ALMOST EVERYWHERE CONVERGENCE OF DYADIC PARTIAL SUMS OF FOURIER SERIES FOR ALMOST PERIODIC FUNCTIONS

A. D. BAILEY

Comments, Corrections and Clarifications

I aim to keep this document up-to-date with corrections to my MPhil thesis as well as clarifications of and comments on any areas that I feel deserve further exposition. In time, I may also include some commentary on further research related to the original work.

Should you discover any additional errors or if you want to contact me about my research, please feel free to email me at maths@andrewdbailey.com.

The latest version of this document and an electronic copy of the original thesis are available through the University of Birmingham's eTheses Repository at `http://etheses.bham.ac.uk/268/`.

1. Definition of the Besicovitch Almost Periodic Function Spaces

The definition of the Besicovitch spaces, B^p, given as part of Definition 1.1.6 on p. 6 is not correct. In [1], spaces defined in this way are given as Definition 5.16 on p. 39 and referred to as the spaces of functions that are almost periodic "in the sense of Doss"*. A similar class is also considered in [2] (see Théorème I of Chapitre VII on p. 69); in this case, an additional assumption of a form of continuity is made.

In its stated form, however, the definition of the Besicovitch spaces given in Definition 1.1.6 does not agree with Besicovitch's original definition and, in particular, when defined in this way, the B^p spaces do not actually satisfy the Fundamental Theorem (Theorem 1.2.1, p. 7) as claimed. To construct Besicovitch's original definition, as given in [3], pp. 77-78, the notion of a satisfactorily uniform set must be introduced:

Definition (Satisfactory Uniformity)**.** A countable set $A \subset \mathbb{R}$ is said to be satisfactorily uniform if there exists $K \in \mathbb{R}^+$ such that

$$\frac{\sup_{x\in\mathbb{R}} |A \cap [x, x+K]|}{\inf_{x\in\mathbb{R}} |A \cap [x, x+K]|} < 2.$$

Given this, Besicovitch's original definition of the B^p spaces is as follows:

Definition (Besicovitch Spaces: Original Definition)**.** For $p \in [1, \infty)$, the Besicovitch spaces, B^p, are defined to be the spaces of all functions $f \in L^p_{\text{loc}}(\mathbb{R})$ such that for all $\varepsilon > 0$, there exists a satisfactorily uniform sequence $(\tau_i)_{i\in\mathbb{Z}} \subset \mathbb{R}$ such that for any $i \in \mathbb{Z}$,

$$\|f(\cdot + \tau_i) - f\|_{B^p} < \varepsilon$$

and for any $c > 0$,

$$\left(\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \limsup_{N\to\infty} \frac{1}{2N+1} \sum_{|i|\leqslant N} \frac{1}{c} \int_x^{x+c} |f(s+\tau_i) - f(s)|^p \, ds \, dx \right)^{\frac{1}{p}} < \varepsilon.$$

The Fundamental Theorem is genuinely satisfied for the Besicovitch spaces when defined in this manner. It is noted that the Besicovitch spaces as defined here are strict subspaces of those as defined in Definition 1.1.6. Additionally, the satisfactory uniformity condition on the distribution of the τ_i is on its own a stronger requirement than the distribution condition given there. For further details, see [1].

Last updated on Tuesday 19th October 2010 at 16:00.

*This attribution appears to be incorrect, however. The cited paper of Doss, [4], requires additional conditions on the functions considered, showing that under these assumptions, the functions obtained are the same as those obtained by Besicovitch's original definition.

2. α–Dependence in Operator Bounds

It should be clarified that there is a dependence on α in the operator bounds of Theorems 3.2.1, 3.5.4 and 3.5.10, as can be seen from the proofs given. These bounds becomes large as α becomes small.

3. Typographical Errors

- On p. 16, the equality concluded in Lemma 1.3.4 only holds if f is such that $|f(x)| = f(x)$ for all $x \in \mathbb{R}$.
- On p. 37, line 11 (answering question 2), "$T : S^p \to S^p$ is weakly bounded $(p - -p)$" should instead read "$T : S^p \to S^p$ is weakly bounded $(p\text{–}p)$".
- On p. 41, in the statement of Theorem 3.2.2, the Fourier transform of ψ_μ should not be normalised with the $\dfrac{1}{2\pi}$ factor. Instead, ζ_μ should be defined as $\zeta_\mu(u) = \displaystyle\int_{\mathbb{R}} \psi_\mu(t)e^{-iut}\,dt = \widehat{\psi}_\mu(u)$.
- On p. 42, in the calculation of the Fourier coefficients of f_μ, on the fourth line, the limits in the integral should be $-T - u$ and $T - u$, not $-T + u$ and $T + u$.
- On the penultimate line of p. 43, "$R_k f = \displaystyle\sum_{n\in\mathbb{N}} \widehat{f}(\lambda_n)\widehat{\phi_k}(\lambda_n)e^{i\lambda_n \cdot}$" should instead read "$R_k f = \displaystyle\sum_{n\in\mathbb{Z}} \widehat{f}(\lambda_n)\widehat{\phi_k}(\lambda_n)e^{i\lambda_n \cdot}$".
- The first inequality on p. 48 is not stated as it was intended, making the equality that follows it incorrect. The absolute value should remain outside the sum in m; the first line is not necessarily less than the second in this case, but it is equivalent to look at this quantity by noting that the quantity $\|(a_n)\|_{\ell^2}$ in the conclusion of the lemma is independent of any unimodular factors attached to the sequence (a_n).
- For greater clarity, all instances of K_l on pages 65 and 66 should read K_0.

References

[1] Andres, J., Bersani, A. M. and Grande, R. F., Hierarchy of Almost-Periodic Function Spaces, *Rendiconti di Matematica e delle sue Applicazioni (Serie VII)* **26**, *no. 2* (2006), pp. 121–88.

[2] Bertrandias, J.-P., Espaces de Fonctions Bornées et Continues en Moyenne Asymptotique d'Ordre p (Spaces of Bounded Continuous Functions on Average Asymptotic of Order p – in French), *Mémoires de la Société Mathématique de France* **5** (1966), pp. 3–106.

[3] Besicovitch, A. S., *Almost Periodic Functions*, Dover, New York, 1954.

[4] Doss, R., On Generalized Almost Periodic Functions, *Annals of Mathematics* **59**, *no. 3* (1954), pp. 477–489.