HARDY-TYPE INEQUALITIES FOR NON-CONVEX DOMAINS

By

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A thesis submitted to the University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY (PH.D.)

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2010

Abstract

This monograph is devoted to a study of certain Hardy-type inequalities for non-convex domains $\Omega \subset \mathbb{R}^n$. We consider inequalities of the following form

$$\mu \int\limits_{\Omega} \frac{|f(x)|^2}{d(x)^2}\, dx \leq \int\limits_{\Omega} |\nabla f(x)|^2 \; dx, \qquad f(x) \in \mathcal{C}_c^{\infty}(\Omega)\,,$$

where

$$d(x) := \min\{|x-y| : y \notin \Omega\}.$$

For convex domains, $\frac{1}{4}$ is the precise value of the constant μ. This thesis is concerned with finding estimates on μ for non-convex multi-dimensional domains. Some estimates were obtained earlier by other authors for simply connected planar domains with the help of complex-analytic methods.

Our aim is to obtain lower bounds for the optimal constant μ, by real-analytic methods, for certain classes of multi-dimensional non-convex domains without the assumption of simple connectedness. To this end, we impose some geometrical conditions on domains $\Omega \subset \mathbb{R}^n;\ n \geq 2$. In fact three types of such conditions are introduced, namely Cone, Exterior Ball and Cylinder conditions. Consequently, new Hardy-type inequalities for non-convex domains are obtained.

Acknowledgments

I acknowledge with deep gratitude the considerable assistance I have received from my supervisor Professor Alexander V. Sobolev in the preparation of this thesis. I gratefully admit his suggestion of this research topic and the helpful ideas arising from our discussions. I have to say that he is the first one who introduced Hardy's inequality to me. I would also like to express my appreciation for his patience, kindness and support, all of which made the research enjoyable.

Also much appreciated are the helpful comments received from my co-supervisor Dr. Jonathan Bennett. I wish to thank him for his magnificent support, professional advice and experience sharing during the research period, his indispensable remarks made this work look much better than before.

I am profoundly grateful to my colleagues Abdulhadi, Carlos and Thomas for their help and support, and Peter for his kind friendship.

I am also much obliged to my parents and my lovely wife for their support and trust throughout these long hard times.

Finally, I am sincerely thankful to my country, Egypt, not only for the financial support but also for making me who I am.

Contents

Chapter 1

Introduction

1.1 Introduction

In this chapter we shed some light on a broad kind of inequality in which some quantity involving the size of a function (typically a norm) is controlled by some quantity involving the size of derivatives of the function (such as the gradient). Perhaps the simplest example would be:

$$\| f \|_{L^\infty} \leq \| f^{'} \|_{L^1},$$

which holds for functions $f \in \mathcal{C}^1([0,\infty))$.[1] This follows directly from the Fundamental Theorem of Calculus. Such inequalities may also be referred to as Sobolev-type inequalities. In the following section two kinds of Sobolev-type inequalities will be presented. Section 1.3 is devoted to introducing Hardy's inequality, which falls into this category, and discussing the aim of this thesis. Section 1.4 describes the structure of the thesis and discusses the most significant results obtained.

[1] For notations used in this thesis please see Appendix A, Page 144.

1.2 Sobolev-type inequalities

Sobolev spaces and embedding theorems for them are of a great significance in functional analysis and partial differential equations. Sobolev-type inequalities are directly implying Sobolev embedding theorems, giving inclusions between certain Sobolev spaces, see for example R. A. Adams ([1, Chapter V]), E. B. Davies ([16, Page 40]), E. H. Lieb and M. Loss ([37, Chapter 8]) and V. G. Maz'ja [40]. There are many inequalities which are ascribed to Sobolev, but here we shall introduce the most basic ones. It is sufficient to mention the classical Sobolev inequality for gradients and the logarithmic Sobolev inequality. Sobolev's inequality for gradients is as follows:

Theorem 1.2.1 (see [37] or [51], Sobolev's inequality for gradients)**.**
Let $n \geq 3$, and f be any real (or complex) valued function defined on $\mathbb{R}^n$, sufficiently smooth and decaying fast enough at infinity. Then $f \in L^q(\mathbb{R}^n)$ with $q = \frac{2n}{n-2}$ and the following inequality holds:

$$\left(\int\limits_{\mathbb{R}^n} |f(x)|^q \, dx \right)^{2/q} \leq S_n \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx, \tag{1.1}$$

where

$$S_n = \frac{4}{n(n-2)} |\mathbb{S}^n|^{-2/n} = \frac{2^{2-2/n}}{n(n-2)} \pi^{-(1+1/n)} \Gamma \left(\frac{n+1}{2} \right)^{2/n}. \tag{1.2}$$

There is equality in (1.1) *if and only if f is a multiple of the function $(\mu^2 + (x-a)^2)^{-(n-2)/2}$ with $\mu > 0$ and $a \in \mathbb{R}^n$ arbitrary.*

Inequality (1.1) is an important tool in proving the existence of a ground state for the one-particle Schrödinger equation, for more details and proofs see [37, Chapters 8, 11].
Many questions have been raised concerning inequality (1.1). Does (1.1) hold for L^p norms instead of L^2 norm of the gradient? What is the constant in this case? Since (1.1) is only valid for $n \geq 3$, then what is the Sobolev inequality in dimensions one and two?
The answers to the first two questions were found in [51]. It was shown that for all $1 < p <$

n, with $n \geq 3$, the following inequality holds:

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \, dx \right)^{1/q} \leq S_{p,n} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx \right)^{1/p}, \quad \text{with } q = \frac{np}{n-p}, \tag{1.3}$$

where the sharp constant $S_{p,n}$ is given by

$$S_{p,n} = \left(\frac{p-1}{n-p} \right)^{1-1/p} \pi^{-1/2} n^{-1/p} \left(\frac{\Gamma(1+n/2)\,\Gamma(n)}{\Gamma(n/p)\,\Gamma(1+n-n/p)} \right)^{1/n}. \tag{1.4}$$

Sharp constants play an essential role because they contain geometric and probabilistic information. This is why we find that many authors are concerned with the constants in Sobolev's inequalities (see for instance [2, 10, 50]).
Inequality (1.3) is referred to as Gagliardo-Nirenberg-Sobolev inequality, see [21, Page 262], and it directly implies the Sobolev embedding

$$W^{1,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n).$$

Concerning the question about Sobolev's inequalities in $\mathbb{R}$ and $\mathbb{R}^2$, there are many different forms, see for instance [1, Chapter V], [37, Page 205], [44, Chapter 1] and [45]. There are many different generalisations in the literature on the classical Sobolev inequality (1.1). One possible generalisation is to replace the first derivatives by higher derivatives. One can also consider more general domains $\Omega \subset \mathbb{R}^n$. Such generalisations can be found e.g. in [1], [37, Chapter 8] and [40].

The classical Sobolev inequality (1.1) helps giving information about the size of a function when measured in the $L^q(\mathbb{R}^n)$ norm, where $q = \frac{2n}{n-2}$, in terms of the derivatives of the function. Natural question here is whether one can replace $L^q(\mathbb{R}^n)$ by a different class of functions which wouldn't depend on n. One such example is given by the so-called logarithmic Sobolev inequality proved first in [49]. Many other inequalities known as logarithmic Sobolev inequalities have appeared like the following one (see [37, Page 223, Theorem 8.14]):

Theorem 1.2.2 (The logarithmic Sobolev inequality).
Let f be any function in $H^1(\mathbb{R}^n)$ and let $a > 0$ be any number. Then the following inequality holds:

$$\int\limits_{\mathbb{R}^n} |f(x)|^2 \log\left(\frac{|f(x)|^2}{\|f(x)\|_2^2}\right) dx + n(1+\log a)\|f(x)\|_2^2 \leq \frac{a^2}{\pi} \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx. \tag{1.5}$$

Moreover, there is equality if and only if f is, up to translation, a multiple of $\exp\{-\pi|x|^2/2a^2\}$.

The logarithmic Sobolev inequality (1.5), and its variations have played an important role in the analysis of the heat kernels associated with many second order elliptic operators, see for instance [14], [15] and [37, Chapter 8]. Besides, they have been exploited to study spectral properties of some differential operators such as Sturm-Liouville and Schrödinger operators, see [27], [46] and the references therein.

Based on what have been stated above, one sees the importance of inequalities that bound the gradient norms from below by some norms of the function. Among these inequalities is Hardy's inequality, which is the main focus of this thesis.

1.3 Hardy's inequality: history and problem formulation

Hardy's inequality is one of those inequalities which turns information about derivatives of functions into information about the size of the function. This section is concerned with introducing the original form of Hardy's inequality through a historical remark followed by a brief discussion of the scope of this thesis.

1.3.1 Historical remark

To talk about the history of the original form of Hardy's inequality, it is necessary to introduce Hilbert's theorems which deal with special cases of some bilinear forms, see [29, Chapter IX].

Theorem 1.3.1.

Suppose that $p > 1$, $p' = p/(p-1)$, and that $K(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has the following properties:

(1) K is non-negative, and homogeneous of degree -1, i.e. $K(tx,ty) = t^{-1}K(x,y)$ for any $x, y \in \mathbb{R}$ and $t \neq 0$ real.

(2) $$\int_0^\infty K(x,1)\, x^{-1/p} dx = \int_0^\infty K(1,y)\, y^{-1/p'} dy = k, \text{ and}$$

(3) $K(x,1)\, x^{-1/p}$ is a strictly decreasing function of x, and $K(1,y)\, y^{-1/p'}$ of y.

Then, for non-negative sequences (a_m) and (b_n) we have

(i) $$\sum_{n=1}^\infty \sum_{m=1}^\infty K(m,n)\, a_m\, b_n < k \left(\sum_{m=1}^\infty a_m^p\right)^{1/p} \left(\sum_{n=1}^\infty b_n^{p'}\right)^{1/p'},$$
unless (a_m) or (b_n) is null;

(ii) $$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty K(m,n)\, a_m\right)^p < k^p \sum_{m=1}^\infty a_m^p,$$
unless (a_m) is null;

(iii) $$\sum_{m=1}^\infty \left(\sum_{n=1}^\infty K(m,n)\, b_n\right)^{p'} < k^{p'} \sum_{n=1}^\infty b_n^{p'},$$
unless (b_n) is null.

The analogous theorem for integrals corresponding to Theorem 1.3.1 is

Theorem 1.3.2.

Suppose that $p > 1$, $p' = p/(p-1)$, that $K(x,y)$ is non-negative and homogeneous of *degree* -1, *and that*

$$\int_0^\infty K(x,1)\, x^{-1/p} dx = \int_0^\infty K(1,y)\, y^{-1/p'} dy = k.$$

Then, for non-negative functions f *and* g *we have*

(i) $$\int_0^\infty\int_0^\infty K(x,y)\, f(x)\, g(y) dx\, dy \le k \left(\int_0^\infty f(x)^p dx\right)^{1/p} \left(\int_0^\infty g(x)^{p'} dx\right)^{1/p'},$$

(ii) $$\int_0^\infty dy \left(\int_0^\infty K(x,y)\, f(x) dx\right)^p \le k^p \int_0^\infty f(x)^p dx,$$

(iii) $$\int_0^\infty dx \left(\int_0^\infty K(x,y)\, g(y) dy\right)^{p'} \le k^{p'} \int_0^\infty g(y)^{p'} dy.$$

If $K(x,y)$ *is positive, then there is inequality in* (ii) *unless* $f \equiv 0$, *in* (iii) *unless* $g \equiv 0$, *and in* (i) *unless either* $f \equiv 0$ *or* $g \equiv 0$.

If we take $K(x,y) = \frac{1}{x+y}$, Theorems 1.3.1 and 1.3.2 give the following theorems, known as ‘Hilbert’s theorems’.

Theorem 1.3.3 (Hilbert’s inequality: discrete form)**.**
If $p > 1,\ \ p' = p/(p-1)$, *then for non-negative sequences* (a_m) *and* (b_n) *we have*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m\, b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^\infty a_m^p\right)^{1/p} \left(\sum_{n=1}^\infty b_n^{p'}\right)^{1/p'}, \tag{1.6}$$

unless (a_m) *or* (b_n) *is null.*

Theorem 1.3.4 (Hilbert’s inequality: integral form)**.**
If $p > 1,\ \ p' = p/(p-1)$, *then for non-negative integrable functions* $f(x)$ *and* $g(x)$ *we have*

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f(x)^p dx\right)^{1/p} \left(\int_0^\infty g(x)^{p'} dx\right)^{1/p'}, \tag{1.7}$$

unless $f \equiv 0$ *or* $g \equiv 0$.

The case $p = p' = 2$ of Theorem 1.3.3 is known as 'Hilbert's double series theorem'. For applications, generalisations and proofs of Hilbert's theorems, especially a proof that the constant $\frac{\pi}{\sin(\pi/p)}$ therein is the best possible, see for example [29, Chapter IX] and [43].

Hilbert's double series theorem, Theorem 1.3.3 with $p = p' = 2$, was proved first by D. Hilbert in his lectures on integral equations. The extensions to general p are due to G. H. Hardy and M. Riesz, see [28]. In the literature there are many proofs of Theorem 1.3.3. Among these is a simple proof due to G. H. Hardy, we provide this proof since its main idea is the inequality which now bears his name. Divide the double series in (1.6) into two parts as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m\,b_n}{m+n} = \sum_{m\leq n}\sum\frac{a_m\,b_n}{m+n} + \sum_{m>n}\sum\frac{a_m\,b_n}{m+n}. \tag{1.8}$$

Then for the first term on the right hand side of (1.8), we have

$$\sum_{m\leq n}\sum\frac{a_m\,b_n}{m+n} \leq \sum_{m\leq n}\sum\frac{a_m\,b_n}{n} = \sum_{n=1}^{\infty}\frac{A_n}{n}\,b_n,$$

where $A_n = a_1 + a_2 + \cdots + a_n$.

Since $\sum\limits_{n=1}^{\infty} b_n^{p'}$ is convergent, then by Young's inequality we find that the series $\sum\limits_{n=1}^{\infty}\frac{A_n}{n}\,b_n$ is convergent whenever $\sum\limits_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p$ is convergent. Consequently, to prove the convergence of the first term on the right hand side of (1.8), we need to prove the convergence of $\sum\limits_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p$ using the information given. The convergence of the second term in (1.8) could then be shown in the same way.

In 1920, G. H. Hardy proved the convergence of $\sum\limits_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p$ as a consequence of that of $\sum\limits_{n=1}^{\infty} a_n^p$. His result as stated in [29, Page 239] is as follows:

Theorem 1.3.5 (The discrete Hardy inequality)**.**
If $p > 1$, $a_n \geq 0$, *and* $A_n = a_1 + a_2 + \cdots + a_n$, *then*

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \tag{1.9}$$

unless all the a_i*'s are zero. The constant is the best possible.*

The integral analogue of Theorem 1.3.5 is

Theorem 1.3.6 (The integral Hardy inequality)**.**
If $p > 1$, $f(x) \geq 0$, *and* $F(x) = \int_0^x f(t)dt$, *then*

$$\int_0^{\infty} \left(\frac{F}{x} \right)^p dx < \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f^p dx, \tag{1.10}$$

unless $f \equiv 0$. *The constant is the best possible.*

For proofs and more discussion on Theorems 1.3.5 and 1.3.6 see [29, Chapter IX].

1.3.2 Problem formulation

This thesis is devoted to integral Hardy-type inequalities which are considered as extensions of the one-dimensional inequality (1.10) to higher dimensions. Namely, we study inequalities of the following type (see for example [4, 18]):

$$\mu \int_{\Omega} \frac{|f(x)|^2}{d(x)^2} dx \leq \int_{\Omega} |\nabla f(x)|^2 \, dx, \quad f \in \mathcal{C}_c^{\infty}(\Omega), \tag{1.11}$$

where

$$d(x) := \min\{|x - y| : y \notin \Omega\},$$

is the distance from the point $x \in \Omega$ to the boundary $\partial\Omega$ of the domain Ω, and μ is a positive constant, which will be referred to as Hardy's constant. Inequality (1.11) is the multi-dimensional version of (1.10) with $p = 2$, and the integrals are over regions Ω in $\mathbb{R}^n$; $n \geq 2$. In Hardy's one-dimensional inequality, (1.10), there is no geometrical problem coming from the domain, and the constant is 4, for $p = 2$. However, in regard to Hardy's inequalities for domains in $\mathbb{R}^n$ the situation is more complicated. That is why there are many limitations when studying Hardy's inequalities for such domains. Consequently, the best constant $\mu = \mu(\Omega)$ in (1.11) depends on the domain Ω. It is known that for convex domains $\Omega \subset \mathbb{R}^n$, $\mu(\Omega) = \frac{1}{4}$ and it is sharp, see for instance [17, Chapter 5], but there are smooth domains such that $\mu(\Omega) < \frac{1}{4}$ (see for example [19, 38, 39]).
However, the sharp constant for non-convex domains is unknown, although for planar simply-connected domains Ω, A. Ancona ([4]) proved, using the Koebe one-quarter Theorem, that the constant in (1.11) is greater than or equal to $\frac{1}{16}$. A. Laptev and A. Sobolev ([35]) considered classes of domains for which there is a stronger version of the Koebe Theorem, which in turn implied better estimates for the constant appearing in (1.11). Other specific examples of non-convex domains were presented by E. B. Davies in [18].

Our objective is to derive Hardy-type inequalities for non-convex domains $\Omega \subset \mathbb{R}^n$; $n \geq 2$, and investigate how the constants $\mu(\Omega)$ depend on the non-convexity parameters. In contrast with defining the convexity of a domain, 'measuring' the non-convexity can be done in many ways. Therefore, we state some geometrical conditions, to act as 'non-convexity measures', under which we achieve our goal. In fact, we present three different conditions:

- Cone condition, which assumes that we can touch every point on the boundary of the domain Ω with a 'spike'. This condition is applicable to non-smooth domains, and the non-convexity is regulated by the angle of the cone.

- Exterior Ball condition, which presumes that we can touch every point on the boundary of the domain Ω with a ball of radius $R > 0$. This condition is applicable to smooth domains, and the non-convexity depends on the radius of that ball. In particular, when R tends to ∞ we obtain a convex case.

- $(n, k)-$Cylinder condition, which for certain values of its parameters covers the Exterior Ball condition and the convex case.

In all of the above conditions, the degree of non-convexity is regulated by some parameters. In particular, for limiting values of those parameters, we allow the domains to become convex. One important issue studied in the thesis is to find the limiting behaviour of the Hardy constants in the obtained inequalities, when the domains become convex in the appropriate limit. For instance under the Exterior Ball Condition in $\mathbb{R}^3$, we obtain the following asymptotic form for the constant in Hardy, inequality:

$$\mu(x, R) = \frac{1}{4} - \frac{d(x)}{2R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{\frac{3}{2}}\right),$$

which tends to $\frac{1}{4}$ as R tends to ∞ linearly in $\frac{d(x)}{R}$.

Hardy-type inequalities, which we will sometimes refer to as H-I, as well as their improved versions and extensions, have a great importance in the development of certain branches of mathematics. For example, in [16, Page 32], [17, Chapter 5] and [42] we find interesting usage of H-I in Spectral Theory, Fourier Analysis and Interpolation Theory. In Chapter 2, we present examples of exciting applications of H-I. We show how they can be used as technical tools in the study of the spectrum of elliptic operators as well as an application in proving the existence and uniqueness of solutions in the theory of viscous incompressible flow.

Since 1920 H-I have always been in the centre of interest of the analysists. However, in the last two or three decades the interest in various versions of H-I has been increased. This fact is confirmed by a great number of published books and papers. For instance, some authors are interested in the best constants in H-I for various domains (e.g. [18, 48]), while others are concerned with improving these inequalities by adding terms including norms other than L^p-norms (see [9, 23]). On the other hand, the book 'Hardy-type Inequalities' by B. Opic and A. Kufner ([42]) is entirely devoted to an extensive study of this kind of inequalities. It, as described in [30], investigates the following wide, key question: under what restrictions on the domain $\Omega \subset \mathbb{R}^n$, on the functions f, on the weights $w, v_1, v_1, \cdots, v_n$, and on the parameters $p,\ q$ does the inequality

$$c\left(\int\limits_{\Omega} w(x)|f(x)|^q dx\right)^{1/q} \leq \left(\Sigma_{i=1}^n \int\limits_{\Omega} \left|\frac{\partial f(x)}{\partial x_i}\right|^p v_i(x)dx\right)^{1/p}, \tag{1.12}$$

hold? Note that for $p = q = 2$, $w(x) = \frac{1}{d(x)^2}$, $v_1 = v_1 = \cdots = v_n = 1$ and $f(x) \in \mathcal{C}_c^{\infty}(\Omega)$, inequality (1.12) gives the Hardy inequality (1.11), whereas it produces the Gagliardo-Nirenberg-Sobolev inequality (1.3), if $w(x) = 1$, $v_1 = v_2 = \cdots = v_n = 1$, $q = np/(n-p)$ with $1 \leq p < n$, and $\Omega \equiv \mathbb{R}^n$.

In fact, inequality (1.12) covers a large spectrum of inequalities for certain choices of the weight functions, p and q. Apart from the Gagliardo-Nirenberg-Sobolev inequality, (1.3), it also covers Gagliardo-Nirenberg inequality, Nash's inequality and Poincaré's inequality (see for example [21]).

1.4 Thesis structure

The thesis is organized as follows: Chapter 2 is largely background but concentrates on a specific topic, the Hardy-type inequalities, presenting some generalisations, improvements and applications of them. Most of the material therein is quite familiar to a reader who is working on Hardy's inequalities.

Chapter 3 may be described as the key chapter of the thesis, in which we establish all the basic notions and notations that will be utilized throughout the following chapters. In addition, it contains two-dimensional geometrical conditions under which we obtain new forms of H-I for domains in $\mathbb{R}^2$.

Chapter 4 is concerned with generalisations of the conditions formed in Chapter 3 to the three-dimensional case, in addition to a new condition associated with higher-dimensional domains. For each single one of these conditions we prove a new formula of H-I for domains in $\mathbb{R}^3$. This chapter serves in understanding the $n-$dimensional study which appears in Chapter 5 where all conditions and results found in Chapter 4 will be extended to $n-$dimensional domains.

Chapter 6 summarises what have been attained through the thesis.

The thesis is concluded with some appendices containing some definitions and concepts used within the thesis.

Our main contribution to the study of Hardy-type inequalities for non-convex domains can be seen through a group of theorems, namely Theorems 3.3.1, 3.4.1, 4.3.1, 4.3.3, 4.4.1, 4.5.1, 5.3.1, 5.3.3, 5.4.1, 5.5.2, where in each theorem we obtained a new Hardy type inequality. Of equivalent significance are the remarks related to those theorems, explicitly Remarks 3.3.2, 3.4.2, 4.3.2, 4.3.4, 4.4.2, 4.5.2, 5.3.2, 5.3.4, 5.4.2, 5.5.3. Although some of those theorems give relatively complex forms for μ (which sometimes depends on x), the related remarks give some simplifications for those forms. For instance, Remark 3.3.2, Part 3 and Theorem 5.3.1 are of a special importance since they give 'simple' forms for μ in $\mathbb{R}^2$ and $\mathbb{R}^n$ respectively. Furthermore, Theorem 5.5.2 is of particular significance since it covers Theorems 5.4.1, consequently Theorem 4.4.1, and 4.5.1.

Chapter 2

Literature review

2.1 Introduction

In this chapter we present some historical background and a literature review of the work done on Hardy-type inequalities. Different aspects and generalisations of the original integral Hardy inequality (Theorem 1.3.6) will be discussed. Moreover, some applications and improvements will be presented. An extensive study of Hardy-type inequalities can be found in [32] and [42].

2.2 The Hardy inequality: various forms

This section is devoted to giving different versions of the following integral Hardy inequality:

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx < \mu \int_0^\infty f^p dx, \tag{2.1}$$

where $F(x) = \int_0^x f(t)\,dt$. Inequality (2.1) was proved by G. H. Hardy in 1920 and it can be seen as the L^p boundedness of the averaging operator H where $Hf(x) = \frac{1}{x}\int_0^x f(t)dt$, with $1 < p \leq \infty$. In 1926 Landau ([34]) obtained the exact value of the optimal constant μ which is $\left(\frac{p}{p-1}\right)^p$. This value of μ can also be seen in [38] or simply from Theorem 1.3.2 by setting

$$K(x,y) = \begin{cases} \frac{1}{y} & , \quad x \le y, \\ 0 & , \quad x > y, \end{cases}$$

and then if $p > 1$ we obtain

$$k = \int_0^\infty K(x,1) x^{\frac{-1}{p}}\, dx = \int_0^1 x^{\frac{-1}{p}}\, dx = \frac{p}{p-1},$$

and all the conditions on K are verified.

We are interested in the case when $p = 2$ in (2.1) for domains in $\mathbb{R}^n$; $n \ge 2$. Although inequality (2.1) was stated and proved for functions on a half-line, many extensions and generalisations of it in different aspects have appeared in the literature since 1920. For extensive study and numerous generalisations of (2.1), see for instance [29, Chapter IX] and [42]. In what follows we give some ‘simple’ examples of such extensions, starting with the following weighted one-dimensional integral Hardy inequality (see E. B. Davies ([17, Page 104])):

Lemma 2.2.1.

Let $0 < b < \infty$ and let f be a $\mathcal{C}^1$ function defined on $[0, b]$ which vanishes in some neighborhood of 0. Then

$$\frac{(1-\alpha)^2}{4} \int_0^b x^{\alpha-2} |f(x)|^2\, dx \le \int_0^b x^\alpha \left| f(x)' \right|^2 dx, \tag{2.2}$$

provided $-\infty < \alpha < 1$.

If we apply Lemma 2.2.1 to the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ with $\alpha = 0$, one can obtain the following inequality, with the distance function as a weight function (see [17, Page 105]):

Corollary 2.2.2.

Let $a < b$ and put $d(x) := \min\{x-a, b-x\}$ for all $a < x < b$. Then

$$\frac{1}{4}\int_a^b \frac{|f(x)|^2}{d(x)^2}dx \leq \int_a^b \left|f(x)'\right|^2 dx,$$

for all $f \in \mathcal{C}_c^\infty((a,b))$.

A two-dimensional version of (2.1), with a 'pseudodistance' function as a weight function, is stated in [17, Page 107] as follows:

Theorem 2.2.3.

Let Ω be a domain (open connected) *in $\mathbb{R}^2$ and let $f \in \mathcal{C}_c^\infty(\Omega)$. Then*

$$\frac{1}{2}\int_\Omega \frac{|f(x)|^2}{m(x)^2}\,dx \leq \int_\Omega |\nabla f(x)|^2\,dx, \tag{2.3}$$

where the 'pseudodistance' $m(x)$ is defined by

$$\frac{1}{m(x)^2} := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{d_\theta(x)^2}, \tag{2.4}$$

and $d_\theta : \Omega \to (0, +\infty]$ is defined by

$$d_\theta(x) := \min\left\{|s| : x + se^{i\theta} \notin \Omega\right\}. \tag{2.5}$$

Remark 2.2.4.

The function $d_\theta(x)$, defined by (2.5), is a periodic function in θ with period π.

An n-dimensional version of Theorem 2.2.3 is (see E. B. Davies [16, Page 27] or [19]):

Theorem 2.2.5.

Let Ω be a domain in $\mathbb{R}^n$ and let $f \in \mathcal{C}_c^\infty(\Omega)$. Then

$$\frac{n}{4}\int_\Omega \frac{|f(x)|^2}{m(x)^2}\,dx \le \int_\Omega |\nabla f(x)|^2\,dx, \tag{2.6}$$

where $m(x)$ is defined by

$$\frac{1}{m(x)^2} := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2}dS(u), \tag{2.7}$$

and

$$d_u(x) := \min\{|t| : x + tu \notin \Omega\},$$

for every unit vector $u \in \mathbb{S}^{n-1}$ and $x \in \Omega$. Here $|\mathbb{S}^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the surface area of the unit sphere in $\mathbb{R}^n$.

We underline here that inequalities (2.3) and (2.6) contain the 'pseudodistance' $m(x)$ defined by (2.4) and (2.7) respectively. A central question here is, how can one derive similar inequalities containing the 'true' distance $d(x)$, the distance to the boundary of the domain Ω, which is defined to be

$$d(x) := \text{dist}\,(x, \partial\Omega) = \min\{|x-y| : y \notin \Omega\}? \tag{2.8}$$

Another question which can be asked here, as a consequence of the above question, is in what form will the constants be?

Due to E. B. Davies [17, Exercise 5.7], we can use Theorem 2.2.3 to prove the following inequality for planar convex domains.

Theorem 2.2.6.

Let Ω be a convex domain in $\mathbb{R}^2$, then

$$\frac{1}{4}\int_{\Omega}\frac{|f(x)|^2}{d(x)^2}\,dx \le \int_{\Omega}|\nabla f(x)|^2\,dx, \tag{2.9}$$

for all $f \in \mathcal{C}_c^{\infty}(\Omega)$.

Proof. To prove this theorem, we first obtain, for arbitrary $x \in \Omega$, a lower bound on the function $\frac{1}{m(x)^2}$, defined by (2.4), relevant to the 'true' distance $d(x)$, given by (2.8). More precisely, we prove for any convex domain $\Omega \in \mathbb{R}^2$ that the following inequality holds:

$$\frac{1}{m(x)^2} \ge \frac{1}{2d(x)^2}.$$

Then an application of Theorem 2.2.3 will complete the proof.

Suppose that a is the closest point of $\partial\Omega$ (the boundary of the domain Ω)[1] to the point x. Let $d(x)$ be the distance from $x \in \Omega$ to a, $d_\theta(x)$ be as defined in (2.5), i.e. the distance

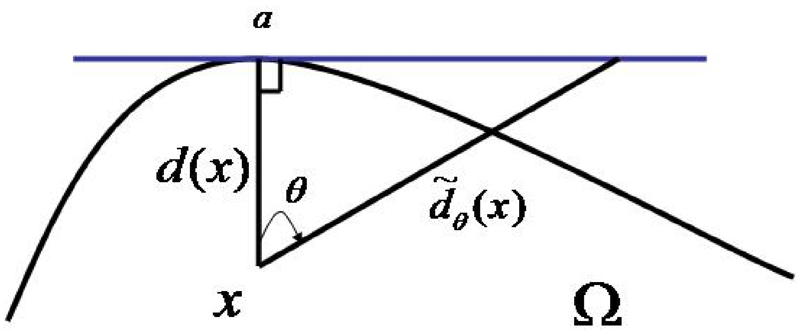

Figure 2.1: A convex domain in $\mathbb{R}^2$

from $x \in \Omega$ to $\partial\Omega$ in the direction θ, and $\tilde{d}_\theta(x)$ be the distance from $x \in \Omega$ to the tangent

[1] $\partial\Omega = \bar{\Omega} \setminus \Omega$

of $\partial\Omega$ (at the point a) in the direction θ, where θ is the angle between the line segments representing $d(x)$ and $\tilde{d}_\theta(x)$, see Fig. 2.1.

Then clearly the relation between $d_\theta(x)$ and $\tilde{d}_\theta(x)$ is such that $d_\theta(x) \leq \tilde{d}_\theta(x)$ and for any convex domain $\Omega \subset \mathbb{R}^2$ we have $\tilde{d}_\theta(x) = \frac{d(x)}{\cos\theta}$, which implies that $\frac{1}{d_\theta(x)^2} \geq \frac{\cos^2\theta}{d(x)^2}$. Thus we obtain

$$\begin{aligned}
\frac{1}{m(x)^2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{d_\theta(x)^2} \\
&\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos^2\theta d\theta}{d(x)^2} = \frac{1}{2\pi d(x)^2}\int_{-\pi}^{\pi} \cos^2\theta d\theta \\
&= \frac{1}{2\pi d(x)^2}\int_{-\pi}^{\pi} \frac{1}{2}\left(1+\cos 2\theta\right) d\theta = \frac{2\pi}{4\pi d(x)^2} \\
&= \frac{1}{2d(x)^2}. \qquad (2.10)
\end{aligned}$$

Therefore, inequality (2.10) yields

$$\frac{1}{4}\int_\Omega \frac{|f(x)|^2}{d(x)^2}\,dx \leq \frac{1}{2}\int_\Omega \frac{|f(x)|^2}{m(x)^2}\,dx. \qquad (2.11)$$

Using Theorem 2.2.3 with (2.11) gives

$$\frac{1}{4}\int_\Omega \frac{|f(x)|^2}{d(x)^2}\,dx \leq \int_\Omega |\nabla f(x)|^2\,dx,$$

as required. $\square$

Now let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary. It is known that the following extension of Hardy's inequality holds (see for example [17], [18] and [19]):

$$\mu\int_\Omega \frac{|f(x)|^2}{d(x)^2}\,dx \leq \int_\Omega |\nabla f(x)|^2\,dx, \qquad \forall f \in \mathcal{C}_c^\infty(\Omega), \qquad (2.12)$$

where μ is a positive constant and $d(x) = \text{dist}\,(x, \partial\Omega)$. The value of μ in (2.12) depends on the domain Ω, whether it is convex or non-convex. Before we show that the constant μ in (2.12) is equal to $\frac{1}{4}$ for any convex domain Ω in $\mathbb{R}^n$, we need to prove the following theorem.

Theorem 2.2.7 ([31]).

Let Ω be a convex domain in $\mathbb{R}^n$. Then, using notations of Theorem 2.2.5, we have

$$\frac{1}{m(x)^2} = \int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2}\, dS(u) \,\geq\, \frac{1}{nd(x)^2}, \tag{2.13}$$

where the measure $dS(u)$ is normalized to have unit total mass.

Proof. Considering the definition of $\frac{1}{m(x)^2}$, (2.7), an application of (C.10)[2] gives

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{1}{|\mathbb{S}^{n-1}|\, d(x)^2} \int_0^\pi \cos^2\theta \sin^{n-2}\theta d\theta \int_{\mathbb{S}^{n-2}} dw \\
&= \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|\, d(x)^2} \int_0^\pi \cos^2\theta \sin^{n-2}\theta d\theta.
\end{aligned} \tag{2.14}$$

Hence, using (C.13), (C.14) and (C.15) in (2.14) for even n leads to

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{2\,(2\pi)^{\frac{n-2}{2}}}{1.3.\,...\,(n-3)} \frac{2.4.\,...\,(n-2)}{(2\pi)^{\frac{n}{2}} d(x)^2} \frac{1.3.5....(n-3)}{2.4.6.\ldots(n-2)n}\, \pi \\
&= 2\,(2\pi)^{\frac{n-2}{2}} \frac{1}{(2\pi)^{\frac{n}{2}} d(x)^2} \frac{1}{n}\pi \\
&= \frac{1}{nd(x)^2},
\end{aligned}$$

the same result holds for odd n. This completes the proof. $\square$

It is appropriate to indicate here that one main step in proving all theorems in the following chapters is to derive lower bounds for the function $\frac{1}{m(x)^2}$ 'similar' to (2.13) for non-

[2] see Appendix C, Page 153.

convex domains i.e., in terms of the distance function $d(x)$.

Now the proof that μ in (2.12) is equal to $\frac{1}{4}$ for any convex domain $\Omega \subset \mathbb{R}^n$ follows directly from Theorem 2.2.5 and Theorem 2.2.7, i.e.,

$$\frac{1}{4}\int\limits_{\Omega} \frac{|f(x)|^2}{d(x)^2}\,dx \le \frac{n}{4}\int\limits_{\Omega} \frac{|f(x)|^2}{m(x)^2}dx \le \int\limits_{\Omega} |\nabla f(x)|^2\,dx.$$

The constant $\mu = \frac{1}{4}$ in (2.12) is sharp. This fact has been proved by many authors, see for example [3] and [13]. One of these proofs is presented below, and is given in [54]. The author used the following identity for $\Omega \subset \mathbb{R}^n$:

$$\int\limits_{\Omega} |\nabla f(x)|^2\,dx = \int\limits_{\Omega} (\operatorname{div} F - |F|^2)\,|f(x)|^2\,dx + \int\limits_{\Omega} |\nabla f(x) + F\,f(x)|^2\,dx, \tag{2.15}$$

where $F = (F_1, ..., F_n) \in W^{1,\infty}(\Omega)$ and $f \in W_0^{1,2}(\Omega)$, to deduce that the following inequality holds:

$$\int\limits_{\Omega} |\nabla f(x)|^2\,dx \ge \int\limits_{\Omega} (\operatorname{div}\ F - |F|^2)\,|f(x)|^2\,dx, \quad f(x) \in W_0^{1,2}(\Omega). \tag{2.16}$$

Obviously, (2.15) gives (2.16) by omitting the positive term $\int_\Omega |\nabla f(x) + F\,f(x)|^2\,dx$, hence the main idea of the equality in (2.15) is to have an exact form for the remainder term in (2.16). Consequently, if we use (2.16) for a specific choice of F we can find a minimizer of (2.16) by solving the system of PDE's obtained by putting the remainder term of (2.15) equal zero, i.e.

$$\nabla f(x) + F\,f(x) = 0. \tag{2.17}$$

If we put $F(x) = -\frac{1}{2}\frac{\nabla d(x)}{d(x)}$ in (2.16), bearing in mind that the distance function $d(x)$ satisfies[3] $|\nabla d(x)| = 1$ almost everywhere and for convex domains $\Delta d(x) \le 0$ (in the dis-

[3]See Appendix D.

tributional sense), then Hardy's inequality (2.12) follows immediately. On the other hand, by solving (2.17) for our choice of F we easily find that $f(x) = d(x)^{\frac{1}{2}}$ is a minimizer of (2.12). However, this function is not in $W_0^{1,2}(\Omega)$ (because the L^2 norm of $|\nabla f|^2$ is not necessarily bounded). Nevertheless, we can obtain a minimizing family to (2.12) by setting $f_\epsilon(x) = d(x)^{\frac{1}{2}+\epsilon}$. Precisely, one sees that $|\nabla f_\epsilon(x)| = (\frac{1}{2}+\epsilon)d(x)^{-\frac{1}{2}+\epsilon}$ and hence,

$$\begin{aligned}\int_\Omega |\nabla f_\epsilon(x)|^2\, dx &= (\frac{1}{2}+\epsilon)^2 \int_\Omega \frac{d(x)^{1+2\epsilon}}{d(x)^2} dx \\ &= (\frac{1}{2}+\epsilon)^2 \int_\Omega \frac{|f(x)|^2}{d(x)^2} dx,\end{aligned}$$

as $\epsilon \to 0$ the constant in the right hand side tends to the sharp one, i.e. $\frac{1}{4}$.

On the other hand, for non-convex domains not many results have been obtained, for example, the sharp constants for Hardy's inequality are still unknown. Nevertheless, in 1986 A. Ancona ([4]) used the Koebe one-quarter theorem, to prove for simply-connected planar domains that the constant μ, in Hardy's inequality (2.12), is greater than or equal to $\frac{1}{16}$. Other interesting results, related to simply-connected planar domains, were obtained by A. Laptev and A. Sobolev, see [35].

The authors in [35] studied some non-convex domains in $\mathbb{R}^2$ subject to some geometrical conditions. This trend is, to some extent, what we are going to do in the next chapters. Therefore, let us have a closer look at their work. The following two conditions were introduced in [35].

Let $\Lambda \subset \mathbb{C}$ be a simply-connected domain such that $0 \in \partial\Lambda$. Denote by $\Lambda\left(w,\phi\right)$ the set

$$\Lambda\left(w,\phi\right) = e^{i\phi}\Lambda + w = \left\{z \in \mathbb{C} : e^{-i\phi}\left(z-w\right) \in \Lambda\right\}.$$

Denote by $K_\theta \subset \mathbb{C}$, $\theta \in (0, \pi]$ the sector

$$K_\theta = \{z \in \mathbb{C} :\mid \arg z \mid < \theta\} ,$$

which is an open sector symmetric with respect to the real axis, with angle 2θ at the vertex. For $R > 0$ and $\theta \in [0, \pi)$, introduce the domains

$$\tilde{D}_R = \{z \in \mathbb{C} :\mid z \mid > R \text{ and } \mid \arg z \mid \neq \pi\} , \; D_{R,\theta} = \tilde{D}_R \left(-Re^{i\theta}, 0\right) .$$

The domain $\tilde{D}_R$ is the exterior of a disk of radius R centred at the origin with an infinite cut along the negative real semi-axis.

Condition 2.2.8.

There exists a number $\theta \in [0, \pi]$ such that for each $w \in \Omega^c$ one can find a $\phi = \phi_w \in (-\pi, \pi]$ such that

$$\Omega \subset K_\theta \left(w, \phi_w\right) .$$

To describe another way of characterizing the non-convexity of a planar domains, the authors in [35] introduced the following condition:

Condition 2.2.9.

There exist numbers $R > 0$ and $\theta_0 \in [0, \pi)$ such that for any $w \in \partial\Omega$ one can find a $\phi = \phi_w \in (-\pi, \pi]$ and $\theta \in [-\theta_0, \theta_0]$ such that

$$\Omega \subset D_{R,\theta} \left(w, \phi_w\right) .$$

Under Condition 2.2.8, the following theorem have been proved:

Theorem 2.2.10 ([35])**.**

Suppose that the domain $\Omega \subset \mathbb{R}^2$, $\Omega \neq \mathbb{R}^2$ satisfies Condition 2.2.8 with some $\theta \in [\frac{\pi}{2}, \pi]$.

Then for any $f \in \mathcal{C}_c^1(\Omega)$ *the Hardy inequality* (2.12) *holds with*

$$\mu = \mu(\theta) = \left(\frac{\pi}{4\theta}\right)^2. \tag{2.18}$$

According to Condition 2.2.9, A. Laptev and A. Sobolev have proved the following theorem, which applies only to domains with a finite inradius

$$\delta_{\text{in}} = \delta_{\text{in}}(\Omega) = \sup_{x \in \Omega} \text{dist}(x, \partial\Omega). \tag{2.19}$$

Theorem 2.2.11 ([35]).
Let $\Omega \subset \mathbb{R}^2$, $\Omega \neq \mathbb{R}^2$ *be a domain such that* $\delta_{\text{in}} < \infty$. *Suppose that* Ω *satisfies Condition 2.2.9 with some* $\theta_0 \in [0, \pi)$ *and that*

$$2\delta_{\text{in}} \leq R_0(R), \quad R_0(R) = \frac{R}{2\left(2^{\frac{1}{2}} \mid \tan\left(\frac{\theta_0}{2}\right) \mid +1\right)}.$$

Then the Hardy inequality (2.12) *holds with*

$$\mu = \mu(\theta) = \frac{1}{4}\left[1 - \frac{2\delta_{\text{in}}}{R_0(R)}\right]^2. \tag{2.20}$$

The main objective of Chapter 3 is to find different 'non-convexity measures' under which new formulas of Hardy-type inequalities can be obtained.
Before we proceed let us mention some of the improvements and applications of Hardy-type inequalities for domains which are not necessarily simply-connected.

2.3 Improved Hardy-type inequalities

In this section we shed some light on a couple of improved Hardy-type inequalities. First of all, the following generalisation of Hardy's inequality holds for arbitrary $1 < p < \infty$, where

$\left|\frac{n-p}{p}\right|^p$ is the best constant (see for example [29, Chapter IX] and [42]):

$$\left|\frac{n-p}{p}\right|^p \int\limits_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p}\, dx \le \int\limits_{\mathbb{R}^n} |\nabla f(x)|^p\, dx, \;\; f(x) \in \mathcal{C}_c^\infty\left(\mathbb{R}^n \setminus \{0\}\right). \tag{2.21}$$

For convex domains $\Omega \subset \mathbb{R}^n$, $n \ge 1$, with smooth boundary, Hardy's inequality takes the form (see [39])

$$\left(\frac{p-1}{p}\right)^p \int\limits_{\Omega} \frac{|f(x)|^p}{d(x)^p}\, dx \le \int\limits_{\Omega} |\nabla f(x)|^p\, dx, \;\; f(x) \in \mathcal{C}_c^\infty(\Omega), \tag{2.22}$$

where $\left(\frac{p-1}{p}\right)^p$ is the best constant and $d(x)$ is the distance function defined in (2.8).

Improved Hardy's inequality means having extra terms on the left hand side of (2.21) or (2.22) that either contain integrals of $|f|^p$ with weights depending on $|x|$ or integrals of $|\nabla f|^q$ with $p < q$, see for instance [9], [23], [24], and [55]. In particular H. Brezis and M. Marcus ([13, Theorem I]) proved the following theorem:

Theorem 2.3.1.

For every smooth domain Ω, there exists a constant $\gamma = \gamma(\Omega) \in \mathbb{R}$ such that

$$\frac{1}{4}\int\limits_{\Omega} \frac{|f(x)|^2}{d(x)^2}\, dx + \gamma(\Omega)\int\limits_{\Omega} |f(x)|^2\, dx \le \int\limits_{\Omega} |\nabla f(x)|^2\, dx, \;\; f(x) \in H_0^1(\Omega). \tag{2.23}$$

In [38] and [39], some examples that confirm the existence of smooth domains with $\gamma \le 0$ have been given. However, for bounded convex domains $\Omega \subset \mathbb{R}^n$, H. Brezis and M. Marcus ([13, Theorem II]) proved that

$$\gamma(\Omega) \ge \frac{1}{4\,\mathrm{diam}^2(\Omega)}. \tag{2.24}$$

Then they asked whether the diameter of Ω in (2.24) can be replaced by an expression depending on the volume of Ω, namely, whether $\gamma(\Omega) \ge \alpha(\mathrm{vol}(\Omega))^{-\frac{2}{n}}$ for some universal constant $\alpha > 0$. This question was later answered in affirmative as the following theorem states (see [31]).

Theorem 2.3.2.

For any convex domain $\Omega \subset \mathbb{R}^n$ *and any* $f(x) \in H_0^1(\Omega)$ *we have*

$$\frac{1}{4}\int\limits_{\Omega} \frac{|f(x)|^2}{d(x)^2}\,dx + \frac{\gamma(n)}{|\Omega|^{\frac{2}{n}}}\int\limits_{\Omega} |f(x)|^2\,dx \leq \int\limits_{\Omega} |\nabla f(x)|^2\,dx, \tag{2.25}$$

where

$$\gamma(n) = \frac{n^{\frac{n-2}{n}}\,|\mathbb{S}^{n-1}|^{\frac{2}{n}}}{4}.$$

As an attempt to improve the constant $\gamma(\Omega)$ in (2.24) for convex domains, S. Filippas, V. Maz'ya and A. Tertikas ([23]) proved the following theorem:

Theorem 2.3.3.

If $\Omega \subset \mathbb{R}^n$ *is a convex domain, then for any* $\alpha > -2$, *the following inequality holds:*

$$\frac{1}{4}\int\limits_{\Omega} \frac{|f(x)|^2}{d(x)^2}\,dx + \frac{C_\alpha}{D_{\text{int}}^{2+\alpha}(\Omega)}\int\limits_{\Omega} |f(x)|^2\,d(x)^\alpha\,dx \leq \int\limits_{\Omega} |\nabla f(x)|^2\,dx, \quad f(x) \in H_0^1(\Omega), \tag{2.26}$$

where $D_{\text{int}} := 2\sup\limits_{x\in\Omega} d(x) =$ the interior diameter of Ω, *with*

$$C_\alpha = \begin{cases} 2^\alpha(\alpha+2)^2 & , \textit{if} \quad -2 < \alpha < -1 \\ 2^\alpha(2\alpha+3) & , \textit{if} \quad \alpha \geq -1. \end{cases} \tag{2.27}$$

Obviously, for $\alpha = 0$, (2.27) gives $\mu(\Omega) = 3\mathrm{D}_{\text{int}}^{-2}$, which is indeed better than (2.24) since $\mathrm{D}_{\text{int}} \leq \frac{\text{diam}(\Omega)}{2}$.

To obtain a sharp form of the inequality (2.23) connected with the inradius, defined in (2.19), F. G. Avkhadiev and K-J Wirths ([7]) proved the following theorem:

Theorem 2.3.4.

Let Ω be an open convex set in $\mathbb{R}^n$. If the inradius $\delta_{\text{in}} := \delta_{\text{in}}(\Omega)$ is finite, then

$$\frac{1}{4}\int\limits_{\Omega} \frac{|f(x)|^2}{d(x)^2}\,dx + \frac{\lambda_0^2}{\delta_{\text{in}}^2}\int\limits_{\Omega} |f(x)|^2\,dx \le \int\limits_{\Omega} |\nabla f(x)|^2\,dx, \quad \forall f(x) \in H_0^1(\Omega), \tag{2.28}$$

where λ_0 is the Lamb constant[4]. *The inequality* (2.28) *is sharp for all dimensions $n \ge 1$.*

This estimate (2.28) is better than (2.26), since $\lambda_0 = 0.940$ and $\mathrm{D}_{\text{int}} = 2\delta_{\text{in}}$, meaning that for $\alpha = 0$, $0.75\frac{1}{\delta_{\text{in}}^2} < 0.88\frac{1}{\delta_{\text{in}}^2}$.
Some other forms of Hardy-type inequalities using the inradius have been proved in [5], [6] and [23]. In a generalisation of (2.25) to any $p > 1$, we have the following theorem:

Theorem 2.3.5 ([53])**.**

For any convex domain $\Omega \subset \mathbb{R}^n$ and any $f(x) \in W_0^{1,p}(\Omega)$, we have

$$\left(\frac{p-1}{p}\right)^p \int\limits_{\Omega} \frac{|f(x)|^p}{d(x)^p}\,dx + \frac{a(p,n)}{|\Omega|^{\frac{p}{n}}}\int\limits_{\Omega} |f(x)|^p\,dx \le \int\limits_{\Omega} |\nabla f(x)|^p\,dx,$$

where

$$a(p,n) = \frac{(p-1)^{p+1}}{p^p}.\left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\frac{p}{n}} \cdot \frac{\sqrt{\pi}\cdot\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}.$$

The results given in [13], [31], and [53] concerning the improved Hardy inequality

$$\int\limits_{\Omega} \frac{|f(x)|^p}{d(x)^p}\,dx + |\Omega|^{\frac{-p}{n}}\int\limits_{\Omega} |f(x)|^p\,dx \le \mu \int\limits_{\Omega} |\nabla f(x)|^p\,dx,$$

have been extended in [20] by showing that the class of domains for which the inequality holds is larger than that of all bounded convex domains. Now we highlight a couple of

[4]This is the first zero in $(0, \infty)$ of the function

$$J_0(x) - 2xJ_1(x) \equiv J_0(x) + 2xJ_0^{'}(x),$$

where J_0 and J_1 are the Bessel functions of order 0 and 1 respectively.

applications of Hardy-type inequalities.

2.4 Some applications of Hardy-type inequalities

In fact there are many applications of Hardy-type inequalities not only in different mathematical branches such as PDE's, spectral theory, function space theory, but also in some physical branches. Some of these applications will be presented in this section.

One important application of Hardy's inequality is using it as a technical tool in the study of elliptic operators, since it can be used to determine whether or not 0 belongs to the spectrum of a non-negative, self-adjoint operator. For instance, the following theorem was proved in [17, Page 109], using the two-dimensional version of Hardy's inequality (2.3).

Theorem 2.4.1.
Let $\Omega \subseteq \mathbb{R}^2$ be regular[5]*, and let H be the Friedrichs extension*[6] *of $-\Delta$ initially defined on $\mathcal{C}_c^\infty(\Omega)$. Then $0 \in Spec(H)$ if and only if the inradius of Ω is infinite.*

We present only one direction of the proof, which uses the Hardy-type inequality (2.3).

Proof. [one sided implication of Theorem 2.4.1]
Suppose Ω is regular with constant c_0 and that it has a finite inradius $r := \sup\{d(x) : x \in \Omega\}$.
Then

$$\frac{1}{d(x)} \geq \frac{1}{m(x)} \geq \frac{1}{c_0 d(x)} \geq \frac{1}{c_0 r}. \tag{2.29}$$

Now use Hardy's inequality (2.3) with (2.29) to obtain

$$\frac{1}{2(c_0 r)^2} \int\limits_\Omega |f(x)|^2 \, dx \leq \frac{1}{2} \int\limits_\Omega \frac{|f(x)|^2}{m(x)^2} \, dx \leq \int\limits_\Omega |\nabla f(x)|^2 \, dx. \tag{2.30}$$

[5] It is said that Ω is regular if there exists a constant $c < \infty$ such that $d(x) \leq m(x) \leq cd(x)$, for all $x \in \Omega$, the first inequality being automatic, where the function $m(x)$ is defined by (2.4) and $d(x)$ is defined by (2.8).

[6] See Appendix B.1, Page 148.

Equivalently, inequality (2.30) can be written as

$$\frac{1}{2(c_0 r)^2} \parallel f(x) \parallel_2^2 \leq \frac{1}{2} \int\limits_{\Omega} \frac{|f(x)|^2}{m(x)^2}\, dx \leq \langle Hf(x),\, f(x) \rangle \,,$$

which implies that,

$$\langle Hf(x),\, f(x) \rangle \geq \frac{1}{2(c_0 r)^2} \parallel f(x) \parallel_2^2 . \tag{2.31}$$

Inequality (2.31) is equivalent [7] to

$$\text{Spec}(H) \subseteq \left[\frac{1}{2(c_0 r)^2}, \infty\right) ,$$

which indicates that if $0 \in$ Spec(H) then the inradius r is infinite. □

Again, in [17, Page 109], the one-dimensional Hardy inequality (2.2) was used to obtain an initial spectral classification of some elliptic operators as the following theorem illustrates:

Theorem 2.4.2.

Let H be the Friedrichs extension on $L^2(-1,1)$ of the symmetric degenerate elliptic operator defined initially on $\mathcal{C}_c^\infty(-1,1)$ by

$$Hf := -\frac{d}{dx}\left\{a(x)\frac{df(x)}{dx}\right\} ,$$

where the coefficient function $a(x)$ is $\mathcal{C}^1$ on $(-1,1)$ and satisfies $a(x) \geq c\,(1-x^2)^\alpha$ for some $c > 0$, some $\alpha \in (0,1)$ and all $x \in (-1,1)$. Then Spec$(H) \subseteq [\mu, \infty)$, where

$$\mu := \frac{c}{4}\,(1-\alpha)^2 > 0.$$

[7] Let H be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and let $c \in \mathbb{R}$. The following conditions are equivalent:

1. One has $\langle Hf, f \rangle \geq c \parallel f \parallel^2$ for all $f \in Dom(H)$.
2. The spectrum of H is contained in $[c, \infty)$.

Proof. Notice that, for all $f(x) \in \mathcal{C}_c^\infty(-1,1)$ we have

$$\langle Hf(x), f(x)\rangle = \int_{-1}^{1} Hf(x)\ \overline{f(x)}dx = \int_{-1}^{1} -\frac{d}{dx}\left\{a(x)\frac{df(x)}{dx}\right\}\ \overline{f(x)}dx.$$

Integrating by parts gives

$$\begin{aligned}\langle Hf(x), f(x)\rangle &= \overline{f(x)}(-a(x)f(x)')\Big|_{-1}^{1} + \int_{-1}^{1} a(x)f(x)'\overline{f}(x)'dx \\ &= \int_{-1}^{1} a(x)\left|f(x)'\right|^2 dx \quad \forall f(x) \in \mathcal{C}_c^\infty(-1,1).\end{aligned}$$

Using the inequality $1 - x^2 \geq 1 - |x| \geq 0$, for all $x \in [-1,1]$, keeping in mind that $a(x) \geq c(1-x^2)^\alpha$, leads to

$$\langle Hf(x), f(x)\rangle \geq c\int_{-1}^{1}(1-|x|)^\alpha \left|f(x)'\right|^2 dx\ \ \forall f(x) \in \mathcal{C}_c^\infty(-1,1). \tag{2.32}$$

Now use the Hardy inequality (2.2) with (2.32) to obtain

$$\langle Hf(x), f(x)\rangle \geq c\frac{(1-\alpha)^2}{4}\int_{-1}^{1}(1-|x|)^{\alpha-2}\left|f(x)\right|^2 dx\ \ \forall f(x) \in \mathcal{C}_c^\infty(-1,1). \tag{2.33}$$

However, $0 < \alpha < 1$ so $\alpha - 2 < 0$, and $0 < 1 - |x| \leq 1$ hence, $(1-|x|)^{\alpha-2} \geq 1$. Thus inequality (2.33) takes the following form:

$$\begin{aligned}\langle Hf(x), f(x)\rangle &\geq c\ \frac{(1-\alpha)^2}{4}\int_{-1}^{1}\left|f(x)\right|^2 dx \\ &= c\ \frac{(1-\alpha)^2}{4} \parallel f(x) \parallel^2 \quad \forall f(x) \in \mathcal{C}_c^\infty(-1,1),\end{aligned} \tag{2.34}$$

which is equivalent to the inclusion of the spectrum of H in $\left[c\frac{(1-\alpha)^2}{4}, \infty\right)$. This completes the proof. $\square$

Theorems 2.4.1 and 2.4.2 are a few examples of how certain Hardy-type inequalities may be applied in the spectral theory. For more examples see [16], [17], and [19].

Now we give examples of how improved Hardy-type inequalities can be used as tools in theorems concerning existence and uniqueness of solutions in the theory of viscous incompressible flow. For instance, the improved Hardy-type inequality (2.25) for a convex domain $\Omega \subset \mathbb{R}^3$, i.e.

$$\frac{1}{4}\int\limits_{\Omega} \frac{|f(x)|^2}{d(x)^2}\,dx + \left(\frac{3\pi^2}{4\,|\Omega|^2}\right)^{\frac{1}{3}} \int\limits_{\Omega} |f(x)|^2\,dx \leq \int\limits_{\Omega} |\nabla f(x)|^2\,dx, \tag{2.35}$$

was used in [54] as follows:
The author defined the stationary Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^3$ to be

$$\begin{cases} -\nu\Delta\mathbf{v} + \Sigma_{k=1}^{n} v_k \frac{\partial \mathbf{v}}{\partial x_k} & = -\nabla p + \mathbf{f}(\mathbf{x}), \\ \nabla \cdot \mathbf{v} & = 0, \end{cases} \tag{2.36}$$

where $\mathbf{v}(x) = (v_1, ..., v_n)$ is an unknown vector, $p(x)$ is an unknown scalar-function, $\mathbf{f} = (f_1, ..., f_n)$ is a given vector-function and ν is a given positive constant. The vector valued function $\mathbf{v}(\mathbf{x})$ belongs to the space $\mathbf{H}(\boldsymbol{\Omega})$ which is the closure of

$$\mathbf{S} = \left\{\mathbf{v} = (v_1, v_2, v_3) : \nabla \cdot \mathbf{v} = 0,\ v_k \in \mathcal{C}_0^\infty(\Omega), k = 1, 2, 3\right\},$$

equipped with the scalar product

$$(\mathbf{u}, \mathbf{v})_1 = \sum_{k=1}^{3} \int\limits_{\Omega} \frac{\partial \mathbf{u}}{\partial x_k}\frac{\partial \mathbf{v}}{\partial x_k}\,dx.$$

In fact, our discussion here is restricted to the case when $\mathbf{v}(\mathbf{x})$ is zero at the boundary of Ω. The vector $\mathbf{v} \in \mathbf{H}(\Omega)$ is said to be a generalised solution to the problem (2.36) if it satisfies the following equation:

$$\sum_{k=1}^{3} \int_{\Omega} \nu \frac{\partial \mathbf{v}}{\partial x_k} \frac{\partial \mathbf{w}}{\partial x_k} + v_k \frac{\partial \mathbf{v}}{\partial x_k} \cdot \mathbf{w} = \int_{\Omega} \mathbf{f} \cdot \mathbf{w}\, dx; \quad \mathbf{w} \in \mathbf{S}.$$

There are two fundamental theorems that enable us to investigate the existence and uniqueness of a generalised solution for (2.36). These theorems are as follows (see [33]):

Theorem 2.4.3.

Let Ω be bounded. Then the equations (2.36)*, with $\mathbf{v}|_{\partial\Omega} = 0$, have at least one generalised solution if $\mathbf{f}$ is such that $\int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx$ defines a linear functional of $\mathbf{w} \in \mathbf{H}(\Omega)$.*

Theorem 2.4.4.

Let Ω be bounded. Then we cannot have more than one solution to the problem described in Theorem 2.4.3 if

$$\frac{2\sqrt{3}}{\lambda_1^{\frac{1}{4}} \nu^2} \sup_{\mathbf{w} \in \mathbf{H}(\Omega), \mathbf{w} \neq 0} \frac{|\int_{\Omega} \mathbf{f} \cdot \mathbf{w} dx|}{(\mathbf{w}, \mathbf{w})_1^{\frac{1}{2}}} < 1, \tag{2.37}$$

where λ_1 is the smallest eigenvalue of the Dirichlet Laplacian $-\Delta$ on Ω.

Hardy-type inequalities are useful when we want to fulfill the conditions of Theorems 2.4.3 and 2.4.4. In particular, the Hardy-type inequality (2.35) with Theorem 2.4.4 provide us with a sufficient condition for uniqueness of solution of (2.36). This criteria is shown in the following lemma:

Lemma 2.4.5 ([54])**.**

The solution of (2.36) *is unique if*

$$\frac{4\sqrt{6}\lambda_1^{\frac{-1}{4}} \nu^{-2}}{\frac{2}{\|\mathbf{f}\|_{L^2}} \left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{6}} + \frac{1}{\|d(x)\,\mathbf{f}\|_{L^2}}} < 1, \tag{2.38}$$

where $d(x)$ is the distance function.

Proof. Let $b \in \mathbb{R}$ be such that $0 \leq b \leq 1$. We have

$$\begin{aligned} \left| \int_\Omega \mathbf{f} \cdot \mathbf{w} dx \right|^2 &\leq \left(\int_\Omega |\mathbf{f}| \, |\mathbf{w}| \, dx \right)^2 \\ &= \left(b \int_\Omega |d(x)\, \mathbf{f}| \, \frac{|\mathbf{w}|}{d(x)} dx + (1-b) \int_\Omega |\mathbf{f}| \, |\mathbf{w}| \, dx \right)^2 . \end{aligned} \tag{2.39}$$

Use the inequality

$$(a+b)^2 \leq 2a^2 + 2b^2; \quad a, b \geq 0,$$

to write (2.39) as follows:

$$\left| \int_\Omega \mathbf{f} \cdot \mathbf{w} dx \right|^2 \leq 2b^2 \left(\int_\Omega |d(x)\, \mathbf{f}| \, \frac{|\mathbf{w}|}{d(x)} dx \right)^2 + 2(1-b)^2 \left(\int_\Omega |\mathbf{f}| \, |\mathbf{w}| \, dx \right)^2 .$$

Hence, by Cauchy's inequality we obtain

$$\begin{aligned} \left| \int_\Omega \mathbf{f} \cdot \mathbf{w} dx \right|^2 &\leq 2b^2 \int_\Omega |d(x)\, \mathbf{f}|^2 \int_\Omega \frac{|\mathbf{w}|^2}{d(x)^2} dx + 2(1-b)^2 \int_\Omega |\mathbf{f}|^2 \, dx \int_\Omega |\mathbf{w}|^2 \, dx \\ &= 2b^2 \|d(x)\, \mathbf{f}\|_{L^2}^2 \left\| \frac{\mathbf{w}}{d(x)} \right\|_{L^2}^2 + 2(1-b)^2 \|\mathbf{f}\|_{L^2}^2 \|\mathbf{w}\|_{L^2}^2 . \end{aligned} \tag{2.40}$$

Now set

$$b = \frac{\|\mathbf{f}\|_{L^2}}{\|\mathbf{f}\|_{L^2} + 2\|d(x)\, \mathbf{f}\|_{L^2} \left(\frac{3\pi^2}{4|\Omega|^2} \right)^{\frac{1}{6}}} .$$

This implies that inequality (2.40) becomes

$$\begin{aligned}
\left|\int\limits_\Omega \mathbf{f}\cdot\mathbf{w}dx\right|^2 &\le 8\|\mathbf{f}\|_{L^2}^2\|d(x)\,\mathbf{f}\|_{L^2}^2\left(\|\mathbf{f}\|_{L^2}+2\|d(x)\,\mathbf{f}\|_{L^2}\left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{6}}\right)^{-2}\\
&\quad\times\left(\frac{1}{4}\left\|\frac{\mathbf{w}}{d(x)}\right\|_{L^2}^2+\left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{3}}\|\mathbf{w}\|_{L^2}^2\right)\\
&= 8\left(\frac{1}{\|d(x)\,\mathbf{f}\|_{L^2}}+\frac{2}{\|\mathbf{f}\|_{L^2}}\left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{6}}\right)^{-2}\\
&\quad\times\left(\frac{1}{4}\left\|\frac{\mathbf{w}}{d(x)}\right\|_{L^2}^2+\left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{3}}\|\mathbf{w}\|_{L^2}^2\right).
\end{aligned}$$

Use Hardy's inequality (2.35) to obtain

$$\left|\int\limits_\Omega \mathbf{f}\cdot\mathbf{w}dx\right|^2 \le 8\left(\frac{1}{\|d(x)\,\mathbf{f}\|_{L^2}}+\frac{2}{\|\mathbf{f}\|_{L^2}}\left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{6}}\right)^{-2}\cdot\int\limits_\Omega|\nabla\mathbf{w}|^2dx. \tag{2.41}$$

Inequality (2.39) gives

$$\sup_{\mathbf{w}\in\mathbf{H}(\Omega),\mathbf{w}\neq0}\frac{\left|\int_\Omega\mathbf{f}\cdot\mathbf{w}dx\right|^2}{\int_\Omega|\nabla\mathbf{w}|^2dx}\le 8\left(\frac{1}{\|d(x)\,\mathbf{f}\|_{L^2}}+\frac{2}{\|\mathbf{f}\|_{L^2}}\left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{6}}\right)^{-2}.$$

Now using Theorem 2.4.4 indicates that the solution of (2.36) is unique if

$$\frac{2\sqrt{3}}{\lambda_1^{\frac{1}{4}}\nu^2}\sup_{\mathbf{w}\in\mathbf{H}(\Omega),\mathbf{w}\neq0}\frac{|\int_\Omega\mathbf{f}\cdot\mathbf{w}dx|}{\left(\int_\Omega|\nabla\mathbf{w}|^2dx\right)^{\frac{1}{2}}}<1.$$

Therefore, in order to have a unique solution of (2.36) it is suffices to have

$$\frac{2\sqrt{3}}{\lambda_1^{\frac{1}{4}}\nu^2}8\left(\frac{1}{\|d(x)\,\mathbf{f}\|_{L^2}}+\frac{2}{\|\mathbf{f}\|_{L^2}}\left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{6}}\right)^{-1}<1,$$

i.e., when

$$\frac{4\sqrt{6}\lambda_1^{\frac{-1}{4}}\nu^{-2}}{\frac{2}{\|\mathbf{f}\|_{L^2}}\left(\frac{3\pi^2}{4|\Omega|^2}\right)^{\frac{1}{6}}+\frac{1}{\|d(x)\,\mathbf{f}\|_{L^2}}}<1.$$

This completes the proof. □

Lemma 2.4.5 extends the class of functions for which the solution of Navier-Stokes equations is unique, since it is still applicable for those functions $f(x)$ that do not have L^2-norms but for which the L^2-norm of the product $d(x)f(x)$ exists. As an example of such functions is the function $f(x)=\frac{1}{1-x^2}$ defined on $[-1,1]$. It is not in $L^2((-1,1))$ since $\frac{1}{(1-x^2)^2}$ is not integrable on $(-1,1)$, because

$$\int\limits_{-1}^{1}|f(x)|^2dx \;=\; \int\limits_{-1}^{1}\frac{1}{(1-x^2)^2}dx=\infty.$$

On the other hand, $d(x)f(x)\in L^2((-1,1))$, where $d(x)=\min\{|1-x|,|1+x|\}$ is in $L^2((-1,1))$. Indeed, if $x\in(-1,0)$ then $d(x)f(x)=\frac{1}{1-x}$, which implies that $\int_{-1}^{0}\frac{1}{(1-x)^2}dx=\frac{1}{2}<\infty$ hence $d(x)f(x)\in L^2((-1,0))$. Similarly, if $x\in(0,1)$ then $d(x)f(x)=\frac{1}{1+x}$ so $\int_0^1\frac{1}{(1+x)^2}dx=\frac{1}{2}<\infty$ meaning that $d(x)f(x)\in L^2((0,1))$, thus $\|d(x)f(x)\|$ is finite.

Some forms of the Hardy-type inequality (2.21) can be used in the analysis of Schrödinger operators. For instance, the following Hardy-type estimate (see [36]):

$$\int\limits_{\mathbb{R}^2}\frac{|f(x)|^2}{|x|^2}dx\le c\int\limits_{\mathbb{R}^2}|(i\nabla+\mathbf{a})f(x)|^2\,dx,\qquad f(x)\in\mathcal{C}_c^{\infty}\left(\mathbb{R}^2\setminus\{0\}\right),\tag{2.42}$$

where $\mathbf{a}$ is a magnetic vector potential[8] and c might depend on $\mathbf{a}$, can be used in the study of the negative spectrum of two-dimensional Schrödinger operators (see for instance [11] and the references therein).

For the purpose of the next application suppose that the boundary of Ω is Lipchitz. Let $d(x)$

[8]It is a vector field whose *curl* is the magnetic field.

be the distance from $x \in \mathbb{R}^2$ to Ω. Then one has the following Hardy inequality (see)

$$\mu \int\limits_{\Omega^c} \frac{|f(x)|^2}{d(x)^2} dx \leq \int\limits_{\Omega^c} |\nabla f(x)|^2 \, dx, \qquad \forall f(x) \in H_0^1 (\Omega^c), \tag{2.43}$$

where $\Omega^c = \mathbb{R}^2 \setminus \Omega$. In their course to obtain a lower bound for the magnetic form

$$h\left[f(x)\right] = \int |(-i\nabla - \mathrm{a})\, f(x)|^2 \, dx, \qquad f(x) \in \mathcal{C}_c^1 \left(\mathbb{R}^2\right), \tag{2.44}$$

with an appropriate vector-potential $\mathrm{a} \in L^2 (\mathbb{R}^2)$ having two real-valued components, the authors of [8] used (2.43). inequality: Inequality (2.43) gives the following inequality:

$$\mu \int\limits_{\Omega^c} \frac{|f(x)|^2}{d(x)^2} dx \leq \int\limits_{\Omega^c} |(-i\nabla - \mathrm{a})\, f(x)|^2 \, dx, \qquad \forall f(x) \in \mathcal{C}_c^1 (\Omega^c) \tag{2.45}$$

which can be seen as a magnetic inequality of Hardy-type. Inequality (2.45) played an important role in proving (under certain conditions) the following lower bound for the magnetic form (2.44):

$$h[f(x)] \geq \frac{\mu}{2} \int \frac{|f(x)|^2}{\ell(x)^2 + d(x)^2} dx, \qquad \forall f(x) \in D[h];$$

where $\ell(x)$ is a positive continuous function satisfying the conditions

$$\ell \in \mathcal{C}^1(\mathbb{R}^2); \quad |\nabla \ell(x)| \leq 1, \quad \forall x \in \mathbb{R}^2.$$

Hardy's inequality (2.1) was used in [12] to improve standard versions of Poincaré's inequality, which has many applications in analysis, especially in differential equations, see for example [56].

All the above applications of Hardy-type inequalities and their improvements are just some examples of how these kinds of inequalities can be used in different mathematical branches. On the other hand, we have noticed that most of these applications depend mainly on the domain under investigation and its properties. Consequently, if we can obtain some forms of Hardy-type inequalities for some non-convex domains we can think of applying such forms in a similar way to the above applications for those non-convex domains.

Chapter 3

Hardy's inequalities for planar non-convex domains

3.1 Introduction

The main goal of this chapter is to obtain new Hardy-type inequalities for some non-convex domains in $\mathbb{R}^2$, assuming that these domains satisfy certain geometrical conditions. In fact, we introduce two different conditions. The first condition is referred to as the 'Truncated Sectorial Region' (TSR) condition, which covers three different cases, while the second condition is called the 'Exterior Disk' condition.

It is relevant to mention here that Theorem 2.2.3 plays a fundamental role in deriving Hardy-type inequalities obtained in this chapter. For that reason, and for the reader's convenience, we recall it here.

Proposition 3.1.1 (E. B. Davies [17]).

Let Ω be a region in $\mathbb{R}^2$ and let $f(x) \in \mathcal{C}_c^\infty(\Omega)$. Then

$$\frac{1}{2} \int\limits_{\Omega} \frac{|f(x)|^2}{m(x)^2}\, dx \leq \int\limits_{\Omega} |\nabla f(x)|^2\, dx, \tag{3.1}$$

where the 'pseudodistance' $m(x) > 0$ is defined by

$$\frac{1}{m(x)^2} := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{d_\theta(x)^2}, \tag{3.2}$$

and $d_\theta : \Omega \to (0, \infty]$ is defined by

$$d_\theta(x) := \min\{|s| : x + se^{i\theta} \notin \Omega\,, s \in \mathbb{R}\}. \tag{3.3}$$

Recall that we use the symbol $d(x)$ to refer to the distance from $x \in \Omega$ to the boundary $\partial\Omega$, i.e.,

$$d(x) := \text{dist}\,(x, \partial\Omega) = \min\{|x - y| : y \notin \Omega\}\,. \tag{3.4}$$

In order to achieve our objective we follow almost the same approach constructed in the proof of Theorem 2.2.6. To be more precise, we obtain lower bounds for the function $\frac{1}{m(x)^2}$, defined in (3.2), in terms of $d(x)$ then apply Proposition 3.1.1 to these lower bounds.
Now let us introduce our geometrical conditions.

3.2 Notation and conditions

Throughout this section we introduce notation which will be used during this chapter, also we state and discuss the 'Truncated Sectorial Region' and 'Exterior Disk' conditions. We start with the so called 'Truncated Sectorial Region' (TSR) condition.
Denote by $K_{h,\psi} \subset \mathbb{C}$, where $\psi \in (0, \pi]$ and $h \geq 0$, the set

$$K_{h,\psi} = \{z \in \mathbb{C} : |\arg\, z| < \psi, \Re\, z > -h\}\,.$$

In other words, $K_{h,\psi}$ is the intersection of the symmetric (with respect to the real axis) open sector, with an internal angle 2ψ, with the half space $\Re z > -h$.

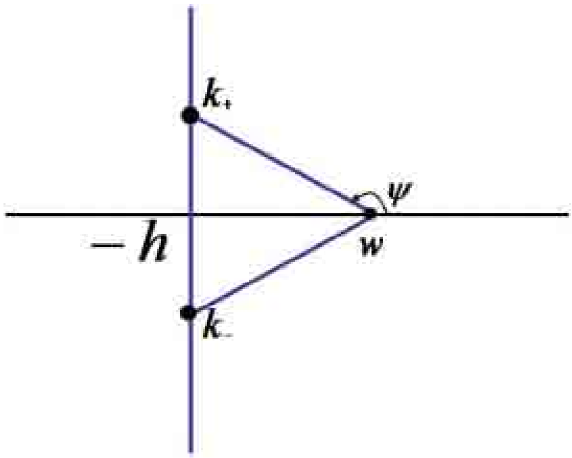

Figure 3.1: The points $k_\pm$

Denote by $K_{h,\psi}(w,\phi) = e^{i\phi}K_{h,\psi} + w$ the transformation of $K_{h,\psi}$ by rotation by an angle $\phi \in (-\pi, \pi]$ in the positive direction and translation by $w \in \mathbb{C}$:

$$K_{h,\psi}(w,\phi) = \left\{ z \in \mathbb{C} : e^{-i\phi}(z - w) \in K_{h,\psi} \right\}.$$

For $\psi > \frac{\pi}{2}$, denote by $k_\pm \in \mathbb{C}$ the two points with $\Re\, k_\pm = -h$, $\arg\, k_\pm = \pm\psi$ (see Fig. 3.1). Accordingly, $k_\pm(w,\phi)$ are the two points $k_\pm e^{i\phi} + w$.

Our assumption on the domain $\Omega \subset \mathbb{R}^2$ is the following:

Condition 3.2.1. (Truncated Sectorial Region (TSR) Condition)

We say that $\Omega \subset \mathbb{R}^2$ satisfies the TSR Condition if there exist numbers $\psi \in \left[\frac{\pi}{2}, \pi\right]$ and $h \geq 0$ such that for each $w \in \partial\Omega$ one can find a number $\phi = \phi_w \in (-\pi, \pi]$ such that

$$\Omega \subset K_{h,\psi}(w, \phi_w).$$

Roughly speaking, the TSR Condition means that the domain Ω satisfies the exterior cone condition with a cut at height h.

Remark 3.2.2.

If a domain $\Omega \subset \mathbb{R}^2$ satisfies the TSR Condition for some ψ, then it is clear that $\psi \geq \frac{\pi}{2}$ and equality holds if Ω is a convex domain.

Notations 3.2.3.

Let us introduce some notation associated with the truncated sectorial region $K = K_{h,\psi}(w, \phi_w)$. For any point $x \in \Omega \subset K$ where Ω satisfies Condition 3.2.1, denote by $d(x,w)$ the Euclidean distance from the point x to the vertex w of K. For $\theta \in (-\pi, \pi]$, define

$$\tilde{d}_\theta(x) = \min\{|s| : x + se^{i(\theta + \arg(x-w))} \notin K, s \in \mathbb{R}\}, \tag{3.5}$$

so that $\tilde{d}_0(x) = |x - w|$ i.e., $\tilde{d}_\theta(x)$ is the distance from the point $x \in K$ to the boundary ∂K of the truncated sectorial region K in the direction θ. Note also that the function $\tilde{d}_\theta(x)$ is a periodic function with period π, i.e., $\tilde{d}_\theta(x) = \tilde{d}_{\theta'}(x)$; $\theta' = \theta + \pi \mod 2\pi$.

If $\psi > \frac{\pi}{2}$, denote by $\theta_\pm \in \left(\arg(x-w) - \frac{\pi}{2}, \arg(x-w) + \frac{\pi}{2}\right)$ the angles such that

$$\tilde{d}_{\theta_\pm}(x) = \left|k_\pm(w, \phi) - x\right|. \tag{3.6}$$

It is clear that at least one of the two angles is non-zero. More precisely, θ_+ (respectively θ_-) is not zero and given by (3.6) if k_+ (respectively k_-) is visible[1] from x, otherwise we define it as 0. Let

$$\theta_0 = \max\left(|\theta_+|, |\theta_-|\right), \tag{3.7}$$

and when $\psi = \frac{\pi}{2}$ we set $\theta_0 = \frac{\pi}{2}$.

[1] A point y is visible from the point $x \in K$ if $y \in K_x \subset K$, where

$$K_x = \{z \in K : x + t(z-x) \in K \ \forall t \in [0,1]\}.$$

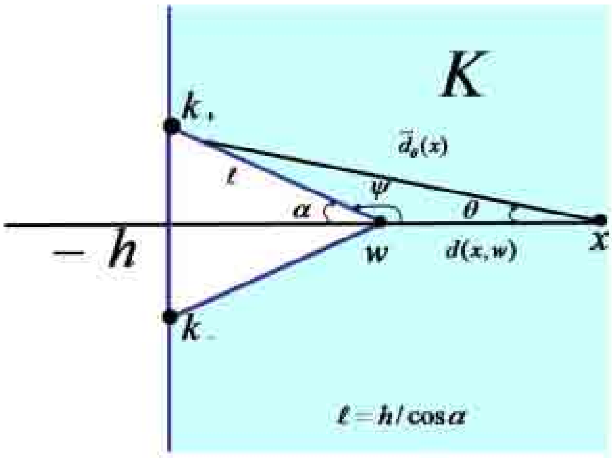

Figure 3.2: x lies on the symmetry axis of K

Considering the above notation we can identify three different cases which stemming from the TSR Condition according to the position of the point $x \in K$.

Case 1.

If the point $x \in \Omega \subset K$ lies on the symmetry axis of K and $\alpha =: \pi - \psi < \frac{\pi}{2}$ (see Fig. 3.2), then we have $\frac{\sin\theta_0}{\ell} = \frac{\sin(\pi-\alpha)}{\tilde{d}_{\theta_0}(x)}$ which implies that $\sin\theta_0 = \frac{h}{\tilde{d}_{\theta_0}(x)}\tan\alpha$. On the other hand, $\tilde{d}_{\theta_0}(x) = \frac{h+d(x,w)}{\cos\theta_0}$ so θ_0 satisfies the following relation:

$$\tan\theta_0 = \frac{h}{h+d(x,w)}\tan\alpha. \tag{3.8}$$

The relation (3.8) is well defined since both h and $d(x,w)$ are positive and $\alpha < \frac{\pi}{2}$. In this case both θ_+ and θ_- are well defined and $\theta_+ = -\theta_-$. If $\alpha = \frac{\pi}{2}$, we set $\theta_0 = \frac{\pi}{2}$, which is consistent with (3.8). Note also that the relation between $\tilde{d}_\theta(x)$ and $d(x,w)$ varies according to the angle α as follows:

1. For $\alpha < \frac{\pi}{2}$ we have

   (a) If $\theta \in (0,\theta_0)$ then, using the Sine Law, we obtain $\frac{\tilde{d}_\theta(x)}{\sin(\pi-\alpha)} = \frac{d(x,w)}{\sin(\alpha-\theta)}$, which implies that

$$\tilde{d}_\theta(x) = \frac{d(x,w)\ \sin\alpha}{\sin(\alpha-\theta)}. \tag{3.9}$$

The relation (3.9) is well defined and the right hand side is positive because $0 \leq \theta < \theta_0$, therefore $\theta < \alpha$ i.e., $0 < \alpha - \theta$. It is also clear that $\alpha - \theta < \frac{\pi}{2}$, so $\sin(\alpha - \theta) > 0$.

(b) If $\theta \in \left[\theta_0, \frac{\pi}{2}\right)$ or $\theta \in \left(-\frac{\pi}{2}, -\theta_0\right]$ then we find that

$$\tilde{d}_\theta(x) = \frac{h + d(x,w)}{\cos\theta}. \tag{3.10}$$

This relation is well defined and the right hand side is positive since $\theta < \frac{\pi}{2}$.

2. For $\alpha = \frac{\pi}{2}$, we have $0 \leq \theta < \theta_0 < \frac{\pi}{2}$. Consequently, the relation between $\tilde{d}_\theta(x)$ and $d(x,w)$ is

$$\tilde{d}_\theta(x) = \frac{d(x,w)}{\cos\theta}. \tag{3.11}$$

Remark 3.2.4.

For a fixed $\alpha < \frac{\pi}{2}$, relation (3.8) gives the following limits:

1. When h tends to ∞, we have

$$\lim_{h\to\infty} \tan\theta_0 = \lim_{h\to\infty} \frac{h}{h + d(x,w)} \tan\alpha = \tan\alpha \lim_{h\to\infty} \frac{1}{1 + \left(\frac{d(x,w)}{h}\right)} = \tan\alpha, \tag{3.12}$$

and hence θ_0 tends to α as h tends to ∞.

2. When h tends to 0, we obtain

$$\lim_{h\to 0} \tan\theta_0 = \lim_{h\to 0} \frac{h}{h + d(x,w)} \tan\alpha = 0,$$

therefore θ_0 tends to 0 as h tends to 0.

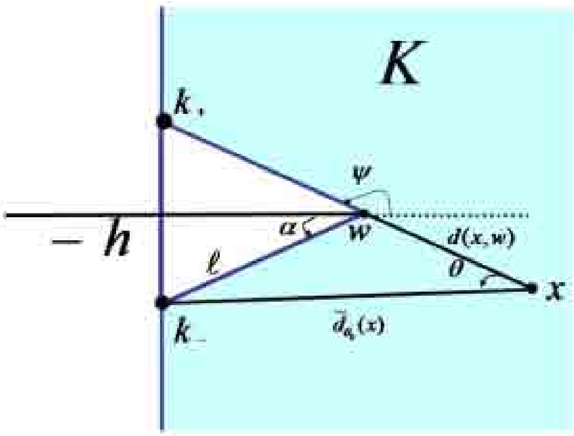

Figure 3.3: x lies on the extension of the line $z = \ell\, e^{i\psi}$

Case 2.

In this case, $x \in \Omega \subset K$ lies on the extension of the line $z = \ell\, e^{i\psi}$ $\left(\text{or } z = \ell\, e^{-i\psi}\right)$ (see Fig. 3.3). Although initially $0 < \alpha \leq \frac{\pi}{2}$, we will see later (see the discussion after Remark 3.2.5) that Case 2 makes sense only for $\alpha \leq \frac{\pi}{4}$. Therefore, throughout this discussion we assume $0 < \alpha \leq \frac{\pi}{4}$. Accordingly, we easily see that $\sin\theta_0 = \frac{h \sin 2\alpha}{\tilde{d}_{\theta_0}(x)\cos\alpha}$. On the other hand, $\tilde{d}_{\theta_0}(x) = \frac{(\ell + d(x,w))\sin\left(\frac{\pi}{2}-\alpha\right)}{\sin\left(\frac{\pi}{2}+\alpha-\theta_0\right)} = \frac{h + d(x,w)\cos\alpha}{\cos(\alpha-\theta_0)}$, thus θ_0 satisfies the following relation:

$$\frac{\sin\theta_0}{\cos\left(\alpha-\theta_0\right)} = \frac{h\, \sin 2\alpha}{\cos\alpha\left(h + d(x,w)\cos\alpha\right)}. \tag{3.13}$$

Relation (3.13) is well defined because $\alpha \leq \frac{\pi}{4}$ and $\theta_{\pm} \in \left(\pm\alpha - \frac{\pi}{2}, \pm\alpha + \frac{\pi}{2}\right)$, which imply that $\alpha - \theta_0 \neq \frac{\pi}{2}$. The denominator in the right hand side of (3.13) is always non-zero. In Case 2, $\theta_+ = 0$ $\left(\text{or } \theta_- = 0\right)$. Note also that:

1. For $\theta \in \left(0, \theta_0\right)$, we have $\frac{\tilde{d}_\theta(x)}{\sin(\pi-2\alpha)} = \frac{d(x,w)}{\sin(2\alpha-\theta)}$, which yields

$$\tilde{d}_\theta(x) = \frac{d(x,w)\sin 2\alpha}{\sin\left(2\alpha-\theta\right)}. \tag{3.14}$$

Relation (3.14) is well defined and the right hand side is positive since $\theta < \alpha$ implies that $\theta < 2\alpha$ which in turn gives $0 < 2\alpha - \theta$. Furthermore $2\alpha - \theta < \frac{\pi}{2}$, and therefore we have $\sin(2\alpha - \theta) > 0$.

2. For $\theta \in \left(\theta_0, \frac{\pi}{2} + \alpha\right)$, we have $\frac{\tilde{d}_\theta(x)}{\sin\left(\frac{\pi}{2}-\alpha\right)} = \frac{d(x,w)+\ell}{\sin\left(\frac{\pi}{2}+\alpha-\theta\right)}$, which implies that

$$\tilde{d}_\theta(x) = \frac{h + d(x,w)\cos\alpha}{\cos(\alpha - \theta)}. \tag{3.15}$$

Relation (3.15) is well defined and the right hand side is positive since $\alpha - \theta < \frac{\pi}{2}$.

3. For $\theta \in \left(-\frac{\pi}{2} + \alpha, 0\right)$, we have negative angles. The Sine Law, however, deals with positive angles. Hence, the relation between $\tilde{d}_\theta(x)$ and $d(x,w)$ is $\frac{\tilde{d}_\theta(x)}{\sin\left(\frac{\pi}{2}+\alpha\right)} = \frac{d(x,w)+\ell}{\sin\left(\frac{\pi}{2}-(\alpha+|\theta|)\right)}$, which means that

$$\tilde{d}_\theta(x) = \frac{h + d(x,w)\cos\alpha}{\cos(\alpha - \theta)}. \tag{3.16}$$

The relation (3.16) is well defined and its right hand side is positive as well, since $\alpha \leq \frac{\pi}{4}$.

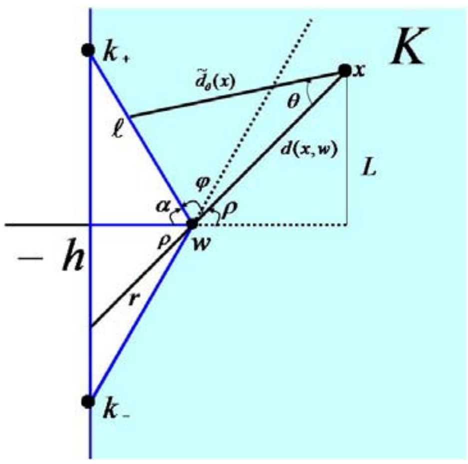

Figure 3.4: x lies at distance L from the symmetry axis

Case 3.

If the point $x \in \Omega \subset K$ lies at distance L above (or below) the symmetry axis of K and $\alpha < \frac{\pi}{2}$ (see Fig. 3.4), then it is clear that $\frac{\sin\theta_0}{\ell} = \frac{\sin(\pi-\alpha-\rho)}{\tilde{d}_{\theta_0}(x)}$ which implies that $\sin\theta_0 = \frac{h}{\cos\alpha}\frac{\sin(\alpha+\rho)}{\tilde{d}_{\theta_0}(x)}$. On the other hand, $\tilde{d}_{\theta_0}(x) = \frac{(r+d(x,w))\sin\left(\frac{\pi}{2}-\rho\right)}{\sin\left(\frac{\pi}{2}+\rho-\theta_0\right)}$ which leads to $\tilde{d}_{\theta_0}(x) = \frac{h+d(x,w)\,\cos\,\rho}{\cos(\rho-\theta_0)}$, thus θ_0 satisfies

$$\frac{\sin\theta_0}{\cos(\rho-\theta_0)} = \frac{h\sin(\alpha+\rho)}{\cos\alpha\,(h+d(x,w)\cos\rho)}, \tag{3.17}$$

where

$$\rho = \sin^{-1}\frac{L}{d(x,w)}. \tag{3.18}$$

It is obvious that the angle ρ is positive. Later on (see the discussion after Remark 3.2.5) we will see that the angle ρ satisfies

$$0 < \rho \leq \frac{\pi}{2} - \alpha.$$

Note also that for $0 \leq \rho < \alpha$, both θ_+ and θ_- are non-zero, so we have the following (see Fig. 3.4):

1. For $\theta \in \left(-\frac{\pi}{2}+\rho, \theta_-\right)$, i.e., $\theta < 0$, we have $\tilde{d}_\theta(x) = \frac{(r+d(x,w))\sin\left(\frac{\pi}{2}+\rho\right)}{\sin\left(\frac{\pi}{2}-\rho-|\theta|\right)}$, thus

$$\tilde{d}_\theta(x) = \frac{h+d(x,w)\cos(\rho)}{\cos(\rho-\theta)}. \tag{3.19}$$

2. For $\theta \in (\theta_-, 0)$, we have $\tilde{d}_\theta(x) = \frac{d(x,w)\sin(\pi-\alpha+\rho)}{\sin(\alpha-\rho-|\theta|)}$, hence

$$\tilde{d}_\theta(x) = \frac{d(x,w)\sin(\alpha-\rho)}{\sin(\alpha-\rho+\theta)}. \tag{3.20}$$

3. For $\theta \in (0, \theta_+)$, we find

$$\tilde{d}_\theta(x) = \frac{d(x,w)\sin(\alpha+\rho)}{\sin(\alpha+\rho-\theta)}. \tag{3.21}$$

4. For $\theta \in \left(\theta_+, \frac{\pi}{2} + \rho\right)$, we have

$$\tilde{d}_\theta(x) = \frac{h + d(x,w)\cos(\rho)}{\cos(\rho - \theta)}. \tag{3.22}$$

Due to the restrictions on ρ ($0 \leq \rho < \alpha$ and $\rho \leq \frac{\pi}{2} - \alpha$) and keeping in mind that $\alpha < \frac{\pi}{2}$, we can see that (3.19), (3.20), (3.21), and (3.22) are well defined.

Now for $\alpha \leq \rho \leq \frac{\pi}{2}$, we have either $\theta_- = 0$ or $\theta_+ = 0$. Let us discuss the case when $\theta_- = 0$, and the discussion of the other case will be similar. For $\theta_- = 0$, it follows from (3.7) that $\theta_0 = |\theta_+|$, hence we have

1. For $\theta \in (0, \theta_0)$, we have $\tilde{d}_\theta(x)$ exactly as given in (3.21).

2. For $\theta \in \left(\theta_0, \frac{\pi}{2} + \rho\right)$, we have $\tilde{d}_\theta(x)$ as given by (3.22).

3. For $\theta \in \left(-\frac{\pi}{2} + \rho, 0\right)$, we have $\tilde{d}_\theta(x)$ as given by (3.19).

Remark 3.2.5.

Case 3 is the more general and complex case than the first two cases. We should therefore expect that any study concerning the first two cases will be simpler than the one concerning the third case since we can extract the first two cases from Case 3 as shown below:

- If x lies on the symmetry axis of K i.e., $L = 0$, then (3.17) gives $\frac{\sin\theta_0}{\cos(-\theta_0)} = \frac{h\sin(\alpha)}{\cos\alpha(h+d(x,w))}$ which implies that

  $$\tan\theta_0 = \frac{h}{h + d(x,w)}\tan\alpha,$$

  as obtained in (3.8).

- If x lies on the extension of one of the lines $z = \ell\, e^{\pm i\psi}$ i.e., $\rho = \sin^{-1}\frac{L}{d(x,w)} = \alpha$, then (3.17) gives

  $$\frac{\sin\theta_0}{\cos(\alpha - \theta_0)} = \frac{h\sin 2\alpha}{\cos\alpha\,(h + d(x,w)\cos\alpha)},$$

  as obtained in (3.13).

Let us discuss the implications of Condition 3.2.1 on the values of α in Case 2 and ρ in Case 3. Let us fix a point $x \in \Omega \subset K$, and let $w \in \partial\Omega$ be such that $|x-w| = d(x)$. Denote by K the domain $K_{h,\psi}(w, \phi_w)$ defined in Condition 3.2.1. Thus, using the above notation we see that $d(x, w) = d(x)$. Note also that this equality limits the possible size of the angle $\rho = \sin^{-1} \frac{L}{d(x)}$, which appear in (3.17), in the following way:

$$\rho \leq \frac{\pi}{2} - \alpha = \psi - \frac{\pi}{2}. \tag{3.23}$$

This is because of the following: If $\rho > \frac{\pi}{2} - \alpha$ then the angle φ (see Fig. 3.4) will be less than $\frac{\pi}{2}$, which means that the distance from x to the boundary ∂K will be less than $d(x)$, which contradicts the fact that $d(x)$ is the 'shortest' distance to $\partial\Omega$. Therefore, in light of (3.23), as α tends to $\frac{\pi}{2}$, ρ tends to 0. The converse is not true.
The bound (3.23) on ρ illustrates the assumption on α in Case 2, since $\rho = \alpha$ thus (3.23) gives $\alpha \leq \frac{\pi}{2} - \alpha$ which means $\alpha \leq \frac{\pi}{4}$.

As a result of the relative 'simplicity' of formula (3.8) for θ_0 (which is simpler than (3.13) and (3.17)), it is natural to introduce a special case of the Condition 3.2.1 related to Case 1.

Condition 3.2.6.

For each $x \in \Omega$ there exists an element $w \in \partial\Omega$ such that $d(x) = |w - x|$ and $\Omega \subset K_{h,\psi}(w, \phi_w)$ with $\phi_w = \arg(x - w) + \pi$.

Condition 3.2.6 means that for every point $x \in \Omega$ we can always find a truncated sectorial region $K_{h,\psi}(w, \phi_w)$ such that x lies on its symmetry axis. This condition is applicable if the domain Ω possesses the following geometrical property:
Let n_w be an exterior normal to Ω (which is smooth) at the point $w \in \partial\Omega$. If Ω is bounded then the half-line $\{sn_w,\ s > 0\}$ does not intersect with $\partial\Omega$ for all $w \in \partial\Omega$.
Again because of the relative 'simplicity' of formula (3.13) for θ_0 (which is simpler than (3.17)), it is natural to introduce another special case of the Condition 3.2.1 related to Case 2.

Condition 3.2.7.

For each $x \in \Omega$ there exists an element $w \in \partial\Omega$ such that $d(x) = |w - x|$ and $\Omega \subset K_{h,\psi}(w, \phi_w)$ with $\phi_w = \arg(x - w) - \alpha + \pi$ or $\phi_w = \arg(x - w) + \alpha + \pi$, with $\alpha \leq \frac{\pi}{4}$.

Condition 3.2.7 means that for every point $x \in \Omega$ we always find a truncated sectorial region $K_{h,\psi}(w, \phi_w)$ such that x lies on one of the extensions of its sides. A simple example of such domains that satisfy Condition 3.2.7 is a disk.

Remark 3.2.8.

If a domain Ω satisfies any of the Conditions 3.2.1, 3.2.6, or 3.2.7 for some $h_0 \geq 0$, then the same domain Ω satisfies that condition for any $h \geq h_0$.

Now let us introduce another 'non-convexity measure', but this time we introduce a 'smooth' condition which we call the 'Exterior Disk' condition.
Denote by $B^2(a, R)$ the open disk in $\mathbb{R}^2$ of centre a and radius R i.e.,

$$B^2(a, R) = \{y \in \mathbb{R}^2 : |y - a| < R\}.$$

Our assumption on the domain $\Omega \subset \mathbb{R}^2$ is the following:

Condition 3.2.9. (Exterior Disk Condition)

We say that $\Omega \subset \mathbb{R}^2$ satisfies the Exterior Disk Condition if there exists a number $R > 0$ such that for each $w \in \partial\Omega$, one can find a point $a \in \mathbb{R}^2$ such that $B^2(a, R) \cap \Omega = \varnothing$ with $|w - a| = R$.

Condition 3.2.9 means that the disk $B^2(a, R)$ does not have any points in common with the interior of the domain Ω. However, the intersection of the disk $B^2(a, R)$ with the boundary $\partial\Omega$ of the domain Ω is not empty. In other words, the domain Ω (which is an open set) completely lies in the complement of the disk $B^2(a, R)$.
Observe here that the TSR Condition allows for an 'inward conical point' in the domain Ω.

On the other hand, the Exterior Disk condition implies more regularity of Ω. In particular 'inward conical points' are not admissible.

In the course of discussing the Exterior Disk Condition, we almost use the same notation introduced before in Notation 3.2.3 with slight differences. More precisely, $d(x)$ denotes the

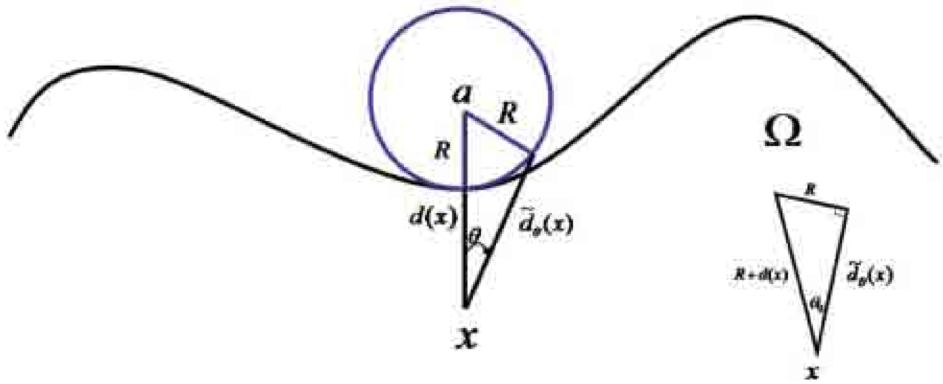

Figure 3.5: A domain in $\mathbb{R}^2$ which satisfies the exterior disk condition

distance from the point $x \in \Omega$ to the boundary $\partial\Omega$ of the domain Ω, as defined in (3.4). The symbol $d_\theta(x)$ denotes the distance from $x \in \Omega$ to the boundary $\partial\Omega$ of the domain Ω in the direction θ, as defined in (3.3), whereas $\tilde{d}_\theta(x)$ is for the distance from $x \in \Omega$ to the boundary $\partial B^2(a, R)$ of the disk $B^2(a, R)$ in the direction θ, i.e.,

$$\tilde{d}_\theta(x) := \min\{|s| : x + se^{i\theta} \in \partial B^2(a, R)\}.$$

Finally, by $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ we denote the angle at which the line segment representing $\tilde{d}_\theta(x)$ leaves the boundary $\partial B^2(a, R)$ of the disk $B^2(a, R)$ to infinity.

If the domain $\Omega \subset \mathbb{R}^2$ satisfies the Exterior Disk Condition, see Fig. 3.5, then it is clear that the angle $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ satisfies

$$\sin\theta_0 = \frac{R}{R + d(x)} = \frac{1}{1 + \frac{d(x)}{R}}. \tag{3.24}$$

Furthermore, the Cosine Law gives

$$R^2 = (R + d(x))^2 + \tilde{d}_\theta(x)^2 - 2\,(R + d(x))\,\tilde{d}_\theta(x)\cos\theta,$$

which produces the following quadratic equation in $\tilde{d}_\theta(x)$:

$$\tilde{d}_\theta(x)^2 - 2\,(R + d(x))\cos\theta\,\tilde{d}_\theta(x) + d(x)^2 + 2Rd(x) = 0. \tag{3.25}$$

The quadratic equation (3.25) has the following two solutions:

$$\tilde{d}_\theta(x) = (R + d(x))\cos\theta \pm \sqrt{(R + d\,(x))^2\cos^2\theta - (d(x)^2 + 2Rd(x))}. \tag{3.26}$$

If $\theta = 0$, the positive sign in (3.26) gives

$$\tilde{d}_\theta(x) = 2R + d(x),$$

which is not true, since at $\theta = 0$ we should have obtained

$$\tilde{d}_\theta(x) = d_\theta(x) = d(x).$$

On the other hand, the minus sign gives

$$\tilde{d}_\theta(x) = d(x),$$

as expected. Therefore, the only acceptable solution for (3.25) is

$$\begin{aligned}
\tilde{d}_\theta(x) &= (R + d(x))\cos\theta - \sqrt{R^2\cos^2\theta - d(x)^2\,(1 - \cos^2\theta) - 2Rd(x)\,(1 - \cos^2\theta)} \\
&= \cos\theta\left(R + d(x) - \sqrt{R^2 - (d(x)^2 + 2Rd(x))\tan^2\theta}\right),
\end{aligned}$$

which yields that

$$\frac{1}{\tilde{d}_\theta(x)^2} = \frac{1}{\cos^2\theta\left(R + d(x) - \sqrt{R^2 - (d(x)^2 + 2Rd(x))\tan^2\theta}\right)^2}. \tag{3.27}$$

Remark 3.2.10.

1. For all $\theta \in [0, \theta_0]$, we have

$$\left(\frac{d(x)^2}{R^2} + 2\frac{d(x)}{R}\right)\tan^2\theta \leq 1. \tag{3.28}$$

   Since

$$\sin\theta_0 = \frac{1}{1 + \frac{d(x)}{R}},$$

   thus

$$\tan^2\theta_0 = \frac{\sin^2\theta_0}{\cos^2\theta_0} = \frac{\sin^2\theta_0}{1 - \sin^2\theta_0} = \frac{1}{\frac{1}{\sin^2\theta_0} - 1} = \frac{1}{\left(1 + \frac{d(x)}{R}\right)^2 - 1},$$

   which leads to

$$\left(\frac{d(x)^2}{R^2} + 2\frac{d(x)}{R}\right)\tan^2\theta_0 = 1.$$

   Now (3.28) follows, since $\tan\theta$ is positive and increasing on $[0, \theta_0]$.

2. Relation (3.24) indicates that, if R tends to ∞, i.e. the domain under investigation approaches convexity, then θ_0 tends to $\frac{\pi}{2}$.

3.3 Hardy's inequalities under the TSR Condition

Through this section new Hardy-type inequalities will be obtained for some non-convex domains Ω in $\mathbb{R}^2$ which satisfy one of the Conditions 3.2.1, 3.2.6, or 3.2.7.

As an emphasis we would like to remind the reader here that our strategy in achieving this aim is to obtain lower bounds for the function $\frac{1}{m(x)^2}$, given by (3.2), in terms of the distance function $d(x)$, defined in (3.4). Then we apply Proposition 3.1.1 to these lower bounds.

Let us start with the case in which the point $x \in \Omega$ lies on the symmetry axis of the region $K_{h,\psi}(w, \phi_w)$. In what follows, $d_\theta(x)$ refers to the distance from $x \in \Omega$ to the boundary of Ω in the direction θ, as has been given in (3.3) and $\tilde{d}_\theta(x)$ denotes the distance from $x \in \Omega$ to the boundary of $K_{h,\psi}(w, \phi_w)$ in the direction θ, as has been defined in (3.5). The result is presented in the following theorem.

Theorem 3.3.1.

Suppose that the domain $\Omega \subset \mathbb{R}^2$ satisfies Condition 3.2.6 with some $\psi \in \left[\frac{\pi}{2}, \pi\right)$. Then for any function $f \in C_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$\int_\Omega \mu_1(x,\alpha,h) \frac{|f(x)|^2}{(h+d(x))^2} dx + \int_\Omega \mu_2(x,\alpha,h) \frac{|f(x)|^2}{d(x)^2} dx \quad \leq \quad \int_\Omega |\nabla f(x)|^2 \, dx, \tag{3.29}$$

where

$$\mu_1(x,\alpha,h) = \frac{\pi - 2\tan^{-1}(a(x)\,\tan\alpha) - \sin\left(2\tan^{-1}(a(x)\,\tan\alpha)\right)}{4\pi}, \tag{3.30}$$

$$\mu_2(x,\alpha,h) = \frac{1}{2\pi}\left[\frac{\tan^{-1}(a(x)\,\tan\alpha)}{\sin^2\alpha} - \frac{a(x)\,(\cos\alpha\cos 2\alpha + a(x)\,\sin\alpha\sin 2\alpha)}{\sin\alpha\left(\cos^2\alpha + a(x)\,\sin^2\alpha\right)}\right], \tag{3.31}$$

and

$$a(x) = \frac{h}{h+d(x)}.$$

If $\psi = \alpha = \frac{\pi}{2}$, then $\mu_1(x,\alpha,h) = 0$ and $\mu_2(x,\alpha,h) = \frac{1}{4}$.

Proof. By (3.2), the definition of $\frac{1}{m(x)^2}$, and the facts that the function $d_\theta(x)$ is a periodic function with period π, $d_\theta(x) \leq \tilde{d}_\theta(x)$ and that the function $\tilde{d}_\theta(x)$ is a symmetric function with respect to the symmetry axis of $K_{h,\psi}(w, \phi_w)$, we have

$$\begin{aligned}
\frac{1}{m(x)^2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{d_\theta(x)^2}d\theta = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{d_\theta(x)^2}d\theta \\
&\geq \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\tilde{d}_\theta(x)^2}d\theta = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\frac{1}{\tilde{d}_\theta(x)^2}d\theta \\
&= \frac{2}{\pi}\left[\int_{0}^{\theta_0} + \int_{\theta_0}^{\frac{\pi}{2}}\right]\frac{1}{\tilde{d}_\theta(x)^2}d\theta. \qquad (3.32)
\end{aligned}$$

Since Ω satisfies Condition 3.2.6, see Fig. 3.6, then θ_0 satisfies

$$\tan\theta_0 = \frac{h}{h + d(x)}\tan\alpha, \qquad (3.33)$$

as stated in (3.8). Recall also that, for $\alpha < \frac{\pi}{2}$, if $\theta \in [0, \theta_0)$ then the relation between $\tilde{d}_\theta(x)$ and $d(x)$ is

$$\tilde{d}_\theta(x) = \frac{d(x)\sin\alpha}{\sin(\alpha - \theta)},$$

as given in (3.9). Furthermore, if $\theta \in [\theta_0, \frac{\pi}{2})$ then the relation is

$$\tilde{d}_\theta(x) = \frac{h + d(x)}{\cos\theta},$$

as obtained in (3.10). On the other hand, for $\alpha = \frac{\pi}{2}$ we have

$$\tilde{d}_\theta(x) = \frac{d(x)}{\cos\theta}.$$

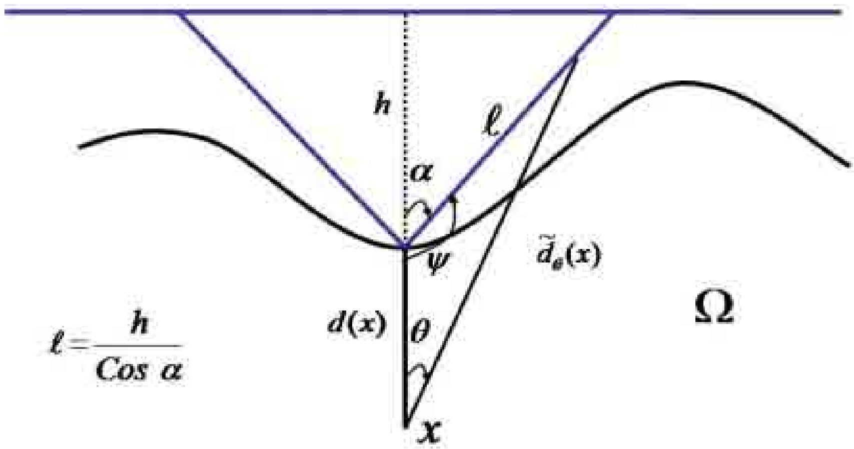

Figure 3.6: x lies on the symmetry axis of K for a domain in $\mathbb{R}^2$

Therefore, inequality (3.32) can be written as follows:

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{2}{\pi}\left[\frac{\int_0^{\theta_0}\sin^2(\alpha-\theta)\,d\theta}{d(x)^2\sin^2\alpha}+\frac{\int_{\theta_0}^{\frac{\pi}{2}}\cos^2\theta d\theta}{(h+d(x))^2}\right] \\
&= \frac{2}{\pi}\left[\frac{\int_0^{\theta_0}(1-\cos 2(\alpha-\theta))\,d\theta}{2d(x)^2\sin^2\alpha}+\frac{\int_{\theta_0}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta}{2(h+d(x))^2}\right] \\
&= \frac{1}{\pi}\left[\frac{\left(\theta+\frac{\sin 2(\alpha-\theta)}{2}\right)\Big|_0^{\theta_0}}{d(x)^2\sin^2\alpha}+\frac{\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_{\theta_0}^{\frac{\pi}{2}}}{(h+d(x))^2}\right] \\
&= \frac{1}{\pi}\left[\frac{\theta_0+\frac{\sin 2(\alpha-\theta_0)}{2}-\frac{\sin 2\alpha}{2}}{d(x)^2\sin^2\alpha}+\frac{\frac{\pi}{2}-\theta_0-\frac{\sin 2\theta_0}{2}}{(h+d(x))^2}\right] \\
&= \frac{\theta_0-\cos(\theta_0-2\alpha)\sin\theta_0}{\pi\, d(x)^2\,\sin^2\alpha}+\frac{\pi-2\theta_0-\sin 2\theta_0}{2\pi\,(h+d(x))^2}. \qquad (3.34)
\end{aligned}$$

Applying Proposition 3.1.1 to (3.34) gives the following inequality:

$$\frac{1}{4\pi}\int\limits_\Omega\frac{(\pi-2\theta_0-\sin 2\theta_0)\,|f(x)|^2}{(h+d(x))^2}dx+\frac{1}{2\pi\sin^2\alpha}\int\limits_\Omega\frac{(\theta_0-\cos(\theta_0-2\alpha)\sin\theta_0)\,|f(x)|^2}{d(x)^2}dx$$
$$\leq\frac{1}{2}\int\limits_\Omega\frac{|f(x)|^2}{m(x)^2}dx\leq\int\limits_\Omega|\nabla f(x)|^2\,dx,$$

which implies that

$$\int_{\Omega} \mu_1^*(x,\theta_0) \frac{|f(x)|^2}{(h+d(x))^2} dx + \int_{\Omega} \mu_2^*(x,\theta_0) \frac{|f(x)|^2}{d(x)^2} dx \quad \leq \quad \int_{\Omega} |\nabla f(x)|^2 \, dx,$$

where

$$\mu_1^*(x,\theta_0) = \frac{\pi - 2\theta_0 - \sin 2\theta_0}{4\pi},$$

and

$$\mu_2^*(x,\theta_0) = \frac{\theta_0 - \cos(\theta_0 - 2\alpha)\sin\theta_0}{2\pi \sin^2 \alpha}.$$

Using (3.33), we can write both functions $\mu_1^*(x,\theta_0)$ and $\mu_2^*(x,\theta_0)$ as functions of α, h and x as follows:

$$\mu_1(x,\alpha,h) = \frac{\pi - 2\tan^{-1}(a(x)\ \tan\alpha) - \sin(2\tan^{-1}(a(x)\ \tan\alpha))}{4\pi},$$

and

$$\mu_2(x,\alpha,h) = \frac{1}{2\pi}\left[\frac{\tan^{-1}(a(x)\ \tan\alpha)}{\sin^2\alpha} - \frac{a(x)\ (\cos\alpha\cos 2\alpha + a(x)\ \sin\alpha\sin 2\alpha)}{\sin\alpha\left(\cos^2\alpha + a(x)\ \sin^2\alpha\right)}\right],$$

as stated in (3.30) and (3.31) respectively. Furthermore, when $\psi = \alpha = \frac{\pi}{2}$, then $\theta_0 = \frac{\pi}{2}$ as well, thus we have $\mu_1(x,\alpha,h) = 0$ and $\mu_2(x,\alpha,h) = \frac{1}{4}$ which completes the proof. □

Remark 3.3.2.

1. If Ω is convex then $\psi = \alpha = \frac{\pi}{2}$, thus (3.30) and (3.31) give $\mu_1(x,\alpha,h) = 0$ and $\mu_2(x,\alpha,h) = \frac{1}{4}$ respectively. Therefore (3.34) yields

$$\frac{1}{m(x)^2} \geq \frac{1}{2d(x)^2},$$

as expected (see Theorem 2.2.7). Thus the Hardy-type inequality (3.29) reproduces the well-known bound (see Theorem 2.2.6)

$$\frac{1}{4}\int_{\Omega}\frac{|f(x)|^2}{d(x)^2}dx \leq \int_{\Omega}|\nabla f(x)|^2\,dx, \tag{3.35}$$

for convex domains.

2. The functions $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$, given by (3.30) and (3.31) respectively, are uniformly bounded functions in their variables ($x\in\Omega$, $\alpha\in\left(0,\frac{\pi}{2}\right]$ and $h\in(0,\infty)$) since

$$\begin{aligned}
|\mu_1(x,\alpha,h)| &= \frac{1}{4\pi}\left|\pi-2\tan^{-1}(a(x)\tan\alpha)-\sin\left(2\tan^{-1}(a(x)\tan\alpha)\right)\right| \\
&\leq \frac{1}{4\pi}\left(\pi+2\left|\tan^{-1}(a(x)\tan\alpha)\right|+\left|\sin\left(2\tan^{-1}(a(x)\tan\alpha)\right)\right|\right) \\
&\leq \frac{1}{4}+\frac{1}{2\pi}\frac{\pi}{2}+\frac{1}{4\pi}<\frac{3}{4},
\end{aligned}$$

so $\mu_1(x,\alpha,h)$ is bounded. In respect of $\mu_2(x,\alpha,h)$, since it is a continuous function in its variables and the only concern about this function is how it behaves when $\alpha\to 0$, we look into the following limit:

$$\begin{aligned}
\lim_{\alpha\to 0}\mu_2(x,\alpha,h) &= \lim_{\alpha\to 0}\frac{1}{2\pi}\left[\frac{\tan^{-1}(a(x)\,\tan\alpha)}{\sin^2\alpha}-\frac{a(x)\,(\cos\alpha\cos 2\alpha+a(x)\,\sin\alpha\sin 2\alpha)}{\sin\alpha\left(\cos^2\alpha+a(x)\,\sin^2\alpha\right)}\right] \\
&= \lim_{\alpha\to 0}\frac{1}{2\pi\left(\cos^2\alpha+a(x)\,\sin^2\alpha\right)}\left[a(x)\tan^{-1}(a(x)\,\tan\alpha)-a(x)^2\sin(2\alpha)\right. \\
&\qquad \left.+\cot^2\alpha\tan^{-1}(a(x)\,\tan\alpha)-a(x)\cot\alpha\cos(2\alpha)\right].
\end{aligned}$$

This limit, by using L'Hôpital's rule twice, gives

$$\lim_{\alpha\to 0}\mu_2(x,\alpha,h) = 0.$$

It follows that $\mu_2(x,\alpha,h)$ is a bounded function as well.

3. For fixed α, as h tends to ∞ we have $a(x)$ tends to 1. Consequently, as a result of Remark 3.2.4, we have the following limit for $\mu_2(x, \alpha, h)$:

$$\begin{aligned}
\lim_{h\to\infty} \mu_2(x,\alpha,h) &= \lim_{h\to\infty} \frac{1}{2\pi}\left[\frac{\tan^{-1}(a(x)\,\tan\alpha)}{\sin^2\alpha} - \frac{a(x)\,(\cos\alpha\cos 2\alpha + a(x)\,\sin\alpha\sin 2\alpha)}{\sin\alpha\left(\cos^2\alpha + a(x)\,\sin^2\alpha\right)}\right]\\
&= \frac{1}{2\pi}\left[\frac{\alpha}{\sin^2\alpha} - \frac{\cos\alpha\cos 2\alpha + \sin\alpha\sin 2\alpha}{\sin\alpha\left(\cos^2\alpha + \sin^2\alpha\right)}\right]\\
&= \frac{1}{2\pi}\left[\frac{\alpha}{\sin^2\alpha} - \frac{\cos\alpha}{\sin\alpha}\right] = \frac{1}{2\pi\sin^2\alpha}\left[\alpha - \frac{1}{2}\sin 2\alpha\right]\\
&= \frac{1}{2\pi\sin^2\psi}\left[\pi - \psi + \sin\psi\cos\psi\right];\ \alpha = \pi - \psi,
\end{aligned}$$

which depends only on $\psi \in \left[\frac{\pi}{2}, \pi\right)$ and belongs to $\left(0, \frac{1}{4}\right]$. Thus due to Remark 3.2.8 and the fact that all functions (f, μ_1, μ_2) are uniformly bounded, we can pass to the limit under the integral, thus the first term in (3.29) tends to zero. Hence, the Hardy inequality (3.29) takes the form

$$\mu(\psi)\int_\Omega \frac{|f(x)|^2}{d(x)^2}dx \le \int_\Omega |\nabla f(x)|^2\,dx, \tag{3.36}$$

where

$$\mu(\psi) = \frac{1}{2\pi\sin^2\psi}\left[\pi - \psi + \sin\psi\cos\psi\right]. \tag{3.37}$$

This constant $\mu(\psi)$, given by (3.37), is close to the constant (2.18) (see Laptev-Sobolev result, i.e. Theorem 2.2.10) for values of the angles ψ, θ in a neighbourhood of $\frac{\pi}{2}$, see Fig. 3.7.

Although their result seems to be more accurate, their approach is applicable only for simply-connected planar domains and results for higher dimensional domains are not mentioned. Our approach, however, can be extended to higher dimensional domains (see Chapters 4 and 5).

On the other hand, when h tends to 0, we have $a(x)$ tends to 0 as well, thus it is clear

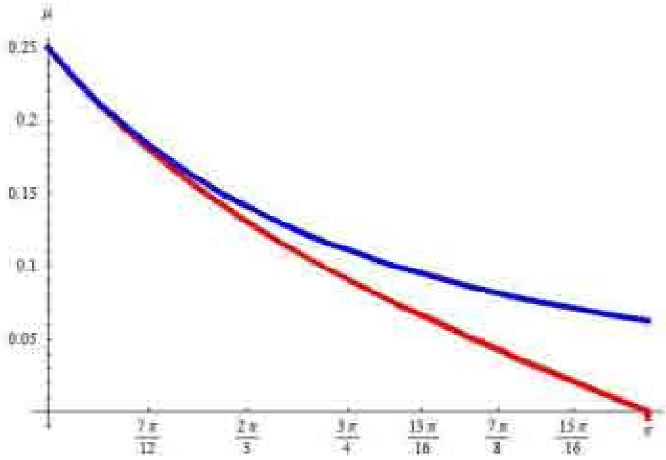

Figure 3.7: Our result in red vs Laptev-Sobolev in blue

that $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ tend to $\frac{1}{4}$ and 0 respectively.

4. As $\alpha \nearrow \frac{\pi}{2}$, the domain Ω approaches convexity, and hence it is natural to compare $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$, given by (3.30) and (3.31) respectively, with their values for the convex case. To this end we use a Taylor expansion to expand $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ in powers of $\left(\frac{\pi}{2}-\alpha\right)$, keeping in mind that for fixed h and $\alpha=\frac{\pi}{2}$, we have $\theta_0=\tan^{-1}(a(x)\tan\alpha)=\frac{\pi}{2}$. Consequently, regarding $\mu_1(x,\alpha,h)$ we have

$$\mu_1\left(x,\frac{\pi}{2},h\right)=0.$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mu_1(x,\alpha,h) &= \frac{1}{4\pi}\left[-2\frac{a(x)\sec^2\alpha}{1+a(x)^2\tan^2\alpha}-\cos\left(2\tan^{-1}(a(x)\tan\alpha)\right)\frac{2a(x)\sec^2\alpha}{1+a(x)^2\tan^2\alpha}\right]\\ &= -\frac{a(x)}{2\pi}\left(\frac{1+\cos\left(2\tan^{-1}(a(x)\tan\alpha)\right)}{\cos^2\alpha+a(x)^2\sin^2\alpha}\right),\end{aligned}$$

which implies that

$$\frac{\partial}{\partial\alpha}\mu_1\left(x,\frac{\pi}{2},h\right) = -\frac{1}{2\pi a(x)}[1-1]=0.$$

Moreover,

$$
\begin{aligned}
\frac{\partial^2}{\partial \alpha^2}\mu_1(x,\alpha,h) = &-\frac{a(x)}{2\pi\left(\cos^2\alpha + a(x)^2\sin^2\alpha\right)^2}\Bigg[\left(\cos^2\alpha + a(x)^2\sin^2\alpha\right) \\
&\left(-\sin\left(2\tan^{-1}\left(a(x)\tan\alpha\right)\right)\frac{2a(x)\sec^2\alpha}{1+a(x)^2\tan^2\alpha}\right) \\
&-\left(1+\cos\left(2\tan^{-1}\left(a(x)\tan\alpha\right)\right)\right)\sin 2\alpha\left(a(x)^2-1\right)\Bigg] \\
= &-\frac{a(x)}{2\pi\left(\cos^2\alpha + a(x)^2\sin^2\alpha\right)^2}\left[-2a(x)\sin\left(2\tan^{-1}\left(a(x)\tan\alpha\right)\right)\right. \\
&\left.-\left(1+\cos\left(2\tan^{-1}\left(a(x)\tan\alpha\right)\right)\right)\sin 2\alpha\left(a(x)^2-1\right)\right],
\end{aligned}
$$

which gives

$$
\frac{\partial^2}{\partial \alpha^2}\mu_1\left(x,\frac{\pi}{2},h\right) = 0.
$$

Similarly, we find

$$
\frac{\partial^3}{\partial \alpha^3}\mu_1\left(x,\frac{\pi}{2},h\right) = -\frac{2}{\pi a(x)^3} = -\frac{2\left(h+d(x)\right)^3}{\pi h^3}, \ \cdots,
$$

we continue taking higher derivatives in this fashion. Accordingly, $\mu_1(x,\alpha,h)$ can be written as follows:

$$
\mu_1(x,\alpha,h) = \frac{(h+d(x))^3}{3\pi h^3}\left(\frac{\pi}{2}-\alpha\right)^3 + \mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^4\right). \tag{3.38}
$$

Concerning the function $\mu_2(x,\alpha,h)$ we have

$$
\mu_2\left(x,\frac{\pi}{2},h\right) = \frac{1}{4}.
$$

Besides,

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\mu_2(x,\alpha,h) \;=\; & \frac{1}{2\pi}\left[\frac{\sin^2\alpha\,\frac{a(x)}{\cos^2\alpha+a(x)^2\sin^2\alpha}-\tan^{-1}(a(x)\tan\alpha)\sin 2\alpha}{\sin^4\alpha}\right.\\
& -\frac{1}{\sin^2\alpha\left(\cos^2\alpha+a(x)^2\sin^2\alpha\right)^2}\Big\{\sin\alpha\left(\cos^2\alpha+a(x)^2\sin^2\alpha\right)\\
& \times\left[a(x)\left(-2\cos\alpha\sin 2\alpha-\sin\alpha\cos 2\alpha+a(x)\left(2\sin\alpha\cos 2\alpha\right.\right.\right.\\
& \left.\left.\left.+\cos\alpha\sin 2\alpha\right)\right)\right]-a(x)\left(\cos\alpha\cos 2\alpha+a(x)\sin\alpha\sin 2\alpha\right)\\
& \left.\left(\sin\alpha\left(a(x)^2-1\right)\sin 2\alpha+\cos\alpha\left(a(x)^2\sin^2\alpha+\cos^2\alpha\right)\right)\Big\}\right],
\end{aligned}
$$

which leads to

$$
\frac{\partial}{\partial\alpha}\mu_2\left(x,\frac{\pi}{2},h\right) \;=\; \frac{1}{\pi}.
$$

In the same way we find

$$
\frac{\partial^2}{\partial\alpha^2}\mu_2\left(x,\frac{\pi}{2},h\right)=\frac{1}{2},\;\frac{\partial^3}{\partial\alpha^3}\mu_2\left(x,\frac{\pi}{2},h\right)=\frac{1}{2},\;\cdots\;\text{and so on.}
$$

Thus $\mu_2(x,\alpha)$ can be written as follows:

$$
\mu_2(x,\alpha,h)=\frac{1}{4}+\frac{1}{\pi}\left(\alpha-\frac{\pi}{2}\right)+\frac{1}{4}\left(\alpha-\frac{\pi}{2}\right)^2+\mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^3\right). \tag{3.39}
$$

In (3.39), h appears in the terms of order greater than or equal three. If α tends to $\frac{\pi}{2}$, then relations (3.38) and (3.39) lead to the same inequality as in (3.35) and indicate that the value of h will not matter. They also show that the second term in (3.29) is the effective term when talking about the convex case, since $\mu_1(x,\alpha,h)$ tends to zero while $\mu_2(x,\alpha,h)$ tends to $\frac{1}{4}$ when α tends to $\frac{\pi}{2}$.

5. Imposing the half space $(\Re z > -h)$ in Condition 3.2.6 enabled us to obtain the improved Hardy-type inequality (3.29) in the above theorem, as well as some measurement for how “deep” the non-convexity is.

The question now is: What if x does not lie on the symmetry axis or equivalently Ω does not satisfy Condition 3.2.6?

As an attempt to answer this question partially we study a slightly different case, the case when x lies on the extension of one of the two lines $z = \ell\, e^{i\psi}$ or $z = \ell\, e^{-i\psi}$.

Theorem 3.3.3.

Suppose that the domain $\Omega \subset \mathbb{R}^2$ satisfies Condition 3.2.7 with some $\psi \in \left[\frac{3\pi}{4}, \pi\right)$. Then for any function $f \in \mathcal{C}_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$\int_\Omega \mu_1(x,\alpha,h)\,\frac{|f(x)|^2}{(h+d(x)\cos\alpha)^2}dx + \int_\Omega \mu_2(x,\alpha,h)\,\frac{|f(x)|^2}{d(x)^2}dx \le \int_\Omega |\nabla f(x)|^2\,dx, \quad (3.40)$$

where

$$\mu_1(x,\alpha,h) = \frac{2\pi - \sin 2\alpha + \sin 2(\alpha-\theta_0) - 2\theta_0}{8\pi}, \quad (3.41)$$

and

$$\mu_2(x,\alpha,h) = \frac{2\theta_0 + \sin(4\alpha - 2\theta_0) - \sin 4\alpha}{8\pi \sin^2 2\alpha}. \quad (3.42)$$

The angle $\theta_0 = \theta_0(x,\alpha,h)$ is defined by (3.13).

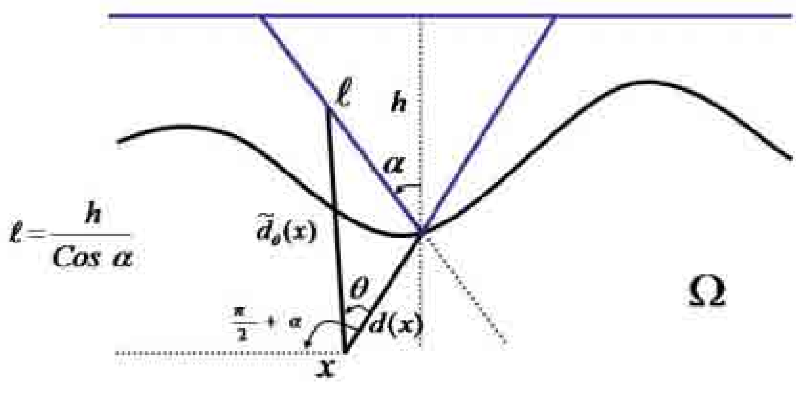

Figure 3.8: x lies on the extension of the line $z = \ell\, e^{i\psi}$ in a domain in $\mathbb{R}^2$

Proof. By (3.2), the definition of $\frac{1}{m(x)^2}$, and the facts that the function $d_\theta(x)$ is a periodic function with period π and $d_\theta(x) \leq \tilde{d}_\theta(x)$, we have

$$\begin{aligned}
\frac{1}{m(x)^2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{d_\theta(x)^2} d\theta = \frac{1}{\pi}\int_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}+\alpha} \frac{1}{d_\theta(x)^2} d\theta \\
&\geq \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}+\alpha}^{0} + \int_{0}^{\theta_0} + \int_{\theta_0}^{\frac{\pi}{2}+\alpha}\right] \frac{1}{\tilde{d}_\theta(x)^2} d\theta. \qquad (3.43)
\end{aligned}$$

Since the domain under investigation satisfies Condition 3.2.7, see Fig. 3.8, then for $\theta \in (0, \theta_0)$, the relation between $\tilde{d}_\theta(x)$ and $d(x)$ is given by

$$\tilde{d}_\theta(x) = \frac{d(x)\sin 2\alpha}{\sin(2\alpha - \theta)},$$

as obtained in (3.14). In addition, for $\theta \in \left(\theta_0, \frac{\pi}{2} + \alpha\right)$ the relation is

$$\tilde{d}_\theta(x) = \frac{h + d(x)\cos\alpha}{\cos(\alpha - \theta)},$$

as obtained in (3.15). Moreover, for $\theta \in \left(-\frac{\pi}{2} + \alpha, 0\right)$ we have

$$\tilde{d}_\theta(x) = \frac{h + d(x)\cos\alpha}{\cos(\alpha - \theta)},$$

as obtained in (3.16). Recall also that the angle θ_0 satisfies

$$\frac{\sin\theta_0}{\cos(\alpha - \theta_0)} = \frac{h\,\sin 2\alpha}{\cos\alpha\,(h + d(x)\cos\alpha)},$$

as stated in (3.13). Therefore, inequality (3.43) gives the following lower bound on the function $\frac{1}{m(x)^2}$:

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{1}{\pi}\left[\int\limits_{-\frac{\pi}{2}+\alpha}^{0} \frac{\cos^2(\alpha-\theta)}{(h+d(x)\cos\alpha)^2}d\theta + \int\limits_{0}^{\theta_0} \frac{\sin^2(2\alpha-\theta)}{d(x)^2\sin^2 2\alpha}d\theta \right.\\
&\qquad \left. + \int\limits_{\theta_0}^{\frac{\pi}{2}+\alpha} \frac{\cos^2(\alpha-\theta)}{(h+d(x)\cos\alpha)^2}d\theta\right]\\
&= \frac{1}{\pi}\left[\frac{\pi-2\alpha-\sin 2\alpha}{4(h+d(x)\cos\alpha)^2} + \frac{2\theta_0+\sin(4\alpha-2\theta_0)-\sin 4\alpha}{4d(x)^2\sin^2 2\alpha}\right.\\
&\qquad \left. + \frac{\pi+2\alpha+\sin 2(\alpha-\theta_0)-2\theta_0}{4(h+d(x)\cos\alpha)^2}\right]\\
&= \frac{2\theta_0+\sin(4\alpha-2\theta_0)-\sin 4\alpha}{4\pi d(x)^2\sin^2 2\alpha}\\
&\qquad + \frac{2\pi-\sin 2\alpha+\sin 2(\alpha-\theta_0)-2\theta_0}{4\pi(h+d(x)\cos\alpha)^2}
\end{aligned} \tag{3.44}$$

Applying Proposition 3.1.1 to (3.44) leads to the following Hardy-type inequality:

$$\int\limits_{\Omega} \mu_1(x,\alpha,h)\frac{|f(x)|^2}{(h+d(x)\cos\alpha)^2}dx + \int\limits_{\Omega} \mu_2(x,\alpha,h)\frac{|f(x)|^2}{d(x)^2}dx \leq \int\limits_{\Omega} |\nabla f(x)|^2\,dx,$$

where

$$\mu_1(x,\alpha,h) = \frac{2\pi-\sin 2\alpha+\sin 2(\alpha-\theta_0)-2\theta_0}{8\pi},$$

and

$$\mu_2(x,\alpha,h) = \frac{2\theta_0+\sin(4\alpha-2\theta_0)-\sin 4\alpha}{8\pi\sin^2 2\alpha},$$

as stated in (3.41) and (3.42) respectively. □

Remark 3.3.4.

To extract Hardy's inequality for convex domains from Theorem 3.3.3, we need to take the limits of the functions $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ given by (3.41) and (3.42) respectively, as α and h tend to 0. However, because of relation (3.13), if α tends to 0, then θ_0 tends to 0

as well. Therefore,

$$\lim_{\alpha \to 0} \mu_1(x, \alpha, h) = \lim_{\alpha \to 0} \frac{2\pi - \sin 2\alpha + \sin 2(\alpha - \theta_0) - 2\theta_0}{8\pi} = \frac{1}{4}.$$

To find the limit of $\mu_2(x, \alpha, h)$, note that relation (3.13) gives:

$$\tan \theta_0 = \frac{\sin\, 2\alpha}{1 + \frac{d(x)}{h} \cos \alpha - \sin\, 2\alpha\, \tan \alpha},$$

which implies that

$$\left. \frac{\partial \theta_0}{\partial\, \alpha} \right|_{\alpha=0} = \frac{2}{1 + \frac{d(x)}{h}} \quad \text{and} \quad \left. \frac{\partial^2 \theta_0}{\partial\, \alpha^2} \right|_{\alpha=0} = 0.$$

Therefore, applying L'Hôpital's rule twice with respect to α leads to

$$\lim_{\alpha \to 0} \mu_2(x, \alpha, h) = 0.$$

Consequently for small α, the Hardy-type inequality (3.40) produces the following bound:

$$\frac{1}{4} \int_{\Omega} \frac{|f(x)|^2}{(h + d(x))^2} dx \leq \int_{\Omega} |\nabla f(x)|^2\, dx,$$

from which if h tends to 0 we obtain the well known inequality for convex domains i.e.,

$$\frac{1}{4} \int_{\Omega} \frac{|f(x)|^2}{d(x)^2} dx \leq \int_{\Omega} |\nabla f(x)|^2\, dx.$$

In the above discussion it does not really matter if we take the limit when h tends to zero first and then the limit when α tends to zero. The reason behind this is the relation between θ_0, α and h, relation (3.13), since for any order of limits we have θ_0 tends to 0.

Suppose that Conditions 3.2.6 and 3.2.7 are not satisfied, so we go to the basic condition i.e., Condition 3.2.1. The result of such case is given in the following theorem.

Theorem 3.3.5.

Suppose that $\Omega \subset \mathbb{R}^2$ satisfies Condition 3.2.1 with some $\psi \in \left(\frac{\pi}{2}, \pi\right)$. Then for any function $f \in \mathcal{C}_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$\int_\Omega \mu_1(x) \frac{|f(x)|^2}{(h + d(x)\cos\rho)^2} dx + \int_\Omega \mu_2(x,\alpha) \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2 \, dx,$$

where

$$\mu_1(x) = \frac{1}{8\pi}\left(2\pi + \sin 4\rho + \sin 2(\rho + \theta_-) + \sin 2(\rho - \theta_+) + 2(\theta_- - \theta_+)\right),$$

$$\mu_2(x,\alpha) = \frac{1}{4\pi}\left(\frac{\cos(2\alpha - 2\rho - \theta_-)\sin\theta_- - \theta_-}{\sin^2(\alpha - \rho)} + \frac{\theta_+ - \cos(2(\alpha+\rho) - \theta_+)\sin\theta_+}{\sin^2(\alpha+\rho)}\right),$$

with $\theta_\pm(x) \in \left(\pm\rho - \frac{\pi}{2}, \pm\rho + \frac{\pi}{2}\right)$, and $\rho = \rho(x)$ is given by (3.18).

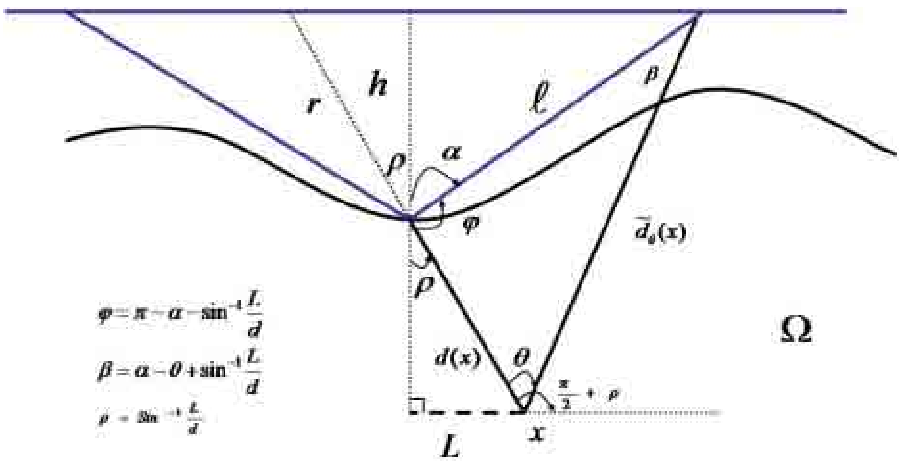

Figure 3.9: x lies at distance L from the symmetry axis for a domain in $\mathbb{R}^2$

Proof. Considering (3.2), the definition of $\frac{1}{m(x)^2}$, and the facts that the function $d_\theta(x)$ is a periodic function with period π and $d_\theta(x) \leq \tilde{d}_\theta(x)$, we obtain

$$\frac{1}{m(x)^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{d_\theta(x)^2} d\theta \geq \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\tilde{d}_\theta(x)^2} d\theta. \tag{3.45}$$

Since the domain under consideration satisfies Condition 3.2.1, let us consider an arbitrary point $x \in \Omega$ to be at distance L from the symmetry axis of $K_{h,\psi}(w, \phi_w)$, as shown in Fig. 3.9. Then for $0 \leq \rho < \alpha$, inequality (3.45) leads to

$$\begin{aligned} \frac{1}{m(x)^2} &\geq \frac{1}{\pi} \int_{-\frac{\pi}{2}+\rho}^{\frac{\pi}{2}+\rho} \frac{1}{\tilde{d}_\theta(x)^2} d\theta \\ &= \frac{1}{\pi} \left[\int_{-\frac{\pi}{2}+\rho}^{\theta_-} + \int_{\theta_-}^{0} + \int_{0}^{\theta_+} + \int_{\theta_+}^{\frac{\pi}{2}+\rho} \right] \frac{1}{\tilde{d}_\theta(x)^2} d\theta. \end{aligned} \tag{3.46}$$

On the other hand, the relation between $\tilde{d}_\theta(x)$ and $d(x)$ differs for each range of the integrals in (3.46), as have been discussed earlier in Section 3.2. It is therefore convenient to recall the following: For $\theta \in \left(-\frac{\pi}{2} + \rho, \theta_-\right)$, the relation is

$$\tilde{d}_\theta(x) = \frac{h + d(x)\cos(\rho)}{\cos(\rho - \theta)}, \tag{3.47}$$

as stated in (3.19).

Note that if x lies on the symmetry axis of K i.e., $L = 0$, we have $\tilde{d}_\theta(x) = \frac{h+d(x)}{\cos\theta}$, and if x lies on the extension line of the side of $K_{h,\psi}(w, \phi_w)$, i.e. $\rho = \alpha$, then $\tilde{d}_\theta(x) = \frac{h+d(x)\cos(\alpha)}{\cos(\alpha-\theta)}$. In this case $\theta_- = 0$ and the integration - when computing the lower bound of $\frac{1}{m(x)^2}$ - will be from $-\frac{\pi}{2} + \alpha$ to 0.

For $\theta \in (\theta_-, 0)$, we find

$$\tilde{d}_\theta(x) = \frac{d(x)\sin(\alpha - \rho)}{\sin(\alpha - \rho + \theta)}, \tag{3.48}$$

as stated in (3.20).

If x lies on the symmetry axis of K, (3.48) gives $\tilde{d}_\theta(x) = \frac{d(x)\sin(\alpha)}{\sin(\alpha+\theta)}$. On the other hand, if x lies on the extension line of the side of $K_{h,\psi}(w, \phi_w)$, then $\theta_- = 0$, so there won't be any contribution from this part when computing the lower bound of $\frac{1}{m(x)^2}$.

For $\theta \in (0, \theta_+)$, we get

$$\tilde{d}_\theta(x) = \frac{d(x) \sin(\alpha + \rho)}{\sin(\alpha + \rho - \theta)}, \tag{3.49}$$

as stated in (3.21).

If x lies on the symmetry axis of K, (3.49) yields $\tilde{d}_\theta(x) = \frac{d(x)\sin(\alpha)}{\sin(\alpha-\theta)}$, however if x lies on the extension line of the side of $K_{h,\psi}(w, \phi_w)$, i.e., $\rho = \alpha$, then $\tilde{d}_\theta(x) = \frac{d(x)\sin 2\alpha}{\sin(2\alpha-\theta)}$.

For $\theta \in \left(\theta_+, \frac{\pi}{2} + \rho\right)$, we have

$$\tilde{d}_\theta(x) = \frac{h + d(x) \cos(\rho)}{\cos(\rho - \theta)}, \tag{3.50}$$

as stated in (3.22).

If x lies on the symmetry axis of K, (3.50) leads to $\tilde{d}_\theta(x) = \frac{h+d(x)}{\cos\theta}$, and if x lies on the extension line of the side of $K_{h,\psi}(w, \phi_w)$, $\tilde{d}_\theta(x) = \frac{h+d(x)\cos(\alpha)}{\cos(\alpha-\theta)}$.

Consequently, we can conclude that for $\alpha \leq \rho \leq \frac{\pi}{2}$, inequality (3.46) takes the form

$$\frac{1}{m(x)^2} \geq \frac{1}{\pi} \left[\int_0^{\theta_0} + \int_{\theta_0}^{\frac{\pi}{2}+\rho} + \int_{\frac{\pi}{2}+\rho}^{\pi} \right] \frac{1}{\tilde{d}_\theta(x)^2} d\theta, \tag{3.51}$$

which would lead to an inequality similar to (3.43) in the proof of Theorem 3.3.3. Therefore, we focus only on the case when $0 \leq \rho < \alpha$.

Now use (3.47), (3.48) (3.49), and (3.50), into (3.46) to obtain

$$\begin{aligned} \frac{1}{m(x)^2} \quad \geq \quad & \frac{1}{\pi} \left[\int_{-\frac{\pi}{2}+\rho}^{\theta_-} \frac{\cos^2(\rho - \theta)}{(h + d(x)\cos(\rho))^2} d\theta + \int_{\theta_-}^{0} \frac{\sin^2(\alpha - \rho + \theta)}{d(x)^2 \sin^2(\alpha - \rho)} d\theta \right. \\ & \left. + \int_0^{\theta_+} \frac{\sin^2(\alpha + \rho - \theta)}{d(x)^2 \sin^2(\alpha + \rho)} d\theta + \int_{\theta_+}^{\frac{\pi}{2}+\rho} \frac{\cos^2(\rho - \theta)}{(h + d(x)\cos(\rho))^2} d\theta \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[\frac{\left(\theta-\frac{\sin 2(\rho-\theta)}{2}\right)\Big|_{-\frac{\pi}{2}+\rho}^{\theta_-}}{2\left(h+d(x)\cos(\rho)\right)^2}+\frac{\left(\theta-\frac{\sin 2(\alpha-\rho+\theta)}{2}\right)\Big|_{\theta_-}^{0}}{2d(x)^2\sin^2(\alpha-\rho)}\right.\\
&\qquad\left.+\frac{\left(\theta+\frac{\sin 2(\alpha+\rho-\theta)}{2}\right)\Big|_{0}^{\theta_+}}{2d(x)^2\sin^2(\alpha+\rho)}+\frac{\left(\theta-\frac{\sin 2(\rho-\theta)}{2}\right)\Big|_{\theta_+}^{\frac{\pi}{2}+\rho}}{2\left(h+d(x)\cos(\rho)\right)^2}\right]\\
&= \frac{\pi-2\rho+\sin 2\left(-\rho+\theta_-\right)+2\theta_-}{4\pi\left(h+d(x)\cos\rho\right)^2}\\
&\quad+\frac{-\theta_-+\cos\left(2\alpha-2\rho+\theta_-\right)\sin\theta_-}{2\pi d(x)^2\sin^2(\alpha-\rho)}\\
&\quad+\frac{\theta_+-\cos\left(2\alpha+2\rho-\theta_+\right)\sin\theta_+}{2\pi d(x)^2\sin^2(\alpha+\rho)}\\
&\quad+\frac{\pi+2\rho+\sin 2\left(\rho-\theta_+\right)-2\theta_+}{4\pi\left(h+d(x)\cos\rho\right)^2}. \qquad (3.52)
\end{aligned}
$$

The dominant factor here is the bound on ρ, see (3.23). Apply Proposition 3.1.1 to (3.52), the lower bound on the function $\frac{1}{m(x)^2}$, to obtain the following Hardy-type inequality:

$$
\int_{\Omega}\mu_1(x)\frac{|f(x)|^2}{\left(h+d(x)\cos\rho\right)^2}dx+\int_{\Omega}\mu_2(x,\alpha)\frac{|f(x)|^2}{d(x)^2}dx\le\int_{\Omega}|\nabla f(x)|^2\,dx,
$$

where

$$
\mu_1(x)=\frac{1}{8\pi}\left(2\pi+\sin 2\left(-\rho+\theta_-\right)+\sin 2\left(\rho-\theta_+\right)+2\left(\theta_--\theta_+\right)\right),
$$

and

$$
\mu_2(x,\alpha)=\frac{1}{4\pi}\left(\frac{\cos\left(2\alpha-2\rho+\theta_-\right)\sin\theta_--\theta_-}{\sin^2(\alpha-\rho)}+\frac{\theta_+-\cos\left(2\left(\alpha+\rho\right)-\theta_+\right)\sin\theta_+}{\sin^2(\alpha+\rho)}\right).
$$

This completes the proof. □

Remark 3.3.6.

1. If x lies on the symmetry axis of the truncated sectorial region $K_{h,\psi}(w,\phi_w)$ i.e., $\rho=0$,

then (3.52) gives

$$\begin{aligned}\frac{1}{m(x)^2} \quad \geq \quad & \frac{2\pi + \sin 2\theta_- + 2\theta_- - \sin 2\theta_+ - 2\theta_+}{4\pi\left(h + d(x)\right)^2} \\ & + \frac{\cos\left(2\alpha + \theta_-\right)\sin\theta_- - \theta_- + \theta_+ - \cos\left(2\alpha - \theta_+\right)\sin\theta_+}{2\pi d(x)^2 \sin^2\alpha}.\end{aligned}$$

However, in the symmetric case we have $|\theta_+| = |-\theta_-| = \theta_0$, thus

$$\frac{1}{m(x)^2} \quad \geq \quad \frac{\pi - \sin 2\theta_0 - 2\theta_0}{2\pi\left(h + d(x)\right)^2} + \frac{\theta_0 - \cos\left(2\alpha - \theta_0\right)\sin\theta_0}{\pi d(x)^2 \sin^2\alpha},$$

as obtained in (3.34).

2. If α tends to $\frac{\pi}{2}$, then according to (3.23), the relation between α and ρ, ρ tends to 0 as well, which means that x lies on the symmetry axis. In this case using (3.8) indicates that θ_0 tends to $\frac{\pi}{2}$, thus (3.52) gives

$$\frac{1}{m(x)^2} \geq \frac{1}{2d(x)^2},$$

as expected from Theorem 2.2.7.

3. If x lies on the extension of one of the lines $z = \ell e^{\mp i\psi}$, then $\rho = \alpha$, $\theta_- = 0$ (respectively $\theta_+ = 0$) , and $|\theta_+| = \theta_0$ (respectively $|\theta_-| = \theta_0$). Consider the case when $\theta_- = 0$, then $|\theta_+| = \theta_0$, hence (3.52) gives

$$\begin{aligned}\frac{1}{m(x)^2} \quad \geq \quad & \frac{\theta_0 - \cos\left(4\alpha - \theta_0\right)\sin\theta_0}{2\pi d(x)^2 \sin^2\left(2\alpha\right)} \\ & + \frac{\pi - 2\alpha + \sin 2(-\alpha) + \pi + 2\alpha + \sin 2\left(\alpha - \theta_0\right) - 2\theta_0}{4\pi\left(h + d(x)\cos\alpha\right)^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\theta_0 - \cos(4\alpha - \theta_0)\sin\theta_0}{2\pi d(x)^2 \sin^2(2\alpha)} \\
&\quad + \frac{2\pi - \sin 2\alpha + \sin 2(\alpha - \theta_0) - 2\theta_0}{4\pi\left(h + d(x)\cos\alpha\right)^2} \\
&= \frac{2\theta_0 + \sin(4\alpha - 2\theta_0) - \sin 4\alpha}{4\pi d(x)^2 \sin^2 2\alpha} \\
&\qquad + \frac{2\pi - \sin 2\alpha + \sin 2(\alpha - \theta_0) - 2\theta_0}{4\pi\left(h + d(x)\cos\alpha\right)^2}
\end{aligned}
$$

as obtained in (3.44).

3.4 Hardy's inequalities under the Exterior Disk Condition

In this section, domains in $\mathbb{R}^2$ which satisfy the Exterior Disk Condition are investigated in order to obtain Hardy-type inequalities for such non-convex domains. In what follows, $d_\theta(x)$ denotes the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$ of the domain Ω in the direction θ, as defined in (3.3) and $\tilde{d}_\theta(x)$ refers to the distance from $x \in \Omega$ to the boundary $\partial B^2(a, R)$ of the disk $B^2(a, R)$ i.e.,

$$
\tilde{d}_\theta(x) := \min\{|s| : x + s\, e^{i\theta} \in \partial B^2(a, R)\}.
$$

The main result is stated in the following theorem.

Theorem 3.4.1.

Suppose that $\Omega \subset \mathbb{R}^2$ satisfies Condition 3.2.9. Then for any function $f \in \mathcal{C}_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$
\int_\Omega \mu(x, R) \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2\, dx,
$$

where

$$\mu(x,R) = \frac{R\left[\pi R + 2\sqrt{d(x)\left(d(x)+2R\right)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)\right]}{2\pi\left(d(x)+2R\right)^2}. \tag{3.53}$$

Proof. By (3.2), the definition of $\frac{1}{m(x)^2}$, and the fact that $d_\theta(x)$ is a periodic function with period π satisfying $d_\theta(x) \leq \tilde{d}_\theta(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{d_\theta(x)^2}d\theta = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{d_\theta(x)^2}d\theta \geq \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\tilde{d}_\theta(x)^2}d\theta. \tag{3.54}$$

Since the domain under consideration satisfies Condition 3.2.9, then the function $\frac{1}{\tilde{d}_\theta(x)^2}$ vanishes for $\theta \in (-\frac{\pi}{2}, -\theta_0)$ and $\theta \in (\theta_0, \frac{\pi}{2})$. Therefore, inequality (3.54) becomes

$$\frac{1}{m(x)^2} \geq \frac{1}{\pi}\int_{-\theta_0}^{\theta_0}\frac{1}{\tilde{d}_\theta(x)^2}d\theta. \tag{3.55}$$

On the other hand, as have been discussed earlier in Section 3.2, the function $\frac{1}{\tilde{d}_\theta(x)^2}$ can be written in terms of $d(x)$ using (3.27). Consequently, inequality (3.55) becomes

$$\frac{1}{m(x)^2} \geq \frac{1}{\pi}\int_{-\theta_0}^{\theta_0}\frac{1}{\cos^2\theta\left(R + d(x) - \sqrt{R^2 - \left(d(x)^2 + 2Rd(x)\right)\tan^2\theta}\right)^2}d\theta,$$

which can be rewritten as

$$\frac{1}{m(x)^2} \geq \frac{1}{\pi}\int_{-\theta_0}^{\theta_0}\frac{1}{\cos^2\theta\left(a(x) - \sqrt{R^2 - \left(a(x)^2 - R^2\right)\tan^2\theta}\right)^2}d\theta, \tag{3.56}$$

where

$$a(x) = R + d(x).$$

Now using the substitution

$$\tan\theta = t,$$

into (3.56) produces the following bound:

$$\frac{1}{m(x)^2} \geq \frac{1}{\pi}\int_{-t_0}^{t_0} \frac{1}{\left(a(x) - R\sqrt{1-\left(\frac{a(x)^2-R^2}{R^2}\right)t^2}\right)^2}dt.$$

However, setting

$$s = \frac{\sqrt{a(x)^2 - R^2}}{R}\ t,$$

leads to

$$\frac{1}{m(x)^2} \geq \frac{1}{\pi}\frac{R}{\sqrt{a(x)^2-R^2}}\int_{-s_0}^{s_0}\frac{1}{\left(a(x)-R\sqrt{1-s^2}\right)^2}ds. \tag{3.57}$$

Let us simplify the integrand in (3.57):

$$\begin{aligned}
\frac{1}{\left(a(x)-R\sqrt{1-s^2}\right)^2} &= \frac{1}{a(x)^2 - 2a(x)R\sqrt{1-s^2} + R^2(1-s^2)} \\
&= \frac{1}{a(x)^2 - 2a(x)R\sqrt{1-s^2} + R^2(1-s^2)} \\
&\qquad \times\frac{a(x)^2 + 2a(x)R\sqrt{1-s^2} + R^2(1-s^2)}{a(x)^2 + 2a(x)R\sqrt{1-s^2} + R^2(1-s^2)} \\
&= \frac{a(x)^2 + 2a(x)R\sqrt{1-s^2} + R^2(1-s^2) + R^2 - R^2 + R^2s^2 - R^2s^2}{\left(R^2s^2 + a(x)^2 - R^2\right)^2} \\
&= \frac{2a(x)R\sqrt{1-s^2}}{\left(R^2s^2 + a(x)^2 - R^2\right)^2} + \frac{2\left(R^2 - R^2s^2\right)}{\left(R^2s^2 + a(x)^2 - R^2\right)^2} \\
&\qquad + \frac{1}{R^2s^2 + a(x)^2 - R^2}.
\end{aligned} \tag{3.58}$$

Hence, by (3.58) the integral in (3.57) can be written as follows:

$$\int_{-s_0}^{s_0} \frac{1}{\left(a\left(x\right) - R\sqrt{1-s^2}\right)^2} ds \;=\; I_1(x,R) + I_2(x,R) + I_3(x,R), \tag{3.59}$$

where

$$I_1(x,R) = \int_{-s_0}^{s_0} \frac{2a\left(x\right) R\sqrt{1-s^2}}{\left(R^2 s^2 + a\left(x\right)^2 - R^2\right)^2}\, ds,$$

$$I_2(x,R) = \int_{-s_0}^{s_0} \frac{2\left(R^2 - R^2 s^2\right)}{\left(R^2 s^2 + a\left(x\right)^2 - R^2\right)^2}\, ds,$$

and

$$I_3(x,R) = \int_{-s_0}^{s_0} \frac{1}{R^2 s^2 + a\left(x\right)^2 - R^2}\, ds.$$

The integral $I_3(x,R)$ can be evaluated as follows:

$$I_3(x,R) = \frac{1}{a\left(x\right)^2 - R^2} \int_{-s_0}^{s_0} \frac{1}{1 + \left(\frac{Rs}{\sqrt{a(x)^2 - R^2}}\right)^2}\, ds,$$

and we may use the substitution

$$u = \frac{Rs}{\sqrt{a\left(x\right)^2 - R^2}},$$

to obtain

$$\begin{aligned} I_3(x,R) &= \frac{1}{R\sqrt{a\left(x\right)^2 - R^2}} \tan^{-1}\left(u\right)\Big|_{\frac{-Rs_0}{\sqrt{a(x)^2-R^2}}}^{\frac{Rs_0}{\sqrt{a(x)^2-R^2}}} \\ &= \frac{2}{R\sqrt{a\left(x\right)^2 - R^2}} \tan^{-1}\left(\frac{Rs_0}{\sqrt{a\left(x\right)^2 - R^2}}\right). \end{aligned} \tag{3.60}$$

Concerning $I_2(x,R)$, we have

$$\begin{aligned} I_2(x,R) &= \int_{-s_0}^{s_0} \frac{2a(x)^2}{\left(R^2s^2+a(x)^2-R^2\right)^2}\,ds - \int_{-s_0}^{s_0} \frac{2}{R^2s^2+a(x)^2-R^2}\,ds \\ &= \frac{2a(x)^2}{\left(a(x)^2-R^2\right)^2}\int_{-s_0}^{s_0} \frac{1}{\left(1+\left(\frac{Rs}{\sqrt{a(x)^2-R^2}}\right)^2\right)^2}\,ds - 2I_3(x,R). \qquad (3.61) \end{aligned}$$

Regarding the first integral in (3.61), put

$$u = \frac{Rs}{\sqrt{a(x)^2-R^2}},$$

then use the standard identity

$$\int \frac{dx}{\left(1+x^2\right)^2} = \frac{1}{2}\left[\frac{x}{1+x^2} + \tan^{-1} x + C\right]. \qquad (3.62)$$

Thus, $I_2(x,R)$ becomes

$$\begin{aligned} I_2(x,R) = &\left[\frac{a(x)^2\,s}{\left(a(x)^2-R^2\right)\left(a(x)^2+R^2\left(s^2-1\right)\right)} + \frac{a(x)^2\tan^{-1}\left(\frac{Rs}{\sqrt{a(x)^2-R^2}}\right)}{R\left(a(x)^2-R^2\right)^{\frac{3}{2}}}\right. \\ &\left.\left. - \frac{2\tan^{-1}\left(\frac{Rs}{\sqrt{a(x)^2-R^2}}\right)}{R\sqrt{a(x)^2-R^2}}\right]\right|_{-s_0}^{s_0} \\ = &\frac{1}{R}\left[\frac{a(x)^2\,Rs}{\left(a(x)^2-R^2\right)\left(a(x)^2+R^2\left(s^2-1\right)\right)} + \frac{\left(2R^2-a(x)^2\right)\tan^{-1}\left(\frac{Rs}{\sqrt{a(x)^2-R^2}}\right)}{\left(a(x)^2-R^2\right)^{\frac{3}{2}}}\right]\Bigg|_{-s_0}^{s_0} \end{aligned}$$

$$= \frac{2}{R}\left[\frac{a(x)^2 Rs_0}{\left(a(x)^2 - R^2\right)\left(a(x)^2 + R^2\left(s_0^2 - 1\right)\right)} + \frac{\left(2R^2 - a(x)^2\right)\tan^{-1}\left(\frac{Rs_0}{\sqrt{a(x)^2 - R^2}}\right)}{\left(a(x)^2 - R^2\right)^{\frac{3}{2}}}\right]. \quad (3.63)$$

Finally, regarding $I_1(x,R)$ we have

$$I_1(x,R) = \frac{2Ra(x)}{\left(a(x)^2 - R^2\right)^2}\int_{-s_0}^{s_0}\frac{\sqrt{1-s^2}}{\left(1+\frac{R^2}{a(x)^2-R^2}s^2\right)^2}ds.$$

Using the substitution

$$u = \frac{\sqrt{1+A}}{\sqrt{1-s^2}}s; \text{ where } A = \frac{R^2}{a(x)^2 - R^2},$$

gives

$$I_1(x,R) = \frac{2Ra(x)}{\left(a(x)^2 - R^2\right)^2\sqrt{1+A}}\int_{-\frac{\sqrt{1+A}}{\sqrt{1-s_0^2}}s_0}^{\frac{\sqrt{1+A}}{\sqrt{1-s_0^2}}s_0}\frac{1}{(1+u^2)^2}du.$$

Now use (3.62) to obtain

$$\begin{aligned}
I_1(x,R) &= \frac{2Ra(x)}{\left(a(x)^2 - R^2\right)^2\sqrt{1+A}}\left[\frac{s_0\sqrt{1+A}}{\sqrt{1-s_0^2}\left(1+\frac{(1+A)s_0^2}{1-s_0^2}\right)} + \tan^{-1}\left(\frac{\sqrt{1+A}}{\sqrt{1-s_0^2}}s_0\right)\right] \\
&= \frac{2Ra(x)}{\left(a(x)^2 - R^2\right)^2}\left[\frac{s_0\sqrt{1-s_0^2}}{1+As_0^2} + \frac{\tan^{-1}\left(\frac{\sqrt{1+A}}{\sqrt{1-s_0^2}}s_0\right)}{\sqrt{1+A}}\right] \\
&= \frac{2Ra(x)\,s_0\sqrt{1-s_0^2}}{\left(a(x)^2 - R^2\right)\left(a(x)^2 + R^2\left(s_0^2-1\right)\right)} + \frac{2R\tan^{-1}\left(\frac{a(x)s_0}{\sqrt{a(x)^2-R^2}\sqrt{1-s_0^2}}\right)}{\left(a(x)^2 - R^2\right)^{\frac{3}{2}}}. \quad (3.64)
\end{aligned}$$

Using (3.64), (3.63), and (3.60), in (3.59) leads to

$$\begin{aligned}
\int_{-s_0}^{s_0} \frac{1}{\left(a(x) - R\sqrt{1-s^2}\right)^2} ds &= 2R\left[\frac{a(x)\sqrt{a(x)^2 - R^2}\, s_0\sqrt{1-s_0^2}}{\left(a(x)^2 - R^2\right)^{\frac{3}{2}}\left(a(x)^2 + R^2\left(s_0^2 - 1\right)\right)}\right. \\
&+ \left.\frac{\left(a(x)^2 + R^2\left(s_0^2-1\right)\right)\tan^{-1}\left(\frac{a(x)s_0}{\sqrt{a(x)^2-R^2}\sqrt{1-s_0^2}}\right)}{\left(a(x)^2 - R^2\right)^{\frac{3}{2}}\left(a(x)^2 + R^2\left(s_0^2-1\right)\right)}\right] \\
&+ \frac{2}{R}\left[\frac{a(x)^2 R\sqrt{a(x)^2 - R^2}\, s_0}{\left(a(x)^2 - R^2\right)^{\frac{3}{2}}\left(a(x)^2 + R^2\left(s_0^2-1\right)\right)}\right. \\
&+ \left.\frac{\left(2R^2 - a(x)^2\right)\left(a(x)^2 + R^2\left(s_0^2-1\right)\right)\tan^{-1}\left(\frac{Rs_0}{\sqrt{a(x)^2-R^2}}\right)}{\left(a(x)^2 - R^2\right)^{\frac{3}{2}}\left(a(x)^2 + R^2\left(s_0^2-1\right)\right)}\right] \\
&+ \frac{2}{R\sqrt{a(x)^2 - R^2}}\tan^{-1}\left(\frac{Rs_0}{\sqrt{a(x)^2 - R^2}}\right) \\
&= \frac{2}{\left(a(x)^2 - R^2\right)^{\frac{3}{2}}\left(a(x)^2 + R^2\left(s_0^2-1\right)\right)}\left[a(x)^2\sqrt{a(x)^2 - R^2}\, s_0\right. \\
&+ a(x)R\sqrt{a(x)^2 - R^2}\, s_0\sqrt{1-s_0^2} + \left(a(x)^2 + R^2\left(s_0^2 - 1\right)\right)\times \\
&\left.\left(R\tan^{-1}\left(\frac{Rs_0}{\sqrt{a(x)^2 - R^2}}\right) + R\tan^{-1}\left(\frac{a(x)\, s_0}{\sqrt{a(x)^2 - R^2}\sqrt{1-s_0^2}}\right)\right)\right].
\end{aligned} \tag{3.65}$$

On the other hand, $a(x) = R + d(x)$, and regarding (3.24) we have

$$s_0 = \frac{\sqrt{d(x)^2 + 2Rd(x)}}{R}\tan\theta_0 = \frac{\sqrt{d(x)^2 + 2Rd(x)}}{R}\frac{R}{\sqrt{d(x)^2 + 2Rd(x)}} = 1,$$

and hence (3.65) becomes

$$\int_{-1}^{1} \frac{1}{\left(a\left(x\right)-R\sqrt{1-s^2}\right)^2}ds$$
$$= \frac{\pi R + 2\sqrt{d\left(x\right)\left(d\left(x\right)+2R\right)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)}{\left(d\left(x\right)\left(d\left(x\right)+2R\right)\right)^{\frac{3}{2}}}. \quad (3.66)$$

Thus using (3.66) with (3.57) produces,

$$\frac{1}{m\left(x\right)^2} \geq \frac{R\left[\pi R + 2\sqrt{d\left(x\right)\left(d\left(x\right)+2R\right)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)\right]}{d\left(x\right)^2\pi\left(d\left(x\right)+2R\right)^2}. \quad (3.67)$$

Applying Proposition 3.1.1 to (3.67) gives the following Hardy-type inequality:

$$\int_{\Omega} \mu\left(x,R\right)\frac{\left|f(x)\right|^2}{d(x)^2}dx \leq \int_{\Omega} \left|\nabla f(x)\right|^2 dx,$$

where

$$\mu\left(x,R\right) = \frac{R\left[\pi R + 2\sqrt{d\left(x\right)\left(d\left(x\right)+2R\right)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)\right]}{2\pi\left(d\left(x\right)+2R\right)^2},$$

as stated in (3.53). □

Remark 3.4.2.

1. If Ω is a convex domain then the Exterior Disk Condition is satisfied for all $R > 0$. Therefore, as R tends to ∞, (3.67) gives

$$\frac{1}{m(x)^2} \geq \frac{1}{2d(x)^2},$$

as expected from Theorem 2.2.7.

2. The function $\mu(x,R)$, given by (3.53), can be asymptotically expanded in powers of $\frac{d(x)}{R}$ whenever $\frac{d(x)}{R}$ is bounded. Rewrite $\mu(x,R)$ as follows:

$$\begin{aligned}\mu(x,R) &= \frac{R\sqrt{d(x)(d(x)+2R)}}{\pi(d(x)+2R)^2}+\frac{\frac{\pi}{2}+\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)}{\pi}\frac{R^2}{(d(x)+2R)^2}\\ &= \frac{\sqrt{d(x)}}{\pi 2^{3/2}\sqrt{R}\left(1+\frac{d(x)}{2R}\right)^{3/2}}\\ &\qquad +\frac{\pi+2\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)}{8\pi}\frac{1}{\left(1+\frac{d(x)}{2R}\right)^2}. \end{aligned} \tag{3.68}$$

Concerning the first term on the right hand side of (3.68), using the Binomial Series leads to:

$$\begin{aligned}\frac{\sqrt{d(x)}}{\pi 2^{3/2}\sqrt{R}\left(1+\frac{d(x)}{2R}\right)^{3/2}} &= \frac{1}{\pi 2^{3/2}}\sqrt{\frac{d(x)}{R}}\left(1+\frac{d(x)}{2R}\right)^{-3/2}\\ &= \frac{1}{\pi 2^{3/2}}\sqrt{\frac{d(x)}{R}}\left(1-\frac{3}{2}\frac{d(x)}{2R}+\frac{1}{2}\frac{3}{2}\frac{5}{2}\left(\frac{d(x)}{2R}\right)^2-\mathcal{O}\left(\left(\frac{d(x)}{2R}\right)^3\right)\right)\\ &= \frac{1}{\pi 2^{3/2}}\sqrt{\frac{d(x)}{R}}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right). \end{aligned} \tag{3.69}$$

To expand the second term in (3.68), recall that according to relation (3.24) we have

$$\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)=\theta_0,$$

and

$$\sin\theta_0 = 1-\frac{d(x)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^2\right).$$

Rewriting,

$$\begin{aligned}\sin\theta_0 = \cos\left(\theta_0 - \frac{\pi}{2}\right) &= \sum_{n=0}^{\infty} \frac{(-1)^n(\theta_0 - \frac{\pi}{2})^{2n}}{(2n)!} \\ &= 1 - \frac{(\theta_0 - \frac{\pi}{2})^2}{2} + \mathcal{O}\left(\left(\theta_0 - \frac{\pi}{2}\right)^4\right),\end{aligned}$$

we conclude that

$$\theta_0 = \frac{\pi}{2} - \sqrt{\frac{2d(x)}{R}} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right).$$

Therefore, the second term in (3.68) can be expanded as follows:

$$\begin{aligned}\frac{\pi + 2\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)}{8\pi\left(1 + \frac{d(x)}{2R}\right)^2} &= \frac{\pi + 2\theta_0}{8\pi}\left(1 - \frac{d\,(x)}{R} + 3\left(\frac{d\,(x)}{2R}\right)^2 - \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^3\right)\right) \\ &= \frac{\pi + 2\left(\frac{\pi}{2} - \sqrt{\frac{2d(x)}{R}} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right)}{8\pi} \times \\ &\quad \left(1 - \frac{d\,(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^2\right)\right) \\ &= \frac{1}{4}\left(1 - \frac{d\,(x)}{R}\right) - \frac{1}{2^{3/2}\pi}\sqrt{\frac{d\,(x)}{R}} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right).\end{aligned} \tag{3.70}$$

Consequently, substituting (3.69) and (3.70) into (3.68) yields

$$\mu\,(x, R) = \frac{1}{4} - \frac{d\,(x)}{4R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right), \tag{3.71}$$

which is approximately linear in $\frac{d(x)}{R}$ for large R and tending to $\frac{1}{4}$ as $\frac{d(x)}{R}$ tends to 0.

3. It is natural to compare the above asymptotic given by (3.71) with Laptev-Sobolev result given in Theorem 2.2.11. Since $d(x) \leq \delta_{\text{in}}$, and $R \geq R_0$, then formula (3.71)

gives

$$\mu\left(x,R\right)=\frac{1}{4}-\mathcal{O}\left(\frac{\delta_{\text{in}}}{R_0(R)}\right),$$

which agrees with (2.20). However, (3.71) is more precise, as it contains $d(x)$ instead of the inradius δ_{in}, and it does not require the simple connectedness of Ω. Besides their approach is only applicable to planar domains while our approach is applicable for higher dimension domains as will be shown in Chapters 4 and 5.

Chapter 4

Hardy's inequalities for three-dimensional non-convex domains

4.1 Introduction

The central aim of this chapter is to derive new Hardy-type inequalities for functions defined on three-dimensional non-convex domains in $\mathbb{R}^3$. This three-dimensional study will assist the reader in understanding the $n-$dimensional case given in Chapter 5.

To this end some 'non-convexity measures' are needed. That is why we impose a certain geometrical condition on each of the domains under investigation. More precisely, four different conditions are stated in Section 4.2. The first condition is referred to as the 'Exterior Cone' condition, while the second one is called the 'Truncated Conical Region' (TCR) condition and is considered a generalisation of the TSR condition. Afterwards, the 'Exterior Disk' condition, Condition 3.2.9, is generalised to form our third condition, the 'Exterior Ball' condition and finally, we establish the fourth condition, the 'Cylinder' condition.

To complete our goal we pursue the same strategy used in Chapter 3 with a necessary difference, which is applying Theorem 2.2.5 instead of Theorem 2.2.3. To be more specific, we try to attain lower bounds, containing the 'true' distance $d(x)$, for the function $\frac{1}{m(x)^2}$ defined in (2.7). We then apply Theorem 2.2.5 to those lower bounds to obtain Hardy-type inequalities in terms of $d(x)$.

In fact, throughout this chapter and Chapter 5, Theorem 2.2.5 is an indispensable tool in deriving Hardy-type inequalities for n-dimensional domains with $n \geq 3$. For that reason, and for the reader's convenience, we mention it again here.

Proposition 4.1.1 (E. B. Davies [16, 19])**.**

Let Ω be a bounded region in $\mathbb{R}^n$ and let $f(x) \in \mathcal{C}_c^\infty(\Omega)$. Then

$$\frac{n}{4} \int_\Omega \frac{|f(x)|^2}{m(x)^2}\, dx \leq \int_\Omega |\nabla f(x)|^2\, dx, \tag{4.1}$$

where $\frac{1}{m(x)^2}$, the (L^2 harmonic) mean distance of $x \in \Omega$ from $\partial\Omega$, is defined by

$$\frac{1}{m(x)^2} := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \frac{dS\,(u)}{d_u\,(x)^2}, \tag{4.2}$$

and $d_u\,(x)$ is defined for every unit vector $u \in \mathbb{S}^{n-1}$ and $x \in \Omega$ by

$$d_u(x) := \min\{|s| : x + su \notin \Omega\}, \tag{4.3}$$

where $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere in $\mathbb{R}^n$.

We now present the notations and geometrical conditions, which will be used throughout this chapter.

4.2 Notations and conditions

The aim of this section is to introduce some notations that will help establish the conditions needed to study some non-convex domains in $\mathbb{R}^3$, which will in turn help to extract some new Hardy-type inequalities.

Let ω be a point in $\mathbb{R}^3$ and ν be a unit vector. For $\alpha \in \left(0, \frac{\pi}{2}\right)$, denote by $C_0(\nu, \alpha)$ the set

$$C_0(\nu, \alpha) = \left\{x \in \mathbb{R}^3 : x \cdot \nu \geq |x| \cos \alpha\right\}.$$

In words, $C_0(\nu, \alpha)$ is a three-dimensional cone with vertex at 0 and symmetry axis in the ν direction with angle 2α at the vertex. Consequently, the set

$$C_0^{(-)}(\nu, \alpha) = \left\{x \in \mathbb{R}^3 : -x \cdot \nu \geq |x| \cos \alpha\right\},$$

is a three-dimensional cone with vertex at 0 and symmetry axis in the opposite ν direction with angle 2α at the vertex.

Denote by

$$C_\omega(\nu, \alpha) = C_0(\nu, \alpha) + \omega,$$

the translation of $C_0(\nu, \alpha)$ by $\omega \in \mathbb{R}^3$, i.e.,

$$C_\omega(\nu, \alpha) = \left\{x \in \mathbb{R}^3 : (x - \omega) \cdot \nu \geq |x - \omega| \cos \alpha\right\},$$

which is a three-dimensional cone with vertex at ω and symmetry axis parallel to the ν direction with angle 2α at the vertex. On the other hand, the set

$$C_\omega^{(-)}(\nu, \alpha) = C_0^{(-)}(\nu, \alpha) + \omega,$$

is a three-dimensional cone with vertex at ω and symmetry axis in the direction $-\nu$ with angle 2α at the vertex.

Now for $h \geq 0$, define the half-space $\mathbf{\Pi}_h(\nu)$ by

$$\mathbf{\Pi}_h(\nu) = \left\{x \in \mathbb{R}^3 : x \cdot \nu \geq h\right\}.$$

In words $\mathbf{\Pi}_h(\nu)$ is a half-space 'of distance h' from the origin in the ν direction. Denote by

$$\mathbf{\Pi}_{h,\omega}(\nu) = \mathbf{\Pi}_h(\nu) + \omega,$$

the translation of $\mathbf{\Pi}_h(\nu)$ by $\omega \in \mathbb{R}^3$, i.e.

$$\mathbf{\Pi}_{h,\omega}(\nu) = \left\{x \in \mathbb{R}^3 : (x-\omega) \cdot \nu \geq h\right\},$$

which is a half-space 'of height h' from the point ω in the ν direction.

Finally, define the 'Truncated Conical Region', $K_{h,\omega}(\nu, \alpha)$ to be

$$K_{h,\omega}(\nu, \alpha) = C_\omega(\nu, \alpha) \cup \mathbf{\Pi}_{h,\omega}(\nu).$$

Considering the above notations we can formulate the first geometrical condition on the domain $\Omega \subset \mathbb{R}^3$.

Condition 4.2.1. (Exterior Cone Condition)

We say that $\Omega \subset \mathbb{R}^3$ satisfies the Exterior Cone Condition if for each $x \in \Omega$ there exists an element $\omega \in \partial\Omega$ such that $d(x) = |\omega - x|$ and $\Omega \subset C^c_\omega(\nu, \alpha)$, with $(x - \omega) \cdot \nu = -|x|$.

Condition 4.2.1 means that for every point $x \in \Omega$ we can always find a cone $C_\omega(\nu, \alpha)$ such that x lies on its symmetry axis.

As a development of the above condition, we establish the following condition.

Condition 4.2.2. (Truncated Cone Region (TCR) Condition)

We say that $\Omega \subset \mathbb{R}^3$ satisfies the TCR Condition if for each $x \in \Omega$ there exist an element

$\omega \in \partial\Omega$ and $h \geq 0$ such that $d(x) = |\omega - x|$ and $\Omega \subset K^c_{h,\omega}(\nu, \alpha)$ with $(x - \omega) \cdot \nu = -|x|$.

Condition 4.2.2 means that for every point $x \in \Omega$ we can always find a truncated conical region $K_{h,\omega}(\nu, \alpha)$ such that x lies on its symmetry axis, which is the symmetry axis of $C_\omega(\nu, \alpha)$.

Because of the nature of the 'Exterior Cone' Condition and the TCR Condition we will use the terminology 'Cone Conditions' to refer to any of them.

Now we introduce the third non-convexity 'measure': Define the three-dimensional open ball with centre $a \in \mathbb{R}^3$ and radius $R > 0$, $B^3(a, R)$, to be the following set:

$$B^3(a, R) = \{y \in \mathbb{R}^3 : |y - a| < R\}.$$

Condition 4.2.3. (Exterior Ball Condition)

We say that $\Omega \subset \mathbb{R}^3$ satisfies the Exterior Ball Condition if there exists a number $R > 0$ such that for each $w \in \partial\Omega \subset \mathbb{R}^3$, one can find a point $a \in \mathbb{R}^3$ such that $|w - a| = R$ and

$$B^3(a, R) \cap \Omega = \emptyset.$$

The above condition means that we can touch every point on the boundary $\partial\Omega$ of the domain Ω with a ball of some radius R.

Finally, let us introduce the fourth non-convexity condition, which will be referred to as the Cylinder condition. We define the cylinder by fixing its axis and radius. Let $\ell \subset \mathbb{R}^3$ be a straight line, and let $R > 0$. Then

$$\mathcal{Z}(\ell, R) = \{x \in \mathbb{R}^3 : \text{dist}(x, \ell) \leq R\},$$

is a cylinder with axis ℓ and radius R.

Condition 4.2.4. (Cylinder Condition)
We say that $\Omega \subset \mathbb{R}^3$ satisfies the Cylinder Condition if there is a number $R > 0$ such that for each $\omega \in \partial\Omega$ there exists a straight line ℓ such that

$$\omega \in \mathcal{Z}(\ell, R) \text{ and } \mathcal{Z}(\ell, R) \cap \Omega = \emptyset.$$

Now all the non-convexity 'measures', needed to achieve our objective, have been established. Here we recall that our plan to extract Hardy-type inequalities for three-dimensional domains, that satisfy one of the above conditions, is: To obtain a suitable lower bound for the function $\frac{1}{m(x)^2}$ given by (4.2). In order to do so, we reduce the number of variables in the integral (4.2), with $n = 3$, accordingly a two-dimensional picture appears. Consequently, we can make use of all results and conclusions derived in Section 3.2. Then we apply Proposition 4.1.1.

Let us start with domains $\Omega \in \mathbb{R}^3$ which satisfy one of the 'Cone Conditions', Conditions 4.2.1 or 4.2.2.

4.3 Hardy's inequalities under the Cone Conditions

In this section we obtain Hardy-type inequalities in $\Omega \subset \mathbb{R}^3$, which satisfy either Condition 4.2.1 or Condition 4.2.2. Let us begin with the case in which the domain Ω satisfies Condition 4.2.1.
Before we proceed, we would like to emphasize that the symbol $d(x)$ denotes the Euclidean distance from the point $x \in \Omega$ to the boundary $\partial\Omega$, i.e.

$$d(x) := \text{dist}\,(x, \partial\Omega) = \min\left\{|x - y| : y \notin \Omega\right\}. \tag{4.4}$$

Whereas, the symbol $d_u(x)$ refers to the distance from the point $x \in \Omega$ to the boundary $\partial\Omega$ in the direction u, i.e.

$$d_u(x) := \min\{|s| : x + su \notin \Omega\}. \tag{4.5}$$

Finally, we use $\tilde{d}_u(x)$ to denote the distance from the point $x \in \Omega$ to the boundary $\partial C_\omega(\nu, \alpha)$ of the cone $C_\omega(\nu, \alpha)$ in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial C_\omega(\nu, \alpha)\}.$$

Theorem 4.3.1.

Suppose that the domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.2.1 with some $\alpha \in \left(0, \frac{\pi}{2}\right)$. Then for any $f \in \mathcal{C}_c^\infty(\Omega)$ the following Hardy-type inequality holds:

$$\mu(\alpha) \int\limits_\Omega \frac{|f(x)|^2}{d(x)^2} dx \le \int\limits_\Omega |\nabla f(x)|^2 \, dx, \tag{4.6}$$

where

$$\mu(\alpha) = \frac{1}{4} \tan^2 \frac{\alpha}{2}. \tag{4.7}$$

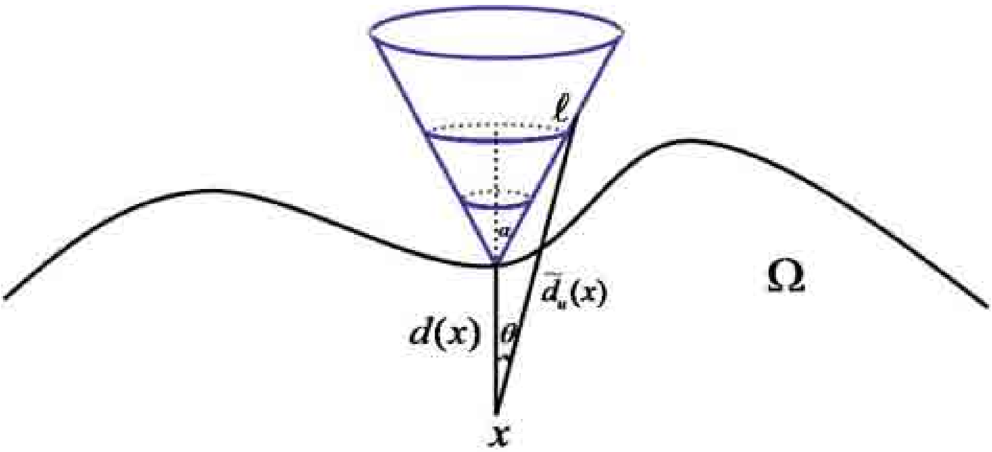

Figure 4.1: x lies on the symmetry axis of $C_\omega(\nu, \alpha)$ for a 3-D domain

Proof. By definition (4.2) of the function $\frac{1}{m(x)^2}$ and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi}\int\limits_{\mathbb{S}^2} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{4\pi}\int\limits_{\mathbb{S}^2} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{4.8}$$

Since $\tilde{d}_u(x)$ is a symmetric function, with respect to the rotation about the symmetry axis of the cone $C_\omega(\nu, \alpha)$, then using spherical coordinates, (r, θ, ϕ) where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, leads to $u = u(\theta, \phi)$, and $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x, \theta)$ instead of $\tilde{d}_u(x)$. Therefore, in light of relation (C.10), inequality (4.8) becomes

$$\frac{1}{m(x)^2} \geq \frac{1}{4\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta \, d\theta d\phi = \frac{2\pi}{4\pi} 2 \int\limits_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta \, d\theta. \tag{4.9}$$

However, the angle θ can not exceed α, thus inequality (4.9) takes the following form:

$$\frac{1}{m(x)^2} \geq \int\limits_0^{\alpha} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta \, d\theta. \tag{4.10}$$

Since $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.2.1 and, in Fig. 4.1, if we consider the two-dimensional cross section that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}(x, \theta)$, we conclude that

$$\tilde{d}(x, \theta) = \frac{d(x) \sin\alpha}{\sin(\alpha - \theta)},$$

as derived in (3.9).

Thus, the lower bound (4.10), on the function $\frac{1}{m(x)^2}$, can be written as follows:

$$\begin{aligned} \frac{1}{m(x)^2} &\geq \frac{\int_0^\alpha \sin^2(\alpha - \theta) \sin\theta d\theta}{d(x)^2 \sin^2\alpha} \\ &= \frac{\int_0^\alpha (\sin\theta - \sin\theta \cos 2(\alpha - \theta)) \, d\theta}{2d(x)^2 \sin^2\alpha} \end{aligned}$$

$$
\begin{aligned}
&= \frac{-\cos\theta|_0^\alpha - \frac{1}{2}\int_0^\alpha \left(\sin\left(2\alpha-\theta\right)+\sin\left(3\theta-2\alpha\right)\right)d\theta}{2d(x)^2\sin^2\alpha} \\
&= \frac{1-\cos\alpha-\frac{1}{2}(\cos\alpha-\cos 2\alpha-\frac{1}{3}\cos\alpha+\frac{1}{3}\cos 2\alpha}{2d(x)^2\sin^2\alpha} \\
&= \frac{1-\frac{4}{3}\cos\alpha+\frac{1}{3}\cos 2\alpha}{2d(x)^2\sin^2\alpha} \\
&= \frac{(1-\cos\alpha)^2}{3\,d(x)^2(1-\cos^2\alpha)} = \frac{1}{3\,d(x)^2}\cdot\frac{1-\cos\alpha}{1+\cos\alpha} \\
&= \frac{\tan^2\frac{\alpha}{2}}{3\,d(x)^2}. \qquad (4.11)
\end{aligned}
$$

Here we have used the trigonometric identity $\sin^2 A = \frac{1}{2}(1-\cos 2A)$. Apply Proposition 4.1.1 to this lower bound in (4.11) to obtain the following Hardy-type inequality:

$$
\mu\left(\alpha\right)\int\limits_{\Omega}\frac{\left|f(x)\right|^2}{d(x)^2}dx \leq \int\limits_{\Omega}\left|\nabla f(x)\right|^2 dx,
$$

where

$$
\mu\left(\alpha\right) = \frac{1}{4}\tan^2\frac{\alpha}{2},
$$

this completes the proof. $\square$

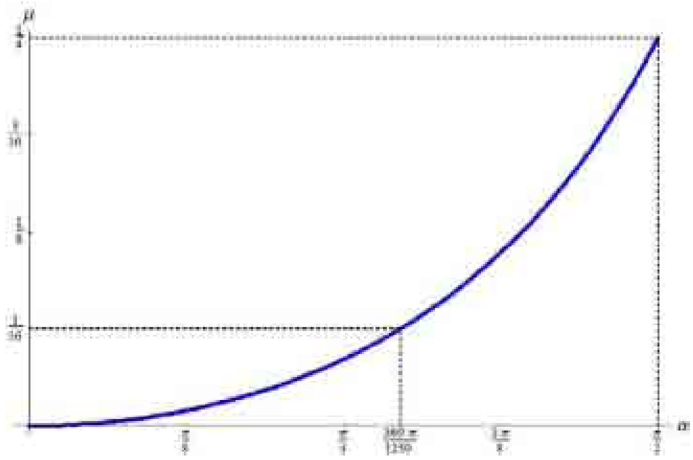

Figure 4.2: μ growth with α

Remark 4.3.2.

If the domain Ω in Theorem 4.3.1 is convex, then $\alpha = \frac{\pi}{2}$. Thus the function $\mu(\alpha)$ defined by (4.7) will be $\mu(\frac{\pi}{2}) = \frac{1}{4}$ as known for convex domains.

Now let us consider domains that satisfy the TCR Condition. Recall that $d(x)$ and $d_u(x)$ are defined as in (4.4) and (4.5) respectively, while $\tilde{d}_u(x)$ is used to denote the distance from the point $x \in \Omega$ to the boundary $\partial K_{h,\omega}(\nu, \alpha)$ in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial K_{h,\omega}(\nu, \alpha)\}.$$

Theorem 4.3.3.

Suppose that the domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.2.2. Then for any $f \in \mathcal{C}_c^\infty(\Omega)$ the following Hardy-type inequality holds:

$$\int\limits_\Omega \mu_1(x, \alpha, h) \frac{|f(x)|^2}{(h + d(x))^2} dx + \int\limits_\Omega \mu_2(x, \alpha, h) \frac{|f(x)|^2}{d(x)^2} dx \le \int\limits_\Omega |\nabla f(x)|^2 \, dx, \tag{4.12}$$

where

$$\mu_1(x; \alpha, h) = \frac{1}{4} \cos^3\left(\tan^{-1}(a(x) \, \tan\alpha)\right), \tag{4.13}$$

and

$$\begin{aligned} \mu_2(x, \alpha, h) \;=\; & \frac{1}{4\sin^2\alpha} \Big[3 - \cos 2\left(\alpha - \left(\tan^{-1}(a(x) \, \tan\alpha)\right)\right) \\ & -2\cos\left(2\alpha - \left(\tan^{-1}(a(x) \, \tan\alpha)\right)\right)\Big] \sin^2 \frac{(\tan^{-1}(a(x) \, \tan\alpha))}{2}, \end{aligned} \tag{4.14}$$

with $a(x) = \frac{1}{1 + \frac{d(x)}{h}}$.

Proof. By (4.2), the definition of the function $\frac{1}{m(x)^2}$, and the fact that $\tilde{d}_u(x) \ge d_u(x)$, we

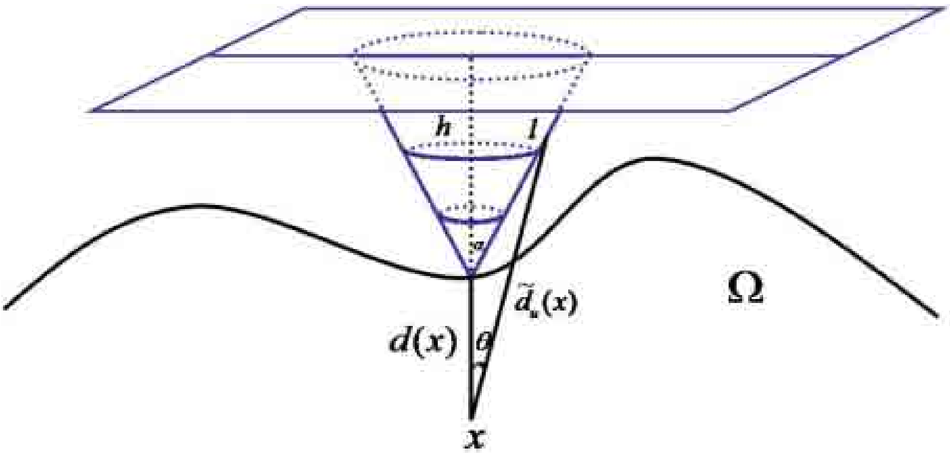

Figure 4.3: x lies on the symmetry axis of $K_{h,\omega}(\nu,\alpha)$ for a 3-D domain

have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\frac{1}{d_u(x)^2}dS(u) \geq \frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\frac{1}{\tilde{d}_u(x)^2}dS(u). \tag{4.15}$$

Since the function $\tilde{d}_u(x)$ is symmetric, with respect to the rotation about the symmetry axis of the domain $K_{h,\omega}(\nu,\alpha)$, then using spherical coordinates, (r,θ,ϕ) where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, leads to $u = u(\theta,\phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x,\theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (4.15) becomes

$$\frac{1}{m(x)^2} \geq \frac{1}{4\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{1}{\tilde{d}(x,\theta)^2}\sin\theta\, d\theta d\phi = \frac{2\pi}{4\pi}2\int\limits_0^{\frac{\pi}{2}}\frac{1}{\tilde{d}(x,\theta)^2}\sin\theta\, d\theta.$$

Since $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.2.2 and, in Fig. 4.3, if we consider the two-dimensional cross section that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}_u(x)$, we can divide the above interval into two intervals considering the relation between $\tilde{d}(x,\theta)$ and $d(x)$, see Section 3.2 Case 1. Thus, for $\theta \in (0,\theta_0)$, the function $\tilde{d}(x,\theta)$ can be expressed in the following form:

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha-\theta)},$$

as derived in (3.9). Besides, for $\theta \in (\theta_0, \frac{\pi}{2})$, the function $\tilde{d}(x,\theta)$ can be written in as

$$\tilde{d}(x,\theta) = \frac{h+d(x)}{\cos\theta},$$

as established in (3.10), where θ_0 is defined by (3.7) and satisfies

$$\tan\theta_0 = \frac{1}{1+\frac{d(x)}{h}}\tan\alpha, \tag{4.16}$$

as obtained in (3.8). Moreover, for $\alpha = \frac{\pi}{2}$ (for which Ω attains the convex case) we have

$$\tilde{d}(x,\theta) = \frac{d(x)}{\cos\theta}.$$

Thus, the function $\frac{1}{m(x)^2}$ is bounded from below as follows:

$$\frac{1}{m(x)^2} \geq \frac{\int_0^{\theta_0}\sin^2(\alpha-\theta)\sin\theta d\theta}{d(x)^2\sin^2\alpha} + \frac{\int_{\theta_0}^{\frac{\pi}{2}}\cos^2\theta\sin\theta d\theta}{\left(h+d(x)\right)^2}.$$

Using the trigonometric identity $\sin^2 A = \frac{1-\cos 2A}{2}$ in the first integral and the substitution $u = \cos\theta$ in the second integral produces

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{-\cos\theta\big|_0^{\theta_0} - \frac{1}{2}\int_0^{\theta_0}\left(\sin\left(2\alpha-\theta\right)+\sin\left(3\theta-2\alpha\right)\right)d\theta}{2d(x)^2\sin^2\alpha} + \frac{\cos^3\theta_0}{3\left(h+d(x)\right)^2} \\
&= \frac{1-\cos\theta_0 - \frac{\cos(2\alpha-\theta_0)}{2} + \frac{\cos(3\theta_0-2\alpha)}{6} + \frac{\cos(2\alpha)}{3}}{2d(x)^2\sin^2\alpha} + \frac{\cos^3\theta_0}{3\left(h+d(x)\right)^2} \\
&= \frac{\left(3-\cos 2\left(\alpha-\theta_0\right) - 2\cos\left(2\alpha-\theta_0\right)\right)\sin^2\frac{\theta_0}{2}}{3d(x)^2\sin^2\alpha} + \frac{\cos^3\theta_0}{3\left(h+d(x)\right)^2}.
\end{aligned} \tag{4.17}$$

Applying Proposition 4.1.1 to this lower bound in (4.17) leads to

$$\int_{\Omega} \mu_1^*(\theta_0) \frac{|f(x)|^2}{(h+d(x))^2} dx + \int_{\Omega} \mu_2^*(\theta_0, \alpha) \frac{|f(x)|^2}{d(x)^2} dx \leq \int_{\Omega} |\nabla f(x)|^2 \, dx,$$

where

$$\mu_1^*(\theta_0) = \frac{\cos^3 \theta_0}{4},$$

and

$$\mu_2^*(\theta_0, \alpha) = \frac{(3 - \cos 2(\alpha - \theta_0) - 2\cos(2\alpha - \theta_0)) \sin^2 \frac{\theta_0}{2}}{4 \sin^2 \alpha}.$$

Now using (4.16), the relation between θ_0 and α, enables us to write $\mu_1^*(\theta_0)$ and $\mu_2^*(\theta_0, \alpha)$ as functions of x, α, and h as follows:

$$\begin{aligned} \mu_1(x, \alpha, h) &= \frac{1}{4} \cos^3 \left(\tan^{-1}(a(x) \tan \alpha)\right), \text{ and} \\ \mu_2(x, \alpha, h) &= \frac{1}{4 \sin^2 \alpha} \Big[3 - \cos 2\left(\alpha - \left(\tan^{-1}(a(x) \tan \alpha)\right)\right) \\ &\qquad -2 \cos\left(2\alpha - \left(\tan^{-1}(a(x) \tan \alpha)\right)\right)\Big] \sin^2 \frac{(\tan^{-1}(a(x) \tan \alpha))}{2}, \end{aligned}$$

where $a(x) = \frac{1}{1 + \frac{d(x)}{h}}$. This completes the proof.

$\square$

Remark 4.3.4.

1. If Ω is a convex domain then $\alpha = \theta_0 = \frac{\pi}{2}$. Thus for a convex domain $\Omega \in \mathbb{R}^3$, inequality (4.17) gives

$$\frac{1}{m(x)^2} \geq \frac{1}{3d(x)^2},$$

as expected from Theorem 2.2.7. Besides, for convex domains with $a(x) \neq 0$, i.e. $h \not\to 0$, we have $\mu_1\left(x, \frac{\pi}{2}, h\right) = 0$ and $\mu_2\left(x, \frac{\pi}{2}, h\right) = \frac{1}{4}$, thus the Hardy-type inequality

(4.12) reproduces the following well-known bound (see Theorem 2.2.6):

$$\frac{1}{4}\int\limits_{\Omega}\frac{|f(x)|^2}{d(x)^2}dx\leq\int\limits_{\Omega}|\nabla f(x)|^2\,dx. \tag{4.18}$$

2. As $\alpha\nearrow\frac{\pi}{2}$, the domain Ω approaches the convexity case, and hence it is natural to compare $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ given by (4.13) and (4.14) respectively, with their values for the convex case. To this end we use the Taylor expansion to expand $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ in powers of $\left(\frac{\pi}{2}-\alpha\right)$. Keeping in mind that for fixed h we have $\theta_0=\tan^{-1}(a(x)\tan\alpha)=\frac{\pi}{2}$ where $\alpha=\frac{\pi}{2}$, see Section 3.2, Case 1. Consequently, for $\mu_1(x,\alpha,h)$, we have

$$\mu_1\left(x,\frac{\pi}{2},h\right)=0.$$

On the other hand

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mu_1(x,\alpha,h)&=-\frac{3}{4}\cos^2\left(\tan^{-1}(a(x)\tan\alpha)\right)\sin\left(\tan^{-1}(a(x)\tan\alpha)\right)\frac{a(x)\sec^2\alpha}{1+a(x)^2\tan^2\alpha}\\&=-\frac{3a(x)}{4}\frac{\cos^2(\tan^{-1}(a(x)\tan\alpha))\sin(\tan^{-1}(a(x)\tan\alpha))}{\cos^2\alpha+a(x)^2\sin^2\alpha},\end{aligned}$$

which gives

$$\frac{\partial}{\partial\alpha}\mu_1\left(x,\frac{\pi}{2},h\right)=-\frac{3a(x)}{4}\frac{0\cdot 1}{a(x)^2}=0.$$

In addition,

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2}\mu_1(x,\alpha,h)=-&\frac{3a(x)}{4\left(\cos^2\alpha+a(x)^2\sin^2\alpha\right)^2}\left[a(x)\left(\cos^3\left(\tan^{-1}(a(x)\tan\alpha)\right)-\right.\right.\\&\left.\sin\left(2\tan^{-1}(a(x)\tan\alpha)\right)\sin\left(\tan^{-1}(a(x)\tan\alpha)\right)\right)\\&\left.-\cos^2\left(\tan^{-1}(a(x)\tan\alpha)\right)\sin\left(\tan^{-1}(a(x)\tan\alpha)\right)\sin 2\alpha\left(a(x)^2-1\right)\right],\end{aligned}$$

which leads to

$$\frac{\partial^2}{\partial\alpha^2}\mu_1\left(x,\frac{\pi}{2},h\right)=0.$$

Similarly, we can find

$$\frac{\partial^3}{\partial\alpha^3}\mu_1\left(x,\frac{\pi}{2},h\right)=-\frac{3}{2a(x)^3}=-\frac{3\left(h+d(x)\right)^3}{2h^3},\cdots \text{ and so on.}$$

Thus $\mu_1\left(x,\alpha,h\right)$ can be written as follows:

$$\mu_1\left(x,\alpha,h\right)=\frac{(h+d(x))^3}{4h^3}\left(\frac{\pi}{2}-\alpha\right)^3+\mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^4\right). \tag{4.19}$$

In the same way, for $\mu_2\left(x,\frac{\pi}{2},h\right)$, we have

$$\begin{aligned}\mu_2\left(x,\frac{\pi}{2},h\right) &= \frac{1}{4},\\ \frac{\partial}{\partial\alpha}\mu_2\left(x,\frac{\pi}{2},h\right) &= \frac{1}{2},\cdots \text{ and so on.}\end{aligned}$$

Thus $\mu_2\left(x,\alpha,h\right)$ can be written as follows:

$$\mu_2\left(x,\alpha,h\right)=\frac{1}{4}+\frac{1}{2}\left(\alpha-\frac{\pi}{2}\right)+\mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^2\right). \tag{4.20}$$

For $\alpha=\frac{\pi}{2}$, we have $\mu_1\left(x,\alpha,h\right)=0$ and $\mu_2\left(x,\alpha,h\right)=\frac{1}{4}$, thus we obtain the same bound as in (4.18).

Relations (4.19) and (4.20) show that the second term in inequality (4.12) is the effective term when talking about the convex case, since $\mu_1\left(x,\alpha,h\right)$ decays rapidly to zero while $\mu_2\left(x,\alpha,h\right)$ tends to $\frac{1}{4}$, when α tends to $\frac{\pi}{2}$.

3. For fixed α, if h tends to ∞ then $a(x)=\frac{h}{h+d(x)}$ tends to 1. Thus the first term in inequality (4.12) tends to zero. Then, because of Remark 3.2.4, we have $\theta_0=$

$\tan^{-1}(a(x)\,\tan\alpha)$ tends to α. Therefore, the function $\mu_2(x,\alpha,h)$ has the following limit as h tends to ∞:

$$\begin{aligned}
\lim_{h\to\infty}\mu_2(x,\alpha,h) &= \lim_{h\to\infty}\frac{1}{4\sin^2\alpha}\Big[\left(3-\cos 2\left(\alpha-\left(\tan^{-1}(a(x)\,\tan\alpha)\right)\right)\right.\\
&\qquad \left.-2\cos\left(2\alpha-\left(\tan^{-1}(a(x)\,\tan\alpha)\right)\right)\right)\sin^2\frac{(\tan^{-1}(a(x)\,\tan\alpha))}{2}\Big]\\
&= \frac{1}{4\sin^2\alpha}\left[(2-2\cos\alpha)\sin^2\frac{\alpha}{2}\right]=\frac{1}{2\sin^2\alpha}\left[(1-\cos\alpha)\sin^2\frac{\alpha}{2}\right]\\
&= \frac{1}{4}\tan^2\frac{\alpha}{2},
\end{aligned}$$

which runs over the half-open interval $\left(0,\frac{1}{4}\right]$, since $\alpha\in(0,\frac{\pi}{2}]$. As a result of Remark 3.2.8, and the functions $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ being bounded, the Hardy-type inequality (4.12) takes the form

$$\mu(\alpha)\int\limits_{\Omega}\frac{|f(x)|^2}{d(x)^2}dx\le\int\limits_{\Omega}|\nabla f(x)|^2\,dx,$$

as $h\to\infty$, where

$$\mu(\alpha)=\frac{1}{4}\tan^2\frac{\alpha}{2},\quad\text{or in terms of }\psi\text{ this is equal to }\quad\frac{1}{4}\cot^2\frac{\psi}{2},$$

which is the same result as Theorem 4.3.1.

Moreover, if h tends to 0, then $a(x)=\frac{h}{h+d(x)}$ tends to 0 as well. Then it is clear that $\mu_1(x,\alpha,h)$ and $\mu_2(x,\alpha,h)$ tend to $\frac{1}{4}$ and 0 respectively.

Now let us investigate domains $\Omega\subset\mathbb{R}^3$ that satisfy the Exterior Ball Condition, Condition 4.2.3, in order to find out formulas for Hardy-type inequalities.

4.4 Hardy's inequalities under the Exterior Ball Condition

In this section we seek forms of the Hardy-type inequalities in $\Omega \subset \mathbb{R}^3$, which satisfy Condition 4.2.3. We remind ourselves here that $d(x)$ denotes the Euclidean distance from the point $x \in \Omega$ to the boundary $\partial\Omega$, whereas $d_u(x)$ refers to the distance from the point $x \in \Omega$ to the boundary $\partial\Omega$ in the direction u, i.e.

$$d_u(x) := \min\{|s| : x + su \notin \Omega\}.$$

Moreover, $\tilde{d}_u(x)$ denotes the distance from the point $x \in \Omega$ to the boundary $\partial B^3(a, R)$ in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial B^3(a, R)\}.$$

To prove the following theorem, we follow the strategy explained at the end of Section 4.2.

Theorem 4.4.1.

Suppose that the domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.2.3. Then for any $f \in \mathcal{C}_c^\infty(\Omega)$ the following Hardy-type inequality holds:

$$\int\limits_\Omega \mu(x, R) \frac{|f(x)|^2}{d(x)^2} dx \leq \int\limits_\Omega |\nabla f(x)|^2 \, dx, \tag{4.21}$$

where

$$\mu(x, R) \quad = \quad \frac{(R - d(x)) \sqrt{d(x)^2 + 2Rd(x)} + d(x)^2}{4(R + d(x)) \sqrt{d(x)^2 + 2Rd(x)}}. \tag{4.22}$$

Proof. By (4.2), the definition of the function $\frac{1}{m(x)^2}$, and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi} \int\limits_{\mathbb{S}^2} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{4\pi} \int\limits_{\mathbb{S}^2} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{4.23}$$

Using spherical coordinates, (r, θ, ϕ) where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, leads to $u = u(\theta, \phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x, \theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (4.23) becomes

$$\frac{1}{m(x)^2} \geq \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta \, d\theta d\phi = \int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin\theta \, d\theta. \tag{4.24}$$

Since $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.2.3, and considering the two-dimensional cross section that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}(x, \theta)$, we can express $\frac{1}{\tilde{d}(x,\theta)^2}$ in terms of x, θ, and R the same way we derived (3.27). Therefore, inequality (4.24) takes the following form:

$$\begin{aligned} \frac{1}{m(x)^2} &\geq \int_0^{\theta_0} \frac{\sin\theta}{\cos^2\theta \left(R + d(x) - \sqrt{R^2 - \left(d(x)^2 + 2Rd(x) \right) \tan^2\theta} \right)^2} d\theta \\ &= \frac{1}{(R + d(x))^2} \int_0^{\theta_0} \frac{\sin\theta}{\cos^2\theta \left(1 - \sqrt{\frac{R^2}{(R+d(x))^2} - \left(\frac{d(x)^2 + 2Rd(x)}{(R+d(x))^2} \right) \tan^2\theta} \right)^2} d\theta. \end{aligned} \tag{4.25}$$

To compute this integral we set:

$$\sec\theta = \frac{R + d(x)}{\sqrt{d(x)^2 + 2Rd(x)}} t,$$

which gives

$$\begin{aligned} \frac{\sin\theta}{\cos^2\theta} d\theta &= \frac{R + d(x)}{\sqrt{d(x)^2 + 2Rd(x)}} dt, \\ \tan^2\theta &= \frac{(R + d(x))^2}{d(x)^2 + 2Rd(x)} t^2 - 1, \end{aligned}$$

and with (3.24) leads to

$$\begin{aligned} t_0 &= \frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)} \cdot \sec\theta_0 \\ &= \frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)} \cdot \frac{R+d(x)}{\sqrt{d(x)^2+2Rd(x)}} = 1. \end{aligned}$$

Thus, inequality (4.25) becomes

$$\begin{aligned} \frac{1}{m(x)^2} &\geq \frac{1}{(R+d\,(x))^2} \cdot \frac{R+d(x)}{\sqrt{d(x)^2+2Rd(x)}} \times \\ & \int\limits_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \frac{dt}{\left(1-\sqrt{\frac{R^2}{(R+d(x))^2} - \left(\frac{d(x)^2+2Rd(x)}{(R+d(x))^2}\right)\left(\frac{(R+d(x))^2}{d(x)^2+2Rd(x)}t^2-1\right)}\right)^2} \\ &= \frac{1}{(R+d(x))\sqrt{d(x)^2+2Rd(x)}} \int\limits_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \frac{dt}{\left(1-\sqrt{1-t^2}\right)^2}. \end{aligned} \tag{4.26}$$

The integrand in (4.26) can be simplified as follows:

$$\begin{aligned} \frac{1}{\left(1-\sqrt{1-t^2}\right)^2} &= \frac{2-t^2+2\sqrt{1-t^2}}{(1-1+t^2)^2} = \frac{2-t^2+2\sqrt{1-t^2}}{t^4} \\ &= \frac{2}{t^4} - \frac{1}{t^2} + \frac{2\sqrt{1-t^2}}{t^4}. \end{aligned} \tag{4.27}$$

Hence, using (4.27) in (4.26) produces the following inequality:

$$\begin{aligned} \frac{1}{m\,(x)^2} &\geq \frac{1}{(R+d(x))\sqrt{d(x)^2+2Rd(x)}} \int\limits_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \left[\frac{2}{t^4} - \frac{1}{t^2} + \frac{2\sqrt{1-t^2}}{t^4}\right] dt \\ &= \frac{2}{(R+d(x))\sqrt{d(x)^2+2Rd(x)}} \times \left(I_1(x,R) - I_2(x,R) + I_3(x,R)\right), \end{aligned} \tag{4.28}$$

where

$$
\begin{aligned}
I_1(x,R) &= \int\limits_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \frac{dt}{t^4},\\
I_2(x,R) &= \int\limits_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \frac{dt}{2t^2},\\
\text{and } I_3(x,R) &= \int\limits_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \frac{\sqrt{1-t^2}}{t^4}dt.
\end{aligned}
$$

Concerning the first integral $I_1(x,R)$, we have

$$
\begin{aligned}
I_1(x,R) &= -\frac{1}{3}\left(1-\frac{(R+d(x))^3}{(d(x)^2+2Rd(x))^{\frac{3}{2}}}\right)\\
&= \frac{(R+d(x))^3-\left(d(x)^2+2Rd(x)\right)^{\frac{3}{2}}}{3\left(d(x)^2+2Rd(x)\right)^{\frac{3}{2}}}.
\end{aligned}
\tag{4.29}
$$

Moreover, the integral $I_2(x,R)$ gives

$$
\begin{aligned}
I_2(x,R) &= \frac{1}{2}\left(\frac{R+d(x)}{\sqrt{d(x)^2+2Rd(x)}}-1\right)\\
&= \frac{R+d(x)-\sqrt{d(x)^2+2Rd(x)}}{2\sqrt{d(x)^2+2Rd(x)}}.
\end{aligned}
\tag{4.30}
$$

Finally, set $r = \frac{1}{t}$ for $I_3(x,R)$ to obtain

$$
\begin{aligned}
I_3(x,R) &= \int\limits_{\frac{R+d(x)}{\sqrt{d(x)^2+2Rd(x)}}}^{1} \frac{\sqrt{1-\frac{1}{r^2}}}{\frac{1}{r^4}} \cdot \left(\frac{-1}{r^2}\right) dr \\
&= -\int\limits_{\frac{R+d(x)}{\sqrt{d(x)^2+2Rd(x)}}}^{1} r \cdot \sqrt{r^2-1}\, dr = -\left.\frac{(r^2-1)^{\frac{3}{2}}}{3}\right|_{\frac{R+d(x)}{\sqrt{d(x)^2+2Rd(x)}}}^{1} \\
&= \frac{1}{3}\left(\frac{(R+d(x))^2}{d(x)^2+2Rd(x)}-1\right)^{\frac{3}{2}} \\
&= \frac{R^3}{3\left(d(x)^2+2Rd(x)\right)^{\frac{3}{2}}}. \qquad (4.31)
\end{aligned}
$$

Using (4.29), (4.30), and (4.31) in (4.28), yields the following lower bound on $\frac{1}{m(x)^2}$:

$$
\begin{aligned}
\frac{1}{m(x)^2} \geq\ & \frac{2}{(R+d(x))\sqrt{d(x)^2+2Rd(x)}}\left(\frac{(R+d(x))^3-\left(d(x)^2+2Rd(x)\right)^{\frac{3}{2}}}{3\left(d(x)^2+2Rd(x)\right)^{\frac{3}{2}}}\right. \\
&\left. -\frac{R+d(x)-\sqrt{d(x)^2+2Rd(x)}}{2\sqrt{d(x)^2+2Rd(x)}}+\frac{R^3}{3\left(d(x)^2+2Rd(x)\right)^{\frac{3}{2}}}\right) \\
=\ & \frac{(R-d(x))\sqrt{d(x)^2+2Rd(x)}+d(x)^2}{3d(x)^2(R+d(x))\sqrt{d(x)^2+2Rd(x)}}. \qquad (4.32)
\end{aligned}
$$

Applying Proposition 4.1.1 to (4.32) returns the following Hardy-type inequality:

$$
\int\limits_{\Omega} \mu(x,R)\frac{|f(x)|^2}{d(x)^2}dx \leq \int\limits_{\Omega} |\nabla f(x)|^2\, dx,
$$

where

$$
\mu(x,R) = \frac{(R-d(x))\sqrt{d(x)^2+2Rd(x)}+d(x)^2}{4(R+d(x))\sqrt{d(x)^2+2Rd(x)}},
$$

this completes the proof. □

Remark 4.4.2.

1. As R tends to ∞, the domain Ω approaches the convexity case, and hence it is natural to compare $\mu(x,R)$, given by (4.22), with its value for the convex case. To this end we expand $\mu(x,R)$ in powers of $\frac{d(x)}{R}$. To achieve this we consider

$$\mu(x,R) = \frac{R-d(x)}{4(R+d(x))} + \frac{d(x)^2}{4(R+d(x))\sqrt{d(x)^2+2Rd(x)}}. \tag{4.33}$$

We note that the first term in (4.33) can be expanded as follows:

$$\begin{aligned}\frac{1}{4}\frac{R-d(x)}{R+d(x)} &= \frac{1}{4}\left(1-\frac{d(x)}{R}\right)\left(1+\frac{d(x)}{R}\right)^{-1}\\ &= \frac{1}{4}\left(1-\frac{d(x)}{R}\right)\left(1-\frac{d(x)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^2\right)\right)\\ &= \frac{1}{4}\left(1-2\frac{d(x)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^2\right)\right).\end{aligned} \tag{4.34}$$

On the other hand, the second term in (4.33) can be expanded as follows:

$$\begin{aligned}\frac{d(x)^2}{4(R+d(x))\left(d(x)^2+2Rd(x)\right)^{\frac{1}{2}}} &= \frac{1}{4\sqrt{2}}\left(\frac{d(x)}{R}\right)^{\frac{3}{2}}\left(1+\frac{d(x)}{R}\right)^{-1}\left(1+\frac{d(x)}{2R}\right)^{-\frac{1}{2}}\\ &= \frac{1}{4\sqrt{2}}\left(\frac{d(x)}{R}\right)^{\frac{3}{2}}\left(1-\frac{5}{4}\frac{d(x)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^2\right)\right)\\ &= \frac{1}{4\sqrt{2}}\left(\frac{d(x)}{R}\right)^{\frac{3}{2}}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{\frac{5}{2}}\right).\end{aligned} \tag{4.35}$$

Using (4.34) and (4.35) in (4.33) implies that

$$\mu(x,R) = \frac{1}{4}-\frac{d(x)}{2R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{\frac{3}{2}}\right), \tag{4.36}$$

which tends to $\frac{1}{4}$ as R tends to ∞ linearly in $\frac{d(x)}{R}$.

2. If Ω is a convex domain, then the Exterior Ball Condition is satisfied for all R, thus as R tends to ∞, inequality (4.32) yields

$$\frac{1}{m(x)^2} \geq \frac{1}{3d(x)^2},$$

as expected from Theorem 2.2.7. Consequently, inequality (4.21) reproduces the well-known bound for convex domains

$$\frac{1}{4}\int_{\Omega} \frac{|f(x)|^2}{d(x)^2} dx \leq \int_{\Omega} |\nabla f(x)|^2 \, dx.$$

Now let us establish Hardy-type inequalities for domains $\Omega \subset \mathbb{R}^3$ that verify the Cylinder Condition.

4.5 Hardy's inequalities under the Cylinder Condition

In this section we derive new Hardy-type inequalities where we have $\Omega \subset \mathbb{R}^3$ satisfying Condition 4.2.4. If the Cylinder Condition is satisfied then the non-convexity manifests itself in the 'limited number of dimensions'. Thus the Hardy inequality is expected to be 'closer' to the convex case than the one obtained under the Exterior Ball Condition. The next theorem confirms this expectation.

It is worthy to stress here that $d(x)$ and $d_u(x)$ are as stated beforehand, i.e. the first represents the Euclidean distance from the point $x \in \Omega$ to the boundary $\partial\Omega$, whilst the latter refers to the distance from the point $x \in \Omega$ to the boundary $\partial\Omega$ in the direction u. Moreover, we use $\tilde{d}_u(x)$ to denote the distance from the point $x \in \Omega$ to the boundary $\partial\mathcal{Z}(\ell, R)$ in the direction

u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial\mathcal{Z}(\ell, R)\}.$$

To achieve our objective we follow the strategy described at the end of Section 4.2 .

Theorem 4.5.1.

Suppose that the domain $\Omega \subset \mathbb{R}^3$ satisfies Condition 4.2.4 for some $R > 0$. Then for any function $f \in \mathcal{C}_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$\int_\Omega \mu(x, R) \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2 dx, \tag{4.37}$$

where

$$\mu(x, R) = \frac{R\left[\pi R + 2\sqrt{d(x)(d(x) + 2R)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)\right]}{2\pi(d(x) + 2R)^2}. \tag{4.38}$$

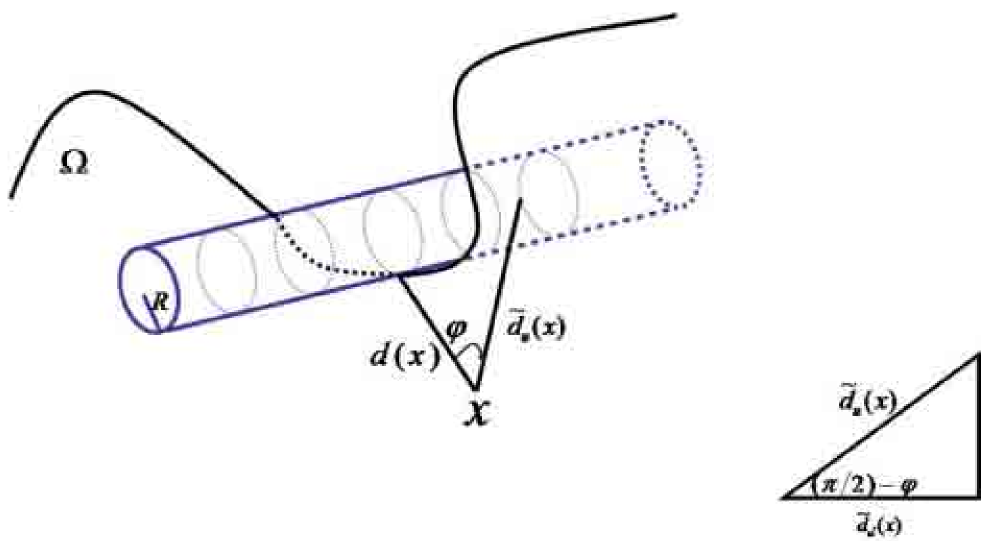

Figure 4.4: A three-dimensional domain which satisfies the Cylinder Condition

Proof. By (4.2), the definition of the function $\frac{1}{m(x)^2}$, and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi}\int\limits_{\mathbb{S}^2} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{4\pi}\int\limits_{\mathbb{S}^2} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{4.39}$$

Consider a two-dimensional cross section of the cylinder $\mathcal{Z}(\ell, R)$ and the domain Ω by the plane Λ, which is orthogonal to ℓ and containing x. Let $u^{'} \in \Lambda$ be the projection of u onto Λ. Therefore, we now have a planar 'picture' in which we have the point x, a disk of radius R with centre that belongs to ℓ, and the line segments representing the distance from x to that disk as well as the distance from x to the boundary of that disk in the direction $u^{'}$. Let $\tilde{d}_{u^{'}}(x)$ be the distance from x to the the boundary of that disk in the direction $u^{'}$. Let θ, ϕ be the standard spherical coordinates of the vector u such that $\theta \in [-\pi, \pi]$ is the angle in the plane Λ and $\phi \in [0, \pi]$. Therefore, and because of the planar 'picture' we have, we can follow the same argument as in Section 3.2 namely the discussion after Condition 3.2.9, to obtain

$$\tilde{d}_{u'}(x) = \tilde{d}_{u'}(x, \theta) = \cos\theta \left(R + d(x) - \sqrt{R^2 - (d(x)^2 + 2Rd(x)) \tan^2\theta} \right), \tag{4.40}$$

where $-\theta_0 \leq \theta \leq \theta_0$. On the other hand, it is clear, see Fig. 4.4, that

$$\tilde{d}_u(x, \theta)^2 = \tilde{d}_{u'}(x, \theta)^2 + \tilde{d}_{u'}(x, \theta)^2 \cot^2\phi, \tag{4.41}$$

where $0 \leq \phi \leq \pi$. Therefore, using the spherical coordinates with (4.41) in (4.39) implies

$$\begin{aligned} \frac{1}{m(x)^2} &\geq \frac{2}{4\pi} \int\limits_0^{\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\phi}{\tilde{d}_{u'}(x, \theta)^2 (1 + \cot^2\phi)} d\theta d\phi \\ &= \frac{1}{2\pi} \int\limits_0^{\pi} \int\limits_{-\theta_0}^{\theta_0} \frac{\sin^3\phi}{\tilde{d}_{u'}(x, \theta)^2} d\theta d\phi. \end{aligned} \tag{4.42}$$

Now use (4.40) in (4.42) to obtain

$$\frac{1}{m(x)^2} \geq \frac{1}{2}\int_0^{\pi} \sin^3\phi\, d\phi \cdot I(x) = \frac{2}{3}\cdot I(x),$$

where

$$I(x) = \frac{1}{\pi}\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\cos^2\theta\left(R + d(x) - \sqrt{R^2 - (d(x)^2 + 2Rd(x))\tan^2\theta}\right)^2},$$

this integral has been evaluated previously in our course to prove Theorem 3.4.1. Thus,

$$\frac{1}{m(x)^2} \geq \frac{2R\left[\pi R + 2\sqrt{d(x)(d(x)+2R)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)\right]}{3d(x)^2\,\pi\,(d(x)+2R)^2}. \quad (4.43)$$

Apply Proposition 4.1.1 to (4.43) to obtain the following Hardy-type inequality:

$$\int_\Omega \mu(x,R)\frac{|f(x)|^2}{d(x)^2}dx \leq \int_\Omega |\nabla f(x)|^2\, dx,$$

where

$$\mu(x,R) = \frac{R\left[\pi R + 2\sqrt{d(x)(d(x)+2R)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)\right]}{2\pi(d(x)+2R)^2},$$

as stated in (4.37) and (4.38) respectively. □

Remark 4.5.2.

1. The function $\mu(x,R)$, given by (4.38), is exactly the same as the function $\mu(x,R)$, given by (3.53) under the Exterior Disk Condition. The reason behind this is that the Cylinder Condition means that the domain $\Omega \subset \mathbb{R}^3$ has a ‘two-dimensional non-convexity’ while it is flat in the third dimension. Therefore, the function $\mu(x,R)$

obtained under the Cylinder Condition will have the same asymptotic expansion in powers of $\frac{d(x)}{R}$ as in the two-dimensional case, see (3.71), i.e., $\mu(x,R)$ can be written in the following form:

$$\mu\left(x,R\right) \;=\; \frac{1}{4}-\frac{1}{4}\frac{d\left(x\right)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right), \tag{4.44}$$

which is approximately linear in $\frac{1}{R}$ for large R and tending to $\frac{1}{4}$ as $\frac{d(x)}{R}$ tends to 0.

2. As R tends to ∞, the domain Ω approaches the convexity case and for this limit inequality (4.43) gives

$$\frac{1}{m(x)^2}\geq\frac{1}{3d(x)^2},$$

as expected from Theorem 2.2.7. Accordingly, inequality (4.37) reproduces the well-known bound for convex domains

$$\frac{1}{4}\int\limits_{\Omega}\frac{\left|f(x)\right|^2}{d(x)^2}dx\leq\int\limits_{\Omega}\left|\nabla f(x)\right|^2dx.$$

3. To compare the results obtained under the Cylinder Condition with those obtained under the Exterior Ball Condition, we compare the second terms in the asymptotic forms of the function $\mu(x,R)$ in each case as R tends to ∞. Clearly, in (4.44) (obtained under the Cylinder Condition) the coefficient by $\frac{d(x)}{R}$ is smaller than the corresponding one in (4.36) (obtained under the Exterior Ball Condition), which means that the former is 'closer' to the convex case than the latter in the limit $R\to\infty$.

Chapter 5

Hardy's inequalities for $n-$dimensional non-convex domains

5.1 Introduction

This chapter concerns deriving new Hardy-type inequalities for functions defined on n-dimensional non-convex domains in $\mathbb{R}^n$ with $n \geq 3$. Each of these domains satisfies a certain geometrical condition, which are generalisations of the conditions established in Chapter 4. Specifically, we introduce four different conditions, which may be regarded as 'non-convexity measures'. The first two conditions are generalisations of the so-called 'Cone Conditions', i.e. Condition 4.2.1 and Condition 4.2.2. The third and fourth conditions are generalisations of the so-called 'Exterior Ball Condition' and the 'Cylinder Condition'. A fundamental tool to attain our goal is the application of Theorem 2.2.5. Let us, therefore, and for the reader's convenience mention it again here.

Proposition 5.1.1 (E. B. Davies, [16, 19])**.**

Let Ω be a domain in $\mathbb{R}^n$ and let $f(x) \in \mathcal{C}_c^\infty(\Omega)$. Then

$$\frac{n}{4}\int\limits_{\Omega} \frac{|f(x)|^2}{m(x)^2}\,dx \leq \int\limits_{\Omega} |\nabla f(x)|^2\,dx, \tag{5.1}$$

where $\frac{1}{m(x)^2}$, the (harmonic) mean distance of $x \in \Omega$ from $\partial\Omega$, is defined by

$$\frac{1}{m(x)^2} := \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\mathbb{S}^{n-1}} \frac{dS\,(u)}{d_u\,(x)^2}, \tag{5.2}$$

and $d_u\,(x)$ is defined for every unit vector $u \in \mathbb{S}^{n-1}$ and $x \in \Omega$ by

$$d_u(x) := \min\{|s| : x + su \notin \Omega\}. \tag{5.3}$$

Here $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

We would like to emphasize that, in order to accomplish this chapter's target, the strategy used in Chapter 4 will be followed here. To be more specific, we try to attain lower bounds, containing the 'true' distance $d(x)$, for the function $\frac{1}{m(x)^2}$, defined in (5.2), we then apply Proposition 5.1.1 to those lower bounds to obtain Hardy-type inequalities in terms of $d(x)$. Given in the next section are notations and geometrical conditions we will utilize from now on.

5.2 Notations and conditions

The purpose of this section is to introduce the geometrical conditions that will help measure the non-convexity of domains under investigation.
Let w be a point in $\mathbb{R}^n$ and ν be a unit vector. For $\alpha \in \left(0, \frac{\pi}{2}\right)$ define

$$C_0(\nu, \alpha) = \{x \in \mathbb{R}^n : x \cdot \nu \geq |x| \cos\alpha\},$$

which is a cone in the Euclidean space $\mathbb{R}^n$ with vertex at 0 and symmetry axis in the ν direction. Denote by $C_w(\nu, \alpha) = C_0(\nu, \alpha) + w$, the transition of $C_0(\nu, \alpha)$ by $w \in \mathbb{R}^n$, i.e.

$$C_w(\nu, \alpha) = \{x \in \mathbb{R}^n : (x - w) \cdot \nu \geq |x - w| \cos\alpha\},$$

which can be seen as an n-dimensional cone with vertex at w and symmetry axis parallel to the ν direction with angle 2α at the vertex.

Now for $h \geq 0$, define the half-space $\mathbf{\Pi}_h(\nu)$ by

$$\mathbf{\Pi}_h(\nu) = \{x \in \mathbb{R}^n : x \cdot \nu \geq h\} .$$

Denote by $\mathbf{\Pi}_{h,w}(\nu) = \mathbf{\Pi}_h(\nu) + w$, the transition of $\mathbf{\Pi}_h(\nu)$ by $w \in \mathbb{R}^n$, i.e.

$$\mathbf{\Pi}_{h,w}(\nu) = \{x \in \mathbb{R}^n : (x - w) \cdot \nu \geq h\} ,$$

which is a half-space of 'height h' from the point w in the ν direction.

Define the region $K_{h,w}(\nu, \alpha)$ to be

$$K_{h,w}(\nu, \alpha) = C_w(\nu, \alpha) \cup \mathbf{\Pi}_{h,w}(\nu).$$

With the notations given above we now state the conditions or 'non-convexity measures' we use throughout the rest of the chapter. These are the $n-$dimensional versions of the conditions stated in Chapter 4. Therefore, they are given exactly as in Chapter 4, taking into account that the domains Ω are in $\mathbb{R}^n$.

Condition 5.2.1. (Exterior Cone Condition)

We say that $\Omega \subset \mathbb{R}^n$ satisfies the Exterior Cone Condition if for each $x \in \Omega$ there exists an element $w \in \partial\Omega$ such that $d(x) = |w - x|$ and $\Omega \subset C^c_w(\nu, \alpha)$, with $(x - w) \cdot \nu = -|x|$.

As an $n-$dimensional generalisation of Condition 4.2.2, we have the following condition:

Condition 5.2.2. (Truncated Cone Region (TCR) Condition)

We say that $\Omega \subset \mathbb{R}^n$ satisfies the TCR Condition if for each $x \in \Omega$ there exists an element $w \in \partial\Omega$ such that $d(x) = |w - x|$ and $\Omega \subset K^c_{h,w}(\nu, \alpha)$, for some $h \geq 0$, with $(x - w) \cdot \nu = -|x|$.

Now we introduce the third non-convexity condition which is a generalisation of Condition 4.2.3. The $n-$dimensional open ball with centre a and radius R is defined by

$$B^n(a,R) = \{y \in \mathbb{R}^n : |y-a| < R\}.$$

Condition 5.2.3. (Exterior Ball Condition)

We say that $\Omega \subset \mathbb{R}^n$ satisfies the Exterior Ball Condition if there exists a number $R > 0$ such that for each $w \in \partial\Omega \subset \mathbb{R}^n$, one can find a point $a \in \mathbb{R}^n$ such that $|w-a| = R$ and

$$B^n(a,R) \cap \Omega = \emptyset.$$

Clearly, the Exterior Ball Condition means that we can touch every point on the boundary $\partial\Omega$ of the domain Ω with a ball of some radius R.

To introduce the cylinder condition in $\mathbb{R}^n$: Let Π be a $k-$dimensional subspace of $\mathbb{R}^n$ and let $R > 0$. We refer to the set

$$\mathcal{Z}\,(\Pi, R) = \{y \in \mathbb{R}^n : \operatorname{dist}(y, \Pi) < R\},$$

as an $(n,k)-$cylinder of radius R.

With the above notations we formulate the $(n,k)-$Cylinder Condition for domains $\Omega \subset \mathbb{R}^n$ as follows:

Condition 5.2.4. ($(n,k)-$Cylinder Condition)

We say that $\Omega \subset \mathbb{R}^n$ satisfies the $(n,k)-$Cylinder Condition if there is a number $R > 0$ such that for each $w \in \partial\Omega$ there exists a $k-$dimensional subspace Π of $\mathbb{R}^n$ such that

$$w \in \mathcal{Z}\,(\Pi, R) \;\text{ and }\; \mathcal{Z}\,(\Pi, R) \cap \Omega = \emptyset.$$

Observe here that the $(n, k)-$Cylinder Condition gives the Exterior Ball Condition for $k = 0$ and is equivalent to the convexity of Ω when $k = n - 1$.
We begin by considering domains $\Omega \in \mathbb{R}^n$, which satisfy one of the 'Cone Conditions'. Afterwards, a study of domains which satisfy the generalised 'Exterior Ball Condition' will follow. Finally, we end this chapter by deriving Hardy type inequalities for domains which satisfy the $(n, k)-$Cylinder Condition.

5.3 Hardy's inequalities under the Cone Conditions

Throughout this section we investigate domains $\Omega \subset \mathbb{R}^n$; $n \geq 3$ which satisfy one of the 'Cone Conditions', Condition 5.2.1 or Condition 5.2.2, in order to obtain some formulas for Hardy-type inequalities. Let us start with the simplest case, where the domain under investigation satisfies Condition 5.2.1.
Before we state and prove the first theorem we note that the symbol $d(x)$ denotes the Euclidean distance from the point $x \in \Omega$ to the boundary $\partial\Omega$, i.e.

$$d(x) := \text{dist}\,(x, \partial\Omega) = \min\left\{|x - y| : y \notin \Omega\right\}. \tag{5.4}$$

By $d_u(x)$ we denote the distance from the point $x \in \Omega$ to the boundary $\partial\Omega$ in the direction u, i.e.

$$d_u(x) := \min\{|s| : x + su \notin \Omega\}, \tag{5.5}$$

and $\tilde{d}_u(x)$ denotes the distance from the point $x \in \Omega$ to the boundary $\partial C_\omega(\nu, \alpha)$ of the $n-$dimensional cone $C_\omega(\nu, \alpha)$ in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial C_\omega(\nu, \alpha)\}.$$

Theorem 5.3.1.

Suppose that the domain $\Omega \subset \mathbb{R}^n$; $n \geq 3$, satisfies Condition 5.2.1. Then for any function $f \in \mathcal{C}_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$\mu(n,\alpha) \int_\Omega \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2 \, dx, \tag{5.6}$$

where

$$\mu(n,\alpha) = \frac{1}{2\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \left(\left((n-1)\cot^2\alpha + 1\right) \int_0^\alpha \sin^{n-2}\theta d\theta - \sin^{n-3}\alpha\cos\alpha \right). \tag{5.7}$$

Proof. By definition (5.2) of the function $\frac{1}{m(x)^2}$ and the fact that $\tilde{d}_u(x) \geq d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{5.8}$$

Because of the definition of $\tilde{d}_u(x)$ and by using spherical coordinates, (r,θ,ϕ) where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, we have $u = (\theta,\phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x,\theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (5.8) becomes

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{1}{|\mathbb{S}^{n-1}|} \int_0^\pi \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta \int_{\mathbb{S}^{n-2}} dw \\
&= 2\frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta.
\end{aligned}$$

However, the angle θ can not exceed the value $\alpha < \frac{\pi}{2}$, hence

$$\frac{1}{m(x)^2} \geq 2\frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} \int_0^\alpha \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta. \tag{5.9}$$

Since Ω satisfies Condition 5.2.1, i.e., we have symmetry with respect to the axis of $C_\omega(\nu,\alpha)$, we consider the two-dimensional cross section that contains the point $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}(x,\theta)$, so we have

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha-\theta)},$$

as derived in (3.9). Thus inequality (5.9) can be rewritten as follows:

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{2\left|\mathbb{S}^{n-2}\right|}{\left|\mathbb{S}^{n-1}\right| d(x)^2 \sin^2\alpha} \int_0^\alpha \sin^2(\alpha-\theta)\sin^{n-2}\theta d\theta \\
&= \frac{\left|\mathbb{S}^{n-2}\right|}{\left|\mathbb{S}^{n-1}\right| d(x)^2 \sin^2\alpha} \int_0^\alpha \left(\sin^{n-2}\theta - \sin^{n-2}\theta\cos 2(\alpha-\theta)\right) d\theta \\
&= \frac{\left|\mathbb{S}^{n-2}\right|}{\left|\mathbb{S}^{n-1}\right| d(x)^2 \sin^2\alpha} \left(\int_0^\alpha \sin^{n-2}\theta d\theta - I_1(\alpha) \right),
\end{aligned} \tag{5.10}$$

where

$$I_1(\alpha) = \int_0^\alpha \sin^{n-2}\theta \cos 2\left(\alpha-\theta\right) d\theta.$$

There are many ways to evaluate $I_1(\alpha)$. One way is to use (C.11) (see Appendix C), which allows us to evaluate $I_1(\alpha)$ as follows: Rewrite $I_1(\alpha)$ as

$$\begin{aligned}
I_1(\alpha) &= \cos 2\alpha \int_0^\alpha \sin^{n-2}\theta\cos 2\theta d\theta + \sin 2\alpha \int_0^\alpha \sin^{n-2}\theta \sin 2\theta d\theta \\
&= \cos 2\alpha \left[\int_0^\alpha \sin^{n-2}\theta\cos^2\theta d\theta - \int_0^\alpha \sin^n\theta d\theta \right] + 2\sin 2\alpha \int_0^\alpha \sin^{n-1}\theta\cos\theta d\theta \\
&= \cos 2\alpha \left[\int_0^\alpha \sin^{n-2}\theta d\theta - 2\int_0^\alpha \sin^n\theta d\theta \right] + \frac{2}{n}\sin 2\alpha \sin^n\alpha \\
&= \cos 2\alpha \left[\int_0^\alpha \sin^{n-2}\theta d\theta + \frac{2}{n}\sin^{n-1}\alpha\cos\alpha - \frac{2n-2}{n}\int_0^\alpha \sin^{n-2}\theta d\theta \right] + \frac{4}{n}\sin^{n+1}\alpha\cos\alpha
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2-n}{n}\cos 2\alpha \int_0^\alpha \sin^{n-2}\theta d\theta + \frac{2}{n}(1-2\sin^2\alpha)\sin^{n-1}\alpha\cos\alpha + \frac{4}{n}\sin^{n+1}\alpha\cos\alpha \\
&= \frac{2-n}{n}\cos 2\alpha \int_0^\alpha \sin^{n-2}\theta d\theta + \frac{2}{n}\sin^{n-1}\alpha\cos\alpha. \qquad (5.11)
\end{aligned}
$$

Thus using (5.11), inequality (5.10) produces the following lower bound on $\frac{1}{m(x)^2}$:

$$
\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{|\mathbb{S}^{n-2}|}{n\, d(x)^2\ |\mathbb{S}^{n-1}|\sin^2\alpha}\left[n\int_0^\alpha \sin^{n-2}\theta d\theta + (n-2)\cos 2\alpha\int_0^\alpha \sin^{n-2}\theta d\theta - 2\sin^{n-1}\alpha\cos\alpha\right] \\
&= \frac{|\mathbb{S}^{n-2}|}{n\, d(x)^2\ |\mathbb{S}^{n-1}|\sin^2\alpha}\left[(n+(n-2)\cos 2\alpha)\int_0^\alpha \sin^{n-2}\theta d\theta - 2\sin^{n-1}\alpha\cos\alpha\right] \\
&= \frac{|\mathbb{S}^{n-2}|}{n\, d(x)^2\ |\mathbb{S}^{n-1}|\sin^2\alpha}\left[2((n-1)\cos^2\alpha+\sin^2\alpha)\int_0^\alpha \sin^{n-2}\theta d\theta - 2\sin^{n-1}\alpha\cos\alpha\right] \\
&= \frac{2\,|\mathbb{S}^{n-2}|}{d(x)^2\, n\,|\mathbb{S}^{n-1}|}\left[((n-1)\cot^2\alpha+1)\int_0^\alpha \sin^{n-2}\theta d\theta - \sin^{n-3}\alpha\cos\alpha\right]. \qquad (5.12)
\end{aligned}
$$

Apply Proposition 5.1.1 to the lower bound (5.12) to obtain the following Hardy-type inequality:

$$
\mu(n,\alpha)\int_\Omega \frac{|f(x)|^2}{d(x)^2}dx \leq \int_\Omega |\nabla f(x)|^2\, dx,
$$

where

$$
\mu(n,\alpha) = \frac{|\mathbb{S}^{n-2}|}{2\,|\mathbb{S}^{n-1}|}\left[((n-1)\cot^2\alpha+1)\int_0^\alpha \sin^{n-2}\theta d\theta - \sin^{n-3}\alpha\cos\alpha\right].
$$

However, relations (C.14) and (C.15) give

$$
\frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} = \frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \qquad (5.13)
$$

Therefore,

$$\mu(n,\alpha)=\frac{1}{2\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\left[\left((n-1)\cot^2\alpha+1\right)\int\limits_0^{\alpha}\sin^{n-2}\theta d\theta-\sin^{n-3}\alpha\cos\alpha\right],$$

this completes the proof. □

Remark 5.3.2.

1. For convex domains we have $\alpha=\frac{\pi}{2}$. In this case, the function $\mu(n,\alpha)$, given by (5.7), becomes

$$\mu\left(n,\frac{\pi}{2}\right)=\frac{1}{2\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\int\limits_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta. \tag{5.14}$$

   However, using the identity (C.9) (see Appendix C) gives

$$\mu\left(n,\frac{\pi}{2}\right)=\tfrac{1}{4}\quad\text{for any } n,$$

   as expected for a convex case.

2. For $n=3$, the function $\mu(n,\alpha)$, given by (5.7), becomes

$$\begin{aligned}\mu(3,\alpha)&=\frac{1}{2\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}\cdot\left[\left(2\cot^2\alpha+1\right)(1-\cos\alpha)-\cos\alpha\right]\\&=\frac{1}{4}\left[\left(\frac{2\cos^2\alpha+1-\cos^2\alpha}{1-\cos^2\alpha}\right)(1-\cos\alpha)-\cos\alpha\right]\\&=\frac{1}{4}\left[\frac{\cos^2\alpha+1-\cos\alpha-\cos^2\alpha}{1+\cos\alpha}\right]=\frac{1}{4}\left[\frac{1-\cos\alpha}{1+\cos\alpha}\right]\\&=\frac{1}{4}\tan^2\frac{\alpha}{2}\end{aligned}$$

   exactly as obtained in (4.7).

For the advantage of 'measuring how deep the dent' inside the domain is, let us consider domains $\Omega\subset\mathbb{R}^n$ that satisfy Condition 5.2.2. In the following theorem, $d(x)$ and $d_u(x)$ are defined as in (5.4) and (5.5) respectively, while $\tilde{d}_u(x)$ is the distance from the point $x \in \Omega$ to the boundary $\partial K_{h,\omega}(\nu, \alpha)$ of the region $K_{h,\omega}(\nu, \alpha)$ in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial K_{h,\omega}(\nu, \alpha)\}.$$

Theorem 5.3.3.

Suppose that the domain $\Omega \subset \mathbb{R}^n$; $n \geq 3$, satisfies Condition 5.2.2. Then for any function $f \in \mathcal{C}_c^\infty(\Omega)$, the following Hardy-type inequality holds:

$$\int_\Omega \mu_1(n, x, \alpha, h) \frac{|f(x)|^2}{(h + d(x))^2} dx + \int_\Omega \mu_2(n, x, \alpha, h) \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2 dx, \quad (5.15)$$

where

$$\mu_1(n, x, \alpha, h) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \left(-\sin^{n-1}\theta_0 \cos\theta_0 + \int_{\theta_0}^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta \right), \quad \text{and} \quad (5.16)$$

$$\begin{aligned}\mu_2(n, x, \alpha, h) \quad = \quad & \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{\sin^2\alpha} \left(\left((n-1)\cot^2\alpha + 1\right) \int_0^{\theta_0} \sin^{n-2}\theta d\theta \right. \\ & \left. - \sin^{n-1}\theta_0 \cos(2\alpha - \theta_0) \right), \quad (5.17)\end{aligned}$$

where $\theta_0 = \theta_0(x, \alpha, h)$ satisfies (3.8). *In particular, when $\alpha = \frac{\pi}{2}$, we have $\mu_1(n, x, \alpha, h) = 0$ and $\mu_2(n, x, \alpha, h) = \frac{1}{4}$.*

Proof. By (5.2), the definition of the function $\frac{1}{m(x)^2}$, and the relation between $\tilde{d}_u(x)$ and $d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2} dS(u) \geq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2} dS(u). \quad (5.18)$$

Using spherical coordinates leads to $u = u(\theta, \phi)$, and $\tilde{d}_u(x)$ depending on θ only. Thus, slightly abusing the notation, we write $\tilde{d}(x,\theta)$ instead of $\tilde{d}_u(x)$. Consequently, inequality (5.18) becomes

$$\begin{aligned}\frac{1}{m(x)^2} &\geq \frac{1}{|\mathbb{S}^{n-1}|}\int_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta \int_{\mathbb{S}^{n-2}} dw \\ &= 2\frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|}\int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta. \qquad (5.19)\end{aligned}$$

Since Ω satisfies Condition 5.2.2, we consider the cross section containing the point $x \in \Omega$ and the line segments representing $d(x)$ and $\tilde{d}(x,\theta)$, then according to the relation between $\tilde{d}(x,\theta)$ and $d(x)$, we can rewrite inequality(5.19) as follows:

$$\frac{1}{m(x)^2} \geq 2b\left[I_1(n,\theta_0) + I_2(n,\theta_0)\right]; \;\; b = \frac{|\mathbb{S}^{n-2}|}{|\mathbb{S}^{n-1}|} = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}, \qquad (5.20)$$

where

$$I_1(n,\theta_0) = \int_0^{\theta_0} \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta,$$

$$I_2(n,\theta_0) = \int_{\theta_0}^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2}\sin^{n-2}\theta d\theta,$$

and $0 \leq \theta_0 < \frac{\pi}{2}$ satisfies

$$\tan\theta_0 = \frac{h}{h+d(x)}\tan\alpha.$$

However, for all angles $\alpha < \frac{\pi}{2}$, recall that: For $\theta \in [0,\theta_0)$, the relation between $\tilde{d}(x,\theta)$ and θ is

$$\tilde{d}(x,\theta) = \frac{d(x)\sin\alpha}{\sin(\alpha-\theta)},$$

and for $\theta \in [\theta_0, \frac{\pi}{2})$, we have

$$\tilde{d}(x,\theta) = \frac{h + d(x)}{\cos\theta}.$$

On the other hand, for $\alpha = \frac{\pi}{2}$ the relation between $\tilde{d}(x,\theta)$ and θ is

$$\tilde{d}(x,\theta) = \frac{d(x)}{\cos\theta}.$$

Therefore, we can evaluate the first integral $I_1(n,\theta_0)$ as follows:

$$\begin{aligned} I_1(n,\theta_0) &= \frac{1}{d(x)^2 \sin^2\alpha} \int_0^{\theta_0} \sin^2(\alpha-\theta)\sin^{n-2}\theta d\theta \\ &= \frac{1}{2d(x)^2 \sin^2\alpha} \left(\int_0^{\theta_0} \sin^{n-2}\theta d\theta - I_3(n,\theta_0) \right), \end{aligned} \tag{5.21}$$

where

$$I_3(n,\theta_0) = \int_0^{\theta_0} \sin^{n-2}\theta \cos 2\,(\alpha-\theta)\, d\theta.$$

Using (C.11) and (C.12) (see Appendix C) we can rewrite $I_3(n,\theta_0)$ as

$$\begin{aligned} I_3(n,\theta_0) &= \cos 2\alpha \int_0^{\theta_0} \sin^{n-2}\theta \cos 2\theta d\theta + \sin 2\alpha \int_0^{\theta_0} \sin^{n-2}\theta \sin 2\theta d\theta \\ &= \cos 2\alpha \left[\int_0^{\theta_0} \sin^{n-2}\theta \cos^2\theta d\theta - \int_0^{\theta_0} \sin^n\theta d\theta \right] + 2\sin 2\alpha \int_0^{\theta_0} \sin^{n-1}\theta\cos\theta d\theta \\ &= \cos 2\alpha \left[\frac{2}{n}\sin^{n-1}\theta_0\cos\theta_0 + \frac{2-n}{n}\int_0^{\theta_0} \sin^{n-2}\theta d\theta \right] + \frac{2}{n}\sin 2\alpha \sin^n\theta_0 \ . \end{aligned} \tag{5.22}$$

Substituting (5.22) into (5.21) produces

$$
\begin{aligned}
I_1(n,\theta_0) &= \frac{1}{2d(x)^2\,\sin^2\alpha}\left(\int_0^{\theta_0}\sin^{n-2}\theta d\theta-\frac{2}{n}\cos 2\alpha\sin^{n-1}\theta_0\cos\theta_0\right.\\
&\qquad\left.-\frac{2-n}{n}\cos 2\alpha\int_0^{\theta_0}\sin^{n-2}\theta d\theta-\frac{2}{n}\sin 2\alpha\sin^{n}\theta_0\right)\\
&= \frac{1}{nd(x)^2\,\sin^2\alpha}\left(\frac{1}{2}\left(n\left(1+\cos 2\alpha\right)-2\cos 2\alpha\right)\int_0^{\theta_0}\sin^{n-2}\theta d\theta\right.\\
&\qquad\left.-\cos 2\alpha\sin^{n-1}\theta_0\cos\theta_0-\sin 2\alpha\sin^{n}\theta_0\right)\\
&= \frac{1}{nd(x)^2\,\sin^2\alpha}\left(\left(n\cos^2\alpha-\cos 2\alpha\right)\int_0^{\theta_0}\sin^{n-2}\theta d\theta\right.\\
&\qquad\left.-\sin^{n-1}\theta_0(\cos 2\alpha\cos\theta_0+\sin 2\alpha\sin\theta_0)\right)\\
&= \frac{1}{nd(x)^2}\left(\left((n-1)\cot^2\alpha+1\right)\int_0^{\theta_0}\sin^{n-2}\theta d\theta\right.\\
&\qquad\left.-\sin^{n-1}\theta_0\frac{\cos\left(2\alpha-\theta_0\right)}{\sin^2\alpha}\right). \qquad (5.23)
\end{aligned}
$$

Concerning $I_2(n,\theta_0)$, use (C.12) (see Appendix C) to have

$$
\begin{aligned}
I_2(n,\theta_0) &= \frac{1}{\left(h+d(x)\right)^2}\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta\cos^2\theta d\theta\\
&= \frac{1}{\left(h+d(x)\right)^2}\left[\left.\frac{\sin^{n-1}\theta\cos\theta}{n}\right|_{\theta_0}^{\frac{\pi}{2}}+\frac{1}{n}\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta\right]\\
&= \frac{1}{n\left(h+d(x)\right)^2}\left[-\sin^{n-1}\theta_0\cos\theta_0+\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta\right]. \qquad (5.24)
\end{aligned}
$$

Therefore, substituting (5.24) and (5.23) into (5.20) gives the following lower bound on the function $\frac{1}{m(x)^2}$:

$$\begin{aligned}\frac{1}{m(x)^2} \geq \frac{2b}{n}\Bigg[\frac{1}{d(x)^2}\Bigg(&\left((n-1)\cot^2\alpha+1\right)\int_0^{\theta_0}\sin^{n-2}\theta d\theta \\ &-\sin^{n-1}\theta_0\frac{\cos(2\alpha-\theta_0)}{\sin^2\alpha}\Bigg) \\ &+\frac{1}{(h+d(x))^2}\left(\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta-\sin^{n-1}\theta_0\cos\theta_0\right)\Bigg].\end{aligned} \tag{5.25}$$

We apply Proposition 5.1.1 to the lower bound (5.25) to obtain the following Hardy-type inequality:

$$\int_\Omega \mu_1(n,x,\alpha,h)\frac{|f(x)|^2}{(h+d(x))^2}dx+\int_\Omega \mu_2(n,x,\alpha,h)\frac{|f(x)|^2}{d(x)^2}dx \leq \int_\Omega |\nabla f(x)|^2\,dx,$$

where

$$\mu_1(n,x,\alpha,h)=\frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(\int_{\theta_0}^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta-\sin^{n-1}\theta_0\cos\theta_0\right),$$

and

$$\begin{aligned}\mu_2(n,x,\alpha,h) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\Bigg(&\left((n-1)\cot^2\alpha+1\right)\int_0^{\theta_0}\sin^{n-2}\theta d\theta \\ &-\sin^{n-1}\theta_0\frac{\cos(2\alpha-\theta_0)}{\sin^2\alpha}\Bigg),\end{aligned}$$

as stated in (5.16) and (5.17) respectively. On the other hand, when $\psi=\alpha=\frac{\pi}{2}$, we have

$\theta_0 = \frac{\pi}{2}$ as well, this implies

$$\begin{aligned}
\mu_1(n,x,\alpha,h) &= 0, \text{ and} \\
\mu_2(n,x,\alpha,h) &= \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta \\
&= \frac{1}{4} \quad \text{for any } n.
\end{aligned}$$

This completes the proof. □

Remark 5.3.4.

1. If Ω is a convex domain then $\psi = \alpha = \theta_0 = \frac{\pi}{2}$. Thus the lower bound (5.25) on the function $m(x)$ reproduces the known bound,

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{2}{n} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \cdot \left[\frac{1}{d(x)^2} \int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta\right] \\
&= \frac{2}{nd(x)^2} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)} = \frac{1}{n\, d(x)^2},
\end{aligned}$$

as expected from Theorem 2.2.7. Consequently, the Hardy-type inequality (5.15) reproduces the well-known bound (see Theorem 2.2.6)

$$\frac{1}{4}\int_\Omega \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2\, dx, \tag{5.26}$$

for any convex domain $\Omega \subset \mathbb{R}^n$.

2. When $\alpha \nearrow \frac{\pi}{2}$, the domain Ω approaches the convexity case. Therefore, it is natural to compare $\mu_1(n,x,\alpha,h)$ and $\mu_2(n,x,\alpha,h)$, given by (5.16) and (5.17) respectively, with their values for the convex case. Keeping in mind that when $\alpha = \frac{\pi}{2}$ we set $\theta_0 = \theta_0(x,\alpha) = \tan^{-1}(a(x)\tan\alpha) = \frac{\pi}{2}$, and for fixed h, expressions for $\mu_1(n,x,\alpha,h)$

and $\mu_2(n,x,\alpha,h)$ can be written as powers of $\left(\alpha-\frac{\pi}{2}\right)$. Since

$$\mu_1\left(n,x,\frac{\pi}{2},h\right)=0.$$

In addition,

$$\frac{\partial}{\partial\alpha}\mu_1(n,x,\alpha,h)=\frac{a(x)}{\cos^2\alpha+a(x)^2\sin^2\alpha}\left[\sin^n\theta_0-\sin^{n-2}\theta_0\left((n-1)\cos^2\theta_0+1\right)\right],$$

which implies that

$$\frac{\partial}{\partial\alpha}\mu_1\left(n,x,\frac{\pi}{2},h\right)\;=\;0.$$

Besides,

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2}\mu_1(n,x,\alpha,h)=\frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\Bigg\{&\frac{a(x)^2}{\left(\cos^2\alpha+a(x)^2\sin^2\alpha\right)^2}\left\{n\sin^{n-1}\theta_0\cos\theta_0\right.\\&+(n-1)\sin^{n-2}\theta_0\sin2\theta_0\\&\left.-(n-2)\sin^{n-3}\theta_0\left[(n-1)\cos^3\theta_0+\cos\theta_0\right]\right\}\\-&\frac{a(x)\left(a(x)^2-1\right)\sin2\alpha}{\left(\cos^2\alpha+a(x)^2\sin^2\alpha\right)^2}\left\{\sin^n\theta_0-\sin^{n-2}\theta_0\left[(n-1)\cos^2\theta_0+1\right]\right\}\Bigg\},\end{aligned}$$

which gives

$$\frac{\partial^2}{\partial\alpha^2}\mu_1\left(n,x,\frac{\pi}{2},h\right)=0.$$

Similarly, we can find

$$\frac{\partial^3}{\partial\alpha^3}\mu_1\left(n,x,\frac{\pi}{2},h\right)=-n\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{1}{a(x)^3},\;\cdots\;\text{and so on.}$$

Therefore, the function $\mu_1(n,x,\alpha,h)$ can be written as follows:

$$\mu_1(n,x,\alpha,h)=\frac{n\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\frac{(h+d(x))^3}{6h^3}\left(\frac{\pi}{2}-\alpha\right)^3+\mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^4\right). \tag{5.27}$$

On the other hand, for $\mu_2\left(n,x,\frac{\pi}{2},h\right)$, we have

$$\mu_2\left(n,x,\frac{\pi}{2},h\right)=\frac{1}{4},\ \frac{\partial}{\partial\alpha}\mu_2\left(n,x,\frac{\pi}{2},h\right)=\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)},\ \cdots \text{ and so on.}$$

Thus, $\mu_2(n,x,\alpha,h)$ can be written as follows:

$$\mu_2(n,x,\alpha,h)=\frac{1}{4}+\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(\alpha-\frac{\pi}{2}\right)+\mathcal{O}\left(\left(\alpha-\frac{\pi}{2}\right)^2\right). \tag{5.28}$$

Notice here that, when $\alpha=\frac{\pi}{2}$, the expressions (5.27) and (5.28) lead to the same inequality as obtained in (5.26). Relations (5.27) and (5.28) show that the second term in Hardy-type inequality (5.15) is the effective term when talking about the convex case, since $\mu_1(n,x,\alpha,h)$ tends to zero while $\mu_2(n,x,\alpha,h)$ tends to $\frac{1}{4}$ as α tends to $\frac{\pi}{2}$.

3. For fixed α, as h tends to ∞, $a(x)$ tends to 1, which means implicitly that θ_0 tends to α. Therefore, we obtain the following limit for $\mu_2(n,x,\alpha,h)$ as h tends to ∞:

$$\begin{aligned}\lim_{h\to\infty}\mu_2(n,x,\alpha,h) &= \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\left(\left((n-1)\cot^2\alpha+1\right)\int_0^\alpha\sin^{n-2}\theta d\theta\right.\\ &\qquad\left.-\sin^{n-3}\alpha\cos(\alpha)\right)=\mu(n,\alpha).\end{aligned} \tag{5.29}$$

Since all functions (f,μ_1,μ_2) are uniformly bounded, we can pass to the limit under the integral, thus the first term in Hardy-type inequality (5.15) tends to zero and we obtain the following inequality:

$$\mu(n, \alpha) \int_{\Omega} \frac{|f(x)|^2}{d(x)^2} dx \leq \int_{\Omega} |\nabla f(x)|^2 \, dx,$$

exactly as obtained in Theorem 5.3.1. On the other hand, as h tends to 0, $a(x)$ tends to 0, which leads to the tendency of θ_0 to 0 as well. This implies that $\mu_1(n, x, \alpha, h) \to \frac{1}{4}$ (use (C.9), see Appendix C) and $\mu_2(n, x, \alpha, h) \to 0$.

5.4 Hardy's inequalities under the Exterior Ball Condition

Throughout this section we consider domains $\Omega \subset \mathbb{R}^n$; $n \geq 3$, which fulfill the Exterior Ball Condition, and obtain a new Hardy-type inequality for such non-convex domains. We have not been able to find a simple analytic expression for the function $\mu(x, R)$, however, we content ourselves with the asymptotic result stated in the following theorem.

We note that $d(x)$ denotes the Euclidean distance from the point $x \in \Omega$ to the boundary $\partial\Omega$, i.e.

$$d(x) := \text{dist}\,(x, \partial\Omega) = \min \{|x - y| : y \notin \Omega\},$$

and $d_u(x)$ refers to the distance from the point $x \in \Omega$ to the boundary $\partial\Omega$ in the direction u, i.e.

$$d_u(x) := \min\{|s| : x + su \notin \Omega\}.$$

Furthermore, $\tilde{d}_u(x)$ is for the distance from the point $x \in \Omega$ to the boundary $\partial B^n(a, R)$ in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial B^n(a, R)\}.$$

Theorem 5.4.1.

Suppose $\Omega \subset \mathbb{R}^n$; $n \geq 3$, satisfies Condition 5.2.3. Then there exists a positive function

$\mu = \mu(x, R)$ such that for any function $f \in \mathcal{C}_c^{\infty}(\Omega)$ the following Hardy-type inequality holds:

$$\int_{\Omega} \mu(x,R)\frac{|f(x)|^2}{d(x)^2}dx \leq \int_{\Omega} |\nabla f(x)|^2\, dx, \tag{5.30}$$

if x is such that $\frac{d(x)}{R} \leq \epsilon$ with some $\epsilon \in (0,1)$, then

$$\mu(x,R) = \frac{1}{4} - \left(\frac{n-1}{4}\right)\frac{d(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right). \tag{5.31}$$

In particular, if the inradius $\delta_{\text{in}} < R$, then

$$\mu(x,R) \geq \frac{1}{4} - \left(\frac{n-1}{4}\right)\frac{\delta_{\text{in}}}{R} + \mathcal{O}\left(\left(\frac{\delta_{\text{in}}}{R}\right)^{3/2}\right), \tag{5.32}$$

uniformly for all $x \in \Omega$.

Proof. By (5.2), the definition of the function $\frac{1}{m(x)^2}$, and the relation between $d(x)$ and $\tilde{d}_u(x)$, we obtain

$$\begin{aligned}\frac{1}{m(x)^2} &= \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2}dS(u)\\ &\geq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2}dS(u) = \frac{1}{|\mathbb{S}^{n-1}|} \times J_n(x),\end{aligned} \tag{5.33}$$

where

$$J_n(x) = \int_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2}dS(u). \tag{5.34}$$

Now the aim is to estimate $J_n(x)$: Using spherical coordinates leads to $u = u(\theta, \phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write

$\tilde{d}(x,\theta)$ instead of $\tilde{d}_u(x)$. Therefore, $J_n(x)$ can be rewritten as follows:

$$\begin{aligned} J_n(x) &= \int_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta \int_{\mathbb{S}^{n-2}} d\omega \\ &= \left|\mathbb{S}^{n-2}\right| \int_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta \\ &= 2\left|\mathbb{S}^{n-2}\right| \int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta. \end{aligned}$$

Since $\Omega \subset \mathbb{R}^n$ satisfies Condition 5.2.3, and using (3.27), the relation between $\tilde{d}_u(x)$ and θ, implies that

$$\begin{aligned} J_n(x) &= 2\left|\mathbb{S}^{n-2}\right| \int_0^{\theta_0} \frac{\sin^{n-2}\theta}{\cos^2\theta\left(R + d(x) - \sqrt{R^2 - \left(d(x)^2 + 2Rd(x)\right)\tan^2\theta}\right)^2} d\theta \\ &= \frac{2\left|\mathbb{S}^{n-2}\right|}{R^2} \int_0^{\theta_0} \frac{\sin^{n-2}\theta}{\cos^2\theta\left(1 + b(x) - \sqrt{1 - (b(x)^2 + 2b(x))\tan^2\theta}\right)^2} d\theta, \end{aligned} \tag{5.35}$$

where $b(x) = \frac{d(x)}{R}$ and $\theta_0 \in (0, \frac{\pi}{2})$ satisfies (3.24) , i.e. $\sin\theta_0 = \frac{1}{1+b(x)}$. The integrand in (5.35) can be simplified as follows: multiplying both the numerator and the denominator of the integrand by $\left(1 + b(x) + \sqrt{1 - (b(x)^2 + 2b(x))\tan^2\theta}\right)^2$, gives

$$\begin{aligned} J_n(x) &= \frac{2\left|\mathbb{S}^{n-2}\right|}{R^2} \int_0^{\theta_0} \frac{\sin^{n-2}\theta\left(1 + b(x) + \sqrt{1 - (b(x)^2 + 2b(x))\tan^2\theta}\right)^2}{\cos^2\theta\left((1 + b(x))^2 - (1 - (b(x)^2 + 2b(x))\tan^2\theta)\right)^2} d\theta \\ &= \frac{2\left|\mathbb{S}^{n-2}\right|}{R^2} \int_0^{\theta_0} \frac{\sin^{n-2}\theta\left(1 + b(x) + \sqrt{1 - (b(x)^2 + 2b(x))\tan^2\theta}\right)^2}{\cos^2\theta\left(1 + 2b(x) + b(x)^2 - 1 + (b(x)^2 + 2b(x))\tan^2\theta\right)^2} d\theta \end{aligned}$$

$$
\begin{aligned}
&= \frac{2\left|\mathbb{S}^{n-2}\right|}{R^2\left(2b(x)+b(x)^2\right)^2}\int_0^{\theta_0}\frac{\sin^{n-2}\theta\left(1+b(x)+\sqrt{1-\left(b(x)^2+2b(x)\right)\tan^2\theta}\right)^2}{\cos^2\theta\left(1+\tan^2\theta\right)^2}d\theta \\
&= \frac{2\left|\mathbb{S}^{n-2}\right|}{R^2} \times \\
&\quad \underbrace{\frac{1}{\left(2b(x)+b(x)^2\right)^2}\int_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\left(1+b(x)+\sqrt{1-\left(b(x)^2+2b(x)\right)\tan^2\theta}\right)^2 d\theta}_{I_1(b(x))}
\end{aligned}
\tag{5.36}
$$

Now let us estimate $I_1(b(x))$: since $\theta_0 \in (0, \frac{\pi}{2})$ then we can write $\theta_0 = \frac{\pi}{2} - \tau_1$ where $c_1\sqrt{b(x)} \leq \tau_1 \leq c_2\sqrt{b(x)}$ with some positive constants $c_1, c_2; c_1 < c_2$. Moreover, since

$$
0 \leq \left(b(x)^2 + 2b(x)\right)\tan^2\theta \leq 1,
$$

then using the expansion

$$
\sqrt{1-t} = 1 - \frac{t}{2} + \mathcal{O}(t^2); \quad 0 \leq t \leq 1,
$$

the integral $I_1(b(x))$ can be rewritten as

$$
\begin{aligned}
I_1(b(x)) &= \frac{1}{\left(2b(x)+b(x)^2\right)^2}\int_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\Bigg(1+b(x) \\
&\qquad +1-\left(\frac{b(x)^2}{2}+b(x)\right)\tan^2\theta+\mathcal{O}\left(\left(b(x)\tan^2\theta\right)^2\right)\Bigg)^2 d\theta \\
&= \frac{1}{\left(2b(x)+b(x)^2\right)^2}\int_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta \times \\
&\qquad \left(2+b(x)-\frac{b(x)}{2}(b(x)+2)\tan^2\theta+\mathcal{O}\left(\left(b(x)\tan^2\theta\right)^2\right)\right)^2 d\theta \\
&= \frac{1}{b(x)^2\left(b(x)+2\right)^2}\int_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\left((b(x)+2)(1-\frac{b(x)}{2}\tan^2\theta)+\mathcal{O}\left(\left(b(x)\tan^2\theta\right)^2\right)\right)^2 d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{b(x)^2}\int_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\left(1-b(x)\tan^2\theta\right)d\theta \\
&\quad \underbrace{+\frac{1}{b(x)^2\left(b(x)+2\right)^2}\int_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\ \mathcal{O}\left(\left(b(x)\tan^2\theta\right)^2\right)d\theta}_{I_2(b(x))}. \qquad (5.37)
\end{aligned}
$$

The second term in (5.37), $I_2(b(x))$, can be estimated as follows: since $\frac{1}{b(x)^2(b(x)+2)^2} \le \frac{1}{b(x)^2}$, and $|\sin\theta| \le 1$, then

$$
I_2(b(x)) \le \frac{1}{b(x)^2}\int_0^{\theta_0} b(x)^2\frac{\sin^{n+2}\theta}{\cos^2\theta}d\theta \le \int_0^{\theta_0}\frac{1}{\cos^2\theta}d\theta = \int_0^{\theta_0}\frac{1}{\sin^2\left(\frac{\pi}{2}-\theta\right)}d\theta.
$$

However, for $0 \le x \le \frac{\pi}{2}$, we have $\sin x \ge \frac{2x}{\pi}$, hence,

$$
I_2(b(x)) \le \frac{\pi^2}{4}\int_0^{\theta_0}\frac{1}{\left(\frac{\pi}{2}-\theta\right)^2}d\theta \le \frac{\pi^2}{4}\cdot\frac{1}{\underbrace{\frac{\pi}{2}-\theta_0}_{\tau_1}} \le \frac{\pi^2}{4}\cdot\frac{1}{c_1\sqrt{b(x)}} = \mathcal{O}\left(\frac{1}{\sqrt{b(x)}}\right).
$$

Thus, the integral $I_1(b(x))$ takes the following form:

$$
\begin{aligned}
I_1(b(x)) &= \frac{1}{b(x)^2}\int_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta d\theta - \frac{1}{b(x)}\int_0^{\theta_0}\sin^n\theta d\theta + \mathcal{O}\left(\frac{1}{\sqrt{b(x)}}\right) \\
&= \frac{1}{b(x)^2}\int_0^{\frac{\pi}{2}}\cos^2\theta\sin^{n-2}\theta d\theta - \frac{1}{b(x)^2}\int_{\theta_0}^{\frac{\pi}{2}}\cos^2\theta\sin^{n-2}\theta d\theta \\
&\quad -\frac{1}{b(x)}\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta + \frac{1}{b(x)}\int_{\theta_0}^{\frac{\pi}{2}}\sin^n\theta d\theta + \mathcal{O}\left(\frac{1}{\sqrt{b(x)}}\right).
\end{aligned}
$$

However, the second and fourth terms in the above equation have the following estimates: since for any $\theta \in (0,\theta_0)$ we have $\theta = \frac{\pi}{2} - \tau_2$ with $c_1\sqrt{b(x)} \le \tau_2 \le \tau_1 \le c_2\sqrt{b(x)}$ which implies $\cos\theta = \sin\tau_2 \leq c_2\sqrt{b(x)}$, therefore we obtain $\cos^2\theta \leq cb(x)$, then

$$\left|\frac{1}{b(x)^2}\int_{\theta_0}^{\frac{\pi}{2}} \cos^2\theta \sin^{n-2}\theta d\theta\right| \leq \frac{cb(x)}{b(x)^2}\int_{\theta_0}^{\frac{\pi}{2}} d\theta = \frac{c}{b(x)}\left(\frac{\pi}{2}-\theta_0\right) = \frac{c\tau_1}{b(x)} \leq \frac{cc_2}{\sqrt{b(x)}} = \mathcal{O}\left(\frac{1}{\sqrt{b(x)}}\right),$$

and,

$$\left|\frac{1}{b(x)}\int_{\theta_0}^{\frac{\pi}{2}} \sin^n\theta d\theta\right| \leq \frac{1}{b(x)}\int_{\theta_0}^{\frac{\pi}{2}} d\theta = \frac{1}{b(x)}\left(\frac{\pi}{2}-\theta_0\right) \leq \mathcal{O}\left(\frac{1}{\sqrt{b(x)}}\right).$$

Therefore, $I_1(b(x))$ becomes

$$\begin{aligned} I_1(b(x)) &= \frac{1}{b(x)^2}\int_0^{\frac{\pi}{2}} \cos^2\theta\sin^{n-2}\theta d\theta - \frac{1}{b(x)}\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta + \mathcal{O}\left(\frac{1}{\sqrt{b(x)}}\right) \\ &= \frac{1}{b(x)^2}\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta - \left(\frac{1}{b(x)^2}+\frac{1}{b(x)}\right)\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta + \mathcal{O}\left(\frac{1}{\sqrt{b(x)}}\right) \end{aligned} \tag{5.38}$$

Thus, using (5.38) in (5.36), we have the following form for $J_n(x)$:

$$\begin{aligned} J_n(x) &= \frac{2\left|\mathbb{S}^{n-2}\right|}{R^2}\left[\frac{1}{b(x)^2}\left(\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta - \int_0^{\frac{\pi}{2}}\sin^n\theta d\theta\right) - \frac{1}{b(x)}\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta + \mathcal{O}\left(\frac{1}{\sqrt{b(x)}}\right)\right] \\ &= \frac{2\left|\mathbb{S}^{n-2}\right|}{R^2 b(x)^2}\left[\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta - \int_0^{\frac{\pi}{2}}\sin^n\theta d\theta - b(x)\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta + \mathcal{O}\left(b(x)^{\frac{3}{2}}\right)\right] \\ &= 2\frac{\left|\mathbb{S}^{n-2}\right|}{d(x)^2}\left[\int_0^{\frac{\pi}{2}}\sin^{n-2}\theta d\theta - \int_0^{\frac{\pi}{2}}\sin^n\theta d\theta - \frac{d(x)}{R}\int_0^{\frac{\pi}{2}}\sin^n\theta d\theta + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right]. \end{aligned} \tag{5.39}$$

On the other hand, using the identity (C.9) (see Appendix C) leads to the following:

$$\begin{aligned} J_n(x) &= \frac{2}{d(x)^2}\cdot\frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}-\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}-\frac{d(x)}{R}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\right. \\ &\qquad\left.+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right] \\ &= \frac{2}{d(x)^2}\cdot\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\left[1-\frac{n-1}{n}-\frac{d(x)}{R}\cdot\frac{n-1}{n}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right] \\ &= \frac{1}{n\,d(x)^2}\cdot\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\left[1-(n-1)\cdot\frac{d(x)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right]. \end{aligned} \tag{5.40}$$

Now use (5.40) in (5.33) to obtain the following lower bound on the function $\frac{1}{m(x)^2}$:

$$\frac{1}{m(x)^2} \geq \frac{1}{n\,d(x)^2}\left[1-(n-1)\cdot\frac{d(x)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right]. \tag{5.41}$$

Applying Proposition 5.1.1 to (5.41) leads to the following Hardy-type inequality:

$$\int_{\Omega}\mu(x,R)\frac{|f(x)|^2}{d(x)^2}dx \leq \int_{\Omega}|\nabla f|^2\,dx,$$

where

$$\mu(x,R)=\frac{1}{4}-\left(\frac{n-1}{4}\right)\frac{d(x)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right),$$

as stated in (5.31). On the other hand, (5.32) follows immediately from (5.31) since $d(x) \leq \delta_{\text{in}}$, this completes the proof. □

Remark 5.4.2.

1. For the three-dimensional case, the asymptotic relation (5.31) gives

$$\mu(x,R)=\frac{1}{4}-\frac{1}{2}\frac{d(x)}{R}+\mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right),$$

exactly as obtained in (4.36).

2. As R tends to ∞, the domain Ω approaches convexity. In this case, the lower bound (5.41) on $\frac{1}{m(x)^2}$ gives
$$\frac{1}{m(x)^2} \geq \frac{1}{nd(x)^2},$$
as expected from Theorem 2.2.7.

5.5 Hardy's inequalities under the $(n,k)-$Cylinder Condition

This section is dedicated to obtaining a formula for a Hardy-type inequality in $n-$dimensional domains Ω in $\mathbb{R}^n$; $n \geq 3$, which fulfill the $(n,k)-$Cylinder Condition. To prove our next theorem we need the following proposition.

Proposition 5.5.1.
Let $n-k \geq 2$. For any function $f(\zeta) = f(\zeta_1, \zeta_2)$ of the angular variable $\zeta \in \mathbb{S}^{n-1}$, where ζ_1 and ζ_2 are projections of ζ on the $(n-k)$-dimensional subspace V (which we identify with $\mathbb{R}^{n-k}$) and its orthogonal complement (which we identify with $\mathbb{R}^k$) respectively, the following formula holds:

$$\int_{\mathbb{S}^{n-1}} f(\zeta)d\zeta = \int_{\mathbb{S}^{k-1}} \int_{\mathbb{S}^{n-k-1}} \int_0^{\frac{\pi}{2}} f(\eta \sin\phi, \xi\cos\phi) \sin^{n-k-1}\phi \cos^{k-1}\phi \, d\phi \, d\eta \, d\xi.$$

Proof. Let $g(x) = f(x|x|^{-1})$ for all non-zero $x \in \mathbb{R}^n$. We use the following formula:

$$n^{-1}\int_{\mathbb{S}^{n-1}} f(\zeta)d\zeta = \int_{|x|<1} g(x)dx =: I.$$

We consider $x = (y, z)$, with $y \in \mathbb{R}^{n-k}$ and $z \in \mathbb{R}^k$, so that

$$I = \int_{|z|<1} \int_{|y|<\sqrt{1-|z|^2}} g(y, z)\, dy\, dz,$$

and introduce spherical coordinates:

$$y = (\rho, \eta), z = (t, \xi), \rho = |y|, t = |z|, \eta \in \mathbb{S}^{n-k-1}, \xi \in \mathbb{S}^{k-1}.$$

Thus

$$I = \int_{\mathbb{S}^{k-1}} \int_{\mathbb{S}^{n-k-1}} \int_{t<1} \int_{\rho<\sqrt{1-t^2}} g(\rho\eta, t\xi) \rho^{n-k-1} t^{k-1}\, d\rho\, dt\, d\eta\, d\xi.$$

Now we view the variables (t, ρ) as coordinates on the plane and introduce the polar coordinates:

$$v = \sqrt{\rho^2 + t^2}, \rho = v \sin\phi, t = v\cos\phi, \phi \in (0, \pi/2),$$

so

$$I = \int_{\mathbb{S}^{k-1}} \int_{\mathbb{S}^{n-k-1}} \int_0^1 v^{n-1} \int_0^{\frac{\pi}{2}} g(v\eta\sin\phi, v\xi\cos\phi) \sin^{n-k-1}\phi \cos^{k-1}\phi\, dv\, d\phi\, d\eta\, d\xi.$$

By the definition of g,

$$g(v\eta\sin\phi, v\xi\cos\phi) = f(\eta\sin\phi, \xi\cos\phi),$$

and hence this function is independent of v. Integrating in v, we get

$$I = \frac{1}{n} \int_{\mathbb{S}^{k-1}} \int_{\mathbb{S}^{n-k-1}} \int_0^{\frac{\pi}{2}} f(\eta\sin\phi, \xi\cos\phi) \sin^{n-k-1}\phi \cos^{k-1}\phi\, d\phi\, d\eta\, d\xi,$$

which leads to the required formula. $\square$

Now let us extract the Hardy-type inequality for domains that satisfy the $(n, k)-$Cylinder

Condition. The formula of such an inequality is stated in the following theorem in which we content ourselves with the asymptotic form of the function $\mu(x, R)$. Before we proceed, recall that we use the symbol $d(x)$ to denote the Euclidean distance from $x \in \Omega$ to $\partial\Omega$, as in (5.4), $d_u(x)$ to refer to the distance from $x \in \Omega$ to $\partial\Omega$ in the direction u, as in (5.5), and $\tilde{d}_u(x)$ to denote the distance from $x \in \Omega$ to the boundary $\partial\mathcal{Z}(\Pi, R)$ of the $(n,k)-$cylinder $\mathcal{Z}(\Pi, R)$ in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial\mathcal{Z}(\Pi, R)\}.$$

Theorem 5.5.2.

Suppose that the domain $\Omega \subset \mathbb{R}^n$; $n \geq 3$, satisfies Condition 5.2.4. Then there exists a positive function $\mu = \mu(x, R)$ such that for any function $f \in \mathcal{C}_c^\infty(\Omega)$ the following Hardy-type inequality holds:

$$\int_\Omega \mu(x, R) \frac{|f(x)|^2}{d(x)^2} dx \leq \int_\Omega |\nabla f(x)|^2 dx, \tag{5.42}$$

if x is such that $\frac{d(x)}{R} \leq \epsilon$ with some $\epsilon \in (0,1)$, then

$$\mu(x, R) = \frac{1}{4} - \left(\frac{n-k-1}{4}\right)\frac{d(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right); \quad n-k \geq 2. \tag{5.43}$$

In particular, if the inradius $\delta_{\text{in}} < R$, then

$$\mu(x, R) \geq \frac{1}{4} - \left(\frac{n-k-1}{4}\right)\frac{\delta_{\text{in}}}{R} + \mathcal{O}\left(\left(\frac{\delta_{\text{in}}}{R}\right)^{3/2}\right); \quad n-k \geq 2. \tag{5.44}$$

uniformly for all $x \in \Omega$.

Proof. A lower bound for the function $\frac{1}{m(x)^2}$, defined in (5.2), is sought. Since $\tilde{d}_u(x) \geq$

$d_u(x)$, then we have

$$\frac{1}{m(x)^2} = \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2}dS(u) \geq \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2}dS(u). \tag{5.45}$$

Now let us consider an $(n-k)-$dimensional cross section of the cylinder $\mathcal{Z}\,(\Pi, R)$ and the domain Ω by the $(n-k)$ plane Λ, which is orthogonal to $\mathbb{R}^k$ and containing x. Then this cross section is exactly the sphere $\mathbb{S}^{n-k-1}$.

Considering the above cross section, the vector $\underline{\tilde{d}}_u(x) = \gamma\, u$; $\gamma = |\underline{\tilde{d}}_u(x)|$, representing the distance $\tilde{d}_u(x)$, can be written as a sum of two orthogonal vectors. One is the projection of $\underline{\tilde{d}}_u(x)$ onto the $k-$dimensional subspace $\mathcal{V}$, which is parallel to the subspace $\mathbb{R}^k$, and the other is in the orthogonal subspace $\mathcal{V}^{\perp}$, which is parallel to $\mathbb{R}^{n-k}$. Accordingly, we have

$$\left|\underline{\tilde{d}}_u(x)\right|^2 = \left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathcal{V}}\right|^2 + \left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathcal{V}^{\perp}}\right|^2.$$

In order to evaluate $\left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathcal{V}}\right|^2$ and $\left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathcal{V}^{\perp}}\right|^2$, we decompose the vector $u \in \mathbb{S}^{n-1}$ into two orthogonal components as follows:

$$u = (\eta\, \sin\phi, \xi\, \cos\phi)\,;\, \xi \in \mathbb{S}^{k-1} \subset \mathcal{V},\, \eta \in \mathbb{S}^{n-k-1} \subset \mathcal{V}^{\perp} \text{ and } \phi \in \left(0, \frac{\pi}{2}\right).$$

Denote $\tilde{d}_\eta(x) = \left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathcal{V}}\right|$, clearly this is the distance from x to the sphere $\mathbb{S}^{n-k-1}$ in the direction η (which is the projection of u onto the $(n-k)$ subspace).

Since, the two coordinates, $\eta\, \sin\phi$ and $\xi\, \cos\phi$, representing the vector u are orthogonal coordinates, one in the $(n-k)-$dimensional subspace and the other in the $k-$dimensional subspace respectively, we have $|\gamma\, \eta\, \sin\phi| = \gamma\, \sin\phi$, which is the distance from x to the $(n-k)-$dimensional cross section of the cylinder, i.e., to the sphere $\mathbb{S}^{n-k-1}$. Therefore, we have

$$\gamma^2 = \frac{\tilde{d}_\eta(x)^2}{\sin^2\phi}. \tag{5.46}$$

Consequently, inequality (5.45), using Proposition 5.5.1 with relation (5.46), produces the following bound:

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2}\,dS(u)\\
&= \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{k-1}}\int_{\mathbb{S}^{n-k-1}}\int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(\eta\sin\phi,\xi\cos\phi)^2}\sin^{n-k-1}\phi\cos^{k-1}\phi\,d\phi\,d\eta\,d\xi\\
&= \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{k-1}}\int_{\mathbb{S}^{n-k-1}}\int_0^{\frac{\pi}{2}} \frac{\sin^2\phi}{\tilde{d}_\eta^2}\sin^{n-k-1}\phi\cos^{k-1}\phi\,d\phi\,d\eta\,d\xi\\
&= \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{k-1}}d\xi\int_{\mathbb{S}^{n-k-1}}\frac{1}{\tilde{d}_\eta^2}\,d\eta\int_0^{\frac{\pi}{2}}\sin^{n-k+1}\phi\cos^{k-1}\phi d\phi\\
&= I_{n,k}\times J_{n-k}(x),
\end{aligned}\tag{5.47}$$

where

$$I_{n,k} = \frac{|\mathbb{S}^{k-1}|}{|\mathbb{S}^{n-1}|}\int_0^{\frac{\pi}{2}}\sin^{n-k+1}\phi\cos^{k-1}\phi d\phi = \frac{|\mathbb{S}^{k-1}|}{2|\mathbb{S}^{n-1}|}\beta\left(\frac{k}{2},\frac{n-k+2}{2}\right),$$

where β is the Beta function, and

$$J_{n-k}(x) = \int\limits_{\mathbb{S}^{n-k-1}}\frac{1}{\tilde{d}_\eta(x)^2}d\eta.$$

Recall that, $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ and $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, so we have

$$\begin{aligned}
I_{n,k} &= \frac{2\pi^{k/2}\Gamma(n/2)}{2\Gamma(k/2)\,2\pi^{n/2}}\cdot\frac{\Gamma(k/2)\,\Gamma(\frac{n-k+2}{2})}{\Gamma(\frac{n+2}{2})}\\
&= \frac{\Gamma(n/2)\,\Gamma(\frac{n-k+2}{2})}{2\Gamma(\frac{n+2}{2})}\pi^{\frac{k-n}{2}}.
\end{aligned}\tag{5.48}$$

On the other hand, for the integral $J_{n-k}(x)$, we follow the same argument applied to $J_n(x)$, defined in (5.34), which results in

$$J_{n-k}(x) = \frac{1}{(n-k)\,d(x)^2} \cdot \frac{2\pi^{\frac{n-k}{2}}}{\Gamma\left(\frac{n-k}{2}\right)} \left[1-(n-k-1)\frac{d(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right] \tag{5.49}$$

Use (5.49) and (5.48) in (5.47) to obtain

$$\begin{aligned}
\frac{1}{m(x)^2} &\geq \frac{\Gamma(n/2)\,\Gamma(\frac{n-k+2}{2})\,\pi^{\frac{k-n}{2}}}{2\Gamma(\frac{n+2}{2})} \times \frac{2\pi^{\frac{n-k}{2}}}{d(x)^2\,(n-k)\,\Gamma\left(\frac{n-k}{2}\right)} \\
&\qquad \times \left[1-(n-k-1)\frac{d(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right] \\
&= \frac{\Gamma(n/2)\,\left(\frac{n-k}{2}\right)\Gamma(\frac{n-k}{2})}{d(x)^2(n-k)\,\frac{n}{2}\Gamma(\frac{n}{2})\,\Gamma(\frac{n-k}{2})} \left[1-(n-k-1)\frac{d(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right] \\
&= \frac{1}{n\,d(x)^2}\left[1-(n-k-1)\frac{d(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right].
\end{aligned} \tag{5.50}$$

We apply Proposition 5.1.1 to the lower bound (5.50) to obtain

$$\int_\Omega \mu(x,R)\frac{|f(x)|^2}{d(x)^2}dx \leq \int_\Omega |\nabla f|^2\,dx,$$

where

$$\mu(x,R) = \frac{1}{4} - \left(\frac{n-k-1}{4}\right)\frac{d(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right),$$

as stated in (5.43). On the other hand, (5.44) follows immediately from (5.43) since $d(x) \leq \delta_{\text{in}}$, this completes the proof. $\square$

Remark 5.5.3.

1. If $n = 3$ and $k = 1$, then (5.43) gives

$$\mu(x,R) = \frac{1}{4} - \frac{1}{4}\frac{d(x)}{R} + \mathcal{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right),$$

   exactly as obtained in (4.44) for the three-dimensional case. On the other hand if R

tends to ∞, which means Ω is convex, then $\mu(x, R)$ tends to $\frac{1}{4}$ as known.

2. Observe that, if $k = 0$ in (5.43), then one immediately obtains (5.31), which confirms the fact that the Exterior Ball Condition is a special case of the $(n, k)-$Cylinder Condition. Moreover, (5.43) agrees with the convex result for $n - k = 1$ up to the linear term. Comparing the asymptotic expression (5.43) with (5.31), the corresponding expression under the Exterior Ball Condition, indicates that (5.43) gives a better result concerning the coefficient of the second term.

Chapter 6

Summary

This thesis has begun with an introductory chapter, Chapter 1: ‘Introduction’, in which some kind of inequalities, that controls the size of a function by the derivatives of the function itself, has been introduced through Sobolev’s inequality. Considering that kind, Hardy’s inequality has been presented with a historical hint on its origin, followed by a discussion of the main aim of this research.

Various forms of Hardy-type inequalities have been mentioned in the second chapter, ‘Literature review’. Among those forms, two crucial theorems due to E. B. Davies, Theorems 2.2.3 and Theorem 2.2.5, have been given. It is absolutely correct to say that those two theorems have played a fundamental role in proving the main theorems stated in this monograph. Theorem 2.2.3 has been the key factor in deriving all Hardy-type inequalities obtained in Chapter 3, whereas Theorem 2.2.5 has been exploited indispensably to prove Hardy-type inequalities for higher dimensions.

Theorem 2.2.6 has given a ‘simple’ proof of the fact that the constant in Hardy’s inequality is $\frac{1}{4}$ for any convex domain Ω in $\mathbb{R}^2$. This proof has been followed by a proof that the value of that constant remains $\frac{1}{4}$ for any convex domain Ω in $\mathbb{R}^n$, using Theorem 2.2.5 in conjunction with Theorem 2.2.7.

A proof by Tidblom ([54]) that the constant $\frac{1}{4}$ is sharp has been introduced using some properties of the distance function proved in Appendix D. Some existing results concerning planar non-convex domains have been referred to as well. Section 2.3 has introduced some improved Hardy-type inequalities in different directions. Section 2.4 has ended Chapter 2 by giving some profound applications of Hardy-type inequalities in spectral theory and fluid mechanics.

Chapter 3 has dealt with the main purpose of this thesis for domains in $\mathbb{R}^2$. In Section 3.1, the target of that chapter has been briefly explained and guidelines of proofs strategy have been clarified. Some non-convexity 'measures', namely the TSR and the Exterior Disk conditions, have been introduced in Section 3.2. The study of the TSR condition has yielded some special cases. Thus, we ended up with three cases, Case 1 on Page 42, Case 2 on Page 44, and Case 3 on Page 45, with three correlative conditions, Condition 3.2.6 on Page 48, Condition 3.2.7 on Page 49, and Condition 3.2.1 on Page 40, respectively.
Three different Hardy-type inequalities related to the cases stemmed from the TSR condition have been obtained in Section 3.3. Those inequalities have been proved in Theorems 3.3.1, 3.3.3, and 3.3.5. Although, those forms may appear relatively complicated, limiting cases have been given through three corresponding remarks, Remark 3.3.2, Remark 3.3.4, and Remark 3.3.6.
Another Hardy-type inequality related to the Exterior Disk condition has been derived in Theorem 3.4.1 followed by Remark 3.4.2 comparing the results obtained here with some known results.

In Chapter 4, four non-convexity conditions have been established for domains in $\mathbb{R}^3$. Two of them are generalizations of the conditions initially introduced in Chapter 3, precisely the TCR condition on Page 85 which generalized the TSR condition and the Exterior Ball condition on Page 86 which generalized the Exterior Disk condition. The other two conditions are the Exterior Cone condition on Page 85 and the Cylinder condition on Page 87.

Consequently, four different Hardy-type inequalities have been obtained in Theorems 4.3.1, 4.3.3, 4.4.1 and 4.5.1 corresponding to each condition. Each theorem of them has been followed by a remark, see Remarks 4.3.2, 4.3.4, 4.4.2 and 4.5.2. Those remarks have showed some limiting cases and compared the obtained results with the existing results.

Finally, Chapter 5 has generalized all conditions and inequalities studied in Chapter 4 to $n-$dimensional domains.

Appendices

Appendix A

Notation

A.1 Geometric notation

(i) $\mathbb{R}^n = n$-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$.

(ii) $e_i = (0, ..., 0, 1, 0, ..., 0) = i^{th}$ standard coordinate vector.

(iii) A typical point in $\mathbb{R}^n$ is $x = (x_1, ..., x_n)$.

(iv) $\partial\Omega = \bar{\Omega} \setminus \Omega =$ boundary of $\Omega \subset \mathbb{R}^n$.

(v) $|\mathbb{S}^{n-1}| =$ surface area of unit sphere in $\mathbb{R}^n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.
$\omega_n =$ volume of unit ball in $\mathbb{R}^n = \frac{\left|\mathbb{S}^{n-1}\right|}{n}$.

(vi) $\mathbb{C}^n = n$-dimensional complex space, $\mathbb{C} =$ complex plane.
If $z \in \mathbb{C}$, we write $\Re\, z$ for the real part of z, Im z for the imaginary part of z, and $\arg\ z$ for the argument of $z \neq\ 0$.

A.2 Notation for functions

(i) Suppose $f : \Omega \to \mathbb{R}$. We say f is smooth if it has continuous derivatives up to some desired order over its domain. The number of continuous derivatives necessary for a function to be considered smooth depends on the problem at hand. A function for which all orders of derivatives are continuous is called a C-infinity function.

(ii) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic with period p if

$$f(x) = f(x+p) \;\; \forall x \in \mathbb{R}.$$

(iii) Support of a function:

The closure of the set of points in the domain of the function f for which f is not zero, i.e., for the function f defined on $\mathbb{R}^n$, the support of f, denoted by supp(f), is

$$\text{supp}(f) := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

A.3 Notation for derivatives

Assume that $f : \Omega \to \mathbb{R},\; x \in \Omega$.

(i) $\frac{\partial f}{\partial x_i} = \lim\limits_{h \to 0} \frac{f(x+he_i)-f(x)}{h}$, provided this limit exists.

(ii) Multi-index notation:

(a) A vector of the form $\alpha = (\alpha_1, ..., \alpha_n)$, where each component α_i is a non-negative integer, is called a multi-index of order

$$\mid \alpha \mid = \alpha_1 + ... + \alpha_n.$$

(b) Given a multi-index α, define

$$D^\alpha f(x) := \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f(x).$$

(c) If k is a non-negative integer, then

$$D^k f(x) := \{D^\alpha f(x) : \mid \alpha \mid = k\},$$

is the set of all partial derivatives of order k.

(iii) Weak derivative:

Suppose $f,\ g \in L^1_{\text{loc}}(\Omega)$, and α is a multi-index. We say that g is the α^{th}-weak partial derivative of f, written

$$D^\alpha f = g,$$

provided

$$\int_\Omega f D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega g \phi dx, \tag{A.1}$$

for all test functions $\phi \in \mathcal{C}_c^\infty(\Omega)$.

In other words, if we are given f and if there happens to exist a function g which verifies (A.1) for all ϕ, we say that $D^\alpha f = g$ in the weak sense. If there does not exist such a function g, then f does not possess a weak α^{th}-partial derivative.

A.4 Function spaces

(i) $\mathcal{C}^k = \mathcal{C}^k(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is } k\text{−times continuously differentiable}\}$.

(ii) $\mathcal{C}^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is infinitely differentiable}\} = \cap_{k=0}^\infty \mathcal{C}^k(\Omega)$.

(iii) $\mathcal{C}_c^\infty(\Omega)$ is the space of infinitely differentiable functions $\phi : \Omega \to \mathbb{R}$, with compact support in Ω. A function ϕ belonging to $\mathcal{C}_c^\infty(\Omega)$ is called a test function.

(iv) $L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is Lebesgue measurable, } \| f \|_{L^p(\Omega)} < \infty \right\}$, where

$$\| f \|_{L^p(\Omega)} = \left(\int_\Omega | f |^p \, dx \right)^{1/p}, \quad 1 \le p < \infty.$$

$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is Lebesgue measurable, } \| f \|_{L^\infty(\Omega)} < \infty \right\}$, where

$$\| f \|_{L^\infty(\Omega)} = \operatorname{ess}\sup_\Omega | f | .$$

The notation $\| f \|_p$ is used instead of $\| f \|_{L^p}$.

(v) $W^{k,p}(\Omega)$, $H^k(\Omega)$; $k = 0, 1, 2, ..., \ 1 \le p \le \infty$, denote Sobolev spaces (see Appendix B for the definition).

Appendix B

Functional Analysis

B.1 Linear operators

(i) Non-negativity of an operator:

It is said that a symmetric operator A with domain D is non-negative if

$$\langle Af, f\rangle \geq 0$$

for all $f \in D$.

(ii) Friedrichs extension:

Every non-negative symmetric operator A has at least one non-negative self-adjoint extension. If A is not essentially self-adjoint then this extension, called the Friedrichs extension, is one of the infinitely many possible self-adjoint extensions.

B.2 Sobolev space

Fix $1 \leq p \leq \infty$ and let k be a non-negative integer. The Sobolev space, denoted by $W^{k,p}(\Omega)$, consists of all locally summable functions $f : \Omega \to \mathbb{R}$, such that for each multi-index α with

$| \alpha | \leq k$, $D^\alpha f$ exists in the weak sense and belongs to $L^p(\Omega)$.

If $p = 2$, we usually write

$$H^k(\Omega) = W^{k,2}(\Omega); \quad k = 0, 1, \dots .$$

The letter H is used, since $H^k(\Omega)$ is a Hilbert space. Note that $H^0(\Omega) = L^2(\Omega)$.

The closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by

$$W_0^{k,p}(\Omega).$$

Thus $f \in W_0^{k,p}(\Omega)$ if and only if there exist functions $f_m \in \mathcal{C}_c^\infty(\Omega)$ such that $f_m \to f$ in $W^{k,p}(\Omega)$.

It is customary to write

$$H_0^k(\Omega) = W_0^{k,2}(\Omega).$$

Appendix C

Special Functions and Useful Identities

C.1 Gamma and Beta functions

The Gamma function, denoted by $\Gamma(z)$, is defined as follows:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt.$$

This integral was first introduced and studied by L.P. Euler $(1707 - 1783)$.

Example: Let $z = n + 1$. Then

$$\Gamma(n+1) = \int_0^\infty e^{-t}\, t^{n}\, dt = n!.$$

This means that the Gamma function is a generalization of the factorial. It also indicates that

$$0! = \Gamma(1) = 1. \tag{C.1}$$

Let us mention here some relations satisfied by the Gamma function:

$$\begin{aligned}\Gamma(z+1) &= z\Gamma(z), && \text{(C.2)}\\ \Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin\,\pi z}, && \text{(C.3)}\\ 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) &= \sqrt{\pi}\Gamma(2z).\end{aligned}$$

Several integrals can be expressed in terms of the Gamma function. For instance, consider

$$\Gamma(x)\Gamma(y) = \int_0^\infty\int_0^\infty e^{-t_1-t_2}t_1^{x-1}t_2^{y-1}\,dt_1 dt_2,$$

under the substitutions $t_1 := r\,\cos^2\phi$ and $t_2 := r\,\sin^2\phi \Rightarrow J = 2r\sin\phi\,\cos\phi$, so we obtain

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 2\int_0^\infty e^{-r}\,r^{x+y-1}\,dr\int_0^{\frac{\pi}{2}}\cos^{2x-1}\phi\,\sin^{2y-1}\phi\,d\phi\\ &= \Gamma(x+y)\int_0^1\zeta^{x-1}\,(1-\zeta)^{y-1}\,d\zeta;\;\cos^2\phi=\zeta. && \text{(C.4)}\end{aligned}$$

The integral in the right hand side of (C.4) is called Beta function and is denoted by $\beta(x,y)$. Hence,

$$\beta(x,y) = \int_0^1\zeta^{x-1}\,(1-\zeta)^{y-1}\,d\zeta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{C.5}$$

The formula (C.5) of the Beta function, for $x=y=\frac{1}{2}$, gives

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \left(\beta\left(\frac{1}{2},\frac{1}{2}\right)\right)^{\frac{1}{2}} = \left(\int_0^1\frac{1}{\sqrt{\zeta-\zeta^2}}d\zeta\right)^{\frac{1}{2}}\\ &= \left(\int_0^{\pi/2}\frac{2\sin\theta\,\cos\theta}{\sqrt{\sin^2\theta\,\cos^2\theta}}d\theta\right)^{\frac{1}{2}} = \left(\int_0^{\pi/2}2\,d\theta\right)^{\frac{1}{2}} = \sqrt{\pi}, && \text{(C.6)}\end{aligned}$$

which agrees with (C.3) for $z = \frac{1}{2}$. Combining the relations (C.1), (C.2) and (C.6) results in

$$\begin{aligned}
\Gamma(\tfrac{3}{2}) &= \Gamma(\tfrac{1}{2}+1) &&= \tfrac{1}{2}\Gamma(\tfrac{1}{2}) &&= \tfrac{\sqrt{\pi}}{2},\\
\Gamma(2) &= \Gamma(1+1) &&= 1\Gamma(1) &&= 1,\\
\Gamma(\tfrac{5}{2}) &= \Gamma(\tfrac{3}{2}+1) &&= \tfrac{3}{2}\Gamma(\tfrac{3}{2}) &&= \tfrac{3\sqrt{\pi}}{4},
\end{aligned}$$

and so on.

Beta function has many other forms than (C.5), for instance:

$$\beta(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta, \qquad \Re(x) > 0,\ \Re(y) > 0, \tag{C.7}$$

$$\beta(x,y) = \int_0^{\infty} \frac{\zeta^{x-1}}{(1+\zeta)^{x+y}}d\zeta, \qquad \Re(x) > 0,\ \Re(y) > 0.$$

Under the substitution $y = \frac{1}{2}$, relation (C.7) results in

$$\beta\left(x, \frac{1}{2}\right) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}d\theta,$$

which, with $x = \frac{n-1}{2}$, gives

$$\beta\left(\frac{n-1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} (\sin\theta)^{n-2}d\theta. \tag{C.8}$$

Combine relations (C.8) and (C.5) to obtain

$$\int_0^{\pi/2} (\sin\theta)^{n-2}d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{C.9}$$

On the other hand, under the substitutions $y = \frac{3}{2}$ and then $x = \frac{n-1}{2}$, relation (C.7) with (C.5) gives

$$\int_0^\pi (\sin\theta)^{n-2} (\cos\theta)^2 \, d\theta = \beta\left(\frac{n-1}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)}.$$

C.2 Some useful identities

(i) Recall, the next standard formula, see [26], switches integrals from Cartesian coordinates to Spherical coordinates:

$$\int_{R\mathbb{S}^{n-1}} f(x)\ d\sigma(x) = \int_{\varphi_1=0}^{\pi} \ldots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f\left(x\left(\varphi\right)\right)\ J\left(n,R,\varphi\right)\ d\varphi_{n-1}\ldots d\varphi_1, \tag{C.10}$$

where

$$\begin{aligned}
x_1 &= R\cos\varphi_1,\\
x_2 &= R\sin\varphi_1\ \cos\varphi_2,\\
x_3 &= R\sin\varphi_1\ \sin\varphi_2\ \cos\varphi_3,\\
&\cdots\\
x_{n-1} &= R\sin\varphi_1\ \sin\varphi_2 \ldots \sin\varphi_{n-2}\ \cos\varphi_{n-1},\\
x_n &= R\sin\varphi_1\ \sin\varphi_2 \ldots \sin\varphi_{n-2}\ \sin\varphi_{n-1},
\end{aligned}$$

and $0 \le \varphi_1, \ldots, \varphi_{n-2} \le \pi$, $0 \le \varphi_{n-1} = \theta \le 2\pi$. Note here that if $n = 2$ then we have only one angle φ_1 such that $0 \le \varphi_1 = \theta \le 2\pi$,

$$x\left(\varphi\right) = \left(x_1\left(\varphi_1, \ldots, \varphi_{n-1}\right), \ldots, x_n\left(\varphi_1, \ldots, \varphi_{n-1}\right)\right),$$

and

$$J\left(n,R,\varphi\right)=R^{n-1}\left(\sin\varphi_1\right)^{n-2}...\left(\sin\varphi_{n-3}\right)^2\,\left(\sin\varphi_{n-2}\right),$$

is the Jacobian of the transformation.

(ii) The standard identity, see [52]

$$\int\left(\sin\,ax\right)^n\,dx=\frac{-\sin^{n-1}ax\,\cos ax}{an}+\frac{n-1}{n}\int\left(\sin\,ax\right)^{n-2}\,dx, \tag{C.11}$$

with (C.9), leads to

$$\int_0^\pi\left(\sin\,x\right)^n\,dx=\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}=\begin{cases}\frac{1.3.5.\,...\,(n-1)}{2.4.6.\,...\,n}\pi & \text{for } n \text{ even} \geq 2,\\ 2\,\frac{2.4.6.\,...(n-1)}{3.5.7.\,...\,n} & \text{for } n \text{ odd} \geq 3,\\ 2 & \text{for } n = 1.\end{cases}$$

(iii) The standard identity

$$\begin{aligned}\int\left(\sin ax\right)^n\left(\cos ax\right)^m dx &= \frac{\sin^{n+1}ax\,\cos^{m-1}ax}{a(n+m)}\\ &\quad+\frac{m-1}{m+n}\int\left(\sin ax\right)^n\left(\cos ax\right)^{m-2}dx,\end{aligned} \tag{C.12}$$

where $m\neq-n$, gives

$$\int_0^\pi\left(\sin ax\right)^n\left(\cos ax\right)^m dx = \frac{m-1}{m+n}\int_0^\pi\left(\sin ax\right)^n\left(\cos ax\right)^{m-2}dx,\ m\neq1,$$

which implies that

$$\int_0^\pi (\sin x)^{n-2} (\cos x)^2 \, dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{n\,\Gamma\left(\frac{n}{2}\right)} = \begin{cases} \frac{1.3.5....(n-3)}{2.4.6.\,...\,n}\,\pi & \text{for } n \text{ even} \quad \geq \quad 4, \\ 2\,\frac{2.4.6.\,...(n-3)}{3.5.7.\,...\,n} & \text{for } n \text{ odd} \quad \geq \quad 5, \\ \frac{2}{3} \text{ for } n = 3, \quad \frac{1}{2} \text{ for } n = 2. \end{cases} \tag{C.13}$$

(iv) Finally, the surface area of the unit sphere in $\mathbb{R}^n$ can be written as follows:

$$\left|\mathbb{S}^{n-1}\right| = \frac{2(\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}\right)} = \begin{cases} \frac{(2\pi)^{n/2}}{2.4.\,...\,(n-2)} & \text{for } n \text{ even} \quad \geq \quad 4, \\ \frac{2\,(2\pi)^{(n-1)/2}}{1.3.\,...\,(n-2)} & \text{for } n \text{ odd} \quad \geq \quad 3. \end{cases} \tag{C.14}$$

Consequently,

$$\left|\mathbb{S}^{n-2}\right| = \frac{2(\sqrt{\pi})^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} = \begin{cases} \frac{2\,(2\pi)^{(n-2)/2}}{1.3.\,...\,(n-3)} & \text{, for } n \text{ even} \quad \geq \quad 4 \\ \frac{(2\pi)^{(n-1)/2}}{2.4.\,...\,(n-3)} & \text{, for } n \text{ odd} \quad > \quad 3. \end{cases} \tag{C.15}$$

Appendix D

Distance function

Let $\Omega \subset \mathbb{R}^n$ be an open set. Then the distance function is defined as

$$d(x) = \min\{|x - y| : y \notin \Omega\}.$$

In this appendix we aim to prove two useful properties of the distance function $d(x)$ over convex domains Ω. More precisely, we want to show, for convex domains Ω, that:

- The gradient of the distance function has length one, i.e.

$$|\nabla d(x)| = 1. \tag{D.1}$$

- The Laplacian of the distance function is non-positive almost everywhere, i.e.

$$\Delta d(x) \leq 0 \qquad \text{almost everywhere.} \tag{D.2}$$

To prove (D.1), we first prove the following lemma:

Lemma D.0.1.

The distance function $d(x)$ is uniformly 1-Lipschitz continuous function.

Proof. Let $x, y \in \mathbb{R}^n$ and choose $z \in \partial\Omega$ such that $|y - z| = d(y)$. Then

$$d(x) = \min_{z \in \partial\Omega} |x - z| \leq |x - z| \leq |x - y| + d(y),$$

so by interchanging x and y we have

$$|d(x) - d(y)| \leq |x - y|,$$

as required. □

Recall that, Rademacher's Theorem states (see [21, Page 281]): a Lipschitz map $f : \Omega \longrightarrow \mathbb{R}^m$, where Ω is an open set in $\mathbb{R}^n$, is differentiable almost everywhere in Ω. Moreover, if L is the Lipschitz constant of f, then $|\nabla f| \leq L$.

Hence, from Lemma D.0.1 and Rademacher's Theorem, we conclude that the distance function is differentiable almost everywhere in Ω and

$$|\nabla d(x)| \leq 1. \tag{D.3}$$

Thus, if we can show that $|\nabla d(x)| \geq 1$, then we are done.

Lemma D.0.2. [41]

Let $\Omega \subset \mathbb{R}^n$ be an open set. Then for almost all $x \in \Omega$,

$$|\nabla d(x)| \geq 1. \tag{D.4}$$

Proof. Let us fix a point x at which the function d is differentiable , and let $y \notin \Omega$ be such that

$$d(x) = |x - y|.$$

Denote

$$\mathbf{e} = \frac{y-x}{s}\,,\ \ s = |x-y|.$$

Since $d(x)$ is differentiable at x, all its directional derivatives also exist.

The directional derivative of the function d in the direction $\mathbf{e}$ is

$$\begin{aligned} d'_{\mathbf{e}}(x) &= \lim_{t\to 0} \frac{d(x+t\mathbf{e}) - d(x)}{t} \\ &= \lim_{t\to 0} \min_{z} \frac{|x+t\mathbf{e}-z| - |x-y|}{t} \\ &\leq \lim_{t\to 0} \frac{|x+t\mathbf{e}-y| - |x-y|}{t}. \end{aligned}$$

The fraction in the right hand side can be rewritten as follows:

$$\frac{|-s\mathbf{e}+t\mathbf{e}| - s}{t} = \frac{|-\mathbf{e}|(s-t) - s}{t} = \frac{s-t-s}{t} = -1.$$

Therefore,

$$d'_{\mathbf{e}}(x) \leq -1.$$

This implies that

$$|\nabla d(x)| \geq 1,\ \text{ almost everywhere},$$

as required. □

Now the proof of (D.1) follows immediately from (D.3) and Lemma D.0.2.

Now to prove the second property, i.e. to prove (D.2), recall that: a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called concave if

$$f((1-t)x+ty) \geq (1-t)f(x) + tf(y),\ \forall t \in [0,1] \text{ and } x, y \in \mathbb{R}^n.$$

This definition merely states that for every z between x and y, the point $(z, f(z))$ on the graph of f is above the straight line joining the points $(x, f(x))$ and $(y, f(y))$. Notice also that if $f(x)$ is concave then $-f(x)$ is convex.

Lemma D.0.3.

The distance function $d(x)$, on a convex domain Ω, is concave.

Proof. Denote $z = tx+(1-t)y$. Let $z_1 \in \partial\Omega$ be a point such that $|z-z_1| = d(z)$. Denote by H a support hyperplane for the set Ω at the point z_1, i.e. a hyperplane such that $z_1 \in H$ and Ω lies entirely on one side of H, or, alternatively, $z_1 \in H, \Omega \cap H = \emptyset$. Clearly, $d(z) = d(z, H)$. Moreover,

$$d(z) = d(tx + (1 - t)y, H) = td(x, H) + (1 - t)d(y, H).$$

On the other hand, $d(x, H) \geq d(x)$ and $d(y, H) \geq d(y)$, so that

$$d(z) \geq td(x) + (1 - t)d(y),$$

as required. □

On the other hand, in [25] we find the following helpful theorem:

Theorem D.0.4.

Let f be a convex real-valued function defined in an open convex subset Ω of $\mathbb{R}^n$, then f is twice differentiable almost everywhere and the gradient of f is differentiable almost everywhere in Ω.

Actually, this fact has been proved earlier by many authors. From [47, Page 32] we quote ‘A convex function is twice differentiable almost everywhere. For $n = 1$, this is a consequence of the differentiability almost everywhere of a monotone function, as was first pointed out by Jessen (1929). For $n = 2$, it was proved by Busemann and Feller (1936), and by using their result and an induction argument, Aleksandrov (1939) obtained the general case'. On the other hand, according to Aleksandrov's Theorem the twice differentiability almost everywhere of the convex function $f : \mathbb{R}^n \to \mathbb{R}$ is interpreted as follows (see [22]):

$$\left| f(y) - f(x) - \mathrm{D}f(x) \cdot (y-x) - \frac{1}{2}(y-x)^T \cdot \mathrm{D}^2 f(x) \cdot (y-x) \right| = o(|y-x|^2) \quad \text{as} \quad y \to x, \tag{D.5}$$

where $\mathrm{D}f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$ and $\mathrm{D}^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$.

Lemma D.0.5. *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex then*

$$\left\langle D^2 f(x)\, \xi, \, \xi \right\rangle \geq 0,$$

almost everywhere for any $\xi \in \mathbb{R}^n$.

Proof. Let x be a point where f is twice differentiable. So by the convexity of the function f we obtain for $y = x + h$ that:

$$f(x + (1-t)h) \leq t f(x) + (1-t) f(x+h),$$

so that

$$\frac{f(x + (1-t)h) - f(x)}{1-t} \leq f(x+h) - f(x).$$

Taking $t \uparrow 1$, we get

$$f(x+h) - f(x) \geq \lim_{t \to 1} \frac{f(x + (1-t)h) - f(x)}{1-t} = \nabla f(x) \cdot h. \tag{D.6}$$

Put $h = \lambda\xi$, where $0 \neq \xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, so (D.6) together with (D.5) implies that

$$0 \leq f(x+\lambda\xi) - f(x) - \lambda \mathrm{D}f(x)\cdot\xi = \frac{\lambda^2}{2}\left\langle \mathrm{D}^2 f(x)\,\xi,\,\xi\right\rangle + o(\lambda^2\xi^2), \text{ as } \lambda \to 0,$$

and hence

$$\left\langle \mathrm{D}^2 f(x)\,\xi,\,\xi\right\rangle + \lambda^{-2}\,o(\lambda^2\xi^2) \geq 0, \text{ as } \lambda \to 0.$$

By definition of $o(\cdot)$, we have $\lambda^{-2}\,o(\lambda^2\xi^2) \to 0$ as $\lambda \to 0$, this completes the proof. □

Therefore, for the distance function we can conclude that

$$\Delta d(x) \leq 0, \quad \text{almost everywhere},$$

as required.

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