VERTICES FOR IWAHORI–HECKE ALGEBRAS OF THE SYMMETRIC GROUP

by

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Abstract

In this thesis we explore the notions of relative projectivity and vertices for $\mathscr{H}_n$, the Iwahori–Hecke algebra related to the symmetric group. We begin by generalising notions from local representation theory of finite groups, such as a Green correspondence and a Brauer correspondence for the blocks of these algebras. Once this is achieved, we look into further detail about the blocks and specific modules in these blocks, to give a classification of the vertices of blocks of $\mathscr{H}_n$, and use this classification to resolve the Dipper–Du conjecture regarding the structure of vertices of indecomposable $\mathscr{H}_n$-modules. We then apply these results to compute the vertices of some Specht modules, in particular all Specht modules of $\mathscr{H}_e$ (where e is the quantum characteristic of $\mathscr{H}_n$), and hook Specht modules when $e \nmid n$ (generalising results from the symmetric group). After considering signed permutation modules to give a method of computing the vertex of signed Young modules, we conclude by looking at possible generalisations of these results to the Iwahori–Hecke algebra of type B.

Dedicated to the memory of Anton Evseev.

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CONTENTS

CHAPTER 1

INTRODUCTION

The Iwahori–Hecke algebras of type A_{n-1}, denoted $\mathscr{H}_n$, arise naturally in many areas of representation theory, generalising the group algebra of the symmetric group $\mathfrak{S}_n$. In [7], Dipper and James showed that the representation theory of $\mathscr{H}_n$ gave a q-analogue to that of $F\mathfrak{S}_n$, and many results about the symmetric group could be recovered in this way.

Relative projectivity and vertices of Hecke algebras were first introduced by Jones in [25], generalising the results from local representation theory of finite groups (see for example [1]). If $\lambda \vDash n$, and M is a $\mathscr{H}_n$-module, we say that $\mathfrak{S}_\lambda$ is a **vertex** of M if M is relatively $\mathscr{H}_\lambda$-projective, and if for any $\mu \vDash n$ with M relatively $\mathscr{H}_\mu$-projective, then a conjugate of $\mathfrak{S}_\lambda$ is a subgroup of $\mathfrak{S}_\mu$.

We show in Chapter 2 various properties of vertices, before extending the Green correspondence in [13] (analogous to the classical correspondence in the local representation theory of finite groups) to bimodules. In Chapter 3, we further extend this to blocks of $\mathscr{H}_n$, giving a Brauer correspondence:

Theorem (Brauer correspondence for Hecke algebras)**.** *Let $n = a + de$, with $\mu = (a, de)$, and $\lambda = (1^a, \lambda_1, \ldots, \lambda_s)$, where $(\lambda_1, \ldots, \lambda_s) \vDash de$ and $\lambda_i \neq 1$ for all i. Then there is a one-to-one correspondence between blocks of $\mathscr{H}_\mu$ with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ and blocks of $\mathscr{H}_n$ with the same vertex.*

Given this, we are able to explicitly compute vertices of the blocks of $\mathscr{H}_n$, by computing the vertex of a block of some maximal parabolic subalgebra $\mathscr{H}_\mu$, and identifying its Brauer correspondent. To do this, we need the following definitions. Given $n \in \mathbb{N}$, write n as its e-p-adic expansion:

$$n = a_{-1} + a_0 e + a_1 ep + \cdots + a_t ep^t,$$

where $0 \leqslant a_{-1} < e$ and $0 \leqslant a_i < p$, for $i \geqslant 0$. If n has the above e-p-adic expansion, the standard maximal e-p-parabolic subgroup of $\mathfrak{S}_n$ is the subgroup corresponding to the composition:

$$(1^{a_{-1}}, e^{a_0}, (ep)^{a_1}, \ldots, (ep^t)^{a_t}) \vDash n.$$

A general e-p-parabolic subgroup of $\mathfrak{S}_n$ corresponds to $\tau = (\tau_1, \ldots, \tau_s) \vDash n$, which has for each i either $\tau_i = 1$ or $\tau_i = ep^{r_i}$ for some $r_i \geqslant 0$.

Theorem (Classification of vertices of blocks of Hecke algebras)**.** *Let F be an algebraically closed field, $q \in F^\times$ with quantum characteristic $e > 0$, and $B = B_{\rho,d}$ the block of $\mathscr{H}_n := \mathscr{H}_n(F, q)$ corresponding to the e-core ρ and e-weight d. If $d = 0$, then B is a projective $(\mathscr{H}_n, \mathscr{H}_n)$-bimodule. Otherwise, let $\tau = (\tau_1, \ldots, \tau_s)$ be the composition corresponding to the e-p-adic expansion of de, and define $\lambda = (1^{|\rho|}, \tau_1, \ldots, \tau_s)$. Then the vertex of B as a $(\mathscr{H}_n, \mathscr{H}_n)$-bimodule is $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$.*

In [6], Dipper and Du showed that for trivial and alternating source modules of $\mathscr{H}_n$, the vertex will always be an e-p-parabolic subgroup, and conjectured that this should hold for any indecomposable $\mathscr{H}_n$-module. This was shown to be true if $p = 0$ in [13], and proved for blocks of finite representation type (i.e. by [15, Theorem 1.2] blocks of e-weight 1) in [31]. As a corollary to the previous theorem, we are able to resolve this conjecture at the end of Chapter 3, proving:

Conjecture (Dipper–Du)**.** *Let F be an (algebraically closed) field of prime characteristic, $n \in \mathbb{N}$, and $q \in F^\times$ with quantum characteristic $e > 0$. Then the vertices of indecomposable $\mathscr{H}_n(F, q)$-modules are e-p-parabolic.*

Results from the beginning of Chapter 2, Section 2.3 and Chapter 3 have recently been published in [34].

In Chapter 4, we explore some consequences of this conjecture, by using it to compute the vertices of certain Specht modules. We show that hook Specht modules for $\mathscr{H}_e$ all have full vertex $\mathfrak{S}_e$ and show that if $e \nmid n$, then the Specht module corresponding to the hook $(r, 1^{n-r})$ has the maximal e-p-parabolic subgroup of $\mathfrak{S}_{r-1} \times \mathfrak{S}_{n-r}$ as its vertex. This gives a result for Hecke algebras corresponding to [35, Theorem 2].

Inspired by [20, Theorem 4.2], which states that all irreducible Specht modules for the symmetric group are signed Young modules (summands of a module induced from a trivial module tensored with a sign module), in Chapter 5 we explore signed permutation modules for the Hecke algebra. We do this by looking at the q-Schur superalgebra $\mathcal{S}_q(m_0|m_1, n)$ for some $m_i \geqslant 0$ and giving a method of assigning a defect group $D(f)$ to a primitive idempotent f in prime characteristic. This lets us generalise [14, Theorem 10.2] to fields of any characteristic:

Theorem. *Let F be a field of characteristic $p \geqslant 0$, and f a primitive idempotent of $\mathcal{S}_q(m_0|m_1, n)$. Then $D(f)$ is the vertex of the corresponding signed Young module.*

Next we look at possible extensions of this work to the Hecke algebra of $\mathfrak{W}_n$, the Weyl group of type B_n. We give similar combinatorial methods to those in [26, Proposition 3.3, Proposition 4.4] which allow us to compute minimal right and double coset representatives. Finally we compute the vertex of the sign module for this algebra, providing evidence towards a possible version of the Dipper–Du conjecture for Hecke algebras of type B.

1.1 The Hecke algebra

Denote by $\mathfrak{S}_n$ the symmetric group on $\{1, \dots, n\}$, generated by the elementary transpositions $s_i = (i, i+1)$ for $1 \leqslant i < n$. Let F be an algebraically closed field of characteristic $p \geqslant 0$, pick $q \in F^\times$, and denote by $\mathscr{H}_n := \mathscr{H}_n(F, q)$ the associative algebra over F given by the following generators and relations:

$$\{T_i : i = 1, \dots, n-1\},$$
$$(T_i + 1)(T_i - q) = 0 \text{ for } 1 \leqslant i < n,$$
$$T_i T_j = T_j T_i \text{ if } |i - j| > 1,$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } 1 \leqslant i < n-1.$$

This is the **Iwahori–Hecke algebra of type** A_{n-1} corresponding to F and q, henceforth just known as a Hecke algebra. Note that if we choose $q = 1$, then $\mathscr{H}_n$ is isomorphic to the group algebra $F\mathfrak{S}_n$. In fact, by for example [26, Theorem 1.13], both algebras have a basis indexed by elements of the symmetric group.

Theorem 1.1. *$\mathscr{H}_n$ has an F-basis $\{T_w : w \in \mathfrak{S}_n\}$ where:*

$$T_w = T_{i_1} \dots T_{i_t},$$

if $w = s_{i_1} \dots s_{i_t}$ is any reduced expression for w. This does not depend on the choice of reduced expression.

Under this convention $T_{s_i} = T_i$. As T_w is a product of elementary T_i, we get from [26, Lemma 1.12] the following multiplication formula for $\mathscr{H}_n$, which we will apply liberally throughout this thesis.

$$T_w T_i = \begin{cases} T_{ws_i} & \text{if } \ell(ws_i) > \ell(w), \\ (q-1)T_w + qT_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \end{cases}$$

where ℓ is the usual length function on a Weyl group. Again, one can see that if $q = 1$, then these multiplication rules simplify to those of $F\mathfrak{S}_n$.

Let e be the smallest integer such that $1 + q + \cdots + q^{e-1} = 0$ if it exists, otherwise set $e = 0$, and call e the **quantum characteristic** of $\mathscr{H}_n$. If $e > 1$ and $q \neq 1$, by minimality q is a primitive e-th root of unity in F. This quantum characteristic plays a similar role for $\mathscr{H}_n$ as the field characteristic p plays for $F\mathfrak{S}_n$. One example of this is in [26, Corollary 3.44]:

Theorem 1.2. *$\mathscr{H}_n$ is semi-simple if and only if $e = 0$ or $e > n$.*

This mirrors Maschke's theorem (for example [1, Theorem 3.1]) when applied to $\mathfrak{S}_n$, where $F\mathfrak{S}_n$ is semi-simple if and only if $p = 0$ or $p \nmid |\mathfrak{S}_n|$, i.e. $p > n$. As such, in this thesis we will restrict ourselves to the case where $e > 1$, i.e. q is a root of unity in F. If our field F has characteristic $p > 0$, we can focus on two cases.

Proposition 1.3. *If F has characteristic $p > 0$, then either $q = 1$ and $e = p$, or q is a primitive e-th root of unity, and* $\mathrm{hcf}(e, p) = 1$.

1.2 Parabolics, partitions, and tableaux

Throughout, we use the conventions and notation from [26]. In particular note that multiplication of elements in $\mathfrak{S}_n$ proceeds from left to right, as we will be using right actions throughout. Say that λ is a **composition** of n (and write $\lambda \vDash n$) if $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a tuple of positive integers with $\sum_{i=1}^{s} \lambda_i = n$. If in addition $\lambda_i \geqslant \lambda_{i+1}$ for $1 \leqslant i < s$, we say λ is a **partition** of n (and write $\lambda \vdash n$). We denote the unique composition of 0 by $\varnothing$. For $\lambda = (\lambda_1, \ldots, \lambda_s) \vDash n$, define the **parabolic subgroup** $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$ as follows:

$$\begin{aligned}\mathfrak{S}_\lambda &= \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \ldots \times \mathfrak{S}_{\{\left(\sum_{i=1}^{s-1}\lambda_i\right)+1,\ldots,\sum_{i=1}^{s}\lambda_s\}}, \\ &\cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \ldots \times \mathfrak{S}_{\lambda_s}.\end{aligned}$$

Alternatively, a parabolic subgroup can also be defined by choosing a subset of the elementary transpositions and taking the subgroup generated by them. In other texts, for example [23], this notion defines a standard parabolic subgroup, and a general parabolic subgroup is one that is conjugate to some standard parabolic subgroup. For our purposes, we are only interested in these subgroups up to conjugacy, and thus this definition suffices.

Given a composition λ, we can also define its corresponding **parabolic subalgebra** $\mathscr{H}_\lambda$ of $\mathscr{H}_n$ as the following F-span:

$$\mathscr{H}_\lambda = \langle T_w : w \in \mathfrak{S}_\lambda \rangle.$$

This is a subalgebra from our previous multiplication rules; if $w, v \in \mathfrak{S}_\lambda$, then v is made up of elementary transpositions all of which lie in $\mathfrak{S}_\lambda$, and if $w, s_i \in \mathfrak{S}_\lambda$, then $ws_i \in \mathfrak{S}_\lambda$. Note that we can implicitly identify $\mathscr{H}_\lambda$ with $\mathscr{H}_{\lambda_1} \otimes \ldots \otimes \mathscr{H}_{\lambda_s}$, the s-fold tensor product over F in the following way. Let T_j be a generator of $\mathscr{H}_\lambda$ with $j = \sum_{i=1}^{k-1} \lambda_i + l$, for $1 \leqslant l < \lambda_k$. Then we identify T_j with the following simple tensor:

$$\underbrace{1 \otimes \ldots \otimes 1}_{k-1} \otimes T_l \otimes \underbrace{1 \otimes \ldots \otimes 1}_{s-k},$$

so T_l lies in the k-th part of the tensor product. We will do this implicitly throughout this thesis.

Given a composition $\lambda \vDash n$, its **Young diagram** $[\lambda]$ is a left-justified diagram with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. By filling these boxes with the numbers $\{1, \ldots, n\}$, each with multiplicity one, we get a **Young tableau** of shape λ. Some particularly important tableaux include the **row standard** tableaux, where the numbers in each row increase from left to right, and if in addition λ is a partition, the **standard** tableaux, where they also increase down each column. Denote the set of standard tableaux of shape λ by $\mathrm{Std}(\lambda)$. Finally we have the unique tableau $\mathfrak{t}^\lambda \in \mathrm{Std}(\lambda)$ where the numbers $\{1, \ldots n\}$ are placed in increasing order first along the first row, then the second row, and so on.

Relating this back to the symmetric group, if $\lambda \vDash n$ we have a right action of $\mathfrak{S}_n$ on the set of λ-tableaux, where for $w \in \mathfrak{S}_n$, we replace the number i in its box by $(i)w$. We illustrate this in the following example, where $\lambda = (4, 3, 1) \vdash 8$.

$$\mathfrak{t}^\lambda \cdot (3,6,8,5) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \cdot (3,6,8,5) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 4 \\ \hline 3 & 8 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Note also that under this action, the parabolic subgroup $\mathfrak{S}_\lambda$ is the row stabiliser of $\mathfrak{t}^\lambda$. As there are $n!$ possible λ-tableaux, for any $\lambda \vDash n$ we have a bijection d between tableaux of shape λ, and elements of $\mathfrak{S}_n$, defined by:

$$\mathfrak{t}^\lambda \cdot d(\mathfrak{t}) = \mathfrak{t}.$$

There are two important partial orders relating to compositions and row standard tableaux. If $\lambda, \mu \vDash n$, we define the **dominance order** on compositions via:

$$\lambda \trianglerighteq \mu \text{ if } \sum_{j=1}^{i} \lambda_j \geqslant \sum_{j=1}^{i} \mu_j \text{ for all } i \geqslant 1,$$

where we use the convention that $\lambda_j = 0$ if $j > s$, the number of non-zero parts of λ. For a row standard λ-tableau $\mathfrak{t}$ and some $m \leqslant n$, let $\mathrm{Shape}(\mathfrak{t}, m)$ be the shape of the row standard tableau obtained by deleting entries in $\mathfrak{t}$ that are greater than m. Then we can give a dominance ordering on row standard tableaux by saying that for a λ-tableau $\mathfrak{t}$ and μ-tableau $\mathfrak{s}$:

$$\mathfrak{t} \trianglerighteq \mathfrak{s} \text{ if } \operatorname{Shape}(\mathfrak{t}, m) \trianglerighteq \operatorname{Shape}(\mathfrak{s}, m),$$

for all $1 \leqslant m \leqslant n$. In particular note that the standard tableau $\mathfrak{t}^\lambda$ is maximal out of all row standard λ-tableaux with respect to this order.

The final concepts we require of diagrams are the notions of e-hooks and e-cores. Given $\lambda \vdash n$, an **e-hook** is a chain of boxes of length e that can be removed from the rim of $[\lambda]$ to get another Young diagram which corresponds to some $\rho \vdash n - e$. The **e-core** of λ is the partition associated to the diagram gained from $[\lambda]$ by recursively removing as many e-hooks as possible. This core does not depend on the choice of hooks removed, for example see [24, Theorem 2.7.16], so the e-core of a partition is well-defined. Finally, the **e-weight** of a partition, is the number of e-hooks you need to remove to reach its e-core. As we will see in Theorem 1.14, e-cores and e-weights help us to determine the blocks of $\mathscr{H}_n$.

1.3 Combinatorics of the symmetric group

Here we collect some important notions of the symmetric group which will be crucial throughout the thesis.

Let $\sigma, \lambda, \nu \vDash n$ be compositions with both $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_\sigma$, and $\mathfrak{S}_\nu \subseteq \mathfrak{S}_\sigma$. Denote by $\mathcal{R}_\lambda^\sigma$ the set of **minimal right coset representatives** of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_\sigma$, denote by $\mathcal{L}_\lambda^\sigma$ the set of **minimal left coset representatives** of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_\sigma$, and denote $\mathcal{D}_{\lambda,\nu}^\sigma$ to be the set of **minimal double coset representatives** of $\mathfrak{S}_\lambda$ and $\mathfrak{S}_\nu$ in $\mathfrak{S}_\sigma$. If $\sigma = (n)$, then we may just write $\mathcal{R}_\lambda$ etc. By a minimal coset representative, we mean the unique element in that coset which is shortest with respect to ℓ. We can relate these minimal coset representatives and row standard tableaux in the following way (combining [26, Proposition 3.3, Proposition 4.4]).

Proposition 1.4. *Let $\lambda, \mu \vDash n$. Then if $\mathcal{R}_\lambda^{-1} := \{d^{-1} : d \in \mathcal{R}_\lambda\}$:*

$$\mathcal{L}_\lambda = \mathcal{R}_\lambda^{-1},$$

$$\mathcal{R}_\lambda = \{d \in \mathfrak{S}_n : \mathfrak{t}^\lambda d \textit{ is row standard}\},$$

$$\mathcal{D}_{\lambda,\mu} = \mathcal{R}_\lambda \cap \mathcal{R}_\mu^{-1} = \mathcal{R}_\lambda \cap \mathcal{L}_\mu.$$

Finally every element $w \in \mathfrak{S}_n$ can be written uniquely as $w = vdu$ where $v \in \mathfrak{S}_\lambda$, $d \in \mathcal{D}_{\lambda,\mu}$ and $u \in \mathcal{R}_\nu^\mu$ where $\mathfrak{S}_\nu = \mathfrak{S}_\lambda^d \cap \mathfrak{S}_\mu$. Furthermore $\ell(w) = \ell(v) + \ell(d) + \ell(u)$.

As a consequence of these properties, we can determine minimal double coset representatives for maximal parabolics, as stated in the following lemma.

Lemma 1.5. *Let $\mu = (a, b) \vDash a + b = n$. Then:*

$$\mathcal{D}_{\mu,\mu} = \left\{ d_k = \prod_{i=1}^{k} (a - k + i, a + i) \;:\; k = 0, \ldots, \min(a, b) \right\}.$$

Proof. First note each d_k is well-defined (with $d_0 = 1$) and $d_k = d_k^{-1}$, since it is the product of non-intersecting transpositions. Applying the previous proposition, we just need to show that $d_k \in \mathcal{R}_\mu$ to show $d_k \in \mathcal{D}_{\mu,\mu}$. Each d_k swaps the last k elements on the first row of the tableau $\mathfrak{t}^\mu$ with the first k elements on the second row, preserving their order. Hence $\mathfrak{t}^\mu \cdot d_k$ is row standard. Again by Proposition 1.4, $d_k \in \mathcal{R}_\mu$, and hence $d_k \in \mathcal{D}_{\mu,\mu}$ for $0 \leqslant k \leqslant \min(a, b)$.

To show we've found all of $\mathcal{D}_{\mu,\mu}$, we take $w \in \mathfrak{S}_n$ and show there exists some k with $w \in \mathfrak{S}_\mu d_k \mathfrak{S}_\mu$. Equivalently, by minimality, we show that for all $y \in \mathcal{R}_\mu$ that $y \in d_k \mathfrak{S}_\mu$ for some k. If $y \in \mathcal{R}_\mu$, then $\mathfrak{t}^\mu y$ is a row standard tableau with two rows. Suppose we have k elements from $\{a + 1, \ldots, m\}$ in the top row of $\mathfrak{t}^\mu y$ (and thus k elements from $\{1, \ldots, a\}$ in the bottom row). As it is row standard, these must lie in the last k boxes of the first row and first k boxes of the second row respectively, similar to that of $\mathfrak{t}^\mu d_k$. As entries from $\{1, \ldots, a\}$ and $\{a+1, \ldots, m\}$ lie in the same sections in both tableaux, there exists some $z \in \mathfrak{S}_\mu$ with $t^\mu d_k z = t^\mu y$, so $y = d_k z$ as required. □

Another key property of minimal coset representatives comes from how they behave with the length function. We expect the following may already be known.

Lemma 1.6. *Let $\lambda \vDash n$ and $w \in \mathcal{R}_\lambda$. Then for $s_i \in \mathfrak{S}_n$ either:*

- *$ws_i \in \mathcal{R}_\lambda$.*
- *$ws_i \notin \mathcal{R}_\lambda$ and $\ell(ws_i) > \ell(w)$. Furthermore there exists $s_k \in \mathfrak{S}_\lambda$ with $ws_i = s_k w$.*

Proof. We begin by showing that if $\ell(ws_i) < \ell(w)$, then $ws_i \in \mathcal{R}_\lambda$. By minimality there exists $t \in \mathfrak{S}_\lambda$ and $v \in \mathcal{R}_\lambda$ with $ws_i = tv$, thus $\ell(w) - 1 = \ell(t) + \ell(v)$ by minimality of v. Similarly, $t^{-1}w = vs_i$, so $\ell(t^{-1}w) = \ell(t) + \ell(w) = \ell(v) \pm 1$ by minimality of w. Combining these shows that $\ell(t) \leqslant 0$, so t is the identity element and $v = ws_i \in \mathcal{R}_\lambda$. Taking the contrapositive, if $ws_i \notin \mathcal{R}_\lambda$, then $\ell(ws_i) > \ell(w)$.

Continuing in this case, we have $\ell(ws_i) = \ell(w) + 1$. Again there exists $v \in \mathcal{R}_\lambda$ and $t \in \mathfrak{S}_\lambda$ with $ws_i = tv$, but now with $\ell(t) + \ell(v) = \ell(w) + 1$. In this setup, again $t^{-1}w = vs_i$, and as w is minimal, $\ell(t) + \ell(w) = \ell(v) \pm 1$, as we do not know if s_i increases or decreases the length of v. Rearranging these gives the following system of equations:

$$\ell(w) = \ell(v) + \ell(t) - 1,$$
$$\ell(w) = \ell(v) - \ell(t) \pm 1,$$

which simplifies to:

$$2\ell(t) = 1 \pm 1.$$

The only possible solution here as $ws_i \notin \mathcal{R}_\lambda$ is that $\ell(t) = 1$, and so $t = s_k$ for some $s_k \in \mathfrak{S}_\lambda$. This also tells us that $\ell(w) = \ell(v)$. We now proceed with the deletion condition [23, Theorem 1.7c] to show that $w = v$.

Let $w = s_{i_1} \dots s_{i_r}$ be a reduced expression for w. Then $ts_{i_1} \dots s_{i_r} s_i$ is an expression for v of length $r + 2$, so by the deletion condition, we get a reduced expression for v by deleting two of these reflections. Suppose we delete s_{i_j} and s_{i_l} for some $j < l$. Note that $w = tvs_i$, meaning that by pre-multiplying v by t and post-multiplying v by s_i, we have an expression for w of length $r - 2$ which is impossible. Thus we delete at most one of the s_{i_j} to get a reduced expression for v.

Again, let us suppose we delete some s_{i_j}, and s_i to get a reduced expression for v (a similar method holds if we delete t instead). Then $ts_{i_1} \dots s_{i_{j-1}} s_{i_{j+1}} \dots s_{i_r}$ is a reduced expression for v, and thus $s_{i_1} \dots s_{i_{j-1}} s_{i_{j+1}} \dots s_{i_r}$ is an expression for $tv = ws_i$ of length $r - 1$. However, ws_i has length $r + 1$, so again this is impossible. Thus we must delete both t and s_i to get a reduced expression for v, so:

$$w = s_{i_1} \dots s_{i_r} = v. \qquad \square$$

Note that this proof works for any finite Weyl group, and not just for $\mathfrak{S}_n$. For $\mathfrak{S}_n$, the second part of this Lemma can also be proved directly using tableaux.

The next proposition gives a method of computing the smallest parabolic containing the normaliser of a given parabolic subgroup.

Proposition 1.7. *Let $\lambda = (\lambda_1, \dots, \lambda_r) \vDash n$. Write $\mathfrak{S}_\lambda$ as follows:*

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,\dots,i_1\}} \times \mathfrak{S}_{\{i_1+1,\dots,i_2\}} \times \dots \times \mathfrak{S}_{\{i_{r-1}+1,\dots,i_r\}}$$

where $i_k = \sum_{j=1}^{k} \lambda_j$. Then the minimal parabolic subgroup of $\mathfrak{S}_n$ containing $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda)$ is generated by the following elementary transpositions:

- $(i, i+1)$ *where* $(i, i+1) \in \mathfrak{S}_\lambda$,
- $(i_j, i_j + 1)$ *where there is some* $1 \leqslant m \leqslant j < k \leqslant r$ *with* $\lambda_m = \lambda_k$.

Proof. Denote by $\mathfrak{S}_\mu$ the parabolic subgroup of $\mathfrak{S}_n$ generated by the above transpositions, and note that by definition, $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_\mu$. By [21, Corollary 3], $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda)$ can be written as $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda) = \mathfrak{S}_\lambda \cdot \mathcal{X}_\lambda$ where:

$$\mathcal{X}_\lambda = \{x \in \mathfrak{S}_n : s_i^x \in \mathfrak{S}_\lambda \text{ for all } s_i \in \mathfrak{S}_\lambda\}.$$

Now elements of $\mathcal{X}_\lambda$ must send each part of $\mathfrak{S}_\lambda$ to another part of the same length. Thus as to do this we need all elementary transpositions between parts of the same length, we have $\mathcal{X}_\lambda$, and hence $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda) \subseteq \mathfrak{S}_\mu$.

Furthermore, for any parabolic containing the normaliser, we need the element $y \in \mathcal{X}_\lambda$ which swaps the first row of length l with the last row of length l, for all possible lengths l to lie inside it. As we need all the transpositions in the definition to write this y, we have that $\mathfrak{S}_\mu$ is minimal amongst parabolics with this property. □

1.4 Modules for the Hecke algebra

Throughout this thesis, we will unless stated, assume that modules are finite-dimensional right modules. Similarly, we will write module homomorphisms on the right. As with the symmetric group, key modules of interest are the permutation modules and the Specht modules. We recap these notions as stated in [26, §3].

For $\lambda \vDash n$, let $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$. Then the **permutation module** corresponding to λ is given by:

$$M^\lambda := x_\lambda \mathscr{H}_n.$$

As (see for example [26, Lemma 3.2]) $x_\lambda T_w = q^{\ell(w)} x_\lambda$, for $w \in \mathfrak{S}_\lambda$, we have $F x_\lambda$ is a 1-dimensional $\mathscr{H}_\lambda$-module, which specialises to the trivial module for $\mathfrak{S}_\lambda$ when $q = 1$. By [25, Lemma 2.19], $M^\lambda \cong F x_\lambda \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n$, thus these permutation modules correspond to the usual notion of permutation modules for $\mathfrak{S}_n$.

Multiplication in these permutation modules can be stated in terms of row standard tableaux [26, Corollary 3.4]:

Lemma 1.8. *Let $\lambda \models n$. Then $\{x_\lambda T_{d(\mathfrak{t})} : \mathfrak{t}$ is a row standard λ-tableau$\}$ is an F-basis of M^λ, and for $1 \leqslant i < n$:*

$$x_\lambda T_{d(\mathfrak{t})} T_i = \begin{cases} q x_\lambda T_{d(\mathfrak{t})} & \text{if } \mathfrak{t}s_i \text{ is not row standard,} \\ x_\lambda T_{d(\mathfrak{t}s_i)} & \text{if } \mathfrak{t}s_i \text{ is row standard and } \ell(d(\mathfrak{t}s_i)) > \ell(d(\mathfrak{t})), \\ (q-1)x_\lambda T_{d(\mathfrak{t})} + q x_\lambda T_{d(\mathfrak{t}s_i)} & \text{if } \mathfrak{t}s_i \text{ is row standard and } \ell(d(\mathfrak{t}s_i)) < \ell(d(\mathfrak{t})). \end{cases}$$

1.4.1 Specht Modules

Let $\lambda \vdash n$ and for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$, denote $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})^{-1}} x_\lambda T_{d(\mathfrak{t})}$. By [26, Theorem 3.20], $\mathscr{H}_n$ is a cellular algebra with cellular basis (known as the Murphy basis) given below.

$$\{m_{\mathfrak{s}\mathfrak{t}} : \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some } \lambda \vdash n\}.$$

Let $\check{\mathscr{H}}^\lambda$ be the two-sided ideal of $\mathscr{H}_n$ with basis

$$\{m_{\mathfrak{u}\mathfrak{v}} : \mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\nu) \text{ for some } \nu \rhd \lambda\}.$$

With this cellular structure, we consider the cell modules S^λ for $\lambda \vdash n$, which are called the **Specht modules**. Denoting $m_\mathfrak{t} = m_{\mathfrak{t}^\lambda \mathfrak{t}} + \check{\mathscr{H}}^\lambda$ for $\mathfrak{t} \in \mathrm{Std}(\lambda)$, the Specht module S^λ can be viewed via [26, Proposition 3.22] as the right $\mathscr{H}_n$-module which is free as an F-module with basis:

$$\{m_\mathfrak{t} : \mathfrak{t} \in \mathrm{Std}(\lambda)\}.$$

Note that these Specht modules correspond to the dual of the Specht modules used by Dipper and James in [8]. Rules for multiplication by elements of $\mathscr{H}_n$ in these modules can be gained by taking Lemma 1.8 and [26, Corollary 3.21] modulo $\check{\mathscr{H}}^\lambda$, as summarised in the following corollary.

Corollary 1.9. *Let $\lambda \vdash n$, and $\mathfrak{t} \in \mathrm{Std}(\lambda)$. Then in S^λ:*

$$m_\mathfrak{t} T_i = \begin{cases} q m_\mathfrak{t} & \text{if } \mathfrak{t}s_i \text{ is not row standard,} \\ m_{\mathfrak{t}s_i} & \text{if } \mathfrak{t}s_i \text{ is standard and } \ell(d(\mathfrak{t}s_i)) > \ell(d(\mathfrak{t})), \\ (q-1)m_\mathfrak{t} + q m_{\mathfrak{t}s_i} & \text{if } \mathfrak{t}s_i \text{ is standard and } \ell(d(\mathfrak{t}s_i)) < \ell(d(\mathfrak{t})). \end{cases}$$

If i and $i+1$ lie in the same column, then there exists $r_\mathfrak{v} \in F$ with:

$$m_\mathfrak{t} T_i = -m_\mathfrak{t} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda), \\ \mathfrak{v} \rhd \mathfrak{t}}} r_\mathfrak{v} m_\mathfrak{v}.$$

In particular, note that $S^{(n)}$ is the trivial $\mathscr{H}_n$-module (all generators of $\mathscr{H}_n$ act by multiplication by q), and $S^{(1^n)}$ is the sign module (all generators act as multiplication by -1) as there is only one standard (1^n)-tableau.

From the cellular structure, we have a symmetric associative bilinear map $\langle\ ,\ \rangle$ on each S^λ, defined for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ by:

$$m_{\mathfrak{u}\mathfrak{s}} m_{\mathfrak{t}\mathfrak{v}} \equiv \langle m_\mathfrak{s}, m_\mathfrak{t} \rangle m_{\mathfrak{u}\mathfrak{v}} \mod \check{\mathscr{H}}^\lambda,$$

where $\mathfrak{u}$ and $\mathfrak{v}$ are any elements of $\mathrm{Std}(\lambda)$. This is independent of the choice of $\mathfrak{u}$ and $\mathfrak{v}$ by [19, Lemma 1.7]. Let $\mathrm{rad}\, S^\lambda$ be the radical of this form, then for each partition define:

$$D^\lambda = S^\lambda / \mathrm{rad}\, S^\lambda.$$

We say that a partition $\lambda \vdash n$ is e**-restricted** if $\lambda_i - \lambda_{i+1} < e$ for all admissible i. With this, we can classify the irreducible modules for $\mathscr{H}_n$ [26, Theorem 3.43].

Theorem 1.10. *A complete set of non-isomorphic irreducible $\mathscr{H}_n$-modules is given by:*

$$\{D^\lambda : \lambda \textit{ is an e-restricted partition of } n\}.$$

In addition if λ is not e-restricted, then $D^\lambda = 0$. The cellular structure of $\mathscr{H}_n$ lets us say more about e-restricted Specht modules.

Lemma 1.11. *Let $\lambda \vdash n$ be an e-restricted partition. Then S^λ is an indecomposable $\mathscr{H}_n$-module.*

Proof. By [19, Corollary 2.6′] and [26, §2]) we get that $\mathrm{End}_{\mathscr{H}_n}(S^\lambda) \cong F$, and thus S^λ is indecomposable. □

In fact when $e > 2$, by [32, Theorem 1.44] we can say even more:

Theorem 1.12. *If $e > 2$, and $\lambda \vdash n$, then S^λ is indecomposable.*

Returning to the semi-simple case, we have by [26, Corollary 3.44]:

Corollary 1.13. *If $\mathscr{H}_n$ is semi-simple, then $S^\lambda = D^\lambda$ for all $\lambda \vdash n$.*

Thus we can see that the Specht modules for $\mathscr{H}_n$ play a similar role to the Specht modules for the symmetric group (see for example [30, Theorem 2.4.6]).

By Nakayama's conjecture (as stated in [24, Theorem 6.1.21]), the blocks of the group algebra $F\mathfrak{S}_n$ can be parameterised by p-cores and p-weights. This includes the cases where $p = 0$ or $p > n$, where every partition is a p-core, and thus lies in its own block.

Similarly for $\mathscr{H}_n$, the quantum characteristic e determines a labelling of the blocks with [26, Corollary 5.38] giving a $\mathscr{H}_n$-version of Nakayama's conjecture.

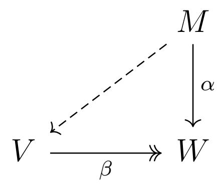
Theorem 1.14 (Nakayama's conjecture for $\mathscr{H}_n$)**.** *Let $\lambda, \mu \vdash n$. Then the Specht modules S^λ and S^μ lie in the same block if and only if λ and μ have the same e-core.*

Note that even though the Specht modules may not be indecomposable when $e = 2$, the cellular structure (see for example [26, Corollary 2.22]) still guarantees that they will lie in a single block. Thus we can label the blocks of $\mathscr{H}_n$ by e-cores and e-weights, denoting the block of $\mathscr{H}_n$ corresponding to e-core ρ and e-weight d by $B_{\rho,d}$.

CHAPTER 2

VERTICES FOR HECKE ALGEBRAS

Let A be an F-algebra with subalgebra $A' \subseteq A$. Recall that an A-module M is **relatively A'-projective** (or just A'-projective) if for any A-modules V and W with A-module homomorphisms α and β as in the below diagram, the existence of an A'-module homomorphism from M to V making the diagram commute, implies there is also an A-module homomorphism from M to V making the diagram commute.



Note that if we take $A' = F$, we obtain our usual notion of projectivity. A more practical definition of relative projectivity for A-modules is given by the equivalences in the theorem below (see for example [25, Theorem 2.34] for the Hecke algebra version). For two modules M and N, we use the notation $M \mid N$ to say that M is isomorphic to a direct summand of N.

Theorem 2.1. *Let $A' \subseteq A$ be F-algebras, and let M be a right A-module. Then the following are equivalent:*

(a) M is A'-projective,

(b) $M \mid M \otimes_{A'} A$,

(c) $M \mid U \otimes_{A'} A$ *for some* A'*-module* U.

We have the following corollaries. First of all, by definition, it is clear that if we have subalgebras $A'' \subseteq A' \subseteq A$, and M is an A''-projective A-module, then it is also an A'-projective A-module. Similarly we have:

Corollary 2.2. *Let* $A'' \subseteq A' \subseteq A$ *be* F*-algebras. Then for an* A*-module* M*, if* M *is relatively* A'*-projective as an* A*-module, and relatively* A''*-projective as an* A'*-module, then* M *is relatively* A''*-projective as an* A*-module.*

We also have the following corollary about how relative projectivity behaves when tensoring two modules over F.

Corollary 2.3. *Let* $A' \subseteq A$ *and* $B' \subseteq B$ *be* F*-algebras,* M *an* A'*-projective* A*-module, and* N *a* B'*-projective* B*-module. Then* $M \otimes N$ *is* $A' \otimes B'$*-projective as an* $A \otimes B$*-module.*

Proof. By Theorem 2.1, $M \mid M \otimes_{A'} A$ and $N \mid N \otimes_{B'} B$. Therefore tensoring together over F gives us as $A \otimes B$-modules:

$$M \otimes N \mid (M \otimes_{A'} A) \otimes (N \otimes_{B'} B).$$

It is straightforward to verify that the natural map φ defined on elementary tensors as

$$(m \otimes a) \otimes (n \otimes b) \mapsto (m \otimes n) \otimes (a \otimes b),$$

for $m \in M$, $n \in N$, $a \in A$ and $b \in B$ gives an $A \otimes B$-module isomorphism:

$$\varphi : (M \otimes_{A'} A) \otimes (N \otimes_{B'} B) \to (M \otimes N) \otimes_{A' \otimes B'} (A \otimes B).$$

As such, we can conclude by Theorem 2.1. $\square$

In [25, Theorem 2.29], a Mackey formula for Hecke algebras was given, and as a consequence, Jones was able to make concrete the notion of a **vertex** of a $\mathscr{H}_n$-module [25, Theorem 2.35]. For a $\mathscr{H}_n$-module M, this is a parabolic subgroup $\mathfrak{S}_\lambda$ (for some $\lambda \models n$) such that M is $\mathscr{H}_\lambda$-projective, and for any $\mu \models n$, if M is $\mathscr{H}_\mu$-projective, then a conjugate of $\mathfrak{S}_\lambda$ is contained in $\mathfrak{S}_\mu$. This is not unique, but it is determined up to conjugation in $\mathfrak{S}_n$. We use the notation $\mathfrak{S}_\lambda \subseteq_{\mathfrak{S}_n} \mathfrak{S}_\mu$ to say that a $\mathfrak{S}_n$-conjugate of $\mathfrak{S}_\lambda$ is contained in $\mathfrak{S}_\mu$.

Combining the notion of a vertex with our previous corollary, we can show that the vertex of a module also behaves as one would expect when taking tensor products. Throughout the rest of this chapter, we may assume that we are working with $\mathscr{H}_\sigma$-modules where $\sigma \models n$, instead of $\mathscr{H}_n$-modules. All definitions and results carry across in the same way, and this helps us work in more generality later on. We will also in future say that a module M is $\mathfrak{S}_\lambda$-projective instead of $\mathscr{H}_\lambda$-projective to mirror the notation used in [1].

Theorem 2.4. *Let $\tau_1, \sigma_1 \models m$ and $\tau_2, \sigma_2 \models n$, with $\mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\sigma_i}$ for $i = 1, 2$. If M is a $\mathscr{H}_{\sigma_1}$-module with vertex $\mathfrak{S}_{\tau_1}$, and N is a $\mathscr{H}_{\sigma_2}$-module with vertex $\mathfrak{S}_{\tau_2}$, then $M \otimes N$ has vertex $(\mathfrak{S}_{\tau_1} \times \mathfrak{S}_{\tau_2})$ as a $\mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2}$-module.*

Proof. By Corollary 2.3, $M \otimes N$ is $(\mathfrak{S}_{\tau_1} \times \mathfrak{S}_{\tau_2})$-projective as a $\mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2}$-module. Suppose that $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2}$ is a vertex of $M \otimes N$ as a $\mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2}$-module. Thus $\mathfrak{S}_{\lambda_i} \subseteq_{\mathfrak{S}_{\sigma_i}} \mathfrak{S}_{\tau_i}$ for both i. As a $\mathscr{H}_{\sigma_1}$-module, $M \otimes N$ is $\mathfrak{S}_{\lambda_1}$-projective since:

$$M \otimes N \mid (M \otimes N) \otimes_{\mathscr{H}_{\lambda_1} \otimes \mathscr{H}_{\sigma_2}} \mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2} \cong (M \otimes N) \otimes_{\mathscr{H}_{\lambda_1}} \mathscr{H}_{\sigma_1},$$

as $\mathscr{H}_{\sigma_1}$-modules, as $\mathscr{H}_{\sigma_1}$ only acts on the part induced from M. Here we used the fact that $M \otimes N$ is $(\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\sigma_2})$-projective as $\mathfrak{S}_{\lambda_2} \subseteq \mathfrak{S}_{\sigma_2}$.

Furthermore, as a $\mathscr{H}_{\lambda_1}$-module, $M \otimes N \cong M^{\oplus \dim N}$, and thus M too is $\mathfrak{S}_{\lambda_1}$-projective as a $\mathscr{H}_{\sigma_1}$-module. So, $\mathfrak{S}_{\tau_1} \subseteq_{\mathfrak{S}_{\sigma_1}} \mathfrak{S}_{\lambda_1}$ as M has vertex $\mathfrak{S}_{\tau_1}$. As we already know that a $\mathfrak{S}_{\sigma_1}$ conjugate of $\mathfrak{S}_{\lambda_1}$ is contained in $\mathfrak{S}_{\tau_1}$, we conclude that $\mathfrak{S}_{\lambda_1}$ is a conjugate of $\mathfrak{S}_{\tau_1}$.

Repeating on the other side with N, gives us that $\mathfrak{S}_{\lambda_2}$ is a conjugate of $\mathfrak{S}_{\tau_2}$, and hence $(\mathfrak{S}_{\tau_1} \times \mathfrak{S}_{\tau_2})$ is a vertex of $M \otimes N$ as a $\mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2}$-module. □

We explore how the vertex behaves under other module operations in the next section, before considering a way of lower bounding the vertex.

2.1 Module operations

Recall that as we are dealing with right modules, we are writing our module homomorphisms on the right. Denote by ${}^* : \mathscr{H}_n \to \mathscr{H}_n$ the anti-automorphism defined by $T_i^* = T_i$ (so for $w \in \mathfrak{S}_n$, by Theorem 1.1 we have $T_w^* = T_{w^{-1}}$). For a right $\mathscr{H}_n$-module M, we have that $\mathrm{Hom}_F(M, F)$ is a left $\mathscr{H}_n$-module by $(m)(hf) = (mh)f$ for $f \in \mathrm{Hom}_F(M, F)$, $h \in \mathscr{H}_n$ and $m \in M$. Using the anti-automorphism $*$, we can turn this into a right $\mathscr{H}_n$-module and define $M^* := \mathrm{Hom}_F(M, F)$ with multiplication:

$$(m)(fh) = (mh^*)f.$$

This is the **dual module to** M. We also have an automorphism ${}^\# : \mathscr{H}_n \to \mathscr{H}_n$ defined by

$$T_w \mapsto (-q)^{\ell(w)}(T_{w^{-1}})^{-1}.$$

Note in particular that $T_i^\# = -T_i + (q-1)$, and thus:

$$T_i^{\#\#} = (-T_i + (q-1))^\# = -(T_i)^\# + (q-1) = -(-T_i + (q-1)) + (q-1) = T_i.$$

This lets us define another module $M^\#$ which is isomorphic to M as a vector space, with multiplication given by $m \cdot h = mh^\#$, as in [9, §2]. When $q = 1$, we have that $T_i^\# = -T_i$ for any i, thus this is a generalisation of tensoring with the sign module (as such we will refer to this operation by that name). As in [28, Theorem 5.2], if $\lambda \vdash n$, and λ' is the conjugate partition gained by reflecting the corresponding diagram in its leading diagonal, then we have:

$$(S^{\lambda})^{*} \cong (S^{\lambda'})^{\#}.$$

That is for $\lambda \vdash n$, we can go from S^{λ} to $S^{\lambda'}$ via the maps * and $^\#$. Thus if (as in the finite group case), these operations commute with induction, we can use Theorem 2.1 to show that S^{λ} and $S^{\lambda'}$ have the same vertex.

2.1.1 Duals and induction

Let M be a $\mathscr{H}_n$-module, and take a basis $\{m_j\}$ of M. Then we can define the **dual basis** of M^* as $\{m^j\}$ where $(m_j)m^i = \delta_{i,j}$. Here we will show that the action of inducing M from a parabolic subalgebra, commutes with the action of taking duals, as it does in the finite group case.

Theorem 2.5. *Let $\lambda \vDash n$, and M a $\mathscr{H}_\lambda$-module, with basis $\{m_j\}$ and dual basis $\{m^j\}$. Then the map $\varphi : (M^* \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n) \to (M \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n)^*$ defined for $w \in \mathcal{R}_\lambda$ by $m^i \otimes T_w \mapsto f_{i,w}$ where*

$$(m_j \otimes T_v) f_{i,w} = (q^{\ell(w)} \delta_{w,v} m_j) m^i,$$

is a $\mathscr{H}_n$-module isomorphism.

Proof. Denote the standard dual basis of $(M \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n)^*$ by $g^{i,w}$ where $(m_j \otimes T_v) g^{i,w} = \delta_{i,j}\delta_{v,w}$. As $f_{i,w} = q^{\ell(w)} g^{i,w}$, it is clear that the map φ is a vector space isomorphism, and thus it remains to show that φ is a $\mathscr{H}_n$-module homomorphism.

Let $s \in \{s_1, \ldots, s_{n-1}\}$. We will show for all i, j, and for all $v, w \in \mathcal{R}_\lambda$ that:

$$(m_j \otimes T_v)(m^i \otimes T_w T_s)\varphi = (m_j \otimes T_v)\left((m^i \otimes T_w)\varphi T_s\right). \tag{2.1}$$

We begin by computing the left-hand side of this equation. Here we get three cases depending on w and s.

1. Suppose $ws \in \mathcal{R}_\lambda$ and $\ell(w) < \ell(ws)$. Then:

$$\begin{aligned}(m_j \otimes T_v)(m^i \otimes T_w T_s)\varphi &= (m_j \otimes T_v)(m^i \otimes T_{ws})\varphi, \\ &= q^{\ell(ws)}(\delta_{ws,v} m_j)m^i, \\ &= q^{\ell(w)+1}\delta_{ws,v}\delta_{i,j}. \end{aligned} \tag{2.2}$$

2. Now suppose $ws \in \mathcal{R}_\lambda$ and this time $\ell(w) > \ell(ws)$. Then:

$$\begin{aligned}(m_j \otimes T_v)(m^i \otimes T_w T_s)\varphi &= (m_j \otimes T_v)(m^i \otimes ((q-1)T_w + qT_{ws}))\varphi, \\ &= (q-1)q^{\ell(w)}(\delta_{w,v} m_j)m^i + q \cdot q^{\ell(ws)}(\delta_{ws,v} m_j)m^i, \\ &= (q-1)q^{\ell(w)}\delta_{w,v}\delta_{i,j} + q^{\ell(w)}\delta_{ws,v}\delta_{i,j}. \end{aligned} \tag{2.3}$$

3. Our third case is when $ws \notin \mathcal{R}_\lambda$, in particular this means by Lemma 1.6 that there exists some elementary transposition $t \in \{s_1, \ldots, s_{n-1}\} \cap \mathfrak{S}_\lambda$ with $ws = tw$ and $T_w T_s = T_{ws} = T_t T_w$. Then:

$$\begin{aligned}(m_j \otimes T_v)(m^i \otimes T_w Ts)\varphi &= (m_j \otimes T_v)(m^i \otimes T_t T_w)\varphi, \\ &= (m_j \otimes T_v)(m^i T_t^* \otimes T_w)\varphi, \\ &= q^{\ell(w)}(\delta_{w,v} m_j)(m^i \cdot T_t), \\ &= q^{\ell(w)}(\delta_{w,v} m_j T_t)m^i, \end{aligned} \tag{2.4}$$

by the right module structure on M^*. We now attempt to compute the right-hand side of (2.1). Note that due to the action we have that:

$$(m_j \otimes T_v)\left((m^i \otimes T_w)(\varphi T_s)\right) = (m_j \otimes T_v T_s)(m^i \otimes T_w)\varphi.$$

We split into cases relating v and w, and show in all of the above situations on w and s that (2.1) holds.

a) First of all suppose $v = w$. Then in case 1:

$$\begin{aligned}(m_j \otimes T_w T_s)(m^i \otimes T_w)\varphi &= (m_j \otimes T_{ws})(m^i \otimes T_w)\varphi, \\ &= q^{\ell(w)}(\delta_{w,ws} m_j)m^i, \\ &= 0,\end{aligned}$$

which agrees with (2.2), proving the statement in this case. When we are in the next case:

$$\begin{aligned}(m_j \otimes T_w T_s)(m^i \otimes T_w)\varphi &= (m_j \otimes (q-1)T_w + qT_{ws})(m^i \otimes T_w)\varphi, \\ &= (q-1)q^{\ell(w)}(\delta_{w,w} m_j)m^i + q \cdot q^{\ell(ws)}(\delta_{w,ws} m_j)m^i, \\ &= (q-1)q^{\ell(w)}\delta_{i,j},\end{aligned}$$

which lines up with (2.3) when $w = v$. Finally, for the third case, we have:

$$\begin{aligned}(m_j \otimes T_w T_s)(m^i \otimes T_w)\varphi &= (m_j \otimes T_t T_w)(m^i \otimes T_w)\varphi, \\ &= (m_j T_t \otimes T_w)(m^i \otimes T_w)\varphi, \\ &= q^{\ell(w)}(m_j T_t)m^i,\end{aligned}$$

agreeing with (2.4), showing that (2.1) holds in all cases when $v = w$.

b) Now suppose that $v = ws$. Then if $\ell(v) > \ell(w)$ we have:

$$\begin{aligned}(m_j \otimes T_v T_s)(m^i \otimes T_w)\varphi &= (q-1)q^{\ell(w)}(\delta_{v,w} m_j)m^i + q \cdot q^{\ell(w)}(\delta_{vs,w} m_j)m^i, \\ &= q^{\ell(w)+1}(m_j)m^i, \\ &= q^{\ell(w)+1}\delta_{i,j},\end{aligned}$$

which lines up with (2.2). Similarly if $\ell(v) < \ell(w)$ then:

$$\begin{aligned}(m_j \otimes T_v T_s)(m^i \otimes T_w)\varphi &= (m_j \otimes T_{vs})(m^i \otimes T_w)\varphi,\\ &= q^{\ell(w)}(m_j)m^i,\\ &= q^{\ell(w)}\delta_{i,j},\end{aligned}$$

which agrees with (2.3).

c) Finally, suppose $v \notin \{w, ws\}$. Note that in all cases our left-hand side evaluates to 0. Then:

$$(m_j \otimes T_v T_s)(m^i \otimes T_w)\varphi = \begin{cases} q^{\ell(w)}(\delta_{vs,w} m_j)m^i & \text{if } \ell(v) < \ell(vs), vs \in \mathcal{R}_\lambda,\\ q^{\ell(w)}([(q-1)\delta_{w,v} + q\delta_{w,vs}]m_j)m^i & \text{if } \ell(v) > \ell(vs), vs \in \mathcal{R}_\lambda,\\ q^{\ell(w)}(\delta_{w,v} m_j T_u)m^i & \text{if } vs \notin \mathcal{R}_\lambda, vs = uv.\end{cases}$$

As $v \neq w$ and $v \neq ws$, all of these evaluate to zero too, proving this final case.

Thus in all cases we have shown that φ is a $\mathscr{H}_n$-module homomorphism, hence is a $\mathscr{H}_n$-module isomorphism. □

Now we can show that taking the dual preserves relative projectivity.

Corollary 2.6. *Let $\lambda \vDash n$ and let M be a relatively $\mathfrak{S}_\lambda$-projective $\mathscr{H}_n$-module. Then M^* is relatively $\mathfrak{S}_\lambda$-projective too. Furthermore M and M^* have the same vertex as $\mathscr{H}_n$-modules.*

Proof. Using Theorem 2.1, we can assume there exists an $\mathscr{H}_\lambda$-module X such that:

$$M \mid X \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n.$$

Taking the dual of both sides and using Theorem 2.5:

$$M^* \mid (X \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n)^* \cong X^* \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n.$$

Thus M^* is isomorphic to a direct summand of the module X^* induced from $\mathscr{H}_\lambda$, and is therefore $\mathfrak{S}_\lambda$-projective. By the definition of vertex, as $M^{**} \cong M$ as $\mathscr{H}_n$-modules, we must have that M and M^* have the same vertex. □

2.1.2 Tensoring with the sign module

We now present a similar result linking induction to the $^\#$ operation. We expect the corresponding result is known for the symmetric group.

Theorem 2.7. *Let $\lambda \vDash n$, and M a $\mathscr{H}_\lambda$-module, with basis $\{m_i\}$. Then*

$$\psi : M^\# \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n \to (M \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n)^\#$$

defined by $m_i \otimes T_w \mapsto m_i \otimes T_w^\#$ for $w \in \mathcal{R}_\lambda$, is a $\mathscr{H}_n$-module isomorphism.

Proof. We begin by showing that ψ is a vector space isomorphism. As shown before, $T_i^{\#\#} = T_i$, which means that as $^\#$ is a $\mathscr{H}_n$-module homomorphism, that $(T_w)^{\#\#} = T_w$ for any $w \in \mathfrak{S}_n$. Thus for $m \in M$ and $w \in \mathfrak{S}_n$ we have:

$$(m \otimes (T_w^\#))\psi = m \otimes (T_w^{\#\#}) = m \otimes T_w.$$

Therefore ψ is a surjective vector space homomorphism. Additionally:

$$\dim M^\# \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n = |\mathcal{R}_\lambda| \dim M^\# = |\mathcal{R}_\lambda| \dim M = \dim M \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n = \dim (M \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n)^\#,$$

since for any $\mathscr{H}_n$-module N we have $\dim N = \dim N^\#$ by definition. Thus as a surjective map between vector spaces of the same dimension, ψ is a vector space isomorphism. It remains to show that ψ is a $\mathscr{H}_n$-module homomorphism.

Let $i \in \{1, \ldots, n-1\}$ and $w \in \mathcal{R}_\lambda$. Then:

$$\begin{aligned}(m_i \otimes T_w)\psi \cdot T_i &= (m_i \otimes T_w^{\#}) \cdot T_i, \\ &= m_i \otimes T_w^{\#} T_i^{\#}, \\ &= m_i \otimes (T_w T_i)^{\#}, \\ &= (m_i \otimes T_w T_i)\psi,\end{aligned}$$

as $^{\#}$ is a $\mathscr{H}_n$-module homomorphism. Thus ψ is a $\mathscr{H}_n$-module homomorphism (and hence isomorphism) as well, proving the theorem. □

Corollary 2.8. *Let $\lambda \models n$ and let M be a relatively $\mathfrak{S}_\lambda$-projective $\mathscr{H}_n$-module. Then $M^{\#}$ is relatively $\mathfrak{S}_\lambda$-projective too. Furthermore M and $M^{\#}$ have the same vertex as $\mathscr{H}_n$-modules.*

We omit the proof as it follows in the exact same way as the proof of Corollary 2.6, as $M^{\#\#} \cong M$ as $\mathscr{H}_n$-modules. Given these two properties, we can finally prove that for $\lambda \vdash n$, the vertices of S^λ and $S^{\lambda'}$ coincide. Again, the analogous result for the symmetric group is well known.

Theorem 2.9. *Let $\lambda \vdash n$. Then S^λ and $S^{\lambda'}$ have the same vertex as $\mathscr{H}_n$-modules.*

Proof. By Corollary 2.6, S^λ and $(S^\lambda)^*$ have the same vertex. Similarly, by Corollary 2.8, $S^{\lambda'}$ and $(S^{\lambda'})^{\#}$ have the same vertex. Thus applying [28, Theorem 5.2] again, the result follows as:

$$(S^\lambda)^* \cong (S^{\lambda'})^{\#}.$$

□

Furthermore, if we denote the Specht module defined by Dipper and James in [7, §4] by S_λ, then we have $S_\lambda \cong (S^{\lambda'})^* \cong (S^\lambda)^{\#}$ (see [28, Theorem 5.3] or remarks following [26, Corollary 3.21]). Thus these results will allow us to translate results on relative projectivity between both types of Specht modules.

2.2 Broué's theorem

Another key notion when considering relative projectivity for Hecke algebras is the **relative trace** for bimodules, as stated in [25]. Let $\tau, \lambda \vDash n$ with $\mathfrak{S}_\tau \subseteq \mathfrak{S}_\lambda$ and suppose B is a $(\mathscr{H}_\lambda, \mathscr{H}_\lambda)$-bimodule. Then the relative trace of $b \in B$ is:

$$\mathrm{Tr}_\tau^\lambda(b) = \sum_{w \in \mathcal{R}_\tau^\lambda} q^{-\ell(w)} T_{w^{-1}} b T_w.$$

Given $\mathscr{H}_n$-modules M and N, we have a $(\mathscr{H}_n, \mathscr{H}_n)$-bimodule structure on $\mathrm{Hom}_F(M, N)$ in the usual way. Then we say that $\phi \in \mathrm{Hom}_{\mathscr{H}_n}(M, N)$ is **$\mathfrak{S}_\tau$-projective** if there exists $\psi \in \mathrm{Hom}_{\mathscr{H}_\tau}(M, N)$ such that $\phi = \mathrm{Tr}_\tau^n(\psi)$. One of the key reasons we are interested in this trace, is due to its relevance when studying relative projectivity of modules. We summarise this in the following extension to Theorem 2.1, proved in [25, Theorem 2.34]. We will refer to the equivalences given in Theorems 2.1 and 2.10 as Higman's criterion.

Theorem 2.10 (Higman's criterion)**.** *Let M be a right $\mathscr{H}_n$-module. Then the following are equivalent for $\tau \vDash n$:*

(a) M is $\mathfrak{S}_\tau$-projective,

(b) $\mathrm{Tr}_\tau^n(\mathrm{Hom}_{\mathscr{H}_\tau}(M, M)) = \mathrm{Hom}_{\mathscr{H}_n}(M, M)$,

(c) The identity map on M is $\mathfrak{S}_\tau$-projective.

For a bimodule B, and $\tau \vDash n$, define the following set:

$$Z_B(\mathscr{H}_\tau) = \{b \in B : hb = bh, \text{ for all } h \in \mathscr{H}_\tau\}.$$

As one might guess based on the definition of relative trace in terms of coset representatives, and the relation via Higman's criterion to induced modules, we have the following crucial properties from [25, Lemma 2.12, Proposition 2.13].

Lemma 2.11. *Let B be a $(\mathscr{H}_n, \mathscr{H}_n)$-bimodule, and $b \in B$. Then for $\tau, \lambda \models n$ with $\mathfrak{S}_\tau \subseteq \mathfrak{S}_\lambda$:*

- $\mathrm{Tr}_\tau^n(b) = \mathrm{Tr}_\lambda^n\left(\mathrm{Tr}_\tau^\lambda(b)\right)$,
- $\mathrm{Tr}_\tau^\lambda(Z_B(\mathscr{H}_\tau)) \subseteq Z_B(\mathscr{H}_\lambda)$.

Now suppose M is a (right) $\mathscr{H}_\lambda$-module. Then we can make it a $(\mathscr{H}_\lambda, \mathscr{H}_\lambda)$-bimodule by giving M the trivial left action, i.e. $T_i m = qm$ for $i = 1, \ldots, n-1$. Applying the relative trace in this manner lets us define a trace for modules given by the formula:

$$\mathrm{Tr}_\tau^\lambda(m) = \sum_{w \in \mathcal{R}_\tau^\lambda} mT_w,$$

for $m \in M$, which we refer to as the **module trace**. We will denote this trace by MTr to emphasise the fact that we are applying the trace to a module. Similarly, under this identification, we get that:

$$Z_M(\mathscr{H}_\tau) = \{m \in M : mT_w = q^{\ell(w)}m \text{ for all } w \in \mathfrak{S}_\lambda\}.$$

We can recover similar properties of the module trace to the bimodule trace.

Corollary 2.12. *Let $\tau, \lambda \models n$ with $\mathfrak{S}_\tau \subseteq \mathfrak{S}_\lambda$, and suppose M is a $\mathscr{H}_n$-module. Then for $m \in Z_M(\mathscr{H}_\tau)$, we have $\mathrm{MTr}_\tau^\lambda(m) \in Z_M(\mathscr{H}_\lambda)$.*

Proof. Apply the previous lemma when the left action is trivial. □

Our aim is to use the module trace to prove a Hecke algebra version of [3, (1.3)], a theorem due to Broué which gives a lower bound on the vertex of a $F\mathfrak{S}_n$-module. Recall the following from [25, Theorem 2.30]:

Theorem 2.13. *Let B be a $(\mathscr{H}_n, \mathscr{H}_n)$-bimodule. Suppose $b \in Z_B(\mathscr{H}_\lambda)$, $d \in \mathcal{D}_{\lambda,\tau}$, and let $\nu(d) \vDash n$ be defined by $\mathfrak{S}_{\nu(d)} = \mathfrak{S}_\lambda^d \cap \mathfrak{S}_\tau$. Then we have $q^{-\ell(d)} T_{d^{-1}} b T_d \in Z_B(\mathscr{H}_{\nu(d)})$, and:*

$$\mathrm{Tr}_\lambda^n(b) = \sum_{d \in \mathcal{D}_{\lambda,\tau}} \mathrm{Tr}_{\nu(d)}^\tau \left(q^{-\ell(d)} T_{d^{-1}} b T_d\right).$$

Take M a $\mathfrak{S}_\lambda$-projective $\mathscr{H}_n$-module, and suppose that $\tau \vDash n$. Denote $A = \mathrm{End}_k(M)$ as a bimodule. Then $Z_A(\mathscr{H}_\lambda) = \mathrm{End}_{\mathscr{H}_\lambda}(M)$, and from the previous theorem:

$$\mathrm{Tr}_\lambda^n(Z_A(\mathscr{H}_\lambda)) \subseteq \sum_{d \in \mathcal{D}_{\lambda,\tau}} \mathrm{Tr}_{\nu(d)}^\tau(Z_A(\mathscr{H}_{\nu(d)})).$$

By Higman's criterion, the identity $\mathscr{H}_n$-homomorphism on M (denoted by $\mathbb{1}_M$) lies in the left-hand side of this expression, thus for $d \in \mathcal{D}_{\lambda,\tau}$, there exist $\mathscr{H}_n$-homomorphisms $\psi_d \in Z_A(\mathscr{H}_{\nu(d)}) = \mathrm{End}_{\mathscr{H}_{\nu(d)}}(M)$ with $\mathbb{1}_M = \sum_{d \in \mathcal{D}_{\lambda,\tau}} \mathrm{Tr}_{\nu(d)}^\tau(\psi_d)$.

Now let $m \in Z_M(\mathscr{H}_\tau)$, so for all $s_i \in \mathfrak{S}_\tau$, we have $mT_i = qm$. Then:

$$\begin{aligned}
m = (m)\mathbb{1}_M &= \sum_{d \in \mathcal{D}_{\lambda,\tau}} (m)\,\mathrm{Tr}_{\nu(d)}^\tau(\psi_d), \\
&= \sum_{d \in \mathcal{D}_{\lambda,\tau}} \sum_{x \in \mathcal{R}_{\nu(d)}^\tau} (m)(q^{-\ell(x)} T_{x^{-1}} \psi_d T_x), \\
&= \sum_{d \in \mathcal{D}_{\lambda,\tau}} \sum_{x \in \mathcal{R}_{\nu(d)}^\tau} (m q^{-\ell(x)} T_{x^{-1}}) \psi_d T_x), \\
&= \sum_{d \in \mathcal{D}_{\lambda,\tau}} \sum_{x \in \mathcal{R}_{\nu(d)}^\tau} (m q^{-\ell(x)} q^{\ell(x)}) \psi_d T_x), \\
&= \sum_{d \in \mathcal{D}_{\lambda,\tau}} \sum_{x \in \mathcal{R}_{\nu(d)}^\tau} (m) \psi_d T_x, \\
&= \sum_{d \in \mathcal{D}_{\lambda,\tau}} \mathrm{MTr}_{\nu(d)}^\tau((m)\psi_d).
\end{aligned}$$

Now as $\psi_d \in Z_A(\mathscr{H}_{\nu(d)})$, then for $s_i \in \mathfrak{S}_{\nu(d)}$, we have $(m)\psi_d T_i = (mT_i)\psi_d = q(m)\psi_d$, so $(m)\psi_d \in Z_M(\mathscr{H}_{\nu(d)})$. This proves:

Proposition 2.14. *Let M be a $\mathscr{H}_\lambda$-projective $\mathscr{H}_n$-module. Then for $\tau \models n$:*

$$Z_M(\mathscr{H}_\tau) \subseteq \sum_{d \in \mathcal{D}_{\lambda,\tau}} \mathrm{MTr}^{\tau}_{\nu(d)}(Z_M(\mathscr{H}_{\nu(d)})),$$

where $\mathfrak{S}_{\nu(d)} = \mathfrak{S}^d_\lambda \cap \mathfrak{S}_\tau$.

Corollary 2.15. *Let M, τ and λ be as in the previous proposition. Then*

$$Z_M(\mathscr{H}_\tau) = \sum_{d \in \mathcal{D}_{\lambda,\tau}} \mathrm{MTr}^{\tau}_{\nu(d)}(Z_M(\mathscr{H}_{\nu(d)})).$$

Proof. Proposition 2.14 shows us that the left-hand side is contained in the right-hand side, and Corollary 2.12 shows us that each part of the sum on the right-hand side is contained in the left-hand side. □

We can now generalise Broué's result [3, (1.3)] to Hecke algebras.

Theorem 2.16. *Let $\tau, \lambda \models n$ and suppose M is a $\mathfrak{S}_\lambda$-projective $\mathscr{H}_n$-module. Define:*

$$M(\tau) = Z_M(\mathscr{H}_\tau) / \sum_{\substack{\gamma \models n \\ \mathfrak{S}_\gamma \subsetneq \mathfrak{S}_\tau}} \mathrm{MTr}^{\tau}_{\gamma}(Z_M(\mathscr{H}_\gamma)).$$

Then $M(\tau) \neq 0$ implies that a conjugate of $\mathfrak{S}_\tau$ is contained in $\mathfrak{S}_\lambda$.

Proof. Suppose that no conjugate of $\mathfrak{S}_\tau$ is contained in $\mathfrak{S}_\lambda$. Then for all $d \in \mathcal{D}_{\lambda,\tau}$, we have $\mathfrak{S}_{\nu(d)} \subsetneq \mathfrak{S}_\tau$ (where again $\nu(d) \models n$ is defined by $\mathfrak{S}_{\nu(d)} = \mathfrak{S}^d_\lambda \cap \mathfrak{S}_\tau$). Applying the previous corollary tells us that:

$$Z_M(\mathscr{H}_\tau) = \sum_{d \in \mathcal{D}_{\lambda,\tau}} \mathrm{MTr}^{\tau}_{\nu(d)}(Z_M(\mathscr{H}_{\nu(d)})) \subseteq \sum_{\substack{\gamma \models n \\ \mathfrak{S}_\gamma \subsetneq \mathfrak{S}_\tau}} \mathrm{MTr}^{\tau}_{\gamma}(Z_M(\mathscr{H}_\gamma)),$$

implying that $M(\tau) = 0$. Thus if $M(\tau) \neq 0$, we must have that a conjugate of $\mathfrak{S}_\tau$ is contained in $\mathfrak{S}_\lambda$. □

This gives us a way of finding a lower bound for the vertex of a $\mathscr{H}_n$-module, since every module M is relatively projective for its vertex. Therefore applying the theorem with $\mathfrak{S}_\lambda$ as the vertex, if one can find a τ with $M(\tau)$ non-zero, then we know that a conjugate of $\mathfrak{S}_\tau$ is contained in $\mathfrak{S}_\lambda$. Thus we have $\mathfrak{S}_\tau$ is a lower bound for the vertex.

2.3 A Green correspondence for bimodules

Given the notion of relative projectivity for modules, we can extend it to bimodules in the following way. Let A, B be F-algebras with subalgebras $A' \subseteq A$ and $B' \subseteq B$. Then an (A, B)-bimodule is the same as a left $A \otimes B^{\mathrm{op}}$-module. Hence we will say that an (A, B)-bimodule is relatively (A', B')-projective if as a left $A \otimes B^{\mathrm{op}}$-module, M is relatively $A' \otimes (B')^{\mathrm{op}}$-projective.

2.3.1 Relative projectivity of bimodules

Using this, we can extend Higman's criterion and its corollaries to bimodules of Hecke algebras. Let $\sigma_1, \sigma_2 \vDash n$, and denote $\underline{\mathscr{H}}_{\sigma_1,\sigma_2} := \mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2}^{\mathrm{op}}$, so a $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule is the same as a left $\underline{\mathscr{H}}_{\sigma_1,\sigma_2}$-module. Similarly use $\underline{T}_{w_1,w_2}$ to denote $T_{w_1} \otimes T_{w_2} \in \underline{\mathscr{H}}_{\sigma_1,\sigma_2}$, for $w_i \in \mathfrak{S}_{\sigma_i}$. Note that under this notation if we have a $(\mathscr{H}_{\lambda_1}, \mathscr{H}_{\lambda_2})$-bimodule M, then

$$\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\lambda_1,\lambda_2}} M \cong \mathscr{H}_{\sigma_1} \otimes_{\mathscr{H}_{\lambda_1}} M \otimes_{\mathscr{H}_{\lambda_2}} \mathscr{H}_{\sigma_2}$$

as $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodules. This can be seen either using the transitivity of induction, or by the associativity formula given in [4, §9, Proposition 2.1]. This gives a useful result if our bimodule is a block of $\mathscr{H}_n$.

Proposition 2.17. *Let B be an indecomposable direct summand of $\mathscr{H}_n$ as a $(\mathscr{H}_n, \mathscr{H}_n)$-bimodule. If B is either $(\mathfrak{S}_\tau, \mathfrak{S}_n)$-projective or $(\mathfrak{S}_n, \mathfrak{S}_\tau)$-projective for some $\tau \vDash n$, then B is $(\mathfrak{S}_\tau, \mathfrak{S}_\tau)$-projective.*

Proof. We only prove the first case as the second follows in the same way. By Higman's criterion, $B \mid \mathscr{H}_n \otimes_{\mathscr{H}_\tau} B \otimes_{\mathscr{H}_n} \mathscr{H}_n$. Since B is a direct summand of $\mathscr{H}_n$ as a $(\mathscr{H}_n, \mathscr{H}_n)$-bimodule, it is also a direct summand of $\mathscr{H}_n$ as a $(\mathscr{H}_\tau, \mathscr{H}_n)$-bimodule. Hence,

$$B \mid \mathscr{H}_n \otimes_{\mathscr{H}_\tau} \mathscr{H}_n \otimes_{\mathscr{H}_n} \mathscr{H}_n \cong \mathscr{H}_n \otimes_{\mathscr{H}_\tau} \mathscr{H}_n \cong \mathscr{H}_n \otimes_{\mathscr{H}_\tau} \mathscr{H}_\tau \otimes_{\mathscr{H}_\tau} \mathscr{H}_n,$$

thus by Higman's criterion again, B is $(\mathfrak{S}_\tau, \mathfrak{S}_\tau)$-projective. $\square$

Recall from Section 2.1 that $\mathscr{H}_n$ (and hence $\mathscr{H}_{\sigma_i}$) has an anti–automorphism which maps $T_w \mapsto T_{w^{-1}}$ for $w \in \mathfrak{S}_n$. Hence as F-algebras:

$$\mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2}^{\mathrm{op}} \cong \mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2} \cong \mathscr{H}_\sigma,$$

where $\sigma \vDash 2n$ is given by the concatenation of σ_1 and σ_2. Thus we can conclude from [25, Theorem 2.29] a Mackey formula for bimodules.

Theorem 2.18 (Mackey formula for bimodules)**.** *For $i = 1, 2$, let $\mathfrak{S}_{\lambda_i}, \mathfrak{S}_{\mu_i}$ be parabolic subgroups of $\mathfrak{S}_{\sigma_i}$, and denote $\mathcal{D}_i = \mathcal{D}^{\sigma_i}_{\lambda_i,\mu_i}$. Then for any left $\underline{\mathscr{H}}_{\lambda_1,\lambda_2}$-module M, we have that as $\underline{\mathscr{H}}_{\mu_1,\mu_2}$-modules:*

$$\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\lambda_1,\lambda_2}} M \cong \bigoplus_{d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2} \underline{\mathscr{H}}_{\mu_1,\mu_2} \otimes_{\underline{\mathscr{H}}_{\nu(d_1),\nu(d_2)}} \left(\underline{T}_{d_1^{-1},d_2} \otimes_{\underline{\mathscr{H}}_{\lambda_1,\lambda_2}} M \right)$$

where $\nu(d_i) \vDash n$ is defined for $i = 1, 2$ by:

$$\mathfrak{S}_{\nu(d_i)} = \mathfrak{S}^{d_i}_{\lambda_i} \cap \mathfrak{S}_{\mu_i}.$$

Note that in this statement $\left(\underline{T}_{d_1^{-1},d_2} \otimes_{\underline{\mathscr{H}}_{\lambda_1,\lambda_2}} M\right) \cong T_{d_1^{-1}} \otimes_{\mathscr{H}_{\lambda_1}} M \otimes_{\mathscr{H}_{\lambda_2}} T_{d_2}$ is indeed a $(\mathscr{H}_{\nu(d_1)}, \mathscr{H}_{\nu(d_2)})$-bimodule. To see this, let $w \in \mathfrak{S}_{\nu(d_1)}$, $m \in M$, then:

$$T_w T_{d_1^{-1}} \otimes m \otimes T_{d_2} = T_{wd_1^{-1}} \otimes m \otimes T_{d_2}$$

since d_1^{-1} is a minimal right coset representative for $\mathfrak{S}_{\mu_1}$. As, $d_1wd_1^{-1} \in \mathfrak{S}_{\lambda_1}$ and d_1^{-1} is a minimal left coset representative for $\mathfrak{S}_\lambda$, we have that $T_{wd_1^{-1}} = T_{d_1^{-1}d_1wd_1^{-1}} = T_{d_1^{-1}}T_{d_1wd_1^{-1}}$. Thus we can pull $T_{d_1wd_1^{-1}}$ across the tensor product to M. Doing something similar on the right confirms our claim.

As before, using again the fact that $\mathscr{H}_{\sigma_i}$ possesses an anti–automorphism, as a consequence of [25, Theorem 2.35], we can define vertices for indecomposable $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodules.

Theorem 2.19. *Let M be a $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule. Then there exist a pair of parabolic subgroups $\mathfrak{S}_{\lambda_i} \subseteq \mathfrak{S}_{\sigma_i}$ for $i = 1, 2$, such that M is relatively $(\mathfrak{S}_{\lambda_1}, \mathfrak{S}_{\lambda_2})$-projective and if for any parabolic subgroups $\mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\sigma_i}$ with M relatively $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$-projective, then $\mathfrak{S}_{\lambda_i} \subseteq_{\mathfrak{S}_{\sigma_i}} \mathfrak{S}_{\tau_i}$, again for $i = 1, 2$. We call the pair $(\mathfrak{S}_{\lambda_1}, \mathfrak{S}_{\lambda_2})$ a* ***vertex*** *of M as a $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule.*

Using this, we get the following consequences of [13, Lemma 3.2].

Lemma 2.20. *Let M be an indecomposable $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$ for $\tau_1, \tau_2 \vDash n$, and let $\lambda_1, \lambda_2 \vDash n$ with $\mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\lambda_i} \subseteq \mathfrak{S}_{\sigma_i}$ for $i = 1, 2$. Then there are indecomposable $(\mathscr{H}_{\lambda_1}, \mathscr{H}_{\lambda_2})$-bimodules X and Y, both with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$ such that:*

(a) $X \mid M$ as $(\mathscr{H}_{\lambda_1}, \mathscr{H}_{\lambda_2})$-bimodules,

(b) $M \mid \underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\lambda_1,\lambda_2}} Y$.

Note that in this situation, Y corresponds to the notion of a **source** for M (see for example [1, §9]). The final lemma we state in this section is a consequence of [13, Lemma 3.3] using Theorem 2.18.

Lemma 2.21. *Let τ_i, λ_i, $\sigma_i \vDash n$ with $\mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\lambda_i} \subseteq \mathfrak{S}_{\sigma_i}$ for $i = 1, 2$. If N is a $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$-projective $(\mathscr{H}_{\lambda_1}, \mathscr{H}_{\lambda_2})$-bimodule, then we get as $(\mathscr{H}_{\lambda_1}, \mathscr{H}_{\lambda_2})$-bimodules:*

$$\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\lambda_1,\lambda_2}} N \cong N \oplus Y,$$

where each indecomposable summand of Y has a vertex contained in:

$$(\mathfrak{S}_{\tau_1}^{d_1} \cap \mathfrak{S}_{\lambda_1}, \mathfrak{S}_{\tau_2}^{d_2} \cap \mathfrak{S}_{\lambda_2})$$

for some $d_i \in \mathcal{D}_{\tau_i,\lambda_i}^{\sigma_i}$ with $(d_1, d_2) \neq (1, 1)$.

Proof. By Higman's criterion, there exists an $\underline{\mathscr{H}}_{\tau_1,\tau_2}$-module V with $N \mid \underline{\mathscr{H}}_{\lambda_1,\lambda_2} \otimes_{\underline{\mathscr{H}}_{\tau_1,\tau_2}} V$. Therefore $\underline{\mathscr{H}}_{\lambda_1,\lambda_2} \otimes_{\underline{\mathscr{H}}_{\tau_1,\tau_2}} V \cong N \oplus T$ for some $\underline{\mathscr{H}}_{\lambda_1,\lambda_2}$-module T. Inducing up to $\underline{\mathscr{H}}_{\sigma_1,\sigma_2}$, then restricting back down to $\underline{\mathscr{H}}_{\lambda_1,\lambda_2}$ via the Mackey formula, after collecting terms we have:

$$\left(\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\tau_1,\tau_2}} V\right) \cong N \oplus Y \oplus T \oplus X,$$

where $N \oplus Y$ is the restriction to $\underline{\mathscr{H}}_{\lambda_1,\lambda_2}$ of $\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\lambda_1,\lambda_2}} N$, and $T \oplus X$ is the restriction of $\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\lambda_1,\lambda_2}} T$. Now inducing V from $\underline{\mathscr{H}}_{\tau_1,\tau_2}$ up to $\underline{\mathscr{H}}_{\sigma_1,\sigma_2}$ and restricting down to $\underline{\mathscr{H}}_{\lambda_1,\lambda_2}$ gives us via the Mackey formula:

$$\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\tau_1,\tau_2}} V \cong \left(\underline{\mathscr{H}}_{\lambda_1,\lambda_2} \otimes_{\underline{\mathscr{H}}_{\tau_1,\tau_2}} V\right) \oplus U \cong (N \oplus T) \oplus U,$$

where summands of U are $(\mathfrak{S}_{\tau_1}^{d_1} \cap \mathfrak{S}_{\lambda_1}, \mathfrak{S}_{\tau_2}^{d_2} \cap \mathfrak{S}_{\lambda_2})$-projective for some double coset representatives d_1 and d_2 with not both $d_1 = d_2 = 1$. Comparing these two expressions means that by the Krull–Schmidt theorem (see for example [26, Theorem A6]) we get $X \oplus Y \cong U$, and thus $Y \mid U$, so each indecomposable summand of Y has a vertex contained in a pair of subgroups of the correct form. □

2.3.2 The correspondence

We now begin to prove a Green correspondence for $(\mathscr{H}_n, \mathscr{H}_n)$-bimodules, as in [13, §3] for $\mathscr{H}_n$-modules, or as done in [1, §11] for finite groups. Let us fix some notation. Take $\lambda_i, \mu_i, \sigma_i$ compositions of n for $i = 1, 2$, with:

$$\mathfrak{S}_{\lambda_i} \subseteq N_{\mathfrak{S}_{\sigma_i}}(\mathfrak{S}_{\lambda_i}) \subseteq \mathfrak{S}_{\mu_i} \subseteq \mathfrak{S}_{\sigma_i}. \tag{2.5}$$

Denote the following set:

$$\mathscr{P} = \{(H_1, H_2) : H_i \text{ is a parabolic subgroup of } \mathfrak{S}_{\sigma_i} \text{ for } i = 1, 2\}.$$

For any subset $\mathscr{Q} \subseteq \mathscr{P}$, we say a $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule is **relatively $\mathscr{Q}$-projective** (or just $\mathscr{Q}$-projective), if each of its indecomposable summands is relatively projective for some pair of parabolic subgroups in $\mathscr{Q}$. Let $(P_1, P_2), (G_1, G_2) \in \mathscr{P}$. Then say that $(P_1, P_2) \in_{(G_1,G_2)} \mathscr{Q}$ if there are elements $x_i \in G_i$ with $(P_1^{x_1}, P_2^{x_2}) \in \mathscr{Q}$. Now we are ready to define the sets used in our version of the Green correspondence.

$$\underline{\mathscr{X}}^2 = \{(H_1, H_2) \in \mathscr{P} : H_i \subseteq \mathfrak{S}_{\lambda_i}^{d_i} \cap \mathfrak{S}_{\lambda_i} \text{ for } (d_1, d_2) \in (\mathfrak{S}_{\sigma_1}, \mathfrak{S}_{\sigma_2}) - (\mathfrak{S}_{\mu_1}, \mathfrak{S}_{\mu_2})\},$$

$$\underline{\mathscr{Y}}^2 = \{(H_1, H_2) \in \mathscr{P} : H_i \subseteq \mathfrak{S}_{\lambda_i}^{d_i} \cap \mathfrak{S}_{\mu_i} \text{ for } (d_1, d_2) \in (\mathfrak{S}_{\sigma_1}, \mathfrak{S}_{\sigma_2}) - (\mathfrak{S}_{\mu_1}, \mathfrak{S}_{\mu_2})\},$$

$$\underline{\mathscr{Z}}^2 = \{H = (H_1, H_2) \in \mathscr{P} : H_1 \subseteq \mathfrak{S}_{\lambda_1}, H_2 \subseteq \mathfrak{S}_{\lambda_2}, H \notin_{\mathfrak{S}_{\sigma_1}, \mathfrak{S}_{\sigma_2}} \underline{\mathscr{X}}^2\}.$$

Note that in the definitions of $\underline{\mathscr{X}}^2$ and $\underline{\mathscr{Y}}^2$, we require that $d = (d_1, d_2)$ cannot have both $d_1 \in \mathfrak{S}_{\mu_1}$ and $d_2 \in \mathfrak{S}_{\mu_2}$, but we could have say $d_1 \in \mathfrak{S}_{\mu_1}$ as long as $d_2 \notin \mathfrak{S}_{\mu_2}$. This follows from Lemma 2.21, where at most one of the d_i in that formula can be the identity element. These sets are bimodule analogues of the sets used in both [1, §11] and [13, §3]. As in both the classical Green correspondence and in [13, Theorem 3.6], we have the following conditions linking our sets.

Lemma 2.22. *If $\mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\lambda_i}$ are parabolic subgroups for $i = 1, 2$, then the following are* *equivalent:*

(a) $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in_{(\mathfrak{S}_{\sigma_1}, \mathfrak{S}_{\sigma_2})} \underline{\mathscr{X}}^2$,

(b) $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{X}}^2$,

(c) $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{Y}}^2$,

(d) $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in_{(\mathfrak{S}_{\mu_1}, \mathfrak{S}_{\mu_2})} \underline{\mathscr{Y}}^2$.

Again, this follows as a consequence of [13, Lemma 3.4]. Alternatively, it can be seen by the fact that $(H_1, H_2) \in \underline{\mathscr{X}}^2$ if and only if one of H_1 or H_2 lies in the corresponding set $\mathscr{X}$ from [13, §3]. We now need the following corollary, which corresponds to [13, Corollary 3.5].

Corollary 2.23. *If M is a $\underline{\mathscr{X}}^2$-projective $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule, then as a $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodule, M is $\underline{\mathscr{Y}}^2$-projective.*

Proof. Let L be an indecomposable summand of M as a $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule, with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{X}}^2$. Note that this means that $\mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\lambda_i}$ for both i by the definition of $\underline{\mathscr{X}}^2$. Thus $L \mid \underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\mathscr{H}_{\tau_1,\tau_2}} L$, and applying our Mackey formula says that as a $\underline{\mathscr{H}}_{\mu_1,\mu_2}$-module, each indecomposable summand of L is $(\mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\gamma_2})$-projective, where $\mathfrak{S}_{\gamma_i} \subseteq \mathfrak{S}_{\tau_i}^{d_i} \cap \mathfrak{S}_{\mu_i}$ for some $d_i \in \mathcal{D}^{\sigma_i}_{\tau_i,\mu_i}$. Note that for both i:

$$\mathfrak{S}_{\gamma_i} \subseteq \mathfrak{S}_{\tau_i}^{d_i} \cap \mathfrak{S}_{\mu_i} \subseteq \mathfrak{S}_{\lambda_i}^{d_i} \cap \mathfrak{S}_{\mu_i}.$$

If at least one $d_i \neq 1$, we have $(\mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\gamma_2}) \in \underline{\mathscr{Y}}^2$ as this $d_i \notin \mathfrak{S}_{\mu_i}$, and thus $(d_1, d_2) \notin (\mathfrak{S}_{\mu_1}, \mathfrak{S}_{\mu_2})$.

Finally if both $d_1 = d_2 = 1$, then we have $(\mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\gamma_2}) \in \underline{\mathscr{X}}^2$, as each $\mathfrak{S}_{\gamma_i} \subseteq \mathfrak{S}_{\tau_i}$ and $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{X}}^2$. As $\mathfrak{S}_{\gamma_i} \subseteq \mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\lambda_i}$, we can conclude with Lemma 2.22 that $(\mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\gamma_2}) \in \underline{\mathscr{Y}}^2$.

Thus in all cases indecomposable summands of L are relatively $\underline{\mathscr{Y}}^2$-projective as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules, and hence so is M. $\square$

We can now fully state our Green correspondence for bimodules of Hecke algebras, generalising [13, Theorem 3.6]. Although we are working with more general λ_i, μ_i, and with σ_i instead of (n), the proof follows through in the same way as we still have the key relationship (2.5) between our subgroups, and our Lemma 2.22 and Corollary 2.23 take the place of [13, Lemma 3.4, Corollary 3.5] respectively. Thus the double sum in the Mackey formula is fully accounted for. We include the proof here for completeness.

Theorem 2.24 (Green correspondence for bimodules)**.** *Let*

$$\mathfrak{S}_{\lambda_i} \subseteq N_{\mathfrak{S}_{\sigma_i}}(\mathfrak{S}_{\lambda_i}) \subseteq \mathfrak{S}_{\mu_i} \subseteq \mathfrak{S}_{\sigma_i},$$

for $i = 1, 2$. We have the following correspondence:

(a) Let M be an indecomposable $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{Z}}^2$. Then there is a unique indecomposable summand $f(M)$ of M as a $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodule, with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$, and

$$M \cong f(M) \oplus Y,$$

as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules, where each indecomposable summand of Y has a vertex in $\underline{\mathscr{Y}}^2$.

(b) Let N be an indecomposable $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodule with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{Z}}^2$. Then there is a unique indecomposable summand $g(N)$ of $\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\mu_1,\mu_2}} N$ as a $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule, with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$ and

$$\underline{\mathscr{H}} \otimes_{\underline{\mathscr{H}}_{\mu_1,\mu_2}} N \cong g(N) \oplus X,$$

where each indecomposable summand of X has a vertex in $\underline{\mathscr{X}}^2$.

(c) Furthermore for M and N as described above, $f(g(N)) \cong N$, and $g(f(M)) \cong M$.

Hence this gives a bijection between isomorphism classes of $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodules, and isomorphism classes of $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules which have vertices in $\underline{\mathscr{Z}}^2$.

Proof. We start by proving (a). Using Lemma 2.20, there exists an indecomposable $\underline{\mathscr{H}}_{\mu_1,\mu_2}$-module V with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$ and:

$$M \mid \left(\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\mu_1,\mu_2}} V\right). \tag{2.6}$$

Restricting both sides down to $\underline{\mathscr{H}}_{\mu_1,\mu_2}$ using Lemma 2.21 gives us that $M \mid V \oplus Y$ where Y is $\underline{\mathscr{Y}}^2$-projective. Hence, by the the Krull–Schmidt theorem, the restriction of M is isomorphic to $V \oplus Y_1$ or Y_1 for some (possibly zero) summand Y_1 of Y. Using Lemma 2.20, we know M has an indecomposable summand V' with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$ when considered as a $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-module. Let V' be any such indecomposable summand.

Suppose V' is isomorphic to some summand of Y. Then by the definition of vertex, $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in_{(\mathfrak{S}_{\mu_1}, \mathfrak{S}_{\mu_2})} \underline{\mathscr{Y}}^2$. Lemma 2.22 then says $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in_{(\mathfrak{S}_{\sigma_1}, \mathfrak{S}_{\sigma_2})} \underline{\mathscr{X}}^2$ which contradicts the fact that $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{Z}}^2$. Hence $V \cong V'$, so $M \cong V \oplus Y_1$ as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules, and this V is uniquely determined by M. Therefore setting $f(M) = V$ gives a well-defined map.

We now move on to proving (b). Take N as in the statement, and decompose the induced module into indecomposable $\underline{\mathscr{H}}_{\sigma_1,\sigma_2}$-modules as follows:

$$\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\mu_1,\mu_2}} N = M_1 \oplus \cdots \oplus M_t. \tag{2.7}$$

Using Lemma 2.21, $(\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\mu_1,\mu_2}} N) \cong N \oplus Y$ as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules where each indecomposable summand of Y is $\underline{\mathscr{Y}}^2$-projective. Write $Y = Y_1 \oplus \cdots \oplus Y_t$ to get (after reordering if necessary):

$$M_1 \cong N \oplus Y_1, \tag{2.8}$$

$$M_j \cong Y_j \text{ for } 1 < j \leqslant t,$$

as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules. Note for all j, $M_j \mid \underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\tau_1,\tau_2}} N$, so by definition of relative projectivity, each M_j has a vertex contained in $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \subseteq (\mathfrak{S}_{\lambda_1}, \mathfrak{S}_{\lambda_2})$.

We now split into cases $j = 1$ and $j > 1$. Let $(\mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\gamma_2})$ be a vertex of M_1 with $\mathfrak{S}_{\gamma_i} \subseteq \mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\lambda_i}$ for $i = 1, 2$. If M_1 is relatively $\underline{\mathscr{X}}^2$-projective as an $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule, then $N \oplus Y_1$ and hence N is relatively $\underline{\mathscr{Y}}^2$-projective as a $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodule by Corollary 2.23. Hence $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in_{(\mathfrak{S}_{\mu_1}, \mathfrak{S}_{\mu_2})} \underline{\mathscr{Y}}^2$. Applying Lemma 2.22, this means that our vertex has a conjugate in $\underline{\mathscr{X}}^2$ which is a contradiction as $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{Z}}^2$. So M_1 is not relatively $\underline{\mathscr{X}}^2$-projective, and thus has vertex in $\underline{\mathscr{Z}}^2$.

Now let $j > 1$, and $(\mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\gamma_2})$ a vertex of M_j, with again $\mathfrak{S}_{\gamma_i} \subseteq \mathfrak{S}_{\tau_i} \subseteq \mathfrak{S}_{\lambda_i}$. Lemma 2.20 gives us some summand of Y_j with the same vertex. But we know $Y_j \mid Y$ which is relatively $\underline{\mathscr{Y}}^2$-projective, hence $(\mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\gamma_2}) \in_{(\mathfrak{S}_{\mu_1}, \mathfrak{S}_{\mu_2})} \underline{\mathscr{Y}}^2$, so again by Lemma 2.22, $(\mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\gamma_2}) \in \underline{\mathscr{X}}^2$. Hence M_j is relatively $\underline{\mathscr{X}}^2$-projective for $j > 1$. Therefore $\underline{\mathscr{H}} \otimes_{\underline{\mathscr{H}}_{\mu_1,\mu_2}} N$ has a unique indecomposable summand which is $\underline{\mathscr{Z}}^2$-projective, and the rest are $\underline{\mathscr{X}}^2$-projective. Define $g(N)$ as this unique summand.

We will prove only the first part of (c); the second follows similarly. Take N as in statement (b). Then by (2.7) and the proof of (b), $\underline{\mathscr{H}}_{\sigma_1,\sigma_2} \otimes_{\underline{\mathscr{H}}_{\mu_1,\mu_2}} N = g(N) \oplus X$ where X is $\underline{\mathscr{X}}^2$-projective, and $g(N)$ has vertex in $\underline{\mathscr{Z}}^2$. Using the proof of part (a), $f(g(N))$ is the unique summand of $g(N)$ as a $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodule with vertex in $\underline{\mathscr{Z}}^2$. However, by (2.8), $g(N) \cong N \oplus Y_1$ as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules, where Y_1 is $\underline{\mathscr{Y}}^2$-projective. Hence we must have $f(g(N)) \cong N$, and by applying (2.6) we see that $g(N)$ and N share the same vertex. So this gives the required one-to-one correspondence. □

Although this correspondence will hold for any $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \in \underline{\mathscr{Z}}^2$, we will typically use it in the simpler case when $\tau_i = \lambda_i$.

We can strengthen our Green correspondence, as the ideas of [1, §12] happily carry over to bimodules of Hecke algebras, affording us the following analogue of [1, Theorem 12.2].

Theorem 2.25. *Let M be an indecomposable $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule with vertex $(\mathfrak{S}_{\lambda_1}, \mathfrak{S}_{\lambda_2})$, and indecomposable $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodule $f(M)$ its Green correspondent. If U is an indecomposable $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule and $f(M) \mid U$ as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules, then $M \cong U$.*

We can also form the following corollary which will be useful in the next chapter.

Corollary 2.26. *Let M and $f(M)$ be as in Theorem 2.25. If U is a $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodule, then $M \mid U$ as $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodules, if and only if $f(M) \mid U$ as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules.*

Proof. Take $U = U_1 \oplus \cdots \oplus U_t$ a decomposition of U into indecomposable direct summands as $(\mathscr{H}_{\sigma_1}, \mathscr{H}_{\sigma_2})$-bimodules. As M is indecomposable, then $M \mid U$ means that $M = U_i$ some $1 \leqslant i \leqslant t$. Hence we get $f(M) \mid U_i \mid U$ as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules. For the other direction if $f(M) \mid U$ as $(\mathscr{H}_{\mu_1}, \mathscr{H}_{\mu_2})$-bimodules, then $f(M) \mid U_i$ for some i, hence by Theorem 2.25, $M \cong U_i$ and so $M \mid U$. □

CHAPTER 3

VERTICES OF BLOCKS

Now we have a Green correspondence for bimodules, the next logical step is to form a type of Brauer correspondence for blocks of Hecke algebras, giving results akin to Brauer's first main theorem (see for example [1, Theorem 14.2]). To begin this process, we start with the following definition, an analogue of the one given for finite groups in [1, §14].

Definition 3.1. For $\mu \vDash n$, let b be a block of $\mathscr{H}_\mu$, and B a block of $\mathscr{H}_n$. We say B is the **Brauer correspondent** of b, and write $b^{\mathscr{H}_n} = B$, if $b \mid B$ as $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodules, and B is the unique block of $\mathscr{H}_n$ with this property.

As $\mathscr{H}_\mu \mid \mathscr{H}_n$ as $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodules (consider the decomposition given by $\mathfrak{S}_\mu$-$\mathfrak{S}_\mu$ double coset representatives), b will always occur as a direct summand in the restriction of at least one block. However, there is no prior guarantee that its Brauer correspondent will exist, as b may occur in the restriction of more than one block. We first state some general properties of Brauer correspondents, omitting the proofs as they are largely identical to those in [1, Lemma 14.1].

Lemma 3.2. *Let b be a block of $\mathscr{H}_\mu$ for $\mu \vDash n$ with vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$ as a $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule. Then if $b^{\mathscr{H}_n}$ is defined, $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2})$ is contained in a vertex of $b^{\mathscr{H}_n}$.*

Lemma 3.3. *Let $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_\mu \subseteq \mathfrak{S}_n$ be a chain of parabolic subgroups of $\mathfrak{S}_n$. If b is a block of $\mathscr{H}_\lambda$, and all three of $b^{\mathscr{H}_n}$, $b^{\mathscr{H}_\mu}$ and $(b^{\mathscr{H}_\mu})^{\mathscr{H}_n}$ are defined, then $(b^{\mathscr{H}_\mu})^{\mathscr{H}_n} = b^{\mathscr{H}_n}$.*

In this chapter we show that for certain maximal parabolics that we can guarantee the existence of Brauer correspondents, and use them to compute the vertices of all blocks of $\mathscr{H}_n$.

3.1 A Brauer correspondence for Hecke algebras

Let $a \geqslant 0$, $d \geqslant 1$, and $n = a + de$. Define compositions of n:

$$\mu = (a, de),$$
$$\alpha = (a, 1^{de}),$$
$$\tau = (1^a, de),$$

so $\mathfrak{S}_\mu = \mathfrak{S}_\alpha \times \mathfrak{S}_\tau$. Recall, from Lemma 1.5, we have the following description of $\mathcal{D}_{\mu,\mu}$:

$$\mathcal{D}_{\mu,\mu} = \left\{ d_k = \prod_{i=1}^{k} (a - k + i, a + i) \; : \; k = 0, \ldots, \min(a, de) \right\}.$$

This tells us for $i = 0, \ldots, \min(a, de)$ that $\mathfrak{S}_{\nu_i} := \mathfrak{S}_\mu^{d_i} \cap \mathfrak{S}_\mu$ has corresponding composition $\nu_i = (a - i, i, i, de - i)$. We define compositions:

$$\tau_i = (1^{a+i}, de - i),$$
$$\tau_i' = (1^a, i, 1^{de-i}),$$
$$\tilde{\tau}_i = (1^a, i, de - i),$$
$$\alpha_i = (1^{a-i}, i, 1^{de}),$$
$$\tilde{\alpha}_i = (a - i, i, 1^{de}).$$

Note in particular that $\mathfrak{S}_{\tilde{\tau}}$ is the group generated by the elementary transpositions from $\mathfrak{S}_{\tau_i}$ and $\mathfrak{S}_{\tau_i'}$. Similarly, the group generated by the collection of elementary transpositions from $\mathfrak{S}_{\alpha_i}$ and $\mathfrak{S}_{\tilde{\tau}_i}$ is $\mathfrak{S}_{\nu_i}$. This lets us present the following technical lemma:

Lemma 3.4. *For $0 \leqslant i \leqslant \min(a, de)$, as a $(\mathscr{H}_\tau, \mathscr{H}_\tau)$-module, $\mathscr{H}_\mu T_{d_i} \mathscr{H}_\mu$ is $(\mathfrak{S}_{\tau_i}, \mathfrak{S}_\tau)$-projective.*

Proof. By Proposition 1.4, every element $w \in \mathfrak{S}_n$ can be uniquely represented as a product $w = g d_i h$ for $g \in \mathfrak{S}_\mu$, $d_i \in \mathcal{D}_{\mu,\mu}$ and $h \in \mathcal{R}^\mu_{\nu_i}$, with $\ell(w) = \ell(g) + \ell(d_i) + \ell(h)$. Hence the following gives us an F-basis for $\mathscr{H}_\mu T_{d_i} \mathscr{H}_\mu$:

$$\{T_{g d_i h} = T_g T_{d_i} T_h : g \in \mathfrak{S}_\mu, h \in \mathcal{R}^\mu_{\nu_i}\}.$$

Furthermore, as $\mathfrak{S}_\mu = \mathfrak{S}_\alpha \times \mathfrak{S}_\tau$, we can further categorise our basis (as for each $g \in \mathfrak{S}_\mu$, there exists unique $x \in \mathfrak{S}_\alpha$ and $y \in \mathfrak{S}_\tau$ with $g = xy$, and $\ell(g) = \ell(x) + \ell(y)$). In the same vein, $\mathcal{R}^\mu_{\nu_i} = \mathcal{R}^\alpha_{\tilde{\alpha}_i} \times \mathcal{R}^\tau_{\tilde{\tau}_i}$, so $h \in \mathcal{R}^\mu_{\nu_i}$ can be written uniquely as $h = h_1 h_2$ with $h_1 \in \mathcal{R}^\alpha_{\tilde{\alpha}_i}$, $h_2 \in \mathcal{R}^\tau_{\tilde{\tau}_i}$, and $\ell(h) = \ell(h_1) + \ell(h_2)$. In particular, $T_{h_1} T_{h_2} = T_{h_2} T_{h_1}$ as h_1 commutes with $\mathfrak{S}_\tau$. Also, as $\mathfrak{S}_{\alpha_i} \subseteq \mathfrak{S}_\alpha$, we can write $x \in \mathfrak{S}_\alpha$ uniquely as $x = x_1 x_2$ with $x_1 \in \mathcal{L}^\alpha_{\alpha_i}$, $x_2 \in \mathfrak{S}_{\alpha_i}$ and $\ell(x) = \ell(x_1) + \ell(x_2)$. Therefore our F-basis for $\mathscr{H}_\mu T_{d_i} \mathscr{H}_\mu$ can be written as:

$$\{T_{x_1} T_{x_2} T_y T_{d_i} T_{h_1} T_{h_2} : x_1 \in \mathcal{L}^\alpha_{\alpha_i}, x_2 \in \mathfrak{S}_{\alpha_i}, y \in \mathfrak{S}_\tau, h_1 \in \mathcal{R}^\alpha_{\tilde{\alpha}_i}, h_2 \in \mathcal{R}^\tau_{\tilde{\tau}_i}\}.$$

Now for some fixed $x_1 \in \mathcal{L}^\alpha_{\alpha_i}$ and $h_1 \in \mathcal{R}^\alpha_{\tilde{\alpha}_i}$, consider the vector subspace

$$M_{x_1,h_1} := \langle T_{x_1} T_{x_2} T_y T_{d_i} T_{h_1} T_{h_2} : x_2 \in \mathfrak{S}_{\alpha_i}, y \in \mathfrak{S}_\tau, h_2 \in \mathcal{R}^\tau_{\tilde{\tau}_i}\rangle.$$

We show that this is closed under left and right multiplication by elements of $\mathscr{H}_\tau$, so is a $(\mathscr{H}_\tau, \mathscr{H}_\tau)$-bimodule.

Let $s_j \in \mathfrak{S}_\tau$. Multiplying basis element $m_{x_2,y,h_2} := T_{x_1} T_{x_2} T_y T_{d_i} T_{h_1} T_{h_2}$ by T_j on the left:

$$\begin{aligned}
T_j m_{x_2,y,h_2} &= T_j T_{x_1} T_{x_2} T_y T_{d_i} T_{h_1} T_{h_2},\\
&= T_{x_1} T_{x_2} T_j T_y T_{d_i} T_{h_1} T_{h_2},\\
&= \begin{cases} T_{x_1} T_{x_2} T_{s_j y} T_{d_i} T_{h_1} T_{h_2} & \text{if } \ell(s_j y) > \ell(y),\\ (q-1) T_{x_1} T_{x_2} T_y T_{d_i} T_{h_1} T_{h_2} + q T_{x_1} T_{x_2} T_{s_j y} T_{d_i} T_{h_1} T_{h_2} & \text{if } \ell(s_j y) < \ell(y), \end{cases}
\end{aligned}$$

as $\mathscr{H}_\alpha$ and $\mathscr{H}_\tau$ commute. As s_j, y and hence $s_j y \in \mathfrak{S}_\tau$, we have that M_{x_1,h_1} is a left $\mathscr{H}_\tau$-module. We now need to check right multiplication by T_j.

$$\begin{aligned}
m_{x_2,y,h_2} T_j &= T_{x_1} T_{x_2} T_y T_{d_i} T_{h_1} T_{h_2} T_j,\\
&= T_{x_1} T_{x_2} T_y T_{d_i} T_{h_2} T_j T_{h_1},\\
&= \begin{cases} T_{x_1} T_{x_2} T_y T_{d_i} T_{h_2 s_j} T_{h_1} & \text{if } \ell(h_2 s_j) > \ell(h_2),\\ (q-1) T_{x_1} T_{x_2} T_y T_{d_i} T_{h_1} T_{h_2} + q T_{x_1} T_{x_2} T_y T_{d_i} T_{h_2 s_j} T_{h_1} & \text{if } \ell(h_2 s_j) < \ell(h_2). \end{cases}
\end{aligned}$$

Hence it suffices to show that $T_{x_1} T_{x_2} T_y T_{d_i} T_{h_2 s_j} T_{h_1} \in M_{x_1,h_1}$. To do this we split into cases dependent on whether $h_2 s_j \in \mathcal{R}^\tau_{\tilde{\tau}_i}$ or not. First of all, if $h_2 s_j \in \mathcal{R}^\tau_{\tilde{\tau}_i}$, then by definition we are in M_{x_1,h_1}. If not, then by Lemma 1.6, there exists $s_k \in \mathfrak{S}_{\tilde{\tau}_i}$ with $s_k h_2 = h_2 s_j$, and thus by Proposition 1.4, $T_{h_2 s_j} = T_k T_{h_2}$. We now split depending on which part of $\mathfrak{S}_{\tilde{\tau}_i}$ contains s_k.

- If $s_k \in \mathfrak{S}_{\tau_i}$, then it commutes with T_{d_i} and:

$$T_{x_1} T_{x_2} T_y T_{d_i} T_{s_k} T_{h_1} T_{h_2} = T_{x_1} T_{x_2} T_y T_{s_k} T_{d_i} T_{h_1} T_{h_2},$$

  which lies in M_{x_1,h_1} as before.

- If $s_k \in \mathfrak{S}_{\tau_i'}$, then:

$$T_{x_1} T_{x_2} T_y T_{d_i} T_{s_k} T_{h_1} T_{h_2} = T_{x_1} T_{x_2} T_y T_{s_{k-i}} T_{d_i} T_{h_1} T_{h_2}.$$

Note $s_{k-i}d_i = d_i s_k$ as d_i is both a minimal left and right coset representative of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$. Furthermore, by minimality $\ell(s_{k-i}d_i) = 1 + \ell(d_i) = \ell(d_i s_k)$. Then:

$$T_{x_1}T_{x_2}T_yT_{s_{k-i}}T_{d_i}T_{h_1}T_{h_2} = T_{x_1}(T_{x_2}T_{s_{k-i}})T_yT_{d_i}T_{h_1}T_{h_2},$$

and as $x_2, s_{k-i} \in \mathfrak{S}_{\alpha_i}$, we have an element in M_{x_1,h_1}.

Thus M_{x_1,h_1} is closed under right multiplication by $\mathscr{H}_\tau$. As multiplication in $\mathscr{H}_n$ is associative, for a fixed x_1 and h_1, M_{x_1,h_1} is a $(\mathscr{H}_\tau, \mathscr{H}_\tau)$-submodule of $\mathscr{H}_\mu T_{d_i}\mathscr{H}_\mu$. Furthermore, since we have described bases for the bimodules involved as vector spaces, we have a direct sum decomposition of $\mathscr{H}_\mu T_{d_i}\mathscr{H}_\mu$ as a $(\mathscr{H}_\tau, \mathscr{H}_\tau)$-bimodule:

$$\mathscr{H}_\mu T_{d_i}\mathscr{H}_\mu = \bigoplus_{x_1 \in \mathcal{L}^\alpha_{\alpha_i}, h_1 \in \mathcal{R}^\alpha_{\tilde{\alpha}_i}} M_{x_1,h_1} \cong \bigoplus_{x_1 \in \mathcal{L}^\alpha_{\alpha_i}, h_1 \in \mathcal{R}^\alpha_{\tilde{\alpha}_i}} M_{1,1},$$

as our above calculations show that the T_{x_1} and T_{h_1} have no effect on left or right multiplication by $\mathscr{H}_\tau$. Therefore for our purposes, it suffices to show that $M_{1,1}$ is $(\mathfrak{S}_{\tau_i}, \mathfrak{S}_\tau)$-projective as a $(\mathscr{H}_\tau, \mathscr{H}_\tau)$-module.

To do this, consider the vector space $N := \langle T_xT_yT_{d_i}T_h : x \in \mathfrak{S}_{\alpha_i}, y \in \mathfrak{S}_{\tau_i}, h \in \mathcal{R}^\tau_{\tilde{\tau}_i}\rangle$. This is a $(\mathscr{H}_{\tau_i}, \mathscr{H}_\tau)$-bimodule, by similar calculations to those above. Looking at the bases of N and $M_{1,1}$, we conclude via [25, Lemma 2.19] that:

$$M_{1,1} \cong \mathscr{H}_\tau \otimes_{\mathscr{H}_{\tau_i}} N,$$

as $(\mathscr{H}_\tau, \mathscr{H}_\tau)$-bimodules. Thus $M_{1,1}$ is $(\mathfrak{S}_{\tau_i}, \mathfrak{S}_\tau)$-projective as a $(\mathscr{H}_\tau, \mathscr{H}_\tau)$-bimodule, hence the same holds for $\mathscr{H}_\mu T_{d_i}\mathscr{H}_\mu$. □

Corollary 3.5. *In the situation of Lemma 3.4, let M be a direct summand of $\mathscr{H}_\mu T_{d_i}\mathscr{H}_\mu$ as a $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule. Then if M is $(\mathfrak{S}_\tau, \mathfrak{S}_\tau)$-projective, it is also $(\mathfrak{S}_{\tau_i}, \mathfrak{S}_\tau)$-projective.*

Proof. This follows from Lemma 3.4 and the bimodule analogue of Corollary 2.2. □

We now introduce the following type of parabolic subgroup.

Definition 3.6. A parabolic subgroup $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_n$ is **fixed-point-free** if the corresponding composition $\lambda = (\lambda_1, \ldots, \lambda_s) \models n$ has $\lambda_i > 1$ for all $1 \leqslant i \leqslant s$.

Suppose that we have a composition $\gamma = (1^a, b) \models n$. We say a parabolic subgroup $\mathfrak{S}_\lambda$ is a fixed-point-free subgroup of $\mathfrak{S}_\gamma$ if the corresponding composition $\lambda = (1^a, \lambda_1, \ldots, \lambda_s)$ has $\lambda_i > 1$ for all $1 \leqslant i \leqslant s$.

This corresponds to the notion that no element of $\{1, \ldots, n\}$ (or $\{a+1, \ldots, n\}$ in the second case) is fixed by all elements of $\mathfrak{S}_\lambda$. Note that for any fixed-point-free subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_\tau$, we have $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda) \subseteq \mathfrak{S}_\mu$ by Proposition 1.7. Hence by Theorem 2.24, in this case, we have a bijection between $(\mathscr{H}_n, \mathscr{H}_n)$ and $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodules with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$.

Theorem 3.7. *Let $1 \neq d_i \in \mathcal{D}_{\mu,\mu}$, and $\mathfrak{S}_\lambda$ be a fixed-point-free parabolic subgroup of $\mathfrak{S}_\tau$. Then no indecomposable summand of $\mathscr{H}_\mu T_{d_i} \mathscr{H}_\mu$ as a $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$.*

Proof. Consider M a direct summand of $\mathscr{H}_\mu T_{d_i} \mathscr{H}_\mu$ as a $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule. If M has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$, then as $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_\tau$, we get that M is $(\mathfrak{S}_\tau, \mathfrak{S}_\tau)$-projective by transitivity of induction. Corollary 3.5 then tells us that M is $(\mathfrak{S}_{\tau_i}, \mathfrak{S}_\tau)$-projective as a $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule. However, M has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ which means that some conjugate of $\mathfrak{S}_\lambda$ is contained in $\mathfrak{S}_{\tau_i}$. As $\mathfrak{S}_\lambda$ is fixed-point-free in $\mathfrak{S}_\tau$, it contains an element of cycle type $\lambda_1 \ldots \lambda_s$. No elements in $\mathfrak{S}_{\tau_i}$ can have this cycle type as $\mathfrak{S}_{\tau_i}$ fixes at least one point, hence M cannot have vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. □

Corollary 3.8. *Let b be a block of $\mathscr{H}_\mu$ with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$, where $\mathfrak{S}_\lambda$ is a fixed-point-free parabolic subgroup of $\mathfrak{S}_\tau$. Then $b^{\mathscr{H}_n}$ exists.*

Proof. Decomposing $\mathscr{H}_n$ as a $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule using our minimal double cosets gives:

$$\mathscr{H}_n = \mathscr{H}_\mu \oplus \left(\bigoplus_{i=1}^{\min(a,de)} \mathscr{H}_\mu T_{d_i} \mathscr{H}_\mu \right).$$

Now b occurs once as a summand of $\mathscr{H}_\mu$, and does not appear as a direct summand of any $\mathscr{H}_\mu T_{d_i} \mathscr{H}_\mu$ for $i \geqslant 1$ by Theorem 3.7, as no indecomposable summands of this have the required vertex. Therefore b occurs exactly once in this direct sum decomposition of $\mathscr{H}_n$, so there must be a unique block of $\mathscr{H}_n$ which restricts to contain b. □

This finally lets us state our Brauer correspondence. Note that this is not as general as the Brauer correspondence stated in [1, Theorem 14.2], as we require $\mathfrak{S}_\mu$ to have two parts of the required form, and need $\mathfrak{S}_\lambda$ to be a fixed-point-free subgroup of $\mathfrak{S}_\tau$. This is instead of only requiring $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda) \subseteq \mathfrak{S}_\mu$ in the classical Brauer correspondence. Nevertheless as we will show in the remainder of this chapter, all blocks have vertices satisfying this condition, and thus it will give a complete characterisation of the vertices for the blocks of $\mathscr{H}_n$.

Theorem 3.9 (Brauer correspondence for Hecke algebras)**.** *Let $n = a + de$, with $\mu = (a, de)$, $\tau = (1^a, de)$ and $\mathfrak{S}_\lambda$ a fixed-point-free parabolic subgroup of $\mathfrak{S}_\tau$. Then there is a one-to-one correspondence between blocks of $\mathscr{H}_\mu$ with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ and blocks of $\mathscr{H}_n$ with the same vertex.*

Proof. First let b be a block of $\mathscr{H}_\mu$ with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. Then $b^{\mathscr{H}_n}$ exists by Corollary 3.8. As $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_\tau$ is fixed-point-free, we have $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda) \subseteq \mathfrak{S}_\mu$. Hence we can use Theorem 2.24 to show that b has a Green correspondent, and by Theorem 2.25, this Green correspondent must be $b^{\mathscr{H}_n}$. This correspondence gives us that $b^{\mathscr{H}_n}$ has the same vertex as b, and as the Green correspondence is a bijection, in particular the map $b \mapsto b^{\mathscr{H}_n}$ must be injective.

Now let B be a block of $\mathscr{H}_n$ with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. By Lemma 2.20, there is an indecomposable $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule N with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$, and $N \mid B$ as $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodules. Theorem 3.7 tells us that N must be a direct summand of $\mathscr{H}_\mu$ and hence is a block of $\mathscr{H}_\mu$. Therefore by Corollary 3.8, $N^{\mathscr{H}_n}$ exists, and by the first part of this proof, $N^{\mathscr{H}_n} = B$. This shows us that the map $b \mapsto b^{\mathscr{H}_n}$ is surjective onto blocks with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$, and hence defines the required one-to-one correspondence. □

In particular, note that Brauer corresponding blocks are also Green correspondents in the sense of Theorem 2.24.

3.1.1 Finding Brauer correspondents

Now we know they exist, we want to be able to identify the Brauer correspondent of a given block. We begin by proving a theorem which links Brauer correspondents to Green correspondents, similar to [1, Corollary 14.4], and to do this, denote the central idempotent of a block b by f_b.

Theorem 3.10. *Let $\mu \vDash n$ and b a block of $\mathscr{H}_\mu$ whose Brauer correspondent $B = b^{\mathscr{H}_n}$ exists. Let $\lambda \vDash n$ with $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda) \subseteq \mathfrak{S}_\mu$, and suppose N is an indecomposable $\mathscr{H}_\mu$-module lying in b, with vertex $\mathfrak{S}_\lambda$. Then $g(N)$, the Green correspondent of N, lies in B.*

Proof. Note first that the Green correspondent of N exists by [13, Theorem 3.6]. Thus

$$N \otimes_{\mathscr{H}_\mu} \mathscr{H}_n \cong g(N) \oplus X,$$

where $g(N)$ is indecomposable, has vertex $\mathfrak{S}_\lambda$, and the indecomposable summands of X all have vertices that are strictly smaller than $\mathfrak{S}_\lambda$. Suppose that $g(N)f_B = 0$. Then:

$$N \otimes_{\mathscr{H}_\mu} B = (N \otimes_{\mathscr{H}_\mu} \mathscr{H}_n)f_B \cong g(N)f_B \oplus Xf_B = Xf_B,$$

and hence each indecomposable summand of $N \otimes_{\mathscr{H}_\mu} B$ has vertex strictly smaller than $\mathfrak{S}_\lambda$ as a $\mathscr{H}_n$-module. Thus its restriction down to $\mathscr{H}_\mu$ must also have vertices strictly smaller than $\mathfrak{S}_\lambda$, by the Mackey formula. By the definition of Brauer correspondents, $B \cong b \oplus Y$ as $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodules, for some $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule Y. Thus as $\mathscr{H}_\mu$-modules:

$$N \otimes_{\mathscr{H}_\mu} b \mid N \otimes_{\mathscr{H}_\mu} B.$$

However,

$$N \otimes_{\mathscr{H}_\mu} b = N \otimes_{\mathscr{H}_\mu} f_b \mathscr{H}_\mu = N f_b \otimes_{\mathscr{H}_\mu} \mathscr{H}_\mu \cong N f_b = N,$$

since N lies in the block b, so $N \mid N \otimes_{\mathscr{H}_\mu} B$ as $\mathscr{H}_\mu$-modules. This is a contradiction, as the indecomposable summands of $N \otimes_{\mathscr{H}_\mu} B$ have vertices strictly smaller than $\mathfrak{S}_\lambda$ as $\mathscr{H}_\mu$-modules. Hence $g(N)f_B = g(N)$ and so $g(N)$ lies in the block B of $\mathscr{H}_n$. □

Thus searching for Green correspondents of modules in our block b gives a way to identify $b^{\mathscr{H}_n}$. We summarise this test in the following corollary.

Corollary 3.11. *Let $\mu = (a, de), \tau = (1^a, de), \gamma \vDash n$, and $\mathfrak{S}_\lambda$ a fixed-point-free parabolic subgroup of $\mathfrak{S}_\tau$. Suppose $\mathfrak{S}_\gamma \subseteq N_{\mathfrak{S}_n}(\mathfrak{S}_\gamma) \subseteq \mathfrak{S}_\mu$, and let b be a block of $\mathscr{H}_\mu$ with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. If N is an indecomposable $\mathscr{H}_\mu$-module lying in b with vertex $\mathfrak{S}_\gamma$, and its Green correspondent $g(N)$ in $\mathscr{H}_n$ lies in B, then $B = b^{\mathscr{H}_n}$.*

Proof. Theorem 3.9 guarantees that $b^{\mathscr{H}_n}$ exists, and by the preceding theorem, $g(N)$ lies in $b^{\mathscr{H}_n}$. □

Before concluding this section, we present one last theorem to aid us when computing the vertex of a particular block; in effect this gives a lower bound on the possible vertex.

Theorem 3.12. *Let B be a block of $\mathscr{H}_n$ with vertex $(\mathfrak{S}_{\lambda_1}, \mathfrak{S}_{\lambda_2})$, and M an indecomposable right $\mathscr{H}_n$-module that lies in B with vertex $\mathfrak{S}_\gamma$. Then $\mathfrak{S}_\gamma \subseteq_{\mathfrak{S}_n} \mathfrak{S}_{\lambda_2}$.*

Proof. As M has vertex $\mathfrak{S}_\gamma$, there exists some $\mathscr{H}_\gamma$-module N with $M \mid N \otimes_{\mathscr{H}_\gamma} \mathscr{H}_n$ by Lemma 2.20. Multiplying both sides by f_B:

$$M = M f_B \mid N \otimes_{\mathscr{H}_\gamma} \mathscr{H}_n f_B = N \otimes_{\mathscr{H}_\gamma} B.$$

By Higman's criterion, $B \mid \mathscr{H}_n \otimes_{\mathscr{H}_{\lambda_1}} B \otimes_{\mathscr{H}_{\lambda_2}} \mathscr{H}_n$ as $(\mathscr{H}_n, \mathscr{H}_n)$-bimodules. By restricting down on the left, the same holds true as $(\mathscr{H}_\gamma, \mathscr{H}_n)$-bimodules. Combining this with our previous statement, as N is a $\mathscr{H}_\gamma$-module, means that as $\mathscr{H}_n$-modules:

$$M \mid N \otimes_{\mathscr{H}_\gamma} (\mathscr{H}_n \otimes_{\mathscr{H}_{\lambda_1}} B \otimes_{\mathscr{H}_{\lambda_2}} \mathscr{H}_n) \cong (N \otimes_{\mathscr{H}_\gamma} (\mathscr{H}_n \otimes_{\mathscr{H}_{\lambda_1}} B)) \otimes_{\mathscr{H}_{\lambda_2}} \mathscr{H}_n,$$

by associativity. Setting $V = N \otimes_{\mathscr{H}_\gamma} (\mathscr{H}_n \otimes_{\mathscr{H}_{\lambda_1}} B)$, a $\mathscr{H}_{\lambda_2}$-module, we get

$$M \mid V \otimes_{\mathscr{H}_{\lambda_2}} \mathscr{H}_n,$$

hence M is relatively $\mathfrak{S}_{\lambda_2}$-projective, and thus some conjugate of $\mathfrak{S}_\gamma$ lies inside $\mathfrak{S}_{\lambda_2}$. □

3.2 Restricting Specht modules

The aim of this section is to explore the vertices of a special type of Specht module, with a view to using them in conjunction with our test, Corollary 3.11. We start by considering whether the Specht modules we care about are indecomposable or projective.

Lemma 3.13. *Let $\rho \vdash n$ be an e-core, or $\rho = (1^n)$. Then S^ρ is an indecomposable $\mathscr{H}_n$-module.*

Proof. As $S^{(1^n)}$ is one-dimensional, it must be indecomposable. When $\rho \vdash n$ is an e-core, we conclude with Lemma 1.11. □

Proposition 3.14. *If $\rho \vdash n$ is an e-core, then S^ρ is projective.*

Proof. If ρ is an e-core, then it lies in a block of e-weight zero, which is semi-simple by [15, Theorem 1.2], and thus all modules in this block are projective. □

This means that if $\rho \vdash a$ is an e-core, then $S^\rho \otimes S^{(1^m)}$ will be an indecomposable $\mathscr{H}_a \otimes \mathscr{H}_m$-module with vertex $\mathfrak{S}_\lambda$ contained in $\mathfrak{S}_{(1^a,m)}$. If $\mathfrak{S}_\lambda$ is a fixed-point-free subgroup of $\mathfrak{S}_{(1^a,m)}$, then by Proposition 1.7 we have $N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda) \subseteq \mathfrak{S}_a \times \mathfrak{S}_m$. Thus $S^\rho \otimes S^{(1^m)}$ is potentially a good candidate for having a Green correspondent in $\mathscr{H}_n$. The following definition gives us somewhere to look for this Green correspondent.

Definition 3.15. Let $\rho = (\rho_1, \dots, \rho_s) \vdash a$ for some positive integer a, and let m be another positive integer and $n = a + m$. Define the **extended partition** $\tilde{\rho} = (\rho_1, \dots, \rho_s, 1^m) \vdash n$ and say that $\mathfrak{t} \in \mathrm{Std}(\tilde{\rho})$ has an **m-tail** if the integers $\{a+1, \dots, a+m\}$ lie in the last m rows of $\mathfrak{t}$. Finally define

$$\mathrm{Std}(\rho, m) := \{\mathfrak{t} \in \mathrm{Std}(\tilde{\rho}) : \mathfrak{t} \text{ has an } m\text{-tail}\}.$$

Fix $n = a+m$. We have an obvious bijection between $\mathrm{Std}(\rho)$ and $\mathrm{Std}(\rho, m)$ by adding or removing the m-tail. We denote this by sending $\mathfrak{t}$ to $\tilde{\mathfrak{t}}$. In fact, the following lemma is easy to verify.

Lemma 3.16. *Let $\rho \vdash a$, and $\mathfrak{t}, \mathfrak{s} \in \mathrm{Std}(\rho)$. Then*

$$\mathfrak{t} \rhd \mathfrak{s} \iff \tilde{\mathfrak{t}} \rhd \tilde{\mathfrak{s}}.$$

Furthermore, if $\mathfrak{t} \in \mathrm{Std}(\rho, m)$, and $\mathfrak{v} \in \mathrm{Std}(\tilde{\rho})$, then $\mathfrak{v} \rhd \mathfrak{t}$ implies $\mathfrak{v} \in \mathrm{Std}(\rho, m)$.

For the rest of this section we fix $\rho \vdash a$, and $\mu = (a, m) \models n$. The following lemma gives an interesting $\mathscr{H}_\mu$-submodule of $S^{\tilde{\rho}}$.

Lemma 3.17. *Let $S^{\rho,m}$ be the vector space inside $S^{\tilde{\rho}}$ spanned by basis elements $m_\mathfrak{t}$ where $\mathfrak{t} \in \mathrm{Std}(\rho, m)$. Then $S^{\rho,m}$ is a $\mathscr{H}_\mu$-submodule of $S^{\tilde{\rho}}$.*

Proof. We show that $S^{\rho,m}$ is closed under multiplication by T_i, for $s_i \in \mathfrak{S}_\mu$. Let $\mathfrak{t} \in \mathrm{Std}(\rho, m)$, so $\mathfrak{t}$ has an m-tail, and consider $m_\mathfrak{t} T_i$. If i and $i+1$ lie in the same column of $\mathfrak{t}$, then we can conclude using the final part of Corollary 1.9 and Lemma 3.16.

Otherwise we split into cases, depending on whether or not i and $i+1$ are in the same row of $\mathfrak{t}$. If they are in the same row, then $\mathfrak{s} = \mathfrak{t}s_i$ is not row standard, and hence again by Corollary 1.9, $m_{\mathfrak{t}}T_i = qm_{\mathfrak{t}}$, and thus lies in $S^{\rho,m}$. If i and $i+1$ are not in the same row, then $\mathfrak{s}$ is standard, and contains an m-tail (as $i < a$ for this to occur). Using Corollary 1.9:

$$m_{\mathfrak{t}}T_i = \begin{cases} m_{\mathfrak{s}} & \text{if } \ell(d(\mathfrak{s})) > \ell(d(\mathfrak{t})), \\ qm_{\mathfrak{s}} + (q-1)m_{\mathfrak{t}} & \text{otherwise,} \end{cases}$$

and so in both cases $m_{\mathfrak{t}}T_i \in S^{\rho,m}$. □

Theorem 3.18. *As $\mathscr{H}_\mu$-modules, $S^{\rho,m} \cong S^\rho \otimes S^{(1^m)}$.*

Proof. Let $\{m_{\tilde{\mathfrak{t}}} : \mathfrak{t} \in \mathrm{Std}(\rho)\}$ be our standard basis of $S^{\rho,m}$, and $\{n_{\mathfrak{t}} \otimes \xi : \mathfrak{t} \in \mathrm{Std}(\rho)\}$ be the basis of $S^\rho \otimes S^{(1^m)}$ gained from taking the standard basis of S^ρ and tensoring with single basis element ξ of $S^{(1^m)}$. Define a map $\phi : S^\rho \otimes S^{(1^m)} \to S^{\rho,m}$ by $n_{\mathfrak{t}} \otimes \xi \mapsto m_{\tilde{\mathfrak{t}}}$ extended linearly. To show ϕ is a $\mathscr{H}_\mu$-module isomorphism, it suffices to show that the map is a $\mathscr{H}_\mu$-module homomorphism, that is:

$$((n_{\mathfrak{t}} \otimes \xi)T_i)\phi = (n_{\mathfrak{t}} \otimes \xi)\phi \cdot T_i,$$

for all $s_i \in \mathfrak{S}_\mu$ and $\mathfrak{t} \in \mathrm{Std}(\rho)$.

First suppose that i and $i+1$ are in the same row of $\tilde{\mathfrak{t}}$. By necessity, this means $s_i \in \mathfrak{S}_a$, thus $(n_{\mathfrak{t}} \otimes \xi)T_i = n_{\mathfrak{t}}T_i \otimes \xi$, and by Corollary 1.9, $n_{\mathfrak{t}}T_i = qn_{\mathfrak{t}}$. By the same reasoning, $m_{\tilde{\mathfrak{t}}}T_i = qm_{\tilde{\mathfrak{t}}}$ when i and $i+1$ are in the same row, and thus

$$((n_{\mathfrak{t}} \otimes \xi)T_i)\phi = (n_{\mathfrak{t}} \otimes \xi)\phi \cdot T_i.$$

Now suppose i and $i+1$ are not in the same column, and are not in the same row (again necessarily $s_i \in \mathfrak{S}_a$). Similarly using Corollary 1.9, we get that $((n_{\mathfrak{t}} \otimes \xi)T_i)\phi = (n_{\mathfrak{t}} \otimes \xi)\phi \cdot T_i$, since $\mathfrak{s} = \mathfrak{t}s_i$ is standard, and $\tilde{\mathfrak{s}} = \tilde{\mathfrak{t}}s_i$.

It remains to deal with the case where i and $i+1$ are in the same column of $\tilde{\mathfrak{t}}$, and we split into further cases based on whether $s_i \in \mathfrak{S}_a$ or $s_i \in \mathfrak{S}_m$.

- First suppose that $s_i \in \mathfrak{S}_a$. Note that when viewed as elements of $\mathscr{H}_n$, we have that $m_\rho = m_{\tilde{\rho}}$ and $T_{d(\mathfrak{t})} = T_{d(\tilde{\mathfrak{t}})}$. So using [26, Corollary 3.21], we have that in $\mathscr{H}_a$:

$$m_{\mathfrak{t}^\rho \mathfrak{t}} T_i \equiv -m_{\mathfrak{t}^\rho \mathfrak{t}} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} r_{\mathfrak{v}} m_{\mathfrak{t}^\rho \mathfrak{v}} \mod \check{\mathscr{H}}^\rho.$$

  Now if $m_{\mathfrak{u}\mathfrak{w}}$ is a basis element of $\check{\mathscr{H}}^\rho$, for $\mathfrak{u}, \mathfrak{w} \in \mathrm{Std}(\lambda)$ for some $\lambda \rhd \rho$, then

$$m_{\mathfrak{u}\mathfrak{w}} = m_{\tilde{\mathfrak{u}}\tilde{\mathfrak{w}}}$$

  is a basis element of $\check{\mathscr{H}}^{\tilde{\rho}}$ as we know that $\mu \rhd \rho \implies \tilde{\mu} \rhd \tilde{\rho}$. Similarly the fact that $\mathfrak{v} \rhd \mathfrak{t} \iff \tilde{\mathfrak{v}} \rhd \tilde{\mathfrak{t}}$ gives us that:

$$m_{\mathfrak{t}^{\tilde{\rho}} \tilde{\mathfrak{t}}} T_i \equiv -m_{\mathfrak{t}^{\tilde{\rho}} \tilde{\mathfrak{t}}} + \sum_{\tilde{\mathfrak{v}} \rhd \tilde{\mathfrak{t}}} r_{\mathfrak{v}} m_{\mathfrak{t}^{\tilde{\rho}} \tilde{\mathfrak{v}}} \mod \check{\mathscr{H}}^{\tilde{\rho}}.$$

  Thus again multiplication is the same in both modules.

- Finally when i and $i+1$ both lie in the m-tail (so $T_{d(\mathfrak{t})}$ commutes with T_i):

$$((n_{\mathfrak{t}} \otimes \xi) T_i)\phi = (n_{\mathfrak{t}} \otimes (\xi T_i))\phi = (-n_{\mathfrak{t}} \otimes \xi)\phi = -m_{\tilde{\mathfrak{t}}}.$$

  So it suffices to show that $m_{\tilde{\mathfrak{t}}} T_i = -m_{\tilde{\mathfrak{t}}}$, i.e. $x_\rho T_{d(\tilde{\mathfrak{t}})}(1 + T_i) \in \check{\mathscr{H}}^{\tilde{\rho}}$. Writing $\tilde{\rho} = (\rho_1, \ldots, \rho_s, 1^l, 1^m)$ where each $\rho_i > 1$, we have that:

$$\begin{aligned} x_\rho T_{d(\tilde{\mathfrak{t}})}(1 + T_i) &= \left(\sum_{w \in \mathfrak{S}_{\tilde{\rho}}} T_w \right) (1 + T_i) T_{d(\tilde{\mathfrak{t}})} \\ &= \left(\sum_{w \in \mathfrak{S}_\nu} T_w \right) T_{d(\tilde{\mathfrak{t}})} = x_\nu T_{d(\tilde{\mathfrak{t}})}, \end{aligned}$$

where ν is the composition of n given by:

$$\nu = (\rho_1, \dots, \rho_s, 1^{l+(i-a)-1}, 2, 1^{m-(i-a)-1}).$$

Let $\lambda = (\rho_1, \dots, \rho_s, 2, 1^{l+m-2})$, the partition of n gained by reordering ν. As $x_\nu = m_{\mathfrak{t}^\nu \mathfrak{t}^\nu}$, we can apply [26, Lemma 3.10] to write $m_{\mathfrak{t}^\nu \mathfrak{t}^\nu}$ as an F-linear combination of elements of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\lambda)$. Since $\lambda \rhd \tilde{\rho}$, these elements lie in $\check{\mathscr{H}}^{\tilde{\rho}}$, and hence $x_\nu \in \check{\mathscr{H}}^{\tilde{\rho}}$. By [26, Lemma 2.3], $\check{\mathscr{H}}^{\tilde{\rho}}$ is a two-sided ideal, therefore $x_\nu T_{d(\tilde{\mathfrak{t}})} = x_\rho T_{d(\tilde{\mathfrak{t}})}(1 + T_i) \in \check{\mathscr{H}}^{\tilde{\rho}}$. Thus in $S^{\tilde{\rho}}$ we have $m_{\tilde{\mathfrak{t}}} T_i = -m_{\tilde{\mathfrak{t}}}$.

So in all possible cases we have shown that $((n_{\mathfrak{t}} \otimes \xi) T_i)\phi = (n_{\mathfrak{t}} \otimes \xi)\phi \cdot T_i$, and hence $S^{\rho,m} \cong S^\rho \otimes S^{(1^m)}$ as $\mathscr{H}_\mu$-modules. □

Recall the following version of the Littlewood–Richardson rule from [18, 13.7]. Let $\pi \vdash n$, $n = a + m$, $\mu = (a, m) \vDash n$, and suppose that $\mathscr{H}_n$ is semi-simple. Then as $\mathscr{H}_\mu$-modules:

$$S^\pi = \bigoplus_{\lambda,\nu} (S^\lambda \otimes S^\nu)^{\oplus c^\pi_{\lambda\nu}},$$

where the sum is over all $\lambda \vdash a$ and $\nu \vdash m$, and $c^\pi_{\lambda\nu}$ are the Littlewood–Richardson coefficients for $\mathfrak{S}_n$. Our ultimate goal in this section is to show that as $\mathscr{H}_\mu$-modules, when ρ is an e-core, that $S^\rho \otimes S^{(1^m)}$ is a direct summand of $S^{\tilde{\rho}}$ for any Hecke algebra, not just the semi-simple ones. Computing the relevant Littlewood–Richardson coefficients with [30, Theorem 4.9.4] gives us this result when $\mathscr{H}_n$ is semi-simple.

Lemma 3.19. *Let $\rho \vdash a$ an e-core and $\tilde{\rho} \vdash a + m$ as before. Then for $\nu \vdash m$:*

$$c^{\tilde{\rho}}_{\rho\nu} = \begin{cases} 1 & \text{if } \nu = (1^m), \\ 0 & \text{otherwise.} \end{cases}$$

We now tackle the general case.

Theorem 3.20. *Let $n = a + m$, $\rho \vdash a$ an e-core, and $\mu = (a, m) \vDash n$. Then as $\mathscr{H}_\mu$-modules:*

$$S^\rho \otimes S^{(1^m)} \mid S^{\tilde{\rho}}.$$

Proof. Let $\mathcal{O}$ be the localisation of $F[x]$ at the maximal ideal generated by $(x - q)$ and K the field of fractions of $\mathcal{O}$. Consider three related Hecke algebras $\mathscr{H}_a(K, x)$, $\mathscr{H}_a(\mathcal{O}, x)$ and $\mathscr{H}_a(F, q)$. As K is a field, and x has quantum characteristic zero, (and thus each partition is its own 0-core), by [26, Corollary 2.21] $\mathscr{H}_a(K, x)$ is semi-simple. As in [8, §5], we have an inclusion homomorphism between $\mathscr{H}_a(\mathcal{O}, x)$ and $\mathscr{H}_a(K, x)$, induced by the inclusion of $\mathcal{O}$ into K, and a map:

$$\overline{\cdot} : \mathscr{H}_a(\mathcal{O}, x) \to \mathscr{H}_a(F, q),$$

induced by $x \mapsto q$. We use the notation S^ν_K to mean the Specht module corresponding to ν in $\mathscr{H}_a(K, x)$, and similarly for $\mathcal{O}$ and F. Following the notation in [8, §5], we can define idempotents H^b in $\mathscr{H}_a(K, x)$, labelled by the blocks of $\mathscr{H}_a(F, q)$ (i.e. representatives of tableaux which have the same e-core), which act as the identity on Specht modules in that block, and zero on all the other Specht modules. As ρ is an e-core, and as such is the only Specht module in its block, denote the idempotent corresponding to this block by H^ρ. Therefore for $\nu \vdash a$:

$$S^\nu_K H^\rho = \begin{cases} S^\rho_K & \text{if } \nu = \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with the Littlewood–Richardson rule and Lemma 3.19:

$$S^{\tilde{\rho}}_K (H^\rho \otimes 1) = S^\rho_K \otimes S^{(1^m)}_K.$$

By [8, Theorem 5.3], we know that $H^b \in \mathscr{H}_a(\mathcal{O}, x)$ for any block b of $\mathscr{H}_a(F, q)$, (even though it is defined in $\mathscr{H}_a(K,x)$). Furthermore, by [8, Theorem 5.4]:

$$\{\overline{H^b} : b \text{ is a block of } \mathscr{H}_a(F,q)\},$$

is a complete set of central orthogonal primitive idempotents of $\mathscr{H}_a(F,q)$, i.e. $\overline{H^b} = f_b$ is the block idempotent of b. Therefore $\overline{H^\rho}$ acts as the identity on S_F^ρ, and 0 on all other Specht modules. By Lemma 3.17 and Theorem 3.18:

$$S_F^\rho \otimes S_F^{(1^m)} = \left(S_F^\rho \otimes S_F^{(1^m)}\right)(\overline{H^\rho} \otimes 1) \subseteq S_F^{\tilde{\rho}}(\overline{H^\rho} \otimes 1). \tag{3.1}$$

For simplicity of notation, let $V = S_K^{\tilde{\rho}}(H^\rho \otimes 1)$, a $\mathscr{H}_\mu(K,x)$-module and $M = S_{\mathcal{O}}^{\tilde{\rho}}(H^\rho \otimes 1)$ a $\mathscr{H}_\mu(\mathcal{O},x)$-module. As $\mathcal{O}$ is a principal ideal domain, and M is an $\mathcal{O}$-submodule of the finite-dimensional $\mathcal{O}$-module $S_{\mathcal{O}}^{\tilde{\rho}}$, it must have a finite $\mathcal{O}$-basis.

In particular, as $S_{\mathcal{O}}^{\tilde{\rho}} \otimes_{\mathcal{O}} K \cong S_K^{\tilde{\rho}}$ as $\mathscr{H}_\mu(K,x)$-modules, and H^ρ is central in both $\mathscr{H}_\mu(K,x)$ and $\mathscr{H}_\mu(\mathcal{O},x)$, we get that $M \otimes_{\mathcal{O}} K \cong V$. Using the relevant analogue of [5, Proposition 16.12], we get that M is a free $\mathscr{H}_\mu(\mathcal{O},x)$-lattice in V, as defined in [5, §16]. In particular each $\mathcal{O}$-basis of M is a K-basis of V. Hence:

$$\dim_{\mathcal{O}}(M) = \dim_K(V).$$

Note that as $\overline{S_{\mathcal{O}}^{\tilde{\rho}}} = S_F^{\tilde{\rho}}$, and as reducing modules via the map $\overline{\cdot}$ commutes with multiplication from the Hecke algebra:

$$\overline{M} = \overline{S_{\mathcal{O}}^{\tilde{\rho}}(H^\rho \otimes 1)} = \overline{S_{\mathcal{O}}^{\tilde{\rho}}}(\overline{H^\rho \otimes 1}) = S_F^{\tilde{\rho}}(\overline{H^\rho} \otimes 1).$$

By the discussion preceding [5, Proposition 16.16]:

$$\dim_F(\overline{M}) = \dim_{\mathcal{O}}(M),$$

therefore:

$$\dim_F(S^\rho_F \otimes S^{(1^m)}_F) = \dim_K(S^\rho_K \otimes S^{(1^m)}_K) = \dim_K(S^{\tilde{\rho}}_K(H^\rho \otimes 1)) = \dim_F(S^{\tilde{\rho}}_F(\overline{H^\rho} \otimes 1)).$$

Coupling this with (3.1) shows that $S^\rho_F \otimes S^{(1^m)}_F = S^{\tilde{\rho}}_F(\overline{H^\rho} \otimes 1)$, and hence $S^\rho_F \otimes S^{(1^m)}_F$ is a direct summand of $S^{\tilde{\rho}}_F$ as a $\mathscr{H}_\mu$-module. □

As a result of this theorem, we know that if $S^\rho \otimes S^{(1^{de})}$ has a Green correspondent M in $\mathscr{H}_n$, then M will lie in the block $B_{\rho,d}$ as it is a direct summand of $S^{\tilde{\rho}}$ by the module version of Corollary 2.26. In the next two sections, dependent on characteristic, we compute both the vertex of $S^{(1^{de})}$ and the block containing $S^\rho \otimes S^{(1^{de})}$ (the block $B_{\rho,0} \otimes B_{\varnothing,d}$) to see if we can use this result with Corollary 3.11 to compute Brauer correspondents.

3.3 Blocks in characteristic zero

Throughout this section we will assume that the (algebraically closed) field F has characteristic 0. A key type of parabolic we will need are the e-**parabolic** subgroups, which are any parabolic subgroups isomorphic to a product of copies of $\mathfrak{S}_e$. These are relevant due to the following theorem [13, Theorem 3.1], providing a Hecke algebra analogue to the fact from local representation theory that vertices are always p-groups.

Theorem 3.21. *Let F be a field of characteristic zero. If M is a finitely generated indecomposable $\mathscr{H}_n$-module, then its vertex is an e-parabolic subgroup of $\mathfrak{S}_n$.*

We will use this theorem to compute the vertex of blocks of $\mathscr{H}_n$ with empty core.

3.3.1 Vertices of Sign Modules

We start by looking at the vertex of the sign module. By Theorem 3.21, the sign module $S^{(1^e)}$ for $\mathscr{H}_e$ either has fixed-point-free vertex $\mathfrak{S}_e$ or is projective as an $\mathscr{H}_e$-module. We first prove a more general result about e-restricted partitions. Recall from Theorem 1.10 that the non-isomorphic irreducible modules for $\mathscr{H}_n$ are given by:

$$\{D^\lambda : \lambda \vdash n \text{ is } e\text{-restricted}\}.$$

If $\lambda \vdash n$ is an e-core, then it is necessarily e-restricted, and since D^λ lies in a block of weight 0, we conclude by [15, Theorem 1.2] that D^λ is projective. The next lemma shows the opposite is true when λ is not an e-core.

Lemma 3.22. *Let $\lambda \vdash n$ be e-restricted, but not an e-core. Then D^λ is not projective.*

Proof. Note that as λ is e-restricted, D^λ is a non-zero irreducible module. Denote $d_{\mu\lambda} = [S^\mu : D^\lambda]$ for $\mu \vdash n$, i.e. the multiplicity of D^λ as a composition factor of S^μ. Assume that D^λ is projective, then in particular $P^\lambda = S^\lambda = D^\lambda$, where P^λ is the corresponding projective indecomposable module. Thus $[P^\lambda : D^\lambda] = 1$. From [26, Theorem 2.20], we have:

$$[P^\lambda : D^\lambda] = \sum_{\mu \vdash n} d_{\mu\lambda}^2 \geqslant 1,$$

as $d_{\lambda\lambda} = 1$. As λ is not an e-core, it is not the only partition in its block. By [26, Corollary 2.22], there exists another partition μ in the same block which shares a simple composition factor with S^λ. As $S^\lambda = D^\lambda$, this must be D^λ itself, and thus we've found a $\mu \neq \lambda$ with $d_{\mu\lambda} \geqslant 1$. Hence $[P^\lambda : D^\lambda] \geqslant 2$ giving the required contradiction. □

Corollary 3.23. *$S^{(1^e)}$ has vertex $\mathfrak{S}_e$ as an $\mathscr{H}_e$-module.*

Proof. By Theorem 3.21, $S^{(1^e)}$ is either projective or has vertex $\mathfrak{S}_e$. As $S^{(1^e)} = D^{(1^e)}$ since it is one-dimensional, it cannot be projective by the preceding lemma as it is not an

e-core. □

We can now extend this result to larger Hecke algebras.

Theorem 3.24. *Let $\lambda = (e^d) \vdash de$ for $d \geqslant 1$. Then the $\mathscr{H}_{de}$-module $S^{(1^{de})}$ has vertex $\mathfrak{S}_\lambda$.*

Proof. Note that $\mathfrak{S}_\lambda \cong \prod_{i=1}^{d} \mathfrak{S}_e$, and as $\mathscr{H}_\lambda$-modules we have

$$S^{(1^{de})} \cong (S^{(1^e)})^{\otimes d}.$$

The latter has vertex $\mathfrak{S}_\lambda$ as a $\mathscr{H}_\lambda$-module by repeated applications of Theorem 2.4 and Corollary 3.23. As we know the vertex of $S^{(1^{de})}$ is e-parabolic, it must be contained in $\mathfrak{S}_\lambda$. By [13, Lemma 3.2], as $S^{(1^{de})}$ is simple both as a $\mathscr{H}_n$ and $\mathscr{H}_\lambda$-module, they share the same vertex. □

Corollary 3.25. *Let $n = a + de$ where $0 \leqslant a < e$, and let $\lambda = (1^a, e^d)$. Then the $\mathscr{H}_n$-module $S^{(1^n)}$ has vertex $\mathfrak{S}_\lambda$.*

This follows immediately from the preceding theorem and Theorem 3.21, which lets us immediately restrict down to $\mathscr{H}_{(1^a,de)}$. Returning to $\mathscr{H}_{de}$, for $\lambda = (e^d)$, we get a lower bound of $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ on the vertex of the empty core block by Theorem 3.12 and Proposition 2.17. We will now bound this vertex from above by showing that this block, and in fact all of $\mathscr{H}_{de}$, is $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$-projective as a $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule.

3.3.2 Relative projectivity of the empty core block

Note that to preserve convention, we write our bimodule homomorphisms on the right as well.

Theorem 3.26. *$\mathscr{H}_{de}$ is relatively $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$-projective as a $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule.*

Proof. We will show as $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodules that:

$$\mathscr{H}_{de} \mid \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de} \cong \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_\lambda \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de}.$$

To do this, we define $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule homomorphisms $\varphi : \mathscr{H}_{de} \to \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de}$ and $\psi : \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de} \to \mathscr{H}_{de}$ such that $\varphi \circ \psi = \mathbb{1}_{\mathscr{H}_{de}}$, the identity bimodule homomorphism on $\mathscr{H}_{de}$.

As $1 \otimes 1 \in Z_{\mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de}}(\mathscr{H}_\lambda)$ (as we can push elements of $\mathscr{H}_\lambda$ across the tensor product), we can define:

$$x := \mathrm{Tr}_\lambda^{(de)}(1 \otimes 1) = \sum_{w \in \mathcal{R}_\lambda^{(de)}} q^{-\ell(w)} T_{w^{-1}} \otimes T_w,$$

and $x \in Z_{\mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de}}(\mathscr{H}_{de})$ by Lemma 2.11. Thus we have a $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule homomorphism $\varphi : \mathscr{H}_{de} \to \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de}$ given by:

$$h \mapsto xh = hx.$$

Now define $\tilde{x} \in \mathscr{H}_{de}$ as:

$$\tilde{x} := \mathrm{Tr}_\lambda^{(de)}(1) = \sum_{w \in \mathcal{R}_\lambda^{(de)}} q^{-\ell(w)} T_{w^{-1}} T_w.$$

By [13, Theorem 2.7], $\tilde{x}$ is invertible, and $\tilde{x} \in Z(\mathscr{H}_{de})$ (again by Lemma 2.11). As $\tilde{x}$ is central, so is $\tilde{x}^{-1}$. Now we can define a $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule homomorphism $\psi : \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de} \to \mathscr{H}_{de}$ via:

$$a \otimes b \mapsto ab\tilde{x}^{-1},$$

extended linearly, for $a, b \in \mathscr{H}_{de}$.

Finally, we show that $\varphi \circ \psi$ is the identity map on $\mathscr{H}_{de}$. Note that by the definition of both x and $\tilde{x}$, we have $(x)\psi = \tilde{x}\tilde{x}^{-1} = 1$. Thus for $h \in \mathscr{H}_{de}$:

$$(h)\varphi\psi = (xh)\psi = (x)\psi \cdot h = 1 \cdot h = h,$$

completing the proof. $\square$

Corollary 3.27. *Let B be a block of $\mathscr{H}_{\mathrm{de}}$, and $\lambda = (e^d) \vdash de$. Then B is relatively $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$-projective as a $(\mathscr{H}_{\mathrm{de}}, \mathscr{H}_{\mathrm{de}})$-bimodule.*

At this point, we have all the machinery required to show that $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ is the vertex of the empty core block of $\mathscr{H}_{de}$ when our field has characteristic zero. However, we defer the proof to Section 3.5, where we can cover all cases on the characteristic of F at once.

3.4 Blocks in prime characteristic

Throughout this section, let F have prime characteristic $p > 0$. Recall from Proposition 1.3 that when e is non-zero, then we can assume either $\mathrm{hcf}(e, p) = 1$ and q is a primitive e-th root of unity, or $e = p$ and $q = 1$. Given n, write n in its ***e*-*p*-adic expansion** as:

$$n = a_{-1} + a_0 e + a_1 ep + \cdots + a_t ep^t,$$

where $0 \leqslant a_{-1} < e$ and $0 \leqslant a_i < p$, for $i \geqslant 0$. If n has the above e-p-adic expansion, the **standard maximal *e*-*p*-parabolic subgroup** of $\mathfrak{S}_n$, denoted by $\mathfrak{P}(\mathfrak{S}_n)$, is the subgroup corresponding to the composition:

$$(1^{a_{-1}}, e^{a_0}, (ep)^{a_1}, \ldots, (ep^t)^{a_t}) \vDash n.$$

A general ***e*-*p*-parabolic subgroup** of $\mathfrak{S}_n$ corresponds to $\tau = (\tau_1, \ldots, \tau_s) \vDash n$ which has for each i, either $\tau_i = 1$ or $\tau_i = ep^{r_i}$ for some $r_i \geqslant 0$. For a general parabolic subgroup

$\mathfrak{S}_\lambda$, we can form its maximal e-p-parabolic subgroup $\mathfrak{P}(\mathfrak{S}_\lambda)$ by repeating the above steps on each component of $\mathfrak{S}_\lambda$.

When $p = 0$, then we recover our previous notion of e-parabolic subgroups. Thus it is our hope that replacing the maximal e-parabolics seen in the previous section with maximal e-p-parabolic subgroups might allow us to achieve corresponding results in prime characteristic.

3.4.1 Vertices of sign modules

Our first aim is to prove a lower bound for the vertex of the empty core block of $\mathscr{H}_n$, and once again we do this by considering the vertex of the sign module and using Theorem 3.12. Let $\tau \vDash n$, and define $N_\tau = \sum_{w \in \mathcal{R}_\tau} q^{-\ell(w)}$.

Proposition 3.28. *Let $\tau \vDash n$. Then $S^{(1^n)}$ is $\mathfrak{S}_\tau$-projective if and only $N_\tau \neq 0$.*

Proof. Suppose $N_\tau \neq 0$, so it is invertible in F. Denote the identity map on $S^{(1^n)}$ as a $\mathscr{H}_n$-module by $\mathbb{1}_n$, and as a $\mathscr{H}_\tau$-module by $\mathbb{1}_\tau$. Let $S^{(1^n)}$ be generated by the element ξ (so $\xi T_i = -\xi$ for all $i = 1, \dots, n-1$). Then:

$$\begin{aligned}
(\xi)\,\mathrm{Tr}_\tau^{(n)}\left(\frac{1}{N_\tau}\mathbb{1}_\tau\right) &= \frac{1}{N_\tau}\sum_{w \in \mathcal{R}_\tau} q^{-\ell(w)}\xi \cdot T_{w^{-1}}\mathbb{1}_\tau T_w,\\
&= \frac{1}{N_\tau}\sum_{w \in \mathcal{R}_\tau} q^{-\ell(w)}(-1)^{\ell(w^{-1})}\xi \cdot T_w,\\
&= \frac{1}{N_\tau}\sum_{w \in \mathcal{R}_\tau} q^{-\ell(w)}(-1)^{\ell(w^{-1})+\ell(w)}\xi,\\
&= \frac{1}{N_\tau}N_\tau\xi,\\
&= \xi.
\end{aligned}$$

Hence $\mathrm{Tr}_\tau^{(n)}(\frac{1}{N_\tau}\mathbb{1}_\tau) = \mathbb{1}_n$. Therefore by Higman's criterion, we conclude that $S^{(1^n)}$ is $\mathfrak{S}_\tau$-projective.

Now suppose that $S^{(1^n)}$ is $\mathfrak{S}_\tau$-projective. Again using Higman's criterion, there exists a $\mathscr{H}_\tau$-homomorphism ψ such that $\mathbb{1}_n = \mathrm{Tr}_\tau^{(n)}(\psi)$. Since $S^{(1^n)}$ is an irreducible $\mathscr{H}_\tau$-module, $\psi = c\mathbb{1}_\tau$ for some $c \in F$. Then the above calculation shows that:

$$\mathbb{1}_n = \mathrm{Tr}_\tau^{(n)}(c\mathbb{1}_\tau) = cN_\tau\mathbb{1}_n,$$

hence $cN_\tau = 1$, so N_τ must be non-zero. □

Therefore relative projectivity of $S^{(1^n)}$ relies entirely upon these N_τ. Consider the following polynomial in $(\mathbb{Z}/p\mathbb{Z})[u]$:

$$a(u) = \sum_{w \in \mathcal{R}_\tau} u^{\ell(w)},$$

and notice that $N_\tau = a(q^{-1})$. By [23, §1.11], $a = P_n/P_\tau$ where $P_n(u) = \sum_{w\in\mathfrak{S}_n} u^{\ell(w)}$ is the Poincaré polynomial of $\mathfrak{S}_n$, and $P_\tau(u) = \sum_{w\in\mathfrak{S}_\tau} u^{\ell(w)}$ is the Poincaré polynomial of $\mathfrak{S}_\tau$. So to check if $S^{(1^n)}$ is relatively $\mathfrak{S}_\tau$-projective, we count the zeroes of P_n and P_τ at q^{-1}.

Definition 3.29. For q a primitive e-th root of unity in F (or $q = 1$ if $e = p$) and $P \in F[u]$, define $z(P)$ to be largest integer l such that $(u - q^{-1})^l \mid P(u)$ in $F[u]$.

This gives us the following test:

Corollary 3.30. *For $\tau \vDash n$, $N_\tau \neq 0$ if and only if $z(P_n) = z(P_\tau)$. Hence $S^{(1^n)}$ is $\mathfrak{S}_\tau$-projective if and only if $z(P_n) = z(P_\tau)$*

From [6, §2] we know that:

$$P_n(u) = \prod_{i=1}^{n} \frac{u^i - 1}{u - 1} = \prod_{i=2}^{n}(1 + \cdots + u^{i-1}). \tag{3.2}$$

We also know that for any i

$$u^i - 1 = \prod_{d|i} \Phi_d(u),$$

where Φ_d is the d-th cyclotomic polynomial. Now denote:

$$C_i(u) := 1 + \cdots + u^{i-1} = \prod_{d|i, d>1} \Phi_d(u), \tag{3.3}$$

so that $P_n(u) = \prod_{i=2}^{n} C_i(u)$. As we can write each P_n as a product of cyclotomic polynomials, we only need to compute $z(\Phi_m)$ for Φ_m involved in P_n.

Recall the notion of the resultant $\mathrm{Rslt}(f, g)$ of two polynomials $f, g \in R[u]$ for some ring R, see for example [2, §2]. This has the property that $\mathrm{Rslt}(f, g) = 0$ if and only if f and g share a common factor. Using [2, Theorems 3 and 4], we can compute the resultant of two cyclotomic polynomials. We reproduce these results below. Without loss of generality let $m > n > 1$. Then:

$$\mathrm{Rslt}(\Phi_m, \Phi_n) = \mathrm{Rslt}(\Phi_n, \Phi_m) = \begin{cases} s^{\mathcal{E}(n)} & \text{if } m/n \text{ is a power of some prime } s, \\ 1 & \text{otherwise,} \end{cases}$$

where $\mathcal{E}$ is Euler's totient function. This allows us to compute $z(\Phi_n)$ for general n.

Theorem 3.31. *Let q have quantum characteristic e, and let $n > 1$. Then $\Phi_n(q^{-1}) = 0$ if and only if $n = ep^r$ for some $r \geqslant 0$. In particular:*

- *If $\mathrm{hcf}(e, p) = 1$, then $z(\Phi_{ep^r}) = p^r - p^{r-1}$ for $r \geqslant 1$, and $z(\Phi_e) = 1$.*
- *If $e = p$ and $q = 1$, then $z(\Phi_{p^r}) = p^r - p^{r-1}$ for $r \geqslant 1$.*

Proof. First of all, if $n < e$, then $z(\Phi_n) = 0$ as Φ_e is the smallest cyclotomic polynomial which can be zero at q^{-1}. Now suppose $\Phi_n(q^{-1}) = 0$, and $n > e$. Consider the resultant of Φ_n with Φ_e. This resultant must be zero, as $(u - q^{-1})$ is a common factor of both by assumption. As $n > e$, by the above result from [2, Theorems 3 and 4], we can only have $\mathrm{Rslt}(\Phi_n, \Phi_e) = 0$ in F if n/e is a power of p, i.e. $n = ep^r$ for some $r \geqslant 1$. Including the possibility when $n = e$, this gives one direction of our first assertion.

It remains to show that Φ_{ep^r} is zero at q^{-1} for all $r \geqslant 0$, and to compute $z(\Phi_n)$ in these cases. Recall from [29, §1 Equations 4,5] that:

$$\Phi_{np}(u) = \begin{cases} \Phi_n(u^p)/\Phi_n(u) & \text{if } p \nmid n, \\ \Phi_n(u^p) & \text{if } p \mid n. \end{cases}$$

When $\text{hcf}(e, p) = 1$, and $n = ep^r$ for $r \geqslant 1$:

$$\Phi_n(u) = \Phi_{ep^r}(u) = \Phi_{ep^{r-1}}(u^p) = \cdots = \Phi_{ep}(u^{p^{r-1}}) = \Phi_e(u^{p^r})/\Phi_e(u^{p^{r-1}}).$$

As F has characteristic p:

$$\Phi_n(u) = \Phi_e(u^{p^r})/\Phi_e(u^{p^{r-1}}) = \Phi_e(u)^{p^r}/\Phi_e(u)^{p^{r-1}} = \Phi_e(u)^{p^r - p^{r-1}}.$$

Thus as $\Phi_e(q^{-1}) = 0$, we get that $\Phi_n(q^{-1}) = 0$. As $z(\Phi_e) = 1$ (its roots are the primitive e-th roots of unity each with multiplicity one), we also get that $z(\Phi_n) = p^r - p^{r-1}$ if $n = ep^r$ for $r \geqslant 1$.

Similarly when $e = p$ and $q = 1$ (so $q = q^{-1}$):

$$\Phi_n(u) = \Phi_{p^r}(u) = \Phi_{p^{r-1}}(u^p) = \cdots = \Phi_p(u^{p^{r-1}}) = \Phi_p(u)^{p^{r-1}},$$

so again $\Phi_{p^r}(1) = 0$, and $z(\Phi_{p^r}) = p^{r-1}z(\Phi_p) = p^r - p^{r-1}$, since $z(\Phi_p) = p - 1$. □

3.4.2 Computing $z(P_n)$

We begin by computing the following preliminary expressions.

Lemma 3.32. *Let $i > 1$.*

- *If* $\mathrm{hcf}(e,p)=1$*:*

$$z(C_i) = \begin{cases} p^r & \textit{if } r \textit{ is the largest integer such that } ep^r \mid i, \\ 0 & e \nmid i. \end{cases}$$

- *If* $e=p$*:*

$$z(C_i) = \begin{cases} p^r - 1 & \textit{if } r \textit{ is the largest integer such that } p^r \mid i, \\ 0 & p \nmid i. \end{cases}$$

Proof. This follows from counting the number of zeroes at q^{-1} in the product (3.3). When $\mathrm{hcf}(e,p)=1$, then if $e \nmid i$, we have $z(C_i)=0$ as no Φ_{ep^r} appear in the product (3.3). Otherwise, if r is the largest integer such that $ep^r \mid i$, then $\Phi_e, \dots, \Phi_{ep^r}$ are the only factors which are zero at q^{-1}. Thus:

$$z(C_i) = 1 + (p-1) + \cdots + (p^r - p^{r-1}) = p^r.$$

If $e=p$, then if $p \nmid i$, there are no zeroes at $q=1$, otherwise we only have factors $\Phi_p, \dots, \Phi_{p^r}$ which are zero at q, where r is the largest integer such that $p^r \mid i$. Thus:

$$z(C_i) = (p-1) + \cdots + (p^r - p^{r-1}) = p^r - 1,$$

completing the proof. □

This lets us compute $z(P_n)$ directly.

Lemma 3.33. *Suppose* $\mathrm{hcf}(e,p)=1$ *and* $r \geqslant 0$ *is the largest integer such that* $ep^r \leqslant n$*. Then*

$$z(P_n) = \sum_{l=0}^{r-1} \left(\left\lfloor \frac{n}{ep^l} \right\rfloor - \left\lfloor \frac{n}{ep^{l+1}} \right\rfloor \right) p^l + \left\lfloor \frac{n}{ep^r} \right\rfloor p^r.$$

If $e = p$ *and* $q = 1$, *and* $r > 1$ *is the largest integer with* $p^r \leqslant n$. *Then*

$$z(P_n) = \sum_{l=1}^{r-1} \left(\left\lfloor \frac{n}{p^l} \right\rfloor - \left\lfloor \frac{n}{p^{l+1}} \right\rfloor \right) (p^l - 1) + \left\lfloor \frac{n}{p^r} \right\rfloor (p^r - 1).$$

Proof. First suppose that $\mathrm{hcf}(e, p) = 1$. Recall that $P_n = \prod_{i=2}^{n} C_i$, hence $z(P_n) = \sum_{i=2}^{n} z(C_i)$. Now each C_i contributes no zeroes if no ep^l divides i, or $z(C_i)$ zeroes if it does. If it contributes zeroes, it contributes according to the largest l such that $ep^l \mid i$. Hence we need to count how many times this occurs. For each l, the number of times that ep^l divides n is $\lfloor \frac{n}{ep^l} \rfloor$. In $\lfloor \frac{n}{ep^{l+1}} \rfloor$ of those times, we also have ep^{l+1} dividing n. Hence the total number of times l is the largest integer such that ep^l divides i for $i = 2, \ldots, n$ is $\lfloor \frac{n}{ep^l} \rfloor - \lfloor \frac{n}{ep^{l+1}} \rfloor$ for $0 \leqslant l \leqslant r - 1$, or $\lfloor \frac{n}{ep^r} \rfloor$ when $l = r$. Summing all these occurrences of zeroes and using the values from Lemma 3.32, gives the result as required. The case where $e = p$ follows similarly using $z(C_{p^r}) = p^r - 1$ instead. □

For the rest of this section let $n > 1$ and fix notation for its e-p-adic expansion as:

$$n = a_{-1} + a_0 e + a_1 ep + \cdots + a_r ep^r,$$

where $0 \leqslant a_{-1} < e$ and $0 \leqslant a_i < p$ for $i = 0, \ldots, r$. If $e = p$, the e-p-adic expansion is just the usual p-adic expansion, and we will simplify notation in this setting by writing

$$n = b_0 + b_1 p + \cdots + b_r p^r,$$

where $0 \leqslant b_i < p$ for $i = 0, \ldots, r$. The previous lemma lets us compute $z(P_n)$ based on these expansions:

Theorem 3.34. *Suppose* $\mathrm{hcf}(e, p) = 1$. *Then:*

$$z(P_n) = a_0 + \sum_{l=1}^{r} a_l \Big((l+1)p^l - lp^{l-1} \Big).$$

Suppose $e = p$ and $q = 1$. Then:

$$z(P_n) = \sum_{l=1}^{r} b_l l(p^l - p^{l-1}).$$

Proof. We first deal with the case when $\mathrm{hcf}(e, p) = 1$. To get this result from Lemmas 3.32 and 3.33, we first compute $\lfloor \frac{n}{ep^l} \rfloor - \lfloor \frac{n}{ep^{l+1}} \rfloor$ for $0 \leqslant l \leqslant r - 1$.

$$\begin{aligned}\left(\left\lfloor \frac{n}{ep^l} \right\rfloor - \left\lfloor \frac{n}{ep^{l+1}} \right\rfloor\right) &= a_l + a_{l+1}p + \cdots + a_r p^{r-l} - (a_{l+1} + a_{l+2}p + \cdots + a_r p^{r-l-1}), \\ &= (a_l - a_{l+1}) + (a_{l+1} - a_{l+2})p + \ldots (a_{r-1} - a_r)p^{r-l-1} + a_r p^{r-l}.\end{aligned}$$

Collecting terms by the $a_l - a_{l+1}$ in the sum gives us:

$$z(P_n) = \sum_{l=0}^{r-1} \Big((a_l - a_{l+1})(z(C_{ep^l}) + pz(C_{ep^{l-1}}) + \cdots + p^l z(C_{ep^0})) + a_r p^{r-l} z(C_{ep^l})\Big) + a_r p^r,$$

and using the fact that $z(C_{ep^j}) = p^j$ for all $j \geqslant 0$, this expression simplifies to

$$\begin{aligned} z(P_n) &= \sum_{l=0}^{r-1} \Big((a_l - a_{l+1})(p^l + \cdots + p^l) + a_r p^r\Big) + a_r p^r, \\ &= \sum_{l=0}^{r-1} \Big((l+1)(a_l - a_{l+1})p^l\Big) + (r+1)a_r p^r, \\ &= a_0 - a_1 + \left(\sum_{l=1}^{r-1} a_l (l+1)p^l\right) - \left(\sum_{l=1}^{r-1} a_{l+1}(l+1)p^l\right) + (r+1)a_r p^r, \\ &= a_0 + \sum_{l=1}^{r} a_l \Big((l+1)p^l - lp^{l-1}\Big), \end{aligned}$$

if we collect by the coefficients a_i.

Now suppose that $e = p$. To get this result from Lemmas 3.32 and 3.33, we similarly compute $\lfloor \frac{n}{p^l} \rfloor - \lfloor \frac{n}{p^{l+1}} \rfloor$ for $1 \leqslant l \leqslant r - 1$.

$$\left(\left\lfloor \frac{n}{p^l} \right\rfloor - \left\lfloor \frac{n}{p^{l+1}} \right\rfloor\right) = \sum_{i=l}^{r-1} (b_i - b_{i+1})p^{i-l} + b_r p^{r-l}.$$

Applying the previous lemma:

$$
\begin{aligned}
z(P_n) &= \left(\sum_{l=1}^{r-1}\left[b_r p^{r-l} + \sum_{i=l+1}^{r-1}(b_i - b_{i+1})p^{i-l}\right](p^l-1)\right) + b_r(p^r-1),\\
&= b_r(p^r-1) + \sum_{l=1}^{r-1}\left((b_l - b_{l+1}) + \cdots + (b_{r-1}p^{r-1-l} - b_r p^{r-1-l}) + b_r p^{r-l}\right)(p^l-1),\\
&= b_r(p^r-1) + \sum_{l=1}^{r-1}\left[b_l(p^l-1) + \sum_{i=l+1}^{r} b_i(p-1)(p^{i-1} - p^{i-1-l})\right],\\
&= \left(\sum_{l=1}^{r} b_l(p^l-1)\right) + \left[\sum_{l=1}^{r}\sum_{i=l+1}^{r} b_i(p-1)(p^{i-1} - p^{i-1-l})\right].
\end{aligned}
$$

We focus on the second bracket and collect by the b_i:

$$
\begin{aligned}
\sum_{l=1}^{r}\sum_{i=l+1}^{r} b_i(p-1)(p^{i-1} - p^{i-1-l}) &= \sum_{l=2}^{r}(p-1)b_l(p^{l-1} - p^{l-2} + p^{l-1} - p^{l-3} + \cdots + p^{l-1} - 1),\\
&= \sum_{l=2}^{r} b_l[(p-1)((l-1)p^{l-1} - (1 + \cdots + p^{l-2}))],\\
&= \sum_{l=2}^{r} b_l[(l-1)(p^l - p^{l-1}) + 1 - p^{l-1}],\\
&= \sum_{l=2}^{r} b_l[(l-1)p^l - lp^{l-1} + 1].
\end{aligned}
$$

Combining everything together:

$$
\begin{aligned}
z(P_n) &= \left(\sum_{l=1}^{r} b_l(p^l-1)\right) + \left(\sum_{l=2}^{r} b_l[(l-1)p^l - lp^{l-1} + 1]\right),\\
&= b_1(p-1) + \sum_{l=2}^{r} b_l\left(p^l - 1 + (l-1)p^l - lp^{l-1} + 1\right),\\
&= b_1(p-1) + \sum_{l=2}^{r} b_l l(p^l - p^{l-1}),\\
&= \sum_{l=1}^{r} b_l l(p^l - p^{l-1}).
\end{aligned}
$$

□

Corollary 3.35. *If* $\operatorname{hcf}(e,p) = 1$ *and* $r \geqslant 0$*, then* $z(P_{ep^r}) = (r+1)p^r - rp^{r-1}$*. If* $e = p$ *and* $r \geqslant 1$*, then* $z(P_{p^r}) = r(p^r - p^{r-1})$*.*

We can use Theorem 3.34 to show that if λ is the composition with $\mathfrak{S}_\lambda = \mathfrak{P}(\mathfrak{S}_n)$, then $S^{(1^n)}$ is $\mathfrak{S}_\lambda$-projective.

Proposition 3.36. *Let $n > 1$, and let $\mathfrak{S}_\lambda = \mathfrak{P}(\mathfrak{S}_n)$. Then $S^{(1^n)}$ is $\mathfrak{S}_\lambda$-projective.*

Proof. We show that $z(P_n) = z(P_\lambda)$, and again we start by proving this when $\mathrm{hcf}(e,p) = 1$. We already have a formula for $z(P_n)$, so we compute $z(P_\lambda)$. As $P_\lambda = \prod_{i=0}^{r}(P_{ep^i})^{a_i}$:

$$z(P_\lambda) = \sum_{i=0}^{r} a_i z(P_{ep^i}) = a_0 + \sum_{i=1}^{r}\Big(a_i(i+1)p^i - a_i i p^{i-1}\Big) = z(P_n).$$

Similarly if $e = p$ and $q = 1$, and as $P_\lambda = \prod_{i=1}^{r}(P_{p^i})^{b_i}$:

$$z(P_\lambda) = \sum_{i=1}^{r} b_i i(p^i - p^{i-1}) = z(P_n).$$

Applying Corollary 3.30 gives the result. □

So we have obtained an upper bound for the vertex of $S^{(1^n)}$ for general n. We now prove the special case of the vertex of $S^{(1^n)}$ where $n = ep^r$ for some $r \geqslant 0$, so in particular $\mathfrak{P}(\mathfrak{S}_n) = \mathfrak{S}_n$. By [6, Theorem 2.9] the vertex of $S^{(1^n)}$ is e-p-parabolic, so these are the only τ we need to check.

Lemma 3.37. *Suppose either* $\mathrm{hcf}(e,p) = 1$ *and $n = ep^r$ for $r \geqslant 0$, or $e = p$ and $n = p^r$ for $r > 0$. Then if $\mathfrak{S}_\tau \subsetneq \mathfrak{S}_n$ is an e-p-parabolic subgroup, $z(P_n) > z(P_\tau)$. Hence $S^{(1^n)}$ has vertex $\mathfrak{S}_n = \mathfrak{P}(\mathfrak{S}_n)$.*

Proof. We once again start by proving this statement when $\mathrm{hcf}(e,p) = 1$. Let $\mathfrak{S}_\tau$ be the e-p-parabolic subgroup corresponding to the expression $n = c_{-1} + c_0 e + \cdots + c_t e p^t$ for natural numbers c_i. In particular as $\mathfrak{S}_\tau \subsetneq \mathfrak{S}_n$, we have that $t < r$. Then we have by

Corollary 3.35 that:

$$\begin{aligned} z(P_\tau) &= c_0 + \sum_{i=1}^{t} \Big(c_i(i+1)p^i - c_i i p^{i-1} \Big), \\ &= \sum_{i=0}^{t} c_i p^i + \sum_{i=1}^{t} c_i i p^{i-1}(p-1). \end{aligned}$$

As $n = c_{-1} + \sum_{i=0}^{t} c_i e p^i = ep^r$, we get immediately that $\sum_{i=0}^{t} c_i p^i \leqslant p^r$, and hence $\sum_{i=1}^{t} c_i p^{i-1} \leqslant p^{r-1}$. This tells us that

$$\begin{aligned} z(P_\tau) &= \sum_{i=0}^{t} c_i p^i + \sum_{i=1}^{t} c_i i p^{i-1}(p-1), \\ &\leqslant p^r + (p-1)\sum_{i=1}^{t} c_i i p^{i-1}, \\ &< p^r + r(p-1)\sum_{i=1}^{t} c_i p^{i-1}, \\ &\leqslant p^r + r(p-1)p^{r-1}, \\ &= z(P_n). \end{aligned}$$

Thus if $\mathfrak{S}_\tau \subsetneq \mathfrak{S}_n$, we have N_τ is zero and the vertex of $S^{(1^n)}$ as an $\mathscr{H}_n$-module must be $\mathfrak{S}_n = \mathfrak{S}_{ep^r}$.

Now consider when $e = p$, and let $\mathfrak{S}_\tau$ be the p-parabolic subgroup corresponding to the expression $n = c_0 + c_1 p + \cdots + c_t p^t$, where again we must have $t < r$. Now as $\sum_{i=0}^{t} c_i p^i = n$, we have $\sum_{i=1}^{i} c_i p^i \leqslant n = p^r$ and thus $\sum_{i=1}^{i} c_i p^{i-1} \leqslant p^{r-1}$. Now using our formula for $z(P_\tau)$:

$$z(P_\tau) = \sum_{i=1}^{t} c_i i(p^i - p^{i-1}) < (p-1)r(\sum_{i=1}^{t} c_i p^{i-1}) \leqslant (p-1)rp^{r-1} = z(P_n),$$

so again we conclude that $\mathfrak{S}_n = \mathfrak{S}_{p^r}$ is the vertex of $S^{(1^n)}$. $\square$

We now compute the vertex of $S^{(1^n)}$ for any n.

Theorem 3.38. *Let $n > 1$. Then $S^{(1^n)}$ has vertex $\mathfrak{S}_\lambda = \mathfrak{P}(\mathfrak{S}_n)$ as a $\mathscr{H}_n$-module.*

Proof. Proposition 3.36 gives $\mathfrak{S}_\lambda$ as an upper bound for the vertex. Now suppose that $S^{(1^n)}$ has vertex $\mathfrak{S}_\tau$ which is strictly contained in $\mathfrak{S}_\lambda$. We once again can assume that $\mathfrak{S}_\tau$ is e-p-parabolic by [6, Theorem 2.9]. Then by Corollary 3.30, $z(P_n) = z(P_\tau)$, and in particular $z(P_\lambda) = z(P_\tau)$. Given $\lambda = (\lambda_1, \ldots, \lambda_s)$, as $\mathfrak{S}_\tau \subsetneq \mathfrak{S}_\lambda$, there exist compositions $\tau^{(i)}$ such that $\tau^{(i)} \vDash \lambda_i$ and $\prod_{i=1}^{s} \mathfrak{S}_{\tau^{(i)}} \cong \mathfrak{S}_\tau$.

For each i, as $P_{\lambda_i}/P_{\tau^{(i)}}$ is a non-zero polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$, we have $z(P_{\lambda_i}) \geqslant z(P_{\tau^{(i)}})$. As $\mathfrak{S}_\tau$ is strictly contained in $\mathfrak{S}_\lambda$, then there exists some j with $\mathfrak{S}_{\tau^{(j)}} \subsetneq \mathfrak{S}_{\lambda_j}$. Since $S^{1^{(\lambda_j)}}$ is not $\mathfrak{S}_{\tau^{(j)}}$-projective by Lemma 3.37, applying Corollary 3.30 tells us that $z(P_{\lambda_j}) > z(P_{\tau^{(j)}})$. Thus $z(P_\lambda) > z(P_\tau)$, giving a contradiction. Hence we must have that $\mathfrak{S}_\tau$ cannot be strictly contained in $\mathfrak{S}_\lambda$, and thus $\mathfrak{S}_\lambda$ must be the vertex of $S^{(1^n)}$ as a $\mathscr{H}_n$-module. ☐

These results, in conjunction with Theorem 3.12, give $(\mathfrak{P}(\mathfrak{S}_{de}), \mathfrak{P}(\mathfrak{S}_{de}))$ as a lower bound for the vertex of the empty core block of $\mathscr{H}_{de}$.

3.4.3 Relative projectivity of empty core blocks

Here we prove an upper bound for the vertex of blocks with empty core. We cannot fully generalise Theorem 3.26, however we can state a similar theorem which only covers the empty core block itself. Denote the central primitive idempotent associated to a block B by f_B, and let $\mathrm{End}_B\,(B)$ be the ring of (B, B)-bimodule homomorphisms on B. This is a local ring by [1, Theorem 4.2] as B is an indecomposable (B, B)-bimodule. Furthermore, as in the proof of [13, Lemma 2.3], $\mathrm{End}_B\,(B) \cong Z(B)$, (the centre of B) and hence $Z(B)$ is local. Thus $x \in Z(B)$ is invertible if and only if its image $\overline{x} \in Z(B)/J(Z(B))$ is non-zero (in a local ring R, its Jacobson radical $J(R)$ consists of all the non-units).

As we have a canonical isomorphism $\theta : Z(B)/J(Z(B)) \to F$, if M is a one-dimensional $Z(B)$-module, the action of $Z(B)/J(Z(B))$ must coincide with the action of the field. That is, for $x \in Z(B)$ and $m \in M$, if $m\overline{x} = \beta m$, then $(\overline{x})\theta = \beta$. Thus x is invertible in $Z(B)$ if and only if $(\overline{x})\theta = \beta \neq 0$.

Denote $B = B_{\varnothing,d}$ the block of $\mathscr{H}_{de}$ with empty core and e-weight d, let $\mathfrak{S}_\lambda = \mathfrak{P}(\mathfrak{S}_{de})$, and define

$$x_B := \mathrm{Tr}_\lambda^{(de)}(f_B) = \sum_{w \in \mathcal{R}_\lambda} q^{-\ell(w)} T_{w^{-1}} f_B T_w.$$

As f_B is central, then $x_B \in Z(B)$ by Lemma 2.11.

Lemma 3.39. *x_B is invertible in $Z(B)$, and hence in B.*

Proof. Take ξ, a generator of $S^{(1^{de})}$, the one-dimensional sign $\mathscr{H}_{de}$-module, and compute $\xi \cdot x_B$. As multiplication by f_B is the identity map on B, using the same calculations from the proof of Proposition 3.28 we obtain:

$$\xi \cdot x_B = \mathrm{Tr}_\lambda^{(de)}(\xi) = \left(\sum_{w \in \mathcal{R}_\lambda} q^{-\ell(w)} \right) \xi.$$

Hence under the isomorphism between $Z(B)/J(Z(B))$ and F:

$$(\overline{x_B})\theta = \sum_{w \in \mathcal{R}_\lambda} q^{-\ell(w)} = N_\lambda,$$

which is non-zero by Proposition 3.28 and Proposition 3.36. Thus by the preceding discussion, x_B is invertible in $Z(B)$, and hence in B. □

We can now generalise the proof of Theorem 3.26 to fields of prime characteristic when only focusing on the empty core block.

Theorem 3.40. *Let $B = B_{\varnothing,d}$ the empty core block of $\mathscr{H}_{de}$, and $\lambda \models de$ the composition of de corresponding to $\mathfrak{P}(\mathfrak{S}_{de})$. Then as $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodules, $B \mid B \otimes_{\mathscr{H}_\lambda} B$.*

Proof. Define a map $\varphi : B \to B \otimes_{\mathscr{H}_\lambda} B$ by $h \mapsto \mathrm{Tr}_\lambda^{(de)}(f_B \otimes f_B)h$ and $\psi : B \otimes_{\mathscr{H}_\lambda} B \to B$ by $a \otimes b \mapsto abx_B^{-1}$ extended linearly.

As in the proof of Theorem 3.26, both are well-defined $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule homomorphisms, and $\varphi \circ \psi = \mathbb{1}_B$ as for $h \in B$:

$$(h)\varphi\psi = (\mathrm{Tr}_\lambda^{(de)}(f_B \otimes f_B)h)\psi = (\mathrm{Tr}_\lambda^{(de)}(f_B \otimes f_B))\psi h = x_B x_B^{-1} h = h.$$

Thus B is a direct summand of $B \otimes_{\mathscr{H}_\lambda} B$ as $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodules. □

Corollary 3.41. *As a $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule, B is relatively $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$-projective.*

Proof. By definition, $B \mid \mathscr{H}_{de}$ as a $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule, and therefore as both $(\mathscr{H}_\lambda, \mathscr{H}_{de})$ and $(\mathscr{H}_{de}, \mathscr{H}_\lambda)$-bimodules as well. By the previous theorem:

$$B \mid B \otimes_{\mathscr{H}_\lambda} B \mid \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} B \mid \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de} \cong \mathscr{H}_{de} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_\lambda \otimes_{\mathscr{H}_\lambda} \mathscr{H}_{de},$$

showing B is relatively $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$-projective. □

3.5 Computing vertices of blocks

In the previous two sections, we showed that in all characteristics, the empty core block of $\mathscr{H}_{de}$ was $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$-projective, where $\mathfrak{S}_\lambda = \mathfrak{P}(\mathfrak{S}_{de})$ is the standard maximal e-p-parabolic subgroup of $\mathfrak{S}_{de}$. We also found a module in that block ($S^{(1^{de})}$) which had vertex $\mathfrak{S}_\lambda$ too. We will first show that $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ is actually the vertex of this block, before applying our Brauer correspondence from Section 3.1 to compute the vertices of all blocks.

Proposition 3.42. *Let $B = B_{\varnothing,d}$ be the block of $\mathscr{H}_{de}$ with empty core, and $\mathfrak{S}_\lambda = \mathfrak{P}(\mathfrak{S}_{de})$. Then as a $(\mathscr{H}_{de}, \mathscr{H}_{de})$-bimodule, B has no vertex strictly contained in $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$.*

Proof. Suppose that B has a vertex $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_2}) \subsetneq (\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. By Corollary 3.11, as $S^{(1^{de})}$ lies in this block and has vertex $\mathfrak{S}_\lambda$ as a right $\mathscr{H}_{de}$-module (by Theorem 3.24 or Theorem 3.38), there must be some $g \in \mathfrak{S}_n$ with $\mathfrak{S}_\lambda^g \subseteq \mathfrak{S}_{\tau_2} \subseteq \mathfrak{S}_\lambda$, thus $\mathfrak{S}_{\tau_2} = \mathfrak{S}_\lambda$.

By earlier assumption, $\mathfrak{S}_{\tau_1} \subsetneq \mathfrak{S}_\lambda$. In particular, B is $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_n)$-projective and hence by Proposition 2.17, it is also $(\mathfrak{S}_{\tau_1}, \mathfrak{S}_{\tau_1})$-projective. This means that B has a vertex where the second subgroup in the pair is strictly contained in $\mathfrak{S}_\lambda$. This cannot happen by the preceding argument, so B has no vertex strictly contained within $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. □

Theorem 3.43 (Vertex of empty core blocks)**.** *Let b be the block of $\mathscr{H}_{de}$ with empty core and e-weight d. Then b has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ where $\lambda \vDash de$ corresponds to $\mathfrak{P}(\mathfrak{S}_{de})$.*

Proof. By Corollary 3.27 or Corollary 3.41 (depending on the characteristic), b is $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$-projective, and hence has a vertex contained in $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. Proposition 3.42 says b cannot have a vertex strictly contained in $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$, finishing the proof. □

Proposition 3.44. *Let ρ be an e-core, $\mu = (|\rho|, de) \vDash n$, $\tau = (1^{|\rho|}, de)$, and $\mathfrak{S}_\lambda$ the standard maximal e-p-parabolic subgroup of $\mathfrak{S}_\tau$. Let $b_{\rho,0}$ be the block of $\mathscr{H}_{|\rho|}$ corresponding to ρ, and $b_{\varnothing,d}$ the block of $\mathscr{H}_{de}$ with empty core. Denote $b := b_{\rho,0} \otimes b_{\varnothing,d}$ a block of $\mathscr{H}_\mu = \mathscr{H}_{|\rho|} \otimes \mathscr{H}_{de}$. Then b has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$, and thus $b^{\mathscr{H}_n}$ exists.*

Proof. Since blocks of e-weight 0 are projective (they are semi-simple from [15, Theorem 1.2]), as a $(\mathscr{H}_\mu, \mathscr{H}_\mu)$-bimodule, b has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ by Theorem 3.43 and Theorem 2.4. Then $b^{\mathscr{H}_n}$ exists by Theorem 3.9, as $\mathfrak{S}_\lambda$ is a fixed-point-free subgroup of $\mathfrak{S}_\tau$ because $e \mid de$. □

So we have shown that there exists a block of $\mathscr{H}_n$ with vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. We now need to identify this block, and show that all blocks can be found in this way.

Theorem 3.45 (Classification of vertices of blocks of Hecke algebras)**.** *Let ρ be an e-core, $\mu = (|\rho|, de) \vDash n$, $\tau = (1^{|\rho|}, de)$, and $\mathfrak{S}_\lambda$ the standard maximal e-p-parabolic subgroup of*

$\mathfrak{S}_\tau$. *Denote by* $B = B_{\rho,d}$*, the block of* $\mathscr{H}_n$ *with* e*-core* ρ *and* e*-weight* d*, as in Section 1.4.1. Then* B *has vertex* $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ *as a* $(\mathscr{H}_n, \mathscr{H}_n)$*-bimodule.*

Proof. When $d = 0$, our block is semi-simple by [15, Theorem 1.2], thus is projective and hence has trivial vertex as required. Now suppose $d > 0$.

Consider the block $b = b_{\rho,0} \otimes b_{\varnothing,d}$ of $\mathscr{H}_\mu$. By the previous proposition this has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$, and has a Brauer correspondent; we will show that this is $B_{\rho,d}$, by applying Corollary 3.11.

$S^\rho \otimes S^{(1^{de})}$ is an indecomposable module which lies in b with vertex $\mathfrak{S}_\lambda$ by Theorem 3.24 or Theorem 3.38. Applying [13, Theorem 3.6], it has a Green correspondent M in $\mathscr{H}_n$. By Theorem 3.20 $S^\rho \otimes S^{(1^e)} \mid S^{\tilde{\rho}}$ as $\mathscr{H}_\mu$-modules, so applying the right-module version of Corollary 2.26 (i.e. setting $\sigma_1 = (1)$ and $\sigma_2 = (n)$), tells us that $M \mid S^{\tilde{\rho}}$ as $\mathscr{H}_n$-modules, thus M lies in $B_{\rho,d}$. As such, we conclude with Corollary 3.11 that $B_{\rho,d} = b^{\mathscr{H}_n}$ and hence has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$. □

Note that by Theorem 1.12, if $e > 2$ this shows that $S^{\tilde{\rho}}$ and $S^\rho \otimes S^{(1^{de})}$ are Green correspondents, as the former is indecomposable. When $e = 2$, this is not necessarily the case.

3.6 The Dipper–Du conjecture

One application of our classification of the vertices of blocks, is resolving the Dipper–Du conjecture, first stated as [6, Conjecture 1.9]. This was shown to be true for Young modules in [6, §5], for fields of characteristic zero in [13, Theorem 3.1], and in blocks of e-weight 1 in [31, Theorem 18.1.13]. Note that in [22], a supposed counter-example was given to this conjecture when $p = 2$ and $e = 3$. Here, an indecomposable $\mathscr{H}_3$-module M is found, which is $\mathscr{H}_{(2,1)}$-projective as a $\mathscr{H}_3$-module. However, as $\mathscr{H}_{(2,1)}$ is semi-simple when

$e = 3$, M is a projective $\mathscr{H}_{(2,1)}$-module, and hence by Corollary 2.2, is projective as a $\mathscr{H}_3$-module. This contradicts the earlier statement in [22] that M could not be projective [22, Theorem 2.2 Part (2)]. We are able to use our classification to prove this conjecture:

Theorem 3.46 (Dipper–Du conjecture). *Let F be an (algebraically closed) field of characteristic $p > 0$, $n \in \mathbb{N}$, and $q \in F^{\times}$ with quantum characteristic $e > 0$. Then the vertices of indecomposable $\mathscr{H}_n$-modules are e-p-parabolic.*

Proof. Let M be an indecomposable (right) $\mathscr{H}_n$-module with vertex $\mathfrak{S}_\tau$, where $\tau = (\tau_1, \dots, \tau_s) \vDash n$. By [13, Lemma 3.2], there is an indecomposable $\mathscr{H}_\tau$-module N such that $M \mid N \otimes_{\mathscr{H}_\tau} \mathscr{H}_n$ and N has vertex $\mathfrak{S}_\tau$. As N is indecomposable, N must belong to a block b of $\mathscr{H}_\tau$, with

$$b = b_{\rho_1,d_1} \otimes \cdots \otimes b_{\rho_s,d_s},$$

where b_{ρ_i,d_i} is the block of $\mathscr{H}_{\tau_i}$ corresponding to some e-core ρ_i and e-weight d_i. By Theorem 3.45, b has vertex $(\mathfrak{S}_\lambda, \mathfrak{S}_\lambda)$ where $\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda^1} \times \cdots \times \mathfrak{S}_{\lambda^s}$, and:

$$\mathfrak{S}_{\lambda^i} = \mathfrak{P}(\mathfrak{S}_{(1^{|\rho_i|})} \times \mathfrak{S}_{d_i e}) \subseteq \mathfrak{S}_{\tau_i}.$$

As N lies in the block b, by Theorem 3.12 we get $\mathfrak{S}_\tau \subseteq_{\mathfrak{S}_\tau} \mathfrak{S}_\lambda$, and thus $\mathfrak{S}_\lambda = \mathfrak{S}_\tau$. In particular, for each i, we get $\mathfrak{S}_{\lambda^i} = \mathfrak{S}_{\tau_i}$.

Therefore each $(\tau_i) \vDash \tau_i$ is an e-p-parabolic composition, so either $\tau_i = ep^r$ for some $r \geqslant 0$, or $\tau_i = 1$. Hence $\tau = (\tau_1, \dots, \tau_s)$ is an e-p-parabolic composition, and thus $\mathfrak{S}_\tau$ is an e-p-parabolic subgroup. $\square$

CHAPTER 4

VERTICES OF SPECHT MODULES

Now we have proved the Dipper–Du conjecture, we can use this powerful tool to compute the vertices of some indecomposable modules. These simple corollaries follow immediately from the conjecture using Higman's criterion:

Corollary 4.1. *Suppose* $\lambda, \mu \vDash n$ *with* $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_\mu$*, and* $\mathfrak{S}_\tau = \mathfrak{P}(\mathfrak{S}_\lambda)$*. Then any indecomposable* $\mathscr{H}_\mu$*-module that is* $\mathfrak{S}_\lambda$*-projective is also* $\mathfrak{S}_\tau$*-projective.*

Proof. As the maximal e-p-parabolic of $\mathfrak{S}_\lambda$, a conjugate of every e-p-parabolic subgroup of $\mathfrak{S}_\lambda$ is contained in $\mathfrak{S}_\tau$. Thus as the vertex is e-p-parabolic, we conclude using the transitivity of induction. □

Corollary 4.2 (Dipper–Du for relative traces)**.** *Let* M *be a* $\mathfrak{S}_\lambda$*-projective* $\mathscr{H}_n$*-module for* $\lambda \vDash n$ *and let* $\mathfrak{S}_\tau = \mathfrak{P}(\mathfrak{S}_\lambda)$*. Then for any* $\phi \in \operatorname{End}_{\mathscr{H}_\lambda}(M)$*, there exists* $\psi \in \operatorname{End}_{\mathscr{H}_\tau}(M)$ *with* $\operatorname{Tr}^n_\lambda(\phi) = \operatorname{Tr}^n_\tau(\psi)$*.*

Proof. By the previous corollary, M is $\mathfrak{S}_\tau$-projective as a $\mathscr{H}_\lambda$-module. Higman's criterion then gives $\psi \in \operatorname{End}_{\mathscr{H}_\tau}(M)$ with $\operatorname{Tr}^\lambda_\tau(\psi) = \phi$, and we conclude using the transitivity of relative traces. □

These corollaries let us immediately push down to the maximal e-p-parabolic subgroup when considering relative projectivity from either point of view.

In this chapter, we will begin by applying the Dipper–Du conjecture to compute the vertices of Specht modules for $\mathscr{H}_n$ where $n \leqslant e$. By the following corollary to Lemma 1.11, we know that all of these Specht modules are indecomposable.

Corollary 4.3. *Let $n \leqslant e$ and $\lambda \vdash n$. Then S^λ is an indecomposable $\mathscr{H}_n$-module.*

Proof. As $n \leqslant e$, all partitions of n are e-restricted except for $(e) \vdash e$, the trivial partition. As $S^{(e)}$ is one-dimensional, it is indecomposable as well. □

Alternatively we could use Theorem 1.12 when $e \neq 2$, and in the case where $e = 2$, then all Specht modules for both $\mathscr{H}_1$ and $\mathscr{H}_2$ are one-dimensional.

When $n < e$, either by the Dipper–Du conjecture or the fact that $\mathscr{H}_n$ is semi-simple, all Specht modules are projective. Thus the first interesting case occurs when $n = e$, where we now know that the vertices of indecomposable modules are either trivial, or $\mathfrak{S}_e$.

After considering Specht modules for $\mathfrak{S}_e$, we proceed to compute the vertex of Specht modules corresponding to hook partitions when $e \nmid n$.

4.1 Vertices of Specht modules when $n = e$

Partitions $\lambda \vdash e$ can be categorised in one of two ways. Either λ is an e-core, or λ is an e-hook. In the first case, by our classification of the vertex of blocks, we get that S^λ is projective. Thus it is the latter case that we will focus on. We begin by setting up some notation.

For $n \geqslant 1$, and $1 \leqslant r \leqslant n$ define $\gamma_{r,n} = (r, 1^{n-r}) \vdash n$. In this section we will show the vertex of $S^{\gamma_{i,e}}$ is $\mathfrak{S}_e$ for all $1 \leqslant i \leqslant e$. Note that for $i = 1$ and $i = e$, these cases have already been proved (either by [6, Theorem 5.8] for the trivial module $S^{\gamma_{e,e}}$ and Corollary 3.23 or Lemma 3.37 for the sign module $S^{\gamma_{1,e}}$ depending on the characteristic of F). As such, throughout the rest of this section we will assume that $1 < i < e$. In addition we assume $e > 2$ as the first e with non-trivial hooks is $e = 3$.

The main proposition we will use to compute these vertices is as follows:

Proposition 4.4. *Let M be an indecomposable $\mathscr{H}_e$-module. Then M is projective as a $\mathscr{H}_e$-module if and only if M is $\mathfrak{S}_{e-1}$-projective as a $\mathscr{H}_e$-module.*

Proof. The forward direction follows from transitivity of induction; if M is projective as a $\mathscr{H}_e$-module, it is also projective for any parabolic subgroup containing the trivial subgroup, thus is $\mathfrak{S}_{e-1}$-projective.

For the other direction, as $\mathscr{H}_{e-1}$ is semi-simple, M is a projective $\mathscr{H}_{e-1}$-module. Thus by Corollary 2.2, M is projective as a $\mathscr{H}_e$-module. □

We also have the following fact about the endomorphism rings of these modules, from [19, Corollary 2.6′] (since when $1 \leqslant i \leqslant e-1$, $S^{\gamma_{i,e}}$ is e-restricted).

Lemma 4.5. *Let $\lambda \vdash n$ be e-restricted. Then $\mathrm{End}_{\mathscr{H}_n}(S^\lambda) \cong F$.*

As $\mathscr{H}_{e-1}$ is semi-simple, we can apply the branching rule from [26, Proposition 6.1, Corollary 6.2]:

$$S^{\gamma_{i,e}} \cong S^{\gamma_{i,e-1}} \oplus S^{\gamma_{i-1,e-1}}, \tag{4.1}$$

as $\mathscr{H}_{e-1}$-modules, writing $S^{\gamma_{i,e}}$ as the sum of two irreducible, non-isomorphic $\mathscr{H}_{e-1}$-modules. Combining this with the previous lemma, tells us that $\mathrm{End}_{\mathscr{H}_{e-1}}(S^{\gamma_{i,e}})$ is two-dimensional, and $\mathrm{End}_{\mathscr{H}_e}(S^{\gamma_{i,e}})$ is one-dimensional. Thus by taking relative traces of a basis of $\mathrm{End}_{\mathscr{H}_{e-1}}(S^{\gamma_{i,e}})$ at a non-zero point, we can show using Higman's criterion whether $S^{\gamma_{i,e}}$ is $\mathfrak{S}_{e-1}$-projective as a $\mathscr{H}_e$-module or not.

4.1.1 Multiplication in $S^{\gamma_{i,e}}$

Recall that $S^{\gamma_{i,e}}$ has a basis $\{m_{\mathfrak{t}} : \mathfrak{t} \in \operatorname{Std}(\gamma_{i,e})\}$. Denote for $0 \leqslant j \leqslant e-i$ the tableau:

$$\mathfrak{t}^{i+j} = \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & \cdots\cdots\cdots & i-1 & i+j \\
\hline
i \\
\cline{1-1}
\vdots \\
\cline{1-1}
i+j-1 \\
\cline{1-1}
i+j+1 \\
\cline{1-1}
\vdots \\
\cline{1-1}
e \\
\cline{1-1}
\end{array}$$

and set $m_{i+j} = m_{\mathfrak{t}^{i+j}}$. Note that $\mathfrak{t}^{\gamma_{i,e}} = \mathfrak{t}^i$. To compute $m_{i+j}T_k$ we use Corollary 1.9, which gives explicit formulae when k and $k+1$ are not both in the first column. To deal with this case, we need the following corollary to [26, Lemma 3.17] which gives an explicit formula in this last case.

Lemma 4.6. *Let $\lambda \vdash n$, and $\mathfrak{t} \in \operatorname{Std}(\lambda)$ with k and $k+1$ in the same column. Then there exists some Garnir tableau $\mathfrak{g}$, and an element $w \in \mathfrak{S}_n$ with $\ell(d(\mathfrak{t}s_k)) = \ell(d(\mathfrak{g})) + \ell(w)$, the property that $\mathfrak{t}s_k = \mathfrak{g}w$, and a unique $\mathfrak{h} \in \operatorname{Std}(\lambda)$ with $\mathfrak{g} = \mathfrak{h}s$ for some elementary transposition s. Then in the Specht module S^λ:*

$$m_{\mathfrak{t}}T_k = -m_{\mathfrak{h}}T_w - \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda), \mathfrak{v} \rhd \mathfrak{h}} m_{\mathfrak{v}}T_w.$$

Proof. The proof of [26, Lemma 3.17] guarantees the existence of $\mathfrak{g}, w, \mathfrak{h}$ and s satisfying the conditions of this theorem. Taking $\mathfrak{s} = \mathfrak{t}^\lambda$ in that proof, gives us that in $\mathscr{H}_n$:

$$m_{\mathfrak{t}^\lambda \mathfrak{t}}T_k = m_{\mathfrak{t}^\lambda \mathfrak{g}}T_w = hT_w - m_{\mathfrak{t}^\lambda \mathfrak{h}}T_w - \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda), \mathfrak{v} \rhd \mathfrak{h}} m_{\mathfrak{t}^\lambda \mathfrak{v}}T_w,$$

where $h \in m_\mu \mathscr{H}_n$ for some $\mu > \lambda$. Thus $h \in \check{\mathscr{H}}_\lambda$ and taking the quotient to get an expression in the Specht module:

$$m_{\mathfrak{t}}T_k = m_{\mathfrak{t}^\lambda}T_{d(\mathfrak{g})}T_w = -m_{\mathfrak{h}}T_w - \sum_{\mathfrak{v}\in\mathrm{Std}(\lambda),\mathfrak{v}\rhd\mathfrak{h}} m_{\mathfrak{v}}T_w. \qquad \square$$

Proposition 4.7. *Let $0 \leqslant j \leqslant e-i$, and either $k > i+j$ or $i \leqslant k < i+j-1$. Then:*

$$m_{i+j}T_k = -m_{i+j}.$$

Proof. First of all note that our conditions on k mean that both k and $k+1$ are in the first column of t^{i+j}, thus we can apply Lemma 4.6. We claim that the required Garnir tableau is the tableau where the entries in the rows that k and $k+1$ occupy in $\mathfrak{t}^{i+j}$ are swapped. Let k lie in row $l-i+1$ of $\mathfrak{t}^{i+j}$. Then $\mathfrak{g} = \mathfrak{t}^i s_l$. This is a Garnir tableau by the definition in [26, §3.2].

If $\mathfrak{t}^{i+j}s_k = \mathfrak{g}w$, then as $d(\mathfrak{t}^{i+j}) = s_i\cdots s_{i+j-1}$, this means we will have $w = s_l s_i\cdots s_{i+j-1}s_k$. We split on cases to show that $\ell(d(\mathfrak{t}s_k)) = \ell(d(\mathfrak{g})) + \ell(w)$.

When $k > i+j$, then we have $l = k$, and since s_k commutes with $s_i\cdots s_{i+j-1}$:

$$w = s_k s_i\cdots s_{i+j-1}s_k = s_i\cdots s_{i+j-1}s_k^2 = s_i\cdots s_{i+j-1} = d(\mathfrak{t}^{i+j}).$$

Thus $\ell(d(\mathfrak{t}^{i+j}s_k)) = 1 + \ell(d(\mathfrak{t}^{i+j})) = \ell(\mathfrak{g}) + \ell(w)$.

If $i \leqslant k < i+j-1$, then $l = k+1$. Then:

$$\begin{aligned} w &= s_{k+1}s_i\cdots s_{i+j-1}s_k = s_i\cdots s_{k-1}s_{k+1}s_k s_{k+1}s_k s_{k+2}\cdots s_{i+j-1}, \\ &= s_i\cdots s_{k-1}s_k s_{k+1}s_k^2 s_{k+2}\cdots s_{i+j-1} = s_i\cdots s_{i+j-1} = d(\mathfrak{t}^{i+j}). \end{aligned}$$

As in the previous case, $\ell(d(\mathfrak{t}^{i+j}s_k)) = \ell(\mathfrak{g}) + \ell(w)$ as required. Thus in all cases we get that $w = d(\mathfrak{t}^{i+j})$, and $\mathfrak{h} = \mathfrak{t}^i$. Applying the previous lemma:

$$m_{\mathfrak{t}^{i+j}} T_k = -m_i T_{d(\mathfrak{t}^{i+j})} = -m_{i+j},$$

as the standard tableau $\mathfrak{t}^i$ is maximal in $\mathrm{Std}(\gamma_{i,e})$ with respect to the dominance ordering.

□

Now define the following element of $\mathscr{H}_e$:

$$D_i := 1 + T_{i-1} + T_{i-1}T_{i-2} + \cdots + T_{i-1}\cdots T_1.$$

When considering relative traces from $\mathscr{H}_{e-1}$ to $\mathscr{H}_e$, the following calculation will be crucial.

Proposition 4.8. *Let $1 \leqslant j \leqslant e-i$. Then $m_i D_i = (1 + q + \cdots + q^{i-1}) m_i$, and $m_{i+j} D_i = (-1)^j m_i$.*

Proof. The first statement follows as each T_k acts as q on m_i when $k < i$.

We briefly set up some more notation. For $k \leqslant i$, $j \geqslant 1$, denote by m^k_{i+j} the basis element corresponding to the following tableau.

1	2	$\cdots\cdots$	$k-1$	$k+1$	$\cdots\cdots$	i	$i+j$
k							
$i+1$							
$\vdots$							
$i+j-1$							
$i+j+1$							
$\vdots$							
e							

Then as $d(\mathfrak{t}^{i+j}) = s_i \cdots s_{i+j-1}$, whenever we multiply by an element $s_{i-1}\cdots s_k$ for $1 \leqslant k \leqslant$

$i-1$, (taking us from m_{i+j} to m^k_{i+j}) we are increasing the length of the group element. Thus by Corollary 1.9:

$$m_{i+j}D_i = m_{i+j} + m^{i-1}_{i+j} + \cdots + m^2_{i+j} + m^2_{i+j}T_1.$$

We now need to evaluate the final term. Here we need $\mathfrak{g}$, the Garnir tableau with first row $\{2, \ldots, i+1\}$, 1 in the second row, and the rest of the elements as in $\mathfrak{t}^{\gamma_{i,e}}$. The corresponding w is $w = s_{i+1} \cdots s_{i+j-1}$, and $\mathfrak{h}$ is given by the tableau corresponding to m^2_{i+1}. Thus applying Lemma 4.6:

$$m^2_{i+j}T_1 = -m^2_{i+j} - \sum_{\mathfrak{v} \rhd \mathfrak{h}} m_{\mathfrak{v}} T_w.$$

All the tableau that dominate m^2_{i+1} are either the standard tableau or m^k_{i+1} for $2 < k \leqslant i$. Now $m^k_{i+1}T_w = m^k_{i+j}$ and $m_i T_w = (-1)^{j-1} m_i$ by Proposition 4.7. Bringing everything together:

$$\begin{aligned} m_{i+j}D_i &= m_{i+j} + m^{i-1}_{i+j} + \cdots + m^2_{i+j} + m^2_{i+j}T_1, \\ &= m_{i+j} + \sum_{k=2}^{i-1} m^k_{i+j} - m^2_{i+j} - \sum_{k=3}^{i} m^k_{i+j} - m_i T_w. \end{aligned}$$

As $m_{i+j} = m^i_{i+j}$ most of the terms in this expression cancel giving:

$$m_{i+j}D_i = -m_i T_w = -(-1)^{j-1} m_i = (-1)^j m_i. \qquad \square$$

4.1.2 Finding a basis of $\mathrm{End}_{\mathscr{H}_{e-1}}(S^{\gamma_{i,e}})$

We showed previously that $\mathrm{End}_{\mathscr{H}_{e-1}}(S^{\gamma_{i,e}})$ is two-dimensional. In addition, a basis is given by the identity map, and projection onto one of the simple modules in the decomposition (4.1). Take m_i, the basis element corresponding to the standard tableau. Note that m_i

generates $S^{\gamma_{i,e}}$ as a $\mathscr{H}_e$-module. Now $m_i\mathscr{H}_{e-1}$ has a nice description in terms of our basis:

$$m_i\mathscr{H}_{e-1} = \langle m_\mathfrak{t} : e \text{ is in the bottom box of } \mathfrak{t}\rangle.$$

Furthermore, $m_i\mathscr{H}_{e-1} \cong S^{\gamma_{i,e-1}}$ as $\mathscr{H}_{e-1}$-modules, so we have a nice description of one of our indecomposable direct summands. To describe the other summand, we need the following. Define:

$$M_e := m_e + \sum_{j=0}^{e-i-1} (-q)^{e-i-j} m_{i+j}.$$

This element is going to be the key to our projection map.

Theorem 4.9. *As $\mathscr{H}_{e-1}$-modules:*

$$S^{\gamma_{i,e}} = M_e\mathscr{H}_{e-1} \oplus m_i\mathscr{H}_{e-1}.$$

Before we prove this, we show how M_e behaves under multiplication by T_j.

Proposition 4.10. *If $1 \leqslant j < i-1$, then $M_eT_j = qM_e$. If $i \leqslant j < e-1$, then $M_eT_j = -M_e$.*

Proof. First consider when $1 \leqslant j < i-1$. Then for any $l \geqslant 0$, $m_{i+l}T_j = qm_{i+1}$. Thus $M_eT_j = qM_e$ as required.

Now suppose $0 \leqslant j < e-i-1$ and consider M_eT_{i+j}.

$$M_eT_{i+j} = m_eT_{i+j} + \sum_{l=0}^{e-i-1} (-q)^{e-i-l} m_{i+l}T_{i+j}$$

Note that by our previous multiplication rules, if $k \neq j, j+1$, then $m_{i+k}T_{i+j} = -m_{i+k}$.

Therefore:

$$\begin{aligned} M_e T_{i+j} &= -m_e - \left(\sum_{l=0, l \neq j, j+1}^{e-i-1} (-q)^{e-i-l} m_{i+l} \right) \\ &\qquad + (-q)^{e-i-j} m_{i+j+1} + q(-q)^{e-i-j-1} m_{i+j} + (q-1)(-q)^{e-i-j-1} m_{i+j+1}, \\ &= -M_e. \end{aligned}$$

□

Proof of Theorem 4.9. We begin by showing that $M_e \mathscr{H}_{e-1} \neq S^{\gamma_{i,e}}$. Due to the above multiplication rules, if $\mu = (i-1, e-i)$, then we have:

$$M_e \mathscr{H}_{e-1} = \langle M_e T_w : w \in \mathcal{R}_\mu^{e-1} \rangle_F.$$

Now $|\mathcal{R}_\mu^{e-1}| = \binom{e-1}{i-1} = |\operatorname{Std}(\gamma_{i,e})| = \dim S^{\gamma_{i,e}}$. If $M_e \mathscr{H}_{e-1} = S^{\gamma_{i,e}}$, then $\{M_e T_w : w \in \mathcal{R}_\mu^{e-1}\}$ is a linearly independent set. However:

$$\begin{aligned} M_e D_i &= m_{i+(e-i)} D_i + (-q)^{e-i} m_i D_i + \left(\sum_{j=1}^{e-i-1} q^{e-i-j} (-1)^{e-i-j} m_{i+j} D_i \right), \\ &= (-1)^{e-i} \left(m_i + q^{e-i}(1 + q + \cdots + q^{i-1}) m_i + \sum_{j=1}^{e-i-1} q^{e-i-j} (-1)^{-j} (-1)^j m_i \right), \\ &= (-1)^{e-i} \left(1 + q^{e-i} + q^{e-i+1} + \cdots + q^{e-1} + \sum_{j=1}^{e-i-1} q^{e-i-j} \right) m_i, \\ &= (-1)^{e-i} m_i \left(1 + q^{e-i} + \cdots + q^{e-1} + q^{e-i-1} + \cdots + q \right), \\ &= 0, \end{aligned}$$

as q is a primitive e-th root of unity. Thus as $D_i \in \langle T_w : w \in \mathcal{R}_\mu^{e-1} \rangle_F$ by Proposition 1.4, $M_e \mathscr{H}_{e-1}$ is a proper $\mathscr{H}_{e-1}$-submodule of $S^{\gamma_{i,e}}$.

As $M_e \notin m_i \mathscr{H}_{e-1}$ (e does not appear in the bottom box of the tableau corresponding to m_e), then $M_e \mathscr{H}_{e-1} \cap m_i \mathscr{H}_{e-1}$ is a proper submodule of $m_i \mathscr{H}_{e-1}$. Since $m_i \mathscr{H}_{e-1} \cong S^{\gamma_{i,e-1}}$ as $\mathscr{H}_{e-1}$-modules, it is indecomposable, hence irreducible as $\mathscr{H}_{e-1}$ is semi-simple. Therefore this intersection is zero. Thus we have:

$$m_i\mathscr{H}_{e-1} + M_e\mathscr{H}_{e-1} = m_i\mathscr{H}_{e-1} \oplus M_e\mathscr{H}_{e-1} \subseteq S^{\gamma_{i,e}}.$$

Comparing to (4.1) with the Krull–Schmidt theorem tells us that we must have equality here, as $S^{\gamma_{i,e}}$ has only two summands in its direct sum decomposition as a $\mathscr{H}_{e-1}$-module. Note that this also shows that $M_e\mathscr{H}_{e-1} \cong S^{\gamma_{i-1,e-1}}$. □

Corollary 4.11. *Let π denote the projection map onto $m_i\mathscr{H}_{e-1}$. Then:*

$$(m_i)\pi = m_i,$$

$$(m_e)\pi = -\left(\sum_{j=0}^{e-i-1}(-q)^{e-i-j}m_{i+j}\right).$$

4.1.3 Computing traces

We now begin to compute the relative trace from $\mathscr{H}_{e-1}$ to $\mathscr{H}_e$ of both $\mathbb{1}$ (the identity map on $S^{\gamma_{i,e}}$) and π. By applying Proposition 1.4:

$$\mathcal{R}_{(e-1,1)} = \{1\} \cup \{s_{e-1}\cdots s_j : j = 1, \ldots, e-1\},$$

hence for $f \in \mathrm{End}_{\mathscr{H}_{e-1}}(S^{\gamma_{i,e}})$ and $x \in S^{\gamma_{i,e}}$:

$$(x)\operatorname{Tr}_{e-1}^{e}(f) = (x)f + \sum_{j=1}^{e-1}\frac{1}{q^{e-j}}(xT_j\cdots T_{e-1})fT_{e-1}\cdots T_j.$$

We now start computing this trace for some special f at m_i.

Proposition 4.12. *Let $f \in \mathrm{End}_{\mathscr{H}_{e-1}}(S^{\gamma_{i,e}})$, with $(m_i)f = m_i$. Then:*

$$(m_i)\operatorname{Tr}_{e-1}^{e}(f) = \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}}\right)m_i + \frac{1}{q^{e-i}}(m_e)fT_{e-1}\cdots T_iD_i.$$

Proof. We start computing the trace, splitting the sum around i:

$$\begin{aligned}
(m_i)\operatorname{Tr}_{e-1}^{e}(f) = (m_i)f &+ \sum_{j=1}^{i-1}\frac{1}{q^{e-j}}(m_iT_j\cdots T_{e-1})fT_{e-1}\cdots T_j \\
&+ \frac{1}{q^{e-i}}(m_iT_i\cdots T_{e-1})fT_{e-1}\cdots T_i \\
&+ \sum_{j=i+1}^{e-1}\frac{1}{q^{e-j}}(m_iT_j\cdots T_{e-1})fT_{e-1}\cdots T_j, \\
= (m_i)f &+ \sum_{j=1}^{i-1}\frac{1}{q^{e-i}}(m_e)fT_{e-1}\cdots T_j + \frac{1}{q^{e-i}}(m_e)fT_{e-1}\cdots T_i \\
&+ \sum_{j=i+1}^{e-1}\frac{1}{(-q)^{e-j}}(m_i)fT_{e-1}\cdots T_j, \\
= (m_i)f &+ \frac{1}{q^{e-i}}(m_e)fT_{e-1}\cdots T_iD_i + \sum_{j=i+1}^{e-1}\frac{1}{(-q)^{e-j}}(m_i)fT_{e-1}\cdots T_j.
\end{aligned}$$

Using the fact that $(m_i)f = m_i$:

$$\begin{aligned}
(m_i)\operatorname{Tr}_{e-1}^{e}(f) &= m_i + \sum_{j=i+1}^{e-1}\frac{1}{(-q)^{e-j}}m_iT_{e-1}\cdots T_j + \frac{1}{q^{e-i}}(m_e)fT_{e-1}\cdots T_iD_i, \\
&= m_i + \sum_{j=i+1}^{e-1}\frac{1}{q^{e-j}}m_i + \frac{1}{q^{e-i}}(m_e)fT_{e-1}\cdots T_iD_i, \\
&= \left(1+\frac{1}{q}+\cdots+\frac{1}{q^{e-i-1}}\right)m_i + \frac{1}{q^{e-i}}(m_e)fT_{e-1}\cdots T_iD_i.
\end{aligned}$$ □

We show that for both $f = \mathbb{1}$ and $f = \pi$, that $(m_i)\operatorname{Tr}_{e-1}^{e}(f) = 0$. Note that both of these maps have $(m_i)f = m_i$ as required in the previous proposition. Thus to fully compute these, we need to look at $(m_e)f$.

Proposition 4.13.

$$(m_i)\operatorname{Tr}_{e-1}^{e}(\mathbb{1}) = 0.$$

Proof. We start by computing $m_e T_{e-1} \cdots T_i D_i$:

$$\begin{aligned} m_e T_{e-1} \cdots T_i D_i &= (q m_{e-1} + (q-1) m_e) T_{e-2} \cdots T_i D_i, \\ &= (q^2 m_{e-2} + q(q-1) m_{e-1} - (q-1) m_e) T_{e-3} \cdots T_i D_i, \\ &\vdots \\ &= (q^{e-i} m_i - (q-1) q^{e-i-1} m_{i+1} + \cdots + (-1)^{e-i-1} (q-1) m_e) D_i, \end{aligned}$$

as multiplication by these terms either decrease the length but keep us standard (so we are using Corollary 1.9), or we are swapping numbers that lie in the first column (contributing a minus sign by Proposition 4.7). Continuing using Proposition 4.8:

$$\begin{aligned} m_e T_{e-1} \cdots T_i D_i &= q^{e-i} \left(m_i D_i + \sum_{j=1}^{e-i} (q-1) \frac{1}{q^j} (-1)^{j-1} m_{i+j} D_i \right), \\ &= q^{e-i} \left((1 + \cdots + q^{i-1}) m_i - \left(\sum_{j=1}^{e-i} (q-1) \frac{1}{q^j} \right) m_i \right), \\ &= q^{e-i} \left(\left(1 + \cdots + q^{i-1} + \frac{1}{q} + \cdots + \frac{1}{q^{e-i}} \right) - \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}} \right) \right) m_i. \end{aligned}$$

Now using the fact that q is an e-th root of unity (so for example $\frac{1}{q} = q^{e-1}$):

$$\begin{aligned} m_e T_{e-1} \cdots T_i D_i &= q^{e-i} \left((1 + \cdots + q^{i-1} + q^i + \cdots + q^{e-1}) - \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}} \right) \right) m_i, \\ &= -q^{e-i} \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}} \right) m_i. \end{aligned}$$

Bringing it all together:

$$\begin{aligned} (m_i) \operatorname{Tr}^e_{e-1}(\mathbb{1}) &= \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}} \right) m_i + \frac{1}{q^{e-i}} (m_e) \mathbb{1} T_{e-1} \cdots T_i D_i, \\ &= \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}} \right) m_i - \frac{1}{q^{e-i}} q^{e-i} \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}} \right) m_i, \\ &= \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}} \right) m_i - \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^{e-i-1}} \right) m_i, \\ &= 0. \end{aligned}$$

□

Proposition 4.14.

$$(m_i)\operatorname{Tr}^e_{e-1}(\pi) = 0.$$

Proof. Recall from earlier that $(m_e)\pi = -\left(\sum_{j=0}^{e-i-1}(-q)^{e-i-j}m_{i+j}\right)$. First of all note that for $j \geqslant 1$:

$$\begin{aligned}
m_{i+j}T_{e-1}\cdots T_iD_i &= (-1)^{e-j-i-1}m_{i+j}T_{i+j}\cdots T_iD_i,\\
&= (-1)^{e-j-i-1}m_{i+j+1}T_{i+j-1}\cdots T_iD_i,\\
&= (-1)^{e-j-i-1}(-1)^j m_{i+j+1}D_i,\\
&= (-1)^{e-i-1}m_{i+j+1}D_i,\\
&= (-1)^{e-i-1}(-1)^{j+1}m_i,\\
&= (-1)^{e-i+j}m_i.
\end{aligned}$$

Thus bringing it all together:

$$\begin{aligned}
(m_i)\operatorname{Tr}^e_{e-1}(\pi) &= \left(1+\frac{1}{q}+\cdots+\frac{1}{q^{e-i-1}}\right)m_i - \frac{1}{q^{e-i}}\left(\sum_{j=0}^{e-i-1}(-q)^{e-i-j}m_{i+j}\right)T_{e-1}\cdots T_iD_i,\\
&= \left(1+\frac{1}{q}+\cdots+\frac{1}{q^{e-i-1}}\right)m_i - \left(\sum_{j=0}^{e-i-1}(-1)^{e-i-j}\frac{1}{q^j}m_{i+j}\right)T_{e-1}\cdots T_iD_i,\\
&= \left(1+\frac{1}{q}+\cdots+\frac{1}{q^{e-i-1}}\right)m_i - \left(\sum_{j=0}^{e-i-1}(-1)^{e-i-j+e-i+j}\frac{1}{q^j}\right)m_i,\\
&= \left(1+\frac{1}{q}+\cdots+\frac{1}{q^{e-i-1}}\right)m_i - \left(\sum_{j=0}^{e-i-1}\frac{1}{q^j}\right)m_i,\\
&= \left(1+\frac{1}{q}+\cdots+\frac{1}{q^{e-i-1}}\right)m_i - \left(1+\frac{1}{q}+\cdots+\frac{1}{q^{e-i-1}}\right)m_i,\\
&= 0.
\end{aligned}$$

□

We can now compute the vertex of $S^{\gamma_{i,e}}$.

Theorem 4.15. *Let $1 \leqslant i \leqslant e$. Then $S^{\gamma_{i,e}}$ has vertex $\mathfrak{S}_e$ as a $\mathscr{H}_e$-module.*

Proof. As $\mathbb{1}$ and π are linearly independent, and $\mathrm{End}_{\mathscr{H}_{e-1}}(S^{\gamma_{i,e}})$ is two-dimensional, we've shown that when we evaluate the trace of any $\mathscr{H}_{e-1}$-endomorphism at m_i, we get 0. Thus as $\mathrm{End}_{\mathscr{H}_e}(S^{\gamma_{i,e}})$ is one-dimensional by Lemma 4.5, this means the trace of all $\mathscr{H}_{e-1}$-endomorphisms are the zero map, so $S^{\gamma_{i,e}}$ is not $\mathfrak{S}_{e-1}$-projective as a $\mathscr{H}_e$-module. Applying Proposition 4.4, means that $S^{\gamma_{i,e}}$ is not a projective $\mathscr{H}_e$-module, and thus by the Dipper–Du conjecture has vertex $\mathfrak{S}_e$. □

An alternative proof is given by appealing to the known decomposition numbers for blocks of weight one. If $S^{\gamma_{i,e}}$ is projective, then $[P^{\gamma_{i,e}} : D^{\gamma_{i,e}}] = [S^{\gamma_{i,e}} : D^{\gamma_{i,e}}] = d_{\gamma_{i,e}\gamma_{i,e}} = 1$. However, from [26, §6.4 Rule 13] we have $d_{\gamma_{i+1,e}\gamma_{i,e}} = 1$ so by [26, Theorem 2.20]:

$$[P^{\gamma_{i,e}} : D^{\gamma_{i,e}}] = \sum_{\mu \vdash n} d^2_{\mu\gamma_{i,e}} \geqslant d^2_{\gamma_{i,e}\gamma_{i,e}} + d^2_{\gamma_{i+1,e}\gamma_{i,e}} = 2,$$

giving a contradiction. Thus $S^{\gamma_{i,e}}$ cannot be projective, and applying Theorem 3.46 means that it must have vertex $\mathfrak{S}_e$.

4.2 Vertices of hook Specht modules

Having computed the vertex of "small" hook Specht modules in the previous section, we use the relatively (compared to other Specht modules) nice multiplication rules for these modules to examine relative projectivity for hook Specht modules when $n > e$. Throughout the rest of this chapter, we continue to assume $e > 2$. In the symmetric group case, when $p \nmid n$, we have from [35, Theorem 2]:

Theorem 4.16. *Suppose $p \nmid n$ and assume that $S^{\gamma_{r,n}}$ is indecomposable (i.e. $p \neq 2$). Then the vertex of $S^{\gamma_{r,n}}$ is a Sylow p-subgroup of $\mathfrak{S}_{r-1} \times \mathfrak{S}_{n-r}$.*

Ideally we want a corresponding result for Hecke algebras, which hopefully will replace the phrase "Sylow p-subgroup" with "maximal e-p-parabolic subgroup".

When $e \nmid n$, the maximal e-p-parabolic subgroup of $\mathfrak{S}_n$ is contained in $\mathfrak{S}_{(1,n-1)}$. If M is an indecomposable $\mathscr{H}_n$-module, then by Theorem 3.46 and Corollary 4.1, M is relatively $\mathfrak{P}(\mathfrak{S}_n)$-projective, and hence has a vertex contained in $\mathfrak{S}_{(1,n-1)}$. Thus by Higman's criterion and the definition of vertex:

Lemma 4.17. *Suppose $e \nmid n$ and M is an indecomposable $\mathscr{H}_n$-module. Then M is $\mathfrak{S}_{(1,n-1)}$-projective and:*

$$M \mid M \otimes_{\mathscr{H}_{(1,n-1)}} \mathscr{H}_n.$$

Fix $1 < r < n$, so $\gamma_{r,n}$ is a non-trivial hook partition. By applying Corollary 1.9, Lemma 4.6, and the calculations from the proof of Proposition 4.7, we have the following multiplication table for $S^{\gamma_{r,n}}$ as a $\mathscr{H}_{(1,n-1)}$-module. Let $\mathfrak{t} \in \operatorname{Std}(\gamma_{r,n})$ and $i \geqslant 2$. Note that in this setup, $\mathfrak{t}s_i$ is standard if and only if one of i and $i+1$ lies in the first row, and the other lies in the first column, thus the following provides a complete multiplication table.

$$m_{\mathfrak{t}}T_i = \begin{cases} qm_{\mathfrak{t}} & \text{if } i, i+1 \text{ are in the first row of } \mathfrak{t}, \\ -m_{\mathfrak{t}} & \text{if } i, i+1 \text{ are in the first column of } \mathfrak{t}, \\ m_{\mathfrak{t}s_i} & \text{if } \mathfrak{t}s_i \text{ is standard and } \ell(d(\mathfrak{t})) < \ell(d(\mathfrak{t}s_i)), \\ (q-1)m_{\mathfrak{t}} + qm_{\mathfrak{t}s_i} & \text{if } \mathfrak{t}s_i \text{ is standard and } \ell(d(\mathfrak{t})) > \ell(d(\mathfrak{t}s_i)). \end{cases} \tag{4.2}$$

Let $\theta = (1, r-1, n-r)$. Denote by $\mathcal{N}_r$ the following one-dimensional $\mathscr{H}_\theta$-module:

$$\mathcal{N}_r := S^{(1)} \otimes S^{(r-1)} \otimes S^{(1^{n-r})},$$

isomorphic to the trivial module for $\mathscr{H}_{r-1}$ tensored with the sign module for $\mathscr{H}_{n-r}$. Fix ϵ a generator of $\mathcal{N}_r$. Then $\epsilon T_i = q\epsilon$ if $2 \leqslant i < r$ and $\epsilon T_i = -\epsilon$ if $i > r$. Consider the $\mathscr{H}_{(1,n-1)}$-module:

$$\mathcal{N}_r \otimes_{\mathscr{H}_\theta} \mathscr{H}_{(1,n-1)}.$$

This has a basis indexed by $\epsilon \otimes T_w$ where $w \in \mathcal{R}_\theta^{(1,n-1)}$. These w can be described, similarly to Proposition 1.4, by $w \in \mathfrak{S}_n$ such that $\mathfrak{t}^\theta w$ is row standard, and $(1)w = 1$ (so $w \in \mathfrak{S}_{(1,n-1)}$). Multiplication rules in this induced module are given by the following proposition.

Proposition 4.18. *Let $w \in \mathcal{R}_\theta^{(1,n-1)}$ and $i \geqslant 2$. Then:*

$$(\epsilon \otimes T_w)T_i = \begin{cases} q(\epsilon \otimes T_w) & \text{if } i \text{ and } i+1 \text{ lie in the second row of } \mathfrak{t}^\theta w, \\ -(\epsilon \otimes T_w) & \text{if } i \text{ and } i+1 \text{ lie in the third row of } \mathfrak{t}^\theta w, \\ (\epsilon \otimes T_w) & \text{if } ws_i \in \mathcal{R}_\theta^{(1,n-1)} \text{ and } \ell(w) < \ell(ws_i), \\ (q-1)(\epsilon \otimes T_w) + q(\epsilon \otimes T_{ws_i}) & \text{if } ws_i \in \mathcal{R}_\theta^{(1,n-1)} \text{ and } \ell(w) > \ell(ws_i). \end{cases}$$

Proof. First of all, this is a full multiplication table for $\mathcal{N}_r \otimes_{\mathscr{H}_\theta} \mathscr{H}_{(1,n-1)}$ as the first two cases cover all possibilities where $ws_i \notin \mathcal{R}_\theta^{(1,n-1)}$ and the second two cases consider all cases when $ws_i \in \mathcal{R}_\theta^{(1,n-1)}$. The latter two cases follow immediately from the usual multiplication rules for $\mathscr{H}_{(1,n-1)}$, so we focus on the case where $ws_i \notin \mathcal{R}_\theta^{(1,n-1)}$.

For $\mathfrak{t}^\theta ws_i$ to not be row standard, either i and $i+1$ both lie in the second row of $\mathfrak{t}^\theta w$, or both lie in the third row (they cannot be in the first row as this only has one box). By Lemma 1.6, there exists a unique $s_j \in \mathfrak{S}_\theta$ with $T_wT_i = T_jT_w$, so $\epsilon \otimes T_wT_i = \epsilon T_j \otimes T_w$. In particular, this j is given by $(i)w^{-1}$, as the fact that $(j)w = i$ and $(j+1)w = i+1$ (since i and $i+1$ must be adjacent in $\mathfrak{t}^\theta w$) means that $s_jw = ws_i$.

Thus if i and $i+1$ lie in the second row of $\mathfrak{t}^\theta w$, then $j < r$ and if i and $i+1$ lie in the third row, then $j > r$. We can now conclude using the action of $\mathscr{H}_{(1,n-1)}$ on ϵ. □

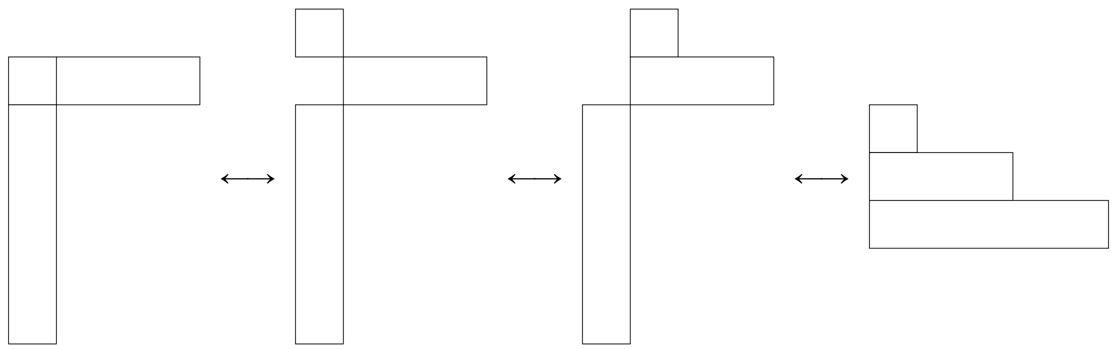
We draw attention to the similarities between the multiplication tables in (4.2) and Proposition 4.18. In fact as $\mathscr{H}_{(1,n-1)}$-modules, $S^{\gamma_{r,n}}$ and $\mathcal{N}_r \otimes_{\mathscr{H}_\theta} \mathscr{H}_{(1,n-1)}$ are isomorphic.

Theorem 4.19. *As $\mathscr{H}_{(1,n-1)}$-modules, $S^{\gamma_{r,n}} \cong \mathcal{N}_r \otimes_{\mathscr{H}_\theta} \mathscr{H}_{(1,n-1)}$.*

Proof. Define a map $\phi : S^{\gamma_{r,n}} \to \mathcal{N}_r \otimes_{\mathscr{H}_\theta} \mathscr{H}_{(1,n-1)}$ on basis elements of $S^{\gamma_{r,n}}$ by:

$$m_{\mathfrak{t}} \mapsto \epsilon \otimes T_{d(\mathfrak{t})}.$$

We first show this map is a bijection between our two bases, that is $\mathfrak{t} \in \mathrm{Std}(\gamma_{r,n})$ if and only if $d(\mathfrak{t}) \in \mathcal{R}_\theta^{(1,n-1)}$. The following construction taking a Young diagram of shape $\gamma_{r,n}$ to a Young diagram of shape θ gives a bijection between tableaux of shape $\gamma_{r,n}$ and tableaux of shape θ.



i.e. the first row of $[\gamma_{r,n}]$ (without its first box) becomes the second row of $[\theta]$ and the first column of $[\gamma_{r,n}]$ (again without its first box) becomes the third row of $[\theta]$.

Suppose $\mathfrak{t} \in \mathrm{Std}(\gamma_{r,n})$ and let $w = d(\mathfrak{t})$. In particular, we must have 1 in the first box of $\mathfrak{t}$, so $w \in \mathfrak{S}_{(1,n-1)}$. As $\mathfrak{t}$ is standard, the numbers increase along the first row and down the first column, so $(2)w < \cdots < (r)w$ and $(r+1)w < \cdots < (n)w$. These are the entries in the second and third rows of $\mathfrak{t}^\theta w$ respectively, and thus $\mathfrak{t}^\theta w$ is a row standard tableaux of shape θ, with 1 in the first row. Therefore $w \in \mathcal{R}_\theta^{(1,n-1)}$. As this construction gives a bijection between standard $\gamma_{r,n}$-tableaux and row standard θ-tableaux with 1 in the first row, ϕ is a bijection between our two bases, and thus is a vector space isomorphism.

It remains to show that ϕ is a $\mathscr{H}_{(1,n-1)}$-module homomorphism. Suppose $\mathfrak{t} \in \mathrm{Std}(\gamma_{r,n})$ and $i \geqslant 2$. Then i and $i+1$ lie in the first row of $\mathfrak{t}$ if and only if i and $i+1$ lie in the second row of $\mathfrak{t}^\theta d(\mathfrak{t})$. Similarly i and $i+1$ lie in the first column of $\mathfrak{t}$ if and only if i and $i+1$ lie in the third row of $\mathfrak{t}^\theta d(\mathfrak{t})$. Finally as $\mathfrak{t}s_i$ is standard if and only if $d(\mathfrak{t})s_i \in \mathcal{R}_\theta^{(1,n-1)}$, we conclude by comparing the tables from (4.2) and Proposition 4.18. □

We can use this isomorphism to examine relative projectivity of $S^{\gamma_{r,n}}$ when $e \nmid n$.

Corollary 4.20. *Suppose $e \nmid n$. As a $\mathscr{H}_n$-module, $S^{\gamma_{r,n}}$ is $\mathfrak{S}_\theta$-projective.*

Proof. As $\mathscr{H}_{(1,n-1)}$-modules $S^{\gamma_{r,n}} \cong \mathcal{N}_r \otimes_{\mathscr{H}_\theta} \mathscr{H}_{(1,n-1)}$. Inducing up to $\mathscr{H}_n$:

$$S^{\gamma_{r,n}} \otimes_{\mathscr{H}_{(1,n-1)}} \mathscr{H}_n \cong \mathcal{N}_r \otimes_{\mathscr{H}_\theta} \mathscr{H}_n. \tag{4.3}$$

By Lemma 4.17, $S^{\gamma_{r,n}}$ is isomorphic to a direct summand of $S^{\gamma_{r,n}} \otimes_{\mathscr{H}_{(1,n-1)}} \mathscr{H}_n$. Therefore by the isomorphism (4.3):

$$S^{\gamma_{r,n}} \mid \mathcal{N}_r \otimes_{\mathscr{H}_\theta} \mathscr{H}_n,$$

and thus $S^{\gamma_{r,n}}$ is $\mathfrak{S}_\theta$-projective as a $\mathscr{H}_n$-module. □

So when $e \nmid n$, we have an upper bound of $\mathfrak{P}(\mathfrak{S}_\theta)$ for the vertex of $S^{\gamma_{r,n}}$, by Corollary 4.1. We now use $\mathcal{N}_r$ to bound the vertex from below.

Lemma 4.21. *As $\mathscr{H}_\theta$-modules, $\mathcal{N}_r \mid S^{\gamma_{r,n}}$.*

Proof. By (4.2) we can see that $Fm_{\mathfrak{t}^{\gamma_{r,n}}}$ is a one-dimensional $\mathscr{H}_\theta$-submodule of $S^{\gamma_{r,n}}$ that is isomorphic to $\mathcal{N}_r$. It remains to show that this submodule is a direct summand.

We do this by showing that for any $\mathfrak{t} \in \mathrm{Std}(\gamma_{r,n})$ with $\mathfrak{t} \neq \mathfrak{t}^{\gamma_{r,n}}$, that $\mathfrak{t}^{\gamma_{r,n}}$ does not appear with a non-zero coefficient when writing $m_\mathfrak{t} T_i$ in the standard basis, for any $s_i \in \mathfrak{S}_\theta$. Again by (4.2) this can only happen if $\mathfrak{t}s_i = \mathfrak{t}^{\gamma_{r,n}}$, i.e. $\mathfrak{t}^{\gamma_{r,n}} s_i = \mathfrak{t}$. The only $s_i \in \mathfrak{S}_n$ with $\mathfrak{t}^{\gamma_{r,n}} s_i$ standard is s_r, which as $s_r \notin \mathfrak{S}_\theta$, completes the proof. □

We can now proceed using the Mackey formula to show that when $e \nmid n$, the vertex of $S^{\gamma_{r,n}}$ is $\mathfrak{P}(\mathfrak{S}_\theta)$. This gives a version of Theorem 4.16 for Hecke algebras.

Theorem 4.22. *Suppose $e \nmid n$ and $1 < r < n$. Then the vertex of $S^{\gamma_{r,n}}$ as a $\mathscr{H}_n$-module is the maximal e-p-parabolic subgroup of $\mathfrak{S}_{r-1} \times \mathfrak{S}_{n-r}$.*

Proof. As before, let $\theta = (1, r-1, n-r)$, so $\mathfrak{S}_\theta \cong \mathfrak{S}_{r-1} \times \mathfrak{S}_{n-r}$. By Corollary 4.20 $S^{\gamma_{r,n}}$ is $\mathfrak{S}_\theta$-projective, thus by Corollary 4.1, it is also $\mathfrak{P}(\mathfrak{S}_\theta)$-projective, and we can choose a vertex $\mathfrak{S}_\lambda$ of $S^{\gamma_{r,n}}$ with $\mathfrak{S}_\lambda \subseteq \mathfrak{P}(\mathfrak{S}_\theta)$.

By Higman's criterion, $S^{\gamma_{r,n}} \mid S^{\gamma_{r,n}} \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n$, and thus restricting both sides down to $\mathscr{H}_\theta$ using the Mackey formula and the previous lemma:

$$\mathcal{N}_r \mid S^{\gamma_{r,n}} \mid \bigoplus_{d \in \mathcal{D}_{\lambda,\theta}} (S^{\gamma_{r,n}} \otimes_{\mathscr{H}_\lambda} T_d) \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\theta,$$

where as usual, $\mathfrak{S}_{\nu(d)} = \mathfrak{S}_\lambda^d \cap \mathfrak{S}_\theta$. Therefore there exists $d \in \mathcal{D}_{\lambda,\theta}$ and some $\mathscr{H}_\theta$-module X with:

$$\mathcal{N}_r \mid X \otimes_{\mathscr{H}_{\nu(d)}} \mathscr{H}_\theta,$$

hence $\mathcal{N}_r$ is $\mathfrak{S}_{\nu(d)}$-projective as a $\mathscr{H}_\theta$-module. Now we know $\mathcal{N}_r$ has vertex $\mathfrak{P}(\mathfrak{S}_\theta)$ as we know the vertices of both the trivial and sign modules (from Theorem 3.38 and Corollary 3.25 for the sign module, and using Theorem 2.9 for the trivial module), so $\mathfrak{P}(\mathfrak{S}_\theta) \subseteq_{\mathfrak{S}_\theta} \mathfrak{S}_{\nu(d)} \subseteq \mathfrak{S}_\lambda^d$. Thus we have:

$$\mathfrak{P}(\mathfrak{S}_\theta) \subseteq_{\mathfrak{S}_n} \mathfrak{S}_\lambda \subseteq \mathfrak{P}(\mathfrak{S}_\theta),$$

meaning that $\mathfrak{S}_\lambda =_{\mathfrak{S}_n} \mathfrak{P}(\mathfrak{S}_\theta)$, and thus is a vertex of $S^{\gamma_{r,n}}$ as a $\mathscr{H}_n$-module. □

Therefore if $e \nmid n$, we can combine this result with our known vertices of the sign and trivial module to get the vertex of $S^{\gamma_{r,n}}$ for any $1 \leqslant r \leqslant n$.

When $e \mid n$, little is known about the vertices of $S^{\gamma_{r,n}}$. In the symmetric group (thus we are considering when $p \mid n$), we can lower bound the vertex using [16, Theorem A] to show that a Sylow p-subgroup of $\mathfrak{S}_{r-1} \times \mathfrak{S}_{n-r}$ has a conjugate contained in the vertex.

This is proved using Broué's result [3, (1.3)], however we cannot use our Hecke algebra version (Theorem 2.16) due to the bound being generated by subgroups which aren't standard parabolics.

In fact this bound does not look to be attained generally if $p \mid n$. By [17, Theorem 1.2], when $k \equiv 1 \mod p$ and $k \not\equiv 1 \mod p^2$, then the vertex of $S^{(kp-p,1^p)}$ is a Sylow p-subgroup of $\mathfrak{S}_{kp}$. For example, take $p = 3$ and $k = 4$. Then the Sylow 3-subgroup of $\mathfrak{S}_{12}$ has order 3^5, whereas the Sylow 3-subgroup of $\mathfrak{S}_8 \times \mathfrak{S}_3$ has order 3^3, so is strictly contained in the vertex.

In the next section we consider an alternative way of finding $S^{\gamma_{r,n}}$ as a direct summand of an induced module in the hopes that it might shed some more light on the situation when $e \mid n$.

4.3 An alternative approach to hook Specht modules

Once again fix $1 < r < n$ and now let $\mu = (n-r+1, r-1) \vDash n$. The aim of this section is to see under what conditions $S^{\gamma_{r,n}}$ is a direct summand of the sign module for $\mathscr{H}_{n-r+1}$ tensored with the trivial module for $\mathscr{H}_{r-1}$ induced from $\mathscr{H}_\mu$ up to $\mathscr{H}_n$, i.e. we will identify the conditions under which:

$$S^{\gamma_{r,n}} \mid (S^{(1^{n-r+1})} \otimes S^{(r-1)}) \otimes_{\mathscr{H}_\mu} \mathscr{H}_n. \tag{4.4}$$

Hopefully "twisting" and finding $S^{\gamma_{r,n}}$ as a direct summand in this way may help us explore the case when $e \mid n$, as this fixed-point-free setup allows us to avoid using Lemma 4.17.

To simplify notation, let $\mathcal{N}_{r,n} = (S^{(1^{n-r+1})} \otimes S^{(r-1)})$, a $\mathscr{H}_\mu$-module. As before, we begin by describing bases of the two modules in (4.4). We use the previously given multiplication table for $S^{\gamma_{r,n}}$ from (4.2) when multiplying by T_i for $i \geqslant 2$. When $i = 1$, then for $\mathfrak{t} \in \mathrm{Std}(\gamma_{r,n})$, either 2 lies in the first row of $\mathfrak{t}$ and $m_{\mathfrak{t}}T_1 = qm_{\mathfrak{t}}$, or 2 lies in the first column and $m_{\mathfrak{t}}T_1$ can be computed via Lemma 4.6 and the other rules.

We will denote a basis of $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$ as: $\{n_w : w \in \mathcal{R}_\mu\}$, where $n_w = \kappa \otimes T_w$ for a fixed generator κ of $\mathcal{N}_{r,n}$. Rules for multiplication in this induced module are proved in the same way as Proposition 4.18, and thus we omit the proof.

Proposition 4.23. *Let $w \in \mathcal{R}_\mu^n$, and $i \in \{1, \ldots, n-1\}$. Then:*

$$n_w T_i = \begin{cases} qn_w & \text{if } i \text{ and } i+1 \text{ are in the bottom row of } \mathfrak{t}^\mu w, \\ -n_w & \text{if } i \text{ and } i+1 \text{ are in the top row of } \mathfrak{t}^\mu w, \\ n_{ws_i} & \text{if } ws_i \in \mathcal{R}_\mu \text{ and } \ell(w) < \ell(ws_i), \\ (q-1)n_w + qn_{ws_i} & \text{if } ws_i \in \mathcal{R}_\mu \text{ and } \ell(w) > \ell(ws_i). \end{cases}$$

We proceed by searching for all $\mathscr{H}_n$-homomorphisms between $S^{\gamma_{r,n}}$ and $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$.

4.3.1 An inclusion map Φ

We start by defining an inclusion map from $S^{\gamma_{r,n}}$ into $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$. First of all we show that if such a map exists, then it is unique up to scalar. To do this, we need the following coset representatives. For $i \in \{1, \ldots, r\}$ let w_i be the element of $\mathcal{R}_\mu$ such that:

$$\mathfrak{t}^\mu w_i = \begin{array}{|c|c|c|c|c|c|} \hline i & r+1 & \cdots & n \\ \hline 1 & \cdots & i-1 & i+1 & \cdots & r \\ \hline \end{array}$$

and denote $n_i := n_{w_i}$.

Theorem 4.24. *Suppose $\phi : S^{\gamma_{r,n}} \to \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$ is a $\mathscr{H}_n$-module homomorphism. Then (after scaling if necessary):*

$$(m_{\mathfrak{t}^{\gamma_{r,n}}})\phi = n_1 + \cdots + n_r.$$

Proof. Suppose that $(m_{\mathfrak{t}^{\gamma_{r,n}}})\phi = \sum_w a_w n_w$ for $a_w \in F$. Then for $1 \leqslant i < r$, we have that $m_{\mathfrak{t}^{\gamma_{r,n}}} T_i = q m_{\mathfrak{t}^{\gamma_{r,n}}}$, and so $(m_{\mathfrak{t}^{\gamma_{r,n}}} T_i)\phi = q \sum_w a_w n_w$. Then:

$$\sum_w a_w n_w T_i = \sum_{\substack{i,i+1 \text{ in} \\ \text{top row}}} -a_w n_w + \sum_{\substack{i,i+1 \text{ in} \\ \text{bottom} \\ \text{row}}} q a_w n_w + \sum_{\substack{w: ws_i \in \mathcal{R}^n_\mu, \\ \ell(w) < \ell(ws_i)}} \big(a_w n_{ws_i} + a_{ws_i}(q-1) n_{ws_i} + a_{ws_i} q n_w \big).$$

Comparing coefficients tells us that if i and $i+1$ lie in the top row of $\mathfrak{t}^\mu w$, for any $1 \leqslant i < r$, then $a_w = 0$. Similarly, if i is in one row and $i+1$ is in the other, then again comparing coefficients, we have $a_w = a_{ws_i}$, i.e. swapping i and $i+1$ means the coefficient has to be the same.

Now if we have $1 \leqslant i < j \leqslant r$ in the top row of $\mathfrak{t}^\mu w$, with i in the box next to j (so no other numbers from $\{1, \ldots, r\}$ are between i and j), then we have:

$$a_w = a_{ws_{j-1}} = \cdots = a_{ws_{j-1} \ldots s_{i+1}}.$$

Now $\mathfrak{t}^\mu w s_{j-1} \ldots s_{i+1}$ has both i and $i+1$ in the top row, and therefore $a_{ws_{j-1} \ldots s_{i+1}} = 0$. So if there are two or more numbers from $\{1, \ldots, r\}$ in the top row, then we have a zero coefficient. Thus:

$$(m_{\mathfrak{t}^\mu})\phi = \sum_{i=1}^{r} a_i n_i,$$

for some a_i. Finally as $w_i s_i = w_{i+1}$, we get that $a_1 = a_2 = \cdots = a_r$, thus after rescaling:

$$(m_{\mathfrak{t}^\mu})\phi = \sum_{i=1}^{r} n_i. \qquad \square$$

As $S^{\gamma_{r,n}}$ is generated by $m_{\mathfrak{t}^{\gamma_{r,n}}}$ as a $\mathscr{H}_n$-module, this theorem shows that if there exists a $\mathscr{H}_n$-homomorphism $\phi : S^{\gamma_{r,n}} \to \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$, then it is unique up to scalar. We still have to show that this actually does give a $\mathscr{H}_n$-module homomorphism, i.e. $\mathrm{Hom}_{\mathscr{H}_n}(S^{\gamma_{r,n}}, \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n) \neq 0$. To do this, we will use the following theorem [7, Theorem 2.6]:

Theorem 4.25 (Nakayama relation)**.** *Let $\lambda \models n$. Suppose N is a $\mathscr{H}_\lambda$-module, and M is a $\mathscr{H}_n$-module. Then:*

$$\operatorname{Hom}_{\mathscr{H}_\lambda}(M, N) \cong \operatorname{Hom}_{\mathscr{H}_n}(M, N \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n).$$

Furthermore this isomorphism is given by $\phi \mapsto \overline{\phi}$ where for $m \in M$:

$$(m)\overline{\phi} = \sum_{d \in \mathcal{R}_\lambda} q^{-\ell(d)}(mT_{d^{-1}})\phi \otimes T_d.$$

We begin by exhibiting a $\mathscr{H}_\mu$-module homomorphism from $S^{\gamma_{r,n}}$ to $\mathcal{N}_{r,n}$.

Lemma 4.26. *Denote by $\mathfrak{t}_{\gamma_{r,n}}$ the standard $\gamma_{r,n}$-tableau with $1, \ldots, n-r+1$ in the first column. Then $\theta : S^{\gamma_{r,n}} \to \mathcal{N}_{r,n}$ defined by:*

$$\left(\sum_{\mathfrak{t} \in \operatorname{Std}(\gamma_{r,n})} a_\mathfrak{t} m_\mathfrak{t} \right) \theta = a_{\mathfrak{t}_{\gamma_{r,n}}} \kappa,$$

is a $\mathscr{H}_\mu$-module homomorphism.

Proof. To begin, we see for which $\mathfrak{t}$ and i does $m_{\mathfrak{t}_{\gamma_{r,n}}}$ appear with a non-zero coefficient when writing $m_\mathfrak{t} T_i$ in the standard basis. Clearly it appears if $\mathfrak{t} = \mathfrak{t}_{\gamma_{r,n}}$, and the only other case that can appear is if $\mathfrak{t} = \mathfrak{t}_{\gamma_{r,n}} s_{n-r+1}$ and $i = n-r+1$, thus this case does not come up as $s_{n-r+1} \notin \mathfrak{S}_\mu$.

Therefore for $s_i \in \mathfrak{S}_\mu$, we have $(\sum a_\mathfrak{t} m_\mathfrak{t} T_i)\theta = (a_{\mathfrak{t}_{\gamma_{r,n}}} m_{\mathfrak{t}_{\gamma_{r,n}}} T_i)\theta$, so it suffices to show that $(m_{\mathfrak{t}_{\gamma_{r,n}}} T_i)\theta = (m_{\mathfrak{t}_{\gamma_{r,n}}})\theta T_i$. We split into cases on i:

- If $i = 1$, then:

$$(m_{\mathfrak{t}_{\gamma_{r,n}}} T_1)\theta = (-m_{\mathfrak{t}_{\gamma_{r,n}}} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} r_\mathfrak{v} m_\mathfrak{v})\theta = -\kappa = \kappa \cdot T_1 = (m_{\mathfrak{t}_{\gamma_{r,n}}})\theta T_1.$$

- If $1 < i < n-r+1$:

$$(m_{\mathfrak{t}_{\gamma_{r,n}}} T_i)\theta = (-m_{\mathfrak{t}_{\gamma_{r,n}}})\theta = -\kappa = \kappa \cdot T_i = (m_{\mathfrak{t}_{\gamma_{r,n}}})\theta T_i.$$

- Finally if $i > n-r+1$:

$$(m_{\mathfrak{t}_{\gamma_{r,n}}} T_i)\theta = (q m_{\mathfrak{t}_{\gamma_{r,n}}})\theta = q\kappa = \kappa \cdot T_i = (m_{\mathfrak{t}_{\gamma_{r,n}}})\theta T_i.$$

Thus θ does define a $\mathscr{H}_\mu$-module homomorphism. □

Corollary 4.27. *The vector space homomorphism $\Phi : S^{\gamma_{r,n}} \to \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$ defined by:*

$$(m_{\mathfrak{t}})\Phi = (n_1 + \cdots + n_r) T_{d(\mathfrak{t})},$$

extended linearly, is a $\mathscr{H}_n$-homomorphism.

Proof. Applying Theorem 4.25 gives us as vector spaces:

$$\mathrm{Hom}_{\mathscr{H}_n}(S^{\gamma_{r,n}}, \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n) \cong \mathrm{Hom}_{\mathscr{H}_\mu}(S^{\gamma_{r,n}}, \mathcal{N}_{r,n}).$$

The previous lemma tells us that the latter is at least one-dimensional, thus there exists a non-zero $\mathscr{H}_n$-homomorphism from $S^{\gamma_{r,n}}$ to $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$. Theorem 4.24 tells us that any such map must agree with Φ on $m_{\mathfrak{t}^{\gamma_{r,n}}}$, thus Φ is a $\mathscr{H}_n$-homomorphism as $m_{\mathfrak{t}} = m_{\mathfrak{t}^{\gamma_{r,n}}} T_{d(\mathfrak{t})}$ for any $\mathfrak{t} \in \mathrm{Std}(\gamma_{r,n})$. □

Therefore we have a unique (up to scalar) $\mathscr{H}_n$-homomorphism $\Phi : S^{\gamma_{r,n}} \to \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$.

4.3.2 A surjective map Ψ

We now attempt to find all $\mathscr{H}_n$-module homomorphisms $\psi : \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n \to S^{\gamma_{r,n}}$. Before doing this, we need to make note of some special tableaux in $\operatorname{Std}(\gamma_{r,n})$.

For $2 \leqslant i \leqslant r$ and $r+1 \leqslant j \leqslant n$ define $\mathfrak{t}_{i,j}$ to be the tableau:

$$\mathfrak{t}_{i,j} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & \cdots\cdots & i-1 & i+1 & \cdots\cdots & r & j \\ \hline i \\ \cline{1-1} r+1 \\ \cline{1-1} \vdots \\ \cline{1-1} j-1 \\ \cline{1-1} j+1 \\ \cline{1-1} \vdots \\ \cline{1-1} n \\ \cline{1-1} \end{array}$$

i.e. has i missing from the top row, and j at the end of the top row. Denote $m_{i,j} = m_{\mathfrak{t}_{i,j}}$.

Lemma 4.28. *Let $\psi : \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n \to S^{\gamma_{r,n}}$ be a $\mathscr{H}_n$-module homomorphism. Then there exist $a, b \in F$ with:*

$$(n_1)\psi = a m_{\mathfrak{t}^{\gamma_{r,n}}} + b \left(\sum_{\substack{2 \leqslant i \leqslant r \\ r+1 \leqslant j \leqslant n}} (-q)^{n-j} m_{i,j} \right).$$

Proof. As in Theorem 4.24, we suppose $(n_1)\psi = \sum_{\mathfrak{t} \in \operatorname{Std}(\gamma_{r,n})} a_\mathfrak{t} m_\mathfrak{t}$ and look for relations between the coefficients.

If $i \neq 1$, then:

$$(n_1)\psi T_i = \sum_{\substack{i,i+1 \\ \text{in first} \\ \text{row}}} q a_\mathfrak{t} m_\mathfrak{t} + \sum_{\substack{i,i+1 \\ \text{in first} \\ \text{column}}} -a_\mathfrak{t} m_\mathfrak{t} + \sum_{\substack{\text{one of } i,i+1 \\ \text{in each part} \\ \ell(d(\mathfrak{t})) < \ell(d(\mathfrak{t})s_i)}} \left(a_\mathfrak{t} m_{\mathfrak{t}s_i} + a_{\mathfrak{t}s_i}(q-1) m_{\mathfrak{t}s_i} + a_{\mathfrak{t}s_i} q m_\mathfrak{t} \right).$$

As in the proof of Theorem 4.24, for $2 \leqslant i < r$, as $n_1 T_i = q n_1$, then by comparing coefficients, if both i and $i+1$ are in the first column, then $a_{\mathfrak{t}} = 0$, and if one of i and $i+1$ is in the first row, and the other in the first column, then $a_{\mathfrak{t}} = a_{\mathfrak{t}s_i}$. Note that this means that $a_{\mathfrak{t}_{i,j}} = a_{\mathfrak{t}_{i+1,j}}$. Similarly for $r+1 \leqslant i$, we have if both i and $i+1$ are in the first row then again $a_{\mathfrak{t}} = 0$, and if one is in each part, and $\ell(d(\mathfrak{t})) < \ell(d(\mathfrak{t})s_i)$, then $a_{\mathfrak{t}} = -qa_{\mathfrak{t}s_i}$.

A similar argument to the proof of Theorem 4.24 shows that if 2 or more elements from $\{2, \ldots, r\}$ are in the first column, then the coefficient $a_{\mathfrak{t}}$ is zero as well. Thus the only tableaux with non-zero coefficients are the ones in the expression from the theorem, that is $m_{\mathfrak{t}^{\gamma_{r,n}}}$ and $m_{i,j}$ for $2 \leqslant i \leqslant r$ and $j \geqslant r+1$. As $m_{i,j}s_j = m_{i,j+1}$ and $\ell(d(\mathfrak{t}_{i,j+1})) = \ell(d(\mathfrak{t}_{i,j})) + 1$, we get for a fixed i that $a_{\mathfrak{t}_{i,j}} = -qa_{\mathfrak{t}_{i,j+1}}$. Collecting all this information together gives $(n_1)\psi$ in the above form. $\square$

Unfortunately, this is not enough to fully specify the map. To find the relationship between a and b, we need to first look at $(n_r)\psi$. This allows us to pick up the final relation we need by considering $(n_r)\psi T_r$.

Lemma 4.29. *Let ψ be a $\mathscr{H}_n$-homomorphism with $(n_1)\psi$ as given previously. Then:*

$$(n_r)\psi = \left[aq^{r-1} + bq^{r-2}(-1)^{n-r}\left(\sum_{j=r+1}^{n} q^{n-j}\right)\right] m_{\mathfrak{t}^{\gamma_{r,n}}} - bq^{r-2}\left(\sum_{j=r+1}^{n} (-q)^{n-j} m_{r,j}\right).$$

Proof. As $n_r = n_1 T_1 \ldots T_{r-1}$, if ψ is a $\mathscr{H}_n$-homomorphism, then $(n_r)\psi = (n_1)\psi T_1 \ldots T_{r-1}$. Now $m_{\mathfrak{t}^{\gamma_{r,n}}} T_i = qm_{\mathfrak{t}^{\gamma_{r,n}}}$ for all $1 \leqslant i < r$, therefore:

$$am_{\mathfrak{t}^{\gamma_{r,n}}} T_1 \ldots T_{r-1} = aq^{r-1} m_{\mathfrak{t}^{\gamma_{r,n}}}.$$

To compute the rest notice first that for $k \in \{1, \ldots, r-1\} - \{i-1, i\}$ we have $m_{i,j}T_k = qm_{i,j}$, and $m_{i,j}T_{i-1} = m_{i-1,j}$. Then if $i \neq 2$:

$$m_{i,j}T_1 \ldots T_{r-1} = q^{i-2} m_{i,j} T_{i-1} \ldots T_{r-1} = q^{i-2} m_{i-1,j} T_i \ldots T_{r-1} = q^{r-2} m_{i-1,j}.$$

Now if $i = 2$ we first have to look at $m_{2,j}T_1$. Note that $\mathfrak{t}_{2,j}$ is the same tableau we defined in the proof of Proposition 4.8 as $\mathfrak{t}_j^2$. Thus by the calculations in this proof we have:

$$m_{2,j}T_1 = \left(\sum_{i=2}^{r} -m_{i,j}\right) + (-1)^{j-r} m_{\mathfrak{t}^{\gamma_{r,n}}}.$$

As we know how to multiply the rest of the terms by $T_2 \dots T_{r-1}$, we need to concentrate on $m_{2,j}$. Using the fact that $m_{i,j}T_i = (q-1)m_{i,j} + qm_{i+1,j}$ we get:

$$\begin{aligned}
m_{2,j}T_2 \dots T_{r-1} &= (q-1)m_{2,j}T_3 \dots T_{r-1} + qm_{3,j}T_3 \dots T_{r-1}, \\
&= q^{r-3}m_{2,j} + q(q-1)m_{3,j}T_4 \dots T_{r-1} + q^2 m_{4,j}T_4 \dots T_{r-1}, \\
&\vdots \\
&= q^{r-3}(q-1)m_{2,j} + q^{r-3}(q-1)m_{3,j} + \cdots + q^{r-3}(q-1)m_{r-1,j} + q^{r-2}m_{r,j}, \\
&= q^{r-2}m_{r,j} + q^{r-3}(q-1)\sum_{i=2}^{r-1} m_{i,j}, \\
&= q^{r-2}m_{r,j} + q^{r-2}\sum_{i=2}^{r-1} m_{i,j} - q^{r-3}\sum_{i=2}^{r-1} m_{i,j}, \\
&= q^{r-2}\sum_{i=2}^{r} m_{i,j} - q^{r-3}\sum_{i=2}^{r-1} m_{i,j}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
m_{2,j}T_1 \dots T_{r-1} &= \left[-\sum_{i=2}^{r} m_{i,j} + (-1)^{j-r} m_{\mathfrak{t}^{\gamma_{r,n}}}\right] T_2 \dots T_{r-1}, \\
&= (-1)^{j-r}q^{r-2}m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=3}^{r} q^{r-3}m_{i-1,j} - m_{2,j}T_2 \dots T_{r-1}, \\
&= (-1)^{j-r}q^{r-2}m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{r-1} q^{r-3}m_{i,j} - \left[q^{r-2}\sum_{i=2}^{r} m_{i,j} - q^{r-3}\sum_{i=2}^{r-1} m_{i,j}\right], \\
&= (-1)^{j-r}q^{r-2}m_{\mathfrak{t}^{\gamma_{r,n}}} - q^{r-2}\sum_{i=2}^{r} m_{i,j}.
\end{aligned}$$

Now writing $M_j = \sum_{i=2}^{r} m_{i,j}$, we can collect the previous work to show:

$$M_j T_1 \dots T_{r-1} = q^{r-2}\left((-1)^{j-r} m_{\mathfrak{t}^{\gamma_{r,n}}} - m_{r,j}\right).$$

Finally bringing everything together:

$$\begin{aligned}
(n_r)\psi &= (n_1)\psi T_1 \dots T_r, \\
&= \left(a m_{\mathfrak{t}^{\gamma_{r,n}}} + b \left(\sum_{j=r+1}^{n} (-q)^{n-j} M_j \right) \right) T_1 \dots T_{r-1}, \\
&= a q^{r-1} m_{\mathfrak{t}^{\gamma_{r,n}}} + b q^{r-2} \left(\sum_{j=r+1}^{n} (-1)^{j-r} (-q)^{n-j} m_{\mathfrak{t}^{\gamma_{r,n}}} - (-q)^{n-j} m_{r,j} \right), \\
&= a q^{r-1} m_{\mathfrak{t}^{\gamma_{r,n}}} + b q^{r-2} \left(\sum_{j=r+1}^{n} (-1)^{n-r} q^{n-j} m_{\mathfrak{t}^{\gamma_{r,n}}} - (-q)^{n-j} m_{r,j} \right), \\
&= \left[a q^{r-1} + b q^{r-2} (-1)^{n-r} \left(\sum_{j=r+1}^{n} q^{n-j} \right) \right] m_{\mathfrak{t}^{\gamma_{r,n}}} - b q^{r-2} \left(\sum_{j=r+1}^{n} (-q)^{n-j} m_{r,j} \right). \square
\end{aligned}$$

We can now use this element to show that there is at most a single map ψ up to scalar.

Theorem 4.30. *Let $\psi : \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n \to S^{\gamma_{r,n}}$ be a non-zero $\mathscr{H}_n$-module homomorphism. Then, after scaling if necessary, we have:*

$$(n_r)\psi = q^{n-1}(-1)^{n-r-1} m_{\mathfrak{t}^{\gamma_{r,n}}} - q^{r-1} \sum_{j=r+1}^{n} (-q)^{n-j} m_{r,j}.$$

Proof. From the previous lemma, we have an idea of what $(n_r)\psi$ looks like depending on two parameters a and b. Now consider $(n_r)\psi T_r$. As both r and $r+1$ are in the top row of $\mathfrak{t}^\mu w_r$, then $n_r T_r = -n_r$. Similarly $m_{\mathfrak{t}^{\gamma_{r,n}}} T_r = m_{r,r+1}$, if $j \neq r+1$ then $m_{r,j} T_r = -m_{r,j}$ and $m_{r,r+1} T_r = (q-1) m_{r,r+1} + q m_{\mathfrak{t}^{\gamma_{r,n}}}$. Thus to have $(n_r T_r)\psi = (n_r)\psi T_r$, by looking at the coefficient of $\mathfrak{t}^{\gamma_{r,n}}$, we must have:

$$-\left[q^{r-1} a + b q^{r-2} (-1)^{n-r} \left(\sum_{j=r+1}^{n} q^{n-j} \right) \right] = -q b q^{r-2} (-q)^{n-r-1}.$$

Cancelling and simplifying this equation, we get that:

$$qa = (-1)^{n-r-1} b (1 + q + \dots + q^{n-r}),$$

and setting $b = q$, and substituting this into the previous expression for $(n_r)\psi$ gives the required result. $\square$

Thus there is at most a unique non-zero $\mathscr{H}_n$-homomorphism $\psi : \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n \to S^{\gamma_{r,n}}$ (up to scalar), and if ψ is such a $\mathscr{H}_n$-homomorphism, it satisfies the given expressions for $(n_1)\psi$ and $(n_r)\psi$. We now will demonstrate some nice properties of $(n_r)\psi$ as given in the previous theorem.

Lemma 4.31. *Define $(n_r)\psi$ as in Theorem 4.30. Then:*

$$(n_r)\psi T_i = \begin{cases} q(n_r)\psi & \text{if } 1 \leqslant i < r-1, \\ -(n_r)\psi & \text{if } i \geqslant r. \end{cases}$$

Proof. This follows as for each $m_{r,j}$ and $m_{\mathfrak{t}^{\gamma_{r,n}}}$, if $1 \leqslant i < r-1$, then i and $i+1$ lie in the top row of each tableau, thus multiplication by T_i is multiplication by q. For $i = r$, we've already seen in Theorem 4.30 that $(n_r)\psi T_r = -(n_r)\psi$, and for $i > r$, we have $m_{\mathfrak{t}^{\gamma_{r,n}}} T_i = -m_{\mathfrak{t}^{\gamma_{r,n}}}$ dealing with the leading term. For $j \neq i, i+1$, then i and $i+1$ are both in the first column of $\mathfrak{t}_{r,j}$ and thus multiplication by T_i here is also multiplication by -1. It remains to show multiplication by T_i is multiplication by -1 on the remaining two terms:

$$\begin{aligned}(-q)^{n-i-1}(m_{r,i+1} - qm_{r,i})T_i &= (-q)^{n-i-1}\left(qm_{r,i} + (q-1)m_{r,i+1} - qm_{r,i+1}\right), \\ &= (-q)^{n-i-1}\left(qm_{r,i} - m_{r,i+1}\right), \\ &= -(-q)^{n-i-1}(m_{r,i+1} - qm_{r,i}).\end{aligned}$$

Therefore bringing everything together: $(n_r)\psi T_i = -(n_r)\psi$. $\square$

We can now prove the existence of this unique $\mathscr{H}_n$-module homomorphism.

Theorem 4.32. *Define a vector space homomorphism* $\Psi : \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n \to S^{\gamma_{r,n}}$ *by:*

$$(n_w)\Psi = (n_r)\psi(T_{w_r})^{-1}T_w,$$

where $(n_r)\psi$ *is defined as in Theorem 4.30. Then* Ψ *is a* $\mathscr{H}_n$*-homomorphism, and is the unique* $\mathscr{H}_n$*-homomorphism up to scalar from* $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$ *to* $S^{\gamma_{r,n}}$*.*

Proof. Note that this map agrees with our previous map on n_r. We begin by exploring how T_{w_r} interacts with T_i. Based on the multiplication rules of the $n_w = \kappa \otimes T_w$ previously given we have:

$$T_{w_r}T_i = \begin{cases} T_jT_{w_r} & \text{for some } 1 \leqslant j < n-r+1 \text{ if } i \geqslant r, \\ T_jT_{w_r} & \text{for some } j > n-r+1 \text{ if } i < r-1, \\ (q-1)T_{w_r} + qT_{w_{r-1}} & \text{if } i = r-1. \end{cases}$$

Therefore if $j < n-r+1$, we have that $(T_{w_r})^{-1}T_j = T_i(T_{w_r})^{-1}$ for some $i \geqslant r$, and if $j > n-r+1$, then $(T_{w_r})^{-1}T_j = T_i(T_{w_r})^{-1}$ for some $i < r-1$. In fact this mapping between i and j gives a bijection between $i \in \{1, \ldots n\} - \{r-1\}$ and $j \in \{1, \ldots, n\} - \{n-r+1\}$. We split the proof into three cases based on w.

- First suppose $ws_i \in \mathcal{R}^n_\mu$ and $\ell(w) < \ell(ws_i)$. Then:

$$(n_wT_i)\Psi = (n_{ws_i})\Psi = (n_r)\psi(T_{w_r})^{-1}T_{ws_i} = (n_r)\psi(T_{w_r})^{-1}T_wT_i = (n_w)\Psi T_i.$$

- Next consider when $ws_i \in \mathcal{R}^n_\mu$ and $\ell(w) > \ell(ws_i)$.

$$(n_wT_i)\Psi = (q-1)n_w + qn_{ws_i})\Psi = (n_r)\psi(T_{w_r})^{-1}\left((q-1)T_w + qT_{ws_i}\right) = (n_w)\Psi T_i.$$

- Now suppose $ws_i = s_jw$ where $j < n-r+1$, the case where $j > n-r+1$ follows similarly, just replacing -1 with q instead. In this case $T_wT_i = T_jT_w$, and

$n_wT_i = -n_w$. Then $(n_wT_i)\Psi = -(n_w)\Psi$. Similarly:

$$(n_w)\Psi T_i = (n_r)\psi(T_{w_r})^{-1}T_wT_i = (n_r)\psi(T_{w_r})^{-1}T_jT_w = (n_r)\psi T_k(T_{w_r})^{-1}T_w,$$

for some $k \geqslant r$. Then using Lemma 4.31 this is $-(n_r)\psi(T_{w_r})^{-1}T_w$ i.e. $-(n_w)\Psi$.

Thus for all w the statement holds. □

Corollary 4.33. *Let Ψ be the unique (up to scalar) $\mathscr{H}_n$-homomorphism described previously. Then:*

$$(n_1)\Psi = am_{\mathfrak{t}^{\gamma_{r,n}}} + b\left(\sum_{\substack{2\leqslant i\leqslant r\\ r+1\leqslant j\leqslant n}}(-q)^{n-j}m_{i,j}\right),$$

where $b = q$ and $a = (-1)^{n-r-1}(1 + q + \cdots + q^{n-r})$.

This follows from Lemma 4.28 and Theorem 4.32. So now we have found the unique $\mathscr{H}_n$-homomorphism $\Psi : \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n \to S^{\gamma_{r,n}}$ (up to scalar).

4.3.3 A direct summand

For $\mathscr{H}_n$-modules M and N, we have that $M \mid N$ if and only if there exists homomorphisms $\alpha : M \to N$ and $\beta : N \to M$ such that $\alpha \circ \beta$ is the identity map on M. Thus to check if $S^{\gamma_{r,n}}$ is a direct summand of $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$, it suffices to look at the composition $\Phi \circ \Psi$ at the generator $m_{\mathfrak{t}^{\gamma_{r,n}}}$ of $S^{\gamma_{r,n}}$, as these are the unique (up to scalar) $\mathscr{H}_n$-homomorphisms in either direction. Recall that:

$$(m_{\mathfrak{t}^{\gamma_{r,n}}})\Phi = n_1 + \cdots + n_r = n_1(1 + T_1 + \cdots + T_1 \dots T_{r-1}).$$

Define $E_r = (1 + T_1 + \cdots + T_1 \dots T_{r-1})$, then:

$$((m_{\mathfrak{t}^{\gamma_{r,n}}})\Phi)\Psi = (n_1 + \cdots + n_r)\Psi = (n_1E_r)\Psi = (n_1)\Psi E_r.$$

As we know from Corollary 4.33 what $(n_1)\Psi$ is, it remains to compute $m_{\mathfrak{t}}E_r$ for the tableaux $\mathfrak{t}$ involved in $(n_1)\Psi$.

We begin with the leading term $m_{\mathfrak{t}^{\gamma_{r,n}}}$.

Lemma 4.34. $m_{\mathfrak{t}^{\gamma_{r,n}}}E_r = (1+q+\cdots+q^{r-1})m_{\mathfrak{t}^{\gamma_{r,n}}}$.

Proof. For T_i involved in E_r, we have i and $i+1$ lie in the first row of $\mathfrak{t}^{\gamma_{r,n}}$, and therefore each T_i acts as q, proving the result. □

Lemma 4.35. *Let $i \neq 2$. Then* $m_{i,j}E_r = (1+q+\cdots q^{i-2})m_{i,j} + (q^{i-2}+\cdots+q^{r-2})m_{i-1,j}$.

Proof. Using our known multiplication rules for $m_{i,j}$ we proceed:

$$\begin{aligned}
m_{i,j}E_r &= m_{i,j}(1+T_1+\cdots+T_1\dots T_{r-1}),\\
&= (1+q+\cdots+q^{i-2})m_{i,j} + m_{i,j}T_1\dots T_{i-1}(1+\cdots+T_i\dots T_{r-1}),\\
&= (1+q+\cdots+q^{i-2})m_{i,j} + q^{i-2}m_{i-1,j}(1+\cdots+T_i\dots T_{r-1}),\\
&= (1+q+\cdots+q^{i-2})m_{i,j} + q^{i-2}(1+q+\cdots+q^{r-i})m_{i-1,j},\\
&= (1+q+\cdots+q^{i-2})m_{i,j} + (q^{i-2}+\cdots+q^{r-2})m_{i-1,j}.
\end{aligned}$$ □

The last terms we have to consider are the most complicated.

Lemma 4.36.

$$m_{2,j}E_r = m_{2,j} + (1+q+\cdots+q^{r-2})\left((-1)^{j-r}m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{r} m_{i,j}\right).$$

Proof. First of all, remember that $m_{2,j}T_1 = (-1)^{j-r}m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{j} m_{i,j}$. Then:

$$m_{2,j}E_r = m_{2,j} + (-1)^{j-r}m_{\mathfrak{t}^{\gamma_{r,n}}}(1+T_2+\cdots+T_2\dots T_{r-1}) - \sum_{i=2}^{r} m_{i,j}(1+T_2+\cdots T_2\dots T_{r-1}).$$

As before when $i \neq 2$, then:

$$m_{i,j}(1 + T_2 + \cdots + T_2 \ldots T_{r-1}) = (1 + q + \cdots + q^{i-3})m_{i,j} + (q^{i-3} + \cdots + q^{r-3})m_{i-1,j}.$$

Thus we get that:

$$\begin{aligned}\sum_{i=3}^{r} m_{i,j}(1 + T_2 + \cdots T_2 \ldots T_{r-1}) &= \sum_{i=3}^{r}(1 + q + \cdots + q^{i-3})m_{i,j} + \sum_{i=2}^{r}(q^{i-2} + \cdots + q^{r-3})m_{i,j}, \\ &= \sum_{i=2}^{r}(1 + q + \cdots + q^{r-3})m_{i,j}.\end{aligned}$$

Denoting $E_{i,r} = 1 + T_i + \cdots + T_i \ldots T_{r-1}$, similarly:

$$m_{\mathfrak{t}^{\gamma_{r,n}}} E_{2,r} = (1 + q + \cdots + q^{r-2})m_{\mathfrak{t}^{\gamma_{r,n}}}.$$

Finally, the last part we need is:

$$\begin{aligned} m_{2,j}E_{2,r} &= m_{2,j} + \left[(q-1)m_{2,j} + qm_{3,j}\right]E_{3,r}, \\ &= \left[1 + (q-1)(1 + q + \cdots + q^{r-3})\right]m_{2,j} + qm_{3,j}E_{3,r}, \\ &= q^{r-2}m_{2,j} + qm_{3,j}E_{3,r}, \\ &\vdots \\ &= \sum_{i=2}^{r} q^{r-2}m_{i,j}. \end{aligned}$$

Bringing everything back together, we get that:

$$\begin{aligned} m_{2,j}E_r &= m_{2,j} + (-1)^{j-r}m_{\mathfrak{t}^{\gamma_{r,n}}}E_{2,r} - \sum_{i=2}^{r} m_{i,j}E_{2,r}, \\ &= m_{2,j} + (-1)^{j-r}(1 + \cdots + q^{r-2})m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{r} q^{r-2}m_{i,j} - \sum_{i=2}^{r}(1 + \cdots + q^{r-3})m_{i,j}, \\ &= m_{2,j} + (1 + q + \cdots + q^{r-2})\left((-1)^{j-r}m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{r} m_{i,j}\right). \end{aligned}$$

□

We are now ready to compute $((m_{\mathfrak{t}^{\gamma_{r,n}}})\Phi)\Psi = (n_1)\Psi E_r$. From here, we shall use the shorthand $C_i = C_i(q) = 1 + q + \cdots + q^{i-1}$ as defined in (3.3) to simplify the expressions involved.

Theorem 4.37. *Recall $(n_1)\Psi$ from Corollary 4.33. Then:*

$$(n_1)\Psi E_r = (-1)^{n-r-1} C_n m_{\mathfrak{t}^{\gamma_{r,n}}}.$$

Proof. Let $M_j = \sum_{i=2}^{r} m_{i,j}$. Now combining the previous two lemmas:

$$\begin{aligned}
M_j E_r &= m_{2,j} + C_{r-1}\left((-1)^{j-r} m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{r} m_{i,j}\right) + \sum_{i=3}^{r}\left(C_{i-1} m_{i,j} + q^{i-2} C_{r-i+1} m_{i-1,j}\right), \\
&= m_{2,j} + C_{r-1}\left((-1)^{j-r} m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{r} m_{i,j}\right) + \left(\sum_{i=3}^{r} C_{i-1} m_{i,j}\right) + \left(\sum_{i=2}^{r-1} q^{i-1} C_{r-i} m_{i,j}\right), \\
&= C_{r-1}\left((-1)^{j-r} m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{r} m_{i,j}\right) + \sum_{i=3}^{r} C_{r-1} m_{i,j} + m_{2,j} + q C_{r-2} m_{2,j}, \\
&= C_{r-1}\left((-1)^{j-r} m_{\mathfrak{t}^{\gamma_{r,n}}} - \sum_{i=2}^{r} m_{i,j}\right) + \sum_{i=2}^{r} C_{r-1} m_{i,j}, \\
&= (-1)^{j-r} C_{r-1} m_{\mathfrak{t}^{\gamma_{r,n}}},
\end{aligned}$$

as for $3 \leqslant i < r$ we have $C_{i-1} + q^{i-1} C_{r-i} = C_{r-1}$. We now compute $(n_1)\Psi E_r$ using the form from Corollary 4.33, so $b = q$ and $a = (-1)^{n-r-1} C_{n-r+1}$. We continue to use a and b rather than their values for now to simplify the expressions involved.

$$\begin{aligned}
(n_1)\Psi E_r &= a m_{\mathfrak{t}^{\gamma_{r,n}}} E_r + b\left(\sum_{j=r+1}^{n} (-q)^{n-j} M_j E_r\right), \\
&= a C_r m_{\mathfrak{t}^{\gamma_{r,n}}} + b\left(\sum_{j=r+1}^{n} (-q)^{n-j} (-1)^{j-r} C_{r-1} m_{\mathfrak{t}^{\gamma_{r,n}}}\right), \\
&= \left[a C_r + b(-1)^{n-r} C_{r-1}\left(\sum_{j=r+1}^{n} q^{n-j}\right)\right] m_{\mathfrak{t}^{\gamma_{r,n}}}, \\
&= \left[a C_r + b(-1)^{n-r} C_{r-1} C_{n-r}\right] m_{\mathfrak{t}^{\gamma_{r,n}}}.
\end{aligned}$$

We can now substitute in our values for a and b, before simplifying.

$$\begin{aligned} aC_r + b(-1)^{n-r}C_{r-1}C_{n-r} &= (-1)^{n-r-1}C_{n-r+1}C_r + q(-1)^{n-r}C_{r-1}C_{n-r}, \\ &= (-1)^{n-r-1}\left[(1+q+\cdots+q^{r-1})C_{n-r+1} - (q+\cdots q^{r-1})C_{n-r}\right], \\ &= (-1)^{n-r-1}\left[C_{n-r+1} + (q+\cdots+q^{r-1})(C_{n-r+1}-C_{n-r})\right], \\ &= (-1)^{n-r-1}\left[C_{n-r+1} + (q+\cdots+q^{r-1})q^{n-r}\right], \\ &= (-1)^{n-r-1}\left[(1+q+\cdots+q^{n-r}) + (q^{n-r+1}+\cdots+q^{n-1})\right], \\ &= (-1)^{n-r-1}C_n. \end{aligned}$$

Thus $(n_1)\Psi E_r = (-1)^{n-r-1}C_n m_{\mathfrak{t}^{\gamma_{\mathfrak{r},\mathfrak{n}}}}$. □

Corollary 4.38. *$S^{\gamma_{r,n}}$ is a direct summand of $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$ as a $\mathscr{H}_n$-module if and only if $e \nmid n$. As a consequence of this, $S^{\gamma_{r,n}}$ is relatively $\mathfrak{S}_\mu$-projective when $e \nmid n$.*

Proof. Recall that $S^{\gamma_{r,n}}$ is a direct summand if and only if we have $\mathscr{H}_n$-homomorphisms $\alpha : S^{\gamma_{r,n}} \to \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$ and $\beta : \mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n \to S^{\gamma_{r,n}}$ whose composition is a non-zero multiple of the identity. We will show for our Φ and Ψ, that this holds if and only if $e \nmid n$.

From Theorem 4.37, we have that

$$((m_{\mathfrak{t}^{\gamma_{\mathfrak{r},\mathfrak{n}}}})\Phi)\Psi = (-1)^{n-r-1}C_n m_{\mathfrak{t}^{\gamma_{\mathfrak{r},\mathfrak{n}}}},$$

Clearly this composition will give a multiple of the identity map (as $m_{\mathfrak{t}^{\gamma_{r,n}}}$ generates $S^{\gamma_{r,n}}$ as an $\mathscr{H}_n$-module) and as $C_n \neq 0$ if and only if $e \nmid n$. □

Thus when $e \nmid n$, we gain a new way of finding the Specht module as a direct summand of an induced module. In fact when $e \nmid n-r+1$, we can extend this to show that $S^{\gamma_{r,n}}$ is $\mathfrak{S}_{(1,n-r,r-1)}$-projective by restricting the sign part, and then use methods similar to the proofs of Lemma 4.21 and Theorem 4.22 to once again show that the vertex of $S^{\gamma_{r,n}}$ is $\mathfrak{P}(\mathfrak{S}_{r-1} \times \mathfrak{S}_{n-r})$.

Unfortunately, this does not further shed any light on the case when $e \mid n$. Although Corollary 4.38 says we cannot find $S^{\gamma_{r,n}}$ as a direct summand in this way, this does not necessarily mean that $S^{\gamma_{r,n}}$ is not relatively $\mathfrak{S}_\mu$-projective, and further research is required to give a definitive answer.

CHAPTER 5

SIGNED YOUNG MODULES

Throughout this chapter, we continue to assume that $e > 2$. As seen previously, the modules induced from a copy of the sign module tensored with the trivial module play a key role in finding the vertex of certain Specht modules.

Denote by $\Lambda(m, n)$ the set of compositions of n which have exactly m parts (for the purpose of this chapter we allow parts to have size zero), write $|\lambda| = n$ if $\lambda \in \Lambda(m, n)$, and let:

$$\Lambda(m_0|m_1, n) = \{\lambda = (\lambda(0)|\lambda(1)) : \lambda(i) \in \Lambda(m_i, n_i), n_0 + n_1 = n\} \cong \Lambda(m_0 + m_1, n).$$

For $\lambda \in \Lambda(m_0|m_1, n)$ we use the convention $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda(0)} \times \mathfrak{S}_{\lambda(1)}$ where:

$$\begin{aligned} \mathfrak{S}_{\lambda(0)} &= \mathfrak{S}_{\{1,\ldots,\lambda(0)_1\}} \times \mathfrak{S}_{\{\lambda(0)_1+1,\ldots,\lambda(0)_1+\lambda(0)_2\}} \times \cdots \times \mathfrak{S}_{\{\sum_{i<m_0} \lambda(0)_i+1, |\lambda(0)|\}} \subseteq \mathfrak{S}_{\{1,\ldots,|\lambda(0)|\}}, \\ \mathfrak{S}_{\lambda(1)} &= \mathfrak{S}_{\{|\lambda(0)|+1,\ldots,|\lambda(0)|+\lambda(1)_1\}} \times \cdots \times \mathfrak{S}_{\{|\lambda(0)|+\sum_{i<m_1} \lambda(1)_i+1, |\lambda(0)|+|\lambda(1)|\}} \subseteq \mathfrak{S}_{\{|\lambda(0)|+1,\ldots,n\}}, \end{aligned}$$

and correspondingly $\mathscr{H}_\lambda \cong \mathscr{H}_{\lambda(0)} \otimes \mathscr{H}_{\lambda(1)}$. For $\lambda \in \Lambda(m_0|m_1, n)$, the **signed permutation module** N^λ is given by:

$$N^\lambda \cong \left[\left(\bigotimes_{i=1}^{m_0} S^{(\lambda(0)_i)} \right) \otimes \left(\bigotimes_{i=1}^{m_1} S^{(1^{\lambda(1)_i})} \right) \right] \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n,$$

the module induced from the trivial module for $\mathscr{H}_{\lambda(0)}$ tensored with the sign module for $\mathscr{H}_{\lambda(1)}$. Both $\mathcal{N}_r \otimes_{\mathscr{H}_{(1,r-1,n-r)}} \mathscr{H}_n$ and $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_{n-r+1,r-1}} \mathscr{H}_n$ from the previous chapter are examples of signed permutation modules. We say that an indecomposable $\mathscr{H}_n$-module M is a **signed Young module** if there exists $\lambda \in \Lambda(m_0|m_1, n)$ for some $m_0, m_1 \geqslant 0$ with $M \mid N^{\lambda}$. Examples of signed Young modules when $e \nmid n$ are $S^{\gamma_{r,n}}$ by Corollary 4.38 as these are indecomposable by Theorem 1.12.

In [20, Theorem 4.2], Hemmer proves the following result for $F\mathfrak{S}_n$:

Theorem 5.1. *Let F be a field of characteristic $p > 2$. Then all irreducible Specht modules are isomorphic to signed Young Modules.*

As a result, these signed Young modules are of key interest when looking at Specht modules. In [11], Donkin classified the signed Young modules for $F\mathfrak{S}_n$, and showed that they were labelled by partitions λ and μ such that $|\lambda| + p|\mu| = n$. Furthermore, by [11, §2 (8)] the signed Young module labelled by λ and μ can be found as a direct summand of $N^{\lambda|p\mu}$ (here $p\mu = (p\mu_1, \ldots, p\mu_s)$), and the other indecomposable summands of $N^{\lambda|p\mu}$ are labelled by pairs of partitions τ and γ where $(\tau|p\gamma)$ dominates $(\lambda|p\mu)$.

In [14], Du et al consider primitive idempotents of the q-Schur superalgebra, which correspond to signed Young modules for $\mathscr{H}_n$. When $p = 0$, they give a classification of the primitive idempotents of this superalgebra (and hence the signed Young modules), and show that the labelling set involved is equivalent to that of Donkin when $e = p$ and $q = 1$.

A key step in proving this classification is the notion of a defect group of a primitive idempotent, defined in [14, Definition 10.1] when $p = 0$. [14, Theorem 10.2] shows that this defect group is the vertex of the corresponding signed Young module, and thus is of great interest to us.

Throughout this chapter, we follow [14] and explore signed permutation modules for

$\mathscr{H}_n$, their relationship to the q-Schur superalgebra and we conclude by proving an analogue of [14, Theorem 10.2] for fields of prime characteristic. This gives us a method of computing the vertex of a signed Young module, if we know how to write its corresponding primitive idempotent in a certain basis of the q-Schur superalgebra, and ideally is one of the first steps to giving a classification of signed Young modules in prime characteristic.

5.1 Signed permutation modules

Throughout this chapter, for $\lambda, \mu \in \Lambda(m_0|m_1, n)$ and $d \in \mathcal{D}_{\lambda,\mu}$ we will use the notation $\lambda d \cap \mu$ (from [14]) to represent the composition with corresponding parabolic subgroup $\mathfrak{S}_{\lambda d \cap \mu} = \mathfrak{S}_\lambda^d \cap \mathfrak{S}_\mu$. This is in contrast to our previous notation of $\nu(d)$, so as to highlight the role of λ and μ.

For $\lambda = (\lambda(0), \lambda(1)) \in \Lambda(m_0|m_1, n)$ define:

$$x_{\lambda(0)} = \sum_{w \in \mathfrak{S}_{\lambda(0)}} T_w, \qquad y_{\lambda(1)} = \sum_{w \in \mathfrak{S}_{\lambda(1)}} (-q)^{-\ell(w)} T_w,$$

$$z_\lambda = x_{\lambda(0)} y_{\lambda(1)}.$$

Then the following proposition gives an alternative characterisation of the signed permutation module N^λ.

Proposition 5.2. *As $\mathscr{H}_n$-modules, $z_\lambda \mathscr{H}_n \cong N^\lambda$.*

Proof. By [25, Lemma 2.19], it suffices to show that $F z_\lambda$ is a one-dimensional $\mathscr{H}_\lambda$-module isomorphic to $\left(\bigotimes_{i=1}^{m_0} S^{(\lambda(0)_i)}\right) \otimes \left(\bigotimes_{i=1}^{m_1} S^{(1^{\lambda(1)_i})}\right)$. As $z_\lambda = x_{\lambda(0)} y_{\lambda(1)} = y_{\lambda(1)} x_{\lambda(0)}$, we just need to show for $s_i \in \mathfrak{S}_{\lambda(0)}$, that $x_{\lambda(0)} T_i = q x_{\lambda(0)}$ and for $s_i \in \mathfrak{S}_{\lambda(1)}$, that $y_{\lambda(1)} T_i = -y_{\lambda(1)}$.

The first follows from [26, Lemma 3.2], and for the second:

$$\begin{aligned} y_{\lambda(1)}T_i &= \sum_{\substack{w\in\mathfrak{S}_{\lambda(1)}\\ \ell(w)<\ell(ws_i)}} (-q)^{-\ell(w)}T_wT_i + (-q)^{-(\ell(w)+1)}T_{ws_i}T_i,\\ &= \sum_{\substack{w\in\mathfrak{S}_{\lambda(1)}\\ \ell(w)<\ell(ws_i)}} (-q)^{-\ell(w)}T_{ws_i} + (-q)^{-(\ell(w)+1)}(q-1)T_{ws_i} + (-q)^{-(\ell(w)+1)}qT_w,\\ &= -y_{\lambda(1)}. \end{aligned}$$

□

We proceed as in [7, Theorem 3.4] to describe a basis of $\mathrm{Hom}_{\mathscr{H}_n}(N^\lambda, N^\mu)$, for $\lambda, \mu \in \Lambda(m_0|m_1, n)$. For this, we need the following standard theorem from [7, Theorem 2.8].

Theorem 5.3. *Let $\lambda, \mu \vDash n$, and suppose M is a $\mathscr{H}_\mu$-module and N is a $\mathscr{H}_\lambda$-module. Then:*

$$\mathrm{Hom}_{\mathscr{H}_n}(M \otimes_{\mathscr{H}_\mu} \mathscr{H}_n, N \otimes_{\mathscr{H}_\lambda} \mathscr{H}_n) \cong \bigoplus_{d\in\mathcal{D}_{\lambda,\mu}} \mathrm{Hom}_{\mathscr{H}_{\lambda d\cap\mu}}(M, N \otimes T_d).$$

We will apply Theorem 5.3 when M and N are Fz_μ and Fz_λ respectively. As these are one-dimensional, we know $\mathrm{Hom}_{\mathscr{H}_{\lambda d\cap\mu}}(Fz_\mu, Fz_\lambda \otimes T_d)$ is at most one-dimensional, and is one-dimensional if and only if $Fz_\mu \cong Fz_\lambda \otimes T_d$ as $\mathscr{H}_{\lambda d\cap\mu}$-modules. Thus it is prudent to identify which $d \in \mathcal{D}_{\lambda,\mu}$ give this non-zero set of homomorphisms.

Definition 5.4. Let $\lambda, \mu \in \Lambda(m_0|m_1, n)$. Let $\mathcal{D}^\circ_{\lambda,\mu}$ be the set of $d \in \mathcal{D}_{\lambda,\mu}$ with the property that for all $s_j \in \mathfrak{S}_{\lambda d\cap\mu}$ we have for $i = 0, 1$:

$$s_j \in \mathfrak{S}_{\mu(i)} \iff s_j^{d^{-1}} \in \mathfrak{S}_{\lambda(i)}.$$

Note that this definition gives the same set described in [14, (2.2.2)]. The next lemma shows these are exactly the coset representatives we need.

Lemma 5.5. *Let $\lambda, \mu \in \Lambda(m_0|m_1, n)$, and $d \in \mathcal{D}_{\lambda,\mu}$. Then $Fz_\mu \cong Fz_\lambda \otimes T_d$ as $\mathscr{H}_{\lambda d\cap\mu}$-modules if and only if $d \in \mathcal{D}^\circ_{\lambda,\mu}$.*

Proof. It suffices to show that for $s_j \in \mathfrak{S}_{\lambda d \cap \mu}$, that if $z_\mu T_j = \alpha_j z_\mu$ and $z_\lambda \otimes T_d T_j = \beta_j z_\lambda \otimes T_d$, then $\alpha_j = \beta_j$ if and only if $d \in \mathcal{D}^\circ_{\lambda,\mu}$. Note $T_d T_j = T_k T_d$ where $s_k = s_j^{d^{-1}} \in \mathfrak{S}_\lambda$, by Lemma 1.6. Therefore if $\alpha_j = \beta_j$, then we must have that T_k and T_j act as the same constant, i.e. $s_j \in \mathfrak{S}_{\mu(i)}$ if and only if $s_k \in \mathfrak{S}_{\lambda(i)}$. Similarly, if $d \in \mathcal{D}^\circ_{\lambda,\mu}$, then by definition we must have $\alpha_j = \beta_j$, proving the opposite direction. □

Theorem 5.6. *Let $\lambda, \mu \in \Lambda(m_0|m_1, n)$. Then a basis of $\mathrm{Hom}_{\mathscr{H}_n}(N^\mu, N^\lambda)$ is given by $\{\varphi^d_{\mu\lambda} : d \in \mathcal{D}^\circ_{\lambda,\mu}\}$, where under the identification $N^\mu \cong z_\mu \mathscr{H}_n$, we determine $\varphi^d_{\mu\lambda}$ by:*

$$(z_\mu)\varphi^d_{\mu\lambda} = \sum_{\substack{w = w_0 w_1 \in \mathcal{R}^\mu_{\lambda d \cap \mu} \\ w_i \in \mathfrak{S}_{\mu(i)}}} (-q)^{-\ell(w_1)} z_\lambda T_d T_w.$$

Proof. Let $\psi_d : F z_\mu \to F z_\lambda \otimes T_d$ be the $\mathscr{H}_{\lambda d \cap \mu}$-isomorphism indexed by $d \in \mathcal{D}^\circ_{\lambda,\mu}$, and determined by $z_\mu \mapsto z_\lambda \otimes T_d$. Theorem 4.25 gives $\overline{\psi}_d \in \mathrm{Hom}_{\mathscr{H}_\mu}(F z_\mu, (F z_\lambda \otimes T_d) \otimes_{\mathscr{H}_{\lambda d \cap \mu}} \mathscr{H}_\mu)$ defined by:

$$\begin{aligned}
(z_\mu)\overline{\psi}_d &= \sum_{w \in \mathcal{R}^\mu_{\lambda d \cap \mu}} q^{-\ell(w)} (z_\mu T_{w^{-1}}) \psi_d \otimes T_w, \\
&= \sum_{\substack{w = w_0 w_1 \in \mathcal{R}^\mu_{\lambda d \cap \mu} \\ w_i \in \mathfrak{S}_{\mu(i)}}} q^{-\ell(w)} (q^{\ell(w_0)} (-1)^{\ell(w_1)} z_\mu) \psi_d \otimes T_w, \\
&= \sum_{\substack{w = w_0 w_1 \in \mathcal{R}^\mu_{\lambda d \cap \mu} \\ w_i \in \mathfrak{S}_{\mu(i)}}} q^{-\ell(w_1)} (-1)^{\ell(w_1)} z_\lambda \otimes T_d \otimes T_w, \\
&= \sum_{\substack{w = w_0 w_1 \in \mathcal{R}^\mu_{\lambda d \cap \mu} \\ w_i \in \mathfrak{S}_{\mu(i)}}} (-q)^{-\ell(w_1)} z_\lambda \otimes T_d \otimes T_w,
\end{aligned}$$

and again by Theorem 4.25, $\{\overline{\psi}_d\}$ is a basis of $\mathrm{Hom}_{\mathscr{H}_\mu}(F z_\mu, (F z_\lambda \otimes T_d) \otimes_{\mathscr{H}_{\lambda d \cap \mu}} \mathscr{H}_\mu)$. Now via the Mackey formula, we have that as $\mathscr{H}_\mu$-modules:

$$N^\lambda \cong \bigoplus_{d \in \mathcal{D}_{\lambda,\mu}} (F z_\lambda \otimes T_d) \otimes_{\mathscr{H}_{\lambda d \cap \mu}} \mathscr{H}_\mu),$$

thus:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{H}_\mu}(Fz_\mu, N^\lambda) &\cong \bigoplus_{d\in\mathcal{D}_{\lambda,\mu}} \mathrm{Hom}_{\mathscr{H}_\mu}(Fz_\mu, (Fz_\lambda \otimes T_d)\otimes_{\mathscr{H}_{\lambda d\cap\mu}} \mathscr{H}_\mu)\\ &= \bigoplus_{d\in\mathcal{D}^\circ_{\lambda,\mu}} \mathrm{Hom}_{\mathscr{H}_\mu}(Fz_\mu, (Fz_\lambda \otimes T_d)\otimes_{\mathscr{H}_{\lambda d\cap\mu}} \mathscr{H}_\mu),\end{aligned}$$

as from before the only non-zero Hom spaces correspond to $d \in \mathcal{D}^\circ_{\lambda,\mu}$. Under this identification, we have $\overline{\psi}_d \in \mathrm{Hom}_{\mathscr{H}_\mu}(Fz_\mu, N^\lambda)$ given by:

$$(z_\mu)\overline{\psi}_d = \sum_{\substack{w=w_0w_1\in\mathcal{R}^\mu_{\lambda d\cap\mu}\\ w_i\in\mathfrak{S}_{\mu(i)}}} (-q)^{-\ell(w_1)} z_\lambda T_d T_w,$$

form a basis of $\mathrm{Hom}_{\mathscr{H}_\mu}(Fz_\mu, N^\lambda)$. Finally we conclude using Frobenius reciprocity, from [7, Theorem 2.5], to get a basis $\{\varphi^d_{\mu\lambda} : d \in \mathcal{D}^\circ_{\lambda,\mu}\}$ of $\mathrm{Hom}_{\mathscr{H}_n}(N^\mu, N^\lambda)$ determined by:

$$(z_\mu)\varphi^d_{\mu\lambda} = (z_\mu)\overline{\psi}_d = \sum_{\substack{w=w_0w_1\in\mathcal{R}^\mu_{\lambda d\cap\mu}\\ w_i\in\mathfrak{S}_{\mu(i)}}} (-q)^{-\ell(w_1)} z_\lambda T_d T_w. \qquad \square$$

Taking the sum over all signed permutation modules:

Corollary 5.7. *Let $\mathcal{S} = \mathrm{End}_{\mathscr{H}_n}\left(\bigoplus_{\lambda\in\Lambda(m_0|m_1,n)} N^\lambda\right)$. Then a basis of $\mathcal{S}$ is given by*

$$\{\varphi^d_{\mu\lambda} : \lambda, \mu \in \Lambda(m_0|m_1, n), d \in \mathcal{D}^\circ_{\lambda,\mu}\}.$$

Note that if $m_1 = 0$, then $\mathcal{S}$ given above is the q-Schur algebra as defined in [9]. In the next section, we show that $\mathcal{S}$ is the q-Schur superalgebra as in [14], before using it to examine signed Young modules.

5.2 The q-Schur superalgebra

Let $V(m_0|m_1)$ be a (m_0+m_1)-dimensional vector space over F with basis $v_1, \ldots, v_{m_0+m_1}$. We can form a parity map $\widehat{\cdot} : \{1, \ldots, m_0+m_1\} \to \{0,1\}$ by:

$$\widehat{i} = \begin{cases} 0 & \text{if } 1 \leqslant i \leqslant m_0, \\ 1 & \text{if } m_0+1 \leqslant i \leqslant m_0+m_1. \end{cases}$$

This makes $V(m_0|m_1)$ into a $\mathbb{Z}_2$-graded vector space (where the even subspace is spanned by the v_i with $\widehat{i}=0$, and the odd part is spanned by the v_i with $\widehat{i}=1$). Let $I(m_0|m_1,n)$ be the set of n-tuples of integers from $\{1,\ldots,m_0+m_1\}$. Then $V(m_0|m_1)^{\otimes n}$ has a basis indexed by $\mathbf{i} \in I(m_0|m_1,n)$ with $v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_n}$. There is an action of $\mathfrak{S}_n$ on $I(m_0|m_1,n)$ given by:

$$\mathbf{i}w = (i_{(1)w^{-1}}, \ldots, i_{(n)w^{-1}}),$$

for $w \in \mathfrak{S}_n$ and $\mathbf{i} = (i_1, \ldots, i_n)$. For $\lambda \in \Lambda(m_0|m_1,n)$, we define:

$$\mathbf{i}_\lambda = (\underbrace{1,\ldots,1}_{\lambda(0)_1}, \ldots, \underbrace{m_0,\ldots,m_0}_{\lambda(0)_{m_0}}, \underbrace{m_0+1,\ldots,m_0+1}_{\lambda(1)_1}, \ldots, \underbrace{m_0+m_1,\ldots,m_0+m_1}_{\lambda(1)_{m_1}}).$$

Thus an alternative basis is given by: $v_{\lambda d} := v_{\mathbf{i}_\lambda d}$ for $\lambda \in \Lambda(m_0|m_1,n)$ and $d \in \mathcal{R}_\lambda$, as the composition λ gives the content of the tuple and the minimal right coset representative gives us the ordering.

The next lemma gives a $\mathscr{H}_n$-module structure on $V(m_0|m_1)^{\otimes n}$. This was first proved in [27, Theorem 2.1], and we provide an outline of the proof here using our notation and conventions. We choose some fixed square root of q, denoted $\sqrt{q}$.

Lemma 5.8. *$V(m_0|m_1)^{\otimes n}$ is a $\mathscr{H}_n$-module with the following action:*

$$v_{\mathbf{i}}T_j = \begin{cases} (-1)^{\widehat{i_j}\widehat{i_{j+1}}}\sqrt{q}v_{\mathbf{i}s_j} & \text{if } i_j < i_{j+1}, \\ qv_{\mathbf{i}} & \text{if } i_j = i_{j+1} \leqslant m_0, \\ -v_{\mathbf{i}} & \text{if } i_j = i_{j+1} \geqslant m_0+1, \\ (-1)^{\widehat{i_j}\widehat{i_{j+1}}}\sqrt{q}v_{\mathbf{i}s_j} + (q-1)v_{\mathbf{i}} & \text{if } i_j > i_{j+1}. \end{cases}$$

Proof. We check that this action satisfies the relations for the Hecke algebra. First we need to show that for all $\mathbf{i}$ and j that:

$$v_{\mathbf{i}}T_j^2 = qv_{\mathbf{i}} + (q-1)v_{\mathbf{i}}T_j.$$

Suppose first that $i_j < i_{j+1}$. Then:

$$\begin{aligned} v_{\mathbf{i}}T_j^2 &= (-1)^{\widehat{i_j}\widehat{i_{j+1}}}\sqrt{q}v_{\mathbf{i}s_j}T_j = (-1)^{\widehat{i_j}\widehat{i_{j+1}}}\sqrt{q}\left[(-1)^{\widehat{i_j}\widehat{i_{j+1}}}\sqrt{q}v_{\mathbf{i}} + (q-1)v_{\mathbf{i}s_j}\right], \\ &= qv_{\mathbf{i}} + (-1)^{\widehat{i_j}\widehat{i_{j+1}}}\sqrt{q}(q-1)v_{\mathbf{i}s_j} = qv_{\mathbf{i}} + (q-1)v_{\mathbf{i}}T_j. \end{aligned}$$

If $i_j = i_{j+1} \leqslant m_0$, then:

$$v_{\mathbf{i}}T_j^2 = q^2v_{\mathbf{i}} = qv_{\mathbf{i}} + (q-1)qv_{\mathbf{i}} = qv_{\mathbf{i}} + (q-1)v_{\mathbf{i}}T_j,$$

and similarly if $i_j = i_{j+1} \geqslant m_0 + 1$:

$$v_{\mathbf{i}}T_j^2 = v_{\mathbf{i}} = qv_{\mathbf{i}} - (q-1)v_{\mathbf{i}} = qv_{\mathbf{i}} + (q-1)v_{\mathbf{i}}T_j.$$

Finally, if $i_{j+1} < i_j$:

$$v_{\mathbf{i}}T_j^2 = \left[(-1)^{\widehat{i_j}\widehat{i_{j+1}}}\sqrt{q}v_{\mathbf{i}s_j} + (q-1)v_{\mathbf{i}}\right]T_j = q(-1)^{2\widehat{i_j}\widehat{i_{j+1}}}v_{\mathbf{i}} + (q-1)v_{\mathbf{i}}T_j = qv_{\mathbf{i}} + (q-1)v_{\mathbf{i}}T_j.$$

If $|j-k| \geqslant 2$, then as $v_{\mathbf{i}s_js_k} = v_{\mathbf{i}s_ks_j}$, we get that $v_{\mathbf{i}}T_jT_k = v_{\mathbf{i}}T_kT_j$. Finally to prove that the final relation holds, we need to consider 8 cases (after considering symmetries) on i_j, i_{j+1} and i_{j+2}. These are:

$$i_j = i_{j+1} = i_{j+2}, \quad i_j < i_{j+1} = i_{j+2}, \quad i_j = i_{j+2} < i_{j+1}, \quad i_j = i_{j+1} < i_{j+2},$$

$$i_{j+1} < i_j = i_{j+2}, \quad i_j < i_{j+1} < i_{j+2}, \quad i_j < i_{j+2} < i_{j+1}, \quad i_{j+1} < i_j < i_{j+1},$$

where in the first 5 cases, we also need to consider the parities involved. The proofs of all of these cases follow broadly the same lines, so we will just prove one as an example.

Suppose $i_{j+1} < i_j = i_{j+2}$, and let $\alpha_k = q$ if $\widehat{i_k} = 0$ and $\alpha_k = -1$ if $\widehat{i_k} = 1$, for $k \in \{j, j+1, j+2\}$. Note that $\alpha_j = \alpha_{j+2}$. Then:

$$\begin{aligned}
v_{\mathbf{i}}T_jT_{j+1}T_j &= \left[(-1)^{\widehat{i_j}\widehat{i_{j+1}}}\sqrt{q}v_{\mathbf{i}s_j} + (q-1)v_{\mathbf{i}}\right]T_{j+1}T_j, \\
&= \left[(-1)^{\widehat{i_j}\widehat{i_{j+1}}}\alpha_j\sqrt{q}v_{\mathbf{i}s_j} + (-1)^{\widehat{i_{j+1}}\widehat{i_{j+2}}}\sqrt{q}(q-1)v_{\mathbf{i}s_{j+1}}\right]T_j, \\
&= (-1)^{2\widehat{i_j}\widehat{i_{j+1}}}\alpha_j q v_{\mathbf{i}} + \alpha_j(-1)^{\widehat{i_{j+1}}\widehat{i_{j+2}}}\sqrt{q}(q-1)v_{\mathbf{i}s_{j+1}}, \\
&= \alpha_j q v_{\mathbf{i}} + \alpha_j(-1)^{\widehat{i_{j+1}}\widehat{i_{j+2}}}\sqrt{q}(q-1)v_{\mathbf{i}s_{j+1}}.
\end{aligned}$$

Computing the other side:

$$\begin{aligned}
v_{\mathbf{i}}T_{j+1}T_jT_{j+1} &= (-1)^{\widehat{i_{j+1}}\widehat{i_{j+2}}}\sqrt{q}v_{\mathbf{i}s_{j+1}}T_jT_{j+1}, \\
&= \alpha_j(-1)^{\widehat{i_{j+1}}\widehat{i_{j+2}}}\sqrt{q}v_{\mathbf{i}s_{j+1}}T_{j+1}, \\
&= \alpha_j(-1)^{\widehat{i_{j+1}}\widehat{i_{j+2}}}\sqrt{q}\left[(-1)^{\widehat{i_{j+1}}\widehat{i_{j+2}}}\sqrt{q}v_{\mathbf{i}} + (q-1)v_{\mathbf{i}s_{j+1}}\right], \\
&= \alpha_j q v_{\mathbf{i}} + \alpha_j(-1)^{\widehat{i_{j+1}}\widehat{i_{j+2}}}\sqrt{q}(q-1)v_{\mathbf{i}s_{j+1}}, \\
&= v_{\mathbf{i}}T_jT_{j+1}T_j.
\end{aligned}$$

□

We can now define the q-Schur superalgebra.

Definition 5.9. Let $m_0, m_1 \geqslant 0$. Then the **q-Schur superalgebra** is the superalgebra:

$$\mathcal{S}_q(m_0|m_1, n) := \mathrm{End}_{\mathscr{H}_n}\left(V(m_0|m_1)^{\otimes n}\right).$$

For more information on these q-Schur superalgebras, in particular a q-Schur–Weyl duality, we refer to [27]. From [14, Lemma 2.8], there is an $\mathscr{H}_n$-module isomorphism:

$$\Xi : \bigoplus_{\lambda \in \Lambda(m_0|m_1,n)} N^\lambda \to V(m_0|m_1)^{\otimes n},$$

sending $z_\lambda T_d$ to $\pm(\sqrt{q})^{\ell(d)} v_{\lambda d}$, where the sign is determined by [14, Definition 2.7], and depends on λ and d. Throughout the rest of this chapter, we will implicitly identify these modules via this isomorphism. Thus as superalgebras:

$$\mathcal{S}_q(m_0|m_1, n) \cong \mathrm{End}_{\mathscr{H}_n}\left(\bigoplus_{\lambda \in \Lambda(m_0|m_1,n)} N^\lambda\right) \cong \bigoplus_{\lambda,\mu \in \Lambda(m_0|m_1,n)} \mathrm{Hom}_{\mathscr{H}_n}(N^\mu, N^\lambda),$$

so under this isomorphism Corollary 5.7 gives a basis of $\mathcal{S}_q(m_0|m_1, n)$. There is an alternative basis which will be useful in the next section. For $\mathbf{i}$ and $\mathbf{j} \in I(m_0|m_1, n)$, define an F-linear map $e_{\mathbf{i},\mathbf{j}}$ on $V(m_0|m_1)^{\otimes n}$ by:

$$(v_{\mathbf{k}})e_{\mathbf{i},\mathbf{j}} = \delta_{\mathbf{i},\mathbf{k}} v_{\mathbf{j}}.$$

These form a basis of $\mathrm{End}_F\left(V(m_0|m_1)^{\otimes n}\right)$. For $\lambda, \mu \in \Lambda(m_0|m_1, n)$, and $d \in \mathcal{R}_\lambda$, $d' \in \mathcal{R}_\mu$, then we use the notation $e_{\mu d', \lambda d} := e_{\mathbf{i}_\mu d', \mathbf{i}_\lambda d}$.

If $d \in \mathcal{D}^\circ_{\lambda,\mu}$, then for $s_j \in \mathfrak{S}_{\lambda d \cap \mu}$, we have $T_d T_j = T_k T_d$ for some $s_k \in \mathfrak{S}_\lambda$, by Lemma 1.6. Furthermore, by the definition of $\mathcal{D}^\circ_{\lambda,\mu}$, if $z_\mu T_j = \alpha z_\mu$, then $z_\lambda T_d T_j = z_\lambda T_k T_d = \alpha z_\lambda T_d$, that is T_k and T_j act as the same constant on z_λ and z_μ respectively. Thus under the identification Ξ given previously, we have that $e_{\mu,\lambda d}$ is a $\mathscr{H}_{\lambda d \cap \mu}$-module homomorphism, so we can apply the relative trace to get $\mathrm{Tr}^n_{\lambda d \cap \mu}(e_{\mu,\lambda d}) \in \mathcal{S}_q(m_0|m_1, n)$. The following theorem (combining [14, Theorems 5.1, 5.4]) gives the importance of these homomorphisms.

Theorem 5.10. *The following is an F-basis of $\mathcal{S}_q(m_0|m_1,n)$:*

$$\{\mathrm{Tr}^n_{\lambda d\cap\mu}(e_{\mu,\lambda d}) : \mu,\lambda \in \Lambda(m_0|m_1,n), d \in \mathcal{D}^\circ_{\lambda,\mu}\}.$$

Furthermore, for each choice of $\lambda,\mu \in \Lambda(m_0|m_1,n)$ and $d \in \mathcal{D}^\circ_{\lambda,\mu}$ there exists a constant c (given explicitly in [14, Theorem 5.4]) with:

$$\mathrm{Tr}^n_{\lambda d\cap\mu}(e_{\mu,\lambda d}) = c\varphi^d_{\mu\lambda}.$$

Thus this new basis is up to scalar multiple the same as our previous basis. In particular this shows that each $\varphi^d_{\mu\lambda}$ is relatively $\mathfrak{S}_{\lambda d\cap\mu}$-projective as a $\mathscr{H}_n$-module homomorphism. We will use both bases, dependent on whichever is most appropriate at the time.

5.3 Vertices of signed Young modules

Let f be a primitive idempotent of $\mathcal{S}_q(m_0|m_1,n)$. Then $V(m_0|m_1)^{\otimes n}f$ is an indecomposable summand of $V(m_0|m_1)^{\otimes n}$, and thus under the identification Ξ, is an indecomposable summand of $\bigoplus_{\lambda\in\Lambda(m_0|m_1,n)} N^\lambda$, so is a signed Young module. Hence primitive idempotents of the q-Schur superalgebra correspond to the signed Young modules we are interested in. In [14, §7, §10], over a field of characteristic zero the authors take a primitive idempotent f and assign to it a parabolic subgroup $D(f)$ of $\mathfrak{S}_n$, which they call its **defect group**. A key result is [14, Theorem 10.2]:

Theorem 5.11. *Let F be a field of characteristic 0. If $f \in \mathcal{S}_q(m_0|m_1,n)$ is a primitive idempotent, then the defect group $D(f)$ of f is the vertex of the indecomposable $\mathscr{H}_n$-module $V(m_0|m_1)^{\otimes n}f$.*

In this section, we prove the corresponding result for fields of any characteristic. To begin, we need to generalise their notion of a defect group.

5.3.1 The defect group of an idempotent

Let $\mathfrak{P}$ be an e-p-parabolic subgroup of $\mathfrak{S}_n$. Then we define a subspace of $\mathcal{S}_q(m_0|m_1,n)$ in the following way:

$$I(\mathfrak{P}) = \langle \varphi^d_{\mu\lambda} : \lambda, \mu \in \Lambda(m_0|m_1,n), d \in \mathcal{D}^\circ_{\lambda,\mu}, \mathfrak{P}(\mathfrak{S}_{\lambda d \cap \mu}) \subseteq_{\mathfrak{S}_n} \mathfrak{P}\rangle_F,$$

where as before, $\mathfrak{P}(H)$ is the maximal e-p-parabolic subgroup of a parabolic subgroup H. By [14, Corollary 6.6], $I(\mathfrak{P})$ is a two-sided ideal of $\mathcal{S}_q(m_0|m_1,n)$. In the characteristic zero case (see [14, (6.6.1)]), we get a chain of ideals:

$$0 \subseteq I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{\lfloor \frac{n}{e} \rfloor} = \mathcal{S}_q(m_0|m_1,n), \tag{5.1}$$

where $I_j := I(\mathfrak{S}_e^j)$. The authors define $D(f)$ to be $\mathfrak{S}_e^j$ where $f \in I_j$ and $f \notin I_{j-1}$. Recall from Theorem 3.21 that each indecomposable $\mathscr{H}_n$-module has vertex of this form when F has characteristic zero.

In the more general case when F has characteristic $p \geqslant 0$, we know from Theorem 3.21 and Theorem 3.46 that vertices of indecomposable $\mathscr{H}_n$-modules have to be e-p-parabolic. As $\mathfrak{S}_e^k$ is the maximal e-0-parabolic of $\mathfrak{S}_{ke}$, one might think that it would suffice to replace the ideals in this chain with the ideals corresponding to the maximal e-p-parabolic subgroups of $\mathfrak{S}_{je}$ for $0 \leqslant j \leqslant \lfloor \frac{n}{e} \rfloor$. The following example shows that this is not sufficient when $p > 0$.

Example 5.12. Suppose $e > 2$ and let $\lambda = (e+1, 1^{(p-1)e}) \vdash pe+1$. Then as $e \nmid pe+1$, S^λ has vertex $\mathfrak{P}(\mathfrak{S}_e \times \mathfrak{S}_{(p-1)e}) \cong \mathfrak{S}_e^p$ by Theorem 4.22. As S^λ is indecomposable by Theorem 1.12, we can use either the proof of Corollary 4.20 or Corollary 4.38 to see that

S^λ is a signed Young module. Thus the vertices of signed Young modules are not always in the form $\mathfrak{P}(\mathfrak{S}_{ke})$ for some k.

As such, we have more possibilities for defect groups, and have to consider more ideals.

Let t be the largest integer such that $ep^t \leqslant n$, and suppose $\underline{k} = (k_0, k_1, \ldots, k_t)$ is a $(t+1)$-tuple of non-negative integers. Define an e-p-parabolic $\mathfrak{P}_{\underline{k}}$ as follows:

$$\mathfrak{P}_{\underline{k}} := \mathfrak{S}_e^{k_0} \times \mathfrak{S}_{ep}^{k_1} \times \cdots \times \mathfrak{S}_{ep^i}^{k_i} \times \cdots \times \mathfrak{S}_{ep^t}^{k_t},$$

and define $I_{\underline{k}} := I(\mathfrak{P}_{\underline{k}})$. Then

$$\mathcal{K} := \left\{ \underline{k} : \sum_{i=0}^{t} k_i(ep^i) \leqslant n \right\},$$

under the correspondence $\underline{k} \mapsto \mathfrak{P}_{\underline{k}}$, gives a set of representatives of all e-p parabolic subgroups of $\mathfrak{S}_n$, up to conjugacy. Note that for any parabolic subgroup $\mathfrak{P}$ with representative $\underline{k} \in \mathcal{K}$, we have that $I(\mathfrak{P}) = I_{\underline{k}}$. We denote $\mathcal{I} = \{I_{\underline{k}} : \underline{k} \in \mathcal{K}\}$.

We give a relation on $\mathcal{K}$ by saying that $\underline{k} \preceq \underline{l}$ if $\mathfrak{P}_{\underline{k}} \subseteq_{\mathfrak{S}_n} \mathfrak{P}_{\underline{l}}$. By the earlier definition of our ideals we have the following consequence:

$$\underline{k} \preceq \underline{l} \iff I_{\underline{k}} \subseteq I_{\underline{l}}, \tag{5.2}$$

and our first proposition shows that we can impose a poset structure on $\mathcal{K}$ (and hence $\mathcal{I}$).

Proposition 5.13. *The relation $\preceq$ is reflexive, transitive and anti-symmetric. Therefore $(\mathcal{K}, \preceq)$ (and consequently $(\mathcal{I}, \subseteq)$) is a partially-ordered set.*

Proof. For any $\underline{k} \in \mathcal{K}$, it is clear that $\mathfrak{P}_{\underline{k}}$ is a subgroup of itself, giving reflexivity. If $\mathfrak{P}_{\underline{k}}$ is conjugate to a subgroup of $\mathfrak{P}_{\underline{l}}$ and $\mathfrak{P}_{\underline{l}}$ is conjugate to a subgroup of $\mathfrak{P}_{\underline{j}}$ then we must also have $\mathfrak{P}_{\underline{k}}$ is conjugate to a subgroup of $\mathfrak{P}_{\underline{j}}$, proving transitivity. Finally, if conjugates of $\mathfrak{P}_{\underline{k}}$ and $\mathfrak{P}_{\underline{l}}$ are contained in each other, then they must be conjugate in $\mathfrak{S}_n$. Thus as each conjugacy class of e-p-parabolic subgroups of $\mathfrak{S}_n$ has a unique representative in $\mathcal{K}$, we must have $\underline{l} = \underline{k}$. □

Therefore $(\mathcal{I}, \subseteq)$ also forms a poset. However this is not enough to assign to each idempotent an e-p-parabolic subgroup. To achieve this, we show that $\mathcal{K}$ (and hence $\mathcal{I}$) forms a lattice. To this end, we give an alternative characterisation of $\preceq$. We denote $\Gamma_i(\underline{k}) := k_i + pk_{i+1} + \cdots + p^{t-i}k_t$, and in particular note that $\Gamma_t(\underline{k}) = k_t$.

Lemma 5.14. *Let $\underline{k}, \underline{l} \in \mathcal{K}$. Then $\underline{k} \preceq \underline{l}$ if and only if for all $i = 0, \dots, t$:*

$$\Gamma_i(\underline{k}) \leqslant \Gamma_i(\underline{l}).$$

Proof. When $i = t$, the condition gives $k_t \leqslant l_t$, which implies that $\mathfrak{S}_{ep^t}^{k_t} \subseteq_{\mathfrak{S}_n} \mathfrak{S}_{ep^t}^{l_t}$. When $i = t-1$, then the condition says that both $k_t \leqslant l_t$ and $k_{t-1} + pk_t \leqslant l_{t-1} + pl_t$. Thus $\mathfrak{S}_{ep^{t-1}}^{k_{t-1}} \times \mathfrak{S}_{ep^t}^{k_t} \subseteq_{\mathfrak{S}_n} \mathfrak{S}_{ep^{t-1}}^{l_{t-1}} \times \mathfrak{S}_{ep^t}^{l_t}$ as there are enough copies of $\mathfrak{S}_{ep^t}$ left in $\mathfrak{S}_{ep^{t-1}}^{l_{t-1}} \times \mathfrak{S}_{ep^t}^{l_t - k_t}$ such that it contains $\mathfrak{S}_{ep^{t-1}}^{k_{t-1}}$ (even though we may have $k_{t-1} > l_{t-1}$). Continuing in this manner all the way down to when $i = 0$, completes the proof. □

Theorem 5.15. *Let $\underline{k}, \underline{l} \in \mathcal{K}$. Then we recursively define $(\underline{k} \vee \underline{l})$ by:*

$$(\underline{k} \vee \underline{l})_i = \begin{cases} \max\{\Gamma_t(\underline{k}), \Gamma_t(\underline{l})\} & \text{if } i = t, \\ \max\{\Gamma_i(\underline{k}), \Gamma_i(\underline{l})\} - p\Gamma_{i+1}(\underline{k} \vee \underline{l}) & \text{if } i < t. \end{cases}$$

Similarly define $(\underline{k} \wedge \underline{l})$ by:

$$(\underline{k} \wedge \underline{l})_i = \begin{cases} \min\{\Gamma_t(\underline{k}), \Gamma_t(\underline{l})\} & \text{if } i = t, \\ \min\{\Gamma_i(\underline{k}), \Gamma_i(\underline{l})\} - p\Gamma_{i+1}(\underline{k} \wedge \underline{l}) & \text{if } i < t. \end{cases}$$

Then $(\mathcal{K}, \preceq)$ with join function $\vee$ and meet function $\wedge$ is a lattice.

Proof. Note that these are well-defined, as $(\underline{k} \vee \underline{l})_t$ is defined from just $\underline{k}$ and $\underline{l}$, and $(\underline{k} \vee \underline{l})_i$ is defined given that $(\underline{k} \vee \underline{l})_{i+1}$ is defined. We need to show that $\underline{k} \vee \underline{l}$ is an upper bound for both $\underline{k}$ and $\underline{l}$, and that it is the least upper bound (i.e. less than any other upper bound).

When $i = t$, we get that $(\underline{k} \vee \underline{l})_t = \max\{k_t, l_t\}$. This is clearly the smallest integer at least that of both k_t and l_t. At each stage with $i < t$, to have $\underline{k} \preceq (\underline{k} \vee \underline{l})$ and $\underline{l} \preceq (\underline{k} \vee \underline{l})$, we need $\Gamma_i(\underline{k}), \Gamma_i(\underline{l}) \leqslant \Gamma_i(\underline{k} \vee \underline{l})$. Since $\Gamma_i(\underline{a}) = a_i + p\Gamma_{i+1}(\underline{a})$ for $i < t$ and $\underline{a} \in \mathcal{K}$, after rearranging this, we need:

$$(\underline{k} \vee \underline{l})_i \geqslant \Gamma_i(\underline{k}) - p\Gamma_{i+1}(\underline{k} \vee \underline{l}),$$

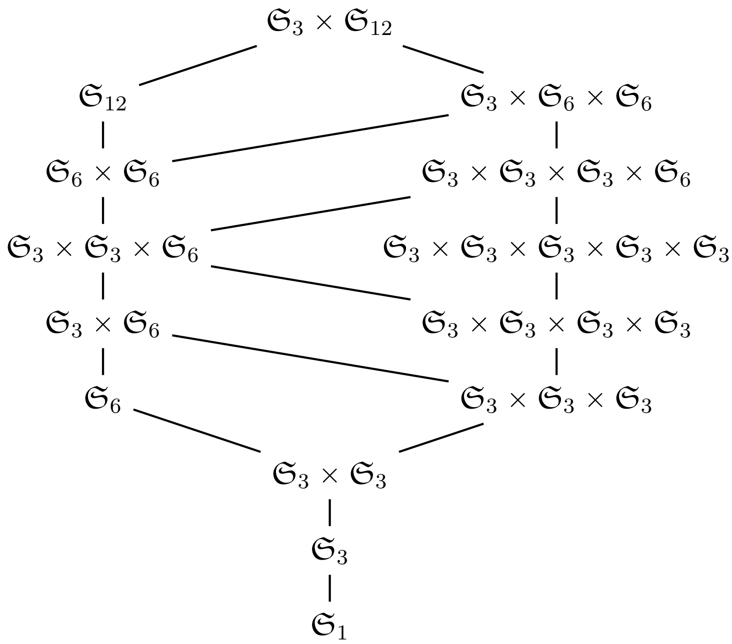
$$(\underline{k} \vee \underline{l})_i \geqslant \Gamma_i(\underline{l}) - p\Gamma_{i+1}(\underline{k} \vee l),$$

for $(\underline{k} \vee \underline{l})$ to be an upper bound. Clearly the definition given above gives us the least integer satisfying this, meaning if we have another upper bound, then it is greater than $(\underline{k} \vee \underline{l})$.

In a similar manner, $(\underline{k} \wedge \underline{l})$ is the greatest lower bound of both $\underline{k}$ and $\underline{l}$, as we take the largest integer satisfying the conditions of Lemma 5.14 at each point. Therefore with these operations, $(\mathcal{K}, \preceq)$ is a lattice. □

We illustrate this lattice in the following example.

Example 5.16. Suppose $e = 3$, $p = 2$ and $n = 15$. Then the following Hasse diagram gives the 3-2-parabolic subgroups of $\mathfrak{S}_{15}$:



Let $\underline{k} = (1,1,0)$, and $\underline{l} = (5,0,0)$, so they correspond to the e-p-parabolic subgroups $\mathfrak{S}_3 \times \mathfrak{S}_6$, and $\mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_3$ respectively. Denote $\underline{a} = (\underline{k} \vee \underline{l})$. Computing $\underline{a}$:

$$a_2 = \max\{0,0\} = 0,$$

$$a_1 = \max\{1 + 0p, 0 + 0p\} - 0p = 1,$$

$$a_0 = \max\{1 + p + 0p^2, 5 + 0p + 0p^2\} - p - 0p^2 = 5 - 2 = 3,$$

giving $\underline{a} = (3,1,0)$ corresponding to $\mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_6$, which we can see from the diagram is the least upper bound of $\underline{l}$ and $\underline{k}$. Similarly denoting $\underline{b} = \underline{k} \wedge \underline{l}$:

$$b_2 = \min\{0,0\} = 0,$$

$$b_1 = \min\{1 + 0p, 0 + 0p\} - 0p = 0,$$

$$b_0 = \min\{1 + p + 0p^2, 5 + 0p + 0p^2\} - 0p - 0p^2 = 3 - 0 = 3,$$

giving the greatest lower bound as $\underline{b} = (3,0,0)$, that is $\mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_3$.

Given this lattice structure on $(\mathcal{K}, \preceq)$, we get a corresponding structure on $(\mathcal{I}, \subseteq)$ by

(5.2). For a basis element $\varphi^d_{\mu\lambda}$ of $\mathcal{S}_q(m_0|m_1, n)$ we can define $\underline{k}_{\lambda d\cap\mu}$ such that $\mathfrak{P}_{\underline{k}_{\lambda d\cap\mu}} \cong \mathfrak{P}(\mathfrak{S}_{\lambda d\cap\mu})$. By definition of $I_{\underline{k}}$, this is the least element $\underline{k}$ of $\mathcal{K}$ with $\varphi^d_{\mu\lambda} \in I_{\underline{k}}$.

Proposition 5.17. *Let $\underline{k}, \underline{l} \in \mathcal{K}$. Then $I_{\underline{k}} \cap I_{\underline{l}} = I_{\underline{k}\wedge\underline{l}}$.*

Proof. Let $\varphi^d_{\mu\lambda} \in I_{\underline{k}\wedge\underline{l}}$. Then as $\mathfrak{P}_{\underline{k}_{\lambda d\cap\mu}} \subseteq_{\mathfrak{S}_n} \mathfrak{P}_{\underline{k}\wedge\underline{l}}$, we have $\underline{k}_{\lambda d\cap\mu} \preceq (\underline{k} \wedge \underline{l})$, and hence it is at most that of both $\underline{l}$ and $\underline{k}$. Therefore $\varphi^d_{\mu\lambda}$ is in both $I_{\underline{k}}$ and $I_{\underline{l}}$ and thus lies in the intersection. Similarly, if $\varphi^d_{\mu\lambda}$ lies in the intersection of $I_{\underline{k}}$ and $I_{\underline{l}}$, then by definition we have $\underline{k}_{\lambda d\cap\mu} \preceq \underline{k}, \underline{l}$ and therefore is at most the meet of the two, giving the other inclusion. □

Corollary 5.18. *Let $x \in \mathcal{S}_q(m_0|m_1, n)$. Then there is a unique $\underline{k} \in \mathcal{K}$ such that $x \in I_{\underline{k}}$ and if for $\underline{l} \in \mathcal{K}$, we have $x \in I_{\underline{l}}$, then $I_{\underline{k}} \subseteq I_{\underline{l}}$.*

Proof. Define $\underline{k}$ as the meet of all $\underline{l}$ with $x \in I_{\underline{l}}$. □

An alternative proof of this corollary comes from the fact that if $x = \sum_{\mu,\lambda,d} a^d_{\mu\lambda}\varphi^d_{\mu\lambda}$ for $a^d_{\mu\lambda} \in F$, then for $\underline{k} \in \mathcal{K}$ we have that:

$$x \in I_{\underline{k}} \iff \varphi^d_{\mu\lambda} \in I_{\underline{k}} \text{ whenever } a^d_{\mu\lambda} \neq 0,$$

as $I_{\underline{k}}$ is spanned by basis elements of this form. Thus the minimal $I_{\underline{k}}$ such that $x \in I_{\underline{k}}$ is also the join over all $\underline{k}_{\lambda d\cap\mu}$ such that $a^d_{\mu\lambda} \neq 0$.

With this corollary, we can now define the defect group $D(f)$ of a primitive idempotent f in any characteristic.

Definition 5.19. Let f be a primitive idempotent of $\mathcal{S}_q(m_0|m_1, n)$. Then the **defect group of** f is the e-p-parabolic subgroup $\mathfrak{P}_{\underline{k}}$, where $\underline{k} \in \mathcal{K}$ is the minimal element in $\mathcal{K}$ (with respect to $\preceq$) such that $f \in I_{\underline{k}}$.

If $p = 0$, then the lattice $\mathcal{K}$ once more becomes the chain (5.1), and we regain the definition of defect group given in [14, Definition 10.1].

5.3.2 Properties of the ideals $I_{\underline{k}}$

Now we have the notion of a defect group for our primitive idempotents, to generalise [14, Theorem 10.2], we first need to generalise the results of [14, §7]. One of the main tools we will use here is Theorem 3.46, which allows us to immediately restrict our attention to the maximal e-p-parabolic subgroup when talking about relative projectivity.

The theorem we prove here is a characteristic $p \geqslant 0$ version of [14, Theorem 7.4].

Theorem 5.20. *Let $\underline{k} \in \mathcal{K}$, and let $\theta \vdash n$ such that $\mathfrak{S}_\theta =_{\mathfrak{S}_n} \mathfrak{P}_{\underline{k}}$. Then:*

$$I_{\underline{k}} = \mathrm{Tr}_\theta^n(\mathrm{End}_{\mathscr{H}_\theta}\left(V(m_0|m_1)^{\otimes n}\right)).$$

As in [14, Lemma 7.1] we start by showing that Theorem 5.20 holds for the trivial parabolic $\mathfrak{P}_{\underline{0}} = \mathfrak{S}_{(1^n)}$. The proof is broadly similar except we need to use Corollary 4.2 (and thus Theorem 3.46) to conclude.

Lemma 5.21.

$$I_{\underline{0}} = \mathrm{Tr}_1^n(\mathrm{End}_F\left(V(m_0|m_1)^{\otimes n}\right)).$$

Proof. We show first that the right-hand side is contained in the left-hand side. Take the following basis of $\mathrm{End}_F\left(V(m_0|m_1)^{\otimes n}\right)$:

$$\{e_{\mu d', \lambda d} \mid \lambda, \mu \in \Lambda(m_0|m_1, n), d' \in \mathcal{R}_\mu, d \in \mathcal{R}_\lambda\}.$$

We apply the relative trace to one of these basis elements, and use the fact from [14, Lemma 4.1] that $\mathrm{Tr}_{\mu}^{n}(e_{\mu,\mu})$ is the identity map on N^{μ}.

$$\begin{aligned}\mathrm{Tr}_1^n(e_{\mu d',\lambda d}) &= \mathrm{Tr}_{\mu}^{n}(e_{\mu,\mu})\,\mathrm{Tr}_1^n(e_{\mu d',\lambda d}),\\ &= \left(\sum_{w\in\mathcal{R}_\mu} q^{-\ell(w)}T_{w^{-1}}e_{\mu,\mu}T_w\right)\mathrm{Tr}_1^n(e_{\mu d',\lambda d}),\\ &= \left(\sum_{w\in\mathcal{R}_\mu} q^{-\ell(w)}T_{w^{-1}}e_{\mu,\mu}\,\mathrm{Tr}_1^n(e_{\mu d',\lambda d})T_w\right),\\ &= \mathrm{Tr}_{\mu}^{n}\left(e_{\mu,\mu}\,\mathrm{Tr}_1^n(e_{\mu d',\lambda d})\right).\end{aligned}$$

Using [14, Lemma 3.3] or Theorem 2.13, transitivity, and linearity of the relative trace we can further manipulate to show:

$$\begin{aligned}\mathrm{Tr}_1^n(e_{\mu d',\lambda d}) &= \mathrm{Tr}_{\mu}^{n}\left(e_{\mu,\mu}\sum_{x\in\mathcal{L}_\mu}\mathrm{Tr}_1^{\mu}(q^{-\ell(x)}T_{x^{-1}}e_{\mu d',\lambda d}T_x)\right),\\ &= \mathrm{Tr}_{\mu}^{n}\left(\sum_{x\in\mathcal{L}_\mu}\mathrm{Tr}_1^{\mu}(e_{\mu,\mu}q^{-\ell(x)}T_{x^{-1}}e_{\mu d',\lambda d}T_x)\right),\\ &= \sum_{x\in\mathcal{L}_\mu} q^{-\ell(x)}\,\mathrm{Tr}_1^n(e_{\mu,\mu}T_{x^{-1}}e_{\mu d',\lambda d}T_x).\end{aligned}$$

Now the inner map $e_{\mu,\mu}T_{x^{-1}}e_{\mu d',\lambda d}T_x$ is non-zero on the basis element $v_{\mathbf{i}}$ if and only if some constituent of $(v_{\mathbf{i}})e_{\mu,\mu}T_{x^{-1}}$ lies in $\langle v_{\mu d'}\rangle$ for $\mathbf{i}\in I(m_0|m_1,n)$. This can only happen if $v_{\mu x^{-1}} = v_{\mu d'}$, which as $x\in\mathcal{L}_\mu$, means that $x = d'^{-1}$ for this to be non-zero. Therefore we have shown that:

$$\mathrm{Tr}_1^n(e_{\mu d',\lambda d}) = q^{-\ell(d')}\,\mathrm{Tr}_1^n(e_{\mu,\mu}T_{d'}e_{\mu d',\lambda d}T_{d'^{-1}}).$$

Now $e_{\mu,\mu}T_{d'}e_{\mu d',\lambda d}T_{d'^{-1}}$ is a map that is only non-zero on v_μ, and maps into $N^\lambda = v_\lambda\mathscr{H}_n$, thus there exist coefficients $a_y\in F$ for $y\in\mathcal{R}_\lambda$ with:

$$\mathrm{Tr}_1^n(e_{\mu d',\lambda d}) = \mathrm{Tr}_1^n\left(\sum_{y\in\mathcal{R}_\lambda} a_y e_{\mu,\lambda y}\right) = \sum_{y\in\mathcal{R}_\lambda} a_y\,\mathrm{Tr}_1^n(e_{\mu,\lambda y}).$$

Thus we are reduced to showing that $\mathrm{Tr}_1^n(e_{\mu,\lambda y})\in I_{\underline{0}}$ for $y\in\mathcal{R}_\lambda$. By [14, Lemma 6.3], we can assume that $y \in \mathcal{D}_{\lambda,\mu}$, and by [14, Corollary 4.5], we can further assume that $y \in \mathcal{D}^{\circ}_{\lambda,\mu}$, otherwise the trace is zero. We now claim that under these conditions, the following holds:

$$\mathrm{Tr}_1^{\lambda y \cap \mu}(e_{\mu,\lambda y}) = P_{\lambda y \cap \mu(0)}(q) P_{\lambda y \cap \mu(1)}(q^{-1}) e_{\mu,\lambda y},$$

where, as in Chapter 3, $P_{\lambda y \cap \mu(i)}$ is the Poincaré polynomial of $\mathfrak{S}_{\lambda y \cap \mu(i)} = \mathfrak{S}^d_{\lambda(i)} \cap \mathfrak{S}_{\mu(i)}$ for $i = 0, 1$. Note that by the definition in [14, (2.2.2)] that $\mathfrak{S}^d_{\lambda(i)} \cap \mathfrak{S}_{\mu(j)}$ is trivial when $i \neq j$, so $\mathfrak{S}_{\lambda d \cap \mu} = \mathfrak{S}_{\lambda d \cap \mu(0)} \times \mathfrak{S}_{\lambda d \cap \mu(1)}$. By definition,

$$\mathrm{Tr}_1^{\lambda y \cap \mu}(e_{\mu,\lambda y}) = \sum_{w \in \mathfrak{S}_{\lambda y \cap \mu}} q^{-\ell(w)} T_{w^{-1}} e_{\mu,\lambda y} T_w.$$

For $(v_{\mathbf{i}})\, \mathrm{Tr}_1^{\lambda y \cap \mu}(e_{\mu,\lambda y})$ to be non-zero, we need v_μ to appear with a non-zero coefficient in $v_{\mathbf{i}} T_{w^{-1}}$ for some $w \in \mathfrak{S}_{\lambda d \cap \mu}$. Clearly this means $\mathbf{i}$ and $\mathbf{i}_\mu$ have the same content. By the action of $\mathfrak{S}_\mu$ on $I(m_0|m_1, n)$, basis terms that appear when multiplying by T_w^{-1} look like $v_{\mathbf{i}}$ but with each section of $\mathbf{i}$ which corresponds to a part of μ permuted. Thus v_μ appears if and only if $\mathbf{i} = \mathbf{i}_\mu$, and we only have to evaluate this trace at v_μ:

$$\begin{aligned}
(v_\mu)\, \mathrm{Tr}_1^{\lambda y \cap \mu}(e_{\mu,\lambda y}) &= \sum_{w \in \mathfrak{S}_{\lambda y \cap \mu}} v_\mu q^{-\ell(w)} T_{w^{-1}} e_{\mu,\lambda y} T_w, \\
&= \sum_{w_i \in \mathfrak{S}_{\lambda y \cap \mu(i)}} q^{-\ell(w_0)-\ell(w_1)} v_\mu T_{w_0^{-1}} T_{w_1^{-1}} e_{\mu,\lambda y} T_{w_0} T_{w_1}, \\
&= \sum_{w_i \in \mathfrak{S}_{\lambda y \cap \mu(i)}} q^{-\ell(w_1)} (-1)^{\ell(w_1)} v_\mu e_{\mu,\lambda y} T_{w_0} T_{w_1}, \\
&= \sum_{w_i \in \mathfrak{S}_{\lambda y \cap \mu(i)}} q^{-\ell(w_1)} (-1)^{\ell(w_1)} v_{\lambda y} T_{w_0} T_{w_1}, \\
&= \sum_{w_i \in \mathfrak{S}_{\lambda y \cap \mu(i)}} q^{\ell(w_0)} q^{-\ell(w_1)} (-1)^{2\ell(w_1)} v_{\lambda y}, \\
&= \sum_{w_i \in \mathfrak{S}_{\lambda y \cap \mu(i)}} q^{\ell(w_0)} q^{-\ell(w_1)} v_{\lambda y}, \\
&= P_{\lambda y \cap \mu(0)}(q) P_{\lambda y \cap \mu(1)}(q^{-1}) v_{\lambda y},
\end{aligned}$$

proving the claim. Thus we have:

$$\mathrm{Tr}_1^n(e_{\mu,\lambda y}) = \mathrm{Tr}_{\lambda y\cap\mu}^n(\mathrm{Tr}_1^{\lambda y\cap\mu}(e_{\mu,\lambda y})) = P_{\lambda y\cap\mu(0)}(q)P_{\lambda y\cap\mu(1)}(q^{-1})\,\mathrm{Tr}_{\lambda y\cap\mu}^n(e_{\mu,\lambda y}).$$

If $\mathfrak{P}(\mathfrak{S}_{\lambda y\cap\mu}) = \mathfrak{P}_{\underline{0}}$, then $\mathrm{Tr}_{\lambda y\cap\mu}^n(e_{\mu,\lambda y})$ lies in $I_{\underline{0}}$ by definition, meaning $\mathrm{Tr}_1^n(e_{\mu,\lambda y}) \in I_{\underline{0}}$. If $\mathfrak{P}(\mathfrak{S}_{\lambda y\cap\mu}) \neq \mathfrak{P}_{\underline{0}}$, then at least one of the $\mathfrak{P}(\mathfrak{S}_{\lambda y\cap\mu(i)})$ is a non-trivial e-p-parabolic subgroup. Without loss of generality (as for these Poincaré polynomials q is a root if and only if q^{-1} is a root) suppose this is satisfied for $i = 0$. Then $P_{\lambda y\cap\mu(0)}(q) = 0$ by the definition of Poincaré polynomial and (3.3), hence $\mathrm{Tr}_1^n(e_{\mu,\lambda y}) = 0 \in I_{\underline{0}}$ as well.

To show that $I_{\underline{0}} \subseteq \mathrm{Tr}_1^n(\mathrm{End}_F(V(m_0|m_1)^{\otimes n}))$, let $\varphi_{\mu\lambda}^d \in I_{\underline{0}}$. This means that $\mathfrak{P}(\mathfrak{S}_{\lambda d\cap\mu})$ is trivial. By Theorem 5.10, there exists $c' \in F$ with $\varphi_{\mu\lambda}^d = c'\,\mathrm{Tr}_{\lambda d\cap\mu}^n(e_{\mu,\lambda d})$. By the proof of Corollary 4.2, there exists some $\alpha \in \mathrm{End}_F(V(m_0|m_1)^{\otimes n})$ with $\mathrm{Tr}_1^{\lambda d\cap\mu}(\alpha) = e_{\mu,\lambda d}$. Therefore using transitivity of the relative trace, $\varphi_{\mu\lambda}^d = c'\,\mathrm{Tr}_1^n(\alpha)$, proving the opposite inclusion. □

In the following results, we will not only consider $\mathcal{S}_q(m_0|m_1,n)$, but also larger q-Schur superalgebras $\mathcal{S}_q(m_0'|m_1',n)$ for $m_i' \geqslant m_i$. We can view $\mathcal{S}_q(m_0|m_1,n)$ as a subalgebra of $\mathcal{S}_q(m_0'|m_1',n)$ by considering $\lambda \in \Lambda(m_0|m_1,n)$ as an element of $\Lambda(m_0'|m_1',n)$ by adding zeros at the end of $\lambda(0)$ and $\lambda(1)$ until they have m_0' and m_1' parts respectively.

Let $\eta = \sum_{\lambda\in\Lambda(m_0|m_1,n)} \varphi_{\lambda\lambda}^1$. By the definition of $\varphi_{\lambda\lambda}^1$, we have that η is the identity $\mathscr{H}_n$-homomorphism on $V(m_0|m_1)^{\otimes n}$, as it sends each N^λ to itself. When η is viewed as an element of $\mathcal{S}_q(m_0'|m_1',n)$, then $(N^\lambda)\eta = 0$ for any $\lambda \in \Lambda(m_0'|m_1',n) - \Lambda(m_0|m_1,n)$. Thus we have that $\eta\mathcal{S}_q(m_0'|m_1',n)\eta = \mathcal{S}_q(m_0|m_1,n)$. Similarly, if for $\underline{k} \in \mathcal{K}$ we define the ideal:

$$I_{\underline{k}}' = \langle \varphi_{\mu\lambda}^d : \lambda,\mu \in \Lambda(m_0'|m_1',n), d \in \mathcal{D}_{\lambda,\mu}^\circ, \mathfrak{P}(\mathfrak{S}_{\lambda d\cap\mu}) \subseteq_{\mathfrak{S}_n} \mathfrak{P}_{\underline{k}}\rangle_F,$$

that is the ideal defined in the same way as $I_{\underline{k}}$ but in $\mathcal{S}_q(m_0'|m_1',n)$, we have that $\eta I_{\underline{k}}'\eta = I_{\underline{k}}$, as $\eta\varphi_{\mu\lambda}^d\eta$ is non-zero if and only if both λ and μ are in $\Lambda(m_0|m_1,n)$.

The following technical lemma, generalising [14, Lemma 7.2], gives a way of taking a partition $\theta \vdash n$ and fitting it into this q-Schur superalgebra framework in a useful way.

Lemma 5.22. *Let* $\underline{k} \in \mathcal{K}$ *and* $\theta \vdash n$ *such that* $\mathfrak{P}_{\underline{k}} =_{\mathfrak{S}_n} \mathfrak{S}_\theta$*. If* $\lambda \in \Lambda(m_0|m_1, n)$*, and* $d \in \mathcal{D}_{\lambda,\theta}$ *with* $\mathfrak{S}_\theta \subseteq \mathfrak{S}_\lambda^d$*, then there exists* $\theta' \in \Lambda(m_0'|m_1', n)$ *(for some* $m_0' \geqslant m_0$ *and* $m_1' \geqslant m_1$*) and* $w \in \mathcal{D}_{\theta',\theta}$ *such that* $\mathfrak{S}_\theta = \mathfrak{S}_{\theta'}^w$ *and*

$$e_{\lambda d, \theta' w} \in \mathrm{End}_{\mathscr{H}_\theta}\left(V(m_0'|m_1')^{\otimes n}\right).$$

Furthermore, $\mathrm{Tr}_\theta^n(e_{\lambda d,\theta' w}) \in I'_{\underline{k}}$*.*

Proof. By definition of $\underline{k}$, we have that θ has $s = \sum_{c=0}^t k_c$ parts of length divisible by e, followed by $r = n - \sum_{c=0}^t k_c e p^c$ ones. As $d \in \mathcal{D}_{\lambda,\theta}$ and $\mathfrak{S}_\theta \subseteq \mathfrak{S}_\lambda^d$, we can apply [12, Lemma 1.3] to see that:

$$\mathbf{i}_\lambda d = (\underbrace{i_1, \ldots, i_1}_{\theta_1}, \underbrace{i_2, \ldots, i_2}_{\theta_2}, \ldots, \underbrace{i_s, \ldots, i_s}_{\theta_s}, j_1, \ldots, j_r),$$

for $i_l, j_l \in \{1, \ldots, m_0 + m_1\}$. Let

$$a_c = \left|\left\{ j : \sum_{l=c+1}^t k_l < j \leqslant \sum_{l=c}^t k_l, \text{ and } \widehat{i_j} = 0 \right\}\right|,$$
$$b_c = \left|\left\{ j : \sum_{l=c+1}^t k_l < j \leqslant \sum_{l=c}^t k_l, \text{ and } \widehat{i_j} = 1 \right\}\right|.$$

Then a_c is the number of runs of repeated numbers in the first $n - r$ entries of $\mathbf{i}_\lambda d$ of length ep^c with even parity, and $b_c = k_c - a_c$ is the number of runs of length ep^c with odd parity. Let $\theta'(0)$ be the partition given by $((ep^t)^{a_t}, \ldots, (ep)^{a_1}, e^{a_0})$, and $\theta'(1)$ given by $((ep^t)^{b_t}, \ldots, (ep)^{b_1}, e^{b_0}, 1^r)$. Then we have that (after padding by zero at the end of $\theta(i)'$ if necessary) $\theta' = (\theta'(0)|\theta'(1)) \in \Lambda(m_0'|m_1', n)$ for some $m_0' \geqslant m_0$ and $m_1' \geqslant m_1$.

Let $a = \sum_{c=0}^t a_c$ and $b = \sum_{c=0}^t b_c$. Then $a+b = s$. Let $\Omega = \{1, \ldots, a, m_0'+1, \ldots, m_0'+b\}$, so $|\Omega| = s$. Then $j \in \Omega$ if and only if $\mathfrak{S}_{\theta'_j}$ is a non-trivial component of $\mathfrak{S}_{\theta'}$. Consider the tuple $\mathbf{i}_{\theta'} \in I(m_0'|m_1', n)$:

$$(\underbrace{1, \ldots, 1}_{\theta_1'}, \ldots, \underbrace{a, \ldots, a}_{\theta_a'}, \underbrace{m_0' + 1, \ldots, m_0' + 1}_{\theta'_{m_0'+1}}, \ldots, \underbrace{m_0' + b, \ldots, m_0' + b}_{\theta'_{m_0'+b}}, m_0' + b + 1, \ldots, m_0' + b + r).$$

Note that $1 \leqslant j \leqslant m_0' + m_1'$ appears more than once in this tuple if and only if $j \in \Omega$. Suppose π is a bijection from Ω to $\{1, \ldots, s\}$ with the property that if $(j)\pi = k$, then $\theta_j' = \theta_k$. Then we get another tuple in $I(m_0'|m_1', n)$:

$$\mathbf{i}_{\pi,\theta} = (\underbrace{(1)\pi^{-1}, \ldots, (1)\pi^{-1}}_{\theta_1}, \ldots, \underbrace{(s)\pi^{-1}, \ldots, (s)\pi^{-1}}_{\theta_s}, m_0' + b + 1, \ldots, m_0' + b + r).$$

We define an element $w_\pi \in \mathfrak{S}_n$ by:

$$\left(\left(\sum_{i<j} \theta_i' \right) + l \right) w_\pi = \left(\sum_{i<\pi(j)} \theta_i \right) + l,$$

for $j = 1, \ldots, s$ and $1 \leqslant l \leqslant \theta_j'$, and $(k)w_\pi = k$ for $k \geqslant n - r + 1$. By construction we get the following properties of w_π:

- $\mathbf{i}_{\pi,\theta} = \mathbf{i}_{\theta' w_\pi}$.
- For each $1 \leqslant j \leqslant s$, we have $\mathfrak{S}_{\{\sum_{i<\pi(j)} \theta_i + 1, \ldots, \sum_{i \leqslant \pi(j)} \theta_i\}} = \mathfrak{S}^{w_\pi}_{\{\sum_{i<j} \theta_i' + 1, \ldots, \sum_{i \leqslant j} \theta_i'\}}$. Therefore $\mathfrak{S}_\theta = \mathfrak{S}_{\theta'}^{w_\pi}$.
- $w_\pi \in \mathcal{R}_\theta'$. This can be seen as $\mathfrak{t}^{\theta'} w_\pi$ is row standard.
- Finally due to the previous two points, $w_\pi \in \mathcal{D}_{\theta',\theta}$. This is because $w_\pi \in \mathcal{R}_{\theta'}$ and for all $s_i \in \mathfrak{S}_\theta$, there exists $s_j = w_\pi s_i w_\pi^{-1} \in \mathfrak{S}_{\theta'}$ with:

$$\ell(w_\pi s_i) = \ell(w_\pi s_i w_\pi^{-1} w_\pi) = \ell(s_j w_\pi) = \ell(s_j) + \ell(w_\pi) = 1 + \ell(w_\pi). \tag{5.3}$$

Thus any such bijection π gives an element $w_\pi \in \mathcal{D}_{\theta',\theta}$ with $\mathfrak{S}_\theta = \mathfrak{S}_\lambda^d \cap \mathfrak{S}_{\theta'}^{w_\pi}$. Finally if in addition, we choose a bijection π such that $\widehat{(j)\pi^{-1}} = \widehat{i_j}$ for all $j \in \Omega$ (such a bijection always exists by the definition of θ'), then the runs of $\mathbf{i}_\lambda d$ and $\mathbf{i}_{\theta'} w_\pi$ have the same parity. Thus $\mathfrak{S}^d_{\lambda(1)} \cap \mathfrak{S}^{w_\pi}_{\theta'(0)} = \mathfrak{S}^d_{\lambda(0)} \cap \mathfrak{S}^{w_\pi}_{\theta'(1)} = \{1\}$, meaning that:

$$\mathfrak{S}_\theta = \left(\mathfrak{S}^d_{\lambda(0)} \cap \mathfrak{S}^{w_\pi}_{\theta'(0)}\right) \times \left(\mathfrak{S}^d_{\lambda(1)} \cap \mathfrak{S}^{w_\pi}_{\theta'(1)}\right).$$

We can now apply [14, Lemma 4.2] to conclude that:

$$e_{\lambda d, \theta' w} \in \operatorname{End}_{\mathscr{H}_\theta}\left(V(m'_0|m'_1)^{\otimes n}\right).$$

To show that $\operatorname{Tr}^n_\theta(e_{\lambda d,\theta' w}) \in I'_{\underline{k}}$, note that $\operatorname{Tr}^n_\theta(e_{\lambda d,\theta' w})$ is a map from N^λ to $N^{\theta'}$. As such there exists $a_x \in F$ for $x \in \mathcal{D}^\circ_{\theta',\lambda}$ with:

$$\operatorname{Tr}^n_\theta(e_{\lambda d,\theta' w}) = \sum_{x \in \mathcal{D}^\circ_{\theta',\lambda}} a_x \varphi^x_{\lambda\theta'}.$$

As $\mathfrak{S}^x_{\theta'} \cap \mathfrak{S}_\lambda \subseteq_{\mathfrak{S}_n} \mathfrak{S}_{\theta'}$, and as $\mathfrak{P}_{\underline{k}} =_{\mathfrak{S}_n} \mathfrak{S}_{\theta'}$ we have that each of the basis elements $\varphi^x_{\lambda\theta'}$, and thus $\operatorname{Tr}^n_\theta(e_{\lambda d,\theta' w})$ lies in $I'_{\underline{k}}$. □

We illustrate the construction from this lemma in the following example when $e = 3$ and $p = 2$.

Example 5.23. Let $\lambda = (10, 4|15, 6, 1) \in \Lambda(2|3, 36)$ and $\underline{k} = (3, 2, 1, 0)$ with corresponding partition $\theta = (12, 6^2, 3^3, 1^3) \vdash 36$. Let:

$$\begin{aligned} d = (&1, 19, 5, 23, 9, 30, 13, 33, 16, 2, 20, 6, 24, 10, 35,\\ &18, 4, 22, 8, 29, 27, 25, 11, 31, 14, 36, 34, 17, 3, 21, 7, 28, 26, 12, 32, 15). \end{aligned}$$

Then $d \in \mathcal{R}_\lambda$, as we can see below $\mathfrak{t}^\lambda d$ is row standard. Similarly the tableau shows that

$\mathfrak{S}_\theta \subseteq \mathfrak{S}_\lambda^d$. Combining these gives $d \in \mathcal{D}_{\lambda,\theta}$ (via the argument in (5.3)).

$\mathfrak{t}^\lambda d =$

19	20	21	22	23	24	28	29	30	35					
31	32	33	36											
1	2	3	4	5	6	7	8	9	10	11	12	25	26	27
13	14	15	16	17	18									
34														

Now we can look at the corresponding tuples:

$$\mathbf{i}_\lambda = (\underbrace{1,\ldots,1}_{10},\underbrace{2,\ldots,2}_{4},\underbrace{3,\ldots,3}_{15},\underbrace{4,\ldots,4}_{7},5),$$

$$\mathbf{i}_\lambda d = (\underbrace{3,\ldots,3}_{12},\underbrace{4,\ldots,4}_{6},\underbrace{1,\ldots,1}_{6},\underbrace{3,3,3},\underbrace{1,1,1},\underbrace{2,2,2},5,1,2).$$

This gives us $\theta'(0) = (6,3,3)$ and $\theta'(1) = (12,6,3,1,1,1)$ and so:

$$\theta' = (6,3,3|12,6,3,1,1,1) \in \Lambda(3|6,36).$$

Continuing, we have $a = b = 3$, so $\Omega = \{1,2,3,4,5,6\}$. To have a bijection $\pi : \{1,\ldots,6\} \to \{1,\ldots,6\}$ with the property that if $(j)\pi = k$, then $\theta'_j = \theta_k$, then π must have $(4)\pi = 1$, $(\{1,5\})\pi = \{2,3\}$ and $(\{2,3,6\})\pi = \{4,5,6\}$. To have $\widehat{(j)\pi^{-1}} = \widehat{i_j}$ for all j, we must have $(1)\pi = 3$, $(5)\pi = 2$, $(6)\pi = 4$ and $(\{2,3\})\pi = \{5,6\}$. We choose $(2)\pi = 6$ and $(3)\pi = 5$. With this we have the tuples:

$$\mathbf{i}_{\theta'} = (\underbrace{1,\ldots,1}_{6},\underbrace{2,2,2},\underbrace{3,3,3},\underbrace{4,\ldots,4}_{12},\underbrace{5,\ldots,5}_{6},\underbrace{6,6,6},7,8,9),$$

$$\mathbf{i}_{\pi,\theta} = (\underbrace{4,\ldots,4}_{12},\underbrace{5,\ldots,5}_{6},\underbrace{1,\ldots,1}_{6},\underbrace{4,4,4},\underbrace{3,3,3},\underbrace{2,2,2},7,8,9).$$

Clearly these have the same content, and note that corresponding runs in $\mathbf{i}_\lambda d$ and $\mathbf{i}_{\pi,\theta}$

have the same length and parity. w_π is given by:

$$w_\pi:\ \begin{array}{lllllllllllllllllll}
1 & \mapsto & 19, & 7 & \mapsto & 31, & 10 & \mapsto & 28, & 13 & \mapsto & 1, & 25 & \mapsto & 13, & 31 & \mapsto & 25,\\
2 & \mapsto & 20, & 8 & \mapsto & 32, & 11 & \mapsto & 29, & 14 & \mapsto & 2, & 26 & \mapsto & 14, & 32 & \mapsto & 26,\\
3 & \mapsto & 21, & 9 & \mapsto & 33, & 12 & \mapsto & 30, & 15 & \mapsto & 3, & 27 & \mapsto & 15, & 33 & \mapsto & 27,\\
4 & \mapsto & 22, &&&&&&& 16 & \mapsto & 4, & 28 & \mapsto & 16, &&&\\
5 & \mapsto & 23, &&&&&&& 17 & \mapsto & 5, & 29 & \mapsto & 17, &&&\\
6 & \mapsto & 24, &&&&&&& 18 & \mapsto & 6, & 30 & \mapsto & 18, &&&\\
&&&&&&&&& 19 & \mapsto & 7, &&&&&&\\
&&&&&&&&& 20 & \mapsto & 8, &&&&&&\\
&&&&&&&&& 21 & \mapsto & 9, &&&&&&\\
&&&&&&&&& 22 & \mapsto & 10, &&&&&&\\
&&&&&&&&& 23 & \mapsto & 11, &&&&&&\\
&&&&&&&&& 24 & \mapsto & 12, &&&&&&
\end{array}$$

and fixes 34, 35, and 36. With this we can see that all the claims in the bullet points are satisfied. Finally computing conjugates:

$$\begin{aligned}
\mathfrak{S}_{\theta'(0)}^{w_\pi} &= \mathfrak{S}_{\{19,\ldots,24\}} \times \mathfrak{S}_{\{31,32,33\}} \times \mathfrak{S}_{\{28,29,30\}},\\
\mathfrak{S}_{\lambda(0)}^{d} &= \mathfrak{S}_{\{19,\ldots,24,28,\ldots,30,35\}} \times \mathfrak{S}_{\{31,32,33,36\}},\\
\mathfrak{S}_{\theta'(1)}^{w_\pi} &= \mathfrak{S}_{\{1,\ldots,12\}} \times \mathfrak{S}_{\{13,\ldots,18\}} \times \mathfrak{S}_{\{25,26,27\}},\\
\mathfrak{S}_{\lambda(1)}^{d} &= \mathfrak{S}_{\{1,\ldots,12,25,26,27\}} \times \mathfrak{S}_{13,\ldots,18\}}.
\end{aligned}$$

Thus we can see that if $i \neq j$, then $\mathfrak{S}_{\lambda(i)}^{d} \cap \mathfrak{S}_{\theta'(j)}^{w_\pi}$ is trivial, and we proceed with [14, Lemma 4.2] to conclude that $e_{\lambda d,\theta' w_\pi} \in \mathrm{End}_{\mathscr{H}_\theta}(V(3|6)^{\otimes 36})$.

The proof of the next corollary follows in the same way as [14, Corollary 7.3], just substituting the previous lemma for [14, Lemma 7.2]. Therefore we omit the proof.

Corollary 5.24. *Let $\underline{k} \in \mathcal{K}$ and $\theta \vdash n$ such that $\mathfrak{P}_{\underline{k}} =_{\mathfrak{S}_n} \mathfrak{S}_\theta$. For $\lambda, \mu \in \Lambda(m_0|m_1, n)$,*

$d \in \mathcal{D}_{\lambda,\theta}$ and $d' \in \mathcal{D}_{\mu,\theta}$, assume that $\mathfrak{S}_\theta \subseteq \left(\mathfrak{S}^d_{\lambda(0)} \cap \mathfrak{S}^{d'}_{\mu(0)}\right) \times \left(\mathfrak{S}^d_{\lambda(1)} \cap \mathfrak{S}^{d'}_{\mu(1)}\right)$. Then there exists $\theta' \in \Lambda(m'_0|m'_1, n)$ for some $m'_0 \geqslant m_0$, $m'_1 \geqslant m_1$ and $w \in \mathcal{D}_{\theta',\theta}$ such that $\mathfrak{S}_\theta = \mathfrak{S}^w_{\theta'}$ and when viewed as an element of $\mathcal{S}_q(m'_0|m'_1, n)$:

$$\mathrm{Tr}^n_\theta(e_{\mu d',\lambda d}) = \mathrm{Tr}^n_\theta(e_{\mu d',\theta' w})\,\mathrm{Tr}^n_\theta(e_{\theta' w,\lambda d}).$$

We can now prove Theorem 5.20. Again, this generalises [14, Theorem 7.4], and although the proof follows the same outline, we include it here in full to highlight the necessary modifications for it to work in any characteristic.

Proof of Theorem 5.20. We first show that $\mathrm{Tr}^n_\theta(\mathrm{End}_{\mathscr{H}_\theta}(V(m_0|m_1)^{\otimes n}))$ is contained in $I_{\underline{k}}$.

Let $\lambda, \mu \in \Lambda(m_0|m_1, n)$, $d \in \mathcal{D}_{\lambda,\theta}$ and $d' \in \mathcal{D}_{\mu,\theta}$. Define $\alpha = \lambda d \cap \theta$, $\beta = \mu d' \cap \theta$, and consider $\alpha y \cap \beta$ for some $y \in \mathcal{D}^\circ_{\alpha,\beta}$. Note that $\mathcal{D}^\circ_{\alpha,\beta}$ is well-defined by [14, (2.2.2)] as $\alpha(i)$ and $\beta(i)$ are defined from λ and μ. Thus $\mathfrak{S}_{\alpha y \cap \beta} = \mathfrak{S}^{dy}_\lambda \cap \mathfrak{S}^y_\theta \cap \mathfrak{S}^{d'}_\mu \cap \mathfrak{S}_\theta$. Then by [14, Theorem 5.3] we have:

$$\{\mathrm{Tr}^\theta_{\alpha y \cap \beta}(e_{\mu d',\lambda dy}) : \lambda, \mu \in \Lambda(m_0|m_1, n), d \in \mathcal{D}_{\lambda,\theta}, d' \in \mathcal{D}_{\mu,\theta}, y \in \mathcal{D}^\circ_{\alpha,\beta} \cap \mathfrak{S}_\theta\}, \tag{5.4}$$

is a basis of $\mathrm{End}_\theta(V(m_0|m_1)^{\otimes n})$. Thus it suffices to show for a choice of these parameters that:

$$\mathrm{Tr}^n_{\alpha y \cap \beta}(e_{\mu d',\lambda dy}) \in I_{\underline{k}}.$$

We proceed by induction in $\mathcal{K}$. For the base case suppose $\underline{k} = \underline{0}$, so $\mathfrak{S}_\theta$ is the trivial group. Then we must have $y = 1$ and $\mathfrak{S}_{\alpha y \cap \beta} = \mathfrak{S}_1$, thus:

$$\mathrm{Tr}^n_{\alpha y \cap \beta}(e_{\mu d',\lambda dy}) = \mathrm{Tr}^n_1(e_{\mu d',\lambda d}) \in I_{\underline{0}},$$

by Lemma 5.21.

Now assume that $\underline{k} \neq \underline{0}$ and by induction, the statement is true for all $\underline{l} \precneqq \underline{k}$.

By definition, $\mathfrak{S}_{\alpha y \cap \beta} \subseteq \mathfrak{S}_\theta$. Let $\mathfrak{S}_\tau = \mathfrak{P}(\mathfrak{S}_{\alpha y \cap \beta})$ and suppose that $\mathfrak{S}_\tau \subsetneq \mathfrak{S}_\theta$. Then there exists $\underline{l}$ with $\mathfrak{P}_{\underline{l}} =_{\mathfrak{S}_n} \mathfrak{S}_\tau$, and $\underline{l} \precneqq \underline{k}$. By Corollary 4.2, there is some $\psi \in \operatorname{End}_{\mathscr{H}_\tau}(V(m_0|m_1)^{\otimes n})$ with:

$$\operatorname{Tr}^n_{\alpha y \cap \beta}(e_{\mu d', \lambda d y}) = \operatorname{Tr}^n_\tau(\psi) \in I_{\underline{l}} \subseteq I_{\underline{k}},$$

by the induction hypothesis and the definition of our poset $\mathcal{I}$.

We now consider the case when $\mathfrak{S}_\tau = \mathfrak{S}_\theta$. For this to occur we must have $\mathfrak{S}_\theta = \mathfrak{S}^{dy}_\lambda \cap \mathfrak{S}^y_\theta \cap \mathfrak{S}^{d'}_\mu \cap \mathfrak{S}_\theta$, meaning that $\mathfrak{S}_\alpha = \mathfrak{S}_\beta = \mathfrak{S}_\theta$ and $y = 1$. Therefore by (5.4), we have $1 \in \mathcal{D}^\circ_{\alpha,\beta}$. Hence:

$$\begin{aligned}
\mathfrak{S}_\theta &= (\mathfrak{S}^1_{\alpha(0)} \cap \mathfrak{S}_{\beta(0)}) \times (\mathfrak{S}^1_{\alpha(1)} \cap \mathfrak{S}_{\beta(1)}) \\
&= \left((\mathfrak{S}^d_{\lambda(0)} \cap \mathfrak{S}_\theta) \cap (\mathfrak{S}^{d'}_{\mu(0)} \cap \mathfrak{S}_\theta)\right) \times \left((\mathfrak{S}^d_{\lambda(1)} \cap \mathfrak{S}_\theta) \cap (\mathfrak{S}^{d'}_{\mu(1)} \cap \mathfrak{S}_\theta)\right), \\
&= \left[(\mathfrak{S}^d_{\lambda(0)} \cap \mathfrak{S}^{d'}_{\mu(0)}) \times (\mathfrak{S}^d_{\lambda(1)} \cap \mathfrak{S}^{d'}_{\mu(1)})\right] \cap \mathfrak{S}_\theta.
\end{aligned}$$

Therefore we have:

$$\mathfrak{S}_\theta \subseteq (\mathfrak{S}^d_{\lambda(0)} \cap \mathfrak{S}^{d'}_{\mu(0)}) \times (\mathfrak{S}^d_{\lambda(1)} \cap \mathfrak{S}^{d'}_{\mu(1)}),$$

and we can apply Corollary 5.24 to get $m'_0 \geqslant m_0$, $m'_1 \geqslant m_1$, $\theta' \in \Lambda(m'_0|m'_1, n)$, and $w \in \mathcal{D}_{\theta',\theta}$ such that when viewed as elements of $\mathcal{S}_q(m'_0|m'_1, n)$:

$$\operatorname{Tr}^n_\theta(e_{\mu d', \lambda d}) = \operatorname{Tr}^n_\theta(e_{\mu d', \theta' w}) \operatorname{Tr}^n_\theta(e_{\theta' w, \lambda d}).$$

By the final statement in Lemma 5.22, we get that $\operatorname{Tr}^n_\theta(e_{\mu d', \theta' w}) \in I'_{\underline{k}} \subseteq \mathcal{S}_q(m'_0|m'_1, n)$. Additionally, as $I'_{\underline{k}}$ is a two-sided ideal, $\operatorname{Tr}^n_\theta(e_{\mu d', \lambda d}) \in I'_{\underline{k}}$. Finally, as $\lambda, \mu \in \Lambda(m_0|m_1, n)$, we have:

$$\operatorname{Tr}^n_\theta(e_{\mu d', \lambda d}) = \eta \operatorname{Tr}^n_\theta(e_{\mu d', \lambda d}) \eta \in \eta I'_{\underline{k}} \eta = I_{\underline{k}},$$

completing the proof of this inclusion.

We now prove that $I_{\underline{k}} \subseteq \mathrm{Tr}_{\theta}^{n}(\mathrm{End}_{\mathscr{H}_\theta}(V(m_0|m_1)^{\otimes n}))$. Let $\lambda, \mu \in \Lambda(m_0|m_1, n)$ and $d \in \mathcal{D}^{\circ}_{\lambda,\mu}$ with $\varphi^{d}_{\mu\lambda} \in I_{\underline{k}}$, so for $\mathfrak{S}_\tau = \mathfrak{P}(\mathfrak{S}_{\lambda d\cap\mu})$, we have $\mathfrak{S}_\tau \subseteq_{\mathfrak{S}_n} \mathfrak{S}_\theta$. As before, there exists some $c \in F$ with $\varphi^{d}_{\mu\lambda} = c\,\mathrm{Tr}^{n}_{\lambda d\cap\mu}(e_{\mu,\lambda d})$, so it suffices to show that:

$$\mathrm{Tr}^{n}_{\lambda d\cap\mu}(e_{\mu,\lambda d}) \in \mathrm{Tr}^{n}_{\theta}(\mathrm{End}_{\mathscr{H}_\theta}\big(V(m_0|m_1)^{\otimes n}\big)).$$

As $\mathrm{Tr}^{n}_{\lambda d\cap\mu}(e_{\mu,\lambda d})$ is relatively $\mathfrak{S}_{\lambda d\cap\mu}$-projective as a $\mathscr{H}_n$-module homomorphism, if $\mathfrak{S}_{\lambda d\cap\mu} \subseteq \mathfrak{S}_\theta$ then we are done using transitivity of the relative trace. Thus we now need to consider the case when we only have a conjugate of $\mathfrak{S}_{\lambda d\cap\mu}$ is contained in $\mathfrak{S}_\theta$.

Let $z \in \mathcal{D}_{\tau,\theta}$ with $\mathfrak{S}_\gamma = \mathfrak{S}^{z}_{\tau} \subseteq \mathfrak{S}_\theta$. As in the second half of the proof of [14, Theorem 7.4], let

$$M = \mathrm{Hom}_F(v_\mu\mathscr{H}_n, Fv_{\lambda d} \otimes_{\mathscr{H}_\tau} \mathscr{H}_n),$$

a $(\mathscr{H}_n, \mathscr{H}_n)$-bimodule with a $(\mathscr{H}_n, \mathscr{H}_\tau)$-bisubmodule $N = \mathrm{Hom}_F(v_\mu\mathscr{H}_n, Fv_{\lambda d})$ and a $(\mathscr{H}_n, \mathscr{H}_\gamma)$-bisubmodule $N' = \mathrm{Hom}_F(v_\mu\mathscr{H}_n, Fv_{\lambda d} \otimes_{\mathscr{H}_\tau} T_z)$. Then

$$M \cong N \otimes_{\mathscr{H}_\tau} \mathscr{H}_n \cong N' \otimes_{\mathscr{H}_\gamma} \mathscr{H}_n,$$

as $(\mathscr{H}_n, \mathscr{H}_n)$-bimodules. Applying [14, Lemma 3.5] gives us that:

$$\mathrm{Tr}^{n}_{\tau}(Z_N(\mathscr{H}_\tau)) = Z_M(\mathscr{H}_n) = \mathrm{Tr}^{n}_{\gamma}(Z_{N'}(\mathscr{H}_\gamma)),$$

that is, as subspaces of $\mathrm{End}_F\,(V(m_0|m_1)^{\otimes n})$:

$$\mathrm{Tr}^{n}_{\tau}(\mathrm{Hom}_{\mathscr{H}_\tau}(v_\mu\mathscr{H}_n, Fv_{\lambda d})) = \mathrm{Tr}^{n}_{\gamma}(\mathrm{Hom}_{\mathscr{H}_\gamma}(v_\mu\mathscr{H}_n, Fv_{\lambda d} \otimes_{\mathscr{H}_\tau} T_z)).$$

Thus for all $h \in \mathrm{Hom}_{\mathscr{H}_\tau}(v_\mu \mathscr{H}_n, F v_{\lambda d})$, there is some homomorphism h' with:

$$h' \in \mathrm{Hom}_{\mathscr{H}_\gamma}(v_\mu \mathscr{H}_n, v_{\lambda d} \otimes_{\mathscr{H}_\tau} T_z) \subseteq \mathrm{End}_{\mathscr{H}_\gamma}\left(V(m_0|m_1)^{\otimes n}\right),$$

such that $\mathrm{Tr}_\tau^n(h) = \mathrm{Tr}_\gamma^n(h')$. We can now finish using Corollary 4.2, as this guarantees some $h \in \mathrm{Hom}_{\mathscr{H}_\tau}(v_\mu \mathscr{H}_n, F v_{\lambda d})$ with:

$$\mathrm{Tr}_{\lambda d \cap \mu}^n(e_{\mu,\lambda d}) = \mathrm{Tr}_\tau^n(h) = \mathrm{Tr}_\gamma^n(h') = \mathrm{Tr}_\theta^n(\mathrm{Tr}_\gamma^\theta(h')).$$

Finally, as $h' \in \mathrm{End}_{\mathscr{H}_\gamma}(V(m_0|m_1)^{\otimes n})$, then $\mathrm{Tr}_\gamma^\theta(h') \in \mathrm{End}_{\mathscr{H}_\theta}(V(m_0|m_1)^{\otimes n})$, and thus $\mathrm{Tr}_{\lambda d \cap \mu}^n(e_{\mu,\lambda d}) = \mathrm{Tr}_\theta^n(\mathrm{Tr}_\gamma^\theta(h'))$ lies in $\mathrm{Tr}_\theta^n(\mathrm{End}_{\mathscr{H}_\theta}(V(m_0|m_1)^{\otimes n}))$ as required. □

5.3.3 Relating defect groups and vertices

We can now give our main theorem about vertices of signed Young modules, generalising [14, Theorem 10.2] to fields of any characteristic.

Theorem 5.25 (Vertices of signed Young Modules). *Let F be a field of characteristic $p \geqslant 0$, and f a primitive idempotent in $\mathcal{S}_q(m_0|m_1, n)$. Then $D(f)$ is the vertex of the indecomposable $\mathscr{H}_n$-module $V(m_0|m_1)^{\otimes n} f$.*

Before proving this theorem, we describe how it can be used in practice to compute vertices. Given a primitive idempotent f, suppose we can write it in terms of the basis given in Corollary 5.7 as

$$f = \sum_{\substack{\lambda,\mu \in \Lambda(m_0|m_1,n) \\ d \in \mathcal{D}^\circ_{\lambda,\mu}}} a^d_{\lambda,\mu} \varphi^d_{\mu\lambda}.$$

For each λ, μ, d with $a^d_{\lambda,\mu} \neq 0$, we then take its corresponding tuple $\underline{k}_{\lambda d \cap \mu} \in \mathcal{K}$ and compute $\underline{b} = \bigvee_{a^d_{\lambda,\mu} \neq 0} \underline{k}_{\lambda d \cap \mu}$. Then $\mathfrak{P}_{\underline{b}}$ is the vertex of $V(m_0|m_1)^{\otimes n} f$. Thus the problem of finding the vertex of $V(m_0|m_1)^{\otimes n} f$ is reduced to that of writing f in the given basis of

$\mathcal{S}_q(m_0|m_1,n)$.

Note that the proof of Theorem 5.25 is largely similar to that of [14, Theorem 10.2] and we include it here for completeness.

Proof of Theorem 5.25. Let $D(f) = \mathfrak{P}_{\underline{k}} =_{\mathfrak{S}_n} \mathfrak{S}_\theta$ for $\underline{k} \in \mathcal{K}$, $\theta \vdash n$ and $\mathfrak{S}_\gamma$ a vertex of $V(m_0|m_1)^{\otimes n}f$. Let $\underline{l} \in \mathcal{K}$ such that $\mathfrak{P}_{\underline{l}} =_{\mathfrak{S}_n} \mathfrak{S}_\gamma$. Then by relative projectivity, we have:

$$\mathrm{Tr}_\gamma^n(\mathrm{End}_{\mathscr{H}_\gamma}\left(V(m_0|m_1)^{\otimes n}f\right)) = \mathrm{End}_{\mathscr{H}_n}\left(V(m_0|m_1)^{\otimes n}f\right).$$

Then:

$$\begin{aligned} f\mathcal{S}_q(m_0|m_1,n)f &= f\,\mathrm{End}_{\mathscr{H}_n}\left(V(m_0|m_1,n)^{\otimes n}\right)f = \mathrm{End}_{\mathscr{H}_n}\left(V(m_0|m_1)^{\otimes n}f\right), \\ &= \mathrm{Tr}_\gamma^n(\mathrm{End}_{\mathscr{H}_\gamma}\left(V(m_0|m_1)^{\otimes n}f\right)) = f\,\mathrm{Tr}_\gamma^n(\mathrm{End}_{\mathscr{H}_\gamma}\left(V(m_0|m_1)^{\otimes n}\right))f, \\ &= fI_{\underline{l}}f, \end{aligned}$$

by Theorem 5.20. Since $f \in f\mathcal{S}_q(m_0|m_1,n)f$, the above calculation shows that $f \in fI_{\underline{l}}f$ therefore $f \in I_{\underline{l}}$ as $I_{\underline{l}}$ is a two-sided ideal ($fI_{\underline{l}}f = I_{\underline{l}}$). Thus we must have $\underline{k} \preceq \underline{l}$, and $D(f) \subseteq_{\mathfrak{S}_n} \mathfrak{S}_\gamma$.

Now by the definition of $D(f)$, we have $f \in I_{\underline{k}}$, so we have:

$$f \in fI_{\underline{k}}f = f\,\mathrm{Tr}_\theta^n(\mathrm{End}_{\mathscr{H}_\theta}\left(V(m_0|m_1,n)^{\otimes n}\right))f = \mathrm{Tr}_\theta^n(\mathrm{End}_{\mathscr{H}_\theta}\left(V(m_0|m_1)^{\otimes n}f\right)).$$

Therefore there exists some $h \in \mathrm{End}_{\mathscr{H}_\theta}(V(m_0|m_1)^{\otimes n}f)$ with $\mathrm{Tr}_\theta^n(h) = f$, the identity map on $V(m_0|m_1)^{\otimes n}f$. Thus by our second version of Higman's criterion, $V(m_0|m_1)^{\otimes n}f$ is $D(f)$-projective, and by the definition of vertex a conjugate of $\mathfrak{S}_\gamma$ is contained in $D(f)$.

This means $D(f)$ is a conjugate of $\mathfrak{S}_\gamma$, and thus $D(f)$ is a vertex of $V(m_0|m_1)^{\otimes n}f$. □

The following corollary gives a quick sanity check on the previous theorem.

Corollary 5.26. *Suppose for $\lambda \in \Lambda(m_0|m_1, n)$, the signed permutation module N^λ is indecomposable. Then its vertex is $\mathfrak{P}(\mathfrak{S}_\lambda)$.*

Proof. Using the notation of the q-Schur superalgebra, then $N^\lambda \cong V(m_0|m_1)^{\otimes n}\varphi^1_{\lambda\lambda}$. By Theorem 5.6, $\varphi^1_{\lambda\lambda}$ is an idempotent, and the fact that N^λ is indecomposable, means that $\varphi^1_{\lambda\lambda}$ must be primitive. Thus the vertex of N^λ is $D(\varphi^1_{\lambda,\lambda})$. Computing this:

$$D(\varphi_{\lambda\lambda^1}) = \mathfrak{P}(\mathfrak{S}^1_\lambda \cap \mathfrak{S}_\lambda) = \mathfrak{P}(\mathfrak{S}_\lambda).$$ □

This gives an alternative proof (given Theorem 3.46) that the vertex of the trivial module for $\mathscr{H}_n$ is $\mathfrak{P}(\mathfrak{S}_n)$, as for $\lambda = (n, 0, \ldots, 0)$ we have $S^{(n)} = N^\lambda$ is indecomposable.

CHAPTER 6

THE HECKE ALGEBRA OF TYPE B

Having explored the notion of vertices for Iwahori–Hecke algebras related to $\mathfrak{S}_n$, i.e. the Weyl group of type A_{n-1}, a potential further avenue is to explore how many of these results are applicable for Hecke algebras related to other Weyl groups. In this chapter we focus on $\mathfrak{W}_n$, the Weyl group of type B_n. We will think of this group as the subgroup of $\mathfrak{S}_{\{\pm 1,\ldots,\pm n\}}$ which respects signs, i.e. for $w \in \mathfrak{W}_n$, if $(i)w = j$, then $(-i)w = -j$, using the shorthand $[i_1, \ldots, i_s] := (i_1, \ldots, i_s)(-i_1, \ldots, -i_s)$. This is generated by elements $t_0 = (-1, 1)$ and $t_i = [i, i+1]$ for $i = 1, \ldots, n-1$. Note that we will think of $\mathfrak{W}_1$ as the group generated by t_0, and $\mathfrak{W}_0$ as the trivial group.

Let $Q, q \in F^\times$ and let $\mathcal{H}_n = \mathcal{H}_n(Q, q)$ be the F-algebra generated by $T_0, T_1, \ldots, T_{n-1}$ with relations:

$$(T_0 + 1)(T_0 - Q) = 0,$$

$$(T_i + 1)(T_i - q) = 0 \text{ for } 1 \leqslant i < n,$$

$$T_i T_j = T_j T_i \text{ for } |i - j| > 1,$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } 1 \leqslant i < n-1.$$

We call $\mathcal{H}_n$ the **Iwahori–Hecke algebra of type** B_n (which we will refer to just as the Hecke algebra of type B). Let e be the quantum characteristic of q, and say that $\mathcal{H}_n$ is Q**-connected** if there exists some positive integer s with $Q = -q^s$. Throughout, as before, we assume that e is finite, and as the proof of Proposition 1.3 carries over, either $\mathrm{hcf}(e, p) = 1$ and q is a primitive e-th root of unity, or $e = p$ and $q = 1$. Therefore if we are Q-connected, either $0 \leqslant s < e$, or $Q = -1$ respectively. For a full discussion of the representation theory of $\mathcal{H}_n$, we recommend [10].

We describe parabolic subalgebras of $\mathcal{H}_n$ in the following way. A **pointed composition** of n is given by $\lambda = (\lambda_0; \lambda_1, \ldots, \lambda_r)$ where $\sum_{i=0}^{r} \lambda_i = n$, $\lambda_0 \geqslant 0$ and $\lambda_i > 0$ for $1 \leqslant i \leqslant r$. We write $\lambda \Vdash n$. Given a pointed composition λ, we can associate to it a parabolic subgroup $\mathfrak{W}_\lambda$ of $\mathfrak{W}_n$ given by

$$\begin{aligned}\mathfrak{W}\lambda &= \mathfrak{W}_{\{1,\ldots,\lambda_0\}} \times \mathfrak{S}_{\{\lambda_0+1,\ldots,\lambda_0+\lambda_2\}} \times \cdots \times \mathfrak{S}_{\{\left(\sum_{i=0}^{r-1} \lambda_i\right)+1,\ldots,\sum_{i=0}^{r} \lambda_r\}}, \\ &\cong \mathfrak{W}_{\lambda_0} \times \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}.\end{aligned}$$

As in Theorem 1.1, $\mathcal{H}_n$ has a basis indexed by elements of $\mathfrak{W}_n$ (see for example [10, §3]), and we can define parabolic subalgebras:

$$\mathcal{H}_\lambda = \langle T_w : w \in \mathfrak{W}_\lambda \rangle \cong \mathcal{H}_{\lambda_0} \otimes \mathscr{H}_{\lambda_1} \otimes \cdots \otimes \mathscr{H}_{\lambda_r}.$$

Note that here we have $\mathcal{H}_{(0;n)} \cong \mathscr{H}_n$, so we can invoke our knowledge of the type A case.

As both $\mathscr{H}_n$ and $\mathcal{H}_n$ have structure inherited from the relevant Weyl groups, a lot of notions carry over happily to $\mathscr{H}_n$, most notably from the methods of [25] we get a Mackey formula, Higman's criterion, and a definition of vertices for both indecomposable $\mathcal{H}_n$-modules, and indecomposable $(\mathcal{H}_n, \mathcal{H}_n)$-bimodules. Unfortunately, at this point, our methods become less applicable, in particular the version of the Green correspondence that follows is not particularly useful.

To be precise, for a Green correspondence, we would need pointed compositions λ and μ, leading to a chain of subgroups as in (2.5):

$$\mathfrak{W}_\lambda \subseteq N_{\mathfrak{W}_n}(\mathfrak{W}_\lambda) \subseteq \mathfrak{W}_\mu \subseteq \mathfrak{W}_n.$$

In this necessary situation, the Green correspondence would give a bijection between $\mathcal{H}_\mu$-modules and $\mathcal{H}_n$-modules with a particular vertex contained in $\mathfrak{W}_\lambda$. We can consider two cases; either t_{n-1} is not in $\mathfrak{W}_\lambda$, or it is inside a copy of $\mathfrak{S}_m$ for some m (assuming that $\mathfrak{W}_\lambda$ is a proper subgroup of $\mathfrak{W}_n$). In the first case, $(n, -n)$ lies in the normaliser, and thus to have a parabolic containing the normaliser, we need all of $\mathfrak{W}_n$. In the second case, consider $\pi_{n,m} = (n, -n)(n-1, -(n-1)) \dots (n-m, -(n-m))$. This is self-inverse, and commutes with all parts of $\mathfrak{W}_\lambda$ that are not the final copy of $\mathfrak{S}_m$. For $i = n-m+1, \dots, n-1$:

$$\begin{aligned} t_i^{\pi_{n,m}} &= \pi_{n,m}^{-1}[i, i+1]\pi_{n,m}, \\ &= (i, -i)((i+1), -(i+1))(i, i+1)(-i, -(i+1))(i, -i)((i+1), -(i+1)), \\ &= (i, i+1)(-i, -(i+1)) = t_i, \end{aligned}$$

as all the other transpositions involved commute and cancel out. Thus we have showed that $\pi_{n,m} \in N_{\mathfrak{W}_n}(\mathfrak{W}_\lambda)$, and again we require all of $t_0, \dots, t_{n-1}$ to create this term. So in all cases, we can only ever apply the Green correspondence if $\mathfrak{W}_\mu = \mathfrak{W}_n$, in which case the result becomes trivial.

Thus the parabolic subgroups of $\mathfrak{W}_n$ are not fine enough to give us results akin to the type A case. A potentially better system of subgroups is defined as follows.

A **pseudo-composition** of n is $\lambda = (\lambda_1, \dots, \lambda_s)$ where $\lambda_i \in \mathbb{Z} - \{0\}$, and $\sum_{i=1}^{s} |\lambda_i| = n$. This defines a **pseudo-parabolic** subgroup in the following way:

$$\mathfrak{A}_\lambda \cong \mathfrak{A}_{\lambda_1} \times \cdots \times \mathfrak{A}_{\lambda_s},$$

where $\mathfrak{A}_{\lambda_i} = \mathfrak{S}_{\lambda_i}$ if $\lambda_i > 0$ and $\mathfrak{A}_{\lambda_i} = \mathfrak{W}_{-\lambda_i}$ if $\lambda_i < 0$. Here the $\mathfrak{S}_{\lambda_i}$ parts are generated by the usual transpositions, and the $\mathfrak{W}_{-\lambda_i}$ parts are generated by $t_{k+1}, \ldots, t_{k+|\lambda_i|-1}$ for some k as well as the element $t_k t_{k-1} \ldots t_1 t_0 t_1 \ldots t_{k-1} t_k$, which corresponds to $(-k, k)$.

Despite behaving better when taking normalisers, these pseudo-parabolics are no more useful than regular parabolics for the Green correspondence. In fact they are arguably worse as there are no corresponding pseudo-parabolic subalgebras, as illustrated in the following example.

Example 6.1. Consider $\lambda = (1, -1)$. Then $\mathfrak{A}_\lambda = \{1, s_1 s_0 s_1\}$. We will show that $\langle 1, T_1 T_0 T_1 \rangle$ is not generally a subalgebra of $\mathcal{H}_2$. To see this, consider $(T_1 T_0 T_1)^2$:

$$\begin{aligned}
(T_1T_0T_1)^2 &= T_1T_0T_1^2T_0T_1 \\
&= (q-1)T_1T_0T_1T_0T_1 + qT_1T_0^2T_1, \\
&= (q-1)T_0T_1T_0T_1^2 + q(Q-1)T_1T_0T_1 + qQT_1^2, \\
&= (q-1)^2T_0T_1T_0T_1 + q(q-1)T_0T_1T_0 + q(Q-1)T_1T_0T_1 + qQ(q-1)T_1 + q^2Q.
\end{aligned}$$

As each of these expressions are reduced, we can see that as long as both q and Q are not equal to 1 (i.e. $\mathcal{H}_2 \cong F\mathfrak{W}_2$), we have terms that will not disappear. Therefore in most cases $\langle 1, T_1 T_0 T_1 \rangle$ is not a subalgebra of $\mathcal{H}_2$.

Despite not having a non-trivial Green correspondence (and thus no Brauer correspondence), there are still things we can say about $\mathcal{H}_n$. In the next section we look at the combinatorics of minimal coset representatives for $\mathfrak{W}_n$, including results on these pseudo-parabolics, before concluding by (as in Chapter 3) computing the vertex of the sign module for $\mathcal{H}_n$.

6.1 Minimal coset representatives for type B

As with the symmetric group, we can allow $\mathfrak{W}_n$ to act on tableaux as well. For a pointed composition $\lambda \Vdash n$, we can define a **signed tableau** of shape λ (note that as λ is pointed, the "0-th" row can be empty) as a tableau of shape λ where each of the boxes is assigned a positive or negative sign, i.e. we have a diagram filled up with entries from $\{\pm 1, \ldots, \pm n\}$ such that if i is in the tableau, $-i$ is not in the tableau. For each λ, there are $2^n n! = |\mathfrak{W}_n|$ such possible signed tableaux, and we have an action on these signed tableaux of $\mathfrak{W}_n$ by permuting the boxes. For example t_0 swaps 1 out for -1 and vice-versa, and t_i swaps i with $i+1$ and $-i$ with $-(i+1)$ (of course only one of i and $-i$ can be in the tableau at any one time).

For a signed tableau $\mathfrak{s}$, we will denote by $d(\mathfrak{s})$ the unique element of $\mathfrak{W}_n$ sending the standard signed tableau $\mathfrak{s}^\lambda$ (which coincides with standard tableau $\mathfrak{t}^\lambda$) to $\mathfrak{s}$. We say that a signed tableau is **row standard** if the entries in each row increase from left to right. This is illustrated in the following example where $\lambda = (3; 2, 1)$ and $d(\mathfrak{s}) = [1, -6][3, 4, -5]$ gives a row standard signed tableau $\mathfrak{s}$ (even though $|-6| > 2$).

$$\mathfrak{s} = \mathfrak{s}^\lambda d(\mathfrak{s}) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \cdot [1, -6][3, 4, -5] = \begin{array}{|c|c|c|}\hline -6 & 2 & 4 \\ \hline -5 & -3 & \multicolumn{1}{c}{} \\ \cline{1-2} -1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Finally, we say that $\mathfrak{s}$ has a **positive 0-th row** if all the elements in the first row (corresponding to λ_0) are positive. We will use these tableaux to give a combinatorial description (for the maximal parabolics an algebraic description is found in [33]) of minimal right coset representatives for parabolic subgroups akin to that of Proposition 1.4.

6.1.1 Right cosets

Similar to the type A case, the row standard tableaux are key to finding the minimal right coset representatives.

Theorem 6.2. *Let $\mathfrak{W}_\lambda$ be a parabolic subgroup of $\mathfrak{W}_n$. Then the following is a complete set of right coset representatives:*

$$\mathcal{R}_\lambda = \{d(\mathfrak{t}) : \mathfrak{t} \text{ is row standard with a positive 0-th row}\}.$$

Furthermore, these representatives have minimal length in their coset and for $w \in \mathcal{R}_\lambda$ and $v \in \mathfrak{W}_\lambda$:

$$\ell(vw) = \ell(v) + \ell(w).$$

To prove this, we first need a generalisation of [26, Lemma 1.2, Proposition 1.3 and Corollary 1.4] which give a way of checking if multiplication by some t_i increases or decreases the length of an element $w \in \mathfrak{W}_n$.

Definition 6.3. Define $\mathcal{T}_1 = \{(-i, i) : 1 \leqslant i \leqslant n\}$, $\mathcal{T}_2 = \{[i, j] : -n \leqslant i < j \leqslant n, |i| < j\}$ and $\mathcal{T} = \mathcal{T}_1 \sqcup \mathcal{T}_2$ (where $\sqcup$ denotes disjoint union). Then $\mathcal{T}$ is the **set of reflections** of $\mathfrak{W}_n$.

Note that if $S = \{t_0, t_1, \dots, t_{n-1}\}$, then $\mathcal{T} = \bigcup_{w\in\mathfrak{W}_n} wSw^{-1}$, but writing it in this way helps to avoid double counting.

Definition 6.4. Let $w \in \mathfrak{W}_n$ and define:

$$N_1(w) = \{(-i, i) \in \mathcal{T}_1 : (i)w < 0\},$$

$$N_2(w) = \{[i, j] \in \mathcal{T}_2 : (i)w > (j)w\},$$

$$N(w) = N_1(w) \sqcup N_2(w).$$

We move on to generalising [26, Lemma 1.2]. This (and the following corollary) corresponds to the more general results in [23, §1.6] about Coxeter groups, but we give a direct proof for $\mathfrak{W}_n$ here.

Lemma 6.5. *Let $v, w \in \mathfrak{W}_n$. Then:*

$$N(vw) = N(v) \ominus vN(w)v^{-1},$$

where $\ominus$ denotes the symmetric difference of two sets.

Proof. As $N(vw)$ is the disjoint union of $N_1(vw)$ and $N_2(vw)$, it suffices to show that the result holds for these sets. As in the proof of [26, Lemma 1.2], we only have to show the result in the case when $v = t_i$, as the general case follows from this by induction on $\ell(v)$ using associativity of $\ominus$, for example when $i \neq j$:

$$\begin{aligned} N(t_it_jw) &= N(t_i) \ominus t_iN(t_jw)t_i^{-1}, \\ &= N(t_i) \ominus t_i(N(t_j) \ominus t_jN(w)t_j^{-1})t_i^{-1}, \\ &= \left(N(t_i) \ominus t_iN(t_j)t_i^{-1}\right), \ominus t_it_jN(w)t_j^{-1}t_i^{-1}, \\ &= N(t_it_j) \ominus (t_it_j)N(w)(t_it_j)^{-1}. \end{aligned}$$

We begin by showing the result holds when $i = 0$. We consider $N_1(t_0w)$ first. Note that $N_1(t_0) = \{t_0\}$, so we need to show that $t_0 \in N_1(t_0w)$ if and only if it is not in $t_0N_1(w)t_0$, and that $(j, -j) \in N_1(t_0w)$ if and only if $(j, -j) \in t_0N_1(w)t_0$. For the first, $t_0 \in N_1(t_0w)$ if and only if $(1)t_0w < 0$ which is true if and only if $(-1)w < 0$, i.e. $(1)w > 0$. Thus by the definition, $t_0 \notin N_1(w)$, and as $t_0t_0t_0 = t_0$, we also have $t_0 \notin t_0N_1(w)t_0$. The same equivalence shows that $t_0 \notin t_0N_1(w)t_0$ implies $t_0 \in N_1(t_0w)$.

Now suppose $j \neq 1$ and $(j, -j) \in N_1(t_0w)$. This is true if and only if $(j)t_0w > 0$, i.e. $(j)w > 0$ (as $(j)t_0 = j$) which by definition is equivalent to $(j, -j) \in N_1(w)$. Again as

$(j)t_0 = j$, this gives us $(j, -j) \in t_0 N_1(w) t_0$, thus proving the result for N_1 when $i = 0$.

Note that $N_2(t_0) = \varnothing$, so we don't need to worry about the symmetric difference here and just prove $N_2(t_0 w) = t_0 N_2(w) t_0$. Suppose we have $[j, k] \in N_2(t_0 w)$ with $j \neq \pm 1$. Then by definition, $(j)t_0 w > (k)t_0 w$ i.e. $(j)w > (k)w$ so $[j, k] \in N_2(w)$. Again this then means that $[j, k] \in t_0 N_2(w) t_0$. Now if $j = \pm 1$, then $(j)t_0 w > (k)t_0 w$, that is $(-j)w > (k)w$ so $[-j, k] \in N_2(w)$, and after conjugating $[j, k] \in t_0 N_2(w) t_0$ as required.

We now consider when $i > 0$. Here we have the opposite: $N_1(t_i) = \varnothing$ now. Showing equality: $(j, -j) \in N_1(t_i w)$ if and only if $(j)t_i w < 0$. Now as $j > 0$, then $(j)t_i$ is positive and $((j)t_i)w < 0$, therefore $((j)t_i, -(j)t_i) \in N_1(w)$, and after conjugating, $(j, -j) \in t_i N_1(w) t_i$.

Our final case is when we are looking at N_2 for $i > 0$. Here again we have $N_2(t_i) = \{t_i\}$, so we first need to consider t_i. If $t_i \in N_2(t_i w)$, then $(i)t_i w > (i+1)t_i w$, that is $(i+1)w > (i)w$, so $t_i \notin N_2(w)$, and after conjugating $t_i \notin t_i N_2(w) t_i$.

Now consider $[j, k] \in N_2(t_i w)$ with $[j, k] \neq t_i$. As t_i preserves signs, and $|j| < k$, then we have $|(j)t_i| < (k)t_i$ unless $j = -i$ and $k = i + 1$. If we are not in this case, then $(j)t_i w > (k)t_i w$ means that $[(j)t_i, (k)t_i] \in N_2(w)$, and conjugating gives the result. If $j = -i$ and $k = i + 1$, then $[-i, i+1] \in N_2(t_i w)$ if and only if $(-i)t_i w > (i+1)t_i w$, so $-(i+1)w > (i)w$. Thus $(-i)w > (i+1)w$, so $[-i, i+1] \in N_2(w)$. Conjugating preserves this reflection giving $[-i, i+1] \in t_i N_2(w) t_i$.

Thus we have shown that for all w, i and for both $k = 1, 2$ that:

$$N_k(t_i w) = N_k(t_i) \ominus t_i N_k(w) t_i,$$

and taking the disjoint union of these gives the result. □

The proof of [26, Proposition 1.3] follows directly, showing us that:

$$N(w) = \{t \in \mathcal{T} : \ell(tw) < \ell(w)\}, \tag{6.1}$$

allowing us to get a $\mathfrak{W}_n$ version of [26, Corollary 1.4].

Corollary 6.6. *Let $w \in \mathfrak{W}_n$. Then for $i \geqslant 1$:*

$$\ell(t_i w) = \begin{cases} \ell(w) + 1 & \text{if } (i)w < (i+1)w, \\ \ell(w) - 1 & \text{if } (i)w > (i+1)w, \end{cases}$$

and

$$\ell(t_0 w) = \begin{cases} \ell(w) + 1 & \text{if } (1)w > 0, \\ \ell(w) - 1 & \text{if } (1)w < 0. \end{cases}$$

Proof. We prove the case for $i = 0$, the other case follows similarly from the definition of $N(w)$. If $(1)w < 0$, then $t_0 \in N(w)$, and thus by our above characterisation of $N(w)$, $\ell(t_0 w) < \ell(w)$, i.e. $\ell(t_0 w) = \ell(w) - 1$. If $(1)w > 0$, then $t_0 \notin N(w)$ and so $\ell(t_0 w) \geqslant \ell(w)$, thus $\ell(t_0 w) = \ell(w) + 1$. □

This gives us a test on whether left-multiplying by a generator of $\mathfrak{W}_n$ increases the length of an element of $\mathfrak{W}_n$ or not. We can now use this to prove Theorem 6.2.

Proof of Theorem 6.2. We first show that each of these representatives lie in a distinct right coset. This can be seen because premultiplying any of these representatives by an element of $\mathfrak{W}_\lambda$ will just swap the positions and signs of the elements in the 0-th row, and swap the positions of the elements in any other row. Thus we have exactly one representative from each right coset.

To show that they are minimal, it is enough to show with Corollary 6.6 that we increase the length by multiplying on the left by any $t_i \in \mathfrak{W}_\lambda$.

Suppose $t_0 \in \mathfrak{W}_\lambda$. Then as $d(\mathfrak{t})$ has a positive 0-th row, we will have $(1)d(\mathfrak{t}) > 0$. Thus by Corollary 6.6 we get $\ell(t_0 w) = \ell(w) + 1 = \ell(w) + \ell(t_0)$.

Now suppose $t_i \in \mathfrak{W}_\lambda$ for $i \geqslant 1$. Then i and $i+1$ must lie in the same row of λ, so again as $\mathfrak{t}$ is row standard, we have that $(i)d(\mathfrak{t}) < (i+1)d(\mathfrak{t})$. Thus again by Corollary 6.6 we get $\ell(t_i w) = \ell(w) + 1 = \ell(w) + \ell(t_i)$.

Therefore we have shown that these elements are minimal in their coset, proving the required result. □

6.1.2 Double cosets

One of the key components in getting a Brauer correspondence for type A was understanding the double cosets given by a maximal parabolic. Here we show that we can similarly categorise the double cosets of a maximal parabolic in type B. We first describe these, before showing that they are in fact a complete set of minimal double coset representatives.

Definition 6.7. Let $\mu = (a; m) \Vdash a + m = n$. Then for $0 \leqslant i \leqslant m$ and for $0 \leqslant j \leqslant \min\{a, m-i\}$ define $w_{i,j}$ to be the element sending $\mathfrak{s}^\mu$ to the following tableau:

- The first row contains in increasing order: $1, \ldots, a-j, a+i+1, \ldots, a+i+j$.
- The second row contains in increasing order: $-(a+i), \ldots, -(a+1), a-j+1, \ldots, a, a+i+j+1, \ldots, a+m$

To illustrate we use $a = 5, m = 6$ in the following example to show the signed tableau afforded by $w_{2,2} = [4,8][5,9][6,-7]$.

1	2	3	8	9	
−7	−6	4	5	10	11

Note that each of these signed tableau are row standard, and the 0-th row contains positive elements, so $w_{i,j} \in \mathcal{R}_\lambda$ by Theorem 6.2. Similarly, as each $w_{i,j}$ is the product of non-intersecting transpositions (we swap $\pm(a-j+k)$ with $\pm(a+i+k)$ for $1 \leqslant k \leqslant j$ and $\pm(a+k)$ with $\mp(a+i+1-k)$ for $1 \leqslant k \leqslant i$), it is self inverse, and thus is both a minimal left coset representative, and a minimal double coset representative. Therefore to show that this gives a complete set of minimal double coset representatives, it suffices to show that we have found all of them. We do this using the same idea as in Lemma 1.5.

Theorem 6.8. *Let $\mu = (a; m) \Vdash n$. Then:*

$$\mathcal{D}_{\mu,\mu} = \{w_{i,j} : 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant \min\{a, m-i\}\},$$

is a complete set of minimal double $\mathfrak{W}_\mu$-$\mathfrak{W}_\mu$ coset representatives in $\mathfrak{W}_n$.

Proof. We show that every $w \in \mathfrak{W}_n$ lies in $\mathfrak{W}_\mu w_{i,j} \mathfrak{W}_\mu$ for some i and j. As each w can be written as tv where $t \in \mathfrak{W}_\mu$ and $v \in \mathcal{R}^n_\mu$, it suffices to show for each $v \in \mathcal{R}^n_\mu$ that $v \in w_{i,j}\mathfrak{W}_\mu$ for some i and j.

We describe the general form of a signed tableau corresponding to these v. The first row can be split into two parts

a_1	$\cdots$	a_k	b_1	$\cdots$	b_j

where $1 \leqslant a_1 < \cdots < a_k \leqslant a$ and $a+1 \leqslant b_1 < \cdots < b_j \leqslant a+m$. Note that $k = a - j$. The second row can be split into four parts,

$-c_1$	$\cdots$	$-c_i$	$-d_1$	$\cdots$	$-d_g$	e_1	$\cdots$	e_h	f_1	$\cdots$	f_r

where first we have the negative elements coming from the second row $\{-c_1, \ldots, -c_i\}$ where $a+1 \leqslant c_i < c_{i-1} < \cdots < c_1 \leqslant a+m$, then negative elements from the top row: $\{-d_1, \ldots, -d_g\}$ where $1 \leqslant d_g < d_{g-1} < \cdots < d_1 \leqslant a$. Following this we have positive elements from the top row: $\{e_1, \ldots, e_h\}$ with $1 \leqslant e_1 < \cdots < e_h \leqslant a$ and finally positive elements originating in the bottom row: $\{f_1, \ldots, f_r\}$ with $a+1 \leqslant f_1 < \cdots < f_r \leqslant a+m$. Note that we must have $g+h=j$, and thus $r = m-i-j$. We now claim that this lies in the left coset created by the element $w_{i,j}$.

Define $w_1 \in \mathfrak{W}_\mu$ by $d_x \mapsto -d_x$. This lies in $\mathfrak{W}_\mu$ as we can map any of the first a numbers to their minus signs.

Define $w_2 \in \mathfrak{W}_\mu$ by:

$$\begin{aligned}
\pm a_x &\mapsto \pm x, \\
\pm b_x &\mapsto \pm(a+i+x), \\
\pm c_x &\mapsto \pm(a+i+1-x), \\
\pm d_x &\mapsto \pm(a-j+x), \\
\pm e_x &\mapsto \pm(a-h+x), \\
\pm f_x &\mapsto \pm(a+i+j+x).
\end{aligned}$$

By definition, $w_2 \in \mathfrak{W}_\mu$: if $|k| \leqslant a$, then $|(k)w| \leqslant a$, and if $|k| > a$, then $|(k)w| > a$. Furthermore, w_2 preserves signs.

Now we consider the action of w_1 and w_2 on $\mathfrak{s}^\mu v$. First of all the tableau corresponding to vw_1 simply swaps the signs of the negative elements on the bottom row that were less than or equal to a. The tableau corresponding to vw_1w_2 now has the elements $1, \ldots, a-j$ at the beginning of the first row, followed by $a+i+1, \ldots, a+i+j$. In the second row we begin with have $-(a+i), \cdots -(a+1)$ followed by $a-j+1, \ldots, a$ and then $a+i+j+1, \ldots, a+m$. Thus we have that $\mathfrak{s}^\mu vw_1w_2 = \mathfrak{s}^\mu w_{i,j}$, so $vw_1w_2 = w_{i,j}$ and $v = w_{i,j}w_2^{-1}w_1^{-1}$. Therefore $v \in w_{i,j}\mathfrak{W}_\mu$ as needed. □

As a corollary of this description, we can see what happens when we conjugate $\mathfrak{W}_\mu$ by a double coset representative, and then intersect with itself.

Corollary 6.9. *Let* $\mu = (a; m) \Vdash n$ *and* $w_{i,j} \in \mathcal{D}_{\mu,\mu}$. *Then the pointed composition* $\nu_{i,j}$ *defined by:*

$$\mathfrak{W}_{\nu_{i,j}} := \mathfrak{W}_{\mu}^{w_{i,j}} \cap \mathfrak{W}_{\mu},$$

is given by $\nu_{i,j} = (a - j; j, i, j, m - i - j)$.

Thus we can mimic the set-up of Section 3.1, even if we cannot generalise those results.

6.1.3 Right cosets for pseudo-parabolics

Despite the fact that pseudo-parabolic subgroups are not useful for our purposes, we can still generalise the preceding results to give a characterisation of a nice set of right coset representatives for these subgroups. Before doing this, we say a signed tableau $\mathfrak{s}$ is in **parity** with pseudo-composition λ if for each i with $\lambda_i < 0$, then all entries in the i-th row of $\mathfrak{s}$ are positive.

Proposition 6.10. *Let* λ *be a pseudo-composition of* n. *Then a complete set of right coset representatives of* $\mathfrak{A}_\lambda$ *in* $\mathfrak{W}_n$ *is given by:*

$$\overline{\mathcal{R}_\lambda} = \{d(\mathfrak{s}) : \mathfrak{s} \text{ is in parity with } \lambda\}.$$

Proof. This is similar to the proof of Theorem 6.2, except we can now have multiple rows not containing negative numbers, instead of just the first row. □

Note that we have not made any claims of minimality here. As a corollary to (6.1), we have for $w \in \overline{\mathcal{R}_\lambda}$ that $\ell(tw) > \ell(w)$ for all reflections $t \in \mathfrak{W}_\lambda$. However we no longer have the (arguably more useful) minimality property that for any of our right coset representatives w and $t \in \mathfrak{W}_\lambda$ that $\ell(tw) = \ell(t) + \ell(w)$.

6.2 Sign modules in type B

We can define the **sign module** sgn_n for $\mathcal{H}_n$ as a one-dimensional module $\langle\xi\rangle$ where each of the T_i act on ξ via multiplication by -1. This is a $\mathcal{H}_n$-module by the defining relations of $\mathcal{H}_n$. Similarly for a parabolic subalgebra $\mathcal{H}_\lambda$, we can define

$$\mathrm{sgn}_\lambda := \mathrm{sgn}_{\lambda_0} \otimes S^{(1^{\lambda_1})} \otimes \cdots \otimes S^{(1^{\lambda_t})},$$

where $S^{(1^j)}$ is the sign module for $\mathscr{H}_j$ seen previously. In type A, the sign module had the largest possible vertex of any $\mathscr{H}_n$-module. We hope that the sign module for $\mathcal{H}_n$ may behave similarly, and thus give evidence towards a version of the Dipper–Du conjecture for Hecke algebras of type B.

6.2.1 Restricting and inducing

In line with the definition of relative projectivity, to find the vertex of the sign module, we begin by asking when $\mathrm{sgn}_n \mid \mathrm{sgn}_n \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n$ for some $\lambda \Vdash n$. Note that $\mathrm{sgn}_n \cong \mathrm{sgn}_\lambda$ as $\mathcal{H}_\lambda$-modules, so we can induce sgn_λ up to $\mathcal{H}_n$ instead, and then look for a copy of sgn_n. To do this, we will first find a submodule of $\mathrm{sgn}_\lambda \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n$ isomorphic to sgn_n for any λ, show it is unique, and then use this submodule to find the vertex of sgn_n.

Before stating the main result of this section, we need the following definition. Let $\ell_0(w)$ be the number of times t_0 appears in a reduced expression for w. Note that this is well-defined as we can get from any reduced expression to any other reduced expression for w only by using the braid relations, and these preserve the number of occurrences of t_0.

Theorem 6.11. *Let $\lambda \Vdash n$, and for $w \in \mathcal{R}_\lambda$ define $b_w = (-1/Q)^{\ell_0(w)}(-1/q)^{(\ell(w)-\ell_0(w))}$.*

Then if:

$$\beta := \sum_{w \in \mathcal{R}_\lambda} b_w \xi \otimes T_w,$$

we get $\beta T_i = -\beta$ *for all* i*, and thus the one-dimensional* $\mathcal{H}_n$*-submodule of* $\operatorname{sgn}_\lambda \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n$ *generated by* β *is isomorphic to* sgn_n*.*

The $\mathfrak{W}_n$ version of Lemma 1.6 is key to the proof of this theorem. Although the original proof worked in the generality of any Coxeter group, we can now give an elementary proof of this using signed tableaux. This illustrates how signed tableaux fill a similar role for $\mathfrak{W}_n$ to that of tableaux for $\mathfrak{S}_n$.

Lemma 6.12. *Let* $\lambda \Vdash n$*, and* $w \in \mathcal{R}_\lambda$*. If* $wt_i \notin \mathcal{R}_\lambda$*, then* $\ell(wt_i) > \ell(w)$*, and there exists* $t_k \in \mathfrak{W}_\lambda$ *with* $wt_i = t_k w$*.*

Proof. As in the proof of Lemma 1.6, it follows in the same way that if $\ell(wt_i) < \ell(w)$, then $wt_i \in \mathcal{R}_\lambda$. Taking the contrapositive says that if $wt_i \notin \mathcal{R}_\lambda$, then $\ell(wt_i) > \ell(w)$.

We now want to show that if $wt_i \notin \mathcal{R}_\lambda$, there exists some $t_k \in \mathfrak{W}_\lambda$ with $wt_i = t_k w$. Here we use our row standard signed tableaux. As $w \in \mathcal{R}_\lambda$, and $wt_i \notin \mathcal{R}_\lambda$, then $\mathfrak{s}^\lambda w$ is row standard with a positive zero row, while $\mathfrak{s}^\lambda wt_i$ is not. We first consider the case when $i = 0$. Then swapping 1 for -1 after applying w keeps us row standard, so we cannot any longer have a positive zero row. Thus we must have 1 in the first box of the first row, so we have $t_0 \in \mathfrak{W}_\lambda$. Thus this box is unchanged by the effect of w, and we can either swap it for -1 before or after applying w with no difference. That is $wt_0 = t_0 w$ and the lemma is proven.

Next suppose $i > 0$. For $\mathfrak{s}^\lambda wt_i$ not to be row standard, we must be in one of the following settings. Either i and $i+1$ are in the same row, or $-(i+1)$ and $-i$ are in the same row (all other possibilities on $\pm i$ and $\pm(i+1)$ stay row standard after applying t_i). In the first case, suppose i is in the box afforded by k in $\mathfrak{t}^\lambda$. Then $i+1$ must be in the box to the right, i.e. afforded by $k+1$, so swapping k and $k+1$, then applying w is the same as applying w, then swapping i and $i+1$, that is $t_k w = wt_i$. As k and $k+1$ lie in the same row of $\mathfrak{s}^\lambda$, then $t_k \in \mathfrak{W}_\lambda$ as required. Similarly, in the second case, let $-(i+1)$ lie in the box afforded by k, and then again we get $t_k w = wt_i$, and $t_k \in \mathfrak{W}_\lambda$, completing the proof. □

Now to prove that $\beta T_i = -\beta$ for all i, we use Lemma 6.12 to write $\beta = \beta_{i,1} + \beta_{i,2}$ where:

$$\beta_{i,1} = \sum_{w \in \mathcal{R}^n_\lambda, wt_i \in \mathcal{R}^n_\lambda} b_w \xi \otimes T_w, \qquad \beta_{i,2} = \sum_{w \in \mathcal{R}^n_\lambda, wt_i \notin \mathcal{R}^n_\lambda} b_w \xi \otimes T_w.$$

We consider both of these sums separately.

Lemma 6.13. *For any i, $\beta_{i,1} T_i = -\beta_{i,1}$.*

Proof. We will only prove this for $i = 0$, to prove for $i > 0$, just replace Q with q. Let $w \in \mathcal{R}_\lambda$ with $wt_0 \in \mathcal{R}_\lambda$ and $\ell(w) < \ell(wt_0)$. Then:

$$\begin{aligned}(b_w \xi \otimes T_w + b_{wt_0} \xi \otimes T_{wt_0}) T_0 &= b_w \xi \otimes T_{wt_0} + (Q-1) b_{wt_0} \xi \otimes T_{wt_0} + Q b_{wt_0} \xi \otimes T_w, \\ &= Q b_{wt_0} \xi \otimes T_w + (b_w + (Q-1) b_{wt_0}) \xi \otimes T_{wt_0}.\end{aligned}$$

Now by the definition of b_w, we have $b_{wt_0} = (-Q)^{-1} b_w$, i.e. $b_w = -Q b_{wt_0}$, which when combined with the above shows us that

$$(b_w \xi \otimes T_w + b_{wt_0} \xi \otimes T_{wt_0}) T_0 = -(b_w \xi \otimes T_w + b_{wt_0} \xi \otimes T_{wt_0}).$$

As we can split the sum for $\beta_{0,1}$ into terms of this form (corresponding to pairing up w and wt_0), the result follows. □

Lemma 6.14. *For any i, $\beta_{i,2} T_i = -\beta_{i,2}$.*

Proof. Again we show for $i = 0$ only. Let $w \in \mathcal{R}_\lambda$ with $wt_0 \notin \mathcal{R}_\lambda$. Then from before, there exists $t_k \in \mathfrak{W}_\lambda$ with $wt_0 = t_k w$. Thus:

$$b_w\xi \otimes T_w T_0 = b_w\xi \otimes T_{wt_0} = b_w\xi \otimes T_k T_w = b_w\xi T_k \otimes T_w = -b_w\xi \otimes T_w,$$

as $T_{wt_0} = T_k T_w$ by Lemma 6.12 and the fact that w is a minimal right coset representative. Summing again gives the required result. □

Combining these two lemmas gives a proof of Theorem 6.11.

6.2.2 Uniqueness

We next want to see if this is the only copy of sgn_n sitting inside this induced module. If so, then this helps us with our relative projectivity as there will be a unique inclusion map, which coupled with the unique $\mathcal{H}_n$-module homomorphism from $\mathrm{sgn}_\lambda \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n$ to sgn_n (given by $\xi \otimes h \mapsto \xi h$), gives us an easy way of checking if this submodule is a direct summand (and hence is $\mathfrak{W}_\lambda$-projective or not). The following shows that this is true.

Theorem 6.15. *For $\lambda \Vdash n$, there is a unique submodule of $\mathrm{sgn}_\lambda \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n$ which is isomorphic to sgn_n.*

Proof. First suppose $\gamma \in \mathrm{sgn}_\lambda \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n$ generates a copy of sgn_n, i.e. $\gamma T_i = -\gamma$ for each i, and write:

$$\gamma = \sum_{w \in \mathcal{R}_\lambda} c_w \xi \otimes T_w.$$

We will show the following relationship has to hold between the coefficients. If $w, wt_i \in \mathcal{R}_\lambda$ and $\ell(w) > \ell(wt_i)$, then:

$$c_w = \begin{cases} \frac{-c_{wt_i}}{Q} & \text{if } i = 0, \\ \frac{-c_{wt_i}}{q} & \text{if } i > 0. \end{cases}$$

This will prove the result, as when we write a reduced expression for w, we can multiply by the corresponding t_i in reverse order to get a chain of coset representatives with this property ending at the identity element. This means every c_w is determined by the choice of c_1, and thus we get a unique generator of the submodule (up to scalar). To show this relationship consider:

$$\left(\sum_{w\in\mathcal{R}_\lambda} c_w \xi \otimes T_w\right) T_i = -\sum_{w\in\mathcal{R}_\lambda} c_w \xi \otimes T_w, \tag{6.2}$$

and look at the coefficient of $\xi \otimes T_w$ on both sides of (6.2). We do this for $i > 0$, the case when $i = 0$ follows similarly. On the right-hand side of (6.2), the coefficient of $\xi \otimes T_w$ is $-c_w$. On the left-hand side, by Lemma 6.12 either both w and ws_i occur in the sum, or only w does and T_w commutes with T_i to give a minus sign. As the latter only appears when $\ell(wt_i) > \ell(w)$, and we are only considering w where $\ell(w) > \ell(wt_i)$, we are in the prior case. Thus $\xi \otimes T_w$ can only appear in the expression

$$c_w \xi \otimes T_w T_i + c_{wt_i} \xi \otimes T_{wt_i} T_i = c_{wt_i} \xi \otimes T_w + (q-1) c_w \xi \otimes T_w + q c_w \xi \otimes T_{wt_i}.$$

Collecting and comparing the coefficients of $\xi \otimes T_w$, we get that $-c_w = c_w(q-1) + c_{wt_i}$, giving the result after simplifying. □

Hence we've shown there is a unique submodule of $\mathrm{sgn}_\lambda \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n$ which is isomorphic to sgn_n as $\mathcal{H}_n$-modules, and by Theorem 6.11 it is generated by β. This gives a unique (up to scalar) inclusion map $\mathrm{sgn}_n \to \mathrm{sgn}_\lambda \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n$ generated by $\xi \mapsto \beta$. Therefore to see if this is a direct summand, we have to check if the composition of this inclusion with the unique surjection $\psi : \mathrm{sgn}_\lambda \otimes_{\mathcal{H}_\lambda} \mathcal{H}_n \to \mathrm{sgn}_n$ given by $\xi \otimes h \mapsto \xi h$ (for $h \in \mathcal{H}_n$) gives a non-zero multiple of the identity map. That is, we need to see if ψ sends β to a non-zero multiple of ξ.

Lemma 6.16. *Let* $\mathcal{P}_\lambda(x, y) := \sum_{w\in\mathcal{R}_\lambda^n} y^{\ell_0(w)} x^{\ell(w)-\ell_0(w)} \in F[x, y]$. *Then:*

$$(\beta)\psi = \mathcal{P}_\lambda(1/q, 1/Q)\xi.$$

Proof. We show directly:

$$\begin{aligned}
(\beta)\psi &= \left(\sum_{w \in \mathcal{R}_\lambda^n} b_w \xi \otimes T_w \right) \psi, \\
&= \sum_{w \in \mathcal{R}_\lambda} b_w \xi T_w, \\
&= \sum_{w \in \mathcal{R}_\lambda} (-1)^{\ell(w)} b_w \xi, \\
&= \sum_{w \in \mathcal{R}_\lambda} (-1)^{\ell(w)} (-1/Q)^{\ell_0(w)} (-1/q)^{\ell(w)-\ell_0(w)} \xi, \\
&= \sum_{w \in \mathcal{R}_\lambda} (1/Q)^{\ell_0(w)} (1/q)^{\ell(w)-\ell_0(w)} \xi, \\
&= \mathcal{P}_\lambda(1/q, 1/Q)\xi. \qquad \square
\end{aligned}$$

As in Chapter 3, we notice that once again the relative projectivity of the sign module depends on a polynomial evaluated at the reciprocal of our parameters. Furthermore, using the more general definition of Poincaré polynomial from [33]:

$$P_n(x, y) = \sum_{w \in \mathfrak{W}_n} x^{\ell(w)-\ell_0(w)} y^{\ell_0(w)},$$

we get the following corollary:

Corollary 6.17. *Let* $\lambda \vDash n$ *a pointed composition. Then* sgn_n *is relatively* $\mathfrak{W}_\lambda$*-projective if and only if*

$$\frac{P_n(1/q, 1/Q)}{P_\lambda(1/q, 1/Q)} = \mathcal{P}_\lambda(1/q, 1/Q) \neq 0.$$

Thus as in Proposition 3.28, once more relative projectivity of the sign module comes down to zeroes of Poincaré polynomials, which we begin to compute in the next subsection.

6.2.3 Vertices for sign modules

We begin by assuming that $p = 0$ or $p > 0$ and $\mathrm{hcf}(e, p) = 1$.

From [33, Corollary 8], we have that:

$$P_n(x, y) = \prod_{i=1}^{n}(1 + yx^{i-1})(1 + x + \cdots + x^{i-1}) = \left(\prod_{i=1}^{n}(1 + yx^{i-1})\right) P_{\mathfrak{S}_n}(x),$$

where $P_{\mathfrak{S}_n}(x)$ is the Poincaré polynomial for $\mathfrak{S}_n$. Using what we know about the Poincaré polynomial for $\mathfrak{S}_n$ from (3.2), we have the following consequences.

Proposition 6.18. sgn_n *is* $\mathfrak{W}_{n-1}$*-projective if and only if:*

$$(1 + (1/Q)(1/q)^{n-1})(1 + (1/q) + \cdots + (1/q)^{n-1}) \neq 0.$$

Thus sgn_n *is* $\mathfrak{W}_{n-1}$ *projective unless* $e \mid n$*, or if* $e \mid n + s - 1$ *in the case where* $\mathcal{H}_n$ *is* Q*-connected.*

Proof. Dividing $P_n(1/q, 1/Q)$ by $P_{n-1}(1/q, 1/Q)$ gives the above expression. The second factor can only be zero if $e \mid n$, whereas the first factor is zero if and only if $-Q = (1/q)^{n-1}$, i.e. we must be Q-connected. Thus here, if $Q = -q^s$, then we get that $e \mid s + n - 1$. □

Proposition 6.19. sgn_n *is* $\mathfrak{S}_n$*-projective if and only if* $\prod_{i=1}^{n}(1 + (1/Q)(1/q)^{i-1}) \neq 0$*. That is* sgn_n *is* $\mathfrak{S}_n$*-projective if and only if* $\mathcal{H}_n$ *is not* Q*-connected or* $\mathcal{H}_n$ *is* Q*-connected and* $n + s - 1 < e$*.*

Proof. Again we get that expression by dividing through by the Poincaré polynomial for $\mathfrak{S}_n$, before noticing that this can only be non-zero in the cases described above. □

Corollary 6.20. *If* $\mathcal{H}_n$ *is not* Q*-connected, then the vertex of* sgn_n *as a* $\mathcal{H}_n$*-module is the maximal* e*-*p*-parabolic subgroup of* $\mathfrak{S}_n$*. If* $\mathcal{H}_n$ *is* Q*-connected with* $n + s - 1 < e$*, then the vertex is* $\mathfrak{W}_0$*.*

Proof. This follows from the previous proposition, transitivity of vertices, and the previously found vertex for the sign module of $\mathfrak{S}_n$. $\square$

So when computing vertices, we can assume that we are in the Q-connected case, and either $e \mid n$ or $e \mid n+s-1$, as otherwise we can use transitivity of the vertex and consider sgn_{n-1} for $\mathcal{H}_{n-1}$. We will show that for these n, that sgn_n will in fact have vertex $\mathfrak{W}_n$.

Theorem 6.21. *Suppose $Q = -q^s$, and either $e \mid n$ or $e \mid n+s-1$. Then sgn_n has vertex $\mathfrak{W}_n$.*

Proof. We proceed by showing that for any pointed composition $\lambda = (m; n-m)$ for $0 \leqslant m < n$ (so λ corresponds to a maximal parabolic), that P_n/P_λ is still zero when evaluated at $1/Q$ and $1/q$. Note that if we define $R^n_m(x,y) = \prod_{i=m+1}^{n}(1 + yx^{i-1})$, then:

$$(P_n/P_\lambda)(x,y) = R^n_m(x,y)\frac{P_{\mathfrak{S}_n}(x)}{P_{\mathfrak{S}_m}(x)P_{\mathfrak{S}_{n-m}}(x)}.$$

We first consider the case when $e \mid n+s-1$, i.e. $n+s-1 \equiv 0 \mod e$. Here, we have that $(1 + yx^{n-1})$ is a factor of R^n_m, and when evaluated at $1/Q$ and $1/q$ this gives zero, thus $P_n/P_\lambda(1/q, 1/Q) = 0$ as well. Now let us suppose that $n = de$ and $n+s-1 \equiv k \mod e$, for $0 < k < e$. Furthermore we can assume that $n - m < e$, otherwise we have $(1 + yx^{j-1})$ divides R^n_m for some $j \equiv 1 - s \mod e$ and again we get zero.

In this case, we have that $R^n_m(1/q, 1/Q) \neq 0$, so to see if $(P_n/P_\lambda)(1/q, 1/Q)$ is zero or not, we have to consider the zeroes of $P_{\mathfrak{S}_n}$ and $P_{\mathfrak{S}_m} \times P_{\mathfrak{S}_{n-m}}$ at $1/q$, which we have done previously. As $n - m < e$, and $P_{\mathfrak{S}_e}$ is the smallest Poincaré polynomial to have a zero here, $P_{\mathfrak{S}_{n-m}}$ has no zeroes at $1/q$. Thus we are just comparing the zeroes of $P_{\mathfrak{S}_n}$ with $P_{\mathfrak{S}_m}$ at $1/q$.

Let

$$n = a_0 e + a_1 ep + \cdots + a_r ep^r,$$

be the e-p-adic expansion of n. Then from Theorem 3.34, we have that:

$$z(P_{\mathfrak{S}_n}) = a_0 + \sum_{l=1}^{r} a_l((l+1)p^l - lp^{l-1}).$$

First of all assume that $a_0 > 0$, so we have an e-p-adic expansion for m given by $m = (e-(n-m))+(a_0-1)e+a_1ep+\dots a_rep^r$. Theorem 3.34 gives us that $z(P_{\mathfrak{S}_m}) = z(P_{\mathfrak{S}_n})-1$, and thus $P_{\mathfrak{S}_n}/P_{\mathfrak{S}_m}(1/q) = 0$.

Now let's assume that $a_0 = 0$, and suppose that i is the smallest index with $a_i > 0$, so:

$$z(P_{\mathfrak{S}_n}) = \sum_{l=i}^{r} a_l((l+1)p^l - lp^{l-1}).$$

Suppose $k = ep^i - (n-m)$ has e-p-adic expansion $k = b_{-1} + b_0e + \dots b_{i-1}ep^{i-1}$. Then as the sign module for $\mathscr{H}_{ep^i}$ has full vertex, we know that $z(P_{\mathfrak{S}_k}) < z(P_{\mathfrak{S}_{ep^i}})$. Now we can write out an e-p-adic expansion for m as:

$$m = b_{-1} + b_0e + \dots + b_{i-1}ep^{i-1} + (a_i-1)ep^i + a_{i+1}ep^{i+1} + \dots + a_rep^r,$$

allowing us to show $z(P_{\mathfrak{S}_m}) < z(P_{\mathfrak{S}_n})$ and thus $P_{\mathfrak{S}_n}/P_{\mathfrak{S}_m}(1/q) = 0$.

$$\begin{aligned}
z(P_{\mathfrak{S}_m}) &= z(P_{\mathfrak{S}_k}) + (a_i-1)((i+1)p^i - ip^{i-1}) + \left(\sum_{l=i+1}^{r} a_l((l+1)p^l - lp^{l-1})\right), \\
&< z(P_{\mathfrak{S}_{ep^i}}) + (a_i-1)((i+1)p^i - ip^{i-1}) + \left(\sum_{l=i+1}^{r} a_l((l+1)p^l - lp^{l-1})\right), \\
&= (i+1)p^i - ip^{i-1} + (a_i-1)((i+1)p^i - ip^{i-1}) + \left(\sum_{l=i+1}^{r} a_l((l+1)p^l - lp^{l-1})\right), \\
&= a_i((i+1)p^i - ip^{i-1}) + \left(\sum_{l=i+1}^{r} a_l((l+1)p^l - lp^{l-1})\right), \\
&= \sum_{l=i}^{r} a_l((l+1)p^l - lp^{l-1}), \\
&= z(P_{\mathfrak{S}_n}).
\end{aligned}$$

To show that $\mathfrak{W}_n$ is the vertex, suppose we have a vertex $\mathfrak{W}_\tau$ with $\tau \neq (n)$. Then $\mathfrak{W}_\tau$ is contained within some maximal parabolic $\mathfrak{W}_\lambda$, and thus sgn_n is also $\mathfrak{W}_\lambda$-projective by transitivity. As a result $P_n/P_\lambda(1/q, 1/Q) \neq 0$, which gives us the required contradiction. □

When $e = p$ and $q = 1$ by the calculation in Proposition 6.19, sgn_n is relatively $\mathfrak{S}_n$-projective if and only if $\mathcal{H}_n$ is not Q-connected. In this case, similarly, we use our $\mathfrak{S}_n$ knowledge to get $\mathfrak{P}(\mathfrak{S}_n)$ as our vertex. When $\mathcal{H}_n$ is Q-connected, then as $q = 1$ (so $Q = -1$) we have for any maximal parabolic $\mathfrak{W}_\lambda$ corresponding to some $(m; n-m)$, that $P_n/P_\lambda(Q, 1) = R^n_m(Q, 1)\binom{n}{m}$. As $R^n_m(-1, 1) = 0$ for all m, this quotient is always zero, meaning that the vertex of sgn_n as a $\mathcal{H}_n$-module is all of $\mathfrak{W}_n$. One can think about this situation as $\mathcal{H}_n$ is Q-connected with all s sufficing.

We summarise these results in the following theorem.

Theorem 6.22. *The vertex of* sgn_n *as a* $\mathcal{H}_n$*-module is given by:*

- *The maximal* e*-*p*-parabolic subgroup of* $\mathfrak{S}_n$ *if* $\mathcal{H}_n$ *is not* Q*-connected,*
- $\mathfrak{W}_k$ *where* $k \leqslant n$ *is the largest integer such that either* $e \mid k$ *or* $e \mid k + s - 1$*, when* $\mathcal{H}_n$ *is* Q*-connected and* $\mathrm{hcf}(e, p) = 1$.
- $\mathfrak{W}_n$ *when* $\mathcal{H}_n$ *is* Q*-connected and* $e = p$.

Given the role the sign module plays in vertices for type A (i.e. being the $\mathscr{H}_n$-module with the largest possible vertex), we conjecture the following version of the Dipper–Du conjecture for Hecke algebras of type B.

Conjecture 6.23. *If* $\mathcal{H}_n$ *is not* Q*-connected, then the vertices of indecomposable* $\mathcal{H}_n$*-modules are* e*-*p*-parabolic subgroups of* $\mathfrak{S}_n$*. If* $\mathcal{H}_n$ *is* Q*-connected, then vertices of indecomposable* $\mathcal{H}_n$*-modules are of the form* $\mathfrak{W}_k \times \mathfrak{P}$*, where* $\mathfrak{P}$ *is an* e*-*p*-parabolic subgroup of* $\mathfrak{S}_n$*, and if* $\mathrm{hcf}(e, p) = 1$*, either* $e \mid k$ *or* $e \mid k + s - 1$.

LIST OF SYMBOLS

Chapter 1

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Chapter 4

w_i	Element of $\mathcal{R}_\mu$ for $\mu = (n-r+1, r-1)$ which corresponds to the tableau with second row $\{1,\dots,r\} - \{i\}$	98
n_i	Basis element of $\mathcal{N}_{r,n} \otimes_{\mathscr{H}_\mu} \mathscr{H}_n$ corresponding to $w_i \in \mathcal{R}_\mu$	98
$\mathfrak{t}_{\gamma_{r,n}}$	Standard $\gamma_{r,n}$-tableau with $\{1,\dots,n-r+1\}$ in the column	100
$\mathfrak{t}_{i,j}$, $m_{i,j}$	Standard $\gamma_{r,n}$ tableau with $\{1,\dots,r,j\} - \{i\}$ in the first row for $2 \leqslant i \leqslant r$ and $j \geqslant r+1$, and corresponding basis element of $S^{\gamma_{r,n}}$	102
E_i	$1 + T_1 + \cdots + T_1 \dots T_{i-1}$	108
C_i	Shorthand for $C_i(q) = 1 + q + \cdots + q^{i-1}$ as defined in Chapter 3	111

Chapter 5

$\Lambda(m_0\|m_1,n)$	Compositions of n divided into two sections with exactly m_0 and m_1 parts respectively	114
N^λ	Signed permutation module corresponding to $\lambda \in \Lambda(m_0\|m_1,n)$	114
$\lambda d \cap \mu$	Composition given by $\mathfrak{S}_\lambda^d \cap \mathfrak{S}_\mu$ for $\lambda,\mu \in \Lambda(m_0\|m_1,n)$ and $d \in \mathcal{D}_{\lambda,\mu}$	116
$\mathcal{D}^\circ_{\lambda,\mu}$	Subset of $\mathcal{D}_{\lambda,\mu}$ with the property that for all $s_j \in \mathfrak{S}_{\lambda d\cap\mu}$, then for $i = 0,1$ we have $s_j \in \mathfrak{S}_{\mu(i)} \iff s_j^{d^{-1}} \in \mathfrak{S}_{\lambda(i)}$	117
$V(m_0\|m_1)$	$\mathbb{Z}_2$-graded (m_0+m_1)-dimensional vector space over F	120
$\hat{\cdot}$	Parity map on $\{1,\dots,m_0+m_1\}$	120
$I(m_0\|m_1,n)$	Set of n-tuples of integers from $\{1,\dots,m_0+m_1\}$	120
$v_{\mathbf{i}}$	Basis of $V(m_0\|m_1)^{\otimes n}$ indexed by $\mathbf{i} \in I(m_0\|m_1,n)$	120
$\mathbf{i}_\lambda$	Element of $I(m_0\|m_1,n)$ with content λ for $\lambda \in \Lambda(m_0\|m_1,n)$	120
$v_{\lambda d}$	Basis of $V(m_0\|m_1)^{\otimes n}$ indexed by $\lambda \in \Lambda(m_0\|m_1,n)$ and $d \in \mathcal{R}_\lambda$	120

Chapter 6

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