and thereby regular, as the nomenclature suggests. For this we employ a result that has many useful consequences.

Theorem 3.5.8. *Let $\mathcal{M}$ be a regular d-frame and $h : \mathcal{M} \to \mathcal{L}$ a d-frame homomorphism. Then the components h_- and h_+ determine each other.*

Proof. We claim that the following diagram commutes, where the maps $(-)_{\mathsf{con}}$ and $(-)^{\mathsf{tot}}$ are the maps of the Fundamental Lemma of d-lattices 2.3.2.

$$\begin{array}{ccc}
\operatorname{Filt} M_- & \xrightarrow{\operatorname{Filt}(h_-)} & \operatorname{Filt} L_- \\
{\scriptstyle (-)^{\mathsf{tot}}} \uparrow & & \downarrow {\scriptstyle (-)_{\mathsf{con}}} \\
\operatorname{Idl} M_+ & & \operatorname{Idl} L_+ \\
{\scriptstyle \twoheaddownarrow} \uparrow & & \downarrow {\scriptstyle \bigsqcup} \\
M_+ & \xrightarrow[h_+]{} & L_+
\end{array} \tag{3.2}$$

Let m be an element of the component frame M_+. We show that the value $h_+(m)$ can be computed knowing only the other component map h_-. Expanding the definition yields

$$(\mathrm{Filt}(h_-)((\downarrow m)^{\mathsf{tot}}))_{\mathsf{con}} = \{x \in L_+ \mid \exists \psi \in M_-.\, x \mathsf{con} h_-(\psi),\ \psi \mathsf{tot} m\}\,. \tag{3.3}$$

By regularity of $\mathcal{M}$ we know that $h_+(m)$ is the join of the set

$$\left\{h_+(m') \,\middle|\, \exists \psi \in M_-.\, m' \mathsf{con} \psi \mathsf{tot} m\right\}. \tag{3.4}$$

Since the d-frame homomorphism h preserves con we know that the set (3.4) is contained in the set (3.3). From preservation of tot we deduce that every element x of the set (3.3) satisfies $x \lhd h_+(m)$. Together this yields both inequalities of the desired identity. □

Corollary 3.5.9. *A completely prime filter of a regular d-frame's component frame extends to a d-point in at most one way.*

We can say a bit more about the correspondence between points and d-points. Recall that the spectrum of any d-frame carries two specialisation orders $\sqsubseteq_-$ and $\sqsubseteq_+$ that arise from the inclusion orders between completely prime filters in the first and second component of d-points.

Proposition 3.5.10. *The spectrum of a regular d-frame is an order-separated space:*

1. *The specialisation orders of the topologies τ_- and τ_+ on the spectrum of a regular d-frame are dual.*
2. *Two distinct points of the spectrum can be separated by disjoint opens, where one open is chosen from τ_- and one from τ_+.*

Proof. Examining Equation (3.3) in Theorem 3.5.8 one finds that the larger the component frame homomorphism h_- (in the point-wise order), the smaller the other component h_+ must be. This proves claim (1).

If (F_-, F_+) and (G_-, G_+) are two d-points represented as pairs of completely prime filters, assume without loss of generality that $F_+ \not\subseteq G_+$. Using Lemma 3.5.5 find $F_+ \ni x \lhd y \notin G_+$. Let $x \mathsf{con} \phi \mathsf{tot} y$. Then $\ulcorner x \urcorner^+$ and $\ulcorner \phi \urcorner^-$ are the disjoint opens of the spectrum with the desired properties. □

Corollary 3.5.11. *The spectrum of a regular d-frame is a regular bitopological space.*

The corollary has a converse.

Theorem 3.5.12. *Every regular bitopological space is d-sober. The dual adjunction between bitopological spaces and d-frames restricts to a dual adjunction between the category of regular bitopological spaces and the category of regular d-frames in such a way that the "d-soberification" endofunctor on regular bitopological spaces is equivalent to the identity.*

Proof. Let X be a regular bitopological space and $\mathcal{O}X$ be the regular d-frame derived from it. It suffices to show that X is bihomeomorphic to the spectrum of $\mathcal{O}X$. For this we show that every point of $\operatorname{pt}\mathcal{O}X$ arises from a point of X, because the natural map $X \to \operatorname{pt}\mathcal{O}X$ is injective for pairwise Hausdorff spaces. Let (F_-, F_+) be a d-point of $\mathcal{O}X$. Consider the meet-prime open sets $\phi = \bigsqcup(L_- \setminus F_-)$ and $u = \bigsqcup(L_+ \setminus F_+)$. We know that ϕ can not be total with u. According to the way the totality relation on the d-frame $\mathcal{O}X$ was defined, there exists a point $x \in X$ that is neither an element of ϕ nor of u. Let (G_-, G_+) be the d-point of $\mathcal{O}X$ associated with x (that is, G_- and G_+ are the neighbourhood filters of x in the two topologies). As $\phi \notin G_-$ we know that $G_- \subseteq F_-$, so by Proposition 3.5.10 we have $F_+ \subseteq G_+$. But as $u \notin G_+$ we also have $G_+ \subseteq F_+$ and thereby $F_+ = G_+$. A dual argument shows that $F_- = G_+$. Hence every d-point of $\mathcal{O}X$ arises from a point of X. □

3.5.2 The patch construction on regular d-frames

The join topology of a regular bitopological space is Hausdorff. Using Theorem 3.5.12 and Corollary 3.4.2 we can say a bit more about the join topology, because any such join topology is the patch frame of a regular d-frame.

Lemma 3.5.13. *The frame homomorphisms* $\ulcorner - \urcorner^-$ *and* $\ulcorner - \urcorner^+$ *from the component frames of a d-frame into its patch frame preserve the well-inside relation.*

Proof. Follows directly from the definition of the patch frame. □

Proposition 3.5.14. *The patch frame of a regular d-frame is regular.*

Proof. By the preceding lemma we know that every generator $\ulcorner \phi \urcorner^-$ or $\ulcorner x \urcorner^+$ of the patch frame is the join of generators well inside it. Those elements form a subbasis of the patch frame whence by Lemma 3.5.3 the patch frame is regular. □

Proposition 3.5.15. *If* L *is a regular d-frame then for any* $x \in L$ *the generators* $\ulcorner x \urcorner^-$ *and* $\ulcorner x \urcorner^+$ *of the patch frame* $\operatorname{Patch} L_=$ *are equal. Consequently,* L *is isomorphic to* $\operatorname{Patch} L_=$.

Proof. We use the same technique as in the proof of Lemma 2.5.5 (2). In order to show $\ulcorner x \urcorner^- \sqsubseteq \ulcorner x \urcorner^+$ we consider the generators $\ulcorner y \urcorner^-$ where y is well inside x in the frame L. By definition of the well-inside relation this means that there exists a witness $z \in L$ with $y \sqcap z = 0$ and $y \sqcup x = 1$. We obtain

$$\begin{aligned} 0 = \ulcorner y \urcorner^- \sqcap \ulcorner z \urcorner^+ &\sqsubseteq \ulcorner x \urcorner^+ \sqcap \ulcorner z \urcorner^+, \\ \ulcorner y \urcorner^- \sqcup \ulcorner z \urcorner^+ &\sqsubseteq \ulcorner x \urcorner^+ \sqcup \ulcorner z \urcorner^+ = 1. \end{aligned}$$

Here, the first inequality holds because z is consistent with y in the symmetric d-frame $L_=$, whereas the second inequality holds because $\ulcorner - \urcorner^+$ preserves joins. With the lattice distributive law one obtains $\ulcorner y \urcorner^- \sqsubseteq \ulcorner x \urcorner^+$ and using the fact that $\ulcorner x \urcorner^-$ is the join of all the $\ulcorner y \urcorner^-$ we conclude $\ulcorner x \urcorner^- \sqsubseteq \ulcorner x \urcorner^+$, as desired. A symmetric argument shows that also the reverse inequality holds. □

Now let us examine the right adjoint to the patch functor and consider the symmetric d-frames that arise from regular frames. By the way the consistency and totality relation on a symmetric d-frame is defined, whenever x is well inside y in a frame L then both $\ulcorner x \urcorner^{+} \lhd \ulcorner y \urcorner^{+}$ and $\ulcorner y \urcorner^{-} \rhd \ulcorner x \urcorner^{-}$ hold in the symmetric d-frame $L_{=}$. Therefore the symmetric d-frame of a regular frame is regular. But even more is true:

Theorem 3.5.16. *The category of regular frames is equivalent to the category of symmetric regular d-frames via the functors* Patch *and* $(-)_{=}$.

Proof. From Proposition 3.5.14 we know that the patch frame of a regular d-frame is regular, and we convinced ourselves that the symmetric d-frame of a regular frame is regular. Because of Theorem 3.5.8 the symmetric d-frame functor is full onto the category of symmetric regular d-frames. In Proposition 3.5.15 we proved that the composite $\mathrm{Patch} \circ (-)_{=}$ is equivalent to the identity functor on regular d-frames. From this it follows that the composite $(-)_{=} \circ \mathrm{Patch}$, too, is equivalent to the identity functor on symmetric regular d-frames. □

3.6 Compact d-frames

It is our declared long-term goal to find a category of d-frames that may serve as an appropriate analogue of the category KHaus of compact Hausdorff spaces. In Section 3.5 we explored a category of d-frames whose spectra are Hausdorff in the join topology. Now we turn towards a class of d-frames that include those arising from compact bitopological spaces. To be more precise, let us review when a bitopological space is called compact. It is not enough to require that a bitopological space (X, τ_-, τ_+) is compact in each topology separately. As an example, consider the punctured unit interval $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ with the upper and lower order topologies. This is a d-sober bitopological space. Any open cover of this space by lower opens must cover the point 1 and has therefore a singleton subcover. Dually, any open cover by upper opens must cover the point 0 and has a singleton subcover. However, we certainly do not want to call such a space compact, as for example the sequence $(\frac{1}{2} + \frac{-1^n}{2^n})_{n \geq 1}$ does not converge. Instead it is common to declare a bitopological space compact if it is compact in its join topology. Kopperman calls such spaces *joincompact* in [38]. Using the Alexander Subbase Lemma, joincompactness of a bitopological space (X, τ_-, τ_+) is equivalent to the following assertion. Whenever $\{(u_i, v_i)\}_{i \in I} \subseteq \tau_- \times \tau_+$ is a directed family of pairs of opens with the property $\left(\bigcup_{i \in I} u_i\right) \cup \left(\bigcup_{i \in I} v_i\right) = X$ then $u_i \cup v_i = X$ for some $i \in I$ already. This motivates the following definition.

Definition 3.6.1. A d-frame $(L_-, L_+, \mathsf{con}, \mathsf{tot})$ is *compact* if for every directed family $\{(\phi_i, x_i)\}_{i \in I}$ of the product $L_- \times L_+$ the following holds. Whenever $\bigsqcup_{i \in I} \phi_i$ is total with $\bigsqcup_{i \in I} x_i$ then there is some $i \in I$ such that ϕ_i is total with x_i already. In the language of domain theory, the compact d-frames are those for which the relation tot is Scott open in the product frame $L_- \times L_+$.

Example 17. Examples of compact d-frames include:

- All finite d-frames,
- The open ideal completion of any d-lattice,
- The d-frame derived from a d-lattice described in Example 10,
- The d-frame $\mathcal{O}X$ of opens derived from a joincompact bitopological space.

Example 18. There are non-compact d-frames whose spectrum is compact. For instance, consider the d-frame of Example 13. Its spectrum is empty and thereby trivially compact. However, for any $x \in (0,1)$ there is a total pair of opens $((0,x),(x,1))$ which we can write as a directed join $\left(\bigcup_{y<x}(0,y), \bigcup_{z>x}(z,1)\right)$ where no pair $((0,y),(z,1))$ is total. Hence the totality relation is not Scott open.

Perhaps the most useful consequence of compactness is this: Given an element of a component frame of a compact d-frame, the filter of elements total with it is Scott open. In particular this applies to the least element of the component frame, whereby the singleton $\{1\} = 0^{\mathsf{tot}}$ of each component frame is a Scott open filter. We record:

Lemma 3.6.1. *Let $\mathcal{L}$ be a compact d-frame. Then the well-inside relation on each component frame is contained in the frame's way-below relation.*

Proof. Suppose x, y are elements of the component frame L_+ of the compact d-frame $\mathcal{L}$. If x is well inside y then there is some witness $\phi \in L_-$ such that $x \mathsf{con} \phi \mathsf{tot} y$. By compactness the filter ϕ^{tot} is Scott open. Since ϕ^{tot} contains y and x is a lower bound for this filter, we conclude that x is way below y. □

Corollary 3.6.2. *The component frames of a compact d-frame are compact, that is, the top element is way below itself.*

Proof. In any d-frame the top element 1 is well inside itself. □

We arrive at another important example of compact d-frames.

Lemma 3.6.3. *A frame L is compact if and only if its symmetric d-frame $L_=$ is compact.*

Proof. By the preceding corollary, compactness of $L_=$ implies compactness of L. For the reverse implication, use the fact that any directed set of the product $L \times L$ gives rise to a directed set in L by mapping a pair (x, y) to the binary join $x \sqcup y$. Therefore the totality relation $\mathsf{tot}_=$ of the symmetric d-frame $L_=$ is Scott open in $L \times L$ whenever the singleton $\{1\}$ is Scott open in L. □

Lemma 3.6.4. *Suppose $F_+ = L_+ \setminus {\downarrow} p$ is a completely prime filter of the second component frame of a d-frame $\mathcal{L}$. If the filter $p^{\mathsf{tot}} \subseteq L_-$ is Scott open, then F_+ extends to a d-point of $\mathcal{L}$.*

Proof. Let $F_+ \subseteq L_+$ be a completely prime filter of a component frame of the d-frame $\mathcal{L}$. We can write $F_+ = L_+ \setminus \downarrow p$ for some meet-prime element p. Recall that a completely prime filter $F_- \subseteq L_-$ extends F_+ to a d-point if and only if it separates the ideal $(F_+)_{\mathsf{con}}$ from the filter p^{tot}. As the ideal $(F_+)_{\mathsf{con}}$ is always disjoint from the Scott open filter p^{tot}, the join of the ideal $(F_+)_{\mathsf{con}}$ is still not in the filter p^{tot}. Now apply the Scott Open Filter Theorem[3] and obtain a completely prime filter F_- that contains p^{tot} and does not contain $\bigsqcup(F_+)_{\mathsf{con}}$. We have found a d-point (F_-, F_+) extending the original completely prime filter. □

Remark. The elements p with p^{tot} Scott open can for good reasons be called *cocompact*. The d-points of a regular d-frame are in bijective correspondence with the cocompact meet-primes of each component frame.

Proposition 3.6.5. *In a compact d-frame, every completely prime filter of a component frame extends to a d-point.*

Proof. In a compact d-frame, every filter of the form p^{tot} is Scott open, so Lemma 3.6.4 applies. □

3.7 Compact regular d-frames

From the spectral theory point of view, neither the regular nor the compact d-frames are entirely satisfying. Any regular bitopological space is the spectrum of a regular d-frame, but the class of regular d-frames contains highly non-spatial members. The compact d-frames possess at least as many d-points as their component frames have points, but then it is easy to construct a compact d-frame whose component frames do not have any points. Both properties combined, however, lead to a most satisfying class of d-frames.

To begin with, the component frames of a compact regular d-frame are stably continuous. Indeed, recall from Lemma 3.6.1 that in a compact d-frame the well-inside relation is contained in the way-below relation. With regularity and Lemma 3.5.6 we obtain the reverse inclusion. It follows that the component frames of a compact regular d-frame are domains. Furthermore, the way-below relation inherits the algebraic properties of the well-inside relation, whereby the component frames are stably continuous.

Every stably continuous frame has a stably continuous Lawson dual. Proposition 1.9.1 gave a proof of this fact by means of interaction algebras. And indeed we do not have to look far to find the Lawson dual of a compact regular d-frame's component frame:

Proposition 3.7.1. *Let $\mathcal{L}$ be a compact regular d-frame.*

[3] In a distributive complete lattice, a point and a Scott open filter can be separated by a completely prime filter. See [22, Lemma I-3.4] or [30, VII-4.3].

1. *The component frame L_- is order-isomorphic to the Lawson dual of the component frame L_+ via the assignments*

$$\phi \mapsto \phi^{\mathsf{tot}} : L_- \to (L_+)^\wedge, \quad F \mapsto \bigsqcup F_{\mathsf{con}} : (L_+)^\wedge \to L_-.$$

2. *Under the isomorphism of (1) the relation $\phi \mathsf{tot} x$ holds precisely when x is an element of the Scott open filter ϕ and $x \mathsf{con} \phi$ holds precisely when x is a lower bound for the Scott open filter ϕ.*

Proof. As we remarked above, the way-below relation on each component frame coincides with the well-inside relation. By Definition 3.6.1 and the observation at the end of Section 2.4 every compact regular d-frame is therefore normal and in fact a Stage 5 interaction algebra. Recall that any domain is isomorphic to the domain of its round ideals with respect to the way-below relation (Proposition 6.1.9). Now (1) follows from Theorem 1.3.4 and the claim (2) follows from Proposition 1.3.5. □

A priori it is not clear whether there is a restriction on the stably continuous frames that feature as component frames of compact regular d-frames. This question can be put aside easily:

Proposition 3.7.2. *For every stably continuous frame L the Stage 5 interaction algebra*

$$\operatorname{Ialg} L = L^\wedge \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} L$$

is a compact regular d-frame.

Proof. Both L and the Lawson dual $L^\wedge$ are stably continuous frames. As for the axioms of a d-frame, only (con-$\bigsqcup$) needs checking. The pseudocomplement of an element $x \in L$ is the filter $\mathop{\Uparrow} x$ of elements way above x. Observe that for this to be a filter, stable continuity is essential. The pseudocomplement of a Scott open filter is simply its meet in L. Compactness and regularity are straightforward to check. □

We arrive at the somewhat surprising conclusion that a compact regular d-frame is entirely determined by one component frame. This corresponds to a well-known fact about stably compact spaces and the de Groot dual [38, Lemma 4.6]: For a T_0 space there is at most one topology that completes it to a compact regular bitopological space, and in that case both topologies must be stably compact and duals of each other. But even more is true: From Theorem 3.5.8 we know that the d-frame homomorphisms between compact regular d-frames are determined by one component map already. Regarding the first component frame as the set of Scott open filters of the second component, one finds that the image of a Scott open filter under the d-frame homomorphism is nothing but the (upper closure of the) forward image of that filter under the second component map.

Does every semi-open frame homomorphism between stably continuous frames give rise to a d-frame homomorphism between compact regular d-frames? The answer is yes, and in due course we will see that it is irrelevant whether the semi-open homomorphism preserves directed joins.

Lemma 3.7.3. *The open ideal completion of a normal d-lattice is a compact regular d-frame.*

Proof. As we remarked already, the open ideal completion of any d-lattice is a compact d-frame because of the way the totality relation is defined. In Proposition 2.4.5 we showed that the component frames of a normal d-lattice's open ideal completion are stably continuous frames. The remarks following that proposition show that the consistency and totality relations of the open ideal completion are of the form described in Proposition 3.7.1 (2). □

Observe that a compact regular d-frame $\mathcal{L}$ is in particular a normal d-lattice. The preceding lemma combined with Proposition 3.7.1 yields:

Proposition 3.7.4. *A compact regular d-frame is isomorphic to its own open ideal completion.*

Let us return to the homomorphisms between compact regular d-frames. Suppose $h : L \to M$ is a map between stably continuous frames that preserves all finite meets and joins as well as the way-below relation. Extending both frames to compact regular d-frames $\mathcal{L} = (L^\wedge, L, \mathsf{con}, \mathsf{tot})$ and $\mathcal{M} = (M^\wedge, M, \mathsf{con}, \mathsf{tot})$ turns h into a well-inside-preserving lattice homomorphism between the components of normal d-lattices. By Proposition 2.3.9 the map h extends to a d-frame homomorphism between the open ideal completions of $\mathcal{L}$ and $\mathcal{M}$. But these are isomorphic to the d-frames $\mathcal{L}$ and $\mathcal{M}$ themselves, so the extension of h can be regarded as a d-frame homomorphism $\mathcal{L} \to \mathcal{M}$. Concretely, the d-frame homomorphism generated by h maps an element $x \in L$ to the join of the forward image of $\twoheaddownarrow x$ under h, and a Scott open filter $\phi \in L^\wedge$ to the upper closure of its forward image under h. Evidently, if h was a frame homomorphism in the first place, then the process described gives back h, because $h(x) = h\left(\bigsqcup \twoheaddownarrow x\right) = \bigsqcup \mathrm{Idl}(h)(\twoheaddownarrow x)$. Let us gather our findings in a theorem.

Theorem 3.7.5. *The category* KRdFrm *of compact regular d-frames is equivalent to the category* SCFrm_π *of stably continuous frames and semi-open frame homomorphisms.*

Compact regular frames

The Stone duals of compact Hausdorff spaces are the compact regular frames, that are frames where the top element is way below itself and the well-inside relation is approximating. Recall from Theorem 3.5.16 that the category of regular frames is equivalent to the category of symmetric regular d-frames. In particular the well-inside relation of a regular frame L is the same as the d-frame-theoretic well-inside relation of the symmetric

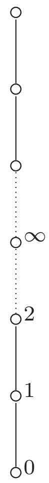


Figure 3.4: A stably continuous frame that is isomorphic to its own Lawson dual, but is not regular.

regular d-frame $L_=$. Further recall from Lemma 3.6.3 that a frame is compact if and only if its symmetric d-frame $L_=$ is compact. Together with Proposition 3.7.1 we obtain:

Proposition 3.7.6. *A frame L is compact regular if and only if the assignment $x \mapsto \{y \in L \mid x \sqcup y = 1\}$ constitutes an isomorphism between the frame and its Lawson dual.*

Observe that the existence of an arbitrary isomorphism with the Lawson dual is not enough. The frame depicted in Figure 3.4 provides a counterexample. Together with Theorem 3.7.5 we conclude:

Proposition 3.7.7. *The category* KRFrm *of compact regular frames and frame homomorphisms is equivalent to the category* $\mathsf{KRdFrm}_=$ *of symmetric compact regular d-frames.*

Proof. Since the way-below relation on a compact regular frame coincides with the frame-theoretic well-inside relation and all frame homomorphisms preserve the latter relation, the category KRFrm is a full subcategory of SCFrm_π. $\square$

3.7.1 Spectral theory of compact regular d-frames

Stably compact spaces

It is well known in domain theory that the stably continuous frames are precisely the Stone duals of stably compact spaces. The semi-open frame homomorphisms on the algebraic side correspond to proper continuous maps on the spatial side, that is, maps with the extra property that preimages of compact saturated sets are compact again. We claim that the best view on this duality is the bitopological one. Indeed, recall that the compact saturated sets are the basic closed sets of the cocompact topology. Hence a proper continuous map is actually bicontinuous when the source and target spaces are endowed with the original topology and its cocompact counterpart.

Let us examine how the duality is presented by d-frames.

Lemma 3.7.8. *In a compact regular d-frame, every frame point of a component frame extends to a unique d-point.*

Proof. Combine Corollary 3.5.9 and Proposition 3.6.5. □

Proposition 3.7.9. *Compact regular d-frames are spatial.*

Proof. As every continuous frame is spatial (this is shown using the Scott Open Filter Theorem), the preceding lemma yields that elements of a component frame can be separated by d-points. It remains to show the two conditions of spatiality concerned with the relations con and tot. For this is it convenient to use Proposition 3.7.1 and view the first component frame of a compact regular d-frame as the Lawson dual of the second component. Let us write $L^\wedge$ for the first and L for the second component frame of the compact regular d-frame $\mathcal{L}$. A completely prime filter $F = L \setminus \downarrow p$ on the second component corresponds to the completely prime filter $\Phi_p := \{\phi \in L^\wedge \mid p \in \phi\}$ of the Lawson dual.

Suppose $x \in L$ is not a lower bound for the Scott open filter ϕ. Then there exists some element $y \in \phi$ such that x is not below y. As the frame L is spatial, we can find a completely prime filter $F \subseteq L$ that contains x and not y. Notice that $y \notin F$ implies $y \sqsubseteq p = \bigsqcup(L \setminus F)$ whereby (Φ_p, F) is a d-point with $\phi \in \Phi_p$ and $x \in F$, as desired.

Now suppose that x is not an element of ϕ. This is the classical case for the Scott Open Filter Theorem; we obtain a completely prime filter $F \subseteq L$ containing ϕ that x is not a member of. From $\phi \subseteq F$ we also get $p = \bigsqcup(L \setminus F) \notin \phi$ whence (Φ_p, F) is a d-point with $\phi \notin \Phi_p$ and $x \notin F$. □

In contrast to spectra in general, the bitopological spectrum of a compact regular d-frame is sober in each of the individual topologies. The Hofmann-Mislove Theorem tells us that the compact saturated subsets of a sober space are in order-reversing bijection with the Scott open filters of open sets. This yields an order-isomorphism between the opens of the cocompact topology and the Lawson dual of a stably compact topology. We arrive at:

Theorem 3.7.10. *The spectrum of a compact regular d-frame consists of a space endowed with a stably compact topology and its cocompact dual. Every compact regular d-frame arises as the d-frame of opens of such a bitopological space. The category* KRdFrm *of compact regular d-frames is dually equivalent to the category* SCTop *of stably compact spaces and proper maps.*

Compact ordered Hausdorff spaces

There is yet another manifestation of compact regular d-frames that brings us closer to our long-term goal of compactifications. Recall from Proposition 3.5.10 that the spectrum of every regular d-frame is an order-separated space whose order is the specialisation order of the second topology. If the d-frame is in addition compact, then using spatiality and the Alexander Subbase Lemma one shows that the join topology on the spectrum is compact Hausdorff. As continuous maps are automatically monotone with respect to the specialisation order, d-frame homomorphisms give rise to monotone continuous maps between compact ordered Hausdorff spaces. In fact, every compact ordered Hausdorff space arises as the patch of a stably compact space (see for example the Compendium [22]). Hence we have yet another category that is dual to compact regular d-frames.

Theorem 3.7.11. *The category* KRdFrm *of compact regular d-frames is dually equivalent to the category* KOrdHaus *of compact ordered Hausdorff spaces and monotone continuous maps.*

3.8 Notes on Chapter 3

The material presented in this chapter has large overlaps with what Jung and Moshier developed in their technical report [33]. Some sections of the technical report are missing here, as our treatment of d-frames is geared towards compact regular d-frames and the theory of point-free compactifications. What we call d-frames here are *reasonable* d-frames in the technical report; the duality with bitopological spaces actually works without requiring that the consistency and totality relations satisfy the rules of Table 3.1. However, every d-frame that arises from a bitopological space obeys the rules of Table 3.1 as well as the cut rules, hence the term "reasonable". Practically all results of Section 3.2 were known before.

Jung and Moshier exhibit other algebraic structures that are equivalent to d-frames. In a nutshell, instead of the full product of the two component frames, one can restrict the attention to the consistency predicate only and model the totality predicate as an auxiliary relation on this lattice. This presentation is particularly suited for the equivalence between compact regular d-frames and strong proximity lattices.

Yet there are aspects in the present work which are not found in the work of Jung and Moshier. The view on the consistency relation as being generated by pseudocomplements is new, and so is Proposition 3.5.2. The patch construction for d-frames was originally conceived for the adjunction between d-frames and biframes, so Theorem 3.4.1 was only stated implicitly. Therefore this work is the first that exploits the fact that the adjunction between frames and d-frames "transports the dualising objects" in the form of Corollary 3.4.2. For the patch constructions for stably (locally) compact spaces and locales using various representations, consult [12, 5, 19, 44]. The equivalence between regular frames and symmetric regular d-frames and in particular Proposition 3.5.15 and Theorem 3.5.8 are new. However, its consequence Proposition 3.5.10 was shown in [33, Prop. 6.4] and Theorem 3.5.12 was proved in greater generality in [33, Theorem 4.13].

Proposition 3.6.5 was proved by Achim Jung and the author, although for compact regular d-frames this extension result already was known by different means.

Section 3.7 contains no new results, as most of its contents are well-known facts about stably compact spaces and the patch topology. The importance of stably compact spaces and their abundance in domain theory nevertheless lets it appear worthwhile to develop tools that make calculations with stably compact spaces easy.

Future work and open problems

In contrast to the patch construction of d-lattices, it is not known whether the generator maps from the components of a d-frame into its patch frame are injective in general. Connected to this question is the conjecture that there are compact d-frames whose patch frame and spectrum are not compact. The reason why we believe that such structures exist is that compactness relies entirely on the totality relation. Therefore it is conceivable that one can start with a joincompact bitopological space, enlarge the consistency relation on its d-frame of opens and thereby remove just enough d-points to render the spectrum non-compact.

The patch frame might provide another approach to the T_1 separation axiom for locales. The adjunction Patch $\dashv (-)_=$ of Theorem 3.4.1 gives rise to a monad on Loc whose fixed points may for good reasons be called T_1 locales[4].

The categorical structure of the category dFrm is yet to be explored. While the construction of products seems straightforward and analogous to the construction of products of d-lattices, coproducts are not as easy to describe. As of the time of writing it seems that Moshier has solved this problem.

Subobjects in the category of locales have several equivalent presentations, each of them with its advantages and disadvantages. A sublocale is a regular monomorphism in Loc, which can be shown to be the same as a surjective frame homomorphism. Using the fact that every frame homomorphism has a right adjoint one arrives at *nuclei*. These are inflationary idempotent self-maps on a frame which preserve finite meets. The collection of all nuclei on a frame forms another frame in the point-wise order. When moving to d-frames, the difficulty is how to accommodate for the consistency and totality relations. A reasonable generalisation seems that a d-frame should have a *d-frame* of nuclei. An alternative approach to sublocales is via congruence preorders on frames. As we demonstrated, it is entirely possible to change the spectrum of a d-frame without altering the component frames at all. Hence a congruence on a d-frame should extend both the order on the frame $L_- \times L_+$ to a congruence preorder as well as the consistency and totality relations to the their "preorder" counterparts. The difficulty in this approach is that one must understand the rules of Table 3.1 as production rules which mutually enlarge the three sets under consideration.

[4] The standard notion defines a locale X to be T_1 if the exponential $X^{\mathbb{S}}$ is isomorphic to X, where $\mathbb{S}$ is the Sierpinski locale. In the spectrum of a frame $L \cong \mathrm{Patch}(L_=)$, distinct points can be separated either by disjoint open or by disjoint closed sets.

Chapter 4

Compactifications of d-frames

Let us review the notion of compactification in classical point-set topology. Suppose one has specified a continuous function on a space X of easily described elements. For example, a function from the rationals $\mathbb{Q}$ to the reals or a function μ assigning a measure to closed intervals. One would like to extend this function to a set of more complicated elements, such as the reals or a sigma algebra of sets, respectively. The extension of the function should be uniquely determined, which can be ensured by requiring the extended domain to be Hausdorff and the original space X to be dense in it, because then every element y of the extension is the unique limit of some net (x_i) in X; the function is extended as $f(y) = \lim_i f(x_i)$. Moreover, every net (x_i) in X should give rise to some some element y of the extension. The latter is achieved by requiring the extension of X to be compact.

Thus the problem of extending the space X to a space Y with the listed properties becomes the *compactification problem*.

The compactification problem. Given a topological space X, find a compact Hausdorff space Y such that X can be identified with a dense subspace of Y.

A compactification of X is a dense topological embedding of X into some compact Hausdorff space Y. This means there exists a continuous map $e : X \to Y$ with the properties

- e is an embedding: The map e is a homeomorphism onto its image.
- The image of e is dense: Any neighbourhood of any point of Y contains some point of the form $e(x)$.

In fact one may describe a compactification purely in terms of the open sets. The map $e : X \to Y$ is an embedding if and only if the preimage map $e^{-1} : \mathcal{O}Y \to \mathcal{O}X$ is surjective. The map e is dense precisely when $e^{-1}(U) = \emptyset$ implies that $U = \emptyset$. We turn this observation into a definition:

Definition 4.0.1. A compactification of a frame L is a surjective frame homomorphism $h : M \twoheadrightarrow L$ from some compact regular frame M onto L such that $h(x) = 0$ implies $x = 0$.

The Stone duals of compact Hausdorff spaces are precisely the compact regular frames. Every space that admits a compactification is Tychonoff and thereby sober. The Stone duals of Tychonoff spaces are the completely regular frames, whence the definition above subsumes the compactifications of spaces.

Surprisingly, the dense embedding $e : X \to Y$ can be described internally in the space X as follows[1]. Consider those pairs (A_0, A_1) of closed sets in X where the closure $\overline{e(A_0)}$ is well inside the closure $\overline{e(A_1)}$ in the space Y. This yields a relation $\prec$ between opens of X that is stronger than set inclusion. The somewhat unexpected result, to our knowledge first published in Russian, is the following.

Theorem 4.0.1. *The compactification $e : X \hookrightarrow Y$ of X can be reconstructed from the relation $\prec$ derived from the dense embedding e. Concretely, the frame of opens $\mathcal{O}Y$ is isomorphic to the frame $\mathrm{Idl}^{\prec}\,\mathcal{O}X$ of round ideals of opens and the frame homomorphism $e^{-1} : \mathcal{O}Y \to \mathcal{O}X$ is isomorphic to the join map $\bigsqcup : \mathrm{Idl}^{\prec}\,\mathcal{O}X \to \mathcal{O}X$. Moreover, every compactification arises this way.*

Once more let us emphasise that the theorem does not mention points at all, and so it is not surprising that there is an analogous statement for the compactifications of frames, due to Banaschewski [4].

Theorem 4.0.2. *Let $h : M \twoheadrightarrow L$ be a compactification of the frame L. Define a relation $\prec$ on the frame L as*

$$x_0 \prec x_1 \text{ iff } h_*(x_0) \ll h_*(x_1)$$

where h_* is the right adjoint to h. Then the frame M is isomorphic to the round ideal completion $\mathrm{Idl}^{\prec} L$ and the frame homomorphism h is isomorphic to the join map $\bigsqcup : \mathrm{Idl}^{\prec} L \to L$. Moreover, every compactification of L arises this way.

Those relations $\prec$ arising from a compactification of a frame are characterised by the properties listed in Table 4.1 and are called *proximities*. Every proximity gives rise to a compactification. In this chapter we set out to answer the following questions.

1. What is the appropriate notion of compactification of a d-frame?
2. Which d-frames admit a compactification? (What are the analogues of completely regular frames?)
3. What is the appropriate notion of proximity on a d-frame? Can the compactifications of d-frames be characterised by proximities?
4. Is there a largest compactification? (What is the Stone-Čech compactification of a d-frame?)
5. Which d-frames possess a smallest compactification?

[1] If $A = X \setminus U$ is closed then the closure of $e(A)$ is the complement of the largest open set $V \subseteq Y$ with $e^{-1}(V) = U$. Hence the use of the right adjoint in Theorem 4.0.2 below.

Auxiliary relation	(i)	$\prec$ is contained in the poset order $\sqsubseteq$.
	(ii)	$x' \sqsubseteq x \prec y \sqsubseteq y'$ implies $x' \prec y'$.
Finite meets and joins	(iii)	If X is a finite set and for all $x \in X$ the relation $x \prec y$ holds, then $\bigsqcup X \prec y$.
	(iv)	If Y is a finite set and for all $y \in Y$ the relation $x \prec y$ holds, then $x \prec \bigsqcap Y$.
Interpolation	(v)	Whenever $x \prec z$ then there exists some y with $x \prec y \prec z$.
Approximation	(vi)	The relation $\prec$ is approximating: every element y is the join of the set $\{x \mid x \prec y\}$.
Well-inside	(vii)	The relation $\prec$ is stronger than the well-inside relation: Whenever $x \prec y$ then $x \lhd y$.
Complements	(viii)	The relation $\prec$ respects complements: Whenever $x \prec y$ then $\neg y \prec \neg x$.

Table 4.1: The axioms characterising a proximity on a frame. For general frames, read $\neg$ as the Heyting complement. If the frame is the powerset of a space, read $\neg$ as set complement.

4.1 Complete regularity and compactifications of d-frames

Our declared goal is to extend the theory of compactification of spaces to a theory of *order-preserving* compactification of *ordered* spaces. Consequently, the category of compact regular frames is replaced by the category of compact regular d-frames, as the latter is dually equivalent to compact ordered Hausdorff spaces (see Theorem 3.7.11). Hence our tentative definition is: *A compactification of a d-frame $\mathcal{L}$ is a d-frame homomorphism $h : \mathcal{M} \to \mathcal{L}$ where $\mathcal{M}$ is compact regular, both component maps are surjective and $h(\phi, x) = (0, 0)$ implies that $(\phi, x) = (0, 0)$.* While surjectivity of the component maps seems to be an obvious criterion, the implicit definition of density we used here turns out to be too weak. The reason is that the spectrum of a compact regular d-frame carries two stably compact topologies, which are in general not Hausdorff. For spaces that are not T_1 the classical version of density makes no sense. Consider, for example, a domain in its Scott topology and suppose the domain has a top element $\top$. Then the singleton subset $\{\top\}$ is dense in the domain, as every non-empty Scott open set is in particular an upper set and therefore contains the top element. But the domain can be arbitrarily large and we certainly do not want the singleton $\{\top\}$ to pass as dense.

Hence we need to find a strengthening of the classical notion of density that works for stably continuous frames. Luckily such a notion already exists. M. Smyth explored the problem of embedding a space into a stably compact space in his influential paper [50], from where we have derived the definitions of the following subsection.

4.1.1 Basis embeddings and dense homomorphisms

Recall from Section 6.1.9 that every frame homomorphism h has a right adjoint which is typically denoted by h_*. Observe that a frame homomorphism h is dense in the classical sense if and only if $h_*(0) = 0$.

Definition 4.1.1. A surjective frame homomorphism $h : M \twoheadrightarrow L$ from a continuous frame M onto a frame L is called *dense* if its right adjoint is a basis embedding with respect to the way-below relation. That is, whenever $a \ll b$ in M there exists some $x \in L$ such that $a \ll h_*(x) \ll b$.

Observe that because of $0 \ll 0$ any surjective frame homomorphism that is dense in the sense of Definition 4.1.1 is also dense in the usual sense of locale theory. Our definition clearly rules out the example of the domain with a top element we mentioned earlier. Equipped with the strengthened version of density we dare to give a definition of compactification.

Definition 4.1.2. A *compactification* of a d-frame $\mathcal{L}$ is a d-frame homomorphism $h : \mathcal{M} \to \mathcal{L}$ where $\mathcal{M}$ is compact regular and the component maps of h are surjective and dense frame homomorphisms.

4.1.2 Completely regular d-frames

The question naturally arises whether every d-frame admits a compactification. In due course we will see that this is not so, and just as the topological spaces that admit a compactification are the Tychonoff spaces and the frames that admit a compactification are the completely regular frames, the class of d-frames that admit a compactification share a property that for good reasons can be called complete regularity.

Consider a compactification $h : \mathcal{M} \to \mathcal{L}$ of a d-frame $\mathcal{L}$. Recall from Section 3.7 that the component frames of the compact regular d-frame $\mathcal{M}$ are stably continuous frames and their well-inside relation coincides with the way-below relation. But even more is true: Any compact regular d-frame is normal, whereby the well-inside relation is interpolative. From Lemma 2.4.3 we know that the well-inside relation on a normal d-lattice coincides with the really-inside relation we introduced in Definition 2.4.3. Just as the well-inside relation, the really-inside relation is preserved by d-lattice homomorphisms and in particular by d-frame homomorphisms. Since the component maps h_- and h_+ of the compactification are surjective, for every element $x \in L_+$ there exists some $a \in M_+$ with $x = h_+(a)$. By regularity of $\mathcal{M}$ the element a is the join of the elements well inside it, and h_+ preserves the really inside relation and directed joins. It follows that x, too, is the join of elements really inside it, namely $x = \bigsqcup\{h_+(a') \mid a' \lessdot a\}$. A similar argument applies to elements $\phi \in L_-$ of the other component frame. We conclude that any d-frame that admits a compactification has component frames on which the really-inside relation is approximating. Let us give a name to this property.

Definition 4.1.3. A d-frame is *completely regular* if the really-inside relation on each component frame is approximating.

Proposition 4.1.1. *If a d-frame admits a compactification, then it is completely regular.*

Complete regularity of d-frames and complete regularity of frames are connected as follows.

Lemma 4.1.2. *Both the symmetric d-frame functor and the patch frame functor preserve regularity and complete regularity.*

Proof. This follows from the fact that both functors transform one version of the well-inside relation into the other. If x is well inside y in the frame L then x is well inside y in the component frame of the symmetric d-frame $L_=$. The other direction was shown in Proposition 3.5.14. □

There is, of course, a notion of complete regularity for bitopological spaces that is defined analogously. Since the well-inside relation and the really-inside relation derived form it are point-free concepts, a bitopological space is completely regular if and only if the d-frame derived from it is completely regular. As an example, let us prove a strengthening of Proposition 3.5.4.

Proposition 4.1.3. *Any continuous poset with the Scott- and weak lower topology is a completely regular bitopological space.*

Proof. The Scott topology on a continuous poset is a domain and thus its way-below relation is interpolative. Recall from Theorem 6.1.12 that $U_0 \ll U_1$ holds in σD if and only if $U_0 \lhd U_1$ with a witness in ωD that is a basic open. Therefore the really-inside relation on σD is approximating.

For the weak lower topology let us prove the desired fact directly. We showed in Proposition 3.5.4 that whenever $x_0 \ll x_1$ then the subbasic open $D \setminus \uparrow x_0$ is well inside the subbasic open $D \setminus \uparrow x_1$. Again, using the interpolation property one builds a dyadic-indexed chain $\{x_d\}_{d \in \mathbb{D}}$ where $d < e$ implies $x_d \ll x_e$ and thus obtains a scale between the basic opens. As the subbasic open $D \setminus \uparrow x_1$ is the union of subbasic opens $D \setminus \uparrow x_0$ we conclude that on ωD the really-inside relation, too, is approximating. □

4.1.3 A Urysohn Lemma for d-frames

In point-set topology the Tychonoff spaces are those where a point can be separated from a closed set by a bounded real-valued function. By "separate" one means that, given a closed set $A \subseteq X$ of a Tychonoff space and a point $x \in X \setminus A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and f restricted to A is constant 1. This separation property is also known as $\mathrm{T}_{3\frac{1}{2}}$.

The Urysohn Lemma states that at the next stronger separation property T_4 one can do the same where the point x is replaced by another closed set B that is disjoint from A. The standard proof of the Urysohn Lemma does in fact exploit methods from locale theory, see the sketch of the proof in the appendix. Let us reformulate the problem.

Lemma 4.1.4. *The following are equivalent for two disjoint closed sets A and B of a T_4 space X.*

1. *There is a continuous map $f : X \to [0, 1]$ separating A and B.*
2. *There is a some open neighbourhood U_0 of B that is well inside the open set $U_1 = X \setminus A$ and some continuous map $f : X \to [0, 1]$ that is constant zero on U_0 and constant one outside U_1.*

The bitopological unit interval

The standard proof of the Urysohn Lemma internally makes use of the unit interval with the bitopology of upper and lower semicontinuity. What is its d-frame representation?

Definition 4.1.4. On the dyadic rationals $\mathbb{D}$ declare a binary relation $\prec$ that is the strict order $<$ together with the pairs $(0, 0)$ and $(1, 1)$. Let $\mathcal{D}$ denote the d-lattice built in the fashion of Example 4, pictured in Figure 4.1. We call this the d-lattice of dyadic rationals.

Interpret an element $d \in \mathbb{D}$ as the lower set $\downarrow d := \{e \in \mathbb{D} \mid e \prec d\}$ and an element δ of the order dual $\mathbb{D}^\partial$ as the set $\uparrow \delta := \{e \in \mathbb{D} \mid \delta \prec e\}$. The element d is defined to be consistent with δ if and only if the set $\downarrow d$ consists of lower bounds of the set $\uparrow \delta$, which is equivalent to requiring $d \leq \delta$ in $\mathbb{D}$. Dually, define the element δ to be total with d whenever $\uparrow \delta \cup \downarrow d$ covers the dyadic rationals, that is to say $\delta \prec d$. Let us denote the d-lattice $(\mathbb{D}^\partial, \mathbb{D}, \leq, \prec)$ by $\mathcal{D}$. This d-lattice is normal, essentially because the relation $\prec$ has the interpolation property.

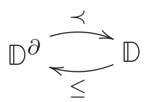


Figure 4.1: The d-lattice of dyadic rationals. Its open ideal completion is the d-frame of the unit interval, pictured on the right.

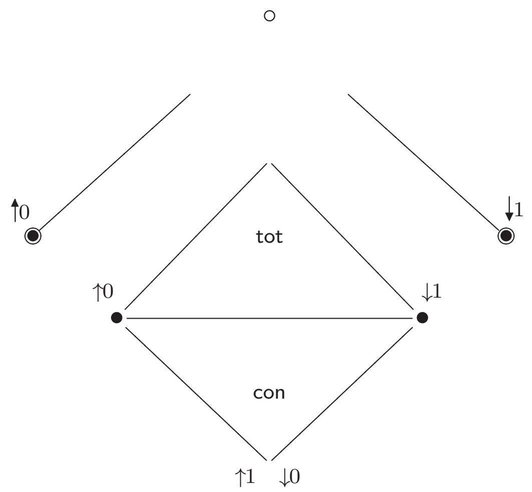


Figure 4.2: The d-frame of the bitopological unit interval. Each component frame is isomorphic to the unit interval with an additional top element. Consistent pairs are in the lower triangle (including the horizontal line) and the pairs marked with •. Total pairs are in the upper triangle (not including the horizontal line), the two diagonal lines and pairs marked with a circle.

Definition 4.1.5. The d-frame $\mathcal{I}$ is the open ideal completion of the d-lattice $\mathcal{D}$ of dyadic rationals. Its first component frame is the frame of open filters $\mathrm{Filt}^{\prec}\,\mathbb{D}$ and its second component frame is the frame of open ideals $\mathrm{Idl}^{\prec}\,\mathbb{D}$. It is pictured in Figure 4.2.

Let us show that this definition represents what it is supposed to. The open ideals of the second component lattice $\mathbb{D}$ of the d-lattice $\mathcal{D}$ take one of two forms. There are the least and greatest ideals $\downarrow 0 = \{0\}$ and $\downarrow\!\!\bullet 1 = \mathbb{D}$ which have a greatest element, and for each real number $0 < x \leq 1$ there is the open ideal $\downarrow x := \{d \in \mathbb{D} \mid d < x\}$. Notice that the lattice $\mathbb{D}$ order-embeds into the frame of open ideals as $d \mapsto \downarrow d$. Dually, the elements of the first component frame of $\mathcal{I}$ are round filters of dyadics with respect to $\prec$, so there are the least filter $\uparrow 1$ and the greatest filter $\twoheaduparrow 0$ as well as an open filter $\uparrow x := \{d \in \mathbb{D} \mid x < d\}$ for every real number $0 \leq x < 1$. Again, the order dual of the dyadics order-embeds into this frame of filters as $\delta \mapsto \uparrow \delta$. Similarly to the d-lattice of dyadic rationals, one can describe the consistency relation on $\mathcal{I}$ as $I \mathsf{con} F$ iff the open ideal $I \subseteq \mathbb{D}$ consists of lower bounds of the open filter $F \subseteq \mathbb{D}$. The filter F is total with the ideal I precisely when the two subsets of $\mathbb{D}$ intersect.

From Lemma 3.7.3 we know that the d-frame $\mathcal{I}$ is compact regular and therefore spatial. The description of its d-points that offers the most insight is via pairs of meet primes. As the component frames of $\mathcal{I}$ are chains, every ideal of $\mathbb{D}$ with the exception of the maximal $\twoheaddownarrow 1$ is meet prime and we know that every such meet prime extends to a d-point. For the ideal $\downarrow x$ the set of open filters total with it is identical with the set of filters intersecting that open ideal. The largest open filter not intersecting the ideal $\downarrow x$ is the round filter $\uparrow x$. We conclude that the d-points of $\mathcal{I}$ are in bijective correspondence with pairs (I, F) of subsets of $\mathbb{D}$ with the properties

1. Both I and F are non-empty,
2. $I \ni d$ and $e \in F$ implies $d \prec e$,
3. $d \in I$ if and only if there exists some $d' \in I$ with $d \prec d'$ and dually for F,
4. I and F are disjoint,
5. Whenever $d \prec e$ then either $d \in I$ or $e \in F$.

In other words, the d-points of $\mathcal{I}$ are in bijective correspondence with *Dedekind cuts* of the totally ordered set $(\mathbb{D}, \prec)$.

Remark. Our cuts differ from Dedekind's original definition [15] insofar as the relation $\prec$ we used is not the strict order everywhere. This is necessary because Dedekind sought to construct the real line from the rationals, where end-points were not desired.

The order on the meet prime opens is dual to the specialisation order on d-points because the larger the completely prime filter, the smaller the maximal element of its complement. Hence it makes sense to equate the Dedekind cut $(\downarrow x, \uparrow x)$ with the point $1 - x \in [0, 1]$. In this reading, an element $\downarrow y$ of the second component frame of $\mathcal{I}$ corresponds to the lower open $[0, y)$ and dually the element $\uparrow y$ of the first component frame corresponds to the upper open $(y, 1]$. Note that in particular the largest elements not equal to the top are $\downarrow 1 \cong [0, 1)$ and $\uparrow 0 \cong (0, 1]$, respectively.

Real-valued functions separating opens

Now that we have defined our d-frame version of the unit interval, we can turn towards the Urysohn Lemma. Recall that our goal is to separate two opens U_0, U_1 by a real-valued function f in the sense that the function is constant zero on U_0 and constant one outside U_1. Using open sets only, one formulates: The preimage of the upper open set $(0,1]$ under f is disjoint from the open U_0 and the preimage of the lower open set $[0,1)$ is contained in the open U_1. This motivates the following definition.

Definition 4.1.6. Let $\mathcal{L}$ be a d-frame and x_0, x_1 elements of the component frame L_+. A d-frame homomorphism $f : \mathcal{I} \to \mathcal{L}$ is said to *separate* x_0 *from* x_1 if the element x_0 is consistent with $f_-(\uparrow 0)$ and the element $f_+(\downarrow 1)$ is below x_1. Dually, the homomorphism f separates ϕ_0 from ϕ_1 in the component frame L_- if ϕ_0 is consistent with $f_+(\downarrow 1)$ and $f_-(\uparrow 0)$ is below ϕ_1.

Finally, we are able to state the Urysohn Lemma for d-frames.

Theorem 4.1.5 (The Urysohn Lemma for d-frames)**.** *Let $\mathcal{L}$ be a d-frame and x_0, x_1 elements of the second component frame where x_0 is really inside x_1. (This holds in particular when $\mathcal{L}$ is normal and x_0 is well inside x_1.) Then there is a d-frame homomorphism from the d-frame of the unit interval into $\mathcal{L}$ that separates x_0 from x_1.*

For the proof of this theorem we need one more lemma.

Notation. For a d-frame $\mathcal{L}$ we write $\varepsilon_{\mathcal{L}}$ for the pair of join maps $\bigsqcup : \mathrm{Idl}_\circ L_- \to L_-$ and $\bigsqcup : \mathrm{Idl}_\circ L_+ \to L_+$

Lemma 4.1.6. *For any d-frame $\mathcal{L}$, the pair $\varepsilon_{\mathcal{L}}$ of join maps is a d-frame homomorphism from the open ideal completion of $\mathcal{L}$ to $\mathcal{L}$.*

Proof. Recall from Lemma 2.3.4 that the frame of open ideals of a d-frame's component frame forms a sub-frame of the frame of all ideals. Since the join operation $\bigsqcup : \mathrm{Idl}\, L \to L$ is a frame homomorphism for any frame L, we obtain a pair of frame homomorphisms

$$(\bigsqcup, \bigsqcup) : \mathrm{Idl}_\circ \mathcal{L} \to \mathcal{L}.$$

It remains to show that this pair preserves consistency and totality. Recall that a pair of open ideals is consistent iff their product is a subset of the consistency relation of $\mathcal{L}$. As the latter set is Scott closed, the join of the ideals remains a consistent pair. A pair of open ideals is total iff the product of the two ideals intersects the totality relation of $\mathcal{L}$, so the joins of the ideals are certainly a total pair of $\mathcal{L}$. □

Proof of Theorem 4.1.5. By definition x_0 is really inside x_1 if there exists a scale between the two elements, that is a dyadic-indexed chain $\{x_d\}_{d \in \mathbb{D}}$ such that $d < e$ in $\mathbb{D}$ implies $x_d \lhd x_e$. Define a lattice homomorphism $h : \mathbb{D} \to L_+$ by letting $h(d) = x_d$ for dyadic rationals $0 < d < 1$ and further $h(0) = 0$, $h(1) = 1$. Observe that monotonicity and preservation of the constants is enough to render h a lattice homomorphism, because $\mathbb{D}$ is a chain. Interpreting $\mathbb{D}$ as the second component lattice of the d-lattice $\mathcal{D}$ of dyadic rationals puts us into the situation of Proposition 2.3.9. Hence h extends to a d-frame homomorphism

$\mathcal{I} = \mathrm{Idl}_\circ \mathcal{D} \to \mathrm{Idl}_\circ \mathcal{L}$ where the second component map takes the open ideal $\downarrow x \subseteq \mathbb{D}$ to the lower set of its forward image under h. In particular, the open ideal $\downarrow 1$ maps to the lower set of $\{x_d\}_{d<1}$. The first component map of the d-frame homomorphism maps the open filter $\uparrow 0$ (which is an open ideal of the order dual) to the ideal $\{\phi \in L_- \mid \exists d > 0.\, x_d \mathsf{con} \phi\}$. Now apply Lemma 4.1.6 and obtain a d-frame homomorphism $f : \mathcal{I} \to \mathcal{L}$. Here $f_+(\downarrow 1) = \bigsqcup_{d<1} x_d$ is certainly below x_1. For any $d > 0$ we have $x_0 \sqsubseteq x_d$ whence $x_d \mathsf{con} \phi$ implies $x_0 \mathsf{con} \phi$. It follows that x_0 is also consistent with the join of all such ϕ. This shows that f has the desired property: It separates x_0 from x_1. □

Remark. One might think that a similar result holds for the really-inside relation on d-lattices, where one replaces the d-frame $\mathcal{I}$ with the d-lattice $\mathcal{D}$. This conjecture turns out to be false, because completeness of the component lattices is a crucial ingredient.

The Urysohn Lemma for d-frames has a converse which is almost trivial to prove because of the way we defined the d-frame of the unit interval.

Lemma 4.1.7. *If the element $x_0 \in L_+$ can be separated from the element x_1 by a d-frame homomorphism $f : \mathcal{I} \to \mathcal{L}$ then x_0 is really inside x_1.*

Proof. Suppose $f : \mathcal{I} \to \mathcal{L}$ is a d-frame homomorphism with $x_0 \mathsf{con} f_-(\uparrow 0)$ and $f_+(\downarrow 1) \sqsubseteq x_1$. Recall that the dyadic rationals embed into the frame $\mathrm{Idl}^{\prec} \mathbb{D}$ via $d \mapsto \downarrow d$. By post-composing this embedding with the component map f_+ one obtains elements $x_d := f_+(\downarrow d)$ for dyadic rationals $0 < d < 1$. To finish the proof, observe that whenever the dyadic rational d is strictly smaller than e then the open ideal $\downarrow d$ is well inside the open ideal $\downarrow e$ in the d-frame $\mathcal{I}$. □

The Urysohn Lemma for d-frames and the preceding lemma combined yield a characterisation of the really-inside relation.

Theorem 4.1.8. *An element of a d-frame's component frame is really inside another if and only if the former can be separated from the latter by a d-frame homomorphism on the d-frame $\mathcal{I}$ of the unit interval.*

Some authors use separation by real-valued functions rather than the really-inside relation to define complete regularity, see for example [42].

4.2 The Stone-Čech compactification of a d-frame

Complete regularity is a necessary criterion for a d-frame admitting a compactification, but so far we do not know of any specific example (apart from the identity on compact regular d-frames). This section is to remedy this lack of examples and to exhibit the category of compact regular d-frames as a coreflective subcategory of the category of all d-frames. Dually it is well-known that the category of compact Hausdorff spaces is a reflective subcategory of Top and the reflection functor takes a topological space to its largest compactification[2], which is known as the Stone-Čech compactification. The Tychonoff spaces are precisely those spaces that embed into their Stone-Čech compactification.

4.2.1 Regular normal d-frames and the normal coreflection

In Example 11 we sketched the proof that the category of d-frames coreflects into the subcategory of normal d-frames. This coreflection preserves complete regularity. Indeed, if $\mathcal{L} = (L_-, L_+, \mathsf{con}, \mathsf{tot})$ is a d-frame then the normal coreflection functor shrinks the totality relation to $\mathsf{tot}; \leqslant$ and leaves the other data unchanged. As a result, the well-inside relation on the normal d-frame coincides with the really-inside relation on the original d-frame. If the really inside relation was approximating on the original d-frame's component frames, then so will be the well-inside relation on the normal coreflection. Finally, recall that in the presence of normality any regular d-frame is also completely regular. Let us record:

Proposition 4.2.1. *The normal coreflection functor restricts to the subcategory of completely regular d-frames.*

The preceding proposition can be used to obtain a somewhat surprising result about the patch frame. One must reach the conclusion that the concept of normality for d-frames is much more inclusive than normality for frames.

Proposition 4.2.2. *Every completely regular frame is the patch frame of a regular normal d-frame.*

Proof. Given a completely regular frame L, its symmetric d-frame $L_=$ is known to be completely regular by Lemma 4.1.2. The normal coreflection $(L_=)^{\leqslant}$ has the same component frames and is regular by Proposition 4.2.1. With Proposition 3.5.15 we obtain an isomorphic copy of L via the patch frame of the regular normal d-frame $(L_=)^{\leqslant}$. □

Next let us examine the open ideal completion of a regular normal d-frame more closely. In Lemma 4.1.6 we introduced a d-frame homomorphism $\varepsilon_{\mathcal{L}} : \mathrm{Idl}_\circ \mathcal{L} \to \mathcal{L}$ that maps an open ideal to its join. In general the d-frame might have only a few open ideals, but regularity together with normality provides plenty of them.

Lemma 4.2.3. *If $\mathcal{L}$ is a regular normal d-frame, then the d-frame homomorphism $\varepsilon_{\mathcal{L}}$ is dense in the sense of Definition 4.1.1.*

Proof. For a regular normal d-frame the components of $\varepsilon_{\mathcal{L}}$ are surjective because $\twoheaddownarrow$ takes values in the open ideals (Lemma 2.4.1 (3)) and $\bigsqcup \circ \twoheaddownarrow$ is the identity on component frames. One finds that $\bigsqcup \dashv \twoheaddownarrow$ is an adjunction between a component frame and the frame of its open ideals. Recall from Proposition 2.4.5 that in the stably continuous frame $\mathrm{Idl}^{\lhd} L_+$ an

[2] Here the term "compactification" is to be understood in a wider sense where the map into the compact space is dense but not necessarily injective.

ideal $I' \ll I$ is equivalent to $I' \subseteq \downarrow x$ for some $x \in I$. Since both ideal in question are open, this is equivalent to the assertion that $I' \ll \downarrow x \ll I$ for some $x \in L_+$. Therefore the frame homomorphism $\bigsqcup$ is dense. □

Corollary 4.2.4. *Every completely regular d-frame admits a compactification.*

Proof. From Proposition 4.2.1 we know that the normal coreflection $\mathcal{L}^{\leqslant}$ of a completely regular d-frame is regular and normal. Pre-compose the identity map $\mathcal{L}^{\leqslant} \to \mathcal{L}$ with the compactification of Lemma 4.2.3 above. □

Remark. For a regular normal d-frame $\mathcal{L}$, the pair of right adjoints $(\downarrow, \downarrow)$ to the compactification morphism $\varepsilon_{\mathcal{L}}$ preserves and reflects the consistency and totality relations, in the sense that $\phi \mathsf{tot} x$ holds in $\mathcal{L}$ if and only if $\downarrow\phi \mathsf{tot}_\circ \downarrow x$ holds in the open ideal completion $\mathrm{Idl}_\circ \mathcal{L}$ and likewise for consistency. In fact this characterises the open ideal completion among all compactifications up to isomorphism. A similar fact was observed by Čech in [53] about the Stone-Čech compactification of a normal space.

4.2.2 The compact regular coreflection

Inspecting the compactification of a completely regular d-frame whose existence we proved in Corollary 4.2.4, one finds that the component frames of the compact regular d-frame are those ideals of the original d-frame's component frames that are round with respect to the really-inside relation. It is well-known in locale theory that for a locale X the frame $\mathrm{Idl}^{\leqslant} \mathcal{O}X$ of round ideals of opens is its Stone-Čech compactification [30, IV-2.2]. In due course we will see that the situation is the same with d-frames.

Proposition 4.2.5. *The open ideal completion functor, when restricted to the category of normal d-frames, is right adjoint to the inclusion functor from compact regular d-frames to normal d-frames. The family ε of homomorphisms defined in Lemma 4.1.6 is its counit.*

Proof. Recall from Lemma 3.7.3 that the open ideal completion of a normal d-frame is compact regular. Next notice that the homomorphism $\epsilon_{\mathcal{L}} : \mathrm{Idl}_\circ \mathcal{L} \to \mathcal{L}$ is indeed natural in the parameter $\mathcal{L}$. This is because the action of the open ideal completion functor on the components of d-frame homomorphisms is essentially the same as the ideal completion functor, and we know that the join map is a natural transformation from the ideal completion functor to the identity in the category of frames.

It remains to show that if $\mathcal{L}$ is a normal d-frame and $\mathcal{M}$ a compact regular d-frame, every d-frame homomorphism $h : \mathcal{M} \to \mathcal{L}$ factors uniquely through the map $\varepsilon_{\mathcal{L}}$. We prove every statement for the positive component only, because the negative component works analogously. Consider the following diagram.

$$\begin{array}{ccc} \mathrm{Idl}^{\lhd} M_+ & \xrightarrow{\mathrm{Idl}_\circ(h_+)} & \mathrm{Idl}^{\lhd} L_+ \\ \uparrow{\scriptstyle \downarrow} & \nearrow_{\tilde{h}_+} & \downarrow{\scriptstyle \bigsqcup = (\varepsilon_{\mathcal{L}})_+} \\ M_+ & \xrightarrow[h_+]{} & L_+ \end{array} \tag{4.1}$$

By hypothesis $\mathcal{M}$ is compact regular, so the composition $\bigsqcup \circ \downarrow$ is the identity on M_+. Using this identity and the fact that the homomorphism h_+ preserves joins of ideals we obtain $\bigsqcup \circ \mathrm{Idl}_\circ(h_+) \circ \downarrow = h_+ \circ \bigsqcup \circ \downarrow = h_+$ and so the square in (4.1) commutes. In general the map $\downarrow : M_+ \to \mathrm{Idl}\, M_+$ is not a frame homomorphism, but since $\mathcal{M}$ is compact regular it is actually an isomorphism of frames (see Proposition 3.7.4). Hence the composite $\tilde{h}_+ := \mathrm{Idl}_\circ(h_+) \circ \downarrow$ is a frame homomorphism with the required factorisation property $h_+ = \bigsqcup \circ \tilde{h}_+$. An immediate consequence of this factorisation is that the open ideal completion functor is faithful on morphisms whose domain is regular, since $\mathrm{Idl}_\circ(h_+) = \mathrm{Idl}_\circ(g_+)$ implies $h_+ = h_+ \circ \bigsqcup \circ \downarrow = \bigsqcup \circ \mathrm{Idl}_\circ(h_+) \circ \downarrow = \bigsqcup \circ \mathrm{Idl}_\circ(g_+) \circ \downarrow = g_+$. Faithfulness of the open ideal completion functor now implies that the factorisation of h_+ in the diagram (4.1) is unique. Indeed, if $f : M_+ \to \mathrm{Idl}^{\lhd} L_+$ is any map with $\bigsqcup \circ f = h_+$ then $\mathrm{Idl}_\circ(\bigsqcup) \circ \mathrm{Idl}_\circ(f) = \mathrm{Idl}_\circ(h_+)$. But $\mathrm{Idl}_\circ(\bigsqcup)$ is an isomorphism of the frames $\mathrm{Idl}^{\lhd} L_+$ and $\mathrm{Idl}^{\lhd} \mathrm{Idl}^{\lhd} L_+$ whence there can be only one such $\mathrm{Idl}_\circ(f)$. □

Theorem 4.2.6 (Stone-Čech compactification)**.** *The category of d-frames coreflects into the subcategory of compact regular d-frames. The coreflection factors through the category of normal d-frames as normal coreflection followed by open ideal completion.*

Proof. The composition of two coreflections is a coreflection again. Combine Theorem 2.4.4 with Proposition 4.2.5. □

For obvious reasons the following name is chosen for the coreflection of Theorem 4.2.6.

Definition 4.2.1. Given any d-frame $\mathcal{L}$, the compact regular d-frame $\mathrm{Idl}_\circ(\mathcal{L}^{\leqslant})$ is called the *Stone-Čech compactification* of $\mathcal{L}$.

Remark. The Stone-Čech compactification of a d-frame has the same description as the open ideal completion in Lemma 2.3.6, with the well-inside relation replaced by the really-inside relation.

4.3 Classification of compactifications

Now that we have characterised which d-frames admit a compactification and constructed the largest such compactification, we turn towards the problem of classifying all compactifications. In Section 4.2 we constructed the Stone-Čech compactification of a completely regular d-frame using the really-inside relation on the component frames. This relation has the properties (i)–(vi) of Table 4.1. Property (vii) holds if one replaces the well-inside relation of frames with the d-frame-theoretic one and property (viii) is satisfied jointly by the really-inside relations on both component frames if one interprets the negation as the pseudocomplement of d-frames.

Definition 4.3.1. A relation satisfying properties (i)–(v) of Table 4.1 is called a *quasi-proximity*. A *proximity* on a d-frame is an approximating quasi-proximity $\prec$ on the second component frame that is stronger than the well-inside relation. Furthermore the relational inverse of

$$\succ := (\mathsf{tot}; \prec; \mathsf{con}) \tag{4.2}$$

is required to be approximating on the first component frame. In the equation above we followed our custom to write relations on the first component frame in the greater-than-style.

Observe that the relation defined in equation (4.2) satisfies $\succ \subseteq \rhd$ because $\prec \subseteq \lhd$ holds by definition. These facts are analogous to the property (vii) of Table 4.1. In order to relate proximities of d-frames to proximities of frames, let us show that a proximity on a d-frame obeys the negation rules

$$\frac{x \prec y}{\neg x \succ \neg y} \qquad \frac{\phi \succ \psi}{\neg \phi \prec \neg \psi}$$

that are analogous to property (viii) of Table 4.1. First we need a lemma.

Lemma 4.3.1. *For any proximity $\prec$ the identity $\prec = (\lhd; \prec)$ holds.*

Proof. Because of the interpolation property, $\prec$ is the same as $\prec; \prec$. As $\prec$ is contained in the well-inside relation we obtain the inclusion $\prec \subseteq (\lhd; \prec)$. For the reverse inclusion, use the fact that $\lhd$ is contained in the frame order $\sqsubseteq$ and the composite $\sqsubseteq; \prec$ is identical to $\prec$ because it is an auxiliary relation. □

If now $x \prec y$, then using interpolation and the preceding lemma write $x \lhd z \prec y$ whereby $\neg x \mathsf{tot} z \prec y \mathsf{con} \neg y$ and so by definition $\neg x \succ \neg y$. Dually, if $\phi(\mathsf{tot}; \prec; \mathsf{con})\psi$ then $\neg\phi(\mathsf{con}; \mathsf{tot}; \prec)\neg\psi$ and using the lemma again arrive at $\neg\phi \prec \neg\psi$.

To summarise our brief study of the internal structure of proximities, every such proximity is determined by a relation on one component frame that has the same properties as those listed in Table 4.1, except that the meanings of "well-inside" and "pseudocomplement" are re-defined in d-frame-theoretic terms.

Lemma 4.3.2. *1. Any proximity on a d-frame is contained in the really-inside relation.*

2. The really-inside relation $\lessdot$ of a completely regular d-frame is a proximity.

Proof. (1) Let $\prec$ be a proximity and suppose $x_0 \prec x_1$. Using the interpolation property one extends this to a dyadic-indexed chain $\{x_d\}_{d \in \mathbb{D}}$ where $d < e$ implies $x_d \prec x_e$. Since $\prec$ is contained in the well-inside relation we conclude that x_0 is really inside x_1.

(2) According to the remark following Theorem 2.4.4 any d-frame obeys the identity $\geqslant = (\mathsf{tot}; \leqslant; \mathsf{con})$. □

Corollary 4.3.3. *A d-frame admits a proximity if and only if it is completely regular.*

Proof. Any proximity is stronger than the really-inside relation by Lemma 4.3.2 (1) and by definition approximating, whence the really-inside relation itself is approximating. Hence the existence of a proximity implies complete regularity. The converse is Lemma 4.3.2 (2). □

4.3.1 The proximity derived from a compactification

Suppose $f : \mathcal{M} \to \mathcal{L}$ is a compactification of the (completely regular) d-frame $\mathcal{L}$. According to the construction sketched in the introduction to this chapter, one should obtain a proximity on $\mathcal{L}$ by pushing the way-below relation of the stably continuous component frames of $\mathcal{M}$ along the homomorphism f.

Definition 4.3.2. Given a compactification $f : \mathcal{M} \to \mathcal{L}$ of the d-frame $\mathcal{L}$, define a relation on L_+ as follows. With f_{+*} denoting the right adjoint to the component map f_+ of f, define Let $x_0 \prec x_1$ if and only if $f_{+*}(x_0) \ll f_{+*}(x_1)$.

Of course we need to check that the definition above indeed produces a proximity. That $\prec$ is a quasi-proximity follows from Lemma 6.1.16. It is stronger than the well-inside relation because $\ll$ coincides with $\lhd$ on M_+, $f_+ \circ f_{+*}$ is the identity on L_+ and f_+ preserves the well-inside relation. The fact that $\prec$ is approximating is derived knowing that M_+ is a domain and f_+ is surjective, and similarly for the relation $\succ = (\mathsf{tot}; \prec; \mathsf{con})$.

Definition 4.3.2 follows the ideas of Banaschewski [4]. Smyth [50] uses a different, but equivalent, definition as Lemma 6.1.17 shows.

4.3.2 The compactification derived from a proximity

Knowing how to construct a proximity from a compactification, we want an inverse that takes a proximity and produces a compactification. An obvious requirement is that the really-inside relation should give rise to the Stone-Čech compactification. Recall from Theorem 4.2.6 that the Stone-Čech compactification factors through the category of normal d-frames. This was because the really-inside relation satisfies the requirements of Lemma 2.4.2.

Lemma 4.3.4. *Let $\mathcal{L}$ be a (completely regular) d-frame and $\prec$ be a proximity on it. The d-frame $\mathcal{L}^{\prec}$ defined according to Lemma 2.4.2 is regular and normal.*

Proof. The proximity $\prec$ satisfies all requirements of Lemma 2.4.2, whence the d-frame $\mathcal{L}^{\prec} = (L_-, L_+, \mathsf{con}, (\mathsf{tot}; \prec))$ is normal. Regularity of this d-frame follows from the characterisation of the well-inside relations on $\mathcal{L}^{\prec}$ given in the proof of Lemma 2.4.2. □

With the Stone-Čech compactification in mind, the following definition comes at no surprise.

Definition 4.3.3. Given a proximity $\prec$ on a (completely regular) d-frame $\mathcal{L}$, the compactification derived from $\prec$ is the d-frame homomorphism

$$(\bigsqcup, \bigsqcup) : \mathrm{Idl}_\circ(\mathcal{L}^\prec) \to \mathcal{L}.$$

A little thought is needed to check that the join maps on round ideals are indeed surjective dense frame homomorphisms. The same argument that we employed in Proposition 2.4.5 to show that $\mathrm{Idl}^\lhd L_+$ is a sub-frame of $\mathrm{Idl}\, L_+$ can be used to show that the set of ideals that are round with respect to $\prec$ is a sub-frame of the frame of all ideals. The join map restricted to $\mathrm{Idl}^\prec L_+$ is again a frame homomorphism, and surjective precisely because the relation $\prec$ is approximating. The way-below relation on the frame $\mathrm{Idl}^\prec L_+$ has the same characterisation as on the frame of open ideals. One finds that the right adjoint to the join map on round ideals is the assignment $x \mapsto \{y \in L_+ \mid y \prec x\}$. The last two facts combined yield that the join map of round ideals is a dense homomorphism in the sense of Definition 4.1.1. Thus Definition 4.3.3 indeed specifies a compactification.

Notice that because of Lemma 4.3.2 the regular normal d-frame $\mathcal{L}^\prec$ has a totality relation that is contained in the totality relation $\mathsf{tot};\leqslant$ of the normal coreflection of $\mathcal{L}$. Thus one obtains a d-frame homomorphism $\mathcal{L}^\prec \to \mathcal{L}^\leqslant$ that gives rise to a d-frame homomorphism $\mathrm{Idl}_\circ\, \mathcal{L}^\prec \hookrightarrow \mathrm{Idl}_\circ\, \mathcal{L}^\leqslant$ with injective component maps. We interpret this fact as the Stone-Čech compactification being the largest possible compactification.

4.3.3 The classification theorem

Finally we are able to state and prove the classification theorem of d-frame compactifications, the main result of this thesis.

Theorem 4.3.5. *There is an order-preserving bijection between the compactifications of a d-frame and its proximities. The operations of Definitions 4.3.2 and 4.3.3 are mutually inverse.*

The term *order-preserving* in the statement of the theorem requires some explanation. We order the set of proximities on a d-frame by inclusion of relations. According to Lemma 4.3.2 the really-inside relation gives rise to the largest proximity in this ordering. The collection of compactifications of a fixed d-frame is a class that is pre-ordered in the following way. Say that a compactification $\mathcal{M} \to \mathcal{L}$ is smaller than the compactification $\mathcal{N} \to \mathcal{L}$ if one can regard $\mathcal{M}$ as a sub-d-frame of $\mathcal{N}$. This means that there is a d-frame homomorphism $\lesssim$ as in the diagram below making it commute. The components of this homomorphism are required to be injective. Further one requires that $\lesssim$ reflects the consistency and totality relations in the sense that $x \mathsf{con} \phi$ in $\mathcal{M}$ holds if and only if the image of the pair (x, ϕ) under $\lesssim$ is consistent and similarly for total pairs of $\mathcal{M}$.

$$\begin{array}{ccc} \mathcal{M} & \xhookrightarrow{\ \lesssim\ } & \mathcal{N} \\ & \searrow \quad \swarrow & \\ & \mathcal{L} & \end{array} \tag{4.3}$$

As we remarked above, any ideal that is round with respect to $\prec$ is also round with respect to the really-inside relation. Thus any compactification arising from a proximity as specified in Definition 4.3.3 is smaller than the Stone-Čech compactification presented in Theorem 4.2.6.

Proof of Theorem 4.3.5. We need to show that the operations specified in Definitions 4.3.2 and 4.3.3 are mutually inverse and preserve the order.

Given a compactification $f : \mathcal{M} \to \mathcal{L}$ and the induced proximity $\prec$ on $\mathcal{L}$ we show that the d-frame $\mathcal{M}$ is isomorphic to $\mathrm{Idl}_\circ(\mathcal{L}^{\prec})$. Recall from Section 3.7 that a compact regular d-frame is completely determined by one of its component frames, so it suffices to show that the component frame M_+ of $\mathcal{M}$ is isomorphic to $\mathrm{Idl}^{\prec} L_+$. The relation $\prec$ is the one induced on L_+ by the right adjoint f_{+*} on the domain M_+ as in the statement of Lemma 6.1.25, whence the desired isomorphism follows. Therefore the compactification $f : \mathcal{M} \to \mathcal{L}$ is isomorphic to the compactification induced by $\prec$.

Now suppose there are two comparable compactifications as in the diagram (4.3) above. Using Lemma 6.1.17, $x_0 \prec x_1$ in L_+ holds whenever there exist elements $y_0 \lhd y_1$ in M_+ such that $x_0 \sqsubseteq f_+(y_0)$ and $f_+(y_1) \sqsubseteq x_1$. Since the d-frame homomorphism $\lesssim$ preserves the well-inside relation and f factors through $\lesssim$, we conclude that the quasi-proximity $\prec'$ induced by the larger compactification $\mathcal{N} \to \mathcal{L}$ has $x_0 \prec' x_1$ as well. Thus the assignment from compactifications to proximities is order-preserving.

Now suppose $<$ is a proximity on $\mathcal{L}$ and let $\prec$ be the quasi-proximity on L_+ induced by the dense frame homomorphism $\bigsqcup : \mathrm{Idl}^{<} L_+ \to L_+$. We show that $<$ and $\prec$ are identical. Recall that the right adjoint to $\bigsqcup$ sends an element x of L_+ to the round ideal $\downarrow x := \{y \in L_+ \mid y < x\}$. By definition $x_0 \prec x_1$ holds if and only if the round ideal $\downarrow x_0$ is way below the round ideal $\downarrow x_1$. This is the case precisely when $\downarrow x_1$ contains an upper bound of $\downarrow x_0$. In that situation we know that the join of the ideal $\downarrow x_0$ is an element of $\downarrow x_1$, and since $<$ is approximating, said join equals x_0. We conclude that $x_0 \prec x_1$ implies that $x_0 < x_1$. For the reverse implication, observe that the assignment $x \mapsto \downarrow x$ transforms the relation $\prec$ into the way-below relation.

If $<' \subseteq <$ is another proximity then any ideal of L_+ that is round with respect to $<'$ is also round with respect to $<$. Therefore the sub-frame embedding $\mathrm{Idl}^{<'} L_+ \hookrightarrow \mathrm{Idl}^{<} L_+$ extends to the d-frame homomorphism $\lesssim$ witnessing the fact that the compactification induced by $<'$ is smaller than the compactification induced by $<$. □

4.3.4 The spectrum of a compactification

If one uses the compactification of d-frames to obtain compactifications of bitopological spaces, then it is of interest how exactly the d-frame compactification acts on d-points. Clearly every d-point $\mathcal{L} \to \mathbf{2}$ gives rise to a d-point of a compactification via composition: If $\mathcal{M} \to \mathcal{L}$ is a compactification and $\mathcal{L} \to \mathbf{2}$ is a d-point then $\mathcal{M} \to \mathcal{L} \to \mathbf{2}$ is a d-point of the compactification. Since the component maps of the compactification homomorphism are surjective, the compactification map is clearly an epimorphism whereby the d-points of $\mathcal{L}$ injectively map into the d-points of $\mathcal{M}$. The same argument on the level of frames shows that the frame points $L_+ \to 2$ of a component frame injectively map to frame points $M_+ \twoheadrightarrow L_+ \to 2$ of the component frame of the compactification. But recall from Lemma 3.7.8 that every such frame point of M_+ extends to a unique d-point of $\mathcal{M}$. We conclude:

Proposition 4.3.6. *If X is a completely regular bitopological space, then the spectrum of any d-frame compactification of $\mathcal{O}X$ contains the soberification of X with respect to each of the two topologies on X.*

As round ideals of open sets are not easy to understand and to work with, a description in terms of objects of lower cardinality might be handy. In locale theory one has the (classically valid) one-to-one correspondence between completely prime filters and meet prime elements of a frame. We used this correspondence extensively when working with regular d-frames, where the relationship between the components of a d-point is most easily described as a translation from completely prime filters of one component frame to meet prime elements of the other component and vice versa. In the literature on proximity lattices, such as [50] or [23], the notion of meet prime is generalised to so-called *weak primes*.

Definition 4.3.4. Let L be a bounded distributive lattice and $\prec$ be a quasi-proximity on L.

1. An element $p \in L$ is called $\prec$-prime (or *weakly prime* if it the quasi-proximity is clear from the context) if for every finite set $A \subseteq L$ the relation $\bigsqcap A \prec p$ implies that A intersects $\Downarrow p$.
2. A filter $F \subseteq L$ is called $\prec$-prime or weakly prime if it is round with respect to $\prec$ and for any finite set $\{\Downarrow a\}_{a \in A}$ of principal round ideals whose join in $\mathrm{Idl}^{\prec} L$ intersects F the finite set A intersects F.

Notice that in case the quasi-proximity $\prec$ is the lattice order, the weakly prime elements are precisely the meet primes of L and the weakly prime filters are precisely the prime filters. Classical Stone duality for bounded distributive lattices tells us that the frame homomorphisms $\mathrm{Idl}\, L \to 2$ are in bijective correspondence with the prime filters of L. More generally, Smyth showed in [50, Proposition 8] that the spectrum of the frame

$\mathrm{Idl}^{\prec} L$ of round ideals is a stably compact space whose points are the weakly prime filters of L. A round ideal $I \in \mathrm{Idl}^{\prec} L$ is a neighbourhood of a weakly prime filter $F \subseteq L$ if and only if I and F intersect.

Since any quasi-proximity is stronger than the lattice order, every meet prime element is weakly prime and every round prime filter is a weakly prime filter. In particular, if $\prec$ is an approximating quasi-proximity on a frame then all Scott open filters are round, whence every completely prime filter is weakly prime. This gives another proof for Proposition 4.3.6. Gierz and Keimel show in [23] that every weak prime p of a proximity lattice $(L, \prec)$ gives rise to a round ideal $\downarrow p$ that is a meet prime in the frame $\mathrm{Idl}^{\prec} L$ of round ideals.

4.4 Compactifications of frames and spaces

The topological spaces that can be embedded densely into compact Hausdorff spaces are the Tychonoff spaces. Every Tychonoff space is in particular Hausdorff and thereby sober. Consequently a compactification $e : X \hookrightarrow Y$ is completely described by the associated compactification of frames $e^{-1} : \mathcal{O}Y \twoheadrightarrow \mathcal{O}X$. In this section we demonstrate that compactifications of d-frames subsume compactifications of frames and thereby implicitly also compactifications of spaces. The main tool for this will be the adjunction between frames and d-frames laid out in Section 3.4.

From [4] we know that there is an order-preserving bijection between compactifications of a completely regular frame and proximities on it, that are relations with the properties listed in Table 4.1. Any such proximity $\prec$ on a completely regular frame L is a proximity on the symmetric d-frame $L_=$ where $\succ$ is the relational inverse of $\prec$. Indeed, recall that the d-frame-theoretic well-inside relation on the symmetric d-frame $L_=$ is the same as the well-inside relation in the frame-theoretic sense and similarly the d-frame pseudocomplement $\neg : L \to L$ on $L_=$ is the same as the Heyting negation of the frame L. Conversely, given a proximity on the symmetric d-frame $L_=$ one finds that it is a proximity in the frame-theoretic sense. We conclude:

Lemma 4.4.1. *There is a bijection between the proximities on a frame L and proximities on the symmetric d-frame $L_=$.*

Once we know the proximity $\prec$ on the frame L is a proximity on the symmetric d-frame $L_=$, we can consider the associated d-frame compactification $\mathrm{Idl}_\circ(L_=^{\preceq}) \to L_=$. The component frames of the compact regular d-frame $\mathrm{Idl}_\circ(L_=^{\preceq})$ are both equal to the frame $\mathrm{Idl}^{\prec} L$ of round ideals with respect to the proximity $\prec$. It is, however, not clear a priori that this d-frame is symmetric again.

Lemma 4.4.2. *If $\prec$ is a proximity on a frame L, then the compact regular d-frame $\mathrm{Idl}_\circ(L_=^{\preceq})$ is identical with the symmetric d-frame $(\mathrm{Idl}^{\prec} L)_=$.*

Proof. By definition of the normal coreflection, two round ideals $I, J \in \mathrm{Idl}^{\prec} L$ are total if and only if the product $I \times J$ intersects the totality relation $\mathsf{tot}_=;\prec$ of the normal d-frame $L_=^{\preceq}$. The join $I \vee J$ in the frame of ideals equals the top element $\downarrow 1$ precisely when there is a pair $(x, y) \in I \times J$ with $x \sqcup y = 1$. Evidently, the latter is equivalent to $I \times J$ intersecting the relation $\mathsf{tot}_=$, and using the fact that the ideals under consideration are round, we obtain the desired equivalence $I \vee J = \downarrow 1 \Leftrightarrow (I \times J) \between (\mathsf{tot}_=;\prec)$. We showed that the totality relation of the d-frame $\mathrm{Idl}_\circ(L_=^{\preceq})$ coincides with the totality relation of the symmetric d-frame $(\mathrm{Idl}^{\prec} L)_=$.

The ideals I and J are consistent in the d-frame $\mathrm{Idl}_\circ(L_=^{\preceq})$ if and only if their product is contained in the relation $\mathsf{con}_=$ of the symmetric d-frame $L_=$. This means that for all pairs $(x, y) \in I \times J$ the identity $x \sqcap y = 0$ holds in L. Since finite meets of ideals are computed as element-wise meet, we conclude that $I \wedge J$ is the smallest ideal $\downarrow 0$. The converse holds as well: If $I \wedge J = \downarrow 0$ then the product $I \times J$ is contained in $\mathsf{con}_=$. Thus the consistency relation of the d-frame $\mathrm{Idl}_\circ(L_=^{\preceq})$ is the same as the consistency relation of the symmetric d-frame $(\mathrm{Idl}^{\prec} L)_=$. □

Remark. Recall from Theorem 3.7.10 that every compact regular d-frame arises from a stably compact space and its de-Groot dual. The compact Hausdorff spaces are precisely those stably compact spaces where the topology coincides with its cocompact dual. Therefore the component frame $\mathrm{Idl}^{\prec} L$ of the preceding lemma is the topology of a compact Hausdorff space.

Proposition 4.4.3. *There is a bijection between (equivalence classes of) compactifications $M \to L$ of a frame L and the compactifications $\mathcal{M} \to L_=$ of the symmetric d-frame $L_=$.*

Proof. From Theorem 4.3.5 we know that any compactification $\mathcal{M} \to L_=$ arises from a proximity of $L_=$. Because of Lemma 4.4.1 we know every such proximity to be associated with a proximity $\prec$ on the frame L. Using the preceding lemma one shows that the d-frame homomorphism $\mathcal{M} \to L_=$ is isomorphic to a homomorphism in the subcategory of symmetric regular d-frames, which we showed in Theorem 3.5.16 to be equivalent to the category of regular frames. □

To conclude our account of frame compactifications, let us consider the compact regular coreflection on the category of frames. From Theorem 3.4.1 we know that the symmetric d-frame functor $(-)_=$ is right adjoint to the patch frame functor. The normal coreflection $(-)^{\leqslant}$ of d-frames is right adjoint to the inclusion functor of the subcategory NdFrm of normal d-frames into dFrm. The inclusion functor $\mathsf{KRdFrm} \hookrightarrow \mathsf{NdFrm}$ of compact regular d-frames into normal d-frames has as right adjoint the open ideal completion functor, as we showed in Proposition 4.2.5. Finally, Theorem 3.7.5 tells us that the category of compact regular d-frames is equivalent to the category of stably continuous frames and perfect frame homomorphisms. We obtain a chain of adjunctions:

$$\mathsf{Frm} \underset{(-)_{=}}{\overset{\text{Patch}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{dFrm} \underset{(-)^{\leqslant}}{\underset{\longrightarrow}{\overset{\hookleftarrow}{\bot}}} \mathsf{NdFrm} \underset{\mathrm{Idl}_{\circ}}{\underset{\longrightarrow}{\overset{\hookleftarrow}{\bot}}} \mathsf{KRdFrm} \quad \cong \quad \mathsf{SCFrm}_{\pi} \tag{4.4}$$

While the normal coreflection of a symmetric d-frame is not symmetric in general, its open ideal completion is symmetric again, as we showed in Lemma 4.4.2. Hence the chain of adjunctions in the diagram (4.4) restricts to a chain of adjunctions involving categories of symmetric d-frames

$$\mathsf{Frm} \underset{(-)_{=}}{\overset{\text{Patch}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{dFrm}_{=} \underset{\mathrm{Idl}_{\circ}(-)^{\leqslant}}{\underset{\longrightarrow}{\overset{\hookleftarrow}{\bot}}} \mathsf{KRdFrm}_{=} \quad \cong \quad \mathsf{KRFrm} \tag{4.5}$$

where the equivalence of categories on the right was shown in Proposition 3.7.7. This demonstrates that the compact regular coreflection of frames factors through the category of d-frames.

4.5 Applications

The compactifications of frames and spaces via d-frames presented in Section 4.4 is a somewhat degenerate case. Below we list some truly bitopological and order-preserving compactifications.

4.5.1 The Fell compactification

A locally compact T_0 space (more generally a core compact T_0 space) has a topology which is a continuous lattice, whence said lattice is a compact Hausdorff space under the Lawson topology. Trivially the same Lawson topology induces a topology on the lattice of closed subsets of a locally compact space. The assignment $x \mapsto \overline{\{x\}}$ sending a point to the closure of its singleton set is injective precisely when the space is T_0. In [21] Fell defined what is now known as the Fell compactification of a locally compact T_0 space X as the closure of the image of X under the embedding $x \mapsto \overline{\{x\}}$ into the lattice of closed sets under the Lawson topology. The Lawson topology on $\mathcal{O}X$ is described by two kinds of subbasic opens:

$$\begin{aligned} \Box k &= \{v \in \mathcal{O}X \mid k \subseteq v\} \\ \Diamond u &= \{v \in \mathcal{O}X \mid u \nsubseteq v\} \end{aligned}$$

where k ranges over the compact (saturated[3]) subsets of the locally compact space X and u ranges over the opens. Notice that the first kind of subbasic opens generates the Scott topology on $\mathcal{O}X$ whereas the second kind generates the weak lower topology, so we are

[3] A compact set k and its saturation generate the same subbasic open.

dealing with a bitopological space. Using closed sets instead of opens sets, the open $\Box k$ consists of all closed sets disjoint from k, whereas the open set $\Diamond u$ consists of all closed sets that intersect u. Observe that $\overline{\{x\}}$ intersects an open u if and only if x is an element of u, whence the preimage of the subbasic open $\Diamond u$ under the injection map is nothing but the open u itself. Dually, the closure $\overline{\{x\}}$ is disjoint from the compact saturated set k precisely when x is not an element of k, whereby the preimage of the subbasic open $\Box k$ under the injection map is just the complement of k. It follows that the embedding $X \hookrightarrow (\mathcal{O}X)^\partial$ is continuous if one refines the original topology on X to its patch topology.

In what follows, instead of a locally compact topology $\mathcal{O}X$ we consider a continuous frame L and construct a compact regular d-frame whose spectrum is the Fell compactification of the spectrum of L.

Definition 4.5.1. We write $\lll\!\lhd$ for the *multiplicative closure* of the way-below relation of a complete lattice. $x \lll\!\lhd y$ holds if and only if there is a finite set A such that x is way below every element of A and the meet of A is below y.

Observe that in a continuous frame (more generally in any continuous preframe) the relation $x \lll\!\lhd y$ holds if and only if there is a finite set Φ of Scott open filters such that x is a lower bound of every member of Φ and y is an element of the filter generated by the union $\bigcup \Phi$. Therefore the multiplicative closure of the way-below relation can be realised as the well-inside relation of the following d-frame.

Definition 4.5.2. Given a continuous frame L, define a d-frame $\mathcal{F}L$ by the following data. The second component frame of $\mathcal{F}L$ is the frame L itself. The first component frame is the smallest sub-frame of $\operatorname{Filt} L$ that contains the preframe of Scott open filters. (Every element of the first component frame is a join of Scott open filters.) The consistency and totality relations are defined just as in the interaction algebra $\operatorname{Ialg} L$, that is, a filter ϕ is total with an element $x \in L$ iff $\phi \ni x$ and an element $x \in L$ is consistent with the filter ϕ iff x is a lower bound for ϕ. We call the d-frame $\mathcal{F}L$ the *Fell d-frame* of L.

Proposition 4.5.1. *Let L be a continuous frame.*

1. *The Fell d-frame $\mathcal{F}L$ is regular normal.*

2. *The well-inside relation induced on L as the component frame of its Fell d-frame coincides with $\lll\!\lhd$.*

3. *The first component frame of the Fell d-frame $\mathcal{F}L$ consists of precisely those filters of L that are round with respect to $\lll\!\lhd$.*

4. *The first component frame of the Fell d-frame $\mathcal{F}L$ is stably continuous and its well-inside relation coincides with the way-below relation.*

If $\mathcal{L}$ is any d-frame, $\prec$ a proximity on it and $\succ$ is the relation defined in equation (4.2) then the inclusions $(\ll) \subseteq (\prec) \subseteq (\lhd)$ and $(\gg) \subseteq (\succ) \subseteq (\rhd)$ hold. With Proposition

4.5.1 (4) it follows that the Fell d-frame of a continuous frame admits precisely one compactification. It can be pictured as

$$\mathrm{Filt}^{\lll} L \underset{\mathsf{con}}{\overset{\mathsf{tot}}{\rightleftarrows}} \mathrm{Idl}^{\lll} L$$

where a round filter is total with a round ideal if and only if the two sets intersect, and a round ideal is consistent with a round filter whenever the ideal consists of lower bounds of the filter. A notable instance of this is a result we alluded to in the introduction:

Proposition 4.5.2. *The (normal) d-frame $\mathbb{O}\mathbb{R}$ of the real line with the topologies of upper and lower semicontinuity admits precisely one compactification. Its spectrum is the extended real line.*

Proof. Each of the two topologies of semicontinuity on the real line is order-isomorphic to the extended real line $[-\infty, \infty]$. On each of these frames, the d-frame-theoretic well-inside relation coincides with the way-below relation except that $\infty \lhd \infty$ but not $\infty \ll \infty$. Thus the top element gives rise to two open ideals $\twoheaddownarrow\infty$ and $\twoheaddownarrow\infty$, both of which are meet-prime in $\mathrm{Idl}^{\lhd}[-\infty, \infty]$. □

The Fell compactification appears in numerous places in the literature. Some instances we deem worth mentioning are:

- The relation $\lll$ is the smallest approximating quasi-proximity on the topology $\mathcal{O}X$ of a locally compact space. Thus the frame $\mathrm{Idl}^{\lll} \mathcal{O}X$ is the topology of the smallest stable compactification of X in the sense of [50].
- Any continuous poset is a locally compact T_0 space in its Scott topology. In [26], Hoffmann shows that the Fell compactification of a continuous poset can be obtained by embedding it into its injective hull and taking the closure in the Lawson topology[4].
- If a locally compact space is Hausdorff, then it is stably locally compact, whence the way-below relation on the lattice of open sets is almost multiplicative except $X \not\lll X$. In this situation the Fell compactification coincides with the Alexandrov one-point compactification.

4.5.2 The Stone-Čech compactification of an algebraic poset

As a second application we turn towards the problem of a maximal order-preserving extension of an algebraic domain. The standard Scott model of PCF interprets a type as a certain algebraic domain. While all of these domains are in fact stably compact in the

[4]The construction is very similar. The injective hull is the complete lattice in $\mathrm{Filt}\,\mathcal{O}X$ that is generated by neighbourhood filters of points.

Scott topology and are contained in a cartesian closed subcategory of Alg, there are algebraic domains which are not stably compact. In this subsection we seek an extension βD of an algebraic domain D with the properties

1. The extension βD is a dcpo,
2. The embedding $D \hookrightarrow \beta D$ is Scott continuous and preserves the way-below relation,
3. The embedding $D \hookrightarrow \beta D$ is dense.

Recall that any algebraic domain D is isomorphic to the ideal completion of the poset of its compact elements and the Scott topology on D is isomorphic to the upper Alexandrov topology on the compact elements. Therefore we reformulate the problem as to find a bitopologically compact regular extension of the locally compact T_0 space $(P, \mathrm{Up}\, P)$ where P is any poset. As the way-below relation on $\mathrm{Up}\, P$ is witnessed by the finitely generated upper sets, the natural partner for the upper Alexandrov topology is the weak lower topology ωP. Thus we set out to describe the Stone-Čech compactification of the d-frame $\mathcal{O}(P, \omega P, \mathrm{Up}\, P)$.

Theorem 4.5.3. *On the set* $\mathrm{Fin}\, P$ *of finite subsets of a poset* P *define a preorder* $A \leq B$ *iff* $\uparrow B \subseteq \uparrow A$. *The weak lower topology* ωP *on* P *is order-isomorphic to the MacNeille completion of the preorder* $(\mathrm{Fin}\, P, \leq)$.

Proof. The argument is more clear if one considers the complete lattice L of closed sets with respect to the weak lower topology. By definition $(\mathrm{Fin}\, P)^\partial$ is meet-dense in this lattice, as every closed set is the intersection of basic closed sets $\uparrow A$ where $A \in \mathrm{Fin}\, P$. But as every upper set is the (filtered) union of its finite subsets, the finitely generated upper sets are also join-dense in L. By the universal property of the MacNeille completion, ωP must be the MacNeille completion of $(\mathrm{Fin}\, P, \leq)$. □

We find it convenient to represent an element W of ωP as the associated element of the MacNeille completion of $\mathrm{Fin}\, P$. Such an element we write as $(\mathbb{L}, \mathbb{U})$ where $\mathbb{L}$ is the lower set of finite sets with $\uparrow A \cup W = P$ and $\mathbb{U}$ is the upper set of finite sets with $\uparrow A \cap W = \emptyset$. Since every weak lower open is in particular a lower set, it has a complement in $\mathrm{Up}\, P$, whereby the d-frame $(\omega P, \mathrm{Up}\, P, \mathsf{con}, \mathsf{tot}) = \mathcal{O}(P, \omega P, \mathrm{Up}\, P)$ is normal for trivial reasons. Further it is obviously regular because on the first component frame the well-inside relation coincides with the frame order, the second component frame is continuous and its well-inside relation contains the way-below relation. By Theorem 4.2.6 the d-frame of the Stone-Čech compactification is the compact regular d-frame determined by the ideal completion of ωP. Hence its spectrum is not only stably compact but even coherent.

From Proposition 4.3.6 and Theorem 3.7.10 we know that the spectrum of the frame $\mathrm{Idl}\, \omega P$ contains both the soberification of the spaces $(P, \omega P)$ and $(P, \mathrm{Up}\, P)$, the latter of which is the ideal completion $\mathrm{Idl}\, P$ in its Scott topology. Concretely, the ideal completion of P embeds into the space $\mathrm{pt}\,\mathrm{Idl}\,\omega P$ as follows. Consider the chain of monotone maps $P \to \mathrm{Fin}\,P \to \omega P$ where the left map is the singleton embedding $x \mapsto \{x\}$ and the right map is to be understood as the embedding of $\mathrm{Fin}\,P$ into its MacNeille completion. This map lifts to a map $\mathrm{Idl}\,P \to \mathrm{Idl}\,\omega P$ which preserves the way-below relation (see Theorem 6.1.13) and transforms an ideal of P into a prime ideal of ωP. The prime ideals of ωP are by classical Stone duality precisely the points of the frame $\mathrm{Filt}\,\omega P$, which is the second component frame of our bitopological Stone-Čech compactification. Finally, recall that the points of any sober space form a dcpo in the specialisation order, whence we declare the desired extension βD of $D = \mathrm{Idl}\,P$ to be the coherent space $\mathrm{pt}\,\mathrm{Filt}\,\omega P$.

4.6 Notes on Chapter 4

As mentioned earlier, the methods used for the classification theorem are not new but stem from the work of Smyth [50] and Banaschewski [4]. In particular the construction of the Stone-Čech compactification of frames has been known for longer, see Johnstone [30] and Banaschewski and Mulvey [7]. The spatial analogue of our Stone-Čech compactification 4.2.1 is called the *Nachbin-Stone-Čech compactification* in [42]. The proof of classification theorem 4.3.5 follows closely [50, Theorem 2]. The bitopological version of complete regularity as well as material on stably compact spaces can be found in [38] and the fact that the Urysohn Lemma works in a bitopological localic setting seems to be folklore among topologists. Stably compact extensions of T_0 spaces can also be obtained via quasi-uniformities, and this is the method of choice in [42, 38, 39]. Theorem 4.5.3, although it seems obvious, has to our knowledge not appeared explicitly in the literature up to now. The author was first made aware of the fact by Andrew Moshier, but the proof we present in this thesis has been communicated to the author by Marcel Erné.

We believe that the bitopological framework explains neatly why Smyth's largest stable compactification is not a reflection of categories as one would expect from a compactification. Without the second topology to limit what kind of proximities are allowed, the topology of the largest stable compactification is simply the ideal completion of the lattice of open sets, which is idempotent only in trivial cases.

What sets the d-frame theoretic compactification apart from the localic and point-set theoretic compactifications is the existence of the normal coreflection through which every compactification factors. This intermediate step is invisible outside the category of d-frames, as we remarked in Proposition 4.2.2. If one traces the chain of adjunctions in equation (4.4) using the patch functor, then the transition from completely regular d-frames to regular normal d-frames is the identity on the level of patch frames.

Future work and open problems

At the time of writing no handy description of the extension βD of Subsection 4.5.2 is known. In particular it is unknown whether the extension is a continuous dcpo again.

Among the real-world examples of completely regular bitopological spaces is the Minkowski space of special relativity. Here the order is given by causality: The upper set of any point in Minkowski space is the light cone emerging from that point. Knowing the order-preserving compactifications of Minkowski space is desirable because it gives a handle to describing the long-term behaviour of relativistic systems.

Chapter 5

Conclusion

We demonstrated that at the heart of the classical theory of compactifications are compactifications of frames and thereby domain-theoretical techniques. Likewise, Kopperman's bitopological separation axioms and compactifications of bitopological spaces, as well as Smyth's stable compactifications, are at the core point-free constructions. This point-free bitopological content we extracted and presented in a clean, efficient manner that has obvious advantages over existing bitopological point-free techniques such as Banaschewski, Brümmer and Hardie's biframes. While d-frames have been used in the literature before [34], the present work is the first that systematically explores the separation axioms for d-frames above T_1 and uses the streamlined notation that makes working with the well-inside relations easy.

We met interpolation properties at numerous places in the thesis: The way-below relation of domains, the really-inside relation on frames and bounded distributive lattices and the normality axiom for lattices and d-lattices that rendered the well-inside relation interpolative. The normality axiom was the basic axiom upon which we built a novel presentation for domains. We made a point that the normality axiom should be considered as more fundamental and more versatile than the interpolation property of the way-below relation.

Using the novel presentation of domains, we obtained short proofs for previously known domain-theoretical facts, in particular many results involving the Lawson dual and the Smyth and Hoare power constructions. More pleasingly, these new proofs are of a finitary nature because the directed complete structure of domains is only present implicitly in our presentations.

Although our notion of morphism in Chapter 1 is more complicated than the morphisms between information systems or abstract bases, and keeping track of two different kinds of tokens does occasionally require extra effort in the proofs, our morphisms resemble the morphisms of the multi-lingual sequent calculus. We gave ways of translating between the categories Tok, Abs and $\mathsf{Infosys}$.

In the notes to the preceding chapters we already hinted at open problems and entry points for further work. We hope that d-frames provide a tool for both topologists and lattice theorists in which to explore bitopological concepts in an efficient way.

Chapter 6

Appendix

This appendix contains definitions and known facts about structures we refer to from within the main chapters. Although there is nothing in this chapter that was not known before, the definitions and formulations are stated as to match the terminology of the main chapters.

6.1 Partial orders and preorders

Anyone who has come in contact with some mathematics should have grasped the concept of the less-than relation $<$ between numbers. It has two crucial properties. Firstly, if a number x is less than another number y and this number is in turn less than a third number z, then x is also less than z. In short, from $x < y < z$ one can conclude $x < z$. Secondly, given any number x, there is no number y such that y is less than x and x is less than y. One says that the order $<$ is *strict* and *antisymmetric*. For the mathematician it is more convenient to weaken the less-than relation to the less-than-or-equal relation $\leq$. This inherits the first property from the strict order, and the second property is split into two others: Any x is less than or equal to itself, and from $x \leq y$ and $y \leq x$ one can conclude $x = y$.

Definition 6.1.1. Let X be a set and $\leq$ be a binary relation on X. The pair $(X, \leq)$ is called a *preordered set* and $\leq$ a *preorder* if it is

reflexive $x \leq x$ for any x,

transitive $x \leq y \leq z$ implies $x \leq z$.

If the preorder is in addition

antisymmetric $x \leq y$ and $y \leq x$ implies $x = y$

then the relation $\leq$ is called a *partial order* and $(X, \leq)$ a *partially ordered set* or *poset* for short. A function $f : (X, \leq) \to (Y, \sqsubseteq)$ between preordered or partially ordered sets is called *monotone* if $x \leq x'$ implies that $f(x) \sqsubseteq f(x')$. The category of preordered sets and monotone functions is denoted by Preord and the category of partially ordered sets and monotone functions is denoted by Poset.

Obviously every poset is a preordered set and the category Poset is a full subcategory of Preord. Any preordered set $(X, \leq)$ can be regarded as a small category where the objects are the elements of X and there is an arrow $x \to y$ whenever $x \leq y$. Reflexivity of $\leq$ means that every object has an identity arrow and transitivity means that one can compose arrows. We will use this analogy below to phrase properties of posets in the language of category theory.

Definition 6.1.2. A poset is called *flat* or an *antichain* if the partial order is equality. A poset is called a *chain* or is said to be *linearly ordered* if for any two elements x and y either $x \leq y$ or $y \leq x$ holds.

Sometimes the preordered set $(X, \leq)$ is itself called a preorder, and if it is clear from the context one omits the symbol $\leq$ and just states "X is a preorder".

Definition 6.1.3. A monotone map $f : X \to Y$ between preordered sets is called an *order embedding* if for any $x, x' \in X$ the relation $f(x) \leq f(x')$ implies that $x \leq x'$. Two preordered sets are called *order-isomorphic* if there exits a bijection between them consisting of monotone maps.

Notice that the maps involved in an order-isomorphism are necessarily order embeddings.

Definition 6.1.4. A monotone function $f : X \to X$ on a preordered set is called *inflationary* if for all elements $x \leq f(x)$. Dually, if for all elements $f(x) \leq x$ then the function is called *deflationary*. An inflationary map which is idempotent, meaning $f \circ f = f$, is called a *closure operator*. In categorical terms, a closure operator is a monad when the poset is considered as a small category.

6.1.1 Constructions on preorders

There is an obvious forgetful functor $\mathsf{Preord} \to \mathsf{Set}$ which forgets the preorder relation. The free preorder over a set is the flat preorder.

Subobjects

If $(X, \leq)$ is a preorder and Y a subset of X then the relation $\leq$ restricted to $Y \times Y$ renders Y a preorder. We say that this order on Y is the one induced by X.

Order dual

There is a covariant involution $(-)^\partial$ on the category Preord that is the identity on morphisms and maps the pair $(X, \leq)$ to the pair $(X, \geq)$ where $\geq$ is shorthand for the relational inverse of $\leq$. We say that $(X, \geq)$ is the *order dual* of $(X, \leq)$. A monotone map $f : X \to Y^\partial$ is said to be an *antitone* map from X to Y.

Posets from preorders

Given a preordered set $(X, \leq)$ one defines the relation $\sim$ to be the intersection of $\leq$ and $\geq$, that is $x \sim y$ whenever $x \leq y \leq x$. The relation $\sim$ is an equivalence relation. On the set of equivalence classes $X/_\sim$ one can define a partial order $\lesssim$ by $[x] \lesssim [y]$ iff $x \leq y$.

Preorders induced by functions

If X is a set and $f : X \to (Y, \leq)$ is a map into a preorder then the preorder on X induced by f is defined as $x \leq x'$ iff $f(x) \leq f(x')$ in Y. Observe that with respect to the preorder on X thus defined, the function f is always an order-embedding. Even if $(Y, \leq)$ is a poset, the induced relation is in general only a preorder, unless the function f is injective.

Products and coproducts

The categories Preord and Poset have arbitrary products and coproducts. If $\{(X_i, \leq_i)\}_{i \in I}$ is a family of preorders then one defines a preorder on the cartesian product $\prod_{i \in I} X_i$ by letting $(x_i)_{i \in I} \leq (y_i)_{i \in I}$ iff $\forall i \in I.\, x_i \leq y_i$. On the disjoint sum $\coprod_{i \in I} X_i$ define a preorder by letting $x \leq y$ if and only if both x and y are elements of the same component X_i and $x \leq_i y$.

Order-enriched categories

The categories Preord and Poset are enriched over themselves. The set of monotone functions between preordered sets X and Y is preordered point-wise: $f \leq g$ if and only if $\forall x \in X.\, f(x) \leq g(x)$. For any three objects X, Y, Z the composition map

$$\circ : \mathsf{Preord}(X, Y) \times \mathsf{Preord}(Y, Z) \to \mathsf{Preord}(X, Z)$$

is monotone in both arguments. In general, a category C is called order-enriched if every hom-set $\mathsf{C}(X, Y)$ carries a preorder such that composition of morphisms is a monotone operation. See the categorical notion of "enriched category" for a more detailed axiomatisation. Typical examples of order-enriched categories are the subcategories of Poset that we define in the subsections below.

6.1.2 Joins and meets

Definition 6.1.5. Let $(X, \leq)$ be a preordered set and $Y \subseteq X$ be a subset. An element $x \in X$ is said to be an *upper bound* for Y if for all $y \in Y$ the relation $y \leq x$ holds. In case that any upper bound x' of Y has $x \leq x'$ we call the element x a *least upper bound*, *supremum* or *join* of Y. In a poset, least upper bounds are unique, but in a preordered set there may be several, all of which are equivalent by the equivalence relation $\sim$ defined above. Lower bounds and greatest lower bounds (infima, meets) of subsets are defined dually.

Notice that the join of a subset Y is the coproduct of Y if $(X, \leq)$ is viewed as a category. Dually the meet of Y is its categorical product. Joins and meets need not always exist. For example, an antichain has joins and meets only for singleton subsets. In due course we shall define a hierarchy of subcategories of Poset that are characterised by the existence of certain joins or meets.

6.1.3 Adjoints

If one regards posets as small categories then monotone maps are functors. A natural transformation $\eta : f \to g$ between two monotone maps with the same source and target is nothing but the assertion that f is below g in the point-wise order. The notion of an adjunction is precisely that of category theory:

Definition 6.1.6. An adjunction between posets $(X, \leq)$ and $(Y, \leq)$ is a pair of monotone maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that for every $x \in X$ and $y \in Y$ the relation $f(x) \leq y$ holds if and only if $x \leq g(y)$ holds. The map f is called *left adjoint* to g and g is *right adjoint* to f. One writes $f \dashv g$.

Equivalently, the pair (f, g) is adjoint if $g \circ f$ is above the identity on X and $f \circ g$ is below the identity on Y. General category theory tells us that

1. Adjoints are unique.
2. A left adjoint preserves all existing joins and a right adjoint preserves all existing meets.
3. A monotone map $f : X \to Y$ has a right adjoint if and only if for all $y \in Y$ the right-hand side in the identity below exists and f preserves all such joins.[1]

$$g(y) = \bigvee \{x \mid f(x) \leq y\}$$

[1] In general category theory this is known as the Freyd Adjoint Functor Theorem. A formulation that resembles the one for posets more closely was observed by Escardó: Let $F : X \to A$ be a functor. Then F has a right adjoint if and only if for every object a in A the forgetful functor $U : (F \downarrow a) \to X$ has a colimit and F preserves it. Here $(F \downarrow a)$ denotes the comma category.

Definition 6.1.7. An adjunction between posets X and Y^∂ is called a *Galois connection*. It comprises a pair of antitone maps with the property $x \leq g(y) \Leftrightarrow y \leq f(x)$.

Equivalently, a Galois connection is a pair of antitone maps (f, g) between two posets such that both $f \circ g$ and $g \circ f$ are above the identity.

6.1.4 Semilattices

Definition 6.1.8. A *semilattice* is a set X together with a binary operation $\cdot : X^2 \to X$ that is

idempotent $x \cdot x = x$ for any $x \in X$,

associative $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in X$,

commutative $x \cdot y = y \cdot x$ for all $x, y \in X$.

The semilattice $(X, \cdot)$ is *bounded* if there exists a neutral element e for the binary operation, meaning $e \cdot x = x$ for all $x \in X$. A semilattice homomorphism is a function between semilattices that preserves the semilattice operation, i.e. $f(x \cdot y) = f(x) \cdot f(y)$. A homomorphism of bounded semilattices in addition preserves the neutral element. The category of bounded semilattices and their homomorphisms is denoted by sLat.

Any semilattice carries a natural partial order. Define $x \leq y$ whenever $x \cdot y = x$. This relation is reflexive because $\cdot$ is idempotent, transitive because $\cdot$ is associative and antisymmetric because $\cdot$ is commutative. The neutral element e is easily seen to be unique and the greatest element of the poset $(X, \leq)$. In the order $\leq$ the element $x \cdot y$ is the meet of the set $\{x, y\}$ whence in this reading the semilattice $(X, \cdot)$ is said to be a meet-semilattice and its neutral element is typically denoted by 1. By associativity of the semilattice operation, the meet of any non-empty finite subset of X is well-defined, and the neutral element 1 serves as the meet of the empty set.

Dually one can define a partial order $x \leq y$ iff $x \cdot y = y$. This partial order is the relational inverse of the one we defined earlier, whence a semilattice under this order is called join-semilattice. Its neutral element is commonly denoted by 0.

Example 19. Let X be a set and let $\mathrm{Fin}\, X$ denote the set of finite subsets of it. Then $\mathrm{Fin}\, X$ is a semilattice when endowed with set union as binary operation. Its neutral element is the empty set.

We shall see that the example above is the generic semilattice.

Semilattices as Eilenberg-Moore algebras

The finite powerset operation Fin is a functor on the category Set. For any function $f : X \to Y$ one defines $\mathrm{Fin}(f)(A) = \{f(a) \mid a \in A\}$, i.e. the action of $\mathrm{Fin}(f)$ on finite subsets is forward image under f. This endofunctor extends to a monad where the unit $X \to \mathrm{Fin}\, X$ is given by the singleton operation $x \mapsto \{x\}$ and the multiplication $\mathrm{Fin}^2\, X \to X$ is given by the union operation $\mathbb{A} \mapsto \bigcup \mathbb{A}$.

Theorem 6.1.1. *The category* sLat *of bounded (join-)semilattices and semilattice homomorphisms is equivalent to the category of Eilenberg-Moore algebras for the finite powerset monad on* Set*.*

Proof. As we noted above, a bounded join-semilattice $(X, \cdot, 0)$ has a join operation $\bigvee :$ $\mathrm{Fin}\, X \to X$ defined inductively on the cardinality of the subset:

$$\begin{aligned} \bigvee \emptyset &= 0 \\ \bigvee \{x\} &= x \\ \bigvee (\{x\} \cup A) &= x \cdot \bigvee A \end{aligned}$$

The unit law for $\bigvee$ as a Fin-algebra map is precisely the second identity above. The associative law holds because both $\bigcup$ and $\cdot$ are associative in the algebraic sense. Observe that the join-operation $\bigvee$ is monotone in the size of its argument. Indeed, if $A \subseteq B$ are two finite subsets of X then $(\bigvee A) \cdot (\bigvee B) = \bigvee B$.

Conversely, if a set X has an algebra map $\alpha : \mathrm{Fin}\, X \to X$ for the finite powerset monad then define a binary operation $x \cdot y := \alpha(\{x, y\})$. This operation is idempotent because $\{x, x\} = \{x\}$, commutative because $\{x, y\} = \{y, x\}$ and the associative law of $\cdot$ is a consequence of the associative law for α as a monad algebra. Indeed, the associative law for α requires that $\alpha \circ \mathrm{Fin}(\alpha) = \alpha \circ \bigcup$. Using this identity and the definition of $\cdot$ one finds

$$\begin{aligned} x \cdot (y \cdot z) &= \alpha(\{x, \alpha(\{y, z\})\}) \\ &= \alpha(\{\alpha(\{x\}), \alpha(\{y, z\})\}) \\ &= (\alpha \circ \mathrm{Fin}(\alpha))(\{\{x\}, \{y, z\}\}) \\ &= (\alpha \circ \bigcup)(\{\{x\}, \{y, z\}\}) \\ &= \alpha(\{x, y, z\}) \\ &= (\alpha \circ \bigcup)(\{\{x, y\}, \{z\}\}) \\ &= (x \cdot y) \cdot z \end{aligned}$$

□

General category theory tells us that the free bounded semilattice over a set X is $(\mathrm{Fin}\, X, \cup)$.

Semilattices over posets

One can adapt the technique of Theorem 6.1.1 to construct the free semilattice over a poset, in other words to add finite joins to a poset in a free manner.

Let $(X, \leq)$ be a poset. On the semilattice $(\mathrm{Fin}\, X, \cup)$ define a preorder

$$A \lesssim B \text{ iff } \forall a \in A \exists b \in B.\, a \leq b$$

This preorder contains the inclusion relation $\subseteq$ of finite subsets because $\leq$ is reflexive. Furthermore it is evident from the definition that for any three sets $A, B, C \in \mathrm{Fin}\, X$ the relation $A \cup B \lesssim C$ holds precisely when $A \lesssim C$ and $B \lesssim C$ hold. While in the join-semilattice $(\mathrm{Fin}\, X, \cup)$ the image of X under the singleton map is an antichain, the singleton map is an order-embedding into $(\mathrm{Fin}\, X, \lesssim)$.

Let $\mathrm{SL}(X)$ denote the poset obtained by quotienting the preorder $(\mathrm{Fin}\, X, \lesssim)$ by the equivalence $\sim = (\lesssim \cap \gtrsim)$.

Theorem 6.1.2. *For any poset* $(X, \leq)$ *the poset* $\mathrm{SL}(X)$ *is a join-semilattice where joins of (equivalence classes of) finite sets are computed as set union. The operation* SL *extends to a monad on* Poset *in the same manner as* Fin *is a monad on* Set*. The Eilenberg-Moore algebras for this monad are the bounded join-semilattices.*

An interesting fact is that while a set X can potentially carry many distinct semilattice structures (i.e. a set may have many algebras for the finite powerset monad), the algebras for the monad SL are unique, because an SL-algebra map on a poset $(X, \leq)$ must compute the joins of finite subsets with respect to the partial order $\leq$. Another reason for uniqueness is that the structure map $\bigvee$ of a join-semilattice is left adjoint to the unit of the SL monad. Indeed, the join of a finite set $A \in \mathrm{Fin}\, X$ is below an element x precisely when x is an upper bound for A, which we can express as $A \lesssim \{x\}$.

Remark. One constructs the free meet-semilattice over a poset by taking the order dual, constructing its free join-semilattice and then taking order dual again.

6.1.5 Complete lattices

Definition 6.1.9. A poset $(X, \leq)$ is a *complete lattice* if every subset has a join.

A fundamental fact about complete lattices is that a poset has joins for all subsets if and only if it has meets for all subsets. Indeed, by definition the meet of a subset is the join of its lower bounds. Hence the order dual transforms a complete lattice into a complete lattice.

Another way of obtaining new complete lattices from old is via closure operators.

Proposition 6.1.3. *Let* $(L, \leq)$ *be a complete lattice and* $\nu : L \to L$ *a monotone inflationary map (In particular, closure operators on* L *satisfy these requirements). Then the set* *of fixed points* $\mathrm{fix}(\nu) \subseteq L$ *of* ν *under the induced order is a complete lattice where meets coincide with those of* L *and the join of a set of fixed points* $Y \subseteq \mathrm{fix}(\nu)$ *is calculated using the join of* L *as the least fixed point of* ν *above* $\bigvee Y$*.*

The proof can be found in textbooks like [14]. An important instance is the case where L is the powerset of some set with the inclusion order.

Categorical properties

A join-preserving map between complete lattices is also called *linear*. The category CL of complete lattices and linear maps is closed under all products, but not coproducts. It is enriched over itself, as the point-wise join of linear maps is again linear.

Complete lattices as Eilenberg-Moore algebras

The covariant powerset functor $\mathcal{P}$ extends to a monad on Set where the unit is the singleton operation and the multiplication is set union. Just as the finite powerset monad yields posets with joins for all finite subsets, the full powerset monad $(\mathcal{P}, \{-\}, \bigcup)$ yields complete lattices.

Theorem 6.1.4. *The Eilenberg-Moore category of the powerset monad on* Set *is equivalent to the category* CL *of complete lattices and join-preserving maps.*

Instead of constructing free complete lattices as powersets over sets, one can do the same over posets.

Definition 6.1.10. Let $(X, \leq)$ be a poset. Define an operation on subsets of X as

$$\downarrow Y = \{x \in X \mid \exists y \in Y.\, x \leq y\}\,.$$

We abbreviate the set $\downarrow\{x\}$ by $\downarrow x$. The set $\downarrow Y$ is called the *lower closure* or *downward closure* of Y and the subsets of X which are their own lower closures are called the *lower sets* of X. In particular the set $\downarrow x$ is called the *principal ideal* generated by x. We denote the set of lower sets of X, ordered by inclusion, by $\mathrm{Lo}\,X$. A monotone function f lifts to lower sets as forward image followed by lower closure.

Dually one defines the *upper closure* operator $\uparrow$ and the poset $\mathrm{Up}\,X$ of upper closed subsets of a poset. The action of Up on morphisms is forward image followed by upper closure.

Evidently the operator $\downarrow$ is a closure operator on the powerset of a poset, whence by Proposition 6.1.3 the poset $\mathrm{Lo}\,X$ is a complete lattice. In fact the lower sets share all joins and meets with the powerset and the map $\mathrm{Lo}(f)$ derived from a monotone map f preserves all unions, but not necessarily intersections. The operation Lo extends to a monad on Poset, the lower set monad. Its unit is the principal ideal map $\downarrow$ and the multiplication is union of lower sets.

Theorem 6.1.5. *Algebras for the lower set monad are unique. The Eilenberg-Moore category for the lower set monad on* Poset *is equivalent to the category* CL *of complete lattices and join-preserving maps.*

Remark. The preorder $\lesssim$ we defined on the finite powerset of a poset in order to compute the free join-semilattice over it can be characterised in terms of lower closure. One has $A \lesssim B$ precisely when ${\downarrow}A \subseteq {\downarrow}B$.

6.1.6 Directed complete partial orders

Definition 6.1.11. Let $(X, \leq)$ be a preordered set. A subset $I \subseteq X$ is called an *ideal* if it is

lower closed $x \leq y \in I$ implies $x \in I$,

directed Any finite subset of I has an upper bound in I.

An ideal of X^∂ is called a *filter*. The set of ideals of X, ordered by inclusion, is denoted by $\mathrm{Idl}\, X$ and called the *ideal completion* of X. One extends this operation to monotone maps by letting $\mathrm{Idl}(f)$ send an ideal to the lower closure of the forward image under f. Dually one defines $\mathrm{Filt}\, X$ as the set of filters of X under inclusion and the map $\mathrm{Filt}(f)$ sending a filter to the upward closure of the forward image under f.

Notice that ideals are never empty, since directedness requires that the empty set has an upper bound in any ideal. A crucial fact is that the directed union of ideals is again an ideal.

Proposition 6.1.6. *The assignment* $f \mapsto \mathrm{Idl}(f)$ *constitutes a functor* $\mathsf{Preord} \to \mathsf{Poset}$ *and restricts to a monad on* Poset *where the unit is the principal ideal map* $x \mapsto {\downarrow}x$ *and the multiplication is (directed) union of ideals.*

Definition 6.1.12. A *directed complete partial order* or dcpo for short is a poset where every ideal has a join. A monotone map between dcpos is called *continuous* if it preserves joins of ideals. The category of dcpos and continuous maps is denoted by Dcpo.

Proposition 6.1.7. *The category* Dcpo *is equivalent to the Eilenberg-Moore category of the ideal completion monad on* Poset*. Consequently, the ideal completion is the free dcpo over a poset.*

Categorical properties

The category Dcpo has arbitrary products and coproducts which are computed as in the category Poset. The category Dcpo is cartesian closed, as the point-wise join of a directed set of continuous maps is continuous.

6.1.7 Domains

Definition 6.1.13. A domain is a a directed complete partial order D with a left adjoint $\Downarrow : D \to \mathrm{Idl}\, D$ to the join operation on ideals. The interpolative relation $\ll$ induced by this map as $x \ll y \Leftrightarrow x \in \Downarrow y$ is called the *way-below relation*. The category Dom is the full subcategory of Dcpo where objects are domains and morphisms are continuous maps.

The way-below relation

Being the left adjoint to the join operation $\bigvee$, we know that $\bigvee \circ \Downarrow$ is above the identity. But as $\Downarrow \dashv \bigvee \dashv \downarrow$ we also know that $\bigvee \circ \Downarrow$ is below the identity, whence any element x of a domain is the join of the ideal $\Downarrow x$.

A very useful fact is that, being a left adjoint, the map $\Downarrow$ is a homomorphism of Idl-algebras. We can write

$$\Downarrow z = \left(\bigcup \circ \mathrm{Idl}(\Downarrow) \circ \Downarrow \right) z$$

Which means that the ideal $\Downarrow z$ is the union of ideals $\bigcup \{ \Downarrow y \mid y \ll z \}$. This shows:

Proposition 6.1.8. *The way-below relation on a domain has the interpolation property, meaning that whenever $x \ll z$ then $x \ll y \ll z$ for some element y.*

A less abstract definition of the way-below relation that applies to arbitrary posets is the following. An element x is way below an element y if for all ideals I that possess a join the implication $y \sqsubseteq \bigsqcup I \Rightarrow x \in I$ holds. A poset where every element is the directed join of elements way below it is called a *continuous poset*. The preceding proposition remains true for continuous posets.

Proposition 6.1.9. *For any domain D there is an order-isomorphism between the points of D and ideals that are round with respect to the way-below relation.*

Proof. Similar to the proof of Lemma 6.1.18. □

Algebraic domains

Definition 6.1.14. An element x of a domain is called *compact* or *finite* if it is way below itself. A domain is called *algebraic* if every element is the directed join of compact elements. With Alg we denote the full subcategory of Dom whose objects are algebraic domains.

Proposition 6.1.10. *The class of algebraic domains is the image of the ideal completion functor* $\mathrm{Idl} : \mathsf{Poset} \to \mathsf{Dcpo}$. *In other words, the algebraic domains are precisely the ideal completions of posets.*

Proof. Every algebraic domain is isomorphic to the ideal completion of the poset of compact elements. Observe that the restriction of the way-below relation to the set of compact elements of a domain is a partial order. Then apply Lemma 6.1.25. □

Categorical constructions

The category Dom has finite products and arbitrary coproducts, but an infinite product of domains may fail to be a domain unless all but finitely many factors have a least element. In contrast to general dcpos, domains are not closed under the function space construction. The full subcategory of algebraic domains, however, is cartesian closed.

The Scott topology

Definition 6.1.15. A subset $U \subseteq D$ of a dcpo is called *Scott open* if for all ideals $I \in \mathrm{Idl}\, D$ the join of I is an element of U precisely when I intersects U. The set of Scott open subsets of D, ordered by inclusion, is denoted by σD.

The following result justifies our name for the morphisms of Dcpo.

Proposition 6.1.11. *The set of Scott open sets of any dcpo is a T_0 topology. A monotone map between dcpos preserves joins of ideals precisely when it is continuous with respect to the Scott topologies on source and target.*

For domains, we can say even more about the Scott topology:

Theorem 6.1.12.

1. *For any domain D, the Scott topology has a basis consisting of sets of the form $\Uparrow x := \{y \in D \mid x \ll y\}$.*
2. *For two subsets x, y of a domain, $x \ll y$ holds precisely when there exists a Scott open set U containing y and having x as a lower bound.*
3. *The Scott topology of a domain is itself a domain, where $U' \ll U$ if there exists a finite subset $A \subseteq U$ such that U' is contained in the upper closure of A.*
4. *A morphism f between domains preserves the way-below relation if and only if the operation $\mathrm{Up}(f)$, that is forward image followed by upper closure, maps Scott open sets to Scott open sets.*

For algebraic domains, the results above specialise further:

Theorem 6.1.13. *If D is an algebraic domain and P is the poset of its compact elements then the following hold:*

1. *The Scott topology of D is isomorphic to the frame $\mathrm{Up}\, P$ of upper sets of compact elements.*
2. *$x \ll y$ holds if and only if there exists some compact element k with $x \sqsubseteq k \sqsubseteq y$.*
3. *The morphisms between algebraic domains that preserve the way-below relation are precisely the maps of the form $\mathrm{Idl}(f)$ for some monotone function between compact elements.*

6.1.8 Lattices

Definition 6.1.16. A *lattice* is a poset with two semilattice operations $\vee$, $\wedge$ such that the partial order induced by $\vee$ is dual to the order induced by $\wedge$. If both semilattice operations have a neutral element then the lattice is called *bounded*.

In other words, a lattice is a poset that has meets and joins for all finite non-empty subsets, and a bounded lattice is a poset with meets and joins for all finite subsets. The order on a lattice $(L, \vee, \wedge)$ is typically the one induced by $\wedge$, that is $x \leq y$ iff $x \wedge y = x$. Clearly the order dual of a lattice is a lattice.

If $(L, \vee, \wedge)$ is a lattice then for any $x \in L$ the maps $y \mapsto x \vee y$ and $y \mapsto x \wedge y$ are monotone, so in particular the inequality $x \wedge (y \vee y') \geq (x \wedge y) \vee (x \wedge y')$ holds for all $x, y, y' \in L$.

Definition 6.1.17. A lattice $(L, \vee, \wedge)$ is called *distributive* if for every $x \in L$ the meet operation $y \mapsto x \wedge y$ is a join-semilattice homomorphism, that is, the inequality $x \wedge (y \vee y') \leq (x \wedge y) \vee (x \wedge y')$ holds for all $x, y, y' \in L$. The category of bounded distributive lattices and lattice homomorphisms is denoted by Lat.

Categorical constructions

Products of lattices are computed as in the category Poset. Coproducts of lattices are computed using the free lattice construction, see the following proposition.

Proposition 6.1.14. *The free bounded lattice over a set is the free join-semilattice over the free meet-semilattice over the set. The free bounded lattice is distributive, hence also the free bounded distributive lattice.*

6.1.9 Frames

Definition 6.1.18. A meet-semilattice $(L, \wedge)$ is called *meet-continuous* if L is a dcpo in the order induced by $\wedge$ and moreover for every $x \in L$ the meet operation $y \mapsto x \wedge y$ is Scott continuous.

Definition 6.1.19. A *frame* is a bounded distributive, meet-continuous lattice. A frame homomorphism is a Scott continuous lattice homomorphism. The category of frames and frame homomorphism is denoted by Frm.

Alternatively one can describe frames as complete lattices where arbitrary joins distribute over finite meets. However, the distinction between finite and directed joins often has to be made in proofs and so we prefer the characterisation in the definition above.

Proposition 6.1.15.

1. *The ideal completion of a bounded distributive lattice is a frame.*
2. *The join operation $\bigvee : \mathrm{Idl}\, L \to L$ of a frame is a frame homomorphism.*

Proof. The ideal completion of any poset is a dcpo where directed joins are computed as set union. The ideal completion of a bounded join-semilattice has finite joins where the empty join of ideals is the ideal $\downarrow 0$ (here 0 denotes the neutral element for $\vee$), and binary joins are computed as the element-wise join of ideals followed by lower closure. The ideal completion of a bounded meet-semilattice has finite meets that are computed as set intersection. Clearly directed joins distribute over finite meets. In a distributive lattice the binary meet of ideals can be computed as element-wise meet. Knowing this, the finite distributive law for the ideal completion follows from the finite distributive law of the lattice itself.

The join operation $\bigvee : \operatorname{Idl} L \to L$ on a frame, being left adjoint to the principal ideal operation, preserves all existing joins. Meet-continuity of the frame implies that the join of ideals preserves finite meets. $\square$

Heyting implication and pseudocomplements

If $x \in L$ is an element of a frame, then $y \mapsto x \wedge y$ preserves all joins and therefore has a right adjoint.

Definition 6.1.20. For elements x, y of a frame L the element $x \to y$ is defined as the join of the set $\{z \in L \mid x \wedge y = x \wedge z\}$. The binary map $\to$ is called the Heyting arrow of the frame L.

Definition 6.1.21. For any element x of a frame L the element $x \to 0$ is abbreviated as $\neg x$ and called the *pseudocomplement* of x. The unary operation $\neg$ is called Heyting negation.

Right adjoints and basis embeddings

Every frame homomorphism has a right adjoint, because it preserves all joins. The right adjoint to a frame homomorphism h is commonly denoted by h_*. If $h : M \twoheadrightarrow L$ is a surjective frame homomorphism then the composite $h_* \circ h$ is a closure operator that preserves finite meets because h preserves finite meets and h_* preserves all meets. Moreover, in that situation h_* is injective and $h \circ h_*$ is the identity on L.

Lemma 6.1.16. *If $h : M \twoheadrightarrow L$ is a surjective frame homomorphism, $\prec_M$ is a quasi-proximity on M and the right adjoint h_* is a basis embedding with respect to $\prec_M$ then the relation on L defined as*

$$x_0 \prec_L x_1 :\Leftrightarrow h_*(x_0) \prec_M h_*(x_1)$$

is a quasi-proximity on L.

Proof. First we show that $\prec_L$ is interpolative whenever $\prec_M$ is. Suppose $h_*(x_0) \prec_M h_*(x_1)$. Since h_* is a basis embedding with respect to $\prec_M$ there exists some $x \in L$ with $h_*(x_0) \prec_M h_*(x) \prec_M h_*(x_1)$ and thereby $x_0 \prec_L x \prec_L x_1$. The right adjoint h_* is an order embedding into M because $h \circ h_*$ is the identity on L. Since $\prec_M$ is stronger than the frame order, so must be $\prec_L$. Further it is obvious that $\prec_L$ is downward closed on the left and upward closed on the right because h_* is monotone and $\prec_M$ has the desired properties, which makes $\prec_L$ an auxiliary relation. With these properties it is easy to show that $\prec_L$ is closed under finite joins on the left and finite meets on the right. □

Lemma 6.1.17. *Let $h : M \twoheadrightarrow L$ be a surjective frame homomorphism such that h_* is a basis embedding with respect to $\ll$ (see Definition 6.1.25). The following are equivalent:*

1. $h_*(x_0) \ll h_*(x_1)$.
2. *There exist $y_0 \ll y_1$ with $x_0 \leq h(y_0)$ and $h(y_1) \leq x_1$.*

Proof. For surjective frame homomorphisms $h : M \to L$ the composite $h \circ h_*$ is the identity on L. Therefore, if $h_*(x_0) \ll h_*(x_1)$ then one can choose $y_0 = h_*(x_0)$ and $y_1 = h_*(x_1)$ and obtain the implication (1) $\Rightarrow$ (2). For the reverse implication, suppose that $x_0 \leq h(y_0)$,$y_0 \ll y_1$ and $h(y_1) \leq x_1$. Since h is dense, there exists some $x \in L$ with $y_0 \ll h_*(x) \ll y_1$. The way-below relation is contained in the order $\leq$ and h preserves the order, whence $x_0 \leq x \leq x_1$. The right adjoint h_* is monotone as well, so $h_*(x_0) \leq h_*(x)$. Now use the fact that h_* is the right adjoint to h and deduce $y_1 \leq h_*(x_1)$. Together we have $h_*(x_0) \leq h_*(x) \ll y_1 \leq h_*(x_1)$ and thereby $h_*(x_0) \ll h_*(x_1)$. □

6.1.10 Completely distributive frames

As we have seen above, the category of complete lattices and join-preserving maps can be regarded as the Eilenberg-Moore category for the lower set monad on Poset. The lower set monad Lo, just as the ideal completion monad Idl, has the special property that any monad algebra map $\bigvee : \mathrm{Lo}\, P \to P$ must be left adjoint to the unit $\downarrow : P \to \mathrm{Lo}\, P$ and is therefore unique for any given poset. Just as domains among the dcpos play a special role, those complete lattices which have a left adjoint for the algebra map take a special place among the complete lattices.

We begin with an obvious counterpart for the way-below relation.

Definition 6.1.22. Let L be a complete lattice and $x, y \in L$. We say that x is *completely below* y and write $x \lll y$ if for any lower set $S \in \mathrm{Lo}\, L$ the relation $y \leq \bigvee S$ implies that $x \in S$. An element $x \in L$ which is completely below itself is called *completely compact*.

Just as the way-below relation on dcpos, the completely-below relation is an auxiliary relation, meaning $x \lll y$ implies $x \leq y$ and $x' \leq x \lll y \leq y'$ implies $x' \lll y'$. In contrast to the way-below relation, it is not closed under finite joins on the left in general[2] In particular, the bottom element 0 of a complete lattice is never completely below itself, because $0 = \bigvee \emptyset$ and certainly $0 \notin \emptyset$.

[2]In fact, a poset where the way-below relation coincides with the completely-below relation must be a chain.

We call a subset $S \in \mathrm{Lo}\, L$ a *round lower set* with respect to $\lll$ if $s \in S$ implies that there exists some $s' \in S$ with $s \lll s'$.

Lemma 6.1.18. *In a complete lattice L, any round lower set S is of the form $\downarrow\!\!\!\downarrow x = \{y \in L \mid y \lll x\}$ for some $x \in L$.*

Proof. Suppose S is a round lower set. Define $x = \bigvee S$. For any $s \in S$ we have $s \lll s' \leq x$ for some $s' \in S$ whence all elements of S are completely below x. Now suppose y is completely below x. Since $x \leq \bigvee S$ we must have $y \in S$. □

Proposition 6.1.19. *The following are equivalent for a complete lattice L.*

1. *The structure map $\bigvee : \mathrm{Lo}\, L \to L$ has a left adjoint.*
2. *Every element $x \in L$ is the join of the set of elements completely below it.*
3. *The lattice L is order-isomorphic to the lattice of round lower sets with respect to $\lll$.*
4. *The lattice L satisfies the* complete distributive law*, meaning that arbitrary meets distribute over arbitrary joins:*

$$\bigwedge_{S \in \mathcal{S}} \bigvee S = \bigvee \left\{ \bigwedge_{S \in \mathcal{S}} \lambda(S) \,\middle|\, \lambda : \mathcal{S} \to \bigcup \mathcal{S} \text{ choice function} \right\}$$

Remark. The equivalence of the above characterisations requires the Axiom of Choice, guaranteeing a rich enough supply of choice functions $\lambda : \mathcal{S} \to \bigcup \mathcal{S}$. However, in our work we actually never make explicit use of the complete distributive law. The characterisation using the left adjoint to $\bigvee$ is called *constructive complete distributivity* in [60].

If a complete lattice is completely distributive, then the left adjoint to the structure map is $\downarrow\!\!\!\downarrow$, just as we expect. The complete distributive law subsumes the frame distributive law, whence we also refer to completely distributive complete lattices as *completely distributive frames* and denote the category of completely distributive frames and join-preserving maps by CDFrm.

At first glance, completely distributive frames seem rather exotic. But there are many of them:

Proposition 6.1.20.
1. *The free complete lattice* $\mathrm{Lo}\, P$ *over any poset P is completely distributive. Joins and meets are computed as set union and intersection, respectively. The completely-below relation has the characterisation $S' \lll S$ if and only if there exists an element $x \in S$ which is an upper bound for S'.*
2. *For any poset P the complete lattice of upper sets* $\mathrm{Up}\, P$ *is completely distributive where $U' \lll U$ if and only if there exists some $x \in U$ which is a lower bound for U'.*

3. *The Scott topology on any domain D is a completely distributive frame. Its completely-below relation has the same characterisation as on* $\mathrm{Up}\, D$.

Recall that a subset $U \subseteq L$ of a complete lattice is a completely prime upper set if for any lower set $S \in \mathrm{Lo}\, L$

$$\bigvee S \in U \Leftrightarrow S \cap U \neq \emptyset.$$

These sets are in bijective correspondence with the order dual of L, because the complement of a completely prime upper set has a largest element, and any set of the form $L \setminus {\downarrow}x$ is completely prime.

Theorem 6.1.21. *Let L be a completely distributive frame.*

1. *The completely-below relation has the interpolation property.*

2. *An upper set $U \subseteq L$ is completely prime if any only if it is round with respect to $\lll$, meaning*

$$x \in U \Leftrightarrow \exists u \in U.\, x \lll u.$$

3. *The relation $x \lll y$ holds if and only if there is a completely prime upper set U which contains y and has x as a lower bound.*

4. *The set of completely prime upper sets of L is again a completely distributive frame, where joins are computed as set union and meets are computed as $\bigwedge \mathcal{U} = \bigcup_{u \in \bigcap \mathcal{U}} \twoheaduparrow u$. If h is a join-preserving map between completely distributive frames, then h^{-1} restricts to a join-preserving map $h^{\wedge\!\!\wedge}$ between completely prime upper sets. This yields a contravariant involution $(-)^{\wedge\!\!\wedge}$ on* CDFrm.

5. *The completely distributive frame $L^{\wedge\!\!\wedge}$ is order-isomorphic to the order-dual L^{∂} via the maps*

$$\begin{aligned} L^{\wedge\!\!\wedge} \ni U &\mapsto \bigvee (L \setminus U) \\ L \ni x &\mapsto L \setminus {\downarrow}x \end{aligned}$$

In this reading, the duality $(-)^{\wedge\!\!\wedge}$ transforms a join-preserving map h to its right adjoint.

Lemma 6.1.22. *Let $2 = \{0, 1\}$ denote the two-chain object in* CDFrm.

1. *For any completely distributive frame L, the set $L^{\wedge\!\!\wedge}$ of completely prime upper sets is order-isomorphic to the hom-set $\mathsf{CDFrm}(L, 2)$ with the point-wise order. A set $U \in L^{\wedge\!\!\wedge}$ corresponds to its characteristic function $\chi_U(x) = 1 :\Leftrightarrow x \in U$.*

2. *For a homomorphism $h \in \mathsf{CDFrm}(L, M)$ and a completley prime upper set $U \in M^{\wedge\!\!\wedge}$ represented by $\chi_U \in \mathsf{CDFrm}(M, 2)$ we have $U \,\mathrm{Ialg}(h) x$ if and only if $(\chi_U \circ h)(x) = 1$. This means that that* Ialg *is a contravariant functor $\mathsf{CDFrm} \to \mathsf{Tok}_0$ which is presented as $\mathsf{CDFrm}(-, 2)$.*

3. *The interaction algebra* $\operatorname{Ialg} 2$ *is isomorphic to* $\mathbf{1}$*, and* $\Omega\mathbf{1}$ *is isomorphic to* 2.

Proof. Assertion (1) is true for every complete lattice L. For $h : L \to 2$ and any lower set $S \subseteq L$ clearly $h(\bigvee S) = 1$ is equivalent to $S \cap h^{-1}(1) \neq \emptyset$. This shows that $h^{-1}(1)$ is completely prime. A characteristic map χ_U is join-preserving if and only if the set U is a completely prime upper set.

(2) If one represents a completely prime upper set $U \in M^{\text{⩕}}$ by its characteristic function, then it is immediate that $U \operatorname{Ialg}(h) x$ iff $h(x) \in U$ iff $\chi_U(h(x)) = 1$. Therefore Ialg acts on characteristic functions by pre-composition with the homomorphism h. functoriality of Ialg now follows from general categorical nonsense.

(3) We showed in Example 1 that $\operatorname{Ialg} 2$ is isomorphic to the interaction algebra $\mathbf{1}$. Its witness set $\{*\}$ has precisely two round lower sets, namely $\emptyset$ and $\{*\}$. Clearly $\{\emptyset, \{*\}\} \cong 2$. □

Recall that the set of filters of a poset is a dcpo under inclusion. Theorem 6.1.21 above tells us that the set $\operatorname{pt} L$ of completely prime filters of a completely distributive frame L is also a dcpo, and that the preimage of such a completely prime filter under a join-preserving map is a completely prime upper set. Thus we get an interesting restriction of the endofunctor $(-)^{\text{⩕}}$:

Theorem 6.1.23. 1. *The set* $\operatorname{pt} L$ *of completely prime filters of a completely distributive frame is a domain when ordered by inclusion. Its way-below relation is the restriction of the completely-below relation on completely prime upper sets.*

2. *The set of completely prime upper sets of* L *is order-isomorphic to the set of Scott closed subsets of* $\operatorname{pt} L$.

3. *A join-preserving map* $h : L \to M$ *between completely distributive frames induces a Scott continuous map* $h^{\text{⩕}}$ *from* $\operatorname{pt} M$ *into the Scott closed sets of* $\operatorname{pt} L$.

4. *A completely distributive frame* L *is isomorphic to the Scott topology on* $\operatorname{pt} L$*. Concretely, any* $x \in L$ *induces the Scott open set* $\{\phi \in \operatorname{pt} L \mid x \in \phi\}$.

In particular, the completely distributive frames are precisely the Scott topologies on domains.

We conclude our account of completely distributive frames with homomorphisms preserving the completely-below relation. Recall that the functor Up on the category of partial orders and monotone maps transforms a map h to the operation $\operatorname{Up}(h)(U) = \{y \mid \exists x.\, f(x) \leq y\}$. This map is the left adjoint to the preimage operation under h:

$$\operatorname{Up}(h)(U) \subseteq V \Leftrightarrow U \subseteq f^{-1}(V).$$

Proposition 6.1.24. *The following are equivalent for a homomorphism* $h : L \to M$ *in the category* CDFrm.

1. *h preserves the completely-below relation.*

2. *The map* Up(h) *restricts to a (join-preserving) map between completely prime upper sets of L and M.*

3. *The right adjoint to h (which always exists) preserves all joins.*

Proof. If h preserves the completely-below relation, then Up(h) restricts to completely prime upper sets. Indeed, if $U \subseteq L$ is a completely prime upper set and $x \in U$ then $U \ni y \lll x$ for some y. Now $h(y) \lll h(x)$ and so Up(h)(U) is completely prime.

If Up(h) restricts to completely prime upper sets, then it is left adjoint to the map $h^{\wedge\!\!\wedge} : M^{\wedge\!\!\wedge} \to L^{\wedge\!\!\wedge}$. Therefore $h \cong (h^{\wedge\!\!\wedge})^{\wedge\!\!\wedge}$ is left adjoint to $(\mathrm{Up}(h))^{\wedge\!\!\wedge}$ which is a join-preserving map.

Suppose $h \dashv g$ and $g : M \to L$ preserves all joins. Let $x_0 \lll x_1$ in L. We show $h(x_0) \lll h(x_1)$. Suppose $Y \subseteq M$ is a lower set and $h(x_1) \leq \bigvee Y$. Using the adjunction we can express this as $x_1 \leq g\left(\bigvee Y\right)$ and using the fact that g preserves joins we have $x_1 \leq \bigvee_{y \in Y} g(y)$. Now use $x_0 \lll x_1$ and deduce $x_0 \leq g(y)$ for some $y \in Y$. Using the adjunction once more we deduce $h(x_0) \leq y$ which shows $h(x_0) \lll h(x_1)$. □

6.1.11 Auxiliary relations and proximities

Definition 6.1.23. An *auxiliary relation* on a poset $(L, \sqsubseteq)$ is a relation $\prec$ satisfying the axioms (i) and (ii) of Table 4.1. The relation is called

interpolative if it satisfies axiom (v),

approximating if it satisfies axiom (vi).

Any auxiliary relation is transitive. Important examples of interpolative auxiliary relations are the way-below relation on domains and the completely-below relation on completely distributive frames.

Definition 6.1.24. Let L be a set with a binary transitive relation $\prec$. A *round ideal* with respect to $\prec$ is a subset $I \subseteq L$ with the properties

1. $x \prec y \in I$ implies $x \in I$,

2. For every finite subset $A \subseteq I$ there exists an element $y \in I$ such that $x \prec y$ for every $x \in A$.

The poset of round ideals of L, ordered by inclusion, is denoted by $\mathrm{Idl}^{\prec} L$.

Evidently the set of round ideals is a dcpo. Observe that in case $\prec$ is an auxiliary relation on a poset L, the poset of round ideals is a sub-dcpo of the ideal completion.

Basis embeddings

Definition 6.1.25. Let L be a poset with auxiliary relation $\prec$ on it. A map $e : B \to L$ is called a *basis embedding* with respect to $\prec$ if $x \prec y$ in L implies that there exists some $b \in B$ with $x \prec e(b) \prec y$.

An important instance is the case when $\prec$ is approximating, $B \subseteq L$ and e is the inclusion map.

Lemma 6.1.25. *Let D be a domain and $e : B \to D$ be a basis embedding with respect to $\ll$. On B define a relation via $b_0 \prec b_1$ iff $e(b_0) \ll e(b_1)$. then* $\mathrm{Idl}^{\prec} B \cong D$.

Proof. From Proposition 6.1.9 we know that $D \cong \mathrm{Idl}^{\ll} D$. Given a round ideal $I \in \mathrm{Idl}^{\ll} D$, the preimage $e^{-1}(I)$ is a round ideal with respect to $\prec$. Since e is a basis embedding, the set $e(e^{-1}(I))$ is cofinal in I. □

6.2 Information systems and abstract bases

6.2.1 Approximable mappings

Although the structural differences between information systems and abstract bases are marginal, the interpretation of their tokens is literally dual. While a token of an abstract basis is thought of as a basic point of a domain, a token of an information system is thought of as a basic Scott open set. Therefore it is even more notable that the morphisms in both categories, too, differ only in a seemingly insignificant detail. When looking at the definitions in [60] and [1] it surprises that the definitions are both written with the approximation relation in the “greater than”-style. In order to keep to the strict notation in this work, we find it convenient to define the relations contravariantly. This underlines our choice to present the morphisms of interaction algebras as dual to the maps in domain theory.

Definition 6.2.1. [55, Definition 2.18] An *information system* is a set X together with an idempotent binary relation $<$ on it. The morphisms between information systems are called *approximable mappings* and the category of information systems and approximable mappings is denoted by $\mathsf{Infosys}$. An approximable mapping $(X, <) \to (Y, <)$ is a relation $R : Y \to X$ which satisfies

(AM1) $R = R; <$,

(AM2) $<; R \subseteq R$,

(AM3) For all $x, x' \in X$ and $M \subseteq Y$ finite, $(\forall y \in M. yRx < x') \Rightarrow \exists y'. M < y'Rx'$

where $M < y'$ is shorthand for $\forall y \in M. y < y'$. Composition of morphisms is usual relational composition, where $<$ is the identity morphism of each object.

There are two important instances of the axiom (AM3) for approximable mappings. The case $M = \emptyset$ tells us that every token x' has a token y' with $y'Rx'$. The case where M is a singleton amounts to the inclusion $R \subseteq <; R$. Just as for interaction algebras, the round upper sets of tokens are those subsets $U \subseteq X$ with $x \in U \Leftrightarrow \exists x' < x. x' \in U$ and the completely distributive frame of round upper sets is denoted by $\mathrm{Up}^< X$. The upper closure of an arbitrary subset U is denoted by $\Uparrow U$, and similarly for lower closures and round lower sets. The set of all round ideals of tokens, ordered by inclusion, is written as $\mathrm{Idl}^< X$.

Definition 6.2.2. [1, Definition 2.2.27] An *abstract basis* is a set X together with an idempotent binary relation $\prec$ on it. The relation $\prec$ is directed on the left, meaning that for any finite $M \subseteq X$ and $x \in X$,

$$M \prec x \Rightarrow \exists x'. M \prec x' \prec x$$

where $M \prec x$ is shorthand for $\forall m \in M. m \prec x$. Abstract bases form the category Abs where morphisms are *approximable relations*. A morphism $(X, \prec) \to (Y, \prec)$ is a relation $R : Y \to X$ satisfying

(AR1) $R = R; \prec$,

(AR2) $\prec; R \subseteq R$,

(AR3) For all $x \in X$ and $M \subseteq Y$ finite, $(\forall y \in M. yRx) \Rightarrow \exists y'. M \prec y'Rx$

Composition of morphisms is usual relational composition, where $\prec$ is the identity morphism of each object.

Notice that the axiom (AM1) is the same as (AR1) and (AM2) is the same as (AR2). The requirement that $\prec$ is directed on the left can be seen as the manifestation of $\prec$ being an approximable relation from $(X, \prec)$ to $(X, \prec)$.

Clearly every abstract basis is an information system, but not every information system is an abstract basis, as Example 20 demonstrates. Every approximable relation is an approximable mapping: If $M \subseteq Y$ is a finite set and $\forall y \in M. yRx \prec x'$ then by (AR1) we have $\forall y \in M. yRx'$ and using (AR3) we get $M \prec y'Rx'$ for some token y'. Hence R satisfies axiom (AM3). Surprisingly, if $R : (X, \prec) \to (Y, <)$ is an approximable mapping and $(X, \prec)$ is an abstract basis, then R is an approximable relation. Indeed, if $\forall y \in M. yRx'$ then using (AM1) we obtain for every $y \in M$ a token $x_y \in X$ such that $yRx_y \prec x'$. Now use the fact that $\prec$ is directed on the left and obtain $\{x_y\}_{y \in M} \prec x \prec x'$ for some token x. Axiom (AM1) now yields $\forall y \in M. yRx \prec x'$ which makes it possible to apply (AM3) and get $M \prec y'Rx'$.

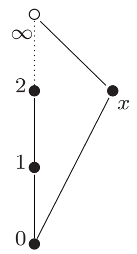
We have shown

Proposition 6.2.1. *The category* Abs *of abstract bases is a full subcategory of the category* $\mathsf{Infosys}$.

Despite one category being a proper subcategory of the other, they have the same expressive power:

Theorem 6.2.2. *Both the categories* Infosys *and* Abs *are equivalent to the category* Dom *of domains and Scott continuous maps via round ideals of tokens. For an information system* $(X, <)$ *the set* $\{\Uparrow\{x\} \mid x \in X\}$ *is a basis for the Scott topology on* $\mathrm{Idl}^< X$*. For an abstract basis* $(Y, \prec)$ *the set* $\{\Downarrow\{y\} \mid y \in Y\}$ *is a basis for the domain* $\mathrm{Idl}^\prec Y$*.*

Example 20. Consider the information system $\mathbb{N}+\{\infty, x\}$ depicted in the diagram below, where filled circles symbolise tokens which are below themselves.



The interpolation property of the relation $<$ holds trivially because all but one token are below themselves. But the round lower set of the token ∞ is $\mathbb{N} + \{x\}$ which is not an ideal. Therefore this information system is not an abstract basis.

However, the token ∞ is not bounded whence one could omit it and obtain an isomorphic information system. In fact, the domain that is represented by this information system has the same shape except that x is not below ∞.

6.3 Topology

Definition 6.3.1. A *topological space* is a pair $(X, \mathcal{O}X)$ where X is a set and $\mathcal{O}X$ is a sub-frame of the powerset of X. The elements of $\mathcal{O}X$ are called *open*. We denote the set of complements of open sets by ΛX and call these *closed*. The elements of $\mathcal{O}X \cap \Lambda X$ are called *clopen*. A function $f : X \to Y$ is called *continuous* if the preimage function $f^{-1} : \mathcal{P}Y \to \mathcal{P}X$ restricts to a frame homomorphism $\mathcal{O}Y \to \mathcal{O}X$. The category Top has topological spaces as objects and continuous functions as morphisms.

We use the symbol $\mathcal{O}$ to denote the contravariant functor Top $\to$ Frm that maps a space to its topology and a continuous function to the associated frame homomorphism f^{-1}. There is an obvious forgetful functor Top $\to$ Set whose left adjoint endows a set X with the *discrete topology* $\mathcal{P}X$.

Definition 6.3.2. A set $\mathcal{B} \subseteq \mathcal{O}X$ is called a *basis* if every open is a union of elements of $\mathcal{B}$. If $\mathcal{B} \subseteq \mathcal{O}X$ has the property that the set of finite intersections of elements of $\mathcal{B}$ is a basis, then $\mathcal{B}$ is called a *subbasis*.

Given any set $\mathcal{B} \subseteq \mathcal{P}X$ there is a smallest topology which has $\mathcal{B}$ as a subbasis, which we call the topology generated by $\mathcal{B}$.

Definition 6.3.3. For any point x of a topological space X, the open sets containing the point x are its *open neighbourhoods*. The collection of open neighbourhoods of x is a completely prime filter in $\mathcal{O}X$ and commonly denoted by $\mathcal{N}(x)$. Likewise, the open neighbourhoods of a set $Y \subseteq X$ are the open sets containing Y.

6.3.1 The specialisation order

Definition 6.3.4. Let X be a topological space. Define the *specialisation preorder* on X by

$$x \sqsubseteq y \;:\Leftrightarrow\; \forall U \in \mathcal{O}X.\, (x \in U \Rightarrow y \in U).$$

If $Y \subseteq X$ is a subset of X, then the *saturation* of Y is the set

$$\{x \in X \mid \exists y \in Y.\, y \sqsubseteq x\}$$

The set Y is called *saturated* if Y is equal to its saturation.

Lemma 6.3.1. *The specialization preorder on any space X is indeed a preorder. The saturation operation is a closure operator on the powerset of X. The saturation of a subset is the intersection of all open sets containing the subset. In particular, all open sets are saturated.*

Note that as all open sets are upper sets in the specialisation order, all closed sets are lower sets. In particular, the closure of a point is the principal down-set $\downarrow x$ in the specialisation preorder.

Proposition 6.3.2. *Every continuous map between topological spaces is monotone with respect to the specialisation preorders. Hence there is a forgetful functor* $\mathsf{Top} \to \mathsf{Preord}$.

Definition 6.3.5. Let X be a set with a preorder $\sqsubseteq$. The largest topology on X that has $\sqsubseteq$ as specialisation order is the (upper) *Alexandrov topology* consisting of all upper sets with respect to $\sqsubseteq$. The smallest topology that has $\sqsubseteq$ as specialisation preorder is the *weak upper topology* that has subbasic opens of the form $X \setminus \downarrow x$ where x ranges over the points of X.

6.3.2 Separation axioms

Definition 6.3.6. A topological space X is called

$\mathbf{T}_0$ if the specialisation preorder is antisymmetric, i.e. a partial order.

$\mathbf{T}_1$ if the specialisation preorder is equality.

T_2 or **Hausdorff** if any two distinct points can be separated by disjoint open neighbourhoods.

regular if every pair (x, A) where $x \notin A \in \Lambda X$ can be separated by a pair of disjoint open neighbourhoods.

T_3 if it is T_1 and regular.

completely regular if for any pair (x, A) as above there exists a continuous function $X \to [0, 1]$ with $f(x) = 0$ and $A \subseteq f^{-1}(\{1\})$.

Tychonoff if it is T_1 and completely regular.

normal if any pair of disjoint closed sets can be separated by disjoint open neighbourhoods.

T_4 if it is T_1 and normal.

Normal topological spaces

Notice that in the definition of normality no explicit reference to points is made, so it does not surprise that one can formulate normality as a property of the lattice of open sets:

Definition 6.3.7. A bounded lattice $(L, \sqcap, \sqcup, 0, 1)$ is *normal* if it obeys the following rule.

$$\frac{x \sqcup y = 1}{\exists x', y'.\, x \sqcup y' = 1,\; y' \sqcap x' = 0, x' \sqcup y = 1}$$

Lemma 6.3.3. *The following are equivalent for a topological space X.*

1. *The space X is normal.*
2. *The lattice $\mathcal{O}X$ of open sets is normal.*
3. *The well-inside relation on the powerset of X is interpolative.*

Proof. (1)⇔(2) Suppose A and B are closed sets of the topological space X. Let $u = X \setminus A$ and $v = X \setminus B$ be their open complements. Then A is disjoint from B if and only if u and v cover X. Hence normality of $\mathcal{O}X$ is equivalent to the assertion that there exist disjoint opens u' and v' with $A \subseteq v'$ and $B \subseteq u'$.

(1)⇔(3) By definition a set A is well inside a set B if the closure of A is contained in the interior of B. Equivalently, A is well inside B if the closure of A is disjoint from the closed complement of the interior of B. With this we see that normality implies the interpolation property of the well-inside relation. Conversely, two closed sets A and B are disjoint if and only if the set A is well inside the open complement of B. The interpolation property of the well-inside relation then yields normality. □

Some important properties of normal spaces are:

- Every compact Hausdorff space is normal.

- The Urysohn Lemma: A disjoint pair of closed subsets can be separated by a bounded real-valued function, that is a function which is constant zero on one closed set and constant one on the other.

- In contrast to weaker separation axioms T_n ($n < 4$) the subcategory of T_4 spaces is not a reflective subcategory of Top, that is, normal spaces are not closed under products and subspaces.

The Urysohn Lemma

The Urysohn Lemma states that normality implies a fact analogous to complete regularity: For any pair (A, B) of disjoint closed sets there exists a continuous function $f : X \to [0, 1]$ with $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$. The following is a sketch of the classical proof of the equivalent formulation stated in Lemma 4.1.4.

Through successive application of the interpolation property one expands $U_0 \lhd U_1$ to a *scale*, that is a dyadic-indexed chain $\{U_d\}_{d\in\mathbb{D}}$ where $d < e$ implies that U_d is well inside U_e. So far the conclusion is just that the well-inside relation on the topology of a normal space is the same as the really-inside relation. In the next step, using the scale one constructs an upper semicontinuous map $f_+ : X \to [0, 1]$ separating U_0 from U_1. What is localic about this step is that one actually constructs the frame homomorphism of this semicontinuous map first: The topology of upper semicontinuity on the unit interval has a basis of opens of the form $[0, x)$ for $x \in [0, 1]$. For every such open, define its preimage under f_+ to be the set $\bigcup_{d<x} U_d$. Likewise, one constructs a lower semicontinuous map $f_- : X \to [0, 1]$ separating U_0 from U_1, again via defining an appropriate frame homomorphism. In fact it is easier to think of this step as specifying the preimages of basic closed sets. Indeed, a basic closed set of the topology of lower semicontinuity is of the form $[0, x]$ for $x \in [0, 1]$. Define the preimage of this closed set under f_- to be $\bigcap_{d>x} U_d$. Why is this a closed set? Whenever a dyadic rational d is strictly greater than the real number x then there exists some dyadic e with $x < e < d$. As U_e is well inside U_d the closure of U_e is contained in U_d. Thus the intersection of opens is actually an intersection of closed sets, hence closed. The final step is to show that f_- and f_+ are actually the same function on points, whereby one obtains the continuous map with the desired properties.

Clearly, there is some bitopology lurking in the background, as one is dealing with a pair of semicontinuous maps. The central idea of the proof is that the dyadic numbers can be regarded as a basis of the topology of upper (or lower) semicontinuity on the unit interval, and specifying a frame homomorphism on a basis determines the frame homomorphism unambiguously.

6.4 Notation

List of Categories

Abs	Abstract bases and approximable relations
Alg	Algebraic domains and Scott continuous maps
BiFrm	Biframes and biframe homomorphisms
BiTop	Bitopological spaces and bitontinuous maps
CCL	Continuous lattices and join-preserving maps
CCSup	Continuous complete sup-lattices and non-emtpy join preserving maps
CFrm	Continuous frames and frame homomorphisms
CDFrm	Completely distributive frames and join-preserving maps
CL	Complete lattices and join-preserving maps
Coh	Coherent spaces and perfect maps
CPreFrm	Continuous preframes (also called continuous semilattices) and preframe homomorphisms
Dcpo	Dcpos and Scott continuous maps
dFrm	d-Frames and d-frame homomorphisms
dLat	d-Lattices and d-lattice homomorphisms
Dom	Domains and Scott continuous maps
Frm	Frames and frame homomorphisms
Infosys	Information systems and approximable mappings
KHaus	Compact Hausdorff spaces and continuous maps
KOrdHaus	Compact ordered Hausdorff spaces and monotone continuous maps
KRFrm	Compact regular frames and frame homomorphisms
KRdFrm	Compact regular d-frames and d-frame homomorphisms
Lat	Bounded distributive lattices and lattice homomorphisms
lcSob	Locally compact sober spaces and continuous maps
Loc	Locales and locale maps
NdFrm	Normal d-frames and d-frame homomorphisms
Poset	Posets and monotone maps
PreFrm	Preframes and preframe homomorphisms

Preord	Preorders and monotone maps
Rel	Sets and relations
SCFrm	Stably continuous frames and frame homomorphisms
SCTop	Stably compact spaces and perfect maps
Set	Sets and functions
sLat	(bounded) semilattices and semilattice homomorphisms
Sob	Sober spaces and continuous maps
Tok	Interaction algebras
Top	Topological spaces and continuous maps

List of Functors

Prefix notation	
Filt	Filter monad on Poset
$\text{Filt}^{\prec}$	Round filters with respect to relation $\prec$
Fin	Finite powerset monad on Set
Flip	Swap polarities in interaction algebras and d-lattices
Ialg	Interaction algebra derived from a domain, completely distributive frame or space
Idl	Ideal monad on Poset
$\text{Idl}_{\circ}$	Open ideal completion on dLat, dFrm
$\text{Idl}^{\prec}$	Round ideals with respect to relation $\prec$
Lo	Lower set monad on Poset
$\mathcal{O}$	Frame derived from a space, d-frame derived from a bitopological space
Ω	Round lower set functor $\mathsf{Tok}_0 \to \mathsf{CDFrm}$
$\mathcal{P}$	Powerset monad on Set
$\mathbb{P}_H$	Hoare poweralgebra comonad on Tok_1
$\mathbb{P}_S$	Smyth poweralgebra comonad on Tok_1
Patch	Patch construction on d-lattices and d-frames
pt	Domain derived from an interaction algebra, space derived from a frame/d-frame
Sym	Symmetric bitopological space over a space
Up	Upper set monad on Poset
$\vee$	Join topology functor on BiTop
Postfix notation	
$(-)^{\wedge}$	Scott open filters of a dcpo (Lawson dual); involution on CPreFrm
$(-)^{\wedge\!\!\wedge\!\!\wedge}$	Completely prime upper sets of a complete lattice; involution on CDFrm
$(-)^{\partial}$	Order dual on Preord and dLat
$(-)^{\leqslant}$	Normal coreflection on dLat, dFrm
$(-)_{=}$	Symmetric d-frame of a frame

List of relations and arrows

Order relations and associated closures

$\leq, \sqsubseteq$	$\downarrow, \uparrow$	Order on posets, specialisation order on spaces.
$\ll$	$\twoheaddownarrow, \twoheaduparrow$	Way-below relation on dcpos, see page 184.
$\lll$	$\twoheaddownarrow$	Completely-below relation on complete lattices, see page 188.
$\prec$	$\downarrow, \uparrow$	Composite relation on tokens of an interaction algebra, see pages 14,15.
$\preccurlyeq, \succcurlyeq$		Order of lower equivalence on tokens/witnesses of an interaction algebra, see page 30.
$\lhd$	$\downarrow, \uparrow$	Well-inside relation on a d-lattice, see page 97.
$\leqslant$		Really-inside relation on a d-lattice, see page 106.

Other relations

con	Consistency relation of a d-lattice, disjointness of sets, see page 95.
tot	Totality relation of a d-lattice, covering relation of sets, see page 95.
$\asymp$	Consistency relation of an interaction algebra, lower-bound-of relation between elements and subsets, see pages 9,13.
$\bigcirc$	Totality relation (identity morphism) of an interaction algebra, containment relation between sets and elements, see pages 9,13.
$\between$	Intersection relation between subsets. $A \between B$ if and only if $A \cap B \neq \emptyset$.

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