QUANTUM GREY SOLITONS IN CONFINING POTENTIALS

by

DOMINIC CARL WADKIN-SNAITH

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Abstract

This thesis deals with grey soliton dynamics in confining potentials and their association with the trapped Lieb II mode. There has been a large body of work into the study of grey solitons and their dissipation in ultra cold trapped Bose gases. It is well known at finite temperature that the standard mechanism for soliton decay in uniform ultra cold gases is scattering of thermal phonons, where the soliton lifetime was found to be proportional to inverse temperature for temperatures greater than the chemical potential [54]. The scattering of thermal phonons becomes no longer efficient for the exactly integrable Gross-Pitaevski equation. The case of non-integrable interactions has been considered and for this case the lifetime of the soliton was found to diverge as the inverse fourth power of temperature T[25]. Thus solitons propagating in a uniform background are found to be stable at zero temperature.

In the presence of a trapping potential, momentum conservation is lost due to the loss of translational invariance and this leads to soliton decay even at zero temperature. Classically it has been shown that as long as the confinement is slowly varying on the length scale of the soliton then conservative dynamics are found [10], [41].

To extend previous findings we use the Born-Sommerfeld rule to quantise the grey soliton in any smooth potential with one minimum. We find a ladder of energy levels for the case of harmonic confinement and associate these states with the trapped Lieb 11 mode. These trapped Lieb 11 states are not eigenstates but are to be considered as quasiparticle resonances. The finite lifetime is due to the radiation of phonons by an accelerating grey soliton, we are able to calculate this lifetime in the semiclassical limit. We show that for a large number of trapped atoms that the quasiparticle states are long lived and can be resolved experimentally.

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Chapter 1

INTRODUCTION

Quantum mechanics naturally divides matter into two species, Fermions and Bosons. These two forms of matter are distinguished by their inherent statistics, an exchange of two Bosons leaves the Boson wavefunction unchanged while an exchange of two Fermions acquires a change of phase such that one picks up a minus sign. This difference in statistics results in deep fundamental differences in the properties of these particles, Fermions carry half integer spin while Bosons carry integer spin. Another consequence of this difference in statistics is occupation of quantum levels, for example only one Fermion can occupy a quantum state characterised by some given quantum numbers. If one tries to place another Fermion into the same state, the wavefunction would be zero for this configuration, of course this is known as the Pauli Exclusion principle.

While generalising the work of Bose [7] to systems of conserved particle number Einstein [18] came across a remarkable discovery for an ideal Bose gas. Einstein noticed that at low temperatures the lowest energy state would become heavily saturated i.e. a large number of Bose particles at these low temperatures would then occupy the lowest energy level. For a period it was believed that this effect would not be observed; this was based on the belief that once interactions were considered the Bosons would behave like a normal system.

That was until 1947 when Bogoliubov [6] demonstrated that a microscopic theory of a weakly interacting Bose gas could be derived on the basis of there being a Bose-Einstein condensate present. From here the number of theoretical works increased and this saw theory ahead of experiment.

In the 1970s experiments started with spin-polarised hydrogen, it was thought of as a good candidate due to its light mass. In these experiments, the hydrogen was cooled to very low temperatures using magnetic and optical traps and cooling mechanisms [60], although a BEC was not to be achieved at the time. Bose-Einstein condensates were first achieved in the landmark experiments [19], [16], [8] in dilute gases of alkali atoms, but it was in 1998 that a Bose-Einstein condensate in spin polarised hydrogen [24] was achieved.

1.1 Solitons

It was found that the Gross-Pitaevskii [32], [59] equation governed the condensate wavefunction while including interactions. This equation after a rescaling is the Nonlinear Schrodinger (NLS) equation commonly used in the study of optics, where one is studying laser beams. The NLS equation has a wealth of interesting properties due to the nonlinear term, it is the competition between the dispersive term and the defocusing nonlinear term that results in exotic solutions such as solitons and vortices.

The history of the soliton is credited with starting with a startling observation made by an engineer named John Scott Russell [15]. He had been observing the motion of a canal boat when it came to a sudden stop, this caused a body of water to accumulate around the prow of the boat. In what appeared to be an unstable state at the prow of the boat, it then suddenly started off leaving the boat behind as it travelled along the canal. Russell proceeded to follow this solitary wave via horseback at what he noted to be around 8 or 9 miles per hour for a couple of miles. He noted that this solitary wave had a remarkable property and that was the preservation of its shape as it travelled, he named this wave form a solitary wave.

After John Scott Russell's observation there was not much thought given to this wave that seemed to maintain its shape without dispersion along the canal. This observation caused some confusion in the scientific community as the existing hydrodynamic theories could not explain this phenomena, it wasn't until 1898 that the dutch physicists Korteweg and de Vries [42] discovered the KdV equation. This nonlinear partial differential equation was shown by the authors to admit continuous wave and solitary wave solutions.

So far we have been using the term solitary wave and not soliton, there is good reason for this as we shall discuss next. Between Russell's first observation of the solitary wave and the work done by Korteweg and de Vries there had not been very much research activity into this phenomena. It wasn't until the mid 20th century that work in this field really started to come into fruition. In 1955, Fermi, Pasta and Ulam [21] studied numerically a 1-dimensional lattice with cubic nonlinearity, the nonlinear term allowed modes of the lattice to interact. They would excite many modes as an initial condition and then wait for the system to thermalise; they believed from thermodynamics that the system would thermalise after long enough time and one would have the standard equipartition of energy. What they found was quite remarkable, after long enough times the system would return to its initial condition, which was very unexpected at the time.

It took another 10 years for the phenomena observed in the Fermi, Pasta and Ulam numerical experiment to be explained. In 1965 Zabusky and Kruskal [74] used the KdV equation as a model for the 1-dimensional nonlinear lattice; they investigated the KdV equation numerically and found an explanation of this recurrence phenomena. They found that a smooth wave initial condition would break up into solitary waves, and used this result to explain the recurrence result from the nonlinear lattice. They proceeded to make a startling observation, a collision of these solitary waves governed by the KdV equation left the solitary waves unchanged. The waves would maintain their shape and velocity after collisions with each other, it would be this condition combined with propagation without dispersion of the solitary wave that would be termed the soliton. We note that the discussion so far on solitons has been in reference to the type of soliton known as a bright soliton which is a localised density maximum travelling with some velocity in the fluid.

Soon after the discovery made by Zabusky and Kruskal more progress was made on obtaining the soliton solutions and demonstrating the collision properties analytically. This progress was handled by the pioneering work of Gardner, Greene, Kruskal and Miura [26]. They demonstrated that the KdV equation could be reduced to two linear problems for some auxiliary variable, one equation governed the scattering dynamics of this variable and the other governed the time dynamics. The equation for scattering was the Schrodinger equation with the potential u being the solution of the original KdV equation that one is solving for. For this potential they were able to construct the scattering data and from this data they were able to build up the solution u of the KdV equation for $t = 0$. The time evolution equation was simple, thus they were able to write down the time dependence of the scattering data and thus deduce that of u. This method is known as the Inverse Scattering Transform (IST) and with this they were able to show that indeed solitons are preserved after collisions.

This then catalysed a flurry of research into the application of the IST to many nonlinear partial differential equations. In 1972 Zakharov and Shabat [75] generalised the IST to solve the NLS equation and found soliton solutions and demonstrated the structure preserving properties of the solitons after collisions. They were also able to demonstrate that one could use the scattering data to generate an infinite set of conservation laws, this is a necessity for a differential equation to be integrable. The Sine-Gordon equation was also found to be solvable by IST and admit soliton solutions by M J Ablowitz et al. [2]; since these early works many more equations have been found to be solved by IST and to admit soliton solutions. It has become an area of research in its own right and still is an active area of research.

1.1.1 Solitons and cold atoms

In 1971 T. Tsuzuki [73] derived a grey soliton solution to the Gross-Pitaevskii equation, which from the previous discussion on the NLS equation should now, be no surprise. However this is an interesting discovery which suggests the possibility of solitons existing in Bose-Einstein condensates.

We need to deviate slightly here and distinguish some of the different forms of solitons from each other so there is no confusion. As we have mentioned bright solitons are localised density peaks in the medium they travel in. Grey solitons on the other hand are localised density minima that travel in some medium with a given velocity. In the limit of zero velocity the grey soliton is of course stationary and is known as a dark soliton. For the NLS equation it is well known that the attractive interactions give rise to bright solitons and repulsive interactions give rise to grey or dark solitons.

1.2 Thesis Overview

This thesis concerns grey solitons in confining potentials, which was born from the question of soliton decay in trapped cold atom clouds. The results and ideas from [41] had left questions about how a soliton modelled as a Landau quasi particle would dissipate in a trapped system and what the mechanism of decay would be at zero temperature. Another question which became related to the soliton decay was the question of soliton quantisation and how one might achieve that.

The thesis is structured as follows: In chapter two we review some basic theoretical results on Bose-Einstein condensed systems. We will consider effective 1-dimensional gases and whether these one-dimensional models are applicable to experiment. We will also demonstrate the existence of a Bose-Einstein condensate in 1-dimension and introduce the concept of a quasicondensate which will be a useful concept and one which can be used for the trapped system as well.

In chapter three we will consider the more formal theory of solitons such as the IST for the NLS equation and discuss the structure preserving collisions of solitons. The integrals of motion will have to be renormalised for grey soliton solution of the NLS equation and the procedure for doing this will be worked through. This will be very important for obtaining a sensible effective theory for a grey soliton in an confining potential. Finally the question of perturbations to the NLS equation with repulsive interactions will be discussed such as obtaining shallow fast moving solitons and demonstrating that they are solutions to the KdV equation. The chapter will finish with an overview of the work from [41], this will set us up well for the discussion of dissipation of a grey soliton modelled as a Landau quasiparticle.

Chapter four is an overview of the quantisation of soliton solutions. We start with a short discussion on the general method of quantisation and then move on to a toy problem of real scalar fields that admit soliton like solutions. We consider a simple toy model to demonstrate the idea behind the quantisation of nontrivial classical solutions, it will also illustrate some of the pitfalls of quantising nontrivial objects such as solitons. A major problem with soliton quantisation is the translational invariance they have which is the cause of infinities in the corresponding quantum theory. The Christ & Lee method will be reviewed, which is a way out of this problem. The Gross-Pitaevskii equation and the quantisation of its soliton solutions will be discussed in the absence of a confining potential. Finally the quantisation of a trapped grey soliton will be presented, this quantisation problem is very different to the previous problems due to the lack of translational invariance in the presence of a trap. We will use the Born-Sommerfeld rule to quantise the soliton and show that these trapped quantum grey solitons can be thought of as Lieb II modes in a trap. The question of when this picture breaks down comes in the next chapter.

The final chapter deals with the nonconservative dynamics of a soliton confined by a smooth external confining potential. The soliton is modelled as a Landau quasiparticle and its interaction with phonons is considered. A derivation of the form of the coupling to phonons is offered which is specific to the soliton and its parameters. We then present a second derivation of the form of the coupling to phonons, only this second derivation is more universal and can apply to impurities immersed in superfluid systems. Once the coupling between the soliton and the phonons has been established we derive an effective action by integrating out the phonons using the Keldysh technique. From this approach we derive Hamilton's equations of motion which we then use to derive the soliton energy loss per period of oscillation within the confining potential. This is used to write down a formula for the lifetime of a harmonically trapped grey soliton.

Chapter 2

EXPERIMENT & SOME THEORETICAL RESULTS

2.1 Some background theory on dilute Bose gases

Within this chapter we will discuss some well known and important properties of dilute bosonic gases as well as briefly discussing the experimental approach to the creation and observation of solitons in trapped Bose-Einstein condensates. We will look into whether it is physically meaningful to study Bose gases in the 1-dimensional regime and whether Bose-Einstein condensation exists in this regime. Certainly it is well known that for a contact interacting 1-dimensional system of infinite extent there is no true Bose-Einstein condensation, as the bosons behave more like fermions [45]. In this context it is also worth noting the mapping of bosons onto Fermions in 1-dimension [30]. We will embark on the concept of a quasi-condensate for homogeneous 1-dimensional systems and will also consider their existence in trapped systems. Thus we will review some important concepts that will set up the correct physical picture for the study of solitons in trapped Bose gases.

2.2 Important scales

To start with we will discuss the meaning of weakly interacting, ultracold and dilute gas. First we look at an important quantity that is known as the thermal de Broglie wavelength of the atoms, which can be expressed in terms of temperature T as

$$\Lambda_T = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{1/2} \propto k_T^{-1}, \tag{2.2.1}$$

where we have introduced the thermal momentum of the atoms, k_T. The next important quantity to introduce is the characteristic radius of interaction between atoms, this quantity is defined as such, when the distance is much larger than this radius R_I then the atoms motion is essentially free. We can now define our first constraint which is known as the dilute limit. We will also consider the mean interatomic spacing to be given by n^{-3}. Then the dilute limit is characterised by the quantity

$$n^{-1/3} \gg R_I \quad \text{or} \quad nR_I^3 \ll 1. \tag{2.2.2}$$

When the interatomic spacing becomes of the order of R_I, one no longer has a gas and we are left with either a solid or a liquid. Next we consider the ultracold limit which occurs when the thermal de Broglie wavelength greatly exceeds the characteristic interaction length, we write the condition as

$$\Lambda_T \gg R_I \quad \text{or} \quad k_TR_I \ll 1. \tag{2.2.3}$$

It is in a dilute ultracold gas that one can consider pair interactions between particles and for the collision dynamics Eq. (2.2.3) permits one to only consider s-wave scattering. We note that the first ultracold gas was created in a gas of spin polarised atomic Hydrogen in 1979. Finally we will establish a condition on when we can use the weakly interacting regime for which one only considers pair interactions. Let us define the mean interparticle spacing $\bar{r}$ as the seperation at which the wavefunction is not affected by interactions. In this regime the interaction energy is just a sum of all the pair interactions which for a large number of particles, $N \gg 1$, is given as $E_{int} = N^2\epsilon_{int}/2$. In this regime we may consider a sphere of radius $\bar{r}$ and on average contains just one particle. If one then considers the weakly interacting regime where the kinetic energy is the dominant energy scale,

$$\frac{\hbar^2}{m\bar{r}^2} \gg n|g|, \tag{2.2.4}$$

where the factor of 2 that normally appears in the denominator is absent as we have used the reduced mass. We can write Eq. (2.2.4) in terms of the scattering length a and this is given below as

$$n|a|^3 \ll 1. \tag{2.2.5}$$

We have written down some useful conditions on energy scales and listed some useful length scales for ultracold dilute bosonic gases.

2.3 The Gross-Pitaevskii equation: Weakly interacting Bose gas

In this section we address Bose-Einstein condensation in weakly interacting Bose gases, where the condition of weak interactions Eq. (2.2.5) is the regime that we will be working in. We know that for a 3-dimensional Bose gas at $T = 0$ all the atoms are condensed into the lowest avaliable quantum state, which for the ideal gas is the state with momentum $p = 0$. We would then expect at $T = 0$ for a weakly interacting Bose gas that the majority of the particles will be condensed. As is well known, the common way to study many-body quantum system is to use second quantised operators or field operators. One can expand the field operators in terms of a single particle basis set as $\hat{\Psi}(r) = \sum_i \psi(r)\hat{a}_i$ where the expansion coefficients $\hat{a}_i$ are now non-commutating second quantised operators. One would like to simplify the expansion so as to deal with condensates in weakly interacting gases, this can be done by following the Bogoliubov prescription. Motivated by the knowledge that the ideal gas condenses and that we are dealing with a weakly interacting gas one can split the field operator up as follows,

$$\hat{\Phi} = \psi_0(r) + \sum_{i\neq 0} \psi_i(r)\hat{a}_i. \tag{2.3.1}$$

The approximation now comes, one assumes a macroscopic occupation of the lowest energy state, for this state the number of particles occupying it is of order $N_0 \gg 1$. The idea is that for lowest energy level, occupation N_0 and $N_0 - 1$ are equivalent up to corrections of order $1/N_0$, one remembers that N_0 is large. This approximation amounts to ignoring the commutation relation between the operators $\hat{a}_0$ and $\hat{a}_0^\dagger$, thus we replace the operators $\hat{a}_0$, $\hat{a}_0^\dagger$ with $\sqrt{N_0}$ and they become c-numbers. Now we can write our field operator in a form that takes into account the possibility of a Bose-Einstein condensate and write

$$\hat{\Phi}(r) = \Psi_0(r) + \delta\hat{\Phi}(r) \tag{2.3.2}$$

where $\Psi_0(r)$ is now a classical field. One can draw an analogy with quantum electrodynamics where one can perform a similar procedure when there are a large number of photons in the same quantum state, in this case one is reduced to studying Maxwells equations.

From the previous discussion we are now ready to derive the Gross-Pitaevskii equation which is very useful in the study of nonuniform Bose gases that are condensed at zero temperature. Let us consider a gas of N interacting bosons in 3-dimensional space, the many-body problem can be studied within the language of second quantisation with the Hamiltonian given in terms of field operators $\hat{\Psi}$ as

$$\hat{H} = \int dr \frac{-\hbar^2}{2m} \hat{\Psi}^\dagger \Delta_r \hat{\Psi} + \int dr \hat{\Psi}^\dagger V_{ext} \hat{\Psi} + \frac{g}{2} \int\int \hat{\Psi}^\dagger \hat{\Psi}'^\dagger V_{int}(r' - r) \hat{\Psi} \hat{\Psi}' dr' dr, \tag{2.3.3}$$

where the field operators have the dependence $\Psi = \Psi(r,t)$. Let us consider the 3-dimensional

Hamiltonian defined in Eq. (2.3.3) in the dilute ultracold limit specified by the quantities Eqs. (2.2.2) and (2.2.3). We also consider a weakly interacting gas such that the interaction potential satisfies $|V_{int}| \ll \hbar^2/mr^2$ for distances the potential acts over. One can use the Born approximation to write the interaction potential as $g_{3d}\delta(r' - r)$, where g_{3d} is defined as $4\pi\hbar^2 a/m$ and a is the s-wave scattering length. The aformentioned condition is applicable to Bose-Einstein condensates, then we can write down the contact interacting Hamiltonian as,

$$\hat{H} = \int \left[\frac{-\hbar^2}{2m} \hat{\Psi}^\dagger \Delta_r \hat{\Psi} + \hat{\Psi}^\dagger V_{ext} \hat{\Psi} + \frac{g}{2} \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \right] dr. \tag{2.3.4}$$

We start by asking the question of what is the time evolution of a field operator govenered by the 3-dimensional Hamiltonian, Eq. (2.3.3). Of course this is a very standard procedure, one uses the Heisenberg equation for time evolution of the quantum field operator $\hat{\Psi}(r,t)$,

$$i\hbar \frac{\partial \hat{\Psi}(r,t)}{\partial t} = \left[\hat{\Psi}(r,t), \hat{H} \right]. \tag{2.3.5}$$

To obtain the Gross-Pitaevskii equation we will substitute the contact interacting Hamiltonian from Eq. (2.3.4) into the expression given in Eq. (2.3.5) to obtain the full quantum time evolution of the field operator $\hat{\Psi}(r,t)$ as

$$i\hbar \frac{\partial \hat{\Psi}(r,t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{ext} + g \left| \hat{\Psi}(r,t) \right|^2 \right) \hat{\Psi}(r,t), \tag{2.3.6}$$

where $|\Psi|$ is the modulus of the field operator. So far we have not taken into account the existence of a condensed state and thus Eq. (2.3.6) governs the exact time evolution of the field operators for the quantum problem. To take into account the possibility of a condensed state we will split the field operator up into two parts as we did in Eq. (2.3.1). We then substitute this representation in Eq. (2.3.1) into Eq. (2.3.6) and keep only terms of order

Ψ_0 we obtain

$$i\hbar\frac{\partial\Psi_0}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + g\left|\Psi_0\right|^2\right)\Psi_0. \tag{2.3.7}$$

This is the celebrated Gross-Pitaevskii equation as obtained by Gross [32] and Pitaevskii [59] for the classical field Ψ_0. Let us again consider our analogy with Maxwells equations, we note that for the Gross-Pitaevskii equation $\hbar$ is present which is not true for Maxwells equations. It is worth observing the nonlinear term as well which results in some striking properties of weakly interacting Bose gases. For example the nonlinear term allows for the existence of non-trivial solutions such as dark and bright solitons.

We can also look for stationary solutions of Eq. (2.3.7) by considering a solution of the form

$$\Psi_0(r,t) = \psi_0(r,t)\exp(-i\mu t/\hbar), \tag{2.3.8}$$

where $\mu = \partial E/\partial N$ and is the chemical potential. We can substitute Eq. (2.3.8) into the Gross-Pitaevskii equation Eq. (2.3.7) to obtain the time independent Gross-Pitaevskii equation,

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + g\left|\Psi_0\right|^2 - \mu\right)\Psi_0 = 0. \tag{2.3.9}$$

Another result that is interesting and easily obtained from Eq. (2.3.7) for 3-dimensions is that for attractive interactions $g < 0$ the gas is unstable and undergoes a collapse. This is obtained by looking at infinitesimal perturbations in density and phase of the condensate. One linearises Eq. (2.3.7) to find that the eigen-frequencies are complex for the density correction and hence an exponential growth in density. One can think of this effect in terms of the Heisenberg uncertaintity relation. Throughout the thesis we will be working with repulsive interactions $g > 0$.

2.4 Healing length

Here we will introduce a characteristic length scale for a Bose-Einstein condensate with repulsive interactions. We shall consider a condensate that is subject to an external potential that is only a function of x and not the other coordinates x, y. The potential $V(x)$ will represent an infinite wall at x=0 and will be zero everywhere else. For simplicity we will pressume that ψ_0 is real, this function should also be continuous and therefore we have the boundary condition, $\psi(x = 0) = 0$. One then integrates the time-independent Gross-Pitaevskii equation Eq. (2.3.9), where the constant of integration is determined by the boundary condition $d\psi_0/dx = 0$ for $x \to \infty$. The solution satisfying these conditions is obtained to be,

$$\psi_0(x) = \pm\sqrt{n_0}\tanh\left(\frac{x}{\xi}\right), \tag{2.4.1}$$

where we have defined the following important quantity

$$\xi = \hbar/\sqrt{mn_0 g} \tag{2.4.2}$$

and is known as the healing length in the literature. We will discuss its significance later in a few sections time.

2.5 External trapping potentials

Let us discuss the case where we have Bose-Einstein condensation in a harmonic trap, which is given by

$$V(r) = \frac{m\omega^2 r^2}{2}. \tag{2.5.1}$$

To consider the condensate in this external potential we need to consider the time independent Gross-Pitaevskii equation from Eq. (2.3.9) with V_{ext} now taking the form of the harmonic potential Eq. (2.5.1). One can easily see that for the non-interacting case the ground state solution is that of a harmonic oscillator, for which the excitations are well known and the energy levels are separated by units of $\hbar\omega$. The harmonic oscillations are characterised by the oscillator length given by

$$l_0 = \left(\frac{\hbar}{m\omega}\right)^{1/2}. \tag{2.5.2}$$

It is well known that in 3-dimensions the ideal gas trapped by a harmonic trapping potential forms a Bose-Einstein condensate at low enough temperatures [60]. Thus if interactions are small in comparison to the oscillator level spacing $n_{max}g \ll \hbar\omega$ then we can consider the non-interacting problem.

An interesting case is when the interactions are important, in fact we are interested when the interaction energy is large in comparison to the oscillator level spacing, $gn_{max} \gg \hbar\omega$. This regime is known as the Thomas-Fermi limit, it proves to be incredibly useful for trapped systems. For the limit of strong interactions we are permitted to ignore the kinetic energy at the center of the trap. In this case Eq. (2.3.9) becomes a simple relation,

$$g|\psi_0|^2 = \mu - \frac{m\omega^2 r^2}{2}, \tag{2.5.3}$$

we immediatly sovle for the condensate configuration ψ_0 to find

$$\psi_0(r) = \sqrt{\frac{\mu}{2}\left(1 - \frac{r^2}{R_{TF}^2}\right)}\Theta(R_{TF} - r). \tag{2.5.4}$$

We have defined the Thomas-Fermi radius as $R_{TF} = \left(\frac{2\mu}{m\omega^2}\right)^{1/2}$, we see that R_{TF} defines the radius of the condensate as $\psi_0 = 0$ for and distance large then R_{TF}. One can then use the normalisation of the Thomas-Fermi configuration ψ_0 to calculate the number of particles contained in the condensate, for 1-dimension this is

$$\int_0^{R_{TF}} dr|\psi_0|^2 = N_0 = \frac{4}{3}\frac{\mu R_{TF}}{g}. \tag{2.5.5}$$

If we substitute the explicit form of the Thomas-Fermi radius into Eq. (2.5.5) we find for the chemical potential, $\mu \propto N_0^{2/3}$ as opposed to the uniform case where $\mu \propto N_0$.

In the study of the dynamics of a trapped gray soliton we will find the Thomas-Fermi limit a necessity for the study of the solitons oscillations and decay. We note that at the boundary of the Thomas-Fermi profile the condensate is properly described by Airy functions [14] and one should note that these two solutions should be pieced together to describe the full extent of the condensate. Near the edge of the Thomas-Fermi profile the curvature comes pronounced and one can no longer neglect the kinetic energy.

2.6 Excitations in Bose-Einstein condensates

In this section we will consider elementary excitations, where only a small number of particles are excited out of the condensate fraction. The task we face is to find the energies and corresponding eigenfunctions of these excitations. We begin by noting that in deriving the Gross-Pitaevskii equation we kept just the condensate part of the field operator and discarded any excitations. Let us now consider the same splitting of the field operator as defined in Eq. (2.3.2) and again substitute this into Eq. (2.3.6) but this time we retain terms up to linear order in $\delta\tilde{\Psi}$ to obtain,

$$i\hbar\frac{\partial\delta\tilde{\Psi}}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta_r + V(r) + 2g|\psi_0|^2 - \mu\right)\delta\tilde{\Psi} + g\Psi_0^2\delta\tilde{\Psi}, \tag{2.6.1}$$

where we have made the definition $\delta\tilde{\Psi} = \delta\Psi\exp(-i\mu t/\hbar)$.

To make progress we will expand $\delta\tilde{\Psi}$ in terms of some normal mode basis set which will allow us to diagonalise the problem, the expansion takes the form

$$\delta\tilde{\Psi} = \sum_j \left[v_j(r)\hat{b}_i \mathrm{e}^{-i\epsilon_i t/\hbar} - \nu_j(r)\hat{b}_i^\dagger \mathrm{e}^{i\epsilon_i t/\hbar} \right]. \tag{2.6.2}$$

The index j denotes the quantum states of the excitations created by $\hat{b}_j$, $\hat{b}_j^\dagger$, the basis is formed by v_j, ν_j and we label the eigenenergies by ϵ_j. Using this expansion in Eq. (2.6.1) and comparing coefficients of $\hat{b}_j$ and $\hat{b}_j^\dagger$ results in

$$\left(-\frac{\hbar^2}{2m}\Delta_r + V(r) + 2g|\psi_0|^2 - \mu\right) v_j - g\psi_0^2\nu_j = \epsilon_j v_j \tag{2.6.3}$$

$$\left(-\frac{\hbar^2}{2m}\Delta_r + V(r) + 2g|\psi_0|^2 - \mu\right) \nu_j - g\left(\psi^*\right)_0^2 v_j = -\epsilon_j \nu_j \tag{2.6.4}$$

and are known as the Bogoliubov-de Gennes equations. Let us consider the contact interacting Hamiltonian Eq. (2.3.4) and expand the field operators to second order in the non-condensed operator $\delta\tilde{\Psi}$, the Hamiltonian to this order will be of the form $H = H^{(0)} + H^{(2)}$. There is no $H^{(1)}$ term in virtue of the Gross-Pitaevskii equation which determines this term to be zero, on using Eqs. (2.6.2), (2.6.3) and (2.6.4) the term $H^{(2)}$ reduces to the following bilinear form

$$H^{(2)} = \sum_j \epsilon_j \hat{b}_j^\dagger \hat{b}_j. \tag{2.6.5}$$

Let us continue with the Bogoliubov-de Gennes equations and consider a homogeneous system and to zero order we put $\Psi_0 = \sqrt{n}$ and $\mu = gn$. For the homogeneous system we can consider eigenfunctions $v_j(r)$, $\nu_j(r)$ as plane waves. For example we can write $v_j(r) = v_k \exp(ikr)/V$, where V is the volume of the system under consideration. Thus the Bogoliubov-de Gennes Eqs. (2.6.3) and (2.6.4) reduce to

$$\left(\frac{\hbar^2 k^2}{2m} + ng\right) v_k - ng\nu_k = \epsilon_k v_k \tag{2.6.6}$$

$$\left(\frac{\hbar^2k^2}{2m}+ng\right)\nu_k - ngv_k \quad = \quad -\epsilon_k\nu_k, \tag{2.6.7}$$

the compatibility condition for Eqs. (2.6.6) and (2.6.7) leads to the excitation spectrum, known in the literature as the Bogoliubov spectrum

$$\epsilon_k = \sqrt{\left(\frac{\hbar^2k^2}{2m}\right)^2 + 2ng\frac{\hbar^2k^2}{2m}}. \tag{2.6.8}$$

We will now make some comments about this important dispersion relation. First let us write the Bogoliubov dispersion Eq. (2.6.8) as $(\hbar^2k/2m)\, sqrtk^2 + 4\xi^2$, where ξ is the healing length defined in Eq. (2.4.2). We now consider the case of small momentum $k \ll \xi^{-1}$, from this consideration we find that the dispersion relation is linear

$$\epsilon_k = \hbar ck, \tag{2.6.9}$$

the linear dispersion is that of phonons which describe the low energy collective excitations of the system which come about due to the theory being interacting. We now consider the opposite limit where the momentum is large $k \gg \xi^{-1}$, then the excitations are particle like and the dispersion relation in this regime becomes

$$\epsilon_k = \frac{\hbar^2k^2}{2m} + gn. \tag{2.6.10}$$

We remark that the presence of the chemical potential $\mu = gn$ indicates that the high energy particle like excitations interact with the condensate. We see that the healing length naturally divides the two regimes of phonon and free particle.

2.6.1 Bose-Einstein condensate and the one-body density matrix

It has been assumed that the number of particles excited out of the condensate is small in comparison to the number of condensed particles in the former derivations. One can formally check this assumption by calculating the density of the non-condensed atoms by calculating the following quantum average

$$\delta n = \langle \hat{\Psi}^\dagger(r,t)\hat{\Psi}(r,t)\rangle. \tag{2.6.11}$$

To calculate this average one just substitutes the expansion Eq. (2.6.2) for $\delta\hat{\Psi}$ and condsiders quantum averages such as $\langle \hat{b}_q^\dagger \hat{b}_k\rangle$. We will not consider the calculation here, although straightforward it is rather time consuming. We note that at zero temperature the Bose distribution function is zero as all particles reside in the ground state. Evaluating all the averages over the excitation operators reduces to evaluating the momentum sum $1/V\sum_k \nu_k^2$, which can be evaluated by converting the momentum sum to an integral leading to the zero temperature,

$$\frac{\delta n}{n} = \frac{8}{3\sqrt{\pi}}\left(na^3\right)^{1/2} \tag{2.6.12}$$

for the ratio of the density of excited particles to condensed particles. One notices that indeed this quantity is small as we have written the ratio in terms of the small quantity $\left(na^3\right)^{1/2}$ which is recognised as the small parameter of the weakly interacting theory.

It is instructive to consider the one-body density matrix defined as

$$g_1(r,r';t,t') = \langle \hat{\Psi}^\dagger(r,t)\hat{\Psi}(r',t')\rangle, \tag{2.6.13}$$

which at equal times and setting $r = r'$ we obtain the diagonal density matrix as defined in Eq. (2.6.11). Here we will consider g_1 for equal times $t = t'$ and only be interested in time independent behaviour so we will put $t = t' = 0$. To simplify matters further we will only consider uniform systems and thus we set $r, r' \to r - r'$. To evaluate g_1 it will prove efficacious to split it into a condensate part and a non-condensed part which we write as

$$g_1(r) = n_0 + \langle \hat{\Psi}^\dagger(r,0)\hat{\Psi}(0,0)\rangle. \tag{2.6.14}$$

One then proceeds in exactly the same fashion as in the previous calculation only this time one has the more complicated momentum sum $1/V \sum_k \nu_k^2 \exp ikr$ to perform. Again one transforms the sum to an integral and then performs an integration by parts to obtain a power expansion in $1/r$. It can then be shown that in the aysmptotic limit $r \to \infty$ one obtains

$$g_1(r) = n_0 + \frac{2}{\sqrt{\pi}} n(na^3)^{1/2} \left(\frac{\xi}{r}\right)^2 \quad \text{for} \quad r \to \infty, \tag{2.6.15}$$

where a is the scattering length and n_0 is the density of the condensed fraction. We remind ourselves that these results so far have been obtained for 3-dimensional systems and we will review 1-dimensional systems shortly. The first observation we make is that g_1 tends to n_0 as $r \to \infty$, we thus see that correlations exist right across the infinite system. This tending to a constant value in the asymptotic limit $r \to \infty$ is known as off-diagonal long range order and is inherent property of superfluid systems. This behaviour results in a spike in the momentum distribution $n_0\delta(k)$ which is a reflection of the macroscopic occupation of the zero momentum state.

2.7 From 3-dimensions to 1-dimension

We are now in a position to consider obtaining the 1-dimensional Hamiltonian from the 3-dimensional Hamiltonian. Let us consider the situation where the 3d gas is tightly trapped in the x, y plane by some harmonic potential $V(\rho) = m\omega_0^2\rho^2/2$ which is stronger than the 2-body interaction between the bosons in this plane. Then the motion in this plane is that of a harmonic oscillator with wavefunction $1/\sqrt{\pi} l_0 \exp(-r^2/2l_0^2)$, where we have specified the harmonic oscillator length by $l_0 = (\hbar/m\omega_0)^{1/2}$. We can now split the field operator up as

$$\hat{\Psi}(\rho, z) = \hat{\Psi}(z)\Psi_\rho, \tag{2.7.1}$$

where z denotes the dimension that is not confined by the tight harmonic potential. We can then proceed by substituting the field operator Eq. (2.7.1) into the 3-dimensional Hamiltonian in Eq. (2.3.3). One can then seperate out the x, y plane from the longitudinal coordinate z. We transform to cylindrical coordinates for the x, y plane with $x^2 + y^2 = \rho^2$ and angular symmetry which allows one to perform the integral over the angle trivially and the integrals over ρ are straightforward, this yields the 1-dimensional Hamiltonian,

$$\hat{H} = \int dz \hat{\Psi}^\dagger(z) \left[\frac{\hbar^2}{2m}\Delta + V_{ext} + \frac{g}{2}\hat{\Psi}^\dagger(z)\hat{\Psi}(z) \right] \hat{\Psi}(z). \tag{2.7.2}$$

The effective 1-dimensional coupling constant g has now been obtained by averaging over the tightly confined harmonic modes, in terms of the harmonic oscillator length l_0 we can write g [1] as

$$g = \frac{g_{3d}}{2\pi l_0^2} = \frac{2\hbar^2 a}{ml^2}. \tag{2.7.3}$$

In the above derivation we have omitted the term $\frac{-\hbar^2}{2m}\Delta_\rho + \frac{1}{2}m\omega_0^2\rho^2$ for the x, y plane, which just renormalises the chemical potential by $\hbar\omega_0$. We should also state that the validity of this averaging over radial modes relies on the constraint $|a| \ll l_0$.

2.7.1 Bose-Einstein condensation in one-dimension

Here we shall address the existence of a condenstate and superfluid state in 1-dimension, this is an important point to address due to reliance of our approach to the soliton on the existence of a superfluid. Here we shall discuss and summarise important results for the uniform and trapped cases, which will set the scene for later work.

As we know from the Mermin-Wagner theorem [53], for a uniform system of infinite extent in one and two dimensions one does not have off-diagonal long range order. This is because any spontaneous symmetry breaking such as having an order parameter with non-zero average will allow for long range fluctuations with very little energy cost that will destroy any long range order. It is also known from the Lieb-Liniger solution for the 1-dimensional contact interacting Bose gas that a Bose-Einstein condensate does not form. So the question arises if one can at all speak about the existence of a condensate in low dimensional systems. It turns out that the answer to this question is yes and we will see below when this is applicable.

For example one can calculate as one does for the 3-dimensional case, the number of particles excited out of the condensate. As previously with the 3-dimensional case we are interested in the density of excitations $\delta n = \langle \delta\hat{\Psi}^\dagger \delta\hat{\Psi} \rangle$ which for 1-dimension is found to give the ratio

$$\frac{\delta n}{n} = \frac{\sqrt{\gamma}}{\pi} \ln\left(\frac{2L}{2.7\pi\xi}\right), \tag{2.7.4}$$

where we have defined the small parameter of the weakly interacting 1-dimensional theory as $\gamma = m|g|/\hbar^2 n$. If one allows the length of the system L to extend to infinity we find that the ratio of excited particles to condensed particles diverges and one cannot indeed speak of a condensate. On the other hand we notice that the small parameter of the theory appears as a factor outside the logarithm, so one might expect for a finite size system the existence of a condensate.

One can calculate the one-body density matrix for equal times, $g_1 = \langle \hat{\Psi}^\dagger(x, 0)\hat{\Psi}(x, 0) \rangle$ for the 1-dimensional case using the density-phase representation. Performing this calculation one finds at $T = 0$ for large distances x, the following asymptotic form,

$$g_1(x) = n\mathrm{e}^{-\frac{1}{2}\langle[\hat{\phi}(x)-\hat{\phi}(0)]^2\rangle} = n\left(\frac{\xi}{|x|}\right)^{\sqrt{\gamma}/2\pi}. \quad (2.7.5)$$

Looking at distances where the one-body density matrix changes significantly leads us to define the phase coherence length, which under considering Eq. (2.7.5) leads to the phase correlation length

$$l_\phi \approx \xi \mathrm{e}^{2\pi/\sqrt{\gamma}}. \quad (2.7.6)$$

We see that it is exponentially large in comparison to the healing length ξ, it allows us to define the quasicondensate. One considers a region of size L such that $\xi \ll L \ll l_\phi$, inside this region we have a true Bose-Einstein condensate. We should note that the phase varies from region to region and it is this situation that we define as the quasicondensate.

One can show that at T=0 for a 1-dimensional weakly interacting Bose gas that is confined by a smooth external potential that there is the existence of a condensate as well [70].

2.8 Soliton propagation in a homogeneous stationary superfluid

Here we shall study a particular solution of the Gross Pitaevskii equation known as the soliton solution, which is characterised as a localised travelling solution. Motivated by the fact that we are dealing with a condensate we shall make the ansatz that the majority of particles sit in the ground state and are described by a phase and density. To this end let us define the dimensionless field

$$\psi\left(z,t\right)=\sqrt{n_0}g\left(z\right)\exp\left(\frac{-i\mu t}{\hbar}\right) \tag{2.8.1}$$

where $z=\left(x-vt\right)/\xi$ as we have moved to the inertial frame of the soliton, and ξ is the healing length. Substituting the ansatz into the Gross-Piteavskii equation and using $mc^2=gn_0$ we get an equation for g

$$2iU\partial_z g=\partial_z^2 g+g\left(1-|g|^2\right), \tag{2.8.2}$$

where the definition

$$U=\frac{mv\xi}{\hbar} \tag{2.8.3}$$

was used. Now we would like to consider the finite density boundary conditions which read as

$$|g|\to 1 \quad , \quad \partial_z g=0 \quad \text{as} \quad |z|\to\infty \tag{2.8.4}$$

We now proceed by multiplying Eq. (2.8.2) by $\bar{g}$ the complex conjugate of g and subtracting the complex conjugate of the resulting equation gives

$$-2iU\partial_z\left|g\right|^2+\partial_z\left(\bar{g}\partial_z g\right)-\partial_z\left(g\partial_z\bar{g}\right)=0, \tag{2.8.5}$$

integrating this equation results in

$$-2iU\left|g\right|^2+\left(\bar{g}\partial_z g\right)-\left(g\partial_z\bar{g}\right)=\mathcal{C}. \tag{2.8.6}$$

The constant of integration $\mathcal{C}$ is determined using the boundary condition $|g|\to 1$ in the asymptotic limit. Thus the boundary conditions at infinity give the constant $\mathcal{C}$ to be $-2iU$

and the above equation is reduced to the form

$$2iU\left(1-|g|^2\right)+(\bar{g}\partial_z g)-(g\partial_z\bar{g})=0 \tag{2.8.7}$$

which is the continuity equation. By construction the function g is complex, so we should express it in a more obvious form such as

$$g=a+ib=\sqrt{a^2+b^2}\exp\left(i\arctan\left[\frac{b}{a}\right]\right), \tag{2.8.8}$$

with the final solution in mind. It is clear what the form of the density and phase are from the above relation, we will now go on to calculate the imaginary part of g. We start by substituting $g = a + ib$ into Eq. (2.8.2) and taking the imaginary part of the resultant expression yields

$$2U\partial_z a=\partial_z^2 b+b\left(1-a^2-b^2\right). \tag{2.8.9}$$

Lets us also introduce this complex form of g into into Eq (2.8.7) and compare the result with the above equation. To get both equations into the same form we are forced to take b equal to a constant of form $\sqrt{2}U$ and transforms Eq. (2.8.9) into

$$\sqrt{2}\partial_z a=\left(1-2U^2-a^2\right)=\left(1-\frac{v^2}{c^2}-a^2\right). \tag{2.8.10}$$

Now we are only left with this integral to do, which is trivial when it is noticed that it can be written as

$$\frac{1}{\sqrt{\beta}}\int\frac{d\acute{a}}{(1-\acute{a}^2)}=\frac{z+z_0}{\sqrt{2}} \tag{2.8.11}$$

where the definitions $\beta=\sqrt{1-\frac{v^2}{c^2}}$ and $a=\sqrt{\beta}\acute{a}$ have been used. The integral is straightforward to do and thus the final solution for ψ is written down as

$$\psi\left(x-vt\right)=\sqrt{n_0}\left(i\frac{v}{c}+\beta\tanh\left[\beta\frac{x-vt-x_0}{\sqrt{2}\xi}\right]\right) \tag{2.8.12}$$

The density profile is depicted for two different velocities in Figure 2.1.

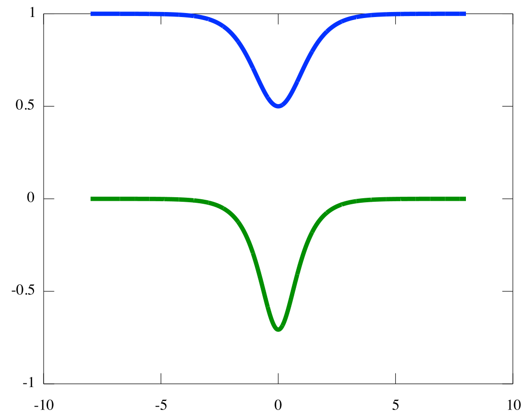

Figure 2.1: The density profile for a grey soliton in a uniform background evaluated at two different velocities.

2.9 Brief review of the Lieb modes

Lieb and Liniger [45] considered the Hamiltonian in Eq. (2.3.4) with no external potential. In the uniform case in the absence of an external trapping potential, the total number of particles operator commutes with the Hamiltonian. This symmetry allows one to reduce the second quantised Hamiltonian to that of first quantised form. The second quantised Hamiltonian given in Eq. (2.3.4) then becomes

$$\hat{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\sum_{i<j=1}^{N}\delta\left(x_i - x_j\right) \tag{2.9.1}$$

in first quantised form. It was in this form that Lieb and Liniger discovered that Eq. (2.9.1) could be diagonalised by the Bethe ansatz. They were able to write down the full wavefunction for the system and found that no two particles could have the same quasimomentum, this being a strange property for bosons.

They found the quasimomentum to be distributed between $-Q$ and Q, where Q is to be considered as a momentum cutoff or a fermi surface. Lieb [44] went on to consider excitations out of this groundstate distribution of quasimomentum. Two distinct types of excitations were found; Lieb I excitations which are particle like and Lieb II excitations which are hole like. The Lieb I excitations are found by adding a particle with momentum $q > Q$, this excitation branch was known from Bogoliubov theory. The Lieb I branch is unbounded and becomes linear for small momentum and thus describes sound waves for small momentum. The Lieb II branch $q < Q$, was unknown to Bogoliubov theory and could only be obtained using Bethe ansatz. This excitation branch is a bounded inverted parabola, for vanishing momentum it to becomes linear and matches the Lieb I branch.

2.10 Experiment

Here we shall discuss briefly the experiments on solitons in trapped Bose-Einstein condensates which allow the study the GP equation for soliton solutions. Dark solitons have been generated in the pioneering experiments [9] where the method of phase imprinting was employed to create the soliton in cold gases confined by cigar shaped traps with strong radial confinement.

Here we shall briefly discuss the method of phase imprinting by considering the methods used in the experiments [5]. We will follow the work in [5] where they created cigar shaped Bose-Einstein condensates made up of ^{87}Rb atoms at extremely low temperatures with the chemical potential being of order 20nK. The number of ^{87}Rb atoms confined to the trap was of order 5×10^4 in the $5^2\mathrm{S}_{1/2}$, F = 1, $m_F = -1$ state. The confinement is provided by an optical dipole trap with the trapping frequencies given by $\omega_z = 2\pi\times5.9$ Hz, $\omega_\perp^{ver} = 2\pi\times85$ Hz and $\omega_\perp^{hor} = 2\pi \times 133$ Hz, thus the trap is highly anisotropic. The typical measured peak atomic densities are of order $5.8\times10^{-7}\mathrm{m}^{-3}$ which implies a sound speed of order $c = 1\mathrm{mms}^{-1}$.

Initially the condensate in this elongated geometry is characterised by a uniform phase and sits in longintudinal trap depth that is very low. The low trap depth is a desired property if one wants to retain very low temperatures, thus any excitations are allowed to leave the trap. It is this constant slight evaporative cooling that allows for the obeservation of no heating for lifetimes longer than the condensate lifetime which is of order 10s. With the condensate trapped in this way a laser is used to phase imprint a phase gradient across the condensate to lead to the creation of a soliton, this method is depicted in Figure 2.2.

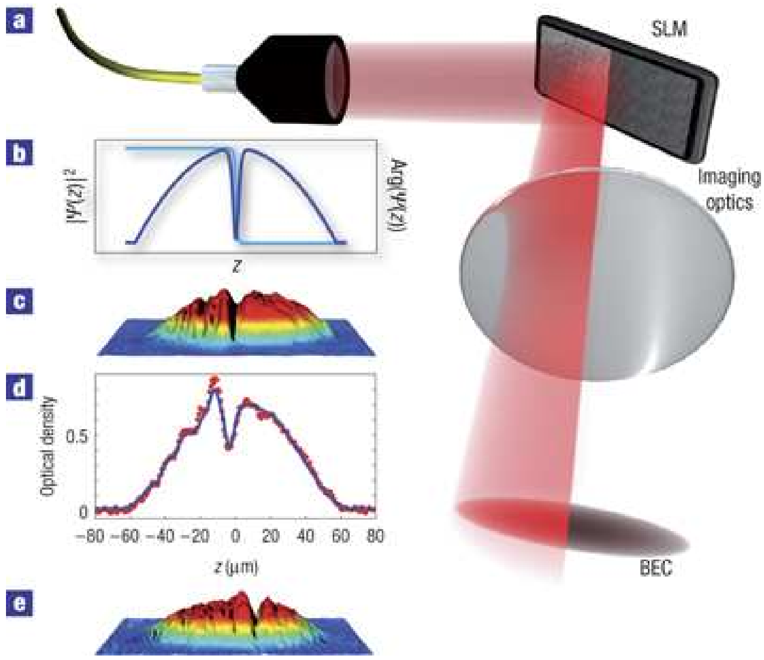

Figure 2.2: **a**, The optical set-up, a spatial light modulator is used to imprint the phase step by only exposing the laser beam to part of of the condensate. The laser is far-detuned so to avoid inducing atomic transitions. **b**, a theoretical plot of the soliton density profile in the trapped condensate where the blue line represents the soliton phase. **c**,A typical absorption image taken after imprinting the soliton and allowing a free expansion of 11ms. **d**, integrated column density of the data with a fit to the data. **e**, Image of soliton after 2.8s of oscillations , the image has been taken from [5].

Part of the condensate is exposed to the dipole potential U_{dip} of the laser beam which is detuned by tens of gigahertz from atomic resonance, the detuning is used to prevent atomic transitions. The exposed part of the condensate takes on a phase evolution of $\Delta\phi = U_{dip}t/(i\hbar)$. A phase step of π is imprinted into the condensate with a pulse time of $t_\pi = 40\mu s$ which is much smaller than the correlation time of $700\mu s$, a short pulse time is chosen as to not induce any density disturbances. The imprinted phase gradient does induce a local superfluid velocity in the condensate which carries with it an associated momentum which contributes to the formation of the density minimum of the dark soliton [9]. It is observed that the generated grey soliton moves in the opposite direction to the superfluid flow.

Absorption images are taken after a time of flight of 11.5 ms, this is undertaken as the soliton size is given to be $l = 0.8\mu$m in the trap and is beyond optical resolution. It is found in the experiment that the initial velocity of the soliton is $V = 0.56$mms^{-1} this gives the soliton amplitude to be $n_s = 0.68 n_0$. Nearly pure dark solitons were detected after times of 2.8 s which was noted to exceed previous measurements by more than a factor of 200. It was noted for the experiment that preparation errors led to fluctuations of the soliton position that made it not possible to observe solition oscillations for times greater than 250 ms.

2.11 Summary

In this chapter we have considered the basic principles of Bose-Einstein condensates. We have discussed important principles such as splitting of the Bose field operators into a condensate part and the excitation part, we found this to be of importance in deriving the Gross-Pitaevskii equation. Importantly for the work on trapped gray solitons we discussed how to work with external trapping potentials using the Thomas-Fermi limit where one discards the kinetic energy term. We have also considered the question of Bose-Einstein condensation in quasi 1-dimensional systems and have found that there is in fact the existence of a quasicondensate which will allow for the use of the Gross-Pitaevskii equation.

Chapter 3

GENERAL SOLITON THEORY

3.1 Inverse Scattering and Dark Solitons

Here we shall review the Inverse Scattering Transform (IST) and follow [20] for the nonlinear Schrodinger equation with a repulsive interaction or commonly known as the Gross Pitaevskii equation in the cold atoms community. The nonlinear Schrodinger equation with attractive interaction has been solved with IST and is known to admit bright soliton solutions corresponding to zeroes in the scattering data. The repulsive interacting case has been shown by Tsuzuki to pertain dark soliton solutions and it is here that we shall show how one can obtain them through the IST method [75]. The advantages of this method are the ability to obtain the N-solitonic solution and the construction of the conservation laws of the said differential equation. Having a repulsive interaction introduces the finite boundary conditions $|\psi(x,t)|^2 \to \text{const}$ as $x \to \pm\infty$, this straight away leads to complications that are not inherent to the attractive case. Throughout this chapter we will set $\hbar$ to one.

3.1.1 Scattering Data

One starts by considering the following set of first order evolution equations

$$\frac{\partial F(x,t)}{\partial x} = U(x,t,\lambda)F(x,t) \tag{3.1.1}$$

$$\frac{\partial F(x,t)}{\partial t} = V(x,t,\lambda)F(x,t). \tag{3.1.2}$$

The equations above are matrix equations where we have defined the matrices as

$$U = \begin{pmatrix} -i\lambda/2 & \psi^* \\ \psi & i\lambda/2 \end{pmatrix}, \tag{3.1.3}$$

for the spatial scattering and for the time evolution we have

$$V = \begin{pmatrix} |\psi|^2 - \rho_0^2 & -\partial_x \psi^* \\ \partial_x \psi & \rho_0^2 - |\psi|^2 \end{pmatrix}, \tag{3.1.4}$$

where F is to be understood as a two component vector and we have introduced the complex parameter λ. A remarkable observation is that the compatibility condition

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0, \tag{3.1.5}$$

which is equivalent to the nonlinear Schrodinger equation with repulsive interactions.

Unlike the case of the attractive interaction an additional complication is introduced due to the potentials being finite at $\pm\infty$ and now having the asymptotic values

$$\lim_{x\to+\infty} \psi(x) = \sqrt{\eta}\exp(i\theta) \tag{3.1.6}$$

$$\lim_{x\to-\infty} \psi(x) = \sqrt{\eta}. \tag{3.1.7}$$

We shall study the first equation in Eq. (3.1.1), known as the scattering problem which contains the solutions of the non-linear Schrodinger equation as potentials.

We will study some properties of Eq. (3.1.1) in the asymptotic limit and demand that the full solutions of Eq. (3.1.1) tend to these solutions in the asymptotic limit. Let us denote $U_-(\lambda)$ and $U_+(\lambda)$ as the evolution matrix from Eq. (3.1.3) in the limits $x \to -\infty$

and $x \to +\infty$ respectively. We denote $\mathcal{E}(x,\lambda)$ as the solution that satisfies the first equation in Eq. (3.1.1) in the limit $x \to -\infty$, which can be written as,

$$\frac{d\mathcal{E}}{dx} = U_-(\lambda)\,\mathcal{E} \tag{3.1.8}$$

and is solved by the matrix solution $\mathcal{E}$ given by

$$\mathcal{E}(x,\lambda) = \begin{pmatrix} 1 & \frac{i(k-\lambda)}{\omega} \\ -\frac{i(k-\lambda)}{\omega} & 1 \end{pmatrix} \exp\left(-\frac{ikx\sigma_3}{2}\right). \tag{3.1.9}$$

Equation in the limit $x \to +\infty$ for the finite density solution can be obtained by using the relation

$$U_+ = \mathcal{Q}^{-1}U_-\mathcal{Q}. \tag{3.1.10}$$

It is to be understood that $k = k(\lambda) = \sqrt{\lambda^2 - \omega^2}$, and from inspection vanishes for $\lambda = \pm\omega$. Thus for these values of λ, the matrix $\mathcal{E}$ is non invertible and therefore we have a gap in the continuous spectrum of $-\omega < \lambda < \omega$. This problem is not encountered in the case of the bright soliton and for this problem forces us to study the two sheet Riemann surface $\Lambda = \Lambda_- \cup \Lambda_+$. For example the sheet Λ_+ is specified by $k(\lambda) > 0$ for Im > 0 and $\lambda > \omega$, we also have $k(\lambda) < 0$ for the $\lambda < -\omega$ case. This is a convention for the definition of the cuts and the same can be done for the sheet Λ_-.

We then take the above asymptotic solution and add to the integral over some undetermined kernel that will in principle solve equation .

Here we will follow the convention of Faddeev et al. [20], in this notation we can finally write the Jost solutions of Eq. (3.1.1) as integral equations,

$$M_+(x,k) = \mathcal{Q}^{-1}\mathcal{E}(x,k) + \int_x^{+\infty} \Gamma_+(x,y)\,\mathcal{Q}^{-1}\mathcal{E}(y,k)\,dy \tag{3.1.11}$$

$$M_-(x,k) = \mathcal{E}(x,k) + \int_{-\infty}^{x} \Gamma_-(x,y)\,\mathcal{E}(y,k)\,dy. \tag{3.1.12}$$

We can derive expressions for the kernels $\Gamma_\pm(x,y)$ via a substition of the expressions from Eq. (3.1.11) into the scattering equation of Eq. (3.1.1). Collecting all the common factors of $\mathcal{E}$ yields the following differential equation for the kernel $\Gamma_\pm(x,y)$

$$\begin{aligned}\partial_x\Gamma_\pm(x,y)+\sigma_3\partial_y\Gamma_\pm(x,y)\,\sigma_3-U_0(x)\,\Gamma_\pm(x,y)\\+\sigma_3\Gamma_\pm(x,y)\,\sigma U_\pm=0,\end{aligned}\tag{3.1.13}$$

where $U_0(x)$ and $U_\pm$ are defined in the footnote[1] and the boundary conditions

$$\begin{aligned}\Gamma_\pm(x,x)-\sigma_3\Gamma_\pm(x,x)\,\sigma_3=\mp\left(U_0(x)-U_\pm\right),\\ \lim_{y\to\pm\infty}\Gamma_\pm(x,y)=0\end{aligned}\tag{3.1.14}$$

were also used in deriving expression Eq. (3.1.13).

The integral Jost solutions have been derived and differential equations for the Jost kernels have also been derived. It is clear that so far this has been following the same path as for inverse scattering for the bright soliton, with the difference being that we have the added complication of the gap in the continuous spectrum.

As we have noted in the footnote, $U_0(x)$ contains the sought after solutions of the NLS equation and thus if we can determine the form of the kernels $\Gamma_\pm(x,y)$, we would then obtain the solutions ϕ and $\bar{\phi}$. As we will show in a summary of the Inverse Scattering Transform this is indeed possible. Using the properties of the asymptotic solutions $\mathcal{E}$ we will determine the properties of the scattering data pertaining to Eq. (3.1.1). From this data we will be able to derive the form of the kernels $\Gamma_\pm(x,y)$ and thus solve the NLS equation for finite boundary conditions.

We shall now list some of the important properties of the Jost functions that we need to in the latter parts of this overview, we begin with the asymptotics of these solutions which

[1] $U_0=\begin{pmatrix}0&\bar{q}\\q&0\end{pmatrix}$ and $U_\pm=U_\pm(\lambda)+\frac{i\lambda}{2}\sigma_3$

are given by

$$M_+(x,\lambda) = \mathcal{Q}^{-1}\mathcal{E}(x,\lambda) \tag{3.1.15}$$

as $x \to +\infty$ and we also have

$$M_-(x,\lambda) = \mathcal{E}(x,\lambda) \tag{3.1.16}$$

as $x \to -\infty$. One can also derive the property that the determinant of the Jost solution coincide with each other with the value given below

$$\det M_+(x,\lambda) = \frac{2k(\lambda-k)}{\omega^2} = \det M_+(x,\lambda), \tag{3.1.17}$$

this property will be useful later. Due to the form of the matrix U we have the following property

$$\bar{M}_\pm(x,\lambda) = \sigma_1 M_\pm(x,\lambda)\sigma_1, \tag{3.1.18}$$

which is known as the involution property. The involution proerty of the Jost funtions implies that the elements of the kernels $\Gamma_\pm$ have the property $\Gamma_{11} = \bar{\Gamma}_{22}$ and $\bar{\Gamma}_{12} = \Gamma_{21}$. We will now outline some asymptotics of the Jost functions which again we will need later, first we would like to discuss the limit $|\lambda| \to \infty$. If one performs an integration by parts in Eq. (3.1.11) such that a factor of $1/k$ is pulled into the integrand then the following aysmptotic forms are obtained for the Jost functions,

$$M_-^{(1)}(x,\lambda) = \begin{pmatrix} 1 \\ \frac{i(\lambda-k)}{\omega} \end{pmatrix} \mathrm{e}^{-ikx/2} + O\left(\frac{1}{|\lambda|}\right) \tag{3.1.19}$$

$$M_+^{(2)}(x,\lambda) = \begin{pmatrix} \frac{i(\lambda-k)}{\omega}\mathrm{e}^{-i\theta/2} \\ \mathrm{e}^{i\theta/2} \end{pmatrix} \mathrm{e}^{ikx/2} + O\left(\frac{1}{|\lambda|}\right), \tag{3.1.20}$$

for $|\lambda| \to \infty$ and λ defined on the upper Riemann surface Λ_+. Importantly we have denoted the first columns of $M_\pm$ as $T_\pm^{(1)}$ and the second columns as $M_\pm^{(2)}$. We can of course make the same manipulations on the remaining columns to get similar relations, we shall omit these results.

Each column of $M_+(x, \lambda)$ represents two solutions to the scattering problem in Eq. (3.1.1) with the correct boundary condition as $x \to \infty$, these solutions are linearly independent as we have seen from the relation in Eq. (3.1.17) where the determinant can be thought of as a Wronskian. The same can be said about the Jost function $M_-(x, \lambda)$, thus the columns of this matrix Jost functions form a basis set and allow a column from, say $M_-(x, \lambda)$ to be expanded in the basis formed from the columns of $M_+(x, \lambda)$. This can be succinctly expressed mathematically as

$$M_- (x, \lambda) = M_+ (x, \lambda) \, T_M (\lambda) \,, \tag{3.1.21}$$

where the matrix $T_M (\lambda)$ is named in the literature as the *reduced monodromy matrix* and has the form

$$T_M (\lambda) = \begin{pmatrix} a (\lambda) & \bar{b} (\lambda) \\ b (\lambda) & \bar{a} (\lambda) \, . \end{pmatrix} \tag{3.1.22}$$

which again satisfies the involution property. We notice that using the fact that the determinant of the Jost functions are equal to each other from Eq. (3.1.17) we can show that the determinant of the monodromy matrix satisfies the normalisation condition

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1. \tag{3.1.23}$$

3.2 Transition Coefficients

We can now write down the transition coefficients in terms of the Jost functions, this follows trivially from Eqs. (3.1.17), Eq. (3.1.21) and yields for $a(\lambda)$

$$a\left(\lambda\right)=\frac{\omega^2}{2k\left(\lambda-k\right)}\det\left[T_-^1,T_+^2\right]. \tag{3.2.1}$$

In a similar manner one can obtain a corresponding formula for $b(\lambda)$, although that shall not be given here. Using the results of the previous section we will establish the analytical properties of $a(\lambda)$, this in turn will lead to the result that $a(\lambda)$ is completely determined by $b(\lambda)$ and its zeroes which occur at λ_j. In addition we will find that the soliton solutions are given for $b(\lambda)=0$ and thus all one needs is λ_j to determine $a(\lambda)$. Let us now establish some of the important properties of $a(\lambda)$, first we will address the asymptotic behaviour for $|\lambda|\to\infty$ which is given as

$$a(\lambda)=\mathrm{e}^{i\theta/2}+O\left(\frac{1}{|\lambda|}\right), \tag{3.2.2}$$

when $\mathrm{Im}\lambda>0$. One can obtain an equivalent expression for $a(\lambda)$ when $\mathrm{Im}\lambda<0$ which to zeroth order in λ is $\mathrm{e}^{-i\theta/2}$. In general it can be shown that $a(\lambda)$ is singular at $\lambda=\pm\omega$, it can also be shown that at these two points $\lim_{\lambda\to\omega}a(\lambda)=\pm ib(\lambda)$. This relation is very useful for the case of a soliton solution $b(\lambda)=0$ and thus we do not have to worry about this singular behaviour and everywhere else $a(\lambda)$ is analytic.

We have one last consideration for the transition coefficent $a(\lambda)$ and that is any zeroes it might possess, these zeroes will correspond to bound states of the scattering problem in Eq. (3.1.1). An instant consequence of $a(\lambda_j)=0$ is that the column solutions of the Jost functions become linearly dependent, from Eq. (3.2.1) we obtain the relation

$$T_-^1=\gamma_jT_+^2. \tag{3.2.3}$$

On recalling some basic propertities of bound state solutions to scattering problems we note that the λ_j corresponding to the zeroes of $a(\lambda)$ form a discrete spectrum. It will not be demonstrated here [20] but one can show that the zeroes are simple, this property completes the discussion on the analytical properties of the transition coefficient $a(\lambda)$. Let us consider the case for solitonic solutions given by the condition $b(\lambda) = 0$, then using the known propertities of $a(\lambda)$ it can be shown that the following is true

$$a(\lambda) = \mathrm{e}^{i\theta/2} \prod_{j=1}^{n} \frac{\lambda + k - \lambda_j - k_j}{\lambda + k - \lambda_j - \bar{k}_j}. \tag{3.2.4}$$

We notice that the zeroes of the above formula are simple and its modulus is one, remember that the coefficient $b(\lambda) = 0$, thus it has the correct properties. For the case of the soliton, we have shown that $a(\lambda)$ is completely determined by the zeroes λ_j and the phase θ.

It is important to note that the phase θ can also be written in terms of the zeroes λ_j by comparing Eq. (3.2.2) and Eq. (3.2.4) in the limit $|\lambda| \to \infty$ one obtains

$$\mathrm{e}^{-i\theta} = \prod_{j=1}^{n} \frac{\lambda_j + k_j}{\lambda_j + \bar{k}_j} \tag{3.2.5}$$

for Im < 0.

We will finish this section with a brief discussion on the time dynamics of the transition coefficients. One can in principle obtain the time dynamics although we shall not go into the details here. The target is to derive time evolution equations for the Jost functions $M_{\pm}(x, \lambda)$, it is clear that one cannot just use the second equation in Eq. (3.1.1). There is of course a neat way of deriving the time evolution, one starts by writing down a Jost solution to Eq. (3.1.1) between the two points x and y, following this through one obtains

$$\frac{\partial M(x, y, \lambda)}{\partial x} = U(x, \lambda) M(x, y, \lambda). \tag{3.2.6}$$

We now differentiate the above equation with respect to time and use the boundary condition

$M(x,y)|_{y\to x} = 1$ to obtain

$$\frac{\partial M(x,y,\lambda)}{\partial t} = V(x,\lambda)M(x,y,\lambda) - M(x,y,\lambda)V(y,\lambda), \tag{3.2.7}$$

where the above dynamical equation resembles the Heisenberg equation for time evolution of quantum operators. Finally the analysis can be completed by multiplying the above equation by $\mathcal{E}(y,\lambda)$ on the right and taking the limit $y \to -\infty$. In the limit taken we find $V(y,\lambda) = -\lambda U_\lambda$ and on using the scattering problem in Eq. (3.1.1) one obtains $\mathcal{E}^{-1}(y,\lambda)U_-(\lambda)\mathcal{E}(y,\lambda) = \frac{-ik\sigma_3}{2}$. Using these observations about asymptotic behaviour we find the following time evolution equations for the Jost functions

$$\frac{\partial M_\pm(x,\lambda)}{\partial t} = V(x,\lambda)M_\pm(x,\lambda) - \frac{ik\lambda}{2}M_\pm(x,\lambda)\sigma_3. \tag{3.2.8}$$

We have already established that when $a(\lambda) = 0$ then the relation in Eq. (3.2.3) holds between the columns $M_-^{(1)}$ and $M_+^{(2)}$ of the Jost solutions. Using the relation from Eq. (3.2.3) and comparing the time evolution equations from Eq. (3.2.8) one obtains the following time dependence for the transition coefficient of the discrete spectrum

$$\gamma_j(t) = \mathrm{e}^{-ik_j\lambda_j t}\gamma_j(0), \quad j = 1, ..., n. \tag{3.2.9}$$

For the time-dependence of $a(\lambda)$ we take a time derivative with respect to time of Eq. (3.1.21) and use Eq. (3.2.8) to obtain time evolution for the monodromy matrix,

$$\partial_t M(\lambda) = \frac{ik\lambda}{2}\left[\sigma_3, M(\lambda)\right]. \tag{3.2.10}$$

The above equation for the time evolution of the monodromy matrix allows one to read off the equations for the time dependence of the transition coefficients corresponding to the continuous spectrum, for example the time-dependence of the transition coefficient $a(\lambda)$ is given by

$$a(\lambda) = a(\lambda, 0). \tag{3.2.11}$$

We will not concern ourselves with the time-dependence of $b(\lambda)$ here as we are only interested in solitonic solutions. It is interesting to mention due to the trivial time dependence given in Eq. (3.2.11) one can use $a(\lambda, 0)$ to generate the conservation laws of the NLS equation.

3.2.1 Gelfand-Levitan-Marchenko (GLM) formulation of the inverse problem

Here we shall highlight the derivation of the GLM equations for inverting the scattering data back to the solutions of interest. We start by looking at Eq. (3.1.21) for just the first column T_-^1 of T_- and rewrite it as

$$\frac{1}{a(z)} T_-^1(x, z) = T_+^1(x, z) + r(z) T_+^2(x, z), \tag{3.2.12}$$

where the following notation $r = \frac{b}{a}$ has been used and is known as the reflection coefficient. The above equation has is jump condition which gives a relation between the jost functions T_-^1, T_+^2 defined on the upper sheet Λ_+ and the Jost solution T_+^1 defined on the lower sheet Λ_-.

We note from Eq. (3.2.3) that the function $\frac{1}{a(z)} T_-^1(x, z)$ has a residue defined as

$$\text{Res}\left[\frac{1}{a(z)} T_-^1(x, z)\,|_{z=z_j}\right] = \frac{\gamma_j}{\dot{a}(z_j)} T_+^{(2)}(x, z_j) = c_j T_+^{(2)}(x, z_j), \tag{3.2.13}$$

where the dot is to be understood as differentiation with respect to z.

We are now in a position to summarise the derivation of the GLM equations. Start by substituting the Jost solutions from Eq. (3.1.11) into Eq. (3.2.12), then substract the first column of $\mathcal{Q}^{-1}\mathcal{E}$ from both sides and finally multiply both sides by $\exp\left(i\frac{y}{4}\left[z - \frac{\omega^2}{z}\right]\right)$.

Integrating the equation that is left, over the variable z, for the real line will leave the correct inversion integral, the residue in Eq. (3.2.13) will also be useful when evaluating these integrals to obtain the non-trivial result.

For brevity the results from operations described above will be given here. The GLM equation is finally given as

$$\Gamma_- (x, y) + \tilde{\Omega} (x + y) + \int_{-\infty}^{x} \Gamma_- (x, s) \tilde{\Omega} (s + y) \, ds = 0 \tag{3.2.14}$$

where the equation is defined for $y \leq x$. The kernel $\tilde{\Omega}(x)$ is given explicitly as

$$\tilde{\Omega} (x) = \begin{pmatrix} \tilde{\xi} (x) & \tilde{\eta} (x) \\ \bar{\tilde{\eta}} (x) & \tilde{\xi} (x) \end{pmatrix}. \tag{3.2.15}$$

The diagonal and off diagonal elements of $\tilde{\Omega}(x)$ are given by

$$\tilde{\xi} (x) = \frac{i\omega}{8\pi} \int_{-\infty}^{\infty} \frac{\tilde{r}(z)}{z} \exp\left[\frac{-ix}{4}\left(z - \frac{\omega^2}{z}\right)\right] dz + \frac{\omega}{4} \sum_{j=1}^{n} \frac{\tilde{c_j}}{z_j} \exp\left[\frac{-ix}{4}\left(z - \frac{\omega^2}{z_j}\right)\right] \tag{3.2.16}$$

and

$$\tilde{\eta} (x) = \frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{\tilde{r}(z)}{z} \exp\left[\frac{-ix}{4}\left(z - \frac{\omega^2}{z}\right)\right] dz + \frac{1}{4i} \sum_{j=1}^{n} \tilde{c_j} \exp\left[\frac{-ix}{4}\left(z - \frac{\omega^2}{z_j}\right)\right], \tag{3.2.17}$$

where the following definitions have been made,

$$\tilde{r}(z) = -\frac{\bar{b}(z)}{a(z)} \tag{3.2.18}$$

$$\tilde{c_j} = \frac{1}{\gamma_j \dot{a}(z_j)}. \tag{3.2.19}$$

Note that a GLM integral equation can be written for the $y \geq x$ case as well.

3.2.2 Soliton Solutions to the GLM Equations

Here we shall show how to solve the GLM equation from Eq. (3.2.14) to retrieve the potential $q(x)$ for the soliton case, which is greatly simplified by a reflectionless potential $b(\lambda) = 0$. With a reflectionless potential, expressions from Eqs. (3.2.16) and (3.2.17) simplify considerably. $\Omega\tilde{(x)}$ takes a new simple form which will be rewritten into the form

$$\tilde{\Omega}(x+y) = M_1 \otimes N_1 \exp\left[\frac{\kappa_1(x+y)}{2}\right], \tag{3.2.20}$$

where

$$\kappa_1 = \sqrt{\omega^2 - \lambda^2} > 0 \tag{3.2.21}$$

and the columns M_1 and N_1 take the form,

$$M_1 = \frac{\sqrt{\tilde{m}_1}}{2}\begin{pmatrix} \omega \\ i\tilde{z}_1 \end{pmatrix} \quad , \quad M_1 = \frac{\sqrt{\tilde{m}_1}}{2}\begin{pmatrix} 1 \\ \frac{z_1}{i\omega} \end{pmatrix}. \tag{3.2.22}$$

We can solve the GLM equation by assuming the following form for the kernel $\Gamma_-(x,y)$

$$\Gamma_-(x,y) = f_1(x)\, N_1^T \exp\frac{\kappa y}{2} \tag{3.2.23}$$

and solving for the unknown function $f_1(x)$ gives,

$$\Gamma_-(x,y) = -\frac{M_1 \otimes N_1 \exp\left[\frac{\kappa_1}{2}(x+y)\right]}{1 + \frac{\tilde{m}_1\omega}{2\kappa_1}\exp[\kappa_1 x]}. \tag{3.2.24}$$

We can finally write down the soliton solution by using the expression from Eq. (3.1.14), the asymptotic form of U_0 and the time evolution of the discrete scattering data $\exp(\lambda_1\kappa_1 t)\,\gamma_1$ which leads to

$$q(x,t) = \rho\frac{1 + \exp(i\theta)\exp[\kappa_1(x - vt - x_0)]}{1 + \exp[\kappa_1(x - vt - x_0)]}. \tag{3.2.25}$$

One can then go from NLS equation back to the Gross-Pitaevskii equation to obtain,

$$|q(x,t)|^2 = n - \frac{m\left(c^2 - v^2\right)}{g\cosh^2\left(\frac{m(c^2-v^2)[x-X(t)]}{\hbar}\right)} \tag{3.2.26}$$

which gives the density we obtained for one dark soliton.

The above analysis for obtaining the one soliton solution naturally generalises to the N-soliton case, one can for this case investigate the phenomena of soliton scattering. We note that for soliton solutions $b(\lambda) = 0$ and thus leads to the reflection coefficient having the property, $r(\lambda) = 0$. Let us consider the situation where the soliton velocities without loss of generality are ordered such that $\lambda_1 > \lambda_2 > ... > \lambda_N$, then in the limit $t \to \pm\infty$ the N-soliton solution can be written as a sum of one soliton solutions by virtue of the reflection coefficient being zero. Investigating the limits $t \to +\infty$ and $t \to -\infty$ results in the observation that the soliton scattering factorises into a product of two soliton shifts given by

$$\begin{aligned} \Delta x_{01} &= \frac{1}{\nu_1}\log\frac{(\lambda_1-\lambda_2)^2+(\nu_1+\nu_2)^2}{(\lambda_1-\lambda_2)^2+(\nu_1-\nu_2)^2} \\ \Delta x_{02} &= -\frac{1}{\nu_2}\log\frac{(\lambda_1-\lambda_2)^2+(\nu_1+\nu_2)^2}{(\lambda_1-\lambda_2)^2+(\nu_1-\nu_2)^2} \end{aligned} \tag{3.2.27}$$

for $\lambda_1 > \lambda_2$. The soliton with larger velocity λ_1 acquires a positive phase shift and the soliton with smaller velocity λ_2 gains a negative phase shift, this tells us that the solitons act as repulsive particles. One can also obtain from Eq. (3.2.27) the relation $\nu_1\Delta x_{01} + \nu_2\Delta x_{02} = 0$, this tells us that the center of mass of the solitons is conserved during the collision.

The Inverse Scattering Transform forms the fundamental theory of solitons governed by the NLS equation. The theory allows one to obtain the full N-soliton solution and show that its scattering factorises as two soliton scattering that results in phase shifts of these solutions. This scattering property is the source of the name soliton and means that in an homogeneous system one can consider the N-soliton case as a sum of non-interacting single solitons. As was mentioned earlier the Inverse Scattering Transform can be used to write down a generating function for the infinite set of conservation laws, this relies on the trivial time-dependence of the transition coefficient of $a(\lambda, t) = a(\lambda, 0)$. This method also allows the development of perturbation theory known as the direct method to solve for corrections in the presence of various integrability breaking terms. We thus see the method is very powerful and gives an insight into solving nonlinear differential equations.

3.3 Renormalised integrals of motion

We will consider the integrals of motion of the Gross-Pitaevskii equation in dimensionless form, they can as has been mentioned be obtained from the Inverse Scattering Transform or from Noethers theorem. We will state the relevant conservation laws below,

$$N = \int_{-\infty}^{\infty} dx|\psi|^2 \tag{3.3.1}$$

$$P = \frac{i}{2}\int_{-\infty}^{\infty} dx \left[\psi\partial_x\bar{\psi} - \bar{\psi}\partial_x\psi\right] \tag{3.3.2}$$

$$E = \frac{1}{2}\int_{-\infty}^{\infty} dx \left[|\partial_x\psi|^2 + |\psi|^4\right], \tag{3.3.3}$$

where N is the total number of particles, P is the momentum of the system and E is the total energy of the system. Considering these equations for the dark soliton one arrives at a problem, the dark soliton solution is finite at spatial infinity and thus the conservation laws are divergent. We will now discuss how to remove this problem, we start by investigating the simplest nontrivial solution of the Gross-Pitaevskii equation. The simplest solution being the plane wave solution $\psi(x,t) = \sqrt{n(x,t)}\exp i\phi(x,t)$, we find from direct substitution,

$$n(x,t) = n_0 = \text{const} \tag{3.3.4}$$

$$\phi(x,t) = mv_0 - \left(\frac{mv_0^2}{2} + gn_0\right), \tag{3.3.5}$$

where we have reintroduced the mass and the interaction parameter g is the chemical potential. It is very important to mention that we have introduced the velocity v_0, which denotes the plane wave velocity in the absence of a soliton. Using the conservation laws we can evaluate the momentum and energy of the plane wave configuration yielding

$$\begin{aligned} P_0 &= L m n_0 v_0 = m N v_0 && (3.3.6) \\ E_0 &= L\left[\frac{g n_0^2}{2} + \frac{m n_0 v_0^2}{2}\right] = \frac{g n_0 N}{2} + \frac{m N v_0^2}{2}, && (3.3.7) \end{aligned}$$

for a system of finite length L, we see in the limit $L \to \infty$ the expressions above for the plane wave solution are divergent. We have to be careful, we need to separate out the background contribution of the plane wave from the soliton; in turn this will give us renormalised integrals of motion that are finite.

We will change tact here, we shall switch to the density phase representation as we shall see this yields a more physical interpretation. For the density-phase representation we write $\psi(x,t) = \sqrt{n(x,t)} \exp i\phi(x,t)$, the conserved energy now takes the form

$$E = \int_{-\infty}^{\infty} dx \left(\frac{1}{2m} \left[\frac{(\partial_x n)^2}{4n} + n(\partial_x \phi)^2 \right] + \frac{g n^2}{2} \right). \tag{3.3.8}$$

The Gross-Pitaevskii equation in this representation takes the hydrodynamic form given below as

$$\begin{aligned} \partial_t n &= -\partial_x \left[\frac{n}{m} \partial_x \phi \right] && (3.3.9) \\ -\partial_t \phi &= \frac{1}{4m} \frac{\partial_x^2 n}{n} - \frac{1}{8m} \frac{(\partial_x n)^2}{n^2} + \frac{1}{2m} (\partial_x \phi)^2 + g n, && (3.3.10) \end{aligned}$$

the first equation is the continuity equation and is independent of the interactions between the bosonic particles, however the second equation contains information about the interactions. Following Tsuzuki [73] one can write down the solutions of Eq. (3.3.10) in the absence of a background current that correspond to a dark soliton and they are given by

$$n_s(z:V,n) = n\left(1-\left[1-V^2/c^2\right]\left[\text{sech}\left(mcz\sqrt{1-V^2/c^2}\right)\right]^2\right) \quad (3.3.11)$$

$$\nu_s(z:V,n) = V\left(1-\frac{n}{n_s(z:V,n)}\right), \quad (3.3.12)$$

where $z = x - Vt$. We point out here that once the soliton density $n_s(z : V, n)$ has been obtained the velocity $\nu_s(z : V, n)$ is straightforward to obtain. For example due to Galilean invariance we have $(x, t) \to (x - Vt)$ for the soliton thus we can re-write each time derivative as $\partial_t = -V\partial_z$, we relate the time derivative to a spatial derivative. Now using this transformation in the continuity equation we are able to write down

$$\partial_z\left[Vn(z) - \frac{n(z)}{m}\partial_z\phi(z)\right] = 0, \quad (3.3.13)$$

which can be formally integrated up to a constant of integration C. We now demand that at spatial infinity the gradient of the phase should be zero i.e. there should be no currents at infinity. This condition will fix our constant of integration C, using the fact that the velocity of the field is given by $\partial_x\phi/m$ and using the obtained value of C leads directly to the soliton velocity $\nu_s(z : V, n)$ given in Eq. (3.3.12). In general we will have a soliton that moves against a background that has velocity v, notice that we do not use v_0 for the velocity of the background as it will be modified by the presence of the soliton. We can make a transformation to the Laboratory frame where the background moves with velocity v, the transformed soliton now becomes

$$n_s(z:V,n,v) = n_s(z:V-v,n) \quad (3.3.14)$$

$$\nu_s(z:V,n,v) = \nu_s(z:V-v,n)+v, \quad (3.3.15)$$

so the soliton moves with velocity $V - v$ with respect to the background fluid. Importantly the soliton is characterised by the phase drop $\Phi_s(V)$ given below as,

$$\begin{aligned}\Phi_s(V) &= -\int_{-\infty}^{\infty} dx m\nu_s(z:V,n) = mV\int_{-\infty}^{\infty} dx \frac{(n-n_s(z:V,n))}{n_s(z:V,n)} \\ &= 2\arctan\left[\frac{c}{V}\sqrt{1-\frac{V^2}{c^2}}\right] = 2\arcsin\left[\sqrt{1-\frac{V^2}{c^2}}\right]. \end{aligned} \tag{3.3.16}$$

Where we used the density given by Eq. (3.3.11), the integral is of standard trigonometric form and thus the result follows. The final expression for the phase was obtained from the identity given in the footnote [2], the latter form will become of use shortly. The other parameter that the soliton is characterised by is the number of particles $N_s(V)$ given by

$$N_s(V) = \int_{-\infty}^{\infty} dx\,(n-n_s(z:V,n)) \tag{3.3.17}$$

$$= \frac{2n}{mc}\sqrt{1-\frac{V^2}{c^2}}, \tag{3.3.18}$$

again the second line is obtained from performing a standard trigonometric integral. We note that the conservation law that results in $N_s(V)$ expressed in Eq. (3.3.17) is the modified version of the conservation law for particle number in Eq. (3.3.1) where have now removed the contribution from the background. The normalised formulas we have derived for the conservation laws so far, allow us to write the number of particles expelled from the soliton core $N_s(V)$ as

$$N_s = \frac{2n}{mc}\sin\left(\frac{\Phi_s(V)}{2}\right), \tag{3.3.19}$$

a formula that we will need in later chapters. We will work with these definitions for the phase jump across the soliton $\Phi_s(V)$ and the number of particles ejected from the soliton core $N_s(v)$.

We note that if we try to calculate the total phase drop across the entire system we run

[2] $\arctan[x] = \arcsin\left[\frac{x}{\sqrt{x^2+1}}\right]$

into another divergence problem, for example the total phase drop is given by

$$\Phi_{tot} = mvL - \Phi_s(V - v) = mv_0 L, \tag{3.3.20}$$

this formula tells us that the difference between the background velocity v in the presence of the soliton and the background velocity v_0 in the absence of the soliton is of order $1/L$. We can also obtain a formula for the total number of particles in the system and this is given by

$$N = nL - N_s(V - v) = n_0 L \tag{3.3.21}$$

and again this results in a difference of order $1/L$ between n and n_0.

We now move on to the conserved momentum in Eq. (3.3.2) and follow the idea of [69], to render this result finite we need to subtract off the momentum contribution from the background given by the expression in Eq. (3.3.6); this will leave us with the momentum of the soliton core and is given below as

$$\begin{aligned} P_s(V, n, v) &= \int_{-\infty}^{\infty} dx m n_s(x : V, n, v) \nu_s(x : V, n, v) - m n_0 v_0 L && (3.3.22) \\ &= n\left(\Phi_s(V - v) - \sin\left(\Phi_s(V - v)\right)\right) && (3.3.23) \\ &= P_s(V - v, n). && (3.3.24) \end{aligned}$$

In deriving the expression above for the soliton momentum we have transformed to the laboratory frame and used Eqs. (3.3.20) and (3.3.21). So we see from the above analysis that the soliton momentum is given by $n\Phi - n\sin(\Phi)$ and it this expression that we will be concerned with. In the spirit of renormalising these divergent expressions we finally deal with the conserved energy. This can be dealt in the same way as the momentum i.e. we subtract off the energy of the background E_0 given in Eq. (3.3.7); this leads straight to the result

$$E_s(V,n,v) = E_s(V-v,n) + P_s(V-v,n)v. \tag{3.3.25}$$

In the absence of a background current the renormalised integral for energy conservation for the soliton takes the form

$$E_s = \int_{-\infty}^{\infty} dx \left(\frac{1}{2m}\left[\frac{(\partial_x n_s)^2}{4n} + n_s(\partial_x\phi)^2\right] + \frac{g}{2}\left(n_s - n_0\right)^2 \right), \tag{3.3.26}$$

where one uses the same consideration as is used for the total number of particles, viz $|\phi|^2 \to |\phi|^2 - n_0$. It is straightforward to verify that the kinetic term does not contribute any divergences. Performing the integral in Eq. (3.3.26) results in the following form for the energy of the soliton

$$E_s(V,n) = \frac{mg^2}{6}N^3, \tag{3.3.27}$$

where in deriving this expression we have used the relation for $N_s(V)$ from Eq. (3.3.19). This formula for the energy tells us that the energy can be completely specified by the number of particles expelled from the soliton core N_s, this will prove to have a nontrivial impact on the study of dissipation of a soliton confined by an external trapping potential.

It is informative to consider the Lagrangian corresponding to the soliton core i.e. where we have removed contributions from the background, combining Eq. (3.3.23) with Eq. (3.3.25) we can write down the Lagrangian for a grey soliton in the laboratory frame as

$$\mathcal{L}_s = P_s(V-v,n)\,(V-v) - E_s(V-v,n). \tag{3.3.28}$$

We note that one should compare this form of the Lagrangian with the expression one would obtain from performing a Galilean transformation. Making a Galilean transformation on some generic Lagrangian leads to the expression,

$$\mathcal{L} = \mathcal{L}'(V - v) + MVv - \frac{Mv^2}{2}. \tag{3.3.29}$$

On comparison with the Lagrangian $\mathcal{L}_s$ we find that the rest mass M of the soliton is zero.

3.4 Perturbations

First we shall discuss an important result first obtained by [73], we will just discuss the main points here and we shall start with the dimensionless form of the NLS equation which takes the form,

$$i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi + |\psi|^2\psi. \tag{3.4.1}$$

We now investigate what happens when the amplitude of the soliton is very narrow, in this case the change in the background density is very small thus we are considering small deviations from the background density n_0. In this case the phase change across the system is very gradual, so we would expect small deviations from not only density but also the phase drop across the background. What one finds in this narrow depth large amplitude limit is that the solitons are in fact governed by the KdV equation. This approximation can prove useful when the exact solitonic solution is not obtainable due to the non-integrable nature of some differential equation.

Here we will pursue the standard route of making a transformation to the density phase representation and substituting this representation into the above dimensionless equation results in the dimensionless form of the hydrodynamic equations in Eq. (3.3.9) and Eq. (3.3.10). Next one assumes slow spatial and temporal variations which necessitates the introduction of slow variables Z $= \epsilon^{1/2}(x - ct)$ and $T = \epsilon^{3/2}t$, ϵ is defined as a small parameter such that $0 < \epsilon \ll 1$. One then proceeds by expanding the density and phase as follows

$$n = n_0 + \epsilon n_1(Z,T) + \epsilon^2 n_2(Z,T) + ..., \tag{3.4.2}$$

$$\phi = -n_0 t + \epsilon^{1/2}\phi_1(Z,T) + \epsilon^{3/2}\phi_2(Z,T) + ..., \tag{3.4.3}$$

one can then substitute the expansions from Eqs. (3.4.2) - (3.4.3), into the dimensionless form of the hydrodynamic equations Eqs. (3.3.9) - (3.3.10) to find a ladder of equations in powers of ϵ. One finds equations of order ϵ and $\epsilon^{3/2}$ which relate the phase and density by $\partial_Z\phi = \frac{c}{n_0}n_1$, the next set of equations is at order ϵ^2 and $\epsilon^{5/2}$ and these equations result in the KdV equation in the form

$$2c\dot{n}_1 + \partial_Z\left(3n_1\partial_Z n_1\right) - \frac{1}{4}\partial_Z^3 n_1 = 0, \tag{3.4.4}$$

along with the condition $c^2 = n_0$. We note that these small amplitude solitons that obey the effective KdV equation need special care if one wants to study the classical behaviour within the trap. One of the requirements of the trap treatment is that the soliton velocity V is small in comparison to the local sound velocity but for the small amplitude solitons $V \approx c$, where c is the sound velocity. In this case one needs to take into account quantum effects, as the soliton has lost its interpretation as a pointlike particle.

3.4.1 Landau dynamics of a grey soliton in a smooth confining potential

Here we shall follow the procedure developed by Konotop and Pitaevskii [41] for trapped solitons in Bose-Einstein Condensates. It had already been shown by Busch and Anglin [11] that to lowest order the soliton will oscillate at $1/\sqrt{2}$ of the trap frequency using a multiple time scale boundary layer theory. As was pointed out this can be rather involved and requires low velocities for the soliton. Konotop and Pitaevskii showed that one can consider the soliton as a Landau quasiparticle to lowest order in correction to the homogeneous problem. We shall now summarise their method as this will serve as the basis for the study of dissipative dynamics of solitons in confining potentials in a later chapter.

First we shall write the one soliton solution in the following way,

$$\psi(x,t) = \sqrt{n}\left(i\frac{V}{c} + \frac{u}{c}\tanh\left[\frac{1}{l}(x - X(t))\right]\right)\mathrm{e}^{-i\mu t/\hbar} \tag{3.4.5}$$

where we use the following definitions, $u = \sqrt{c^2 - V^2}$ and the soliton width $l = \hbar/mu$. The energy of the soliton described by Eq. (3.4.5) is given by Eq. (3.3.26) and in the dimensions that we are working with is given by

$$\mathcal{E} = \frac{4\hbar}{3g}\left(c^2 - V^2\right)^{3/2} = \frac{4\hbar}{3g}u^3. \tag{3.4.6}$$

The approximation comes in the form of the size of the condensate $2L$ in comparison to l which is required to be $L \gg l$; this restriction implies that the condensate is effectively at rest as seen by the soliton. Thus one states that the soliton has a negligible effect on the condensate, in turn one can regard the energy given by Eq. (3.4.6) as an effective Hamiltonian for the soliton i.e. the quasiparticle. In this regime one can make the Thomas-Fermi approximation and write for the sound velocity; $c \to c(X)$ and now we talk of the local sound velocity where X is the soliton coordinate. It follows that if we consider the energy in Eq. (3.4.6) to be the effective Hamiltonian and to be correct in the homogeneous case then to a first approximation the soliton wavefunction retains the same form as in the homogeneous case Eq. (3.4.5) and we have the following conservation of energy,

$$\frac{4\hbar}{3g}\left(c^2(X) - V^2\right)^{3/2} = \mathcal{E} = \mathrm{const}, \tag{3.4.7}$$

which must mean that u is a constant. It is a matter of simple algebra to show that this relation implies $m\dot{X}^2 = c^2(X) - u^2$ which in principle can be integrated for a given $c(X)$ to yield the velocity of the quasiparticle. What is quite remarkable is the observation on using the form of $c(X)$ in the local density approximation which results in the following effective energy for the soliton

$$m\dot{X}^2 + U(X) = mc_0^2 - mu^2. \tag{3.4.8}$$

Let us consider the harmonic trapping potential given by $U(X) = 1/2m\omega^2 X^2$; then we find that the soliton behaves as a quasiparticle that oscillates in the trap with frequency $\omega/\sqrt{2}$.

Chapter 4

QUANTISATION OF CLASSICAL SOLITONS

4.1 Overview of the general procedure

Within this chapter we shall discuss the quantisation of classical solitonic solutions and the problems that occur when trying to carry out this procedure; the first half of this chapter will follow [62]. First we will shall summarise the philosophy behind the quantisation of classical systems to motivate and introduce the following sections.

Let us for illustration consider a potential V(x) that has minima at the coordinates a and b, such that $V(a) < V(b)$. For a classical problem the lowest energy state would be $E_0^{cl} = V(a)$, for the quantum version of this problem this would be contradictory as it would lead to a simultaneous measurement of momentum and position. One expects that the quantum theory will have a ground state of the form $E_0 = V(a) + \delta E$. If the potential is approximately harmonic near the point a, then one can perform a taylor expansion. To be valid one would require $\omega^2 \gg \lambda$ where ω^2 is the coefficient of the $(x-a)^2$ term and λ represents all higher order corrections. If this condition holds then one can expect the energy for the quantum case to take the form

$$E_0 = V(a) + (n + \frac{1}{2})\hbar\omega + O(\lambda) \qquad (4.1.1)$$

and the ground state is found by setting $n = 0$. Of course if there are two minima of the potential then between them there should be an extremum, this we will not consider as it will lead to imaginary frequencies and thus be unstable. One can of course consider this procedure for the local minimum at the point b, if the anharmonic terms with coefficients $\tilde{\lambda}$ are small in comparison to the harmonic term ω' then we may proceed as before and obtain the quantum low energy sector of the problem as

$$\tilde{E}_0 = V(b) + \hbar\omega'(n' + \frac{1}{2}). \tag{4.1.2}$$

This is the quantum corrected energy now built around the potential minimum at the point b, we have also assumed that the anharmonic coefficients λ and $\tilde{\lambda}$ are small enough that the effect of quantum tunelling is small, effectively there is a large energy barrier between the two minima. We also note that the energy $\tilde{E}_0$ is greater than E_0 which should be clear from the above treatment.

The above discussion is very pertinent despite its deceptive simplicity, we will find when we get onto the quantisation of solitons that indeed several minima exist and there will be some vacuum state that corresponds to a global minimum of the problem. The solitonic state will on the other hand correspond to a higher energy state, an excitation of the system, very much like the energy built up around the point b in the previous example.

Finally we need to discuss a shortcoming of this method which is of course unavoidable for the quantum expansion around a classical soliton. For example what if $\omega = 0$, if this is to be the case then we find our weak coupling expansion breaks down completely. No matter how small we demand the anharmonic terms λ to be we just can't satisfy the condition of applicability, thus we cannot proceed with the semi-classical expansion.

Let us first consider a particle moving under a constant potential, for which one does not have a trapping frequency. This problem has the much celebrated translational invariance which as we know from Noethers theorem implies momentum is a conserved quantity.

Straightaway the energy can be written down as

$$E_n = V + \frac{P_n^2}{2m}, \tag{4.1.3}$$

this demonstrates that when a $\omega = 0$ we end up with the momentum associated with that mode in the energy instead. We will find this to be true of the zero modes in connection with solitons and for each zero mode we should expect a conjugate momentum to appear in its place.

We start with a simple discussion of semi-classical quantisation, where using the correspondance principle one looks for corrections to the classical ground state to the system. For example if we took a field theory with a large but finite number of degrees of freedom in a potential $V(x_1, ..., x_N)$ with a local or global minimum then the lowest energy state will be

$$E_0 = V(a) + \sum_{i=1}^{N} \frac{1}{2}\hbar\omega_i + \text{corrections}, \tag{4.1.4}$$

where V(a) represents the minimum of the potential and the expansion around this mimimum has been neglected. One might expect an expression like this, where the second term represents the Heisenberg uncertainty principle and the first term is a classical energy.

Let us now consider a simple field theory for real scalar fields, defined by the Lagrangian

$$L = \int dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left(\nabla \phi \right)^2 - U(\phi) \right], \tag{4.1.5}$$

we can try and expand around a static classical solution. We shall name the sum of the last two terms as the field potential $V(\phi)$, leaving us with the lagrangian in the form $L = 1/2 \int dx (\dot{\phi})^2 - V(\phi)$. We also define the mimimum of the potential to occur at the field configuration $\phi(x) = \phi_0(x)$, then a Taylor expansion around such a minimum gives

$$V(\phi) = V(\phi_0) + \int dx \frac{1}{2} \left[\delta\phi(x) \left(-\nabla^2 + \left(\frac{d^2U}{d\phi^2} \right)_{\phi=\phi_0} \right) \delta\phi(x) \right]. \tag{4.1.6}$$

The standard procedure is to now search for the eigenvalues and eigenvectors of the quadratic term in the expansion of the potential, this with the definition $\phi - \phi_0 = \sum_i c_i(t)\delta\phi_i(x)$ leads us to solve the following eigenvalue problem,

$$\left[-\frac{d^2}{dx^2} + \left(\frac{d^2U}{d\phi^2} \right)_{\phi=\phi_0} \right] \delta\phi_i(x) = \omega_i^2 \delta\phi_i(x). \tag{4.1.7}$$

In the above eigenvalue equation the $\delta\phi_i$ are orthonormal functions and are to be considered as the fluctuations around the classical grounstate ϕ_0. Finally using the orthonormal property and integrating the Lagrangian to quadratic order takes the form

$$L = -V(\phi_0) + \frac{1}{2} \sum_i \left[\dot{c}_i^2 - \omega_i^2 c_i^2 \right] + \text{corrections}. \tag{4.1.8}$$

So we see that the procedure is to take a classical ground state also known as the vacuum and then carrying out an expansion in normal modes to find the correction to the energy. Forcing commutation relations on the normal mode expansion coefficients $c_i(t)$ ensures quantisation of the system and leads to a correction to the classical vacuum.

4.2 Quantisation Of A Simple Model

Let us take the example of the ubiquitous ϕ^2 field theory and choose our vacuum to be the trivial configuration $\phi_0 = 0$ which does indeed solve the field equations. The potential term in Eq. (4.1.5) takes the form

$$U(\phi) = \frac{1}{2} m\phi^2 + \frac{1}{4} \lambda\phi^4 \tag{4.2.1}$$

and for static field solutions one can vary the action to obtain the equation of motion which is solved by $\phi_0 = 0$. We find that this solution is an absolute minimum of the potential $V(\phi)$. Expanding around this minimum we find that the normal mode coordinates satisfy the following eigen-value equation,

$$\left[-\frac{d^2}{dx^2} + m^2\right] c_i = \omega_i^2 c_i. \tag{4.2.2}$$

which is in the form of a Schrodinger equation. Of course this differential equation is trivially solved using a plane wave basis with wave vectors constrained by $k_i = 2\pi N_i/L$ with the limit $L \to \infty$ and eigenvalues $\omega_i^2 = k_i^2 + m^2$. We can see that the energy is corrected by harmonic oscillator levels to give

$$\begin{aligned} E &= V(\phi_0) + \hbar \sum_{k_i} \sqrt{k_i^2 + m^2}(n_i + \frac{1}{2}) + O(\lambda) \\ &= \hbar \sum_{k_i} \sqrt{k_i^2 + m^2}(n_i + \frac{1}{2}) + O(\lambda), \end{aligned} \tag{4.2.3}$$

we see that the vacuum energy is obtained by setting $n_i = 0$ thus obtaining

$$E_{vac} = \frac{1}{2}\hbar \sum_{k_i} \sqrt{k_i^2 + m^2} + O(\lambda). \tag{4.2.4}$$

We could in principle start considering excitations for $k_i = 0$ modes, one then finds the energy increases by units of $\hbar m$. Here we have demonstrated the quantisation of a system around a trivial classical state, these results should be familiar from standard perturbation theory. It has also been assumed that a perturbative expansion in λ is legitimate, this is certainly a requirement of this semi-classical treatment. We have used the ϕ^4 model to illustrate this principle but it does not admit non-trivial classical solutions with finite energy, for the non-trivial case we shall consider a different Lagrangian.

4.3 Quantisation of vacuum and Kink like solutions

Here we shall again consider real scalar fields in (1+1) dimensions for which are now governed by the Lagrangian in Eq. (4.1.5) with $V(\phi)$ given by

$$\int dx V(\phi) = \int dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{\lambda}{4} \left(\phi^2 - \frac{m^2}{\lambda} \right)^2 \right], \tag{4.3.1}$$

we can verify that $\phi(x,t) = \pm m/\sqrt{\lambda}$ is a trivial static solution and minimum of the potential. Also the finite energy kink solution $\phi_k(x,t) = \pm(m/\sqrt{\lambda}) \tanh[m(x-a)/\sqrt{2}]$ is a static solution the the equation of motion. Note that the kink solution tends to the trivial vacuum solution for $x \to \pm\infty$. One may now proceed in the same spirit for the ϕ^4 theory for the trivial vacuum solution $\phi_0 = m/\sqrt{\lambda}$ and obtain the spectrum

$$E_{n_i} = \hbar \sum_{n_i} \sqrt{k_i^2 + 2m^2} \left(n_i + \frac{1}{2} \right) + O(\lambda), \tag{4.3.2}$$

this is of course a familiar and expected result. It is interesting to note that in this example we could have expanded around the $\phi_0 = -m/\sqrt{\lambda}$ state. The two vacuum states violate the $\phi \to -\phi$ symmetry of the system and is known as spontaneous symmetry breaking. So far we have done nothing interesting or surprising but now let us try the same procedure for the kink solution.

Before we outline the method we note that, some of the mathematical details will be omitted here due to relatively complicated nature of their manipulations. We are merely using the following example to highlight an intrinsic problem associated with the quantisation of kink and solitonic like solutions.

4.3.1 The Kink Solution

So we now work with the kink solution given below as

$$\phi_K(x-a) = m/\sqrt{\lambda}\tanh\left[m(x-a)/\sqrt{2}\right], \tag{4.3.3}$$

remember this is a static solution and has energy

$$V(\phi_k) = 2\sqrt{2/3}\left(\frac{m^3}{\lambda}\right). \tag{4.3.4}$$

The kink solution is an extremum of the potential and as before we can expand the potential to second order in normal mode coefficients. The expansion is performed to zero order in the coupling constant λ, we shall discard higher order terms for this example. The expansion of the potential yields

$$V(\phi) = V(\phi_k) + \frac{1}{2}\int dx\eta(x)\left(-\frac{\partial^2}{\partial x^2} - m^2 + 3\lambda\phi_k^2\right)\eta(x), \tag{4.3.5}$$

where we have indeed ignored terms of order λ or higher. One should substitute the explicit form of the solution ϕ_k into the part of the potential that is quadratic in η and upon making the change of variables $z = mx/\lambda$ we arrive at the following eigenvalue problem

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + (3\tanh[z]-1)\right)\tilde{\eta}_n(z) = \frac{\omega_n^2}{m^2}\tilde{\eta}_n(z). \tag{4.3.6}$$

We shall now briefly discuss the solutions and eigenvalues of the eigenvalue problem in Eq. (4.3.6), we shall state them without proof and discuss their significance. First we note that again it is a Schrodinger equation with a Hyperbolic potential, this analogy is useful for finding the solutions. There are two discrete solutions corresponding to bound states listed below

$$\tilde{\eta}_0(z) = \text{sech}^2(z) \tag{4.3.7}$$

$$\tilde{\eta}_1(z) = \sinh(z)\text{sech}^2(z), \tag{4.3.8}$$

where $\omega_0^2 = 0$ and $\omega_1^2 = \frac{3}{2}m^2$. Then there is a continuum of energies given by

$$\omega_n^2 = m^2 \left(\frac{1}{2} n^2 + 2 \right), \tag{4.3.9}$$

where we think of n as a momentum. We have a continuum of oscillator states that correspond to quantum corrections as in the example where we expanded around the trivial vacuum. We have also gained a mode that is a bound state and very importantly we have a mode with zero energy. It is this so called zero mode that is the major deviation from the previous example and as we shall now discuss leads to problems in the quantum theory. We have a certain picture when we quantise by expanding to second order in normal modes which goes as follows: We start with some classical solution which is a minimum of the potential for the physical system we are studying. We then look for a new minimum which will correspond to the quantum problem, the solution smears the classical solution or to put another way climbs the potential from the minimum. A problem with this picture occurs when we have a zero frequency $\omega = 0$, for this case the solution is not constrained to harmonic confinement which keeps the solutions near the classical minimum, along the zero frequency mode the wavefunction spreads. Another troublesome caveat is that of the calculation of higher order corrections in perturbation theory, any mode coupling to the zero mode introduce a zero into the energy denominators and thus lead to divergences.

Alas not all is lost as up to and not including of order λ one can still calculate the quantum ground state, which for the kink example, we shall discuss. Here we write down the energy of the oscillator states localised around the kink solution ϕ_k which is

$$E_{N_n} = \frac{2\sqrt{2}m^3}{3\lambda} + (N_1 + \frac{1}{2})\hbar\sqrt{3/2}m + m\hbar\sum_n (N_n + \frac{1}{2})(\frac{1}{2}n^2 + 2)^{1/2} + O(\lambda). \quad (4.3.10)$$

We identify the first term as the classical kink energy, the second as the discrete bound state associated with n=1 mode and the last term as the continuum contribution. We see that up to this level of accuracy the zero mode does not present any problems. Note that the lowest energy state is that of a kink at rest, this is what is called the quantum kink. One should also stress that this is not a global minimum, the true ground state is the vacuum ϕ_1 whose spectrum is defined in Eq. (4.2.3). The system we have here is that of a quantum kink which is also known as an extended particle in the literature.

Let us now excite into the $N_1 = 1$ state, this is to be interpreted as a direct excitation of the kink particle itself; of course we can add more excitations to this state which will correspond to higher energy states of the kink itself. Finally increasing n will correspond to exciting normal modes in the system which for this example can be called mesons. One can show that asymptotically in space these states feel a phase shift due to the presence of the kink which acts as a stationary potential to these particles. All this information is obtained without any problems arising from the zero mode, although as we have said before higher order corrections would contain divergences. To close this discussion we remark that if one wants to explicilty calculate the kink mass i.e., $N_1 = 0$ and $n = 0$ one needs to use a renormalisation scheme for the mass coming from the need to normal order the Hamiltonian, this is of little interst to us and the example here has highlighted quantisation of a kink like solution.

4.4 Inherent zero modes of kink like solutions

Now we discuss the physical origin of the zero mode and what it implies for the quantum theory. The Lagrangian that we studied in the previous section along with the equation of motion satisfied by the kink solution and the potential from Eq. (4.3.1) are all translationally invariant. Thus a field $\phi(x)$ and its translated relative $\phi(x - X)$ result in the same value for these functionals. We see that the potential $V(\phi)$ by virtue of being translationally invariant has some fixed value defined by $\phi(x)$ and is the same for any arbitary spatial translation, put another way $V(\phi(x - X))$ is independent of X.

We say that the family of translated functions $\phi(x - X)$ form a one-parameter curve in that space for which the potential remains constant, we name this curve an equipotential curve. Of course the vacuum solution $\phi_1(x) = m/\lambda$ of the Lagrangian in Eq. (4.1.5) with the potential from Eq. (4.3.1) leaves the potential invariant for all translations as it is a constant. For this example the one-parameter curve is a single point instead of forming an equipotential curve; the solution cannot move away from the point ϕ_1.

Again we see the same problem with the kink solution $\phi_k(x)$ defined in Eq. (4.3.3), it is also translationally invariant and thus all its translated counterparts leave the potential invariant. Consider translating $\phi_k(x)$ along the equipotential curve, as this corresponds to moving in a valley of the potential that does not change the value of the potential we see that the second and all higher order derivatives must vanish thus resulting in the zero frequency. It is instructive to consider the translation of the zero mode,

$$\Delta\phi_k(x) = \phi_k(x - \delta X) - \phi_k(x) \tag{4.4.1}$$

$$= -\delta X \frac{\partial \phi_k(x)}{\partial x} \tag{4.4.2}$$

$$= -\frac{m^2}{\sqrt{2}\lambda} \text{sech}^2[mx/\sqrt{2}], \tag{4.4.3}$$

where we notice that this infinitesimally translated function is proportional to the zero mode solution from Eq. (4.3.7) which is satisfying. Yet another way of seeing the relationship between the zero mode and the kink solution is to first consider the static equation of motion for the kink solution, given below as

$$-\frac{1}{2}\frac{\partial^2\phi_k}{\partial x^2}+\frac{\partial U}{\partial\phi}\Big|_{\phi_k}=0. \tag{4.4.4}$$

We proceed by taking the spatial derivative of the equation of motion and using the chain rule on the second term results in the following equation for the zero mode,

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2 U}{\partial\phi^2}\Big|_{\phi_k}\right]\frac{\partial\phi_k}{\partial x}=0=\omega_0\frac{\partial\phi_k}{\partial x}. \tag{4.4.5}$$

One may be tempted to consider these modes as Goldstone modes but this would be a wrong conclusion although there are similarities in the origin of these modes. For example both arise from a continuous symmetry of the Lagrangian of the system and of course both result in zero modes. One must stress the differences to avoid confusion, Goldstone modes for instance are massless and the zero frequency is the lower limit of a continuous spectrum. In the case of the zero modes associated with kink and soliton like solutions the zero mode comes from a discrete spectrum and the associated eigenvector has a mass given by the original solution, the kink in our example.

We note that these ideas readily generalise to more complicated systems and higher dimensions, for example one might have a rotational symmetry that leaves the potential unaltered for any rotation. In the case of additional rotational symmetry we would expect another zero mode resulting from this symmetry and of course we conclude that for each symmetry of the Lagrangian we would have a resultant zero mode for each when performing a quantum expansion around a non trivial state.

So the picture we are left with is that of a kink in the function space that specifies a local minimum of a potential. We expand around this local minimum and find a continuum of harmonic oscillator modes confined near to the minimum. Except when we try and expand around the kink solution along the equipotential curve we find that along this curve we cannot have harmonic type fluctuations and as a result we find no localisation of the wavefunction along the equipotential curve.

4.5 Dealing with the zero mode: Christ & Lee method

Here we shall outline the method of Christ & Lee [13] which deals with the zero mode problem that allows one to perform a perturbation theory around soliton and kink like solutions that is free of divergences. They developed a method to deal with zero modes in the canonical operator method which incorporates collective coordinates. In this method one introduces as many collective coordinates as there are zero modes in the problem, this then allows one to make a semiclassical expansion that is free from divergences caused by having normal modes present in the theory. One should be reminded that of course there will be other divergences like the infra-red and ultra-violet although these can be dealt with using normal ordering and a suitable renormalisation procedure.

They considered a Lagrangian density for the N component real scalar field ϕ^i, where the Lagrangian density is given by

$$L = -\frac{1}{2}\sum_i \left[\left(\frac{\partial \phi^i}{\partial t}\right)^2 - \left(\frac{\partial \phi^i}{\partial x}\right)^2 - g^{-2}V(g\phi^i) \right] \tag{4.5.1}$$

and the parameter g is the coupling constant. Naturally one would like to vary the above action to obtain the Euler-Lagrange equations, from these equations one can write down a classical solution known as a soliton. Generally the soliton solution can be written down in terms of the so-called collective coordinates,

$$[\phi^i]_{cl} = g^{-1}\phi_0^i(r, X_1(t), ..., X_K(t)), \tag{4.5.2}$$

one also notes that the time dependence has been moved to the collective coordinates. For example a single soliton in D-dimensional space will need D integration constants to specify each zero mode. To keep a feel for the problem, we could envisage the situation with one zero mode in one dimension, we could then think of $X_1(t)$ as the translation of the soliton center of mass $X(t)$ that leaves the potential invariant. These collective coordinates $X_1(t), ..., X_K(t)$ are introduced to account for the effect of the zero modes which as we will see below are left out of the expansion of the field operator which takes the form

$$\phi^i(r,t) = g^{-1}\phi_0^i(r, X_1(t), ..., X_K(t)) + \sum_{n\geq K+1}^{\infty} c_n(t)\eta_n^i(r, X_1(t), ..., X_K(t)), \tag{4.5.3}$$

$c_n(t)$ are just the expansion coefficients. From our previous discussion on zero modes we know that the eigenvectors responsible are of the form $\frac{\partial\phi_0^i}{\partial X_m}$, we can then say that the functions $\eta_n^i(r, X_1(t), ..., X_K(t))$ form a complete basis set subject to the condition

$$\sum_{i=1}^{N} \int d^D r \eta_n^i \frac{\partial \phi_0^i}{\partial X_m} = 0 \tag{4.5.4}$$

and the familiar orthonormality condition

$$\sum_{i}^{N} \int d^D r \eta_n^i \eta_m^i = \delta_{nm}, \tag{4.5.5}$$

where δ_{nm} is the Kronecker delta. To illustrate the use of this method we shall consider the $(1+1)$ version of the Lagrangian in Eq. (4.5.1) for a single component real scalar field with one zero mode due to translational invariance. In this case the field operator becomes

$$\phi(x,t) = \phi_0(x - X(t)) + g\sum_{n=1}^{\infty} c_n(t)\eta_n(x - X(t)). \tag{4.5.6}$$

Substitution of this expansion into the Lagrangian leads to the following expression

$$L = \sum_{n,m=0}^{\infty} \dot{u}_n D_{nm}(c_n)\dot{u}_m - M_{cl} - \frac{1}{2}\sum_{n=1}^{\infty} c_n^2\omega_n^2 \tag{4.5.7}$$

where we have defined $u_0 = X(t)$, $u_{n>1} = c_n$ and we have used the expression for the soliton classical rest energy or known as the classical soliton mass given by

$$M_{cl} = \frac{1}{g^2}\int d^D r\left[\frac{1}{2}\left(\frac{\partial\phi_0}{\partial x}\right)^2 + U(\phi_0)\right]. \tag{4.5.8}$$

The matrix elements D_{ij} follow straightforwardly from the substitution of the expansion in Eq. (4.5.6) into the Lagrangian density. The matrix elements of D_{ij} are all translationally invariant integrals and so do not contain the collective coordinate $X(t)$ as is the potential $V(\phi)$, we note that D is also symmetric. We can of course calculate the canonical momentum using

$$\pi_n = \frac{\partial L}{\partial \dot{u}_n} = \sum_m D_{nm}\dot{u}_m, \tag{4.5.9}$$

and then use this formula to write down the Hamiltonian. It is interesting to use the above formula to calculate the conjugate momenta to the collective coordinate X and in turn calculate the total field momentum using $P = -\frac{1}{g^2}\int \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x}dx$, a comparison of the formulas leads to the following relation

$$\pi_0 = D_{00}\dot{X} + \sum_{n=1}^{\infty} D_{0n}\dot{c}_n = P. \tag{4.5.10}$$

We note that the coordinate X is the center of mass coordinate for the total field and thus the conjugate momentum is indeed the total field momentum. Inverting the canonical relation in Eq. (4.5.9) we can obtain the velocities as functions of momentum and write down the classical Hamiltonian as

$$\begin{aligned} H = \; & \pi_0 \quad (D^{-1})_{00}\pi_0 + \sum_{m>0}\pi_0(D^{-1})_{0m}\pi_m + \sum_{n,m>0}\pi_n(D^{-1})_{nm}\pi_m \\ + \; & M_{cl} + \frac{1}{2}\sum_{n=1}^{\infty} c_n^2\omega_n^2. \end{aligned} \tag{4.5.11}$$

In writing down the Hamiltonian in this form we have used the fact that D_{nm} is a symmetric matrix. We are intersted in the low energy sector of the theory so we are prompted to expand up to zero order in the coupling constant g. We note that it is helpful to expand the 3rd term in the Hamiltonian into the diagonal piece $n = m$ and off-diagonal piece $n \neq m$. The diagonal piece does not contain g and the offdiagonal piece is a function of g to the positive power. In fact of the momentum terms the diagonal piece $(D^{-1})_{nn} = \delta_{nn}$, where n is greater than zero is the only one to have power of g lower than order one. So keeping terms that are only of order g^{-2}, g^{-1} and g^0 we write the Hamiltonian as

$$H = M_{cl} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\pi_n^2 + c_n^2\omega_n^2\right) + H_1, \tag{4.5.12}$$

where H_1 contains powers of the coupling constant but is independent of the collective coordinate X. If we neglect H_1, we can perform a simple quantisation procedure and promote the classical dynamical variables to operators that now obey the commutators

$$[c_n(t), \pi_m(t)] = \delta_{nm} \tag{4.5.13}$$

$$[c_n(t), c_m(t)] = [\pi_n(t), \pi_m(t)] = 0. \tag{4.5.14}$$

With the dropping of the term H_1, we recognise that this is just the many particle harmonic oscillator with energy given by

$$E = M_{cl} + \sum_{n=1}^{\infty}\left(N_n + \frac{1}{2}\right). \tag{4.5.15}$$

We have recovered a result that we could have achieved by not treating the zero mode, although we expect to this order in g that the zero mode would not present a problem. What has been achieved is the removal of the zero mode problem altogether as H_1 does not contain X and we have excluded the η_0 from our normal mode expansion. We notice that before neglecting terms higher order in g, the momentum P conjugate to X was explicit in the Hamiltonian, which from earlier discussion on zero frequencies was expected. One can in principle apply a standard perturbative scheme to H_1 although this would require a careful normal ordering of the Hamiltonian which we leave out here.

4.6 Quantising the Non-Linear Schrodinger dark soliton

In this section we shall look at a system relevant to cold atoms experiments which admits soliton solutions; in this section we will reinsert $\hbar$ to make the physical results transparent. We shall apply the Christ & Lee method from the previous section to the Lagrangian of the Non-Linear Schrodinger equation which is given below as

$$
\begin{aligned}
L &= \int dx \left[i\hbar\bar{\phi}\partial_t\phi - \frac{\hbar^2}{2m}|\partial_x\phi|^2 - g|\phi|^4 + \mu|\phi|^2 \right] && (4.6.1)\\
&= \int dx \left[i\hbar\bar{\phi}\partial_t\phi - V(\bar{\phi},\phi) \right], && (4.6.2)
\end{aligned}
$$

where in the second line we have written it in a form that relates to the previous section and we use bar to denote complex conjugation. Being first order in time derivative we expect some differences from the last section, for example the canonical momentum to ϕ is $i\hbar\phi$ and thus one does not need time derivatives to specify the trajectory of a field. We know that the equation of motion following from this Lagrangian admit soliton solutions and for our interest the one-soliton solution in particular. This Lagrangian also has some symmetries that we should discuss briefly as these will be the origin of the zero modes if we try a naive expansion around the solitonic solution. Firstly we have translational invariance such that $\phi(x)$ and $\phi(x - X)$ have the same value when the potential V is evaluated. We also notice that a global rotation leaves the Lagrangian invariant, such that when we multiply the field ϕ by the phase factor $e^{i\alpha}$ we maintain the aforementioned invariance. It will be these two symmetries that we take interest in and apply the method of collective coordinates to. First we will expand the fields into a new basis, where the expansion is given as

$$
\begin{aligned}
\phi(x,t) = \phi_s(x - X(t), \theta - \alpha(t)) \quad &+ \quad \sum_{n=2} c_n(t) u_n(x - X(t), \theta - \alpha(t)) \\
&+ \quad \sum_{n=2} c_n^\dagger(t) \nu_n^*(x - X(t), \theta - \alpha(t)),
\end{aligned}
\tag{4.6.3}
$$

where we have defined ϕ_s as the classical soliton solution and the collective coordinates associated with translation X and phase rotation α have been defined. Notice that the expansion starts from $n = 2$ as the first two modes that result in zero frequencies have been excluded. We now proceed to expand the potential in the Lagrangian in Eq. (4.6.1) up to second order in the expansion coefficients c_n and $c_n^\dagger$ as in the example of the Christ & Lee method. The term linear in expansion coefficents is zero due to the equation of motion and the second order term is $1/2 \int dx \delta\bar{\phi} \mathcal{L}[\bar{\phi}_s, \phi_s] \delta\phi$, where the operator $\mathcal{L}$ is defined by the following eigenvalue equation

$$
\mathcal{L} \begin{pmatrix} u_n \\ \nu_n \end{pmatrix} = \epsilon \begin{pmatrix} u_n \\ \nu_n \end{pmatrix}
\tag{4.6.4}
$$

this is known as the Bogoliubov-de Gennes equation where the non-Hermtian operator $\mathcal{L}$ is defined as

$$
\mathcal{L} = \begin{pmatrix} -\frac{\hbar^2}{2m}\partial_x^2 + 2g|\phi_s|^2 - \mu & +g\phi_s^2 \\ -g\bar{\phi}_s^2 & +\frac{\hbar^2}{2m}\partial_x^2 - 2g|\phi_s|^2 + \mu \end{pmatrix}.
\tag{4.6.5}
$$

For this operator if ϵ is a solution then so is $-\epsilon$,ϵ^* and $-\epsilon^*$, we also note that for a homogeneous system with constant density n_0, the functions u_n and ν_n will be plain waves and one obtains from the Bogoliubov-de Gennes equation, the Bogoliubov dispersion relation for a weakly interacting Bose gas. The solutions to Eq. (4.6.5) are the squared Jost functions obtained from studying perturbations within the Inverse Scattering Method, for example the modes that are oscillatory at infinity are given below as,

$$u = \frac{g\rho}{\sqrt{4\pi\xi}\epsilon_k} \mathrm{e}^{ikx}\mathrm{e}^{-i\Phi} \left[\left((k\xi)^2 + \frac{2\epsilon_k}{g\rho} \right) \left(\frac{k\epsilon}{2} + i \tanh(\frac{x-X}{\xi}) \right) + k\xi \mathrm{sech}^2(\frac{x-X}{\xi}) \right]$$
$$\nu = \frac{g\rho}{\sqrt{4\pi\xi}\epsilon_k} \mathrm{e}^{ikx}\mathrm{e}^{i\Phi} \left[\left((k\xi)^2 - \frac{2\epsilon_k}{g\rho} \right) \left(\frac{k\epsilon}{2} + i \tanh(\frac{x-X}{\xi}) \right) + k\xi \mathrm{sech}^2(\frac{x-X}{\xi}) \right] \quad (4.6.6)$$

and we have defined the total density ρ.

We note that in the limit $(x - X) \rightarrow \pm\infty$ in the above squared Jost solutions, the solutions are just plane waves that experience a phase shift of $\Delta\phi = 2\arctan(2/k\xi)$. The soliton's effect on the phonons is thus to impart a phase shift on to them, the remaining modes are the zero modes which are proportional to $\mathrm{sech}^2(x - X)$. The energy eigen values ϵ which form the continuous spectrum of the eigen modes in Eq. (4.6.6) is given below as

$$\epsilon_k = \sqrt{\hbar^2 c^2 k^2 + \left(\frac{\hbar^2 k^2}{2m} \right)^2}, \quad (4.6.7)$$

which is of course the typical Bogoliubov dispersion for excitations in a weakly-interacting Bose gas. The more interesting terms come from the substitution of the expansion in Eq. (4.6.3) into the time derivative of ϕ, these will result explicitly in the momentum conjugate to the zero mode coordinates. Combining the time derivative term with the expanded potential we obtain the following expression for the classical Lagrangian

$$L = L_s + \hbar N\dot{\alpha} + P\dot{X} + \sum_n \left[i\hbar c_n^* \dot{c}_n - \epsilon_n c_n^* c_n \right]. \quad (4.6.8)$$

We now take some time to explain the notation in the first two terms in the Lagrangian. P is the total field momentum which can be checked by calculating

$$P = \frac{i\hbar}{2} \int \left[\phi \partial_x \bar{\phi} - \bar{\phi} \partial_x \phi \right] dx = \frac{i\hbar}{2} \int \left[\sigma \partial_x \bar{\sigma} - \bar{\sigma} \partial_x \sigma \right] dx + \text{corrections}, \quad (4.6.9)$$

this is indeed the term one obtains from the Lagrangian multiplied by $\dot{X}$. The corrections are terms of first and second order in c_n and its complex conjugate, although relating total field momentum P to the momentum conjugate to the zero mode coordinate X holds to these orders. Thus after this consideration one is led to label the coefficient of $\dot{X}$ as P. Let us now discuss the coefficent of $\dot{\alpha}$ which we have written as N and denotes the total number of particles in the system. The number of particles is another integral of motion and is written down as the integral of the modulus squared of the field solution ϕ,

$$N = \int |\phi|^2 dx = \frac{i}{2} \int \left[\phi \partial_\theta \bar{\phi} - \bar{\phi} \partial_\theta \phi \right]. \tag{4.6.10}$$

Written in the second form one can easily identify the coefficent of $\dot{\alpha}$ as the total number of particles. Finally we observe the sum of harmonic oscillators in the final term of Eq. (4.6.8). Thus the Christ & Lee method has given us all the terms we would expect, we have the momenta conjugate to zero modes appearing explicitly and a sum over normal mode oscillators. We can now perform a Legendre transform to obtain the Hamiltonian written below as

$$H = E_s + \sum_n \epsilon_n c_n^* c_n, \tag{4.6.11}$$

as we might expect due to the firstoscillation order time derivative, there is a notable absence of the momentum terms and E_s is the classical energy of the soliton. One can now safely quantise this Hamiltonian as we have no sum over modes with zero frequencies, in fact we can impose the commutation relations $[\hat{c}_n, \hat{c}_m^\dagger] = \delta(n - m)$ thus elevating the c-numbers to operators. One then obtains from expanding around the dark soliton state the quantum Hamiltonian given as

$$\hat{H} = E_s + \sum_n \epsilon_n \hat{c}_n^\dagger \hat{c}_n. \tag{4.6.12}$$

4.7 Quantisation of dark soliton in a confining potential

In this final section will discuss a very important consideration viz, the quantisation of a single dark soliton confined by an external potential. Surprisingly we shall discover that technically speaking the trap turns out to be easier to deal with and will have a nice analogy with one dimensional Fermi systems. This work will also tie in neatly with the classical dynamics of a single dark soliton within a confining potential, which as we will find enters a moribund state that leads to the inevitable demise of the soliton. To set up this quantisation procedure we shall use the work of Pitaevskii et.al. [41] as our classical discription of a dark soliton undergoing harmonic oscillator type motion within an external traping potential; which will shall now motivate below.

In the present case of working with a confining potential we shall work within the Thomas-Fermi prescription which amounts to writing

$$gn(x) = \mu - U(x) \tag{4.7.1}$$

for the Bose gas density profile. We should stress that the trap does indeed present a fundamental difference in comparison to the homogeneous case which is the violation of translational invariance. Secondly we also note that the trap does not act on the soliton center of mass coordinate X directly.

We note that we have the condition $\mu - U(x) > 0$ which in turn defines the Thomas-Fermi radius R_{TF} by $U(R_{TF}) = \mu$. For a smooth potential and R_{TF} large in comparison to the soliton width we can allow the following substitution $n \rightarrow n(X)$ into the soliton parameters $\Phi(P, n)$, $N(P, n)$ and $E(P, n)$. This transformation allows us to write an effective Hamiltonian in the form below

$$H(P, X) = E(P, n(X)). \tag{4.7.2}$$

Due to the restriction on the solitons velocity we are left with the picture of the soliton being confined in the region $x < R_{TF}$. We will assume in the following that $U(x)$ has one minimum located at $x = 0$ and at this minimum the maximum energy the soliton can take is

$$E_0 = \frac{4}{3}\hbar c(0)n(0) = N_{tot}\frac{\hbar\omega}{\sqrt{2}}. \tag{4.7.3}$$

A soliton with this energy sits at the center of the trap with $V = 0$ with total phase drop $\Phi = \pm\pi$, it is this energy we label E_0. A soliton that is confined within the trap and that undergoes indefinite oscillations is a bound state of the system. This leads one to ask the natural question of how to write down the action variables of this classical problem; we use the formula below to answer this question,

$$I(E) = \frac{1}{2\pi}\oint P\left(E, X\right)dX. \tag{4.7.4}$$

We first notice that considering the action variable as a function of the maximum energy E_0 then $I(E_0) = 0$. This should be easy to see as the soliton corresponding to this state just sits at the center of the trap and thus corresponds to a single point in phase space. We note that a soliton with any other velocity by definition has to have lower energy; in fact we notice that the soliton is absent when $E = 0$ and for this energy the action variable takes its maximum value given by

$$I(0) = \frac{1}{2\pi}\int_{-R_{TF}}^{R_{TF}} 2\pi\hbar n(X)dX = \hbar N_{tot}. \tag{4.7.5}$$

The action variable I has the property that it is smooth and monotonically decreasing function between its maximum value at $E = 0$ and its minimum at $E = E_0$. These results are elegantly summarised in the phase space plot given in Figure 4.1. One can calculate the inverse period of oscillation using the property of the action variable viz, $dI/dE = \omega^{-1}$

which is the classical frequency.

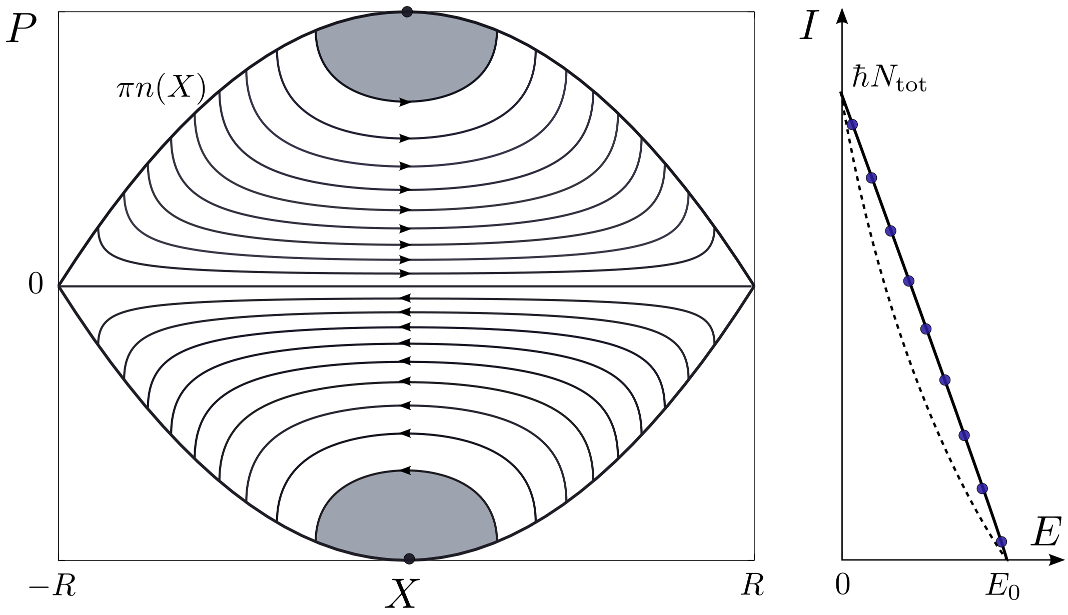

Figure 4.1: The first picture on the left illustrates classical phase space trajectories of a trapped grey soliton for energies between $E = 0$(the horizontal line with $P = 0$) and the singular trajectory for the stationary soliton at rest at the center of the trap with $E = E_0$(for $X = 0$ and $P = \pm\pi n_0$). The shaded area represents an area enclosed by a given classical orbit and it is the enclosed area that corresponds to the action variable given in Eq. (4.7.4). The graph to the right depicts the action variable $I(E)$ as a function of energy E, where the dashed line represents an arbitrary trapping potential $U(x)$ while the straight line corresponds to the harmonic potential. The circles denote quantised energy levels in correspondence with the Born-Sommerfeld rule.

Having evaluated the classical action variable and its phase space we can now look at applying the Born-Sommerfeld rule which uses the action variables with the modification that only a discrete number of bound states should exist, mathematically we write this condition down by modifying Eq. (4.7.4) and writing

$$I(E_n) = \frac{1}{2\pi} \oint P\left(E_n, X\right) dX = \hbar\left(n + 1/2\right) \tag{4.7.6}$$

where n belongs to the non-negative natural numbers. We see taking into account the classical properties of $I(E)$ there will be N_{tot} quantum states for any given potential $U(x)$ with one minimum at $x = 0$. Let us consider the particular case of a harmonic potential

$U(X) = \frac{1}{2}m\omega^2 X^2$, then the effective Hamiltonian in Eq. (4.7.2) describes simple harmonic motion with oscillation frequency $\omega/\sqrt{2}$. Thus we can write down the ladder of quantum energy states for the hamonically trapped soliton as

$$E_n = E_{max} - \left(\frac{\hbar\omega}{\sqrt{2}}\right)\left(n + \frac{1}{2}\right). \tag{4.7.7}$$

It is expected and interesting to note that the classical energy maximum $E_{n=0}$ has its energy reduced by zero point oscillations with energy $\hbar\omega/\sqrt{2}$. It is striking to compare this result with that of the Tonks-Giradeau limit, that of infinitely repulsive Bosons, which can be mapped onto non-interacting Fermions. One sees that in the Fermionic case the Lieb II mode is obtained by creating holes in the completely filled Fermi sea comprising of N_{tot} particles. The energy in this case is given by $E_n = E_F - \hbar\omega\,(n + 1/2)$ and again n is given by the non-negative integers up to N_{tot} corresponding to the fermi energy $E_F = N_{tot}\hbar\omega$.

An important comment is needed here regarding this quantisation procedure as we have neglected to point out how applicable this model of the Lieb II states in a trap is. In the trap the Lieb II states are not eigenstates but are actually narrow quasiparticle resonances with finite lifetime, the analysis of the decay due to the trap will give the constraints on the applicability of Eq. (4.7.7). We will show in the final chapter that the levels of the quantum trapped grey soliton are well defined for most energies and correspond to the Lieb II modes.

Chapter 5

DISSIPATIVE DYNAMICS OF TRAPPED GREY SOLITONS

This chapter will deal with dissipation dynamics of a grey soliton in a 1-dimensional Bose-Einstein condensate due to the presence of an integrability breaking term such as a confining potential. Theoretically the dissipative dynamics of grey solitons has been studied in some detail.

The influence on grey soliton dynamics due to the soliton interacting with a thermal cloud has been studied in [54] where it was found that the viscosity coefficient κ had a linear dependence on temperature, $\kappa \propto T/\mu$, in the regime $T \gg \mu$ where μ is the chemical potential. In reference [54] they considered highly elongated traps that are effectively 1-dimensional geometries, however they recognised that these 3-dimensional elongated traps are not integrable and that the soliton can scatter from thermal excitations. From these considerations they calculated the probability for an excitation to be scattered from the soliton with a given energy and momentum by solving the Bogoliubov-de Gennes equations and found the soliton life time to be given by the inverse of the viscosity coefficient, $\tau \sim (T/\mu)^{-1}$. This case is the classical case of the soliton scattering classical thermal excitations.

The quantum case has also been considered in reference [25], these authors considered the cold regime typified by $T \ll \mu$, here the long-wavelength phonons have to be treated as quantised objects. Again they considered the same system as in [54] where the integrability is broken by virtual excitations to higher transverse states. These virtual transitions to higher quantum transverse states leads to an effective three-body interaction with the coupling constant $\alpha = -12\ln(4/3)g^2/\hbar\omega_\perp$ [51]. It was found in [25] for the regime $T \ll \mu$ that the viscosity coefficient κ was proportional to $(T/\mu)^4$, where the constant of proportionality was also calculated.

Trapped solitons have been studied in the literature, where providing the confining potential is slowly varying the soliton energy obeys $dE_s/dt = -\kappa(dX/dt)^2$, [58], [40], [57], [56] and thus is found to dissipate. We shall now develop the equilibrium quasiparticle picture in [41] to include the radiation of phonons. In this chapter we shall arrive at an effective Lagrangian that models the coupling of the soliton collective coordinates to low energy phonons. We will find that this coupling to low energy phonons in conjunction with an harmonic trap that allows excitations to escape to spatial infinity leads to grey soliton dissipation.

5.1 Effective Lagrangian

We start with the standard Lagrangian for weakly interacting 1-dimensional bosons, which in the density-phase representation takes the following form,

$$L = \int dx\, \left[\phi\, \partial_t n - E(n,\phi)\right]. \tag{5.1.1}$$

Here $n(x,t)$, $\phi(x,t)$ are density and phase fields and the energy density is

$$E(n,\phi) = \frac{1}{2m}\left(\left(\partial_x\sqrt{n}\right)^2 + n\left(\partial_x\phi\right)^2\right) + \frac{g}{2}(n-n_0)^2, \tag{5.1.2}$$

where m is the particle mass and we note that we are interested in a repulsive weakly interacting gas, thus $g > 0$. In Eq. (5.1.2) we have substracted the constant contribution of static background density n_0 as we discussed when considering the renormalised integrals of motion such as the energy in Eq. (3.3.26). Variation with respect to the fields gives the equations of motion.

$$\partial_t n = \frac{\partial E}{\partial \phi} = -\partial_x \left(\frac{n}{m} \partial_x \phi \right) \tag{5.1.3}$$

$$-\partial_t \phi = \frac{\partial E}{\partial n} = -\frac{1}{2m\sqrt{n}} \partial_x^2 \sqrt{n} + \frac{1}{2m} \left(\partial_x \phi\right)^2 + g(n - n_0) \tag{5.1.4}$$

This is the hydrodynamic form of the Nonlinear Schrödinger Equation (NLSE). In addition to Bogoliubov modes which become phonon-like excitations for small momenta and energy, Eqs. (5.1.3), (5.1.4) allow for a family of dark solitons whose density $n_{\mathrm{s}}(x - Vt; \Phi)$ and velocity $u_{\mathrm{s}}(x - Vt; \Phi) = (\partial_x \phi_{\mathrm{s}})/m$ are parametrised by the total phase drop Φ. Following Tsuzuki [73] we write the grey soliton density and velocity profiles in the following form

$$n_{\mathrm{s}}(z; \Phi) = n_0 \left(1 - \frac{\sin^2(\Phi/2)}{\cosh^2\left(zmc\sin(\Phi/2)\right)} \right); \tag{5.1.5}$$

$$u_{\mathrm{s}}(z; \Phi) = V \left(1 - \frac{n_0}{n_{\mathrm{s}}(z; \Phi)} \right). \tag{5.1.6}$$

Substitution of these solutions into Eqs. (5.1.3), (5.1.4) gives the relation between the phase drop and the velocity of the soliton

$$V/c = \cos(\Phi/2). \tag{5.1.7}$$

Here c is the sound velocity obtained from the thermodynamic relation $mc^2 = n_0 \left(\partial \mu / \partial n_0\right) = gn_0$. Eq.(5.1.7) reflects the fact that the velocity of the soliton cannot exceed the sound velocity.

Following Ref.[41] we wish to establish a description of a soliton as an effective particle. To this end we use position $X(t)$ and phase $\Phi(t)$ of the soliton as dynamical variables and substitute $n_s(x-X;\Phi)$, $u_s(x-X;\Phi)$ into the Lagrangian (5.1.1).

$$L_s = P_s(\Phi)\dot{X} - E_s(\Phi), \tag{5.1.8}$$

where we have used the soliton momentum and energy

$$P_s(\Phi) = n_0(\Phi - \sin\Phi), \qquad E_s(\Phi) = \frac{4}{3}cn_0|\sin(\Phi/2)|^3. \tag{5.1.9}$$

To obtain these expressions it is important to substract the contribution of a flat background wih the same total current $n_0\Phi$ and total number of particles $N_s(\Phi)$ as we have explained in chapter three.

Equations of motion for the soliton parameters follow from variation of the action with the Lagrangian given by Eq. (5.1.8). The variation with respect to phase leads to condition $\dot{X}\partial_\Phi P_s = \partial_\Phi E_s$, which provides a relation between the phase and the velocity of the soliton

$$V = \partial_\Phi E_s/\partial_\Phi P_s = c\cos(\Phi/2). \tag{5.1.10}$$

The variation with respect to X results in condition $\Phi = \mathrm{const}$ reflecting the constant velocity and momentum of the soliton in the uniform background.

As in Ref.[41] we assume the background to be slightly non-uniform, given by the Thomas-Fermi density profile

$$gn_0(x) = mc^2(x) = \mu - U(x), \tag{5.1.11}$$

corresponding to the external confining potential $U(x)$. Here $c(x)$ is the local sound velocity. We assume that the parameters such as density and sound velocity change smoothly on the typical length scale of the soliton given by the healing length $\xi = 1/mc$. In this case one can use the solution given by Eqs. (5.1.5), (5.1.6), which depend on the background density

n_0 as a parameter and substitute its value *in the vicinity* of the soliton $n_0 \to n_0(X)$. This results in position-dependent momentum and energy in Eq. (5.1.9). The first equation of motion remains unchanged by this modification ,

$$\frac{\partial P_s}{\partial \Phi}\left(\dot{X} - V\right) = 0 \tag{5.1.12}$$

resulting from a variation of the Lagrangian (5.1.8), this leads one to the same conclusion as in the homogeneous case, viz $\dot{X} = V$. In the Thomas-Fermi approximation the Lagrangian (5.1.8) now becomes a function of X as well as $\dot{X}$, which results in a change in the second equation of motion. The second dynamical equation now has the form

$$\frac{dP_s}{dt} = \frac{\partial P_s}{\partial X}\frac{dX}{dt} - \frac{\partial E_s}{\partial X} \tag{5.1.13}$$

which reduces to

$$\frac{\partial P_s}{\partial \Phi}\dot{\Phi} = -\frac{\partial E_s}{\partial X}, \tag{5.1.14}$$

where we have used $\frac{dP_s}{dt} = \partial_\Phi P_s \dot{\Phi} + \partial_X P_s \dot{X}$. Equation (5.1.14) gives the slow evolution of the phase $\Phi(t)$ as a result of the non-uniform background.

Combining these equations one can show that the total energy of the soliton during the evolution remains constant,

$$\frac{dE_{\mathrm{s}}}{dt} = \dot{X}\,\partial_X E_{\mathrm{s}} + \dot{\Phi}\,\partial_\Phi E_{\mathrm{s}} = \left(\dot{X} - V\right)\partial_X E_{\mathrm{s}} = 0 \tag{5.1.15}$$

and can be used to integrate equations of motion. Following [41] we use Eq. (5.1.10) and the second equation of (5.1.9) to show that $c^2(X) - V^2$ is conserved. Multiplying it by m and employing the Thomas-Fermi relation Eq. (5.1.11) leads to an *effective* Hamiltonian

$$E_{\mathrm{eff}}(V, X) = mV^2 + U(x) - \mu \tag{5.1.16}$$

to be distinguished from the soliton energy E_{s}. Eq. (5.1.16) describes the soliton as an effective particle with mass $2m$ moving in the potential $U(X)$. Due to the positiveness of $c^2(X) - V^2$ the energy E_{eff} is always negative which restricts the motion to the interval where $U(X) < \mu$, *i.e.* inside the atomic cloud.

5.2 Interactions with phonons

The presented picture of solitons as a Landau quasiparticle must be complemented with its interactions with the phonons. Indeed, a soliton accelerating in smooth non-uniform density profile is analogous to an electron moving in a static electric field created by some external sources. It is well known that its acceleration leads to radiation of electromagnetic waves taking with it momentum and energy of the electron known as the phenomenon of *Bremsstrahlung* [43]. In the case of soliton it is the phonons which are emitted which leads to eventual loss of energy and momentum of the soliton. Due to the negative effective mass of the soliton we speak rather of *Beschleunigungsstrahlung* also known as acceleration radiation.

To demonstrate this we consider a combined system of soliton and phonons

$$L_{\mathrm{eff}} = L_{\mathrm{s}} + L_{\mathrm{s-ph}} + L_{\mathrm{ph}} \tag{5.2.1}$$

where the last term in the Lagrangian describes the low-energy phonons in the quadratic harmonic approximation [61] with $\hbar$ set to one for the duration of the calculation,

$$L_{\mathrm{ph}} = \int dx \left[-\rho \partial_t \varphi - \frac{mc^2}{2n_0} \rho^2 - \frac{n_0}{2m} \left(\partial_x \varphi \right)^2 \right] \tag{5.2.2}$$

which depends on slow components of density $\rho(x,t)$ and phase $\varphi(x,t)$. To determine the form of the second term in Eq. (5.2.1) describing the interaction of the soliton with slow phononic modes we consider the energy of the soliton, Eqs. (5.1.9) as a function of the *local* background density $n_0 + \rho(X,t)$ and velocity $u(X,t) = \partial_x \varphi(X,t)/m$ and expand to the first order

$$E_{\rm s} \to E_{\rm s}(n_0 + \rho(X,t), u(X,t)) = E_{\rm s} + \left(\frac{\partial E_{\rm s}}{\partial n_0}\right)_P \rho(X,t) + Pu(X,t) \tag{5.2.3}$$

The coefficient of the velocity expansion follows from the Galilean invariance [4] and the fact that the soliton has zero bare mass. It is important to keep the total momentum P constant in these calculations in order to preserve the Lagrangian structure in (5.1.8). The last two terms in Eq. (5.2.3) represent the desired soliton-phonon interactions and the negative of their sum equals to $L_{\rm s-ph}$ in Eq. (5.2.1).

Consider the state of the soliton for a given value of P. Its motion is coupled with phonons through the linear terms $L_{\rm s}$ and introduces source (current) terms in the wave equation for phonons. Variation with respect to phononic fields yields:

$$\partial_t \varphi + \frac{mc^2}{n_0}\rho = -\left(\frac{\partial E_{\rm s}}{\partial n_0}\right)_P \delta(x - X) \tag{5.2.4}$$

$$\partial_t \rho + \frac{n_0}{m}\partial_x^2 \varphi = -P\delta'(x - X), \tag{5.2.5}$$

where δ' is to be interpreted as differentiating the Dirac delta function with respect to its argument. These are inhomogeneous linear equations. Following the standard prescription we seek a solution as a sum of a solution of the homogeneous system and a particular solution ρ_0, φ_0. Assuming the soliton is moving with the constant velocity $X = Vt$ the latter is

$$\rho_0(x,t) = -R\delta(x - Vt), \qquad \varphi_0(x,t) = -F\theta(x - Vt) \tag{5.2.6}$$

where the constants R, F satisfy

$$\frac{mc^2}{n_0}R - VF = \left(\frac{\partial E_{\rm s}}{\partial n_0}\right)_P, \qquad n_0 F - mVR = P \tag{5.2.7}$$

For a soliton governed by the Nonlinear Schrödinger equation it is possible to show, see [58], that these relations are identical to those for the total number of particles expelled from the soliton, N_s and the total phase drop Φ if one identifies $R = N_s$ and $F = \Phi$. For a more general case, Eqs. (5.2.7) must be regarded as the definitions of parameters R and F through the dispersion of the soliton $E_{\rm s}(P, n_0)$.

We deal with the time-dependent situation which arises, for example, in the non-uniform background due to some confinement potential $U(x)$. It is convenient to redefine phononic fields by excluding the particular solution (zero mode) given by Eqs. (5.2.6). The coefficient $F = \Phi(t)$ should be regarded as a time dependent parameter of the soliton with $R = N_{\rm s}(\Phi) = (2n_0/mc)\sin(\Phi(t)/2)$ fixed by the one-parameter structure of the solitonic solution. To preserve the canonical structure of the equations of motion, the momentum P has to be redefined accordingly. We have therefore the following (gauge) transformation,

$$\begin{aligned} \rho(x,t) &\rightarrow \rho(x,t) - R\,\delta(x-X) \\ \varphi(x,t) &\rightarrow \varphi(x,t) - F\,\theta(x-X) \\ P &\rightarrow P + R\,\partial_x\varphi(X,t) + F\rho(X,t). \end{aligned} \tag{5.2.8}$$

Substituting new coordinates into Eqs. (5.2.1), (5.1.8), (5.2.3), (5.2.2) and collecting terms that are first order in $\varphi(x,t)$ and $\rho(x,t)$ leads to the full interacting Lagrangian

$$L = P_{\rm s}(\Phi)\dot{X} - E_{\rm s}(\Phi) - \dot{N}_{\rm s}\,\varphi(X,t) - \frac{1}{\pi}\dot{\Phi}\,\vartheta(X,t) + L_{\rm ph}[\vartheta,\varphi], \tag{5.2.9}$$

where we have introduced the displacement field $\vartheta(x,t) = \int^x \rho(y,t)dy$ also known as the dual field in the literature; a more general derivation is given in appendix b. In deriving the Lagrangian in Eq. 5.2.9 we collected certain terms as illustrated below,

$$N\partial_t\varphi(X,t) + N\dot{X}\partial_X\varphi(X,t) = N\frac{d}{dt}\varphi(X,t) \tag{5.2.10}$$

$$\frac{\Phi}{\pi}\partial_t\vartheta(X,t)+\frac{\Phi}{\pi}\dot{X}\partial_X\vartheta(X,t)=\frac{\Phi}{\pi}\frac{d}{dt}\vartheta(X,t), \tag{5.2.11}$$

and used the time integration from the action to convert $-\frac{\dot{\Phi}}{\pi}\vartheta(X,t)$ into $\frac{\Phi}{\pi}\partial_t\vartheta(X,t)$. All other terms were constant terms which have no effect on the dynamics of the soliton or phonons. We schematically represent this combination of phonons interacting with an oscillating soliton in Figure 5.1.

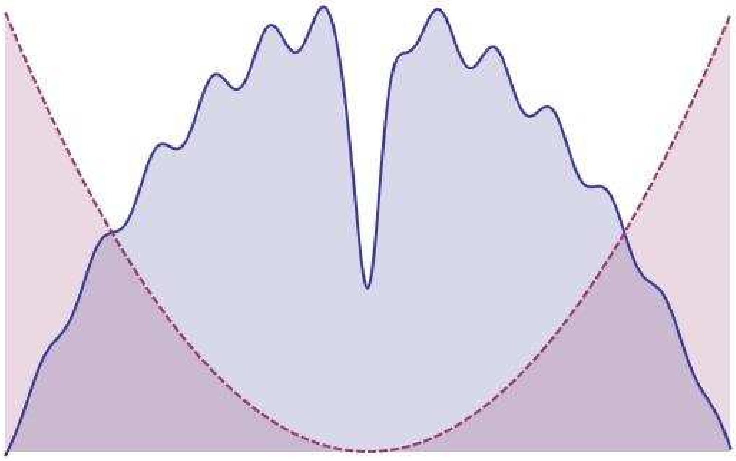

Figure 5.1: A schematic diagram depicting the modelling of the phonons interacting with the soliton.

In Fourier space the phononic action reads as,

$$\mathcal{S}_{\text{ph}}=\int L_{\text{ph}}[\rho,\varphi]\,dt=\frac{1}{2\pi}\sum_{q,\omega}\left(\varphi^*\left(q,\omega\right),\vartheta^*\left(q,\omega\right)\right)\begin{pmatrix}-q^2cK & q\omega\\ q\omega & -q^2c/K\end{pmatrix}\begin{pmatrix}\varphi\left(q,\omega\right)\\ \vartheta\left(q,\omega\right)\end{pmatrix}. \tag{5.2.12}$$

5.3 Induced soliton stability

In this section we consider the situation when evaporative cooling is not performed. This leads us to model the phonons in a finite size system, from which one would deduce that the soliton could be dissipation free due to reabsorption processes.

We will work in the local-density approximation and study a Bose gas confined by an harmonic trap. It is known from the work of Stringari and Menotti [52] that the phonon modes are the Legendre polynomials and are solutions to,

$$-\omega^2 \rho(x) = \frac{1}{m}\partial_x \left[n(x)\, \partial_x \left(\frac{\partial \mu}{\partial n} \rho(x) \right) \right]. \tag{5.3.1}$$

It is more useful for our description to work with the dual field $\vartheta(x,t)$, for which the above equation reduces to

$$-\omega^2 \vartheta(x,t) = \frac{n(x)}{m}\partial_x \left[\frac{\partial \mu}{\partial n} \partial_x \vartheta(x) \right]. \tag{5.3.2}$$

Let us begin with the Luttinger liquid action and derive an action whose equation of motion is that which is obtained from the perturbation procedure in [52].

$$L = \int dx \left[-\partial_x \vartheta \partial_t \varphi - \frac{\nu_N(x)}{2} (\partial_x \vartheta)^2 - \frac{\nu_J(x)}{2} (\partial_x \varphi)^2 \right] \tag{5.3.3}$$

Where $\nu_N(x) = \partial\mu/\partial n$ and $\nu_J(x) = n/m$. To proceed we vary the action with respect to the phase field, this procedure leads to the continuity equation given in the form,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[\nu_J(x) V\right] = 0 \tag{5.3.4}$$

where $V = \partial_x \varphi$, for our means we can write the continuity equation in the more useful form

$$\partial_t \vartheta = -\nu_J(x) \partial_x \varphi \tag{5.3.5}$$

Substituting this expression for $\partial_x \varphi$ into the Luttinger action yields the following action for the dual field dynamics

$$L = \frac{1}{2}\int dx \left[\frac{\dot{\vartheta}^2}{\nu_J(x)} - \nu_N(x) (\partial_x \vartheta)^2 \right]. \tag{5.3.6}$$

Finally varying this action one obtains the desired result for the equation governing the dynamics for the dual field from Eq. (5.3.2).

We would now like to diagonalise the Lagrangian in Eq. (5.3.6). To achieve this, it is useful to write the Lagrangian in the following form,

$$L = \int dx \dot{\rho}\varphi - H, \tag{5.3.7}$$

where the Hamiltonian is given by

$$H = \int dx \left[\frac{\nu_N(x)}{2} (\partial_x \vartheta)^2 + \frac{\nu_J(x)}{2} (\partial_x \varphi)^2 \right]. \tag{5.3.8}$$

Following the hydrodynamic prescription we would like the operators ρ and φ to satisfy the following commutation relations

$$[\varphi(x), \rho(x')] = i\delta(x - x') \tag{5.3.9}$$

evaluated at equal time. We expand the density and phase operators in the trap basis with the constraint that they must satisfy the above commutation relations, this leads to the following expansions

$$\varphi = \frac{i}{\sqrt{2}} \sum_n [A_n^{-1} P_n(x) \hat{b}_n(t) - h.c] \tag{5.3.10}$$

$$\rho = \frac{1}{\sqrt{2}} \sum_n [(A_n) P_n(x) \hat{b}_n(t) + h.c] \tag{5.3.11}$$

where in the second equation B_n became $(A_n)^{-1}$ in order for the commutation relations to be preserved. One should also note that the trap expansion coefficients satisfy $[b_n, b_m^\dagger] = \delta_{nm}$. The above expansions can now be substituted into the rescaled Luttinger action to obtain

$$L = \sum_n \frac{2}{2n+1} b_n^\dagger [\omega_n - R_{TF} c_0 \Omega_n] b_n \tag{5.3.12}$$

where the trap expansion coefficient was chosen to be $A_n = \sqrt{\frac{c_0\Omega_n}{g}}$ to diagonalise the lagrangian.

$$L_{coupling} = \frac{\dot{\Phi}}{\sqrt{2}}\sum_n \sqrt{\frac{c_0\Omega_n}{g}}\frac{P_{n+1}(X) - P_{n-1}(X)}{2n+1}(b_n + b_n^\dagger) \tag{5.3.13}$$

The above expression comes from the relation between density and the dual field $\rho = 1/\pi\partial_x\vartheta$ which leads one to the expression

$$\vartheta(X) = \frac{1}{\sqrt{2}}\sum_n \sqrt{\frac{c_0\Omega_n}{g}} \underbrace{\int_{-1}^{X} P_n(x)dx}_{\frac{P_{n+1}(X)-P_{n-1}(X)}{2n+1}} [b_n + b_n^\dagger] \tag{5.3.14}$$

Integrating out the trap phonons yields the following effective action

$$S_{eff} = \int dtdt' \frac{K}{4} f_n(X)\frac{\Omega_n}{R_{TF}c_0\Omega_n - \omega_n}\dot{\Phi}(t)\dot{\Phi}(t') \tag{5.3.15}$$

where $f_n(X) = (P_{n+1}(X) - P_{n-1}(X))^2/(2n+1)$, the numerator is bounded above by 1 and below by 0 thus it behaves as $1/(2n+1)$.

5.4 Dissipative dynamics for Thomas-Fermi grey soliton

To start our discussion of non-equilibrium dynamics we shall describe the evolution of the dynamic fields on the Schwinger-Keldysh time contour, of which details can be found in Kamenev's exposition on this technique . Here we shall follow Kamenev's convention especially due to it naturally leading to the equation of motion for the system of study, details are given in appendix A. Let us consider the equilibrium many body partition function for our system, written down as follows,

$$\mathcal{Z} = \int \mathcal{D}[\theta(x,t)]\mathcal{D}[\varphi(x,t)]\mathrm{e}^{iS}, \tag{5.4.1}$$

where we have defined the action S as

$$S = \int dt\{L_0 - L_{int} + L_{ph}\}. \tag{5.4.2}$$

It is this action that describes the interaction between the grey soliton and phononic excitations in the trapped bose gas. Let us investigate the interaction part of the action a little closer and make some general comments on its form, which explicitly can be written as

$$L_{int} = -\int dx\{\dot{N}_s\delta(x-X)\varphi(x,t) + \frac{1}{\pi}\dot{\Phi}\delta(x-X)\theta(x,t)\}. \tag{5.4.3}$$

We see that the interaction with the phonon background creates two potential sources located at the position of center of mass for the soliton. Coupling of a linear nature can be dealt with easily in field theory when the phonon lagrangian is quadratic. However we are interested in the non-equilibrium nature of the problem and thus we go over to the Keldysh treatment. We have written the interaction explicitly above to elude to the fact that when one integrates out phonons the interaction is local. This does lead to a different solution to when one would naively integrate with no delta function.

Thus we proceed by extending the time contour to a closed contour followed by a Keldysh rotation on all dynamic variables in the problem resulting in the following form for the partition function,

$$\mathcal{Z} = \int \mathcal{D}[\theta_{cl}(x,t), \theta_q(x,t)]\mathcal{D}[\varphi_{cl}(x,t), \varphi_q(x,t)]\mathrm{e}^{iS} \tag{5.4.4}$$

where the action is now a function of the classical and quantum components.

Below we write our action for the interaction of solitonic coordinates interacting with phononic degrees of freedom on the Schwinger-Keldysh contour,

$$S_{int} = \int_{-\infty}^{\infty} dt\,[L_{int}(P_+, n(X_+)) - L_{int}(P_-, n(X_-))] \tag{5.4.5}$$

Next one would like to make the linear transformation to the quantum-classical discription, to set the notation we shall define the quantum-classical notation for the following fields,

$$P_{\pm} = P_{cl} \pm \frac{1}{2}P_q \quad X_{\pm} = X_{cl} \pm \frac{1}{2}X_q \tag{5.4.6}$$

$$\psi_{\pm} = \psi_{cl} \pm \frac{1}{2}\psi_q \quad \theta_{\pm} = \theta_{cl} \pm \frac{1}{2}\theta_q. \tag{5.4.7}$$

The terms that are purely classical vanish due to the cancelling of the backwards branch against the forwards branch and thus are symmetric. We would now like to expand this expression to first order in quantum components, this will yield corrections to the classical equilibrium state,

$$\begin{aligned} S_{int} = \int dt \Big[& N(P_{cl}, n)\dot{\phi}_q + \partial_P N_{cl}\dot{\phi}_{cl} P_q + \partial_X N_{cl}\dot{\phi} X_q \\ & + \frac{1}{\pi}\Phi(P_{cl}, n)\dot{\phi}_q + \frac{1}{\pi}\partial_P \Phi_{cl}\dot{\phi}_{cl} P_q + \frac{1}{\pi}\partial_X \Phi_{cl}\dot{\phi} X_q \\ & N(P_{cl}, n)\partial_X \dot{\phi}_q X_q + \frac{1}{\pi}\Phi(P_{cl}, n)\partial_X \dot{\theta}_q X_q \Big] \end{aligned} \tag{5.4.8}$$

The expression above is rather cumbersome, some progress is made in going to a vector notation which compactifies the action and makes the proceeding calculation easier. Let us below define the vectors that we will need, it is clear that the best way to proceed is to define the following two component vectors,

$$\dot{\Lambda}_{cl/q} = \begin{pmatrix} \dot{\varphi}_{cl/q} \\ \dot{\theta}_{cl/q} \end{pmatrix}, \qquad \dot{S}_{cl/q} = \begin{pmatrix} \dot{N}_{cl/q} \\ \frac{1}{\pi}\dot{\Phi}_{cl/q} \end{pmatrix}. \tag{5.4.9}$$

Using these vector definitions we can rewrite the action in the following form,

$$S_{int} = \int dt \left[S_{cl}^{\dagger} \cdot \dot{\Lambda}_q + \partial_P S_{cl}^{\dagger} \cdot \dot{\Lambda}_{cl} P_q + \partial_X S_{cl}^{\dagger} \cdot \dot{\Lambda}_{cl} X_q - \dot{S}_{cl}^{\dagger} \cdot \partial_X \Lambda_{cl} X_q \right]. \tag{5.4.10}$$

We are interested in an effective theory for the dark soliton dynamics due to its interaction with the phonon bath, thus we would like to integrate out the phonon degrees of freedom. In this calculation we note that the phonon dynamics are governed by the Luttinger action which is quadratic and thus integrating out the phonons amounts to calculating $\exp\left[\frac{i}{2}\langle S_{int}^2\rangle\right]$, which is exact for a quadratic action. It turns out that using this formula is easier to deal with and keep track of coefficients. Before we write down the result of performing this average we should write down definitions of the Greens functions that we will need, which are defined as the following averages,

$$-i\langle\Lambda_{cl}(x,t)\Lambda_{cl}^{\dagger}(x',t')\rangle = \chi^K(x-x',t-t') \tag{5.4.11}$$

$$-i\langle\Lambda_{q}(x,t)\Lambda_{cl}^{\dagger}(x',t')\rangle = \chi^A(x-x',t-t') \tag{5.4.12}$$

$$-i\langle\Lambda_{cl}(x,t)\Lambda_{q}^{\dagger}(x',t')\rangle = \chi^R(x-x',t-t') \tag{5.4.13}$$

$$-i\langle\Lambda_{q}(x,t)\Lambda_{q}^{\dagger}(x',t')\rangle = 0. \tag{5.4.14}$$

We now perform the averaging process equiped with the definitions made in Eq.5.4.11 and explicitly write down the resulting expression which will need to be manipulated into a workable form. The resulting action post averaging is given below as,

$$\begin{aligned}\frac{i}{2}\langle S_{int}^2\rangle = \int dtdt' \Bigg[& \dot{S}^{\dagger}(t)\frac{d}{dt'}\chi^A(X-X',t-t')\partial_P S(t')P_q(t') \\ & + \partial_P S^{\dagger}(t)P_q(t)\frac{d}{dt}\chi^R(X-X',t-t')\dot{S}(t') \\ & + \dot{S}^{\dagger}(t)\frac{d}{dt'}\chi^A(X-X',t-t')\partial_X S(t')X_q(t') \\ & + \partial_X S^{\dagger}(t)X_q(t)\frac{d}{dt}\chi^R(X-X',t-t')\dot{S}(t') \\ & + \dot{S}^{\dagger}(t)\partial_{X'}\chi^A(X-X',t-t')\dot{S}(t')X_q(t') \\ & + \dot{S}^{\dagger}(t)\partial_X\chi^R(X-X',t-t')\dot{S}(t')X_q(t)\Bigg],\end{aligned} \tag{5.4.15}$$

as mentioned the solution is not of a very desirable form and thus needs simplifying. The ordering of terms in the above expression has been very much deliberate; as we will show, the first two terms can be combined in a neat way as can the following pairs. One has to make the following redefinition of time, $t \to t'$ and $t' \to t$ in each term in Eq. (5.4.15) involving the advanced Greens function. We then use the following property of the two by two matrix Greens function $\chi^A = \left(\chi^A\right)^T$, ie it is a symmetric matrix to obtain the more satisfying form for the effective action as,

$$\begin{aligned}\frac{i}{2}\langle S_{int}^2\rangle = \frac{1}{2}\int dtdt' \Big[&\partial_P S^\dagger(t) P_q(t)\frac{d}{dt}\left(\chi^R(X-X',t-t') - \chi^A(X-X',t-t')\right)\dot{S}(t') \\ &+ \partial_X S^\dagger(t) X_q(t)\frac{d}{dt}\left(\chi^R(X-X',t-t') - \chi^A(X-X',t-t')\right)\dot{S}(t') \\ &+ \dot{S}^\dagger(t)\partial_X\left(\chi^R(X-X',t-t') - \chi^A(X-X',t-t')\right)\dot{S}(t')X_q(t)\Big]\,.\end{aligned} \tag{5.4.16}$$

In deriving the above action we have neglected the Keldysh components, as we want the zero temperature description of the soliton dynamics. We have also been careful in only keeping terms that are linear in quantum components, considering higher orders of these components would lead to quantum corrections to the action. It is interesting to note that it is the spectral function that appears in the dissipative action, in fact derivatives of this function. To proceed we need to calculate the explicit form of the correlation functions, which we shall work though in the next section.

5.4.1 Phonon Dynamics

In this section we consider the Luttinger liquid action, for which we can perform the functional integral exactly owing to its quadratic structure. Written down in real space and time the Green's function looks like

$$\chi^R_{tt'} = \begin{pmatrix} \langle\theta_q(X,t)\,\theta_{cl}(X',t')\rangle & \langle\theta_q(X,t)\,\varphi_{cl}(X',t')\rangle \\ \langle\varphi_q(X,t)\,\theta_{cl}(X',t')\rangle & \langle\varphi_q(X,t)\,\varphi_{cl}(X',t')\rangle \end{pmatrix} \tag{5.4.17}$$

We consider the inverse of the Green's function in Fourier space which is given in the form

$$\chi_{q\omega}^{-1} = \frac{1}{\pi}\begin{pmatrix} -q^2cK & q\omega \\ q\omega & -q^2c/K \end{pmatrix}, \tag{5.4.18}$$

where we have defined the Luttinger parameter, $K = \pi\hbar n/mc$. We are interested in the Green's funtion itself so we invert this matrix to find

$$\chi_{q\omega} = \frac{\pi}{q\left(\omega^2 - q^2c^2\right)}\begin{pmatrix} qc/K & \omega \\ \omega & qcK \end{pmatrix}, \tag{5.4.19}$$

it turns out to be very useful to write the Green's function in the following form

$$\chi_{q\omega} = \frac{\pi}{2cq^2}\left[\frac{1}{\omega - qc} - \frac{1}{\omega + cq}\right]\begin{pmatrix} qc/K & \omega \\ \omega & qcK \end{pmatrix}. \tag{5.4.20}$$

From this expression for the full correlator we can identify the retarded and advanced correlators as

$$\chi^R(q,\omega) = \chi(q,\omega + i0) \tag{5.4.21}$$

$$\chi^A(q,\omega) = \chi(q,\omega - i0). \tag{5.4.22}$$

The Infinitesimal shift off the real line results in the correct time ordering for retarded and advanced Green's functions. We recall that the Green's functions only appear in the combination of time or spatial derivative of the difference $\chi^R - \chi^A$ in Eq. (5.4.16). Thus we will proceed in calculating this quantity which will allow us to then substitute this expression back into Eq. (5.4.16). We calculate the difference between retarded and advanced correlators as

$$\chi^R_{q,\omega} - \chi^A_{q,\omega} = \frac{i(2\pi)^2}{4cq^2}\left[\delta\left(\omega + cq\right) - \delta\left(\omega - qc\right)\right] \begin{pmatrix} qc/K & \omega \\ \omega & qcK \end{pmatrix}, \tag{5.4.23}$$

where we have used the formula defined in [1]. Let us now Fourier invert this expression back to real space and time, the ω integral is a trivial integration over delta functions and leads to the expression

$$\left(\chi^R - \chi^A\right)_{y,\tau} = \frac{i}{4}\int \frac{dq}{q}\left[\mathrm{e}^{iq(V+c)t}\begin{pmatrix} \frac{1}{K} & -1 \\ -1 & K \end{pmatrix} - \mathrm{e}^{iq(V-c)t}\begin{pmatrix} \frac{1}{K} & 1 \\ 1 & K \end{pmatrix}\right]. \tag{5.4.24}$$

where we have put the phonons on the equations of motion such that $X - X' \approx V(t-t') = Vt$. One now deals with the spatial and time derivatives of Eq. (5.4.24), note that $\partial_X = \frac{1}{V\pm c}\partial_t$, the $+/-$ is dependent on which term in Eq. (5.4.24) we act ∂_X on. We obtain the following,

$$\begin{aligned} \partial_X\left(\chi^R - \chi^A\right)_{y,t} &= -\frac{1}{4}\int \frac{dq}{q}\left[\mathrm{e}^{iqt}\frac{1}{V+c}\begin{pmatrix} \frac{1}{K} & -1 \\ -1 & K \end{pmatrix} + \mathrm{e}^{iqt}\frac{1}{c-V}\begin{pmatrix} \frac{1}{K} & 1 \\ 1 & K \end{pmatrix}\right] \\ &= -\pi\delta(t)\frac{1}{c^2 - V^2}\begin{pmatrix} \frac{c}{K} & V \\ V & cK \end{pmatrix}. \end{aligned} \tag{5.4.25}$$

In a similar vein we can can calculate the time derivative of the zero temperature spectral function which is given below as

$$\partial_t\left(\chi^R - \chi^A\right)_{y,t} = -\pi\delta(t)\begin{pmatrix} \frac{1}{K} & 0 \\ 0 & K \end{pmatrix}. \tag{5.4.26}$$

[1] $\frac{1}{x+i0} = \frac{P}{x} - i\pi\delta(x)$, where P is to be understood as taking the principal part of an integral.

5.4.2 Obtaining the equations of motion

We now substitute the expressions from Eqs. (5.4.25), (5.4.26) into the effective action in Eq. (5.4.16) and perform the integration over t' which is trivial due to the presence of the delta function, this leaves the following action,

$$\begin{aligned}\frac{i}{2}\langle S_{int}^2\rangle = \int dt \Bigg[& -\pi \partial_P S^\dagger(t) \begin{pmatrix} \frac{1}{K} & 0 \\ 0 & K \end{pmatrix} \dot{S}(t) P_q(t) \\ & - \pi \partial_X S^\dagger(t) \begin{pmatrix} \frac{1}{K} & 0 \\ 0 & K \end{pmatrix} \dot{S}(t) X_q(t) \\ & - \pi \frac{1}{c^2 - V^2} \dot{S}^\dagger(t) \begin{pmatrix} \frac{c}{K} & V \\ V & cK \end{pmatrix} \dot{S}(t') X_q(t) \Bigg] . \end{aligned} \tag{5.4.27}$$

Writing the above expression out explicitly we arrive at the effective action coming from the influence of phonons, so the part of the action that represents the effect of phonons on the soliton dynamics is given below as,

$$\begin{aligned} S_{eff} = & \left[-\frac{\dot{N}}{2\kappa} \partial_P N - \frac{\kappa \dot{\Phi}}{2} \partial_P \Phi \right] P_q \\ & - \left[\frac{\dot{N}}{2\kappa} \partial_X N - \frac{\kappa \dot{\Phi}}{2} \partial_X \Phi - \frac{1}{2} \frac{1}{c^2 - V^2} \left[c\kappa \dot{\Phi}^2 + 2V \dot{N} \dot{\Phi} + \frac{c \dot{N}^2}{\kappa} \right] \right] X_q . \end{aligned} \tag{5.4.28}$$

If we now consider the functional integral over the remaining quantum components we notice that due to the action being linear in these components we have a functional delta function. Performing this integration leads to the following equations of motion modified by the presence of the phonons

$$\dot{X} = V(P, X) + \frac{\hbar\kappa}{2} \frac{\partial \Phi}{\partial P} \dot{\Phi} + \frac{\hbar}{2\kappa} \frac{\partial N}{\partial P} \dot{N} \tag{5.4.29}$$

$$\dot{P} = -\frac{\partial H}{\partial X} - \frac{\hbar\kappa}{2} \frac{\partial \Phi}{\partial X} \dot{\Phi} - \frac{\hbar}{2\kappa} \frac{\partial N}{\partial X} \dot{N} - \frac{\hbar c}{c^2 - V^2} \left[\frac{\kappa V}{2c} \dot{\Phi}^2 + \dot{\Phi} \dot{N} + \frac{V}{2\kappa c} \dot{N}^2 \right], \tag{5.4.30}$$

where we have reinserted $\hbar$ to make clear the semi-classical nature of these equations of motion. Here we have also defined $\kappa = n/mc$ which is related to the Luttinger parameter $K = \pi\kappa$ of the phononic Lagrangian Eq. (5.2.2), we also note that κ is a large number. We note that $V(P, X) = \partial H/\partial P$ is the velocity of the soliton in the absence of phonons. These equations have been derived in the adiabatic limit, assuming slow deviation of soliton parameters. Similar expressions were obtained in Ref.[58].

5.5 Soliton lifetime

Using the equations of motion we can now calculate the energy loss per period due to the radiation of phononic degrees of freedom, using the expression $\frac{dH}{dt} = \frac{\partial H}{\partial P}\dot{P} + \frac{\partial H}{\partial X}\dot{X}$ we arrive at the following formula for the total time derivative of the Hamiltonian,

$$\frac{dH}{dt} = \{H, H\} - \frac{\hbar\kappa\dot{\Phi}}{2}\{\Phi, H\} - \frac{\hbar\dot{N}}{2\kappa}\{N, H\} - F_r V \tag{5.5.1}$$

where the radiation force is defined by $F_r = \frac{\hbar}{2}\frac{\kappa V^2}{c^2 - V^2}\dot{\Phi}^2$. The expression in Eq. (5.5.1) is the full expression for the time rate of change of the Hamiltonian that governs the dynamics of the trapped soliton interacting with phonons. Unfortunately we are unable to make any progress with this expression due to the following reasons:

In the studies of electron damping due to an external field in QED [43, 29]) the dissipative Eqs. (5.4.29), (5.4.30) admit runaway solutions. One must solve for the soliton trajectory in the absence of the phonons and then to calculate the non-adiabatic corrections perturbatively to avoid runaway solutions.

The only consistent way to treat these equations is by perturbation theory around zero-order solution in the absence of phonons. In the absence of phonons the number of particles ejected from the soliton are conserved and thus the r.h.s. of Eqs. (5.4.29), (5.4.30) simplifies considerably when $\dot{N} = 0$. The energy dissipation rate Eq. (5.5.1) now simplifies to

$$\dot{H} = \dot{P}\partial_P H + \dot{X}\partial_X H = -\frac{\hbar\kappa}{2}\frac{c^2}{c^2 - V^2}\dot{\Phi}^2. \tag{5.5.2}$$

Using Eq. (3.3.16), (3.3.19) it is a matter of straightforward calculus to show that $\dot{\Phi} = -gN\dot{V}/c^2$. We then average Eq. (5.5.2) over one period of oscillation using the harmonic solution $V/c = \sqrt{1-(N/2\kappa)^2}\cos(\omega t/\sqrt{2})$ and we find the slow energy change of the soliton to be

$$\begin{aligned} \dot{E} &= -\frac{1}{T}\int_0^T \dot{H}dt = -\frac{2\kappa}{c^2}\frac{1}{T}\int_0^T \dot{V}^2 dt \\ &= \frac{\kappa\omega^2}{2}\left[1 - \left(\frac{N}{2\kappa}\right)^2\right] = \frac{1}{2}mg^2N^2\dot{N}\,. \end{aligned} \tag{5.5.3}$$

We know that $N = 0$ for an absent soliton, thus we can calculate the life time of a soliton that initailly corresponds to N ejected particles, we then find

$$\tau = \int_0^N \frac{dN}{\dot{N}} = \frac{mg^2}{\kappa\omega^2}\int_0^N \frac{N^2\,dN}{1-(N/2\kappa)^2} = \frac{8\mu}{\hbar\omega^2}F\left(\frac{N}{2\kappa}\right). \tag{5.5.4}$$

In performing the integral we have defined the following function,

$$F(x) = \int_0^x \frac{y^2\,dy}{1-y^2} = \frac{1}{2}\log\frac{1+x}{1-x} - x\,. \tag{5.5.5}$$

We expand $F(x)$ for small x and find that F behaves like $x^3/3$, another limit of interest is near to $x = 1$ where we find F diverges logarithmically as $\log\sqrt{2/(1-x)}$. The energy of the dark soliton is given by, $E_{ds} = \hbar\omega N/\sqrt{2} = 4\hbar nc/3$ and the energy of a grey soliton in the absence of phonons is given by Eq. (3.3.27). Using these expressions for the soliton energy we find $(N/2\kappa)^3 = E/E_{ds}$. Now we are in a position to comment on the lifetime of the soliton in different regimes. We find for high energies, those that are close to the energy of the dark soliton that the lifetime diverges logarithmically $\omega\tau \simeq (4\mu/\hbar\omega)\log\left[6E_{\rm ds}/(E_{\rm ds} - E)\right]$. We also see for low energy states $E \ll E_{ds}$ that the lifetime now obeys $\omega\tau \simeq (8\mu/3\hbar\omega)(E/E_{\rm ds})$. This last expression gives a lower bound on the energy E, we define $E^* = (\hbar\omega/\mu)E_{ds}$ which for energies lower than E^* the quasiparticle Lieb 11 modes can no longer be resolved. This of course corresponds to a soliton that decays before it can complete a single oscillation. We also note from [39] that we have the energy restriction $E \gg E^{**} = E_{ds}K^{3/2} = \mu/K^{1/2}$ which arises from the requirement of the validity of the quasiclassical treatment. Where the Luttinger parameter K is to be evaluated at the center of the trap.

For the values of the experiment [5], that are discussed in chapter two we find $E^* \simeq 0.01E_{ds} \gg E^{**}$ and thus Eq. (4.7.7) accurately describes most of the Lieb 11 modes which are resolvable. The results are schematically summarised in figure 5.2

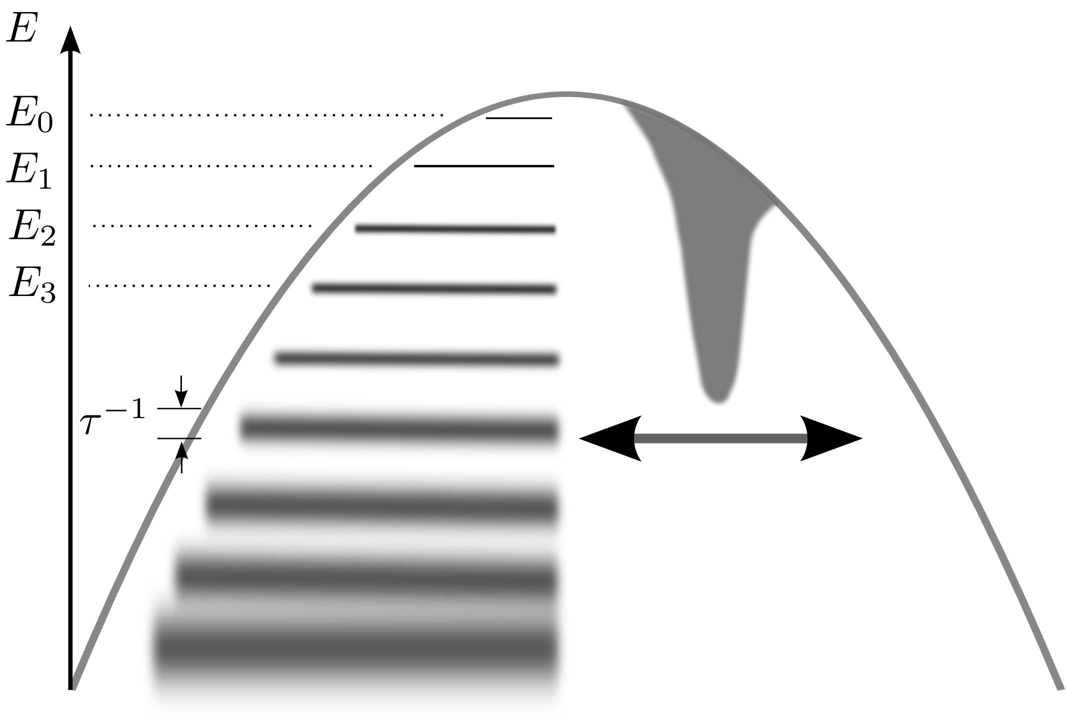

Figure 5.2: A schematic picture of trapped Lieb 11 modes, we see for high energies comparable to E_{ds}, the states are well defined. The states with low energies cannot be resolved and are characterised by their finite lifetime.

Chapter 6

CONCLUSION

In this thesis we have considered the dynamics of grey solitons in weakly interacting Bose liquids confined by smooth external trapping potentials. We have discussed the fundamental properties of grey solitons in homogeneous media by reviewing the Inverse Scattering transform that governs the theory of solitons. We have built up how to consider grey solitons in such a way that one does not have divergences in the integrals of motion such as the soliton energy. We have also reviewed some of the basic techniques that have been developed to deal with small perturbations that render the Gross-Pitaevskii equation no longer integrable.

We have also considered the question of quantisation of grey soliton solutions in uniform systems. For the uniform case we reviewed techniques that have been developed to deal with the inherent zero modes that lead to unwanted divergences when one looks for quantum solutions. The Christ & Lee method allowed for a divergence free quantisation of grey solitons in homogeneous Bose liquids. Using this method we found that for each collective coordinate associated with a zero mode; a conjugate momentum would appear in the Lagrangian. We found that the zero modes of the problem were removed from the Lagrangian and replaced by a momentum term. We went on to consider the question of quantisation of grey solitons in nonuniform systems. Using the Born-Sommerfeld rule we were able to quantise the grey soliton confined by an external potential in a more straightforward manner than in the uniform case. For the case of a harmonic confining potential we found that the energies of the grey soliton formed a ladder of energy states. This ladder of energy states was found to be similar to that of the Tonks-Girardeau limit, where for the soliton we found $E_n = E_{Max} - \left(\frac{\hbar\omega}{\sqrt{2}}\right)\left(n + \frac{1}{2}\right)$. We then associated these quantised states with those of the trapped Lieb II modes.

The last part of the thesis was concerned with the question of dissipative semi-classical dynamics of a grey soliton oscillating in an external trapping potential. We considered as originally conceived in [41] the soliton to be a Landau quasiparticle and then considered nonconservative dynamics. The question not considered in [41] was that of the mechanism of decay which is the question we set out to answer. We derived a coupling between the soliton parameters and phononic degrees of freedom. Motivated by the evaporative cooling used in the cold atoms experiments, we modelled the phonons as a Luttinger liquid. This modelling of interactions and the phonons allowed for the development of a semi-classical theory of trapped grey solitons. We found due to the radiation of phonons that the trapped grey soliton has a finite lifetime given by $\tau = 8\mu/\hbar\omega^2 F\left(N/2\kappa\right)$ where the function F is defined in Eq. (5.5.5). We found for high energies i.e. that of a dark soliton, that the lifetime diverges thus facilitating the predicted observation of trapped Lieb II modes.

Future work includes looking at soliton collisions in confining potentials: it is known that two colliding solitons will experience an effective potential proportional to $1/\sinh^2(z)$ [64], where z represents the separation between the two colliding solitons. We would be interested in how these collisions would affect the soliton lifetime in confining potentials. It is interesting to note that for the harmonic confining potential and small separation between the solitons one ends up having to solve a classical Calogero-Sutherland model for the two solitons. One could then in principle use this solution to calculate the soliton lifetime as was done for the single soliton.

We would also like to calculate the matrix elements for soliton transitions in the trap when one considers the phonons in a finite size system. One could consider sending in phonons with a certain frequency and calculate the amplitude for the quantum grey soliton to make a transition.

We would also like to consider soliton dynamics in double wells with the treatment developed in this thesis, effects such as a grey soliton tunneling into another well, would be an interesting effect to research. We would also like to look into applying the technique developed in this thesis to soliton excitations in other systems such as spin systems.

Appendix A

TECHNIQUES

This chapter reviews some of the Techniques that will be used in probing the soliton in a non-integrable confining potential. We will first give an overview of the Keldysh technique which deals with quantum systems out of equilibrium and can also be used for the study of classical dynamics. We will present this treatment in the language of many body functional integrals, this path is chosen mainly due to its natural links to the classical action and infact its ease for performing calculations. The technique is named after the work of L.V.Keldysh [38] who considered quantum systems out of equilibrium. Other earlier works dealing with the same topic, include [36], [67] and [23]. Interestingly there are also classical counterparts to the Keldysh technique, an example of this can be found in [49].

We shall start with a brief review of the equilibrium technique to set the scene and then go on to motivate why one might want to use the Keldysh technique. There have been several reviews that one can consult in the following references, [63], [71], [66].

Let us define as in Mahan [48] the S matrix as the operator that evolves the wave function $\phi(t')$ to the wave function $\phi(t)$ by the following equation,

$$\phi(t) = S(t, t')\phi(t') \tag{A.0.1}$$

where it can be shown that for an interacting quantum system, the S matrix is given by the following formula,

$$S(t,t') = \mathcal{T}\left[\exp\left(-i\int_{t'}^{t} dt_1 V(t_1)\right)\right]. \tag{A.0.2}$$

We would like to be able to calculate the expectation value of some observable $\mathcal{O}$ in the interacting system, given in the form $\langle GS|\mathcal{O}|GS\rangle$, where the state $|GS\rangle$ is the full interacting ground state. This itself is a problem as we do not know the ground state of some interacting groundstate, often that is a quantity that one is trying to calculate. Let us step back a bit to understand how to proceed: we note that we are working in the interaction representation which splits the hamiltonian into $H = H_0 + V$, where H_0 is the hamiltonian whose eigenvalues we know. What we would like to do is calculate the average in terms of the groundstate of H_0 which we do know. In fact this can be done and was established by Gell-Mann and Low [27], one simply writes the following relation between the interacting and non interacting ground state as follows

$$|GS\rangle = S(t,-\infty)|0\rangle. \tag{A.0.3}$$

The expression in Eq. (A.0.3) describes the adiabatic evolution of the non-interacting state $|0\rangle$ to the interacting ground state $|GS\rangle$. Thus we start with the non-interacting problem and adiabatically switch on the interaction part of the hamiltonian. Then in some time in the future we perform the adiabatic switching off of the interactions and thus we end up with the expectation $\langle 0|\mathcal{O}|0\rangle$. Gell-Mann and Low argued that during this procedure of switching the interactions on and off, there would be a phase factor defined by,

$$\mathrm{e}^{iL} = \langle 0|S(\infty,-\infty)|0\rangle. \tag{A.0.4}$$

With this procedure we can write a time ordered expectation as

$$\langle GS|\mathcal{O}|GS\rangle = \frac{\langle 0|\mathcal{T}\mathcal{O}S(\infty,-\infty)|0\rangle}{\langle 0|S(\infty,-\infty)|0\rangle}. \tag{A.0.5}$$

What we have achieved here, is that the expectation over the interacting ground of some operator has been reduced to an expectation over the ground state that we know from the distant past to the distant future. This is made possible due to the adiabatic procedure that allows the system to return to its initial state at t = ∞.

Although the above method is a very useful and allows for the calculation of many-body averages, it is of no use for a system that is out of equilibrium. In the case of systems out of equilibrium one can no longer guarantee that the final state will be the same as the initial state after interactions have been switched off. We would still like to calculate properties of a given system as is done in the equilibrium case, one way of achieving this is to consider the time evolution from t = $-\infty$ and back to t = $-\infty$. Thus we consider a closed time contour, as in the one depicted in Figure A.1. For the time contour in Figure A.1 it can be shown that a partition function Z when evaluated on this contour will be equal to one. This is very useful in some respects as one does not have to worry about the normalisation; when performing disorder calculations with the equilibrium technique the normalisation presents an issue.

A.1 Keldysh Field Theory

Let us now discuss in more detail the general formalism for the many-body partition function $\mathcal{Z}$, where we will mainly follow references [37], [3]. The target is to set up a continuum field theory for the Keldysh contour in the language of functional integrals. This section will be instructive and therefore to set the scene of how one should use Keldysh. It is actually very illuminating to set up this theory by considering a many-particle bosonic system with only a single quantum state available, $\hat{H} = \omega_0 \hat{a}^\dagger \hat{a}$. We now consider the partition function,

$$\mathcal{Z} = \frac{Tr\left(\hat{U}_C \hat{\rho}\right)}{Tr\left(\hat{\rho}\right)}, \tag{A.1.1}$$

where we have taken the initial density matrix to be the equilibrium density matrix $\hat{\rho} = \hat{\rho}_0 = \exp\left[-\beta\left(\hat{H} - \mu\hat{N}\right)\right]$ for which $\hat{H} = \omega_0\hat{a}^\dagger\hat{a}$ and $\hat{N} = \hat{a}^\dagger\hat{a}$. We are allowed to make this choice due to the freedom of turning on interactions and external perturbations at some later time. Evaluating the trace of the equilibrium density matrix results in the following expression,

$$Tr[\hat{\rho}_0] = \sum_{n=0}^{\infty} \exp\left[-\beta\left[(\omega_0 - \mu)\,\hat{a}^\dagger\hat{a}\right]\right] = (1 - \rho(\omega_0))^{-1}, \tag{A.1.2}$$

where $\rho(\omega_0) = \mathrm{e}^{-\beta(\omega_0-\mu)}$. We now construct a coherent state functional integral by inserting 2N resolutions of identity into a time sliced closed time contour such that $T = \epsilon/N$ as in Figure A.1. One can consult many of the excellent reviews on functional integrals, e.g., [55], [22], [65]

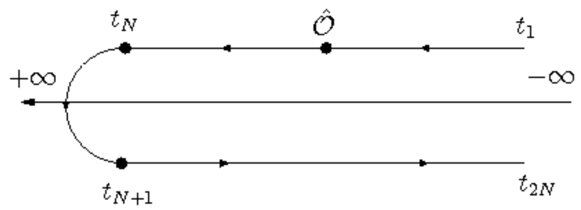

Figure A.1: Closed time contour used in the Keldysh technique, where we have inserted some operator $\mathcal{O}$ at some time t after the interactions have been turned on.

After proceeding in this way we end up with a functional integral in the following form,

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_0}\int \mathcal{D}\left[\bar{\phi}, \phi\right] \exp\left(i\sum_{jj'}^{2N} \bar{\phi}_j G^{-1}_{jj'}\phi_{j'}\right) \tag{A.1.3}$$

where $\phi = (\phi_1^+, ..., \phi_N^+, \phi_{N+1}^-, ..., \phi_{2N}^-)$, where the + denotes being on the forward contour and - denotes being on the backwards contour. We have defined the 2N $\times$ 2N Green's function below,

$$
G^{-1} = i \left[\begin{array}{cccc|cccc}
1 & & & & & & & -\kappa \\
-a_+ & 1 & & & & & & \\
 & \ddots & \ddots & & & & & \\
 & & -a_+ & 1 & & & & \\
\hline
 & & & -1 & 1 & & & \\
 & & & & -a_- & \ddots & & \\
 & & & & & \ddots & 1 & \\
 & & & & & & -a_- & 1
\end{array} \right] \tag{A.1.4}
$$

and made the definition $a_\pm = 1 \mp i\epsilon\omega$. We shall now check the claim that the partition function is indeed one. From the rules of Gaussian integration, the evaluation of Eq. (A.1.3) amounts to calculating the determinant of Eq. (A.1.4). It is trivial to verify that $\det(-iG^{-1}) = 1 - \kappa(a_+a_-)^{N-1}$ and then taking the limit $N \to \infty$ leads to, $\det(-iG^{-1}) = 1 - \rho(\omega) = \mathcal{Z}_0$. This proves that the Keldysh functional integral that we have considered has unit normalisation.

We will now take the continuum limit in Eq. (A.1.3) with G^{-1} defined in Eq. (A.1.4); taking this limit leads to the following form of the action,

$$
S[\bar{\phi}, \phi] = \int_{-\infty}^{\infty} dt \left[\bar{\phi}^+ (i\partial_t - \omega_0)\phi^+ - \bar{\phi}^- (i\partial_t - \omega_0)\phi^- \right] \tag{A.1.5}
$$

where the fields are to be understood as functions of time t. This action is naive and misleading; it implies that the fields on the forwards and backwards contours are not correlated; this is in fact not true due to the nonzero off-diagonal blocks in Eq. (A.1.4). It is not clear that the action is convergent and there is no information about the initial density distribution. The naive continuum limit is not correct, it loses essential information that is there before we take this limit. In the next section we will examine the discrete form of the functional integral in closer detail and then formulate a continuum formalism.

A.1.1 Correlation Functions

This next section is very important and we will spend some time developing a continuum representation of correlation functions and the continuum form of the Keldysh action. We shall calculate these within the functional integral representation and they are represented below as,

$$\langle \phi^{\pm}(t)\bar{\phi}^{\pm}(t')\rangle = \int \mathcal{D}[\bar{\phi},\phi]\phi^{\pm}(t)\bar{\phi}^{\pm}(t')\mathrm{e}^{iS[\bar{\phi},\phi]}, \tag{A.1.6}$$

where the action is given by $i\bar{\phi}G^{-1}\phi$, indices have been dropped. Here we give the explicit form of the Green's function, which written in block form is given by

$$G^{-1} = \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix}^{-1}. \tag{A.1.7}$$

To this end it is useful to invert the above matrix to find G but we will stay in the discrete notation, naively taking a continuum limit will result in the incorrect functional integral. Inverting the block matrix in Eq. (A.1.4) we can write down the following Green's functions in discrete form,

$$G_{ij}^{++} = -ia_{+}^{i-j}\left(\Theta(i-j) + \frac{\kappa(a_{+}a_{-})^{N-1}}{1-\kappa(a_{+}a_{-})^{N-1}}\right) \tag{A.1.8}$$

$$G_{ij}^{--} = -ia_{-}^{i-j}\left(\Theta(i-j) + \frac{\kappa(a_{+}a_{-})^{N-1}}{1-\kappa(a_{+}a_{-})^{N-1}}\right) \tag{A.1.9}$$

$$G_{ij}^{+-} = -i\frac{\kappa a_{-}^{N-j}a_{+}^{i-1}}{1-\kappa(a_{+}a_{-})^{N-1}} \tag{A.1.10}$$

$$G_{ij}^{-+} = -i\frac{\kappa a_{+}^{N-j}a_{-}^{i-1}}{1-\kappa(a_{+}a_{-})^{N-1}} \tag{A.1.11}$$

and $\Theta(z) = 1$ for z $\geq$ 0. There is a very neat interpretation of these expressions, for brevity we will consider expression for G^{++}. The first term represents propagation along the forward time contour from j to i and we pick up $i-j$ hopping amplitudes. The second term represents the more interesting cases, for example we could start from the position j and hop N-j times to the end of the contour picking up a factor of a_+^{N-j}. We then hop N-1 times along the backwards part of the contour picking up the factor a_-^{N-1}, picking up a factor of κ we can hop back to the forward time contour and finally we hop i-1 times with factor a_+^{i-1} to i. This gives the overall amplitude of $a_+^{i-j}(a_+a_-)^{N-1}$, infact we could then sum the trips that contain multiple loops and this would gives the denominator in the second part of G^{++}. The other propagators have a similar interpretations, which are self evident after the previous discussion.

We now take the continuum limit for the correlators in Eqs. A.1.8 to achieve the following continuum representations

$$G^{++}_{tt'} = -i\langle\phi^+(t)\bar{\phi}^+(t')\rangle = -i\mathrm{e}^{-i\omega(t-t')}\left(\Theta(t-t')+n(\omega)\right) \tag{A.1.12}$$

$$G^{--}_{tt'} = -i\langle\phi^-(t)\bar{\phi}^-(t')\rangle = -i\mathrm{e}^{-i\omega(t-t')}\left(\Theta(t'-t)+n(\omega)\right) \tag{A.1.13}$$

$$G^{+-}_{tt'} = -i\langle\phi^+(t)\bar{\phi}^-(t')\rangle = -i\mathrm{e}^{-i\omega(t-t')}n(\omega) \tag{A.1.14}$$

$$G^{-+}_{tt'} = -i\langle\phi^-(t)\bar{\phi}^+(t')\rangle = -i\mathrm{e}^{-i\omega(t-t')}\left(1+n(\omega)\right) \tag{A.1.15}$$

where $n(\omega)$ is defined as the Bose distribution function.

We should note that this correlation functions are not independent, due to correlations between time branches they are dependent and we find the following relation holds

$$G^{++}+G^{--}=G^{+-}+G^{-+}. \tag{A.1.16}$$

Let us encode this information into the block matrix of correlation functions by defining the following linear transformation,

$$\mathcal{L}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{A.1.17}$$

Using this transformation and computing $\mathcal{L}G_{tt'}\mathcal{L}^{\dagger}$ we find the following form for G,

$$G = \begin{pmatrix} G^K & G^+ \\ G^- & 0 \end{pmatrix} = -i\mathrm{e}^{-i\omega(t-t')} \begin{pmatrix} 1+2n(\omega) & \Theta(t-t') \\ -\Theta(t'-t) & 0 \end{pmatrix} \tag{A.1.18}$$

where we can identify G^+ with the retarded Green's function G^R and G^- with the advanced Green's function G^A. We notice that the Keldysh Green's function $-i\mathrm{e}^{-i\omega(t-t')}$ contains information about the initial distribution. Before we finish this analysis we shall write down the transformation of the fields induced by the operator $\mathcal{L}$; the fields change in the following way,

$$\phi_{cl} = \frac{1}{\sqrt{2}}\left(\phi^+ + \phi^-\right) \tag{A.1.19}$$

$$\phi_q = \frac{1}{\sqrt{2}}\left(\phi^+ - \phi^-\right). \tag{A.1.20}$$

In the above equation cl stands for classical and q stands for quantum, this notation will become clear soon. We have now established the true structure of the Green's function in its block structure. We can now write down the functional integral in Eq. (A.1.3) in its continuum form using the field representations ϕ_{cl}, ϕ_q. We also use the matrix representation of the correlation functions from Eq. (A.1.18). We can now write the continuum limit of Eq. (A.1.3) as;

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_I}\int \mathcal{D}[\phi_{cl},\phi_q]\mathrm{e}^{iS[\phi_{cl},\phi_q]}, \tag{A.1.21}$$

where the action is now written as,

$$S[\phi_{cl},\phi_q] = \int\int_{-\infty}^{\infty} \left(\bar{\phi}_{cl},\bar{\phi}_q\right)_{t'} \begin{pmatrix} 0 & (G^{-1})^A \\ (G^{-1})^R & (G^{-1})^K \end{pmatrix}_{tt'} \begin{pmatrix} \phi_{cl} \\ \phi_q \end{pmatrix}_{t'}. \tag{A.1.22}$$

This is the full continuum theory for the non-interacting problem where we notice that the cl-cl component of the action is zero, hence this notation. We remember that the classical field is a symmetric combination of the fields ϕ^+ and ϕ^-. The cl-q components are responsible for the causality of the responce functions and the q-q component contains the information about the initial density distribution and is responsible for the convergence of the functional integral.

A.1.2 Adding Interactions and Semi-Classical Limits

So far all we have done is to formulate the continuum limit for a non-interacting theory; here we demonstrate that interactions can be added very naturally to the current formalism. Let us consider complex Bose fields interacting through a two body interaction $V(r-r')$. We will consider the case of a contact interaction $V(r-r') = g\delta(r-r')$; the interaction part of the action then looks like

$$S_{int} = -\frac{g}{2}\int dt \int_{C_k} dr\bar{\phi}\bar{\phi}\phi\phi = -\frac{g}{2}\int dr \int_{-\infty}^{\infty} dt \left[\left(\bar{\phi}^+\phi^+\right)^2 - \left(\bar{\phi}^-\phi^-\right)^2\right]. \tag{A.1.23}$$

where in the last line we have split the action onto the forward and backward contour respectively. We note that the interactions will be turned on and off at some time after $t = -\infty$, so the initial density distribution is not affected. Let us now use the Keldysh rotation defined in Eq. (A.1.19) to transform the interacting action into the form,

$$S_{int} = -\frac{g}{2}\int dr \int dt \left[\bar{\phi}_{cl}\bar{\phi}_q \left(\phi_{cl}\phi_{cl} + \phi_q\phi_q\right) + c.c\right] \tag{A.1.24}$$

where c.c represents complex conjugate. We notice that $S[\phi_{cl}, 0] = 0$, which we have noted is a general property of the Keldysh action. We will now consider a saddle point solution and identify the classical saddle point with

$$\left.\frac{\partial S}{\partial \phi_q}\right|_{\phi_q=0} = 0. \tag{A.1.25}$$

Evaluating this variation in the action gives the eponymous Gross-Piteavskii equation

$$\left(i\partial_t + \frac{1}{2m}\partial_r^2 + \mu - g|\phi|^2 \right) \phi = 0. \tag{A.1.26}$$

There is an important question to ask; is the normalisation still preserved when the interactions are considered. It can be shown that indeed it is; one can perform a perturbative expansion in the interaction parameter and show that at each order the normalisation is preserved. We omit this calculation here, which is a straightforward calculation to perform. We remind that one can always find a classical point in this manner. We make a variation with respect to the quantum components, then set the remaining quantum components to zero leaving the classical saddle point. We will use this method to study the trapped soliton in later chapters. We shall note that $\partial S/\partial\phi_{cl} = 0$ is another saddle point of the Keldysh action and can always be solved by $\phi_q = 0$.

A.1.3 Some toy problems

In this section we consider some simple problems for which have experience with and intuition for. This will help us in understanding the application of the Keldysh technique to physical problems and in gaining some experience with the tools. Let us consider the many-body problem with one bosonic state on the Keldysh contour; for this single state we may write the following transformation down,

$$\phi(t) = \frac{1}{\sqrt{2\omega_0}}\left[P(t) - i\omega_0 X(t)\right], \qquad \phi(t) = \frac{1}{\sqrt{2\omega_0}}\left[P(t) + i\omega_0 X(t)\right]. \tag{A.1.27}$$

We then substitute the transformation above into a non-interacting coherent state path integral, although now evaluated over the Keldysh time contour. This trivial procedure leads one to the following action,

$$S[X,P] = \int_{C_K} dt \left(\dot{X}(t)P(t) - H(X(t),P(t)) \right), \tag{A.1.28}$$

where the Hamiltonian is defined as $H = \frac{1}{2}P^2 + V(X)$ and for the harmonic oscillator problem we are considering the potential given by $V(X) = \frac{\omega^2}{2}X^2$. We would now like to perform a Keldysh rotation on P and X in the same spirit as in Eq. (A.1.19) along the closed time contour. In keeping with the saddle point expansion we expand the Hamiltonian to first order in quantum components, which for the toy problem of a harmonic oscillator is exact.

We now obtain the following action,

$$S[X_{cl/q}, P_{cl/q}] = \int_{-\infty}^{\infty} dt \left[\left(\dot{X} - \frac{\partial H}{\partial P} \right) P_q - \left(\dot{P} + \frac{\partial H}{\partial X} \right) X_q \right], \tag{A.1.29}$$

where we have performed an integration by parts to move the time derivative onto the classical momentum. We could consider the saddle-point of this action to obtain the classical equation of motion but for this simple non-interacting problem we can do the integral over the quantum components of X and P. The functional integral is linear in these components and amounts to the functional representation of the delta function. Performing the integration over these variable enforces Hamiltons equations,

$$\dot{P} = -\frac{\partial H}{\partial X} \tag{A.1.30}$$

$$\dot{X} = \frac{\partial H}{\partial P} \tag{A.1.31}$$

and one obtains the equation of motion for the harmonic oscillator.

We shall now investigate the final toy problem which will implement some ideas that will be used later. Let us consider the problem of our single quantum particle coupled to a bath of oscillators. Let us assume a linear coupling to the particle's coordinate X, then we have the following additional terms in the action to consider,

$$S_{bath}(\phi_n) = \frac{1}{2}\sum_n \int_{C_K} dt\phi_n \mathcal{D}_n^{-1}\phi_n \tag{A.1.32}$$

$$S_{int}(X,\phi_n) = \sum_n g_n \int_{C_K} dt X\phi_n, \tag{A.1.33}$$

where we have used C_K to represent the Keldysh time contour. We proceed via a Keldysh rotation as in Eq. (A.1.19) for both X and the fields ϕ. It is useful to use a vector notation once we go over to the Keldysh representation due to the doubling of degrees of freedom, in the following we will be using $\underline{X} = (X_{cl}, X_q)^T$ and $\underline{\phi}_n = (\phi_{cl}, \phi_q)^T$. The actions now take on the following form,

$$S_{bath}(\phi_n) = \frac{1}{2}\sum_n \int_{-\infty}^{\infty} dt\underline{\phi}_n^T \mathcal{D}_n^{-1}\underline{\phi}_n \tag{A.1.34}$$

$$S_{int}(X,\phi_n) = \sum_n g_n \int_{-\infty}^{\infty} dt\underline{X}^T\sigma_1\underline{\phi}_n, \tag{A.1.35}$$

where σ_1 is the first Pauli matrix. Notice that the action for ϕ_n is quadratic, the procedure we shall follow here is standard in functional integral methods. We will integrate out the ϕ_n fields to obtain an effective action describing the damped dynamics of the single quantum particle. The integration is trivial and leads to the action responsible for dissipation, expressed below as,

$$S_{diss} = \frac{1}{2}\int_{-\infty}^{\infty} dtdt'\underline{X}^T\mathcal{G}_n(t-t')\sigma_1\underline{X} \tag{A.1.36}$$

where the correlation function now has the two by two structure. One can check that the dissipative action has the correct causality structure i.e., the classical part of the action is zero. We can write down the fourier transform of the retarded component of $\mathcal{D}$ as

$$\mathcal{G} = \sum_n g_n^2\mathcal{D}^R(\epsilon) = -\frac{1}{2}\sum_n \frac{g_n^2}{(\epsilon+i0)^2-\omega_n^2} = \int_0^{\infty}\frac{d\omega}{2\pi}\frac{\omega J(\omega)}{\omega^2-(\epsilon+i0)^2}, \tag{A.1.37}$$

where we have defined the spectral function for the bath as $\pi\sum_n(g_n^2/\omega_n)\delta(\omega-\omega_n)$. For simplicity and foresight into where we want to arrive we shall assume that the spectral density is given by $J(\omega) = 8\gamma\omega$, known in the literature as the Ohmic bath. Of course it is important to note that we can also obtain the advanced component by letting $\epsilon \to \epsilon - i0$. Let us now Fourier transform Eq. (A.1.37) back to the time basis, leading us to consider,

$$\sum_n g_n^2 \mathcal{D}^R(t-t') = 4\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\omega^2 \int_{-\infty}^{\infty} d\epsilon \frac{\mathrm{e}^{-i\epsilon(t-t')}}{\omega^2 - (\epsilon + i0)^2} = -2\gamma\delta(t-t')\partial_{t'} \tag{A.1.38}$$

where we have extended the ω integral to the entire real line. Repeating the calculation for the advanced function one obtains $2\gamma\delta(t-t')$. The Keldysh component is evaluated using the retarded/advanced components to be given by and set by the fluctuation dissipation theorem to be

$$\sum_n g_n^2 \mathcal{D}^K(\epsilon) = \sum_n g_n^2 \left[\mathcal{D}^R - \mathcal{D}^A\right] \coth(\frac{\epsilon}{2T}) = 4i\gamma\epsilon \coth(\frac{\epsilon}{2T}). \tag{A.1.39}$$

We will not Fourier transform the Keldysh component back to the time basis as it is not needed in full for the proceeding discussion. However we will mention that in the time basis the Keldysh component has a non-local in time part given by $-4i\gamma\pi T^2/\sinh^2[\pi T(t-t')]$. This Keldysh component is the coefficient of the ϕ_q^2 term and describes the induced quantum fluctuations from the bath which are damping in nature. The bath's effect on the particle is also manifest in the retarded/advanced components through the time derivative that appears.

We are interested in the semi-classical limit as we believe these give the largest contribution to the action, so let us re-introduce $\hbar$ in the following way. In summary we write $S_{diss} \to S_{diss}/\hbar$, $X_q \to \hbar X_q$ and $T \to T/\hbar$. One would like to now construct an expansion around the classical minimum; to enforce this expansion we consider the limit that $\hbar \to 0$. In this limit we will use the expansion $4i\gamma\epsilon \coth[\frac{\epsilon\hbar}{2T}] \to 8i\gamma T/\hbar$; we are now equiped to write the action as

$$S_{diss} = \frac{1}{2}\int\int_{-\infty}^{\infty} dtdt'(X, X_q)\begin{pmatrix} 0 & 2\gamma\delta(t-t')\partial_{t'} \\ -2\gamma\delta(t-t')\partial_{t'} & 8i\gamma\delta(t-t') \end{pmatrix}\begin{pmatrix} X \\ X_q \end{pmatrix}. \tag{A.1.40}$$

In the above we have dropped the subscript denoting the classical field, finally we do the trivial matrix multiplication in conjunction with an integration by parts to obtain

$$S[X_{cl/q}, P_{cl/q}] = \int_{-\infty}^{\infty} dt\left[-\left(\dot{P} + \frac{\partial H}{\partial X} + 2\gamma\dot{X}\right)X_q + 4i\gamma T X_q^2 + \left(\dot{X} - \frac{\partial H}{\partial P}\right)P_q\right]. \tag{A.1.41}$$

We notice that the delta functions make the non local in time action local. If one considered just the terms first order in the quantum component we would have the standard equation of motion for a harmonic oscillator with a viscous friction force. The equations of motion are now given as

$$\dot{P} = -\frac{\partial V}{\partial X} - 2\gamma\dot{X} \tag{A.1.42}$$

$$\dot{X} = \frac{\partial H}{\partial P} = P, \tag{A.1.43}$$

for which we used the explicit example where the Hamiltonian is given by $H(X, P) = \frac{1}{2}P^2 + V(X)$, where the potential is harmonic. It is worth noting that the term $\dot{X}$ has been derived within the action principle, this is a remarkable feat that is only possible by using the 2N degrees of freedom from the Keldysh rotation. The motivation for working through this example will now be made clearer. We will find that we can derive in a similiar fashion, equations for $\dot{X}$ and $\dot{P}$ for the confined soliton.

We can make further progess by considering the term proportional to X_q^2; we can deal with this term by utilising the Hubbard-Stratonovich transformation (See [3] for a nice introduction and [35] for some of the first work on this transform). We start by introducing the following identity into our partition function,

$$1 = \int \mathcal{D}[\xi(t)] \mathrm{e}^{-\int dt \frac{1}{4\gamma T}\xi^2(t)}, \tag{A.1.44}$$

where the integration measure is defined in such a way that the normalisation is maintained. The trick is to introduce an auxillary field by the linear transformation $\xi \to \xi - AX_q$ for which A is defined as a real or complex constant. We proceed by plugging this transformation into the identity Eq. (A.1.44), importantly the following term arises $-A^2X_q^2/4\gamma T$ in the action, if we choose $A = i4\gamma T$ the term quadratic in the quantum component is removed from the problem at the expense of introducing the field $\xi(t)$. The transformation is summarised in the equation below,

$$1 \cdot \mathrm{e}^{-4\gamma T \int dt X_q^2(t)} = \int \mathcal{D}[\xi(t)] \mathrm{e}^{-\int dt \left[\frac{1}{4\gamma T}\xi^2(t) - 2i\xi(t)X_q\right]}. \tag{A.1.45}$$

We are now left with just the terms that are linear in the quantum component of the particle position, let us write in full the functional integral for this problem as

$$\mathcal{Z} = \int \mathcal{D}[\xi] \mathrm{e}^{-\int dt \frac{1}{4\gamma T}\xi^2(t)} \int \mathcal{D}[X] \int \mathcal{D}[X_q] \mathrm{e}^{-2i\int dt \left[\frac{\ddot{X}}{2} + \gamma \dot{X} + \frac{1}{2}V'(X) - \xi(t)\right] X_q} \tag{A.1.46}$$

$$= \int \mathcal{D}[\xi] \mathrm{e}^{-\int dt \frac{1}{4\gamma T}\xi^2(t)} \int \mathcal{D}[X] \delta \left[\frac{\ddot{X}}{2} + \gamma \dot{X} + \frac{1}{2}V'(X) - \xi(t)\right]. \tag{A.1.47}$$

In the last line the functional delta function enforces the classical equation of motion with a frictional force and an external force. If we consider the auxillary field $\xi(t)$ to have the following correlation properties, i.e., $\langle \xi(t)\xi(t')\rangle = 2\gamma T\delta(t - t')$ we then have a Langevin equation with white noise.

A.2 Luttinger Liquids: Harmonic Fluid Approach

We found that the dark soliton in one spatial dimension couples to phonons known as a Luttinger liquid. Here we will give a brief review of the basics to familiarise with the Luttinger Liquid concept before it is treated in the context of trapped solitons. The method was originally devised to describe the low energy excitations of one dimensional interacting fermion systems, where it had been noted that the particle-hole excitations were bosonic in nature. The work in this field had a rich development [72], [50], [17], [46] throughout its formative years.

Haldane [33], [34] generalised the concept when he realised that one-dimensional systems with gapless and linear dispersions could be described in this context, he named these models "Luttinger Liquids". In one dimension, the longitudinal excitations are greatly enhanced and lead to a prevention of long range order and thus there is no continuous symmetry breaking in the thermodynamic limit, this is even true at zero temperature. One advantage of the harmonic-fluid approach is that it is not a mean field theory and thus does not break continuous symmetries, it can also deal with weak and strongly interacting systems and allows one to calculate important correlation functions for these one dimensional systems. One can think of it as a one-dimensional analogue to Fermi liquid theory where the latter does not apply to one-dimensional systems. There are also many excellent reviews on the harmonic-fluid approach [28], [12], [68], [31] which have been useful in writing the next section.

A.2.1 Harmonic-Fluid Approach

In this section we shall work with second quantised operators $\hat{\Psi}^\dagger(x)$ and the density operator will be given by the usual definition $\hat{\Psi}^\dagger(x)\hat{\Psi}(x)$. The average density is denoted as ρ_0 and is fixed by the chemical potential μ. To describe the low energy sector of the theory it is useful to identify the appropriate coordinates and for a bosonic system it is the density ρ

and phase ϕ; in these variable the field operator takes the form,

$$\hat{\Psi}^{\dagger}(x) = \sqrt{\rho(x)}\mathrm{e}^{-i\phi(x)}. \tag{A.2.1}$$

One can obtain the following commutation relation between these operators using the standard bosonic commutation relations, on substition of the density-phase relation one obtains

$$\mathrm{e}^{i\phi(x)}\rho(x)\mathrm{e}^{-i\phi(x)} - \rho(x) = 1, \tag{A.2.2}$$

for $x = x'$.

We now move forward in our low energy description by splitting $\rho(x) = \rho_{<}(x) + \rho_{>}(x)$ and $\phi(x) = \phi_{<}(x) + \phi_{>}(x)$, a procedure known as coarse-graining. The subscript $<$ refers to slow parts of the fields and are defined over distances $d \gg \rho_0^{-1}$ and the subscript $>$ are known as the fast fields which are defined on a length scale of order or shorter than ρ_0^{-1}. Let us concentrate on the slow fields and particularly the density, we shall make the useful definition $\rho(x) = \rho_0 + \Pi(x)$ where $\Pi(x)$ is the density fluctuation operator. Let us now introduce another operator defined as

$$\frac{1}{\pi}\partial_x\Theta(x) = \rho_0 + \Pi(x), \tag{A.2.3}$$

we can get some physical intuition for this operator by integrating Eq. (A.2.3) from 0 to L resulting in $\Theta(L) - \Theta(0) = \pi N$. Thus the field $\Theta(x)$ seems to count the number of particles along our line of length L; it turns out to be more useful to use the counting field defined as,

$$\vartheta(x) = \Theta(x) - \pi\rho_0 x. \tag{A.2.4}$$

The counting field $\vartheta(x)$ counts the deviations from the average density ρ_0. The slow component of density can now be written in terms of $\Theta(x)$ as

$$\rho(x) = \partial_x \Theta(x) \sum_{n=-\infty}^{\infty} \delta(\Theta(x) - \pi n). \tag{A.2.5}$$

We see that writing the density in this way guarantees that the spatial integral of the density $\rho(x)$ over the system size gives the number of particles in the system, which follows trivially from the integration over the delta function. Using the Poisson summation formula [1] we can re-write the density in a more appealling form, given below as,

$$\rho(x) = \frac{1}{\pi}\partial_x \Theta(x) \sum_{n=-\infty}^{\infty} \mathrm{e}^{2ni\Theta(x)} = \left(\frac{1}{\pi}\partial_x \vartheta(x) + \rho_0\right) \sum_{-\infty}^{\infty} \mathrm{e}^{2ni\vartheta(x)+2\pi i n\rho_0 x}. \tag{A.2.6}$$

We note that in Eq. (A.2.6) for the slow part of the density we have a sum over oscillatory terms corresponding to terms which describe changes in momentum of $2\pi\rho_0$ and so on. These terms are due to a change of momentum from say -q to q which could arise from back scattering, these terms do indeed contribute to the low energy excitations. So we can now write down the field operator in terms of the density and phase in the low energy sector as

$$\Psi^\dagger(x) = \frac{1}{\pi}\partial_x \Theta(x) \sum_{n=-\infty}^{\infty} \mathrm{e}^{2ni\Theta(x)} = \left(\frac{1}{\pi}\partial_x \vartheta(x) + \rho_0\right) \sum_{-\infty}^{\infty} \mathrm{e}^{2ni\vartheta(x)+2\pi i n\rho_0 x - i\phi(x)}, \tag{A.2.7}$$

where the adjoint field naturally follows from the above expression. Finally we would like to obtain a low energy effective Hamiltonian for an interacting bosonic sytem, which we can now go on to derive thanks to the expression in Eq. (A.2.7). We will consider here a many body bosonic Hamiltonian with a two body interaction and assume that the particles sit in the lowest energy level of the transverse confinement, below this Hamiltonian is written as

$$H = \frac{\hbar^2}{2m}\int_0^L dx \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + \frac{1}{2}\int_0^L dx dx' U(x-x')\rho(x)\rho(x'). \tag{A.2.8}$$

[1] $\sum_n f(n) = \sum_n \int dw f(w) \mathrm{e}^{2n\pi i w}$

Furthermore we will consider a contact interaction of the form $g\delta(x - x')$ for the pairwise interaction $U(x - x')$. We now take the field operator defined in Eq. (A.2.7) and substitute into the Hamiltonian in Eq. (A.2.8) and only keep terms that are of quadratic order in ϕ and Θ. We disregard derivatives of third order or higher and arrive at the following effective hamiltonian

$$H_{eff} = \frac{\hbar}{2\pi} \int_0^L dx \left[\nu_J(\partial_x \phi)^2 + \nu_N(\partial_x \Theta - \pi\rho_0)^2\right] = \frac{\hbar}{2\pi} \int_0^L dx \left[\nu_J(\partial_x \phi)^2 + \nu_N(\partial_x \vartheta)^2\right]. \tag{A.2.9}$$

In the above effective action we have defined the phase stiffness as $\nu_J = \frac{\hbar\rho_0\pi}{M}$ which can be related to the superfluid fraction via a twisted boundary condition

$$\Psi^\dagger(x + L) = \mathrm{e}^{\alpha}\Psi(x), \tag{A.2.10}$$

One then evaluates the difference between the hamiltonian with $\alpha \neq 0$ and $\alpha = 0$ to obtain the said relationship. Lastly we define $\nu_N = \frac{g}{\pi\hbar}$ as the inverse compressibility of the bosonic gas, where the relation below is used,

$$\kappa^{-1} = \rho_0^2 \left(\frac{\partial\mu}{\partial\rho}\right)_{\rho=\rho_0} \tag{A.2.11}$$

As a short remark on this section one can expand the fields ϑ and ϕ in terms of normal mode oscillators and show that the Hamiltonian is that of non-interacting bosons with a linear dispersion, this can be done for open and periodic boundary conditions. We should also mention the effect of a confining potential $V_{ex}(x)$ on this low energy description. It is handy to define the Luttinger parameter as $K = \sqrt{\frac{\nu_J}{\nu_N}}$ and the parameter $\nu_s = \sqrt{\nu_J\nu_N}$ then we can write the effective hamiltonian for a nonuniform system as

$$H_{eff} = \frac{\hbar\nu_s}{2\pi} \int_0^L dx \left[K(\partial_x\phi)^2 + \frac{1}{K}(\partial_x\vartheta)^2\right] \tag{A.2.12}$$

$$H_{eff} = \frac{\hbar \nu_s}{2\pi} \int_0^L dx \left[K(x)(\partial_x \phi)^2 + \frac{1}{K(x)}(\partial_x \vartheta)^2 \right], \tag{A.2.13}$$

where in the second form of the Hamiltonian the effect of a slowly varying trap has been included.

Appendix B

A UNIVERSAL DERIVATION OF THE COUPLING TO PHONONS

In Chapter 5 we derived the coupling of the soliton phase Φ and the number of particles ejected from the soliton core N to the phonons. In this section we will demonstrate that this is actually a universal coupling that can descibe a whole host of systems that couple to some background of phonons. We will now consider the general situation of an impurity that moves with some depletion of particles and has a phase associated with it and follow [47]. This will also demonstrate that the soliton can be viewed as an impurity immersed in the Bose gas. However one should bear in mind that the soliton has a very particular dispersion that is specific to itself and does not really have any relation to an impurity dispersion.

We start the discussion by considering the background fluid of the Bose gas in the absence of an impurity, we consider the fluid to be hydrodynamic and follow the prescription given by Popov [61]. We consider a slowly varying chemical potential μ and the fluid density n, it then turns out that the Lagrangian of the fluid can be written as an integral over the local thermodynamic pressure,

$$L_0(\mu, n) = \int dx p_0(\mu, n) = \int dx \left[\mu n - e_0(n)\right]. \tag{B.0.1}$$

We have also defined the energy density of the fluid e_0 which is related to the chemical potential by $\mu = \mu(n) = \partial e_0/\partial n$. We note that the Lagrangian given in Eq. (B.0.1) can be related to the grand-canonical thermodynamic potential of the background fluid by the definition,

$$\Omega_0(\mu) = -L_0(\mu, n(\mu)). \tag{B.0.2}$$

We are now ready to consider the presence of an impurity immersed in the background fluid, to start with we will not consider coupling to the background. The impurity is defined with a mass M, with its position defined by the coordinate X and moves through the background fluid with velocity $\dot{X} = V$ as measured in the laboratory reference frame. It is more convenient to go the the frame where the impurity is at rest and the background fluid moves with the velocity $-V$. This is achieved by performing a Galilean transformation to the impurity frame of reference and results in the impurity experiencing the supercurrent j' while seeing the chemical potential μ'. We note that from now on the primes denote the frame of reference moving with the impurity. The modified supercurrent and chemical potential can be written down using a Galilean transformation and are given by

$$j' = -nV, \qquad \mu' = \mu + \frac{mV^2}{2}, \tag{B.0.3}$$

where one should also note that the energy density has transformed as $e_0' = e_0 + nmV^2/2$. One can then show by using Eqs. (B.0.3), (B.0.1) and (B.0.2) along with the transformed energy density that $\Omega_0'(\mu') = \Omega_0(\mu)$, which one expects from Galilean invariance.

We are now in a position to introduce the impurity into the background fluid, in doing so we will keep j' and μ' fixed and wait for the system to equilibrate. We will now consider the depletion cloud that moves with the impurity, this corresponds to the distortion in the background fluid due to the presence of the impurity. We note that the increase in the thermodynamic potential due to the depletion will be $\Omega_d'(j', \mu') = E_d' - \mu' N_d$, where the energy change associated with the depletion cloud E_0' and change in number of particles N_d

have now been defined. Note that a Galilean transform can be used to relate the energy and number of particles associated with the depletion cloud in the impurity reference frame back to E_d and P_d as measured in the laboratory frame due to the presence of the impurity. This relation is expressed in terms of the energy resulting from a Galilean transform back to the laboratory frame and takes the form

$$E'_d = E_d - P_d V + mN_d V^2/2. \tag{B.0.4}$$

From the above considerations we write down the Lagrangian for the depletion cloud, as,

$$L_d = P_d V - E_d + \mu N_d = -E'_d + \mu' N_d = -\Omega'_d(j', \mu') \tag{B.0.5}$$

and we see that we have written down a remarkable relation between the dynamics and the thermodynamic potential. This approach so far has relied on the existence of a superfluid which allows for us to consider thermodynamic equilibium for non zero current. We would also like to note that the increase in energy E'_d, momentum P_d, number of particles N_d and the grandcanonical potential $\Omega'_d(j', \mu')$ due to the impurity result in finite size corrections to the extensive properties of the system.

So far we have considered the equilibrium situation where the supercurrent and the chemical potential have been kept fixed. Now we consider variations in j' and μ' that will result in the following change in the thermodynamic potential of the depletion cloud,

$$d\Omega'_d = \Phi dj' + N d\mu'. \tag{B.0.6}$$

We will refer to Eq. (B.0.6) as the response function. The response to a change in chemical potential is the number of particles N and for the current study of a depletion cloud is given by $N = -N_d$. The response to a change in the chemical potential is directly related to the number of particles expelled from the cloud in the presence of an impurity. The response Φ

to a change in supercurrent is to be interpreted as the superfluid phase. We note that for the situation where one is considering a soliton, Φ will correspond to the total phase drop across the soliton. In thermodynamic equilibrium Φ and N are fixed to be constant by the supercurrent and chemical potential of the fluid. We also note from Eq. (B.0.3) that this would also imply that one can determine Φ and N fully from the velocity V and density n by writing $\Phi = \Phi_0(V, n)$ and $N = N_0(V, n)$ when we have global thermodynamical equilibrium.

We now turn to the nonequilibrium situation where the supercurrent and the chemical potential are no longer fixed and fluctuate. In this situation we would no longer expect Φ and N to be fixed either, it is useful to now consider the variables Φ and N as independent. We then can use a Legendre transformation to obtain the new thermodynamic potential

$$H_d(\Phi, N) = \Omega'_d - j'\Phi - \mu' N, \qquad dH_d(\Phi, N) = -j' d\Phi - \mu' dN. \tag{B.0.7}$$

It is now the the independent variables Φ and N that describe the state of the depletion cloud in the vicinity of the impurity, for which may or may not be in equilibrium with the values j' and μ'. As we will see shortly the new thermodynamic potential $H_d(\Phi, N)$ has been introduced so we can incorporate interactions of the depletion cloud with long wavelength phonons. A coupling to these phonons can change the number of particles and momentum of the depletion cloud thus forcing the system to equilibrate to new values of Φ and N.

It is interesting to substitute Eq. (B.0.7) into Eq. (B.0.5) and to also introduce the Lagrangian of the free impurity to write the total Lagrangian for the depletion cloud as

$$L_d = \frac{MV^2}{2} - j'\Phi - \mu' N - H_d(\Phi, N), \tag{B.0.8}$$

where M is the impurity mass. We can re-write the total Lagrangian of the depletion cloud Eq. (B.0.8) in terms of the velocity V and density n using the relations defined in Eq.

(B.0.3) to obtain the total Lagrangian of the depletion cloud as

$$L = \frac{1}{2}\left(M - mN\right)V^2 + nV\Phi - \mu N - H_d(\Phi, N). \tag{B.0.9}$$

We have succeeded in writing the total Lagrangian in terms of the variables V, N and Φ which are for the equilibrium case fixed to be constants. From the impurity Lagrangian Eq. (B.0.9) we can define the momentum canonical to the impurity coordinate X as

$$P = \frac{\partial L}{\partial V} = \left(M - mN\right)V + n\Phi. \tag{B.0.10}$$

We see this describes a reduced mass of $M - mN$ moving with velocity V, this tells us that a mass of mN is removed from the vicinity of the impurity. We know for the soliton $M = 0$, we should stress that the impurity mass M is not an effective mass which one would obtain from the second derivative of the dispersion also known as the curvature of the dispersion. Of course one can invert the relation in Eq. (B.0.10) to obtain the velocity $V = V(P, \Phi, N)$. Using Eqs. (B.0.9) and (B.0.10) we can write down the Hamiltonian for the system which takes on the form

$$H(P, \Phi, N) = PV - L = \frac{1}{2}\frac{(P - n\Phi)^2}{(M - mN)} + \mu N + H_d(\Phi, N). \tag{B.0.11}$$

One can then obtain Hamilton's equations, which when generated from the Hamiltonian Eq. (B.0.11) become,

$$\begin{aligned} \dot{P} &= 0 && \text{(B.0.12)} \\ \dot{\Phi} &= 0 = -\frac{mV^2}{2} - \mu - \frac{\partial H_d}{\partial N} && \text{(B.0.13)} \\ \dot{N} &= 0 = -nV - \frac{\partial H}{\partial \Phi}. && \text{(B.0.14)} \end{aligned}$$

We see that the momentum is conserved which one would expect in the absence of external forces and the variables Φ, N have no time dynamics and just provide static constraints.

We now move on to the coupling of the impurity to long wavelength phonons. The general idea is that an external force acting on the impurity will cause it to radiate energy and momentum. The force will act in such a way that the independent variables Φ and N will have to change to try and re-establish equilibrium. We will consider weak external forces that are provided by long wavelength phonons, for example the force could be provided by a confining potential and mediated by phonons.

We shall start by considering the phonon Lagrangian which we consider as a function of the counting field ϑ and the superfluid current ψ which we met in the previous derivation of the coupling. We continue by following the Popov prescription [61] and allow the density to fluctuate as $n \to n + \rho(x,t)$ and the chemical potential becomes under slow fluctuations,

$$\mu \to \mu - \dot{\psi} - \frac{mu^2(x,t)}{2}. \tag{B.0.15}$$

We then substitute these slow changes into the background fluid Lagrangian Eq. (B.0.1) and obtain,

$$L = \int dx \left[(\mu - e_0(n)) + \rho \left(\mu - \frac{\partial e_0}{\partial n} \right) - \dot{\psi}\rho - \frac{nmu^2}{2} - \frac{1}{2}\frac{\partial^2 e_0}{\partial n^2}\rho^2 \right]. \tag{B.0.16}$$

In writing down the above expression for the phonon Lagrangian Eq. (B.0.16) we have disgarded terms of order 3 or more in fluctuating variables. We have not considered these terms as in the case of a soliton in a confining potential one does not need to consider interactions between phonons. The second term in the Lagrangian Eq. (B.0.16) is zero as we know that the condition $\mu = \partial e_0/\partial n$ holds for equilibrium. We remind ourselves that although we have allowed the phonon coordinates to fluctuate we have not considered them to be out of equilibrium. The first term is just the background fluid Lagrangian defined in Eq. (B.0.1) in the absence of any fluctuations. Thus it is the final term that is indeed the phonon Lagrangian that we derived previously. We also note that this procedure so far is very general for a system with a slowly varying background, this is valid for an impurity as well as the soliton.

Now that we have obtained the phonon Lagrangian we can finally move on to obtain the coupling of these phonons to the variables describing the depletion cloud. In the presence of phonons we can write down the supercurrent and chemical potential in the frame of reference that moves with the impurity, these quantities are given as follows,

$$\mu' = \mu - \dot{\psi} - \frac{mu^2}{2} + \frac{(V-u)^2}{2} = \mu + \frac{mV^2}{2} - \left(\dot{\psi} + V\partial_x\psi\right) \tag{B.0.17}$$

$$j' = -(n+\rho)(V-u) = -nV - \frac{1}{\pi}\left(\dot{\vartheta} + V\partial_x\vartheta\right). \tag{B.0.18}$$

The final expression for the modified supercurrent given in Eq. (B.0.18) has been derived by using the continuity equation which is equivalent to $\dot{\theta}/\pi = -(n+\rho)u$ for the counting field. We can now substitute the modified supercurrent and chemical potential Eqs. (B.0.17) and (B.0.18) into Eq. (B.0.9) and subtract off the equilibrium values in the absence of phonons we obtain the coupling derived in the previous section, where we have defined the full time derivative as $d/dt = \partial_t + \dot{X}\partial_x = \partial_t + V\partial_x$.

We have given a second derivation of the coupling to really convince ourselves that it is indeed the soliton parameters Φ and N coupling to the phonon modes and that the coupling comes in the form of a total time derivative of the phonon variables.

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