QUASI-CLASSICAL THEORY OF WEAKLY ANISOTROPIC SUPERCONDUCTORS

by

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Abstract

This thesis starts by reviewing superconductivity in one-dimension where fluctuations cause a loss of supercurrent due to an intrinsic resistance. Solved via the Ginzburg-Landau equations, the theory of thermally activated phase slips given by Langer and Ambegaokar is outlined. In turn this leads to the investigation of superconductivity via a microscopic approach, in particular the quasi-classic green's functions of Eilenberger.

The Eilenberger equations are derived and considered in the dirty and weakly anisotropic limits which provides a simple derivation of the Ginzburg-Landau equations near the transition temperature. This prompts an extended derivation which includes the non-linear terms normally removed in deriving the Ginzburg-Landau equations. This is required for calculating effects at temperatures below the transition temperature.

These quasi-classic equations of weakly anisotropic superconductors are first written for arbitrary temperature and impurity concentration then limited to the pure and dirty cases. The latter being simplified to zero temperature and solved in the context of thermally activated phase slips.

Dedicated to the many people who have aided my knowledge and understanding over the years.

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Chapter 1

THESIS OVERVIEW

1.1 Brief Overview

A reduced overview of the thesis suggests four main sections:

- The first of these outlines Langer and Ambegaokar's (LA) theory of thermally activated phase slips (TAPS). This is done to motivate recent work done using Eilenberger's equations, along with offering techniques which can be used in subsequent chapters.

- The second, much larger part, derives the Eilenberger equations and investigates their usage in areas where superconductivity has a slowly varying spatial dependence. This is the case for the problem of phase slips as well as many other areas of research.

- These investigations lead to the third part in which a quasi-classical extension of the Ginzburg-Landau (GL) equations is proposed. This method includes non-linear terms giving temperature effects below T_c and as such provides a quasi-classical theory of weakly anisotropic superconductors at all temperatures.

- Finally this theory is considered in the context of phase slips at zero temperature.

1.2 Introduction

Many existing texts give a good background in superconductivity and a short literature review of some of these which effected this thesis are provided. The style of the middle two parts of the thesis is such as to fit in with these sources, allowing for extensions and corrections to be made in a familiar manor.
For background phenomena and history the opening chapters of Tinkham [1] and Kopnin's [2] books are good primers.

Moving on to specifically dealing with Ginzburg-Landau theory, an excellent and brief introduction is given by Werthamer [3] and a more extensive overview by Tinkham [4].

Continuing with general theories, for the theory of Bardeen, Cooper and Schrieffer one ought to refer again to Tinkham [5] but since the main focus of this thesis stems from a Green's function approach we must consider sources such as, Fetter [6], Ketterson and Song [7][8], Parks' book has a section by Rickaysen [9] worth reading. Also Kopnin [10] where much of our method comes from. A further set up is found in Bennemann and Ketterson's book [11]. Some standard books of help include AGD [12], and the Landau and Lifshitz books are always nearby for many theorists [13][14].

When including impurities we draw heavily from, Kopnin [15] and Gorkov [16], with comparisons to Ambegaokar [17], AGD [18] and Ketterson and Song [19]. These should be treated with care since their results are limited, as will be explained later.

Finally, for general discussions of phase-slips and fluctuations in books please see, Tinkham [20][21], Larkin [22] and Tidecks [23]. Tidecks' text also contains a background to some experimental details not included in this thesis [24].

With this knowledge in mind we can sketch a logical path through the ideas in this thesis.

Thermally activated phase slips of the kind investigated by Langer and Ambegaokar [25] are the result of fluctuations in the magnitude of the superconducting order parameter. In regions where the amplitude dips to zero, the superconductor becomes a normal metal. In one dimension these boundaries cause portions of supercurrent to be lost. This accounts for the decay of persistent current observed by Little in ring samples [26].

In order to describe this phenomena a spatially dependent theory of superconductivity is required which allows for what can be viewed as a mix of superconducting and normal states. Ginzburg Landau [27] theory describes type-II superconductors and is particularly good at describing slowly varying systems near the critical temperature. These equations were used by LA to develop a theory of how the superconductor behaves in a region which is allowed to fluctuate whilst carrying a current.

Since the GL equations are found by minimising a free-energy functional, the energy cost to force a region into the normal state whilst carrying a current can be calculated. This cost gives the energy required for a thermally activated phase slip to occur and through the use of an Arrhenius law can give a resistance which is responsible for the decay of the current.

The limitations of this theory are due to the GL equations. Being derived near T_c they are simply not valid at lower temperatures. This is a problem because experimental observations show different behaviour in this regime [28]. This led to the need for the theory of TAPS to be revisited at via a microscopic theory. Based on pairings of electrons which cause conventional superconductivity this approach was valid in the whole temperature range and utilises the Eilenberger equations [29].

These equations are similar to those of Gorkov which describe BCS [30] theory through the use of correlation functions. However, Eilenberger used the fact that the superconductor exists in a small gapped region near the Fermi surface to approximate

the correlation functions to ones which have typical momenta given by p_F provided by the condition $\Delta \ll E_F$, i.e. the gapped region is small compared to the Fermi energy. It is these quasi-classicalal Green's functions which the Eilenberger equations give, and from these observable quantities may be calculated. Their solution for pure, one-dimensional systems gives an extension of the LA theory of TAPS to low temperatures. The impure version of these equations are non-linear and much harder to solve in general so approximations are made.

Usadel [31] considered the dirty limit of the Eilenberger equations, here the short mean free path smears out the anisotropy of the system. This led him to consider a form which included a velocity independent term and an additional term, linear in velocity and as such described a small anisotropic correction.

Impurities are not the only cause of this weak anisotropic consideration. Indeed any superconductor with a small spatial dependence could be pictured as being mainly isotropic with small corrections. Usadel could create a set of closed equations by using the symmetry of the impurity terms of the Eilenberger equations. If one were to consider any other cause of weak anisotropy as a perturbation from the isotropic case this would not be possible, since 'order' and not symmetry removes terms as will be seen in chapter 4.

Within the microscopic approach the order parameter is given by a self-consistency equation. For the quasi-classical approximation this is related to the average of the anomalous Green's function. If an expansion for anisotropy is to be considered, then we note that only the even terms would contribute to the order parameter. Indeed the first correction would result from the second order expansion. Kogan [32] used this idea to derive equations for moderately dirty superconductors from Eilenberger's equations. When considered near T_c these equations used in conjunction with the self-consistency equation reduce to those of Ginzburg and Landau.

This suggests a question:

> Can the quasi-classical theory of superconductors be reduced to a theory for slowly varying superconductors at all temperatures less than or equal to the critical temperature?

An answer to the question is given in chapter 5 where the corrections to the Ginzburg-Landau equations are given in full for quasi-classical systems.

These equations can be taken in the limit of $T \to T_c$ to reproduce the GL equations at any impurity concentration. They can also be taken to $T = 0$, though this is only done for the impure case.

This forms the basis for calculating the effects of impurities on thermally activated phase slips at low temperature.

Chapter 2

THERMALLY ACTIVATED PHASE-SLIPS, TAPS

2.1 Introduction

Motivated by Little, the decay of persistent currents is discussed and the calculations of Langer and Ambegaokar reproduced. This is done to clarify points in their calculation, and to provide a method for us to use later in the thesis.

This is then re-examined when low temperature results suggest quantum effects take over. The microscopic theory of thermal phase slips in pure systems presented by Zharov [33] is then outlined suggesting that quasi-classical green's functions may aid calculations far below the critical temperature.

2.2 A Little Idea; The Decay of Persistent Currents

An open problem in the 1960's was that of the lifetime of persistent currents in superconducting rings. Superconductors had already been formalised at this point by both the phenomenological theory of Ginzburg and Landau (GL), valid at the transition temperature and with a microscopic theory attributed to Bardeen, Cooper and Schrieffer (BCS), which described conventional superconductivity in the whole temperature range via the pairing of electrons.

Both theories predicted near infinite lifetimes for the supercurrent in bulk samples since the superconducting resistance was zero. The ring samples did not seem to show this. Indeed this reduced geometry showed a resistance, which was first investigated mathematically by William Little. He suggested that the cause might be thermal fluctuations which block the conducting channel.

Shortly afterwards the theory was developed by Langer and Ambegaokar (LA) in a one dimensional analogue to the ring. We work with the latter theory since it gives certain methods which will be used later in the thesis as such we consider the following model of a wire shown in Figure (2.1),

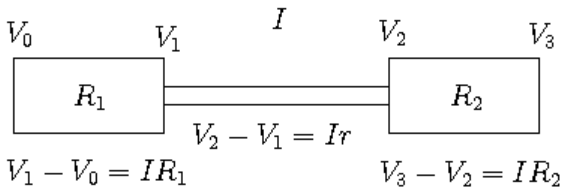

Figure 2.1: Simplified wire set up

Controlling the potential difference across the leads can make a current flow through the wire, there are two possibilities:

$$V_1 = V_2 \Rightarrow r = 0 \tag{2.2.1}$$

$$V_1 \neq V_2 \Rightarrow r \neq 0. \tag{2.2.2}$$

One might assume in the superconducting regime the internal resistance of the wire would be zero, this is not observed near the critical temperature for thin wires (cite parks and groff, etc). In order to understand this we should understand what 'thin' means for a wire, and what can occur near the critical temperature.

For these superconducting samples a thin wire is one for which the two perpendicular dimensions are less than both the coherence length, ξ , and the penetration depth, λ. These natural length scales of superconductors describe the size over which the superconductor can vary and how far an applied field can penetrate respectively. When taken near the critical temperature, regions of the wire can fluctuate into the normal state. Allowing this to happen in the wire would effect a volume $\sim l^2\xi$, in the bulk material it would effect a region $\sim \xi^3$. For a thin wire, with $l < \xi$, the volume is smaller, as such the energy cost to fluctuate a section of wire would be lower than in the bulk making the transition thermodynamically accessible. This region would provide a blockage to the conducting channel, shown in Figure (2.2).

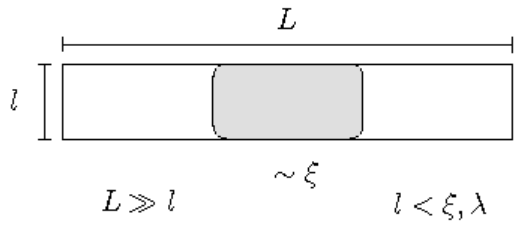

Figure 2.2: Bubble blocking conducting channel

Next we should consider the effect of a potential difference across the superconductor. Josephson's relation is given by,

$$\frac{\partial \Delta\phi_{1,2}}{\partial t} = \frac{2e}{\hbar}\Delta V_{1,2} \tag{2.2.3}$$

with no potential difference we see that the phase difference is constant. However, if there is a potential difference this is shown by an increasing change in the phase

difference shown in Figure (2.3).

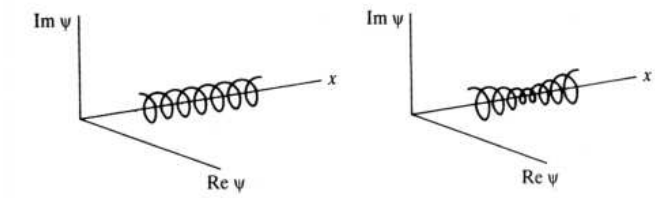

Figure 2.3: Uniform and non-uniform solutions

The energy supplied to the superconductor is taken up by the supercurrent which increases until the critical current is reached. As a steady, uniform state is observed there must be a way to decrease the supercurrent, some intrinsic resistance must exist. The resolution is that at the point where the superconductors amplitude drops to zero due to the fluctuation, an integer number of loops can be removed from the phase. This phase slip is the cause of the resistance which Langer and Ambegaokar calculate and based on the above it would have an energy cost $\sim l^2\xi\frac{H_c^2}{8\pi}$.

2.3 Langer Ambegaokar Theory

2.3.1 Intrinsic Resistive Transitions

In order to explain the resistance needed to reduce the lifetime of the persistent currents LA considered current carrying states to be meta-stable and that a fluctuation can force the system from a state of high current, high energy to one of a lower current at a lower energy.

The rate at which such an event could occur would be given by an Arrhenius law in which some guessed frequency was multiplied by the probability. The probability coming from the Boltzmann factor, k_B, given in terms of the GL free energy.
The two current carrying solutions and the energy needed to force the system between them are therefore given by solutions to the GL equations.

We start from the Ginzburg-Landau free energy in one dimension,

$$F[\psi] = \sigma \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left[\left| \frac{d\psi}{dx} \right|^2 - \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \right] \tag{2.3.1}$$

in terms of the cross-section, σ of a length of wire, L centred on the fluctuation, along with the corresponding equation,

$$\frac{d^2\psi}{dx^2} + \alpha\psi - \beta|\psi|^2\psi = 0. \tag{2.3.2}$$

The complex order parameter is chosen to have the form, $\psi = f(x)e^{i\phi(x)}$ which gives equations for the amplitude and phase,

$$\frac{d^2 f}{dx^2} - f\left(\frac{d\phi}{dx}\right)^2 + \alpha f - \beta f^3 = 0 \tag{2.3.3}$$

$$2\frac{d\phi}{dx}\frac{df}{dx} + f\frac{d^2\phi}{dx^2} = 0. \tag{2.3.4}$$

Multiplication by f in the latter gives a conserved quantity interpreted as the current,

$$\frac{d}{dx}\left[f^2\frac{d\phi}{dx}\right] = 0 \tag{2.3.5}$$

$$f^2\frac{d\phi}{dx} = \text{const} = J \tag{2.3.6}$$

which can be included in (2.3.3),

$$\frac{d^2 f}{dx^2} - \frac{J^2}{f^3} + \alpha f - \beta f^3 = 0. \tag{2.3.7}$$

We picture the order parameter in modulus argument form where the phase is related to the current and the higher the number of loops, the higher the current.

A uniform current carrying solution can be visualised as a cylindric helix along the wire given by,

$$f(x) = f_k \qquad ; \qquad \phi(x) = kx \tag{2.3.8}$$

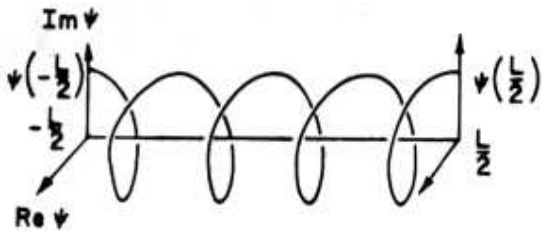

Figure 2.4: LA picture representing uniform solution [25]

Substituting this into (2.3.3) and (2.3.4) gives,

$$f_k^2 = \frac{\alpha - k^2}{\beta} \tag{2.3.9}$$

$$J = f_k^2 k = \frac{\alpha - k^2}{\beta} k. \tag{2.3.10}$$

This can be used to calculate the maximum current the superconductor can have before the superconductivity is destroyed $\frac{\partial J}{\partial k}\big|_{k=k_c} = 0 \Rightarrow J_c = \sqrt{\frac{\alpha}{3}}$. If the current is below this we can picture a local minimum corresponding to a steady current carrying solution as shown in the figure above. A barrier exists for the free energy to pass between uniform states with different values of current, the height of this barrier is the energy needed for the fluctuation to lower the current, i.e. form a resistance.

Josephson gave a relation for the change in the phase difference across a superconductor to a potential applied across it,

$$\frac{\partial \Delta\phi}{\partial t} = \frac{2e}{\hbar} \Delta V$$

If V = const then one would expect an increase in the phase difference. This is pictured as the helix tightening corresponding to an increasing current, which if left unchecked would pass J_c and take the material out of the superconducting phase.

If fluctuations which lower the current are matched to this increase, then a steady solution can be achieved. These fluctuations are known as phase slips for obvious reasons.

We can picture the free energy landscape as containing two minima with a barrier between them. In terms of the wave number k we denote the higher current carrying solution ψ_k, the barrier, $\delta F_0(k)$ and the lower state, $\psi_{k-\frac{2\pi}{L}}$ and the free energy difference between the minima δF_1, shown below in Figure (2.5)

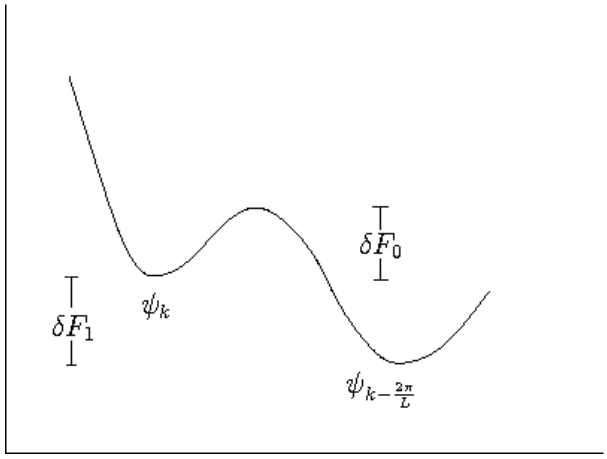

Figure 2.5: A picture showing the free energy landscape

The free energy for a uniform current, k, is given by,

$$\begin{aligned} F[\psi_k] &= \sigma \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left[k^2 f_k^2 - \alpha f_k^2 + \frac{\beta}{2} f_k^4 \right] \\ &= \sigma L \left[(k^2 - \alpha) f_k^2 + \frac{\beta}{2} f_k^4 \right]. \end{aligned} \tag{2.3.11}$$

From which we can find the variation of F with k

$$\frac{dF}{dk} = \frac{\partial F}{\partial k} = 2\sigma L k f_k^2 = 2\sigma L J \tag{2.3.12}$$

and thus the difference between energy states,

$$\delta F_1 = \frac{dF}{dk}\frac{2\pi}{L} = 4\pi\sigma J. \tag{2.3.13}$$

The rates for moving between states given by the Arrhenius law in terms of the expected rate of occurrence, Ω,

$$Rate\left(k \rightleftharpoons k - \frac{2\pi}{L}\right) \cong \Omega \exp\left(-\frac{\delta F_0}{k_B T} \pm \frac{\delta F_1}{2k_B T}\right) \tag{2.3.14}$$

between the two minima respectively. This gives the overall rate of change, given by the difference and simplified by the Josephson relation,

$$\begin{aligned} \frac{2e}{\hbar}V &= 2\Omega \sinh\left(\frac{\delta F_1}{2k_B T}\right)\exp\left(-\frac{\delta F_0}{k_B T}\right) \\ V &= \frac{\hbar\Omega}{e}\sinh\left(\frac{2\pi\sigma J}{k_B T}\right)\exp\left(-\frac{\delta F_0}{k_B T}\right). \end{aligned} \tag{2.3.15}$$

The electrical current I is given by,

$$I = \frac{4e\sigma J}{\hbar}$$

such that,

$$R = \frac{V}{I} = \frac{\hbar\Omega}{Ie}\sinh\left(\frac{\pi\hbar I}{2ek_B T}\right)\exp\left(-\frac{\delta F_0}{k_B T}\right)$$

and in the limit that the current tends to zero,

$$R = \Omega \left(\frac{\pi \hbar^2}{2e^2 k_B T} \right) \exp \left(-\frac{\delta F_0(J \to 0)}{k_B T} \right). \tag{2.3.16}$$

Now we have a form for the resistance, we must calculate the height of the barrier.

2.3.2 The Free-Energy Barrier

Returning to equation (2.3.7), we attempt a solution in analogy to the classical central force problem of mechanics we form an effective potential by multiplying (2.3.7) by $\frac{df}{dx}$ and pulling out the derivative and integrating,

$$\frac{d}{dx} \left[\frac{1}{2} \left(\frac{df}{dx} \right)^2 + \frac{\alpha}{2} f^2 - \frac{\beta}{4} f^4 + \frac{J^2}{2f^2} \right] = 0 \tag{2.3.17}$$

$$\left(\frac{df}{dx} \right)^2 + \alpha f^2 - \frac{\beta}{2} f^4 + \frac{J^2}{f^2} = \text{const} \tag{2.3.18}$$

Plotting the effective potential we see the curve in Figure (2.6),

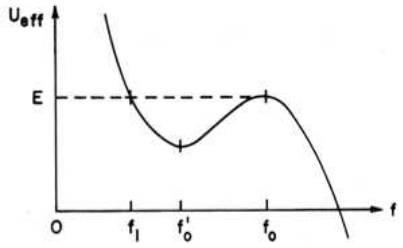

Figure 2.6: LA image showing the barrier [25]

We therefore set the constant by using our uniform solution marked by the line E. Looking at the picture we see that we start at f_0 travel toward the origin and hit the line at f_1 and move back to f_0. In the mechanical analogue we picture a circular orbit spiralling in and back out. In a similar way to spinning on a desk chair swinging your legs in an out, this would vary the angular speed. Here the corresponding physics is that the phase would wind up more quickly as the superconducting density is

reduced. This will be taken account of below.

Returning to the constant, we see that

$$\begin{aligned}\text{const} &= \alpha f_k^2 - \frac{\beta}{2} f_k^4 + \frac{J^2}{f_k^2} \\ &= (\alpha + k^2) f_k^2 - \frac{\beta}{2} f_k^4 \\ &= \frac{1}{\beta}(\alpha^2 - k^4) - \frac{1}{2\beta}(\alpha - k^2)^2 \\ &= \frac{1}{2\beta}(\alpha^2 - 3k^4 + 2\alpha k^2) \\ &= E\end{aligned}$$

$$\left(\frac{df}{dx}\right)^2 = E - \alpha f^2 + \frac{\beta}{2} f^4 - \frac{J^2}{f^2}. \tag{2.3.19}$$

In order to correspond to the LA calculation we change variables in the same way, and therefore define $f^2 = \frac{\alpha}{\beta} u$,

$$\left(\frac{du}{dx}\right)^2 = \frac{4\beta E}{\alpha} u - 4\alpha u^2 + 2\alpha u^3 - \frac{4J^2\beta^2}{\alpha^2} \tag{2.3.20}$$

along with,

$$\epsilon = \frac{\beta E}{\alpha^2} \qquad ; \qquad \frac{2J^2\beta^2}{\alpha^3} = \frac{8}{27} j^2$$

allowing us to write the equation as,

$$\frac{1}{2\alpha}\left(\frac{du}{dx}\right)^2 = u^3 - 2u^2 + 2\epsilon u - \frac{8}{27} j^2 \tag{2.3.21}$$

In order to solve this equation we use the fact that the equation is cubic with a repeated root at the uniform solution, $u = u_0$ we write

$$u^3 - 2u^2 + 2\epsilon u - \frac{8}{27}j^2 = (u - u_1)(u_0 - u)^2$$
$$= u^3 - 2u^2(u_1 + 2u_0) + u(2u_0u_1 + u_0^2) - u_0^2u_1$$
$$u_1 + 2u_0 = 2 \tag{2.3.22}$$
$$2u_0u_1 + u_0^2 = 2\epsilon \tag{2.3.23}$$
$$u_0^2u_1 = \frac{8}{27}j^2. \tag{2.3.24}$$

In terms of these points the equation reads,

$$\frac{1}{2\alpha}\left(\frac{du}{dx}\right)^2 = (u - u_1)(u_0 - u)^2. \tag{2.3.25}$$

Changing variable to around the non-uniform solution, $u = u_1 + t^2$ we find,

$$\frac{1}{2\alpha}\left(\frac{dt}{dx}\right)^2\left(\frac{du}{dt}\right)^2 = t^2(u_0 - u_1 - t^2)^2 \tag{2.3.26}$$
$$\sqrt{\frac{2}{\alpha}}\frac{dt}{dx} = u_0 - u_1 - t^2 \tag{2.3.27}$$

allowing for a simple integration

$$t = \sqrt{(u_0 - u_1)}\tanh\left(\sqrt{\frac{\alpha}{2}(u_0 - u_1)}x\right) \tag{2.3.28}$$

for the variable u this reads,

$$u - u_1 = (u_0 - u_1)\tanh^2\left(\sqrt{\frac{\alpha}{2}(u_0 - u_1)}x\right). \tag{2.3.29}$$

Defining, $\Delta = u_0 - u_1$ we can re-cast this as,

$$u = u_0 - \Delta \text{sech}^2\left(\sqrt{\frac{\alpha}{2}}\Delta x\right) \tag{2.3.30}$$

whereby using equations (2.3.22) and (2.3.24) we can find an equation for Δ,

$$u_1 + 2u_0 = 2 \Rightarrow u_1 = 2(1 - u_0)$$

$$\begin{aligned}\Delta &= u_0 - u_1 \\ &= u_0 - 2(1 - u_0) \\ u_0 &= \frac{\Delta + 2}{3}\end{aligned}$$

$$\begin{aligned}u_1 &= u_0 - \Delta \\ &= \frac{\Delta + 2}{3} - \Delta \\ &= \frac{2}{3}(1 - \Delta)\end{aligned}$$

$$\begin{aligned}u_0^2 u_1 &= \frac{8}{27}j^2 \\ \left(\frac{\Delta + 2}{3}\right)^2 \frac{2}{3}(1 - \Delta) &= \frac{8}{27}j^2 \\ (\Delta + 2)^2(1 - \Delta) &= 4j^2\end{aligned}$$

We have then,

$$f^2 = \frac{1}{\beta}(\alpha - k^2) - \Delta\frac{\alpha}{\beta}\text{sech}^2\left(\sqrt{\frac{\alpha}{2}}\Delta x\right) \tag{2.3.31}$$

$$\frac{d\phi}{dx} = \frac{J}{f^2} \tag{2.3.32}$$

$$(\Delta + 2)^2(1 - \Delta) = 4j^2 \tag{2.3.33}$$

The function Δ can be expressed as a function of k if one wishes. As we said before, in analogue to the mechanical problem when the orbital separation decreases the angular momentum increases. So here the supercurrent increases as the density of superconducting electrons decreases as given by the change in phase,

$$\begin{aligned}
\Delta\phi &= \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left(\frac{d\phi}{dx}\right) \\
&= J\int_{-\frac{L}{2}}^{\frac{L}{2}} dx \frac{1}{f^2} \\
&= \frac{2\beta J}{\alpha}\int_{0}^{\frac{L}{2}} dx \frac{1}{u} \\
&= \frac{2\beta J}{\alpha}\int_{0}^{\frac{L}{2}} dx \left[\frac{1}{u}+\frac{1}{u_0}-\frac{1}{u_0}\right] \\
&= \frac{\beta JL}{\alpha u_0} + \frac{2J\beta}{\alpha}\int_{u_1}^{u_0} du \left(\frac{dx}{du}\right)\frac{u_0-u}{uu_0} \\
&= \frac{\beta JL}{\alpha u_0} + \frac{2J\beta}{\alpha u_0\sqrt{2\alpha}}\int_{u_1}^{u_0} du \frac{1}{u\sqrt{(u-u_1)}} \\
&= \frac{\beta JL}{\alpha u_0} + \frac{2J\beta}{\alpha u_0\sqrt{2\alpha}}\left[\frac{2}{\sqrt{u_1}}\tan^{-1}\left(\frac{\sqrt{u-u_1}}{\sqrt{u_1}}\right)\right]_{u_1}^{u_0} \\
&= \frac{\beta JL}{\alpha u_0} + \frac{2J\beta}{\alpha u_0\sqrt{2\alpha}}\frac{2}{\sqrt{u_1}}\tan^{-1}\left(\frac{\sqrt{u_0-u_1}}{\sqrt{u_1}}\right)
\end{aligned} \tag{2.3.34}$$

Using

$$\begin{aligned}
u_1 &= \frac{2}{3}(1-\Delta) \\
u_0\sqrt{u_1} &= \sqrt{\frac{8}{27}j^2} = \sqrt{\frac{2J^2\beta^2}{\alpha^3}}
\end{aligned}$$

this can be simplified to

$$\Delta\phi = \frac{JL}{f_0^2} + 2\tan^{-1}\left(\sqrt{\frac{3\Delta}{2(1-\Delta)}}\right) \tag{2.3.35}$$

whose terms correspond to the uniform and non-uniform parts. The change in wave-number between a state with current J_i, where the subscript denotes the minima

for the constant current, and the uniform solution J_0 is given by δk,

$$\delta k = k_i - k_0 = \frac{2}{L}\tan^{-1}\left(\sqrt{\frac{3\Delta}{2(1-\Delta)}}\right) \tag{2.3.36}$$

which is of order L^{-1}, and can be used to find the decrease in current. We use this instead of the previously used, $\delta k = \frac{2\pi}{L}$ to include the effects of the current. This allows us to calculate the free energy associated with travelling between states k_i and $k_i - \frac{2\pi}{L}$. In order to make the calculation easier we first simplify the free energy functional by integrating the first term by parts, and then substituting in equation (2.3.2)

$$\begin{aligned} F[\psi] &= \sigma \int dx \left[\left|\frac{d\psi}{dx}\right|^2 - \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4\right] \\ &= \sigma \underbrace{\left[\psi^* \frac{d\psi}{dx}\right]}_{=0} + \sigma \int dx \left[-\psi^* \frac{d^2\psi}{dx^2} - \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4\right] \\ &= \sigma \int dx \left[-\psi^*(-\alpha\psi + \beta|\psi|^2\psi) - \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4\right] \\ &= -\frac{\sigma\beta}{2}\int dx|\psi|^4 \end{aligned} \tag{2.3.37}$$

The height of the barrier is given by,

$$\delta F = -\frac{\sigma\beta}{2}\int dx[f^4(x,J) - f_i^4(J_i)] \tag{2.3.38}$$

since $\delta k \sim \mathcal{O}(L^{-1})$ we taylor expand and keep to linear terms in δk,

$$f_i^4(J_i) = f_0^4(J) + \frac{\partial f_0^4}{\partial k_0}\delta k \ldots$$

and

$$\frac{\partial f_0^4}{\partial k_0} = \frac{d}{dk_0}\left[\left(\frac{(\alpha - k^2)}{\beta}\right)^2\right] = -\frac{4J}{\beta}$$

allowing

$$\delta F = \frac{\sigma\beta}{2}\int dx[f_0^4(J) - f^4(J_i)] - 4J\sigma\tan^{-1}\left(\sqrt{\frac{3\Delta}{2(1-\Delta)}}\right) \tag{2.3.39}$$

This can be integrated and the result written in terms of the free energy per unit volume, $\frac{H_c^2}{8\pi}$, and coherence length, $\xi(T)$

$$\delta F = \frac{8\sqrt{2}}{3}\sigma\xi(T)\frac{H_c^2}{8\pi}\left[\sqrt{\Delta} - \sqrt{\frac{2}{3}}\left(\frac{J}{J_c}\right)\tan^{-1}\left(\sqrt{\frac{3\Delta}{2(1-\Delta)}}\right)\right] \tag{2.3.40}$$

and in the limit that $J \to 0$ we find $\Delta = 1$ and the barrier is given by,

$$\delta F_0 = \frac{8\sqrt{2}}{3}\sigma\xi(T)\frac{H_c^2}{8\pi} \tag{2.3.41}$$

such that the resistance is,

$$R = \Omega\left(\frac{\pi\hbar^2}{2e^2k_BT}\right)\exp\left(-\frac{8\sqrt{2}}{3}\frac{\sigma\xi(T)}{k_BT}\frac{H_c^2}{8\pi}\right) \tag{2.3.42}$$

2.3.3 Simple Free Energy Barrier Calculation

For the zero current case we present a simple calculation of the free energy barrier. Starting with the free energy functional,

$$\begin{aligned} F[f] &= \sigma\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\left[\left(\frac{df}{dx}\right)^2 - \alpha f^2 + \frac{\beta}{2}f^4\right] \\ &= -\frac{\sigma\beta}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx f^4 \end{aligned}$$

and the equation,

$$\frac{d^2f}{dx^2} + \alpha f - \beta f^3 = 0$$

$$\left(\frac{df}{dx}\right)^2 + \alpha f^2 - \frac{\beta}{2} f^4 = \text{const}$$

The uniform case, $f_0 = \text{const} = \sqrt{\frac{\alpha}{\beta}}$ is very useful. First we can use it to set the constant in the second equation which becomes,

$$\frac{df}{dx} = \sqrt{\frac{\beta}{2}} \left(f^2 - f_0^2\right).$$

Secondly it can be used to de-dimensionalise the first equation which leads to defining a natural unit for length, the coherence length, $\xi = \frac{1}{\sqrt{\alpha}}$. Finally it can be used to calculate the condensation energy

$$\frac{H_c^2}{8\pi} = \frac{F[f_0]}{\sigma L} = -\frac{\alpha^2}{2\beta}.$$

We may now calculate the free energy barrier quite simply by changing the integration element to f,

$$\begin{aligned}
\delta F &= F[f] - F[f_0] \\
&= -\frac{\sigma\beta}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \left[f^4 - f_0^4\right] \\
&= -\sigma\beta \int_0^{f_0} df \frac{dx}{df} \left[f^4 - f_0^4\right] \\
&= -\sigma\sqrt{2\beta} \int_0^{f_0} df \left[\frac{f^4 - f_0^4}{f^2 - f_0^2}\right] \\
&= -\sigma\sqrt{2\beta} \int_0^{f_0} df \left(f_0^2 + f^2\right) \\
&= -\sigma\sqrt{2\beta} \Big[f_0^2 f + \frac{f^3}{3}\Big]_0^{f_0} \\
&= -\sigma\sqrt{2\beta} \left(\frac{4}{3} f_0^3\right) \\
&= -\sigma\sqrt{2\beta} \left(\frac{4}{3} \sqrt{\frac{\alpha^3}{\beta^3}}\right) \\
&= -\frac{4}{3}\sqrt{2}\sigma \frac{\alpha^2}{\beta} \frac{1}{\sqrt{\alpha}}
\end{aligned}$$

$$= \frac{8\sqrt{2}}{3}\sigma\xi\frac{H_c^2}{8\pi} \tag{2.3.43}$$

we will use a similar procedure later on.

In both cases we find that the free energy is proportional to the volume of a 'bubble' of the normal metallic state, sitting in the superconducting wire giving an effective volume for the fluctuation of $\frac{8\sqrt{2}}{3}\sigma\xi$.

For the zero current solution we have from equation (2.3.31) that $f = \sqrt{\frac{\alpha}{\beta}}\tanh\left(\sqrt{\frac{\alpha}{2}}x\right)$ which is shown in figure (2.7)

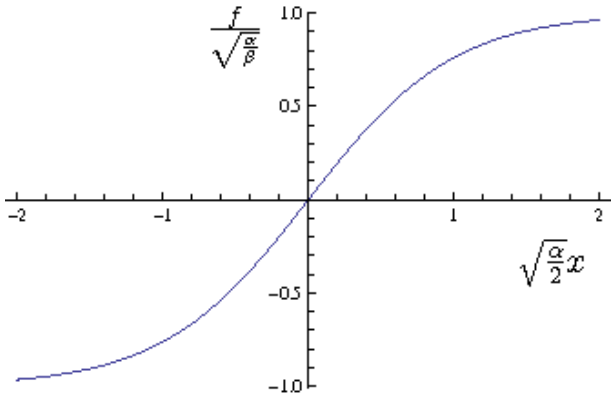

Figure 2.7: Phase slip for zero current

2.4 Beyond LA; LAMH

Although the initial results of LA theory were good it soon became apparent that the guessed frequency was incorrect.

We will not go into detail here on the exact methods used by McCumber and Halperin [34] other than to say they used a time dependent form of the Ginzburg-Landau equations to calculate gaussian fluctuations around the LA saddlepoint solution to find a more reliable form for the resistance in which they reduced the frequency of events by more than 10 orders of magnitude.

2.5 Quantum Phase Slips: QPS

Once again this work lasted a long time with great success. But being based on the GL equations, it does have limitations.

Changes in the form of the resistance at much lower temperatures lead people to consider the role of quantum-phase-slips QPS. Again these cause fluctuations in the magnitude of the order parameter, which if allowed to tend to zero force the superconductor into the normal state. This state will provide a resistance analogous to that of TAPS.

2.6 Microscopic Theory of TAPS

QPS are not going to be treated in this thesis, instead an attempt will be made to construct a microscopic theory of the spatial dependence of the order parameter and not focus on the cause of the fluctuation driving it. This is done after the work of Zharov *et al* [33] who solved the pure situation exactly. We now outline their work.

Following LAMH theory they consider the exact solution of the Eilenberger equations in one dimension. All the equations in this section will be derived more fully later in the thesis, for now we present a stripped down version of their paper to give an idea of how the microscopic theory would be used to solve the problem of TAPS. The pure Eilenberger equations read,

$$v_F \frac{dg}{dx} + \Delta f^\dagger - \Delta^* f = 0 \tag{2.6.1}$$

$$-v_F \frac{df}{dx} - 2\omega_n f + 2\Delta g = 0 \tag{2.6.2}$$

$$v_F \frac{df^\dagger}{dx} - 2\omega_n f^\dagger + 2\Delta^* g = 0 \tag{2.6.3}$$

and satisfy satisfy the non-linear relationship $g^2 + ff^\dagger = 1$. Here f and g are the quasi-classical Green's functions. The Fermi-velocity is $v_F = \pm|v_F|$ and the

Matsubara frequencies are given by $\omega_n = \pi T(2n+1)$, in units where $\hbar, k_B = 1$.

To make the equations easier to work with they rescale variables, $T \to \frac{T}{T_c}$, split the order parameter to real and imaginary parts and allow for a weak current to flow in the form, $\Delta, f \to \Delta e^{ikx}, f e^{ikx}$. Defining the functions $f_\pm = (f \pm f^\dagger)/2$ they recast the Eilenberger equations in the form they would like to solve,

$$\frac{dg}{dx} + 2i\Delta_I f_+ - 2\Delta_R f_- = 0 \tag{2.6.4}$$

$$\frac{df_-}{dx} - 2\Delta_R g + 2\omega_n' f_+ = 0 \tag{2.6.5}$$

$$\frac{df_+}{dx} - 2i\Delta_I g + 2\omega_n' f_- = 0 \tag{2.6.6}$$

where $\omega_n' = \omega_n + ik/2$ includes the weak current.

The self consistency equation needed to calculate the order parameter is given by,

$$\frac{\Delta_R}{\lambda} = \pi T \sum_{\omega_n} \mathrm{Re} f_+(\omega_n, x) \tag{2.6.7}$$

$$\frac{\Delta_I}{\lambda} = \pi T \sum_{\omega_n} \mathrm{Im} f_-(\omega_n, x) \tag{2.6.8}$$

In a similar manor to LA they solve for the homogeneous current carrying case in order to match far from the inhomogeneity. Here they use $g^{(0)}, f_+^{(0)}, f_-^{(0)} = \text{consts}$ and $g^{(0)2} + f_+^{(0)2} - f_-^{(0)2} = 1$ which gives,

$$g^{(0)} = S_n \omega_n' \qquad f_+^{(0)} = S_n \Delta_{R0} \qquad f_-^{(0)} = i S_n \Delta_{I0}^*$$

where $S_n = (\omega_n'^2 + |\Delta_0|^2)^{-\frac{1}{2}}$.

The order parameter and supercurrent calculations give,

$$\pi T \sum_{\omega_n} \left(\mathrm{Re} \frac{1}{\sqrt{\omega_n'^2 + |\Delta_0|^2}} - \frac{1}{|\omega_n|} \right) = \ln T \tag{2.6.9}$$

$$j = -2eT \sum_{\omega_n} \mathrm{Im} \frac{\omega_n'}{\sqrt{\omega_n'^2 + |\Delta_0|^2}} \tag{2.6.10}$$

After manipulation of the Eilenberger equations one can show simply that,

$$i\Delta_I \frac{df_-}{dx} - \Delta_R \frac{df_+}{dx} - \omega_n' \frac{dg}{dx} = 0. \tag{2.6.11}$$

It is at this point that the authors make some strong assumptions, they assume that the imaginary part of the order parameter is constant and that $f_+ = S_n \Delta_R$. There are more general solutions for the pure Eilenberger equations, for instance they can be transformed into a scalar Riccati equation [35], but this is limited when considering impurities as such we stick to the assumptions proposed. These give a first integral via equation (2.6.11). The constant of integration is given in terms of the homogeneous result and upon rewriting gives,

$$\omega_n' g - i\Delta_I f_- = S_n \left(\omega_n'^2 + \Delta_I^2 + \frac{1}{2}(\Delta_{R0}^2 - \Delta_R^2) \right). \tag{2.6.12}$$

Differentiating the Eilenberger equation for $\frac{df_+}{dx}$ and substituting in the other equations allows us to use the above result along with the assumption that $f_+ = S_n \Delta_R$ to provide an equation for Δ_R,

$$\frac{d^2 \Delta_R}{dx^2} + 2\Delta_{R0}^2 \Delta_R - 2\Delta_R^3 = 0 \tag{2.6.13}$$

which bares resemblance to the Ginzburg-Landau equation, being second order and

having cubic and linear terms. It solves to give,

$$\Delta_R = \Delta_{R0}\tanh(\Delta_{R0}x). \tag{2.6.14}$$

The general results for the green's functions are now expressible as,

$$f_+ = S_n\Delta_{R0}\tanh(\Delta_{R0}x) \tag{2.6.15}$$

$$f_- = S_n\left(i\Delta_I - \frac{\Delta_{R0}^2}{2(\omega_n' + i\Delta_I)}\frac{1}{\cosh^2(\Delta_{R0}x)}\right) \tag{2.6.16}$$

$$g = S_n\left(i\omega_n' + \frac{\Delta_{R0}^2}{2(\omega_n' + i\Delta_I)}\frac{1}{\cosh^2(\Delta_{R0}x)}\right). \tag{2.6.17}$$

The previous assumption that the imaginary part of the order parameter is constant leads to

$$\sum_{\omega_n}\mathrm{Im}\left[\frac{1}{\omega_n' + i\Delta_I}\frac{1}{\sqrt{\omega_n'^2 + |\Delta_0|^2}}\right] = 0 \tag{2.6.18}$$

along with $\Delta_{R0} = \sqrt{|\Delta_0|^2 - \Delta_I^2}$ meaning the phase slip is now fully defined.

To continue they use Eilenberger's equation for the free-energy and calculate the barrier by simply taking the difference between the free-energies of the homogeneous and inhomogeneous results. This leads to the general form for the barrier,

$$\delta F = T\mathrm{Re}\sum_{\omega_n}\left(\ln\frac{|\Delta_0|^2 - i\Delta_I\omega_n' + \Delta_{R0}\sqrt{\omega_n'^2 + |\Delta_0|^2}}{|\Delta_0|^2 - i\Delta_I\omega_n' - \Delta_{R0}\sqrt{\omega_n'^2 + |\Delta_0|^2}} - \frac{2\Delta_{R0}}{\sqrt{\omega_n'^2 + |\Delta_0|^2}}\right) \tag{2.6.19}$$

In general this should be solved on computer the results from which are given in their figure, reproduced below in Figure 2.8

Exact results are possible at $T = 0$ and $T = T_c$. At zero temperature they find,

$$\delta F(0,k) = \frac{2}{\pi}\Delta(0)\sqrt{1 + \frac{k}{2}} \tag{2.6.20}$$

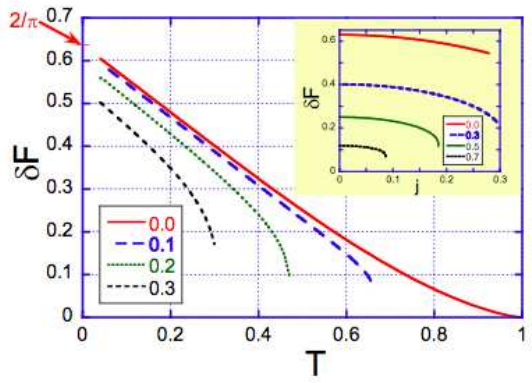

Figure 2.8: Image showing temperature dependence of the barrier in units of $\Delta(0)$ from Zharov *et al* [33]

provided that the current remains below the critical current, $k \approx 0.83$, where $\Delta(0)$ is the BCS order parameter at zero temperature.

Following this their result near $T = T_c$ in the LA regime is,

$$\delta F = C_0 \Delta(0)(1-T)^{\frac{3}{2}} \tag{2.6.21}$$

with, $C_0 = \frac{8\sqrt{2}}{3\sqrt{7\zeta(3)}}\frac{\gamma}{\pi}$ where $\gamma = e^C \approx 1.7811$ in terms of the Euler constant $C \approx 0.5772$ and the Riemann zeta-function, $\zeta(3) \approx 1.202$. Taking this limit to $T = 0$ shows the LA result to be a good approximation at low temperature.

We see that a calculation in the whole temperature range is possible from the Eilenberger equations and ask ourselves, what role impurities take at low temperature? In order to answer this question we need to see how the Eilenberger equations are derived and then extend this to see what part of them the Ginzburg-Landau equations approximate.

2.7 Questions Motivating Thesis

1. How are the Eilenberger equations derived and used?

2. Given that GL breaks down is there a quasi-classical, microscopic approach that can be used to probe phase slips at low temperature which includes the effects of impurities?

3. Can this be used to build an effective model for Quantum Phase slips?

4. Can a time dependent form be constructed?

Chapter 3

GREEN'S FUNCTION APPROACH

3.1 Introduction

In this chapter we will formulate Eilenbergers's quasi-classical theory of superconductivity. We choose this over Larkin's [36] version for reasons of clarity and comparison to the sources given earlier. Starting from the Green's function, we will derive Gorkov's equations for BCS theory we then proceed to define quasi-classical, energy-integrated Green's functions. Along with some useful observables defined in terms of these functions. They will then be used to derive the pure Eilenberger equations in a simple manor. After this impurities are considered in a way that spatially varying superconducting alloys can be investigated.
The general line of reasoning for the work in this chapter is as follows.

Being derived from correlation functions, the Gorkov equations are easier to work with than the full state function. This approximation retains relevant information of some important observable quantities, but is still too complex to solve in general. In particular the full velocity dependence provides an unwanted headache in the form of complicated vertex corrections, due to the non-linear dependence on momentum.

Quasiparticles in conventional superconductors are known to exist in a small energy gap near to the Fermi Surface (FS), as such the momentum of the quasiparticles ought to be near that of the Fermi momentum. This is the reasoning

behind Eilenberger's desire to expand the Gorkov equations at the FS in terms of quasi-classical Green's functions.

These quasi-classical Green's functions are to be considered as being the normal Green's functions of superconductivity, but where the energy related to the 'fast' momenta has been integrated out. The slower momentum associated with larger scale spatial dependence remain allowing for investigation of inhomogeneous systems.

The simplification that comes from using quasi-classical Green's functions is that the troublesome velocity dependence is replaced with a dependence on $\hat{\mathbf{v}}_F$. In turn this allows for a general solution, frequently referred to as the 'normalisation', to be calculated. This solution gives a non-linear relationship between the normal and anomalous quasi-classical Green's functions, which returns some information lost by Eilenberger's method.

The treatment of impurities within an inhomogeneous superconducting system also makes use of the separation of length scales. The presentation of this here differs from many derivations in that the inhomogeneity is treated within the averaging. The applied field, and spatial dependence of the order parameter are not simply added to the results of an impurity averaged homogenous system.

In the chapter following this, the velocity dependence is used to consider how very dirty superconductors behave, along with the physics of weakly-anisotropic superconductors.

3.2 Temperature Green's functions

In this section we will define Green's functions and derive the Gorkov equations for superconductivity. The layout will be similar to that of Kopnin [37] and Abrikosov Gorkov and Dzyaloshinski [38]. Since all authors have a tendency to use different definitions we write ours for completeness.

3.2.1 Definitions and Properties

To start we define the single particle Green's function,

$$G_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = -\langle \hat{T}_\tau \hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle. \tag{3.2.1}$$

written in terms of the modified Heisenberg operators,

$$\begin{aligned}\hat{\psi}_\alpha(\mathbf{r},\tau) &= \exp[\hat{H}'\tau]\hat{\psi}(\mathbf{r})\exp[-\hat{H}'\tau]\\ \hat{\psi}^\dagger_\alpha(\mathbf{r},\tau) &= \exp[\hat{H}'\tau]\hat{\psi}^\dagger(\mathbf{r})\exp[-\hat{H}'\tau],\end{aligned} \tag{3.2.2}$$

where, $\tau = it$, is an imaginary time. We do this in order to calculate certain temperature effects.

The operator $\hat{T}_\tau$ orders the field operators from right to left in order of increasing time and the operation $\langle \dots \rangle$ is an average over the grand canonical distribution $\mathrm{tr}\left\{\exp\left[\beta(\Omega - \hat{H}')\right]\dots\right\}$[1] such that,

$$G_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = -\mathrm{tr}\left\{e^{\beta\Omega}\hat{T}_\tau e^{-\beta\hat{H}'}\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\right\} \tag{3.2.3}$$

where Ω is the thermodynamic potential: $\Omega = F - \mu N$ in terms of the free energy $F = -T\ln\sum_n \exp[-\frac{E_n}{T}]$. The first property to show is that these functions depend only on the 'time' difference $\tau = \tau_1 - \tau_2$. To show this we consider $\tau_1 < \tau_2$ and

[1] Here and afterwards $\hbar = k_B = 1$ and $\beta = \frac{1}{T}$ when not used as an index denoting spin

permute the terms in the trace,

$$\begin{aligned}
G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau_1<\tau_2) &= \mathrm{tr}\left\{e^{\beta\Omega}e^{-\beta\hat{H}'}e^{\hat{H}'\tau_2}\hat{\psi}^\dagger_\beta(\mathbf{r_2})e^{-\hat{H}'\tau_2}e^{\hat{H}'\tau_1}\hat{\psi}_\alpha(\mathbf{r_1})e^{-\hat{H}'\tau_1}\right\} \\
&= -\mathrm{tr}\left\{e^{\beta\Omega}e^{-(\beta+\tau_1-\tau_2)\hat{H}'}\hat{\psi}^\dagger_\beta(\mathbf{r_2})e^{-(\tau_1-\tau_2)\hat{H}'}\hat{\psi}_\alpha(\mathbf{r_1})e^{-\hat{H}'}\right\} \\
&= -\mathrm{tr}\left\{e^{\beta\Omega}e^{-(\beta+\tau)\hat{H}'}\hat{\psi}^\dagger_\beta(\mathbf{r_2})e^{-\tau\hat{H}'}\hat{\psi}_\alpha(\mathbf{r_1})e^{-\hat{H}'}\right\} = G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau) \quad ,\tau<0.
\end{aligned}$$

Repeating for $\tau_1 > \tau_2$ gives,

$$G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau_1>\tau_2) = \mathrm{tr}\left\{e^{\beta\Omega}e^{-\tau\hat{H}'}\hat{\psi}^\dagger_\beta(\mathbf{r_2})e^{-(\beta-\tau)\hat{H}'}\hat{\psi}_\alpha(\mathbf{r_1})e^{-\hat{H}'}\right\} = G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau) \quad ,\tau>0$$

which are both functions of τ alone. Furthermore we see that $G(\tau+\beta) = -G(\tau<0)$. This leads us to consider τ belongs to the interval $\tau \in [-\beta,\beta]$ and may use this fact to define a Fourier series in τ,

$$G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau) = T\sum_{n=-\infty}^{\infty} e^{-i\omega_n\tau}G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\omega_n) \tag{3.2.4}$$

the inverse transform is given by,

$$G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau}G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau). \tag{3.2.5}$$

Including the condition that $G(\tau+\beta) = -G(\tau<0)$ yields the Matsubara frequencies,

$$\begin{aligned}
G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\omega_n) &= \frac{1}{2}\int_0^\beta d\tau e^{i\omega_n\tau}G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau) + \frac{1}{2}\int_{-\beta}^0 d\tau e^{i\omega_n\tau}G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau) \\
&= \frac{1}{2}\int_0^\beta d\tau e^{i\omega_n\tau}G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau) - \frac{1}{2}\int_{-\beta}^0 d\tau e^{i\omega_n\tau}G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau+\beta) \\
&= \frac{1}{2}(1-e^{-i\beta\omega_n})\int_0^\beta d\tau e^{i\omega_n\tau}G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2;\tau).
\end{aligned}$$

This requires the Matsubara frequencies, $\omega_n = (2n+1)\pi T$ which correspond to the poles of the Fermi-Dirac distribution, giving the correct Fermi statistics to the quasiparticles.

3.2.2 Equations of Motion

In order to calculate equations for the Green's functions we consider the time dependencies of the field operators. Differentiating the definitions (3.2.2) with respect to τ we can set up equations of motion for the fields in terms of the commutators,

$$\frac{\partial\hat{\psi}_\alpha}{\partial\tau} = \left[\hat{H}', \hat{\psi}_\alpha\right] \quad \text{and} \quad \frac{\partial\hat{\psi}^\dagger_\alpha}{\partial\tau} = \left[\hat{H}', \hat{\psi}^\dagger_\alpha\right].$$

Working from the superconducting Hamiltonian in the presence of impurities and an external field given by,

$$\hat{H}' = \hat{H}_0 + \hat{V}_1 + \hat{V}_2$$

$$\begin{aligned}\hat{H}' = \int d^3r\hat{\psi}^\dagger_\alpha(\mathbf{r},\tau)\left[-\frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}(\mathbf{r})\right)^2 - \mu\right]\hat{\psi}_\alpha(\mathbf{r},\tau) \\ + \sum_a \int d^3r\hat{\psi}^\dagger_\alpha(\mathbf{r},\tau)U(\mathbf{r}-\mathbf{r}_a)\hat{\psi}_\alpha(\mathbf{r},\tau) \\ - \frac{g}{2}\int d^3r\hat{\psi}^\dagger_\beta(\mathbf{r},\tau)\hat{\psi}^\dagger_\alpha(\mathbf{r},\tau)\hat{\psi}_\alpha(\mathbf{r},\tau)\hat{\psi}_\beta(\mathbf{r},\tau).\end{aligned}$$

Here $U(\mathbf{r}-\mathbf{r}_a)$ is the potential due to impurity atom at $\mathbf{r}_a$, and the last line is the attractive two body term responsible for the coupling required for conventional superconductivity. We consider only the weak coupling limit, in which the coupling constant g is small.

Impurities will be discussed in more detail later, for now we will group the first

two terms via the following Hamiltonian,

$$\hat{\mathcal{H}} = -\frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}(\mathbf{r})\right)^2 - \mu + \sum_a U(\mathbf{r} - \mathbf{r}_a).$$

To proceed we will need know how to permute the field operators, as these are Fermi-fields they obey the normal Fermionic relations,

$$\begin{aligned}\left\{\hat{\psi}_\alpha(\mathbf{r}_1,\tau), \hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau)\right\} &= \delta_{\alpha\beta}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)\\ \left\{\hat{\psi}_\alpha(\mathbf{r}_1,\tau), \hat{\psi}_\beta(\mathbf{r}_2,\tau)\right\} &= \left\{\hat{\psi}^\dagger_\alpha(\mathbf{r}_1,\tau), \hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau)\right\} = 0.\end{aligned}$$

Useful identities for commutators [2] and anticommutators[3] are given at the bottom of the page.

Using these we now calculate the time derivatives of $\hat{\psi}$ and $\hat{\psi}^\dagger$,

$$\frac{\partial\hat{\psi}_\gamma(\mathbf{r},\tau)}{\partial\tau} = \int d^3r' \left[\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\hat{\mathcal{H}}\hat{\psi}_\alpha(\mathbf{r}',\tau) - \frac{g}{2}\hat{\psi}^\dagger_\beta(\mathbf{r}',\tau)\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\hat{\psi}_\alpha(\mathbf{r}',\tau)\hat{\psi}_\beta(\mathbf{r}',\tau), \hat{\psi}_\gamma(\mathbf{r},\tau)\right]$$

by considering the terms in turn. The round parentheses below show which operator the hamiltonian is acting on, this is important when considering the applied field, as it makes $\hat{\mathcal{H}}$ complex.

$$\begin{aligned}\left[\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\Big(\hat{\mathcal{H}}\hat{\psi}_\alpha(\mathbf{r}',\tau)\Big), \hat{\psi}_\gamma(\mathbf{r},\tau)\right] &= \hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\underbrace{\left\{\hat{\mathcal{H}}\hat{\psi}_\alpha(\mathbf{r}',\tau), \hat{\psi}_\gamma(\mathbf{r},\tau)\right\}}_{=0} -\\ &\quad - \left\{\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau), \hat{\psi}_\gamma(\mathbf{r},\tau)\right\}\hat{\mathcal{H}}\hat{\psi}_\alpha(\mathbf{r}',\tau)\\ &= -\hat{\mathcal{H}}\hat{\psi}_\alpha(\mathbf{r}',\tau)\delta_{\alpha\gamma}\delta^{(3)}(\mathbf{r} - \mathbf{r}')\\ &= -\hat{\mathcal{H}}\hat{\psi}_\gamma(\mathbf{r}',\tau)\delta^{(3)}(\mathbf{r} - \mathbf{r}').\end{aligned}$$

[2] $[A,B] = AB - BA$

[3] $\{A,B\} = AB + BA$

$[AB,C] = A\{B,C\} - \{A,C\}B$

$[AB,C] = A[B,C] + [A,C]B$

The other term is only slightly harder. If we consider the first part as two products, one marked by parentheses, and expand. We are left with a commutator containing field operators of one type this will clearly be zero and removed. In expanding the other part one must take care in permuting single operators.

$$\begin{aligned}
&\Big[\hat{\psi}^\dagger_\beta(\mathbf{r}',\tau)\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\Big(\hat{\psi}_\alpha(\mathbf{r}',\tau)\hat{\psi}_\beta(\mathbf{r}',\tau)\Big),\hat{\psi}_\gamma(\mathbf{r},\tau)\Big] = \\
&\hat{\psi}^\dagger_\beta(\mathbf{r}',\tau)\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\underbrace{\Big[\hat{\psi}_\alpha(\mathbf{r}',\tau)\hat{\psi}_\beta(\mathbf{r}',\tau),\hat{\psi}_\gamma(\mathbf{r},\tau)\Big]}_{=0} + \Big[\hat{\psi}^\dagger_\beta(\mathbf{r}',\tau)\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau),\hat{\psi}_\gamma(\mathbf{r},\tau)\Big]\,\hat{\psi}_\alpha(\mathbf{r}',\tau)\hat{\psi}_\beta(\mathbf{r}',\tau) \\
&= \Big(\hat{\psi}^\dagger_\beta(\mathbf{r}',\tau)\Big\{\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau),\hat{\psi}_\gamma(\mathbf{r},\tau)\Big\} - \Big\{\hat{\psi}^\dagger_\beta(\mathbf{r}',\tau),\hat{\psi}_\gamma(\mathbf{r},\tau)\Big\}\hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\Big)\hat{\psi}_\alpha(\mathbf{r}',\tau)\hat{\psi}_\beta(\mathbf{r}',\tau) \\
&= \Big(\hat{\psi}^\dagger_\beta(\mathbf{r}',\tau)\hat{\psi}_\alpha(\mathbf{r}',\tau)\hat{\psi}_\beta(\mathbf{r}',\tau)\delta_{\alpha\gamma} - \hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\hat{\psi}_\alpha(\mathbf{r}',\tau)\hat{\psi}_\beta(\mathbf{r}',\tau)\delta_{\beta\gamma}\Big)\delta^{(3)}(\mathbf{r}-\mathbf{r}') \\
&= -\Big(\hat{\psi}^\dagger_\beta(\mathbf{r}',\tau)\hat{\psi}_\beta(\mathbf{r}',\tau)\hat{\psi}_\gamma(\mathbf{r}',\tau) + \hat{\psi}^\dagger_\alpha(\mathbf{r}',\tau)\hat{\psi}_\alpha(\mathbf{r}',\tau)\hat{\psi}_\gamma(\mathbf{r}',\tau)\Big)\delta^{(3)}(\mathbf{r}-\mathbf{r}')
\end{aligned}$$

taking the sum over α and β and integrating over d^3r' gives the final result,

$$\frac{\partial\hat{\psi}_\gamma(\mathbf{r},\tau)}{\partial\tau} = -\hat{\mathcal{H}}\hat{\psi}_\gamma(\mathbf{r},\tau) + g\hat{\psi}^\dagger_\alpha(\mathbf{r},\tau)\hat{\psi}_\alpha(\mathbf{r},\tau)\hat{\psi}_\gamma(\mathbf{r},\tau) \tag{3.2.6a}$$

repeating for $\hat{\psi}^\dagger_\gamma$ gives,

$$\frac{\partial\hat{\psi}^\dagger_\gamma(\mathbf{r},\tau)}{\partial\tau} = \hat{\mathcal{H}}^*\hat{\psi}^\dagger_\gamma(\mathbf{r},\tau) - g\hat{\psi}^\dagger_\gamma(\mathbf{r},\tau)\hat{\psi}^\dagger_\alpha(\mathbf{r},\tau)\hat{\psi}_\alpha(\mathbf{r},\tau) \tag{3.2.6b}$$

where in order to act to on the other field operator,

$$\hat{\mathcal{H}}^* = -\frac{1}{2m}\left(\nabla + \frac{ie}{c}\mathbf{A}(\mathbf{r})\right)^2 - \mu + \sum_a U(\mathbf{r}-\mathbf{r}_a)$$

has been used.

Two Particle Green's functions and Anomalous Green's Functions

From equations (3.2.6) we can find the time dependencies of the Green's function. To start we write the time operator explicitly,

$$\begin{aligned} G_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) &= -\langle \hat{T}_\tau \hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle \\ &= -\langle \hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle\theta(\tau_1-\tau_2) + \langle \hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\rangle\theta(\tau_2-\tau_1) \end{aligned}$$

then differentiate this with respect to τ_1 by substituting in (3.2.6) where relevant,

$$\begin{aligned} \frac{\partial G_{\alpha\beta}}{\partial\tau_1} &= -\delta(\tau_1-\tau_2)(\underbrace{\langle \hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2) + \hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\rangle}_{\left\{\hat{\psi}_\alpha(\mathbf{r}_1,\tau),\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau)\right\}}) - \left\langle \hat{T}_\tau \frac{\partial\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)}{\partial\tau_1}\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\right\rangle \\ &= -\delta_{\alpha\beta}\delta^{(4)}(X_1-X_2) - \langle \hat{T}_\tau(-\hat{\mathcal{H}}_1\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1) + g\hat{\psi}^\dagger_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1))\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle \\ &= -\delta_{\alpha\beta}\delta^{(4)}(X_1-X_2) - \hat{\mathcal{H}}_1 G_{\alpha\beta} - g\langle \hat{T}_\tau\hat{\psi}^\dagger_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle. \end{aligned}$$

where for brevity $X_i \to \mathbf{r}_i,\tau_i$. This forces us to consider two particle Green's functions of the form,

$$K_{\gamma\delta;\alpha\beta}(X_3,X_4;X_1,X_2) = \langle \hat{T}_\tau\hat{\psi}_\gamma(\mathbf{r}_3,\tau_3)\hat{\psi}_\delta(\mathbf{r}_4,\tau_4)\hat{\psi}^\dagger_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle. \tag{3.2.7}$$

Weak coupling BCS superconductivity being studied here is caused by the paring of electrons, as such we consider operators which create pairs,

$$\hat{\psi}^\dagger_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2).$$

If applied to the ground state of a system this pairing operator would create a superposition of excited states but in the presence of a condensate of Cooper pairs it simply adds another pair. With a sufficiently large number of pairs in the condensate

this will not excite the system, written as,

$$\langle N+2|\hat{\psi}^{\dagger}_{\alpha}(\mathbf{r}_1,\tau_1)\hat{\psi}^{\dagger}_{\beta}(\mathbf{r}_2,\tau_2)|N\rangle.$$

the limit of having a large number of particles implies, the difference of 2 becomes negligible. In this limit the above is called an 'anomalous average' and with this in mind look again at the two-particle Green's function,

$$K_{\gamma\delta;\alpha\beta}(X_3,X_4;X_1,X_2) = \\ \langle N|\hat{T}_{\tau}\hat{\psi}_{\gamma}(\mathbf{r}_3,\tau_3)\hat{\psi}_{\delta}(\mathbf{r}_4,\tau_4)|N+2|\rangle\langle N+2|\hat{T}_{\tau}\hat{\psi}^{\dagger}_{\alpha}(\mathbf{r}_1,\tau_1)\hat{\psi}^{\dagger}_{\beta}(\mathbf{r}_2,\tau_2)|N\rangle \quad (3.2.8)$$

and define the 'Anomalous Green's functions',

$$F_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = \langle\hat{T}_{\tau}\hat{\psi}_{\alpha}(\mathbf{r}_1,\tau_1)\hat{\psi}_{\beta}(\mathbf{r}_2,\tau_2)\rangle \\ F^{\dagger}_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = \langle\hat{T}_{\tau}\hat{\psi}^{\dagger}_{\alpha}(\mathbf{r}_1,\tau_1)\hat{\psi}^{\dagger}_{\beta}(\mathbf{r}_2,\tau_2)\rangle. \quad (3.2.9)$$

We also define the order parameter,

$$\Delta_{\alpha\beta}(\mathbf{r}) = g\langle\hat{\psi}_{\alpha}(\mathbf{r},\tau)\hat{\psi}_{\beta}(\mathbf{r},\tau)\rangle = gF_{\alpha\beta}(\mathbf{r},\mathbf{r}) \\ \Delta^{*}_{\alpha\beta}(\mathbf{r}) = g\langle\hat{\psi}^{\dagger}_{\alpha}(\mathbf{r},\tau)\hat{\psi}^{\dagger}_{\beta}(\mathbf{r},\tau)\rangle = gF^{\dagger}_{\alpha\beta}(\mathbf{r},\mathbf{r}).$$

for the creation and annihilation of paired electrons in terms of the coupling constant g. For calculation purposes, it gives a self-consistent equation in terms of the anomalous Green's function for the gap.

Returning to the equation of motion for the Green's function we use these definitions to write,

$$\frac{\partial G_{\alpha\beta}}{\partial\tau_1} = -\delta_{\alpha\beta}\delta^{(4)}(X_1-X_2) - \hat{\mathcal{H}}_1 G_{\alpha\beta} - g\langle\hat{\psi}^{\dagger}_{\gamma}(\mathbf{r}_1,\tau_1)\hat{\psi}_{\gamma}(\mathbf{r}_1,\tau_1)\hat{\psi}_{\alpha}(\mathbf{r}_1,\tau_1)\hat{\psi}^{\dagger}_{\beta}(\mathbf{r}_2,\tau_2)\rangle$$

here we use Wick's theorem to rewrite the average in terms of the products of paired operators,

$$\begin{aligned}\langle \hat{T}_\tau \hat{\psi}^\dagger_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle \\ = \langle \hat{\psi}^\dagger_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\gamma(\mathbf{r}_1,\tau_1)\rangle\langle \hat{T}_\tau\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle \\ - \langle \hat{\psi}^\dagger_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\rangle\langle \hat{T}_\tau\hat{\psi}_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle \\ + \langle \hat{T}_\tau\hat{\psi}^\dagger_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle\langle \hat{\psi}_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\rangle, \quad (3.2.10)\end{aligned}$$

the first two lines of which are discounted as they do not contribute to the superconductivity and simply normalise the chemical potential.

$$\begin{aligned}\frac{\partial G_{\alpha\beta}}{\partial\tau_1} &= -\delta_{\alpha\beta}\delta^{(4)}(X_1-X_2) - \hat{\mathcal{H}}_1 G_{\alpha\beta} - g\langle \hat{T}_\tau\hat{\psi}_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\rangle\langle \hat{T}_\tau\hat{\psi}^\dagger_\gamma(\mathbf{r}_1,\tau_1)\hat{\psi}_\beta(\mathbf{r}_2,\tau_2)\rangle \\ &= -\delta_{\alpha\beta}\delta^{(4)}(X_1-X_2) - \hat{\mathcal{H}}_1 G_{\alpha\beta} - \Delta_{\gamma\alpha}(\mathbf{r}_1)F^\dagger_{\gamma\beta}\end{aligned}$$

which, upon rearranging becomes,

$$-\left(\frac{\partial}{\partial\tau_1}+\hat{\mathcal{H}}_1\right)G_{\alpha\beta} - \Delta_{\gamma\alpha}(\mathbf{r}_1)F^\dagger_{\gamma\beta} = \delta_{\alpha\beta}\delta^{(4)}(X_1-X_2).$$

Considering spin singlets the gap and anomalous Green's functions change sign when their spin labels are switched. This can be accounted for by using the antisymmetric unit spinor, $I_{\alpha\beta}$

$$\begin{aligned}F_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) &= I_{\alpha\beta}F(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) \\ F^\dagger_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) &= I_{\alpha\beta}F^\dagger(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) \\ I_{\alpha\beta} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i(\sigma_y)_{\alpha\beta}.\end{aligned}$$

Provided any external field doesn't make the number of up- and down-spins signifi-

cantly different then we have $G_{\alpha\alpha} = G_{\beta\beta} = G$ and $G_{\alpha\beta} = 0$ if $\alpha \neq \beta$. This allows the removal of all spin labels,

$$-\left(\frac{\partial}{\partial\tau_1} + \hat{\mathcal{H}}_1\right) G(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) + \Delta(\mathbf{r}_1)F^\dagger(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = \delta^{(4)}(X_1 - X_2). \quad (3.2.11a)$$

Repeating this treatment for the other functions give,

$$\left(\frac{\partial}{\partial\tau_1} - \hat{\mathcal{H}}_1^*\right) F^\dagger(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) - \Delta^*(\mathbf{r}_1)G(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = 0 \quad (3.2.11b)$$

$$\left(\frac{\partial}{\partial\tau_1} + \hat{\mathcal{H}}_1\right) F(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) + \Delta^*(\mathbf{r}_1)G(\mathbf{r}_2,\tau_2;\mathbf{r}_1,\tau_1) = 0 \quad (3.2.11c)$$

$$\left(\frac{\partial}{\partial\tau_1} - \hat{\mathcal{H}}_1^*\right) G(\mathbf{r}_2,\tau_2;\mathbf{r}_1,\tau_1) + \Delta^*(\mathbf{r}_1)F(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = \delta^{(4)}(X_1 - X_2). \quad (3.2.11d)$$

3.2.3 Gorkov Equations

These results, calculated by Gorkov may be written in a single matrix equation,

$$\begin{pmatrix} -\frac{\partial}{\partial\tau_1} - \mathcal{H}_1 & \Delta(\mathbf{r}_1) \\ -\Delta^*(\mathbf{r}_1) & \frac{\partial}{\partial\tau_1} - \mathcal{H}_1^* \end{pmatrix} \begin{pmatrix} G(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) & -F(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) \\ F^\dagger(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) & \bar{G}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) \end{pmatrix} = \check{1}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) \quad (3.2.12)$$

where $\check{1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The same Hamiltonian as before is used

$$\mathcal{H}_i = -\frac{1}{2m}\left(\nabla_i - \frac{ie}{c}\mathbf{A}(\mathbf{r}_i)\right)^2 - \mu + \sum_a U(\mathbf{r}_i - \mathbf{r}_a)$$

another Green's function is defined and is interpreted as corresponding to holes,

$$\overline{G}_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = \langle \hat{T}_\tau \hat{\psi}_\alpha^\dagger(\mathbf{r}_1,\tau_1)\hat{\psi}_\beta(\mathbf{r}_2,\tau_2)\rangle = G_{\alpha\beta}(\mathbf{r}_2,\tau_2;\mathbf{r}_1,\tau_1)$$

this also has the property $\overline{G}_{\alpha\alpha} = \overline{G}_{\beta\beta} = \overline{G}$.

The above procedure can be repeated for the variable τ_2 which gives the corresponding set of equations,

$$\begin{pmatrix} G(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) & -F(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) \\ F^\dagger(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) & \bar{G}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\tau_2} - \mathcal{H}_2^* & \Delta(\mathbf{r}_2) \\ -\Delta^*(\mathbf{r}_2) & -\frac{\partial}{\partial\tau_2} - \mathcal{H}_2 \end{pmatrix} = \check{1}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2). \tag{3.2.13}$$

Earlier we noted that a Fourier series in time existed for the difference variable, τ (3.2.5). Here too we make the change to difference and averaged time variables,

$$\begin{array}{lll} \tau = \tau_1 - \tau_2 & \tau_1 = T + \frac{\tau}{2} & \frac{\partial}{\partial\tau_1} = \frac{1}{2}\frac{\partial}{\partial T} + \frac{\partial}{\partial\tau} \\ T = \frac{1}{2}(\tau_1 + \tau_2) & \tau_2 = T - \frac{\tau}{2} & \frac{\partial}{\partial\tau_2} = \frac{1}{2}\frac{\partial}{\partial T} - \frac{\partial}{\partial\tau} \end{array}$$

since we are considering equilibrium problems we set $\frac{\partial}{\partial T} = 0$ and see that, $\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau_1} = -\frac{\partial}{\partial\tau_2}$. Taking the Fourier series and writing the matrix $\check{G} = \left(\begin{smallmatrix} G & -F \\ F^\dagger & \overline{G} \end{smallmatrix}\right)$ we obtain,

$$\left.\begin{array}{l} \begin{pmatrix} i\omega_n - \mathcal{H}_1 & \Delta(\mathbf{r}_1) \\ -\Delta^*(\mathbf{r}_1) & -i\omega_n - \mathcal{H}_1^* \end{pmatrix} \check{G}_{\omega_n}(\mathbf{r}_1,\mathbf{r}_2) = \check{1}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) \\ \check{G}_{\omega_n}(\mathbf{r}_1,\mathbf{r}_2) \begin{pmatrix} i\omega_n - \mathcal{H}_2^* & \Delta(\mathbf{r}_2) \\ -\Delta^*(\mathbf{r}_2) & -i\omega_n - \mathcal{H}_2 \end{pmatrix} = \check{1}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) \end{array}\right\} \quad \text{Gorkov Equations} \tag{3.2.14}$$

These are equations for the Green's functions of conventional superconductors in a weak applied field, where the superconductivity caused by weakly coupled, s-wave pairings of electrons describes events in equilibrium.

Extensions can be made to more exotic pairings. One could also create a more detailed theory by including the phonon modes, or by considering non-equilibrium effects via the Keldysh technique [39].

We restrict our attention first to this more simple case.

3.2.4 Homogeneous Solutions

Before going further we use these equations to calculate the homogeneous case in the absence of fields. We may now Fourier transform the spatial variables, using $\check{G}(\mathbf{p}) = \check{G}_0 = \int d^3r e^{i\mathbf{pr}}\check{G}(\mathbf{r})$ giving $\mathcal{H} = \frac{p^2}{2m} - \mu = \xi_p$, the order parameter is given by the constant, Δ_0,

$$\begin{pmatrix} i\omega_n - \xi_p & \Delta_0 \\ -\Delta_0^* & -i\omega_n - \xi_p \end{pmatrix} \begin{pmatrix} G_0 & -F_0 \\ F_0^\dagger & \overline{G}_0 \end{pmatrix} = \check{1}. \tag{3.2.15}$$

This solves to give the following,

$$G_0 = -\frac{(i\omega_n + \xi_p)}{\xi_p^2 + \omega_n^2 + |\Delta_0|^2} \qquad \overline{G}_0 = \frac{(i\omega_n - \xi_p)}{\xi_p^2 + \omega_n^2 + |\Delta_0|^2}$$
$$F_0 = \frac{\Delta_0}{\xi_p^2 + \omega_n^2 + |\Delta_0|^2} \qquad F_0^\dagger = \frac{\Delta_0^*}{\xi_p^2 + \omega_n^2 + |\Delta_0|^2}.$$

We may also calculate the normal state for which $|\Delta_0|^2 = 0$,

$$G_0^{(n)} = \frac{1}{i\omega_n - \xi_p}$$

3.3 Quasi-Classical Approximation

Much of the method here is based from that of Kopnin [40], however the notation of his method is a little confusing, forcing certain ways of writing momenta to be preferential. This is simply not the case, and a result of performing impurity calculations incorrectly. As alloys are treated correctly later in the thesis, we choose a notation now which should be both both obvious and in the form that we will use later.

Measurable quantities depend on the Green's functions integrated over momentum. Looking at the results above shows that the Green's function diverges. We must be careful then in considering momentum integrals of Green's functions.

The type of superconductor being considered here, only has quasiparticles in a relatively small gapped region of energy near the Fermi Surface (FS), $\Delta \ll E_F$.

As we have written them the Green's functions are functions of ξ_p, a quantity which varies strongly at the FS. Indeed a change by $d\xi_p$ is of order Δ, but since this is much smaller than E_F the magnitude of the quasiparticle momentum, $\mathbf{p}$ stays close to, $\mathbf{p}_F$.

$$\delta p \sim \frac{\delta \xi_p}{v_F} \sim \xi^{-1}$$

ξ^{-1} being the inverse coherence length.
We must have as a condition that,

$$\frac{\delta p}{p_F} \sim (p_F \xi)^{-1} \ll 1$$

which will force the momentum pre-factors to the Green's functions to take their value at the FS.

We parameterise the momentum integration to take place at the FS,

$$\frac{d^3p}{(2\pi)^3} = \frac{d\xi_p}{v_F}\frac{dS_F}{(2\pi)^3}$$

given in terms of the momentum increment perpendicular to the FS, $\frac{d\xi_p}{v_F}$.
Since the superconductor is gapped at the FS, we use the original normal state FS as an energy reference,

$$\xi_p = E_n(\mathbf{p}) - E_F$$

More detailed electronic band structure is neglected as the superconducting Green's functions are taken to be sufficiently close to the Fermi surface. If we were to consider measurable quantities now, they would depend on the Green's functions integrated over $d\xi_p$ and multiplied by functions dependent on the direction of particle momentum.

The 'Quasi-Classical Approximation' is introduced to consider Green's functions which have had the integral over ξ_p performed.
Doing this will remove the 'fast' oscillations $\sim \mathcal{O}(\mathbf{p}_F^{-1})$ associated with the relative co-ordinate, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ which do not affect the superconductivity.

What is left are the 'slow' momentum dependence which describe the residual inhomogeneities on the scale, $\mathbf{R} = \frac{(\mathbf{r}_1+\mathbf{r}_2)}{2}$.

There is little problem in defining the quasi-classical anomalous Green's functions, they decay as ξ_p^{-2} and do not exist in the normal state. We therefore define,

$$if_{\omega_n}(\hat{\mathbf{p}}_F, \mathbf{k}) = \frac{i}{\pi}\int d\xi_p F_{\omega_n}(\mathbf{p}, \mathbf{k}) = \frac{i}{\pi}\oint d\xi_p F_{\omega_n}(\mathbf{p}, \mathbf{k}) \tag{3.3.1}$$

$$if^{\dagger}_{\omega_n}(\hat{\mathbf{p}}_F, \mathbf{k}) = \frac{i}{\pi}\int d\xi_p F^{\dagger}_{\omega_n}(\mathbf{p}, \mathbf{k}) = \frac{i}{\pi}\oint d\xi_p F^{\dagger}_{\omega_n}(\mathbf{p}, \mathbf{k}) \tag{3.3.2}$$

where $\oint$ shows only contributions from the poles closest to the FS are taken. The momenta $\mathbf{p}$ and $\mathbf{k}$ correspond to the Fourier transforms of $\mathbf{r}$ and $\mathbf{R}$ respectively. Once the perpendicular component has been integrated, the only dependence remaining is its direction, hence the dependence on $\hat{\mathbf{p}}_F$.

The divergence of G presents a problem, however this is removed by adding and subtracting the normal state Green's function, the difference at high energy is then convergent leaving a term which can be calculated from the known, normal state. As such we still define

$$g_{\omega_n}(\hat{\mathbf{p}}_F, \mathbf{k}) = \frac{i}{\pi} \oint d\xi_p G_{\omega_n}(\mathbf{p}, \mathbf{k}) \tag{3.3.3}$$

and consider the full integral of G,

$$\int \frac{d^3p}{(2\pi)^3} G_{\omega_n}(\mathbf{p}, \mathbf{k}) = \int \frac{dS_F}{v_F(2\pi)^3} \int d\xi_p \Big[G^{(n)}_{\omega_n}(\mathbf{p}, \mathbf{k}) + \{ G_{\omega_n}(\mathbf{p}, \mathbf{k}) - G^{(n)}_{\omega_n}(\mathbf{p}, \mathbf{k}) \} \Big]$$

in order to use the normal state Green's function we see

$$\begin{aligned} G^{(n)}_{\omega_n}(\mathbf{p}, \mathbf{k}) &= (2\pi)^3 \delta^{(3)}(\mathbf{k}) G^{(n)}_{\omega_n}(\mathbf{p}) \\ G^{(n)}_{\omega_n}(\mathbf{p}) &= \frac{1}{i\omega_n - \xi_p} = -\frac{\mathcal{P}}{\xi_p} + i\pi\delta(\xi_p)\mathrm{sign}(\omega_n) \\ \oint d\xi_p G^{(n)}_{\omega_n}(\mathbf{p}) &= i\pi \mathrm{sign}(\omega_n). \end{aligned}$$

If we consider the term in curly parentheses we see that it, like the anomalous Green's functions, is well behaved allowing the following approximation to be made,

$$\begin{aligned} \int \frac{d^3p}{(2\pi)^3} G_{\omega_n}(\mathbf{p}, \mathbf{k}) &= \int \frac{dS_F}{v_F(2\pi)^3} \int d\xi_p \Big[G^{(n)}_{\omega_n}(\mathbf{p}, \mathbf{k}) + \{ G_{\omega_n}(\mathbf{p}, \mathbf{k}) - G^{(n)}_{\omega_n}(\mathbf{p}, \mathbf{k}) \} \Big] \\ &\approx \int \frac{dS_F}{v_F(2\pi)^3} \Big[\int d\xi_p G^{(n)}_{\omega_n}(\mathbf{p}, \mathbf{k}) + \oint d\xi_p \{ G_{\omega_n}(\mathbf{p}, \mathbf{k}) - G^{(n)}_{\omega_n}(\mathbf{p}, \mathbf{k}) \} \Big] \\ &= \int \frac{dS_F}{v_F(2\pi)^3} \Big[-(2\pi)^3 \delta^{(3)}(\mathbf{k}) \mathcal{P} \int d\xi_p \frac{1}{\xi_p} + (2\pi)^3 \delta^{(3)}(\mathbf{k}) i\pi \mathrm{sign}(\omega_n) \end{aligned}$$

$$+\frac{\pi}{i}g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})-(2\pi)^3\delta^{(3)}(\mathbf{k})i\pi\mathrm{sign}(\omega_n)\Big]$$
$$=\int\frac{dS_F}{v_F(2\pi)^3}\Big[-(2\pi)^3\delta^{(3)}(\mathbf{k})\mathcal{P}\int d\xi_p\frac{1}{\xi_p}+\frac{\pi}{i}g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})\Big].$$

From this we see the large-ξ_p dependence is governed by the properties of the material in the normal state, while all superconducting properties are given by the functions, $g, f, f^\dagger$ and $\bar{g}$ which are determined near the FS. As a matrix we may write the definitions as,

$$\check{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})=\frac{i}{\pi}\int d\xi_p\check{G}_{\omega_n}(\mathbf{p},\mathbf{k})=\begin{pmatrix} g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) & -if_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) \\ if^\dagger_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) & \bar{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})\end{pmatrix}$$

which of course can be written similarly in a mixed representation for $\check{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{R})$. In considering the superconductor at the Fermi surface we assume that the velocity dependence varies in direction but has magnitude, v_F. This assumption works well for superconductors with reasonably isotropic Fermi surfaces, but is not sufficient to explain more exotic, anisotropic superconductors.

The additional factors of i present on the anomalous terms are included to be consistent with Eilenberger's derivation. Here the equations are written in terms of complex quantities, but the equations themselves do not contain additional complex numbers.

3.3.1 Homogeneous Results

We can now calculate the quasi-classical homogeneous results. For the Green's function,

$$\begin{aligned} g_0 &= \frac{i}{\pi}\int d\xi_p G_0 \\ &= -\frac{i}{\pi}\int d\xi_p \frac{(i\omega_n + \xi_p)}{\xi_p^2 + \omega_n^2 + |\Delta_0|^2} \end{aligned}$$

This can be expressed as two fractions, the first having numerator $i\omega$ is calculated simply from the poles. The second with ξ_p on top is odd and therefore vanishes, this can also be shown by the decomposition given below,

$$\begin{aligned} g_0 = -\frac{i}{\pi}\int d\xi_p \frac{i\omega_n}{(\xi_p + i\sqrt{\omega_n^2 + |\Delta_0|^2})(\xi_p - i\sqrt{\omega_n^2 + |\Delta_0|^2})} \\ - \frac{i}{2\pi}\int d\xi_p \frac{1}{\xi_p + i\sqrt{\omega_n^2 + |\Delta_0|^2}} + \frac{1}{\xi_p - i\sqrt{\omega_n^2 + |\Delta_0|^2}} \\ = \frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta_0|^2}}. \end{aligned}$$

A similar calculation is performed for $\overline{g}_0$ and gives,

$$\overline{g}_0 = -\frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta_0|^2}}. \tag{3.3.4}$$

Finally the functions f_0 and $f_0^\dagger$ are calculated simply

$$\begin{aligned} f_0 &= \frac{1}{\pi}\int d\xi_p \frac{\Delta_0}{(\xi_p + i\sqrt{\omega_n^2 + |\Delta_0|^2})(\xi_p - i\sqrt{\omega_n^2 + |\Delta_0|^2})} \\ &= \frac{\Delta_0}{\sqrt{\omega_n^2 + |\Delta_0|^2}} \end{aligned} \tag{3.3.5}$$

$$f_0^\dagger = \frac{1}{\pi}\int d\xi_p \frac{\Delta_0^*}{(\xi_p + i\sqrt{\omega_n^2 + |\Delta_0|^2})(\xi_p - i\sqrt{\omega_n^2 + |\Delta_0|^2})}$$

$$= \frac{\Delta_0^*}{\sqrt{\omega_n^2 + |\Delta_0|^2}}. \tag{3.3.6}$$

We see that in the homogenous state,

$$g_0 + \overline{g}_0 = 0 \tag{3.3.7}$$

$$g_0^2 + f_0 f_0^\dagger = 1 \tag{3.3.8}$$

which can be written as,

$$\check{g}_0^2 = \check{1} \tag{3.3.9}$$

These results are important and will be used later. The first is related to particle-hole symmetries of equilibrium systems. The second is a non-linear solution which will be shown to be a general solution of the full Eilenberger equations for inhomogeneous and impure superconductors.

3.4 Observables: N, $\mathbf{j}$ and Δ

To derive expressions for measurable quantities we start from the definitions of the Green's functions,

$$G_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) = -\langle \hat{T}_\tau \hat{\psi}_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}^\dagger_\beta(\mathbf{r}_2,\tau_2)\rangle \tag{3.4.1}$$

$$\begin{aligned}\overline{G}_{\alpha\beta}(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2) &= \langle \hat{T}_\tau \hat{\psi}^\dagger_\alpha(\mathbf{r}_1,\tau_1)\hat{\psi}_\beta(\mathbf{r}_2,\tau_2)\rangle \qquad (3.4.2)\\ &= G_{\beta\alpha}(\mathbf{r}_2,\tau_2;\mathbf{r}_1,\tau_1)\end{aligned}$$

$$G_{\alpha\alpha} = G_{\beta\beta} = G$$

$$\Delta_{\alpha\beta}(\mathbf{r}) = g\langle \hat{T}_\tau \hat{\psi}_\alpha(\mathbf{r},\tau)\hat{\psi}_\beta(\mathbf{r},\tau)\rangle = gF_{\alpha\beta}(\mathbf{r},\mathbf{r};\tau)$$

$$\Delta(\mathbf{r}) = gF(\mathbf{r},\mathbf{r};\tau) \tag{3.4.3}$$

and the momentum integrals in terms of quasi-classical Green's functions,

$$\int \frac{d^3p}{(2\pi)^3} G_{\omega_n}(\mathbf{p},\mathbf{k}) = \nu(0)\int \frac{d\Omega_p}{4\pi}\left[\int d\xi_p G_{\omega_n}(\mathbf{p},\mathbf{k}) + \frac{\pi}{i} g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) - i\pi(2\pi)^3\delta^{(3)}(\mathbf{k})\delta(\xi_p) sign(\omega_n)\right] \tag{3.4.4}$$

$$= \nu(0)\int \frac{d\Omega_p}{4\pi}\left[\frac{\pi}{i} g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) - (2\pi)^3\delta^{(3)}(\mathbf{k})\mathcal{P}\int \frac{d\xi_p}{\xi_p}\right] \tag{3.4.5}$$

$$\int \frac{d^3p}{(2\pi)^3} \overline{G}_{\omega_n}(\mathbf{p},\mathbf{k}) = \nu(0)\int \frac{d\Omega_p}{4\pi}\left[\int d\xi_p \overline{G}_{\omega_n}(\mathbf{p},\mathbf{k}) + \frac{\pi}{i} \overline{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) - i\pi(2\pi)^3\delta^{(3)}(\mathbf{k})\delta(\xi_p) sign(\omega_n)\right] \tag{3.4.6}$$

$$= \nu(0)\int \frac{d\Omega_p}{4\pi}\left[\frac{\pi}{i} \overline{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) - (2\pi)^3\delta^{(3)}(\mathbf{k})\mathcal{P}\int \frac{d\xi_p}{\xi_p}\right] \tag{3.4.7}$$

where $\nu(0) = \frac{mp_F}{2\pi^2}$.

3.4.1 Density

The number operator is given by,

$$\hat{\mathcal{N}} = \sum_{\alpha} \int d^3 r \psi_{\alpha}^{\dagger} \psi_{\alpha}$$

such that the number density can be found as,

$$\begin{aligned} N &= \sum_{\alpha} G_{\alpha\alpha}(\mathbf{r}, \tau; \mathbf{r}, \tau + 0) \\ &= 2G(\mathbf{r}, \tau; \mathbf{r}, \tau + 0) \qquad (3.4.8) \\ &= 2\overline{G}(\mathbf{r}, \tau + 0; \mathbf{r}, \tau). \qquad (3.4.9) \end{aligned}$$

Transforming to Matsubara frequencies, $G(\mathbf{r}_1, \mathbf{r}_2) = T \sum_n e^{-i\omega_n \tau} G_{\omega_n}(\mathbf{r}_1, \mathbf{r}_2)$ gives,

$$N = 2T \lim_{\tau \to -0} \sum_n G_{\omega_n}(\mathbf{r}, \mathbf{r}) e^{-i\omega_n \tau} \qquad (3.4.10)$$

$$N = 2T \lim_{\tau \to +0} \sum_n \overline{G}_{\omega_n}(\mathbf{r}, \mathbf{r}) e^{-i\omega_n \tau}. \qquad (3.4.11)$$

Which can be written in terms of the inhomogeneous Green's function in momentum space,

$$N(\mathbf{k}) = 2T \lim_{\tau \to -0} \sum_n \int \frac{d^3 p}{(2\pi)^3} G_{\omega_n}(\mathbf{p}, \mathbf{k}) e^{-i\omega_n \tau} \qquad (3.4.12)$$

$$N(\mathbf{k}) = 2T \lim_{\tau \to +0} \sum_n \int \frac{d^3 p}{(2\pi)^3} \overline{G}_{\omega_n}(\mathbf{p}, \mathbf{k}) e^{-i\omega_n \tau}. \qquad (3.4.13)$$

and the quasi-classical Green's functions,

$$\begin{aligned} N(\mathbf{k}) = 2T \lim_{\tau \to -0} \sum_n \Bigg[\nu(0) \int \frac{d\Omega_p}{4\pi} \Bigg[\int d\xi_p G_{\omega_n}(\mathbf{p}, \mathbf{k}) + \\ \frac{\pi}{i} g_{\omega_n}(\hat{\mathbf{p}}_F, \mathbf{k}) + i\pi (2\pi)^3 \delta^{(3)}(\mathbf{k}) \delta(\xi_p) sign(\omega_n) \Bigg] \Bigg] e^{-i\omega_n \tau} \qquad (3.4.14) \end{aligned}$$

the first term gives the normal-state electron density N_0, and since there is no singularity for the quasi-classicalal Green's function we write the above as,

$$N(\mathbf{k}) = (2\pi)^3\delta^{(3)}(\mathbf{k})N_0 + 2\nu(0)T\sum_n \int \frac{d\Omega_p}{4\pi}\left[\frac{\pi}{i}g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) + i\pi(2\pi)^3\delta^{(3)}(\mathbf{k})\delta(\xi_p)sign(\omega_n)\right]$$
$$= (2\pi)^3\delta^{(3)}(\mathbf{k})N_0 - 2\pi i\nu(0)T\sum_n \int \frac{d\Omega_p}{4\pi}g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) \tag{3.4.15}$$

as ω_n is odd, it makes $\sum_n sign(\omega_n) = 0$.

Similarly in terms of $\overline{g}$ we can write this,

$$N(\mathbf{k}) = (2\pi)^3\delta^{(3)}(\mathbf{k})N_0 - 2\pi i\nu(0)T\sum_n \int \frac{d\Omega_p}{4\pi}\overline{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}). \tag{3.4.16}$$

Adding half of each we see,

$$N(\mathbf{k}) = (2\pi)^3\delta^{(3)}(\mathbf{k})N_0 - \pi i\nu(0)T\sum_n \int \frac{d\Omega_p}{4\pi}(g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) + \overline{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})) \tag{3.4.17}$$

the combination, $g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) + \overline{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})$ vanishes due to particle-hole symmetry as the number of particles is unchanged in transitions between equilibrium states, $N = N_0$.

3.4.2 Current Density

To calculate the current we start from the momentum operator $\hat{p} = -i\nabla$ and its expectation value, $\langle\psi_\alpha^\dagger\hat{p}\psi_\alpha\rangle$, this gives the momentum density,

$$P = -\frac{i}{2}\sum_\alpha\left[\psi_\alpha^\dagger\nabla\psi_\alpha - \nabla\psi_\alpha^\dagger\psi_\alpha\right]$$
$$= -\frac{i}{2}\sum_\alpha\left[-\nabla_1 G_{\alpha\alpha}(\mathbf{r},\tau;\mathbf{r},\tau+0) + \nabla_2 G_{\alpha\alpha}(\mathbf{r},\tau;\mathbf{r},\tau+0)\right]_{r_1=r_2}$$
$$= \quad i(\nabla_1 - \nabla_2)G(\mathbf{r},\tau;\mathbf{r},\tau+0)|_{r_1=r_2} \tag{3.4.18}$$
$$= -i(\nabla_1 - \nabla_2)\overline{G}(\mathbf{r},\tau+0;\mathbf{r},\tau)|_{r_1=r_2} \tag{3.4.19}$$

again this can be transformed in terms of Matsubara frequencies, this time however the combination of differential operators removes the need for the convergence factor $e^{-\omega_n \tau}$ giving,

$$P = \quad iT \sum_n (\nabla_1 - \nabla_2) G_{\omega_n}(\mathbf{r}, \mathbf{r})|_{r_1 = r_2} \tag{3.4.20}$$

$$= -iT \sum_n (\nabla_1 - \nabla_2) \overline{G}_{\omega_n}(\mathbf{r}, \mathbf{r})|_{r_1 = r_2} \tag{3.4.21}$$

the current density is found from $\mathbf{j} = e\mathbf{v}$,

$$\mathbf{j} = \frac{ie}{m} T \sum_n (\nabla_1 - \nabla_2) G_{\omega_n}(\mathbf{r}, \mathbf{r})|_{r_1 = r_2} \tag{3.4.22}$$

including the vector potential,

$$\mathbf{j} = \frac{ie}{m} T \sum_n (\nabla_1 - \nabla_2) G_{\omega_n}(\mathbf{r}, \mathbf{r})|_{r_1 = r_2} - \frac{Ne^2 \mathbf{A}}{mc}. \tag{3.4.23}$$

Using $\nabla_1 - \nabla_2 = 2\nabla_r$ and transforming to momentum space gives,

$$\begin{aligned} \mathbf{j}(\mathbf{k}) &= \frac{ie}{m} T \sum_n \int \frac{d^3 p}{(2\pi)^3} (-2i\mathbf{p}) G_{\omega_n}(\mathbf{p}, \mathbf{k}) - \frac{Ne^2 \mathbf{A}(\mathbf{k})}{mc} \\ &= 2eT \sum_n \int \frac{d^3 p}{(2\pi)^3} \mathbf{v}_F G_{\omega_n}(\mathbf{p}, \mathbf{k}) - \frac{Ne^2 \mathbf{A}(\mathbf{k})}{mc}. \end{aligned} \tag{3.4.24}$$

This can once again be written in terms of the quasi-classical Green's function.It can be shown that the large-ξ_p contribution from the normal-state cancels the term $\frac{Ne^2 \mathbf{A}(\mathbf{k})}{mc}$ [41] giving,

$$\mathbf{j}(\mathbf{k}) = -2\pi i e \nu(0) T \sum_n \int \frac{d\Omega_p}{4\pi} \mathbf{v}_F g_{\omega_n}(\hat{\mathbf{p}}_F, \mathbf{k}) \tag{3.4.25}$$

$$= \quad 2\pi i e \nu(0) T \sum_n \int \frac{d\Omega_p}{4\pi} \mathbf{v}_F \overline{g}_{\omega_n}(\hat{\mathbf{p}}_F, \mathbf{k}) \tag{3.4.26}$$

$$\mathbf{j}(\mathbf{k}) = -\pi i e\nu(0)T\sum_n \int \frac{d\Omega_p}{4\pi}\mathbf{v}_F(g_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k}) - \overline{g}_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})) \tag{3.4.27}$$

$$= -\pi i e\nu(0)T\sum_n \langle \mathrm{tr}\mathbf{v}_F\check{\sigma}_3\check{g}\rangle \tag{3.4.28}$$

3.4.3 Order Parameter: The self consistency equation

Finally we come to the most important observable, the order parameter,

$$\begin{aligned}
\Delta(\mathbf{R}) &= gT\sum_n F_{\omega_n}(\mathbf{r},\mathbf{R}) \\
\Delta(\mathbf{k}) &= gT\sum_n \int \frac{d^3p}{(2\pi)^3}F_{\omega_n}(\mathbf{p},\mathbf{k}) \\
&= \underbrace{g\nu(0)}_{\lambda}\pi T\sum_n \int \frac{d\Omega_p}{4\pi}\int \frac{d\xi_p}{\pi}F_{\omega_n}(\mathbf{p},\mathbf{k}) \\
&= \lambda\pi T\sum_n \int \frac{d\Omega_p}{4\pi}f_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})
\end{aligned}$$

Using the symmetry for ω_n we write an expression for the order parameter in terms of the average over the FS of f,

$$\frac{\Delta(\mathbf{k})}{\lambda} = 2\pi T\sum_{n\geq 0}\langle f_{\omega_n}(\hat{\mathbf{p}}_F,\mathbf{k})\rangle. \tag{3.4.29}$$

Although this is enough, we can calculate the constant λ by use of the homogeneous case,

$$\begin{aligned}
\frac{\Delta_0}{\lambda} &= 2\pi T\sum_{n\geq 0}\langle f^{(0)}\rangle \\
\frac{\Delta_0}{\lambda} &= 2\pi T\sum_{n\geq 0}\left\langle \frac{\Delta_0}{\sqrt{\omega_n^2 + |\Delta_0|^2}}\right\rangle.
\end{aligned} \tag{3.4.30}$$

The term on the right hand side is divergent, this happens because in the BCS theory the coupling is assumed to be independent of energy. We therefore need to introduce a cut-off at some characteristic energy, Ω_{BCS}. This cut-off must be larger

than T_c but much smaller than E_F. For the sums written, this is converted to a number, $N_0 = \frac{\Omega_{BCS}}{2\pi T}$.

Including this and cancelling the Δ_0 we continue to calculate λ,

$$\frac{1}{\lambda} = 2\pi T \sum_{n\geq 0}^{N_0(T)} \frac{1}{\sqrt{\omega_n^2 + |\Delta_0|^2}}. \tag{3.4.31}$$

Near T_c, where the order parameter vanishes we can find a relationship for λ

$$\frac{1}{\lambda} = 2\pi T_c \sum_{n\geq 0}^{N_0(T_c)} \frac{1}{\omega_n}. \tag{3.4.32}$$

For the time being we shall define a temperature-independent frequency $\overline{\omega}_n = \frac{\omega_n}{T}$ with $\omega_n = 2\pi T\left(n + \frac{1}{2}\right)$. We do this since the sum's limit is temperature dependent but the summand itself is not and therefore we avoid any confusion.

$$\frac{1}{\lambda} = 2\pi \sum_{n\geq 0}^{N_0(T_c)} \frac{1}{\overline{\omega}_n} = \sum_{n\geq 0}^{N_0(T_c)} \frac{1}{n + \frac{1}{2}} \approx \ln(N_0(T_c)) + \ln 4 + C. \tag{3.4.33}$$

where $C = 0.5772\ldots$ is the Euler constant.

We would like to use the result near T_c to find a value for λ in the whole temperature range. To do this, we first consider the sum,

$$\begin{aligned} 2\pi \sum_{n\geq 0}^{N_0(T)} \frac{1}{\overline{\omega}_n} &= 2\pi \sum_{n\geq 0}^{N_0(T_c)} \frac{1}{\overline{\omega}_n} + 2\pi \sum_{N_0(T_c)}^{N_0(T)} \frac{1}{\overline{\omega}_n} \\ &= \frac{1}{\lambda} + 2\pi \left\{ \sum_{0}^{N_0(T)} \frac{1}{\overline{\omega}_n} - \sum_{0}^{N_0(T_c)} \frac{1}{\overline{\omega}_n} \right\} \\ &= \frac{1}{\lambda} + \ln\left(\frac{N_0(T)}{N_0(T_c)}\right). \end{aligned}$$

Such that,

$$\begin{aligned}\frac{1}{\lambda} &= 2\pi T \sum_{n\geq 0}^{N_0(T)} \frac{1}{\omega_n} - \ln\left(\frac{N_0(T)}{N_0(T_c)}\right) \\ &= 2\pi T \sum_{n\geq 0}^{N_0(T)} \frac{1}{\omega_n} - \ln\left(\frac{T_c}{T}\right). \end{aligned} \tag{3.4.34}$$

Which can now be used to remove λ

$$\begin{aligned}\frac{\Delta}{\lambda} &= 2\pi T \sum_{n\geq 0} \langle f\rangle \\ \Delta\left\{2\pi T \sum_{n\geq 0}^{N_0(T)} \frac{1}{\omega_n} - \ln\left(\frac{T_c}{T}\right)\right\} &= 2\pi T \sum_{n\geq 0} \langle f\rangle \\ \Delta \ln\left(\frac{T_c}{T}\right) &= 2\pi T \sum_{n\geq 0} \left[\frac{\Delta}{\omega_n} - \langle f\rangle\right] \end{aligned} \tag{3.4.35}$$

where the cut-off is assumed to be used when needed.

Later we will consider an expansion in orders of $\mathbf{v}_F$ for the Green's functions, as such $f = f_0 + f_1 + f_2 + \ldots$. For this case we consider the self consistency written in the following way. Again using the homogeneous result to remove λ,

$$\frac{1}{\lambda} = 2\pi T \sum_{n\geq 0}^{N_0(T)} \frac{1}{\sqrt{\omega_n^2 + |\Delta_0|^2}}$$

we can write,

$$\begin{aligned}\Delta 2\pi T \sum_{n\geq 0}^{N_0(T)} \frac{1}{\sqrt{\omega_n^2 + |\Delta_0|^2}} &= 2\pi T \sum_{n\geq 0} \langle f\rangle \\ &= 2\pi T \sum_{n\geq 0} \langle f_0 + \underbrace{f_1}_{=0} + f_2\rangle \\ &= 2\pi T \sum_{n\geq 0} \langle f_0\rangle + 2\pi T \sum_{n\geq 0} \langle f_2\rangle \end{aligned}$$

$$= 2\pi T \sum_{n\geq 0} \frac{\Delta}{\sqrt{\omega_n^2 + |\Delta|^2}} + 2\pi T \sum_{n\geq 0} \langle f_2 \rangle$$

$$\Delta 2\pi T \sum_{n\geq 0} \left[\frac{1}{\sqrt{\omega_n^2 + |\Delta_0|^2}} - \frac{1}{\sqrt{\omega_n^2 + |\Delta|^2}} \right] = 2\pi T \sum_{n\geq 0} \langle f_2 \rangle$$

$$\alpha \Delta = 2\pi T \sum_{n\geq 0} \langle f_2 \rangle \tag{3.4.36}$$

where $\alpha = 2\pi T \sum_{n\geq 0} \left[\frac{1}{\sqrt{\omega_n^2+|\Delta_0|^2}} - \frac{1}{\sqrt{\omega_n^2+|\Delta|^2}} \right]$.

Special Results; $T = 0, T = T_c$

We can perform the sum in two special limits, first when $T = T_c$ we already know that the order parameter vanishes giving,

$$\begin{aligned} \frac{1}{\lambda} &\approx \ln(N_0(T_c)) + \ln 4 + C \\ &= \ln \frac{2\Omega_{BCS}}{\pi T_c} + C \\ T_c &= \frac{2\Omega_{BCS}\gamma}{\pi} e^{-\frac{1}{\lambda}} \end{aligned} \tag{3.4.37}$$

where $\gamma = e^C \approx 1.711$.

For $T = 0$ we consider the sum in terms of real frequencies and write,

$$\frac{1}{\lambda} = \int_{|\Delta_0|}^{\Omega_{BCS}} \tanh\left(\frac{\epsilon}{2T}\right) \frac{d\epsilon}{\sqrt{\epsilon^2 - |\Delta_0|^2}} \tag{3.4.38}$$

which at $T = 0$ for $\Omega_{BCS} \gg |\Delta_0|$ gives,

$$\frac{1}{\lambda} = \ln \frac{2\Omega_{BCS}}{|\Delta_0|} \tag{3.4.39}$$

so at zero temperature, $|\Delta_0| = \frac{\pi}{\gamma} T_c$, a relationship which will be used later in considering low-temperature effects.

3.5 Eilenberger Equations

Paraphrasing Eilenberger a little, his motivation seems simple: since the Gorkov equations describe superconductivity very well and given that superconductivity occurs very near the Fermi surface, can you approximate the Gorkov equations at the Fermi surface by integrating out the energy dependence?

The answer to this was yes and the result of performing it was to create a set of equations which are easier to work with than Gorkov's original ones.

Eilenberger's method is a little dated, so we present a new one based on standard techniques [42][43]. These techniques will be altered a little when considering impurities, as at this point existing derivations are somewhat insufficient.

3.5.1 Pure Eilenberger Equations

To get a better feel for what is going on, we first restrict our attention to the pure Eilenberger equations and start from the Gorkov equations, (3.2.14) for a superconductor in an applied field,

$$\begin{pmatrix} i\omega_n - \mathcal{H}_1 & \Delta(\mathbf{r}_1) \\ -\Delta^*(\mathbf{r}_1) & -i\omega_n - \mathcal{H}_1^* \end{pmatrix} \check{G}(\mathbf{r}_1, \mathbf{r}_2) = \check{1}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) \tag{3.5.1a}$$

$$\check{G}(\mathbf{r}_1, \mathbf{r}_2) \begin{pmatrix} i\omega_n - \mathcal{H}_2^* & \Delta(\mathbf{r}_2) \\ -\Delta^*(\mathbf{r}_2) & -i\omega_n - \mathcal{H}_2 \end{pmatrix} = \check{1}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) \tag{3.5.1b}$$

where the index ω_n has been dropped for notational reasons and the the Hamiltonian is as before,

$$\mathcal{H}_i = -\frac{1}{2m}\left(\nabla_i - \frac{ie}{c}\mathbf{A}(\mathbf{r}_i)\right)^2 - \mu + \sum_a U(\mathbf{r}_i - \mathbf{r}_a).$$

Step one is to simplify the Hamiltonian and initially we focus on the applied field. We choose a gauge such that $\nabla_r \cdot \mathbf{A}(\mathbf{r}) = 0$, the Fourier transform of which gives $\mathbf{k} \cdot \mathbf{A}(\mathbf{k}) = 0$.

Further to this we assume any applied field is small enough that terms of order A^2 are negligible allowing the Hamiltonian to be written as,

$$\mathcal{H}_i = -\frac{1}{2m}\nabla_i^2 + \frac{ie}{mc}\mathbf{A}(\mathbf{r}_i)\nabla_i - \mu + \sum_a U(\mathbf{r}_i - \mathbf{r}_a).$$

Further approximations involve more harsh spatial restrictions, so we start by thinking about how to describe an inhomogeneous superconductor. The fact that we are describing a superconductor is crucial as it allows us to make use of relevant physical length scales.

When possible a superconductor would prefer being homogeneous as there is an energy associated with varying the order parameter. This cost gives rise to a typical length scale, over which the superconductor varies. This is known as the coherence length, ξ.

We therefore separate out two scales and obtain a small homogeneous scale on which we can Fourier transform and a much larger scale which describes the residual inhomogeneity. To do this we rewrite our equations in terms of centre of mass and difference variables.

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mathbf{r}_1 = \mathbf{R} + \frac{\mathbf{r}}{2}$$
$$\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2) \qquad \mathbf{r}_2 = \mathbf{R} - \frac{\mathbf{r}}{2}$$

where $\mathbf{r} \ll \mathbf{R}$.

The gradients become

$$\nabla_{1,2} = \frac{1}{2}\nabla_R \pm \nabla_r$$

$$\nabla_{1,2}^2 = \frac{1}{4}\nabla_R^2 \pm \nabla_R\nabla_r + \nabla_r^2$$

knowing that the smallest length scale is described by r, we therefore keep to order ∇_r and approximate the gradients as,

$$\nabla_{1,2} = \pm\nabla_r$$
$$\nabla_{1,2}^2 = \nabla_r^2 \pm \nabla_R\nabla_r.$$

Along with this we assert that the functions, $\mathbf{A}$ and Δ are smoothly varying functions of events occurring on the larger scale, i.e. $\mathbf{A}(\mathbf{r}_{1,2}), \Delta(\mathbf{r}_{1,2}) \rightarrow \mathbf{A}\left(\mathbf{R} \pm \frac{\mathbf{r}}{2}\right), \Delta\left(\mathbf{R} \pm \frac{\mathbf{r}}{2}\right) \rightarrow \mathbf{A}(\mathbf{R}), \Delta(\mathbf{R})$.

In the pure case, for which $U = 0$ these conditions allow us to rewrite the Gorkov equations as,

$$\begin{pmatrix} i\omega_n + \frac{\nabla_r^2}{2m} + \frac{\nabla_r}{2m}\cdot\nabla_R + \mu & 0 \\ 0 & -i\omega_n + \frac{\nabla_r^2}{2m} + \frac{\nabla_r}{2m}\cdot\nabla_R + \mu \end{pmatrix} \check{G}(\mathbf{r},\mathbf{R}) + \begin{pmatrix} -\frac{ie}{mc}\mathbf{A}(\mathbf{R})\nabla_r & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & \frac{ie}{mc}\mathbf{A}(\mathbf{R})\nabla_r \end{pmatrix} \check{G}(\mathbf{r},\mathbf{R}) = \check{1}\delta^{(3)}(\mathbf{r}) \quad (3.5.2)$$

$$\check{G}(\mathbf{r},\mathbf{R}) \begin{pmatrix} i\omega_n + \frac{\nabla_r^2}{2m} - \frac{\nabla_r}{2m}\cdot\nabla_R + \mu & 0 \\ 0 & -i\omega_n + \frac{\nabla_r^2}{2m} - \frac{\nabla_r}{2m}\cdot\nabla_R + \mu \end{pmatrix} + \check{G}(\mathbf{r},\mathbf{R}) \begin{pmatrix} -\frac{ie}{mc}\mathbf{A}(\mathbf{R})\nabla_r & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & \frac{ie}{mc}\mathbf{A}(\mathbf{R})\nabla_r \end{pmatrix} = \check{1}\delta^{(3)}(\mathbf{r}). \quad (3.5.3)$$

We know that the superconductor cannot vary on scales smaller than those described

by the coherence length, this allows us to define the Fourier transform,

$$\int d^3 r e^{i\mathbf{pr}} \check{G}(\mathbf{r},\mathbf{R}) = \check{G}(\mathbf{p},\mathbf{R}).$$

Giving the mixed representation,

$$\begin{pmatrix} i\omega_n - \xi_p + \frac{i\mathbf{v}}{2}\cdot\nabla_R & 0 \\ 0 & -i\omega_n - \xi_p + \frac{i\mathbf{v}}{2}\cdot\nabla_R \end{pmatrix} \check{G}(\mathbf{p},\mathbf{R}) + \\ \begin{pmatrix} \frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & -\frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) \end{pmatrix} \check{G}(\mathbf{p},\mathbf{R}) = \check{1} \quad (3.5.4)$$

and

$$\check{G}(\mathbf{p},\mathbf{R}) \begin{pmatrix} i\omega_n - \xi_p - \frac{i\mathbf{v}}{2}\cdot\nabla_R & 0 \\ 0 & -i\omega_n - \xi_p - \frac{i\mathbf{v}}{2}\cdot\nabla_R \end{pmatrix} + \\ \check{G}(\mathbf{p},\mathbf{R}) \begin{pmatrix} \frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & -\frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) \end{pmatrix} = \check{1} \quad (3.5.5)$$

where $\xi_p = \frac{p^2}{2m} - \mu$ and $\mathbf{p} = m\mathbf{v}$.

Eilenberger would like to integrate out the energy dependence, ξ_p, of these equations. The fact that both have a linear dependence on this variable is at first sight discouraging. However, if one were to take the difference between the equations we see that the energy, ξ_p drops out.

$$i\mathbf{v}\cdot\nabla_R\check{G} + i\omega_n(\check{\tau}_3\check{G} - \check{G}\check{\tau}_3) + (\check{M}\check{G} - \check{G}\check{M}) = 0 \quad (3.5.6)$$

with

$$\check{M}(\mathbf{R}) = \begin{pmatrix} \frac{e\mathbf{v}}{c} \cdot \mathbf{A}(\mathbf{R}) & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & -\frac{e\mathbf{v}}{c} \cdot \mathbf{A}(\mathbf{R}) \end{pmatrix}$$

These equations can now be simplified by integrating out the energy to reach the pure, quasi-classical Eilenberger equations,

$$i\mathbf{v}_F \cdot \nabla_R \check{g} + i\omega_n(\check{\tau}_3\check{g} - \check{g}\check{\tau}_3) + (\check{m}\check{g} - \check{g}\check{m}) = 0 \tag{3.5.7}$$

$$\check{m}(\mathbf{R}) = \begin{pmatrix} \frac{e\mathbf{v}_F}{c} \cdot \mathbf{A}(\mathbf{R}) & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & -\frac{e\mathbf{v}_F}{c} \cdot \mathbf{A}(\mathbf{R}) \end{pmatrix}$$

3.5.2 Homogeneous Results

We may recalculate the homogeneous results by setting $\nabla_R \check{g}_0 = 0$ to give,

$$i\omega_n(\check{\tau}_3\check{g}_0 - \check{g}_0\check{\tau}_3) + (\check{m}\check{g}_0 - \check{g}_0\check{m}) = 0 \tag{3.5.8}$$

when expanded this yields,

$$\begin{pmatrix} i(\Delta_0 f_0^\dagger - \Delta_0^* f_0) & 2\omega_n f_0 - 2\Delta_0 g_0 \\ 2\omega_n f_0^\dagger - 2\Delta_0^* g_0 & -i(\Delta_0 f_0^\dagger - \Delta_0^* f_0) \end{pmatrix} = 0 \tag{3.5.9}$$

In order to solve this we must use the condition $\check{g}_0^2 = \check{1}$. We know this is the case in the homogeneous state and again it yields,

$$\begin{aligned} g_0 = S_n\omega_n, \quad f_0 = S_n\Delta_0, \quad f_0^\dagger = S_n\Delta_0^* \\ S_n = \frac{1}{\sqrt{\omega_n^2 + |\Delta_0|^2}} \end{aligned} \tag{3.5.10}$$

The use of $\check{g}_0^2 = \check{1}$ was important, because we lost information by taking the difference between the two transformed Gorkov equations.

We can see by multiplying equation (3.5.7) on the left and right by $\check{g}$ and adding, that $\check{g}^2$ is also a solution, one could choose a solution of the form $\check{g}^2 = \check{A} + \check{B}\check{g}$. In the homogeneous state we showed that $\check{g}_0^2 = \check{1}$, a solution one would like to match to in an inhomogeneous superconductor when far from the inhomogeneity. We therefore take the solution $\check{g}^2 = \check{1}$ as this satisfies the want of the superconductor to behave homogeneously, this is likely since the coherence length would screen out any inhomogeneity. Furthermore, the off-diagonal terms being zero led to the conservation of particle number. Although this is not the most general solution, the non-linear result will provide the necessary information to produce more solutions as will be seen later in chapters 4 and 5.

3.5.3 Theory of Alloys

Having calculated the pure Eilenberger equations we must now consider the effect of impurities.

Several impurity calculations exist, the most relavent for this method being AGD [18] and the extended version in Bennemann and Ketterson [44], Kopnin [15], Lifshitz [13] and Ambegaokar [17]. However, each of the existing methods are unsatisfactory for the theory of inhomogenous superconductors.

In all cases this is due to how the spatial dependence of the order parameter is treated along with performing the entire calculation in momentum space.

Considering AGD as a typical example they perform an impurity average not by considering the exact solution of equations,

$$\left[i\omega + \frac{\nabla^2}{2m} + \mu - \sum_{r_a} u(\mathbf{r} - \mathbf{r}_a)\right] G_\omega(\mathbf{r}, \mathbf{r}') + \Delta(\mathbf{r}) F_\omega^\dagger(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$
$$\left[-i\omega + \frac{\nabla^2}{2m} + \mu - \sum_{r_a} u(\mathbf{r} - \mathbf{r}_a)\right] F_\omega^\dagger(\mathbf{r}, \mathbf{r}') - \Delta^*(\mathbf{r}) G_\omega(\mathbf{r}, \mathbf{r}') = 0$$

but by considering their averages, and therefore averaged Green's functions and an

averaged order parameter.

The spatial dependence of the order parameter presents a problem. The presence of the impurities would in general change the gap, and as this is found self-consistently from terms in the integral equation, complicated vertex corrections could be needed. AGD argue that considering the averaged order parameter, these corrections vanish, allowing the replacement, $\overline{\Delta(\mathbf{r})} = \Delta^{(0)}$.

In averaging, we require the distance between impurity atoms being much larger than the lattice spacing. In the case of the order parameter, the coherence length screens the effect of the impurity justifying the replacement by an averaged value.

The result and indeed method of their calculation is similar to that of an normal metal. Transforming in both $\mathbf{r}$ and $\mathbf{r}'$ the sum over resulting terms requires the calculation of the self energies, marked by overlines, in the following equations,

$$(i\omega - \xi - \overline{G}_\omega)G(\mathbf{p}) + (\Delta^{(0)} + \overline{F^\dagger_\omega})F^\dagger(\mathbf{p}) = 1$$
$$(i\omega + \xi + \overline{G}_{-\omega})F^\dagger(\mathbf{p}) + (\Delta^{(0)} + \overline{F^\dagger_\omega})G(\mathbf{p}) = 0$$

Straight away we notice the single momentum dependence. This is a result of taking the momentum transform of both lengths. But more worryingly the spatial dependency given by the real-space information used in the pure result is no longer present.

Indeed, the fact that AGD now justify their substitution of a 'constant' order parameter via performing the homogeneous calculation seems insufficient as the entire calculation is only valid for the homogenous case where the order parameter is fully delocalised.

In correcting this calculation we must re-include the spatial variation and have enough degrees of freedom in order to calculate inhomogenous effects.

Many derivations of the Eilenberger equations use the above result with the spatial dependence simply re-included, this hints at the cure being largely notational.

We know that length scales are important for the superconductor and, knowing that the impurities are screened by the coherence length lets us separate out the length scales. To that end we use the same lengths as for the pure result, including the potential as a function of $\mathbf{r}-\mathbf{r}_a$. This forces the integral equation to be taken over the smaller length scale, separating out the larger, independent spatial effects.

Further, these larger scale inhomogeneities are not transformed over, since this implies homogeneity. If one chooses to transform on the scale $\mathbf{R}$ this would include an additional $\delta^{(3)}(\mathbf{k})$ which would wipe out the inhomogeneities. This is why when AGD transform in $\mathbf{r}$ and $\mathbf{r}'$ their result is necessarily homogeneous, implying a constant order parameter.

The calculation follows in a similar manor to existing derivations but where the extra degree of freedom for the inhomogeneity is included from the start.

Including the impurity term as a function of $\mathbf{r}-\mathbf{r}_a$ we have,

$$\begin{pmatrix} i\omega_n+\frac{\nabla_r^2}{2m}+\frac{\nabla_r}{2m}\cdot\nabla_R+\mu & 0 \\ 0 & -i\omega_n+\frac{\nabla_r^2}{2m}+\frac{\nabla_r}{2m}\cdot\nabla_R+\mu \end{pmatrix}\check{G}(\mathbf{r},\mathbf{R})+$$
$$\begin{pmatrix} -\frac{ie}{mc}\mathbf{A}(\mathbf{R})\nabla_r & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & \frac{ie}{mc}\mathbf{A}(\mathbf{R})\nabla_r \end{pmatrix}\check{G}(\mathbf{r},\mathbf{R})+$$
$$\sum_a \begin{pmatrix} U(\mathbf{r}-\mathbf{r}_a) & 0 \\ 0 & U(\mathbf{r}-\mathbf{r}_a) \end{pmatrix}\check{G}(\mathbf{r},\mathbf{R})=\check{1}\delta^{(3)}(\mathbf{r}) \quad (3.5.11)$$

Using the Fourier transforms,

$$\check{G}(\mathbf{p},\mathbf{R})=\int d^3r e^{i\mathbf{pr}}\check{G}(\mathbf{r},\mathbf{R})$$
$$U(\mathbf{r}-\mathbf{r}_a)=\int \frac{d^3q}{(2\pi)^3}e^{-i\mathbf{q}(\mathbf{r}-\mathbf{r}_a)}U(\mathbf{q})$$

we write the transformed equation as,

$$\overbrace{\left\{\begin{pmatrix} i\omega_n - \xi_p + \frac{i\mathbf{v}}{2}\nabla_R & 0 \\ 0 & -i\omega_n - \xi_p + \frac{i\mathbf{v}}{2}\nabla_R \end{pmatrix} + \begin{pmatrix} \frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & -\frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) \end{pmatrix}\right\}}^{\check{G}_0^{-1}(\mathbf{p},\mathbf{R})} \check{G}(\mathbf{p},\mathbf{R}) - \sum_a \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{r}_a\mathbf{q}} \begin{pmatrix} U(\mathbf{q}) & 0 \\ 0 & U(\mathbf{q}) \end{pmatrix} \check{G}(\mathbf{p}-\mathbf{q},\mathbf{R}) = \check{1}. \quad (3.5.12)$$

We set about averaging the impurities after symbolically multiplying through by $\check{G}_0(\mathbf{p},\mathbf{R})$ and rearranging to give the integral equation,

$$\check{G}(\mathbf{p},\mathbf{R}) = \check{G}_0(\mathbf{p},\mathbf{R}) + \check{G}_0(\mathbf{p},\mathbf{R}) \sum_a \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{r}_a\mathbf{q}} \check{U}(\mathbf{q}) \check{G}(\mathbf{p}-\mathbf{q},\mathbf{R}). \quad (3.5.13)$$

To solve this equation we assume the impurity potential to be small and solve order by order till a series forms.

Zeroth order,

Taking the first term to be zeroth with respect to the impurities we have,

$$\check{G}^{(0)}(\mathbf{p},\mathbf{R}) = \check{G}_0(\mathbf{p},\mathbf{R}) \quad (3.5.14)$$

This can be represented diagrammatically, if one wishes, as a single line implying the electrons move across the system unhindered.

Figure 3.1: Zeroth order, no interactions

First order,

$$\check{G}^{(1)}(\mathbf{p},\mathbf{R}) = \check{G}_0(\mathbf{p},\mathbf{R}) \sum_a \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{r}_a\mathbf{q}} \check{U}(\mathbf{q}) \check{G}^{(0)}(\mathbf{p}-\mathbf{q},\mathbf{R}) \tag{3.5.15}$$

assuming many impurity sites we replace the sum with an integral,

$$\sum_a \to n \int d^3 r_a$$

averaging over impurities is simply integrating over all sites. Where n is the impurity density. Doing so gives delta functions, the existence of which conserves momenta during collisions.

$$\begin{aligned}\check{G}^{(1)}(\mathbf{p},\mathbf{R}) &= \check{G}_0(\mathbf{p},\mathbf{R}) \int \frac{d^3q}{(2\pi)^3} n \underbrace{\int d^3 r_a e^{i\mathbf{r}_a\mathbf{q}}}_{(2\pi)^3\delta^{(3)}(\mathbf{q})} \check{U}(\mathbf{q}) \check{G}^{(0)}(\mathbf{p}-\mathbf{q},\mathbf{R}) \\ &= \check{G}_0(\mathbf{p},\mathbf{R}) n \check{U}(0) \check{G}^{(0)}(\mathbf{p},\mathbf{R}) \end{aligned} \tag{3.5.16}$$

Again this can be represented by a diagram,

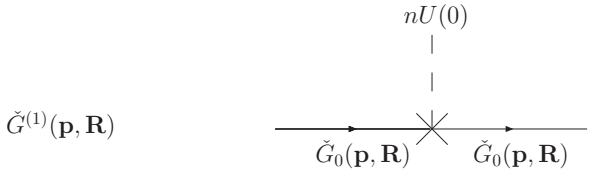

Figure 3.2: Single site interaction

this time the electrons move part way across the system, hit an impurity and move on.

Second order,

$$\check{G}^{(2)}(\mathbf{p},\mathbf{R}) = \check{G}_0(\mathbf{p},\mathbf{R})\sum_a \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{r}_a\mathbf{q}}\check{U}(\mathbf{q})\left\{\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\sum_b \int \frac{d^3q'}{(2\pi)^3} e^{i\mathbf{r}_b\mathbf{q}'}\check{U}(\mathbf{q}')\check{G}^{(0)}(\mathbf{p}-\mathbf{q}-\mathbf{q}',\mathbf{R})\right\}$$
$$= \sum_{a,b}\int \frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3} e^{i\mathbf{r}_a\mathbf{q}}e^{i\mathbf{r}_b\mathbf{q}'}\check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q})\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(\mathbf{q}')\check{G}^{(0)}(\mathbf{p}-\mathbf{q}-\mathbf{q}',\mathbf{R}). \tag{3.5.17}$$

Here there are two possibilities for impurity averaging, either the particle scatters off the same site twice, or two differing sites. Considering first the two single sites we have $a \neq b$. The two exponentials turn to delta functions, $\delta^{(3)}(\mathbf{q}), \delta^{(3)}(\mathbf{q}')$ giving

$$\check{G}^{(2)}(\mathbf{p},\mathbf{R}) = \check{G}_0(\mathbf{p},\mathbf{R})n\check{U}(0)\check{G}_0(\mathbf{p},\mathbf{R})n\check{U}(0)\check{G}^{(0)}(\mathbf{p},\mathbf{R}). \tag{3.5.18}$$

From here we see a series of single site collisions given by $\begin{pmatrix} nU(0) & 0 \\ 0 & nU(0) \end{pmatrix}^i$ exists. Since U is small this can be summed and corresponds to nothing more than a shift of the chemical potential.

Next the two site interaction $a = b$,

$$\check{G}^{(2)}(\mathbf{p},\mathbf{R}) = \sum_a \int \frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3} e^{i\mathbf{r}_a(\mathbf{q}+\mathbf{q}')}\check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q})\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(\mathbf{q}')\check{G}^{(0)}(\mathbf{p}-\mathbf{q}-\mathbf{q}',\mathbf{R})$$
$$= n\int \frac{d^3qd^3q'}{(2\pi)^3}\delta^{(3)}(\mathbf{q}+\mathbf{q}')\check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q})\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(\mathbf{q}')\check{G}^{(0)}(\mathbf{p}-\mathbf{q}-\mathbf{q}',\mathbf{R})$$
$$= n\int \frac{d^3q}{(2\pi)^3}\check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q})\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(-\mathbf{q})\check{G}^{(0)}(\mathbf{p},\mathbf{R})$$

$$= \check{G}_0(\mathbf{p},\mathbf{R})\left\{n\int\frac{d^3q}{(2\pi)^3}|\check{U}(\mathbf{q})|^2\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\right\}\check{G}^{(0)}(\mathbf{p},\mathbf{R}). \tag{3.5.19}$$

Which defines the self-energy

$$\begin{aligned}\check{\Sigma}^{(0)}(\mathbf{p},\mathbf{R}) &= n\int\frac{d^3q}{(2\pi)^3}|\check{U}(\mathbf{q})|^2\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\\ &= n\int\frac{d^3p'}{(2\pi)^3}|\check{U}(\mathbf{p}-\mathbf{p}')|^2\check{G}_0(\mathbf{p}',\mathbf{R})\end{aligned} \tag{3.5.20}$$

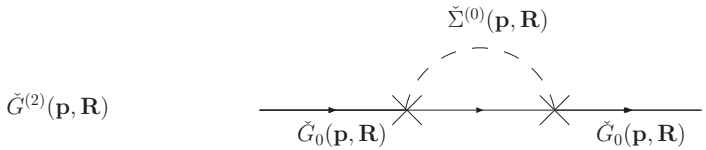

Figure 3.3: Two, same site interactions

The self-energy represents the transfer of momentum from the electron to impurity and back again as such it could transfer a range of momentum making it the key quantity for calculating the effect of impurities.

Third Order

$$\check{G}^{(3)}(\mathbf{p},\mathbf{R}) = \sum_{a,b,c}\int \frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3}\frac{d^3q''}{(2\pi)^3} e^{i\mathbf{r}_a\mathbf{q}} e^{i\mathbf{r}_b\mathbf{q}'} e^{i\mathbf{r}_c\mathbf{q}''} \check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q})\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(\mathbf{q}')\check{G}_0(\mathbf{p}-\mathbf{q}-\mathbf{q}',\mathbf{R})\check{U}(\mathbf{q}'')\times$$
$$\check{G}^{(0)}(\mathbf{p}-\mathbf{q}-\mathbf{q}'-\mathbf{q}'',\mathbf{R}) \quad (3.5.21)$$

the only distinct interaction for three sites is where $a = b = c$ giving,

$$\check{G}^{(3)}(\mathbf{p},\mathbf{R}) = n\int \frac{d^3q d^3q' d^3q''}{(2\pi)^6}\delta^{(3)}(\mathbf{q}+\mathbf{q}'+\mathbf{q}'')\check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q})\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(\mathbf{q}')\check{G}_0(\mathbf{p}-\mathbf{q}-\mathbf{q}',\mathbf{R})\check{U}(\mathbf{q}'')\times$$
$$\check{G}^{(0)}(\mathbf{p}-\mathbf{q}-\mathbf{q}'-\mathbf{q}'',\mathbf{R}) \quad (3.5.22)$$

$$\begin{aligned}\check{G}^{(3)}(\mathbf{p},\mathbf{R}) &= \check{G}_0(\mathbf{p},\mathbf{R})\left\{n\int \frac{d^3q d^3q''}{(2\pi)^6}\check{U}(\mathbf{q})\check{U}(-\mathbf{q}-\mathbf{q}'')\check{U}(\mathbf{q}'')\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{G}_0(\mathbf{p}+\mathbf{q}'',\mathbf{R})\right\}\check{G}^{(0)}(\mathbf{p},\mathbf{R}) \\ &= \check{G}_0(\mathbf{p},\mathbf{R})\Sigma^{(3)}(\mathbf{p},\mathbf{R})\check{G}^{(0)}(\mathbf{p},\mathbf{R}) \quad (3.5.23)\end{aligned}$$

$$\begin{aligned}\Sigma^{(3)}(\mathbf{p},\mathbf{R}) &= n\int \frac{d^3q d^3q''}{(2\pi)^6}\check{U}(\mathbf{q})\check{U}(-\mathbf{q}-\mathbf{q}'')\check{U}(\mathbf{q}'')\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{G}_0(\mathbf{p}+\mathbf{q}'',\mathbf{R}) \\ &= n\int \frac{d^3q_1 d^3q_2}{(2\pi)^6}\check{U}(\mathbf{p}-\mathbf{q}_1)\check{U}(\mathbf{q}_1-\mathbf{q}_2)\check{U}(\mathbf{q}_2-\mathbf{p})\check{G}_0(\mathbf{q}_1,\mathbf{R})\check{G}_0(\mathbf{q}_2,\mathbf{R}) \quad (3.5.24)\end{aligned}$$

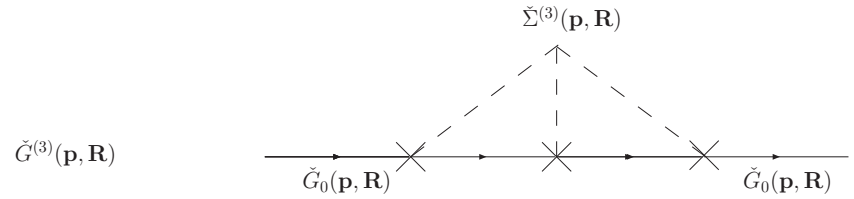

Figure 3.4: Interacting with one site three times

Estimation of this where the the Green's function gives the density of particles $\sim nU^3\nu^2$ which when compared to $\Sigma^0 \sim nU^2\nu$ contains an extra power of $U\nu$. Since this is in turn of order $\frac{u}{E_F} \ll 1$ the process is neglected. We therefore only consider two-site interactions.

Fourth Order

Finally we consider fourth order interactions, made from these two site interactions

$$\check{G}^{(4)}(\mathbf{p},\mathbf{R}) = \sum_a \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{r}_a\mathbf{q}} \check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q}) \sum_b \int \frac{d^3q'}{(2\pi)^3} e^{i\mathbf{r}_b\mathbf{q}} \check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(\mathbf{q}')\times$$
$$\sum_c \int \frac{d^3q''}{(2\pi)^3} e^{i\mathbf{r}_c\mathbf{q}''} \check{G}_0(\mathbf{p}-\mathbf{q}-\mathbf{q}',\mathbf{R})\check{U}(\mathbf{q}'') \sum_d \int \frac{d^3q'''}{(2\pi)^3} e^{i\mathbf{r}_d\mathbf{q}'''} \check{G}_0(\mathbf{p}-\mathbf{q}-\mathbf{q}'-\mathbf{q}'',\mathbf{R})\check{U}(\mathbf{q}''')\times$$
$$\check{G}^{(0)}(\mathbf{p}-\mathbf{q}-\mathbf{q}'-\mathbf{q}''-\mathbf{q}''',\mathbf{R}) \quad (3.5.25)$$

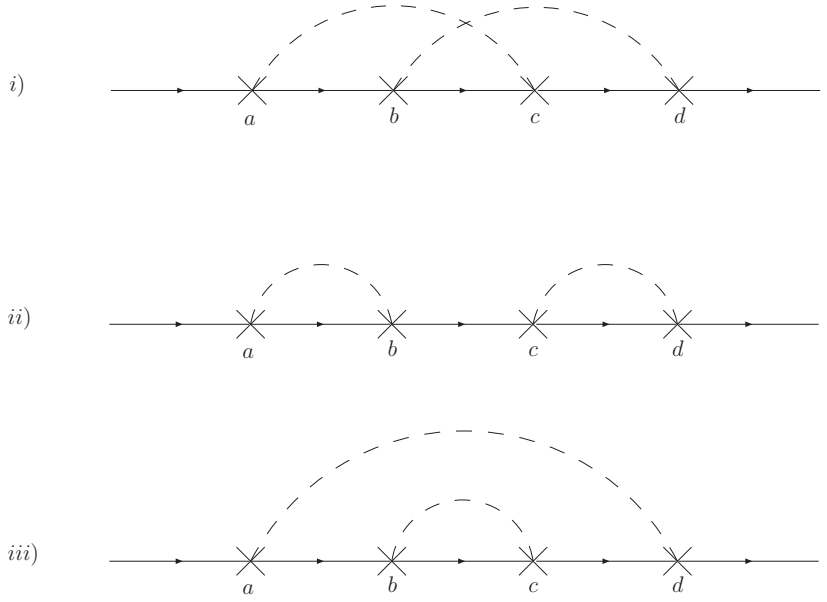

Figure 3.5: Three possible two-site interactions between four sites

Starting with the crossed term where, $a = c, b = d$,

$$\check{G}^{(4)}(\mathbf{p},\mathbf{R}) = \sum_{a,b}\int \frac{d^3q d^3q' d^3q'' d^3q'''}{(2\pi)^{12}} e^{i\mathbf{r}_a(\mathbf{q}+\mathbf{q}'')} e^{i\mathbf{r}_a(\mathbf{q}'+\mathbf{q}''')} \times$$

$$\check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q})\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(\mathbf{q}')\check{G}_0(\mathbf{p}-\mathbf{q}-\mathbf{q}',\mathbf{R})\check{U}(\mathbf{q}'')\check{G}_0(\mathbf{p}-\mathbf{q}-\mathbf{q}'-\mathbf{q}'',\mathbf{R})\check{U}(\mathbf{q}''')\check{G}^{(0)}(\mathbf{p}-\mathbf{q}-\mathbf{q}'-\mathbf{q}''-\mathbf{q}''',\mathbf{R}) \quad (3.5.26)$$

$$\check{G}^{(4)}(\mathbf{p},\mathbf{R}) = n^2 \int \frac{d^3q d^3q'''}{(2\pi)^6} \check{G}_0(\mathbf{p},\mathbf{R})\check{U}(\mathbf{q})\check{G}_0(\mathbf{p}-\mathbf{q},\mathbf{R})\check{U}(-\mathbf{q}''')\check{G}_0(\mathbf{p}-\mathbf{q}+\mathbf{q}''',\mathbf{R})\check{U}(-\mathbf{q})\check{G}_0(\mathbf{p}+\mathbf{q}''',\mathbf{R})\check{U}(\mathbf{q}''')\check{G}^{(0)}(\mathbf{p},\mathbf{R}) \quad (3.5.27)$$

using

$$\begin{aligned}\mathbf{q}_1 &= \mathbf{p}-\mathbf{q}\\ \mathbf{q}_2 &= \mathbf{p}-\mathbf{q}+\mathbf{q}'''\\ &= \mathbf{q}_1+\mathbf{q}'''\end{aligned}$$

$$\check{G}^{(4)}(\mathbf{p},\mathbf{R}) = \check{G}_0(\mathbf{p},\mathbf{R}) n^2 \int \frac{d^3q_1 d^3q_2}{(2\pi)^6} |\check{U}(\mathbf{p}-\mathbf{q}_1)|^2 |\check{U}(\mathbf{p}-\mathbf{q}_2)|^2 \check{G}_0(\mathbf{q}_1,\mathbf{R})\check{G}_0(\mathbf{q}_2,\mathbf{R})\check{G}_0(\mathbf{p}+\mathbf{q}_2-\mathbf{q}_1,\mathbf{R})\check{G}^{(0)}(\mathbf{p},\mathbf{R}) \quad (3.5.28)$$

We are required to have the momentum all lying close to the Fermi-Surface, this places a restriction on the angles of momenta for $\mathbf{p} + \mathbf{q}_2 - \mathbf{q}_1$. In turn, these small angles reduce the contribution of crossed diagrams allowing them to be neglected.

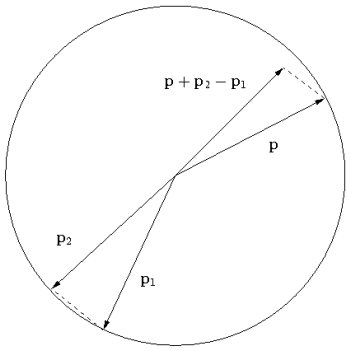

Figure 3.6: Momenta lying close to the Fermi Surface

The other two interactions where, $a = b, c = d$ and $a = d, b = c$ do not have this restriction to momentum and would need to be calculated.

Having removed single-site interactions along with three-site and crossed diagrams we can represent all diagrams in the following manor

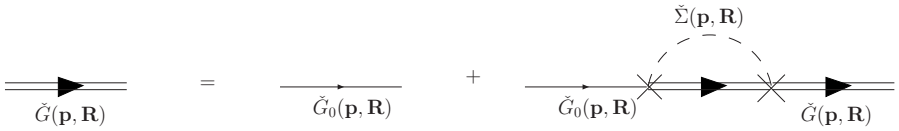

Figure 3.7: Dyson series diagram for impurity interactions

or

$$\check{G}(\mathbf{p}, \mathbf{R}) = \check{G}_0(\mathbf{p}, \mathbf{R}) + \check{G}_0(\mathbf{p}, \mathbf{R})\check{\Sigma}(\mathbf{p}, \mathbf{R})\check{G}(\mathbf{p}, \mathbf{R}) \tag{3.5.29}$$

where the self energy is given by the expression,

$$\check{\Sigma}(\mathbf{p}, \mathbf{R}) = n \int \frac{d^3q}{(2\pi)^3} |\check{U}(\mathbf{q})|^2 \check{G}(\mathbf{p} - \mathbf{q}, \mathbf{R}) \tag{3.5.30}$$

or equivalently

$$\check{\Sigma}(\mathbf{p},\mathbf{R}) = n\int \frac{d^3p'}{(2\pi)^3}|\check{U}(\mathbf{p}-\mathbf{p}')|^2\check{G}(\mathbf{p}',\mathbf{R}) \tag{3.5.31}$$

After impurity averaging we have the equation,

$$\left[\begin{pmatrix} i\omega_n - \xi_p + \frac{i\mathbf{v}}{2}\nabla_R & 0 \\ 0 & -i\omega_n - \xi_p + \frac{i\mathbf{v}}{2}\nabla_R \end{pmatrix} + \begin{pmatrix} \frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & -\frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) \end{pmatrix} - n\int \frac{d^3p'}{(2\pi)^3}|\check{U}(\mathbf{p}-\mathbf{p}')|^2\check{G}(\mathbf{p}',\mathbf{R})\right]\check{G}(\mathbf{p},\mathbf{R}) = \check{1} \tag{3.5.32a}$$

along with,

$$\check{G}(\mathbf{p},\mathbf{R})\left[\begin{pmatrix} i\omega_n - \xi_p - \frac{i\mathbf{v}}{2}\nabla_R & 0 \\ 0 & -i\omega_n - \xi_p - \frac{i\mathbf{v}}{2}\nabla_R \end{pmatrix} + \begin{pmatrix} \frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) & \Delta(\mathbf{R}) \\ -\Delta^*(\mathbf{R}) & -\frac{e\mathbf{v}}{c}\cdot\mathbf{A}(\mathbf{R}) \end{pmatrix} - n\int \frac{d^3p'}{(2\pi)^3}|\check{U}(\mathbf{p}-\mathbf{p}')|^2\check{G}(\mathbf{p}',\mathbf{R})\right] = \check{1}. \tag{3.5.32b}$$

from which we can calculate the impure Eilenberger equations.

3.5.4 Impure Eilenberger Equations

Taking the difference between equations (3.5.32) gives

$$i\mathbf{v}\cdot\nabla_R\check{G} + i\omega_n(\check{\tau}_3\check{G} - \check{G}\check{\tau}_3) + (\check{M}\check{G} - \check{G}\check{M}) = \check{\Sigma}\check{G} - \check{G}\check{\Sigma}. \tag{3.5.33}$$

Returning to the self energy we see that it contains an integral over momentum of the Green's function as such we use the quasi-classical Green's functions,

$$\begin{aligned}\check{\Sigma}(\mathbf{p},\mathbf{R}) &= n\int \frac{d^3q}{(2\pi)^3}|U(p-q)|^2\check{G}(\mathbf{q},\mathbf{R}) \\ &= \int \frac{nmp_F}{(2\pi)^3}|U(p-q)|^2\check{G}(\mathbf{q},\mathbf{R})d\Omega_q d\xi_q.\end{aligned}$$

Writing the differential probability for elastic scattering between two states of momenta, p, q, lying near the Fermi surface in the Born approximation as,

$$W(\mathbf{p},\mathbf{q}) = \frac{mp_F}{(2\pi)^2}|U(\mathbf{p}-\mathbf{q})|^2$$

the self energy becomes

$$\begin{aligned}\check{\Sigma}(\mathbf{p},\mathbf{R}) &= \int \frac{d\Omega_q}{2\pi} nW(\mathbf{p},\mathbf{q})\check{G}(\mathbf{q},\mathbf{R})d\xi_q \\ &= \frac{\pi}{i}\int \frac{d\Omega_q}{2\pi} nW(\mathbf{p},\mathbf{q})\frac{i}{\pi}\int d\xi_q \check{G}(\mathbf{q},\mathbf{R}) \\ &= \frac{1}{2i}\int d\Omega_q nW(\mathbf{p},\mathbf{q})\check{g}(\hat{q}_F,\mathbf{R})\end{aligned}$$

giving Eilenberger's equations,

$$\begin{aligned}&i\mathbf{v}_F\cdot\nabla_R\check{g} + i\omega_n(\check{\tau}_3\check{g}-\check{g}\check{\tau}_3) + (\check{m}\check{g}-\check{g}\check{m}) = \\ &\qquad \frac{1}{2i}\int d\Omega_{q_F} nW(\mathbf{p}_F,\mathbf{q}_F)\Big(\check{g}(\hat{q}_F,\mathbf{R})\check{g}(\hat{p}_F,\mathbf{R}) - \check{g}(\hat{p}_F,\mathbf{R})\check{g}(\hat{q}_F,\mathbf{R})\Big). \quad (3.5.34)\end{aligned}$$

We also consider the following decomposition for isotropic scattering,

$$\begin{aligned}\check{\Sigma}(\mathbf{p},\mathbf{R}) &= \frac{\pi}{i}\int \frac{nmp_F}{(2\pi)^3}|U(\mathbf{p}-\mathbf{q})|^2 d\Omega_q \frac{i}{\pi}\int d\xi_q \check{G}(\mathbf{q},\mathbf{R}) \\ &= \frac{\pi}{i}\nu(0)n\int \frac{d\Omega_q}{4\pi}|U(\mathbf{p}-\mathbf{q})|^2\check{g}(\mathbf{q},\mathbf{R})\end{aligned}$$

only the integral over ξ_p is taken, and any contribution from the normal part is included in the chemical potential.

For isotropic scattering where scattering between the two states is independent of angle the momentum $\mathbf{q}$ becomes a dummy variable,

$$\check{\Sigma}(\mathbf{p},\mathbf{R}) = \frac{\pi}{i}\nu(0)n|U|^2\int \frac{d\Omega_q}{4\pi}\check{g}(\mathbf{q},\mathbf{R})$$

Defining the scattering mean free time as $\frac{1}{\tau} = 2\pi\nu(0)|U|^2 n$, such that,

$$\begin{aligned}\check{\Sigma}(\mathbf{p},\mathbf{R}) &= \frac{1}{2i\tau}\int \frac{d\Omega_q}{4\pi}\check{g}(\mathbf{q},\mathbf{R}) \\ &= \frac{1}{2i\tau}\langle\check{g}\rangle\end{aligned}$$

we get for the isotropic case the equations,

$$i\mathbf{v}_F\cdot\nabla_R\check{g} + i\omega_n(\check{\tau}_3\check{g} - \check{g}\check{\tau}_3) + (\check{m}\check{g} - \check{g}\check{m}) = \frac{1}{2i\tau}(\langle\check{g}\rangle\check{g} - \check{g}\langle\check{g}\rangle) \tag{3.5.35}$$

where $\langle ...\rangle$ is the average over the Fermi surface.

3.5.5 Normalisation

Once again we show that $\check{g}_0^2 = \check{1}$ can be used as a solution by multiplying (3.5.35) by the left and right by $\check{g}$, this time explicitly we find

$$i\mathbf{v}_F\cdot\check{g}(\nabla_R\check{g}) + i\omega_n(\check{g}\check{\tau}_3\check{g} - \check{g}\check{g}\check{\tau}_3) + (\check{g}\check{m}\check{g} - \check{g}\check{g}\check{m}) = \frac{1}{2i\tau}(\check{g}\langle\check{g}\rangle\check{g} - \check{g}\check{g}\langle\check{g}\rangle) \tag{3.5.36}$$

$$i\mathbf{v}_F\cdot(\nabla_R\check{g})\check{g} + i\omega_n(\check{\tau}_3\check{g}\check{g} - \check{g}\check{\tau}_3\check{g}) + (\check{m}\check{g}\check{g} - \check{g}\check{m}\check{g}) = \frac{1}{2i\tau}(\langle\check{g}\rangle\check{g}\check{g} - \check{g}\langle\check{g}\rangle\check{g}) \tag{3.5.37}$$

which when added yield,

$$\begin{aligned}i\mathbf{v}_F\cdot\{\check{g}(\nabla_R\check{g}) + (\nabla_R\check{g})\check{g}\} + i\omega_n(\check{\tau}_3\check{g}\check{g} - \check{g}\check{g}\check{\tau}_3) + (\check{m}\check{g}\check{g} - \check{g}\check{g}\check{m}) \\ = \frac{1}{2i\tau}(\langle\check{g}\rangle\check{g}\check{g} - \check{g}\check{g}\langle\check{g}\rangle)\end{aligned} \tag{3.5.38}$$

$$\begin{aligned}i\mathbf{v}_F\cdot\nabla_R(\check{g}\check{g}) + i\omega_n(\check{\tau}_3\check{g}\check{g} - \check{g}\check{g}\check{\tau}_3) + (\check{m}\check{g}\check{g} - \check{g}\check{g}\check{m}) \\ = \frac{1}{2i\tau}(\langle\check{g}\rangle\check{g}\check{g} - \check{g}\check{g}\langle\check{g}\rangle).\end{aligned} \tag{3.5.39}$$

From this we know $\check{g}$ and $\check{g}\check{g}$ are both solutions and write, for the same reasons as before we use

$$\check{g}^2 = \check{1} \tag{3.5.40}$$

such that

$$g^2 + ff^\dagger = 1 \quad \text{and} \quad g + \overline{g} = 0 \tag{3.5.41}$$

for the spatially dependent, impure case.

3.5.6 Expansion to Coupled Equations

We are now in a place to unpack the Eilenberger equations from the matrices of

$$i\mathbf{v}_F \cdot \nabla_R \check{g} + i\omega_n(\check{\tau}_3\check{g} - \check{g}\check{\tau}_3) + (\check{m}\check{g} - \check{g}\check{m}) = \frac{1}{2i\tau}(\langle\check{g}\rangle\check{g} - \check{g}\langle\check{g}\rangle). \tag{3.5.42}$$

The second term is given by,

$$\begin{aligned}\check{\tau}_3\check{g} - \check{g}\check{\tau}_3 = & \\ & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} g & -if \\ if^\dagger & -g \end{pmatrix} - \begin{pmatrix} g & -if \\ if^\dagger & -g \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ & = \begin{pmatrix} 0 & -2if \\ -2if^\dagger & 0 \end{pmatrix}\end{aligned} \tag{3.5.43}$$

the third by,

$$\check{m}\check{g} - \check{g}\check{m} = \begin{pmatrix} \frac{e\mathbf{v}_F}{c}\mathbf{A} & \Delta \\ -\Delta^* & -\frac{e\mathbf{v}_F}{c}\mathbf{A} \end{pmatrix} \begin{pmatrix} g & -if \\ if^\dagger & -g \end{pmatrix} - \begin{pmatrix} g & -if \\ if^\dagger & -g \end{pmatrix} \begin{pmatrix} \frac{e\mathbf{v}_F}{c}\mathbf{A} & \Delta \\ -\Delta^* & -\frac{e\mathbf{v}_F}{c}\mathbf{A} \end{pmatrix}$$
$$= \begin{pmatrix} i(\Delta f^\dagger - \Delta^* f) & -2\left(\Delta g + \frac{ie\mathbf{v}_F}{c}\mathbf{A}f\right) \\ -2\left(\Delta^* g + \frac{ie\mathbf{v}_F}{c}\mathbf{A}f^\dagger\right) & -i(\Delta f^\dagger - \Delta^* f) \end{pmatrix} \quad (3.5.44)$$

finally the scattering term gives us,

$$\langle\check{g}\rangle\check{g} - \check{g}\langle\check{g}\rangle = \begin{pmatrix} \langle g\rangle & -i\langle f\rangle \\ i\langle f^\dagger\rangle & -\langle g\rangle \end{pmatrix} \begin{pmatrix} g & -if \\ if^\dagger & -g \end{pmatrix} - \begin{pmatrix} g & -if \\ if^\dagger & -g \end{pmatrix} \begin{pmatrix} \langle g\rangle & -i\langle f\rangle \\ i\langle f^\dagger\rangle & -\langle g\rangle \end{pmatrix}$$
$$= \begin{pmatrix} f^\dagger\langle f\rangle - f\langle f^\dagger\rangle & 2i(g\langle f\rangle - f\langle g\rangle) \\ 2i(g\langle f^\dagger\rangle - f^\dagger\langle g\rangle) & -(f^\dagger\langle f\rangle - f\langle f^\dagger\rangle) \end{pmatrix}. \quad (3.5.45)$$

From this the Eilenberger equations for isotropic scattering are found,

$$\mathbf{v}_F\nabla_R g + \Delta f^\dagger - \Delta^* f = \frac{1}{2\tau}(f\langle f^\dagger\rangle - f^\dagger\langle f\rangle) \quad (3.5.46)$$
$$\mathbf{v}_F\left(\nabla_R - \frac{2ie}{c}\mathbf{A}\right) f + 2\omega_n f - 2\Delta g = \frac{1}{\tau}(g\langle f\rangle - f\langle g\rangle) \quad (3.5.47)$$
$$-\mathbf{v}_F\left(\nabla_R + \frac{2ie}{c}\mathbf{A}\right) f^\dagger + 2\omega_n f^\dagger - 2\Delta^* g = \frac{1}{\tau}(g\langle f^\dagger\rangle - f^\dagger\langle g\rangle) \quad (3.5.48)$$

with $g^2 + ff^\dagger = 1$. This can be differentiated to show, $2g\nabla_R g + f\nabla_R f^\dagger + f^\dagger\nabla_R f = 0$, which in turn can be shown by the equations above.

3.5.7 Symmetries of The Eilenberger Equations

From Eilenberger's equations written above we can easily see the following symmetries,

$$
\begin{aligned}
[f(-\omega_n, \mathbf{v}_F, \mathbf{R})]^* &= f^\dagger(\omega_n, \mathbf{v}_F, \mathbf{R}) \quad ; \quad [g(-\omega_n, \mathbf{v}_F, \mathbf{R})]^* = -g(\omega_n, \mathbf{v}_F, \mathbf{R}) \\
[f(\omega_n, -\mathbf{v}_F, \mathbf{R})]^* &= f^\dagger(\omega_n, \mathbf{v}_F, \mathbf{R}) \quad ; \quad [g(\omega_n, -\mathbf{v}_F, \mathbf{R})]^* = \quad g(\omega_n, \mathbf{v}_F, \mathbf{R}) \\
f(-\omega_n, -\mathbf{v}_F, \mathbf{R}) &= f(\omega_n, \mathbf{v}_F, \mathbf{R}) \quad ; \quad g(-\omega_n, -\mathbf{v}_F, \mathbf{R}) = \quad g(\omega_n, \mathbf{v}_F, \mathbf{R})
\end{aligned}
$$

In one dimension the averages are given in terms of $\pm\mathbf{v}_F$ such that

$$
\begin{aligned}
\langle g \rangle &= \frac{1}{2}(g(\omega_n, \mathbf{v}_F, \mathbf{R}) + g(\omega_n, -\mathbf{v}_F, \mathbf{R})) \\
&= \frac{1}{2}(g + g^*) \\
&= \Re(g) \\
\langle f \rangle &= \frac{1}{2}(f(\omega_n, \mathbf{v}_F, \mathbf{R}) + f(\omega_n, -\mathbf{v}_F, \mathbf{R})) \\
&= \frac{1}{2}(f + (f^\dagger)^*) \\
\langle f^\dagger \rangle &= \frac{1}{2}(f^\dagger(\omega_n, \mathbf{v}_F, \mathbf{R}) + f^\dagger(\omega_n, -\mathbf{v}_F, \mathbf{R})) \\
&= \frac{1}{2}(f^\dagger + (f)^*) \\
&= \langle f \rangle^*
\end{aligned}
$$

and this can be extended to higher dimensions.

We can also see that,

$$
\begin{aligned}
\langle f + f^\dagger \rangle &= \frac{1}{2}(f + f^\dagger + (f + f^\dagger)^*) \\
&= \Re(f + f^\dagger) \\
\langle f - f^\dagger \rangle &= \frac{1}{2}(f - f^\dagger - (f - f^\dagger)^*) \\
&= i\Im(f - f^\dagger)
\end{aligned}
$$

These have been considered following the microscopic theory presented earlier in chapter 2 which defined the symmetric and antisymmetric combinations $f_{\pm} = \frac{1}{2}(f \pm f^{\dagger})$.

However much time was spent fruitlessly by the author using these to extend the earlier papers work to the general impure case. It seems quite likely that these combinations respect underlying spatial inversion symmetries leading to possible simplifications in the calculations. However, the equations derived later in this thesis have not been decomposed the same way as the combination makes the matrices required more complicated.

3.5.8 Simple Matrix Representation

A particularly useful representation of the Eilenberger equations is found when one separates out the velocity dependent terms from the matrix form of the Eilenberger equations by introducing the operator,

$$\check{\partial} = \begin{pmatrix} \nabla_R & \Pi \\ \Pi^* & \nabla_R \end{pmatrix}$$

with, $\Pi = \left(\nabla_R - \frac{2ie}{c}\mathbf{A}(\mathbf{R})\right)$.

This operator acts on a matrix $\check{O}$ as,

$$\check{\partial}\check{O} = \begin{pmatrix} \nabla_R O_{11} & \Pi O_{12} \\ \Pi^* O_{21} & \nabla_R O_{22} \end{pmatrix}.$$

Writing the isotropic Eilenberger equations in this way,

$$i\mathbf{v}_F \cdot \check{\partial}\check{g} + [\check{H}, \check{g}] = \frac{1}{2i\tau}[\langle\check{g}\rangle, \check{g}] \tag{3.5.49}$$

with $\check{H} = \begin{pmatrix} i\omega_n & \Delta \\ -\Delta^* & -\omega_n \end{pmatrix}$.
This gives the most succinct way of writing the Eilenberger equations and later will allow us to make an expansion of the equations in orders of the Fermi velocity quite easily.

3.5.9 Eilenberger, what was the point?

In deriving the Eilenberger equations two operations are performed,

1. the difference was taken between the two Gorkov equations in a mixed representation,

2. a quasi-classical approximation which set the energy scale at the Fermi surface was made.

The first of these removes the direct dependence of the equations on momentum, from the point of view of working with the Gorkov equations this step removes the source of complicated vertex corrections, allowing for simple expansions to be made.

The second gives a way of proving a general, non-linear solution for the quasi-classical Green's functions expressible as $\check{g}^2 = \check{1}$. This supplementary information when re-included accounts for the loss in information by taking the difference between equations. Again, from the view of calculations, the non-linear solution gives back the results which would be present by calculating the vertex corrections.

Eilenberger remained uncertain of the boundary conditions to the equations, his feeling was that some physical conditions would keep the equations bounded and finite and based this on considering his equations as transport equations. Some boundary conditions are considered in more detail by Kopnin [45].

Chapter 4

APPLYING THE EILENBERGER EQUATIONS

4.1 Introduction

In this chapter the $\hat{\mathbf{v}}_F$ dependence is used to investigate dirty and weakly anisotropic systems. Much of the method and ideas for this section come from Kopnin [46].

In the dirty system scattering smears out anisotropy, velocity dependence is assumed to form a linear correction. This idea gives rise to the Usadel's equations, and is treated at the start of the chapter. The derivation differs from that of Usadel's original and follows that of Gorkov and Kopnin [47] [48]. The latter's use of the matrix representation giving simplifications to the algebra.

The $\mathbf{v}_F$ term in Eilenberger's equations proceeds the gradient and applied field, as such its value is related to these quantities. When considering weakly anisotropic systems the value for $\mathbf{v}_F$ can be considered to be 'small', coming from either a small field, small persistent currents or superconductors where the spatial gradient is small.

Treating this value as an expansion parameter means a self-contained equation similar to that of Usadel cannot be found at higher orders. Here the self-consistency equation comes to the rescue and allows for the calculation of the order-parameter. Being given in terms of the average of the anomalous Green's function, the first correction comes from the result at second order.

The full first order correction is found by elementary algebra on the expanded equations. The extension to higher orders via the same method seems difficult, so an approximation is taken in which the other parameter is small. Taken in conjunction with the temperature being given by T_c, this provides a simple derivation of the Ginzburg Landau equations.

Again, the use of the non-linear solution gives the results from the more complicated derivations involving vertex corrections.

In the next chapter we will see if a matrix representation can be used to simplify the first correction calculation, and offer a method to calculate a more general second order result.

4.2 Diffuse Limit; The Usadel Equations

When the mean free path is short we can make a simplification by assuming the motion of electrons is isotropic. Usadel used this to calculate the generalisation of de Gennes-Maki theory to the non-linear case.

Strong scattering caused by impurities produces an averaging over momentum directions, the correction to the original Green's function would be linear in $\underline{\mathbf{v}}_F$ and expand,

$$\check{g}(\omega_n, \mathbf{r}, \hat{\mathbf{v}}_F) = \check{g}_0(\omega_n, \mathbf{r}) + \underline{\hat{\mathbf{v}}}_F \cdot \underline{\check{\mathbf{g}}}(\omega_n, \mathbf{r}) \tag{4.2.1}$$

where $\underline{\hat{\mathbf{v}}}_F = \frac{\underline{\mathbf{v}}_F}{v_F}$ is the unit vector in the direction of momentum, $\check{g}_0$ does not depend on directions of $\underline{\mathbf{v}}_F$ and $|\underline{\check{\mathbf{g}}}| \ll \check{g}_0$.

Averaging this and substituting it into the normalisation condition ignoring terms $\mathcal{O}(v_F^2)$ gives,

$$\begin{aligned} \langle \check{g} \rangle &= \check{g}_0 \\ \check{g}_0^2 &= \check{1} \\ \check{g}_0 \underline{\check{\mathbf{g}}} + \underline{\check{\mathbf{g}}} \check{g}_0 &= 0 \\ \Rightarrow [\check{g}_0, \underline{\check{\mathbf{g}}}] &= 2\check{g}_0 \underline{\check{\mathbf{g}}} \end{aligned}$$

Substituting equation (4.2.1) into (3.5.49) we have,

$$i\mathbf{v}_F \check{\partial} \cdot (\check{g}_0 + \underline{\hat{\mathbf{v}}}_F \cdot \underline{\check{\mathbf{g}}}) + [\check{H}, \check{g}_0 + \underline{\hat{\mathbf{v}}}_F \cdot \underline{\check{\mathbf{g}}}] = \frac{1}{2i\tau}[\check{g}_0, \check{g}_0 + \underline{\hat{\mathbf{v}}}_F \cdot \underline{\check{\mathbf{g}}}]. \tag{4.2.2}$$

This can be averaged over momenta to remove the vector dependence of $\mathbf{v}_F$. This is best done via an index notation,

$$iv_F \hat{\mathbf{v}}_\alpha \check{\partial}_\alpha (\check{g}_0 + \hat{\mathbf{v}}_\beta \check{\mathbf{g}}_\beta) + [\check{H}, \check{g}_0 + \hat{\mathbf{v}}_\beta \check{\mathbf{g}}_\beta] = \frac{1}{2i\tau}[\check{g}_0, \check{g}_0 + \hat{\mathbf{v}}_\beta \check{\mathbf{g}}_\beta]$$

$$
\begin{aligned}
iv_F\langle\hat{\mathbf{v}}_\alpha\hat{\mathbf{v}}_\beta\rangle\check{\partial}_\alpha\check{\mathbf{g}}_\beta+[\check{H},\check{g}_0]&=\frac{1}{2i\tau}[\check{g}_0,\check{g}_0]\\
iv_F\left(\frac{\delta_{\alpha,\beta}}{d}\right)\check{\partial}_\alpha\check{\mathbf{g}}_\beta+[\check{H},\check{g}_0]&=0\\
i\frac{v_F}{d}\check{\partial}\underline{\check{\mathbf{g}}}+[\check{H},\check{g}_0]&=0
\end{aligned}
\tag{4.2.3}
$$

where $\langle\hat{\mathbf{v}}_\alpha\hat{\mathbf{v}}_\beta\rangle=\frac{\delta_{\alpha,\beta}}{d}$.

To get another equation we multiply by $\underline{\hat{\mathbf{v}}}_F$ and average again,

$$
\begin{aligned}
iv_F\hat{\mathbf{v}}_\gamma\hat{\mathbf{v}}_\alpha\check{\partial}_\alpha(\check{g}_0+\hat{\mathbf{v}}_\beta\check{\mathbf{g}}_\beta)+\hat{\mathbf{v}}_\gamma[\check{H},\check{g}_0+\hat{\mathbf{v}}_\beta\check{\mathbf{g}}_\beta]&=\frac{\hat{\mathbf{v}}_\gamma}{2i\tau}[\check{g}_0,\check{g}_0+\hat{\mathbf{v}}_\beta\check{\mathbf{g}}_\beta]\\
iv_F\langle\hat{\mathbf{v}}_\gamma\hat{\mathbf{v}}_\alpha\rangle\check{\partial}_\alpha\check{g}_0+\langle\hat{\mathbf{v}}_\gamma\hat{\mathbf{v}}_\beta\rangle[\check{H},\check{\mathbf{g}}_\beta]&=\frac{\langle\hat{\mathbf{v}}_\gamma\hat{\mathbf{v}}_\beta\rangle}{2i\tau}[\check{g}_0,\check{\mathbf{g}}_\beta]\\
iv_F\left(\frac{\delta_{\alpha,\gamma}}{d}\right)\check{\partial}_\alpha\check{g}_0+\left(\frac{\delta_{\gamma,\beta}}{d}\right)[\check{H},\check{\mathbf{g}}_\beta]&=\frac{\left(\frac{\delta_{\gamma,\beta}}{d}\right)}{2i\tau}[\check{g}_0,\check{\mathbf{g}}_\beta]\\
iv_F\check{\partial}\check{g}_0+[\check{H},\underline{\check{\mathbf{g}}}]&=\frac{1}{2i\tau}[\check{g}_0,\underline{\check{\mathbf{g}}}]
\end{aligned}
\tag{4.2.4}
$$

In the dirty limit the term $[\check{H},\underline{\check{\mathbf{g}}}]$ can be neglected leaving

$$
\begin{aligned}
iv_F\check{\partial}\check{g}_0&=\frac{1}{2i\tau}[\check{g}_0,\underline{\check{\mathbf{g}}}]\\
&=\frac{1}{i\tau}\check{g}_0\underline{\check{\mathbf{g}}}\\
-v_F\tau\check{g}_0\check{\partial}\check{g}_0&=\underline{\check{\mathbf{g}}}.
\end{aligned}
\tag{4.2.5}
$$

The assumption where $[\check{H},\underline{\check{\mathbf{g}}}]=0$ should be shown to be correct in whichever system was under consideration. Below we will consider an alternative expansion which includes the effect of this term.

From equation (4.2.3)

$$
\begin{aligned}
i\frac{v_F}{d}\check{\partial}\underline{\check{\mathbf{g}}}+[\check{H},\check{g}_0]&=0\\
i\frac{v_F}{d}\check{\partial}(-v_F\tau\check{g}_0\check{\partial}\check{g}_0)+[\check{H},\check{g}_0]&=0
\end{aligned}
$$

$$i\left(\frac{v_F^2\tau}{d}\right)\check{\partial}\check{g}_0\check{\partial}\check{g}_0 = [\check{H},\check{g}_0]$$

$$iD\check{\partial}\check{g}_0\check{\partial}\check{g}_0 = [\check{H},\check{g}_0] \tag{4.2.6}$$

These are the Usadel equations, which when unpacked give,

$$D\nabla_R\cdot\left(g_0\nabla_R g_0 + f_0\left(\nabla_R + \frac{2ie}{c}\mathbf{A}(\mathbf{R})\right)f_0^\dagger\right) = \Delta f_0^\dagger - \Delta^* f_0 \tag{4.2.7}$$

$$D\left(\nabla_R - \frac{2ie}{c}\mathbf{A}(\mathbf{R})\right)\cdot\left(g_0\left(\nabla_R - \frac{2ie}{c}\mathbf{A}(\mathbf{R})\right)f_0 - f_0\nabla_R g_0\right) = 2\omega_n f_0 - 2\Delta g_0 \tag{4.2.8}$$

$$D\left(\nabla_R + \frac{2ie}{c}\mathbf{A}(\mathbf{R})\right)\cdot\left(g_0\left(\nabla_R + \frac{2ie}{c}\mathbf{A}(\mathbf{R})\right)f_0^\dagger - f_0\nabla_R g_0\right) = 2\omega_n f_0^\dagger - 2\Delta^* g_0 \tag{4.2.9}$$

$$D\nabla_R\cdot\left(g_0\nabla_R g_0 + f_0^\dagger\left(\nabla_R - \frac{2ie}{c}\mathbf{A}(\mathbf{R})\right)f_0\right) = \Delta^* f_0 - \Delta f_0^\dagger \tag{4.2.10}$$

with the diffusion constant, $D = \frac{v_F^2\tau}{d}$.

Though easier to work with than the Eilenberger equations, the loss of information in this approach is quite large. Instead of working in such a dirty regime close to the isotropic state, we could relax the assumptions and consider an expansion in anisotropy not necessarily caused by impurities.

4.3 Weak Anisotropic Expansion; First Method

Instead of using impurities in the diffuse limit to smear out the effect of the vector, $\mathbf{v}_F$ we note that it preceeds the gradient and applied field. Its existence then is due to these causes of inhomogeneity and in considering an expansion is small velocity what we are really doing is to consider an expansion from the isotropic case to one where small variations are present. This is known as weak anisotropy. Weak anisotropic superconductors cover those investigate by Langer and Ambergaokar with small persistent currents, as well as granular systems, or systems in weak fields.

The expansion will appear similar to Usadel's method at first order since the lower orders are highly isotropic. The first order result when expanded for $\tau \to 0$ will give exact agreement with Usadel's method, although no closed equation can be found.

4.3.1 First Order Expansion

This first method again follows Kopnin [49]. Assuming a nearly isotropic superconductor we consider the expansion,

$$\check{g} = \check{g}_0 + \check{g}_1 \tag{4.3.1}$$

where $\check{g}_0$ is velocity independent, and $\check{g}_1$ is the linear correction to this. Taking its average gives,

$$\langle \check{g} \rangle = \check{g}_0 \tag{4.3.2}$$

the normalisation condition gives,

$$\check{g}^2 = 1 \qquad \Rightarrow \qquad \begin{array}{c} \check{g}_0^2 = \check{1} \\ \check{g}_0\check{g}_1 + \check{g}_1\check{g}_0 = 0 \end{array} \tag{4.3.3}$$

which in turn gives

$$2g_0g_1 = -(f_0f_1^\dagger + f_0^\dagger f_1) \tag{4.3.4}$$

$$1 = g_0^2 + f_0f_0^\dagger \tag{4.3.5}$$

Using the equations,

$$\mathbf{v}_F\left(\nabla_R - \frac{2ie}{c}\mathbf{A}\right)f + 2\omega_n f - 2\Delta g = \frac{1}{\tau_{imp}}(g\langle f\rangle - f\langle g\rangle) \tag{4.3.6}$$

$$-\mathbf{v}_F\left(\nabla_R + \frac{2ie}{c}\mathbf{A}\right)f^\dagger + 2\omega_n f^\dagger - 2\Delta^* g = \frac{1}{\tau_{imp}}(g\langle f^\dagger\rangle - f^\dagger\langle g\rangle) \tag{4.3.7}$$

we expand in orders of $\mathbf{v}_F$.

Zeroth order

$$\omega_n f_0 = \Delta g_0 \tag{4.3.8}$$

$$\omega_n f_0^\dagger = \Delta^* g_0 \tag{4.3.9}$$

which is solved via $g_0^2 + f_0f_0^\dagger = 1$ to give,

$$g_0 = \frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta|^2}} \quad f_0 = \frac{\Delta}{\sqrt{\omega_n^2 + |\Delta|^2}} \quad f_0^\dagger = \frac{\Delta^*}{\sqrt{\omega_n^2 + |\Delta|^2}} \tag{4.3.10}$$

Although at first sight this appear to be the same as the homogenous result, the function Δ is still assumed to be spatially dependent. The removal of the operator ∇_R was due to $\mathbf{v}_F$, not the functions being constant.

First order

$$\mathbf{v}_F \Pi f_0 + 2\omega_n f_1 - 2\Delta g_1 = \frac{1}{\tau}(g_1 f_0 - f_1 g_0) \tag{4.3.11}$$

$$-\mathbf{v}_F \Pi f_0^\dagger + 2\omega_n f_1^\dagger - 2\Delta^* g_1 = \frac{1}{\tau}(g_1 f_0^\dagger - f_1^\dagger g_0) \tag{4.3.12}$$

Multiplying (4.3.11) by $f_0^\dagger$ and (4.3.12) by f_0 then adding gives,

$$\begin{aligned}\mathbf{v}_F\{f_0^\dagger \Pi f_0 - f_0 \Pi^* f_0^\dagger\} = \\ 2g_1(\Delta f_0^\dagger + \Delta^* f_0) - 2\omega_n(f_0 f_1^\dagger + f_0^\dagger f_1) \\ + \frac{1}{\tau}(2g_1 f_0 f_0^\dagger - g_0(f_0 f_1^\dagger + f_0^\dagger f_1))\end{aligned} \tag{4.3.13}$$

substituting in the result $f_0 f_1^\dagger + f_0^\dagger f_1 = -2g_0 g_1$

$$\begin{aligned}\mathbf{v}_F\{f_0^\dagger \Pi f_0 - f_0 \Pi^* f_0^\dagger\} = \\ 2g_1(\Delta f_0^\dagger + \Delta^* f_0) + 4\omega_n g_0 g_1 + \frac{1}{\tau}(2g_1 f_0 f_0^\dagger + 2g_0^2 g_1)\end{aligned} \tag{4.3.14}$$

$$\begin{aligned}\mathbf{v}_F\{f_0^\dagger \Pi f_0 - f_0 \Pi^* f_0^\dagger\} = \\ 2g_1\left((\Delta f_0^\dagger + \Delta^* f_0) + 2\omega_n g_0 + \frac{1}{\tau}\underbrace{(f_0 f_0^\dagger + g_0^2)}_{1}\right)\end{aligned} \tag{4.3.15}$$

$$\begin{aligned}\mathbf{v}_F\{f_0^\dagger \Pi f_0 - f_0 \Pi^* f_0^\dagger\} &= 2g_1\left((\Delta f_0^\dagger + \Delta^* f_0) + 2\omega_n g_0 + \frac{1}{\tau}\right) \\ &= 2g_1\left(\Delta\frac{\Delta^*}{\sqrt{\omega_n^2 + |\Delta|^2}} + \Delta^*\frac{\Delta}{\sqrt{\omega_n^2 + |\Delta|^2}} + 2\omega_n\frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta|^2}} + \frac{1}{\tau}\right) \\ &= 4g_1\left(\sqrt{\omega_n^2 + |\Delta|^2} + \frac{1}{2\tau}\right)\end{aligned} \tag{4.3.16}$$

$$g_1 = \frac{\mathbf{v}_F}{4\left(\sqrt{\omega_n^2 + |\Delta|^2} + \frac{1}{2\tau}\right)} \left\{ f_0^\dagger \Pi f_0 - f_0 \Pi^* f_0^\dagger \right\}$$
$$= \frac{\mathbf{v}_F}{4(\omega_n^2 + |\Delta|^2)\left(\sqrt{\omega_n^2 + |\Delta|^2} + \frac{1}{2\tau}\right)} \left\{ \Delta^* \Pi \Delta - \Delta \Pi^* \Delta^* \right\}. \tag{4.3.17}$$

From this we can calculate the current, but we leave this until a later chapter where the same expression for g_1 is calculated by another method.

Taking this to higher orders is complicated. Kogan [32] wrote a paper on an expansion which continued to second order. However, it is unclear whether this paper obtained a valid result, or if the extension to 'moderately dirty' superconductors is correct.

His method is similar to that of Usadel's original paper and as such somewhat cumbersome. Added to this is his assumption that the dirty limit is taken by setting $\tau = 0$ everywhere but the diffusion constant. This seems distinct from considering the dirty limit as the first order, τ expansion of the equations written at arbitrary impurity concentration. Later we will calculate the full second order expansion at arbitrary impurity concentration by another method which seems more sensible and robust.

4.4 The Ginzburg Landau Equations

For the time being we do not go to exact results at higher orders via this method, but instead make an approximation as Kopnin [50] does. We consider the case where $|\Delta|^2 \to 0$ as would be the situation near $T = T_c$. This allows us to expand the square root giving,

$$g_0 \approx \frac{\omega_n}{\omega_n}\left(1 - \frac{|\Delta|^2}{2\omega_n^2}\right) \quad f_0 \approx \frac{\Delta}{\omega_n}\left(1 - \frac{|\Delta|^2}{2\omega_n^2}\right) \quad f_0^\dagger \approx \frac{\Delta^*}{\omega_n}\left(1 - \frac{|\Delta|^2}{2\omega_n^2}\right). \tag{4.4.1}$$

Looking first at g we see that $g_0 = 1$ and the higher order terms are even in powers of Δ one fewer than the corrections to $f, f^\dagger$, we therefore take $g_1 = g_2 = 0$.

Starting from the equation,

$$\mathbf{v}_F \Pi f + 2\omega_n f - 2\Delta g = \frac{1}{\tau}(g\langle f\rangle - f\langle g\rangle) \tag{4.4.2}$$

at first order we have,

$$\mathbf{v}_F \Pi f_0 + 2\omega_n f_1 = -\frac{f_1}{\tau} \tag{4.4.3}$$

$$f_1 = -\frac{\mathbf{v}_F}{2\omega_n + \frac{1}{\tau}} \Pi f_0 \tag{4.4.4}$$

at second order we have both f_2 and $\langle f_2\rangle$

$$\mathbf{v}_F \Pi f_1 + 2\omega_n f_2 = \frac{1}{\tau}(\langle f_2\rangle - f_2) \tag{4.4.5}$$

$$f_2 = \frac{\langle f_2\rangle}{\left(2\omega_n + \frac{1}{\tau}\right)\tau} - \frac{\mathbf{v}_F}{\left(2\omega_n + \frac{1}{\tau}\right)} \Pi f_1 \tag{4.4.6}$$

$$= \frac{\langle f_2\rangle}{\left(2\omega_n + \frac{1}{\tau}\right)\tau} + \frac{\mathbf{v}_F \cdot \mathbf{v}_F}{\left(2\omega_n + \frac{1}{\tau}\right)^2} \Pi^2 f_0 \tag{4.4.7}$$

averaging this over directions at the Fermi Surface gives

$$\langle f_2\rangle = \frac{\langle f_2\rangle}{\left(2\omega_n + \frac{1}{\tau}\right)\tau} + \frac{v_F^2}{d\left(2\omega_n + \frac{1}{\tau}\right)^2} \Pi^2 f_0 \tag{4.4.8}$$

which upon rearranging gives,

$$\langle f_2\rangle = \frac{v_F^2}{2\omega_n d\left(2\omega_n + \frac{1}{\tau}\right)} \Pi^2 f_0. \tag{4.4.9}$$

Substituting in $f_0 = \frac{\Delta}{\omega_n}$, multiplying top and bottom by τ and using the diffusion constant $D = \frac{v_F^2 \tau}{d}$ gives

$$\langle f_2 \rangle = \frac{D}{2\omega_n^2 (2\omega_n \tau + 1)} \Pi^2 \Delta \tag{4.4.10}$$

Up to second order we can now write the average of f,

$$\langle f \rangle = \frac{\Delta}{\omega_n} - \frac{\Delta |\Delta|^2}{2\omega_n^3} + \frac{D}{2\omega_n^2 (2\omega_n \tau + 1)} \Pi^2 \Delta \tag{4.4.11}$$

this can now be substituted into the self consistency equation from section 3.4.3

$$\Delta \ln \left(\frac{T_c^{(0)}}{T} \right) = 2\pi T \sum_{n \geq 0} \left[\frac{\Delta}{\omega_n} - \langle f \rangle \right] \tag{4.4.12}$$

The sums over Matsubara frequencies are given via the Riemann zeta function and the result,

$$\sum_{n \geq 0} \frac{1}{(n + \frac{1}{2})^z} = (2^z - 1)\zeta(z) \tag{4.4.13}$$

In the pure limit, $\tau \to \infty$ and cancels with the τ in D.

Finally the condition that $|\Delta|^2 \to 0$ taken in conjunction with $T \to T_c$ gives us the Ginzburg-Landau equations,

$$\left(1 - \frac{T}{T_c^{(0)}} \right) \Delta - \frac{7\zeta(3)}{8\pi^2 T_c^2} \Delta |\Delta|^2 + \frac{v_F^2 7\zeta(3)}{16 d \pi^2 T_c^2} \Pi^2 \Delta = 0 \tag{4.4.14}$$

In the dirty case, $\tau \to 0$,

$$\left(1 - \frac{T}{T_c^{(0)}} \right) \Delta - \frac{7\zeta(3)}{8\pi^2 T_c^2} \Delta |\Delta|^2 + \frac{D\pi}{8 T_c} \Pi^2 \Delta = 0 \tag{4.4.15}$$

We see that the only difference between these equations occurs on the gradient term and can write,

$$\left(1-\frac{T}{T_c}\right)\Delta-\frac{7\zeta(3)}{8\pi^2T_c^2}\Delta|\Delta|^2+\frac{\tilde{D}\pi}{8T_c}\Pi^2\Delta=0 \tag{4.4.16}$$

where

$$\tilde{D}=\begin{cases}\dfrac{7v_F^2\zeta(3)}{2d\pi^3T_c} & \text{Clean}\\ D & \text{Dirty}\end{cases} \tag{4.4.17}$$

this is consistent with Anderson's theorem.

Comparing with Langer and Ambegaokar we find equations for the coefficients α,β, and coherence length,

$$\alpha=\frac{8(T_c-T)}{\tilde{D}\pi} \tag{4.4.18}$$

$$\beta=\frac{7\zeta(3)}{\pi^3\tilde{D}T_c} \tag{4.4.19}$$

$$\xi=\sqrt{\frac{\tilde{D}\pi}{8(T_c-T)}} \tag{4.4.20}$$

This derivation of the Ginzburg-Landau equations out of a microscopic theory differs from that given by Gorkov. His derivation started from the Gorkov equations and not the Eilenberger equations. The distinctions between these approaches will be highlighted later in comparison to work by Werthamer and Tewordt [51] [52].

Chapter 5

WEAK ANISOTROPY

5.1 Introduction

In the preceding section we calculated the Ginzburg Landau equations via a simple expansion of the Eilenberger equations. In this chapter we use a matrix representation to repeat the calculation.

In doing so we calculate the full second order correction with no assumptions on the size of the order parameter or value of temperature. This is then used with the self-consistency equation to form a second order differential equation for the order parameter.

The result is then expanded to the pure and dirty limits, a simple check can show the $T \to T_c$, small Δ, expansion of the resulting equations limit to the Ginzburg Landau equations of the previous section.

In the following chapter we will consider the equations expanded at zero temperature and attempt a solution in the context of phase slips.

5.2 Weak Anisotropic Expansion; Second Method

Once more our goal is to make an expansion in anisotropy, but this time we use the matrix structure that proved useful in the derivation of Usadel's equations,

$$i\mathbf{v}_F \cdot \check{\partial}\check{g} + [\check{H}, \check{g}] = \frac{1}{2i\tau}[\langle\check{g}\rangle, \check{g}] \tag{5.2.1}$$

and the expansion

$$\check{g} = \check{g}_0 + \check{g}_1 + \check{g}_2 + ...$$

again, $\check{g}_0$ gives the isotropic, velocity independent part, which is still spatially varying.

Keeping terms up to order v_F^2 we expand the normalisation, $\check{g}^2 = \check{1}$ and write the result as commutators,

$$\check{g}^2 = \check{g}_0^2 + \check{g}_1^2 + \check{g}_0\check{g}_1 + \check{g}_1\check{g}_0 + \check{g}_0\check{g}_2 + \check{g}_2\check{g}_0 = 1$$

$$\check{g}_0^2 = 1 \tag{5.2.2}$$

$$\check{g}_0\check{g}_1 + \check{g}_1\check{g}_0 = 0 \quad ; \quad [\check{g}_0, \check{g}_1] = 2\check{g}_0\check{g}_1 \tag{5.2.3}$$

$$\check{g}_1^2 + \check{g}_0\check{g}_2 + \check{g}_2\check{g}_0 = 0 \quad ; \quad [\check{g}_0, \check{g}_2] = \check{g}_1^2 + 2\check{g}_0\check{g}_2. \tag{5.2.4}$$

We write the results as commutators since the equation is given naturally in terms of commutators.

Next we consider the average over the Fermi surface, since the terms are expanded in orders of velocity the odd terms cancel much like before. As before the zeroth order term being velocity independent remains unchanged so that to third order inclusive we write the average,

$$\langle\check{g}\rangle = \check{g}_0 + \langle\check{g}_2\rangle \tag{5.2.5}$$

Next we must work from equations, expanding the full equation (5.2.1) to zeroth, first and second orders gives the equations,

$$[\check{H}, \check{g}_0] = 0 \tag{5.2.6}$$

$$i\mathbf{v}_F\check{\partial}\check{g}_0 + [\check{H}, \check{g}_1] = \frac{1}{2i\tau}[\check{g}_0, \check{g}_1] \tag{5.2.7}$$

$$\begin{aligned} i\mathbf{v}_F\check{\partial}\check{g}_1 + [\check{H}, \check{g}_2] &= \frac{1}{2i\tau}([\check{g}_0, \check{g}_2] + [\langle\check{g}_2\rangle, \check{g}_0]) \\ &= \frac{1}{2i\tau}([\check{g}_0, \check{g}_2] - [\check{g}_0, \langle\check{g}_2\rangle]) \end{aligned} \tag{5.2.8}$$

which we may now solve.

Zeroth Order

$$[\check{H}, \check{g}_0] = 0$$

$$\begin{pmatrix} i\omega_n & \Delta \\ -\Delta^* & -i\omega_n \end{pmatrix}\begin{pmatrix} g_0 & -if_0 \\ if_0^\dagger & -g_0 \end{pmatrix} - \begin{pmatrix} g_0 & -if_0 \\ if_0^\dagger & -g_0 \end{pmatrix}\begin{pmatrix} i\omega_n & \Delta \\ -\Delta^* & -i\omega_n \end{pmatrix} = 0$$

$$\begin{pmatrix} i(\Delta f_0^\dagger - \Delta^* f_0) & 2\omega_n f_0 - 2\Delta g_0 \\ 2\omega_n f_0^\dagger - 2\Delta^* g_0 & -i(\Delta f_0^\dagger - \Delta^* f_0) \end{pmatrix} = 0,$$

which when solved using $\check{g}_0^2 = 1$ gives,

$$\begin{aligned} &g_0 = S_n\omega_n, \quad f_0 = S_n\Delta, \quad f_0^\dagger = S_n\Delta^* \\ &S_n = \frac{1}{\sqrt{\omega_n^2 + |\Delta|^2}} \\ &\check{g}_0 = S_n\begin{pmatrix} \omega_n & -i\Delta \\ i\Delta^* & -\omega_n \end{pmatrix} = \frac{S_n}{i}\begin{pmatrix} i\omega_n & \Delta \\ -\Delta^* & -i\omega_n \end{pmatrix} = -iS_n\check{H} \end{aligned} \tag{5.2.9}$$

We are now at the main point of this method. Previously when considering the first order expansion we solved for the zeroth order, rearranged the first order equation and substituted in the result. This then gave an equation for the first order correction in terms of the functions Δ and ω_n.

Now we think about these functions. Firstly consider ω_n. Although the functions depend on this it is not order dependent and as such was substituted without worry. Secondly, the order parameter is given by the anomalous green's function, naively we may think this be also given by an expansion, $\Delta_0 + \Delta_1 \dots$ but this is not the case. The order parameter is solved self-consistently and must be written in terms of the same function at all orders.

This was done in the previous calculation where the first order equation had a Δ and the zeroth order result substituted in was the same Δ. When combined, they made simplifications to the expression which could be verified by calculation of the current.

In this method though we choose to replace the equation's dependence on Δ and ω_n with the lower order green's functions. That is to say we replace $\check{H}$ with $-\frac{1}{iS_n}\check{g}_0$. What we are doing is simply using the other side of the equality in our perturbation. This is a substitution which allows us to use the commutation relations above.

First Order

In replacing $\check{H}$ by $-\frac{1}{iS_n}\check{g}_0$ we arrive at the first order result swiftly.

$$\begin{aligned} i\mathbf{v}_F\check{\partial}\check{g}_0 + [\check{H},\check{g}_1] &= \frac{1}{2i\tau}[\check{g}_0,\check{g}_1] \\ i\mathbf{v}_F\check{\partial}\check{g}_0 - \frac{1}{iS_n}[\check{g}_0,\check{g}_1] &= \frac{1}{2i\tau}[\check{g}_0,\check{g}_1] \\ i\mathbf{v}_F\check{\partial}\check{g}_0 &= \frac{1}{i}\underbrace{\left\{\frac{1}{S_n}+\frac{1}{2\tau}\right\}}_{\eta}[\check{g}_0,\check{g}_1] \\ -\mathbf{v}_F\check{\partial}\check{g}_0 &= \eta[\check{g}_0,\check{g}_1] = 2\eta\check{g}_0\check{g}_1 \end{aligned}$$

$$\check{g}_1 = -\frac{\mathbf{v}_F}{2\eta}\check{g}_0\check{\partial}\check{g}_0 \tag{5.2.10}$$

this can be used to show other useful relations. Since $\check{g}_0$ and $\check{g}_1$ anticommute we find,

$$\{\check{g}_0, \check{g}_0\check{\partial}\check{g}_0\} = 0$$

multiplying by $\check{g}_0$ gives

$$\{\check{g}_0, \check{\partial}\check{g}_0\} = 0 \tag{5.2.11}$$

This can be used in the manipulation of expressions.

It is at this point we can compare with the diffuse equations, indeed taking the limit $\eta \to \frac{1}{2\tau}$ we have the same result as Usadel.

Though we have made the calculation quite simple, in order to expand these results in terms of $\omega_n, \Delta, \Delta^*$ we must do a lot more work. To do this we first note a few key results,

$$\begin{aligned}
S_n &= (\omega_n^2 + |\Delta|^2)^{-\frac{1}{2}} \\
\partial S_n &= -\frac{S_n^3}{2}(\Delta^* \partial \Delta + \Delta \partial \Delta^*) \\
&= -\frac{S_n^3}{2}\partial |\Delta|^2 \\
\Pi(S_n \Delta) &= S_n \partial \Delta - S_n \frac{2ie\mathbf{A}}{c}\Delta + \Delta \partial S_n \\
&= S_n \Pi \Delta + \Delta \partial S_n \\
\Delta^* \Pi(S_n \Delta) &= |\Delta|^2 \partial S_n + \Delta^* S_n \partial \Delta - S_n |\Delta|^2 \frac{2ie\mathbf{A}}{c}.
\end{aligned}$$

then calculate the following term,

$$\begin{aligned}
\check{g}_0 \check{\partial} \check{g}_0 &= \begin{pmatrix} \omega_n S_n & -iS_n\Delta \\ iS_n\Delta^* & -\omega_n S_n \end{pmatrix} \begin{pmatrix} \omega_n \partial S_n & -i\Pi(S_n\Delta) \\ i\Pi^*(S_n\Delta^*) & -\omega_n \partial S_n \end{pmatrix} \\
&= S_n \begin{pmatrix} \omega_n^2 \partial S_n + \Delta \Pi^*(S_n\Delta^*) & -i\omega_n(\Pi(S_n\Delta) - \Delta \partial S_n) \\ -i\omega_n(\Pi^*(S_n\Delta^*) - \Delta^* \partial S_n) & \omega_n^2 \partial S_n + \Delta^* \Pi(S_n\Delta) \end{pmatrix} \\
&= S_n \begin{pmatrix} (\omega_n^2 + |\Delta|^2)\partial S_n + S_n(\Delta \partial \Delta^* + |\Delta|^2 \frac{2ie\mathbf{A}}{c}) & -i\omega_n S_n \Pi \Delta \\ -i\omega_n S_n \Pi^* \Delta^* & (\omega_n^2 + |\Delta|^2)\partial S_n + S_n(\Delta^* \partial \Delta - |\Delta|^2 \frac{2ie\mathbf{A}}{c}) \end{pmatrix} \\
&= S_n^2 \begin{pmatrix} \frac{1}{S_n^3}\partial S_n + \Delta \partial \Delta^* + |\Delta|^2 \frac{2ie\mathbf{A}}{c} & -i\omega_n \Pi \Delta \\ -i\omega_n \Pi^* \Delta^* & \frac{1}{S_n^3}\partial S_n + \Delta^* \partial \Delta - |\Delta|^2 \frac{2ie\mathbf{A}}{c} \end{pmatrix}
\end{aligned}$$

$$= -\frac{S_n^2}{2}\begin{pmatrix} \Delta^*\partial\Delta - \Delta\partial\Delta^* - |\Delta|^2\frac{4ie\mathbf{A}}{c} & 2i\omega_n\Pi\Delta \\ 2i\omega_n\Pi^*\Delta^* & -(\Delta^*\partial\Delta - \Delta\partial\Delta^* - |\Delta|^2\frac{4ie\mathbf{A}}{c}) \end{pmatrix}$$

$$= -\frac{S_n^2}{2}\begin{pmatrix} \Delta^*\Pi\Delta - \Delta\Pi^*\Delta^* & 2i\omega_n\Pi\Delta \\ 2i\omega_n\Pi^*\Delta^* & -(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*) \end{pmatrix}.$$

We use this to write $\check{g}_1$ explicitly,

$$\begin{aligned} \check{g}_1 &= -\frac{\mathbf{v}_F}{2\eta}\check{g}_0\check{\partial}\check{g}_0 \\ &= \frac{\mathbf{v}_F S_n^2}{4\eta}\begin{pmatrix} \Delta^*\Pi\Delta - \Delta\Pi^*\Delta^* & 2i\omega_n\Pi\Delta \\ 2i\omega_n\Pi^*\Delta^* & -(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*) \end{pmatrix} \\ &= \frac{\mathbf{v}_F}{4(\omega_n^2+|\Delta|^2)\left(\sqrt{\omega_n^2+|\Delta|^2}+\frac{1}{2\tau}\right)}\begin{pmatrix} \Delta^*\Pi\Delta - \Delta\Pi^*\Delta^* & 2i\omega_n\Pi\Delta \\ 2i\omega_n\Pi^*\Delta^* & -(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*) \end{pmatrix} \end{aligned} \tag{5.2.12}$$

reading the upper left component to give g_1 we see that we have calculated the same result as before. Again we can calculate the current and this time we do,

$$\begin{aligned} \mathbf{j} &= -i\pi e\nu(0)T\sum_{n\geq 0}\langle \mathrm{tr}\mathbf{v}_F\check{\sigma}_3\check{g}\rangle \\ &= -i\pi e\nu(0)T\sum_{n\geq 0}\left\langle \frac{v_F^2 S_n^2}{2\eta}(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*)\right\rangle \\ &= e\nu(0)\frac{v_F^2}{d}\Im(\Delta^*\Pi\Delta)\pi T\sum_{n\geq 0}\left(\frac{S_n^2}{\eta}\right) \\ &= e\nu(0)\frac{v_F^2}{d}\Delta^2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\pi T\sum_{n\geq 0}\left(\frac{S_n^2}{\eta}\right) \end{aligned} \tag{5.2.13}$$

valid up to second order inclusive.

Second Order

$$i\mathbf{v}_F\check{\partial}\check{g}_1 + [\check{H}, \check{g}_2] = \frac{1}{2i\tau}([\check{g}_0, \check{g}_2] - [\check{g}_0, \langle\check{g}_2\rangle]) \tag{5.2.14}$$

The equation for second order contains both $\check{g}_2$ and $\langle\check{g}_2\rangle$ but if we average the whole equation we see that the right hand side cancels, the left giving the commutator, $[\check{g}_0, \langle\check{g}_2\rangle]$ this is possible since the first order correction brings its own $\mathbf{v}_F$.
In index notation this reads,

$$iv_\alpha\check{\partial}_\alpha\left(-\frac{v_\beta}{2\eta}\check{g}_0\check{\partial}_\beta\check{g}_0\right) - \frac{1}{iS_n}[\check{g}_0, \check{g}_2] = \frac{1}{2i\tau}([\check{g}_0, \check{g}_2] - [\check{g}_0, \langle\check{g}_2\rangle]) \tag{5.2.15}$$

$$\frac{v_F^2 S_n}{2d}\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right) = [\check{g}_0, \langle\check{g}_2\rangle] \tag{5.2.16}$$

this time we have not used unit vectors for the velocities and each $\mathbf{v}_{\alpha,\beta\ldots}$ has magnitude $\mathbf{v}_F$.

Substituting this in to the right hand side of equation (5.2.15) and grouping as before the second term on the L.H.S. with the first of the R.H.S., then using the expanded normalisation condition gives,

$$\begin{aligned}
-\frac{iv_\alpha v_\beta}{2}\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\beta\check{g}_0\right) &= \frac{\eta}{i}[\check{g}_0, \check{g}_2] - \frac{1}{2i\tau}\left(\frac{v_F^2 S_n}{2d}\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right)\right) \\
[\check{g}_0, \check{g}_2] &= \frac{v_\alpha v_\beta}{2\eta}\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\beta\check{g}_0\right) + \frac{v_F^2 S_n}{4\eta d\tau}\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right) \\
&= \check{g}_1^2 + 2\check{g}_0\check{g}_2 \\
&= \frac{1}{4\eta^2}v_\alpha v_\beta(\check{g}_0\check{\partial}_\alpha\check{g}_0)(\check{g}_0\check{\partial}_\beta\check{g}_0) + 2\check{g}_0\check{g}_2.
\end{aligned}$$

Re ordering the expression we can write the full second order correction to $\check{g}$,

$$2\check{g}_0\check{g}_2 = \frac{v_\alpha v_\beta}{2\eta}\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\beta\check{g}_0\right) + \frac{v_F^2 S_n}{4\eta d\tau}\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right) - \frac{1}{4\eta^2}v_\alpha v_\beta(\check{g}_0\check{\partial}_\alpha\check{g}_0)(\check{g}_0\check{\partial}_\beta\check{g}_0)$$

$$\check{g}_2 = \frac{v_\alpha v_\beta}{4\eta}\check{g}_0\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\beta\check{g}_0\right) + \frac{v_F^2 S_n}{8\eta d\tau}\check{g}_0\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right) - \frac{1}{8\eta^2}v_\alpha v_\beta(\check{\partial}_\alpha\check{g}_0)(\check{g}_0\check{\partial}_\beta\check{g}_0). \tag{5.2.17}$$

This appears to be a very complicated expression and as such we average to find $\langle\check{g}_2\rangle$,

$$\begin{aligned}\langle\check{g}_2\rangle &= \frac{v_F^2}{4d\eta}\check{g}_0\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right) + \frac{v_F^2 S_n}{8\eta d\tau}\check{g}_0\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right) - \frac{1}{8d\eta^2}v_F^2(\check{\partial}_\alpha\check{g}_0)(\check{g}_0\check{\partial}_\alpha\check{g}_0)\\ &= \frac{v_F^2}{4\eta d}\left(1+\frac{S_n}{2\tau}\right)\check{g}_0\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right) - \frac{1}{8d\eta^2}v_F^2(\check{\partial}_\alpha\check{g}_0)(\check{g}_0\check{\partial}_\alpha\check{g}_0)\end{aligned}$$

Lastly $\check{\partial}_\alpha\check{g}_0$ is commuted past the middle $\check{g}_0$ in the final term,

$$\begin{aligned}\langle\check{g}_2\rangle &= \frac{v_F^2 S_n}{4d}\check{g}_0\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}_\alpha\check{g}_0\right) - \frac{1}{8d\eta^2}v_F^2(\check{\partial}_\alpha\check{g}_0)(\check{g}_0\check{\partial}_\alpha\check{g}_0)\\ &= \frac{v_F^2 S_n}{4d}\check{g}_0\left[\check{\partial}_\alpha\left(\frac{1}{\eta}\check{g}_0\check{\partial}\check{g}_0\right) + \frac{1}{2S_n\eta^2}(\check{\partial}_\alpha\check{g}_0)^2\right]\end{aligned} \tag{5.2.18}$$

giving a more compact, averaged result.

5.3 Self Consistency Equation

It is the goal of this section to use the equation for $\langle\check{g}_2\rangle$ to calculate $\langle f_2\rangle$ and substitute this into the self consistency equation in order to ultimately create an equation for the order parameter similar to the Ginzburg-Landau equations. Writing,

$$\langle\check{g}_2\rangle = -i\frac{v_F^2 S_n^2}{4d}\begin{pmatrix} i\omega_n & \Delta \\ -\Delta^* & -i\omega_n\end{pmatrix}\left[\check{\partial}\left(\frac{1}{\eta}\check{g}_0\check{\partial}\check{g}_0\right) + \frac{1}{2S_n\eta^2}(\check{\partial}\check{g}_0)^2\right] \tag{5.3.1}$$

we identify the first term in the square brackets,

$$\check{\partial}\left(\frac{1}{\eta}\check{g}_0\check{\partial}\check{g}_0\right) = -\frac{1}{2}\begin{pmatrix}\partial\left[\frac{S_n^2}{\eta}(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*)\right] & 2i\omega_n\Pi\left[\frac{S_n^2}{\eta}\Pi\Delta\right] \\ 2i\omega_n\Pi^*\left[\frac{S_n^2}{\eta}\Pi^*\Delta^*\right] & -\partial\left[\frac{S_n^2}{\eta}(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*)\right]\end{pmatrix}$$

and see that we must consider acting Π multiple times and on products of functions. From this consideration we write the following 'product rule',

$$\begin{aligned}\Pi[F_1\Pi F_2] &= \Pi[F_1 G] \\ &= \partial[F_1 G] - \frac{2ie}{c}\mathbf{A}F_1 G \\ &= G\partial F_1 + F_1\left(\partial G - \frac{2ie}{c}\mathbf{A}G\right) \\ &= G\partial F_1 + F_1\Pi G \\ &= (\Pi F_2)\partial F_1 + \Pi^2 F_2.\end{aligned}$$

Although ∂ alone acts on the diagonals we consider the following,

$$\Delta^*\Pi[F_1\Pi\Delta] = \Delta^*\{(\Pi\Delta)\partial F_1 + F_1\Pi^2\Delta\}$$

and for real F_1 its conjugate,

$$\Delta\Pi^*[F_1\Pi^*\Delta^*] = \Delta\{(\Pi^*\Delta^*)\partial F_1 + F_1\Pi^{*\,2}\Delta^*\}.$$

The difference between these two expressions gives,

$$\Delta^*\Pi[F_1\Pi\Delta] - \Delta\Pi^*[F_1\Pi^*\Delta^*] = (\partial F_1)(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*) + F_1(\Delta^*\Pi^2\Delta - \Delta^*\Pi^{*\,2}\Delta^*).$$

Returning to the diagonal we now consider,

$$\partial[F_1(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*)] = (\partial F_1)(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*) + F_1\partial(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*).$$

Next we see if the second term on the right of this expression matches that of the one above,

$$\begin{aligned}\partial(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*) &= \partial\left(\Delta^*\partial\Delta - \Delta\partial\Delta^* - \frac{4ie}{c}\mathbf{A}|\Delta|^2\right)\\ &= \Delta^*\partial^2\Delta - \Delta\partial^2\Delta^* - \frac{4ie}{c}\mathbf{A}\left(\Delta^*\partial\Delta + \Delta\partial\Delta^*\right)\\ &= \Delta^*\left(\partial^2\Delta - \frac{4ie}{c}\mathbf{A}\partial\Delta\right) - \Delta\left(\partial^2\Delta^* + \frac{4ie}{c}\mathbf{A}\partial\Delta^*\right)\\ &= \Delta^*\Pi^2\Delta - \Delta\Pi^{*\,2}\Delta^*.\end{aligned}$$

This gives a relationship between the on- and off-diagonal components,

$$\partial[F_1(\Delta^*\Pi\Delta - \Delta\Pi^*\Delta^*)] = \Delta^*\Pi[F_1\Pi\Delta] - \Delta\Pi^*[F_1\Pi^*\Delta^*]. \tag{5.3.2}$$

So may write,

$$\check{\partial}\left(\frac{1}{\eta}\check{g}_0\check{\partial}\check{g}_0\right) = -i\begin{pmatrix}\Im(\Delta^*\Phi_n) & \omega_n\Phi_n\\ \omega_n\Phi_n^* & -\Im(\Delta^*\Phi_n)\end{pmatrix} \tag{5.3.3}$$

$$\Phi_n = \Pi\left[\frac{S_n^2}{\eta}\Pi\Delta\right]. \tag{5.3.4}$$

The other term required for $\langle\check{g}_2\rangle$ is

$$\begin{aligned}(\check{\partial}\check{g}_0)^2 &= -\begin{pmatrix}i\omega_n\partial S_n & \Pi(S_n\Delta)\\ -\Pi^*(S_n\Delta^*) & -i\omega_n\partial S_n\end{pmatrix}\begin{pmatrix}i\omega_n\partial S_n & \Pi(S_n\Delta)\\ -\Pi^*(S_n\Delta^*) & -i\omega_n\partial S_n\end{pmatrix}\\ &= \check{1}\Big((\omega_n\partial S_n)^2 + |\Pi(S_n\Delta)|^2\Big)\\ &= \check{1}\Big((\omega_n\partial S_n)^2 + (\Delta\partial S_n + S_n\Pi\Delta)(\Delta^*\partial S_n + S_n\Pi^*\Delta^*)\Big)\\ &= \check{1}\Big(\underbrace{(\omega_n^2 + |\Delta|^2)}_{S_n^{-2}}(\partial S_n)^2 + S_n^2|\Pi\Delta|^2 + S_n(\Delta^*\Pi\Delta + \Delta\Pi^*\Delta^*)\partial S_n\Big)\end{aligned}$$

in the last term the field's cancel leaving $\Delta^*\partial\Delta + \Delta\partial\Delta^* = \partial|\Delta|^2$.

The derivative of S_n is also related to $\partial|\Delta|^2$ as given before and as such,

$$\begin{aligned}(\check{\partial}\check{g}_0)^2 &= \check{1}\Big(S_n^2|\Pi\Delta|^2 + S_n^{-2}(\partial S_n)^2 + S_n\partial|\Delta|^2\partial S_n\Big) \\ &= \check{1}\left(S_n^2|\Pi\Delta|^2 + S_n^4\left(\frac{1}{2}\partial|\Delta|^2\right)^2 - \frac{S_n^4}{2}(\partial|\Delta|^2)^2\right) \\ &= \check{1}S_n^2\left(|\Pi\Delta|^2 - S_n^2\left(\frac{1}{2}\partial|\Delta|^2\right)^2\right)\end{aligned}$$

which allows us to write,

$$\langle\check{g}_2\rangle = -i\frac{v_F^2S_n^2}{4d}\begin{pmatrix} i\omega_n & \Delta \\ -\Delta^* & -i\omega_n\end{pmatrix}\left[-i\begin{pmatrix}\Im(\Delta^*\Phi_n) & \omega_n\Phi_n \\ \omega_n\Phi_n^* & -\Im(\Delta^*\Phi_n)\end{pmatrix} + \frac{\check{1}S_n}{2\eta^2}\left(|\Pi\Delta|^2 - S_n^2\left(\frac{1}{2}\partial|\Delta|^2\right)^2\right)\right]. \quad (5.3.5)$$

Using this and, $\check{g} = \left(\begin{smallmatrix} g & -if \\ if^\dagger & -g\end{smallmatrix}\right)$ allows us to calculate the average of the second order correction to f,

$$\langle f_2\rangle = \frac{v_F^2S_n^2}{4d}\left[\omega_n^2\Phi_n + i\Delta\Im(\Delta^*\Phi_n) + \frac{\Delta S_n}{2\eta^2}\left(|\Pi\Delta|^2 - S_n^2\left(\frac{1}{2}\partial|\Delta|^2\right)^2\right)\right] \quad (5.3.6)$$

with

$$\begin{aligned}\Phi_n &= \Pi\left[\frac{S_n^2}{\eta}\Pi\Delta\right] \\ &= \frac{S_n^2}{\eta}\Pi^2\Delta + \partial\left(\frac{S_n^2}{\eta}\right)\Pi\Delta \\ &= \frac{S_n^2}{\eta}\left[\Pi^2\Delta - 2S_n^2\left(1 + \frac{1}{2S_n\eta}\right)\frac{\partial|\Delta|^2}{2}\Pi\Delta\right] \end{aligned} \quad (5.3.7)$$

5.3.1 Modulus-Argument Form

To continue we choose a modulus-argument form for the complex order parameter

$$\Delta \to \Delta e^{i\phi}$$

which gives,

$$\partial\Delta \to e^{i\phi}(\partial\Delta + i\Delta\partial\phi) \tag{5.3.8}$$

$$\partial^2\Delta \to e^{i\phi}\Big(\partial^2\Delta - \Delta(\partial\phi)^2 + i\big(\Delta\partial^2\phi + 2(\partial\Delta)(\partial\phi)\big)\Big) \tag{5.3.9}$$

$$\Pi = \partial - \frac{2\pi i e}{c}\mathbf{A} \tag{5.3.10}$$

$$\Pi^2 = \partial^2 - \frac{4\pi i e}{c}\mathbf{A}\partial + \mathcal{O}(A^2) \tag{5.3.11}$$

$$\Pi\Delta \to e^{i\phi}\left(\partial\Delta + i\Delta\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\right) \tag{5.3.12}$$

$$\Pi^2\Delta \to e^{i\phi}\left(\partial^2\Delta - \Delta\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi + i\left(\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\partial\Delta\right)\right) \tag{5.3.13}$$

$$\frac{1}{2}\partial|\Delta|^2 \to \Delta\partial\Delta \tag{5.3.14}$$

$$|\Pi\Delta|^2 \to (\partial\Delta)^2 + \Delta^2\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi. \tag{5.3.15}$$

In order to write (5.3.6) in a modulus-argument form we identify the crucial parts of the three main terms in the square brackets

$$|\Pi\Delta|^2 - S_n^2\left(\frac{1}{2}\partial|\Delta|^2\right) \tag{5.3.16}$$

$$\Phi_n \tag{5.3.17}$$

$$\Im(\Delta^*\Phi_n) \tag{5.3.18}$$

and calculate them in order.

$$|\Pi\Delta|^2 - S_n^2\left(\frac{1}{2}\partial|\Delta|^2\right) = (\partial\Delta)^2 - S_n^2\Delta^2(\partial\Delta)^2 + \Delta^2\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi$$
$$= (\omega_n S_n\partial\Delta)^2 + \Delta^2\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi. \tag{5.3.19}$$

Expanding Φ_n as

$$\Phi_n = \frac{S_n^2}{\eta}\left[\Pi^2\Delta - 2S_n^2\left(1 + \frac{1}{2S_n\eta}\right)\Delta\partial\Delta\Pi\Delta\right] \tag{5.3.20}$$

we substitute in the modulus-argument form,

$$\Phi_n = \frac{S_n^2}{\eta}e^{i\phi}\left[\{\partial^2\Delta - \Delta\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi - 2\Delta S_n^2\left(1 + \frac{1}{2S_n\eta}\right)(\partial\Delta)^2\}\right.$$
$$\left. + i\left\{\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\left(1 - \Delta_0^2 S_n^2\left(1 + \frac{1}{2S_n\eta}\right)\right)\partial\Delta\right\}\right]. \tag{5.3.21}$$

Finally,

$$\Im(\Delta^*\Phi_n) = \frac{S_n^2\Delta}{\eta}\left(\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\left(1 - \Delta_0^2 S_n^2\left(1 + \frac{1}{2S_n\eta}\right)\right)\partial\Delta\right). \tag{5.3.22}$$

Putting these results together in $\langle f_2\rangle$,

$$\langle f_2\rangle = \frac{v_F^2 S_n^2}{4d}e^{i\phi}\left[\omega_n^2\frac{S_n^2}{\eta}\left\{\partial^2\Delta - \Delta\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi - 2\Delta S_n^2\left(1 + \frac{1}{2S_n\eta}\right)(\partial\Delta)^2\right.\right.$$
$$\left. + i\left(\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\left(1 - \Delta_0^2 S_n^2\left(1 + \frac{1}{2S_n\eta}\right)\right)\partial\Delta\right)\right\}$$
$$+ i\Delta^2\frac{S_n^2}{\eta}\left(\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\left(1 - \Delta_0^2 S_n^2\left(1 + \frac{1}{2S_n\eta}\right)\right)\partial\Delta\right)$$
$$\left.\Delta\frac{S_n}{2\eta^2}\left((\omega_n S_n\partial\Delta)^2 + \Delta_0^2\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi\right)\right] \tag{5.3.23}$$

then simplifying by separating the real and imaginary parts we find,

$$\langle f_2\rangle = \frac{v_F^2 S_n^4}{4d\eta} e^{i\phi}\left[\omega_n^2\left(\partial^2\Delta - \Delta\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi - 2\Delta S_n^2\left(1+\frac{1}{2S_n\eta}\right)(\partial\Delta)^2\right)\right.$$
$$+\frac{\Delta}{2S_n\eta}\left((\omega_n S_n\partial\Delta)^2 + \Delta_0^2\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi\right)$$
$$\left.+i\left\{\underbrace{(\omega_n^2+\Delta^2)}_{S_n^{-2}}\left(\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\left(1-\Delta^2 S_n^2\left(1+\frac{1}{2S_n\eta}\right)\right)\partial\Delta\right)\right\}\right]. \tag{5.3.24}$$

Using the modulus-argument form in the self consistency equation written as,

$$\alpha\Delta = 2\pi T\sum_{n\geq 0}\langle f_2\rangle \tag{5.3.25}$$

we can find two equations in terms of the real and imaginary parts,

$$\alpha\Delta = 2\pi T\sum_{n\geq 0}\Re(e^{-i\phi}\langle f_2\rangle) \tag{5.3.26}$$

$$0 = 2\pi T\sum_{n\geq 0}\Im(e^{-i\phi}\langle f_2\rangle) \tag{5.3.27}$$

with

$$\alpha = 2\pi T\sum_{n\geq 0}(S_n^{(0)} - S_n)$$

in terms of the homogeneous result, $S_n^{(0)} = (\omega_n^2 + \Delta_0^2)^{-\frac{1}{2}}$.

5.3.2 General Result

We can now write the most general result for the self-consistent order parameter equation,

$$\alpha\Delta = 2\pi T\sum_{n\geq 0}\left[\frac{v_F^2 S_n^4\omega_n^2}{4d\eta}\left(\partial^2\Delta - \Delta\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi - 2\Delta S_n^2\left(1+\frac{1}{2S_n\eta}\right)(\partial\Delta)^2 + \frac{\Delta}{2S_n\eta}\left((S_n\partial\Delta)^2 + \frac{\Delta^2}{\omega_n^2}\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi\right)\right)\right] \quad (5.3.28)$$

$$0 = 2\pi T\sum_{n\geq 0}\left[\frac{S_n^2}{\eta}\left(\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\left(1 - \Delta^2 S_n^2\left(1+\frac{1}{2S_n\eta}\right)\right)\partial\Delta\right)\right] \quad (5.3.29)$$

valid for arbitrary impurity concentration and temperature. This somewhat complicated set of results include the non-linear corrections to Ginzburg-Landau theory under the quasi-classical approximation, $\Delta \ll E_F$. Taking this limit turns $\alpha\Delta$ into the linear and cubic terms, the majority of terms on the right hand side would correspond to gradients of higher order Δ terms and as such be taken as zero. The current is still given by the first order correction,

$$\mathbf{j} = e\nu(0)\frac{v_F^2}{d}\Delta^2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\pi T\sum_{n\geq 0}\left(\frac{S_n^2}{\eta}\right) \quad (5.3.30)$$

in terms of the functions,

$$\begin{aligned}
\alpha &= 2\pi T\sum_{n\geq 0}(S_n^{(0)} - S_n)\\
S_n &= \frac{1}{\sqrt{\omega_n^2+\Delta^2}} \qquad ; \qquad S_n^{(0)} = \frac{1}{\sqrt{\omega_n^2+\Delta_0^2}}\\
\eta &= \frac{1}{S_n} + \frac{1}{2\tau}\\
\omega_n &= \pi T(2n+1)\\
\nu(0) &= \frac{mp_F}{2\pi^2}
\end{aligned}$$

5.3.3 Matsubara Sums

To proceed we must consider taking the Matsubara sums. These come in two forms, $\sum S_n^z$ and $\sum S_n^z \omega_n^2$. The second is relatable to the first by using $S_n^2\omega_n^2 = 1 - \Delta^2 S_n^2$ and as such we define the following functions,

$$\Lambda(z) = 2\pi T \sum_{n\geq 0} S_n^z \tag{5.3.31}$$

$$\Gamma(z) = 2\pi T \sum_{n\geq 0} S_n^z \omega_n^2 \tag{5.3.32}$$

$$= \Lambda(z-2) - \Delta^2 \Lambda(z) \tag{5.3.33}$$

$$\begin{aligned} S_n^z &= (\omega_n^2 + \Delta^2)^{-\frac{z}{2}} \\ &= \Big((\pi T(2n+1))^2 + \Delta^2\Big)^{-\frac{z}{2}} \\ &= (\pi T)^{-z}\Big((2n+1)^2 + x^2\Big)^{-\frac{z}{2}} \end{aligned} \tag{5.3.34}$$

with $x = \frac{\Delta}{\pi T}$.

This gives us formulae for calculating the Matsubara sums,

$$\Lambda(z) = 2(\pi T)^{1-z} \sum_{n\geq 0} \Big((2n+1)^2 + x^2\Big)^{-\frac{z}{2}} \tag{5.3.35}$$

$$\begin{aligned} \Lambda(z-2) &= 2(\pi T)^{3-z} \sum_{n\geq 0} \Big((2n+1)^2 + x^2\Big)^{1-\frac{z}{2}} \\ \Gamma(z) &= 2(\pi T)^{3-z} \sum_{n\geq 0} \Big((2n+1)^2 + x^2\Big)^{1-\frac{z}{2}} - (x\pi T)^2 2(\pi T)^{1-z} \sum_{n\geq 0} \Big((2n+1)^2 + x^2\Big)^{-\frac{z}{2}} \\ &= 2(\pi T)^{3-z} \sum_{n\geq 0} \Big\{ \Big((2n+1)^2 + x^2\Big)^{1-\frac{z}{2}} - x^2\Big((2n+1)^2 + x^2\Big)^{-\frac{z}{2}} \Big\}. \end{aligned} \tag{5.3.36}$$

Both Λ and Γ can be easily calculated for even values of z. For odd z however, the sums are more complicated. Expansions can be made near $T = T_c$ as considered when calculating the GL equations earlier.

5.3.4 Clean and Dirty Limits

Since the general form is quite complicated we consider the special cases of the pure and dirty limits that we have considered before,

Clean limit, $\eta \to \frac{1}{S_n}$

$$\alpha\Delta = \frac{v_F^2}{4d}\left[\Gamma(5)\left(\partial^2\Delta - \Delta\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi\right)\right.$$
$$\left. -\frac{5}{2}\Delta\Gamma(7)(\partial\Delta)^2 + \frac{\Delta^3}{2}\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi\Lambda(5)\right] \quad (5.3.37)$$

$$0 = \Lambda(3)\left(\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\partial\Delta\right) - 3\Delta_0^2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\partial\Delta\Lambda(5) \quad (5.3.38)$$

Dirty limit, $\eta \to \frac{1}{2\tau}, \quad \tau^2 = 0, \quad D = \frac{v_F^2\tau}{d}$

$$\alpha\Delta = \frac{D}{2}\left[\Gamma(4)\left(\partial^2\Delta - \Delta\left(\partial\phi - \frac{4e}{c}\mathbf{A}\right)\partial\phi\right) - 2\Delta(\partial\Delta)^2\Gamma(6)\right] \quad (5.3.39)$$
$$0 = \Lambda(2)\Delta\partial^2\phi + 2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\partial\Delta\Gamma(4). \quad (5.3.40)$$

where the functions Γ and Λ have even arguments and the Matsubara sums are taken,

$$\Lambda(2) = \frac{\pi}{2\Delta}\tanh\left(\frac{\Delta}{2T}\right) \quad (5.3.41)$$
$$\Gamma(4) = \frac{\pi}{8T}\mathrm{sech}^2\left(\frac{\Delta}{2T}\right)\left(1 + \frac{\sinh\left(\frac{\Delta}{T}\right)}{\frac{\Delta}{T}}\right) \quad (5.3.42)$$
$$\Gamma(6) = \frac{\pi}{32\Delta^3}\left[\frac{\Delta}{T}\mathrm{sech}^2\left(\frac{\Delta}{2T}\right)\left\{\frac{\Delta}{T}\tanh\left(\frac{\Delta}{2T}\right) - 1\right\} + 2\tanh\left(\frac{\Delta}{2T}\right)\right] \quad (5.3.43)$$

$$\frac{\Lambda(2)}{\Gamma(4)} = \frac{2\sinh\left(\frac{\Delta}{T}\right)}{\frac{\Delta}{T} + \sinh\left(\frac{\Delta}{T}\right)} \quad (5.3.44)$$

The last of which when used with (5.3.40) yields,

$$\frac{2\sinh\left(\frac{\Delta}{T}\right)}{\frac{\Delta}{T}+\sinh\left(\frac{\Delta}{T}\right)}\Delta\partial^2\phi = -2\left(\partial\phi - \frac{2e}{c}\mathbf{A}\right)\partial\Delta. \tag{5.3.45}$$

Considering the zero field case, $\mathbf{A} = 0$, this can be solved and used to eliminate the ϕ dependence in favour of the conserved current,

$$\frac{1}{\partial\phi}\partial^2\phi = -\frac{\partial\Delta}{\frac{\sinh\left(\frac{\Delta}{T}\right)}{\frac{\Delta}{T}+\sinh\left(\frac{\Delta}{T}\right)}\Delta} \tag{5.3.46}$$

$$\partial\ln\partial\phi = -\frac{\partial\Delta}{\frac{\sinh\left(\frac{\Delta}{T}\right)}{\frac{\Delta}{T}+\sinh\left(\frac{\Delta}{T}\right)}\Delta} \tag{5.3.47}$$

$$\ln\partial\phi = -\ln\left(\Delta\tanh\left(\frac{\Delta}{2T}\right)\right) + \ln C \tag{5.3.48}$$

$$\partial\phi = \frac{C}{\Delta\tanh\left(\frac{\Delta}{2T}\right)}. \tag{5.3.49}$$

Allowing the current to be defined as,

$$0 = \partial\left(2T\Delta\tanh\left(\frac{\Delta}{2T}\right)\partial\phi\right) \tag{5.3.50}$$

$$\mathbf{J} = 2T\Delta\tanh\left(\frac{\Delta}{2T}\right)\partial\phi \tag{5.3.51}$$

$$\partial\mathbf{J} = 0 \tag{5.3.52}$$

where the $2T$ has been included for the $T \to T_c$ limit to correspond to the equations used in Chapter 2 for which $J = \Delta^2\partial\phi$.

Comparing this to the current (5.3.30) for in the dirty, zero-field limit we find

$$\begin{aligned}\mathbf{j} &= e\nu(0)\frac{v_F^2}{d}\Delta^2\partial\phi\pi T\sum_{n\geq 0}\left(\frac{S_n^2}{\eta}\right)\\ &= e\nu(0)D\Delta^2\partial\phi 2\pi T\sum_{n\geq 0}S_n^2\end{aligned}$$

$$= \frac{\pi e \nu(0) D}{4} \underbrace{2T\Delta \tanh\left(\frac{\Delta}{2T}\right) \partial\phi}_{\mathbf{J}} \tag{5.3.53}$$

using this to remove the $\partial\phi$ dependence in (5.3.39),

$$\frac{D}{2}\left[\Gamma(4)\left(\partial^2\Delta - \Delta\left(\frac{\mathbf{J}}{2T\Delta\tanh\left(\frac{\Delta}{2T}\right)}\right)^2\right) - 2\Delta(\partial\Delta)^2\Gamma(6)\right] - \alpha\Delta = 0 \tag{5.3.54}$$

We now have a second order differential equation for the order parameter of type-II, superconductors at arbitrary temperature, in the zero-field, dirty limit.

$$\begin{aligned}\frac{D}{2}\Bigg[&\frac{\pi}{8T}\mathrm{sech}^2\left(\frac{\Delta}{2T}\right)\left(1+\frac{\sinh\left(\frac{\Delta}{T}\right)}{\frac{\Delta}{T}}\right)\left(\partial^2\Delta - \frac{J^2}{4T^2\Delta\tanh^2\left(\frac{\Delta}{2T}\right)}\right)\\ &-(\partial\Delta)^2\frac{\pi}{16\Delta^2}\left[\frac{\Delta}{T}\mathrm{sech}^2\left(\frac{\Delta}{2T}\right)\left\{\frac{\Delta}{T}\tanh\left(\frac{\Delta}{2T}\right)-1\right\}+2\tanh\left(\frac{\Delta}{2T}\right)\right]\Bigg]\\ &-\alpha\Delta = 0\end{aligned} \tag{5.3.55}$$

with the functions,

$$\mathbf{J} = 2T\Delta\tanh\left(\frac{\Delta}{2T}\right)\partial\phi \tag{5.3.56}$$

$$\partial\mathbf{J} = 0 \tag{5.3.57}$$

$$\mathbf{J} = \Delta^2\partial\phi \qquad T \to T_c \tag{5.3.58}$$

$$\alpha = 2\pi T\sum_{n\geq 0}(S_n^{(0)} - S_n) \tag{5.3.59}$$

$$\approx -\left(1 - \frac{T}{T_c^{(0)}}\right) + \frac{7\zeta(3)}{8\pi^2 T_c^2}\Delta^2 \qquad T \to T_c \tag{5.3.60}$$

5.3.5 Comparison to Werthamer

At this point we draw attention to the work of Werthamer [51] who extended the calculation of Gorkov.

He too felt that the restrictions on temperature and therefore the size of Δ were unnecessary and calculated the second order expansion and extended the range of validity for the Ginzburg-Landau equations. His method is similar to Gorkov, and as such requires the calculation of the vertex corrections not present in this derivation. However, it seems possible that the work presented in this thesis would match that of Werthamer in the quasi-classical limit.

In addition to his paper, the section he gives in Parks [3], provides a nice introduction to the Ginzburg-Landau equations not to be overlooked.

Chapter 6

A RETURN TO PHASE SLIPS

6.1 Introduction

Returning to the problem of phase slips we reduce the equations of the last section to zero temperature. This is done for the impure case alone where the functional form resulting from taking the Matsubara sums is known.

Since we do not know the free energy equation, we proceed by manufacturing a function which minimises to these reduced equations. In an analogous manor to the simple barrier calculation we form the free energy barrier for thermal phase slips at zero current.

Due to the temperature dependence of the Arrhenius law we expect the probability of thermal phase slips at zero temperature to be 'unlikely' to say the least. However, an approximate form for the order parameter is given which may shed light on quantum phase slips.

6.2 Zero Temperature Equation

We now consider the expansion of the dirty result to zero temperature. We do this since the exact functional dependence is known,

$$\alpha = 2\pi T \sum_{n\geq 0} \left[\frac{1}{\sqrt{\omega_n^2 + \Delta_0^2}} - \frac{1}{\sqrt{\omega_n^2 + \Delta^2}} \right]$$

$$\lim_{T\to 0} \alpha = \alpha_0$$

$$\alpha_0 = \ln\left(\frac{\Delta}{\Delta_0}\right) \tag{6.2.1}$$

This time since we are taking the $T \to 0$ limit we remove the previously included factor of $2T$, since this would cause a divergence and define the current $\mathbf{h} = \frac{\mathbf{J}}{2T}$ such that,

$$\Delta \tanh\left(\frac{\Delta}{2T}\right) \partial\phi = \text{const}$$

$$\Delta \partial\phi = \mathbf{h} \tag{6.2.2}$$

$$\partial \mathbf{h} = 0. \tag{6.2.3}$$

By comparing this result to that of Langer and Ambegaokar, i.e. when we used the current $\mathbf{J}$, it seems at low temperatures that the current becomes temperature dependent, this is not the case as in the LA calculation $\mathbf{J} = \Delta^2 \partial\phi$ and here $\mathbf{h} = \Delta\partial\phi$. We must now be careful in reducing the equation,

$$\alpha\Delta = \frac{D}{2}\left[\frac{\pi}{8T}\text{sech}^2\left(\frac{\Delta}{2T}\right)\left(1 + \frac{\sinh\left(\frac{\Delta}{T}\right)}{\frac{\Delta}{T}}\right)\{\partial^2\Delta - \Delta(\partial\phi)^2\}\right.$$
$$\left. -\frac{\pi}{16\Delta^2}(\partial\Delta)^2\left\{\frac{\Delta}{T}\text{sech}^2\left(\frac{\Delta}{2T}\right)\left\{\frac{\Delta}{T}\tanh\left(\frac{\Delta}{2T}\right) - 1\right\} + 2\tanh\left(\frac{\Delta}{2T}\right)\right\}\right]. \tag{6.2.4}$$

By using $\text{sech}^2(X)\sinh(X/2) = 2\tanh(X/2)$, the above for $T \to 0$ becomes,

$$\begin{aligned}\alpha_0\Delta &= \frac{D}{2}\left[\frac{\pi}{8\Delta}2\left\{\partial^2\Delta - \frac{h^2}{\Delta}\right\} - \frac{2\pi}{16\Delta^2}(\partial\Delta)^2\right] \\ &= \frac{D\pi}{8\Delta}\left[\partial^2\Delta - \frac{h^2}{\Delta} - \frac{(\partial\Delta)^2}{2\Delta}\right] \end{aligned} \tag{6.2.5}$$

So we can write,

$$\partial^2\Delta - \frac{1}{2\Delta}(\partial\Delta)^2 - \frac{h^2}{\Delta} = \frac{8\Delta^2}{\pi D}\ln\left(\frac{\Delta}{\Delta_0}\right). \tag{6.2.6}$$

If we consider the following second derivative,

$$\begin{aligned}\partial^2[\sqrt{\Delta}] &= \partial\left[\frac{1}{2\sqrt{\Delta}}\partial\Delta\right] \\ &= \frac{1}{2\sqrt{\Delta}}\partial^2\Delta - \frac{1}{4\sqrt{\Delta^3}}(\partial\Delta)^2 \\ 2\sqrt{\Delta}\partial^2[\sqrt{\Delta}] &= \partial^2\Delta - \frac{1}{2\Delta}(\partial\Delta)^2\end{aligned}$$

and substitute this in to our equation along with dividing them by Δ_0 we have,

$$2\sqrt{\frac{\Delta}{\Delta_0}}\partial^2\sqrt{\frac{\Delta}{\Delta_0}} - \frac{h^2}{\Delta_0\Delta} = \frac{8\Delta_0}{\pi D}\left(\frac{\Delta}{\Delta_0}\right)^2\ln\left(\frac{\Delta^2}{\Delta_0^2}\right) \tag{6.2.7}$$

using the substitution, $z = \sqrt{\frac{\Delta}{\Delta_0}}, \Delta = \Delta_0 z^2$ yields finally,

$$\frac{d^2z}{dx^2} - \frac{h^2}{2\Delta_0^2 z^3} = \frac{8\Delta_0}{\pi D}z^3\ln z \tag{6.2.8}$$

where we are now considering the function in one dimension.

Multiplying through by $\frac{dz}{dx}$ allows one to find,

$$\left(\frac{dz}{dx}\right)^2 + \frac{h^2}{2\Delta_0^2 z^2} - \frac{\Delta_0}{\pi D}z^4(\ln z^4 - 1) = \text{const} = E. \tag{6.2.9}$$

As in the case of the LA calculation we must find the uniform solution to these equations, this requires,

$$z = z_0 \qquad \frac{dz_0}{dx} = 0 \qquad \phi = kx$$

using these gives,

$$z_0^2 \ln z_0^2 = -\frac{\pi D}{8\Delta_0} k^2 \tag{6.2.10}$$

solving this transcendental equation is in general difficult, but later we will consider the case where $k = 0$ giving, $z_0 = 0, 1$.

Next we turn to the constant in equation (6.2.9)

$$\begin{aligned} E &= \frac{h^2}{2\Delta_0^2 z_0^2} - \frac{\Delta_0}{\pi D} z_0^4 (\ln z_0^4 - 1) \\ &= \frac{(\Delta_0 z_0^2 k)^2}{2\Delta_0^2 z_0^2} - \frac{2\Delta_0}{\pi D} z_0^2 (z_0^2 \ln z_0^2) + \frac{\Delta_0}{\pi D} z_0^4 \\ &= \frac{z_0^2}{2} k^2 + \frac{2\Delta_0}{\pi D} z_0^2 \frac{\pi D}{8\Delta_0} k^2 + \frac{\Delta_0}{\pi D} z_0^4 \\ &= \frac{3}{4} z_0^2 k^2 + \frac{\Delta_0}{\pi D} z_0^4. \end{aligned} \tag{6.2.11}$$

This gives

$$\begin{aligned} \left(\frac{dz}{dx}\right)^2 &= \frac{3}{4} z_0^2 k^2 + \frac{\Delta_0}{\pi D} z_0^4 - \frac{h^2}{2\Delta_0^2 z^2} + \frac{\Delta_0}{\pi D} z^4 (\ln z^4 - 1) \\ \left(\frac{dz}{dx}\right)^2 &= \frac{\Delta_0}{\pi D} \left[z_0^4 - z^4 (1 - \ln z^4) \right] + \frac{3}{4} z_0^2 k^2 - \frac{h^2}{2\Delta_0^2 z^2} \end{aligned} \tag{6.2.12}$$

which is the general extension of the LA theory to zero temperature. Knowing that the limits are controlled by the transcendental equation for the uniform case we shall restrict our attention to the zero current $k = 0$. Above we noted that this leads to $z_0 = 0, 1$ and indeed this is related to Δ going between $0, \Delta_0$.

Rescaling the spatial variable, $X = \sqrt{\frac{2\Delta_0}{\pi D}}x$ gives,

$$\left(\frac{dz}{dX}\right)^2 = \frac{1}{2}\Big[1 + z^4(\ln z^4 - 1)\Big] \tag{6.2.13}$$

This can be solved numerically and does indeed appear to behave like an inverse tanh, as shown in Figure 6.1.

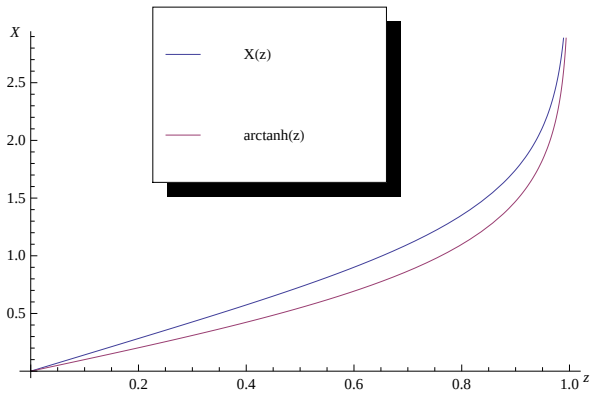

Figure 6.1: Numerical result of integration plotted with inverse tanh

By analogy for the tanh solution,

$$\frac{dz}{dX} = 1 - z^2 \qquad z = \tanh(q) \qquad \frac{dz}{dq} = \operatorname{sech}^2(q)$$

this would give,

$$\frac{dq}{dX} = 1 \qquad z = \tanh(X)$$

We shall continue in a similar way to see to what extent the function on the right hand side differs from one.

$$\frac{dq}{dX} = \sqrt{\frac{1}{2}(\cosh^4(q) + \sinh^4(q)(4\ln(\tanh(q)) - 1))} \tag{6.2.14}$$

Plotting the function on the right hand side we see the result shown in Figure 6.2,

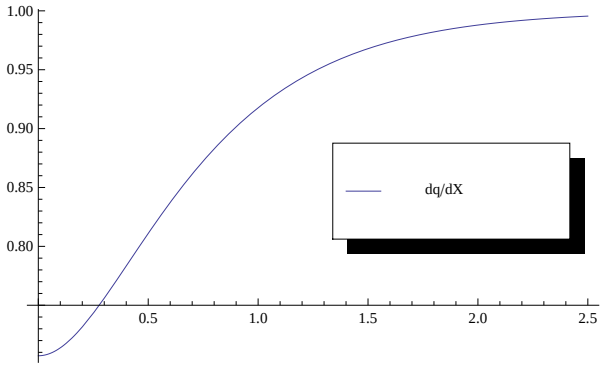

Figure 6.2: The extent to which $\frac{dq}{dX}$ differs from 1

The function is 1 far from the origin, but smoothly goes down to a value of $1/\sqrt{2}$ at $X = 0$. We interpret this as behaving tanh like but more slowly sloped near the origin.

$$z = \tanh(aX) \tag{6.2.15}$$

where a smoothly varies near the origin from $1/\sqrt{2} \to 1$.

For the order parameter we must look at this function squared, i.e.

$$\Delta = \Delta_0 \tanh^2\left(a\sqrt{\frac{2\Delta_0}{\pi D}}x\right) \tag{6.2.16}$$

and shown in Figure (6.3)

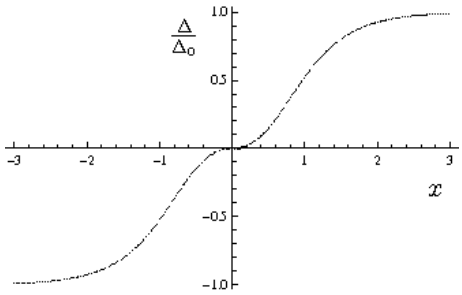

Figure 6.3: Approximate form for Δ/Δ_0 across phase slip

6.3 Free Energy

Though we do not formally have a free energy functional we construct one by finding a functional that when minimised gives us the equations of interest. So as to not cause issue with conserved quantities we restrict our attention to the zero current case. We also know the bounds for integration will be given by 0 and 1. In this situation we have the equations,

$$\frac{d^2z}{dx^2} = \frac{8\Delta_0}{\pi D} z^3 \ln z \tag{6.3.1}$$

$$\left(\frac{dz}{dx}\right)^2 = \frac{\Delta_0}{\pi D}(1 - z^4(1 - \ln z^4)) \tag{6.3.2}$$

we must construct a functional that when minimised gives the equation above,

$$F[z] = \alpha\sigma \int dx \left[\left(\frac{dz}{dx}\right)^2 + \frac{\Delta_0}{\pi D} z^4(\ln z^4 - 1)\right]. \tag{6.3.3}$$

Since the equations must result by minimising a free energy we associate this functional with the free energy. Integrating the first term by parts as was done in the early section we may reduce this functional,

$$F[z] = \alpha\sigma \left(\underbrace{\left[z\frac{dz}{dx}\right]}_{=0} + \int dx \left[-z\left(\frac{d^2z}{dx^2}\right) + \frac{\Delta_0}{\pi D} z^4(\ln z^4 - 1)\right]\right) \tag{6.3.4}$$

$$= -\alpha\sigma \frac{\Delta_0}{\pi D} \int dx z^4(1 + \ln z^4). \tag{6.3.5}$$

we can can calculate the condensation energy as in the LA section,

$$\frac{F[z_0]}{\sigma L} = -\alpha \frac{\Delta_0}{\pi D} = \frac{H_c^2}{8\pi} \tag{6.3.6}$$

$$-\alpha = \frac{\pi D}{\Delta_0} \frac{H_c^2}{8\pi} \tag{6.3.7}$$

Free energy difference between uniform and non-uniform solutions is given by,

$$\begin{aligned}
\delta F &= F[z_0] - F[z] \\
&= -\alpha\sigma\frac{\Delta_0}{\pi D}\int dx\left[1 - z^4(1+4\ln z)\right] \\
&= -2\alpha\sigma\frac{\Delta_0}{\pi D}\int_0^1 dz\frac{dx}{dz}\left[1 - z^4(1+4\ln z)\right] \\
&= -2\alpha\sigma\frac{\Delta_0}{\pi D}\int_0^1 dz\frac{1 - z^4(1+4\ln z)}{\sqrt{\frac{\Delta_0}{\pi D}(1 - z^4(1-4\ln z))}} \\
&= -2\alpha\sigma\sqrt{\frac{\Delta_0}{\pi D}}\underbrace{\int_0^1 dz\frac{1 - z^4(1+4\ln z)}{\sqrt{1 - z^4(1-4\ln z)}}}_{a} \\
&= 2a\sigma\sqrt{\frac{\pi D}{\Delta_0}}\frac{H_c^2}{8\pi} \qquad (6.3.8)
\end{aligned}$$

$$a \approx 1.49276. \qquad (6.3.9)$$

Using the relationships

$$\Delta_0 = \frac{\pi T_c}{\gamma} \qquad \gamma = 1.7811 \qquad \xi = \sqrt{\frac{\pi D}{8T_c}} \qquad (6.3.10)$$

this can be written in comparison to the Langer Ambegaokar result,

$$\delta F = \left(\frac{3}{2}a\sqrt{\frac{\gamma}{\pi}}\right)\frac{8\sqrt{2}}{3}\sigma\xi\frac{H_c^2}{8\pi} \approx 1.686\ \delta F_{LA} \qquad (6.3.11)$$

Since we are using the difference between the free-energies we are still investigating the role of thermal phase slips.

The similarity of results from totally distinct equations goes some way to verifying the equations validity at temperatures much lower than T_c.

Chapter 7

FUTURE WORK AND CONCLUDING REMARKS

7.1 Introduction

In this section we will consider extensions to the work and draw conclusions from the thesis.

7.2 Future Work

The following is a brief exposition of items not addressed in the thesis for considerations in the future.

7.2.1 Free Energy

It is painfully apparent that the formalism presented is incomplete. There is no general formula given for the free-energy. The method given by Kopnin [53] for its calculation is brief to say the least. It is clear that the equations should form from the minimisation of a free-energy functional as is the case for the GL equations. Even the Eilenberger equations have a free-energy, it was this which caused much distraction in the consideration of a free-energy here. Knowledge of this functional would greatly extend the applicability of this work. Allowing for the creation of effective models of QPS.

7.2.2 Momentum Transfer and Paramagnetism

In the derivation of the impurity terms thought was given to the spatial dependence of given functions such as the order parameter and applied fields.

The spatial variable which describes the overall inhomogeneity was considered to be independent of events occurring on the length scale associated with impurities. In truth this is a simple extension of existing derivations. The spatial dependence provides non-linearity, a first approximation is to wipe out the dependence, the second to introduce an independent length scale to what is causing the non-linearity. After this momentum transfer across length scales would be required.

Though not apparently required for the derivation of the Eilenberger equations, this transfer would be present if any field be applied in practice. This is likely to be resolved by considering higher order terms in the convolution between the impurity

potential and Green's function via,

$$A \otimes B(p, R, \omega_n) = \exp\left[\frac{i}{2}\left(\frac{\partial A}{\partial R}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial R}\right)\right] A(p, R, \omega_n)B(p, R, \omega_n).$$

The application of a field would also alter the numbers of electrons in a specific spin configuration. Addition of these paramagnetic effects would require the use of a 4×4 matrix structure from the beginning to account for $G_{\alpha\beta}$ terms along with $G_{\alpha\alpha} \neq G_{\beta\beta}$. The complete description would contain more equations, but might be soluble in a similar expansion.

7.2.3 Non-Equilibrium Phenomena

It would also be beneficial to present a similar expansion which could give extensions to the time dependent Ginzburg Landau equations. This would require early assumptions to be corrected and the Keldysh technique employed.

7.2.4 Computational Investigation

Although a result has been extracted from the equations in a specific limit, investigation of the expressions at higher temperature, or more general impurity concentrations would be preferable. This would no doubt require computational effort, as even the addition of current to the simple case considered shows.

7.3 Conclusion

By considering the physical example of TAPS we have been lead into thinking about Eilenberger's equations for quasi-classical superconducting Green's functions.

We derive Eilenberger's equations in a self contained manor where all quantities are defined along the way. This starts from the definition of the Green's function and the corresponding Gorkov equations, then leads to defining quasi-classical Green's functions which are analogue to the previous Green's functions averaged over the Fermi-Surface (FS). Doing so removes the fast momenta which now take their value at the FS, this approximation leaves the slower momenta which describe residual spatial dependence of Type-II superconductors.

To use these Green's functions Eilenberger subtracted the two Gorkov equations from each other in order to remove the expressions dependence on momenta. This had the side effect of removing the cause of non-linear vertex corrections, the effect of which could be reintroduced via a general non-linear solution.

It is this which is the most useful aspect of the Eilenberger equations and gives them great power. Eilenberger's equations are now less complex than Gorkov's, owing to the removal of the full momentum dependence, but they are much more easy to manipulate due to the removal of vertex corrections.

Once calculated, applications the Eilenberger equations in certain limits were considered. The first of these was the dirty limit investigated by Usadel in which the concentration of impurities smears out the anisotropy. This lead naturally to considering superconductors which were weakly anisotropy in their own right, a limit which could be reduced to that of Usadel.

The calculation of the second order correction to slowly varying quasi-classical Green's functions was performed for the situation where the order parameter itself was vanishing. This is the case near $T = T_c$ and gives a simple derivation of the Ginzburg-Landau equations.

In performing this calculation it was noted that the condition of making the order parameter small was self imposed, and only present in order to simplify the expressions. The general second order expansion should in principle be calculable.

This general second order expansion was then performed, and in an analogous manor to the previous calculation, a set of second order differential equations was derived. These equations form the extension of the GL equations to lower temperatures where the order parameter has no restriction on magnitude.

This idea was also used by Werthamer in his derivation of the Ginzburg-Landau-Gorkov equations. However, his method started from Gorkov's equations and as such contained full information about the momentum. This leads to the complicated vertex corrections not present here. Starting from Eilenberger, the Ginzburg-Landau-Eilenberger equations given in this thesis are the quasi-classical counterpart to the GLG equations. The derivation of the GLE equations far more simple and could in principle be extended to higher orders. The only cost is the assumption that the typical energy scale for the order parameter is significantly lower than the Fermi energy which for conventional superconductors being considered, is indeed the case.

Finally the resulting equations were restricted to the dirty limit and taken to $T = 0$ where these simplifications could allow for an attempted solution in the context of TAPS. An approximate form for the order parameter has been suggested along with the calculation of the free-energy gap for a phase slip event at zero current and temperature.

The free energy can be written in a similar form to that of Langer and Ambegaokar and its value is the same order of magnitude. Calculated from vastly distinct equations, this offers a verification of sorts for the general equations.

The form for the order parameter is found to go as $\tanh^2(X)$, mirrored for negative X providing the phase difference of 2π required at zero current. This differs

from the LA result of $\tanh(X)$, the major difference being that the new solution is flat at the origin. This does not seem to affect the physics much as this point does not form a bound solution.

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