Topics in Trivalent Graphs

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Comments, errors & omissions (May 2008)

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Topics in Trivalent Graphs

My PhD thesis *Topics in Trivalent Graphs* was submitted for examination 29 Sep 2006, defended 15 Dec 2006 and recommended to proceed with minor corrections; the **final version** including corrections as requested by the examiners was handed in 31 Jan 2007, and the PhD was awarded (Spring 2007).

The `dvi` of the final version was subsequently corrected for one typo in the Abstract and a font glitch in the References, but is otherwise identical to the print version deposited in the library, and was uploaded here 2 Feb 2007. Thanks to Bernard Beard for the conversion to `pdf`, uploaded 30 April 2008 and also submitted to the university's new **e-theses** depository.

If you're printing it please do so **two-sided:** saves trees, and looks better as the layout was planned that way. It was designed for A4 paper like we use here in Europe, but also fits within the bounds of US standard paper. Keeping it in electronic form saves even more trees, of course. Either way, better grab a copy of this web page, for the **errors & omissions** below. **Correspondence** is most likely to reach me if addressed to **`gmarijke`** *this bit is only for spam robots* **`@gmail.com`**

`PhDbody.dvi` or **`PhDbody.pdf`** **main body incl. abstract** (also below), **index, references.** As submitted it came with three appendices:

`PhDB.dvi` or **`PhDB.pdf`** **Appendix B**, listings of cycle maps (cycle double covers) tallied by orientability, number of cycles, and cycle size distribution. Covers all trivalent graphs (with $2h$ nodes and $3h$ edges) of $h \leq 4$, prisms of $h \leq 10$, polydiamonds, a few generalised Petersen graphs, a few snarks, some incidence graphs from projective geometry, Platonic solids.

`PhDD.dvi` or **`PhDD.pdf`** **Appendix D**, survey of dimension of cycle space $\cap$ cut space (listing for specific graphs as in Appendix B, and tally for millions of randomly generated graphs).

`PhDH.dvi` or **`PhDH.pdf`** **Appendix H**, essay on hyperbolic space in Minkowski coördinates, and the embedding of infinite 4- and 3-valent trees X_∞ and Y_∞ in it as regular polytopes.

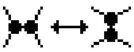
In case you wonder, appendices A, C, ... J either didn't gel in the end or would have knocked the word count over the limit. Some of them may yet see the light of day, one day.

Abstract

Chapter 0 details the notation and terminology used.

Chapter 1 introduces the usual linear algebra over $\mathbb{F}_2$ of edge space **E** and its orthogonal subspaces **Z** (cycle space) and $\mathbf{Z}^{\perp}$ (cut space). **Reduced vectors** are defined as elements of the quotient space $\mathbf{E} / \mathbf{Z}^{\perp}$. Reduced vectors of edges give a simple way of characterising edges that are bridges (their reduced vector is null) or 2-edge cuts (their vectors are equal), and also of spanning trees (the edges outside the tree are a basis) and form to the best of my knowledge a new approach. They are also useful in later chapters to describe Tait colorings, as well as cycle double covers. Perhaps the most important property of $\mathbf{E} / \mathbf{Z}^{\perp}$ is the **Unique graph theorem**: unlike in **E**, a list of which reduced vectors are edges uniquely determines graph structure (if edge connectivity is high enough; that covers certain "solid" components every trivalent graph can be decomposed into).

Chapter 2 gives a brief intoduction to graph embeddings and planar graphs.

Chapter 3 deals specifically with trivalent graphs, listing some of the ways in which they are different from graphs in general. Results here include two versions of **Bipolar growth theorem** which can be used for constructive proofs, and (after defining "halftrees" and a "flipping" operation  between them) a theorem enumerating the set $\boldsymbol{C}_n$ of halftrees of a given size, the **Caterpillar theorem** showing $\boldsymbol{C}_n$ is connected by flipping, and the **Butterfly theorem** derived from it. Graphs referred to here as **solid** are shown to play an important structural rôle.

Chapter 4 deals with the 4-coloring theorem. The first half shows the older results in a unified light using edge spaces over $\mathbb{F}_4$. The second half applies methods from **coding theory** to this. The 4-color theorem is shown to be equivalent to a variety of statements about cycle-shaped words in codes over $\mathbb{F}_4$ or $\mathbb{F}_3$, many of them tantalisingly simple to state (but not, as yet, to prove).

Chapter 5 deals with what has been variously called polyhedral decompositions and (specifically for those using cycles) cycle double covers, as in the cycle double cover conjecture. The more general concept is referred to as a **map** in this paper, and identified with what is termed here **cisness structures**, which is a new approach. There is also a simpler proof of a theorem by Szekeres. Links with the subject of the previous chapter are identified, and some approaches towards proving the conjecture suggested.

Several planned **appendices** were left out of the version submitted for examination because they would make the thesis too big, and/or were not finished. Of the ones that

remain, appendix **H** (on embedding infinite 4- and 3-valent trees X_∞ and Y_∞ in the hyperbolic plane) now seems disjointed from the body of the text (a planned appendix dealt with colorings of finite graphs as the images of homomorphisms from embeddings of Y_∞). Appendix **B** enumerates cycle maps (cycle double covers) on a number of small graphs while appendix **D** investigates dim $\mathbf{Z} \cap \mathbf{Z}^\perp$.

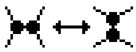
Errors & omissions

Please find more bugs! The ones here were found by myself.

(1.5) Quotient spaces as matroids

The quotient space $\mathbf{E} / \mathbf{Z}^\perp$ referred to as "reduced edge space" above has been remarked on before, though quotient space $\mathbf{E} / \mathbf{Z}$ appears to have been studied more. They are both mentioned by Peter Cameron as examples of *matroids* in an article on his web pages.

(3.3) Flips, what about the flops

Section 3.3 in the thesis defines the operation  on trivalent [half]graph embeddings to be "a **flip** (as opposed to a concept of *flop* introduced later)". The thesis goes on to show that trivalent halftrees of a given size (with labelled halfedges) form a set of Catalan number cardinality and that arranged as a big "meta"graph (with halftrees as "meta"nodes and flipping as "meta"edges) that "meta"graph is connected. The promised flops never made it to the final thesis, so what were they and what were they meant to be for?

Recall a trivalent halftree is a planar embedding of a cycle-free portion of a trivalent graph, a halfedge is an edge crossing the boundary of the planar patch so we only have one of its nodes (these edges are supposed to be distinct in any graph this is a portion of, i. e. we're not looking at two ends of the same edge, so we don't have all the nodes of any cycle).

Pick up the halftree, twist any of its edges by a half turn, put it back in the embedding surface, that's a **flop**. While flips preserve the cyclic order of the labelled halfedges, a flop reverses their order along part of the circumference. Allowing flips as well as flops makes for a much bigger "meta"graph.

Application to Tait colorings (edge 3-colorings such that no adjacent edges have the same color, which in a planar embedding implies the colors go RGB clockwise or anticlockwise at any node): a halftree of n nodes can have its colors oriented either way independently at any node, giving $3{\cdot}2^n$ Tait colorings. A halftree with Tait coloring can only be flipped (preserving color everywhere else) if the two nodes involved have the same color orientation (otherwise each new node gets edges of the same color, in other words the new edge gets color zero in IF_4). Flopping a Tait colored halfgraph is always possible (it reverses the color orientation in part of the halfgraph).

So the idea was this: make an even bigger "meta"graph of all Tait colored planar halfgraphs (of a given number of nodes) as "meta"nodes, only the allowed flips survive as "meta"edges, now put all the flops in as "meta"edges too, is the whole thing still connected? If so then (we already knew all the shapes were connected by walking the colorless "meta"graph with all the flips, but you can't generally take your coloring with you) we would have proven the missing flips don't matter, you can still go from any shape to any shape by a detour via some flops, and end up having taken some tweaked form of your coloring with you.

Ultimately it would all end up being used to investigate how many patterns of colorings we get at the halfedges, if we get enough diversity there it becomes possible to find matching colorings for the two treelike halves of a whole graph.

(2+3+4+5) Planarity, Tait colorings and cycle double covers

An embedding in a surface implies a **circuit map** (double cover of the edges by circuits): each **face** (region of surface bounded by edges) has a circuit boundary, and every edge features two times in a circuit of the map, once for the face on each side. If a face circuit is not a cycle the face borders itself, e.g. in O-O (drawn on a plane or sphere) the little faces have cycle boundaries but the third face (outer face if drawn on a plane) borders itself. If a graph has bridges such a feature is unavoidable; if it doesn't have bridges a circuit map on it may still have such a feature.

If all faces are cycles, which implies the graph is bridge-free, we have a **cycle map** (double cover of the edges by cycles): every edge now features in two distinct cycles. The cycle double cover conjecture says every bridgefree graph has a cycle map. It's enough to prove this for trivalent graphs, see **(5.0)** below.

A trivalent graph of $2h$ nodes and $3h$ edges is **planar** (chapter **2**) if it has a cycle map of $h+2$ faces. That's the maximum number of faces in a cycle map; one with $h+2-\Delta$ faces forms an embedding in a surface of genus $\Delta/2$ (if there are no further topological shenanigans in the interior of a face, and there don't need to be as its boundary is a cycle).

The red-green cycles, blue-red cycles, and green-blue cycles of a Tait coloring (chapter **4**) form a cycle map, albeit not in general a planar ($\Delta=0$) *map*, i.e. fewer than $h+2$ of these cycles. Clearly, each of the cycles (being of alternating color) has an **even** number of edges. Conversely, a such an **even** cycle map implies Tait colorability, if the graph is planar (proven by Szekeres).

So there's a wider issue here that links all these. In a sense most of the thesis is about this but it nowhere spells it out as clearly as it should. The 4-color theorem can be stated as follows: **if** a trivalent graph has a cycle map with the maximum number $h+2$ of faces (that means it's planar and bridgefree) **then** it also has a cycle map where all the faces have an even number of edges (which then means Tait colorable).

(4) Flows

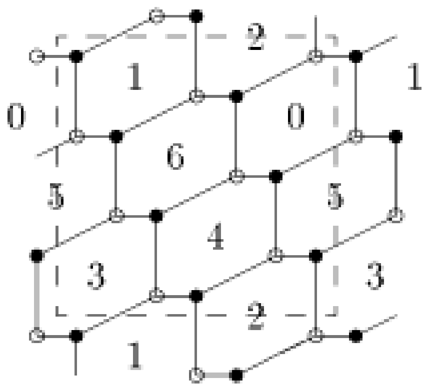
Let a **flow** be an edge coloring (with elements of an abelian group as color names) where direction matters, e.g. an edge being colored +1 in this direction is the same as it being colored -1 in the other direction, and such that outgoing (or incoming) color sums to 0 at every node, and no edge has color 0. For example a Tait coloring is an IF_4-flow, where direction does not matter as IF_4 having characteristic 2 means $-a = +a$ for every element.

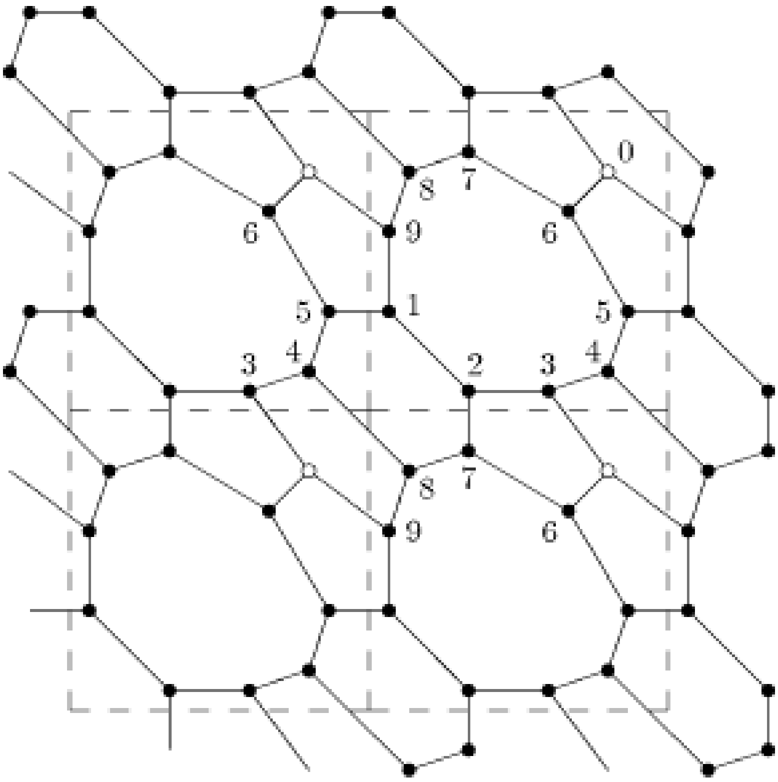
An important theorem (see the chapter on flows in Diestel's book) says that if abelian groups *A* and *B* have the same number of elements then a graph has an *A*-flow iff it has a *B*-flow, regardless of group structure. In particular we have an IF_4-flow iff we have a flow using residue classes (mod 4). Of course direction does matter for a (mod 4)-flow.

By the way, the theorem is easily seen to be true in this particular case. Let a **matching** *M* on 3-valent *G* be a spanning 1-valent subgraph (a set of edges touching each node once, which always exist though this isn't trivial to prove). The complementary set of edges forms a spanning 2-valent subgraph *M** which therefore forms one or more disjoint cycles. Let an even *M** be one that has only even cycles. It is easy to see that both IF_4-flows and (mod 4)-flows exist iff such even *M** do, although the reason why we (in one direction) get, or (in the other direction) need, evenness is different for IF_4 and (mod 4).

(4.3) Heawood's theorem

Tait's theorem is true in one direction, face-4-coloring implies edge-3-coloring, for all trivalent graph embeddings. The opposite direction requires planarity (as correctly stated in the thesis). A nice counterexample (edge-3-colorable, needs 7 face colors) is the seven-hexagon map on the torus (right), the graph here is the Heawood 6-cage, aka the incidence graph of the Fano plane.





Let a Heawood node-2-coloring be one where, for once, adjacent items are not required to have distinct colors but rather the restriction is that in each face cycle the number of black nodes minus the number of white nodes is ≡ 0 (mod 3).

Heawood's theorem is true in one direction, edge-3-coloring implies this particular node-2-coloring, for all orientable trivalent graph embeddings. The opposite direction again requires planarity. The "proof" in the thesis omits to mention this, but the way node color is integrated to edge color along any face cycle implicitely uses a construction where the portion of any cycle already assigned edge color is a single contiguous stretch.

To prove this can always be done (given planarity): Let the two **teriors** of a cycle in a spherical embedding be what on a plane embedding would be the *in*terior and *ex*terior. Construct a Tait edge coloring (derived from Heawood node color) in the same order in which "black-black" edges appear in any instance of bipolar growth (see Bipolar Growth Theorem in the thesis, it starts with no black nodes and makes them all black one by one). This never divides a planar face cycle in 4 pieces black/white/black/white (or 6, 8 etc. pieces) because by the theorem black and white subgraphs each remain connected, so such a division would need one terior to connect the black portions and one to connect the white portions (the connecting pieces cannot cross as every node is black or white), but a *face* cycle is a cycle with one empty terior. o o o

To show it is really necessary to have a single contiguous stretch of any face cycle already-colored, a counterexample (left). The map is again on the torus, with face cycles 123456789, 76032, 19065, 43098, 721548. This has a Heawood node-2-coloring (0 white, other nine black) which could never be turned into an edge-3-coloring, because the graph is the Petersen graph which doesn't have such a coloring.

(5.0) Cycle Double Cover Conjecture

A **cycle map** (cycle double cover, simple polyhedral decomposition) of a graph *G* is a collection *M* of cycles such that for every edge of *G* there are exactly two cycles in the collection that pass through that edge. Here "collection" really means multiset (for instance if *G* is a single cycle the only possible *M* uses it twice).

In the trivalent case it's easy to see each node must be visited thrice (to cover each edge twice), once each via each pair of edges incident to that node (if we take the same corner twice there's no way to cover the third edge). So in particular no cycle occurs more than once in *M* and we can replace "collection" by "set".

Cycle Double Cover Conjecture: every bridge-free *G* has a cycle map.

It would suffice to prove the conjecture for trivalent graphs only, because it then follows for all graphs. The "proof" of that fact in the thesis (p77) cut some corners though. Here's the amended version.

Lemma: we only have to prove the CDC Conjecture for connected graphs ($\kappa \geq 1$).

Let *G* be any graph. Every cycle happens within one component of *G*, so we just need to prove it for each component separately. $_o{}^o{}_o$

Now let a **cutvertex** be a node whose removal would disconnect a connected graph, let **splitting** a cutvertex consist of disconnecting the graph that way but giving each component formed a copy of that node back, and let the **blocks** of *G* be the result of splitting every cutvertex. The endpoints of a bridge are cutvertices, so one kind of block is a bridge (with its two edges) and the other blocks don't have bridges. By construction, all non-bridge blocks have κ at least 2.

Lemma: we only have to prove the CDC Conjecture for graphs with $\kappa \geq 2$.

In the conjecture we don't have bridges, so blocks have $\kappa \geq 2$. Cycles happen wholly within one block, so we only need to prove it for each block separately. $_o{}^o{}_o$

Lemma: we only have to prove the CDC Conjecture for *trivalent* graphs with $\kappa \geq 2$.

Because then, for any graph with $\kappa \geq 2$.

- Remove all 0- and 2-valent nodes.
- If there's a 1-valent node the edge to it is a bridge.
- Replace each node V of valency $v > 3$ (linked by an edge to A, B, C, D...) by trivalent T (linked to A, B, U) and U with valency $v - 1$ (linked to T, C, D...), repeat until every node is trivalent.
- Find a cycle map for the now trivalent graph (which exists by assumption).
- Reverse the process by contracting each T and U into V which preserves the identity of cycles the obvious way; they still cover the remaining edges twice.
- Put the 2-valent nodes back (subdividing edges, letting the cycles of the map run straight through) and put the 0-valent nodes back.

Any of the old edges cannot become a bridge if it wasn't already one (because the same cycles still pass through it, possibly extended with new edges). Any of the new edges cannot be a bridge (because then the node just split was a cutvertex; this is the bit where we need $\kappa \geq 2$). So if we can find a cycle map for the bridge-free trivalent graph that arose, we can find one for the original graph. $_o{}^o{}_o$

Note in general $\kappa \geq 2$ is stronger than $\kappa' \geq 2$ (absence of cutvertices implies absence of bridges but not vice versa), however in the trivalent case &kappa = κ' so the trivalent graphs we are left to prove the conjecture for are all the bridge-free connected ones, and bridge-free is a condition of the conjecture anyway. The detour via κ is merely there to make the proof of the lemma work.

Lemma: we only have to prove the CDC Conjecture for solid trivalent graphs.

This is proven in the thesis by decomposing the graph along 2-edge cuts and non-trivial 3-edge cuts.

(5.2) Circuit maps

The proof that a choice of cisness at each edge always produces circuits (p83) is alright, but the argument that they cover every edge only twice should be fleshed out more. Lack of edge direction means we don't have an easy "next edge" permutation (as in the oriented case) with orbits, so we have to follow a circuit both ways and when it revisits an edge enumerate cases, including the absence of fourth case (p85). I've worked it out in detail now and it all works out; to be added here as time permits.

Software

I've written software to enumerate Tait colorings, including a graphical user interface to grow trivalent graphs along the lines of the Bipolar Growth Theorem, and am adding cycle double cover searches. Currently in assembly for 32-bit Windows. To be uploaded (with source) when i get to a stable version integrating the various functionalities, and/or (in C) when i get it to work across platforms.

cdc.html - older write-up part I (better read the thesis instead)
map.html - older write-up part II with links to ad-hoc software (asm & C source, DOS executables)

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